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HYPER BCK-ALGEBRA

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Abstract: In this note the notion of a hyper BCK-algebra which is a generalization of a BCK-algebra introduced, and some basic properties for this algebra are proved. Then the notions of P-ideals and hyper-congruence relations are introduced, and the relationship between them on a positive implicative hyper BCk-algebra with condition (S), are specified.

1. Introduction and preliminaries

The theory of hypercompositional structures has been introduced by F.Marty, in 1934 [4]. Marty was led to the introduction of this structure from the study of several problems of the noncommutative algebra.

The notion of BCK-algebra first was introduced by, Y. Imai and K.Iseki, in 1966 [1]. This notion is a generalization of properties of the set-difference. Since then many mathematicians have worked in this area and obtained interesting results. For example see [2,3]. Now recall some definitions.

Definition 1.1. Let X be non-empty set with a binary operation "*" and a constant 0. Then X is called a BCK-algebra, if it satisfies the following conditions

(i) $(x * y) * (x * z) \leq z * y$

(ii) $x * (x * y) \leq y$

(iii) $x \leq x$

(iv) $x \leq y$ and $y \leq x$ imply $x = y$

(v) $0 \leq x$ for all $x \in X$

where $x \leq y$ means $x * y = 0$.

Definition 1.2. A non-empty subset I of a BCK-algebra X is said to be an ideal if the following condition holds:

(i) $0 \in I$

(ii) $x * y \in I$, $y \in I$ imply $x \in I$.

Definition 1.3. The non-empty subset A of a BCK-algebra X is called a implicative ideal if:

(i) $0 \in A$

(ii) $(y * x) * z \in A$ and $x * z \in A$ imply that $y * z \in A$.

Definition 1.4. Let $B \neq \emptyset$. A hyperoperation on B is a function from $B \times B$ to $P^*(B) = P(B) \backslash \{\emptyset\}$.

2. Hyper BCK-Algebra

Definition 2.1. Let X be a set with a hyper operation " $*$ " and a constant 0. Then X is called a hyper BCK-algebra, if it satisfies the following Conditions:

$(H1)$ $(x * y) * z = (x * z) * y$,

$(H2)$ $t \in x * (x * y) \Rightarrow t \leq y$,

$(H3)$ $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$,

$(H4)$ $x \leq x$,

$(H5)$ $0 \leq x$,

$(H6)$ $x \leq y, y \leq x \Rightarrow x = y$

for all $x, y, z \in X$ where $x \leq y$ is defined by $0 \in x * y$ and $A \leq B$ is defined by for all $b \in B$, there exists $a \in A$ such that $a \leq b$ for all $x, y \in X$ and $A, B \subseteq X$.

Henceforth $X = (X, *, 0)$ will be denoted a Hyper BCK-algebra.

Example 2.2. Let $X = R^+$ and

$$x * y = \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \\ \{0\} & \text{if } y = 0 \end{cases}$$

Then $(R^+, *, 0)$ is a BCK-algebra.

Notation. By $A \leq x$ and $x \leq A$ we mean $A \leq \{x\}$ and $\{x\} \leq A$, for all $x \in X$ and $A \subseteq X$.

Lemma 2.3. For all $x, y \in X$ and $A \subseteq X$,

(i) $(x * y) * A = (x * A) * y$,

(ii) $x * (x * 0) = \{0\}$.

Proof. The proof is simple.

Theorem 2.4. For all $x, y, z \in X$

(i) $(x * y) * (x * z) \leq z * y$,

(ii) $x \leq y, y \leq z \Rightarrow x \leq z$.

Proof. (i) is proved by Lemma 2.3 (i) , $(H2)$ and $(H3)$.

(ii) The proof follows from $(H3)$, $(H5)$ and $(H6)$.

Theorem 2.5. For all $x, y, z \in X$, and $A, B, C \subseteq X$

(i) $x * y \leq z \Rightarrow x * z \leq y$

(ii) $x * y \leq x$

(iii) $x \leq x * 0$

(iv) $A \leq B, B \leq C \Rightarrow A \leq C$

(v) $A * y \leq A$

(vi) $x * A \leq z \Leftrightarrow x * z \leq A$

(vii) $A \leq B \Rightarrow A * C \leq B * C$ and $C * B \leq C * A$

$(viii)$ $A \leq A * 0$

(ix) $x \in x * 0$

(x) $t \in 0 * 0 \Leftrightarrow t = 0$

(xi) $x * x = \{x\} \Leftrightarrow x = 0$.

Proof. (i) $0 \in t * z \subseteq (x * y) * z = (x * z) * y$ for some $t \in x * y$. Thus there exists $w \in x * z$ such that $0 \in w * y$. In other words $x * z \leq y$.

(ii) $0 \in 0 * y \subseteq (x * x) * y = (x * y) * x$. Then there exists $t \in x * y$ such that $0 \in t * x$. Hence $x * y \leq x$.

(iii) The proof follows from Lemma 2.3 (ii).

(iv) Is proved by Theorem 2.4 (ii).

(v) Let $a \in A$ be an arbitrary element. Then, by (ii), $a * y \leq a$. Hence there is $b \in a * y \leq A * y$ such that $b \leq a$. That is $A * y \leq A$.

(vi) Since $x * A \leq z$, there exists $a \in A$ such that $x * a \leq z$. Hence by (i) $x * z \leq a \leq A$. Thus by (iv) $x * z \leq A$. The proof of converse is simple.

(vii) Let $A \leq B$ and $x \in B * C$. Then $x \in b * c$ for some $b \in B$, $c \in C$. Hence there is $a \in A$ such that $a \leq b$. So, by $(H3)$, $a * c \leq b * c$.

Thus $t \leq x$, for some $t \in a * c \subseteq A * C$. That is $A * C \leq B * C$. Similarly $C * B \leq C * A$.

$(viii)$ The proof follows from (iii).

(ix) Since $x * 0 \leq x$, there is $t \in x * 0$ such that $t \leq x$. Now since $x \leq x * 0$, we have $x \leq t$. Thus $t = x$.

(x) $t \in 0 * 0 \subseteq 0 * (0 * 0)$. Now, by $(H2)$, $t \leq 0$. Hence by, $(H5)$ and $(H6)$, $t = 0$.

(xi) $\{x\} = x * x \subseteq x * (x * 0)$. Hence by $(H5)$ $x \leq 0$. Thus $x = 0$. The proof of converse follows from (x).

3. Hyper relation

Definition 3.1. Let $A \neq \emptyset$, $B \neq \emptyset$. A hyper relation θ on $A \times B$ is a function from $A \times B$ to $P(B)$. If $C, D \in P^*(A)$, then by $\theta(C, D)$ we mean the set $\bigcup_{c \in C, d \in D} \theta(c, d)$. Obviously every relation is a hyper relation.

Definition 3.2. A hyper equivalence relation θ on a non-empty set Y is a hyper relation on $Y \times Y$ satisfying:

(i) $\theta(x, x) = \bigcup_{y, z \in Y} \theta(y, z)$ (reflexive)

(ii) $\theta(x, y) = \theta(y, x)$ (symmetric)

(iii) $\theta(x, z) \bigcap \theta(z, y) \subseteq \theta(x, y)$ (transitive)

for all $x, y, z \in Y$.

Example 3.3. Let $Y = R^+$ and

$$\theta(x, y) = \begin{cases} R^+ & \text{if } x = y \\ [0, min(x, y)] & \text{if } x \neq y \end{cases}$$

Then θ is a hyper equivalence relation.

Definition 3.4. A hyper equivalence relation θ on X is called a hyper congruence relation on X iff

$$\theta(u * x, u * y) \supseteq \theta(x, y), \quad \forall x, y, u \in X. \tag{1}$$

Definition 3.5. Let $\emptyset \neq Y \subseteq X$. Then Y is called subalgebra of X if $y_1 * y_2 \subseteq Y$, for all $y_1, y_2 \in Y$.

Theorem 3.6. If θ is a hyper congruence relation on X and Y is a subalgebra of X, then the subset H of X which is defined as follows:

$$H = \{x \in X : \theta(x, y) = \theta(0, 0), \text{ for some } y \in Y\}$$

is a subalgebra of X.

Proof. The proof is simple.

Theorem 3.7. Let θ be a hyper equivalence relation on X and for all $x_1, x_2, y_1, y_2 \in X$,

$$\theta(x_1 * x_2, y_1 * y_2) \supseteq \theta(x_1, y_1) \cap \theta(x_2, y_2). \qquad (2)$$

Then θ is a hyper congruence relation.

Proof. The proof follows from Definition 3.2 (i).

Notation. Let μ be a function of X to $P(X)$. Then by $\mu(A)$ we mean $\bigcup_{a \in A} \mu(a)$, for all $A \subseteq X$. Clearly

$$\bigcup_{a \in A} \mu(a * B) = \mu(A * B) = \bigcup_{b \in B} \mu(A * b)$$

Definition 3.8. A function μ of X to $P(X)$ is called a P-ideal $(\mu \Delta_p X)$ if

(i) $\mu(0) \supseteq \mu(y)$

(ii) $\mu(x) \supseteq \mu(x * y) \cap \mu(y)$

for all $x, y \in X$.

Definition 3.9. A non-empty subset I of X is said to be a hyper ideal of X. If

(i) $0 \in I$

(ii) $(x * y) \cap I \neq \emptyset$ and $y \in I \Rightarrow x \in I, \forall x, y \in X$.

Clearly each ideal is a hyper ideal.

Example 3.10. Let I be a hyper ideal of X. The function $\mu : X \to P(X)$ which is defined by

$$\mu(x) = \begin{cases} I & x \in I \\ \emptyset & x \notin I \end{cases}$$

is a P-ideal of X.

Definition 3.11. A function μ of X to $P(X)$ is called a P-implicative ideal $(\mu \Delta_{p-i} X)$ if:

(i) $\mu(0) \supseteq \mu(y)$

(ii) $\mu(x * z) \supseteq \mu((x * y) * z) \cap \mu(y * z)$

for all $x, y, z \in X$.

Definition 3.12. If in X, the equality

$$(x * z) * (y * z) = (x * y) * z, \ \forall x, y, z \in X$$

holds, then it is said to be positive implicative.

Definition 3.13. A non-empty subset I of X is called a hyper implicative ideal of X if for any $x, y, z \in X$

(*i*) $0 \in I$

(*ii*) $((x * y) * z) \cap I \neq \emptyset, (y * z) \cap I \neq \emptyset \Rightarrow (x * z) \cap I \neq \emptyset$.

Clearly each implicative ideal is a hyper implicative ideal.

Example 3.14. Let I be a hyper implicative ideal. Then the function which is defined in Example 3.10 is a P-implicative ideal.

Lemma 3.15. Let μ be a P-ideal of X. Then for all $\emptyset \neq A, B \subseteq X, x \in X$

(*i*) $a \leq b \Rightarrow \mu(a) \supseteq \mu(b)$

(*ii*) $A \leq B \Rightarrow \mu(A) \supseteq \mu(B)$

(*iii*) $\mu(x * 0) = \mu(x)$

Proof. (*i*) $\mu(a) \supseteq \mu(a * b) \cap \mu(b) = \mu(0) \cap \mu(b) = \mu(b)$

(*ii*) Let b be an arbitrary element of B. Then there exist $a \in A$ such that $a \leq b$. Hence by (*i*), $\mu(A) \supseteq \mu(a) \supseteq \mu(b)$ for all $b \in B$. Thus $\mu(A) \supseteq \mu(B)$.

(*iii*) By Theorem 2.5, we have $x \leq x*0 \leq x$. Hence $\mu(x) \supseteq \mu(x*0) \supseteq \mu(x)$.

Theorem 3.16. (*i*) If $\mu\Delta_{p-i}X$, then $\mu\Delta_p X$

(*ii*) Let X be positive implicative and $\mu\Delta_p X$, then $\mu\Delta_{p-i}X$.

Proof. (*i*) The proof follows from Lemma 3.15 (*iii*).

(*ii*) Let $x, y, z \in X$. Then by Lemma 3.15 (*ii*) we have

$$\begin{aligned} \mu(x * z) &\supseteq \mu(t) \supseteq \mu(t * w) \cap \mu(w), \ \forall t \in x * z, \ \forall w \in y * z \\ &\supseteq \mu((x * z) * (y * z)) \cap \mu(y * z). \end{aligned}$$

Lemma 3.17. Let X be positive implicative then

$$x * y \leq (x * y) * y, \quad \forall x, y \in X.$$

Proof. For all $x, y \in X$, we have

$$(x * y) * (y * y) = (x * y) * y.$$

Now since $y * y \leq 0$, by Theorem 2.5 (viii), (vii), we get that

$$x * y \leq (x * y) * 0 \leq (x * y) * (y * y).$$

Definition 3.18. Let y, z be fixed elements of X. Consider

$$(x * y) \leq z \tag{3}$$

We suppose that there is a greatest element x satisfying (3). Then X is said satisfy the condition (S). In this case x is denoted by yoz.

Notation. Let $\emptyset \neq A \subseteq X$. Then by xoA and Aox we mean the sets $\{xoa : a \in A\}$ and $\{aox : a \in A\}$ respectively.

Lemma 3.19. Let X be a hyper BCK-algebra with the condition (S). Then for all $x,y \in X$, $\emptyset \neq A \subseteq X$, $(xoy) * A \leq y$ implies that $xoy \leq Aoy$.

Proof. Let $(xoy) * A \leq y$. Then from Theorem 2.5 (vi) we have $(xoy) * y \leq A$. Now let $t \in A$ be arbitrary. Then there is $w \in (xoy) * y$ such that $w \leq t$. In other words $(xoy) * y \leq t$. Hence, by Theorem 2.5 (i) $(xoy) * t \leq y$. Thus $xoy \leq toy$, for all $t \in A$. Consequently $xoy \leq Aoy$.

Definition 3.20. Let θ be a hyper relation on X. Then the function μ, from X to $P(X)$ which is defined by $\mu(x) = \theta(x, 0)$ is called the P-function induced by θ and it is denoted by $I(\theta)$.

Lemma 3.21. Let θ be a hyper congruence relation on X. Then

(i) $x_1 \leq x_2 \Rightarrow I(\theta)(x_1) \supseteq I(\theta)(x_2)$

(ii) $A \leq B \Rightarrow I(\theta)(A) \supseteq I(\theta)(B)$

Proof.

(i) Let $x_1 \leq x_2$. Then $0 \in x_1 * x_2$. Thus

$$\begin{aligned} I(\theta)(x_1) &= \theta(x_1, 0) \supseteq \theta(x_1 * 0, x_1 * x_2) && \text{, by Theorem 2.5}(ix) \\ &= \theta(0, x_2) && \text{, by Definition 3.7} \\ &= \theta(x_2, 0) && \text{, since } \theta \text{ is symmetric} \\ &= I(\theta)(x_2) \end{aligned}$$

(ii) Let $b \in B$ be an arbitrary element. Then there is $a \in A$ such that $a \leq b$. So

$$\theta(A, 0) \supseteq \theta(a, 0) \supseteq \theta(b, 0), \quad \forall b \in B.$$

Definition 3.22. Let θ be a hyper equivalence relation on hyper BCK-algebra with the condition (S). Then θ is called O-congruence relation on X if

$$\theta(x_1oy_1, x_2oy_2) \supseteq \theta(x_1, x_2) \cap \theta(y_1, y_2)$$

If θ is a hyper congruence relation and O-congruence, then it is called a hyper O-congruence relation.

Lemma 3.23. If X is with the condition (S), then $0o0 = 0$.

Proof. By Theorem 2.5 (iii), $0o0 \leq 0o0 * 0 \leq 0$, therefore, $0o0 = 0$.

Lemma 3.24. Let θ be an O-congruence relation on a hyper BCK-algebra X with the condition (S), then

$$\theta(Aoy, 0) \supseteq \theta(A, 0) \cap \theta(y, 0) \tag{4}$$

Proof. Let $a \in A$, be an arbitrary element. Then

$$\theta(Aoy, \theta) \supseteq \theta(aoy, 0) = \theta(aoy, 0o0) \supseteq \theta(a, 0) \cap \theta(y, 0), \ \forall a \in A$$

Hence, (4) holds.

Theorem 3.25. Let X be a positive implicative hyper BCK-algebra with the condition (S) and let θ be a hyper O-congruence relation on X. Then $I(\theta)\Delta_p X$.

Proof. Let $\mu = I(\theta)$. For all $x \in X$ we have

$$\mu(0) = \theta(0,0) = \bigcup_{x,y\in X} \theta(x,y) \supseteq \theta(x,0) = \mu(x).$$

Thus condition (i) of Definition 3.8 holds. Now, let $x, y \in X$. Since X is with the condition (S), $(xoy) * x \leq y$. Hence by Theorem 2.5 (i), $(xoy) * y \leq x$, which implies that $((xoy) * y) * y \leq x * y$ by Theorem 2.5 (vii). Hence by Lemma 3.17, $(xoy) * y \leq ((xoy) * y) * y \leq x * y$. Consequently $(xoy) * (x * y) \leq y$ by Theorem 2.5 (vi). So from Lemma 3.19 we conclude that

$$xoy \leq (x * y)oy, \quad \forall x, y \in X \tag{5}$$

From (5) and Lemma 3.21 (iii) we obtain that

$$\theta(xoy, 0) \supseteq \theta((x * y)oy, 0).$$

By Lemma 3.24 we have

$$\theta(xoy, 0) \supseteq \theta(x * y, 0) \cap \theta(y, 0). \tag{6}$$

Now since $0 \in x * x$, we conclude that $x * x \leq y$. Therefore $x \leq xoy$, by the condition (S). Hence

$$\theta(x,0) \supseteq \theta(xoy,0),\ \forall x, y \in X, \quad \text{by Lemma } 3.21(ii)$$

Thus (6) implies that

$$\theta(x,0) \supseteq \theta(x * y, 0) \cap \theta(y, 0)$$

In other words

$$\mu(x) \supseteq \mu(x * y) \cap \mu(y).$$

Consequently, Condition (ii) of Definition 3.8 holds.

Definition 3.26. Let $\mu : X \to P(X)$ be a function. Then for all $A \subseteq X$, by μ_A we mean the set

$$\mu_A = \{x \in X : \mu(x) \supseteq A\}.$$

Theorem 3.27. Let μ be a function from X to $P(X)$. Then (*i*) If μ is a P-ideal, $A \subseteq \bigcup_{x \in X} \mu(x)$ and $\mu_A \neq \emptyset$ for all $A \subseteq X$, then μ_A is a hyper ideal of X.

(*ii*) If μ_A is a hyperideal of X, for all $A \subseteq X$, then μ is a P-ideal of X.

Proof. (*i*) Since $A \subseteq \bigcup_{x \in X} \mu(x) = \mu(0)$, so $0 \in \mu_A$.

On the other hand, let $x * y \cap \mu_A \neq \emptyset$ and $y \in \mu_A$. Then there exists $t \in x * y$ such that $\mu(t) \supseteq A$. Thus we get

$$\mu(x * y) \supseteq \mu(t) \supseteq A, \mu(y) \supseteq A.$$

Hence $\mu(x) \supseteq \mu(x * y) \cap \mu(y) \supseteq A$ which implies that $x \in \mu_A$. In other words μ_A is a hyper ideal of X.

(*ii*) Let $x \in X$ be an arbitrary element. Then we put $A = \mu(x)$. Thus by hypothesis μ_A is a hyper ideal of X, so $0 \in \mu_A$. That is $\mu(0) \supseteq \mu(x)$, for all $x \in X$. On the other hand, let $x, y \in X$ and $t \in x*y$, be arbitrary. We put $\mu(t) = A$, $\mu(y) = B$ and $A \cap B = C$. Therefore μ_C is a hyper ideal, which implies that $x \in \mu_C$. That is $\mu(x) \supseteq C = \mu(t) \cap \mu(y)$ for all $t \in x * y$. Consequently

$$\mu(x) \supseteq \bigcup_{t \in x*y} (\mu(t) \cap \mu(y)) = (\bigcup_{t \in x*y} (\mu(t)) \cap \mu(y) = \mu(x * y) \cap \mu(y).$$

Thus μ is a P-ideal.

Theorem 3.28. Let $\emptyset \neq A \subseteq X$. We define the function $\varphi_A : X \to P(X)$ by

$$\varphi_A(x) = \begin{cases} A & x \in A \\ \emptyset & x \notin A \end{cases}$$

Then A is a hyper ideal of X if and only if φ_A is a P-ideal of X.

Proof. Let φ_A be a P-ideal and $x \in A$. Since $\varphi_A(0) \supseteq \varphi_A(x) = A$, then we get that $0 \in A$. Now let $(x * y) \cap A \neq \emptyset$ and $y \in A$. Then there exists $t \in x * y$ such that $t \in A$. On the other hand we have

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y) \supseteq \varphi_A(t) \cap \varphi_A(y) = A.$$

Hence $x \in A$. In other words A is a hyper ideal of X.

Conversely, let A be a hyper ideal. We shall that $\varphi_A \Delta_p X$. Since $0 \in A$, we obtain that $\varphi_A(0) \supseteq \varphi_A(x)$, for all $x \in X$. Now let $x, y \in X$. If $(x * y) \cap A = \emptyset$ or $y \notin A$, then obviously

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y).$$

Now, if $(x * y) \cap A \neq \emptyset$ and $y \in A$, then $x \in A$. Hence

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y).$$

That is $\varphi_A \Delta_p X$.

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