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Ordered Polygroups

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Abstract

In this paper, those polygroups which are partially ordered are introduced and some properties and related results are given.

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1 Introduction and Preliminaries

The notion of a hyperstructure and hypergroup, as a generalization of group, was introduced by F. Marty [5] in 1934 at the 8th congress of Scandinavian Mathematicians. In this definition for nonempty set H, a function $\cdot : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, is called a *hyperoperation* on H, and the system $(H, \cdot)$ is called a *hypergroupoid*. If the hypergroupoid H satisfies $a \cdot H = H \cdot a = H$, for all $a \in H$, it is called a *hypergroup*. In a hypergroupoid H, for $A, B \subseteq H$ and $x \in H$, $A \cdot B$ and $A \cdot x$ are defined as

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, A \cdot x = A \cdot \{x\}.$$

An element e of hypergorupoid H is called an *identity* if for all $a \in H$, $a \in a \circ e \cap e \circ a$. An element $a' \in H$ is called an inverse for $a \in H$ if there is an identity $e \in H$ such that $e \in a \circ a' \cap a' \circ a$.

By a *subhypergroupoid* of hypergroupoid H we mean a subset K of H that is closed with respect to the hyperoperation on H, and contains the unique identity of H and the inverses of its elements, provided there exist.

Hyperstructures have many applications to several sectors of both pure and applied sciences. A short review of the theory of hyperstructures appear in [2]. In [3] a wealth of applications can be found, too. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy set and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities. Polygroups are certain subclasses of hypergroups which studied in 1981 by Ioulidis in [4] and are used to study colour algebra.

A *polygroup* is a system $< G, \cdot, ^{-1}, e >$ where $e \in G$, '$^{-1}$' is a unary operation on G and '$\cdot$' is a binary hyperoperation on H satisfying the following:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(2) $e \cdot x = x \cdot e = \{x\}$,

(3) $x \in y \cdot z \;\Leftrightarrow\; y \in x \cdot z^{-1} \;\Leftrightarrow\; z \in y^{-1} \cdot x$.

In any polygroup the following hold:

$$e \in x \cdot x^{-1} \cap x^{-1} \cdot x,\; e^{-1} = e,\; (x^{-1})^{-1} = x,\; (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$$

where $A^{-1} = \{x^{-1} : x \in A\}$.

Some other concepts in polygroups is as follows.

A nonempty subset K of polygroup G is said to be a *subpolygroup* if and only if $e \in K$ and $< K, \cdot, ^{-1}, e >$ is itself a polygroup. Subpolygroup K of polygroup G is said to be *normal* if and only if $a^{-1}Ka \subseteq K$, for all $a \in G$.

From now on, in this paper, $G =< G, \cdot, ^{-1}, e >$ will denote a polygroup.

2 Ordered hyperstructures: Definition and properties

This section is devoted to introduce the concept of a compatible order on a polygroup. It is first introduced the concept of an ordered hypergroupoid and some basic notions. Then, the concept of ordered polygroups is introduced and some related results are given. For more details on compatible orders, specially ordered algebraic structures we refer to [1].

Definition 2.1. Let $(H, \cdot)$ be a hypergroupoid. By a *compatible order* on H we mean an order "$\leq$" with respect to which all translations $x \mapsto x \cdot y$ and $x \mapsto y \cdot x$ are isotone, that is

$$x \leq y \;\text{ implies }\; b \cdot x \cdot a \leq b \cdot y \cdot a,\; \text{ for all } a, b \in H \tag{2.1}$$

where for $A, B \subseteq H$, $A \leq B$ means that for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a \leq b$.

Definition 2.2. By an *ordered hypergroupoid* we mean a hypergroupoid on which is defined a compatible order.

When "$\cdot$" is commutative or associative, H is said to be an *ordered commutative hypergroupoid* or an *ordered semihypergroup*, respectively.

Example 2.3. (1) Consider $\mathbb{R}_1 = [1, \infty)$, the set of all real numbers greater than 1, as a poset with the natural ordering, and define $x \cdot y$ to be the set of all upper bounds of $\{x, y\}$. Thus $(\mathbb{R}_1, \cdot, \leq)$ is an ordered commutative semihypergroup with 1 as the unique identity.

(2) Consider $\mathbb{Z}$, the additive group of all integers which is a chain with the natural ordering. For $m, n \in \mathbb{Z}$, let $m \cdot n$ be the subgroup of $\mathbb{Z}$ generated by $\{m, n\}$. Then $(\mathbb{Z}, \cdot, \leq)$ is an ordered commutative semihypergroup in which 0 is an identity.

(3) Let $(G, \cdot, e, \leq)$ be an ordered group, and let $x \circ y = \langle\{x, y\}\rangle$, the subgroup of G generated by $\{x, y\}$. Then, $(G, \circ, \leq)$ is an ordered commutative hypergroup with an identity e.

(4) Let $(L; \vee, \wedge, 0)$ be a lattice with the least element 0. For $a, b \in L$, let $a \circ b = F(a \wedge b)$, where $F(x)$ is the principal filter generated by $x \in L$. Then, $(L; \circ)$ is an ordered hypergroup. Also, 0 is an identity, and if $x \in L$ be such that $x \wedge y = 0$, for some $y \in L$, then y is an inverse of x.

Definition 2.4. Let H be an ordered hypergroupoid.

(1) For every $x, y \in H$ with $x \leq y$, the set $[x, y] = \{z \in H : x \leq z \leq y\}$ is said to be an *interval* in H.

(2) A subset A of H is said to be *convex* if for all $a, b \in A$, where $a \leq b$, we have $[a, b] \subseteq A$.

Definition 2.5. Let $(E; \leq)$ be an ordered set. A subset D of E is said to be a *down-set* if $y \leq x$ and $x \in D$ imply $y \in D$. Down-set D is said to be *principal* if there exists $x \in D$ such that $D = \{y \in E : y \leq x\}$ denoted by $x^{\downarrow}$.

Definition 2.6. Let $(G; \circ_G, \leq_G)$ and $(H; \circ_H, \leq_H)$ be ordered hypergroupoids and $f : G \longrightarrow H$ be an isotone map, that is $f(x) \leq_H f(y)$ whenever $x \leq_G y$. Then,

(1) f is said to be an *order homomorphism* if f is a homomorphism of hypergroupoids $(G; , \circ_G)$ and $(H; \circ_H)$,

(2) f is an *order isomorphism* if f is an isomorphism of hypergroupiods, and f^{-1} is isotone,

(3) the kernel of f is defined by $kerf = \{(x, y) \in G \times G : f(x) = f(y)\}$.

3 Ordered polygroups

In this section, we assume that $G =< G, \cdot, ^{-1}, e >$ is a polygroup unless otherwise mentioned. Hereafter, in this paper, we use xy for $x \cdot y$, and a for $\{a\}$.

Definition 3.1. By an *ordered polygroup* we mean a polygroup which is also a poset under the binary relation $\leq$ and in which (2.1) holds.

Definition 3.2. Let H be an ordered hypergroupoid with a unique identity e. An element $x \in H$ is called *positive* if $e \leq x$. The set of all positive elements of H is called the *positive cone* of H and is denoted by H^+. $x \in H$ is called *negative* if $x \leq e$. The set of all negative elements of H is called the *negative cone* of H and is denoted by H^-.

By an elementary consequence of translations we have

Proposition 3.3. *In any ordered polygroup G, for each $x, y \in G$, we have*

$$\begin{aligned} x \leq y \;\; &\Leftrightarrow \;\; x^{-1}y \cap G^+ \neq \emptyset \;\Leftrightarrow\; yx^{-1} \cap G^+ \neq \emptyset \;\Leftrightarrow\; xy^{-1} \cap G^- \neq \emptyset \\ &\Leftrightarrow \;\; y^{-1}x \cap G^- \neq \emptyset \Leftrightarrow y^- \leq x^-. \end{aligned}$$

Theorem 3.4. *A subset P of a polygroup G is the positive cone with respect to some compatible order if and only if*

(1) $P \cap P^{-1} = \{e\}$,

(2) $P^2 = P$,

(3) *for all $x \in G$, $xPx^{-1} = P$.*

Moreover, if this order is total, $P \cup P^{-1} = G$.

Proof. $(\Rightarrow)$ Let $\leq$ be a compatible order on G and $P = G^+$, the associated positive cone.

(1) If $x \in P \cap P^{-1}$, on the one hand $e \leq x$, and on the other hand $x = y^{-1}$, for some $y \in P$. Since, $e \leq y$, then $x = y^{-1} \leq e$ proves that $x = e$.

(2) Since $e \in P$, $P = Pe \subseteq PP = P^2$. Now, let $x, y \in P$. Then $e \leq x$ and $e \leq y$ and so $e \leq xy$ which implies that $xy \subseteq P$. Hence, $P^2 \subseteq P$.

(3) Let $y \in P$, and $x \in G$. Then, $e \leq y$ implies that $e \in xex^{-1} \leq xyx^{-1}$ proves that $xyx^{-1} \subseteq P$. Since, this follows for all $x \in G$, replacing x by x^{-1}, we have $x^{-1}Px \subseteq P$ and so $P \subseteq xPx^{-1}$, complete the proof.

($\Leftarrow$) Let P be a subset of G that satisfies properties (1)-(3), and define the relation $\leq$ on G by

$$x \leq y \ \Leftrightarrow \ yx^{-1} \cap P \neq \emptyset.$$

Since, $e \in P$, by (3), $xx^{-1} = xex^{-1} \subseteq xPx^{-1} = P$ implies that $x \leq x$ and so $\leq$ is reflexive. Suppose that $x \leq y$ and $y \leq x$, for $x, y \in G$. Then $yx^{-1} \cap P \neq \emptyset$ and $xy^{-1} \cap P \neq \emptyset$ whence $xy^{-1} \cap P^{-1} \cap P \neq \emptyset$, implies that $e \in xy^{-1}$, i.e., $x = y$ proving $\leq$ is antisymmetric. Now, assume that $x \leq y$ and $y \leq z$, for $x, y, z \in G$. Then $yx^{-1} \cap P \neq \emptyset$ and $zy^{-1} \cap P \neq \emptyset$. Let $u \in yx^{-1} \cap P$ and $v \in zy^{-1} \cap P$. Then $uv \subseteq P^2 = P$. On the other hand, $\in zy^{-1}$ and $v \in yx^{-1}$ imply $y^{-1} \in z^{-1}u$ and $y \in vx$ whence $e \in y^{-1}y \subseteq z^{-1}(uv)x$. Then, there is $t \in uv$ and $s \in tx$ such that $e \in z^{-1}s$. This implies that $z = s \in tx$. Hence, $t \in zx^{-1}$, i.e., $uv \cap zx^{-1} \neq \emptyset$ whence $zx^{-1} \cap P \neq \emptyset$ proving $\leq$ is transitive. Thus, $\leq$ is an order. For compatibility, we first prove that $Px = xP$, for all $x \in G$. Let $z \in G$. Then

$$\begin{aligned} z \in Px \ &\Rightarrow \ z \in yx \text{ for some } y \in P \Rightarrow \ x^{-1}z \subseteq x^{-1}yx = x^{-1}y(x^{-1})^{-1} \subseteq P \\ &\Rightarrow \ z \in xP, \end{aligned}$$

i.e., $Px \subseteq xP$. By a similar way, we can prove that $xP \subseteq Px$. Hence, $xP = Px$, for all $x \in G$. Now, assume that $x \leq y$ and $a, b \in G$. Since, $\leq$ is reflexive, by (3)

$$\begin{aligned} ayb(axb)^{-1} \ &= \ aybb^{-1}x^{-1}a^{-1} \subseteq ayPx^{-1}a^{-1} \subseteq aPyx^{-1}a^{-1} \subseteq aP^2a^{-1} \\ &= \ aPa^{-1} = P \end{aligned}$$

which shows that $axb \leq ayb$. By the definition of $\leq$ we get $x \in P$ if and only if $e \leq x$ and so $P = G^+$.

If G is totally ordered, then $x \leq e$ or $e \leq x$, for all $x \in G$. So, $e \in xx^{-1} \leq ex^{-1} = x^{-1}$ and so $x \in P$ or $x \in P^{-1}$, observe that $x = (x^{-1})^{-1}$. Thus, $G = P \cup P^{-1}$. □

Proposition 3.5. *If G is an ordered polygroup with $|G| > 1$, then G can not have a top element or a bottom element.*

Proof. Let $G = \{e, a\}$. If $e < a$ or $a < e$, then $a = a^{-1} < e$ or $e < a^{-1} = a$, respectively, which is a contradiction. Now, assume that

$|G| > 2$, t be the top element of G and $e \neq a \in G$. Then $a \leq t$ and so $ta \leq t$ whence $t \in te \subseteq taa^{-1} \leq ta^{-1}$. Hence, $t \in ta^{-1}$. Likewise, we conclude that $t \in a^{-1}t$. By the uniqueness, we get $a = e$ which is a contradiction.

The proof of the other case is concluded as well. □

Definition 3.6. An element x of G is said to be of *order* n, $n \in \mathbb{N}$, if $e \in x^n$ where $x^n = (\cdots((\overbrace{x \circ x) \circ x) \circ \cdots) \circ x}^{n\ times})$. If such a natural number does not exist, we say that x is of infinite order.

Theorem 3.7. *Suppose that G is an ordered polygroup in which $G^+ \neq \{e\}$. Then every element of $G^+ \setminus \{e\}$ is of infinite order.*

Proof. Suppose that $x \in G^+ \setminus \{e\}$. We first observe that if $x = x^{-1}$, x can not belong to G^+. Then, $e < x$ implies that $e < x = ex < x^2$. Moreover, this implies that $e \notin x^2$. Similarly, we conclude that $e < x^3$ and $e \notin x^3$. Continuing this process we get $e < x^n$ and $e \notin x^n$, for all $n \in \mathbb{N}$, proving x is not of finite order. □

Corollary 3.8. *Any ordered polygroup in which every nontrivial element is of finite order is an antichain.*

Proof. Let G be an ordered polygroup satisfying the hypothesis. By Theorem 3.7, we know that $G^+ = \{e\}$. Now, if $a, b \in G$ be such that $a \leq b$, then $e \in a^{-1}a \leq a^{-1}b$ and so $e \leq u$, for some $u \in a^{-1}b$. This implies that $u \in G^+$ and so $u = e$. Thus, $e \in a^{-1}b$ whence $a = b$. This means that G is an antichain. □

Corollary 3.9. *Every finite ordered polygroup is an antichain.*

Example 3.10. Let $G = \{e, a\}$. Then G is a polygroup where the hyperoperation is given by the following table:

$\circ$	e	a
e	e	a
a	a	$\{e, a\}$

in which $a^{-1} = a$ i.e., a is an idempotent. Now, if a is a positive element, so $G^+ = \{e, a\}$ and hence $(G^+)^{-1} \cap G^+ \neq \{e\}$. This contradicts Theorem 3.4. This example shows that the converse of Theorem 3.7 does not hold in general.

Definition 3.11. If G is an ordered polygroup, by a *convex subgroup* of G we shall mean a subgroup which is also a convex subset, under the order of G.

Definition 3.12. A nonempty subset H of G is said to be *S-reflexive* if $xy \cap H \neq \emptyset$ implies that $xy \subseteq H$, for all $x, y \in G$.

Theorem 3.13. *If H is a subpolygroup of an ordered polygroup G then $H^+ = H \cap G^+$. Moreover, if H^+ is S-reflexive, the following statements are equivalent:*

(1) *H is convex;*

(2) *H^+ is a down-set of G^+.*

Proof. Since, $e_H = e_G$, it is clear that $H^+ = H \cap G^+$.

(1) $\Rightarrow$ (2) Suppose that $e_H \leq y \leq x$ where $e_H, x \in H^+ \subseteq H$. Then (1) gives $y \in H \cap G^+ = H^+$ and so H^+ is a down-set of G^+.

(2) $\Rightarrow$ (1) Suppose now that $x \leq y \leq z$ where $x, z \in H$. Then $x^{-1}x \leq x^{-1}y \leq x^{-1}z$. Thus, $x^{-1}z \subseteq H^+$ and so there is $a \in H^+$ such that $a \in x^{-1}z$. Hence, there is $b \in x^{-1}y$ such that $b \leq a \in H^+$, and since H^+ is a down-set of G^+, $b \in H^+$, i.e., $x^{-1}y \cap H^+ \neq \emptyset$. Since, H^+ is S-reflexive, so $x^{-1}y \subseteq H^+ \subseteq H$ whence $y \in xH = H$, proving H is convex. $\square$

If G is an ordered polygroup and H is a normal subpolygroup of G, then a natural candidate for a positive cone of G/H is $\natural_H(G^+)$, where $\natural_H : G \longrightarrow G/H$ is the canonical projection. Precisely when this occurs is the substance of the following result.

Theorem 3.14. *Let G be an ordered polygroup and let H be a normal subpolygroup of G. Then $\natural_H(G^+) = \{pH : p \in G^+\}$ is the positive cone of a compatible order on the quotient polygroup G/H if and only if H is convex.*

Proof. Suppose that $Q = \{pH : p \in G^+\}$ is the positive cone of a compatible order on G/H. To show that H is convex, suppose that $c \leq b \leq a$ with $c, a \in H$. Then $(bH)^{-1} = (bH)^{-1} \cdot aH = b^{-1}aH$. On the other hand, $b \leq a$ implies that $b^{-1}a \cap G^+ \neq \emptyset$. Hence $(bH)^{-1} \cap Q \neq \emptyset$ and so $bH \cap Q^{-1} \neq \emptyset$. Similarly, we have $bH = bH \cdot c^{-1}H = bc^{-1}H$ and since $bc^{-1} \cap G^+ \neq \emptyset$, $bH \cap Q \neq \emptyset$. Thus, $bH \cap (Q \cap Q^{-1}) \neq \emptyset$ whence $bH = H$, i.e., $b \in H$.

Conversely, suppose that H is convex and let $Q = \{pH : p \in G^+\}$. It is clear that $Q^2 = Q$. Suppose now that $xH \in Q \cap Q^{-1}$. Then $xH = pH = q^{-1}H$ where $p, q \in G^+$. These equalities also give $pq \cap H \neq \emptyset$. Now, since $p \leq pq$, then $e_H \leq p \leq u$, where $u \in pq \cap H$ whence the convexity of H gives $p \in H$. It follows that $xH = pH = H$ and hence $Q \cap Q^{-1} = \{H\}$. Finally, since G^+ is a normal subsemihypergroup of G it is clear that $Q = \natural_H(G^+)$ is a normal subsemihypergroup of G/H. It now follows by Theorem 3.4 that Q is the positive cone of a compatible order on G/H. $\square$

If H is a convex normal subpolygroup of an ordered polygroup G then the order $\leq_H$ on G/H that corresponds to the positive cone $\{pH : p \in G^+\}$ can be described as in the proof of Theorem 3.4. We have

$$\begin{aligned} xH \leq_H yH &\Rightarrow yx^{-1}H \subseteq Q \\ &\Rightarrow (\forall a \in yx^{-1})(\exists p \in G^+)aH = pH \\ &\Rightarrow (\forall a \in yx^{-1})(\exists p \in G^+)(\exists h \in H)a \in ph \geq h \\ &\Rightarrow (\forall a \in yx^{-1})(\exists h \in H)\ a \geq h \\ &\Rightarrow yx^{-1} \geq h. \end{aligned}$$

From the last inequality and that $y \in ye \subseteq yx^{-1}x$ it follows that $y \geq u$, for some $u \in hx$. Conversely, assume that there exists $h \in H$ and $u \in hx$ such that $y \geq u$, and let $a \in yx^{-1}$. From $yx^{-1} \geq yx^{-1}$ it follows that $a \geq t$, for some $t \in ux^{-1}$ and hence $at^{-1} \geq tt^{-1}$. This implies that $v \geq e$, for some $v \in at^{-1}$ and so

$$vH \in at^{-1}H \cap Q. \tag{3.1}$$

Now, $t \in ux^{-1}$ implies that $t^{-1} \in xu^{-1} \subseteq xx^{-1}h^{-1} \subseteq xx^{-1}H$ and so $at^{-1} \subseteq axx^{-1}H = axHx^{-1} = aH$. Thus, $at^{-1}H \subseteq aH$. Combining (3.1), we get $\{aH\} \cap Q \neq \emptyset$, i.e., $aH \in Q$ and so $aH = pH$, for some $p \in G^+$. This implies $yx^{-1}H \subseteq Q$ and hence $xH \leq_H yH$, completes the proof.

Thus we see that $\leq_H$ can be described by

$$xH \leq_H yH \Leftrightarrow (\exists h \in H)(\exists u \in hx)\ y \geq u.$$

In referring to the ordered quotient polygroup G/H we shall implicitly infer that the order is $\leq_H$ as described above.

Here we give a characterization of polygroup homomorphisms that are isotone.

Theorem 3.15. *Let G and H be ordered polygroups. If $f : G \longrightarrow H$ is a polygroup homomorphism, f is isotone if and only if $f(G^+) \subseteq H^+$.*

Proof. Assume that f is isotone. If $x \in G^+$, i.e., $x \geq e$ then $f(x) \geq f(e_G) = e_H$ means that $f(x) \in H^+$.

Conversely, assume that $x \leq y$ in G. Then $yx^{-1} \subseteq G^+$ and so $f(y)f(x)^{-1} = f(yx^{-1}) \subseteq f(G^+) \subseteq H^+$. This implies that $f(y) \geq f(x)$ proving f is isotone. $\square$

Corollary 3.16. *If G is an ordered polygroup and H is a convex normal subpolygroup of G, then the natural homomorphism $\natural_H : G \longrightarrow G/H$ is isotone.*

Proof. By Theorem 3.15, it is enough to prove that $\natural(G^+) \subseteq (G/H)^+$. For this, let $yH \in \natural(G^+)$. Then $yH = gH$, for some $g \in G^+$ whence $y \in gh \geq h$ for some $h \in H$. This implies that $eH \leq_H yH$ and so $yH \in (G/H)^+$. $\square$

Definition 3.17. Let G and H are ordered polygroups. A mapping $f : G \longrightarrow H$ is said to be *exact* if $f(G^+) = H^+$.

Definition 3.18. Two ordered polygroups G and H are said to be *isomorphic* if there is a polygroup isomorphism $f : G \longrightarrow H$ that is also an order isomorphism.

If two ordered polygroups G and H are isomorphic we write $G \doteq H$.

Theorem 3.19. *For ordered polygroups G and H, the following are equivalent:*

(1) $G \doteq H$,

(2) *there is an exact polygroup isomorphism* $f : G \longrightarrow H$.

Proof. (1) $\Rightarrow$ (2) If G and H are isomorphic, there is a polygroup isomorphism $f : G \longrightarrow H$ which is also an order isomorphism. By Theorem 3.15, $f(G^+) \subseteq H^+$. Let $g = f^{-1}$. Obviously, g satisfies the conditions of Theorem 3.15. Hence, $g(H^+) \subseteq G^+$ whence $H^+ = f(g(H^+)) \subseteq f(G^+)$. Thus $H^+ = f(G^+)$ and so (2) holds.

(2) $\Rightarrow$ (1) It is obvious. $\square$

Theorem 3.20. *Let G and H be ordered polygroups and $f : G \longrightarrow H$ be an exact polygroup homomorphism. Then $Imf \doteq G/kerf$.*

Proof. We first observe that $kerf$ is a convex normal subpolygroup of G and so $G/kerf$ is an ordered polygroup. By first isomorphism theorem of polygroups there is an isomorphism $\phi : G/kerf \simeq Imf$ which $\phi(xK) = f(x)$ where $K = kerf$. It remains that we prove ϕ is exact. Let $xK \in (G/K)^+$. Then $e_G K \leq_K xK$ whence $k \leq x$, for some $k \in K$, and so $e_H = f(k) \leq f(x)$ whence $\phi(xK) = f(x) \in (Imf)^+$. Conversely, if $f(x) \in (Imf)^+ \subseteq H^+$, since f is exact, there exists $g \in G^+$ such that $f(x) = f(g)$. Consequently, $xK = gK$ and so $x \in gk \geq k$, for some $k \in K$. Thus,

$$xK \in (G/K)^+ \Leftrightarrow \phi(xK) = f(x) \in (Imf)^+$$

proving ϕ is exact. It now follows by Theorem 3.19 that $G/kerf \doteq Imf$. $\square$

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