THE TRANSPOSITION AXIOM IN HYPERCOMPOSITIONAL STRUCTURES

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ABSTRACT. The hypergroup (as defined by F. Marty), being a very general algebraic structure, was subsequently quickly enriched with additional axioms. One of these is the transposition axiom, the utilization of which led to the creation of join spaces (join hypergroups) and of transposition hypergroups. These hypergroups have numerous applications in geometry, formal languages, the theory of automata and graph theory.

This paper deals with transposition hypergroups. It also introduces the transposition axiom to weaker structures, which result from the hypergroup by the removal of certain axioms, thus defining the transposition hypergroupoid, the transposition semi-hypergroup and the transposition quasi-hypergroup. Finally, it presents hypercompositional structures with internal or external compositions and hypercompositions, in which the transposition axiom is valid. Such structures emerged during the study of formal languages and the theory of automata through the use of hypercompositional algebra.

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1. THE TRANSPOSITION AXIOM IN HYPERGROUPS

Hypercompositional structures are algebraic structures equipped with multivalued compositions, which are called hyperoperations or hypercompositions. A hypercomposition in a non-void set $H$ is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of $H$. Hypercompositional structures came into being through the notion of the hypergroup. The hypergroup was introduced by F. Marty in 1934, during the 8\textsuperscript{th} congress of the Scandinavian
Mathematicians [18]. F. Marty used hypergroups in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups. A hypergroup, which is a generalization of the group, satisfies the following axioms:

i. \((ab)c = a(bc)\) for all \(a, b, c \in H\) (associativity),

ii. \(aH = Ha = H\) for all \(a \in H\) (reproduction).

Note that, if «» is a hypercomposition in a set \(H\) and \(A, B\) are subsets of \(H\), then \(A \cdot B\) signifies the union \(\bigcup_{(a, b) \in A \times B} a \cdot b\). In both cases, \(aA\) and \(Aa\) have the same meaning as \(\{a\}A\) and \(A\{a\}\) respectively. Generally, the singleton \(\{a\}\) is identified with its member \(a\). In [18], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

\[
\begin{align*}
\left\{ \frac{a}{b} \right\} &= \{x \in H \mid a \in xb\} \quad \text{and} \quad \left\{ \frac{a}{b} \right\} = \{x \in H \mid a \in bx\}.
\end{align*}
\]

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, W. Prenowitz [48] denoted division in commutative hypergroups by \(a/b\). Later on, J. Jantosciak used the notation \(a/b\) for right division and \(b\backslash a\) for left division [14]. Notations \(a:b\) and \(a\cdot b\) have also been used correspondingly for the above two types of division [21].

In [14] and then in [15], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition \(ab\) with the hypercomposition \(ba\). One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds:

*Given a theorem, the dual statement resulting from interchanging the order of hypercomposition “.,” (and, necessarily, interchanging of the left and the right divisions), is also a theorem.*

This principle is used throughout this paper. The following properties are direct consequences of axioms (i) and (ii) and the principle of duality is used in their proofs [see also 20, 21]:

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Property 1.1. $ab \neq \emptyset$ is valid for all the elements $a,b$ of a hypergroup $H$.

Proof. Suppose that $ab = \emptyset$ for some $a,b \in H$. Per reproduction, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \emptyset H = \emptyset$, which is absurd.

Property 1.2. $a / b \neq \emptyset$ and $a \b / \emptyset$ for all the elements $a,b$ of a hypergroup $H$.

Proof. Per reproduction, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a / b$ and, therefore, $a / b \neq \emptyset$. Dually, $a \b / \emptyset$.

Property 1.3. In a hypergroup $H$, the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.

Proof. (i) Per Property 1.1, the result of hypercomposition in $H$ is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y / a$. Hence, $H \subseteq H / a$. Moreover, $H / a \subseteq H$. Therefore, $H = H / a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a / y$. Hence, $H \subseteq a / H$. Moreover, $a / H \subseteq H$. Therefore, $H = a / H$. (ii) follows by duality.

Property 1.4. In a hypergroup $H$ equalities (i) $H = H / a = a / H$ and (ii) $H = a \b / H = H \b / a$ are valid for all $a$ in $H$.

Proof. (i) Per Property 1.1, the result of hypercomposition in $H$ is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y / a$. Hence, $H \subseteq H / a$. Moreover, $H / a \subseteq H$. Therefore, $H = H / a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a / y$. Hence, $H \subseteq a / H$. Moreover, $a / H \subseteq H$. Therefore, $H = a / H$. (ii) follows by duality.

The hypergroup (as defined by F. Marty), being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups.

One of these axioms is the transposition axiom. It was introduced by W. Prenowitz, who used it in commutative hypergroups. W. Prenowitz called the resulting hypergroup join space [48]. Thus, join space (or join hypergroup) is defined as a commutative hypergroups $H$, in which $a / b \cap c / d \neq \emptyset$ implies $ad \cap bc \neq \emptyset$ for all $a,b,c,d \in H$ (transposition axiom) is true. This type of hypergroup has been widely utilized in the study of
Geometry via the use of hypercompositional algebra tools which function without any need of Cartesian or other coordinate-type systems \([48, 49]\). Later, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

\[ b \setminus a \cap c / d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H. \]

He named this particular hypergroup transposition hypergroup and studied its properties in \([14]\).

The transposition axiom also emerged in the hypercompositional structures which surfaced during the study of formal languages through the use of hypercompositional algebra tools [see, for example, 6, 7, 27, 32, 33, 35, 36, 42, 44; see also 7, 12, 13 for other occurrences of the join space]. The manner in which these structures emerged will be discussed in paragraph 3. In the present paragraph we will only deal with the mathematical description of join space classes which resulted from the theory of formal languages and automata. The basic concept which generated these types of join spaces is the incorporation of a special neutral element \(e\) into a transposition hypergroup. This neutral element \(e\) possesses the property \(e x = x e \subseteq \{e, x\}\) for every element \(x\) of the hypergroup and was named strong. Thus, the fortification of transposition hypergroups by an identity element came into being.

Therefore a **fortified transposition hypergroup** is a transposition hypergroup \(H\) for which the following axioms are valid:

i. \(ee = e\),

ii. \(x e = xe\) for all \(x \in H\),

iii. for every \(x \in H - \{e\}\) there exists a unique \(y \in H - \{e\}\), such that \(e \in xy\) and, furthermore, \(y\) satisfies \(e \in yx\).

If the commutativity is valid in \(H\), then \(H\) is called a **fortified join hypergroup**.

**Theorem 1.1.** In a fortified transposition hypergroup \(H\), the identity is strong.

**Proof.** It must be proven that \(e x \subseteq \{e, x\}\) for all \(x \in H\). This is true for \(x = e\). Let \(x \neq e\). Suppose that \(y \in ex\). Then, \(x \in e \setminus y\). However, \(x \in e / x^{-1}\), since \(e \in xx^{-1}\). Thus, \(e y = e / x^{-1}\) and transposition yields \(e = ee \approx yx^{-1}\). Hence, \(y \in \{e, x\}\).

**Theorem 1.2.** In a fortified transposition hypergroup \(H\), the strong identity is unique.
Proof. Suppose that \( u \) is an identity distinct from \( e \). It then follows that there exists \( z \) distinct from \( u \), such that \( u \in ez \). But, \( ez \subseteq \{e, z\} \), so \( u \in \{e, z\} \), which is a contradiction.

It is worth noting that a transposition hypergroup \( H \) becomes a quasicanonical hypergroup, if it incorporates a scalar identity, i.e. an identity \( e \) with the property \( ex = xe = x \) for all \( x \) in \( H \). Moreover, a join hypergroup is a canonical hypergroup, if it contains a scalar identity [14, 20, 23].

A hypergroup \( H \) with a strong identity \( e \) has a natural partition. Let

\[
A = \{x \in H \mid ex = xe = \{e, x\}\} \quad \text{and} \quad C = \{x \in H - \{e\} \mid ex = xe = e\}.
\]

Then, \( H = A \cup C \) and \( A \cap C = \emptyset \). A member of \( A \) is an attractive element and a member of \( C \) is a canonical element. See [39] for the origin of terminology.

Fortified join hypergroups and fortified transposition hypergroups have been studied in a series of papers [see, for example, 15, 22, 33, 37, 39, 43], in which several very interesting properties of these types of hypergroups were revealed. The following was proven, among others [15]:

**Structure Theorem.** A transposition hypergroup \( H \) containing a strong identity \( e \) is isomorphic to the expansion of a quasicanonical hypergroup \( C \cup \{e\} \) by the transposition hypergroup \( A \) of all attractive elements through the idempotent \( e \).

Moreover, from the theory of automata resulted the transposition polysymmetrical hypergroup [24, 42, 45], i.e. a transposition hypergroup \( H \), having an identity (or neutral) element \( e \), such that \( ee = e, x \in ex = xe \) for all \( x \in H \) and also, for every \( x \in H - \{e\} \) there exists at least one element \( x' \in H - \{e\} \), (called symmetric or two-sided inverse of \( x \)), such that \( e \in xx' \) and \( e \in x'x \). The set of the symmetric elements of \( x \) is denoted by \( S(x) \) and is called the symmetric set of \( x \). A commutative transposition polysymmetrical hypergroup is called a join polysymmetrical hypergroup.

**Theorem 1.3.** If a polysymmetrical transposition hypergroup contains a strong identity \( e \), then this identity is unique.

Analytical examples of the above hypergroup types are presented in [28]. A thorough study of transposition hypergroups with idempotent identity is presented [30].
2. THE TRANSPOSITION AXIOM IN HYPERGROUPOIDS

In the previous paragraph it was mentioned that the hypergroup was enriched with further axioms, a fact which led to the creation of specific hypergroup families. However, mathematical research also followed the reverse course. Certain axioms were removed from the hypergroup and the resulting weaker structures were studied. Thus, the pair \((H,\cdot)\), where \(H\) is a non-empty set and "\." a hypercomposition, was named partial hypergroupoid, while it was called hypgroupoid if \(ab \neq \emptyset\) for all \(a, b \in H\). A hypergroupoid in which the associativity is valid, was called semi-hypergroup, while it was called quasi-hypergroup, if only the reproductivity is valid. The quasi-hypergroups in which the weak associativity is valid, i.e. \((ab)c \cap a(bc) \neq \emptyset\) for all \(a, b, c \in H\), were named \(H_v\)-groups [55]. Certain properties of these structures, which are analogous to those of hypergroups, are presented herein.

**Property 2.1.** If the weak associativity is valid in a hypergroupoid, then this hypergroupoid is not partial.

**Proof.** Suppose that \(ab = \emptyset\) for some \(a, b \in H\). Then, \((ab)c = \emptyset\) for any \(c \in H\). Therefore, \((ab)c \cap a(bc) = \emptyset\), which is absurd. Hence, \(ab\) is non-empty.

The following is a direct consequence of the above property:

**Property 2.2.** The result of the hypercomposition in an \(H_v\)-group \(H\) is always a non-empty set.

**Property 2.3.** A hypergroupoid \(H\) is a quasi-hypergroup, if the results of induced hypercompositions in it are non-empty.

**Proof.** Suppose that \(x/a \neq \emptyset\) is valid for all \(x, a \in H\). Then, there exists \(y \in H\), such that \(x = ya\). Therefore, \(x \in Ha\) for all \(x \in H\) and so \(H \subseteq Ha\). But \(Ha \subseteq H\) is also valid for all \(a \in H\). Hence, \(H = Ha\). By duality, \(aH = H\). Thus, \(H\) is a quasi-hypergroup.

**Property 2.4.** \(a/b \neq \emptyset\) and \(b\backslash a \neq \emptyset\) is valid for all the elements \(a, b\) of a quasi-hypergroup \(H\).

**Proof.** Per equality \(H = Hb\), there exists \(y \in H\), such that \(a \in yb\) for every \(a \in H\). Thus, \(ya \in a/b\) and, therefore, \(a/b \neq \emptyset\). \(b\backslash a \neq \emptyset\), per the principle of duality.
Property 2.5. In a quasi-hypergroup $H$, the equalities $H = a / H = H \setminus a$ are valid for all $a$ in $H$.

Proof. Let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a / y$. Hence, $H \subseteq a / H$. Moreover, $a / H \subseteq H$. Therefore, $H = a / H$. The other equality follows by duality.

Property 2.6. In any non-partial hypergroupoid $H$, the equalities $H = H / a = a \setminus H$ are valid for all $a$ in $H$.

Proof. Since the result of the hypercomposition in a non-partial hypergroupoid is always a non-empty set, there exists $y \in H$ such that $y \in xa$ for every $x \in H$. This implies that $x \in y / a$. Hence, $H \subseteq H / a$. Moreover, $H / a \subseteq H$. Therefore, $H = H / a$. The other equality follows by duality.

The following is a direct consequence of Properties 2.5 and 2.6 above:

Property 2.7. In any $H_V$-group $H$, the equalities (i) $H = H / a = a \setminus H$ and (ii) $H = a \setminus H = H \setminus a$ are valid for all $a$ in $H$.

Extensive work has been done on the construction of hypergroupoids, on their enumeration and on the study of their structure (see, for example, [3, 4, 5, 6, 9, 10, 11, 29, 50, 51, 52, 54]). As mentioned above, this direction pertained to researching hypercompositional structures resulting from the weakening of the structure of the hypergoup. The opposite direction pertained to researching hypercompositional structures resulting from the reinforcement of the structure of the hypergoup. These two directions are combined in [31], via the introduction of the transposition axiom into the $H_V$-group, thus leading to the following definition:

Definition 2.1. An $H_V$-group $(H, \cdot)$ is called transposition $H_V$-group, if it satisfies the transposition axiom:

$$b \setminus a \cap c / d \neq \emptyset$$

imply $ad \cap bc \neq \emptyset$ for all $a, b, c, d \in H$.

A transposition $H_V$-group $(H, \cdot)$ is called join $H_V$-group, if $H$ is a commutative $H_V$-group, while it is called weak join $H_V$-group, if $H$ is an $H_V$-commutative group.

The fortified transposition $H_V$-group was also defined in [31], in a manner analogous to the definition of the fortified transposition hypergoup, as follows:

Definition 2.2. A transposition $H_V$-group $(H, \cdot)$ is called fortified, if $H$ contains an element $e$, which satisfies the axioms:

i. $ee = e$,
ii. $xe = ex$ for all $x \in H$,
for every \( x \in H - \{ e \} \) there exists a unique \( y \in H - \{ e \} \), such that \( e \in xy \) and, furthermore, \( y \) satisfies \( e \in yx \).

If "\( \cdot \)" is commutative, then \( H \) is called a **fortified join \( H_V \)-group**.

Properties of the structure above, as well as relevant examples are presented in [31]. The elements of the fortified transposition \( H_V \)-group are partitioned into canonical and attractive, exactly as in hypergroups.

**Proposition 2.1.** Let \( H \) be a fortified transposition \( H_V \)-group and suppose that \( x, y \) are attractive elements with \( y \neq x^{-1} \). Then, \( x, y \in xy \) and \( x, y \in yx \).

**Proof.** Since \( x \) is an attractive element, \( ex = xe = \{ e, x \} \) is valid. Therefore, \( e / x = x \setminus e = \{ e, x^{-1} \} \). Moreover, \( y / y = \{ z \mid y \in zy \} \). Hence, \( e \in y / y \). Thus, \( y / y \cap x \setminus e \neq \emptyset \) which, per the transposition axiom, results into \( ey \cap yx \neq \emptyset \) or, equivalently, \( \{ e, y \} \cap yx \neq \emptyset \). Since \( y \neq x^{-1} \), it follows that \( y \in yx \). Similarly, \( x \in yx \) and, per duality, \( x, y \in xy \).

**Corollary 2.1.** A fortified transposition \( H_V \)-group containing exclusively attractive elements is weakly commutative.

As can be observed, the transposition axiom is not dependent on the two hypergroup axioms (assosiativity and reproduction) and their consequences. Therefore, the transposition axiom can be introduced even into a partial hypergroupoid. Thus, the notions of the **transposition hypergroupoid**, of the **transposition quasi-hypergroup** and of the **transposition semi-hypergroup** emerge. If the commutativity is also valid in the above, the notions of the **join hypergroupoid**, of the **join quasi-hypergroup** and of the **join semi-hypergroup** emerge as well. The following proposition is analogous to the one used in [31] for the construction of transposition \( H_V \)-groups. The proof of this proposition, as well as of Proposition 2.3 below, is quite straightforward, albeit long, since all the possible cases must be verified.

**Proposition 2.2.** Let \( H \) be a hypergroupoid (either partial or non-partial) or a quasi-hypergroup. Also, let an arbitrary subset \( I_{ab} \) of \( H \) be associated to each pair of elements \( (a, b) \in H^2 \). If \( \bigcap_{a, b \in H} I_{ab} \neq \emptyset \), then \( H \) endowed with the hypercomposition: \( a \# b = ab \cup I_{ab} \), \( a, b \in H \) is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or
a join quasi-hypergroup, if the commutativity is valid in $H$ and $I_{ab} = I_{ba}$ for all $a, b \in H$.

Corollary 2.1. If $H$ is a hypergroupoid (either partial or non-partial) or a quasi-hypergroup and $w$ is an arbitrary element of $H$, then $H$ endowed with the hypercomposition

$$x \ast y = xy \cup \{x, y, w\}$$

is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or a join quasi-hypergroup, if the commutativity is valid in $H$.

Proposition 2.3. Let $H$ be a set with more that two elements and let $w$ be an arbitrary element in $H$. Two hypercompositions are defined in $H$ as follows:

$$a \circ_1 b = \{a, w\} \text{ for all } a, b \in H \quad \text{and} \quad a \circ_2 b = \{b, w\} \text{ for all } a, b \in H.$$ 

Then, $(H, \circ_1)$ and $(H, \circ_2)$ are transposition semi-hypergroups.

3. The Transposition Axiom in Hypercompositional Structures with Internal Compositions

M. Krasner was the first to expand hypercompositional structures via the creation of structures containing composition and hypercompositions. Thus, in 1956, he replaced the additive group of a field with a special hypergroup, thereby introducing the hyperfield. He then used the hyperfield as the proper algebraic tool, in order to define a certain approximation of complete valued fields by sequences of such fields [16, 17]. Later, he introduced a more general structure, which relates to hyperfields in the same way rings relate to fields. He called this structure hyperring. Additional hypercompositional structures, similar to the above, introduced by various researchers, soon followed. Examples of those are the superring and the superfield, in which both the addition and the multiplication are hypercompositions [47]. Additionally, the study of formal languages introduced structures in which the hypercompositional component is a join hypergroup.

Indeed, let $A$ be an alphabet, let $A^*$ denote the set of the words defined over $A$ and let $\lambda$ be the empty word. Then, set $A^*$ is a semigroup with regard to the concatenation of the worlds. This semigroup has $\lambda$ as its neutral element, since $\lambda a = a \lambda = a$ for all $a$ in $A^*$. In addition, the expression $a + b$, where $a$ and $b$ are words over $A$, is used in formal languages theory to denote «either $a$ or
Based on the fact that $a+b$ is in essence a biset, hypercomposition $a+b=\{a,b\}$ appears in the word set $A^*$. It has been proven that $A^*$ is a join hypergroup [32, 33] with regard to this hypercomposition. This hypergroup was named $B(\text{iset})$-hypergroup. However, since $A^*$ is a semigroup with regard to world concatenation and since it has been proven that world concatenation is distributive with regard to the hypercomposition, a new hypercompositional structure thus emerged. This structure was named hyperringoid.

**Definition 3.1.** A hyperringoid is a non-empty set $Y$ equipped with an operation “$\cdot$” and a hyperoperation “$+$”, such that:

i) $(Y, +)$ is a hypergroup,

ii) $(Y, \cdot)$ is a semigroup,

iii) the operation “$\cdot$” is distributive on both sides of the hyperoperation “$+$”.

If $(Y, +)$ is a join hypergroup, $(Y, +, \cdot)$ is called join hyperringoid. The join hyperringoid that results from a $B$-hypergroup is called $B$-hyperringoid and the special $B$-hyperringoid that appears in the theory of formal languages is the linguistic hyperringoid. Join hyperringoids are studied in [38, 40, 41].

Another notion in the theory of formal languages is the null word, the introduction of which resulted from the theory of automata. The null word is symbolized with $0$ and is bilaterally absorbing with regard to word concatenation. Therefore, the extension of the composition and of the hypercomposition onto $A^* \cup \{0\}$ results into the following:

$$0a = a0 = 0, \quad 0 + a = a + 0 = \{0, a\}$$

for all $a \in A^*$.

With these extensions, structure $(A^* \cup \{0\}, +, \cdot)$ continues to be a hyperringoid, which, however now also has an absorbing element. The additive structure of these hyperringoid comprises a fortified join hypergroup. Thus, a new hypercompositional structure appeared:

**Definition 3.2.** If the additive part of a hyperringoid is a fortified join hypergroup whose zero element is bilaterally absorbing with respect to the multiplication, then, this hyperringoid is named join hyperring. A join hyperdomain is a join hyperring which has no divisors of zero. A proper join hyperring is a join hyperring which is not a Krasner hyperring. A join hyperring $K$ is called join hyperfield if $K^* = K - \{0\}$ is a multiplicative group.

Join hyperrings are studied in [25, 41].

Moreover, hypercompositional structures having external operations and hyperoperations on hypergroups appeared [see, for example, 1, 2, 19, 56]. The
notions of the set of operators and hyperoperators from a hyperringoid $Y$ over an arbitrary non-void set $M$ were introduced in [33, 34], in order to describe the action of the state transition function in the theory of Automata. $Y$ is a set of operators over $M$, if there exists an external operation from $M \times Y$ to $M$, such that $(s\kappa)\lambda = s(\kappa\lambda)$ for all $s \in M$ and $\kappa, \lambda \in Y$ and, moreover, $s1 = s$ for all $s \in M$, when $Y$ is a unitary hyperringoid. If there exists an external hyperoperation from $M \times Y$ to $P(M)$ which satisfies the above axiom, with the variation that $s \in s1$ when $Y$ is a unitary hyperringoid, then $Y$ is a set of hyperoperators over $M$. If $M$ is a hypergroup and $Y$ is a hyperringoid of operators over $M$, such that, for each $\kappa, \lambda \in Y$ and $s, t \in M$, the axioms: (i) $(s + t)\lambda = s\lambda + t\lambda$, (ii) $s(\kappa + \lambda) \subseteq s\kappa + s\lambda$ hold, then $M$ is called right hypermoduloid over $Y$. If $Y$ is a set of hyperoperators, then $M$ is called right supermoduloid. If the second of the above axioms holds as an equality, then the hypermoduloid is called strongly distributive. There is a similar definition of the left hypermoduloid and the left supermoduloid over $Y$, in which the elements of $Y$ operate from the left side. When $M$ is both right and left hypermoduloid (resp. supermoduloid) over $Y$, it is called $Y$-hypermoduloid (resp. $Y$-supermoduloid) [33, 34]. If $M$ is a canonical hypergroup, the set of operators $Y$ is a hyperring and, if $s1 = s$, $s0 = 0$ for all $s \in M$, then $M$ is named right hypermodule, while it is named right supermodule if $Y$ is a set of hyperoperators [26]. A study of external operations and hyperoperations on hypergroups is carried out in [26].
BIBLIOGRAPHY.


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54. Ch. G. Tsitouras, Ch. G. Massouros: «Enumeration of Rosenberg type Hypercompositional structures defined by binary relations» Submitted for publication.


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