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Closed, Reflexive, Invertible, and Normal Subhypergroups of Special Hypergroups

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Abstract

In [5] J. Jantosciak introduced several special types of subhypergroups (invertible, closed, normal, reflexive) of a general hypergroup and studied their relationship. In this article, the full description of such subhypergroups in hypergroups induced by quasiordered groups is given. Further, it is shown that there are no such non-trivial subhypergroups in quasiorder hypergroups.

Key words: Hypergroup, transposition hypergroup, join space, closed, reflexive, invertible, and normal subhypergroup, quasiordered group, quasiorder hypergroup.

MSC 2010: 20N20.

1 Introduction

We will sum up basic concepts from hypergroup theory and results which will be needed in the following text.

- Let $H \neq \emptyset$. A mapping $*\colon H \times H \to \mathscr{P}^*$ is called *binary hyperoperation* on H. The pair $(H, *)$ is *hypergroupoid.*
- A hypergroupoid $(H, *)$ is *extensive* if $\{a, b\} \subseteq a * b$ for any $a, b \in H$.
- A hypergroupoid $(H, *)$ is called *hypergroup* if the hyperoperation $*$ is associative, i.e

$$(x * y) * z = x * (y * z) \qquad \text{for any } x, y, z \in H,$$

and *the reproduction axiom*

$$a * H = H = H * a \qquad \text{for any } a \in H$$

is satisfied.

- Let $(H, *)$ be a hypergroup and $S \subseteq H$. Assume that $a * b \subseteq S$ for any $a, b \in S$. Thus $(S, *)$ is an associative hypergroupoid (so called *semihypergroup*). If, moreover, it satisfies the reproduction axiom, that is, it is a hypergroup, we say that $(S, *)$ is *subhypergroup* of $(H, *)$.
- A hypergroup $(H, *)$ is called *transposition hypergroup* if *the transposition axiom* is satisfied, that is, for any quadruple of elements $a, b, c, d \in H$ the implication:

  $$\text{If } b\backslash a \approx c/d, \text{ then } a * d \approx b * c$$

  holds, where

  $$b\backslash a = \{x \in H : a \in b * x\},$$
  $$c/d = \{x \in H : c \in x * d\}$$

  are *left* and *right extensions*, respectively.
  (For two set A, B, the symbol $A \approx B$ means that A and B are incident, i.e. $A \cap B \neq \emptyset$.)
- A commutative transposition hypergroup $(H, *)$ is called *join space*.

The following concepts play the key role in the formulation of the main results.

Definition 1.1 *A subhypergroup $(S, *)$ of a hypergroup $(H, *)$ is called*

- closed *if $a/b \subseteq S$ and $b\backslash a \subseteq S$ for any $a, b \in S$,*
- invertible *if $a/b \approx S$ implies $b/a \approx S$ and $b\backslash a \approx S$ implies $a\backslash b \approx S$ for any $a, b \in H$,*
- reflexive *if $a\backslash S = S/a$ for any $a \in H$,*
- normal *if $a * S = S * a$ for any $a \in H$.*

J. Jantosciak (see [5]) proved that

- invertible subhypergroup of any hypergroup is closed,
- closed and normal subhypergroup of a transposition hypergroup is reflexive,
- invertible and normal subhypergroup of a transposition hypergroup is closed and reflexive.

2 Hypergroups Induced by Quasiordered Groups

First the relationship among closed, normal, invertible, and reflexive subhypergroups of hypergroups induced by quasiordering will be tackled.

Definition 2.1 An ordered (quasiordered) group *is a triple* $(H, \cdot, \leq)$*, where* $(H, \cdot)$ *is a group and "*$\leq$*" is an ordering (quasiordering) on* H *having* the substitution property *on* $(H, \cdot)$*, that is for any quadruple* $a, b, c, d \in H$ *such that* $a \leq b$*,* $c \leq d$*, there is* $a \cdot c \leq b \cdot d$*.*

A hyperoperation is naturally induced on each (quasi)ordered group $(H, \cdot, \leq)$. For $x \in H$, let us denote $[x)_\leq$ *the principal end* generated by x, that is $[x)_\leq = \{y \in H : x \leq y\}$. Analogously *the principal beginning* $(x]_\leq$ is defined. For $x, y \in H$, let us denote $x * y = [x \cdot y)_\leq$. Then $(H, *)$ is a hypergroupoid associated with $(H, \cdot, \leq)$. The following result can be proven (see [3, 6, 7]):

Theorem 2.1 *Let* $(H, \cdot, \leq)$ *be a quasiordered group and* $(H, *)$ *the hypergroupoid associated with it. Then* $(H, *)$ *is the transposition hypergroup.*

Let $(G, \cdot, \leq)$ be *a quasiordered group*, $(G, *)$, where $a * b = [a \cdot b)_\leq$, be the induced (transposition) hypergroup. Let $(S, *)$ be its subhypergroup, that is, $[x \cdot y)_\leq \subset S$ holds for $x, y \in S$, namely $x \cdot y \in S$.

For the right and left extensions we have:

$$
\begin{aligned}
a/b &= \{x \in G : a \in x * b\} = \{x \in G : x \cdot b \leq a\} = \\
&= \{x \in G : x \leq a \cdot b^{-1}\}, \\
b\backslash a &= \{x \in G : a \in b * x\} = \{x \in G : b \cdot x \leq a\} = \\
&= \{x \in G : x \leq b^{-1} \cdot a\}.
\end{aligned}
$$

Theorem 2.2 *A subhypergroup* $(S, *)$ *is closed iff* $(S, \cdot)$ *is a subgroup of the group* $(G, \cdot)$ *and* $(a]_\leq \cup [a)_\leq \subset S$ *for any* $a \in S$*.*

Proof. *Necessity:*

For $a \in S$ there is $a/a = (e]_\leq$, hence $e \in S$ (e is the neutral element).

If $a \in S$, then $e/a = (a^{-1}]_\leq$, hence $a^{-1} \in S$.

If $a \in S$, then $a/e = (a]_\leq$, hence $(a]_\leq \subset S$. Because $(S, *)$ is a subhypergroup we also get $a * e = [a)_\leq \subset S$.

Sufficiency: Let $a, b \in S$. Then $a \cdot b^{-1} \in S$, so $(a \cdot b^{-1}]_\leq = a/b \subset S$. Analogously for $b\backslash a$. $\square$

Theorem 2.3 *A subhypergroup $(S, *)$ is closed iff it is invertible.*

Proof. Each invertible subhypergroup is closed.

Now, let us assume that $(S, *)$ is closed. If $a/b \approx S$, there exists $x \in S$ such that $x \leq a \cdot b^{-1}$. Due to the previous theorem $a \cdot b^{-1} \in S$ and also $(a \cdot b^{-1})^{-1} = b \cdot a^{-1} \in S$, therefore $b/a \subset S$. Especially, $b/a \approx S$. Analogously the statement for $b \backslash a$ can be proven. $\square$

Theorem 2.4 *A subhypergroup $(S, *)$ is reflexive iff the following property holds: If for $x, y \in G$ the element $x \cdot y$ is covered by an element of S, then $y \cdot x$ is also covered by an element of S.*

Proof. The following equalities hold:

$$a \backslash S = \bigcup_{b \in S} a \backslash b = \bigcup_{b \in S} (a^{-1} \cdot b]_{\leq} ,$$
$$S/a = \bigcup_{b \in S} b/a = \bigcup_{b \in S} (b \cdot a^{-1}]_{\leq} .$$

Hence, $x \in a \backslash S$ iff $b \in S$ exists such that $x \leq a^{-1} \cdot b$, that is, $a \cdot x \leq b$. Analogously $x \in S/a$ iff $c \in S$ exists such that $x \cdot a \leq c$. This implies the statement. $\square$

If $(S, *)$ is a closed subhypergroup, according to the previous result, the condition "to be comparable with an element of S" is equivalent with the condition "to be in S". Therefore, we get:

Corollary 2.1 *A closed subhypergroup $(S, *)$ is reflexive iff the following property holds: If for $x, y \in G$ the element $x \cdot y \in S$, then also $y \cdot x \in S$.*

Theorem 2.5 *A subhypergroup $(S, *)$ is normal iff the following property holds: If $x \cdot y$, where $x, y \in G$, covers an element of S, then $y \cdot x$ also covers an element of S.*

Proof. The following equalities hold:

$$a * S = \bigcup_{b \in S} a * b = \bigcup_{b \in S} [a \cdot b)_{\leq} ,$$
$$S * a = \bigcup_{b \in S} b * a = \bigcup_{b \in S} [b \cdot a)_{\leq} .$$

Hence, $x \in a * S$ iff $b \in S$ exists such that $a \cdot b \leq x$, that is, $b \leq a^{-1} \cdot x$. Analogously $x \in S * a$ iff $c \in S$ exists such that $c \leq x \cdot a^{-1}$. This implies the statement. $\square$

In case $(S, *)$ is a closed subhypergroup, analogously to reflexive subhypergroups we have:

Corollary 2.2 *A closed subhypergroup $(S, *)$ is normal iff the following property holds: If for $x, y \in G$ the element $x \cdot y \in S$, than also $y \cdot x \in S$.*

Joining the previous two results we get:

Corollary 2.3 *Let $(S, *)$ be a closed subhypergroup. Then $(S, *)$ is normal iff it is reflexive.*

3 Quasiorder Hypergroups

Now the relationship among closed, normal, invertible, and reflexive subhypergroups of quasiorder hypergroups will be tackled.

If R is a quasiordering on H we denote $R(a) = \{x \in H : a\,R\,x\}$. Further, for $A \subseteq H$ we set $R(A) = \bigcup\limits_{a \in A} R(a)$.

Theorem 3.1 *Let (H, R) be a quasiordered set. For any pair $a, b \in H$ we set*

$$a *_R b = R(a) \cup R(b) = R(\{a, b\}).$$

Then $(H, *_R)$ is commutative extensive hypergroup.

For the proof see [3, p. 150, Th. 2.1].

Definition 3.1 *A* hypergroup $(H, *)$ *is said* quasiorder *if the following conditions are satisfied:*

- $a \in a^3 \subseteq a^2$,
- $a * b = a^2 \cup b^2$

for any $a, b \in H$.

The following theorem characterizes all quasiorder hypergroups.

Theorem 3.2 *A hypergroup $(H, *)$ is quasiorder iff a quasiordering R on H exists such that for any $a, b \in H$ there is*

$$a * b = R(a) \cup R(b) = R(\{a, b\}).$$

For the proof see [1, p. 96].

Theorem 3.3 *Let $(G, *)$ be a quasiorder hypergroup. Then any subhypergroup of $(G, *)$ is reflexive and normal. Moreover, $(G, *)$ contains no proper closed or invertible subhypergroup.*

Proof. Suppose that $(G, *)$ is a quasiorder hypergroup and R is a quasiordering from the previous theorem.

The hyperoperation $*$ is commutative, hence any subhypergroup $(S, *)$ is reflexive and normal.

Further,

$$a/b = \{x \in G : a \in b * x\} = \{x \in G : a \in R(b) \cup R(x)\}.$$

Especially $a/a = G$. If $(S, *)$ is closed, then necessarily $S = G$. Therefore, no proper closed subhypergroup exists.

Analogously, any invertible subhypergroup is closed, therefore, no proper invertible subhypergroup exists. □

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