

WEAK HYPERSTRUCTURES ON SMALL SETS¹

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Abstract The class of the weak hyperstructures is larger than the known ones originated from the hypergroup in the sense of Marty. The weak hyperstructures are also called H_V -structures. In this paper we deal with some classes of H_V -groups defined on sets with three elements.

H_V -STRUCTURES ON SMALL SETS

A hyperstructure is called H_V -structure if it satisfies the structure-like axioms where the related axioms are replaced by the weak ones, i.e. the equality is replaced by the non-empty intersection. For example the **weak associativity** in a H_V -group (H, \bullet) is

$$(xy)z \cap x(yz) \neq \emptyset \quad \text{for all } x, y, z \in H$$

and the **weak commutativity** is

$$xy \cap yx \neq \emptyset \quad \text{for all } x, y \in H.$$

Similarly, in a H_V -ring $(H, +, \bullet)$ the left weak distributivity is

$$x(y+z) \cap (xy+xz) \neq \emptyset \quad \text{for all } x, y, z \in H.$$

The main tool to study the H_V -structures remains as in all hyperstructures, the fundamental relation. That is, the smallest equivalence relation such that the quotient would be the corresponding structure. So the quotient of a H_V -

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group by the fundamental relation β^* is a group, the smallest one. Similarly, the quotient of a H_V -ring by the fundamental relation β^* is a ring [2] and so on. According the above one can define the H_V -semigroup, the H_V -quasigroup and so on.

The H_V -structures have been, introduced in [6] and properties as well as basic facts on the topic can be found in the book [4] and for the fundamental relation of hyperstructures in [1].

In the paper [5] the set of all H_V -groups with two elements is given. Moreover the set of H_V -groups with three elements is calculated under the assumption that they contain a scalar unit element. In this paper we determine some more general cases in a set with three elements.

Let (H, \bullet) , $(H, *)$ be two H_V -groups, defined on the same set H . The hyperoperation (\bullet) is called *less or equal* than $(*)$, if and only if there exists a $f \in \text{Aut}(H, *)$ such that

$$xy \subseteq f(x*y) \quad \text{for all } x, y \text{ in } H.$$

Then $(*)$ is called greater than (\bullet) and we say that $(H, *)$ contains (H, \bullet) . A H_V -group is called *minimal* if contains no other H_V -group defined in the same set. A basic consequence of the above definition is the following.

THEOREM 1

Greater hyperoperation from the one of a given H_V -group defines a H_V -group. The weak commutativity is also valid for every greater hyperoperation.

In our study we need to recall the following properties.

PROPERTIES 2

Let (\bullet) is a hyperoperation in the set H , then

- (i) If (\bullet) is weak commutative then

$$a(ba) \cap (ab)a \neq \emptyset \quad \text{for all } a, b \in H.$$

- (ii) If (\bullet) is commutative then for all $a, b \in H$,

$$a(ab) \cap (aa)b \neq \emptyset \quad \text{implies} \quad b(aa) \cap (ba)a \neq \emptyset.$$

- (iii) If e is a unit of the hypergroupoid (H, \bullet) we have

$$(xy)z \cap x(yz) \neq \emptyset \quad \text{for all } x, y, z \in H \text{ such that } e \in \{x, y, z\}.$$

Finally we recall the following theorem from [5], [4].

THEOREM 3

Let $H = \{e, a, b\}$ be a set with three elements. Suppose we want to define in H a hyperoperation (\bullet) for which the element e is a scalar unit. Denote by a four (aa, ab, ba, bb) the products needed to be defined. Then, the set of all H_v -groups defined in H has a set of minimal the following ones, given with their isomorphic:

- $M_1 \quad (e, b, b, \{e, a\}) \cong (\{e, b\}, a, a, e)$
- $M_2 \quad (e, \{e, b\}, \{e, b\}, \{a, b\}) \cong (\{a, b\}, \{e, a\}, \{e, a\}, e)$
- $M_3 \quad (e, H, H, a) \cong (b, H, H, e)$
- $M_4 \quad (e, \{a, b\}, \{a, b\}, e)$
- $M_5 \quad (e, H, H, b) \cong (a, H, H, e)$
- $M_6 \quad (a, \{e, b\}, \{e, b\}, a) \cong (b, \{e, a\}, \{e, a\}, b)$
- $M_7 \quad (a, H, H, b)$
- $M_8 \quad (b, e, e, a)$
- $M_9 \quad (\{e, b\}, \{e, a\}, b, H) \cong (H, a, \{e, b\}, \{e, a\})$
- $M_{10} \quad (\{e, b\}, b, \{e, a\}, H) \cong (H, \{e, b\}, a, \{e, a\})$

In the above theorem we observe that if

$$ee = \{e\} \text{ or } \{e, a\} \text{ or } \{e, b\} \text{ or } H$$

then we need the above fours because the reproductivity must be valid. In other words in these cases the minimal H_v -groups remain the same. Therefore we proved the following.

THEOREM 4

Suppose we want to define in $H = \{e, a, b\}$ a hyperoperation (\bullet) for which

$$\begin{aligned} ex = xe = x \quad \text{for all } x \neq e \quad \text{and} \\ ee = \{e\} \text{ or } \{e, a\} \text{ or } \{e, b\} \text{ or } H \end{aligned}$$

such that (H, \bullet) is H_v -group. Then, we must have greater or equal four (aa, ab, ba, bb) than the ones $M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}$ given in Theorem 3.

THEOREM 5

Suppose we want to define in $H = \{e, a, b\}$ a hyperoperation (\bullet) for which the element e is a two-sided unit and $x^2 = H$, for $x \in \{a, b\}$, such that (H, \bullet) is H_v -group. Denote by the ordered 7-tuples

$$(ee, ea, ae, eb, be, ab, ba)$$

the products needed to be defined. Then, the set of all H_v -groups defined in H has a set of minimals the following ones, given with their isomorphic:

$$\begin{aligned} \Sigma_1 & (e, a, a, b, b, e, e) \\ \Sigma_2 & (e, a, a, b, b, a, a) \cong (e, a, a, b, b, b, b) \\ \Sigma_3 & (e, a, a, b, b, a, b) \\ \Sigma_4 & (e, a, a, b, b, b, a) \\ \Sigma_5 & (e, a, a, b, b, e, \{a,b\}) \cong (e, a, a, b, b, \{a,b\}, e) \\ \Sigma_6 & (e, a, a, b, \{b,e\}, e, a) \cong (e, a, \{a,e\}, b, b, b, e) \\ \Sigma_7 & (e, \{a,e\}, a, b, b, e, b) \cong (e, a, a, \{b,e\}, b, a, e) \end{aligned}$$

Proof

First observe that the reproduction axiom is always valid. To find these 7-tuples we worked in the following procedure. Since we have only to check the weak associativity we start with the smallest possible cases in the 7-tuples and we checked the weak associativity.

In case that the weak associativity is not valid we enlarged the products up to the sets in order the weak associativity to be valid. In this case there is the possibility the obtained hyperoperation is not minimal. Therefore we checked smaller hyperoperations up to the sets the weak associativity is not valid. However, in our special case the last step is not needed since the reproductivity is always valid.

Special case : in the case that e is a scalar unit then the set of minimals is given above $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$.

THEOREM 6

Suppose we want to define in $H = \{e,a,b\}$ a hyperoperation (\bullet) for which e is unit and a is scalar, such that (H, \bullet) is H_v -group. Then, we have two cases

$$(i) \quad aa = b \quad \text{and} \quad (ii) \quad aa = e$$

In the case (i) the only one minimal H_v -group is the group defined in H . In the case (ii) the four (ee, eb, be, bb) is the one needed to be defined. Then the set of minimals is the following:

$$\begin{aligned} F_1 & (e, H, H, b) \\ F_2 & (e, b, b, \{e,a\}) \\ F_3 & (e, \{a,b\}, \{a,b\}, \{e,b\}) \\ F_4 & (e, \{e,b\}, \{e,b\}, \{a,b\}). \end{aligned}$$

Proof

(i) We observe that

$$a(ab) = ae = a \quad \text{and} \quad (aa)b = bb$$

therefore $a \in bb$. So, in this case we obtain the group as the minimal.

(ii) We follow the method given in Theorem 5 where we have always to check the reproduction axiom as well.

We conclude our study with the results for two general cases. In the proofs we followed the method described above however one can see that these cases are more laborious. The first one theorem can be considered as the Boolean type H_v -groups defined in set with three elements.

THEOREM 7

Suppose we want to define in $H = \{e,a,b\}$ a H_v -group (H, \bullet) for which e is unit and the relation $x^2 = x, \forall x \in H$, is valid. Denote by the ordered 6-tuples

$$(ea, ae, eb, be, ab, ba)$$

the products needed to be defined. Then the set of all H_v -groups defined in H has a set of minimal the following ones given with their isomorphic.

- S₁ $(H, H, \{e,b\}, \{e,b\}, a, a) \cong (\{e,a\}, \{e,a\}, H, H, b, b)$
- S₂ $(\{e,a\}, H, \{e,b\}, H, a, b)$
- S₃ $(H, H, \{e,b\}, H, a, e) \cong (\{e,a\}, H, H, H, e, b)$
- S₄ $(\{a,b\}, H, \{e,b\}, b, a, \{e,a\}) \cong (\{e,a\}, a, \{a,b\}, H, \{e,b\}, b)$
- S₅ $(a, H, \{e,b\}, \{a,b\}, a, \{e,b\}) \cong (\{e,a\}, \{a,b\}, b, H, \{e,a\}, b)$
- S₆ $(\{e,a\}, H, \{e,b\}, \{e,b\}, a, \{a,b\}) \cong (\{e,a\}, \{e,a\}, \{e,b\}, H, \{a,b\}, b)$
- S₇ $(a, H, \{e,b\}, b, a, H) \cong (\{e,a\}, a, b, H, H, b)$
- S₈ $(H, \{e,a\}, H, \{e,b\}, b, a)$
- S₉ $(H, \{e,a\}, H, H, b, e) \cong (H, H, H, \{e,b\}, e, a)$
- S₁₀ $(\{a,b\}, \{e,a\}, H, b, b, \{e,a\}) \cong (H, a, \{a,b\}, \{e,b\}, \{e,b\}, a)$
- S₁₁ $(a, \{e,a\}, H, \{a,b\}, b, \{e,b\}) \cong (H, \{a,b\}, b, \{e,b\}, \{e,a\}, a)$
- S₁₂ $(\{e,a\}, \{e,a\}, H, \{e,b\}, b, \{a,b\}) \cong (H, \{e,a\}, \{e,b\}, \{e,b\}, \{a,b\}, a)$
- S₁₃ $(a, \{e,a\}, H, b, b, H) \cong (H, a, b, \{e,b\}, H, a)$
- S₁₄ (H, H, H, H, e, e)
- S₁₅ $(\{a,b\}, H, H, b, e, \{e,a\}) \cong (H, a, \{a,b\}, H, \{e,b\}, e)$
- S₁₆ $(a, H, H, \{a,b\}, e, \{e,b\}) \cong (H, \{a,b\}, b, H, \{e,a\}, a)$
- S₁₇ $(\{e,a\}, H, H, \{e,b\}, e, \{a,b\}) \cong (H, \{e,a\}, \{e,b\}, H, \{a,b\}, e)$
- S₁₈ $(a, H, H, b, e, H) \cong (H, a, b, H, H, e)$
- S₁₉ $(\{a,b\}, \{a,b\}, b, b, \{e,a\}, \{e,a\}) \cong (a, a, \{a,b\}, \{a,b\}, \{e,b\}, \{e,b\})$
- S₂₀ $(a, \{a,b\}, b, \{a,b\}, \{e,a\}, \{e,b\})$

- $S_{21} \quad (\{e,a\},\{a,b\},b,\{e,b\},\{e,a\},\{a,b\}) \cong (a,\{e,a\},\{e,b\},\{a,b\},\{a,b\},\{e,b\})$
 $S_{22} \quad (a,\{a,b\},b,b,\{e,a\},H) \cong (a,a,b,\{a,b\},H,\{e,b\})$
 $S_{23} \quad (\{a,b\},a,\{a,b\},b,\{e,b\},\{e,a\})$
 $S_{24} \quad (\{e,a\},a,\{a,b\},\{e,b\},\{e,b\},\{a,b\}) \cong (\{a,b\},\{e,a\},\{e,b\},b,\{a,b\},\{e,a\})$
 $S_{25} \quad (a,a,\{a,b\},b,\{e,b\},H) \cong (\{a,b\},a,b,b,H,\{e,a\})$
 $S_{26} \quad (\{e,a\},\{e,a\},\{e,b\},\{e,b\},\{a,b\},\{a,b\})$
 $S_{27} \quad (a,\{e,a\},\{e,b\},b,\{a,b\},H) \cong (\{e,a\},a,b,\{e,b\},H,\{a,b\})$
 $S_{28} \quad (a,a,b,b,H,H)$

THEOREM 8

Suppose in $H = \{e,a,b\}$ we want to define a H_v -group (H, \bullet) for which e is unit and xy is a singleton for all $x,y \in \{a,b\}$. Denote by the ordered 9-tuples

$$(ee, ea, eb, ae, aa, ab, be, ba, bb)$$

the products needed to be defined. Then the set of all H_v -groups defined in H has a set of minimal the following ones given with their isomorphic.

- $I_1 \quad (e,H,\{e,b\},H,a,a,\{e,b\},a,a) \cong (e,\{e,a\},H,\{e,a\},b,b,H,b,b)$
 $I_2 \quad (e,H,\{e,b\},H,a,a,\{e,b\},a,b) \cong (e,\{e,a\},H,\{e,a\},a,b,H,b,b)$
 $I_3 \quad (e,\{e,a\},\{e,b\},H,a,a,H,b,b)$
 $I_4 \quad (e,H,H,\{e,a\},a,b,\{e,b\},a,b)$
 $I_5 \quad (e,\{e,a\},\{e,b\},\{e,a\},a,b,\{e,b\},b,a) \cong (e,\{e,a\},\{e,b\},\{e,a\},b,a,\{e,b\},a,b)$
 $I_6 \quad (e,\{a,b\},\{e,b\},\{a,b\},e,a,\{e,b\},a,b) \cong (e,\{e,a\},\{a,b\},\{e,a\},a,b,\{a,b\},b,e)$
 $I_7 \quad (e,a,H,a,e,b,H,b,b) \cong (e,H,b,H,a,a,b,a,e)$
 $I_8 \quad (e,H,H,H,a,e,\{e,b\},a,b) \cong (e,H,H,\{e,a\},a,b,H,e,b)$
 $I_9 \quad (e,\{e,a\},H,H,a,e,H,b,b) \cong (e,H,\{e,b\},H,a,a,H,e,b)$
 $I_{10} \quad (e,H,H,\{a,b\},e,e,\{e,b\},a,b) \cong (e,H,H,\{e,a\},a,b,\{a,b\},e,e)$
 $I_{11} \quad (e,\{e,a\},H,\{a,b\},e,e,H,b,b) \cong (e,H,\{e,b\},H,a,a,\{a,b\},e,e)$
 $I_{12} \quad (e,\{a,b\},\{e,b\},H,e,a,H,e,b) \cong (e,\{e,a\},\{a,b\},H,a,e,H,b,e)$
 $I_{13} \quad (e,\{a,b\},H,\{e,a\},e,b,H,e,b) \cong (e,H,\{a,b\},H,a,e,\{e,b\},a,e)$
 $I_{14} \quad (e,\{a,b\},\{e,b\},\{a,b\},e,a,\{e,b\},a,e) \cong (e,\{e,a\},\{a,b\},\{e,a\},e,b,\{a,b\},b,e)$
 $I_{15} \quad (e,H,\{a,b\},H,a,e,\{a,b\},e,a) \cong (e,\{a,b\},H,\{a,b\},b,e,H,e,b)$
 $I_{16} \quad (e,H,H,H,a,e,H,e,b) \cong (e,H,H,H,a,e,H,e,b)$
 $I_{17} \quad (e,a,b,a,b,e,b,e,a)$
 $I_{18} \quad (e,\{a,b\},\{e,b\},\{a,b\},e,e,\{e,b\},e,a) \cong (e,\{e,a\},\{a,b\},\{e,a\},b,e,\{a,b\},e,e)$
 $I_{19} \quad (e,\{a,b\},H,\{a,b\},e,e,H,e,b) \cong (e,H,\{a,b\},H,a,e,\{a,b\},e,e)$
 $I_{20} \quad (e,H,\{a,b\},\{a,b\},e,e,\{e,b\},a,e) \cong (e,\{a,b\},H,\{e,a\},e,b,\{a,b\},e,e)$

$$\begin{aligned}
 I_{21} & (e, \{e, a\}, \{a, b\}, \{a, b\}, e, e, H, b, e) \cong (e, \{a, b\}, \{e, b\}, H, e, a, \{a, b\}, e, e) \\
 I_{22} & (e, \{a, b\}, \{a, b\}, \{a, b\}, e, e, \{a, b\}, e, e)
 \end{aligned}$$

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