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FUZZY SUB-F-POLYGROUPS

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چکیده: در این مقاله مفاهیم زیر F-پلی گروه فازی و روابط هم‌ارزی فازی یک F-پلی گروه تعریف شده‌اند و قضایایی اثبات گردیده‌اند.

Abstrace: The concepts of scalars, fuzzy sub-F-polygroups and fuzzy congruence relations of an F-polygroup are defined and some theorems are proved.

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1. PRELIMINARIES

A fuzzy subset [3] of a non-empty set A is a function from A to $[0,1]$. Throughout this paper I is the unit interval $[0,1] \subseteq \mathbb{R}$ and I^A is the set of all fuzzy subsets of A. If $\mu \in I^A$, then by $supp(\mu)$ we mean the set $\{x \in A | \mu(x) \neq 0\}$. Let $\mu, \eta \in I^A$. Then $\mu \leq \eta$ iff $\mu(x) \leq \eta(x)$, For all $x \in A$.

Definition 1.1. Let μ, η and $\mu_\alpha \in I^A$ where α is in the index set Λ. We define the fuzzy subsets $\mu \cap \eta, \mu \cup \eta, \bigcap_{\alpha \in \Lambda} \mu_\alpha$ and $\bigcup_{\alpha \in \Lambda} \mu_\alpha$ as follows:

(i) $(\mu \cap \eta)(x) = \min\{\mu(x), \eta(x)\}$,
(ii) $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}$,
(iii) $(\bigcap_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf_{\alpha \in \Lambda} \mu_\alpha(x)$,
(iv) $(\bigcup_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup_{\alpha \in \Lambda} \mu_\alpha(x)$, for all $x \in A$.

Definition 1.2. Let $a \in A, t \in I$. They by a fuzzy point a_t of A we mean the fuzzy subset of A given as:

$$a_t(x) = \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.3 [5] Let $A \neq \emptyset$ and $I_*^A = I^A \setminus \{o\}$, where o is the function which is identically 0. Then

(i) by an F-hyperoperation $*$ on A we mean a function from $A \times A$ to I_*^A; in other words for any $a, b \in A, a * b$ is a non-empty fuzzy =subset of A.

(ii) if $\mu, \eta \in I_*^A$, then $\mu * \eta \in I_*^A$ is defined by:

$$\mu * \eta = \bigcup_{x \in supp(\mu), y \in supp(\eta)} x * y.$$

Notation 1.4. Let $\mu \in I_*^A, B\&C \in P^*(A)$ and $a \in A$. Then

(i) $a * \mu$ and $\mu * a$ denote $\chi_{\{a\}} * \mu$ and $\mu * \chi_{\{a\}}$ respectively,

(ii) $a * B, B * a, \mu * B, B * \mu$ and $B * C$ denote $\chi_{\{a\}} * \chi_B, \chi_B * \chi_{\{a\}}, \mu * \chi_B, \chi_B * \mu$ and $\chi_B * \chi_C$ respectively.

Definition 1.5 [5]. Let $\mathcal{F}$ be a non-empty set and "$*$" an F-hyperoperation on $\mathcal{F}$. Then $(\mathcal{F}, *)$ is called an F-polygroup iff

(i) $(x * y) * z = x * (y * z), \ \forall\, x, y, z \in \mathcal{F}$,

(ii) there exists an element $e_F \in \mathcal{F}$ such that $x \in supp(x * e_F \cap e_F * x), \ \forall\, x \in \mathcal{F}$ (In this case we say e_F is an F-identity element of $\mathcal{F}$.),

(iii) for each $x \in \mathcal{F}$, there exists a unique element $x' \in \mathcal{F}$ such that $e_F \in supp(x * x' \cap x' * x)$, ($x'$ is called the F-inverse of x and is denoted by x_F^{-1}.),

(iv) $z \in supp(x * y) \Rightarrow x \in supp(z * y^{-1}) \Rightarrow y \in supp(x^{-1} * z), \ \forall\, x, y, z \in \mathcal{F}$. (This property is called the F-reversibility of $\mathcal{F}$ with respect to $*$.)

When there is no ambiguity, for simplicity of notation we use e and x^{-1} instead of e_F and x_F^{-1} respectively.

If $\mathcal{F}$ is an F-polygroup and $x * y = y * x, \ \forall\, x, y \in \mathcal{F}$, then $\mathcal{F}$ is said to be Abelian.

Henceforth $\mathcal{F}$ will denote an F-polygroup with hyperoperation "$*$" and e will denote the F-identity of $\mathcal{F}$.

Definition 1.6 [5]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then H is called an F-subpolygroup iff

(i) if $x \in H$, then $x^{-1} \in H$;

(ii) $supp(x * y) \subseteq H, \ \forall\, x, y \in H$.

In this case we write: $H <_{F-P} \mathcal{F}$.

Note that condition (ii) of the above Definition is equivalent to $x * y \leq \chi_H, \ \forall\, x, y \in H$.

Lemma 1.7 [5]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then $H <_{F-P} \mathcal{F}$ if and only if $supp(x * y^{-1}) \subseteq H, \ \forall\, x, y \in H$.

Definition 1.8 [5]. Let $H <_{F-P} \mathcal{F}$. Then

(i) H is said to be weak normal in $\mathcal{F}(H \lhd^{w}_{F-P} \mathcal{F})$ iff $x * H * x^{-1} \leq \chi_H, \ \forall\, x \in \mathcal{F}$,

(ii) H is said to be normal in $\mathcal{F}(H \lhd_{F-P} \mathcal{F})$ iff $x * H * x^{-1} = \chi_H, \ \forall\, x \in \mathcal{F}$.

Notation: Let $H \triangleleft_{F-P} \mathcal{F}$. Then $\mathcal{F}/H = \{x * H : x \in \mathcal{F}\}$.

Theorem 1.9 [6]. Let $H \triangleleft_{F-P} \mathcal{F}$. Define the F-hyperoperation "$\square$" on $\mathcal{F}/H$ as follows:

$$\begin{aligned} \square : \mathcal{F}/H \times \mathcal{F}/H &\longrightarrow I_*^{\mathcal{F}/H} \\ (x * H, y * H) &\longmapsto x * H \square y * H \end{aligned}$$

where

$$(x * H \square y * H)(z * H) = (x * y * H)(z), \quad \forall\, z * H \in \mathcal{F}/H.$$

Then $(\mathcal{F}/H, \square)$ is an F-polygroup called the quotient F-polygroup.

Definition 1.10. A fuzzy binary relation R on a set X (i.e. $R \in I^{X \times X}$) is said to be a fuzzy similarity relation if it satisfies for all $x, y, z \in X$:

(S1) reflexivity: $R(x, x) = 1$;

(S2) symmetry: $R(x, y) = R(y, x)$;

(S3) transitivity: $\min\{R(x, y), R(y, z)\} \leq R(x, z)$.

2. MAIN RESULTS

Definition 2.1. Let $\mu \in I^{\mathcal{F}}$. Then μ is a fuzzy sub-F-polygroup of $\mathcal{F}$ iff

(i) $\mu(z) \geq min\{\mu(x), \mu(y)\}, \forall z \in supp(x * y), \forall x, y \in \mathcal{F}$.

(ii) $\mu(x^{-1}) \geq \mu(x), \forall x \in \mathcal{F}$.

Note that this definition is a generalization of Definition 4.1 of [4]. Clearly (ii) implies that $\mu(x^{-1}) = \mu(x), \forall x \in \mathcal{F}$ and (i) implies that $\mu(e) \geq \mu(x), \forall x \in \mathcal{F}$.

Definition 2.2. Let ξ be a binary fuzzy relation between two F-polygroups $\mathcal{F}, \mathcal{F}'$ (i.e. $\xi \in I^{\mathcal{F} \times \mathcal{F}'}$). Then ξ is called an FP-relation iff:

(i) $(e, e') \in supp(\xi)$, where e and e' are the identity elements of $\mathcal{F}$ and $\mathcal{F}'$ respectively;

(ii) $\xi(a,b) \leq \xi(a^{-1},b^{-1})\ \forall a,b \in \mathcal{F}$,

(iii) $\min\{\xi(a,b),\xi(c,d)\} \leq \xi(x,y), \forall x \in supp(a*c), y \in supp(b*d)$, for all $(a,b),(c,d) \in \mathcal{F} \times \mathcal{F}'$.

Clearly (ii) implies that $\xi(a,b) = \xi(a^{-1},b^{-1}),\ \forall a,b \in \mathcal{F}$.

Theorem. 2.3. If ξ is an FP-relation between $\mathcal{F},\mathcal{F}'$ and if $K <_{F-P} \mathcal{F}'$, then the subset H of $\mathcal{F}$ which is defined as follows:

$$H = \{x \in \mathcal{F} : (x,y) \in supp(\xi), \text{ for some } y \in K\}$$

is an F-subpolygroup of $\mathcal{F}$.

Proof. Since $(e,e') \in supp(\xi)$, so $e \in H$. Now let $x \in H$. Then there is $y \in K$ such that $\xi(x,y) > 0$. Since $\xi(x,y) \leq \xi(x^{-1},y^{-1})$ and $y^{-1} \in K$, we get $x^{-1} \in H$. At present let $x_1,x_2 \in H$ and $t \in supp(x_1*x_2)$. Thus $(x_1,y_1)\&(x_2,y_2) \in supp(\xi)$, for some $y_1,y_2 \in K$. Thus

$$0 < \min\{\xi(x_1,y_1),\xi(x_2,y_2)\} \leq \xi(t,w)$$

where $w \in supp(y_1 * y_2) \subseteq K$. Hence $t \in H$.

Theorem 2.4. If ξ is a reflexive FP-relation on an F-polygroup $\mathcal{F}$, then ξ is a fuzzy similarity relation on $\mathcal{F}$.

Proof. First we prove the condition S_2 of Definition 1.10. Let $(a,b) \in \mathcal{F} \times \mathcal{F}$. Then

$$\begin{aligned}
& \xi(a,b) \\
= \ & \min\{\xi(a,b),\xi(b^{-1},b^{-1})\}, \text{ by reflexivity of } \xi \\
\leq \ & \xi(t,e)\ ,\ \forall t \in supp(a*b^{-1}), \text{ since } e \in supp(b*b^{-1}) \\
\leq \ & \xi(t^{-1},e)\ ,\ \forall t \in supp(a*b^{-1}) \\
= \ & \min\{\xi(t^{-1},e)\ ,\ \xi(a,a)\}, \text{ by reflexivity of } \xi \\
\leq \ & \xi(b,a), \text{ since } b \in supp(t^{-1}*a).
\end{aligned}$$

Similarly $\xi(b,a) \leq \xi(a,b)$. Hence ξ is symmetric. Now we prove the condition S_3 of Definition 1.10. For any $a,b,d \in \mathcal{F}$ we have:

$$\begin{aligned}\min\{\xi(a,b),\xi(b,d)\} &\le \xi(b,d)\\ &= \min\{\xi(b,d),\xi(b^{-1},b^{-1})\}\\ &\le \xi(e,w)\ ,\ \ \forall w\in supp(d*b^{-1}).\end{aligned}$$

Since, $\min\{\xi(a,b),\xi(b,d)\}\le\xi(a,b)$. Therefore

$$\begin{aligned}\min\{\xi(a,b),\xi(b,d)\} &\le \min\{\xi(e,w),\xi(a,b)\}\\ &\le \xi(a,d), \text{since}\quad d\in supp(w*b).\end{aligned}$$

Thus ξ is transitive.

Definition 2.5. An FP-relation on an F-polygroup $\mathcal{F}$ which is also a fuzzy similarity relation is called a fuzzy congruence relation on $\mathcal{F}$.

Definition 2.6. Let ξ be a fuzzy congruence relation on $\mathcal{F}$, then the fuzzy subset $\xi<e>$ of $\mathcal{F}$ is defined by $\xi<e>(x)=\xi(e,x)$, for all $x\in\mathcal{F}$.

Theorem 2.7. If ξ is a fuzsy congruence relation on an F-polygroup $\mathcal{F}$, then the fuzzy subset $\xi<e>$ is a fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. Let $x,y\in\mathcal{F}$. Then $\min\{\xi<e>(x),\xi<e>(y)\}=\min\{\xi(e,x),\xi(e,y)\}\le\xi(e,z)$, $\forall z\in supp(x*y)$.

Also we have $\xi<e>(x)=\xi(e,x)\le\xi(e,x^{-1})=\xi<e>(x^{-1})$. Therefore $\xi<e>$ is a fuzzy sub-F-polygroup of $\mathcal{F}$.

Definition 2.8. Let μ be a fuzzy sub-F-polygroup of $\mathcal{F}$. Then μ is said to be normal iff for all $x,y\in\mathcal{F}$

$$\mu(z)=\mu(z'),\forall z\in supp(x*y),\ \forall z'\in supp(y*x).$$

Note that the above definition generalizes Definition 2.3 of [4].

Remark 2.9. It is obvious that if μ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$, then $\mu(z) = \mu(z')$, $\forall z, z' \in supp(x * y)$, $\forall x, y \in \mathcal{F}$.

Theorem 2.10. Let μ be a fuzzy sub-F-polygroup of $\mathcal{F}$. Then the following condition are equivalent:

(i) μ is normal.

(ii) For all $x, y \in \mathcal{F}, \mu(z) = \mu(y)$, $\forall z \in supp(x * y * x^{-1})$

(iii) For all $x, y \in \mathcal{F}, \mu(z) \geq \mu(y)$, $\forall z \in supp(x * y * x^{-1})$

(iv) For all $x, y \in \mathcal{F}, \mu(z) = \mu(y)$, $\forall z \in supp(x^{-1} * y^{-1} * x * y)$.

Proof. The proof is similar to the proof of Theorem 2.5 of [4].

Corollary 2.11. If ξ is a fuzzy congruence relation on $\mathcal{F}$, then $\xi < e > \in I^{\mathcal{F}}$ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. By Theorem 2.7, $\xi < e >$ is a fuzzy sub-F-polygroup of $\mathcal{F}$. Now let $x, y \in \mathcal{F}$ and $z \in supp(x * y * x^{-1})$. Then $z \in supp(t * x^{-1})$, for some $t \in supp(x * y)$. Hence we have:

$$\begin{aligned}
\xi < e > (z) &= \xi(e, z) \\
&\geq \min\{\xi(x, t), \xi(x^{-1}, x^{-1})\}, \\
&= \xi(x, t), \text{ since } \xi \text{ is reflexive} \\
&\geq \min\{\xi(x, x), \xi(e, y)\}, \text{ since } t \in suup(x * y) \\
&= \xi(e, y), \text{ since } \xi \text{ is reflexive} \\
&= \xi < e > (y).
\end{aligned}$$

Therefore by Theorem 2.10 (iii), $\xi < e >$ is normal.

Corollary 2.12. $H \lhd^{w}_{F-P} \mathcal{F}$ if and only if χ_H is a normal fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. The proof follows from Definition 2.1, Theorem 2.10 (iii).

Corollary 2.13. Let $H \lhd_{F-P} \mathcal{F}$. Then $x * H * x^{-1}$ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$, for all $x \in \mathcal{F}$.

Proof. It is obvious.

Remark 2.14. It is well-known that in the fuzzy group theory, every fuzzy subgroup of an Abelian group is normal. But the following example shows that this is not true in the case of F-polygroups. At first we have the following theorem:

Theorem 2.15 [6]. Let (A, o) be a polygroup [1]. Then $(A, *)$ is an F-polygroup where $x * y = \chi_{xoy}$, $\forall x, y \in A$.

Example 2.16. If $(H = \{e, a, b, c\}, .)$ is Klein's four-group, then (H, o) is a polygroup (see [2]) where the hyperoperation "o" is defined as follows:

$$xoy = \{x, y, x.y\}, \text{ if } x \neq y^{-1}, \quad x, y \neq e,$$
$$xox^{-1} = x^{-1}ox = H, \text{ if } x \neq e,$$
$$xoe = eox = \{x\} \text{ for every } x \in H.$$

Now let " $*$ " be the F-hyperoperatoin induced by "o" (i.e. $x * y = \chi_{xoy}, \forall x, y \in H$). Then by Theorem 2.15 $(H, *)$ is an F-polygroup. Let $\beta, \gamma \in [0, 1]$ such that $\beta < \gamma$. Define a fuzzy subset μ of H as follows:

$$\mu(a) = \mu(b) = \mu(c) = \beta, \mu(e) = \gamma.$$

Then it is easy to see that μ is a fuzzy sub-F-polygroup of $(H, *)$. But since $e \in supp(a * a)$, $a \in supp(a * a)$ and $\mu(e) \neq \mu(a)$, we get μ is not normal.

Theorem 2.17. Let $\mathcal{F}$ be an Abelian F-polygroup and μ a normal fuzzy sub-F-polygroup of $\mathcal{F}$ such that $\mu(e) = 1$. Define $\xi \in I^{\mathcal{F} \times \mathcal{F}}$ as follows:

$$\xi(x, y) = \mu(z), \quad \text{for some arbitrary element } z \in supp(x * y^{-1})$$

Then ξ is a fuzzy congruence relation and $\mu = \xi < e >$.

Proof. By Remark 2.9, ξ is well-defined. Clearly $\xi(e, e) > 0$. Now for all $(x, y) \in \mathcal{F} \times \mathcal{F}$ we have:

$$\xi(x, y) \quad = \quad \mu(w) \text{ , } w \in supp(x * y^{-1})$$

$$\begin{aligned} &= \mu(w^{-1})\ ,\ w^{-1} \in supp(y * x^{-1}) \\ &= supp(x^{-1} * y), \text{ by commutativity of} * \\ &= \xi(x^{-1}, y^{-1}). \end{aligned}$$

Now we show that

$$\min\{\xi(a_1, b_1)\ ,\ \xi(a_2, b_2)\} \leq \xi(x, y), \forall x \in supp(a_1 * a_2) \\ , \forall y \in supp(b_1 * b_2).$$

Let $x \in supp(a_1 * a_2)$, $y \in supp(b_1 * b_2)$ and $t \in supp(x * y^{-1})$ be arbitrary. Then we have: $t \in supp(x * y^{-1}) \subseteq supp(a_1 * b_1^{-1} * a_2 * b_2^{-1})$. Thus $t \in supp(s * w)$, for some $s \in supp(a_1 * b_1^{-1})$ and $w \in supp(a_2 * b_2^{-1})$. Therefore we get

$$\begin{aligned} \xi(x, y) &= \mu(t), \text{ by definition of } \xi \\ &\geq \min\{\mu(s), \mu(w)\}, \text{ since } \mu \text{ is a fuzzy sub-F-polygroup} \\ &= \min\{\xi(a_1, b_1), \xi(a_2, b_2)\}. \end{aligned}$$

Consequently ξ is an FP-relation. Since $\xi(x, x) = \mu(e) = 1$, then ξ is reflexive. Hence by Theorem 2.4 and Definition 2.5, ξ is a fuzzy congruence relation.

Since $\mu(x) = \mu(x^{-1}), \forall x \in \mathcal{F}$, we get that $\mu = \xi < e >$.

Corollary 2.18. Let μ be a fuzzy subset of an Abelian F-polygroup, $\mathcal{F}$. Then μ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$ and $\mu(e) = 1$ if and only if there exists a congruence relation ξ on $\mathcal{F}$ such that $\mu = \xi < e >$.

Proof. The proof follows from Theorem 2.17, and Corollary 2.11.

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