

The general solutions of a functional equation related to information theory

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Abstract. The general solutions of a functional equation, containing two unknown functions, and related to a functional equation characterizing the Shannon entropy and the entropy of degree α , are obtained.

Keywords: Functional equations, continuous solutions, Lebesgue measurable solutions, the Shannon entropy, the nonadditive entropy of degree α , multiplicative functions, additive functions.

1. Introduction

For $n = 1, 2, 3, \dots$, let

$$\Gamma_n = \left\{ (p_1, \dots, p_n) : p_i \geq 0, i = 1, \dots, n; \sum_{i=1}^n p_i = 1 \right\}$$

denote the set of all n -component complete discrete probability distributions with nonnegative elements and let $F : I \rightarrow \mathbb{R}$, \mathbb{R} denoting the set of all real numbers and $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, the unit closed interval.

The functional equation

$$(1.1) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k F(p_i) + \sum_{j=1}^{\ell} F(q_j)$$

with $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_{\ell}) \in \Gamma_{\ell}$ was first studied by T.W. Chaundy and J.B. Mcleod [4]. They proved that if (1.1) holds for integers $k = 2, 3, \dots$ and $\ell = 2, 3, \dots$ and F is continuous on I , then F is of the form

$$(1.2) \quad F(p) = c p \log_2 p, \quad 0 \leq p \leq 1$$

where c is an arbitrary real constant and $0 \log_2 0 = 0$. Later on, J. Aczél and Z. Daróczy [1] proved the same by assuming $k = \ell = 2, 3, \dots$. Z. Daróczy [5] obtained the Lebesgue measurable solutions of (1.1) by fixing $k = 3$, $\ell = 2$ and assuming $F(1) = 0$. Gy. Maksa [13] obtained the solutions of (1.1) by fixing $k = 3$, $\ell = 2$ but assuming F to be bounded on a subset, of I , of positive Lebesgue measure.

If $F\left(\frac{1}{2}\right) = \frac{1}{2}$, then (1.2) gives $c = -1$ and then (1.2) reduces to

$$(1.3) \quad F(p) = -p \log_2 p$$

for all $p \in I$.

For any probability distribution $(p_1, \dots, p_m) \in \Gamma_m$,

$$(1.4) \quad H_m(p_1, \dots, p_m) = - \sum_{i=1}^m p_i \log_2 p_i$$

is known as the Shannon entropy [15] of the probability distribution $(p_1, \dots, p_m) \in \Gamma_m$ and the sequence $H_m : \Gamma_m \rightarrow \mathbb{R}$, $m = 1, 2, \dots$ is known as the sequence of the Shannon entropies.

A generalization of the Shannon entropy with which we shall be concerned in this paper is (with $H_m^\alpha : \Gamma_m \rightarrow \mathbb{R}$, $m = 1, 2, 3, \dots$)

$$(1.5) \quad H_m^\alpha(p_1, \dots, p_m) = (1 - 2^{1-\alpha})^{-1} \left(1 - \sum_{i=1}^m p_i^\alpha \right), \quad \alpha > 0, \alpha \neq 1, 0^\alpha := 0, \alpha \in \mathbb{R}.$$

The entropies (1.5) are due to J. Havrda and F. Charvat [7].

The axiomatic characterization of the entropies (1.5) leads to the study of the functional equation

$$(1.6) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k F(p_i) + \sum_{j=1}^{\ell} F(q_j) + \lambda \sum_{i=1}^k F(p_i) \sum_{j=1}^{\ell} F(q_j)$$

where $(p_1, \dots, p_k) \in \Gamma_k$, $(q_1, \dots, q_\ell) \in \Gamma_\ell$ and $\lambda = 2^{1-\alpha} - 1$, $\alpha \in \mathbb{R}$. Clearly, (1.6) reduces to (1.1) if $\lambda = 0$.

By taking $\lambda = 2^{1-\alpha} - 1$, $\alpha \neq 1$, $\alpha \in \mathbb{R}$, $0^\alpha := 0$, the continuous solutions of (1.6) were obtained by M. Behara and P. Nath [3] for all positive integers $k = 2, 3, \dots$; $\ell = 2, 3, \dots$. Later on PL. Kannappan [10] and D.P. Mittal [14] also obtained the continuous solutions of (1.6) for $\lambda \neq 0$ and $k = 2, 3, \dots$; $\ell = 2, 3, \dots$. For fixed integers $k \geq 3$ and $\ell \geq 2$, L. Losonczi [11] obtained the measurable solutions of (1.6). Also, PL. Kannappan [8] obtained the Lebesgue measurable solutions of both (1.1) and (1.6) for fixed integers $k \geq 3$, $\ell \geq 3$.

It seems that L. Losonczi and Gy. Maksa [12] are the first to obtain the general solutions of (1.6) in both cases, namely $\lambda \neq 0$ and $\lambda = 0$, by fixing integers k and $\ell, k \geq 3$ and $\ell \geq 3$.

There are several generalizations of (1.6), with $\lambda \in \mathbb{R}$, containing at least two unknown functions. Below we list only three important generalizations of (1.6), namely,

$$(1.7) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k H(p_i) + \sum_{j=1}^{\ell} H(q_j) + \lambda \sum_{i=1}^k H(p_i) \sum_{j=1}^{\ell} H(q_j)$$

$$(1.8) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k F(p_i) + \sum_{j=1}^{\ell} H(q_j) + \lambda \sum_{i=1}^k F(p_i) \sum_{j=1}^{\ell} H(q_j)$$

and

$$(1.9) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k G(p_i) + \sum_{j=1}^{\ell} H(q_j) + \lambda \sum_{i=1}^k G(p_i) \sum_{j=1}^{\ell} H(q_j).$$

The object of this paper is to investigate the general solutions of the functional equation (1.7) for fixed integers $k \geq 3$ and $\ell \geq 3$. The corresponding results for the functional equations (1.8) and (1.9) have also been investigated by the authors and shall be presented elsewhere in our subsequent research work.

The process of finding the general solutions of (1.7) requires a detailed study of the following two functional equations :

$$(1.10) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} g(p_i q_j) = \sum_{i=1}^k g(p_i) \sum_{j=1}^{\ell} g(q_j) + \ell(k-1)g(0)$$

and

$$(1.11) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} f(p_i q_j) = \sum_{i=1}^k h(p_i) \sum_{j=1}^{\ell} h(q_j)$$

where $f : [0, 1] \rightarrow \mathbb{R}$, $g : [0, 1] \rightarrow \mathbb{R}$ and $h : [0, 1] \rightarrow \mathbb{R}$.

The functional equation (1.10) is, indeed, a generalization of the multiplicative-type functional equation

$$(1.12) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} g(p_i q_j) = \sum_{i=1}^k g(p_i) \sum_{j=1}^{\ell} g(q_j)$$

whose importance in information theory is well-known (see L. Losonczi and Gy. Maksa [12]). The functional equation (1.6), for $\lambda \neq 0$, can be written in the multiplicative form (1.12) by defining $g : I \rightarrow \mathbb{R}$ as $g(x) = \lambda F(x) + x$ for all $x \in I$. Likewise, each of the functional equations (1.7), (1.8) and (1.9), for $\lambda \neq 0$, can also be written in the corresponding multiplicative forms. This is precisely the reason for paying attention to the functional equations (1.7) to (1.9).

2. The general solutions of functional equation (1.10)

Before investigating the general solutions of (1.10) for fixed integers k and ℓ , $k \geq 3$, $\ell \geq 3$, we need some definitions and results already existing in the literature (see [12]). Let

$$\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}.$$

In other words, Δ denotes the unit closed triangle in

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$

A mapping $a : \mathbb{R} \rightarrow \mathbb{R}$ is said to be additive if it satisfies the equation

$$(2.1) \quad a(x + y) = a(x) + a(y)$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $a : I \rightarrow \mathbb{R}$, $I = [0, 1]$ is said to be additive on the triangle Δ if it satisfies (2.1) for all $(x, y) \in \Delta$.

A mapping $m : [0, 1] \rightarrow \mathbb{R}$ is said to be multiplicative if $m(0) = 0$, $m(1) = 1$ and $m(xy) = m(x)m(y)$ for all $x \in]0, 1[, y \in]0, 1[$.

Now we state :

Lemma 1. *Let $\Psi : I \rightarrow \mathbb{R}$ be a mapping which satisfies the functional equation*

$$(2.2) \quad \sum_{i=1}^n \Psi(p_i) = c$$

for all $(p_1, \dots, p_n) \in \Gamma_n$; c a given constant and $n \geq 3$ a fixed integer. Then there exists an additive mapping $a : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(2.3) \quad \Psi(p) = a(p) + \Psi(0), \quad 0 \leq p \leq 1$$

where

$$(2.4) \quad a(1) = c - n\Psi(0).$$

Conversely, if (2.4) holds, then the mapping $\Psi : I \rightarrow \mathbb{R}$, defined by (2.3), satisfies the functional equation (2.2).

This lemma appears on p-74 in [12].

Lemma 2. *Every mapping $a : I \rightarrow \mathbb{R}$, $I = [0, 1]$, additive on the unit triangle Δ , has a unique additive extension to the whole of \mathbb{R} .*

Note. This unique additive extension to the whole of \mathbb{R} will also be denoted by the symbol a but now $a : \mathbb{R} \rightarrow \mathbb{R}$.

For Lemma 2, See Theorem (0.3.7) on p-8 in [2] or Z. Daróczy and L. Losonczy [6]. Now we prove :

Theorem 1. *Let $k \geq 3$, $\ell \geq 3$ be fixed integers and $g : [0, 1] \rightarrow \mathbb{R}$ be a mapping which satisfies the functional equation (1.10) for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_\ell) \in \Gamma_\ell$. Then g is of the form*

$$(2.5) \quad g(p) = a(p) + g(0)$$

where $a : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function such that $a(1)$ satisfies the equation

$$(2.6) \quad a(1) + k\ell g(0) = [a(1) + k g(0)][a(1) + \ell g(0)] + \ell(k - 1) g(0)$$

or

$$(2.7) \quad g(p) = M(p) - A(p) + g(0)$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function with

$$(2.8) \quad A(1) = \ell g(0)$$

and $M : [0, 1] \rightarrow \mathbb{R}$ is a mapping such that

$$(2.9) \quad M(0) = 0$$

$$(2.10) \quad M(1) = g(1) + (\ell - 1) g(0)$$

and

$$(2.11) \quad M(pq) = M(p)M(q) \quad \text{for all } p \in]0, 1[, q \in]0, 1[.$$

Proof. Let us put $p_1 = 1, p_2 = \dots = p_k = 0$ in (1.10). We obtain

$$(2.12) \quad [1 - g(1) - (k - 1)g(0)] \sum_{j=1}^{\ell} g(q_j) = 0.$$

CASE 1. $1 - g(1) - (k - 1)g(0) \neq 0$. Then (2.12) reduces to

$$(2.13) \quad \sum_{j=1}^{\ell} g(q_j) = 0.$$

Hence, by Lemma 1, g is of the form (2.5) in which $a : \mathbb{R} \rightarrow \mathbb{R}$ is an additive mapping such that $a(1) = -\ell g(0)$ satisfies the equation (2.6).

CASE 2. $1 - g(1) - (k - 1)g(0) = 0$.

The functional equation (1.10) may be written in the form

$$\sum_{j=1}^{\ell} \left[\sum_{i=1}^k g(p_i q_j) - g(q_j) \sum_{i=1}^k g(p_i) \right] = \ell(k - 1)g(0).$$

Hence, by Lemma 1,

$$(2.14) \quad \sum_{i=1}^k g(p_i q) - g(q) \sum_{i=1}^k g(p_i) \\ = A_1(p_1, \dots, p_k, q) - \frac{1}{\ell} A_1(p_1, \dots, p_k, 1) + (k - 1)g(0)$$

where $A_1 : \Gamma_k \times \mathbb{R} \rightarrow \mathbb{R}$ is additive in the second variable. The substitution $q = 0$ in (2.14) gives

$$(2.15) \quad A_1(p_1, \dots, p_k, 1) = \ell g(0) \left[\sum_{i=1}^k g(p_i) - 1 \right].$$

Let $x \in [0, 1]$, $(r_1, \dots, r_k) \in \Gamma_k$. Put $q = xr_t$, $t = 1, \dots, k$ in (2.14); add the resulting k equations; and use the additivity of A_1 . We get

$$(2.16) \quad \sum_{i=1}^k \sum_{t=1}^k g(p_i r_t x) - \sum_{i=1}^k g(p_i) \sum_{t=1}^k g(xr_t) \\ = A_1(p_1, \dots, p_k, x) - \frac{k}{\ell} A_1(p_1, \dots, p_k, 1) + k(k - 1)g(0).$$

Now put $q = x$, $p_1 = r_1, \dots, p_k = r_k$ in (2.14). We obtain

$$(2.17) \quad \sum_{t=1}^k g(xr_t) = g(x) \sum_{t=1}^k g(r_t) + A_1(r_1, \dots, r_k, x) \\ - \frac{1}{\ell} A_1(r_1, \dots, r_k, 1) + (k-1)g(0).$$

From (2.16) and (2.17), it follows that

$$(2.18) \quad \sum_{i=1}^k \sum_{t=1}^k g(p_i r_t x) - g(x) \sum_{i=1}^k g(p_i) \sum_{t=1}^k g(r_t) - k(k-1)g(0) \\ = (k-1)g(0) \sum_{i=1}^k g(p_i) + A_1(r_1, \dots, r_k, x) \sum_{i=1}^k g(p_i) \\ - \frac{1}{\ell} A_1(r_1, \dots, r_k, 1) \sum_{i=1}^k g(p_i) + A_1(p_1, \dots, p_k, x) \\ - \frac{k}{\ell} A_1(p_1, \dots, p_k, 1).$$

The left hand side of (2.18) does not undergo any change if we interchange p_i and $r_i, i = 1, \dots, k$. So, the right hand side of (2.18) must also remain unchanged on interchanging p_i and $r_i, i = 1, \dots, k$. Consequently, we obtain

$$(2.19) \quad A_1(p_1, \dots, p_k, x) \left[\sum_{t=1}^k g(r_t) - 1 \right] - \frac{1}{\ell} A_1(p_1, \dots, p_k, 1) \left[\sum_{t=1}^k g(r_t) - k \right] \\ + (k-1)g(0) \sum_{t=1}^k g(r_t) \\ = A_1(r_1, \dots, r_k, x) \left[\sum_{i=1}^k g(p_i) - 1 \right] - \frac{1}{\ell} A_1(r_1, \dots, r_k, 1) \left[\sum_{i=1}^k g(p_i) - k \right] \\ + (k-1)g(0) \sum_{i=1}^k g(p_i).$$

Now we divide our discussion into two cases depending upon whether $\sum_{t=1}^k g(r_t) - 1$ vanishes identically on Γ_k or does not vanish identically on Γ_k .

CASE 2.1. $\sum_{t=1}^k g(r_t) - 1$ vanishes identically on Γ_k . Then

$$\sum_{t=1}^k g(r_t) = 1$$

for all $(r_1, \dots, r_k) \in \Gamma_k$. By using Lemma 1, it follows that g is of the form (2.5) in which $a(1) = 1 - k g(0)$ satisfies the equation (2.6).

CASE 2.2. $\sum_{t=1}^k g(r_t) - 1$ does not vanish identically on Γ_k .

In this case, there exists a probability distribution $(r_1^*, \dots, r_k^*) \in \Gamma_k$ such that

$$(2.20) \quad \sum_{t=1}^k g(r_t^*) - 1 \neq 0.$$

Putting $r_1 = r_1^*, \dots, r_k = r_k^*$ in (2.19), making use of (2.20) and (2.15); and performing necessary calculations, it follows that

$$(2.21) \quad A_1(p_1, \dots, p_k, x) = A(x) \left[\sum_{i=1}^k g(p_i) - 1 \right]$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$(2.22) \quad A(x) = \left[\sum_{t=1}^k g(r_t^*) - 1 \right]^{-1} A_1(r_1^*, \dots, r_k^*, x)$$

From (2.22) it is easy to conclude that $A : \mathbb{R} \rightarrow \mathbb{R}$ is additive as the mapping $x \mapsto A_1(r_1^*, \dots, r_k^*, x)$ is additive. Also, putting $x = 1$ in (2.22) and making use of (2.15) by taking $p_i = r_i^*$, $i = 1, \dots, k$; (2.8) follows. Also, from (2.14), (2.15), (2.21) and (2.8), it follows that

$$(2.23) \quad \sum_{i=1}^k [g(p_i q) + A(p_i q) - g(0)] - [g(q) + A(q) - g(0)] \\ \times \sum_{i=1}^k [g(p_i) + A(p_i) - g(0)] + [g(q) + A(q) - g(0)](\ell - k) g(0) = 0.$$

Define a mapping $M : I \rightarrow \mathbb{R}$, $I = [0, 1]$, as

$$(2.24) \quad M(p) = g(p) + A(p) - g(0)$$

for all $p \in I$. Then, (2.23) reduces to the equation

$$(2.25) \quad \sum_{i=1}^k [M(p_i q) - M(q) M(p_i) + (\ell - k) g(0) M(q) p_i] = 0.$$

Hence, by Lemma 1,

$$(2.26) \quad M(pq) - M(q) M(p) + (\ell - k) g(0) M(q) p = E_1(p, q) - \frac{1}{k} E_1(1, q)$$

where $E_1 : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is additive in its first variable.

Since $A(0) = 0$ and $A(1) = \ell g(0)$, (2.9) and (2.10) follow from (2.24). Also, putting $p = 0$ in (2.26) and making use of (2.9), it follows that

$$(2.27) \quad E_1(0, q) = 0$$

for all q , $0 \leq q \leq 1$. Consequently,

$$(2.28) \quad E_1(1, q) = 0$$

for all q , $0 \leq q \leq 1$. Now, (2.26) reduces to

$$(2.29) \quad M(pq) - M(p) M(q) = E_1(p, q) - (\ell - k) g(0) M(q) p$$

for all $p \in [0, 1]$ and $q \in [0, 1]$.

Since $M(1) = g(1) + (\ell - 1) g(0)$, from now onwards, we divide our discussion into two subcases, depending upon whether $g(1) + (\ell - 1) g(0) = 1$ or $g(1) + (\ell - 1) g(0) \neq 1$.

CASE 2.2.1. $g(1) + (\ell - 1) g(0) = 1$.

In this case, $1 = g(1) + (\ell - 1) g(0) = g(1) + (k - 1) g(0) + (\ell - k) g(0)$.

Since $g(1) + (k-1)g(0) = 1$, it follows that $(\ell - k)g(0) = 0$. Then, (2.29) reduces to

$$(2.30) \quad M(pq) - M(p)M(q) = E_1(p, q)$$

where $E_1 : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is additive in the first variable and $0 \leq p \leq 1$, $0 \leq q \leq 1$. The left hand side of (2.30) is symmetric in p and q . Hence, $E_1(p, q) = E_1(q, p)$ for all $p \in [0, 1], q \in [0, 1]$. Consequently, E_1 is also additive in second variable. Also, we may suppose that $E_1(p, \cdot)$ has been extended additively to the whole of \mathbb{R} and this extension is unique by Lemma 2.

From (2.30), as on p-77 in [12], it follows that

$$(2.30a) \quad \begin{aligned} M(pqr) - M(p)M(q)M(r) &= E_1(pq, r) + M(r)E_1(p, q) \\ &= E_1(qr, p) + M(p)E_1(q, r) \end{aligned}$$

for all p, q, r in $[0, 1]$. Now, we prove that $E_1(p, q) = 0$ for all p, q , $0 \leq p \leq 1$, $0 \leq q \leq 1$. If possible, suppose there exist p^* and q^* , $0 \leq p^* \leq 1$, $0 \leq q^* \leq 1$, such that $E_1(p^*, q^*) \neq 0$. Then, from (2.30a),

$$M(r) = [E_1(p^*, q^*)]^{-1} [E_1(q^*r, p^*) + M(p^*)E_1(q^*, r) - E_1(p^*q^*, r)]$$

from which it is easy to conclude that M is additive. Now, making use of (2.8), (2.10), (2.20), (2.24), the condition $g(1) + (\ell - 1)g(0) = 1$; and the additivity of A and M , we have

$$1 \neq \sum_{t=1}^k g(r_t^*) = M(1) - A(1) + k g(0) = 1$$

a contradiction. Hence $E_1(p, q) = 0$ for all p and q , $0 \leq p \leq 1$, $0 \leq q \leq 1$. Thus, (2.30) reduces to $M(pq) = M(p)M(q)$ for all p and q , $0 \leq p \leq 1$, $0 \leq q \leq 1$. So, M is a nonconstant multiplicative function. Hence, from (2.24), it follows that g is of the form (2.7).

CASE 2.2.2. $g(1) + (\ell - 1)g(0) \neq 1$.

Since the values of M at 0 and 1 are given by (2.9) and (2.10), our next task is to get some information about $M(r)$ when $0 < r < 1$. For this purpose, we proceed as follows:

Let p, q, r be in $]0, 1[$. Now, from (2.29), one can derive

$$\begin{aligned}
 (2.31) \quad M(pqr) - M(p)M(q)M(r) \\
 &= E_1(r, pq) - (\ell - k)g(0)M(pq)r + M(r)[E_1(p, q) - (\ell - k)g(0)M(q)p] \\
 &= E_1(rq, p) - (\ell - k)g(0)M(p)r + M(p)[E_1(r, q) - (\ell - k)g(0)M(q)r].
 \end{aligned}$$

Now, we prove that $E_1(p, q) - (\ell - k)g(0)M(q)p = 0$ for all $p, q, 0 < p < 1, 0 < q < 1$. If possible, suppose there exist $p^* \in]0, 1[$ and $q^* \in]0, 1[$ such that $E_1(p^*, q^*) - (\ell - k)g(0)M(q^*)p^* \neq 0$. Then, from (2.31), it follows that for all $r \in]0, 1[$,

$$\begin{aligned}
 (2.32) \quad M(r) &= [E_1(p^*, q^*) - (\ell - k)g(0)M(q^*)p^*]^{-1} \\
 &\quad \times [E_1(rq^*, p^*) - (\ell - k)g(0)M(p^*)rq^* \\
 &\quad + M(p^*)\{E_1(r, q^*) - (\ell - k)g(0)M(q^*)r\} - E_1(r, p^*q^*) \\
 &\quad + (\ell - k)g(0)M(p^*q^*)r].
 \end{aligned}$$

Now we prove that $M : [0, 1] \rightarrow \mathbb{R}$ is additive on Δ , that is,

$$(2.33) \quad M(x + y) = M(x) + M(y)$$

for all $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1$.

If $x = 0, 0 \leq y \leq 1$ or $y = 0, 0 \leq x \leq 1$, then (2.33) holds trivially.

If $0 < x < 1, 0 < y < 1, 0 < x + y < 1$, then (2.33) follows from (2.32).

Now consider the case when $0 < x < 1, 0 < y < 1$ but $x + y = 1$. In this case, let us choose $q = 1$ and $p = x + y$ in (2.29) and use the additivity of E_1

with respect to first variable. We obtain

$$\begin{aligned} & M(x+y)\{1-g(1)-(\ell-1)g(0)\} \\ &= \{M(x)+M(y)\}\{1-g(1)-(\ell-1)g(0)\}. \end{aligned}$$

Since $g(1)+(\ell-1)g(0) \neq 1$, (2.33) follows. Thus, M is additive on the triangle Δ . Now, making use of (2.8), (2.10), (2.20), (2.24), the condition $g(1)+(k-1)g(0)=1$; and the additivity of A and M , we have

$$1 \neq \sum_{t=1}^k g(r_t^*) = M(1) - A(1) + k g(0) = 1$$

a contradiction. Hence $E_1(p, q) - (\ell - k) g(0) M(q) p = 0$ for all $p, q, 0 < p < 1, 0 < q < 1$. Thus, (2.29) reduces to (2.11). But in this case M is not multiplicative because $M(1) = g(1) + (\ell - 1) g(0) \neq 1$. Hence from (2.24), the solution (2.7) follows.

Note. It is easy to verify that (2.5); subject to the condition (2.6), satisfies (1.10). However (2.7) also satisfies (1.10). In this case we need to use (2.25) in addition to (2.8) to (2.11).

3. The general solutions of functional equation (1.11)

Now we prove :

Theorem 2. *Let $k \geq 3, \ell \geq 3$ be fixed integers and $f : I \rightarrow \mathbb{R}, h : I \rightarrow \mathbb{R}, I = [0, 1]$, be mappings which satisfy the functional equation (1.11) for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_\ell) \in \Gamma_\ell$. Then any general solution of (1.11) is of the form*

$$(3.1) \quad \begin{cases} f(p) = b(p) + f(0) \\ h(p) = B(p) + h(0) \end{cases}$$

subject to the condition

$$(3.2) \quad b(1) + k\ell f(0) = [B(1) + k h(0)][B(1) + \ell h(0)];$$

or

$$(3.3) \quad \begin{cases} f(p) = [h(1) + (k-1)h(0)]^2 a(p) + A^*(p) + f(0) \\ h(p) = [h(1) + (k-1)h(0)] a(p) + h(0) \end{cases}$$

subject to the condition

$$(3.3a) \quad \begin{aligned} & [h(1) + (k-1)h(0)]^2 a(1) + A^*(1) + k\ell f(0) \\ &= \{[h(1) + (k-1)h(0)]a(1) + k h(0)\} \{[h(1) + (k-1)h(0)]a(1) + \ell h(0)\} \end{aligned}$$

or

$$(3.4) \quad \begin{cases} f(p) = [h(1) + (k-1)h(0)]^2 [M(p) - A(p)] + A^*(p) + f(0) \\ h(p) = [h(1) + (k-1)h(0)] [M(p) - A(p)] + h(0) \\ A(1) = \frac{\ell h(0)}{h(1) + (k-1)h(0)}, \\ A^*(1) = \ell \{[h(1) + (k-1)h(0)] h(0) - k f(0)\} \end{cases}$$

where $A^* : \mathbb{R} \rightarrow \mathbb{R}$, $A : \mathbb{R} \rightarrow \mathbb{R}$, $B : \mathbb{R} \rightarrow \mathbb{R}$, $a : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ are additive functions; $f(0)$ and $h(0)$ are arbitrary constants; and $M : [0, 1] \rightarrow \mathbb{R}$ is a mapping which satisfies (2.9), (2.11) and

$$(3.5) \quad M(1) = \frac{h(1) + (\ell-1)h(0)}{h(1) + (k-1)h(0)}$$

with $h(1) + (k-1)h(0) \neq 0$ in (3.3), (3.3a), (3.4) and (3.5).

To prove this theorem, we need to prove some Lemmas :

Lemma 3. *If a mapping $f : I \rightarrow \mathbb{R}$ satisfies the functional equation*

$$(3.6) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} f(p_i q_j) = 0$$

for all $(p_1, \dots, p_k) \in \Gamma_k$, $(q_1, \dots, q_\ell) \in \Gamma_\ell$, $k \geq 3, \ell \geq 3$ fixed integers; then

$$(3.7) \quad f(p) = b(p) + f(0)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function with $b(1) = -k\ell f(0)$.

Proof. Choose $q_1 = 1, q_2 = \dots = q_\ell = 0$. Then, equation (3.6) reduces to

$$\sum_{i=1}^k f(p_i) = -k(\ell - 1) f(0).$$

Hence, by Lemma 1,

$$(3.8) \quad f(p) = b(p) - \frac{1}{k} b(1) - (\ell - 1) f(0)$$

for all p , $0 \leq p \leq 1$, $b : \mathbb{R} \rightarrow \mathbb{R}$ being any additive function with $b(1) = -k\ell f(0)$. Putting this value of $b(1)$ in (3.8), (3.7) readily follows.

Lemma 4. *Under the assumptions stated in the statement of Theorem 2, the following conclusions hold :*

$$(3.9) \quad f(p) = [h(1) + (k-1)h(0)]h(p) + A^*(p) - [h(1) + (k-1)h(0)]h(0) + f(0)$$

$$(3.10) \quad [h(1) + (k-1)h(0)] \sum_{i=1}^k \sum_{j=1}^{\ell} h(p_i q_j) - \sum_{i=1}^k h(p_i) \sum_{j=1}^{\ell} h(q_j) \\ = \ell(k-1)h(0)[h(1) + (k-1)h(0)]$$

$$(3.11) \quad [h(1) + (\ell-1)h(0)] \sum_{i=1}^k \sum_{j=1}^{\ell} h(p_i q_j) - \sum_{i=1}^k h(p_i) \sum_{j=1}^{\ell} h(q_j) \\ = k(\ell-1)h(0)[h(1) + (\ell-1)h(0)]$$

where $A^* : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function.

Proof. Putting $p_1 = 1, p_2 = \dots = p_k = 0$ in (1.11), we obtain

$$(3.12) \quad \sum_{j=1}^{\ell} \{f(q_j) - [h(1) + (k-1)h(0)]h(q_j)\} = -\ell(k-1)f(0).$$

Hence, by Lemma 1 (changing q to p),

$$(3.13) \quad f(p) = [h(1) + (k-1)h(0)]h(p) + A^*(p) - \frac{1}{\ell}A^*(1) - (k-1)f(0)$$

for all $p, 0 \leq p \leq 1$, $A^* : \mathbb{R} \rightarrow \mathbb{R}$ being an additive function with

$$(3.14) \quad A^*(1) = \ell \{[h(1) + (k-1)h(0)]h(0) - k f(0)\}.$$

From equations (3.13) and (3.14), equation (3.9) follows.

From (3.9) and (3.14), it is easy to see that

$$(3.15) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} f(p_i q_j) = [h(1) + (k-1)h(0)] \sum_{i=1}^k \sum_{j=1}^{\ell} h(p_i q_j) \\ - \ell(k-1)[h(1) + (k-1)h(0)]h(0).$$

From (1.11) and (3.15), we get (3.10). The proof of (3.11) is similar and hence omitted.

Proof of Theorem 2. We divide our discussion into three cases :

CASE 1. $\sum_{i=1}^k h(p_i)$ vanishes identically on Γ_k , that is,

$$(3.16) \quad \sum_{i=1}^k h(p_i) = 0$$

for all $(p_1, \dots, p_k) \in \Gamma_k$. Then, (1.11) reduces to (3.6). So, f is of the form (3.7)

for all $p, 0 \leq p \leq 1$. Also applying Lemma 1 to (3.16), we obtain

$$(3.17) \quad h(p) = B(p) - \frac{1}{k}B(1)$$

for all $p, 0 \leq p \leq 1$, $B : \mathbb{R} \rightarrow \mathbb{R}$ being an additive function with $B(1) = -k h(0)$.

Now (3.17) reduces to

$$(3.18) \quad h(p) = B(p) + h(0).$$

Equations (3.7), (3.18), together with the condition (3.2), constitute the solution (3.1) of (1.11).

CASE 2. $\sum_{j=1}^{\ell} h(q_j)$ vanishes identically on Γ_{ℓ} . In this case, we also get the solution (3.1), subject to the condition (3.2); of (1.11). The proof is omitted as it is similar to that in case 1.

CASE 3. Neither $\sum_{i=1}^k h(p_i)$ vanishes identically on Γ_k nor $\sum_{j=1}^{\ell} h(q_j)$ vanishes identically on Γ_{ℓ} . Then, there exist a $(p_1^*, \dots, p_k^*) \in \Gamma_k$ and a $(q_1^*, \dots, q_{\ell}^*) \in \Gamma_{\ell}$ such that $\sum_{i=1}^k h(p_i^*) \neq 0$ and $\sum_{j=1}^{\ell} h(q_j^*) \neq 0$; and consequently

$$(3.19) \quad \sum_{i=1}^k h(p_i^*) \sum_{j=1}^{\ell} h(q_j^*) \neq 0.$$

Now, we prove that $h(1) + (k-1)h(0) \neq 0$. If possible, suppose $h(1) + (k-1)h(0) = 0$. Then (3.10) reduces to the equation

$$\sum_{i=1}^k h(p_i) \sum_{j=1}^{\ell} h(q_j) = 0$$

valid for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_{\ell}) \in \Gamma_{\ell}$. In particular, $\sum_{i=1}^k h(p_i^*) \sum_{j=1}^{\ell} h(q_j^*) = 0$ contradicting (3.19). Hence $h(1) + (k-1)h(0) \neq 0$.

Similarly, making use of (3.11), we can prove that $h(1) + (\ell-1)h(0) \neq 0$.

Let us consider the case when $h(1) + (k-1)h(0) \neq 0$. In this case, let us define a mapping $g : [0, 1] \rightarrow \mathbb{R}$ as

$$(3.20) \quad g(x) = [h(1) + (k-1)h(0)]^{-1}h(x)$$

for all $x \in [0, 1]$. Then, with the aid of (3.20), (3.10) reduces to the functional

equation (1.10). Also, from (3.20), it is easy to see that $g(1) + (k - 1)g(0) = 1$. Consequently, from the discussion, carried out under this case, in the proof of theorem 1, it follows that g is of the form (2.5), subject to the condition (2.6); and (2.7). From equations (2.5), (2.7), (3.9) and (3.20), the solutions (3.3) subject to the condition (3.3a); and (3.4) of functional equation (1.11) follow. The details are omitted for the sake of brevity.

4. The general solutions of functional equation (1.7) when $\lambda \neq 0$

In this section we prove the following :

Theorem 3. *Let $k \geq 3$, $\ell \geq 3$ be fixed integers and $F : I \rightarrow \mathbb{R}$, $H : I \rightarrow \mathbb{R}$, $I = [0, 1]$, be mappings which satisfy the functional equation (1.7) for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_\ell) \in \Gamma_\ell$. Then, any general solution of (1.7) is of the form*

$$(4.1) \quad F(p) = \frac{b(p) + \lambda F(0) - p}{\lambda}, \quad H(p) = \frac{B(p) + \lambda H(0) - p}{\lambda}$$

subject to the condition

$$(4.2) \quad b(1) + \lambda k \ell F(0) = [B(1) + \lambda k H(0)][B(1) + \lambda \ell H(0)]$$

or

$$(4.3) \quad \begin{cases} F(p) = \frac{[\lambda(H(1) + (k-1)H(0)) + 1]^2 a(p) + A^*(p) + \lambda F(0) - p}{\lambda} \\ H(p) = \frac{[\lambda(H(1) + (k-1)H(0)) + 1] a(p) + \lambda H(0) - p}{\lambda} \end{cases}$$

subject to the condition

$$(4.3a) \quad \begin{aligned} & [\lambda(H(1) + (k-1)H(0)) + 1]^2 a(1) + A^*(1) + \lambda k \ell F(0) \\ &= \{[\lambda(H(1) + (k-1)H(0)) + 1] a(1) + \lambda k H(0)\} \\ & \quad \times \{[\lambda(H(1) + (k-1)H(0)) + 1] a(1) + \lambda \ell H(0)\} \end{aligned}$$

or

$$(4.4) \quad \left\{ \begin{array}{l} F(p) = \frac{\left([\lambda(H(1) + (k-1)H(0)) + 1]^2 [M(p) - A(p)] \right.}{\lambda} \\ \left. + A^*(p) + \lambda F(0) - p \right)}{\lambda} \\ H(p) = \frac{[\lambda(H(1) + (k-1)H(0)) + 1][M(p) - A(p)] + \lambda H(0) - p}{\lambda} \\ A(1) = \frac{\lambda \ell H(0)}{[\lambda(H(1) + (k-1)H(0))]}, \\ A^*(1) = \lambda \ell \{ [\lambda(H(1) + (k-1)H(0)) + 1] H(0) - k F(0) \} \end{array} \right.$$

where $A^* : \mathbb{R} \rightarrow \mathbb{R}$, $A : \mathbb{R} \rightarrow \mathbb{R}$, $B : \mathbb{R} \rightarrow \mathbb{R}$, $a : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ are additive functions; $M : [0, 1] \rightarrow \mathbb{R}$ satisfies (2.9), (2.11) and

$$(4.5) \quad M(1) = \frac{\lambda(H(1) + (\ell - 1)H(0)) + 1}{\lambda(H(1) + (k - 1)H(0)) + 1}$$

with $[\lambda(H(1) + (k - 1)H(0)) + 1] \neq 0$ in (4.3), (4.3a), (4.4) and (4.5).

Proof. Let us write (1.7) in the multiplicative form

$$(4.6) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} [\lambda F(p_i q_j) + p_i q_j] = \sum_{i=1}^k [\lambda H(p_i) + p_i] \sum_{j=1}^{\ell} [\lambda H(q_j) + q_j].$$

Define the mappings $f : I \rightarrow \mathbb{R}$, $h : I \rightarrow \mathbb{R}$ as

$$(4.7) \quad f(x) = \lambda F(x) + x, \quad h(x) = \lambda H(x) + x$$

for all $x \in I$. Then, (4.6) reduces to the functional equation (1.11) whose solutions are given by (3.1) subject to the condition (3.2); (3.3) subject to (3.3a); and (3.4) in which $A^* : \mathbb{R} \rightarrow \mathbb{R}$, $A : \mathbb{R} \rightarrow \mathbb{R}$, $B : \mathbb{R} \rightarrow \mathbb{R}$, $a : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ are additive functions; and $M : [0, 1] \rightarrow \mathbb{R}$ is a mapping which satisfies (2.9), (2.11) and (3.5). Now making use of (4.7) and (3.1) subject to the condition (3.2); (3.3) subject to the condition (3.3a); and (3.4); the required solutions (4.1)

subject to the condition (4.2); (4.3) subject to the condition (4.3a) and (4.4) follow. The details are omitted.

5. The general solutions of functional equation (1.7) when $\lambda = 0$

If $\lambda = 0$, then (1.7) reduces to the functional equation

$$(5.1) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k H(p_i) + \sum_{j=1}^{\ell} H(q_j)$$

where $k \geq 3$, $\ell \geq 3$ are fixed integers and $(p_1, \dots, p_k) \in \Gamma_k$, $(q_1, \dots, q_{\ell}) \in \Gamma_{\ell}$.

The substitutions $p_1 = 1$, $p_2 = \dots = p_k = 0$ in (5.1) yield

$$(5.2) \quad \sum_{j=1}^{\ell} [F(q_j) - H(q_j)] = H(1) + (k-1)H(0) - \ell(k-1)F(0).$$

Hence, by Lemma 1,

$$(5.3) \quad F(p) = H(p) + A_1^*(p) - \frac{1}{\ell} A_1^*(1) + \frac{1}{\ell} \{H(1) + (k-1)H(0) - \ell(k-1)F(0)\}$$

where $A_1^* : \mathbb{R} \rightarrow \mathbb{R}$ is additive with

$$(5.4) \quad A_1^*(1) = H(1) + (k + \ell - 1)H(0) - k\ell F(0).$$

From (5.3) and (5.4), we obtain

$$(5.5) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^k \sum_{j=1}^{\ell} H(p_i q_j) + H(1) - (k-1)(\ell-1)H(0).$$

From (5.1) and (5.5), we obtain

$$(5.6) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} H(p_i q_j) = \sum_{i=1}^k H(p_i) + \sum_{j=1}^{\ell} H(q_j) - \{H(1) - (k-1)(\ell-1)H(0)\}.$$

Define $H_1 : [0, 1] \rightarrow \mathbb{R}$ as

$$(5.7) \quad H_1(x) = H(x) - \{H(1) - (k-1)(\ell-1)H(0)\}x$$

for all $x \in [0, 1]$. Then, equation (5.6) reduces to

$$(5.8) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} H_1(p_i q_j) = \sum_{i=1}^k H_1(p_i) + \sum_{j=1}^{\ell} H_1(q_j).$$

Putting $p_1 = q_1 = 1$ and $p_2 = \dots = p_k = q_2 = \dots = q_{\ell} = 0$ in (5.8), we obtain $H_1(1) = (k-1)(\ell-1)H_1(0)$. Define $H_2 : [0, 1] \rightarrow \mathbb{R}$ as

$$(5.9) \quad H_2(x) = H_1(x) - H_1(0) - [H_1(1) - H_1(0)]x$$

for all $x \in [0, 1]$. Then

$$(5.10) \quad \sum_{i=1}^k \sum_{j=1}^{\ell} H_2(p_i q_j) = \sum_{i=1}^k H_2(p_i) + \sum_{j=1}^{\ell} H_2(q_j)$$

where $H_2(1) = H_2(0) = 0$, and $(p_1, \dots, p_k) \in \Gamma_k$, $(q_1, \dots, q_{\ell}) \in \Gamma_{\ell}$, $k \geq 3$, $\ell \geq 3$ fixed integers. Theorem 2 (p-78 in [12]) may now be written as :

Theorem 4. *Let $k \geq 3$, $\ell \geq 3$ be fixed integers. The mapping $H_2 : [0, 1] \rightarrow \mathbb{R}$ with $H_2(1) = 0$, $H_2(0) = 0$, defined in (5.9) is a solution of (5.10) if and only if*

$$(5.11) \quad H_2(p) = \begin{cases} a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ 0 & \text{if } p = 0 \end{cases}$$

where $a : \mathbb{R} \rightarrow \mathbb{R}$ is additive; $D : \mathbb{R} \times]0, 1] \rightarrow \mathbb{R}$ is additive in the first variable and there exists a function $E : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, additive in both variables such that $E(1, 1) = a(1)$ and, moreover,

$$(5.12) \quad D(pq, pq) - D(pq, p) - D(pq, q) = E(p, q) \quad \text{if } 0 < p \leq 1, 0 < q \leq 1.$$

Making use of corollary 3 on p-81 in [12], it follows that

$$(5.13) \quad H_1(p) = \begin{cases} c + c(k\ell - k - \ell)p + a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ c & \text{if } p = 0 \end{cases}$$

where $c = H_1(0)$ is an arbitrary real constant, $a : \mathbb{R} \rightarrow \mathbb{R}$ is additive, $D : \mathbb{R} \times]0, 1] \rightarrow \mathbb{R}$ is as described above in Theorem 4. Now from (5.7) and

(5.13), we obtain

$$(5.14) \quad H(p) = \begin{cases} c(1-p) + d_1 p + a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ c & \text{if } p = 0 \end{cases}$$

where $c = H(0)$ and $d_1 = H(1)$ are arbitrary real constants, $a : \mathbb{R} \rightarrow \mathbb{R}$ is additive function; $D : \mathbb{R} \times]0, 1] \rightarrow \mathbb{R}$ as described above in Theorem 4. Now from (5.3), (5.4) and (5.14), we obtain

$$(5.15) \quad F(p) = \begin{cases} d_0 + d_1 p - c p + a(p) + A_1^*(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ d_0 & \text{if } p = 0 \end{cases}$$

where $c = H(0)$, $d_0 = F(0)$, $d_1 = H(1)$ are arbitrary real constants; $a : \mathbb{R} \rightarrow \mathbb{R}$, $A_1^* : \mathbb{R} \rightarrow \mathbb{R}$ are additive functions with $A_1^*(1)$ given by (5.4); $D : \mathbb{R} \times]0, 1] \rightarrow \mathbb{R}$ as described above in Theorem 4. Thus, we have proved the following:

Theorem 5. *Let $k \geq 3$, $\ell \geq 3$ be fixed integers. The mappings $F : [0, 1] \rightarrow \mathbb{R}$, $H : [0, 1] \rightarrow \mathbb{R}$ satisfy the equation (5.1) if and only if F and H are respectively of the forms (5.15) and (5.14) with $A_1^*(1)$ given by (5.4) and D as described above in Theorem 4.*

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