Construction of \( k \)-Hyperideals by \( P \)-Hyperoperations

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Abstract

In this note we present a method to construction new \( k \)-hyperideals from given \( k \)-ideals of a semiring \( R \) by using of the \( P \)-hyperoperations. Then we investigate the relationship between them. In particular, we describe all \( k \)-hyperideals of the semihyperring of the nonnegative integers.

Keywords: (semi)hyperring, \( k \)-(hyper)ideal, \( P \)-hyperoperation, weak distributive

1 Introduction

Hyperstructures theory was born in 1934 when Marty [12] defined hypergroups as a generalization of groups. Also Wall in 1937 defined the notion of cyclic hypergroup. This theory has been studied in the following decades and nowadays by many mathematicians. A short review of the theory of hypergroups appears in [2]. A recent books [2], [3] and [15] contain a wealth of applications. There are applications
to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, groups, rational algebraic functions and etc. One of the several contexts which they arise is hyperring. First M. Krasner studied hyperrings, which is a triple \((R, +, \cdot)\), where \((R, +)\) is a canonical hypergroup and \((R, \cdot)\) is a semigroup, such that for all \(a, b, c \in R\), \(a(b + c) = ab + ac, (b + c)a = ba + ca\) ([10]).

The notion of \(k\)-ideals in ordinary semirings was introduced by D. R. Latore in 1965 ([11]). Also M. K. Sen and others worked on one-sided \(k\)-ideals and maximal \(k\)-ideals of semirings ([14], [16]).

The authors in [6] introduced the notion of \(k\)-hyperideals in the sense of Krasner and obtained some related results about this notion. We now follow [6] to introduce a method to construct new \(k\)-hyperideals from given \(k\)-ideals.

In section 2 of this paper, we gather all the preliminaries of (semi)hyperrings and \(k\)-(hyper)ideals which will be used in the next sections. In section 3, we represent some methods for construction semihyperrings from semirings by \(P\)-hyperoperations and then we investigate the relationship between their \(k\)-hyperideals and \(k\)-ideals. As an important result of this section, all \(k\)-hyperideals of the nonnegative integers \(\mathbb{N}^*\) as a semihyperring, constructed by \(P\)-hyperoperations, are described. In section 4, we characterize the \(k\)-hyperideals of product of semihyperrings which are made by \(P\)-hyperoperations and a family of semirings.

## 2 Preliminaries

A map \(\circ : H \times H \longrightarrow P_s(H)\) is called hyperoperation or join operation. A hypergroupoid is a set \(H\) with together a (binary) hyperoperation \(\circ\). A hypergroupoid \((H, \circ)\), which is associative, that is \(x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in H\) is called a semihypergroup.

A hypergroup is a semihypergroup such that \(\forall x \in H\) we have \(x \circ H = H = H \circ x\), which is called reproduction axiom (see [2]).

Let \(H\) be a hypergroup and \(K\) be a nonempty subset of \(H\). Then \(K\) is said to be
a subhypergroup of $H$ if itself is a hypergroup under hyperoperation "$\circ$" restricted to $K$. Hence it is clear that a subset $K$ of $H$ is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on $H$.

**Definition 2.1.** A hyperalgebra $(R, +, \cdot)$ is called a semihyperring if and only if

1. $(R, +)$ is a semihypergroup;
2. $(R, \cdot)$ is a semigroup;
3. $\forall a, b, c \in R, a.(a + b) = a.b + a.c$ and $(b + c).a = b.a + c.a$.

**Remark.** In Definition 2.1, if we replace (iii) by

$$\forall a, b, c \in R, a.(a + b) \subseteq a.b + a.c \text{ and } (b + c).a \subseteq b.c + c.a,$$

we say that $R$ is a weak distributive semihyperring.

A semihyperring $R$ is called with zero element, if there exists an unique element $0 \in R$ such that $0 + x = x = x + 0$ and $0x = 0 = x0$ for all $x \in R$.

A semihyperring $R$ is called additive commutative, if $x + y = y + x$, $\forall x, y \in R$.

A semihyperring $(R, +, \cdot)$ is called a hyperring provided $(R, +)$ is a canonical hypergroup.

**Definition 2.2.** A hyperring $(R, +, \cdot)$ is called

1. commutative if $a.b = b.a$ for all $a, b \in R$;
2. with identity, if there exists an element, say $1 \in R$, such that $1.x = x.1 = x$ for all $x \in R$.

Let $(R, +, \cdot)$ be a hyperring, a nonempty subset $S$ of $R$ is called a subhyperring of $R$ if $(S, +, \cdot)$ is itself a hyperring.

**Definition 2.3.** A subhyperring $I$ of a hyperring $R$ is said to be a (resp. right) left hyperideal of $R$ provided that (resp. $x.r \in I$) $r.x \in I$ for all $r \in R$ and for all $x \in I$. We say that $I$ is a hyperideal if $I$ is both a left and right hyperideal.

**Definition 2.4.**[11] Let $(R, +, \cdot)$ be a semiring. A nonempty subset $I$ of $R$ is called a left $k$-ideal of $R$, if $I$ is a left ideal of $R$ and for $a \in I$ and $x \in R$ we have

$$a + x \in I \text{ or } x + a \in I \implies x \in I.$$
Similarly a right $k$-ideal is defined. A two sided $k$-ideal or simply a $k$-ideal is both a left and right $k$-ideal. We denote $I$ as $k$-ideal (resp. ideal) of $R$ by $I \triangleleft_k R$ (resp. $I \triangleleft R$).

In the sequel, by $R$ we mean a semihyperring, unless otherwise specified.

**Definition 2.5.**[6] Let $(R, +, .)$ be a (weak distributive) semihyperring. A nonempty subset $I$ of $R$ is called

(i) a left (resp. right) hyperideal of $R$ if and only if

(a) $(I, +)$ is a semihypergroup of $(R, +)$; and
(b) $rx \in I$ (resp. $xr \in I$), for all $r \in R$ and for all $x \in I$.

(ii) a hyperideal of $R$ if it is both left and right hyperideal of $R$. The hyperideal $I$ of $R$ is denoted by $I \triangleleft_h R$.

(iii) a left $k$-hyperideal of $R$, if $I$ is a left hyperideal of $R$ and for $a \in I$ and $x \in R$ we have

$$a + x \approx I \text{ or } x + a \approx I \quad \Rightarrow \quad x \in I,$$

where by $A \approx B$ we mean $A \cap B \neq \emptyset$.

(iv) Similarly a right $k$-hyperideal is defined. A two sided $k$-hyperideal or simply a $k$-hyperideal is both a left and right $k$-hyperideal. We denote $I$ as $k$-hyperideal of $R$ by $I \triangleleft_{k,h} R$.

### 3 Construction of $k$-hyperideals by $P$-hyperoperations

In this section we apply three kinds of $P$-hyperoperations (which were introduced for $H_v$-structures in [15]) to construct semihyperrings from semirings. Then we investigate the relationship between their $k$-hyperideals and $k$-ideals.

**Definition 3.1.** Let $(R, +, .)$ be semiring and $\emptyset \neq P \subseteq R$. We define two hyperoperations as follows

$$x \oplus_k y = \{x + t + y \mid t \in P\},$$
\[ x \odot y = x.y = xy, \]

which \( \oplus_c \) is called \textit{centre \( P \)-hyperoperation}.

**Proposition 3.2.** Let \((R, +, .)\) be semiring and \(P \subseteq R\) be a nonempty such that \(PR \subseteq P\) and \(RP \subseteq P\), then \((R, \oplus_c, \odot)\) is a weak distributive semihyperring.

**Proof.** First, we show \((R, \oplus_c)\) is a semihypergroup. For this we prove that \((x \oplus_c y) \oplus_c z = x \oplus_c (y \oplus_c z)\).

For \(x, y, z \in R\) we have
\[
\begin{align*}
\forall a \in (x \oplus_c y) \oplus_c z &\implies \exists a_1 \in x \oplus_c y, \ a \in a_1 \oplus_c z \\
&\implies \exists t_1, t_2 \in P, \ a = a_1 + t_1 + z, \ a_1 = x + t_2 + y \\
&\implies a = x + t_2 + y + t_1 + z \\
&\implies a = x + t_2 + b, \ b = y + t_1 + z \in y \oplus_c z \\
&\implies a \in x \oplus_c b, \ b \in y \oplus_c z \\
&\implies a \in x \oplus_c (y \oplus_c z) \\
&\implies (x \oplus_c y) \oplus_c z \subseteq x \oplus_c (y \oplus_c z).
\end{align*}
\]

Similarly, we obtain that
\[
(x \oplus_c y) \oplus_c z \supseteq x \oplus_c (y \oplus_c z).
\]

Clearly \((R, \odot)\) is a semigroup, since \((R, .)\) is a semigroup and \(x \odot y = xy\).

We now prove weak distributivity, that is
\[
x \odot (y \oplus_c z) \subseteq (x \odot y) \oplus_c (x \odot z) = xy \oplus_c xz.
\]

For this we have
\[
\begin{align*}
\forall a \in x \odot (y \oplus_c z) &\implies \exists a_1 \in y \oplus_c z, \ a = x \odot a_1 = xa_1 \\
&\implies \exists t \in P, \ a = xa_1, \ a_1 = y + t + z \\
&\implies a = x(y + t + z) \\
&\implies xy + xt + xz \in xy \oplus_c xz, \ (RP \subseteq P) \\
&\implies x \odot (y \oplus_c z) \subseteq xy \oplus_c xz.
\end{align*}
\]
Similarly we conclude that \((y \oplus_c z) \odot x \subseteq yx \oplus_c zx\). □

**Definition 3.3.** Let \((R, +, \cdot)\) be a semiring and \(\emptyset \neq P \subseteq R\). We define the following hyperoperations

\[
x \oplus_r y = \{ x + y + t \mid t \in P \}, \quad x \oplus_l y = \{ t + x + y \mid t \in P \},
\]

\[
x \odot y = xy,
\]

which \(\oplus_r\) and \(\oplus_l\) are called *right* \(P\)-hyperoperation and *left* \(P\)-hyperoperation respectively.

**Proposition 3.4.** Let \((R, +, \cdot)\) be a semiring and \(P \subseteq R\) be a nonempty such that \(PR \subseteq P\) and \(RP \subseteq P\) and \(x + P = P + x\), for all \(x \in R\). Then \((R, \oplus_r, \odot)\) and \((R, \oplus_l, \odot)\) are weak distributive semihyperrings.

**Proof.** First, we prove that

\[
(x \oplus_r y) \oplus_r z = x \oplus_r (y \oplus_r z).
\]

For this we have

\[
a \in (x \oplus_r y) \oplus_r z \implies \exists a_1 \in x \oplus_r y, a \in a_1 \oplus_r z
\]

\[
\implies \exists t_1, t_2 \in P, a_1 = x + y + t_1, a = a_1 + z + t_2
\]

\[
\implies \exists t_1, t_2 \in P, a = x + y + t_1 + z + t_2 \quad (1)
\]

also we have

\[
b \in x \oplus_r (y \oplus_r z) \implies \exists b_1 \in y \oplus_r z, b \in x \oplus_r b_1
\]

\[
\implies \exists w_1, w_2 \in P, b_1 = y + z + w_1, b = x + b_1 + w_2
\]

\[
\implies \exists w_1, w_2 \in P, b = x + y + z + w_1 + w_2 \quad (2)
\]

From (1) we have

\[
a = x + y + t_1 + z + t_2 = x + y + z + w_1 + t_2, \exists w_1 \in P \quad (z + P = P + z)
\]

\[
\implies a \in x \oplus_r (y \oplus_r z) \quad \text{(by (2))}
\]

\[
\implies (x \oplus_r y) \oplus_r z \subseteq x \oplus_r (y \oplus_r z).
\]
Similarly we can prove that

\[(x \oplus_r y) \oplus_r z \supseteq x \oplus_r (y \oplus_r z).\]

Clearly \((R, \odot)\) is semigroup, since \((R, \cdot)\) is a semigroup. In a similar way to the Proposition 3.2 we can prove weak distributivity. Therefore \((R, \oplus_r, \odot)\) is a weak distributive semihyperring. Analogously we can prove that \((R, \oplus_l, \odot)\) is a weak distributive semihyperring. □

**Remark.** In Propositions 3.2 and 3.4, if we replace the conditions \(RP \subseteq P\) and \(PR \subseteq P\) by \(rP = P = Pr\) for all \(r \in R\), then \((R, \oplus_c, \odot)\) and \((R, \oplus_r, \odot)\) and \((R, \oplus_l, \odot)\) become semihyperring.

**Theorem 3.5.** Let \((R, +, \cdot)\) be a semiring with zero and \(P\) be the same as Proposition 3.2 such that \(0 \in P\). Then there is a one-to-one correspondence between the \(k\)-ideals of \((R, +, \cdot)\) containing \(P\) and \(k\)-hyperideals of \((R, \oplus_c, \odot)\).

**Proof.** Let \(I\) be a \(k\)-ideal of \((R, +, \cdot)\) containing \(P\). First we prove that \(I \triangleleft_h (R, \oplus_c, \odot)\). Suppose that \(x, y \in I\), we prove \(x \oplus_c y \subseteq I\). For this we have

\[
z \in x \oplus_c y \implies \exists t \in P \subseteq I, \ z = x + t + y \\
\implies z = x + t + y \in I \quad (\text{since } x, t, y \in I) \\
\implies x \oplus_c y \subseteq I.
\]

Also if \(r \in R\) and \(x \in I\), then \(r \odot x = rx \in I\), since \(I \triangleleft (R, +, \cdot)\). Thus \(I\) is a hyperideal of \((R, \oplus_c, \odot)\). We now prove that \(I \triangleleft_{k,h} (R, \oplus_c, \odot)\). For \(r \in R\) and \(x \in I\) we have

\[
r \oplus_c x \approx I \implies \exists z \in r \oplus_c x \approx I \\
\implies \exists t \in P, \ z = r + t + x, \ z \in I \\
\implies r + t + x \in I, \ t + x \in I \\
\implies r \in I \quad (\text{since } I \triangleleft_{k} (R, +, \cdot) ) \\
\implies I \triangleleft_{k,h} (R, \oplus_c, \odot).
\]
Conversely, suppose that $I \triangleleft_{k,h} (R, \oplus, \odot)$. We prove that $I$ is a $k$-ideal of $(R, +, \cdot)$ containing $P$. For this we have

$$x, y \in I \implies x \oplus_c y \subseteq I \quad (I \triangleleft_{k,h} (R, \oplus, \odot))$$

$$\implies \forall t \in P, \ x + t + y \in I$$

$$\implies x + y \in I \quad (0 \in P).$$

On the other hand

$$r \in R, x \in I \implies r \odot x \in I \quad (I \triangleleft_{k,h} (R, \oplus, \odot))$$

$$\implies rx \in I.$$

Also we have

$$r + x \in I, x \in I \implies r + 0 + x \in I, x \in I \quad (0 \in P)$$

$$\implies r \oplus_c x \approx I, x \in I$$

$$\implies r \in I \quad (I \triangleleft_{k,h} (R, \oplus, \odot))$$

$$\implies I \triangleleft_{k} (R, +, \cdot).$$

We have $0 \oplus_c 0 \subseteq I$, then $\{0 + t + 0 \mid t \in P\} \subseteq I$, therefore $P \subseteq I$. □

**Theorem 3.6.** Let $(R, +, \cdot)$ be a semiring with zero and $P$ be the same as Proposition 3.4 such that $0 \in P$. Then there is a one-to-one correspondence between $k$-ideals of $(R, +, \cdot)$ containing $P$ and $k$-hyperideals of $(R, \oplus, \odot)$.

**Proof.** The proof is similar to the proof of Theorem 3.5 by some manipulation. □

**Examples.** (i) Let $\mathbb{N}$ be the set of natural numbers and $2\mathbb{N} = \{2, 4, 6, 8, \ldots\}$. Clearly $(\mathbb{N}, +, \cdot)$ is a semiring and $2\mathbb{N}$ is a $k$-ideal of $(\mathbb{N}, +, \cdot)$. Now if $P = \{4, 8, 12, 16, \ldots\} \subseteq 2\mathbb{N}$, then it is easy to verify that $(\mathbb{N}, \oplus_c, \odot)$ is a weak distributive semihyperring, where for all $m, n \in \mathbb{N}$ we have

$$m \oplus_c n = \{m + k + n \mid k \in P\} \quad \text{and} \quad m \odot n = mn.$$

Thus $2\mathbb{N}$ is a $k$-hyperideal of $(\mathbb{N}, \oplus_c, \odot)$. 
Let $N^* = N \cup \{0\}$ and $N^*[x] = \{f(x) = \sum_{i=1}^{n} a_i x^i \mid a_i \in N^*\}$. Clearly $(N^*[x], +, \cdot)$ is a semiring and $< x > = \{f(x) \in N^*[x] \mid a_0 = 0\}$ is a $k$-ideal of $(N^*[x], +, \cdot)$ generated by $x$. Set $P = < x^m >$ for $m \in \mathbb{N}$. Obviously, $0 \in P \subseteq < x >$. Then by Propositions 3.2 and 3.5, $(N^*[x], \oplus, \odot)$ is a weak distributive semihyperring and $< x >$ is a $k$-hyperideal of $(N^*[x], \oplus, \odot)$.

In the next theorem we describe all $k$-hyperideals of semihyperring of the natural numbers constructed by $P$-hyperoperation. For this we consider the semiring $(\mathbb{N}, +, \cdot)$ of natural numbers by usual ordinary operations.

**Theorem 3.7.** Let $0 \in P \subseteq N^*$ and $PN^* \subseteq P$ and $N^*P \subseteq P$ and $P \subseteq I$. Then $I$ is a $k$-hyperideal of $(N^*, \oplus, \odot)$ if and only if there exists $a \in N^*$ such that $I = \{na \mid n \in N^*\}$.

**Proof.** By Theorem 3.5, $I \triangleleft_{k.h} (N^*, \oplus, \odot)$ if and only if $I \triangleleft_k (N^*, +, \cdot)$. Also by Proposition 4.1 [14], $I \triangleleft_k (N^*, +, \cdot)$ if and only if there exists $a \in N^*$ such that $I = \{na \mid n \in N^*\}$. □

## 4 Product of $k$-hyperideals

In the sequel by $\prod_{i \in I} R_i$, we mean the *cartesian product* of the family $\{R_i\}_{i \in I}$. It means

$$\prod_{i \in I} R_i = \{(x_i)_{i \in I} \mid x_i \in R_i\}.$$  

**Proposition 4.1.** Let $\{R_i\}_{i \in I}$ be a family of semirings and $P_i \subseteq R_i$ be nonempty such that $R_i P_i \subseteq P_i$ and $P_i R_i \subseteq P_i$, for all $i \in I$. For $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} R_i$. Define

$$(x_i)_{i \in I} \oplus_c (y_i)_{i \in I} = \{(x_i + t_i + y_i)_{i \in I} \mid t_i \in P_i\},$$

$$(x_i)_{i \in I} \odot (y_i)_{i \in I} = (x_i y_i)_{i \in I}.$$  

Then $\left(\prod_{i \in I} R_i, \oplus_c, \odot\right)$ is a weak distributive semihyperring.
Proof. First we show that \((\prod_{i \in I} R_i, \oplus_c)\) is a semihypergroup. For this we prove that
\[
(x_i)_{i \in I} \oplus_c [(y_i)_{i \in I} \oplus_c (z_i)_{i \in I}] = [(x_i)_{i \in I} \oplus_c (y_i)_{i \in I}] \oplus_c (z_i)_{i \in I}.
\]
We have \(A \in (x_i)_{i \in I} \oplus_c [(y_i)_{i \in I} \oplus_c (z_i)_{i \in I}]\)
\[
\implies \exists t_i \in P_i, A \in (x_i)_{i \in I} \oplus_c (y_i + t_i + z_i)_{i \in I}
\]
\[
\implies \exists t_i' \in P_i, A = (x_i + t_i' + y_i + t_i + z_i)_{i \in I}
\]
\[
A \in (x_i + t_i' + y_i)_{i \in I} \oplus_c (z_i)_{i \in I}
\]
\[
A \in [(x_i)_{i \in I} \oplus_c (y_i)_{i \in I}] \oplus_c (z_i)_{i \in I}
\]
\[
(x_i)_{i \in I} \oplus_c [(y_i)_{i \in I} \oplus_c (z_i)_{i \in I}] \subseteq [(x_i)_{i \in I} \oplus_c (y_i)_{i \in I}] \oplus_c (z_i)_{i \in I}.
\]
In a similar way, we can prove the reverse inclusion. Therefore, \((\prod_{i \in I} R_i, \oplus_c)\) is a semihypergroup. Clearly \((\prod_{i \in I} R_i, \odot)\) is a semigroup. It is enough we prove weak distributivity. For this we should prove that
\[
(x_i)_{i \in I} \odot [(y_i)_{i \in I} \oplus_c (z_i)_{i \in I}] \subseteq (x_i y_i)_{i \in I} \oplus_c (x_i z_i)_{i \in I}.
\]
We have \(A \in (x_i)_{i \in I} \odot [(y_i)_{i \in I} \oplus_c (z_i)_{i \in I}]\)
\[
\implies \exists t_i \in P_i, A \in (x_i)_{i \in I} \odot (y_i + t_i + z_i)_{i \in I}
\]
\[
A = (x_i y_i + x_i t_i + x_i z_i)_{i \in I}
\]
\[
A \in (x_i y_i)_{i \in I} \oplus_c (x_i z_i)_{i \in I} \quad (R_i P_i \subseteq P_i).
\]
This completes the proof. \(\square\)

**Proposition 4.2.** If \(\{R_i\}_{i \in I}\) is a family of semirings and for all \(i \in I, P_i \subseteq R_i\) is nonempty such that \(R_i P_i \subseteq P_i\) and \(P_i R_i \subseteq P_i\) and \(x_i + P_i = P_i + x_i\), for all \(x_i \in R_i\), then \(\bigotimes_{i \in I} R_i, \odot_r, \odot\) and \(\bigotimes_{i \in I} R_i, \odot, \odot\) are weak distributive semihyperring where
\[
(x_i)_{i \in I} \odot_r (y_i)_{i \in I} = \{(x_i + y_i)_{i \in I} \mid t_i \in P_i\},
\]
\[
(x_i)_{i \in I} \odot_l (y_i)_{i \in I} = \{(t_i + x_i + y_i)_{i \in I} \mid t_i \in P_i\}.
\]
Proposition 4.3. If $(\prod_{i \in I} R_i, \oplus_r, \odot_r)$ is a semihypergroup. For this we prove that

$$(x_i)_{i \in I} \odot_r (y_i)_{i \in I} = (x_i y_i)_{i \in I}.$$ 

**Proof.** First we prove that $(\prod_{i \in I} R_i, \oplus_r, \odot_r)$ is a semihypergroup. For this we prove that

$$(x_i)_{i \in I} \odot_r [(y_i)_{i \in I} \odot_r (z_i)_{i \in I}] = [(x_i)_{i \in I} \odot_r (y_i)_{i \in I}] \odot_r (z_i)_{i \in I}.$$ 

We have $A \in (x_i)_{i \in I} \odot_r [(y_i)_{i \in I} \odot_r (z_i)_{i \in I}]$

$$\Rightarrow \exists t \in P_k, A \in (x_i)_{i \in I} \odot_r (y_i + z_i + t_i)_{i \in I}$$

$$\Rightarrow \exists t' \in P_k, A = (x_i + y_i + z_i + t_i + t'_i)_{i \in I}$$

$$\Rightarrow \exists w \in P_k, A = (x_i + y_i + w_i + z_i + t_i + t'_i)_{i \in I} \quad (\text{since } z_i + P_i = P_i + z_i)$$

$$\in (x_i + y_i + w_i)_{i \in I} \odot_r (z_i)_{i \in I}$$

$$\subseteq [(x_i)_{i \in I} \odot_r (y_i)_{i \in I}] \odot_r (z_i)_{i \in I}$$

$$\Rightarrow (x_i)_{i \in I} \odot_r [(y_i)_{i \in I} \odot_r (z_i)_{i \in I}] \subseteq [(x_i)_{i \in I} \odot_r (y_i)_{i \in I}] \odot_r (z_i)_{i \in I}.$$

Similarly, we can prove that the reverse inclusion.

Clearly $(\prod_{i \in I} R_i, \odot)$ is a semigroup. Also the weak distributivity is obtained similar to the proof of Proposition 4.1. Therefore $(\prod_{i \in I} R_i, \odot_r, \odot)$ is a semihyperring.

Analogously we can prove that $(\prod_{i \in I} R_i, \oplus_l, \odot)$ is a weak distributive semihyperring.

This completes the proof. □

**Remark.** In Propositions 4.1 and 4.2, if we replace the conditions $R_i P_i \subseteq P_i$ and $P_i R_i \subseteq P_i$ by the condition $r_i P_i = P_i = P_i r_i$, for all $r_i \in R_i$ and for all $i \in I$, then $(\prod_{i \in I} R_i, \oplus_c, \odot), (\prod_{i \in I} R_i, \odot_r, \odot)$ and $(\prod_{i \in I} R_i, \oplus_l, \odot)$ will be semihyperrings.

**Proposition 4.3.** If $\{R_j\}_{j \in J}$ is a family of semirings and for all $j \in J$, $P_j \subseteq R_j$ is nonempty such that $R_j P_j \subseteq P_j$ and $P_j R_j \subseteq P_j$. Then $I$ is a $k$-hyperideal of $(\prod_{j \in J} R_j, \odot_c, \odot)$ if and only if $I = \prod_{j \in J} I_j$ such that $I_j \triangleleft_k (R_j, \odot_{c_j}, \odot_j)$, where

$$x_j \odot_{c_j} y_j = \{x_j + t_j + y_j \mid t_j \in P_j\},$$

$$x_j \odot_j y_j = x_j y_j.$$
Proof. ($\implies$) For all $j \in J$ define

$$I_j = \{ x \in R_j \mid (x_i)_{i \in J} \in I, \exists x_i \in R_i, x = x_j \}.$$  

We have

$$x, y \in I \implies \exists x_i, y_i \in R_i, (x_i)_{i \in J}, (y_i)_{i \in J} \in I, x = x_j, y = y_j$$

$$\implies (x_i)_{i \in J} \oplus_c (y_i)_{i \in J} \subseteq I \quad (I \triangleleft_h (\prod_{j \in J} R_i, \oplus_c, \circ))$$

$$\implies \forall t_i \in P_i, (x_i + t_i + y_i)_{i \in J} \in I \quad (\forall i \in J)$$

$$\implies \forall t_j \in P_j, x + t_j + y \in I_j$$

$$\implies x \oplus_c y \subseteq I_j.$$  

Now suppose that

$$r_j \in R_j, x \in I_j \implies \exists r_i \in R_i, (r_i)_{i \in J} \in \prod_{i \in J} R_i \text{ and } \exists x_i \in R_i, (x_i)_{i \in J} \in I, x = x_j$$

$$\implies (r_i)_{i \in J} \odot (x_i)_{i \in J} \in I \quad (I \triangleleft_h (\prod_{i \in J} R_i, \oplus_c, \odot))$$

$$\implies (r_i x_i)_{i \in J} \in I$$

$$\implies r_j x_j \in I_j \quad (\text{by definition of } I_j).$$

Therefore $I_j \triangleleft_h R_j$.

We now show that $I_j \triangleleft_{k,h} R_j$ for all $j \in J$. We have

$$r_j \in R_j, x_j \in I_j, r_j \oplus_{c_j} x_j \approx I_j \implies \exists t_j \in P_j, r_j + t_j + x_j \in I_j$$

$$\implies (r_j)_{j \in J} \oplus_c (x_j)_{j \in J} \approx I,$$

where $(r_j)_{j \in J} \in \prod_{j \in J} R_j$, $(x_j)_{j \in J} \in \prod_{j \in J} I_j$. Then since $I \triangleleft_{k,h} (\prod_{j \in J} R_j, \oplus_c, \odot)$ we have

$$(r_j)_{j \in J} \in I \implies r_j \in I_j, \forall j \in J$$

$$\implies I_j \triangleleft_{k,h} R_j.$$  

($\iff$) Suppose that $I = \prod_{j \in J} I_j$ such that $I_j \triangleleft_{k,h} (R_j, \oplus_{c_j}, \odot_j)$. First we prove $I \triangleleft_h \prod_{j \in J} R_j, \oplus_c, \odot)$. Let $(x_j)_{j \in J}, (y_j)_{j \in J} \in I$, then

$$(x_j)_{j \in J} \oplus_c (y_j)_{j \in J} = \{(x_j + t_j + y_j)_{j \in J} \mid t_j \in P_j\} \subseteq \prod_{j \in J} I_j;$$
also we have

\[ I_j \triangleleft_h (R_j, \oplus_{c_j}, \circ_j) \implies \forall t_j \in P_j, \ x_j + t_j + y_j \in I_j \]

\[ \implies (x_j)_{j \in J} \oplus (y_j)_{j \in J} \subseteq I. \]

Now if \((r_j)_{j \in J} \in \prod_{j \in J} R_j \) and \((x_j)_{j \in J} \in I\), then \((r_j)_{j \in J} \circ (x_j)_{j \in J} = (r_j x_j)_{j \in J} \in \prod_{j \in J} I_j\), since \(r_j x_j \in I_j\) by hypothesis. We now prove that \(I \triangleleft_h (\prod_{j \in J} R_j, \oplus_c, \circ)\). For this we have

\[\begin{align*}
(r_j)_{j \in J} & \in \prod_{j \in J} R_j, \ (x_j)_{j \in J} \in I, \ (r_j)_{j \in J} \oplus_c (x_1, x_2) \approx I \\
\implies & \exists t_j \in P_j, \ (r_j + t_j + x_j)_{j \in J} \in I = \prod_{j \in J} I_j \\
\implies & \exists t_j \in P_j, \ r_j + t_j + x_j \in I_j, \ \forall j \in J \\
\implies & r_j \oplus_c x_j \approx I_j, \ r_j \in R_j, \ x_j \in I_j \\
\implies & r_j \in I_j \quad (I_j \triangleleft_h (R_j, \oplus_{c_j}, \circ_j)) \\
\implies & (r_j)_{j \in J} \in \prod_{j \in J} I_j. \quad \Box
\end{align*}\]

**Proposition 4.4.** Let \(\{R_j\}_{j \in J}\) be a family of semirings. Suppose that \(P_j \subseteq R_j\) be nonempty such that \(R_j P_j \subseteq P_j\) and \(P_j R_j \subseteq P_j\) and \(x_j + P_j = P_j + x_j\), for all \(x_j \in R_j\) and for all \(j \in J\). Then \(I\) is a \(k\)-hyperideal of \((\prod_{j \in J} R_j, \oplus_r, \circ)\) (resp. \((\prod_{j \in J} R_j, \oplus_l, \circ)\)) if and only if \(I = \prod_{j \in J} I_j\) such that for all \(j \in J\), \(I_j \triangleleft_h (R_j, \oplus_{r_j}, \circ_j)\) (resp. \(I_j \triangleleft_h (R_j, \oplus_{l_j}, \circ_j)\)), where

\[x_j \oplus_{r_j} y_j = \{x_j + y_j + t_j \mid t_j \in P_j\},\]

\[x_j \oplus_{l_j} y_j = \{t_j + x_j + y_j \mid t_j \in P_j\},\]

\[x_j \circ_{j} y_j = x_j y_j.\]

**Proof.** The proof is similar to the proof of Proposition 4.3. \(\Box\)

**References**


