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HYPERSTRUCTURES AS TOOL TO COMPARE URBAN PROJECTS[1]

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ABSTRACT - The objective of this work is the application of theories of fuzzy sets and hyperstructures to the problem of selection among alternative urban projects, so to identify those ones pursue a predetermined set of heterogeneous objectives in the best way.

The recourse to fuzzy sets depends on the consciousness that all the planning stages are in condition of uncertainty. So it is necessary to adopt techniques that permit to study the uncertainty about the influence that a decision could have on the validity of the project. The hyperstructures, instead, in this case represent a way to define some rules to choose.

1. The Evaluation in the prospect of a Sustainable Urban Growth

The complex problem of redevelopment planning has consolidated a major consciousness of urgency to adopt suitable evaluation procedures for the projects, so to render "*equitable*" and "*transparent*" the decision process. Besides, we think that it is possible to combine quality and feasibility of architecture and environment through the comparison among different projects.

The quality of places, the respect and exploitation of natural landscapes, the protection of the consolidated urban systems, the preservation of cultural resources are now the hearth of every experience of town planning.

The public interventions in restoration and redevelopment plans have to aim at improving the arrangement of the territory through projects, which tend towards "*integrated maintenance*" of natural and cultural resources. In

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fact we think that an urban development will be achieved not only through economical and social factors, but also through cultural elements, recovering the historical, environmental, architectural values of a place.

For this reason with "*sustainable urban development*" we mean a balanced urban growth, as the result of an urban policy that aims at improving the quality of urban scenery to obtain positive economic effects.

Then the integrated maintenance of physical resources put a set of heterogeneous objectives. In order to achieve these objectives it is necessary to arrange suitable procedures of evaluation about the effects produced by an urban project, in such a way as to analyse all the aspects involved in the urban planning.

In the last National Conference of Architects (Florence, March 1997) the social role of the Architect has been specified: he has to operate on territory, exploiting and watching over the available resources and examining the entire question about the urban transformations.

Public works in urban questions can't be seen only in an economical point of view. Really, we have to examine various impacts of an urban project (environmental, cultural, social, technical, psychological...) to determine the feasibility and to optimise the pursuit of a predetermined set of various objectives.

In other words, the aim of the evaluation will be to define the measure in which an urban project increases the general standard of welfare for the involved community, considering the complexity of the consequences of a redevelopment project on the contest of intervention.

For this reason we think that an accurate evaluation of projects concerning the exploitation of existent resources could be made considering opportunely also the symbolic esthetical-cultural characters of the interested area, getting over the traditional valuing procedures which consider only the monetary aspects.

To introduce also the qualitative aspects into a valuing procedure it is necessary to refer to "*multidimensional techniques*".

In presence of various objectives these procedures represent a rational answer to problems of decision making among a series of alternatives, improving the traditional economical analysis of project whether to estimate the not monetary benefits or to consider the multiplicity of social groups involved in an urban planning.

In this viewpoint we think that the "*global feasibility*" of an urban project could be valued examining some general objectives, within which defining a set of criteria to achieve by carrying out the project.

2. The mathematical model

The problem about the examination of a set of alternative solutions is put to verify the feasibility of each project and to single out the best proposals. It is a question of valuing all the projects according to some parameters.

In order to deal with this problem of evaluation we consider the following mathematical model:

a) We determine a set $F=\{F_1, F_2, ..., F_m\}$, called *set of feasibilities*, whose elements are the objectives that we think any projects must attain.

b) We select a set $C=\{C_1, C_2,..., C_n\}$, called *set of criteria*, whose elements represent the ways to measure the degree in which each project realises each objective.

c) We consider a function W: $\Omega=F\times C\rightarrow[0,1]$, called *weight function*, that to any $\omega_{ij}=(F_i, C_j)\in\Omega$ associates a number w_{ij} that indicate the weight of ω_{ij} in the model. We suppose that the sum of all the w_{ij} is 1. We denote with W also the matrix, called *weight matrix* with elements the w_{ij}. The elements φ_i of the marginal column of W are the weights of the feasibilities and the elements γ_j of the marginal row are the weights of the criteria. In practice it is convenient to consider m functions K_i: $C\rightarrow[0,1]$, called *conditional weight functions*, such that, for any i, the sum of the $K_i(C_j)$, j=1,…,n, is 1. The matrix K with elements $k_{ij}=K_i(C_j)$ is called *conditional weight matrix*. If $\Phi=\text{diag}(\varphi_1, \varphi_2,\ldots,\varphi_m)$ is the diagonal matrix with elements the feasibility weights, we have $W=\Phi K$.

d) For any project P we assign a function E^P: $\Omega=F\times C\rightarrow[0,1]$, called *efficaciousness function of* P, such that to any $\omega_{ij}=(F_i, C_j)\in\Omega$ associates a number e_{ij}^P that indicates the measure in which P satisfies the pair (F_i, C_j).

e) The matrix R^P with generic element $r_{ij}^P=k_{ij}\, e_{ij}^P$ is called *relative utility matrix of* P and the matrix U^P with generic element $u_{ij}^P=w_{ij}\, e_{ij}^P=r_{ij}^P\varphi_i$ is the *utility matrix* or *representative matrix of* P.

In this paper we consider as feasibilities for an urban project the following aspects:

environmental: evaluation of project about the effects on natural and built ambient;

aesthetic-cultural: evaluation of the project about the effects on the historical, artistic and archaeological interests;

economical: evaluation of the project to verify the increase of general welfare in the considered area;

financial: evaluation of the project to control the measure of costs-return ratio;

technical: evaluation of the project about the effects on constraints of construction and regulation;

social: evaluation of the project about the effects on the persons directly and indirectly involved by the redevelopment and the improvement on functional characters of the site;

procedural: evaluation of project about the administrative-procedural aspects, in particular with regard to the conditions of agreements and the modality of entrusting of works.

Besides, for an urban project some of the selected criteria are:

- reduction of the urban decay, through the urbanistic and functional rejoining of vacant sites or urban areas without identity;
- compatibility of project with the morphology and with the geologic conditions of the site;
- offering of new possibility to enjoy the landscape;
- preservation of artistic-historical-architectural values, through the protection of typological, structural and formal values that are peculiar of buildings which have created the historic ancient environment;
- consistent use of colours and materials as regard to the conformation of the urban environment;
- compatibility of the project with the rules of the urbanistic instrumentation;
- improvement in the equipment of social services;
- compatibility of the project with the local and state rules so to simplify the administrative procedures.

Any criterion C_j may be valid only for some feasibilities. In this case we put $w_{ij}=0$ for any F_i that not consider the criterion C_j.

The elements of all the matrices considered are numbers belonging to the interval [0,1]. So any matrix can be seen as fuzzy set with universe set Ω. The matrix K can be seen also as a *fuzzy partition*, if we consider its columns as fuzzy sets with universe the set of feasibilities.

For each $H \subseteq \Omega$, $H \neq \emptyset$, we denote with U_H^P the part of U^P constituted by the elements u_{ij}^P such that $(F_i, C_j) \in H$. U_H^P is also a fuzzy set with the universe set H.

We have a particular case of the treated theory if we consider the matrices of incidence as matrices associated to projects. That is, for each pair (F_i, C_j) and for each project P, we put $u_{ij}^P=1$ if the pair (F_i, C_j) is satisfied by the project P, and $u_{ij}^P=0$ if the pair is not satisfied. Then the matrix U^P of the project P can be interpreted as a subset of Ω, that is the set of the $(F_i, C_j) \in \Omega$ such that $u_{ij}^P=1$ and $U_H^P = U^P \cap H$.

Denote with $\wp$ the family of projects and with $\Im$ the family of fuzzy sets on Ω. Let U be the function that associates $U^P \in \Im$ to each $P \in \wp$.

Generally, for each $H \subseteq \Omega$, $H \neq \emptyset$, let $\Im_H$ be the family of fuzzy sets on H and let U_H be the function that associates to each $P \in \wp$ the $U_H^P \in \Im_H$

For each pair (M, N) of elements of $\Im_H$ we put

$$M \leq_H N \Leftrightarrow m_{ij} \leq n_{ij},\ \forall (F_i, C_j) \in H.$$

The pair $(\Im_H, \leq_H)$ is a lattice. If M and N are any two elements of $\Im_H$ we denote with $M \wedge_H N$ the infimum of N and M, and with $M \vee_H N$ their supremum.

For each pair (P, Q) of projects, we put:

$$P \leq_H Q \Leftrightarrow U_H(P) \leq_H U_H(Q), \tag{2.1}$$

$$P \approx_H Q \Leftrightarrow (P \leq_H Q \text{ and } Q \leq_H P) \tag{2.2}$$

The $\leq_H$ is a preorder relation on $\wp$, called *H-preorder* on $\wp$. If $H=\Omega$ it is called *global preorder*, on the contrary it is called *partial preorder*. The $\approx_H$ is an equivalence relation on $\wp$.

Given n projects $P_1,P_2,...,P_n$, we call *lower bound* of them *relatively to* H or *H-lower bound* every project P such that $P \leq_H P_i$, $\forall i \in \{1,2,..,n\}$ and we denote the set of H-lower bounds with $LBH(P_1,P_2,...,P_n)$. An element P of this set is said to be *maximal* if for each $Q \in LBH(P_1,P_2,...,P_n)$, $P \leq_H Q \Rightarrow P \approx_H Q$. We denote the set of maximal lower bounds relatively to H with $\delta_H(P_1,P_2,...,P_n)$.

We call *upper bound* of $P_1,P_2,...,P_n$ *relatively to* H or *H-upper bound* every project P such that $P_i \leq_H P$, $\forall i \in \{1,2,...,n\}$ and we denote the set of H-upper bounds with $UBH(P_1,P_2,...,P_n)$. An element P of this set is said to be *minimal* if for each $Q \in UBH(P_1,P_2,...,P_n)$, $Q \leq_H P \Rightarrow P \approx_H Q$. Let $\sigma_H(P_1,P_2,...,P_n)$ be the set of minimal upper bounds of $P_1,P_2,...,P_n$ relatively to H.

It is easy to prove the following

$$(P \in \delta_H(P_1,P_2,...,P_n),\ P \approx_H Q) \Rightarrow Q \in \delta_H(P_1,P_2,...,P_n), \tag{2.3}$$

$$(P \in \sigma_H(P_1,P_2,...,P_n),\ P \approx_H Q) \Rightarrow Q \in \sigma_H(P_1,P_2,...,P_n). \tag{2.4}$$

Evidently, if $\varnothing \subset H \subseteq I \subseteq \Omega$ then the $\leq_H$ and $\approx_H$ are, respectively, extensions of the $\leq_I$ e $\approx_I$. So, for each n-tuple of projects $(P_1,P_2,...,P_n)$, we have

$$LBH(P_1,P_2,...,P_n) \supseteq LBI(P_1,P_2,...,P_n) \text{ and } UBH(P_1,P_2,...,P_n) \supseteq UBI(P_1,P_2,...,P_n).$$

3. Hyperstructures associated to projects

Now we show how it is possible to define some particularly very significant hyperstructures of projects for geometrical and practical interpretation of the implications of the preorder relations considered.

At this aim, we recall some definitions about the hyperstructures. We assume the terminology given in (Corsini, [1]), (Vougiouklis, [8]), (Migliorato, [6]) and (Maturo, [5]).

Definition 3.1 A *hypergroupoid* (S, α), is a non empty set S with a function $\alpha: S \times S \rightarrow P^*(S) = P(S) - \{\varnothing\}$, called *hyperoperation*. The image of the pair (x,y) is noted $x\alpha y$ and is called *hyperproduct* by x and y.

For any pair (H,K) of subsets of S different from $\emptyset$, we denote by $H\alpha K$ the union of all the sets $x\alpha y$ with $x\in H$ and $y\in K$. The hyperproducts $a\alpha K$ and $H\alpha a$, $a\in S$, are considered equal, respectively, to $\{a\}\alpha K$ and $H\alpha\{a\}$.

$\forall n\in N$, $\forall(x_1,x_2,\ldots,x_n)\in S^n$, the set $\Im_\alpha(x_1,x_2,..x_n)$ of all the hyperproducts generated by $(x_1,x_2,\ldots,x_n)$ is given, by induction, as follows: $\Im_\alpha(x_1)=\{x_1\}$ and, for $n>1$, $\Im_\alpha(x_1,x_2,\ldots,x_n)$ is the set of all the hyperproducts $K=F\alpha G$, with $F=\{x_1,x_2,..x_h\}$, $G=\{x_{h+1},x_2,..x_n\}$, $h\in\{1,2,..,n-1\}$.

Any element of $\Im_\alpha(x_1,x_2,..x_n)$ is called *block of order n* or *n-block* of S generated by $(x_1,x_2,..x_n)$.

Definition 3.2 A hypergroupoid (S, α) is said to be
(H1) *semihypergroup* if, $\forall x,y,z\in S$, $x\alpha(y\alpha z)=(x\alpha y)\alpha z$;
(H2) *quasihypergroup* if, $\forall x\in S$, $x\alpha S=S\alpha x=S$;
(H3) *commutative* if, $\forall x,y\in S$, $x\alpha y=y\alpha x$;
(H4) *hypergroup* if it is a semihypergroup and a quasihypergroup.

Definition 3.3 A hypergroupoid (S, α) is said to be
(W1) *weak commutative* if, $\forall x,y\in S$, $x\alpha y\cap y\alpha x\neq\emptyset$;
(W2) *weak associative* if, $\forall x,y,z\in S$, $x\alpha(y\alpha z)\cap(x\alpha y)\,\alpha z\neq\emptyset$;
(W3) *feebly associative* if, $\forall n\in N$ and $\forall(x_1,x_2,..x_n)\in S^n$, the intersection of all the blocks belonging to $\Im_\alpha(x_1,x_2,..x_n)$ is different from $\emptyset$.

Definition 3.4 Let (S, α) be a hypergroupoid and let $\emptyset\neq T\subseteq S$. (T, α) is called a *subhypergroupoid* of S if, $\forall x,y\in T$, $x\alpha y\subseteq T$. A subhypergroupoid T of S is called *subhypergroup* if (T, α) is a hypergroup. If $a\alpha b\cap T\neq\emptyset, \forall a,b\in T$, the pair (T, β), with $a\beta b=a\alpha b\cap T$, $\forall a,b\in T$, is called *substructure generated* by T.

Finally any hypergroupoid (T,γ) such that, $\forall a,b\in T$, $a\gamma b\subseteq a\alpha b\cap T$, is called *partial substructure* of (S,α) *with domain* T.

Now, we assume S equal to the set $\wp$ of projects. For any $H\subseteq\Omega$, $H\neq\emptyset$, we consider the following functions:

$$\delta_H\colon (P,Q)\in\wp^2\to\delta_H(P,Q), \tag{3.1}$$

$$\sigma_H\colon (P,Q)\in\wp^2\to\sigma_H(P,Q). \tag{3.2}$$

A project I_H is said to be an H-*inferior project* if

$$\forall P\in\wp,\ U_H(I_H)\leq_H U_H(P). \tag{3.3}$$

A project I is called *inferior project* if it is an H-inferior project for any $H\subseteq\Omega$, $H\neq\emptyset$.

A project S_H is said to be an H-*superior project* if

$$\forall P\in\wp,\ U_H(P)\leq_H U_H(S_H). \tag{3.4}$$

A project S is called *superior project* if it is an H-superior project for any $H \subseteq \Omega$, $H \neq \emptyset$.

Eventually by adding two fictitious projects, we can suppose that there exist, in $\wp$, an H-superior project and an H-inferior project. In this hypothesis δ_H and σ_H are hyperoperations on $\wp$ and so we have the following

Theorem 3.5 If an H-superior project S_H exists, than the pair $(\wp, \sigma_H)$ is a commutative hypergruppoid. Similarly, if an H-inferior project I_H exists, than the pair $(\wp, \delta_H)$ is a commutative hypergruppoid.

Proof Let S_H be an H-superior project. Then, for any $P,Q \in \wp$, S_H is an upper bound of (P,Q) and so $\sigma_H(P,Q) \neq \emptyset$. Since $\sigma_H(P,Q)=\sigma_H(Q,P)$, then $(\wp, \sigma_H)$ is a commutative hypergruppoid. Analogously the existence of an H-inferior project I_H implies that $(\wp, \delta_H)$ is a commutative hypergruppoid.□

Theorem 3.6 If $U_H(\wp)$ is a sublattice of $(\Im_H, \leq_H)$, then the pairs $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative semihypergroups.

Proof Suppose $U_H(\wp)$ is a sublattice of $(\Im_H, \leq_H)$. Since $U_H(\wp)$ is finite then there exist the maximum S_H and the minimum I_H with respect $\leq_H$, that are, respectively, an H-superior project and a H-inferior project. By theorem 3.5 it follows that $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative hypergroupoids.

By hypothesis that $U_H(\wp)$ is a sublattice of $(\Im_H, \leq_H)$, $\forall P,Q \in \wp$, we have,

$$P\delta_H Q=\{R \in \wp : U_H(R) = U_H(P) \wedge_H U_H(Q)\}, \quad (3.5)$$

$$P\sigma_H Q=\{R \in \wp : U_H(R) = U_H(P) \vee_H U_H(Q)\}. \quad (3.6)$$

Since $\wedge_H$ and $\vee_H$ are associative, theorem follows.□

Generally, $U_H(\wp)$ is not a sublattice of $(\Im_H, \leq_H)$ and so it is not assured the existence of the H-superior and H-inferior projects. Therefore, if such projects exist, the associative properties of δ_H and σ_H are not valid. In this case, we have the following

Theorem 3.7 If an H-superior project S_H exists, than the pair $(\wp, \sigma_H)$ is a commutative and weak associative hypergruppoid. Similarly, if an H-inferior project I_H exists, than the pair $(\wp, \delta_H)$ is a commutative and weak associative hypergruppoid.

Proof Let S_H be an H-superior project. By theorem 3.5 we have that the pair $(\wp, \sigma_H)$ is a commutative hypergruppoid. Therefore, for any $P,Q,R \in \wp$, S_H is an upper bound of (P,Q,R) and so $\sigma_H(P,Q,R) \neq \emptyset$. Let Y be any element of $\sigma_H(P,Q,R)$. Then $Y \in UBH(P,Q)$ and so there exists a $Z \in \sigma_H(P,Q)$ such that

$Z \leq_H Y$ and $Y \in UBH(Z,R)$. Let T be an element of $\sigma_H(Z,R)$ such that $T \leq_H Y$. T is an upper bound of (P, Q, R). Since $Y \in \sigma_H(P,Q,R)$ we have $T \approx_H Y$ and so $Y \in \sigma_H(Z,R)$. Then, $\forall P,Q,R \in \wp$, it results $(P\sigma_H Q)\sigma_H R \supseteq \sigma_H(P,Q,R)$. Analogously, we can prove that $P\sigma_H(Q\sigma_H R) \supseteq \sigma_H(P,Q,R)$. So $\forall P,Q,R \in \wp$, we have

$$(P\sigma_H Q)\sigma_H R \cap P\sigma_H(Q\sigma_H R) \supseteq \sigma_H(P,Q,R) \neq \varnothing, \tag{3.7}$$

and the weak associative property of σ_H is proved.

In a similar way we prove that, $\forall P,Q,R \in \wp$,

$$(P\delta_H Q)\delta_H R \cap P\delta_H(Q\delta_H R) \supseteq \delta_H(P,Q,R) \neq \varnothing \tag{3.8}$$

and so we conclude that also δ_H is a weak associative hyperoperation. □

Practically, in the hypotheses of the previous theorem, given two projects P_1 and P_2, it is possible to single out at least a project Q such that, with respect H, is not superior to P_1 and P_2 and is not inferior to any project not superior to P_1 and P_2.

Besides, we can single out at least a project R, such that, with respect H, is not inferior to P_1 and P_2, and is not superior to any project not inferior to P_1 and P_2.

The hyperproduct δ_H gives the set of all the projects that are H-*maximal lower bounds* of P_1 and P_2. They are the projects with the better qualities, with respect H, in the set of the projects not superior to P_1 and P_2. They can be chosen from an economic point of view.

The hyperproduct σ_H gives the set of all the projects that are H-*minimal upper bounds* of P_1 and P_2. Those ones present all the better qualitative characteristics, with respect H, as regard to the pair (P_1, P_2). They are a *compromise* between the solution prefigured by both projects P_1 and P_2 and every proposal P_3 that is not inferior to P_1 and P_2.

In decision making problems, often the economical-financial factor is so important that the decider is induced to choose the H-*minimal upper bounds* projects. Those ones make it possible to achieve all the pre-established objectives in a satisfying measure, at a lower price than the amount implied by another project P_3 that achieves the same objectives.

If the U^P are *incidence matrices*, than $U_H(\wp)$ is a family of subsets of H, the operations $\wedge_H$ and $\vee_H$ in this family are the set operations of intersection and union and the order relation induced by these ones is reduced to the inclusion among the subsets of H belonging to $U_H(\wp)$.

If I and S are, respectively, an inferior and a superior project, then

- the matrix U(I) specifies a set of pairs that must be necessary satisfied, that is $U(P) \supseteq U(I)$, $\forall P \in \wp$;
- the matrix U(S) represents the largest set of pairs that can be satisfied, that is $U(P) \subseteq U(S)$, $\forall P \in \wp$.

Two particular fictitious projects that can be usefully used, respectively, as inferior and superior projects are the following:

a) the *null project* O with efficaciousness matrix E^O having all the elements equal to 0 and so with U^O equal to null matrix;
b) the *fantastic project* F such that E^F has all the elements equal to 1 and so with $U^F=W$.

In many situations it is necessary to consider only projects such that the elements of the efficaciousness matrices are not inferior to fixed values, called *threshold values*. In this case we have to consider as inferior project, a project T, real or fictitious, called *threshold project*, with efficaciousness matrix E^T such that any e_{ij}^T represents the lowest acceptable grade of satisfaction of the pair (F_i, C_j) by any project.

Now we give some properties of hypergruppoids $(\wp, \delta_H)$ and $(\wp, \sigma_H)$. Precisely, given n projects $P_1,P_2,\dots,P_n$, we study the relations among the blocks generated by these elements and the sets $\delta_H(P_1,P_2,\dots,P_n)$ of their maximal lower bounds and $\sigma_H(P_1,P_2,\dots,P_n)$ of their minimal upper bounds.

Theorem 3.8 For each n>1, for each $k\in\{1,2,\dots,n\}$ and for each n-tuple of projects $(P_1,P_2,\dots,P_n)$, it results

$$\delta_H(P_1,P_2,\dots,P_n) \subseteq \delta_H(P_1,P_2,\dots,P_k)\ \delta_H\ \delta_H(P_{k+1},P_{k+2},\dots,P_n), \tag{3.5}$$

$$\sigma_H(P_1,P_2,\dots,P_n) \subseteq \sigma_H(P_1,P_2,\dots,P_k)\ \sigma_H\ \sigma_H(P_{k+1},P_{k+2},\dots,P_n). \tag{3.6}$$

Proof We prove the (3.6). In a similar way the (3.5) is verified. It results evidently

$$\sigma_H(P_1,P_2,\dots,P_n) \subseteq UBH(P_1,P_2,\dots,P_k) \cap UBH(P_{k+1},P_{k+2},\dots,P_n).$$

Now, suppose $P\in\sigma_H(P_1,P_2,\dots,P_n)$. We have

$$\exists Q\in\sigma_H(P_1,P_2,..,P_k),\ \exists R\in\sigma_H(P_{k+1},P_{k+2},..,P_n) : P\in UBH(Q,R).$$

If a project S belongs to UBH(Q,R) and $P\geq_H S$ then we have that $S\in UBH(P_1,P_2,\dots,P_k)\cap UBH(P_{k+1},P_{k+2},\dots,P_n)$and so $S\in UBH(P_1,P_2,\dots,P_n)$.
Since $P\in\sigma_H(P_1,P_2,\dots,P_n)$, $P\geq_H S$, it follows that $S\in\sigma_H(P_1,P_2,\dots,P_n)$ and so $S\approx_H P$.
Then $P\in Q\sigma_H R\subseteq \sigma_H(P_1,P_2,\dots,P_k)\ \sigma_H\ \sigma_H(P_{k+1},P_{k+2},\dots,P_n)$. This implies (4.6).□

For each n≥1 and for each n-tuple of projects $(P_1,P_2,\dots,P_n)$, we denote with $\Im_\delta(P_1,P_2,\dots,P_n)$ the family of blocks associated to the n-tuple as regards the hyperoperation δ_H and with $\Im_\sigma(P_1,P_2,\dots,P_n)$ that one as regards the hyperoperation σ_H. We have the following

Theorem 3.9 $\forall n>1$, $\forall$n-tuple of projects $(P_1,P_2,\dots,P_n)$, $\forall K_\delta\in\Im_\delta(P_1,P_2,\dots,P_n)$ and $\forall K_\sigma\in\Im_\sigma(P_1,P_2,\dots,P_n)$, it results

$$\delta_H(P_1,P_2,\dots,P_n)\subseteq K_\delta, \tag{3.7}$$

$$\sigma_H(P_1,P_2,...,P_n)\subseteq K_\sigma. \qquad (3.8)$$

Proof We prove (4.8). In a similar way (4.7) is verified. The theorem is evident for n=2 because $\Im_\sigma(P_1, P_2)$ has the only element $\sigma_H(P_1,P_2)$. We suppose that the theorem is true for $n\le k$, for a generic $k\ge 2$ and we verify that it is true also for n=k+1.

In fact, suppose $K_\sigma\in\Im_\sigma(P_1,P_2,\ldots,P_{k+1})$. Then $\exists h$: $1\le h\le k$ such that $K_\sigma=K_1\sigma_H K_2$, with $K_1\in\Im_\sigma(P_1,P_2,...,P_h)$ and $K_2\in\Im_\sigma(P_{h+1},P_{h+2},\ldots,P_{k+1})$. For hypothesis of induction

$$\sigma_H(P_1,P_2,...,P_h)\subseteq K_1,\ \sigma_H(P_{h+1},P_{h+2},...,P_{k+1})\subseteq K_2.$$

For the previous theorem we have

$$\sigma_H(P_1,P_2,...,P_{k+1}) \subseteq \sigma_H(P_1,P_2,...,P_h)\ \sigma_H\ \sigma_H(P_{h+1},P_{h+2},...,P_{k+1})$$

and so

$$\sigma_H(P_1,P_2,...,P_{k+1}) \subseteq K_\sigma=K_1\sigma_H K_2. \ \square$$

Theorem 3.10 The pairs $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative and feebly associative hypergruppoids.

Proof By previous theorem we can observe that, $\forall n\in N$ and $\forall P_1,P_2,...,P_n\in\wp$, $\delta_H(P_1,P_2,...,P_n)$ is included in the intersection of all blocks K_δ belonging to $\Im_\delta(P_1,P_2,...,P_n)$ and, similarly, the intersection of all blocks K_σ belonging to $\Im_\sigma(P_1,P_2,...,P_n)$ includes $\sigma_H(P_1,P_2,...,P_n)$. $\square$

By two examples we verify that, generally, the associative property doesn't hold and we can have two possibilities:

a) the intersection of all blocks K_σ belonging to $\Im_\sigma(P_1,P_2,...,P_n)$ is equal to $\sigma_H(P_1,P_2,...,P_n)$;
b) $\sigma_H(P_1,P_2,...,P_n)$ is strictly contained in the intersection of all K_σ belonging to $\Im_\sigma(P_1,P_2,...,P_n)$.

Example 3.11 Let $\wp=\{P_0,P_1,...,P_9\}$ be the set of projects. For each pair A, B of subset of $\wp$ we put $A\le_H B$ if for each $a\in A$ e $b\in B$ we have $a\le_H b$. We impose the following conditions

$$\{P_0\}\le_H\{P_1,P_2,P_3\},\ \{P_1\}\le_H\{P_4,P_5\},\ \{P_2\}\le_H\{P_4,P_5,P_6\},$$
$$\{P_3\}\le_H\{P_5,P_6\},\ \{P_4,P_5\}\le_H\{P_7\},\ \{P_5,P_6\}\le_H\{P_8\},\ \{P_7,P_8\}\le_H\{P_9\}.$$

These conditions determine an antisymmetric relation. The reflexive and transitive closure of this relation is an order relation and for each $A\subseteq\wp$, $\delta_H(A)$ and $\sigma_H(A)$ are not empty.

In particular we have

$$\sigma_H(P_1,P_2,P_3)=\{P_5\},\ P_1\sigma_H(P_2\sigma_H P_3)=\{P_5,P_8\},\ (P_1\sigma_H P_2)\sigma_H P_3=\{P_5,P_7\}.$$

The two blocks generated by (P_1,P_2,P_3) are different, so the associative property doesn't hold. Their intersection is equal to $\sigma_H(P_1,P_2,P_3)$.

Example 3.12 Let $\wp=\{P_0,P_1,\dots,P_7\}$ be the set of projects. We impose the following conditions

$$\{P_0\}\leq_H\{P_1,P_2,P_3\},\ \{P_1\}\leq_H\{P_4,P_5\},\ \{P_2\}\leq_H\{P_4,P_5,P_6\},$$
$$\{P_3\}\leq_H\{P_5,P_6\},\ \{P_4,P_5,P_6\}\leq_H\{P_7\}$$

These conditions determine an antisymmetric relation. The reflexive and transitive closure of this relation is an order relation and for each $A\subseteq\wp$, $\delta_H(A)$ and $\sigma_H(A)$ are not empty.

In particular we have

$$\sigma_H(P_1,P_2,P_3)=\{P_5\},\ P_1\sigma_H(P_2\sigma_H P_3)=\{P_5,P_7\},\ (P_1\sigma_H P_2)\sigma_H P_3=\{P_5,P_7\}.$$

The two blocks generated by (P_1,P_2,P_3) are equal. Their intersection is different from $\sigma_H(P_1,P_2,P_3)$.

References

[1] **Corsini P.,** *Prolegomena of hypergroup theory,* Aviani Ed. Udine, (1993).

[2] **Ferri B, Maturo A.**, *Un modello matematico per la valutazione di fattibilità dei programmi integrati di intervento urbano*, in "Ratio Mathematica" 10, 1996, pp.67-84.

[3] **Ferri B, Maturo A.**, *Confronto fra proposte di intervento urbano utilizzando la teoria dei fuzzy set*, in Atti del Convegno Nazionale "I fondamenti della matematica ed i suoi legami con la società contemporanea", Verona, 28-30 novembre 1996, pp.407-418.

[4] **Ferri B, Maturo A.**, *Fuzzy classification and hyperstructures: an application to evaluation of urban projects*, in "Book of Short Papers" of the Scientific Reunion Scientific IFCS, Pescara, 3-4 luglio 1997, pp.137-140.

[5] **Maturo A.**, *On some hyperstructures of conditional events*, in Proceedings of sixth International Congress on Algebraic Hyperstructures and Applications, pp.115-132, Prague, Czech Republic, September 1-9, 1996.

[6] **Migliorato R.**, *Some topics on the feebly associative hypergroupoids*, in Proceedings of the fifth International Congress on Algebraic Hyperstructures and Applications, pp.67-80, Jasi, Rumania, 1993.

[7] **Prenowitz W., Jantosciak J.,** *Join Geometries*, Springer-Verlag UTM, New York, (1979).

[8] **Vougiouklis T.,** *Hyperstructures and their representations,* Hadronic Press, (1994).