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Information and Intertemporal Choices in Multi-Agent Decision Problems

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Abstract

Psychological evidences of impulsivity and false consensus effect lead results far from rationality. It is shown that impulsivity modifies the discount function of each individual, and false consensus effect increases the degree of consensus in a multi-agent decision problem. Analyzing them together we note that in strategic interactions these two human factors involve choices which change equilibriums expected by rational individuals.

Keywords: Consensus, Intertemporal choice, Decisions

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1.Introduction

In 1937, to compare future alternatives, Samuelson introduced the Discounted Utility Model (DU model), which assumes an exponential delay discount function, with a constant discount rate that implies dynamic consistency and stationary intertemporal preferences. Contrary to this normative economic theory, it has been established that human and animal intertemporal choice behaviors are not rational (i.e., inconsistent). For this reason, recent behavioral decision theory on intertemporal choice has adopted a hyperbolic discount model, in which result preference reversal as time passes (Takahashi, 2009) (Section 2).

Neurobiological and psychological factors have determined individual differences in intertemporal choice and have been explored in recent neuroeconomic and econophysical studies. Takahashi (2007) attempts to dissociate impulsivity and inconsistency in their econophysical studies proposing the *Q-exponential Delay Discount Function*. Other behavioral economists propose Multiple Selves Models attempting to measure the strength of the internal conflict within the decision maker, best known as *Quasi-hyperbolic discount model* first introduced by Laibson (1997) (Section 3).

Thaler and Shefrin (1981), in the field of Multiple Selves Models, consider that the concept of self-control is incorporated in a theory of individual intertemporal choice by modeling the individual as an organization. The individual is treated as if he contained two distinct psyches denoted as *planner* and *doer*. This model can be compared with the principal-agent problem present in any organization, so the individual may adopt many of the same strategies to solve self-control problems in intertemporal choice (Section 4).

In a multi-agent decision context the objective for a group decision is to choose a common decision, among each choice, that is to say an alternative which is judged the best by the majority of the decision makers. So in most strategic decisions, it is important to be able to estimate the characteristics and behavior of others. If the characteristics of other players are unknown, estimating them is a critical task. Moreover, psychological evidence suggests people's own beliefs, values, and habits tend to bias their perceptions of how widely they are shared (false consensus effect). This effect demonstrates an inability of individuals to process information rationally (Section 5).

Therefore when we use the aggregation of the agents' preferences to assess consensus, we obtain a coefficient which includes the false consensus effect that depends on the subjectivity and also increases the degree of consensus. To eliminate this aspect of human judgment vagueness we can use a model defined by ordered weighted averaging (OWA) operators introduced in Yager (1988) (Section 6).

Many decision problems are characterized by interplay between intertemporal considerations and strategic interactions. Two or more agents could have to take a common decision for a future time, in that process they are influenced by false consensus effect and by impulsivity that reveals inconsistency. Finally in order to consider intertemporal choices in a multi-agent decision process needs to study the problem of each agent and the influence of false consensus effect (Section 7). A strategic interaction is mathematical developed with the use of the theory of games, then it is possible to demonstrate the difference of psychological influence between a cooperative interaction (Section 8) and non-cooperative one (Section 9).

2.Intertemporal Discounting

Standard discount model. The standard economic model of discounted utility (DU model) assumes that economic agents make intertemporal choices over consumption profiles $(c_t, \dots, c_T)$ and such preferences can be represented by an intertemporal utility function $U^t(c_t, \dots, c_T)$, which can be described by the following form:

$$U^t(c_t, \dots, c_T) = \sum_{k=0}^{T-t} D(k) u(c_{t+k}) \qquad \text{where} \quad D(k) = \left(\frac{1}{1+\rho}\right)^k$$

So the DU model assumes an exponential temporal discounting function and a constant discount rate (ρ). An important implication of these two features is that a person's intertemporal preferences are time-consistent: if in period t a person prefers c_2 at *t+2* to c_1 at *t+1*, then in period *t+1* she must prefer c_2 at *t+2* to c_1 instantly.

However, several empirical studies, mainly arisen from the field of psychology, have documented various inadequacies of the DU model as a descriptive model of behavior.

The first anomaly found to contradict discounted utility was that, instead of remaining constant over time, observed discount rates appear to decline with time, this reveal decreasing impatience, or *hyperbolic discounting*: a later outcome is discounted less per unit of time than an earlier one (*delay effect*).

Furthermore, other anomalies derive from the fact that, even for a given delay, discount rates vary across different types of intertemporal choices:

- larger outcomes are discounted at a lower rate than smaller outcomes (*magnitude effect*);
- gains are discounted at a higher rate than losses of the same magnitude (*sign effect*);
- increasing sequences of consumption are preferred over decreasing ones even if the total amount is the same (*improving sequence effect*).

Hyperbolic discount model. A hyperbolic discount model can represent the tendency of the individuals to increasingly choose a smaller-sooner reward over a larger-later reward as the delay occurs sooner in time (delay effect).

Many authors proposed different hyperbolic discount functions, in which δ (temporal discount function) increases with the delay to an outcome. In 1992 Loewenstein and Prelec proposed this form:

$$d(t) = \left(\frac{1}{1 + \alpha t}\right)^{\beta/\alpha}$$

where $\beta > 0$ is the degree of discounting and $\alpha > 0$ is the departure from exponential discounting.

A second type of empirical support for hyperbolic discounting comes from experiments on dynamic inconsistency. Several studies report systematic preference reversals between two rewards as the time-distance to these rewards diminishes. A hyperbolic discount model can demonstrate this; in fact, non-exponential time-preference curves can cross (Strotz, 1955/56) and consequently the preference for one future reward over another may change with time.

3. Neuroeconomics: two model to consider impulsivity and inconsistency in intertemporal choice

Behavioral economist have found that there is a number of behavior patterns that violate the rational choice theory (Kahneman et al., 1982; Thaler, 1991); the most important is inconsistent preference, which represent behavior typically seen in psychiatric disorders (alcoholism, drug abuse), but also in more ordinary phenomena (overeating, credit card debt).

Neuroeconomics has found that addicts are more myopic (have large time-discount rates) in comparison to non-addicted populations (Ainslie, 1975; Bickel, et al. 1999), so hyperbolic discounting may explain various human problematic behaviors (Laibson, 1997): loss of self-control, failure in planned abstinence from addictive drugs, etc.

Recently, behavioral neuroeconomic and econophysical studies have proposed two discount models, in order to better describe the neural and behavioral correlates of impulsivity and inconsistency in intertemporal choice.

***Q*-exponential discount model**. Takahashi et al. (2007) have proposed and examined this function for subjective value V(D) of delayed reward:

$$V(D) = \frac{A}{exp_q\left(k_q D\right)} = A/[1 + (1-q)k_q D]^{\frac{1}{1-q}}$$

where D denotes a delay until receipt of a reward, A the value of a reward at D = 0, and k_q a parameter of impulsivity at delay D = 0 (*q*-exponential discount rate) and the *q*-exponential function is defined as:

$$exp_q(x) = \ (1 + (1-q))^{\frac{1}{1-q}}$$

This function can distinctly parametrized impulsivity and inconsistency. If q < 0, the intertemporal choice behavior is more inconsistent than hyperbolic discounting (Ventre and Ventre, 2012).

Quasi-hyperbolic discount model. Behavioral economists have proposed that the inconsistency in intertemporal choice may be attributable to an internal conflict between "multiple selves" within a decision maker. As a consequence, there are (at least) two exponential discounting selves (with two exponential discount rates) in a single human individual; and when delayed rewards are at the distant future (>1 year), the self with a smaller discount rate wins, while delayed rewards approach to the near future (within a year), the self with a larger discount rate wins, resulting in preference reversal over time. This intertemporal choice behavior can be parametrized in a quasi-hyperbolic discount model (also as a β-δ model) (Laibson 1997; O'Donoghue and Rabin, 1999).

For discrete time τ (the unit assumed is one year) it is defined as (Laibson, 1997):

$$F(\tau) = \beta\delta^t \text{ (for } \tau\text{=1,2,3,…)} \quad \text{and} \quad F(0) = 1 \qquad (0 < \beta < \delta < 1).$$

A discount factor between the present and one-time period later (β) is smaller than that between two future time-periods (δ).

In the continuous time, the proposed model is equivalent to the linearly-weighted two-exponential functions (generalized quasi-hyperbolic discounting):

$$V(D) = A[w \exp(-k_1 D) + (1 - w) \exp(-k_2 D)]$$

where w, $0 < w < 1$, is a weighting parameter and k_1 and k_2 are two exponential discount rates ($k_1 < k_2$). Note that the larger exponential discount rate of the two k_2, corresponds to an impulsive self, while the smaller discount rate k_1 corresponds to a patient self (Ventre and Ventre, 2012).

These economists proposed different Multiple Self Models, which often draw analogies between intertemporal choice and a variety of different models of interpersonal strategic interactions.

4. Self-control in intertemporal choices

In many cases a dynamic inconsistent behavior is attributed to the existence of contingent "temptations" that increase impulsivity and induce a deviation from the desirable behavior. What the person knows to be his best long run interest conflict with his short run desires.

Stroz's model. To represent this incoherent purpose, Strotz (1955) proposed two strategies that might be employed by a person who foresees how her preferences will change over time.

The "strategy of pre-commitment": a person can commits to some plan of action. For example, consider a consumer with an initial endowment K_0 of consumer goods which has to be allocated over the finite interval (0, T). At time period *t* he wishes to maximize his utility function:

$$J_0 = \int_0^T \lambda(t-0)U[\bar{c}(t),t]dt \quad \text{subject to} \quad \int_0^T c(t)dt = K_0$$

where $[\bar{c}(t),t]$, is the instantaneous rate of consumption at time period *t*, and $\lambda(t - 0)$ is a discount factor, the value of which depends upon the elapse of time between a past or future date and present. And this implies that the discounted marginal utility of consumption should be the same for all periods. But, at a later date, the consumer may reconsider his consumption plan. The problem then is to maximize

$$J_0 = \int_0^T \lambda(t-\tau)U[c(t),t]dt \quad \text{subject to} \quad \int_\tau^T c(t)dt = K_\tau = K_0 - \int_0^\tau c(t)dt$$

The optimal pattern of consumption will change with changes in τ and if the original plan is altered, the individual is said to display dynamic inconsistency. Strotz showed that individuals will not alter the original plan only if $\lambda(t,\tau)$ is exponential in $|t - \tau|$.

The "strategy of consistent planning": since pre-commitment is not always a feasible solution to the problem of intertemporal conflict, an individual may adopt a different strategy: take into account future changes in the utility function and reject any plan that he will not follow through. His problem is then to find the best plan among those he will actually follow.

Thaler and Shefrin's model. In the setting of Multiple Selves Models, to control impulsivity, Thaler and Shefrin (1981) proposed a "planner-doer" model which draws upon principal-agent theory. They treat an individual as if he contained two distinct psyches: one *planner*, which pursue longer-run results, and multiple *doers*, which are concerned only with short-term satisfactions, so they care only about their own immediate gratification (and have no affinity for future or past doers).

For example, consider an individual with a fixed income stream $y = [y_1, y_2, \dots, y_T]$,where

$$\sum_t y_t = Y$$

which has to be allocated over the finite interval (0, T). The planner would choose a consumption plan to maximize his utility function

$$V(Z_1, Z_2, \dots, Z_T) \quad \text{subject to} \quad \sum_{t=1}^{t} c_t \leq Y$$

in which such Z_t is a function of utility of level consumption in t (c_t).

On the other hand, an unrestrained doer 1 would borrow $Y - y_1$ on the capital market and therefore choose $c_1 = Y$; the resulting consequence is naturally $c_2 = c_3 = \cdots = c_T = 0$. Such action would suggest a complete absence of psychic integration.

Then the model focuses on the strategies employed by the planner to control the behavior of the doers, and it proposes two instruments he can use. (a) He can impose *rules* on the doers' behavior, which operate by altering the constraints imposed on any given doer. Pure rules, like pre-commitment, can be a very effective self-control strategy because they eliminate all choice. The advantage of these strategies is that once in place they require little or no self-enforcement. However, they may be unavailable or too expensive. (b) He can use *discretion* accompanied by some method of altering the incentives or rewards to the doer without any self-imposed constraints. One planner can alter the doer's utility function directly introducing a modification parameter $\theta = \theta_1, \theta_2, \dots, \theta_T$. Z is assumed to be a function of two arguments, c_t and θ_T. If $\theta_T = 0$, then the doer is completely unrestrained. As θ_t increases, both Z and $(\delta Zt)/(\delta ct)$ are reduced. θ might be thought of as a guilt parameter. The higher is θ_t , the more guilt the doer feels for any level of c_t (Ventre and Ventre, 2012).

In conclusion, the essential insight that Multi Selves Model capture is that, much like cooperation in a social dilemma, self-control often requires the cooperation of a series of temporally situated selves. When one "self" defects by opting for immediate gratification, the consequence can be a kind of unraveling or "falling off the wagon" whereby subsequent selves follow the precedent (Frederick, Loewenstein, and O'Donoghue, 2002).

5. Multi-agent decision problem: consensus and false consensus effect

In a multi-agent decision problem an individual needs to take his intertemporal choice considering others' preferences, to the purpose of achieving a consensus on a common decision. Group decision problems, indeed, consist in finding the best alternative(s) from a set of feasible alternatives $A = \{a_1, \dots, a_n\}$ according to the preferences provided by a group of agents $E = \{e_1, \dots, e_m\}$. The objective is to obtain the maximum degree of agreement among the agents' overall performance judgements on the alternatives.

Once the alternatives have been evaluated, the main problem is to compare agents' judgements to verify the consensus among them; in the case of unanimous consensus, the evaluation process ends with the selection of the best alternative(s). However, in real situations humans rarely come to a unanimous agreement: this has led to evaluate not only crisp degrees of consensus (degree 1 for fully and unanimous agreement) but also intermediate degrees between 0 and 1 corresponding to partial agreement among all agents. Furthermore, full consensus (degree = 1) can be considered not necessarily as a result of unanimous agreement, but it can be obtained ever in the case of agreement among a fuzzy majority of agents (Fedrizzi M, Kacprzyk J, Nurmi H., 1992/1993).

The judgements of each agent are frequently based, in part, on intuition or subjective beliefs, rather than detailed data on the preferences of the people being predicted. Such intuitive judgements become more pervasive judgements when people lack necessary data to base their judgements.

Research in others areas of social judgement has revealed that people are egocentric: they judge others in the same way that they judge themselves. Consequently, as pointed out in several experiments, each decision maker overestimates his own opinion. Social psychology has founded that people with a certain preference tend to make higher judgements of the popularity of that preference in others, compared to the judgements of those with different preferences. This empirical result has been termed the *false consensus effect* (Ross et al., 1977; Mullen, et al., 1985). It states that individuals overestimate the number of the people who possess the same attributes as they do. People often believe that others are more like themselves than they really are. Thus, their predictions about others' beliefs or behaviors, based on casual observation, are very likely to err in the direction of their own beliefs or behavior. For example, college students who preferred brown bread estimated that over 50% of all other college students preferred brown bread, while white-bread eaters estimated that 37% showed brown bread preference (Ross et al., 1977).

As the consequence, in multi-agent decision problem we often have to deal with different opinions, different importance of criteria and agents, who are not fully impartial objective. In this sense, the false consensus effect produces partial objectivity and incomplete impartiality, which perturbs the agreements over the evaluation.

6. Assessment of consensus and false consensus effect

In the literature, different methods to compute a degree of a consensus in fuzzy environments have been defined, and some approaches have been proposed to measure consensus in the context of fuzzy preference relations (Fedrizzi, Kacprzyk, Nurmi, 1992-1993). But, as we have seen, the false consensus effect can lead to an absence of objectivity in the evaluation process. Indeed, there may be cases where an agent would not be able to objectively express any kind of preference degree between two or more of the available options caused by the presence of the false consensus effect.

Then just a numerical indication seems not to be sufficient to synthesize the degree of consensus of agents. To put in evidence the lack of objectivity and, consequently, synthesized judgements, a description of the individual opinion should incorporate both the true knowledge generated agent opinion and the subjective component that produces false consensus outputs. The opinion of each agent is decomposed into two components: a vector, made of the ranking of the alternatives, built by means of a classical procedure, e.g., a hierarchical procedure, and a fuzzy component that represents the contribution of the false consensus effect, which we assume to be fuzzy in nature. This allows us to consider aggregation operators, such as OWA operators, useful when synthesis among fuzzy variables is to be built (Squillante and Ventre, 2010).

The formal model considers the set N of decision makers, the set A of the alternatives, and the set C of the criteria. Let any decision maker $I \in N$ be able to assess the relevance of each criterion. Precisely, for every i, a function

$$h_i : C \to [0,1] \quad \text{with} \quad \sum_{c \in C} h_i(c) = 1$$

denoting the evaluation or weight that the decision maker assigns to the criterion c, is defined.

Furthermore, the function

$$g_i: A \times C \to [0,1]$$

is defined, such that $g_i(a,c)$ is the value of the alternative a with respect to the criterion c, in the perspective of i.

Let n, p,and m denote the (positive integer) numbers of the elements of the sets N, C, and A, respectively. The value $h_i(c)_{c \in C}$ denotes the evaluation of the p-tuple of the criteria by the decision maker i and the value $g_i(c,a)_{c \in C, a \in A}$ denotes the matrix $p \times m$ whose elements are the evaluations, made by i, of the alternatives with respect to each criterion in C. Function: $A \to [0,1]$, defined by

$$(f_i(a))_{a \in A} = h_i(c)_{c \in C} \cdot g_i(c,a)_{c \in C, a \in A}$$

is the evaluation, made by i, of the alternative $a \in A$.

An Euclidean metric that acts between couples of decision makers i and j, i.e., between individual rankings of alternatives, is defined by

$$d(f_i, f_j) = \sqrt{\frac{1}{|A|} \sum_{a \in A} (f_i(a) - f_j(a))^2}$$

If the functions h_i, g_i range in [0, 1], then also $0 \leq d(f_i, f_j) \leq 1$.

If we set $d^* = max\{d(f_i, f_j) | i, j \in N\}$, then a degree of consensus δ^* can be defined as the complement to one of the maximum distance between two positions of the agents:

$$\delta^* = 1 - \delta^* = 1 - max\{d(f_i, f_j) | i, j \in N\}.$$

Now to identify the portion of the false consensus effect internal to the consensus-reaching process we have to consider a vector that represents the *components of the consensus* = $p(a)P + q(a)Q$. This polynomial representation of the measure of the effect is composed by a numeric component

$p(a)P$ that contains all quantitative information available derived from the consensus-reaching process, and $q(a)Q$ that reflects the false consensus effect.

Then the measure of the effect is:

$$q(a) = \frac{1}{N(d^*)^2}\sum_{i=1}^{N}(f_i - f_j)^2$$

$$\text{with} \quad 0 \leq q(a) \leq 1, \forall i, j \in N\ .$$

This component can be estimate with OWA operators (a large class of decision support tools for providing heuristic solution to situations where several trade-offs should be taken into consideration). In Yager (1988) is introduced an approach for multiple criteria aggregation, based on ordered weighted averaging (OWA) operators. By ranking the alternatives, the operators provide an enhanced methodology for evaluating actions on a qualitative basis.

7.False consensus effect and intertemporal choice in a multi-agent context

Many decisions are made in condition of strategic interaction, i.e. situations in which consequences of our choices depend on decisions of others interactive. For example, in bidding in auctions or in a bargaining the choice depends not only on one's evaluation of the good but also on the evaluation of other individuals.

Mathematical instrument used to describe these situations is the theory of games. Indeed, a strategic game is considered as an interactive situation where two or more rivals interact and try to obtain an advantage from this interdependence.

In this perspective, the theory of games can be considered as a tool for understanding and forecasting the decision-making processes; according to this theory the outcome of the game coincides with the decision of equilibrium, it occurs when each agent adopts the *best strategy*, which is the one selected on the basis of rational choice.

Rationality is one of the most important assumptions made in theory of games. It implies that every player always maximizes his utility, thus being able to perfectly calculate the probabilistic result of every action. So they have consistent preferences on the final outcome of the decision-making process and their aim is to maximize these preferences.

However, first of all we have showed that intertemporal choices of each individual are influenced by impulsivity and show inconsistency; furthermore we have seen that in a group decision problem each individual tends to overestimate the extent to which other people share one's beliefs, attitudes and behaviors. This means that in a strategic interaction people are not rationales; their choices are not solely a function of the objective response but of their subjective structure. The consequence is that in a strategic interaction, the equilibrium of the decision is the result of an internal process (which not reveals rationality).

Rational choice and equilibrium decision coincide only if decision makers (alone or in group) succeed to fight loss of self-control and to keep out false consensus effect. So these psychological evidences involve new equilibriums in strategic games, which are not justified with rational behaviors.

The consequences are different according to the nature of the interactions; indeed, in theory of games the basic classification of interactions is between non-cooperative games and cooperative ones, consequently we have non-cooperative decision problems and cooperative decision problems too. The first group summarizes the dynamics by which each person pursues his own interests without regard to gains / losses of others. In the second group, subjects form a coalition and assume mutual commitments to share the surplus generated by cooperation.

Psychological aspects of impulsivity and false consensus effect influence in different way these two kinds of interactions. A way to analyze these effects is to identify the portion of the false consensus effect in the equilibrium point (Section 6), and to consider influence of doers in each individual choice (Thaler and Shefrin, 1981).

8. Cooperative decision problems

In a cooperative game a group of players (coalitions) may enforce cooperative behavior; hence the game is a competition between coalitions of players, rather than between individual players.

An example is a coordination game, when players choose the strategies by a consensus decision-making process. Indeed, coordination games are a class of games with multiple pure strategy Nash equilibria in which players choose the same or corresponding strategies. The classic example of coordination game is the “battle-of-the sexes”, where an engaged couple must choose what to do in the evening: the man prefers to attend a baseball game and the women prefers to attend an opera. In term of utility the payoff for each strategy is:

		Man	
		Opera (O)	Baseball (B)
Woman	Opera (O)	3, 1	0, 0
	Baseball (B)	0, 0	1, 3

In this example there are multiple outcomes that are equilibriums: (B,B) and (O,O). However both players would rather do something together than go to separate events, so no single individual has an incentive to deviate if others are conforming to an outcome: the man would attend the opera if he thinks the woman will be there even though he prefers the other equilibrium outcome in which both attend the baseball game.

One of the most commonly suggested criteria for the analysis of games with multiple equilibria is to select the one with the highest payoffs for all, if such a “Paretodominant” outcome exists.

In this context, a consensus decision-making process can be considered as an instrument to choose the best strategy in a coordination game. The final decision is often not the first preference of each individual in the group and they may not even like the final result. But it is a decision to which they all consent because it is the best for the group.

If we follow the Thaler and Shefrin’s model, we can analyze choices in a cooperative game in this way: at period-one the planner of each agent states his preference, which is the *best strategy* because the planner wants maximize his utility function; indeed planners are rational part of each player.

However, the period-one doers of each agent want obtain an immediate gratification, so they drive each agent to act differently from rational program of own planner, thinking that the others make the same by effect of false consensus. But each agent have a different utility function, so each one will select a different choice with degree = 1, and this make impossible the aggregation of the preferences with OWA operators to obtain a common decision. In fact according the model to measure consensus proposed in Section

6 a certain consensus degree $\tilde{\delta} \in (0,1]$ is required in advance, consensus is reached if the constraint $\delta^* \geq \tilde{\delta}$ is satisfied.

Nevertheless, in cooperative decision problem, the influence of doers can be avoid, indeed agents can enforce contracts through parties at period-one, which eliminates the problem of loss of self-control, because it eliminates all choices.

As a consequence the consensus is obtained with the aggregation of preferences of each planner. The planners are rationales, so the final common choice is the best strategy according to the theory of games. However, the result of this aggregation includes a part of the coefficient called the false consensus effect that depends on the subjectivity and also increases the degree of the opinions (Squillante and Ventre, 2010): with cooperation the group utility is higher than real utility of each one derived from strategy chosen. So they have to extract from the degree of consensus the measure of false consensus effect according the model analyzed in Section 6.

This means that at the best solution corresponds an improvement in terms of utility that is overrated as a result of the false consensus.

Then in a cooperative decision problem the influence of false consensus effect is present at period-one, while the loss of self-control of each agent is fought by the imposition of a rule (Thaler and Shefrin, 1981). The rationality of the equilibrium choice of the game is saved by the possibility of making an arrangement among agents, which represents a pure rule to control the behavior of the doers and maintain self-control at later time (Section 4); nevertheless the final decision has a higher consensus degree because it is influenced by the false consensus effect. However this effect acts only on planners, so we can eliminate it in planners' utility functions: the false consensus effect directly influence the discount function of each agent.

For example, consider two person who live together and put in common a part of their monthly income to do the common expenses, this part of each salary form a fixed income stream $y = [y_1, y_2, \dots, y_T]$, where

$$\sum_t y_t = Y$$

which has to be allocated over the finite interval (0, T).

The two agents must agree on how to spend this money. We can eliminate the influence of the doers because both are obliged to deposit in common fund a fixed amount of money, and also because they made the plan of consumption of common expenses at period-one, so they can not use this money for other purpose. In this way we can take into account only each planner and get the consensus about the common choice through the process of evaluation of a multi-agent decision problem.

The planner's preferences are represented by a utility function $V(Z_1, Z_2, \dots, Z_T)$, in which such Z_t is a function of utility of level consumption in t (c_t). Then the planner would choose a consumption common plan to maximize $V(Z_1, Z_2, \dots, Z_T)$, subject to their fixed income stream

$$\sum_{t=1}^{t} c_t \leq Y.$$

The consumption plan chosen by each agent will provide different degrees of preference for different types of consumption according to their preferences, then to reach an agreement it simply suffices aggregate the preferences of each planner (Section 6).

However, the consensual choice obtained will have a greater degree due to the false consensus effect established in the preferences of each planner.

So the utility function of each planner may be released in advance of the false consensus effect by reducing the degree of preference of favorite choices.

The function to maximize will always be $V(Z_1, Z_2, \dots, Z_T)$, but each Z will represent a degree of utility lower for each type of preferred consume.

This example can be analyzed according to the theory of repeated games. The choice of "what we consume with the common fund" can be seen as a choice that is repeated over time. The repeated games study the repetition of the strategic choices over time.

According to the theory of games, if in a repeated game, finitely or infinitely, there are multiple Nash equilibria, then there are many subgame perfect equilibria. Some of these involve the play of strategies that are collectively more profitable for players than the one-shot game Nash equilibria. The economic reasoning that supports this balance is as follows: the players will agree to maximize their utility in the first period, while the actions to be taken in the second period are of two types: a punishment if the rival does not maintain the agreement and a prize (the best Nash equilibrium of the single game) if it is fair. In this case the strategies take into account the history of the game, which makes possible the cooperation. When the agents interact only once, they often have an incentive to deviate from cooperation, but in a repeated interaction, any mutually beneficial outcome can be sustained in an equilibrium. The deviation is not convenient in the long run, since players can make retaliation and this operates especially when the game is repeated infinitely.

According to our theory, the end result is the same: repeating a cooperative game make possible to obtain a common result which is not achievable in a one-period situation (see the battle of the sexes). However, this happens not because the rational player has more convenience to cooperate in the long run, but because through the agreements made at first period they eliminate any temptation to deviate, which is then made impossible. It is necessary set the impossibility to divert, otherwise, in later games, the doer of each player push his agent to deviate, also believing that the others will do the same as a result of the false consensus.

9.Non-cooperative decision problems

In non-cooperative games, also called competitive games, players can not stipulate binding agreements, regardless of their goals. So in a non-cooperative decision problem each agent makes decisions independently, without collaboration or communication with any of the others (J. Nash, 1951), an example is the daily trading on the stock exchange. In this category the solution is given by Nash Equilibrium.

Consequently in this kind of interaction is not possible to implement some pre-commitment to control the doer's actions, as a consequence is not possible recognize the best choice on a rational base.

If we analyze a non-cooperative multi-agent decision problem like the traditional *prisoner's dilemma,* on one temporal interval and with only two alternatives, we see that the agents achieve common decision, and this is the best strategy, because each doer wants obtain the higher advantage which is the same and, for the false consensus effect, each one thinks that other make the same. The doer of each prisoner will choose the strategy of "do not confess".

In the traditional version of the game, the police arrest two suspects (A and B) and interrogate them in separate rooms. Each can either confess, thereby implicating the other, or keep silent. In terms of years in prison the payoff for each strategy are these:

		Agent A	
		Confess (C)	Do not confess (NC)
Agent B	Confess (C)	5, 5	0, 10
	Do not confess (NC)	10, 0	1, 1

According to the theory of games, given this set of payoffs, there is a strong tendency for each to confess. If prisoner A remains silent, prisoner B is better off confessing (because 0 is better than 1 year in jail). However, B is also better off confessing if A confesses (because 5 years is better than 10). Hence, B will tend to confess regardless of what A will do; and by an identical argument, A will also tend to confess.

This line of reasoning implies two rational players with consistent preferences. Actually, when each player has to choose the best strategy every doer drives his agent to make decision that leads him a greater advantage, believing that the other will do the same due to the effect of the false consensus.

Consequently, the decision made by each leads to optimal decision in terms of Pareto, because both have the same utility function and both doers choose the only action that is the best strategy. This creates the paradoxical situation that rational players lead to a poorer outcome than irrational players.

However, it is just a coincidence that the two players have achieved a common strategy.

In other types of non-cooperative problems this can not happen, with the result that you will never achieve a joint decision without a prior agreement.

Consider, for example, a multi-agent decision problem in which the agents set to save money to realize a common purchase. Even agent has a fixed income, Y_A and Y_B, and a nonnegative level of saving, S_A and S_B.

As in cooperative games, the planner of each agent choose the best strategy which maximize his function utility of saving (thinking for future), but the doer of each agent want obtain the highest advantage now, so it would consume Y and therefore choose $= 0$, with a degree =1. Indeed, the doers are impulsives, each one assigns weight=1 at one preference and weight=0 at all the others, thinking that everybody will make in the same way for effect of false consensus. In this case, as we see in cooperative game, is not possible to aggregate the preferences to obtain a common decision.

The plan made in advance by group of agent (to realize a common purchase) is not feasible if they don't set some rule or some method to alter the incentives for the doers.

This type of problem can be represented in the following way:

		Agent A	
		Save (S)	Do not save (NS)
Agent B	Save (S)	10, 10	5, 5
	Do not save (NS)	5, 5	-10, -10

where the payoff represent the utility of each agent for each strategy.

According rational choice we note the Nash equilibrium coincides with the best strategy (S,S). However false consensus effect and impulsivity lead each agent to the worst equilibrium, because utility functions of the agents are different among them (each agent prefers consumptions to savings). This causes the lack of consensus on a common decision.

In conclusion, in a non-cooperative multi-agent decision problem, there are two situations: 1) the doers of each agent have the same preference and they will reach a common decision that is given by the unanimous choice, 2) the doers have different preferences and do not assign any weight to the other preferences, so it is not possible to aggregate the preferences. Then the influence of doers don't affect if their choices are unanimous, and in this case the final decision will be also the best decision in term of Pareto, but if this does not happen is impossible to achieve a common strategy without arresting impulsivity, and when the number of agents increases unanimity becomes increasingly difficult to obtain.

Analyzing this type of decision problem in long time, we note that the influence of the psychological aspects leads to the same conclusion of the theory of games, namely the impossibility of obtaining cooperation over time, but in a different way: according to the theory of games because the dominant strategy prevails, according to our analysis because the doers will divert to their preferences.

Indeed, according to the theory of games a repeated game with a unique Nash equilibrium has the same subgame perfect equilibrium outcome, because in the last stage the strategy which will be played by each player does not depend on the history of the game, that is the strategies of the last stage of game are history independent: every player in last round probably choose the equilibrium of dominant strategy so he betray (playing the last time is like playing a single time). Thus, in finitely repeated games, if you fail to cooperate in the last game you can not do in any other round.

However analyzing the situation according our theory we obtain the same conclusion but for different causes. We can reconsider the example of the two agents who save for common expenses, and continue the game for several years: in the same way, in subsequent periods, the doer of each agent will push to consume all what he has saved.

If we consider two periods, at the first the payoffs are the same, in the second they are the sum:

		Agent A	
		Save (S)	Do not save (NS)
Agent B	Save (S)	20, 20	10, 10
	Do not save (NS)	10, 10	-20, -20

The doers of the second period will want to consume everything and choose $S_2 = 0$, with the result that is not possible achieve the plan and the equilibrium is the worst solution (NS,NS). The planners will establish a consumption plan by discounting the expected future payoff and so smearing the savings over the years, but in every period the doers will deviate their agents for the temptation to consume everything today and save tomorrow, this impulsiveness is psychologically justified by the effect of the false consensus.

In conclusion, even in the long time psychological influence of the doers can not lead to cooperation and to achievement of rational results.

We can affirm that in a non-cooperative decision problem is only a chance obtaining a common decision.

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Valuation of Barrier Options with the Binomial Pricing Model

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Abstract

Derivatives are products of different nature which are becoming increasingly common in financial markets. In certain cases, determining the assessment criteria can sometimes be a difficult task. Specifically, this paper focuses on one type of exotic option: the barrier option. This option has to satisfy some conceptual conditions which are specified at the time of its purchase and define its characteristics. In order to analyze this type of option more deeply, in this paper we choose a specific one, the so-called barrier option cap, whose value is going to be derived by the binomial pricing model.

Keywords: barrier options; exotic options; barrier option cap; binomial model.

2010 AMS subject classification: 62P05, 91G20, 97M30, 05A10.

1. Introduction

In the last years, the interest rate has reached historic low levels. As a consequence, the investment habits have changed and investors are interested in new and more profitable products. For this reason, derivatives have been selected as an alternative to traditional investment products. These are financial products whose price does not only vary according to parameters such as risk, but also depends on the market price of another asset, called the underlying asset (stock, foreign exchange, stock market index, etc.) (Carr, 1998). The option holder is committed to the evolution, up or down, of a certain underlying asset in the securities market. There are different products: options, warrant, futures, etc. The main difference is the way in which the price is derived and the nature of the transaction to which this instrument gives rise, that is to say, how and when the delivery of the asset takes place.

A derivative is a forward contract whose characteristics are established at the agreement moment, whilst the money exchange occurs at a future moment. Derivatives, like financial options, are products with higher profits since the premium is lower than the corresponding to the underlying asset, whereby the results can be multiplied, either in the positive or negative sense, in relation to the premium. Hence they are highly risky products. In order to make them more attractive, exotics options, specifically the barrier option, arise in order to allow taking more control of the operational risk by employing covertures.

Barrier options are very popular but there is a scarce economic research given its novelty and complexity (Rich, 1994). We start with an analysis of the product from a theoretical point of view and by studying the analogies and differences of this type of exotic options with financial standard options (plain vanilla).

They are options whose exercise will depend on whether the underlying asset reaches a pre-set barrier level during a certain period of time. If this occurs, the conditional option becomes a simple call or put option (knock-in options) or, on the other hand, it may cease to exist from the moment that the barrier level is reached (knock-out options).

Once this financial product has been introduced as an alternative to traditional investment products, we present this paper organization. In Section 2, barrier options are described and studied from a mathematical point of view. In Section 3, we focus on the barrier option cap which is a specific barrier option. Then, in Section 4, the financial analysis to derive the value of this type of financial option is presented. Finally, Section 5 summarizes and concludes.

2. Barrier options

Barrier options are derivatives which can be canceled or activated depending on the prices reached by the corresponding underlying asset (Soltes and Rusnakova, 2013). They are available for a predetermined period of time if, during this period, the underlying asset reaches a certain level, the conditional option is converted from that moment into a simple option (knock-in options) or, in another case, if the option already exists, it is canceled from that moment (knock-out options).
These options are similar to a call or a put option with a specified barrier (called *B*). It ensures that the option has a fixed value (called *L*) if the maximum or minimum of the underlying asset price (called *S*) do not touch the barrier (Rubinstein and Reiner, 1991). Below the different types of barrier options are explained.

2.1. Knock-in options

These options only arise if the underlying asset price reaches a certain level, known as barrier level (Fernández, 1996). They can be classified into two types:

1) Up-and-in options: The right to exercise the option is activated when the underlying asset price is above a certain level (B) during the option's life. Its price at maturity, if the strike price is denoted by K, is:

-Call up-and-in option
$$\begin{cases} \max(0; S_t - K), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \max(S, S_1, \ldots, S_t) < B \end{cases}$$

-Put up-and-in option
$$\begin{cases} \max(0; K - S_t), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \max(S, S_1, \ldots, S_t) < B \end{cases}$$

2) Down-and-in options: The right to exercise the option at maturity appears if the underlying asset price falls below the pre-determined barrier (*B*). In this way, we can distinguish between:

-Call down-and-in option
$$\begin{cases} \max(0; S_t - K), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

-Put down-and-in option
$$\begin{cases} \max(0; K - S_t), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ 0, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

2.2. Knock-out options

These options only may be exercised if the underlying asset price does not reach the barrier, that is to say, the right to be exercised disappears if the underlying asset price intersects the barrier at any time of the option's life; at this moment, the option acquires a fix price (*L*) (Fernández, 1996). They can be classified into two types:

1) Up-and-out options: They only make sense if the underlying asset price is above a pre-determined value during the option's life:

-Call up-and-out option
$$\begin{cases} \max(0; S_t - K), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \max(S, S_1, \ldots, S_t) > B \end{cases}$$

-Put up-and-out option
$$\begin{cases} \max(0; K - S_t), & \text{if } \max(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \max(S, S_1, \ldots, S_t) > B \end{cases}$$

2) Down-and-out options: The right to exercise the option disappears if the underlying asset price is below the level established by the barrier.

-Call down-and-out option
$$\begin{cases} \max(0; S_t - K), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \min(S, S_1, \ldots, S_t) < B \end{cases}$$

-Put down-and-out option
$$\begin{cases} \max(0; K - S_t), & \text{if } \min(S, S_1, \ldots, S_t) = B \\ L, & \text{if } \min(S, S_1, \ldots, S_t) > B \end{cases}$$

There is another type of option called “double barrier option” which disappears if the underlying asset does not stay within a certain interval (Kunitomo and Ikeda, 1992 and Fernández and Somalo, 2006).

The main advantage of using barrier options is its lower price, compared to a vanilla equivalent option. The saving of using barrier options versus simple options depends on:

-The proximity of the barrier to the current price of the underlying asset (with greater proximity to savings of the “out” type) and, conversely, to greater distance (in the “in” type).

-The option’s life (the longer the time to maturity, the greater the probability of reaching the barrier and therefore the greater the savings in the “out” and inversely in the “in” type).

-The greater the volatility (greater probability of touching the barrier and, therefore, greater savings in the “out” type, and inversely in the “in” type).

Barrier options can be very useful in hedging commodities providing protection at a lower price than traditional options for coverage of risks (Crespo, 2001).

3. Barrier option cap

In this section, we are going to study a specific barrier option, the so-called barrier option cap. It guarantees a certain profitability called “option level” at maturity, i.e. it guarantees a final sale price independently of the share price, with the only condition that during the option’s life the underlying asset does not reach a certain lower level, called the “barrier”. This product was issue by PNB Paribas Bank in Spain with the name “bonus cap”. It has been marketed for a short time since they were first issued in Spain on June 16, 2010.

The barrier and option level are given by the issuer bank and they are known from the beginning, that is to say, from the issue date and during the barrier option cap life.
In case that the underlying asset reaches the barrier, this does not imply that the option disappears but simply loses the guaranteed price at maturity (the "option level"), for which the barrier option cap will continue being traded with normality being able to give profits if the share has an upward tendency which allows the holder to sell above the level of purchase.
The barrier option cap profit is limited to the "option level" so it should be clarified that in case that the underlying asset quotes above the "option level" at maturity, the holder will receive at most the profitability previously fixed corresponding to the "option level". On the other hand, if the barrier option cap reaches the "option level" before maturity, the holder can get rid of his/her investment since it does not make sense to keep the investment when the highest allowed profitability has been already achieved. In this way, we would have achieved the maximum expected return without waiting to maturity.
Therefore, it can be said that the barrier option cap limits the profits which can be obtained, in exchange for ensuring a known profit provided that the underlying asset price is higher than the barrier value.

3.1. Analogies and differences of barrier options cap with other derivatives

-A **future** contract is an agreement whereby two persons (physical or legal) undertake to sell and to buy, respectively, an asset, called the underlying asset, at a price and at a future date according to the conditions fixed in advance by both parties.
However, the holder of a barrier option cap will never be the owner of the underlying asset; he/she will receive the cash corresponding to the price of the underlying asset.
The future is a compromise, whilst the purchase of a barrier option cap is an option to buy.
-An **option** is an agreement granting the buyer, in return for payment of a price (premium), the right (not the obligation) to buy or sell an underlying asset at a price (strike price) and at a future date, in accordance with the conditions set forth in advance by both parties.
As for the sale of the barrier option cap, it is a liquid product and can be sold at any time, so we could say that it keeps more similarities with the American options since it is not necessary to wait for the expiration to exercise the sale.

Barrier options cap present the following differences with respect to other derivative products which make them a new banking product:

-They present a well-known and bounded return from the moment of contracting, as long as the underlying asset does not reach the barrier during the life of the barrier option cap.
-The underlying asset is not acquired at any time.
-At maturity, the option owner will receive in cash the traded price of the underlying asset in case it reaches the barrier and never exceeding the "option level".
-It is not necessary to wait until maturity to obtain liquidity.

4. Assessment of a barrier option cap

The methodology we are going to use in this paper is the binomial model, introduced by Cox, Ross and Rubinstein (1973) to value financial options. It is a discrete-time model based on the binomial tree, with different possible trajectories. A barrier option cap is a derivative over an underlying asset (usually a share) which is defined by the following elements:

- B: barrier.
- L: option level.
- S_k : price of the underlying asset at moment k ($k = 1, 2, \ldots, n$).

The possible performances of the barrier option cap are the following ones:

-If, at any time, the underlying asset is traded between the barrier and the option level, the barrier option cap guarantees the payment of the option level.

Figure 1. Underlying asset between the barrier and the option level.

Source: Own elaboration from BNP Paribas Bank data.

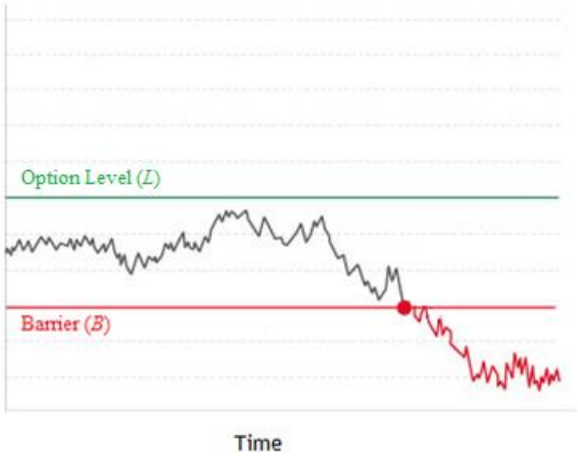
-If, at any moment, the underlying asset quotes above the option level, the option can be sold at that time obtaining, in advance, the maximum amount that could be reached with the option, i.e. the option level.

Figure 2. Underlying asset above option level.

Source: Own elaboration from BNP Paribas Bank data.

-If, at maturity, the asset quotes below the pre-set barrier level, the holder of the option will receive at maturity the price of the share at that time, with limit the level of the option.

Figure 3. Underlying asset below the barrier level.



Source: Own elaboration from BNP Paribas Bank data.

Taking into account the given definition of the barrier option cap, the option price *P* at moment 0 is given by the mathematical expectation of the following random variable (*BOC*) that represents the possible values of the option at that instant:

$$BOC = \begin{cases} L(1+r_f)^{-n}, & \text{if} \quad B < S_k < L, \quad \text{for all } k \\ L(1+r_f)^{-k}, & \text{if} \quad S_k \geq L, \quad \text{for some } k \\ \min\{S_n, L\}(1+r_f)^{-n}, & \text{if} \quad S_k \leq B, \quad \text{for some } k \end{cases}$$

where r_f is the risk-free interest rate. Therefore, $P = E[BOC]$.

In Figure 4, we are going to describe a methodology to calculate the price of a barrier option cap assuming that the underlying asset follows a binomial process with a rising factor *u* and a downward factor *d*, starting from the price volatility of the underlying asset at time 0 (S_0). To do this, we start from an example in which the option maturity is after five periods.

In this case, the value of the barrier option cap is:

$$BOC = \begin{cases} N(1+r_f)^{-5}, & \text{with probability } 1 - p^4 - q^3 - 3pq^4 \\ N(1+r_f)^{-3}, & \text{with probability } p^4 \\ S_0 u^2 d^3 (1+r_f)^{-5}, & \text{with probability } p^2 q^3 \\ S_0 u d^4 (1+r_f)^{-5}, & \text{with probability } 2pq^4 + 3pq^4 \\ S_0 d^5 (1+r_f)^{-5}, & \text{with probability } q^5 \end{cases}$$

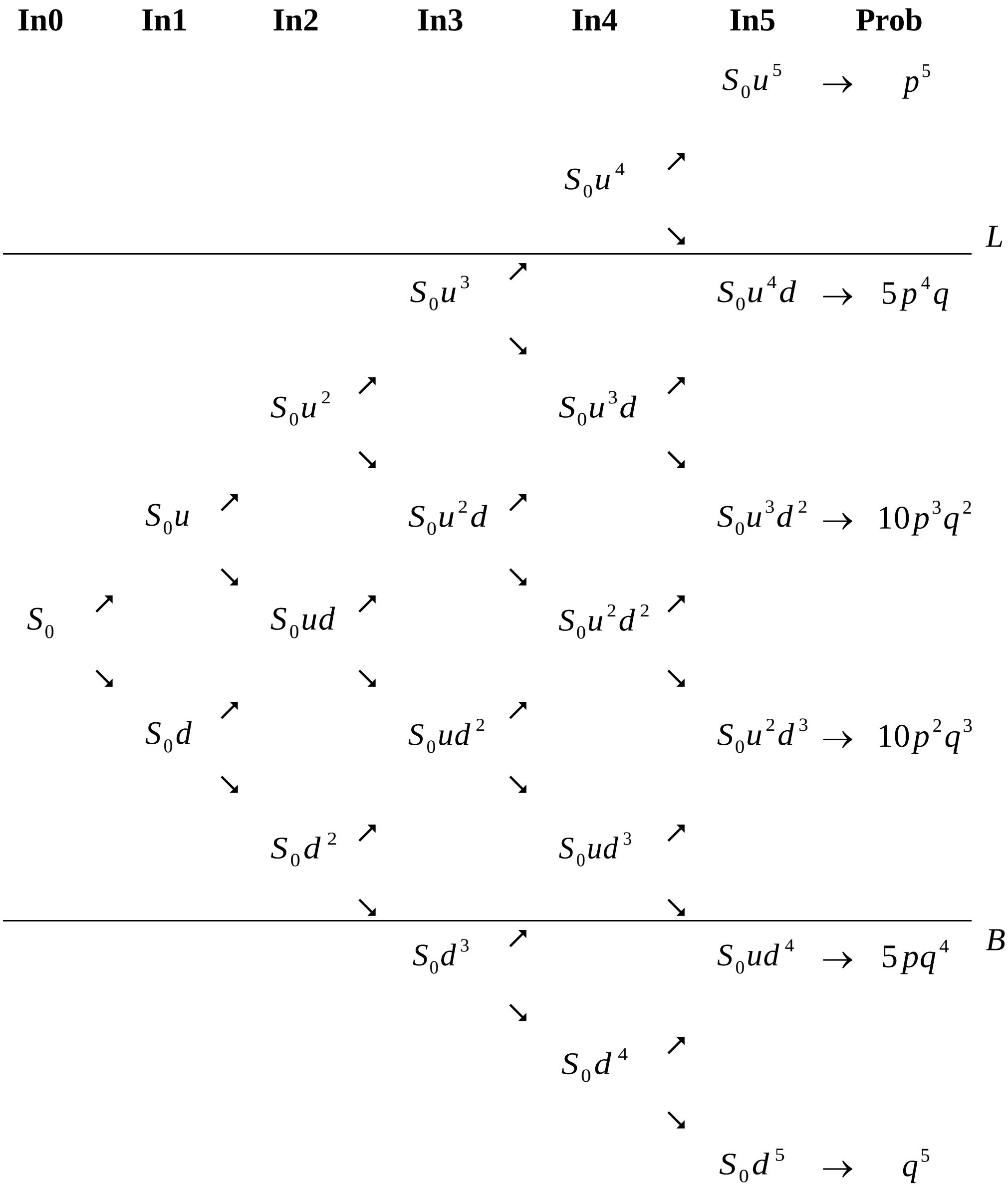
As previously indicated, $P = E[BOC]$.

5. Conclusions

Taken into account the wide offer of financial products with different risks, profitability and liquidity, an accurate analysis of their characteristics and real values is completely necessary.

Despite barrier options increase the covertures of risks, they are cheaper than the equivalent standard option. Specifically, a barrier option cap is a derivative with a given profitability provided that a certain condition is satisfied. In this way, a barrier option cap limits the benefit which can be obtained, in exchange of ensuring a known profit (the option level) if the price of the underlying asset is higher than the barrier value. So, this paper aims to analyze the assessment of this option by employing the binomial options pricing model.

Figure 4: Value of the barrier option cap within 5 periods (Instants).



Source: Own elaboration.

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Max-Min Fuzzy Relation Equations for a Problem of Spatial Analysis

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Abstract

We implement an algorithm that uses a system of max-min fuzzy relation equations (SFRE) for solving a problem of spatial analysis. We integrate this algorithm in a Geographical Information Systems (GIS) tool. We apply our process to determine the symptoms after that an expert sets the SFRE with the values of the impact coefficients related to some parameters of a geographic zone under study. We also define an index of evaluation about the reliability of the results.

Keywords: Fuzzy relation equation, max-min composition, GIS, triangular fuzzy number

2010 AMS subject classification: 03E72, 94D05.

1. Introduction

A Geographical Information System (GIS) is used as a support decision system for problems in a spatial domain. We use a GIS to analyse spatial distribution of data, the impact of event data on spatial areas: this analysis implies the creation of geographic thematic maps. Several authors (cfr., e. g., [3], [4], [7], [8], [25]) solve spatial problems using fuzzy relational calculus. In this paper, we propose an inferential method to solve such problems based on an algorithm for the resolution of a system of fuzzy relation equations (shortly, SFRE) given in [20] (cfr. also [21], [22]) and applied in [10] to solve industrial application problems. Here we integrate this algorithm in the context of a GIS architecture. Usually a SFRE with max-min composition is read as

$$\begin{cases} (a_{11} \wedge x_1) \vee \ldots \vee (a_{1n} \wedge x_n) = b_1 \\ (a_{21} \wedge x_1) \vee \ldots \vee (a_{2n} \wedge x_n) = b_2 \\ \ldots \\ (a_{m1} \wedge x_1) \vee \ldots \vee (a_{mn} \wedge x_n) = b_m \end{cases} \tag{1}$$

The system (1) is said consistent if it has solutions. Sanchez [23] determines its greatest solution, moreover many researchers have found algorithms which determine minimal solutions of (1) (cfr., e. g., [1], [2], [5], [6], [9], [11]÷[24], [26]). In [20] and [21] a method is described for the consistence of the system (1).

This method has been applied in this paper to real spatial problem in which the input data vary for each subzone of the geographical area. The expert starts from a valuation of input data and he uses linguistic labels for the determination of the output results for each subzone. The input data are the facts or symptoms, the parameters to be determined are the causes. For example, let us consider a planning problem. A city planner needs to determine in each subzone the mean state of buildings (x_1) and the mean soil permeability (x_2), knowing the number of collapsed building in the last year (b_1) and the number of flooding in the last year (b_2). The expert creates the SFRE (1) for each subzone by setting the impact matrix A, whose entries a_{ij} (i=1,…,n and j=1,…,m) represent the impact of the j-th cause x_j to the production of the i-th symptom b_i, where the value of b_i is the membership degree in the corresponding fuzzy set and let $B=[b_1,\ldots,b_m]$. In another subzone, the input data vector B and the matrix A can vary.

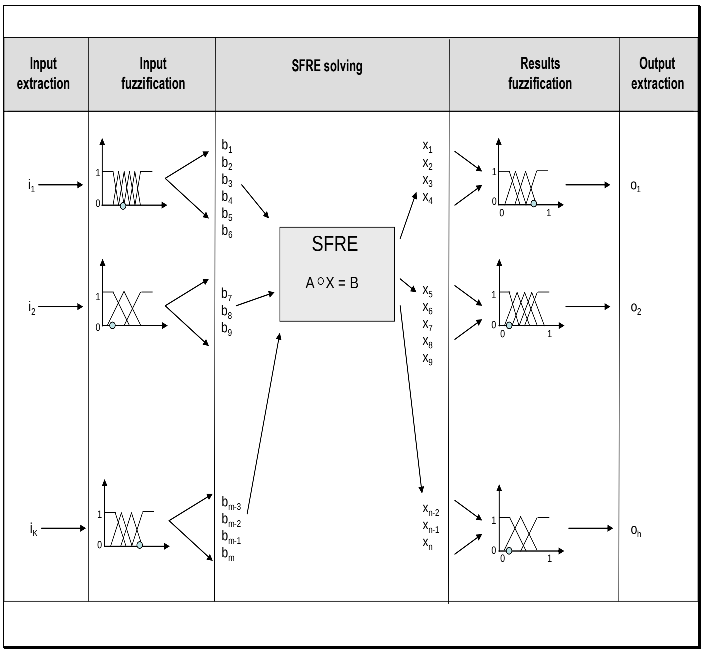

Fig. 1. Resolution process of a SFRE

The process of the resolution of the system (1) is schematized in Fig. 1. We can determine the maximal interval solutions of (1). Each maximal interval solution is an interval whose extremes are the values taken from a lower solution and from the greatest solution. Every value x_i belongs to this interval. If the SFRE (1) is inconsistent, it is possible to determine the rows for which no solution is permitted. If the expert decides to exclude the row for which no solution is permitted, he considers that the symptom b_i (for that row) is not relevant to its analysis and it is not taken into account. Otherwise, the expert can modify the setting of the coefficients of the matrix A to verify if the new system has some solution. In general, the SFRE (1) has T maximal interval solutions $X_{max(1)},\ldots,X_{max(T)}$. In order to describe the extraction process of the solutions, let $X_{max(t)}$, $t\in\{1,\ldots,T\}$, be a maximal interval solution given below, where X^{low} is a lower solution and X^{gr} is the greatest solution. Our aim is to assign the linguistic label of the most appropriate fuzzy sets, usually triangular fuzzy numbers (briefly, TFN), corresponding to the unknown $\{x_{j_1}, x_{j_1}, \ldots, x_{j_s}\}$ related to an output variable o_s, s = 1,…,k. For example, assuming that INF(j), MEAN(j), SUP(j) are the three fundamental values of the generic TFN x_j, $j=j_1, \ldots, j_s$, respectively, we can write their membership functions $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_h}$ as follows:

$$\mu_{j_1} = \begin{cases} 1 & \text{if } INF(j_1) \le x \le MEAN(j_1) \\ \dfrac{SUP(j_1) - x}{SUP(j_1) - MEAN(j_1)} & \text{if } MEAN(j_1) < x \le SUP(j_1) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\mu_{j} = \begin{cases} \dfrac{x - INF(j)}{MEAN(j) - INF(j)} & \text{if } INF(j) \le x \le MEAN(j) \\ \dfrac{SUP(j) - x}{SUP(j) - MEAN(j)} & \text{if } MEAN(j) < x \le SUP(j) \text{ and } j \in \{j_2, ..., j_{s-1}\} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\mu_{j_s} = \begin{cases} \dfrac{x - INF(j_s)}{MEAN(j_s) - INF(j_s)} & \text{if } INF(j_s) \le x \le MEAN(j_s) \\ 1 & \text{if } MEAN(j_s) < x \le SUP(j_s) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

If $XMin_t(j)$ (resp. $XMax_t(j)$) is the min (resp., max) value of every interval corresponding to the unknown x_j, we can calculate the arithmetical mean value

$XMean_t(j)$ of the j-th component of the above maximal interval solution $X_{max(t)}$ as

$$XMean_t(j) = \frac{XMin_t(j) + XMax_t(j)}{2} \tag{5}$$

and we get the vector column $XMean_t = [XMean_t(1),\ldots, XMean_t(n)]^{-1}$. The value given from $\max\{XMean_t(j_1),\ldots,XMean_t(j_s)\}$ obtained for the unknowns $x_{j_1},\ldots,x_{j_s}$ corresponding to the output variable o_s, is the linguistic label of the fuzzy set assigned to o_s and it is denoted by $score_t(o_s)$, defined also as reliability of o_s in the interval solution t. For the output vector $O = [o_1,\ldots,o_k]$, we define the following reliability index in the interval solution t as

$$\mathrm{Re}l_t(O) = \frac{1}{k} \cdot \sum_{s=1}^{k} score_t(o_s) \qquad (6)$$

and then as final reliability index of O, the number $Rel(O)=\max\{Rel_t(O):t=1,\ldots,T\}$.

The reliability of our solution is higher, the more the final reliability index Rel(O) close to 1 is. In Section 2 we give an overview of how finding the whole set of the solutions of a SFRE. In Section 3 we show how the proposed algorithm is applied in spatial analysis. Section 4 contains the results of our simulation and it is divided in five subsections.

2. SFRE: An Overview

The SFRE (1) is abbreviated in the following known form:

$$A \circ X = B$$

where $A = (a_{ij})$, is the matrix of coefficients, $X = (x_1, x_2,\ldots, x_n)^{-1}$ is the column vector of the unknowns and $B = (b_1,b_2,\ldots,b_m)^{-1}$ is the column vector of the known terms, being $a_{ij}, x_j, b_i \in [0,1]$ for each i = 1,…,m and j = 1,…,n. We have the following definitions and terminologies: the whole set of all solutions X of the SFRE (1) is denoted by Ω. A solution $\hat{X} \in \Omega$ is called a minimal solution if $X \leq \hat{X}$ for some $X \in \Omega$ implies $X=\hat{X}$, where "≤" is the partial order induced in Ω from the natural order of [0, 1]. We also recall that the system (1) has the unique greatest (or maximum) solution $X^{gr} = (x_1^{gr}, x_2^{gr},\ldots, x_n^{gr})^{-1}$ if $\Omega \neq \emptyset$ [23]. A matrix interval $X_{interval}$ of the following type:

$$X_{interval} = \begin{pmatrix} [a_1, b_1] \\ [a_2, b_2] \\ [\ldots,\ldots] \\ [a_n, b_n] \end{pmatrix}$$

where $[a_j,b_j] \subseteq [0,1]$ for each j=1,…,n, is called an interval solution of the SFRE (1) if every $X=(x_1,x_2,\ldots,x_n)^{-1}$ such that $x_j \in [a_j, b_j]$ for each j = 1,…,n, belongs to Ω. If a_j is a membership value of a minimal solution and b_j is a membership value of X^{gr} for each j = 1,…,n, then $X_{interval}$ is called a maximal interval solution of the SFRE (1) and it is denoted by $X_{max(t)}$, where t varies from 1 till to the number of minimal solutions. The SFRE (1) is said to be in normal form if $b_1 \geq b_2 \geq \ldots \geq b_m$. The time computational complexity to reduce a SFRE in a normal form is polynomial [20, 22]. Now we consider the matrix $A^* = (a^*_{ij})$ so defined:

$$a^*_{ij} = \begin{cases} 0 & \text{if } a_{ij} < b_i \\ b_i & \text{if } a_{ij} = b_i \\ 1 & \text{if } a_{ij} > b_i \end{cases}$$

where i = 1,…,m and j = 1,…,n, that is a^*_{ij} is S—type coefficient (Smaller) if $a_{ij}<b_i$, E—type coefficient (Equal) if $a_{ij}=b_i$ and G—type coefficient (Greater) if $a_{ij}>b_i$. A^* is called augmented matrix and the system $A^* \circ X = B$ is said associated to the SFRE (1). Without loss of generality, from now on we suppose that the system (1) is in normal form. We also the following definitions and results from [16, 17, 20, 22].

Definition 1. Let SFRE (1) be consistent and $A^*_j = \{a^*_{1j},..., a^*_{mj}\}$. If A^*_j contains G-type coefficients and $k \in \{1,\ldots,m\}$ is the greatest index of row such that $a^*_{kj} = 1$, then the following coefficients in A^*_j are called selected:

- a^*_{ij} for $i \in \{1,\ldots,k\}$ with $a^*_{ij} \geq b_i = b_k$,

- a^*_{ij} for $i \in \{k+1,\ldots,m\}$ with $a^*_{ij} = b_i$.

Definition 2. If A^*_j not contains G-type coefficients, but it contain E-type coefficients and $r \in \{1,\ldots,m\}$ is the smallest index of row such that $a^*_{rj} = b_r$, then any $a^*_{ij} = b_i$ in A^*_j for $i \in \{r,\ldots,m\}$ is called selected.

Theorem 1. Let us consider a SFRE (1). Then

- The SFRE (1) is consistent if and only if there exist at least one selected coefficient for each i-th equation, i=1,…,m.

- The complexity time function for determining the consistency of the SFRE (1) is O(m·n).

Consequently, when a SFRE (1) is inconsistent, the equations for which no element is a selected coefficient, could not be satisfied simultaneously with the other equations having at least one selected coefficient. Furthermore a vector IND=(IND(1),…,IND(m)) is defined by setting IND(i) equal to the number of selected coefficients in the i*th* equation for each *i* = l,...,m. If IND(i) = 0, then all the coefficients in the i*th* equation are not selected and the system is inconsistent. The system is consistent if IND(i) ≠ 0 if for each i = l,...,m and the product

$$PN2 = \prod_{i=1}^{m} IND(i)$$

gives the upper bound of the number of the eventual minimal solutions.

Theorem 2. Let SFRE (1) be consistent. Then
- the SFRE has an unique greatest solution X^{gr} with component $x_j^{gr} = b_k$ if the j*th* column A_j^* contains selected G-type coefficients a_{kj}^* and $x_j^{gr} = 1$ otherwise.

- The complexity time function for computing X^{gr} is O(m·n).

A help matrix H=[h_{ij}], i = 1,…,m and j = 1,…,n, is defined as follows:

$$h_{ij} = \begin{cases} b_i & \text{if } a_{ij}^* \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

Let $|H_i|$ be the number of coefficients h_{ij} in the i*th* equation of the SFRE (1). Then the number of potential minimal solutions cannot exceed the value

$$PN1 = \prod_{i=1}^{m} |H_i|$$

and one has $PN2 \le PN1$.

Definition 3. Let $h_i = (h_{i1}, h_{i2}, ..., h_{in})$ and $h_k = (h_{k1}, h_{k2}, ..., h_{kn})$ be the i*th* and the k*th* rows of the matrix H. If for each j=1,…n, $h_{ij} \ne 0$ implies both $h_{kj} \ne 0$ and $h_{kj} \le h_{ij}$, then the i*th* row (resp. equation) is said dominant over the k*th* row in H (resp. equation) or that the k*th* row (resp. equation) is said dominated by the i*th* row (resp. equation).

If the i*th* equation is dominant over the k*th* equation in (1), then the k*th* equation is a redundant equation of the system. By using Definition 3, we can build a matrix of dimension m×n, called dominance matrix *H**, having components:

$$h_{ij}^* = \begin{cases} 0 & \text{if the ith equation is dominated by another equation} \\ h_{ij} & \text{otherwise} \end{cases}$$

For each i= 1, ...,m, now we set $|H_i^*|$ as the number of coefficients $h_{ij}^* = b_i \neq 0$ in the i*th* row of the dominance matrix *H**. When this value is 0, we set $|H_i^*|$ = 1. Then the number of potential minimal solutions of the SFRE cannot exceed the value

$$PN3 = \prod_{i=1}^{m} |H*_i|$$

being $PN3 \le PN2 \le PN1$ [17, 20 ,22]. There the authors use the symbol $\left\langle \frac{b_i}{j} \right\rangle$ to indicate the coefficients $h_{ij}^* = b_i \neq 0$. We have $h_{ij}^* \wedge x_j = b_i$ if $x_j \in [b_i, 1]$ and $x_j = b_i$ is the j*th* component of a minimal solution. A solution of the i*th* equation can be written as

$$H_i = \sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle$$

In [20,22] the concept of concatenation W is introduced to determine all the components of the minimal solutions and it is given by

$$W = \prod_{i=1}^{m} H_i = \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle \right)$$

We can determine the minimal solutions $X^{low(t)} = (x_1^{low(t)}, x_2^{low(t)}, \ldots, x_n^{low(t)})^{-1}$, $t \in \{1, \ldots, PN(3)\}$, with components

$$x_j^{low(t)} = \begin{cases} b_{i_t} & \text{if } b_{i_t} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

In order to determine if a SFRE is consistent, hence its greatest solution and minimal solutions, we have used the universal algorithm of [20,22] based on the above concepts. For brevity of presentation, here we do not give this algorithm which has been implemented and tested under C++ language. The C++ library has been integrated in the ESRI ArcObject Library of the tool ArcGIS 9.3 for a problem of spatial analysis illustrated in the next Section 3.

3. SFRE in Spatial Analysis

We consider a specific area of study on the geographical map on which we have a spatial data set of “causes” and we want to analyse the possible “symptoms”.

We divide this area in P subzones where a subzone is an area in which the same symptoms are derived by input data or facts, and the impact of a symptom on a cause is the same one as well. It is important to note that even if two subzones have the same input data, they can have different impact degrees of symptoms on the causes. For example, the cause that measures the occurrence of floods may be due with different degree of importance to the presence of low porous soils or to areas subjected to continuous rains. Afterwards the area of study is divided in homogeneous subzones, hence the expert creates a fuzzy partition for the domain of each input variable and he determines the values of the symptoms b_i, as the membership degrees of the corresponding fuzzy sets (cfr., input fuzzification process of Fig. 1) for each subzone on which the expert sets the most significant equations and the values a_{ij} of impact of the j-th cause to the i-th symptom. After the determination of the set of maximal interval solutions, the expert for each interval solution calculates, for each unknown x_j, the mean interval solution $X_{mean(t)}$ with (5). The linguistic label $Rel_t(o_s)$ is assigned to the output variable o_s . Then he calculates the reliability index $Rel_t(O)$, given from formula (6), associated to this maximal interval solution t. After the iteration of this step, the expert determines the reliability index (6) for each maximal interval solution, by choosing the output vector O for which Rel(O) assumes the maximum value. Iterating the process for all the subzones (cfr., Fig. 2), the expert can show the thematic map of each output variable. If the SFRE related to a specific subzone is inconsistent, the expert can decide whether or not eliminate rows to find solutions: in the first case, he decides that the symptoms associated to the rows that make the system inconsistent are not considered and eliminates them, so reducing the number of the equations. In the second case, he decides that the corresponding output variable for this subzone remain unknown and it is classified as unknown on the map.

4. Simulation Results

Here we show the results of an experiment in which we apply our method to census statistical data agglomerated on four districts of the east zone of Naples (Italy). We use the year 2000 census data provided by the ISTAT (Istituto Nazionale di Statistica). These data contain informations on population, buildings, housing, family, employment work for each census zone of Naples. Every district is considered as a subzone with homogeneous input data given in Table 2.

In this experiment, we consider the following four output variables: "o_1 = *Economic prosperity*" (wealth and prosperity of citizens), "o_2 = *Transition into the job*" (ease of finding work), "o_3 = *Social Environment*" (cultural levels of citizens) and "o_4 = *Housing development*" (presence of building and residential dwellings of new construction). For each variable, we create a fuzzy partition composed by three TFNs called "low", "mean" and "high" presented in Table 1.

Moreover, we consider the following seven input parameters: i_1=percentage of people employed=number of people employed/total work force, i_2=percentage of women employed=number of women employed/number of people employed,

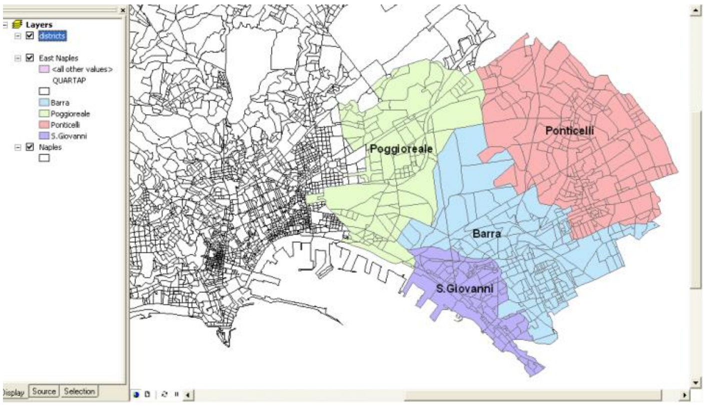

Fig. 2. Area of study: four districts at east of Naples (Italy)

Table 1. Values of the TFNs low, mean, high

Output	low			mean			high		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
o_1	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_2	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_3	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_4	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0

i_3=percentage of entrepreneurs and professionals = number of entrepreneurs and professionals/number of people employed, i_4 = percentage of residents graduated=numbers of residents graduated/number of residents with age > 6 years, i_5=percentage of new residential buildings=number of residential buildings built since 1982/total number of residential buildings, i_6 = percentage of residential dwellings owned=number of residential dwellings owned/ total number of residential dwellings, i_7 = percentage of residential dwellings with central heating system = number of residential dwellings with central heating system/total number of residential dwellings. In Table 4 we show these input data for the four subzones.

Table 2. Input data given for the four subzones

District	i_1	i_2	i_3	i_4	i_5	i_6	i_7
Barra	0.604	0.227	0.039	0.032	0.111	0.424	0.067
Poggioreale	0.664	0.297	0.060	0.051	0.086	0.338	0.149
Ponticelli	0.609	0.253	0.039	0.042	0.156	0.372	0.159
S. Giovanni	0.576	0.244	0.041	0.031	0.054	0.353	0.097

Table 3. TFNs values for the input domains

Input Var	low			Mean			High		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
i_1	0.00	0.40	0.60	0.40	0.60	0.80	0.60	0.80	1.00
i_2	0.00	0.10	0.30	0.10	0.30	0.40	0.30	0.50	1.00
i_3	0.00	0.04	0.06	0.04	0.06	0.10	0.07	0.20	1.00
i_4	0.00	0.02	0.04	0.02	0.04	0.07	0.04	0.07	1.00
i_5	0.00	0.05	0.08	0.05	0.08	0.10	0.08	0.10	1.00
i_6	0.00	0.10	0.30	0.10	0.30	0.60	0.30	0.60	1.00
i_7	0.00	0.10	0.30	0.10	0.30	0.50	0.30	0.50	1.00

Table 4: TFNs for the symptoms $b_1 \div b_{12}$

Subzone	b_1: i_1 = low	b_2: i_1 = mean	b_3: i_1 = high	b_4: i_2 = low	b_5: i_2= mean	b_6: i_2 = high	b_7: i_3 = low	b_8: i_3 = mean	b_9: i_3 = high	b_{10}: i_4 = low	b_{11}: i_4 = mean	b_{12}: i_4 = high
Barra	0.00	0.98	0.02	0.36	0.63	0.00	1.00	0.00	0.00	0.40	0.60	0.00
Poggioreale	0.00	0.93	0.07	0.01	0.99	0.00	0.00	1.00	0.00	0.00	0.63	0.37
Ponticelli	0.00	0.91	0.05	0.23	0.76	0.00	1.00	0.00	0.00	0.00	0.93	0.07
S. Giovanni	0.12	0.88	0.00	0.28	0.72	0.00	0.95	0.05	0.00	0.45	0.55	0.00

The expert indicates a fuzzy partition for each input domain formed from three TFNs labeled “low”, “mean” and “high”, whose values are reported in Table 3. In Tables 4 and 5 we show the values of TFNS for the 21 symptoms $b_1,...,b_{21}$. In order to form the SFRE (1) in each subzone, the expert defines the most significant symptoms.

Table 5: TFNs for the symptoms $b_{13} \div b_{21}$

Subzone	b_{13}: i_5 = low	b_{14}: i_5 = mean	b_{15}: i_5 = high	b_{16}: i_6 = low	b_{17}: i_6 = mean	b_{18}: i_6 = high	b_{19}: i_7 = low	b_{20}: i_7 = mean	b_{21}: i_7 = high
Barra	0.00	0.00	0.10	0.00	0.59	0.41	1.00	0.00	0.00
Poggioreale	0.00	0.70	0.30	0.00	0.87	0.13	0.75	0.25	0.00
Ponticelli	0.00	0.00	1.00	0.00	0.76	0.24	0.70	0.30	0.00
S. Giovanni	0.87	0.13	0.00	0.00	0.82	0.18	1.00	0.00	0.00

4.1 Subzone "Barra"

The expert chooses the significant symptoms b_2, b_4, b_5, b_7, b_{10}, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, by obtaining a SFRE (1) with m = 10 equations and n = 12 unknowns. The matrix A of the impact values a_{ij} has dimensions 10×12 and the vector B of the symptoms b_i has dimension 10×1 and both are given below. The SFRE (1) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain four maximal interval solutions $X_{max(t)}$ (t=1,…,4) and we calculate the vector column $XMean_t$ on each maximal interval solution. Hence we associate to the output variable o_s (s = 1,…,4), the linguistic label of the fuzzy set with the higher value calculated with formula (5) obtained for the corresponding unknowns $x_{j_1},...,x_{j_s}$ and given in Table 6. For determining the reliability of our solutions, we use the index given by formula (6). We obtain that $Rel_t(o_1)$ = $Rel_t(o_2)$ = $Rel_t(o_3)$ = $Rel_t(o_4)$ = 0.6025 for t=1,…,4 and hence Rel(O)=max{Rel_t(O): t=1,…,4}=0.6025 where O={o_1,…o_4}. We note that the same final set of linguistic labels associated to the output variables o_1 = "high", o_2 = "mean", o_3 = "low", o_4 = "low" is obtained as well. The relevant quantities are given below.

$$A=\begin{pmatrix}
0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2\\
0.3 & 0.5 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 & 0.6 & 0.2 & 0.0 & 0.0 & 0.0\\
0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.0 & 0.0 & 0.0\\
1.0 & 0.2 & 0.0 & 0.8 & 0.3 & 0.1 & 0.8 & 0.2 & 0.2 & 0.3 & 0.0 & 0.0\\
0.5 & 0.3 & 0.1 & 0.6 & 0.4 & 0.1 & 0.6 & 0.4 & 0.1 & 0.1 & 0.0 & 0.0\\
0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1\\
0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.3 & 0.3\\
0.2 & 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.2 & 0.5 & 0.1 & 0.3 & 0.7 & 0.3\\
0.1 & 0.4 & 0.4 & 0.1 & 0.4 & 0.4 & 0.1 & 0.5 & 0.5 & 0.2 & 0.4 & 0.5\\
0.5 & 0.2 & 0.0 & 0.4 & 0.3 & 0.0 & 0.4 & 0.3 & 0.0 & 1.0 & 0.1 & 0.0
\end{pmatrix}
\quad B=\begin{pmatrix}0.98\\0.36\\0.63\\1.00\\0.40\\0.60\\0.10\\0.59\\0.41\\1.00\end{pmatrix}$$

$$X_{max(1)}=\begin{pmatrix}[0.40,0.40]\\ [0.36,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.41,0.41]\\ [1.00,1.00]\\ [0.00,0.10]\\ [0.00,0.10]\end{pmatrix}
\quad X_{max(2)}=\begin{pmatrix}[0.40,0.40]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.36,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.41,0.41]\\ [1.00,1.00\\ [0.00,0.10]\\ [0.00,0.10]\end{pmatrix}
\quad X_{max(3)}=\begin{pmatrix}[0.40,0.40]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.36,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.41,0.41]\\ [1.00,1.00]\\ [0.00,0.10]\\ [0.00,0.10]\end{pmatrix}
\quad X_{max(4)}=\begin{pmatrix}[0.40,0.40]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.36,0.36]\\ [0.00,1.00]\\ [0.00,0.36]\\ [0.00,1.00]\\ [0.36,0.36]\\ [0.41,0.41]\\ [1.00,1.00]\\ [0.00,0.10]\\ [0.00,0.10]\end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 0.40 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.18 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.05 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}$$

Table 6. Final linguistic labels for the output variables in the district Barra

Output variable	$score_1(o_s)$	$score_2(o_s)$	$score_3(o_s)$	$score_4(o_s)$
o_1	high	high	high	high
o_2	mean	mean	mean	mean
o_3	low	low	low	low
o_4	low	low	low	low

For determining the reliability of our solutions, we use the index given by formula (6). We obtain $Rel(O_k) = 0.4675$ for k = 1,..,12. Then we obtain two final sets of linguistic labels associated to the output variables: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "low", and o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally.

4.2 Subzone "Poggioreale"

The expert choices the significant symptoms b_2, b_5, b_8, b_{11}, b_{12}, b_{14}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, by obtaining a SFRE (1) with m = 11 equations and n = 12 unknowns. The matrix A of the impact values a_{ij} has sizes dimension 11×12 and the column vector B of the symptoms b_i has sizes 11×1 are given below. The SFRE (7) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 12 maximal interval solutions $X_{max(t)}$ (t=1,…,12) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 7 contains the output variables and the relevant quantities are given below.

$$A=\begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 0.9 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.0 & 0.0 & 0.1 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.3 & 0.4 & 0.6 & 0.2 & 0.4 & 0.6 & 0.0 & 0.1 & 0.2 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 \\ 0.0 & 0.1 & 0.5 & 0.1 & 0.2 & 0.5 & 0.1 & 0.2 & 0.5 & 0.0 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.0 & 0.8 & 0.5 & 0.3 & 0.5 & 0.3 & 0.1 & 0.7 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.3 & 0.6 & 0.2 \end{pmatrix} \quad B=\begin{pmatrix} 0.93 \\ 0.99 \\ 1.00 \\ 0.63 \\ 0.37 \\ 0.70 \\ 0.30 \\ 0.87 \\ 0.13 \\ 0.75 \\ 0.25 \end{pmatrix}$$

$$X_{max(1)}=\begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.13,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{max(2)}=\begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.13,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{max(3)}=\begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{max(4)}=\begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix}$$

$$X_{\max(5)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(6)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(7)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.0] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(8)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix}$$

$$X_{\max(9)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.13,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(10)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.13,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(11)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.13,0.13] \end{pmatrix} \quad X_{\max(12)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.13,0.13] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.130 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.050 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.130 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.130 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.065 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.130 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix}$$

$$XMean_5 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.130 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.05 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.130 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.050 \end{pmatrix} XMean_7 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.130 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.065 \end{pmatrix} \quad XMean_8 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.130 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix}$$

$$XMean_9 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.130 \\ 0.250 \\ 0.125 \\ 0.050 \end{pmatrix} \quad XMean_{10} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.130 \\ 0.125 \\ 0.250 \\ 0.050 \end{pmatrix} XMean_{11} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.130 \end{pmatrix} \quad XMean_{12} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.130 \end{pmatrix}$$

For determining the reliability of our solutions, we use the index given by formula (6). We obtain $Rel(O_k) = 0.4675$ for k = 1,..,12. Then we obtain two final sets of linguistic labels associated to the output variables: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "low", and o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally.

Table 7. Final linguistic labels for the output variables in the district "Poggioreale"

	Linguistic labels associated to											
output variable	$XMean_1$	$XMean_2$	$XMean_3$	$XMean_4$	$XMean_5$	$XMean_6$	$XMean_7$	$XMean_8$	$XMean_9$	$XMean_{10}$	$XMean_{11}$	$XMean_{12}$
o_1	low	low	low	high	low	low	low	high	low	low	low	high
o_2	low	low	low	mean	low	low	low	mean	low	low	low	mean
o_3	low	low	low	low	low	low	low	low	low	low	low	low
o_4	low	mean	low	mean	low	mean	low	mean	low	mean	low	mean

4.3 Subzone: District Ponticelli

The expert choices the significant symptoms b_2, b_4, b_5, b_7, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, obtaining a SFRE (7) with m = 10 equations and n = 12 variables: The matrix A of sizes 10×12 and the column vector B of dimension 10×1 are given by:

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.0 & 0.2 & 0.1 & 0.0 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 1.0 & 0.1 & 0.0 & 0.8 & 0.2 & 0.2 & 0.3 & 0.1 & 0.0 \\ 0.4 & 0.8 & 0.1 & 0.3 & 0.9 & 0.1 & 0.2 & 0.8 & 0.1 & 0.1 & 0.3 & 0.0 \\ 0.0 & 0.1 & 1.0 & 0.1 & 0.3 & 0.7 & 0.1 & 0.3 & 0.7 & 0.0 & 0.1 & 1.0 \\ 0.3 & 0.7 & 0.3 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.3 & 0.7 & 0.3 \\ 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.0 & 0.4 & 0.2 & 0.1 & 0.4 & 0.2 & 0.1 & 0.7 & 0.2 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.3 & 0.5 & 0.1 \end{pmatrix} \quad B = \begin{pmatrix} 0.91 \\ 0.23 \\ 0.76 \\ 1.00 \\ 0.93 \\ 1.00 \\ 0.76 \\ 0.24 \\ 0.70 \\ 0.30 \end{pmatrix}$$

The SFRE (7) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 8 maximal interval solutions $X_{max(t)}$ (t=1,…,8) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 10 contains the output variables and the relevant quantities are given below.

$$X_{\max(1)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{\max(2)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [1.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{\max(3)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix} \quad X_{\max(4)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix}$$

$$X_{\max(5)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{\max(6)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{\max(7)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix} \quad X_{\max(8)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 1.00 \\ 0.38 \\ 1.00 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.5 \\ 0.38 \\ 1.00 \\ 1.00 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 1.00 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix}$$

$$XMean_5 = \begin{pmatrix} 1.00 \\ 0.38 \\ 1.00 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.50 \\ 0.38 \\ 1.00 \\ 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_7 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix} \quad XMean_8 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix}$$

Now we associate to the output variables o_s k = 1,…,4, the linguistic label of the fuzzy set with the higher $XMean_j$ obtained for the corresponding unknowns x_{j_1} ,…, x_{j_s} obtaining:

Table 8. Final linguistic labels for the output variables in the district "Ponticelli"

Linguistic labels associated to								
output variable	$XMean_1$	$XMean_2$	$XMean_3$	$XMean_4$	$XMean_5$	$XMean_6$	$XMean_7$	$XMean_8$
o_1	Low-high	high	low	Low -high	Low -high	high	low	Low -high
o_2	mean	low	mea n	low	Low -high	low	Low -high	low
o_3	Low-high	Low-high	Low -high	Low -high	mea n	mea n	mea n	mea n
o_4	low	low	low	low	low	low	low	low

Here "low-high" indicates that the membership degree of both the fuzzy sets with linguistic labels "low" and "high" have the maximal value for that output variable. We obtain for each solution $Rel(O_1)$ =0.565, $Rel(O_2)$ = 0.625, $Rel(O_3)$

= 0.565 $Rel(O_4) = 0.5$, $Rel(O_5) = 0.565$, $Rel(O_6) = 0.69$, $Rel(O_7) = 0.565$ $Rel(O_8) = 0.565$.

Thus we choice the solution O_6 which have the greatest reliability $Rel(O_6) = 0.69$. Our solution for this subzone is: o_1 = "high", o_2 = "low", o_3 = "mean", o_4 = "low".

4.4 Subzone: district S. Giovanni

The expert choices the significant symptoms b_2, b_4, b_5, b_7, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, obtaining a SFRE (1) with m = 12 equations and n = 12 variables: The matrix A of sizes 12×12 and the column vector B of sizes 12×1 are given by:

$$A = \begin{pmatrix} 0.3 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 & 0.1 & 0.0 & 0.0 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.0 & 0.3 & 0.0 \\ 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.2 & 0.0 \\ 1.0 & 0.2 & 0.0 & 1.0 & 0.1 & 0.0 & 0.9 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 \\ 0.5 & 0.3 & 0.1 & 0.5 & 0.3 & 0.1 & 0.6 & 0.3 & 0.1 & 0.2 & 0.1 & 0.0 \\ 0..3 & 0.6 & 0.2 & 0.2 & 0.5 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.2 & 0.0 \\ 0.6 & 0.3 & 0.1 & 0.5 & 0.2 & 0.1 & 0.5 & 0.2 & 0.1 & 0.8 & 0.2 & 0.0 \\ 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.1 & 0.4 & 0.1 \\ 0.3 & 0.6 & 0.3 & 0.3 & 0.6 & 0.3 & 0.3 & 0.6 & 0.3 & 0.3 & 0.7 & 0.1 \\ 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.1 & 0.5 \\ 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 1.0 & 0.0 & 0.0 \end{pmatrix} \quad B = \begin{pmatrix} 0.12 \\ 0.88 \\ 0.28 \\ 0.72 \\ 0.95 \\ 0.45 \\ 0.55 \\ 0.87 \\ 0.13 \\ 0.82 \\ 0.18 \\ 1.0 \end{pmatrix}$$

The SFRE (1) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 6 maximal interval solutions $X_{max(t)}$ (t=1,…,6) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 11 contains the output variables and the relevant quantities are given below.

$$X_{\max(1)} = \begin{pmatrix} [0.12,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max,(2)} = \begin{pmatrix} [0.12,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max,(3)} = \begin{pmatrix} [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix}$$

$$X_{\max(4)} = \begin{pmatrix} [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max(5)} = \begin{pmatrix} [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max(6)} = \begin{pmatrix} [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 0.12 \\ 0.55 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.12 \\ 0.275 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.55 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 0.06 \\ 0.55 \\ 0.50 \\ 0.12 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix}$$

$$XMean_{41} = \begin{pmatrix} 0.06 \\ 0.275 \\ 0.50 \\ 0.12 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.55 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_5 = \begin{pmatrix} 0.06 \\ 0.55 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.060 \\ 0.275 \\ 0.500 \\ 0.060 \\ 0.720 \\ 0.500 \\ 0.120 \\ 0.550 \\ 0.500 \\ 1.000 \\ 0.130 \\ 0.180 \end{pmatrix}$$

Table 9. Final linguistic labels for the output variables in the district "San Giovanni"

output variable	linguistic label associated to $XMean_1$	linguistic label associated to $XMean_2$	linguistic label associated to $XMean_3$	linguistic label associated to $XMean_4$	linguistic label associated to $XMean_5$	linguistic label associated to $XMean_6$
o_1	mean	high	mean	high	mean	high
o_2	mean	mean	mean	mean	mean	mean
o_3	high	mean	high	mean	high	mean
o_4	low	low	low	low	low	low

We obtain $Rel(O_k) = 0.6925$ for $k = 1,\ldots,6$. Thus we obtain two final sets of linguistic labels associated to the output variables: o_1 = "mean", o_2 = "mean", o_3 = "high", o_4 = "low", and o_1 = "high", o_2 = "mean", o_3 = "mean", o_4 = "low" with the same reliability index value 0.6925. The expert prefers to choose the first solution: o_1 = "mean", o_2 = "mean", o_3 = "high", o_4 = "low", because he considers in this district that in the two years the presence of residents was graduated and consequently, the cultural level of citizens has increased, whereas the average pro capite wealth of citizens has decreased.

4.5 Thematic maps and conclusions

Finally, we obtain four final thematic maps shown in Figs. 3, 4, 5, 6 for the output variable o_1, o_2, o_3, o_4, respectively.

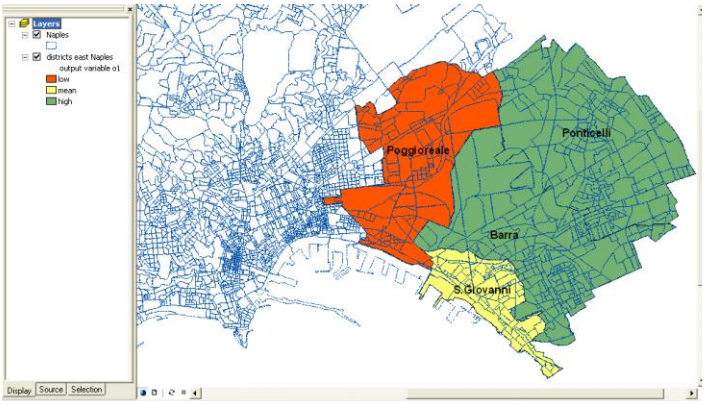

Fig. 3. Thematic map for output variable o_1 (*Economic prosperity*)

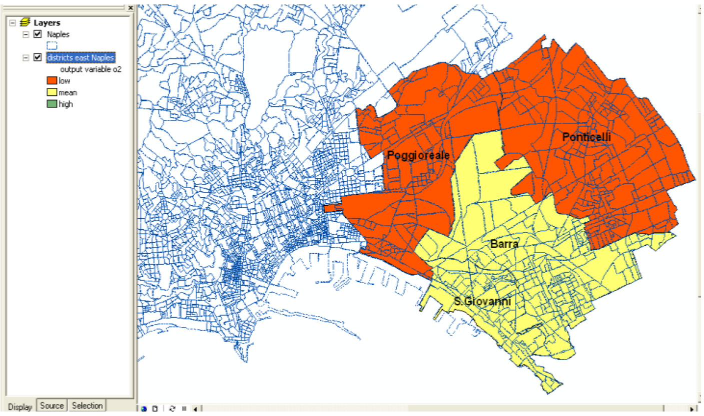

Fig. 4. Thematic map of the output variable o_2 (*Transition into the job*)

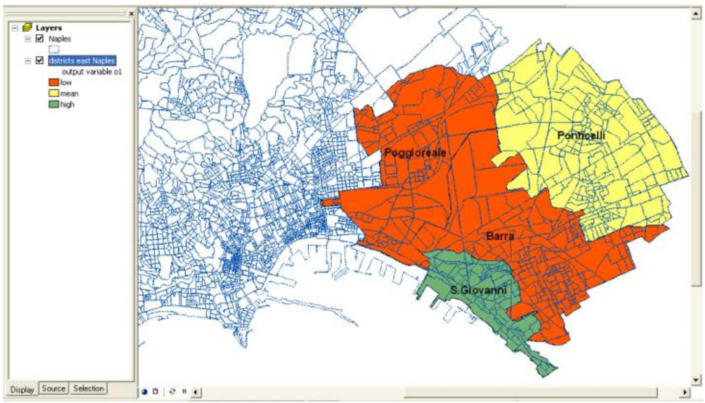

Fig. 5. Thematic map for the output variable o_3 (*Social Environment*)

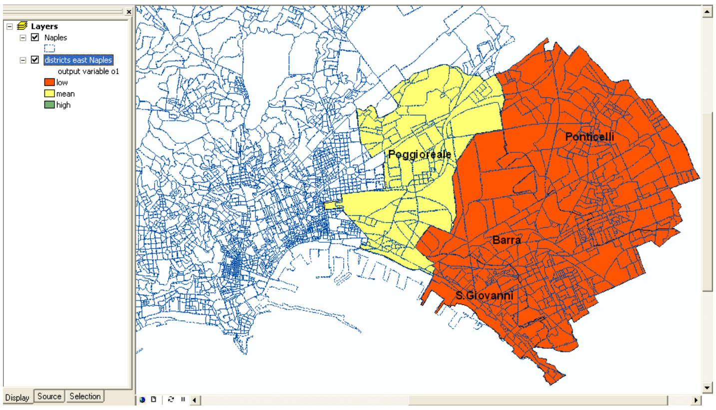

Fig. 6. Thematic map for the output variable o_4 (*Housing development*)

The results show that there was no housing development in the four districts in the last 10 years and there is difficulty in finding job positions. In Fig. 7 we show the histogram of the reliability index Rel(O) for each subzone, where $O=[o_1,o_2,o_3,o_4]$.

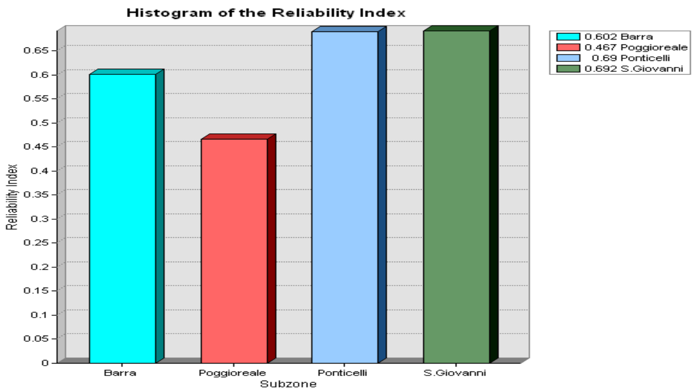

Fig. 7. Histogram of the reliability index Rel(O) for the four subzones.

This paper is a new reformulation of our work titled “Spatial Analysis and Fuzzy Relation Equations” published in Advances in Fuzzy Systems, Volume 2011 (2011), Article ID 429498, 14 pages (http://dx.doi.org/10.1155/2011/429498) (under Common License) where an extended version of the first three sections can be found, indeed an extended version of Section 4 is here more complete with respect to Section 4 presented there.

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Helix-Hopes on Finite Hyperfields

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Abstract

Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We study the helix-hyperstructures on the representations using ordinary fields. The related theory can be faced by defining the hyperproduct on the set of non square matrices. The main tools of the Hyperstructure Theory are the fundamental relations which connect the largest class of hyperstructures, the H_v-structures, with the corresponding classical ones. We focus on finite dimensional helix-hyperstructures and on small H_v-fields, as well.

Keywords*:* hyperstructures, H_v-structures, h/v-structures, hope.

2010 AMS subject classification: 20N20, 16Y99.

1 Introduction

We deal with the largest class of hyperstructures called *H_v-structures* introduced in 1990 [10], [11], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Definitions 1.1 In a set H equipped with a ***hyperoperation*** (which we abbreviate it by ***hope***)

$$\cdot : H\times H\rightarrow P(H)-\{\emptyset\}: (x,y)\rightarrow x\cdot y\subset H$$

we abbreviate by

WASS the *weak associativity*: $(xy)z\cap x(yz)\neq\emptyset,\ \forall x,y,z\in H$ and by

COW the *weak commutativity*: $xy\cap yx\neq\emptyset,\ \forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called H_v-semigroup if it is WASS and is called ***H_v-group*** if it is reproductive H_v-semigroup: $xH=Hx=H,\ \forall x\in H$.

$(R,+,\cdot)$ is called ***H_v-ring*** if (+) and (·) are WASS, the reproduction axiom is valid for (+) and (·) is weak distributive with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\emptyset,\quad (x+y)z\cap(xz+yz)\neq\emptyset,\ \forall x,y,z\in R.$$

For more definitions, results and applications on H_v-structures, see books and the survey papers as [2], [3], [11], [1], [6], [15], [16], [20]. An extreme class is the following: An H_v-structure is *very thin* iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, in a very thin H_v-structure in a set H there exists a hope (·) and a pair $(a,b)\in H^2$ for which ab=A, with cardA>1, and all the other products, with respect to any other hopes (so they are operations), are singletons.

The fundamental relations β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [9], [10], [11], [12], [13]. The main theorem is the following:

Theorem 1.2 Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.

An element is called *single* if its fundamental class is a singleton.

Motivation for H_v-structures:
The quotient of a group with respect to an invariant subgroup is a group.
Marty states that, the quotient of a group by any subgroup is a hypergroup.

Now, the quotient of a group with respect to any partition is an H_v-group.

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is *smaller* than $(\otimes)$, and $(\otimes)$ *greater* than $(\cdot)$, iff there exists automorphism

$$f\in Aut(H,\otimes) \text{ such that } xy\subset f(x\otimes y),\ \forall x\in H.$$

Then $(H,\otimes)$ *contains* $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is *basic* and $(H,\otimes)$ is an *H_b-structure*.

The Little Theorem [11]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.4 [10], [11]. The H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if the quotient R/γ^* is a field.

Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call *reproductive H_v-field* any H_v-field $(R,+,\cdot)$ if

$$x(R-\omega^*)=(R-\omega^*)x=R-\omega^*,\forall x\in R-\omega^*.$$

From this definition, a new class is introduced [15]:

Definition 1.5 The H_v-semigroup $(H,\cdot)$ is ***h/v-group*** if the H/β^* is a group.

Similarly *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces*, are defined. The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then the quotient is reproductive $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x\in H$.

An H_v-group is called *cyclic* [11], if there is element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator. If there exists an element and a special power, the minimum, is the underline set, then the H_v-group is called *single-power cyclic*.

Definitions 1.6 [11], [14]. Let $(R,+,\cdot)$ be an H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope $\cdot: R\times M\to P(M):(a,x)\to ax$, such that, $\forall a,b\in R$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\emptyset,\quad (a+b)x\cap(ax+bx)\neq\emptyset,\quad (ab)x\cap a(bx)\neq\emptyset,$$

then M is called an ***H_v-module*** over R. In the case of an H_v-field F instead of H_v-ring R, then the ***H_v-vector space*** is defined.

Definition 1.7 [17]. Let $(L,+)$ be H_v-vector space on $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map φ': $L\to L/\varepsilon^*$ and denote again 0 the zero of L/ε^*. Consider the *bracket* (commutator) hope:

$$[\ ,\ ] : L\times L\to P(L): (x,y)\to[x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear:

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,\ \forall x,x_1,x_2,y,y_1,y_2\in L \text{ and } \lambda_1,\lambda_2\in F$$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\ \forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in L$

Two well known and large classes of hopes are given as follows [11], [16]:

Definitions 1.8 Let $(G,\cdot)$ be a groupoid, then for every subset $P\subset G$, $P\neq\varnothing$, we define the following hopes, called ***P-hopes***: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y = (xP)y\cup x(Py),$$

$$\underline{P}_r: x\underline{P}_ry= (xy)P\cup x(yP),\qquad \underline{P}_l: x\underline{P}_ly= (Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called ***P-hyperstructures***.

The usual case is for semigroup $(G,\cdot)$, then

$$x\underline{P}y=(xP)y\cup x(Py)=xPy$$

and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes: Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } y\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope, *P_e-hope*. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

Definition 1.9 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), called *theta-hope*, we write *∂-hope*, on G as follows

$$x\partial y = \{f(x)\cdot y,\ x\cdot f(y)\}\ (\text{ resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y))),\ \forall x,y\in G.$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

If $(G,\cdot)$ is groupoid (or hypergroupoid) and $f:G\to P(G)-\{\varnothing\}$ multivalued map. We define the ∂-hope on G as follows: $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G$.

Motivation for the ∂-hope is the map *derivative* where only the product of functions can be used.

Basic property: if $(G,\cdot)$ is semigroup then $\forall f$, the ∂-hope is *WASS*.

2 Some Applications of H_v-Structures

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [2], [3], [7], [18].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond into the hyperstructures introduced by Santilli & Vougiouklis in 1999 [7] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 2.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 2.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0,\ \forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x,\ \forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 2.3 *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$x\otimes y = \{xy, g_1, g_2,\dots\},\ \forall x,y\in G-\{e\},\ \text{where } g_1, g_2,\dots\in G-\{e\}$$

g_1, g_2,… are not necessarily the same for each pair (x,y). (G,⊗) is an H_v-group, it is an H_b-group which contains the (G,·). (G,⊗) is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have x⊗y=xy, then (G,⊗) becomes a strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example 2.4 Consider the quaternion group ***Q***={1,-1, i,-i, j,-j, k,-k} with defining relations $i^2 = j^2 = -1$, ij = -ji = k. Denoting $\underline{i}$={i,-i}, $\underline{j}$={j,-j}, $\underline{k}$={k,-k} we may define a very large number (∗) hopes by enlarging only few products. For example, (-1)∗k=$\underline{k}$, k∗i=$\underline{j}$ and i∗j=$\underline{k}$. Then the hyperstructure (Q,∗) is a strong e-hypergroup.

Mathematicalisation of a problem could make its results recognizable and comparable. This is because representing a research object or a phenomenon with numbers, figures or graphs might be simplest and in a recognizable way of reading the results. In questionnaires Vougiouklis & Vougiouklis proposed the substitution of Likert scales with the *bar* [5], [18].This substitution makes things simpler and easier for both the subjects of an empirical research and the researcher, either at the stage of designing or that of results processing, because it is really flexible. Moreover, the application of *the bar* opens a window towards the use of fuzzy sets in the whole procedure of empirical research, activating in this way more recent findings from different sciences, as well. The bar is closelly related with hyperstructure and fuzzy theories, as well.

More specifically, the following was proposed:

In every question, substitute the Likert scale with the 'bar' whose poles are defined with '0' on the left and '1' on the right:

0 ———————————————— 1

The subjects/participants are asked, instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel best expresses their answer to the specific question.

The suggested length of the bar is approximately 6.18cm, or 6.2cm, following the golden ration on the well known length of 10cm.

3 Small H_v-Numbers. H_v-Matrix Representations

In representations important role are playing the small hypernumbers.

Construction 3.1 On the ring ($\boldsymbol{Z}_4$,+,·) we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal,

(b) COW (non-commutative),
(c) they have the elements 0 and 1, scalars.

Then, we have only the following isomorphic cases $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. Fundamental classes: $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Thus it is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f\otimes d=\{I,b\}$. Moreover, the $(\boldsymbol{X},\otimes)$ is not cyclic.

Construction 3.2 On $(\boldsymbol{Z}_6,+,\cdot)$ we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are:

(a) very thin minimal
(b) COW (non-commutative)
(c) they have the elements 0 and 1, scalars

Then we have the following cases, by giving the only one hyperproduct,

(i) $2\otimes 3=\{0,3\}$ or $2\otimes 4=\{2,5\}$ or $2\otimes 5=\{1,4\}$

$3\otimes 4=\{0,3\}$ or $3\otimes 5=\{0,3\}$ or $4\otimes 5=\{2,5\}$

In all 6 cases the fundamental classes are $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot)$.

(ii) $2\otimes 3=\{0,2\}$ or $2\otimes 3=\{0,4\}$ or $2\otimes 4=\{0,2\}$ or $2\otimes 4=\{2,4\}$ or

$2\otimes 5=\{0,4\}$ or $2\otimes 5=\{2,4\}$ or $3\otimes 4=\{0,2\}$ or $3\otimes 4=\{0,4\}$ or

$3\otimes 5=\{1,3\}$ or $3\otimes 5=\{3,5\}$ or $4\otimes 5=\{0,2\}$ or $4\otimes 5=\{2,4\}$.

In all 12 cases the fundamental classes are $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2\otimes 5=\{1,4\}$ from (i), and the case $3\otimes 5=\{1,3\}$ from (ii).

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations or by H_v-matrices [11], [12], [13], [14].

***H_v-matrix** (or **h/v-matrix**)* is a matrix with entries of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m\times n$ and $n\times r$ respectively, is defined in the usual manner and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS.

The problem of the H_v-matrix (or h/v-group) representations is the following:

Definition 3.3 Let $(H,\cdot)$ be H_v-group (or h/v-group). Find an H_v-ring (or h/v-ring) $(R,+,\cdot)$, a set $M_R=\{(a_{ij})\mid a_{ij}\in R\}$ and a map $T:H\to M_R$: $h\mapsto T(h)$ such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \emptyset,\ \forall h_1,h_2 \in H.$$

T is *H_v-matrix (or h/v-matrix) representation*. If $T(h_1h_2) \subset T(h_1)T(h_2)$, $\forall h_1,h_2 \in H$, then T is called *inclusion*. If $T(h_1h_2)=T(h_1)T(h_2)= \{T(h) \mid h \in h_1h_2\}$, $\forall h_1,h_2 \in H$, then T is *good* and then an induced representation T* for the hypergroup algebra is obtained. If T is one to one and good then it is *faithful*.

The main theorem on representations is [13]:

Theorem 3.4 A necessary condition to have an inclusion representation T of an h/v-group $(H,\cdot)$ by $n\times n$, h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following:

For all classes $\beta^*(x)$, $x \in H$ must exist elements $a_{ij} \in H$, $i,j \in \{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i,j \in \{1,...,n\}\}$$

Inclusion $T:H \rightarrow M_R: a \mapsto T(a)=(a_{ij})$ induces homomorphic representation T* of H/β^* on R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T* is called *fundamental induced* of T.

In representations, several new classes are used:

Definition 3.5 Let $M=M_{m\times n}$ be the module of $m\times n$ matrices over R and $P=\{P_i: i \in I\} \subseteq M$. We define a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M \rightarrow P(M): (A,B) \rightarrow A\underline{P}B=\{AP^t_iB: i \in I\} \subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is bilinear map, is strong associative and inclusion distributive:

$$A\underline{P}(B+C) \subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C \in M$$

Definition 3.6 Let $M=M_{m\times n}$ the $m\times n$ matrices over R and let us take sets

$$S=\{s_k: k \in K\} \subseteq R,\quad Q=\{Q_j: j \in J\} \subseteq M,\quad P=\{P_i: i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S}: R\times M \rightarrow P(M): (r,A) \rightarrow r\underline{S}A = \{(rs_k)A: k \in K\} \subseteq M$$

$$\underline{Q}_+: M\times M \rightarrow P(M): (A,B) \rightarrow A\underline{Q}_+B = \{A+Q_j+B: j \in J\} \subseteq M$$

$$\underline{P}: M\times M \rightarrow P(M): (A,B) \rightarrow A\underline{P}B = \{AP^t_iB: i \in I\} \subseteq M$$

Then $(M,\underline{S},\underline{Q}_+,\underline{P})$ is hyperalgebra on R called *general matrix P-hyperalgebra*.

4 Helix-Hopes and Applications

Recall some definitions from [19], [8], [20], [4]:

Definition 4.1 Let $A=(a_{ij})\in M_{m\times n}$ be m×n matrix and $s,t\in N$ be naturals such that $1\le s\le m$, $1\le t\le n$. We define the map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst}=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$. We call this map *cut-projection* of type $\underline{st}$. Thus $A\underline{cst}$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.

We use cut-projections on all types of matrices to define sums and products.

Definitions 4.2 Let $A=(a_{ij})\in M_{m\times n}$ be an m×n matrix and $s,t\in N$, $1\le s\le m$, $1\le t\le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st}=(\underline{a}_{ij})$ which has as entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s,j+\lambda t} \mid 1\le i\le s,\ 1\le j\le t. \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ \ j+\lambda t\le n\}.$$

Thus we have the map

$$\underline{st}: M_{m\times n} \to M_{s\times t}: A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map *helix-projection* of type $\underline{st}$. $A\underline{st}$ is a set of s×t-matrices $X=(x_{ij})$ such that $x_{ij}\in\underline{a}_{ij}$, $\forall i,j$. Obviously $A\underline{mn}=A$.

Let $A=(a_{ij})\in M_{m\times n}$ be a matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then it is clear that we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-progection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then if $A\underline{st}$ is not a set but one single matrix then we call A *cut-helix matrix* of type s×t. In other words the matrix A is a helix matrix of type s×t, if $A\underline{cst}= A\underline{st}$.

Definitions 4.3

(a) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,u)$. We define a hope, called *helix-addition* or ***helix-sum***, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where

$$(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})= (a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

(b) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=\min(n,u)$. We define a hope, called *helix-multiplication* or ***helix-product***, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}):(A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\Sigma a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

The helix-sum is external hope since it is defined on different sets and the result is also in different set. The commutativity is valid in the helix-sum. For the helix-product we remark that we have $A\otimes B=A\underline{ms}\cdot B\underline{sv}$ so we have either $A\underline{ms}=A$ or $B\underline{sv}=B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the *helix-Lie Algebra* is defined [17], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where $m<n$. We have analogous results if $m>n$ and for $m=n$ we have the classical theory.

Notation. For given $\kappa\in\mathbb{N}-\{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa}=m$ if the remainder is zero. Thus a matrix $A=(a_{\kappa\lambda})\in M_{m\times n}$, $m<n$ is a *cut-helix matrix* if we have $a_{\kappa\lambda}=a_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$.

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$.

Proposition 4.4 For $m<n$ in $(M_{m\times n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.

Proof. Let $A,B\in M_{m\times n}$ then in the helix-multiplication, since $m<n$, we take helix projection of the matrix A, therefore, the result $A\otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m\times m$. Moreover, in order to have $A\otimes B=A\underline{mm}\cdot B=B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$, is necessarily the left scalar unit.

Let $A=(a_{uv})\in M_{m\times n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\leq\kappa\leq m$, $1\leq\lambda\leq n$, of the form

$$\Sigma\underline{a}_{\kappa s}c_{s\lambda} = \Sigma\underline{a}_{\kappa s}\delta_{s\underline{\lambda}}= \underline{a}_{\kappa\underline{\lambda}}\ni a_{\kappa\lambda}.$$

Therefore $A\otimes I_c\ni A$, $\forall A\in M_{m\times n}$. ■

In the following examples of the helix-hope, we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 4.5 Consider the 2×3 matrices of the forms,

$$A_{\kappa\lambda} = E_{11}+E_{13}+\kappa E_{21}+E_{22}+\lambda E_{23},\ \forall \kappa,\lambda\in\mathbb{Z}.$$

Then we obtain $A_{\kappa\lambda}\otimes A_{st} = \{A_{\kappa+s,\kappa+t},\ A_{\kappa+s,\lambda+t},\ A_{\lambda+s,\kappa+t},\ A_{\lambda+s,\lambda+t}\}$.

Moreover $A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+s}, A_{\kappa+s,\lambda+t}, A_{\kappa+t,\lambda+s}, A_{\kappa+t,\lambda+t}\}$, so

$$A_{\kappa\lambda}\otimes A_{st}\cap A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+t}\},\text{ thus } (\otimes) \text{ is COW.}$$

The helix multiplication (⊗) is associative.

Example 4.6 Consider all *traceless* matrices $A=(a_{ij})\in M_{2\times3}$, in the sense that we have $a_{11}+ a_{22}=0$. The cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively.

Proposition. The Lie-bracket of two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.

Example 4.7 Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, on the set $\boldsymbol{Z}_4=\{0,1,2,3\}$, of the case that only $2\otimes3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\quad a=E_{11}+E_{12}+E_{22},\quad b=E_{11}+2E_{12}+E_{22},\quad c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\quad e=E_{11}+E_{12}+3E_{22},\quad f=E_{11}+2E_{12}+3E_{22},\quad g=E_{11}+3E_{12}+3E_{22},$$

A hyper-matrix representation of four dimensional case with helix-hope:

Example 4.8 On the field of real or complex numbers we consider the four dimensional space of all 2×4 matrices of type, called helix-upper triangular matrices,

$$\boldsymbol{A} = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix}$$

This set is closed under the helix-hope. That means that the helix-product of two such matrices is a 2×4 matrix, of the same type. In fact we have

$$\mathbf{A}\otimes\mathbf{A}' = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' & a' & c' \\ 0 & d' & 0 & d' \end{pmatrix} =$$

$$= \begin{pmatrix} a & \{b,c\} \\ 0 & d \end{pmatrix} \bullet \begin{pmatrix} a' & b' & a' & c' \\ 0 & d' & 0 & d' \end{pmatrix} =$$

$$= \begin{pmatrix} aa' & \{ab'+bd',\ ab'+cd'\} & aa' & \{ac'+bd',\ ac'+cd'\} \\ 0 & dd' & 0 & dd' \end{pmatrix}$$

Therefore the result is a set with 4 matrices.

Examples 4.9

(a) On the same type of matrices using the Construction 4.1, on $(\boldsymbol{Z}_4,+,\cdot)$ we take the small h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, where only $2\otimes 3=\{0,2\}$, where we remind that the fundamental classes are $\{0,2\}$, $\{1,3\}$. We take from the set of all matrices

$$\boldsymbol{A} = \begin{pmatrix} a & b & a & c \\ 0 & d & 0 & d \end{pmatrix}$$

the matrix

$$\boldsymbol{X} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then the powers of this matrix are

$$\boldsymbol{X}^2 = \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{X}^3 = \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

We obtain that the generating set is the following

$$\left\{ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\} \cup \begin{pmatrix} 0 & \{1,3\} & 0 & \{1,3\} \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The classes remain the same.

(b) If we take the matrix

$$\boldsymbol{Y} = \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then the powers of this matrix are

$$\boldsymbol{Y}^2 = \begin{pmatrix} 0 & \{0,3\} & 0 & \{1,2\} \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{Y}^3 = \begin{pmatrix} 0 & \boldsymbol{Z}_4 & 0 & \boldsymbol{Z}_4 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

We obtain that the generating set is the following

$$\left\{ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\} \cup \begin{pmatrix} 0 & \boldsymbol{Z}_4 & 0 & \boldsymbol{Z}_4 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

We have only one class.

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A Delayed Mathematical Model to Break the Life Cycle of Anopheles Mosquito

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Abstract

In this paper, we propose a delayed mathematical model to break the life cycle of anopheles mosquito at the larva stage by incorporating a time delay τ at the larva stage that accounts for the period of growth or development to pupa. We prove local stability of the system's equilibrium and find the critical values for Hopf bifurcation to occur. Also, we find that the system's equilibrium undergoes stability switching from stable to periodic to unstable and vice versa when the time delay τ crosses the imaginary axis from the left half plane to the right half plane in the (Re, Im) plane. Finally, we perform some numerical simulations and the results agree well with the analytical analysis. This is the first time such a model is proposed.

Keywords: Delayed model; Anopheles mosquito; Malaria Control; Hopf bifurcation; Larva; Stability analysis

2010 AMS subject classifications: 97U99.

1 Introduction

Every year, one to three million deaths is attributed to malaria parasite in sub-Saharan Africa out of which one third are children. Much work has been done to genetically modify mosquitoes in the laboratory to hinder the parasite from transmission thus, making the mosquitoes refractory. This can be achieved by inserting of genes at appropriate site to create stable germline. The progress in this area is fairly recent.

Malaria is a killer disease, is one of the leading causes of death in many parts of the world. Its devastating effect has persisted for many decades. Despite the longevity of the disease, malaria, which has been brought under control in some developed countries, still constitutes a major health menace in many developing countries, where most areas of high endemic reside. Some African countries, especially countries within sub-Saharan Africa, still feature among the leading areas of high malaria endemic in the world [21]. According to World Health Organization report [34], an estimated of about 225 million malaria clinical cases occurred in 2009, with an estimated 781,000 malaria mortalities. Although these statistics reflect a reduction compared to an estimated 243 million malaria cases, with an estimated 863,000 malaria deaths, 89% of which occurred in Africa in 2008 [35], the reduction is not sufficient. Generally, susceptibility to malaria is universal, that is, any person living in a country where malaria is prevalent is at risk of contracting the disease. However, the impact of malaria is greatest amongst children below five [36], where one in every five childhood deaths is due to the effects of the disease, among pregnant women, and among people from non-malarious regions.

Temperature is known to affect the life stages of the mosquito parasite [3]. There is a general consensus that future changes in climate may alter the prevalence and incidence of malaria; however, there are conflicting views among authors [20], [39], [40], [11]. However, some authors argued that climate and ecology are the main factors the severity of malaria and the difficulty in controlling it [12]. Other factors that have led to difficulties in controlling malaria are socio-economic conditions, population growth, urbanization, drug resistance, deficiencies in health care systems, poor sanitation, lack of information and education, water storage, garbage disposal, unpaved roads, and drainage systems that generate good breeding stagess for malaria transmission close to human settlements [14], [23], [32]. Thus, research in malaria that integrates the disease dynamics with breeding sites/life cycle properties of the vector and the different developmental stages of the parasite may provide novel insights toward disease control and eradication.

Although malaria is deadly, it can be cured by administering anti-malaria drugs. However, in endemic regions, the malaria parasite develops resistance to such drugs and there is no effective vaccine for malaria. Consequently, prevention is the only other option. Prevention can be achieved through the use of prophylactic drugs and vector control strategies. To advance, plan, design, and implement effective or better vector control measures, a clear understanding of mosquito population dynamics, the disease dynamics, and mosquito interaction with the human population is necessary. We introduce a new approach to the development of models for malaria transmission, wherein the mosquito vector is placed at the centre of the transmission process. Our objective is to develop a mathematical model for the dynamics of malaria transmission that takes into consideration the population dynamics of the malaria vector and how these vectors interact with the human population. To do that, an understanding of the vector population demography and dynamics is needed.

The malaria vector undergoes a complete metamorphosis, as it passes through four different life stages in its cycle: egg, larva, pupa and adult. The egg, larva and pupa stages are aquatic, while the adult stage is terrestrial. The entire cycle from egg laying to the emergence of the adult mosquito takes approximately 7-20 days, with 2-3 days spent in the egg stage, 4-10 days spent in the larva stage, and 2-4 days spent in the pupa stage [14]. While the average life span of the adult female mosquito ranges from 2-3 weeks, that of the males is approximately one week. As for the first three life stages, the life span of the adult mosquito depends on the species and ambient temperature. In addition to natural factors, survival of the adult female Anopheles mosquito also depends on its success in acquiring blood meals from humans. Therefore, in this research we propose a delayed model to break the life cycle at larva stage. To this end, we introduce a time delay τ at the larva compartment to account for the control measures (this can be bio-organism eg copecods or chemical substances). This is the firt such a delayed model is proposed.

2 Model derivation

In this section, we derive the delayed model from the life cycle of anopheles mosquito following the approached used in the paper by [22]. We make the following assumptions: The total population of anopheles mosquito is sub-divided into four compartments (Adults, Eggs, Larva, and Pupa). The birth rate b is constant and proportional to the total population b, there is a time delay τ in the growth or development to pupa at the larva stage cause by the introduction of control measures (can be natural enemy e.g bio-organisms or chemical substances) that can slow the growth process. Anopheles mosquito are assumed to transmit malaria only through direct contact.

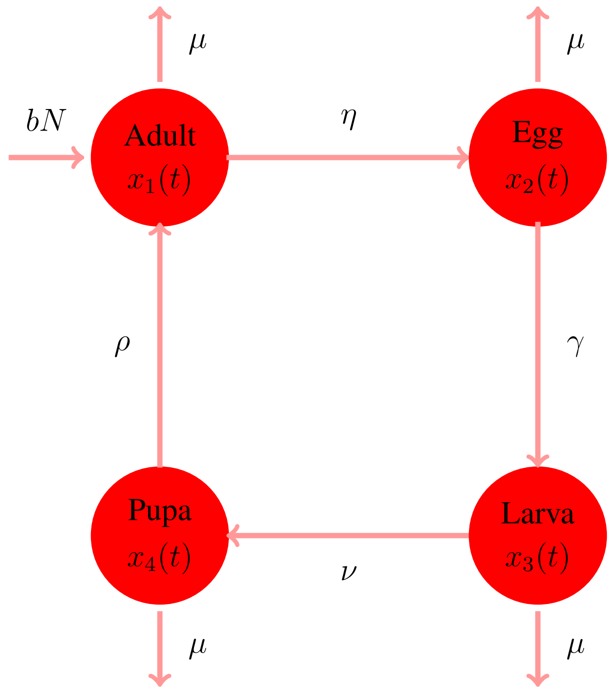


Figure 1: A flow chart of the life cycle of a mosquito

From the model assumptions and the flow chart in figure (1) above, we derive the following model. Let $x_1(t), x_2(t), x_3(t), x_4(t)$ be the number of Adult mosquitoes, Eggs, Larva, and Pupa at time t respectively. Then, the life cycle of anopheles mosquito is represented by the following model:

$$\begin{aligned}
\dot{x}_1(t) &= bN - (\eta + \mu)x_1(t) + \rho x_4(t) \\
\dot{x}_2(t) &= \eta x_1(t) - (\gamma + \mu)x_2(t) \\
\dot{x}_3(t) &= \gamma x_2(t) - \nu x_3(t - \tau) - \mu x_3(t) \\
\dot{x}_4(t) &= \nu x_3(t - \tau) - (\rho + \mu)x_4(t)
\end{aligned} \tag{1}$$

where b is the natural birth rate, η is the rate at which adult mosquitoes oviposit, μ is the natural death rate, γ is the rate at which the eggs hatch, ν is the rate at which larva develops to pupa, ρ is the rate at which pupa develops to adult mosquitoes. The initial data are $x_1(\theta) = \phi_1(\theta), x_2(\theta) = \phi_2(\theta), x_3(\theta) = \phi_3(\theta), x_4(\theta) = \phi_4(\theta))$ for $\tau \in [-\tau, 0]$, where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C([-\tau, 0], \mathbb{R}^4)$ such that $\phi_i \geq 0, i = 1, 2, 3, 4$.

3 Local stability analysis

It is obvious that model (1) has a trivial equilibrium $E^0 = (bN/\mu, 0, 0, 0)$ and a unique positive non-trivial equilibrium $E^* = (x_1^*, x_2^*, x_3^*, x_4^*)$, where

$$x_1^* = \frac{b[(\rho+\mu)(\nu+\mu)N + \rho\nu]}{(\eta+\mu)(\rho+\mu)(\nu+\mu) - \rho\nu\gamma\eta}, \quad x_2^* = \frac{\eta x_1^*}{\gamma+\mu},$$

$$x_3^* = \frac{\gamma\eta x_1^* + b}{\nu+\mu}, \quad x_4^* = \frac{\nu(\gamma\eta x_1^* + b)}{(\nu+\mu)(\rho+\mu)}.$$

The characteristic polynomial equation for the linearised system 1 is

$$\lambda^4 + p_0\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + (q_0\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3)e^{-\lambda\tau} = 0, \tag{2}$$

where

$$\begin{aligned}
p_0 &= 4\,\mu + \rho + \gamma + \eta, \\
p_1 &= \mu\,(\rho + \gamma + \eta + 3\,\mu) + 2\,\eta\,\mu + \eta\,\gamma + 3\,\mu^2 + \eta\,\rho + 2\,\mu\,\rho + \gamma\,\rho + 2\,\mu\,\gamma, \\
p_2 &= \mu\,(2\,\eta\,\mu + \eta\,\gamma + 3\,\mu^2 + \eta\,\rho + 2\,\mu\,\rho + \gamma\,\rho + 2\,\mu\,\gamma) + \eta\,\gamma\,\rho + \mu^3 \\
&\quad + \mu\,\gamma\,\rho + \mu^2\rho + \eta\,\mu^2 + \mu^2\gamma + \eta\,\mu\,\rho + \eta\,\gamma\,\mu, \\
p_3 &= \mu\,(\eta\,\gamma\,\rho + \mu^3 + \mu\,\gamma\,\rho + \mu^2\rho + \eta\,\mu^2 + \mu^2\gamma + \eta\,\mu\,\rho + \eta\,\gamma\,\mu)\,, \\
q_0 &= \nu, \quad q_1 = \nu\,(\rho + \gamma + \eta + 3\,\mu)\,, \\
q_2 &= \nu\,(\mu\,(\rho + \gamma + \eta + 2\,\mu) + \gamma\,\rho + \mu\,\gamma + \eta\,\mu + \eta\,\rho + \mu^2 + \mu\,\rho + \eta\,\gamma)\,, \\
q_3 &= \nu\,\mu\,(\gamma\,\rho + \mu\,\gamma + \eta\,\mu + \eta\,\rho + \mu^2 + \mu\,\rho + \eta\,\gamma)\,.
\end{aligned} \tag{3}$$

If $\tau = 0$ the characteristic equation 2 becomes

$$\lambda^4 + (p_0+q_0)\lambda^3 + (p_1+q_1)\lambda^2 + (p_2+q_2)\lambda + (p_3+q_3) = 0. \tag{4}$$

By Routh-Hurwitz condition, we have the following necessary and sufficient conditions for 4 to have roots with negative real part

$$\begin{aligned}
H_1 &= p_0 + q_0 > 0 \\
H_2 &= (p_0+q_0)(p_1+q_1) - (p_2+q_2) > 0 \\
H_3 &= (p_0+q_0)[(p_1+q_1)(p_2+q_2) - (p_0+q_0)(p_3+q_3)] - (p_2+q_1)^2 > 0 \\
H_4 &= p_3 + q_3 > 0.
\end{aligned} \tag{5}$$

$$H_i > 0, \quad i = 1, 2, 3, 4. \tag{A2}$$

Lemma 3.1.
If A2 is satisfied, then the characteristic equation 4 have roots with negative real part.

The above result is true only when $\tau = 0$.

Now if $\tau > 0$, we let $\lambda = i\xi \ (\xi > 0)$ be a root of the characteristic equation 2, then

$$\xi^4 - ip_0\xi^3 - p_1\xi^2 + ip_2\xi + p_3 + (-iq_0\xi^3 - 2q_1\xi^2 + iq_2\xi + q_3)(\cos(\xi\tau) - i\sin(\xi\tau)) = 0. \tag{6}$$

Separating equation 6 into real and imaginary parts we have

$$\begin{array}{ll} \xi^4 - p_1\xi^2 + p_3 & = (q_1\xi^2 - q_3)\cos(\xi\ (\tau)) + (q_0\xi^3 - q_2\xi)\sin(\xi\ (\tau)), \\ -p_0\xi^3 + p_2\xi + & = (q_0\xi^3 - q_2\xi)\cos(\xi\ (\tau)) - (q_1\xi^2 - q_3)\sin(\xi\ (\tau)). \end{array} \tag{7}$$

Squaring both sides of 7 and adding we have the following

$$\xi^8 + s_0\xi^6 + s_1\xi^4 + s_2^2 + s_3 = 0, \tag{8}$$

where $s_0 = {p_0}^2 - {q_0}^2 - 2\,p_1,\ s_1 = 2\,p_3 + {p_1}^2 + 2\,q_0q_2 - {q_1}^2 - 2\,p_0p_2,$
$s_2 = -2\,p_1p_3 + 2\,q_1q_3 + {p_2}^2 - {q_2}^2,\ s_3 = {p_3}^2 - {q_3}^2$. Let $z = \xi^2$, then

$$z^4 + s_0z^3 + s_1z^2 + s_2z + s_3 = h(z). \tag{9}$$

From 9

$$\frac{dh(z)}{dz} = 4z^3 + 3s_0z^2 + 2s_1z + s_2 = g(z). \tag{10}$$

Let $y = z + \frac{s_0}{4}$ then $g(z) = 0$,

$$\Rightarrow y^3 + ay + b = 0, \tag{11}$$

where $a = \frac{8s_0 - 3s_0^2}{16},\ b = \frac{s_0^3 - 4s_0s_1 + 8s_2}{32}$.

By Cardano's theorem, we have

$$\begin{array}{ll} Q & = \frac{24s_1 - 9s_0^2}{144} \\ R & = \frac{216s_0s_1 - 432s_2 - 54s_0^3}{3456} \\ D & = Q^3 + R^2 \\ K_1 & = \sqrt[3]{R + \sqrt{D}} \\ K_2 & = \sqrt[3]{R - \sqrt{D}} \end{array} \tag{12}$$

and then

$$\begin{array}{ll} z_1 & = K_1 + K_2 - \frac{s_0}{4} \\ z_2 & = -\frac{K_1 + K_2}{2} - \frac{3s_0}{12} + \frac{i\sqrt{3}}{2}(K_1 - K_2) \\ z_3 & = -\frac{K_1 + K_2}{2} - \frac{3s_0}{12} - \frac{i\sqrt{3}}{2}(K_1 - K_2) \end{array} \tag{13}$$

Assume that $D > 0$, then the equation $g(z) = 0$ has one real root namely; $z_1^* = z_1$ and two complex conjugates, if $D = 0$, then all roots of $g(z) = 0$ are real and at least two are equal, namely; $z_1, z_2 = z_3$, where $z_2^* = \max\{z_1, z_2\}$, if $D < 0$, then all roots of $g(z) = 0$ are real and distinct, namely; z_1, z_2, z_3, where $z_3^* = \max\{z_1, z_2, z_3\}$.

According to Lemma 2.2 in Li and Hu [38], we have the following

Lemma 3.2.

1. *If $s_3 < 0$, then equation 9 has at least one positive root.*

2. *If $s_3 \geq 0$, then equation 9 has no positive root if and only if one of these conditions holds:*

$$(a)\ D > 0 \textit{ and } z_1^* \leq 0;\ (b)\ D = 0 \textit{ and } z_2^* \leq 0;\ (c)\ D < 0 \textit{ and } z_3^* \leq 0.$$

3. *If $s_3 \geq 0$, then equation 9 has at least a positive root if and only if one of these conditions holds:*

$$(a)\ D > 0,\ z_1^* > 0, \textit{ and } h(z_1^*) < 0;\ (b)\ D = 0,\ z_2^* > 0 \textit{ and } h(z_2^*) < 0;$$
$$(c)\ D < 0\ z_3^* \leq 0 \textit{ and } h(z_3^*) < 0.$$

Now, suppose that equation 9 have four postive real roots, given by $z_1,\ z_2,\ z_3,\ z_4$, then equation 8 also have positive real roots, namely; $\xi_1 = \sqrt{z_1},\ \xi_2 = \sqrt{z_2},\ \xi_3 = \sqrt{z_3},\ \xi_4 = \sqrt{z_4}$.

From 2, we find the critical time delay τ_0 as follows

$$\tau_n^j = \frac{1}{\xi}\left[\arctan\left\{-\frac{\omega\ \left(q_0\omega^6+(-q_0p_1-q_2+p_0q_1)\,\omega^4+(-p_0q_3+q_2p_1-p_2q_1+q_0p_3)\,\omega^2+p_2q_3-q_2p_3\right)}{(q_1-q_0p_0)\,\omega^6+(-q_3+q_2p_0+q_0p_2-q_1p_1)\,\omega^4+(q_3p_1+q_1p_3-q_2p_2)\,\omega^2-q_3p_3}\right\}+j\pi\right], \tag{14}$$

where $n = 1, 2, 3, 4,\ j = 0, 1, 2, ...$ Then (τ_n^j) are solutions of 6 and $\lambda = \pm i\xi_n$ are a pair of purely imaginary roots of 2 with $\tau = \tau_n^j$. We define

$$\tau_0 = \tau_{n_0}^0 = \min_{1\leq n\leq 4}\{\tau_n^0\},\ \xi_0 = \xi_{n_0},$$

where $n_0 \in \{1, 2, 3, 4\}$. Then τ_0 is the first value of τ such that 2 have purely imaginary roots.

Let $\lambda(\tau) = \alpha(\tau) \pm i\xi(\tau)$ be the root of 2, around $\tau = \tau_n^j$ satisfying $\alpha(\tau_n^j) = 0,\ \xi(\tau_n^j) = \xi_0(n = 1, 2, 3, 4,\ j = 0, 1, 2...)$.

Lemma 3.3.
Suppose $h^{'}(z_n) \neq 0$ $(n = 1, 2, 3, 4)$, where $h(z)$ is defined by 9, then the following transversality condition holds:

$$\frac{d\mathit{Re}\{\lambda(\tau)\}}{d\tau}\bigg|_{\tau=\tau_n^j} \neq 0. \tag{15}$$

Moreover, the sign of $\frac{d\mathit{Re}\{\lambda(\tau)\}}{d\tau}\Big|_{\tau=\tau_n^j}$ is consistent with that of $h^{'}(z_n)$.

Theorem 3.1.
Suppose that A2 holds, we have the following:

1. *The quasi-polynomial 2 have roots with negative real parts and the steady state solution of system 1 is stable if $s_3 \geq 0$ and one of these conditions holds:*

   (a) $D > 0$ *and* $z_1^* \leq 0$; (b) $D = 0$ *and* $z_2^* \leq 0$; (c) $D < 0$ *and* $z_3^* \leq 0$.

2. *The quasi-polynomial 2 have roots with negative real parts and the steady state solution of system 1 is asymptotically stable if $\tau \in [0, \tau_0)0$ for $s_3 < 0$ or $s_3 \geq 0$ and one of these conditions holds:*

   (a) $D > 0,\ z_1^* > 0,$ *and* $h(z_1^*) < 0$; (b) $D = 0,\ z_2^* > 0$ *and* $h(z_2^*) < 0$;
   (c) $D < 0$ $z_3^* \leq 0$ *and* $h(z_3^*) < 0$.

3. *If the conditions in $(2.)$ hold and also $h^{'}(z_n) \neq 0$, then the system 1 have periodic solutions arising from the Hopf bifurcation at $\tau = \tau_n^j(n = 1, 2, 3, 4, j = 0, 1, 2...)$.*

In the figures below, we illustrate the above stability results and also numerically compute real part of the leading eigenvalue of the characteristic equation using traceDDE suite in MATLAB and plot the results in gnuplot. By varying the natural clearance rate μ, we investigate how stability changes in the τ, ν plane, and also the effects on the dynamical behaviour of the system.

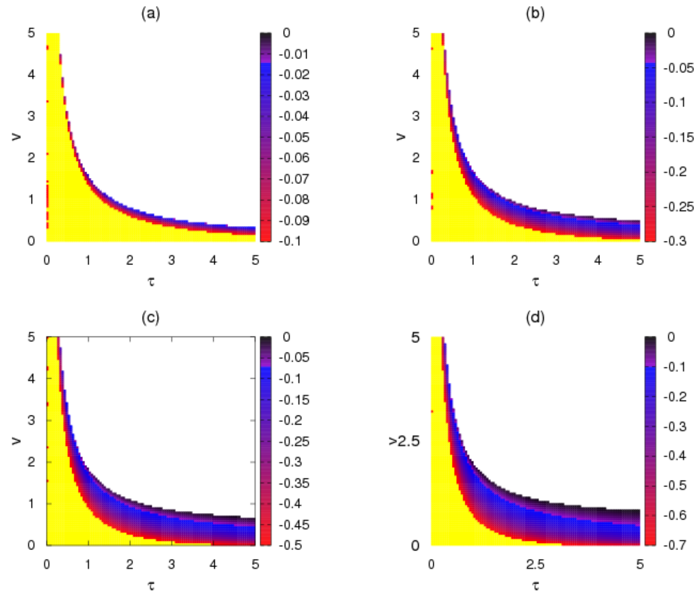
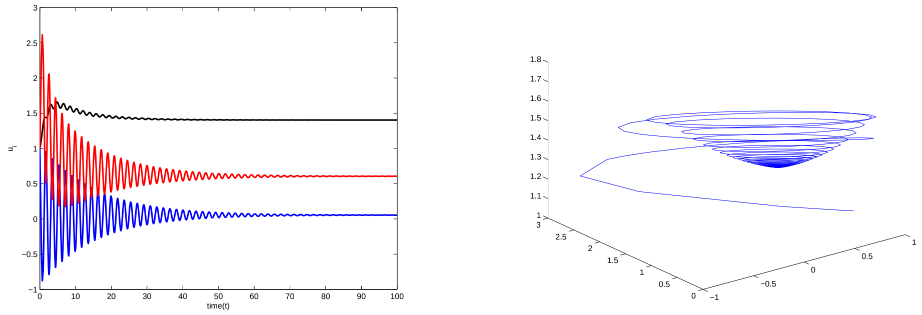


Figure 2: Stability charts: (a) $\mu = 0.1$, (b) $\mu = 0.3$, (c) $\mu = 0.5$, (d) $\mu = 0.7$. The color in the figures corresponds to the real part of the leading eigenvalue of the characteristic quasi-polynomial

From the figures above, we can see that μ played an important role in the dynamical behaviour of the system (1). The colors in the figures stand for: yellow “most stable region”, red “more stable region”, dark-violet “stable region”, black “critical line or Hopf region” and the remaining area (white) corresponds to “unstable region”. As μ increased, the dynamics of the system also increased in the (τ, ν) plane. Again, we observed that change in γ has similar system dynamics as above. Therefore, in general in the (τ, ν) plane, the overall dynamical behaviour of the system is determined by the parameters μ and γ.

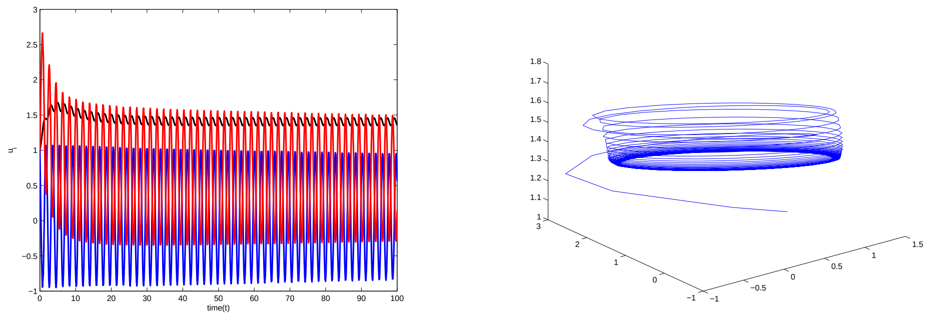
4 Numerical simulation

In this section, we present some numerical simulations using *dde23* suit in Matlab. We will show stable, periodic and unstable solutions as τ is varied. We have stability switches from stable to periodic to unstable and to stable as τ takes on the critical values τ_0 or as τ crosses the imaginary axis. In the first simulation, we take $\tau < \tau_0$, and we have stable solutions ($\tau = 0.45, \tau_0 = 0.85$).



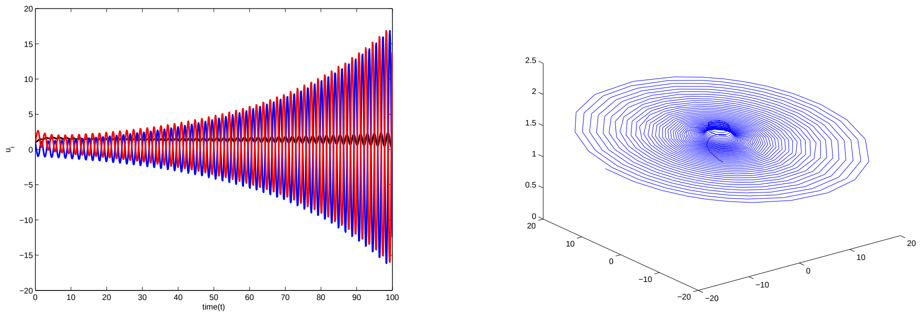
(a) (b)

Figure 3: stable solutions and phase portrait of system 1 for $\tau = 0.45$



(a) (b)

Figure 4: periodic solutions and phase portrait of system 1 for $\tau = 0.85$



(a) (b)

Figure 5: Unstable solutions and phase portrait of system 1 for $\tau = 0.95$

5 Conclusion

In this paper, we derived a mathematical model to break the life cycle of a mosquito that incorporate a time delay at the larva stage that accounts for the period of growth and development to pupa. We prove the local stability of the system's equilibrium and the critical values for Hopf bifurcation to occur. We find that the model undergoes stability switching from stable to periodic and to unstable when the time delay τ crosses the imaginary axis from the left half plane to the right half plane in the (Re, Im) plane. That is, the system's equilibrium E^* is stable if $\tau < \tau_0$ (see figure 3), if $\tau = \tau_0$, E^* loses its stability and a Hopf bifurcation occurs which means, a family of periodic solutions bifurcate from E^* (see figure 4). And if $\tau > \tau_0$, then E^* is unstable as seen in figure 5.

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Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

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Abstract

This paper deals with groups of transformations with finite number of isometries and extends previous studies (Casolaro, F. L. Cirillo and R. Prosperi 2015) which are related to endless groups of transformations with isometrics. In particular, isometries of the tetrahedron and cube, which turn these figures in itself, are presented.

Keywords: Geometric transformations, isometries, symmetry.

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1.Introduction

Compared with the operation of product of isometries, in previous studies, we presented some examples of infinite groups of transformations, whose we highlighted the following properties:

- *The isometries of the space form a group.*
- *The direct isometries of the space form a group, subgroup of the previous group.*
- *The translations of the space form a group, subgroup of the group of direct isometries.*
- *Rotations around a straight form a group, subgroup of direct isometries.*
- *The helical movements all having the same axis form a group, subgroup of the group of direct isometries. In this case, since the helical movements turn out to be products of rotations for translations having the direction of the axis of rotation, also translations (the rotation is reduced to the identity) and rotations (the translation is reduced to the identity) may be considered helical movements.*

It is also possible to obtain groups of transformation with a finite number of isometries.

In particular: about the tetrahedron, we show the *axial symmetry* μ having as an axis line *r*, *rotations* ρ of 120 ° and 240 ° around the height of the tetrahedron outgoing from a fixed vertex, *planar symmetry* σ relative to the plan π passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices; about the cube, *rotations* ρ around a line *r* connecting the centers of two opposite faces, *rotations* ρ around the line *r* joining the midpoints of two opposite edges, *planar symmetry* σ relative to the plan π passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices, *planar symmetry* σ relative to the pane π parallel to two faces passing through the midpoints of the four edges perpendicular to these two faces, *planar symmetries* σ relative to the pane π passing through two opposite edges that do not have face in common and a vertex in common.

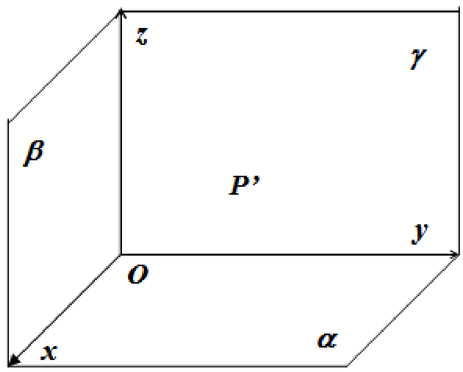

Figure 1

Consider three straight lines x, y, z, passing through the same point O and perpendicular to each other two by two. The three planes α, β, γ, respectively determined by the straight lines x and y, x and z, and y and z, are also perpendicular to each other two by two (Figure 1).

Let be:

I the identity,
s_x the axial symmetry having as an axis the line *x*,
s_y the axial symmetry having as an axis the line *y*,
s_z the axial symmetry having as an axis the line *z*,
s_α the planar symmetry relative to the plane α,
s_β the planar symmetry relative to the plane β,
s_γ the planar symmetry relative to the plane γ,
s_o the symmetry with center *O*,

It occurs that these eight isometries form a group. For this purpose, it is sufficient to prove that the product of any two of them is still one of the eight indicated isometries.

2. Tetrahedron's Isometries

Other examples of finite groups of isometries can be obtained considering all the isometries which leave fixed a given figure *F*, that is, such that in each of them F is united (*F* is transformed into itself). ABCD and A'B'C'D 'are two congruent tetrahedra. Then there exists one and only one isometry that transforms the vertices *A*, *B*, *C*, *D* neatly in the vertices *A '*, *B'*, *C '*, *D'* (Figure 2). This isometry is direct or reverse depending on whether or not the two tetrahedra are equally oriented.

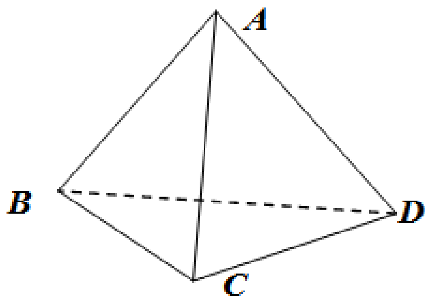

Figure 2

Isometries that turn a tetrahedron *T* into itself are 24 (twenty-four). They form a group S_T , obviously isomorphic to the group S_4 of the 24 permutations on four letters A, B, C, D.

Among the isometries ϕ that transform the tetrahedron *T* into itself, we present the following:

a) The *axial symmetry* μ having as an axis the straight line *r*, joining the midpoints of two opposite sides (*bimedian*), is a rotation of 180 ° around the straight line *r*.
   The symmetries of this type present in the group are 3 (as many as the pairs of opposite sides of the tetrahedron); they have evidently period 2. Therefore there are 3 axial symmetries that leave *T* globally invariant, as many as the pairs of opposite sides.
   A substitution is associated with each of these symmetries (M. Impedovo 1998).
   - With symmetry μ_1 about the line r_1 joining the midpoints of the sides AB and CD, the following substitution is associated:

$$\mu_1 : \begin{pmatrix} A\,B\,C\,D \\ B\,A\,D\,C \end{pmatrix}$$

   - With symmetry μ_2 about the line r_2 joining the midpoints of the sides *AC* e *BD* the following substitution is associated:

$$\mu_2 : \begin{pmatrix} A\,B\,C\,D \\ C\,D\,A\,B \end{pmatrix}$$

   - With symmetry μ_3 about the line r_3 joining the midpoints of the sides *AD* e *BC* the following substitution is associated:

$$\mu_3 : \begin{pmatrix} A\,B\,C\,D \\ D\,C\,B\,A \end{pmatrix}$$

b) The *rotations* ρ of 120 ° and 240 ° around the height of the tetrahedron outgoing from a fixed vertex. For each height of the tetrahedron, you have two rotations of period 3 which hold the summit fixed. Since the tetrahedron heights are 4, these rotations are 8; therefore, there are 8 rotations of this type which transform T into itself, two for each height of the tetrahedron.
A substitution is associated with each of these rotations.

- With rotation ρ_1 about the height outgoing from A the following substitution is associated:

$$\rho_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,C\,D\,B \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_2 : \begin{pmatrix} A\,B\,C\,D \\ A\,D\,B\,C \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_3 about the height outgoing from B the following substitution is associated:

$$\rho_3 : \begin{pmatrix} A\,B\,C\,D \\ C\,B\,D\,A \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_4 : \begin{pmatrix} A\,B\,C\,D \\ D\,B\,A\,C \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_5 about the height outgoing from C the following substitution is associated:

$$\rho_5 : \begin{pmatrix} A\,B\,C\,D \\ B\,D\,C\,A \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_6 : \begin{pmatrix} A\,B\,C\,D \\ D\,A\,C\,B \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_7 about the height outgoing from D the following substitution is associated:

$$\rho_7 : \begin{pmatrix} A\,B\,C\,D \\ B\,C\,A\,D \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_8 : \begin{pmatrix} A\,B\,C\,D \\ C\,A\,B\,D \end{pmatrix}$$ relative to the amplitude of 240°

c) The *planar symmetry* σ relative to the plan π passing through the two vertices of the tetrahedron and the midpoint of the edge that joins the other two vertices. The σ symmetry σ is uniquely determined by the initial vertex. The symmetries of this type are 6 (as many as the pairs of vertices of the tetrahedron), and have period 2.

A substitution is associated with each of these symmetries

- With symmetry about the plane ABM_1, with M_1 medium point of *CD*, the following substitution is associated:

$$\sigma_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,D\,C \end{pmatrix}$$

- With symmetry about the plane ACM_2, with M_2 medium point of *BD*, the following substitution is associated:

$$\sigma_2 : \begin{pmatrix} A\,B\,C\,D \\ A\,D\,C\,B \end{pmatrix}$$

- With symmetry about the plane ADM_3, with M_3 medium point of *BC*, the following substitution is associated:

$$\sigma_3 : \begin{pmatrix} A\,B\,C\,D \\ A\,C\,B\,D \end{pmatrix}$$

- With symmetry about the plane BCM_4, with M_4 medium point of *AD*, the following substitution is associated:

$$\sigma_4 : \begin{pmatrix} A\,B\,C\,D \\ D\,B\,C\,A \end{pmatrix}$$

- With symmetry about the plane BDM_5, with M_5 medium point of AC, the following substitution is associated:

$$\sigma_5 : \begin{pmatrix} A\,B\,C\,D \\ C\,B\,A\,D \end{pmatrix}$$

- With symmetry about the plane CDM_6, with M_6 medium point of AB, the following substitution is associated:

$$\sigma_6 : \begin{pmatrix} A\,B\,C\,D \\ B\,A\,C\,D \end{pmatrix}$$

It is observed that the two sets of isometries described in points a) and b) each supplemented with the identity

$$I : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,C\,D \end{pmatrix}$$

are closed about to the product.

The first set is a G_1 group of order 4 of involutorie transformations. The second set is a G_2 group of order 9 of periodic transformations of order 3.

The union of the two groups is a G_3 group of order 12, which is the group of *direct isometries* of T.

We will now examine the product of three symmetries, or we will fix an isometry σ_k of type c) (planar symmetry), and we will consider an isometry α_t (t = 1, 2, … , 12) variable in the G3 group. The product $\sigma_k \circ \alpha_t$ is still an isometry that changes the tetrahedron T into itself.

They are in number of 12; in fact, if we fix, for example, the isometry

$$\sigma_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,D\,C \end{pmatrix}$$

multiplying each isometry of the G_3 Group for σ_1, we will get 12 *reverse isometries* reverse, which can be summarized as:

$$\sigma_1 \circ \mu_1 = \varphi_1 : \begin{pmatrix} A & B & C & D \\ B & A & C & D \end{pmatrix}, \qquad \sigma_1 \circ \mu_2 = \varphi_2 : \begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix},$$

$$\sigma_1 \circ \mu_3 = \varphi_3 : \begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_1 = \varphi_4 : \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix},$$

$$\sigma \circ \rho_2 = \varphi_5 : \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_3 = \varphi_6 : \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix},$$

$$\sigma \circ \rho_4 = \varphi_7 : \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}, \qquad \sigma_1 \circ \rho_5 = \varphi_8 : \begin{pmatrix} A & B & C & D \\ B & D & A & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_6 = \varphi_9 : \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}, \qquad \sigma_1 \circ \rho_7 = \varphi_{10} : \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_8 = \varphi_{11} : \begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_9 = \varphi_{12} : \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix}.$$

It is easily seen that it results:

$$\phi_{12} = \sigma_1, \quad \phi_5 = \sigma_2, \quad \phi_4 = \sigma_3, \quad \phi_7 = \sigma_4, \quad \phi_6 = \sigma_5, \quad \phi_1 = \sigma_6$$

That is the 12 isometries $\sigma_k \circ \alpha_t$ are given by the 6 planar symmetries σ_k of the type c) and by the 6 antirotations ϕ_k, with period 4. The isometries ϕ_k do not take firm no vertex and no edge of the tetrahedron.

In summary, we can say that the three axial symmetries of the G_1 group, the 8 rotations of the G_2 group, the 6 planar symmetries, the 6 latest found isometries, along with the identity, are the 24 isometries that leave the tetrahedron T globally invariant; their set is the S_T group of isometries of T.

S_T is the group of isometries that change the tetrahedron T in itself.

3.Isometries of Cube

Some examples of finite groups of isometries can be had considering all isometries leaving globally invariant a cube (A. Morelli, 1989).
ABCDEFGH e *A'B'C'D'E'F'G'H'* are two equal cubes. Then there exists one and only one isometry that transforms the vertices *A, B, C, D, E, F, G, H,* neatly in the vertices *A', B', C', D', E', F', G', H'* (Figure 3). This isometry is direct or reverse depending on whether or not the two cubes are equally oriented.

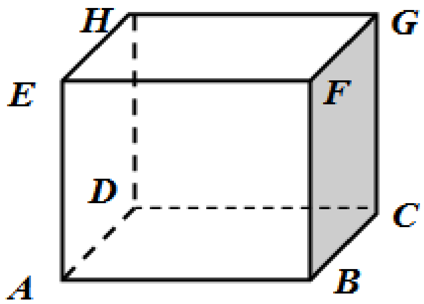

Figure 3

Isometries that transform a C cube to itself are forty eight. They forming a S_c group evidently isomorphic to S_8 group of forty eight permutations on eight letters *A, B, C, D, E, F, G, H.*

Among the isometries that transform the C Cube itself there are obviously the following:

a) The rotations ρ around a straight line r which joins the centers of two opposite faces.
Since the faces of the cube are six, these lines are three; for each of these straight lines the cube is transformed into itself by the amplitude rotations, respectively, 90°, 180°, 270°.
Therefore you have nine rotations of this type which transform C itself.
For each of these rotations it is associated a substitution.
   - To ρ_1 rotation around the straight through M_1M_2, with M_1 the center of the *ABCD* face and M_2 the center of the *EFGH* face, is associated the substitution:

$$\rho_1 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ D\,A\,B\,C\,F\,G\,H\,E \end{pmatrix}$$

relative to the amplitude of 90°

$\rho_2 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ C\,D\,A\,B\,G\,H\,E\,F \end{pmatrix}$ relative to the amplitude of 180°

$\rho_3 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ B\,C\,D\,A\,H\,E\,F\,G \end{pmatrix}$ relative to the amplitude of 270°

- To ρ_4 rotation around the straight through M_3M_4, with M_3 the center of the *ABFE* face and M_4 the center of the *DCGH* face, is associated the substitution:

$\rho_4 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ B\,G\,F\,C\,D\,E\,H\,A \end{pmatrix}$ relative to the amplitude of 90°

$\rho_5 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ G\,H\,E\,F\,C\,D\,A\,B \end{pmatrix}$ relative to the amplitude of 180°

$\rho_6 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ H\,A\,D\,E\,F\,C\,B\,G \end{pmatrix}$ relative to the amplitude of 270°

- To ρ_7 rotation around the straight through M_5M_6, with M_5 the center of the *AEHD* face and M_6 the center of the *BFGC* face, is associated the substitution:

$\rho_7 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ H\,G\,B\,A\,D\,C\,F\,E \end{pmatrix}$ relative to the amplitude of 90°

$\rho_8 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ E\,F\,G\,H\,A\,B\,C\,D \end{pmatrix}$ relative to the amplitude of 180°

$\rho_9 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ D\,C\,F\,E\,H\,G\,B\,A \end{pmatrix}$ relative to the amplitude of 270°

b) The rotations ρ around the straight line r that connects the midpoints of two opposite edges. Since the edges of the cube are twelve, these lines are six; for

each of these straight lines the cube is transformed into itself by rotations of 180 ° amplitude.
For each of these rotations it is associated a substitution.

- To rotation ρ_{10} around the straight line joining the midpoints of AB and EF edges, is associated the substitution:

$$\rho_{10}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix}$$

- To rotation ρ_{11} around the straight line joining the midpoints of CD and HG edges, is associated the substitution:

$$\rho_{11}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix}$$

- To rotation ρ_{12} around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$\rho_{12}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & H \end{pmatrix}$$

- To rotation ρ_{13} around the straight line joining the midpoints of AD and FG edges, is associated the substitution:

$$\rho_{13}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

- To rotation ρ_{14} around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$\rho_{14}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & D \end{pmatrix}$$

- To rotation ρ_{15} around the straight line joining the midpoints of AD and FG edges, is associated the substitution:

$$\rho_{15}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

c) The rotations ρ around the straight line r that contains a diagonal. The number of se lines is four; for each of these straight lines the cube is transformed into itself by the amplitude rotations respectively 120° and 240°. Therefore there are eight rotations of this type which transform C to itself.
For each of these rotations it is associated a substitution.

- To rotation ρ_{16} around the diagonal AF, it is associated the substitution:

$$\rho_{18}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{19}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{18} around the diagonal BE, it is associated the substitution:

$$\rho_{18}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{19}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{20} around the diagonal CH, it is associated the substitution:

$$\rho_{20}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & F & C & B & A & D & E & H \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{21}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & C & F & G & B & A & H \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{21} around the diagonal DG, it is associated the substitution:

$$\rho_{22}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & E & D & A & H & G & B \end{pmatrix}$$ relative to the amplitude of 120°

-

$$\rho_{23}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & H & A & D & C & B & G & F \end{pmatrix}$$ relative to the amplitude of 240°

d) The planar symmetry σ with respect to π plane parallel to two faces through the midpoints of the four edges perpendicular to these two faces. The symmetries of the type indicated are three.
For each of these symmetries it is associated a substitution.

- At the planar symmetry σ_1 with respect to the plane π_1 parallel to *ABGH* and *EFCD* faces, is associated the substitution:

$$\sigma_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}$$

-

- At the planar symmetry σ_2 with respect to the plane π_2 parallel to *ABDC* and *HGEF* faces, is associated the substitution:

$$\sigma_2:\begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}$$

- At the planar symmetry σ_3 with respect to the plane π_3 parallel to *BCGH* and *ADHE* faces, is associated the substitution:

$$\sigma_3:\begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}$$

-

e) The symmetries σ with respect to the π plan through two opposite edges that do not have common face and vertex. The symmetries of the type indicated are six.
For each of these symmetries it is associated a substitution.

- At the planar symmetry σ_4 respect to the π_4 plan through the edges *AD* and *GF* is associated with the substitution:

$$\sigma_4 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

- At the planar symmetry σ_5 respect to the π_5 plan through the edges *BC* and *HE* is associated with the substitution:

$$\sigma_5 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & H \end{pmatrix}$$

-

- At the planar symmetry σ_6 respect to the π_6 plan through the edges *AB* and *EF* is associated with the substitution:

$$\sigma_6 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix}$$

- At the planar symmetry σ_7 with respect to the π_7 plan through the edges *CD* and *HG* is associated with the substitution:

$$\sigma_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix}$$

- At the planar symmetry σ_8 with respect to the π_8 plan through the edges *AH* and *CF* is associated with the substitution:

$$\sigma_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix}$$

- At the planar symmetry σ_9 with respect to the π_9 plan through the edges *BG* and *DE* is associated with the substitution:

$$\sigma_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix}$$

Note that the two sets of isometry described in points a), b) and c), each supplemented with the identity:

$$I:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & C & D & E & F & G & H \end{pmatrix},$$

are closed with respect to the product.

The first set G_1 is a group of order ten, the second set is a group G_2 of order seven, the third set is a group G_3 of order nine. The union of these three groups is a G_4 group of order twenty four which constitutes the group of direct isometries of C.

Let us now examine the product of three symmetries, that is fix an type d) isometry σ_k (planar symmetry), and consider an isometry α_t (t = 1, 2, ..., 24) variable in the G_4 group. The product $\sigma_k \circ \alpha_t$ is still an isometry which changes the C Cube to itself.

The number of these product is twenty four; in fact, it fixed eg. the isometry

$$\sigma_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix},$$

multiplying each isometry of the G_4 group σ_1, you get twentyfour reverse isometries, which can be summarized as:

$$\sigma_1 \circ \rho_1 = \varphi_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix},$$

$$\sigma_1 \circ \rho_2 = \varphi_2:\begin{pmatrix} A & B & C & D & E & F & G & H \\ B & A & D & C & F & E & H & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_3 = \varphi_3:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix},$$

$$\sigma_1 \circ \rho_4 = \varphi_4:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & G & B & A & H & E & D \end{pmatrix},$$

$$\sigma_1 \circ \rho_5 = \varphi_5:\begin{pmatrix} A & B & C & D & E & F & G & H \\ F & E & H & G & B & A & D & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_6 = \varphi_6:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & A & H & G & B & C & F \end{pmatrix},$$

$$\sigma_1 \circ \rho_7 = \varphi_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix},$$

$$\sigma_1 \circ \rho_8 = \varphi_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_9 = \varphi_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix},$$

$$\sigma_1 \circ \rho_{10} = \varphi_{10} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & B & A & D & C & F & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{11} = \varphi_{11} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & F & E & H & G & B & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{12} = \varphi_{12} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{13} = \varphi_{13} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{14} = \varphi_{14} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{15} = \varphi_{15} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{16} = \varphi_{16} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & E & D & A & B & C & F & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_{17} = \varphi_{17} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & G & H & A & D & E & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{18} = \varphi_{18} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & B & G & F & C & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{19} = \varphi_{19} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & G & B & C & D & A & H & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{20} = \varphi_{20} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & C & F & G & H & E & D & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_{21} = \varphi_{21} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & D & E & H & A & B & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_{22} = \varphi_{22} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & F & C & B & G & H & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_{23} = \varphi_{23} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & A & H & E & F & G & B & C \end{pmatrix},$$

$$\sigma_1 \circ I = \varphi_{24} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}.$$

It is easily seen that results:

$$\varphi_{24} = \sigma_1 \,,\; \varphi_8 = \sigma_2 \,,\; \varphi_2 = \sigma_3 \,,\; \varphi_7 = \sigma_6 \,,\; \varphi_9 = \sigma_7 \,,\; \varphi_3 = \sigma_8 \,,\; \varphi_1 = \sigma_9$$

that is, the twentyfour isometries $\sigma_k \circ \alpha_t$ are given from nine symmetries σ_k planar type d), e), and fifteen anti rotations φ_k.

In summary therefore it can be said that the twenty three rotations of the G_4 group, the nine planar symmetries and the latest isometries found, along with the identity, are the forty eight isometries which leave the cube C globally invariant; their set is the S_C group of isometries of the cube C.

S_C is the group of the isometries that change C cube to itself.

Conclusions

As already shown in a previous work (Casolaro, F., Cirillo, L. and Prosperi, R. 2015), the geometric Universe is three-dimensional, so the transformations taking place in it are generated in space. Then, we believe, for a correct analysis of the physical phenomena that occur in the universe, that it is essential to the knowledge of the real transformations that take place in it. Recent results of other branches of mathematics, in particular the modern algebra, have highlighted the interrelationships between movements in the plane and in space with some properties of the Theory of Groups (Casolaro, F. 1992), for which we consider essential to the deepening of these issues both in education and in the field of pure research (Casolaro, F. and Eugeni, F. 1996). Unfortunately, teaching (Casolaro F. 2014) in both the Secondary School that the University has been anchored to old programs that do not take into account the development of mathematics in the last 150 years, so we hope that this work will stimulate teachers and researchers to expand their views.

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