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On some probability concepts in fuzzy framework

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Abstract

In this paper some modalities in which the concept of probability can be fuzzified are investigated in order to obtain new tools useful in the modelization of the risks. Some papers related to this approach are [1-6]. Finally, some open problems are proposed to the reader.

Key words: probability, fuzzy set, fuzzy probability, fuzzy numbers

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1 Classical case

In the set theory the following operations are used:
- the intersection ($A \cap B$);
- the union ($A \cup B$);
- the complement ($\overline{A}$);
- the difference ($A \backslash B = A \cap \overline{B}$);
- the implication ($A \to B = \overline{A \backslash B} = \overline{A} \cup B$);
- the symmetric difference ($A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cap \overline{B}) \cup (\overline{A} \cap B)$);
- the equivalence ($A \leftrightarrow B = (A \to B) \cap (B \to A) = \overline{A \triangle B}$), where A, B are sets.

The empty set is denoted by $\emptyset$, and the set of subsets of a set S will be denoted by $\mathcal{P}(S)$.

Let $A, B, C \in \mathcal{P}(S)$.

Remark 1. We have:

i) $\cap, \cup$ are commutative and associative;

ii) $A \cap A = A$, $A \cup A = A$;

iii) $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$;

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

v) $A \cap \overline{A} = \emptyset$, $A \cup \overline{A} = S$;

vi) $\overline{A \cap B} = \overline{A} \cup \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$;

vii) $\overline{\overline{A}} = A$.

Let $\Omega \neq \emptyset$.

Definition 2. *By field of events (in relation with the space Ω) one intend $K \subseteq \mathcal{P}(\Omega)$ such that*

i) $\Omega \in K$;

ii) $A, B \in K \Rightarrow A \cup B \in K$;

iii) $A \in K \Rightarrow \overline{A} \in K$.

Remark 3. Let K be a field of events. We have

i) $A, B \in K \Rightarrow A \cap B \in K$;

ii) $A, B \in K \Rightarrow A \backslash B \in K$;

iii) $A, B \in K \Rightarrow A \rightarrow B \in K$;

iv) $A, B \in K \Rightarrow A \triangle B \in K$;

v) $A, B \in K \Rightarrow A \leftrightarrow B \in K$;

vi) $\emptyset \in K$.

Let K be a field of events.

Definition 4. *By probability on K one intend*
$P : K \rightarrow [0, 1]$ *such that:*

i) $P(\Omega) = 1$;

ii) $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.

Remark 5. We have

i) $P(\emptyset) = 0$;

ii) $P(\overline{A}) = 1 - P(A)$;

iii) $P(A \backslash B) = P(A) - P(A \cap B)$;

iv) $P(A \cap B) + P(A \cup B) = P(A) + P(B)$;

v) $P(A \to B) = 1 - P(A) + P(A \cap B)$;

v) $P(A) + P(A \to B) = P(B) + P(B \to A)$.

Remark 6. If $P : K \to [0, 1]$ is an application satisfying $P(\emptyset) = 0$, $P(\Omega) = 1$, then the condition ii), from the definition and the condition vi) from the remark are equivalent.

2 Fuzzy case

For the construction which will be given in this case we need the concepts of t-norms and t-conorms.

Definition 7. *A function $t : [0, 1] \times [0, 1] \to [0, 1]$ will be called t-norm if the following conditions are satisfied:*

i) $t(x, 1) = x$, $\forall x \in [0, 1]$*;*

ii) $t(x, y) = t(y, x)$*, for any* $x, y \in [0, 1]$*;*

iii) $t(x, t(y, z)) = t(t(x, y), z)$*, for any* $x, y, z \in [0, 1]$*;*

iv) $x \le z \Rightarrow t(x, y) \le t(z, y)$*,* $\forall y \in [0, 1]$.

Remark 8. We have also:

v) $t(x, 0) = t(1, 0) = t(0, 1) = 0$, $\forall x \in [0, 1]$.

Example 9. i) $p : [0, 1] \times [0, 1] \to [0, 1]$, $p(x, y) = xy$;

ii) $\min : [0, 1] \times [0, 1] \to [0, 1]$, $\min(x, y) = \begin{cases} x, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$

ii) $t_m : [0, 1] \times [0, 1] \to [0, 1]$, $t_m(x, y) = \max\{x + y - 1, 0\}$.

Definition 10. *A function $t^* : [0, 1] \times [0, 1] \to [0, 1]$ will be called t-conorm if the following conditions are satisfied:*

i) $t^*(x,0) = x$, $\forall x \in [0,1]$;

ii) $t^*(x,y) = t^*(y,x)$, *for any* $x, y \in [0,1]$;

iii) $t^*(x, t^*(y,z)) = t^*(t^*(x,y), z)$, *for any* $x, y, z \in [0,1]$.

iv) $x \leq z \Rightarrow t^*(x,y) \leq t^*(z,y)$, $\forall y \in [0,1]$.

Example 11. i) $p^* : [0,1] \times [0,1] \to [0,1]$, $p^*(x,y) = x + y - xy$;

ii) $\max : [0,1] \times [0,1] \to [0,1]$, $\max(x,y) = \begin{cases} x, & \text{if } x \geq y \\ y, & \text{if } x < y \end{cases}$;

iii) $t_m^* : [0,1] \to [0,1] \to [0,1]$, $t_m^*(x,y) = \min\{x + y, 1\}$.

Definition 12. *The* t*-norm* t *and the* t*-conorm* t^* *are called dual each another if for any* $x, y \in [0,1]$

$$t(x,y) = 1 - t^*(1-x, 1-y).$$

For example, p, p^* or $\min, \max$ or t_m, t_m^* are such couples.

Definition 13. *A couple* (U, μ) *where* $U \neq \emptyset$ *and* $\mu : U \to [0,1]$ *is an application will be called fuzzy set (on the universe* U*) or fuzzy subset of* U*.*

The empty fuzzy set is given by $\widetilde{\phi} : U \to [0,1]$, $\widetilde{\phi}(x) = 0$, $\forall x \in U$.

We shall denote $\mu \subseteq \eta$ if $\mu(x) \leq \eta(x)$, $\forall x \in U$.

By $\widetilde{U} : U \to [0,1]$ one intend the application given by $\widetilde{U}(x) = 1$, $\forall x \in U$.

Let $\mathcal{F}(U)$ be the family of fuzzy subsets of U. The operations with fuzzy subsets can be defined in the following way:

for $\mu, \eta : \mathcal{F}(U)$, $\mu \bigcap_t \eta : U \to [0,1]$, $(\mu \bigcap_t \eta)(x) = t(\mu(x), \eta(x))$

$\mu \bigcup_t \eta \to [0,1]$, $(\mu \bigcup_t \eta)(x) = t^*(\mu(x), \eta(x))$.

The complement $\overline{\mu} : U \to [0,1]$ will be given by $\overline{\mu}(x) = 1 - \mu(x)$. In a similar way with the classical case one define $\mu \xrightarrow{t} \eta$, $\mu \xrightarrow{t} \eta$, etc.

$\mu \overset{t}{-} \eta : U \to [0,1]$, $(\mu \overset{t}{-} \eta)(x) = t(\mu(x), 1 - \eta(x))$;

and $\mu \xrightarrow{t} \eta : U \to [0,1]$, $(\mu \xrightarrow{t} \eta)(x) = t^*(1 - \mu(x), \eta(x))$.

For the couples t-norm/conorm described above we obtain: $\odot, \oplus$; $\cap, \cup$; ... More precisely for $\mu, \eta : U \to [0,1]$ we have:

A.

$\mu \odot \eta : U \to [0,1]$, $(\mu \odot \eta)(x) = \mu(x)\eta(x)$;

$\mu \oplus \eta : U \to [0,1]$, $(\mu \oplus \eta)(x) = \mu(x) + \eta(x) - \mu(x)\eta(x)$.

$\overline{\mu} : U \to [0,1]$, $\overline{\mu}(x) = 1 - \mu(x)$; and

$\mu \ominus \eta : U \to [0,1]$, $(\mu \ominus \eta)(x) = \mu(x) - \mu(x)\eta(x)$;

$\mu \bigominus \eta : U \to [0,1]$, $(\mu \bigominus \eta)(x) = 1 - \mu(x) + \mu(x)\eta(x)$;

Remark 14. We have

i) $\odot, \oplus$ are commutative and associative;

ii) $\mu \odot \mu \subseteq \mu$, $\mu \subseteq \mu \oplus \mu$;

iii) $\mu \supseteq \mu \odot (\mu \oplus \eta)$; $\mu \subseteq \mu \oplus (\mu \odot \eta)$;

iv) $\mu \oplus (\eta \odot \tau) \supseteq (\mu \oplus \eta) \odot (\mu \oplus \tau)$; $\mu \odot (\eta \oplus \tau) \subseteq (\mu \odot \eta) \oplus (\mu \odot \tau)$;

v) $(\mu \odot \overline{\mu})(x) \leq \frac{1}{4}$, $(\mu \oplus \overline{\mu})(x) \geq \frac{3}{4}$, $\forall x \in U$;

v) $\overline{\mu \oplus \eta} = \overline{\mu} \odot \overline{\eta}$; $\overline{\mu \odot \eta} = \overline{\mu} \oplus \overline{\eta}$.

B.

$\mu \cap \eta : U \rightarrow [0,1]$, $(\mu \cap \eta) = \min\{\mu(x), \eta(x)\}$;

$\mu \cup \eta : U \rightarrow [0,1]$, $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}$;

$\overline{\mu} : U \rightarrow [0,1]$, $\overline{\mu}(x) = 1 - \mu(x)$;

$\mu - \eta : U \rightarrow [0,1]$, $(\mu - \eta)(x) = \min\{\mu(x), 1 - \eta(x)\}$;

$\mu \rightarrow \eta : U \rightarrow [0,1]$, $(\mu \rightarrow \eta)(x) = 1 - \min\{\mu(x), 1 - \eta(x)\}$;

Remark 15. We have

i) $\cap, \cup$ are commutative and associative;

ii) $\mu \cap \mu = \mu$, $\mu \cup \mu = \mu$

iii) $\mu \cup (\mu \cap \eta) = \mu$; $\mu \cap (\mu \cup \eta) = \mu$;

iv) $\mu \cup (\eta \cap \tau) = (\mu \cup \eta) \cap (\mu \cup \tau) = (\mu \cap \eta) \cup (\mu \cap \tau)$;

v) $(\mu \cap \overline{\mu})(x) \leq \frac{1}{2}$, $(\mu \cup \overline{\mu})(x) \geq \frac{1}{2}$, $\forall x \in U$;

vi) $\overline{\mu \cup \eta} = \overline{\mu} \cap \overline{\eta}$; $\overline{\mu \cap \eta} = \overline{\mu} \cup \overline{\eta}$.

C.

$$\mu \bigtriangledown \eta : U \to [0,1],\ (\mu \bigtriangledown \eta)(x) = \max\{\mu(x) + \eta(x) - 1, 0\};$$

$$\mu \bigtriangleup \eta : U \to [0,1],\ (\mu \bigtriangleup \eta)(x) = \min\{\mu(x) + \eta(x), 1\};$$

$$\overline{\mu} : U \to [0,1],\ \overline{\mu}(x) = 1 - \mu(x);$$

$$\mu \bullet \eta : U \to [0,1],\ (\mu \bullet \eta)(x) = \max\{\mu(x) - \eta(x), 0\};$$

$$\mu \dot{\to} \eta : U \to [0,1],\ (\mu \dot{\to} \eta)(x) = \min\{1 - \mu(x) + \eta(x), 1\}.$$

Remark 16. We have

i) $\bigtriangledown, \bigtriangleup$ are commutative and associative;

ii) $\mu \bigtriangledown \eta \subseteq \mu$, $\mu \subseteq \mu \bigtriangleup \mu$;

iii) $\mu \subseteq \mu \bigtriangleup (\mu \odot \eta)$; $\mu \supseteq \mu \bigtriangledown (\mu \bigtriangleup \eta)$;

iv) $(\mu \bigtriangledown \overline{\mu})(x) = 0$, $(\mu \bigtriangleup \overline{\mu})(x) = 1$, $\forall x \in U$;

v) $\overline{\mu \bigtriangledown \eta} = \overline{\mu \bigtriangleup \eta} = \overline{\mu} \bigtriangledown \overline{\eta}$.

Remark 17. We have $\mu \bigtriangledown \eta \subseteq \mu \odot \eta \subseteq \mu \cap \eta$; $\mu \cup \eta \subseteq \mu \oplus \eta \subseteq \mu \bigtriangleup \eta$ and $\overline{\overline{\mu}} = \mu$.

3 Fuzzy numbers

In the last section of the paper fuzzy number will be used.

Let $\mathbb{R}$ be the field of real numbers.

Definition 18. *By triangular fuzzy number one intend a triple* (a, b, c)*, where* $a, b, c \in \mathbb{R}$*,* $a \le b \le c$*.*

We shall denote $\mathbb{R}_t$ the set of triangular fuzzy numbers.

For $A = (a_1, b_1, c_2)$, $B = (a_2, b_2, c_2)$ from $\mathbb{R}_t$, if $c_1 \le a_2$, or
$a_2 \le c_1$ and $\frac{a_1+2b_1+c_1}{4} < \frac{a_2+2b_2+c_2}{4}$, or
$a_2 \le c_1$, $\frac{a_1+2b_1+c_1}{4} = \frac{a_2+2b_2+c_2}{4}$ and $b_1 < b_2$, or
$a_2 \le c_1$, $\frac{a_1+2b_1+c_1}{4} = \frac{a_2+2b_2+c_2}{4}$, $b_1 = b_2$ and $c_1 - a_1 < c_2 - a_2$,
we shall write $A \lesssim B$ (a special kind of "order" being obtained in this way).

Remark 19. A triangular fuzzy number $(a, b, c) \in \mathbb{R}_t$ is uniquely determined by a triple (λ, b, ρ) where $\lambda = b - a$, $\rho = c - b$ are positive reals called the left, respectively right tolerance.

We will use the notation with the central value on the first place (b, λ, ρ).

We consider the operations (these operations are introduced by the author and was presented for the first time at a conference given at the University of Chieti in 2007 and was published in [6]):

$$(a, \lambda, \rho) \boxplus (b, \lambda', \rho') = (a + b, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\})$$

$$(a, \lambda, \rho) \boxdot (b, \lambda', \rho') = (ab, \max\{\lambda, \lambda'\}, \max\{\rho, \rho'\})$$

and the relation "$\sim$" given by

$$(a, \lambda, \rho) \sim (b, \lambda', \rho') \text{if } \begin{cases} a = b \\ \lambda - \lambda' = \rho - \rho'. \end{cases}$$

One obtains:

Remark 20. We have:

i) $\boxplus, \boxdot$ are commutative and associative;

ii) $\boxdot$ is distributive with respect to $\boxplus$;

iii) $(0, 0, 0)$ is neutral element for $\boxplus$, and $(1, 0, 0)$ is neutral element for $\boxdot$;

iv) $(a, \lambda, \rho) \boxplus (-a, \rho, \lambda) \sim (0, 0, 0)$; if $a \neq 0$

$$(a, \lambda, \rho) \boxdot (\frac{1}{a}, \rho, \lambda) \sim (1, 0, 0).$$

v) "$\sim$" is an equivalence relation on $\mathcal{R}_t$.

4 Fuzzy events

Let be $\Omega \neq \emptyset$ and $\mathcal{F}(\Omega)$.

Definition 21. *By fuzzy field of events one intend $K \subseteq \mathcal{F}(\Omega)$ such that:*

i) $\widetilde{\Omega} \in K$

ii) $\mu, \eta \in K \Rightarrow \mu \bigcup_t \eta \in K$;

iii) $\mu \in K \Rightarrow \overline{\mu} \in K$.

Remark 22. We have:

i) $\widetilde{\phi} \in K$;

ii) $\mu, \eta \in K \Rightarrow \mu \bigcap_t \eta \in K$; $\mu \overset{t}{-} \eta \in K$, $\mu \overset{t}{\to} \eta \in K$;

iii) $(\mu \in K \Rightarrow \overline{\mu} \in K) \Leftrightarrow (\mu, \eta \in K \Rightarrow \mu \overset{t}{-} \eta \in K) \Leftrightarrow (\mu, \eta \in K \Rightarrow \mu \overset{t}{\to} \eta \in K)$.

Let K be a fuzzy field of events.

Definition 23. *By probability on K one intend $P : K \to [0, 1]$ such that*

i) $P(\widetilde{\Omega}) = 1$

ii) $\mu \bigcap_t \eta = \phi \Rightarrow P(\mu \bigcup_t \eta) = P(\mu) + P(\eta)$.

Remark 24. Verify the following:

i) $P(\widetilde{\phi}) = 0$;

ii) $P(\overline{\mu}) = 1 - P(\mu)$;

iii) $\mu \subseteq \eta \Rightarrow P(\eta \overset{t}{-} \mu) = P(\eta)$;

iv) $P(\mu \overset{t}{-} \eta) = P(\mu) - P(\mu \bigcap_t \eta)$;

v) $P(\mu \bigcup_t \eta) + P(\mu \bigcap_t \eta) = P(\mu) + P(\eta)$;

vi) $P(\mu \overset{t}{\to} \eta) = 1 - P(\mu) + P(\mu \bigcap_t \eta)$;

vii) $P(\mu) + P(\mu \overset{t}{\to} \eta) = P(\eta) + P(\eta \overset{t}{\to} \mu)$.

In the case $t = t_m$ we suppose also that $\mu, \eta \in K \Rightarrow \mu \odot \eta \in K$. In this context we shall denote $P(\mu/\eta) = P(\mu \odot \eta)/P(\eta)(P(\eta) \neq 0)$.

Proposition 25. *In the above condition we have:*

$$P(\mu/\eta) = \frac{P(\mu)P(\eta/\mu)}{P(\mu)P(\eta/\mu) + P(\eta)P(\mu/\eta)}.$$

We have also

Proposition 26. *If $\mu_1, \ldots, \mu_n \in K$ are such that $\mu_i \bigcap_t \mu_j = \widetilde{\phi}$ for $i \neq j$, then $P(\mu \bigcup_t, \ldots, \bigcup_t \mu_n) = P(\mu_1) + \ldots + P(\mu_n)$.*

5 Fuzzy probability

The next step is to substitute $[0,1]$ in the definition of the probability (in K) with the

$$I_t = \{(a, \lambda, \rho) \in \mathbb{R}^t / \lambda \le a, \rho \le 1 - a, a \in [0,1]\}.$$

We have two possibilities:

A. We shall use the operations and the equivalence relation given in III.

Remark 27. If (a, λ, ρ) is such that $a \in [0,1]$ then there exists $(a', \lambda', \rho') \in I_t$ such that $(a, \lambda, \rho) \sim (a', \lambda', \rho')$.

Let K be a fuzzy field of events.

Definition 28. *By fuzzy probability on K one intend an application $P : K \to I_t$ such that*

i) $P(\widetilde{\phi}) = 0$*;*

ii) $\mu \bigcap_t \eta = \phi \Rightarrow P(\mu \bigcup_t \eta) \sim P(\mu) \boxplus P(\eta)$*;*

iii) If $P(\mu) = (\alpha, \lambda, \rho)$ *then* $P(\overline{\mu}) = (1 - a, \rho, \lambda)$*.*

Remark 29. In view to obtain more properties ii) can be replaced by ii$'$) $P(\mu) \boxplus P(\eta) \sim P(\mu \bigcap_t \eta) \boxplus P(\mu \bigcup_t \eta)$.

Problem 30. In the case i), ii$'$), iii), verify the following:

i) $P(\widetilde{\Omega}) = (1, 0, 0)$;

ii) $P(\mu \overset{t}{\backslash} \eta) \sim P(\mu) - P(\mu \bigcap_t \eta)$;

iii) $P(\mu \xrightarrow{t} \eta) \sim P(\overline{\mu}) + P(\mu \bigcap_t \eta)$;

iv) $P(\mu) + P(\mu \xrightarrow{t} \eta) \sim P(\eta) + P(\eta \xrightarrow{t} \mu)$.

B. In the following we propose new operations:

$(a, \lambda, \rho) \widetilde{+} (a', \lambda', \rho') = (a + a' - aa', a + a' - aa' - \max\{a + \lambda, a' + \lambda'\}, \min\{a + \rho + a' + \rho', 1\} - a - a' + aa')$

$(a, \lambda, \rho) \widetilde{\cdot} (a', \lambda', \rho) = (aa', aa' - \max\{a - \lambda + a' - \lambda' - 1, 0\} \min\{a + \rho, a' + \rho'\} - aa')$

When the numbers are written in the form (a, b, c) $(a \le b \le c)$, the operation are defined by

$$(a, b, c) \widetilde{+} (a', b', c') = (\max\{a, a'\}, b + b' - bb', \min\{c + c', 1\})$$

$$(a, b, c) \widetilde{\cdot} (a', b', c') = (\max\{a + a' - 1, 0\}, bb', \min\{c, c'\}).$$

Remark 31. The above operations are satisfying

$$0 \leq \max\{a, a'\} \leq b + b' - bb' \leq \min\{c + c', 1\} \leq 1$$

$$0 \leq \max\{a + a' - 1, 0\} \leq bb' \leq \min\{c, c'\} \leq 11.$$

In this frame using the form (a, b, c) we can propose the following

Definition 32. *By fuzzy probability on* K *one intend* $P : K \to I_t$ *such that*

i) $P(\widetilde{\Omega}) = (1, 1, 1)$, $P(\widetilde{\phi}) = (0, 0, 0)$*;*

ii) $P(\mu)\widetilde{+}P(\eta) = P(\mu \bigcap_t \eta)\widetilde{+}P(\mu \bigcup_t \eta)$*;*

iii) $\mu \leq \eta$, $P(\mu) \lesssim P(\eta)$.

Problem 33. Verify the following:

i) $P(\mu \overset{t}{-} \eta) = P(\mu) - P(\mu \bigcap_t \eta)$;

ii) $P(\mu \overset{t}{\to} \eta) = P(\overline{\mu}) + P(\mu \bigcap_t \eta)$;

iii) $P(\mu) + P(\mu \overset{t}{\to} \eta) = P(\eta) + P(\eta \overset{t}{\to} \mu)$.

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