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Fundamental hoop-algebras

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Abstract

In this paper, we investigate some results on hoop algebras and hyper hoop-algebras. We construct a hoop and a hyper hoop on any countable set. Then using the notion of the fundamental relation we define the fundamental hoop and we show that any hoop is a fundamental hoop and then we construct a fundamental hoop on any non-empty countable set.

Keywords: hoop algebras, hyper hoop algebras, (strong) regular relation,fundamental relations.

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1 Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids were originally introduced by Bosbach in [7] under the name of complementary semigroups. It was proved that a hoop is a meet-semilattice. Hoop-algbras then investigated by Büchi and Owens in an unpublished manuscript [8] of 1975, and they have been studied by Blok and Ferreirim[2],[3], and Aglianò et.al.[1]. The study of hoops is motivated by researchers both in universal algebra and algebraic logic.In recent years, hoop theory was enriched with deep structure theorems.

Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic(see Theorem 3.8 of [1]) introduced by Hájek in [13]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops and MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

Hypersructure theory was introduced in 1934[15], by Marty. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography[9]. Many researchers have worked on this area. The authors applied hyper structure theory on hyper hoop and introduced and studied hyper hoop algebras in [17]and[16].

In this paper, we investigate some new results on hoop-algebras and hyper hoop-algebras. We construct a hoop and a hyper hoop on any countable set. Then using the notion of the fundamental relation we define the fundamental hoop.

2 Preliminaries

First, we recall following basic notions of the hypergroup theory from[10]: Let A be a non-empty set. A hypergroupoid is a pair $(A, \odot)$, where $\odot : A \times A \longrightarrow P(A) - \{\emptyset\}$ is a binary hyperoperation on A. If associativity low holds, then $(A, \odot)$ is called a semihypergroup, and it is said to be commutative if $\odot$ is commutative. An element $1 \in A$ is called a unit, if $a \in 1 \odot a \cap a \odot 1$, for all $a \in A$ and is called a scaler unit, if $1 \odot a = a \odot 1 = \{a\}$, for all $a \in A$. Note that if $B, C \subseteq A$, then we consider $B \odot C$ by $B \odot C = \bigcup\limits_{b \in B, c \in C} (b \odot c)$. (See [10])

Definition 2.1. [3] A *hoop-algebra* or briefly *hoop* is an algebra $(A, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that, (HP1): $(A, \odot, 1)$ is a commutative monoid and for all $x, y, z \in A$, (HP2): $x \rightarrow x = 1$, (HP3): $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and (HP4): $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$. On hoop A we define "$x \leq y$" if and only if $x \rightarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A.

Definition 2.2. [17] A *hyper hoop-algebra* or briefly, a *hyper hoop* is a non-empty set A endowed with two binary hyperoperations $\odot, \rightarrow$ and a constant 1 such that, for all $x, y, z \in A$ satisfying the following conditions,
(HHA1) $(A, \odot, 1)$ is a commutative semihypergroup with 1 as the unit,
(HHA2) $1 \in x \rightarrow x$,
(HHA3) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$,
(HHA4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,
(HHA5) $1 \in x \rightarrow 1$,
(HHA6) if $1 \in x \rightarrow y$ and $1 \in y \rightarrow x$ then $x = y$,
(HHA7) if $1 \in x \rightarrow y$ and $1 \in y \rightarrow z$ then $1 \in x \rightarrow z$.

In the sequel we will refer to the hyper hoop $(A, \odot, \rightarrow, 1)$ by its universe A. On hyper hoop A, we define $x \leq y$ if and only if $1 \in x \rightarrow y$. If A is a hyper hoop, it is easy to see that $\leq$ is a partial order relation on A. Moreover, for all $B, C \subseteq A$ we define $B \ll C$ iff there exist $b \in B$ and $c \in C$ such that $b \leq c$ and define $B \leq C$ iff for any $b \in B$ there exists $c \in C$ such that $b \leq c$. A hyper hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.

Proposition 2.3. In any hyper hoop $(A, \odot, \rightarrow, 1)$, if $x \odot y$ and $x \rightarrow y$ are singletons, for any $x, y \in A$, then $(A, \odot, \rightarrow, 1)$ is a hoop. Then hyper hoops are a generalization of hoops and every hoop is a trivial hyper hoop.

Proposition 2.4. [17] Let A be a *hyper hoop*. Then for all $x, y, z \in A$ and $B, C, D \subseteq A$, the following hold,
(HHA8) $x \odot y \ll z \Leftrightarrow x \leq y \rightarrow z$,
(HHA9) $B \odot C \ll D \Leftrightarrow B \ll C \rightarrow D$,
(HHA10) $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x)$,
(HHA11) $z \rightarrow y \ll (x \rightarrow z) \rightarrow (x \rightarrow y)$,
(HHA12) $1 \odot 1 = \{1\}$.

Notations: Let **R** be an equivalence relation on hyper hoop A and $B, C \subseteq A$. Then $B\mathbf{R}C$, $B\overline{\mathbf{R}}C$ and $B\overline{\overline{\mathbf{R}}}C$ denoted as follows,
(i) $B\mathbf{R}C$ if there exist $b \in B$ and $c \in C$ such that $b\mathbf{R}c$,
(ii) $B\overline{\mathbf{R}}C$ if for all $b \in B$ there exists $c \in C$ such that $b\mathbf{R}c$ and for all $c \in C$ there exists $b \in B$ such that $b\mathbf{R}c$,
(iii) $B\overline{\overline{\mathbf{R}}}C$ if for all $b \in B$ and $c \in C$, we have $b\mathbf{R}c$.

Remark 2.5. It is clear that $B\overline{\mathbf{R}}C$ and $C\overline{\mathbf{R}}D$ imply that $B\overline{\mathbf{R}}D$, for all $B, C, D \subseteq A$.

Definition 2.6. [17] Let **R** be an equivalence relation on hyper hoop A. Then **R** is called a *regular relation* on A if and only if for all $x, y, z \in A$,
(i) if $x\mathbf{R}y$, then $x \odot z\overline{\mathbf{R}}y \odot z$,
(ii) if $x\mathbf{R}y$, then $x \rightarrow z\overline{\mathbf{R}}y \rightarrow z$ and $z \rightarrow x\overline{\mathbf{R}}z \rightarrow y$,
(iii) if $x \rightarrow y\mathbf{R}\{1\}$ and $y \rightarrow x\mathbf{R}\{1\}$, then $x\mathbf{R}y$.

Definition 2.7. [17] Let **R** be an equivalence relation on hyper hoop A. Then **R** is called a strong regular relation on A if and only if, for all $x, y, z \in A$,
(i) if $x\mathbf{R}y$, then $x \odot z\overline{\overline{\mathbf{R}}}y \odot z$,
(ii) if $x\mathbf{R}y$, then $x \rightarrow z\overline{\overline{\mathbf{R}}}y \rightarrow z$ and $z \rightarrow x\overline{\overline{\mathbf{R}}}z \rightarrow y$,

Theorem 2.8. *[17] Let **R** be a regular relation on hyper hoop A and $\frac{A}{\mathbf{R}}$ be the set of all equivalence classes respect to **R**, that is $\frac{A}{\mathbf{R}} = \{[x] | x \in A\}$. Then $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$*

is a hyper hoop, which is called the quotient hyper hoop of A respect to $\boldsymbol{R}$*, where for all* $[x], [y] \in \frac{A}{\boldsymbol{R}}$,

$$[x] \otimes [y] = \{[t] | t \in x \odot y\} \quad \textit{and} \quad [x] \hookrightarrow [y] = \{[z] | z \in x \to y\}$$

Theorem 2.9. *[17] Let* $\boldsymbol{R}$ *be a strong regular relation on hyper hoop A. Then* $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ *is a hoop which is called the quotient hoop of A respect to* $\boldsymbol{R}$.

Theorem 2.10. *[4] Let X and Y be two sets such that $|X| = |Y|$. If $(Y, \leq, 0)$ is a well-ordered set, then there exists a binary order relation on X and $x_0 \in X$, such that $(X, \leq, x_0)$ is a well-ordered set.*

Lemma 2.11. *[14] Let X be an infinite set. Then for any set $\{a, b\}$, we have $|X \times \{a, b\}| = |X|$.*

3 Constructing of hoops

In this section, we show that we can construct a hoop on any non-empty countable set.

Lemma 3.1. *Let A and B be two sets such that $|A| = |B|$. If A is a hoop, then we can construct a hoop on B by using of A.*

Proof. *Since $|A| = |B|$, there exists a bijection $\varphi : A \to B$. For any $b_1, b_2 \in B$. We define the binary operations $\odot_B$ and $\to_B$ on B by,*

$$b_1 \odot_B b_2 = \varphi(a_1 \odot_A a_2) \quad \textit{and} \quad b_1 \to_B b_2 = \varphi(a_1 \to_A a_2)$$

where $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$ and $a_1, a_2 \in A$. It is easy to show that $\odot_B$ and $\to_B$ are well-defined. Moreover, for any $b \in B$ we define 1_B as $1_B = \varphi(1_A)$. Now, by some modification we can show that $(B, \odot_B, \to_B, 1_B)$ is a hoop.□

Lemma 3.2. *For any $k \in \mathbb{N}$, we can construct a hoop on $\mathbb{W}_k = \{0, 1, 2, 3, ..., k-1\}$.*

Proof. *Let $k \in \mathbb{N}$. We define the operations "$\odot$" and "$\to$", on $\mathbb{W}_k$ as follows, for all $a, b \in \mathbb{W}_k$,*

$$a \odot b = \begin{cases} 0 & \textit{if } a + b \leq k-1, \\ a + b - k + 1 & \textit{otherwise} \end{cases}$$

$$a \to b = \begin{cases} k-1 & \textit{if } a \leq b, \\ k - 1 - a + b & \textit{otherwise} \end{cases}$$

Now, we show that $(\mathbb{W}_k, \odot, \to, k-1)$ *is a hoop,*
(***HP1***)*: Since,* $+$ *is commutative, hence* $\odot$ *is commutative. Now, we show that* $\odot$ *is associative on* $\mathbb{W}_k$*. For all* $a, b, c \in \mathbb{W}_k$*,*
Case 1*: If* $a+b \leq k-1$ *and* $b+c \leq k-1$*, then* $(a \odot b) \odot c = (0) \odot c = 0$ *and* $a \odot (b \odot c) = a \odot 0 = 0$ *and so* $(a \odot b) \odot c = a \odot (b \odot c)$*.*
Case 2*: If* $a+b > k-1$ *and* $b+c \leq k-1$*, since* $a+b+c \leq 2(k-1)$ *and so* $a+b+c-k+1 \leq k-1$*, we get* $(a \odot b) \odot c = (a+b-k+1) \odot c = 0$*. On the other hand,* $a \odot (b \odot c) = a \odot 0 = 0$ *and then* $(a \odot b) \odot c = a \odot (b \odot c)$*.*
Case 3*: If* $a+b > k-1$ *and* $b+c > k-1$*, then* $(a \odot b) \odot c = (a+b-k+1) \odot c$ *and* $a \odot (b \odot c) = a \odot (b+c-k+1)$*. If* $a+b+c \leq 2k$ *then* $(a \odot b) \odot c = a \odot (b \odot c) = 0$ *and if* $a+b+c > 2k$ *then* $(a \odot b) \odot c = a \odot (b \odot c) = a+b+c-2k+2$*.*
Case 4*: Let* $a+b \leq k-1$ *and* $b+c > k-1$*. This case is similar to the Case 2. Now, we have* $0 \odot k-1 = 0$ *and if* $0 \neq a \in \mathbb{W}_k$*, we have* $a+(k-1) > k-1$ *and so* $a \odot (k-1) = a+k-1-k+1 = a$*. Then* $(k-1)$ *is the identity of* $(\mathbb{W}_k, \odot)$ *and so* $(\mathbb{W}_k, \odot, k-1)$ *is a commutative monoid.*
(***HP2***)*: It is clear that, for all* $a \in \mathbb{W}_k$*,* $a \to a = k-1$*.*
(***HP3***)*: Let* $a, b, c \in \mathbb{W}_k$*. We show that* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 1*: If* $a+b \leq k-1$ *and* $a \leq b \leq c$*, then* $(a \odot b) \to c = 0 \to c = k-1$ *and* $a \to (b \to c) = a \to (k-1) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 2*: If* $a+b \leq k-1$ *and* $a \leq c < b$*,* $(a \odot b) \to c = 0 \to c = k-1$ *and since* $k-1-b+c \geq a$*,* $a \to (b \to c) = a \to (k-1-b+c) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 3*: If* $a+b \leq k-1$ *and* $b \leq a \leq c$*, then* $(a \odot b) \to c = 0 \to c = k-1$ *and* $a \to (b \to c) = a \to (k-1) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 4*: If* $a+b \leq k-1$ *and* $b \leq c < a$*, then* $(a \odot b) \to c = 0 \to c = k-1$ *and* $a \to (b \to c) = a \to (k-1) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 5*: If* $a+b \leq k-1$ *and* $c \leq b \leq a$*, then* $(a \odot b) \to c = 0 \to c = k-1$*. On the other hand since* $a+b \leq k-1$*, we get* $a+b-c \leq k-1$*,* $a \leq (k-1-b+c)$ *and* $a \to (k-1-b+c) = k-1$*. Then* $a \to (b \to c) = a \to (k-1-b+c) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 6*: If* $a+b \leq k-1$ *and* $c \leq a < b$*, then* $(a \odot b) \to c = 0 \to c = k-1$*. On the other hand since* $a+b \leq k-1$*, we get* $a+b-c \leq k-1$*,* $a \leq (k-1-b+c)$ *and* $a \to (k-1-b+c) = k-1$*. Then* $a \to (b \to c) = a \to (k-1-b+c) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 7*: Let* $a+b > k-1$ *and* $a \leq b \leq c$*. Since* $a \leq b \leq c$*, we get* $a+b-c \leq a \leq k-1$ *and so* $a+b-k+1 \leq c$*. Then* $(a \odot b) \to c = (a+b-k+1) \to c = k-1$*. On the other hand,* $a \to (b \to c) = a \to (k-1) = k-1$*. Hence,* $(a \odot b) \to c = a \to (b \to c)$*.*
Case 8*: Let* $a+b > k-1$ *and* $a \leq c < b$*. Since* $a \leq c < b$ *we get* $a+b-c \leq b \leq k-1$ *and so* $a+b-k+1 \leq c$*. Then* $(a \odot b) \to c = (a+b-k+1) \to c = k-1$*. On the other hand, since* $k-1-b+c \geq c \geq a$*, we get* $a \to (b \to c) = a \to$

$(k-1-b+c) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.
***Case 9**: Let $a+b > k-1$ and $b \leq a \leq c$. Since $b \leq a \leq c$, we get $a+b-c \leq a \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \to c = (a+b-k+1) \to c = k-1$. On the other hand since $k-1-b+c \geq c \geq a$, we get $a \to (b \to c) = a \to (k-1-b+c) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.*
***Case 10**: Let $a+b > k-1$ and $b \leq c < a$. Since $b \leq c < a$, we get $a+b-c \leq a \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \to c = (a+b-k+1) \to c = k-1$. On the other hand $a \to (b \to c) = a \to (k-1) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.*
***Case 11**: If $a+b > k-1$ and $c \leq b \leq a$, then $(a \odot b) \to c = (a+b-k+1) \to c$ and $a \to (b \to c) = a \to (k-1-b+c)$. Hence, if $a+b-c \leq k-1$, then $(a \odot b) \to c = a \to (b \to c) = k-1$ and if $a+b-c > k-1$, then $(a \odot b) \to c = a \to (b \to c) = 2k-2-a-b+c$.*
***Case 12**: If $a+b > k-1$ and $c \leq a < b$, then $(a \odot b) \to c = (a+b-k+1) \to c$ and $a \to (b \to c) = a \to (k-1-b+c)$. Hence, if $a+b-c \leq k-1$, then $(a \odot b) \to c = a \to (b \to c) = k-1$ and if $a+b-c > k-1$, then $(a \odot b) \to c = a \to (b \to c) = 2k-2-a-b+c$*
***(HP4)**: Now, we show that $(a \to b) \odot a = (b \to a) \odot b$, for all $a, b \in \mathbb{W}_k$.*
***Case 1**: If $a \leq b$, then $(a \to b) \odot a = (k-1) \odot a = a$ and $(b \to a) \odot b = (k-1-b+a) \odot b = k-1-b+a+b-k+1 = a$. Hence, $(a \to b) \odot a = (b \to a) \odot b$.*
***Case 2**: If $a > b$, then $(a \to b) \odot a = (k-1-a+b) \odot a = k-1-a+b+a-k+1 = b$ and $(b \to a) \odot b = (k-1) \odot b = b$. Hence, $(a \to b) \odot a = (b \to a) \odot b$.*
Therefore, $(\mathbb{W}_k, \odot, \to, k-1)$ is a hoop.□

Theorem 3.3. *Let A be a finite set. Then there exist binary operations $\odot$ and $\to$ and constant 1 on A, such that $(A, \odot, \to, 1)$, is a hoop.*

Proof. *Let A be a finite set. Then, there exists $k \in \mathbb{N}$ such that $|A| = |\mathbb{W}_k|$. Now, by Lemma 3.2, $(\mathbb{W}_k, \odot, \to, 1)$ is a hoop and so by Lemma 3.1, there exist binary operations $\odot$ and $\to$, and constant 1 on A , such that $(A, \odot, \to, 1)$ is a hoop.*□

Lemma 3.4. *Let $1 < n \in \mathbb{Q}$. Then there exist binary operations $\odot$ and $\to$ on $E = \mathbb{Q} \cap [1, n]$, such that $(E, \odot, \to, n)$ is a hoop.*

Proof. *For any $1 < n \in E$, we define the binary operations $\odot$ and $\to$ on E as follows, for all $a, b \in E$,*

$$a \odot b = \begin{cases} 1 & \text{if } ab \leq n, \\ \frac{ab}{n} & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} n & \text{if } a \leq b, \\ \frac{nb}{a} & \text{otherwise} \end{cases}$$

Clearly, $\odot$ and $\to$ are well-defined on E. Now, we show that $(E, \odot, \to, n)$ is a hoop.

(HP1): *For all $a \in E$, if $a \neq 1$, since $an > n$ we have $a \odot n = n \odot a = \frac{an}{n} = a$ and if $a = 1$, we have $a \odot n = 1 \odot n = 1 = a$. Then n is the identity element of $(E, \odot)$. Now, we show that $\odot$ is associative on E. Let $a, b, c \in E$,*
Case 1: *If $ab \leq n$ and $bc \leq n$, then $(a \odot b) \odot c = 1 \odot c = 1$. On the other hand $a \odot (b \odot c) = a \odot (1) = 1$. Then $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case 2: *If $ab \leq n$ and $bc > n$, then $(a \odot b) \odot c = 1 \odot c = 1$. On the other hand $b \odot c = \frac{bc}{n}$ and then $a \odot (b \odot c) = a \odot (\frac{bc}{n})$. Since $\frac{abc}{n} = \frac{ab}{n} c \leq c \leq n$, we get $a \odot (b \odot c) = 1$ and so $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case3: *If $ab > n$ and $bc > n$, then $(a \odot b) \odot c = (\frac{ab}{n}) \odot c$. On the other hand $a \odot (b \odot c) = a \odot (\frac{bc}{n})$. If $\frac{abc}{n} \leq n$, then $(a \odot b) \odot c = a \odot (b \odot c) = 1$ and if $\frac{abc}{n} > n$, then $(a \odot b) \odot c = a \odot (b \odot c) = \frac{abc}{n^2}$. Hence, $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case 4: *Let $ab > n$ and $bc \leq n$. This case is similar to the Case 2.*
It is clear that, for all $a, b \in E$, $a \odot b = b \odot a$. Hence, $(E, \odot, n)$ is a commutative monoid.
(HP2): *It is clear that, for all $a \in E$, we have $a \to a = n$.*
(HP3): *For all $a, b, c \in E$, we have the following cases,*
Case 1: *If $b \leq c$ and $ab \leq n$, then $a \to (b \to c) = a \to n = n$ and $(a \odot b) \to c = 1 \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 2: *If $b \leq c$ and $ab > n$, then $a \to (b \to c) = a \to n = n$ and since $\frac{a}{n} < 1$, we get $\frac{ab}{n} < b \leq c$ and so $(a \odot b) \to c = \frac{ab}{n} \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 3: *If $b > c$ and $ab \leq n$, since $ab \leq n \leq nc$ and so $a \leq \frac{nc}{b}$, then $a \to (b \to c) = a \to \frac{nc}{b} = n$. On the other hand, $(a \odot b) \to c = 1 \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 4: *If $b > c$ and $ab > n$, then $a \to (b \to c) = a \to \frac{nc}{b}$ and $(a \odot b) \to c = \frac{ab}{n} \to c$. We have, $a \leq \frac{nc}{b}$ if and only if $\frac{ab}{n} \leq c$, and so $a \to (b \to c) = (a \odot b) \to c$.*
HP4: *For all $a, b \in E$, we have the following cases,*
Case 1: *If $a \leq b$, then $a \odot (a \to b) = a \odot n = \frac{an}{n} = a$ and $b \odot (b \to a) = b \odot \frac{na}{b} = \frac{bna}{bn} = a$ and so $a \odot (a \to b) = b \odot (b \to a)$.*
Case 2: *If $a > b$, then $a \odot (a \to b) = a \odot \frac{nb}{a} = \frac{anb}{an} = b$ and $b \odot (b \to a) = b \odot n = \frac{bn}{n} = b$ and so $a \odot (a \to b) = b \odot (b \to a)$.*
Therefore, $(E, \odot, \to, n)$ is a hoop.$\square$

Theorem 3.5. *Let A be an infinite countable set. Then there exist binary operations $\odot$ and $\to$ and constant 1 on A, such that $(A, \odot, \to, 1)$ is a hoop.*

Proof. *Let A be an infinite countable set and $E = Q \cap [1, n]$. Then by Lemma 3.4, $(E, \odot, \to, 1)$ is an infinite countable hoop and $|A| = |E|$. Hence, by Lemma 3.1, there exist binary operations $\odot$ and $\to$ and constant 1, such that $(A, \odot, \to, 1)$ is a hoop.*$\square$

Corollary 3.6. For any non-empty countable set A, we can construct a hoop on A.

Proof. Let A be a non-empty countable set. Then, A is a finite set, or an infinite countable set . Then by the Theorems 3.3 and 3.5, the proof is clear.$\Box$

4 Constructing of some hyper hoops

In this section first we show that the Cartesian product of hoops is a hyper hoop and then we construct a hyper hoop by any non-empty countable set.

Theorem 4.1. *Let $(A, \odot_A, \to_A, 1_A)$ and $(B, \odot_B, \to_B, 1_B)$ be two hoops. Then there exist hyperoperations $\odot$, $\to$ and constant 1 on $A \times B$ such that $(A \times B, \odot, \to, 1)$ is a hyper hoop.*

Proof. *For any $(a_1, b_1), (a_2, b_2) \in A \times B$, we define the binary hyperoperations $\odot$, $\to$ on $A \times B$ by,*

$$(a_1, b_1) \odot (a_2, b_2) = \{(a_1 \odot_A a_2, b_1), (a_1 \odot_A a_2, b_2)\},$$
$$(a_1, b_1) \to (a_2, b_2) = \begin{cases} \{(a_1 \to_A a_2, b_2), (a_1 \to_A a_2, 1_B)\} & \text{if } b_1 = b_2, \\ \{(a_1 \to_A a_2, b_2)\} & \text{otherwise} \end{cases}$$

and constant $1 = (1_A, 1_B)$. It is easy to show that the hyperoperations are well-defined. Now, we show that $(A \times B, \odot, \to, 1)$ is a hyper hoop.
*(**HHA1**): Since $\odot_A$, is associative and commutative, we get $\odot$ is associative and commutative. Moreover, for all $(a, b) \in A \times B$, we have $(a, b) \odot (1_A, 1_B) = \{(a \odot_A 1_A, b), (a \odot_A 1_A, 1_B)\} \ni (a, b)$. Then $(A \times B, \odot, \to, 1)$ is a commutative semihypergroup with 1 as the unit, where $1 = (1_A, 1_B)$.*
***(HHA2)**: For all $(a, b) \in A \times B$, we have*

$$(a, b) \to (a, b) = \{(a \to_A a, b), (a \to_A a, 1_B)\} =$$

$$\{(a \to_A a, b), (1_A, 1_B)\} \ni (1_A, 1_B) = 1$$

***(HHA3)**: For all $(a_1, b_1), (a_2, b_2) \in A \times B$, we have the following cases,*
***Case 1**: If $b_1 \neq b_2$, then,*

$$\begin{aligned} ((a_1, b_1) \to (a_2, b_2)) \odot (a_1, b_1) &= \{(a_1 \to a_2, b_2)\} \odot (a_1, b_1) \\ &= \{((a_1 \to a_2) \odot_A a_1, b_1), ((a_1 \to a_2) \odot_A a_1, \\ &\quad b_2)\} \\ &= \{((a_2 \to a_1) \odot_A a_2, b_1), ((a_2 \to a_1) \odot_A a_2, \\ &\quad b_2)\} \\ &= ((a_2, b_2) \to (a_1, b_1)) \odot (a_2, b_2) \end{aligned}$$

***Case 2**: If $b_1 = b_2$, then,*

$$\begin{aligned}((a_1,b_1)\to(a_2,b_2))\odot(a_1,b_1) &= \{(a_1\to a_2,b_2),(a_1\to a_2,1_B)\}\odot(a_1,b_1)\\ &= \{((a_1\to a_2)\odot_A a_1,b_1),((a_1\to a_2)\odot_A a_1,\\ &\quad b_2),((a_1\to a_2)\odot_A a_1,1_B)\}\\ &= \{((a_2\to a_1)\odot_A a_2,b_1),((a_2\to a_1)\odot_A a_2,\\ &\quad b_2),((a_2\to a_1)\odot_A a_2,1_B)\}\\ &= ((a_2,b_2)\to(a_1,b_1))\odot(a_2,b_2)\end{aligned}$$

***(HHA4)**: For all $(a_1,b_1),(a_2,b_2),(a_3,b_3)\in A\times B$, we have the following cases,*
***Case 1**: If $b_1 = b_2 = b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3),((a_2\to_A a_3),\\ &\quad 1_B)\}\\ &= \{(a_1\to_A(a_2\to_A a_3),1_B),(a_1\to_A(a_2\to_A\\ &\quad a_3),b_3)\}\\ &= \{((a_1\odot_A a_2)\to_A a_3,1_B),((a_1\odot_A a_2)\to_A\\ &\quad a_3),b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

***Case 2**: If $b_1 \neq b_2 = b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3),((a_2\to_A a_3),\\ &\quad 1_B)\}\\ &= \{(a_1\to_A(a_2\to_A a_3),1_B),(a_1\to_A(a_2\to_A\\ &\quad a_3),b_3)\}\\ &= \{(a_1\odot_A a_2)\to_A(a_3,1_B),((a_1\odot_A a_2)\to_A\\ &\quad a_3),b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

***Case 3**: If $b_1 = b_2 \neq b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3)\}\\ &= \{a_1\to_A(a_2\to_A a_3),b_3)\}\\ &= \{((a_1\odot_A a_2)\to_A a_3,b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

Case 4: *If* $b_1 \neq b_2 \neq b_3$,

$$\begin{aligned}(a_1,b_1) \to ((a_2,b_2) \to (a_3,b_3)) &= (a_1,b_1) \to \{((a_2 \to_A a_3), b_3)\}\\ &= \{(a_1 \to_A (a_2 \to_A a_3), b_3)\}\\ &= \{((a_1 \odot_A a_2) \to_A a_3, b_3)\}\\ &= ((a_1,b_1) \odot (a_2,b_2)) \to (a_3,b_3)\end{aligned}$$

(HHA5): *For all* $(a,b) \in A \times B$, *we have the following cases,*
Case 1: *If* $b = 1_B$, *then* $(a,b) \to (1_A,1_B) = \{(a \to 1_A, 1_B), (a \to 1_A, b \to 1_B)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
Case 2: *If* $b \neq 1_B$, *then* $(a,b) \to (1_A,1_B) = \{(a \to 1_A, 1_B)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
(HHA6): *For all* $(a_1,b_1),(a_2,b_2) \in A \times B$, *if* $(1_A,1_B) \in (a_1,b_1) \to (a_2,b_2)$ *and* $(1_A,1_B) \in (a_2,b_2) \to (a_1,b_1)$, *then we have the following cases,*
Case 1: *If* $b_1 \neq b_2$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_1, b_1)\}$. *Hence,* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_1$ *and* $1_B = b_1 = b_2$. *Since* A *is a hoop, we get* $a_1 = a_2$ *and so* $(a_1,b_1) = (a_2,b_2)$
Case 2: *If* $b_1 = b_2$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2), (a_1 \to_A a_2, 1_B)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_1, b_1), (a_2 \to_A a_1, 1_B)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_1$. *Since* A *is a hoop, we get* $a_1 = a_2$ *and by assumption, we have* $b_1 = b_2$. *So* $(a_1,b_1) = (a_2,b_2)$.
(HHA7): *For all* $(a_1, b1),(a_2,b_2),(a_3,b_3) \in A \times B$, *let* $(1_A,1_B) \in (a_1,b_1) \to (a_2,b_2)$ *and* $(1_A,1_B) \in (a_2,b_2) \to (a_3,b_3)$. *Then we consider the following cases:*
Case 1: *If* $b_1 = b_2 = b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, 1_B), (a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, 1_B), (a_2 \to_A a_3, b_3)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3), (a_1 \to_A a_3, 1_B)\} = \{(1_A,b_3),(1_A,1_B)\} \ni (1_A,1_B)$.
Case 2: *If* $b_1 \neq b_2 = b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, 1_B), (a_2 \to_A a_3, b_3)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$ *and* $b_2 = b_3 = 1_B$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
Case 3: *Let* $b_1 = b_2 \neq b_3$. *Then proof is similar to the Case 2.*
Case 4: *If* $b_1 \neq b_2 \neq b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, b_3)\}$. *Hence,* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$ *and* $b_2 = b_3 = 1_B$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.

Therefore,$(A \times B, \odot, \to, 1)$ *is a hyper hoop, where* $1 = (1_A,1_B)$.□

Lemma 4.2. *Let* A *and* B *be two sets such that* $|A| = |B|$. *If* $(A, \odot_A, \to_A, 1_A)$ *is a hyper hoop, then there exist hyperoperations* $\odot_B$, $\to_B$ *and constant* 1_B *on* B, *such that* $(B, \odot_B, \to_B, 1_B)$ *is a hyper hoop and* $(A, \odot_A, \to_A, 1_A) \cong (B, \odot_B, \to_B, 1_B)$.

Proof. *Since $|A| = |B|$, then there exists a bijection $\varphi : A \to B$. For any $b_1, b_2 \in B$, there exist $a_1, a_2 \in A$ such that $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$. Then we define the hyperoperations $\odot_B, \to_B$ on B by, $b_1 \odot_B b_2 = \{\varphi(a) | a \in a_1 \odot a_2\}$, and $b_1 \to_B b_2 = \{\varphi(a) | a \in a_1 \to a_2\}$. It is easy to show that $\odot_B, \to_B$ are well-defined and $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop, where $1_B = \varphi(1_A)$. Now, we define the map $\theta : (A, \odot_A, \to_A, 1_A) \to (B, \odot_B, \to_B, 1_B)$ by $\theta(x) = \varphi(x)$. Since φ is a bijection then θ is a bijection and it is easy to see that θ is a homomorphism and so it is an isomorphism.*$\square$

Corollary 4.3. For any non-empty countable set A and any hoop B, we can construct a hyper hoop on $A \times B$.

Proof. By Corollary 3.6, we can construct a hoop on A and by Theorem 4.1, we can construct a hyper hoop on $A \times B$.$\square$

Corollary 4.4. Let A be an infinite countable set. We can construct a hyper hoop on A.

Proof. Let A be an infinite countable set. Then by Corollary 3.6, we can construct a hoop on A. Now, By Theorem 3.3, for arbitrary elements x, y not belonging to A, we can define operations $\odot$ and $\to$ on the set $\{x, y\}$, such that $(\{x, y\}, \odot, \to)$ is a hoop. Then by Theorem 4.1, we can construct a hyper hoop on $A \times \{x, y\}$. Then by Lemma 2.11 and 4.2, there exists a hyper hoop on A.$\square$

5 Fundametal hoops

In this section we apply the β^* relation to the hyper hoops and obtain some results. Then we show that any hoop is a fundamental hoop.

Let $(A, \odot, \to, 1)$ be a hyper hoop and $U(A)$ denote the set of all finite combinations of elements of A with respect to $\odot$ and $\to$. Then, for all $a, b \in A$, we define $a\beta b$ if and only if $\{a, b\} \subseteq u$, where $u \in U(A)$, and $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ with $z_1 = a, z_{m+1} = b$ such that $\{z_i, z_{i+1}\} \subseteq u_i \subseteq U(A)$, for $i = 1, ..., m$ (In fact β^* is the transitive closure of the relation β).

Theorem 5.1. *Let A be a hyper hoop. Then β^* is a strong regular relation on A.*

Proof. *Let $a\beta^* b$, for $a, b \in A$. Then there exist $x_1, ..., x_{n+1} \in A$ with $x_1 = a, x_{n+1} = b$ and $u_i \in U(A)$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, for $1 \leq i \leq n$. Let $z_i \in x_i \to c$, for all $1 \leq i \leq n+1, c \in A$. Then we have,*

$$\{z_i, z_{i+1}\} \subseteq (x_i \to c) \cup (x_{i+1} \to c) \subseteq u_i \to c \subseteq U(A), \textit{for all } 1 \leq i \leq n.$$

Hence, $z_1 \beta^* z_{n+1}$, where $z_1 \in a \to c$ and $z_{n+1} \in b \to c$ and so $a \to c \overline{\overline{\beta^*}} b \to c$. Similarly, we can show that $c \to a \overline{\overline{\beta^*}} c \to b$. Now, by the same way we can prove *that $a\beta^*b$ implies $a\odot c\overline{\overline{\beta^*}}b\odot c$, for all $c \in A$. Hence, β^* is a strong regular relation on A.*$\square$

Corollary 5.2. Let A be a hyper hoop. Then $(\frac{A}{\beta^*}, \otimes, \hookrightarrow)$ is a hoop, where $\otimes$ and $\hookrightarrow$ are defined by Theorem 2.8.

Proof. By Theorem 2.9 the proof is clear.$\square$

Theorem 5.3. *Let A be a hyper hoop. Then the relation β^* is the smallest equivalence relations γ defined on A such that the quotient $\frac{A}{\gamma}$ is a hoop with operations*

$$\gamma(x) \otimes \gamma(y) = \gamma(t) : t \in x \odot y \quad \textit{and} \quad \gamma(x) \hookrightarrow \gamma(y) = \gamma(z) : z \in x \to y$$

where $\gamma(x)$ is equivalence class of x with respect to the relation γ.

Proof. *By Corollary 5.2, $\frac{A}{\beta^*}$ is a hoop. Now, let γ be an equivalence relation on A such that $\frac{A}{\gamma}$ is a hoop. Let $x\beta y$, for $x, y \in A$ and $\pi : A \to \frac{A}{\gamma}$ be the natural projection such that $\pi(x) = \gamma(x)$. It is clear that π is a homomorphism of hyper hoops. Then there exists $u \in U(A)$ such that $\{x, y\} \subseteq u$. Since π is a homomorphism of hyper hoops, we get $|\pi(u)| = |\gamma(u)| = 1$. Since $\{\pi(x), \pi(y)\} \subseteq \pi(u)$ and $|\pi(u)| = 1$, we get $\pi(x) = \pi(y)$ and so $\gamma(x) = \gamma(y)$ i.e. $x\gamma y$. Hence, $\beta \subseteq \gamma$. Now, let $a\beta^*b$, for $a, b \in A$. Then there exist $x_1, ..., x_{n+1} \in A$, such that $a = x_1\beta x_2, ..., \beta x_{n+1} = b$. Since $\beta \subseteq \gamma$, we get $a = x_1\gamma x_2, ..., \gamma x_{n+1} = b$. Then since γ is a transitive relation on A, we get $a\gamma b$ and so $\beta^* \subseteq \gamma$.*$\square$

Corollary 5.4. The relation β^* is the smallest strong regular relation on hyper hoop A.

Proof. The proof is straightforward.$\square$

Lemma 5.5. *If A_1 and A_2 are two hyper hoops, then the Cartesian product $A_1 \times A_2$ is a hyper hoop with the unit $(1_{A_1}, 1_{A_2})$ by the following hyperoperations, for $(x_1, y_1), (x_2, y_2) \in A_1 \times A_2$,*

$$(x_1, y_1) \odot (x_2, y_2) = \{(a, b) | a \in x_1 \odot x_2, b \in y_1 \odot y_2\},$$
$$(x_1, y_1) \to (x_2, y_2) = \{(a', b') | a' \in x_1 \to x_2, b' \in y_1 \to y_2\}$$

Proof. *The proof is straightforward.*$\square$

Lemma 5.6. *Let A_1 and A_2 be two hyper hoops. Then, for $a, c \in A_1$ and $b, d \in A_2$, we have $(a, b)\beta^*_{A_1\times A_2}(c, d)$ if and only if $a\beta^*_{A_1}c$ and $b\beta^*_{A_2}d$.*

Proof. *We know that $u \in U(A_1 \times A_2)$ if and only if there exist $u_1 \in U(A_1)$ and $u_2 \in U(A_2)$ such that $u = u_1 \times u_2$. Then $(a, b)\beta^*_{A_1\times A_2}(c, d)$ if and only if there exist $u_1 \in U(A_1)$ and $u_2 \in U(A_2)$ such that $\{(a, b), (c, d)\} \subseteq u_1 \times u_2$ if and only if $\{a, c\} \subseteq u_1$ and $\{b, d\} \subseteq u_2$ if and only if $a\beta^*_{A_1}c$ and $b\beta^*_{A_2}d$.*$\square$

Theorem 5.7. *Let A_1 and A_2 be two hyper hoops. Then $\frac{A_1\times A_2}{\beta^*_{A_1\times A_2}} \cong \frac{A_1}{\beta^*_{A_1}} \times \frac{A_2}{\beta^*_{A_2}}$.*

Proof. *Let $\varphi : \frac{A_1\times A_2}{\beta^*} \to \frac{A_1}{\beta^*_{A_1}} \times \frac{A_2}{\beta^*_{A_2}}$ be defined by $\varphi(\beta^*(x,y)) = (\beta^*_{A_1}(x), \beta^*_{A_2}(y))$, where $\beta^* = \beta^*_{A_1\times A_2}$ By Lemma 5.5, $\frac{A_1\times A_2}{\beta^*}$ is well-define. It is clear that φ is onto. By Lemma 5.6, we have $\beta^*(x_1,y_1) = \beta^*(x_2,y_2)$ if and only if $\beta^*_{A_1}(x_1) = \beta^*_{A_2}(x_2)$ and $\beta^*_{A_2}(y_1) = \beta^*_{A_2}(y_2)$, for any $(x_1,y_1),(x_2,y_2) \in A_1\times A_2$. So, φ is well defined and one to one. Also, by considering the hyperoperations $\otimes$ and $\hookrightarrow$ defined in Theorem 2.8, we have,*

$$\begin{aligned}
\varphi(\beta^*(x_1,y_1) \hookrightarrow \beta^*(x_2,y_2)) &= \varphi(\{\beta^*(a,b) | a \in x_1 \to x_2, b \in y_1 \to y_2\}) \\
&= \{\varphi(\beta^*(a,b)) | a \in x_1 \to x_2, b \in y_1 \to y_2\} \\
&= \{(\beta^*_{A_1}(a), \beta^*_{A_2}(b)) | a \in x_1 \to x_2, b \in y_1 \to y_2\} \\
&= (\beta^*_{A_1}(x_1) \hookrightarrow \beta^*_{A_1}(x_2), \beta^*_{A_2}(y_1) \hookrightarrow \beta^*_{A_2}(y_2)) \\
&= (\beta^*_{A_1}(x_1), \beta^*_{A_2}(y_1)) \hookrightarrow (\beta^*_{A_1}(x_2), \beta^*_{A_2}(y_2)) \\
&= \varphi(\beta^*(x_1,y_1)) \hookrightarrow \varphi(\beta^*(x_2,y_2))
\end{aligned}$$

Similarly, we can show that $\varphi(\beta^*(x_1,y_1) \otimes \beta^*(x_2,y_2)) = \varphi(\beta^*(x_1,y_1)) \otimes \varphi(\beta^*(x_2, y_2))$. Moreover, it is clear that $\varphi(\beta^*(1_{A_1}, 1_{A_2})) = (\beta^*(1_{A_1}), \beta^*(1_{A_2}))$. Hence, φ is an isomorphism.□

Corollary 5.8. Let $A_1, A_2, ..., A_n$ be hyper hoops. Then,

$$\frac{A_1\times A_2\times ....\times A_n}{\beta^*_{A_1\times A_2\times ....\times A_n}} \cong \frac{A_1}{\beta^*_1} \times \frac{A_2}{\beta^*_2} \times .... \times \frac{A_n}{\beta^*_n}$$

Proof. The proof is straightforward.□

Theorem 5.9. *Let A and B be two sets such that $|A| = |B|$. If $(A, \odot_A, \to_A, 1_A)$ is a hyper hoop, then there exist hyperoperations $\odot_B$ and $\to_B$ and constant 1_B on B such that $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop and $\frac{(A,\odot_A,\to_A,1_A)}{\beta^*_A} \cong \frac{(B,\odot_B,\to_B,1_b)}{\beta^*_B}$.*

Proof. *Since $|A| = |B|$, then by Lemma 4.2, there exist binary hyperoperations $\odot_B$ and $\to_B$, such that $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop. Moreover, there exists an isomorphism $f : (A, \odot_A, \to_A, 1_A) \to (B, \odot_B, \to_B, 1_B)$, such that $f(1_A) = 1_B$. Now, we define $\varphi : \frac{(A,\odot_A,\to_A,1_A)}{\beta^*_A} \to \frac{(B,\odot_B,\to_B,1_B)}{\beta^*_B}$ by $\varphi(\beta^*_A(x)) = \beta^*_B(f(x))$. Since f is an isomorphism, φ is onto. Let y_1 , $y_2 \in B$. Then there exist $a_1, a_2 \in A$ such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then $\beta^*_A(a_1) = \beta^*_A(a_2)$ iff $a_1\beta^*_A a_2$ iff there exists $u \in U(A)$ such that $\{a_1, a_2\} \subseteq u$ iff there existes $f(u) \in U(B) : \{f(a_1), f(a_2)\} \subseteq f(u)$ iff $\beta^*_B(b_1) = \beta^*_B(b_2)$ iff $\beta^*_B(f(a_1)) = \beta^*_B(f(a_2))$. Then φ is well-defined and one to one. Also, by consid-*

ering the hyperoperations $\otimes$ and $\hookrightarrow$ defined in Theorem 2.8, we have,

$$\begin{array}{rcl}\varphi(\beta^*_A(a_1)\otimes\beta^*_A(a_2)) & = & \varphi_{t\in a_1\odot a_2}(\beta^*_A(t))=\beta^*_{t\in a_1\odot a_2}(f(t))\\ & = & \beta^*_{t'\in f(a_1\odot a_2)}(t')=\beta^*_{t'\in f(a_1)\odot f(a_2)}(t')=\beta^*_B(f(a_1))\otimes\beta^*_B\\ & & (f(a_2))\\ & = & \varphi(\beta^*_A(a_1))\otimes\varphi(\beta^*_A(a_2))\end{array}$$

By the same way, we can show that

$$\varphi(\beta^*_A(a_1)\hookrightarrow\beta^*_A(a_2))=\varphi(\beta^*_A(a_1))\hookrightarrow\varphi(\beta^*_A(a_2))$$

Since f is an isomorphism, we get $\varphi(\beta^*_A(1_A))=\beta^*_B(f(1_A))=\beta^*_B(1_B)$. Hence, φ is an isomorphism.$\square$

Definition 5.10. Let A be a hoop algebra. Then A is called a *fundamental hoop*, if there exists a nontrivial hyper hoop B, such that $\frac{B}{\beta^*_B}\cong A$

Theorem 5.11. *Every hoop is a fundamental hoop.*

Proof. *Let A be a hoop. Then by Theorem 4.1, for any hoop B, $A\times B$ is a hyper hoop. By considering the hyperoperations $\odot$ and $\rightarrow$ defined in Theorem 4.1, we get that any finite combination $u\in U(A\times B)$ is the form of $u=\{(a,x_i)|a\in A, x_i\in B\}$. Hence, for any $(a_1,b_1),(a_2,b_2)\in A\times B$,*

$$\begin{array}{c}(a_1,b_1)\beta^*(a_2,b_2)\Leftrightarrow\exists u\in U(A\times B)\ \textit{such that}\\ \{(a_1,b_1),(a_2,b_2)\}\subseteq u\Leftrightarrow a_1=a_2\end{array}$$

Hence, for any $(a,b)\in A\times B$, $\beta^*(a,b)=\{(a,x)|x\in B\}$.
Now, we define the map $\psi:\frac{A\times B}{\beta^*}\rightarrow A$ by, $\psi(\beta^*(a,b))=a$. It is clear that,

$$\beta^*(a_1,b_1)=\beta^*(a_2,b_2)\Leftrightarrow a_1=a_2\Leftrightarrow\psi(\beta^*(a_1,b_1))=\psi(\beta^*(a_2,b_2)).$$

Then, ψ is well defined and one to one. In the following, we show that ψ is a homomorphism. For this we have,

$$\begin{array}{rcl}\psi(\beta^*(a_1,b_1)\otimes\beta^*(a_2,b_2)) & = & \psi(\beta^*(u,v)):(u,v)\in(a_1,b_1)\odot(a_2,b_2)\\ & = & \psi(\beta^*(u,v)):(u,v)\in\{((a_1\odot a_2),b_1),((a_1\odot\\ & & a_2),b_2)\}\\ & = & \{u|u\in a_1\odot a_2\}=a_1\odot a_2\\ & = & \psi(\beta^*(a_1,b_1))\odot\psi(\beta^*(a_2,b_2))\end{array}$$

and similarly, for the operation $\hookrightarrow$*, we have the following cases,*
Case 1*: If* $b_1 \neq b_2$*, then,*

$$\begin{aligned}\psi(\beta^*(a_1,b_1) \hookrightarrow \beta^*(a_2,b_2)) &= \psi(\beta^*(u,v)) : (u,v) \in (a_1,b_1) \to (a_2,b_2)\\ &= \psi(\beta^*(u,v)) : (u,v) \in \{((a_1 \to a_2), b_2)\}\\ &= \{u|u \in a_1 \to a_2\} = a_1 \to a_2\\ &= \psi(\beta^*(a_1,b_1)) \to \psi(\beta^*(a_2,b_2))\end{aligned}$$

Case 2*:If* $b_1 = b_2$*, then,*

$$\begin{aligned}\psi(\beta^*(a_1,b_1) \hookrightarrow \beta^*(a_2,b_2)) &= \psi(\beta^*(u,v)) : (u,v) \in (a_1,b_1) \to (a_2,b_2)\\ &= \psi(\beta^*(u,v)) : (u,v) \in \{((a_1 \to a_2), b_2), ((a_1 \to\\ &\quad a_2), 1_B)\}\\ &= \{u|u \in a_1 \to a_2\} = a_1 \to a_2\\ &= \psi(\beta^*(a_1,b_1)) \to \psi(\beta^*(a_2,b_2))\end{aligned}$$

Clearly, $\psi(\beta^*(1_A, 1_B) = 1_A$ *and* ψ *is onto. Therefore,* ψ *is an isomorphism i.e.* $\frac{A\times B}{\beta^*} \cong A$ *and so* A *is fundamental.*$\square$

Corollary 5.12. For any non-empty countable set A, we can construct a fundamental hoop on A.

Proof. By Corollary 3.6 and Theorem 5.11 the proof is clear.$\square$

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