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Max-Min Fuzzy Relation Equations for a Problem of Spatial Analysis

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Abstract

We implement an algorithm that uses a system of max-min fuzzy relation equations (SFRE) for solving a problem of spatial analysis. We integrate this algorithm in a Geographical Information Systems (GIS) tool. We apply our process to determine the symptoms after that an expert sets the SFRE with the values of the impact coefficients related to some parameters of a geographic zone under study. We also define an index of evaluation about the reliability of the results.

Keywords: Fuzzy relation equation, max-min composition, GIS, triangular fuzzy number

2010 AMS subject classification: 03E72, 94D05.

1. Introduction

A Geographical Information System (GIS) is used as a support decision system for problems in a spatial domain. We use a GIS to analyse spatial distribution of data, the impact of event data on spatial areas: this analysis implies the creation of geographic thematic maps. Several authors (cfr., e. g., [3], [4], [7], [8], [25]) solve spatial problems using fuzzy relational calculus. In this paper, we propose an inferential method to solve such problems based on an algorithm for the resolution of a system of fuzzy relation equations (shortly, SFRE) given in [20] (cfr. also [21], [22]) and applied in [10] to solve industrial application problems. Here we integrate this algorithm in the context of a GIS architecture. Usually a SFRE with max-min composition is read as

$$\begin{cases} (a_{11} \wedge x_1) \vee ... \vee (a_{1n} \wedge x_n) = b_1 \\ (a_{21} \wedge x_1) \vee ... \vee (a_{2n} \wedge x_n) = b_2 \\ ... \\ (a_{m1} \wedge x_1) \vee ... \vee (a_{mn} \wedge x_n) = b_m \end{cases} \tag{1}$$

The system (1) is said consistent if it has solutions. Sanchez [23] determines its greatest solution, moreover many researchers have found algorithms which determine minimal solutions of (1) (cfr., e. g., [1], [2], [5], [6], [9], [11]÷[24], [26]). In [20] and [21] a method is described for the consistence of the system (1).

This method has been applied in this paper to real spatial problem in which the input data vary for each subzone of the geographical area. The expert starts from a valuation of input data and he uses linguistic labels for the determination of the output results for each subzone. The input data are the facts or symptoms, the parameters to be determined are the causes. For example, let us consider a planning problem. A city planner needs to determine in each subzone the mean state of buildings (x_1) and the mean soil permeability (x_2), knowing the number of collapsed building in the last year (b_1) and the number of flooding in the last year (b_2). The expert creates the SFRE (1) for each subzone by setting the impact matrix A, whose entries a_{ij} (i=1,…,n and j=1,…,m) represent the impact of the j-th cause x_j to the production of the i-th symptom b_i, where the value of b_i is the membership degree in the corresponding fuzzy set and let B=[b_1,…,b_m]. In another subzone, the input data vector B and the matrix A can vary.

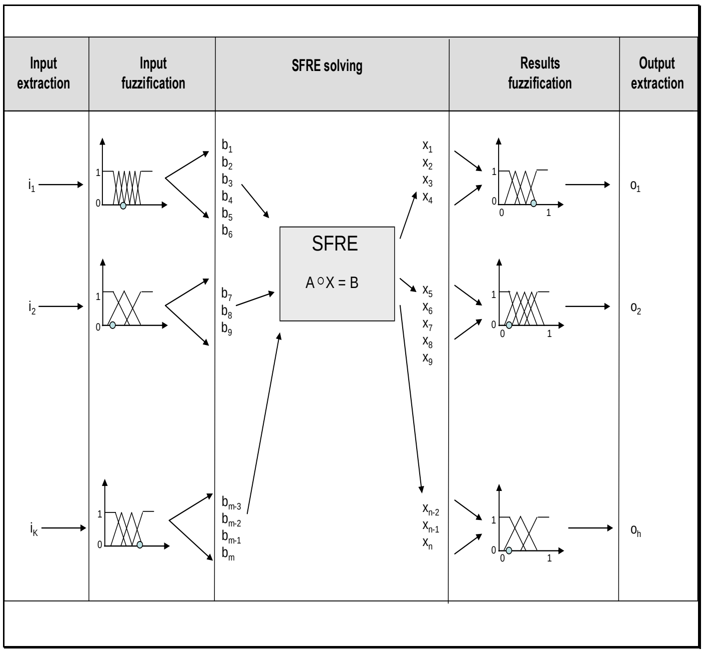

Fig. 1. Resolution process of a SFRE

The process of the resolution of the system (1) is schematized in Fig. 1. We can determine the maximal interval solutions of (1). Each maximal interval solution is an interval whose extremes are the values taken from a lower solution and from the greatest solution. Every value x_i belongs to this interval. If the SFRE (1) is inconsistent, it is possible to determine the rows for which no solution is permitted. If the expert decides to exclude the row for which no solution is permitted, he considers that the symptom b_i (for that row) is not relevant to its analysis and it is not taken into account. Otherwise, the expert can modify the setting of the coefficients of the matrix A to verify if the new system has some solution. In general, the SFRE (1) has T maximal interval solutions $X_{max(1)},\ldots,X_{max(T)}$. In order to describe the extraction process of the solutions, let $X_{max(t)}$, $t\in\{1,\ldots,T\}$, be a maximal interval solution given below, where X^{low} is a lower solution and X^{gr} is the greatest solution. Our aim is to assign the linguistic label of the most appropriate fuzzy sets, usually triangular fuzzy numbers (briefly, TFN), corresponding to the unknown $\{x_{j_1}, x_{j_1},\ldots,x_{j_s}\}$ related to an output variable o_s, s = 1,…,k. For example, assuming that INF(j), MEAN(j), SUP(j) are the three fundamental values of the generic TFN x_j, $j=j_1$, …, j_s, respectively, we can write their membership functions $\mu_{j_1}, \mu_{j_2},\ldots,\mu_{j_h}$ as follows:

$$\mu_{j_1} = \begin{cases} 1 & \text{if } \mathrm{INF}(j_1) \le x \le MEAN(j_1) \\ \dfrac{SUP(j_1)-x}{SUP(j_1)-MEAN(j_1)} & \text{if } MEAN(j_1) < x \le SUP(j_1) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\mu_{j} = \begin{cases} \dfrac{x-INF(j)}{MEAN(j)-INF(j)} & \text{if } \mathrm{INF}(j) \le x \le MEAN(j) \\ \dfrac{SUP(j)-x}{SUP(j)-MEAN(j)} & \text{if } \mathrm{MEAN}(j) < x \le SUP(j) \text{ and } j \in \{j_2,\dots,j_{s-1}\} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\mu_{j_s} = \begin{cases} \dfrac{x-INF(j_s)}{MEAN(j_s)-INF(j_s)} & \text{if } INF(j_s) \le x \le MEAN(j_s) \\ 1 & \text{if } \mathrm{MEAN}(j_s) < x \le SUP(j_s) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

If $XMin_t(j)$ (resp. $XMax_t(j)$) is the min (resp., max) value of every interval corresponding to the unknown x_j, we can calculate the arithmetical mean value $XMean_t(j)$ of the j-th component of the above maximal interval solution $X_{max(t)}$ as

$$XMean_t(j) = \frac{XMin_t(j) + XMax_t(j)}{2} \tag{5}$$

and we get the vector column $XMean_t = [XMean_t(1),\dots, XMean_t(n)]^{-1}$. The value given from $\max\{XMean_t(j_1),\dots,XMean_t(j_s)\}$ obtained for the unknowns $x_{j_1},\dots,x_{j_s}$ corresponding to the output variable o_s, is the linguistic label of the fuzzy set assigned to o_s and it is denoted by $score_t(o_s)$, defined also as reliability of o_s in the interval solution t. For the output vector $O = [o_1,\dots,o_k]$, we define the following reliability index in the interval solution t as

$$\mathrm{Rel}_t(O) = \frac{1}{k} \cdot \sum_{s=1}^{k} score_t(o_s) \tag{6}$$

and then as final reliability index of O, the number $\mathrm{Rel}(O)=\max\{\mathrm{Rel}_t(O):t=1,\ldots,T\}$.

The reliability of our solution is higher, the more the final reliability index Rel(O) close to 1 is. In Section 2 we give an overview of how finding the whole set of the solutions of a SFRE. In Section 3 we show how the proposed algorithm is applied in spatial analysis. Section 4 contains the results of our simulation and it is divided in five subsections.

2. SFRE: An Overview

The SFRE (1) is abbreviated in the following known form:

$$A \circ X = B$$

where $A = (a_{ij})$, is the matrix of coefficients, $X = (x_1, x_2,\ldots, x_n)^{-1}$ is the column vector of the unknowns and $B = (b_1,b_2,\ldots,b_m)^{-1}$ is the column vector of the known terms, being $a_{ij}, x_j, b_i \in [0,1]$ for each $i = 1,\ldots,m$ and $j = 1,\ldots,n$. We have the following definitions and terminologies: the whole set of all solutions X of the SFRE (1) is denoted by Ω. A solution $\hat{X} \in \Omega$ is called a minimal solution if $X \le \hat{X}$ for some $X \in \Omega$ implies $X= \hat{X}$, where "$\le$" is the partial order induced in Ω from the natural order of [0, 1]. We also recall that the system (1) has the unique greatest (or maximum) solution $X^{gr} = (x_1^{gr}, x_2^{gr},\ldots,x_n^{gr})^{-1}$ if $\Omega \ne \emptyset$ [23]. A matrix interval $X_{interval}$ of the following type:

$$X_{interval} = \begin{pmatrix} [a_1, b_1] \\ [a_2, b_2] \\ [\ldots,\ldots] \\ [a_n, b_n] \end{pmatrix}$$

where $[a_j,b_j] \subseteq [0,1]$ for each $j=1,\ldots,n$, is called an interval solution of the SFRE (1) if every $X=(x_1,x_2,\ldots,x_n)^{-1}$ such that $x_j \in [a_j, b_j]$ for each $j = 1,\ldots,n$, belongs to Ω. If a_j is a membership value of a minimal solution and b_j is a membership value of X^{gr} for each $j = 1,\ldots,n$, then $X_{interval}$ is called a maximal interval solution of the SFRE (1) and it is denoted by $X_{max(t)}$, where t varies from 1 till to the number of minimal solutions. The SFRE (1) is said to be in normal form if $b_1 \geq b_2 \geq \ldots \geq b_m$. The time computational complexity to reduce a SFRE in a normal form is polynomial [20, 22]. Now we consider the matrix $A^* = (a^*_{ij})$ so defined:

$$a^*_{ij} = \begin{cases} 0 & \text{if } a_{ij} < b_i \\ b_i & \text{if } a_{ij} = b_i \\ 1 & \text{if } a_{ij} > b_i \end{cases}$$

where i = 1,…,m and j = 1,…,n, that is a^*_{ij} is S—type coefficient (Smaller) if $a_{ij}<b_i$, E—type coefficient (Equal) if $a_{ij}=b_i$ and G—type coefficient (Greater) if $a_{ij}>b_i$. A^* is called augmented matrix and the system $A^* \circ X = B$ is said associated to the SFRE (1). Without loss of generality, from now on we suppose that the system (1) is in normal form. We also the following definitions and results from [16, 17, 20, 22].

Definition 1. Let SFRE (1) be consistent and $A^*_j = \{a^*_{1j},..., a^*_{mj}\}$. If A^*_j contains G-type coefficients and $k \in \{1,\ldots,m\}$ is the greatest index of row such that $a^*_{kj} = 1$, then the following coefficients in A^*_j are called selected:

- a^*_{ij} for $i \in \{1,\ldots,k\}$ with $a^*_{ij} \geq b_i = b_k$,

- a^*_{ij} for $i \in \{k+1,\ldots,m\}$ with $a^*_{ij} = b_i$.

Definition 2. If A^*_j not contains G-type coefficients, but it contain E-type coefficients and $r \in \{1,\ldots,m\}$ is the smallest index of row such that $a^*_{rj} = b_r$, then any $a^*_{ij} = b_i$ in A^*_j for $i \in \{r,\ldots,m\}$ is called selected.

Theorem 1. Let us consider a SFRE (1). Then

- The SFRE (1) is consistent if and only if there exist at least one selected coefficient for each i-th equation, i=1,…,m.

- The complexity time function for determining the consistency of the SFRE (1) is O(m·n).

Consequently, when a SFRE (1) is inconsistent, the equations for which no element is a selected coefficient, could not be satisfied simultaneously with the other equations having at least one selected coefficient. Furthermore a vector IND=(IND(1),…,IND(m)) is defined by setting IND(i) equal to the number of selected coefficients in the i*th* equation for each *i* = 1,...,m. If IND(i) = 0, then all the coefficients in the i*th* equation are not selected and the system is inconsistent. The system is consistent if IND(i) ≠ 0 if for each i = l,...,m and the product

$$PN2 = \prod_{i=1}^{m} IND(i)$$

gives the upper bound of the number of the eventual minimal solutions.

Theorem 2. Let SFRE (1) be consistent. Then
- the SFRE has an unique greatest solution X^{gr} with component $x_j^{gr} = b_k$ if the j*th* column A_j^* contains selected G-type coefficients a_{kj}^* and $x_j^{gr} = 1$ otherwise.

- The complexity time function for computing X^{gr} is O(m·n).

A help matrix H=[h_{ij}], i = 1,…,m and j = 1,…,n, is defined as follows:

$$h_{ij} = \begin{cases} b_i & \text{if } a_{ij}^* \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

Let $|H_i|$ be the number of coefficients h_{ij} in the i*th* equation of the SFRE (1). Then the number of potential minimal solutions cannot exceed the value

$$PN1 = \prod_{i=1}^{m} |H_i|$$

and one has $PN2 \leq PN1$.

Definition 3. Let $h_i = (h_{i1}, h_{i2}, ..., h_{in})$ and $h_k = (h_{k1}, h_{k2}, ..., h_{kn})$ be the i*th* and the k*th* rows of the matrix H. If for each j=1,…n, $h_{ij} \neq 0$ implies both $h_{kj} \neq 0$ and $h_{kj} \leq h_{ij}$, then the i*th* row (resp. equation) is said dominant over the k*th* row in H (resp. equation) or that the k*th* row (resp. equation) is said dominated by the i*th* row (resp. equation).

If the i*th* equation is dominant over the k*th* equation in (1), then the k*th* equation is a redundant equation of the system. By using Definition 3, we can build a matrix of dimension m×n, called dominance matrix *H**, having components:

$$h_{ij}^* = \begin{cases} 0 & \text{if the ith equation is dominated by another equation} \\ h_{ij} & \text{otherwise} \end{cases}$$

For each i= 1, ...,m, now we set $|H_i^*|$ as the number of coefficients $h_{ij}^* = b_i \neq 0$ in the i*th* row of the dominance matrix *H**. When this value is 0, we set $|H_i^*|$ = 1. Then the number of potential minimal solutions of the SFRE cannot exceed the value

$$PN3 = \prod_{i=1}^{m} |H*_i|$$

being $PN3 \leq PN2 \leq PN1$ [17, 20 ,22]. There the authors use the symbol $\left\langle \frac{b_i}{j} \right\rangle$ to indicate the coefficients $h_{ij}^* = b_i \neq 0$. We have $h_{ij}^* \wedge x_j = b_i$ if $x_j \in [b_i, 1]$ and $x_j = b_i$ is the j*th* component of a minimal solution. A solution of the i*th* equation can be written as

$$H_i = \sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle$$

In [20,22] the concept of concatenation W is introduced to determine all the components of the minimal solutions and it is given by

$$W = \prod_{i=1}^{m} H_i = \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle \right)$$

We can determine the minimal solutions $X^{low(t)} = (x_1^{low(t)}, x_2^{low(t)}, ..., x_n^{low(t)})^{-1}$, $t \in \{1, ..., PN(3)\}$, with components

$$x_j^{low(t)} = \begin{cases} \mathrm{b}_{\mathrm{i}_t} & \text{if } \mathrm{b}_{\mathrm{i}_t} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

In order to determine if a SFRE is consistent, hence its greatest solution and minimal solutions, we have used the universal algorithm of [20,22] based on the above concepts. For brevity of presentation, here we do not give this algorithm which has been implemented and tested under C++ language. The C++ library has been integrated in the ESRI ArcObject Library of the tool ArcGIS 9.3 for a problem of spatial analysis illustrated in the next Section 3.

3. SFRE in Spatial Analysis

We consider a specific area of study on the geographical map on which we have a spatial data set of “causes” and we want to analyse the possible “symptoms”.

We divide this area in P subzones where a subzone is an area in which the same symptoms are derived by input data or facts, and the impact of a symptom on a cause is the same one as well. It is important to note that even if two subzones have the same input data, they can have different impact degrees of symptoms on the causes. For example, the cause that measures the occurrence of floods may be due with different degree of importance to the presence of low porous soils or to areas subjected to continuous rains. Afterwards the area of study is divided in homogeneous subzones, hence the expert creates a fuzzy partition for the domain of each input variable and he determines the values of the symptoms b_i, as the membership degrees of the corresponding fuzzy sets (cfr., input fuzzification process of Fig. 1) for each subzone on which the expert sets the most significant equations and the values a_{ij} of impact of the j-th cause to the i-th symptom. After the determination of the set of maximal interval solutions, the expert for each interval solution calculates, for each unknown x_j, the mean interval solution $X_{mean(t)}$ with (5). The linguistic label $Rel_t(o_s)$ is assigned to the output variable o_s . Then he calculates the reliability index $Rel_t(O)$, given from formula (6), associated to this maximal interval solution t. After the iteration of this step, the expert determines the reliability index (6) for each maximal interval solution, by choosing the output vector O for which Rel(O) assumes the maximum value. Iterating the process for all the subzones (cfr., Fig. 2), the expert can show the thematic map of each output variable. If the SFRE related to a specific subzone is inconsistent, the expert can decide whether or not eliminate rows to find solutions: in the first case, he decides that the symptoms associated to the rows that make the system inconsistent are not considered and eliminates them, so reducing the number of the equations. In the second case, he decides that the corresponding output variable for this subzone remain unknown and it is classified as unknown on the map.

4. Simulation Results

Here we show the results of an experiment in which we apply our method to census statistical data agglomerated on four districts of the east zone of Naples (Italy). We use the year 2000 census data provided by the ISTAT (Istituto Nazionale di Statistica). These data contain informations on population, buildings, housing, family, employment work for each census zone of Naples. Every district is considered as a subzone with homogeneous input data given in Table 2.

In this experiment, we consider the following four output variables: "o_1 = *Economic prosperity*" (wealth and prosperity of citizens), "o_2 = *Transition into the job*" (ease of finding work), "o_3 = *Social Environment*" (cultural levels of citizens) and “o_4 = *Housing development*” (presence of building and residential dwellings of new construction). For each variable, we create a fuzzy partition composed by three TFNs called “low”, “mean” and “high” presented in Table 1.

Moreover, we consider the following seven input parameters: i_1=percentage of people employed=number of people employed/total work force, i_2=percentage of women employed=number of women employed/number of people employed,

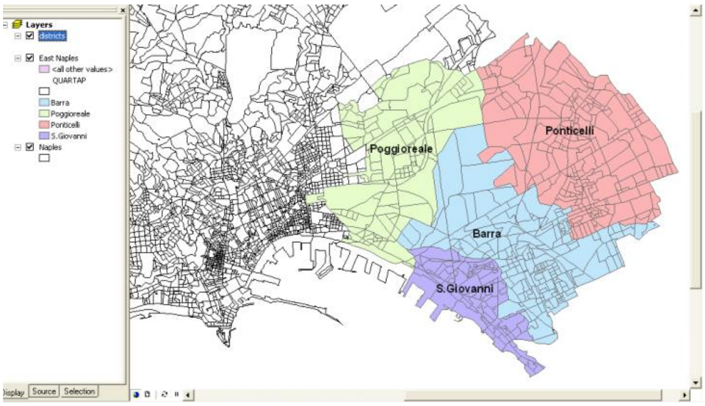

Fig. 2. Area of study: four districts at east of Naples (Italy)

Table 1. Values of the TFNs low, mean, high

Output	low			mean			high		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
o_1	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_2	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_3	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_4	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0

i_3=percentage of entrepreneurs and professionals = number of entrepreneurs and professionals/number of people employed, i_4 = percentage of residents graduated=numbers of residents graduated/number of residents with age > 6 years, i_5=percentage of new residential buildings=number of residential buildings built since 1982/total number of residential buildings, i_6 = percentage of residential dwellings owned=number of residential dwellings owned/ total number of residential dwellings, i_7 = percentage of residential dwellings with central heating system = number of residential dwellings with central heating system/total number of residential dwellings. In Table 4 we show these input data for the four subzones.

Table 2. Input data given for the four subzones

District	i_1	i_2	i_3	i_4	i_5	i_6	i_7
Barra	0.604	0.227	0.039	0.032	0.111	0.424	0.067
Poggioreale	0.664	0.297	0.060	0.051	0.086	0.338	0.149
Ponticelli	0.609	0.253	0.039	0.042	0.156	0.372	0.159
S. Giovanni	0.576	0.244	0.041	0.031	0.054	0.353	0.097

Table 3. TFNs values for the input domains

Input Var	low			Mean			High		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
i_1	0.00	0.40	0.60	0.40	0.60	0.80	0.60	0.80	1.00
i_2	0.00	0.10	0.30	0.10	0.30	0.40	0.30	0.50	1.00
i_3	0.00	0.04	0.06	0.04	0.06	0.10	0.07	0.20	1.00
i_4	0.00	0.02	0.04	0.02	0.04	0.07	0.04	0.07	1.00
i_5	0.00	0.05	0.08	0.05	0.08	0.10	0.08	0.10	1.00
i_6	0.00	0.10	0.30	0.10	0.30	0.60	0.30	0.60	1.00
i_7	0.00	0.10	0.30	0.10	0.30	0.50	0.30	0.50	1.00

Table 4: TFNs for the symptoms $b_1 \div b_{12}$

Subzone	b_1: i_1 = low	b_2: i_1 = me-an	b_3: i_1 = hi-gh	b_4: i_2 = low	b_5: i_2= me-an	b_6: i_2 = hi-gh	b_7: i_3 = low	b_8: i_3 = me-an	b_9: i_3 = hi-gh	b_{10}: i_4 = low	b_{11}: i_4 = me-an	b_{12}: i_4 = hi-gh
Barra	0.00	0.98	0.02	0.36	0.63	0.00	1.00	0.00	0.00	0.40	0.60	0.00
Poggioreale	0.00	0.93	0.07	0.01	0.99	0.00	0.00	1.00	0.00	0.00	0.63	0.37
Ponticelli	0.00	0.91	0.05	0.23	0.76	0.00	1.00	0.00	0.00	0.00	0.93	0.07
S. Giovanni	0.12	0.88	0.00	0.28	0.72	0.00	0.95	0.05	0.00	0.45	0.55	0.00

The expert indicates a fuzzy partition for each input domain formed from three TFNs labeled “low”, “mean” and “high”, whose values are reported in Table 3. In Tables 4 and 5 we show the values of TFNS for the 21 symptoms $b_1,...,b_{21}$. In order to form the SFRE (1) in each subzone, the expert defines the most significant symptoms.

Table 5: TFNs for the symptoms $b_{13} \div b_{21}$

Subzone	b_{13}: i_5 = low	b_{14}: i_5 = mean	b_{15}: i_5 = high	b_{16}: i_6 = low	b_{17}: i_6 = mean	b_{18}: i_6 = high	b_{19}: i_7 = low	b_{20}: i_7 = mean	b_{21}: i_7 = high
Barra	0.00	0.00	0.10	0.00	0.59	0.41	1.00	0.00	0.00
Poggioreale	0.00	0.70	0.30	0.00	0.87	0.13	0.75	0.25	0.00
Ponticelli	0.00	0.00	1.00	0.00	0.76	0.24	0.70	0.30	0.00
S. Giovanni	0.87	0.13	0.00	0.00	0.82	0.18	1.00	0.00	0.00

4.1 Subzone "Barra"

The expert chooses the significant symptoms b_2, b_4, b_5, b_7, b_{10}, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, by obtaining a SFRE (1) with m = 10 equations and n = 12 unknowns. The matrix A of the impact values a_{ij} has dimensions 10×12 and the vector B of the symptoms b_i has dimension 10×1 and both are given below. The SFRE (1) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain four maximal interval solutions $X_{max(t)}$ (t=1,…,4) and we calculate the vector column $XMean_t$ on each maximal interval solution. Hence we associate to the output variable o_s (s = 1,…,4), the linguistic label of the fuzzy set with the higher value calculated with formula (5) obtained for the corresponding unknowns $x_{j_1},...,x_{j_s}$ and given in Table 6. For determining the reliability of our solutions, we use the index given by formula (6). We obtain that $Rel_t(o_1) = Rel_t(o_2) = Rel_t(o_3) = Rel_t(o_4) = 0.6025$ for t=1,…,4 and hence $Rel(O)=\max\{Rel_t(O): t=1,\ldots,4\}=0.6025$ where $O=\{o_1,\ldots o_4\}$. We note that the same final set of linguistic labels associated to the output variables o_1 = "high", o_2 = "mean", o_3 = "low", o_4 = "low" is obtained as well. The relevant quantities are given below.

$$
A = \begin{pmatrix}
0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\
0.3 & 0.5 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 & 0.6 & 0.2 & 0.0 & 0.0 & 0.0 \\
0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.0 & 0.0 & 0.0 \\
1.0 & 0.2 & 0.0 & 0.8 & 0.3 & 0.1 & 0.8 & 0.2 & 0.2 & 0.3 & 0.0 & 0.0 \\
0.5 & 0.3 & 0.1 & 0.6 & 0.4 & 0.1 & 0.6 & 0.4 & 0.1 & 0.1 & 0.0 & 0.0 \\
0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.3 & 0.3 \\
0.2 & 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.2 & 0.5 & 0.1 & 0.3 & 0.7 & 0.3 \\
0.1 & 0.4 & 0.4 & 0.1 & 0.4 & 0.4 & 0.1 & 0.5 & 0.5 & 0.2 & 0.4 & 0.5 \\
0.5 & 0.2 & 0.0 & 0.4 & 0.3 & 0.0 & 0.4 & 0.3 & 0.0 & 1.0 & 0.1 & 0.0
\end{pmatrix}
\quad
B = \begin{pmatrix}
0.98 \\ 0.36 \\ 0.63 \\ 1.00 \\ 0.40 \\ 0.60 \\ 0.10 \\ 0.59 \\ 0.41 \\ 1.00
\end{pmatrix}
$$

$$
X_{\max(1)} = \begin{pmatrix}
[0.40,0.40] \\ [0.36,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.41,0.41] \\ [1.00,1.00] \\ [0.00,0.10] \\ [0.00,0.10]
\end{pmatrix}
\quad
X_{\max(2)} = \begin{pmatrix}
[0.40,0.40] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.36,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.41,0.41] \\ [1.00,1.00 \\ [0.00,0.10] \\ [0.00,0.10]
\end{pmatrix}
\quad
X_{\max(3)} = \begin{pmatrix}
[0.40,0.40] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.36,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.41,0.41] \\ [1.00,1.00] \\ [0.00,0.10] \\ [0.00,0.10]
\end{pmatrix}
\quad
X_{\max(4)} = \begin{pmatrix}
[0.40,0.40] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.36,0.36] \\ [0.00,1.00] \\ [0.00,0.36] \\ [0.00,1.00] \\ [0.36,0.36] \\ [0.41,0.41] \\ [1.00,1.00] \\ [0.00,0.10] \\ [0.00,0.10]
\end{pmatrix}
$$

$$XMean_1 = \begin{pmatrix} 0.40 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.18 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.05 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}$$

Table 6. Final linguistic labels for the output variables in the district Barra

Output variable	$score_1(o_s)$	$score_2(o_s)$	$score_3(o_s)$	$score_4(o_s)$
o_1	high	high	high	high
o_2	mean	mean	mean	mean
o_3	low	low	low	low
o_4	low	low	low	low

For determining the reliability of our solutions, we use the index given by formula (6). We obtain $Rel(O_k) = 0.4675$ for k = 1,..,12. Then we obtain two final sets of linguistic labels associated to the output variables: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "low", and o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally.

4.2 Subzone "Poggioreale"

The expert choices the significant symptoms b_2, b_5, b_8, b_{11}, b_{12}, b_{14}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, by obtaining a SFRE (1) with m = 11 equations and n = 12 unknowns. The matrix A of the impact values a_{ij} has sizes dimension 11×12 and the column vector B of the symptoms b_i has sizes 11×1 are given below. The SFRE (7) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 12 maximal interval solutions $X_{max(t)}$ (t=1,…,12) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 7 contains the output variables and the relevant quantities are given below.

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 0.9 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.0 & 0.0 & 0.1 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.3 & 0.4 & 0.6 & 0.2 & 0.4 & 0.6 & 0.0 & 0.1 & 0.2 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 \\ 0.0 & 0.1 & 0.5 & 0.1 & 0.2 & 0.5 & 0.1 & 0.2 & 0.5 & 0.0 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.0 & 0.8 & 0.5 & 0.3 & 0.5 & 0.3 & 0.1 & 0.7 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.3 & 0.6 & 0.2 \end{pmatrix} \quad B = \begin{pmatrix} 0.93 \\ 0.99 \\ 1.00 \\ 0.63 \\ 0.37 \\ 0.70 \\ 0.30 \\ 0.87 \\ 0.13 \\ 0.75 \\ 0.25 \end{pmatrix}$$

$$X_{\max(1)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.13,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(2)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.13,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} X_{\max(3)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(4)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix}$$

$$X_{\max(5)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(6)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(7)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.0] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(8)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.13,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix}$$

$$X_{\max(9)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.13,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(10)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.13,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.00,0.13] \end{pmatrix} \quad X_{\max(11)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.25,0.25] \\ [0.00,0.25] \\ [0.13,0.13] \end{pmatrix} \quad X_{\max(12)} = \begin{pmatrix} [0.37,0.37] \\ [0.00,0.30] \\ [0.00,0.13] \\ [0.75,0.75] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,1.00] \\ [0.00,0.13] \\ [0.00,0.13] \\ [0.00,0.25] \\ [0.25,0.25] \\ [0.13,0.13] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.130 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.050 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.130 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix} XMean_3 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.130 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.065 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.130 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix}$$

$$XMean_5 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.130 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.05 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.130 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.050 \end{pmatrix} XMean_7 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.130 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.065 \end{pmatrix} \quad XMean_8 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.130 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.065 \end{pmatrix}$$

$$XMean_9 = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.130 \\ 0.250 \\ 0.125 \\ 0.050 \end{pmatrix} \quad XMean_{10} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.130 \\ 0.125 \\ 0.250 \\ 0.050 \end{pmatrix} XMean_{11} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.250 \\ 0.125 \\ 0.130 \end{pmatrix} \quad XMean_{12} = \begin{pmatrix} 0.370 \\ 0.150 \\ 0.065 \\ 0.750 \\ 0.065 \\ 0.065 \\ 0.500 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.250 \\ 0.130 \end{pmatrix}$$

For determining the reliability of our solutions, we use the index given by formula (6). We obtain $Rel(O_k) = 0.4675$ for $k = 1,..,12$. Then we obtain two final sets of linguistic labels associated to the output variables: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "low", and o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally.

Table 7. Final linguistic labels for the output variables in the district "Poggioreale"

	Linguistic labels associated to											
output variable	$XMean_1$	$XMean_2$	$XMean_3$	$XMean_4$	$XMean_5$	$XMean_6$	$XMean_7$	$XMean_8$	$XMean_9$	$XMean_{10}$	$XMean_{11}$	$XMean_{12}$
o_1	low	low	low	high	low	low	low	high	low	low	low	high
o_2	low	low	low	mean	low	low	low	mean	low	low	low	mean
o_3	low	low	low	low	low	low	low	low	low	low	low	low
o_4	low	mean	low	mean	low	mean	low	mean	low	mean	low	mean

4.3 Subzone: District Ponticelli

The expert choices the significant symptoms b_2, b_4, b_5, b_7, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, obtaining a SFRE (7) with m = 10 equations and n = 12 variables: The matrix A of sizes 10×12 and the column vector B of dimension 10×1 are given by:

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.0 & 0.2 & 0.1 & 0.0 & 0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 1.0 & 0.1 & 0.0 & 0.8 & 0.2 & 0.2 & 0.3 & 0.1 & 0.0 \\ 0.4 & 0.8 & 0.1 & 0.3 & 0.9 & 0.1 & 0.2 & 0.8 & 0.1 & 0.1 & 0.3 & 0.0 \\ 0.0 & 0.1 & 1.0 & 0.1 & 0.3 & 0.7 & 0.1 & 0.3 & 0.7 & 0.0 & 0.1 & 1.0 \\ 0.3 & 0.7 & 0.3 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.3 & 0.7 & 0.3 \\ 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.0 & 0.4 & 0.2 & 0.1 & 0.4 & 0.2 & 0.1 & 0.7 & 0.2 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.3 & 0.5 & 0.1 \end{pmatrix} \quad B = \begin{pmatrix} 0.91 \\ 0.23 \\ 0.76 \\ 1.00 \\ 0.93 \\ 1.00 \\ 0.76 \\ 0.24 \\ 0.70 \\ 0.30 \end{pmatrix}$$

The SFRE (7) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 8 maximal interval solutions $X_{max(t)}$ (t=1,…,8) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 10 contains the output variables and the relevant quantities are given below.

$$X_{max(1)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{max(2)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [1.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{max(3)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix} \quad X_{max(4)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix}$$

$$X_{max(5)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{max(6)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [1.00,1.00] \\ [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [0.00,1.00] \end{pmatrix} \quad X_{max(7)} = \begin{pmatrix} [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix} \quad X_{max(8)} = \begin{pmatrix} [0.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.00,0.76] \\ [0.00,1.00] \\ [0.00,1.00] \\ [0.76,0.76] \\ [0.00,1.00] \\ [0.70,1.00] \\ [0.00,0.30] \\ [1.00,1.00] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 1.00 \\ 0.38 \\ 1.00 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.5 \\ 0.38 \\ 1.00 \\ 1.00 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix} \quad XMean_4 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 1.00 \\ 0.76 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix}$$

$$XMean_5 = \begin{pmatrix} 1.00 \\ 0.38 \\ 1.00 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.50 \\ 0.38 \\ 1.00 \\ 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 0.50 \end{pmatrix} XMean_7 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix} \quad XMean_8 = \begin{pmatrix} 0.50 \\ 0.38 \\ 0.50 \\ 1.00 \\ 0.38 \\ 0.50 \\ 0.50 \\ 0.76 \\ 0.50 \\ 0.85 \\ 0.15 \\ 1.00 \end{pmatrix}$$

Now we associate to the output variables o_s k = 1,…,4, the linguistic label of the fuzzy set with the higher $XMean_j$ obtained for the corresponding unknowns x_{j_1} ,…, x_{j_s} obtaining:

Table 8. Final linguistic labels for the output variables in the district "Ponticelli"

Linguistic labels associated to								
output variable	$XMean_1$	$XMean_2$	$XMean_3$	$XMean_4$	$XMean_5$	$XMean_6$	$XMean_7$	$XMean_8$
o_1	Low-high	high	low	Low -high	Low -high	high	low	Low -high
o_2	mean	low	mea n	low	Low -high	low	Low -high	low
o_3	Low-high	Low-high	Low -high	Low -high	mea n	mea n	mea n	mea n
o_4	low	low	low	low	low	low	low	low

Here "low-high" indicates that the membership degree of both the fuzzy sets with linguistic labels "low" and "high" have the maximal value for that output variable. We obtain for each solution $Rel(O_1)$ =0.565, $Rel(O_2)$ = 0.625, $Rel(O_3)$

= 0.565 $Rel(O_4) = 0.5$, $Rel(O_5) = 0.565$, $Rel(O_6) = 0.69$, $Rel(O_7) = 0.565$ $Rel(O_8) = 0.565$.

Thus we choice the solution O_6 which have the greatest reliability $Rel(O_6) = 0.69$. Our solution for this subzone is: o_1 = "high", o_2 = "low", o_3 = "mean", o_4 = "low".

4.4 Subzone: district S. Giovanni

The expert choices the significant symptoms b_2, b_4, b_5, b_7, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, obtaining a SFRE (1) with m = 12 equations and n = 12 variables: The matrix A of sizes 12×12 and the column vector B of sizes 12×1 are given by:

$$A = \begin{pmatrix} 0.3 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 & 0.1 & 0.0 & 0.0 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.0 & 0.3 & 0.0 \\ 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.2 & 0.0 \\ 1.0 & 0.2 & 0.0 & 1.0 & 0.1 & 0.0 & 0.9 & 0.1 & 0.0 & 0.3 & 0.1 & 0.0 \\ 0.5 & 0.3 & 0.1 & 0.5 & 0.3 & 0.1 & 0.6 & 0.3 & 0.1 & 0.2 & 0.1 & 0.0 \\ 0..3 & 0.6 & 0.2 & 0.2 & 0.5 & 0.2 & 0.2 & 0.8 & 0.2 & 0.0 & 0.2 & 0.0 \\ 0.6 & 0.3 & 0.1 & 0.5 & 0.2 & 0.1 & 0.5 & 0.2 & 0.1 & 0.8 & 0.2 & 0.0 \\ 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.1 & 0.4 & 0.1 \\ 0.3 & 0.6 & 0.3 & 0.3 & 0.6 & 0.3 & 0.3 & 0.6 & 0.3 & 0.3 & 0.7 & 0.1 \\ 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.0 & 0.1 & 0.5 \\ 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.0 & 0.4 & 0.1 & 0.0 & 1.0 & 0.0 & 0.0 \end{pmatrix} \qquad B = \begin{pmatrix} 0.12 \\ 0.88 \\ 0.28 \\ 0.72 \\ 0.95 \\ 0.45 \\ 0.55 \\ 0.87 \\ 0.13 \\ 0.82 \\ 0.18 \\ 1.0 \end{pmatrix}$$

The SFRE (1) is inconsistent and eliminating the rows for which the value IND(j) = 0, we obtain 6 maximal interval solutions $X_{max(t)}$ (t=1,…,6) and we calculate the vector column $XMean_t$ on each maximal interval solution. Table 11 contains the output variables and the relevant quantities are given below.

$$X_{\max(1)} = \begin{pmatrix} [0.12,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max,(2)} = \begin{pmatrix} [0.12,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max,(3)} = \begin{pmatrix} [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix}$$

$$X_{\max(4)} = \begin{pmatrix} [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max(5)} = \begin{pmatrix} [0.00,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix} \quad X_{\max(6)} = \begin{pmatrix} [0.00,0.12] \\ [0.00,0.55] \\ [0.00,1.00] \\ [0.00,0.12] \\ [0.72,0.72] \\ [0.00,1.00] \\ [0.12,0.12] \\ [0.55,0.55] \\ [0.00,1.00] \\ [1.00,1.00] \\ [0.13,0.13] \\ [0.18,0.18] \end{pmatrix}$$

$$XMean_1 = \begin{pmatrix} 0.12 \\ 0.55 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_2 = \begin{pmatrix} 0.12 \\ 0.275 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.55 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_3 = \begin{pmatrix} 0.06 \\ 0.55 \\ 0.50 \\ 0.12 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix}$$

$$XMean_{41} = \begin{pmatrix} 0.06 \\ 0.275 \\ 0.50 \\ 0.12 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.55 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_5 = \begin{pmatrix} 0.06 \\ 0.55 \\ 0.50 \\ 0.06 \\ 0.72 \\ 0.50 \\ 0.06 \\ 0.275 \\ 0.50 \\ 1.00 \\ 0.13 \\ 0.18 \end{pmatrix} \quad XMean_6 = \begin{pmatrix} 0.060 \\ 0.275 \\ 0.500 \\ 0.060 \\ 0.720 \\ 0.500 \\ 0.120 \\ 0.550 \\ 0.500 \\ 1.000 \\ 0.130 \\ 0.180 \end{pmatrix}$$

Table 9. Final linguistic labels for the output variables in the district "San Giovanni"

output variable	linguistic label associated to $XMean_1$	linguistic label associated to $XMean_2$	linguistic label associated to $XMean_3$	linguistic label associated to $XMean_4$	linguistic label associated to $XMean_5$	linguistic label associated to $XMean_6$
o_1	mean	high	mean	high	mean	high
o_2	mean	mean	mean	mean	mean	mean
o_3	high	mean	high	mean	high	mean
o_4	low	low	low	low	low	low

We obtain $Rel(O_k) = 0.6925$ for $k = 1,\ldots,6$. Thus we obtain two final sets of linguistic labels associated to the output variables: o_1 = "mean", o_2 = "mean", o_3 = "high", o_4 = "low", and o_1 = "high", o_2 = "mean", o_3 = "mean", o_4 = "low" with the same reliability index value 0.6925. The expert prefers to choose the first solution: o_1 = "mean", o_2 = "mean", o_3 = "high", o_4 = "low", because he considers in this district that in the two years the presence of residents was graduated and consequently, the cultural level of citizens has increased, whereas the average pro capite wealth of citizens has decreased.

4.5 Thematic maps and conclusions

Finally, we obtain four final thematic maps shown in Figs. 3, 4, 5, 6 for the output variable o_1, o_2, o_3, o_4, respectively.

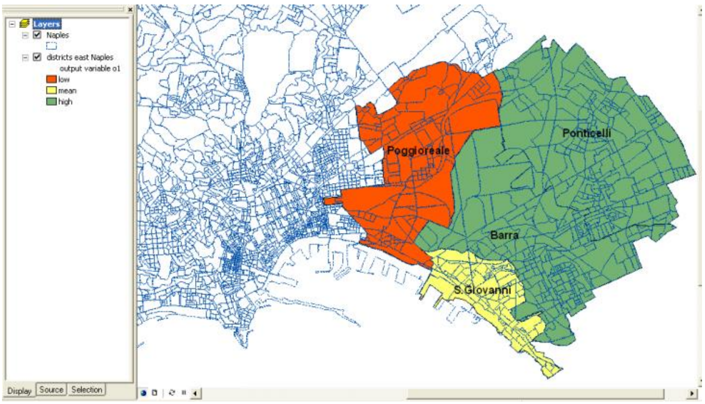

Fig. 3. Thematic map for output variable o_1 (*Economic prosperity*)

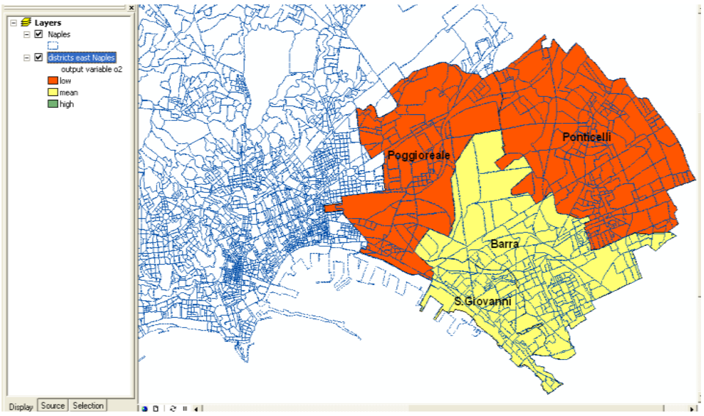

Fig. 4. Thematic map of the output variable o_2 (*Transition into the job*)

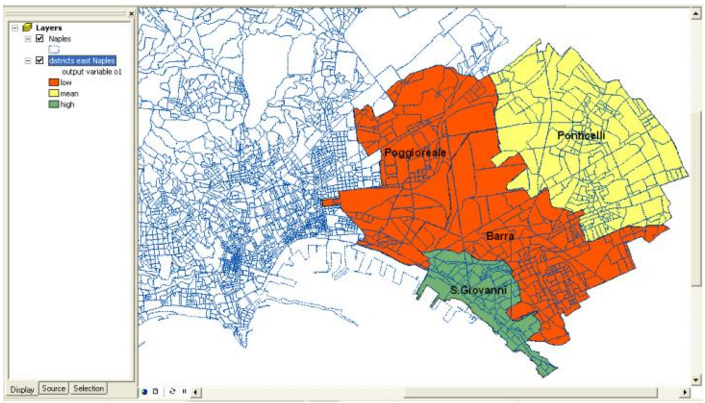

Fig. 5. Thematic map for the output variable o_3 (*Social Environment*)

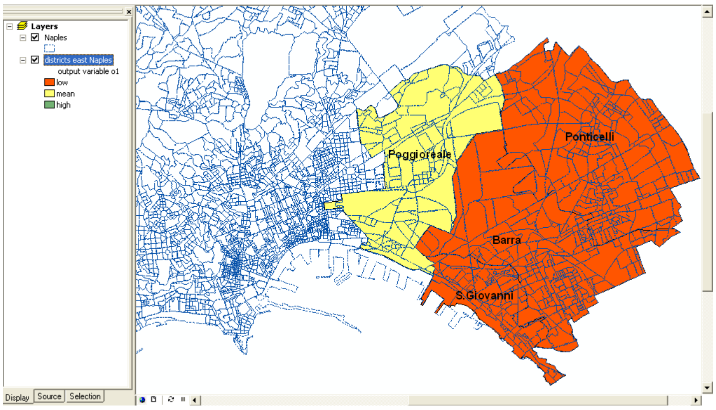

Fig. 6. Thematic map for the output variable o_4 (*Housing development*)

The results show that there was no housing development in the four districts in the last 10 years and there is difficulty in finding job positions. In Fig. 7 we show the histogram of the reliability index Rel(O) for each subzone, where $O=[o_1,o_2,o_3,o_4]$.

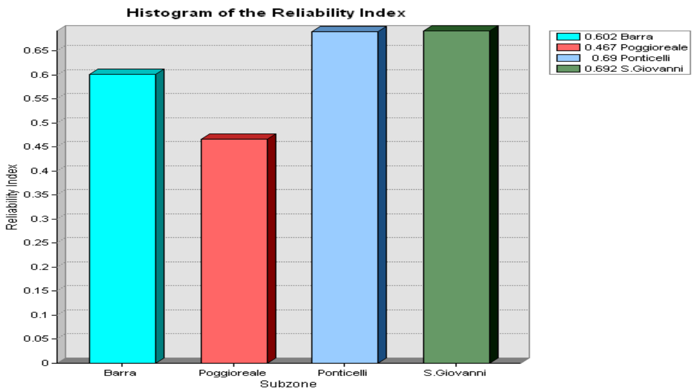

Fig. 7. Histogram of the reliability index Rel(O) for the four subzones.

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References

1. Chen, L., Wang, P.: Fuzzy Relational Equations (I): the General and Specialized Solving Algorithms. Soft Computing 6, 428—435 (2002)

2. De Baets, B.: Analytical Solution Methods for Fuzzy Relational Equations. In: Dubois, D., Prade, H., (eds.) Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Vol. 1, pp. 291—340. Kluwer Academic Publishers, Dordrecht (2000)

3. Di Martino, F., Loia, V., Sessa, S.: A Fuzzy-Based Tool for Modelization and Analysis of the Vulnerability of Aquifers: a Case Study. International Journal of Approximate Reasoning 38, 98—111 (2005)

4. Di Martino, F., Loia, V., Sessa, S., Giordano, M.: An Evaluation of the Reliability of a GIS Based on the Fuzzy Logic in a Concrete Case Study. In: Petry, F.E., Robinson,V.B., Cobb, M.A. (eds.) Fuzzy Modeling with Spatial Information for Geographic Problems, pp. 185—208. Springer, Heidelberg (2005)

5. Di Martino, F., Loia, V., Sessa, S.: Extended Fuzzy C-Means Clustering Algorithm for Hotspot Events in Spatial Analysis, International Journal of Hybrid Intelligent Systems 5 (1), 31—44 (2008)

6. Di Nola, A., Pedrycz, W., Sessa, S., Sanchez, E.: Fuzzy Relation Equations and Their Application to Knowledge Engineering. Kluwer Academic Press, Dordrecht, 1989

7. Groenemans, R., Van Ranst, E., Kerre, E.: Fuzzy Relational Calculi in Land Evaluation. Geoderma 77 (2-4), 283—298 (1997)

8. Hemetsberger, M., Klinger, G., Niederer, S., Benedikt, J.: Risk Assessment of Avalanches - a Fuzzy GIS Application. In: Ruan, D., D'hondt, P., Kerre, E.E. (eds.) Proceedings of 5th International FLINS. Conference Computational Intelligent Systems for Applied Research, pp. 397—402. World Scientific, Singapore (2002)

9. Higashi, M., Klir, G.J.: Resolution of Finite Fuzzy Relation Equations. Fuzzy Sets and Systems 13 (1), 65—82 (1984)

10. Kyosev, Y.: Diagnostics of Sewing Process Using Fuzzy Linear Systems. In: Proceedings

of 5th International Conference Textile Science TEXSCI 2003. CD ROM Edition. Liberec, (2003)

11. Li, P., Fang, S.C..: A Survey on Fuzzy Relational Equations, Part I: Classification and Solvability. Fuzzy Optimation and Decision Making 8, 179—229 (2009)

12. Markovskii, A.V. : On the Relation between Equations with Max-Product Composition and the Covering Problem. Fuzzy Sets and Systems 153, 261—273 (2005)

13. Miyakoshi, M., Shimbo, M.: Minimal Solutions of Systems of Fuzzy Equations. Fuzzy Sets and Systems 19, (1986) 37—46

14. Pappis, C.P., Adamopoulos, G.: A Computer Algorithm for the Solution of the Inverse Problem of Fuzzy Systems. Fuzzy Sets and Systems 39, 279—290 (1991)

15. Pappis, C.P., Sugeno, M.: Fuzzy Relational Equations and the Inverse Problem. Fuzzy Sets and Systems 15, 79—90 (1985)

16. Peeva, K.: Systems of Linear Equations over a Bounded Chain. Acta Cybernetica 7(2), 195-202 (1985)

17. Peeva, K.: Fuzzy Linear Systems. Fuzzy Sets and Systems 49, 339—355 (1992)

18. Peeva, K.: Fuzzy Linear Systems –Theory and Applications in Artificial Intelligence Areas, DSc Thesis, Sofia (2002) (in Bulgarian)

19. Peeva, K.: Resolution of Min–Max Fuzzy Relational Equations. In: Nikravesh, M., Zadeh , L.A., Korotkikh, V. (eds.) Fuzzy Partial Differential Equations and Relational Equations, pp. 153—166. Springer, Heidelberg (2004)

20. Peeva, K.: Universal Algorithm for Solving Fuzzy Relational Equations. Italian Journal of Pure and Applied Mathematics 19, 9—20 (2006)

21. Peeva, K., Kyosev, Y.: Algorithm for Solving Max-product Fuzzy Relational Equations, Soft Computing 11 (7), 593—605 (2007)

22. Peeva, K., Kyosev, Y.: Fuzzy Relational Calculus: Theory, Applications and Software (with CD-ROM). Series Advances in Fuzzy Systems-Applications and Theory, vol. 22, World Scientific, Singapore (2004)

23. Sanchez, E.: Resolution of Composite Fuzzy Relation Equations. Information and Control, 30, 38—48 (1976)

24. Shieh, B.S.: New Resolution of Finite Fuzzy Relation Equations with Max-Min Composition. Internat. J. of Uncertainty, Fuzziness Knowledge Based Systems 16 (1), 19—33 (2008)

25. Sicat,, R.S., Carranza, E.J.M., Nidumolu, U.B.: Fuzzy modeling of farmers' kno- wledge for land suitability classification. Agricultural Systems 83, 49—75 (2005)

26. Wu, Y.K., Guu, S.M.: An efficient procedure for solving a fuzzy relational equati- on with max-Archimedean t-norm composition. IEEE Transactions on Fuzzy Syst- ems 16 (1), 73—84 (2008)