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Operators on Weak Hypervector Spaces

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Abstract

Let X and Y be weak hypervector spaces and $L_w(X,Y)$ be the set of all weak linear operators from X into Y. We prove some algebraic properties of $L_w(X,Y)$.
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1 Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions has been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability and etc. A wealth of applications of this concepts are given in $[1-2]$ and [12].

In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [11]. In [11], Tallini introduced the concept of norm on weak hypervector spaces. We used this definition to extend some theorems of analysis from classic vector spaces to hypervector spaces. For example see $[5-7, 9]$. Moreover, in [4] we defined the concept of dimension of weak hypervector spaces and also authors in [8] introduce the new concept hyperalgebra and quotient hyperalgebra.

Now we want to use some of our defined concepts and prove some algebraic properties of $L_w(X,Y)$, where $L_w(X,Y)$ is the set of all weak linear operators from the weak hypervector space X into the weak hypervector space Y. Note that the hypervector spaces used in this paper are the special case where there is only one hyperoperation, the external one, all the others are ordinary operations. The general hypervector spaces have all operations multivalued also in the hyperfield (see [12]).

2 Preliminaries

We need some Preliminary definitions for to state our results. In this section we state them.

Definition 2.1. [11] *A weak or weakly distributive hypervector space over a field F is a quadruple $(X,+,o,F)$ such that $(X,+)$ is an abelian group and $o : F \times X \longrightarrow P_*(X)$ is a multivalued product such that*

(i) $\forall a \in F, \forall x, y \in X,\ [ao(x+y)] \cap [aox + aoy] \neq \emptyset$,

(ii) $\forall a, b \in F, \forall x \in X,\ [(a+b)ox] \cap [aox + box] \neq \emptyset$,

(iii) $\forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox$,

(iv) $\forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox = -(aox)$,

(v) $\forall x \in X,\ x \in 1ox$.

We call (i) and (ii) weak right and left distributive laws, respectively.

Note that the set $ao(box)$ in (3) is of the form $\cup_{y \in box} aoy$.

Definition 2.2. [11] *Let $(X,+,o,F)$ be a weak hypervector space over a field F, that is the field of real or complex numbers. We define a pseudonorm in X as a mapping $\|.\| : X \longrightarrow R$, of X into the reals such that:*

(i) $\|0\| = 0$,

(ii) $\forall x, y \in X,\ \|x+y\| \leq \|x\| + \|y\|$,

(iii) $\forall a \in F,\ \forall x \in X, \quad \sup \|aox\| = |a|\|x\|$.

Definition 2.3. *Let X and Y be hypervector spaces over F. A map $T : X \longrightarrow Y$ is called*

(i) linear if and only if

$$T(x+y) = T(x) + T(y), \quad T(aox) \subseteq aoT(x), \quad \forall x, y \in X, a \in F$$

(ii) antilinear if and only if

$$T(x+y) = T(x) + T(y), \quad T(aox) \supseteq aoT(x), \quad \forall x, y \in X, a \in F$$

(iii) strong linear if and only if

$$T(x+y) = T(x) + T(y), \quad T(aox) = aoT(x), \quad \forall x, y \in X, a \in F.$$

3 Main results

Before to state our results we describe some fundamental concepts and lemmas from [4]. For more details see [4]. By Lemma 3.1 in [4] we have the following definition. Throughout paper, suppose that X and Y are weak hypervector spaces over a field F.

Definition 3.1. [4] *If $a \in F$ and $x \in X$, then z_{aox} for $0 \neq a$ is that element of aox such that $x \in a^{-1}oz_{aox}$ and for $a = 0$, we define $z_{aox} = 0$.*

As the descriptions in [4], z_{aox} is not unique, necessarily. So the set of all these elements denoted by Z_{aox}. In the mentioned paper we introduced a certain category of weak hypervector spaces. These weak hypervector spaces have been called "normal". We proved that Z_{aox} is singleton in a normal weak hypervector space.

Definition 3.2. [4] *Suppose X satisfy the following conditions:*

(i) $(Z_{a_1ox} + Z_{a_2ox}) \cap Z_{(a_1+a_2)ox} \neq \emptyset, \ \forall x \in X, \ \forall a_1, a_2 \in F,$

(ii) $(Z_{aox_1} + Z_{aox_2}) \cap Z_{ao(x_1+x_2)} \neq \emptyset, \ \forall x_1, x_2 \in X, \ \forall a \in F.$

Then X is called a normal weak hypervector space.

Lemma 3.1. [4] *If $a \in F$, $0 \neq b \in F$ and $x \in X$, then the following properties hold:*

(i) $x \in Z_{1ox}$;

(ii) $aoZ_{box} = abox$;

(iii) $Z_{-aox} = -Z_{aox}$;

(iv) If X is normal, then Z_{aox} is singleton.

In [4], the following lemma stated a criterion for normality of a weak hypervector space.

Lemma 3.2. [4] *X is normal if and only if*

(i) $z_{a_1ox} + z_{a_2ox} = z_{(a_1+a_2)ox}, \ \forall x \in X, \ \forall a_1, a_2 \in F,$

(ii) $z_{aox_1} + z_{aox_2} = z_{ao(x_1+x_2)}, \ \forall x_1, x_2 \in X, \ \forall a \in F.$

Definition 3.3. [6] *Let $T : X \longrightarrow Y$ be an operator. T is said to be bounded if there exists a positive real number K such that we have*

$$\|Tx\| \leq K\|x\| \qquad (\forall x \in X).$$

Definition 3.4. [9] *A map $T : X \longrightarrow Y$ is called weak linear operator if T is additive and satisfies*

$$T(Z_{aox}) \subseteq aoTx, \quad (a \in F, x \in X).$$

Denote the set of all weak linear operators and the set of all bounded weak linear operators from X into Y by $L_w(X,Y)$ and $B_w(X,Y)$, respectively.

Theorem 3.1. [4] *Let X be normal. Then X with the same defined sum and the following scalar product is a classical vector space:*

$$a.x = z_{aox}, \quad \forall a \in F, x \in X.$$

Lemma 3.3. *Let Y be normal, $T \in L_w(X,Y)$ and $a \in F$. Define*

$$aT : X \to Y$$

$$x \mapsto a.Tx$$

Then aT is a weak linear operator, where the operation '.' is the defined scalar product in Theorem 3.1. Moreover, for all $a, b \in F$ and $T, S \in L_w(X,Y)$ we have

$$a(T+S) = aT + aS,$$
$$(a+b)T = aT + bT.$$

Proof. Let $a.u = z_{aou}$, where $u \in Y$. From Theorem ?, we know that Y with this scalar product is a classical vector space. Let $x, y \in X$ and $b \in F$. By Lemma 3.1 we have

$$\begin{aligned} (aT)(z_{box}) &= a.T(z_{box}) \subseteq a.(boTx) \\ &= \{a.u : u \in boTx\} \\ &= \{z_{aou} : u \in boTx\} \\ &= z_{ao(boTx)} = z_{aboTx} \\ &= boz_{aoTx} = bo(a.Tx) = bo(aT)x \end{aligned}$$

and also from the normality of Y, we obtain

$$\begin{aligned}(aT)(x+y)=a.T(x+y) &\subseteq a.(Tx+Ty)\\ &= z_{ao(Tx+Ty)}\\ &= z_{aoTx}+z_{aoTy}\\ &= a.Tx+a.Ty\\ &= (aT)x+(aT)y.\end{aligned}$$

Hence aT is a weak linear operator. Now let $T,S\in L_w(X,Y)$ and $x\in X$. The normality of Y yields

$$\begin{aligned}[a(T+S)]x=a.(T+S)x=z_{ao(T+S)x} &= z_{ao(Tx+Sx)}\\ &= z_{aoTx}+z_{aoSx}\\ &= a.Tx+a.Sx\\ &= (aT)x+(aS)x\\ &= (aT+aS)x\end{aligned}$$

which implies that

$$a(T+S)=aT+aS.$$

The second relation is proved in a similar way.

Theorem 3.2. *Let Y be normal. Then $L_w(X,Y)$ with the following sum and product is a weak hypervector space over F.*

$$(T+S)x=Tx+Sx \quad (T,S\in L_w(X,Y),x\in X)$$

$$aoT=\{S\in L_w(X,Y):Sx\in aoTx,\forall x\in X\} \quad (a\in F,T\in L_w(X,Y)).$$

Proof. First we show that aoT is a nonempty subset of $L_w(X,Y)$. By Lemma 3.3, $aT\in L_w(X,Y)$ and for any $x\in X$ we have

$$(aT)x=a.Tx=z_{aoTx}\in aoTx$$

which imply that $aT\in aoT$. It is easy to check that $(L_w(X,Y),+)$ is an abelian group. We show the correctness the first property of scalar product, the rest properties are obtained in a similar way. By Lemma 3.3, for all $a\in F$ and $T,S\in L_w(X,Y)$ we have

$$a(T+S)=aT+aS.$$

This together with

$$a(T+S)\in ao(T+S),\ aT\in aoT,\ aS\in aoS$$

imply that
$$[ao(T+S)] \cap [aoT + aoS] \neq \emptyset$$
and this completes the proof.

Theorem 3.3. *Let Y be normal. Then the following statements are hold.*

(i) *For all $a \in F$ and $T \in L_w(X,Y)$ we have $z_{aoT} = aT$.*

(ii) *$L_w(X,Y)$ is a normal weak hypervector space.*

Proof. (i) By Definition 3.1 we have
$$z_{aoT} \in aoT, T \in a^{-1}oz_{aoT}$$
which for all $x \in X$ implies
$$z_{aoT}x \in (aoT)x, Tx \in (a^{-1}oz_{aoT})x.$$
Since by Theorem 3.2 we have
$$(aoT)x \subseteq aoTx, (a^{-1}oz_{aoT})x \subseteq a^{-1}oz_{aoT}x,$$
we obtain
$$z_{aoT}x \in aoTx, Tx \in a^{-1}oz_{aoT}x.$$
These relations, by Definition 3.1 yield that $z_{aoT}x = z_{aoTx}$. So we obtain $z_{aoT}x = (aT)x$, for all $x \in X$ and hence $z_{aoT} = aT$.

(ii) The normality of $L_w(X,Y)$ can be concluded from Lemma 3.3 and part (i).

Theorem 3.4. *Let Y be normal. Then $B_w(X,Y)$ with the defined sum and scalar product in Theorem ? is a subhypervector space of $L_w(X,Y)$.*

Proof. It is enough to show that $T+S, aoT \in B_w(X,Y)$ for any $a \in F$ and $T,S \in B_w(X,Y)$ it is easy to check that $T+S \in B_w(X,Y)$. Let $S \in aoT$. Hence $Sx \in aoTx$ and so
$$\|Sx\| \leq |a|\|Tx\| \leq |a|\|T\|\|x\|.$$
This completes the proof.

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