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PROPERTIES OF HYPERPRODUCTS AND THE RELATION β IN QUASIHYPERGROUPS

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Abstract : Some properties of the complete parts in hypergroupoids are established. Applying these properties to the case of quasihypergroups H for which $H/_{\beta^*}$ is a quasigroup, a necessary and sufficient condition for the transitivity of the relation β is proved. Consequently, several classes of quasihypergroups in which β is transitive are obtained (for instance, in any finite quasihypergroup with identity β is a transitive relation). Then, in the case of quasihypergroups having underlying groups, the relation β is completely determined.

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1. INTRODUCTION

Let H be a hypergroupoid, $n \geq 2$ be an integer and $x_1, \ldots, x_n$ be elements of H. If γ is a grouping of the indices $1, 2, \ldots, n$ in this order, then the product of the elements x_i respecting γ is denoted $\prod_{\substack{i=1 \\ (\gamma)}}^{n} x_i$. Denote by $\Gamma(n)$ all the groupings of the indices $1, 2, \ldots, n$ in this order. Using these notations consider :

$P_1(H) = \{\{x\} \mid x \in H\}$,

$P_n(H) = \{ \prod_{\substack{i=1 \\ (\gamma)}}^{n} x_i \mid x_1, \ldots, x_n \in H, \ \gamma \in \Gamma\ (n)\ \}$,

$P_*(H) = \bigcup_{n=1}^{\infty} P_n(H)$.

By means of hyperproducts of $P(H)$ we define on H a chain of relations $(\beta_n)_{n \geq 1}$ as follows : $x\beta_n y$ if and only if there exists $Q \in P_n(H)$ such that $x, y \in Q$.

It is evident that $\beta_1 = Id(H)$ and β_n are symmetrical.

Consider the relation $\beta = \bigcup_{n=1}^{\infty} \beta_n$ which is reflexive and symmetrical. Its transitive closure $\beta^* = \beta \cup \beta \circ \beta \cup \ldots$ is an equivalence relation on H and $H/_{\beta^*}$ is a groupoid. Hence using the relation β^* we can define functors between categories of hypergroupoids and categories of groupoids which permit to reduce some problems on hyperstructures to easier others on *univalent* structures.

The relation β^* in hypergroups has been studied by many authors like M. Koskas ([9]), P. Corsini ([1], [2]), Y . Sureau ([11]), D. Freni ([4], [5], [6]), M. De Salvo ([3]), R. Migliorato ([10]).

There is an interesting problem concerning the relations β and β^* :

PROBLEM 1. When $\beta^* = \beta$? (Find the classes of hypergroupoids for which the corresponding relation β is transitive.)

A first important answer to this problem was obtained in 1991 by D. Freni ([4]). He proved that in hypergroups the relations β and β^* coincide.

The semihypergroups for which the relation β is transitive are characterised in [7].

In connection with Problem 1 we mention the following problem proposed by T. Vougiouklis ([12]).

PROBLEM 2. Do the relations β and β^* coincide in weakly associative quasihypergroups?

In this paper we extend some properties of the complete parts from semihypergroups to hypergroupoids. Using these properties we treat Problem 1 in the particular case of quasihypergroups.

2. COMPLETE PARTS AND THE RELATION β IN HYPERGROUPOIDS

The notion of complete part in hypergroups has been introduced and studied by M. Koskas in [9]. Then P. Corsini

([1], [2]), Y. Sureau ([11]), D. Freni ([4]), M. De Salvo ([3]), R . Migliorato ([10]) have established connections between the complete parts and the heart of a hypergroup.

In the folllowing some properties of the complete parts in (semi) hypergroups (see [1] and [7]) are extended to hypergroupoids. Using complete parts, a characterization of hypergroupoids in which the relation β is transitive, analogues with that for semihypergroups obtained in [7], is given.

Let H be a hypergroupoid. A subset A of H is a *complete part* if for every $Q \in P(H)$ such that $Q \cap A \neq \emptyset$ we have $Q \subset A$.

Remark that $\emptyset$ and H are complete parts of H. Also, the intersection of any family of complete parts of H is a complete part in H.

For a subset X of H denote by $\mathcal{C}(X)$ the intersection of all complete parts of H containing X. It is easy to verify that $\mathcal{C}(X)$ is the smallest complete part of H containing X (called the *complete closure* of X).

The following properties hold :

(1) $X \subset \mathcal{C}(X)$.

(2) If $X \subset X'$ then $\mathcal{C}(X) \subset \mathcal{C}(X')$.

(3) $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$.

(4) $\mathcal{C}(X) = \bigcup_{x \in X} \mathcal{C}(x)$, where $\mathcal{C}(x) = \mathcal{C}(\{x\})$.

As for an associative hyperoperation, we can associate to any subset X of H an ascending chain of subsets $(\mathcal{C}_n(X))_{n \in \mathbb{N}}$ defined by the following two relations :

i) $\mathcal{C}_0(X) = X$

ii) $\mathcal{C}_{n+1}(X) = \cup\{Q \in P(H) \mid Q \cap \mathcal{C}_n(X) \neq \emptyset\}$.

Using the chain $(\mathcal{C}_n(X))_{n\in\mathbb{N}}$ we can obtain the complete closure of X, as it is shown in the next result.

2.1. PROPOSITION. *Let X be a subset in a hypergroupoid H. Then the following properties hold :*

(5) $\mathcal{C}_n(X) = \underset{x\in X}{\cup} \mathcal{C}_n(x)$, where $\mathcal{C}_n(x) = \mathcal{C}_n(\{x\})$.

(6) $\mathcal{C}_n(\mathcal{C}_m(X)) = \mathcal{C}_{n+m}(X)$.

(7) $\mathcal{C}(X) = \underset{n\in\mathbb{N}}{\cup} \mathcal{C}_n(X)$.

PROOF:

(5) As $\mathcal{C}_n(x) \subset \mathcal{C}_n(X)$ whenever $x \in X$, it follows that $\left(\underset{x\in X}{\cup} \mathcal{C}_n(x)\right) \subset \mathcal{C}_n(X)$. We prove the converse inclusion by induction on n. If $n = 0$ then the equality (5) holds. Assume that $\mathcal{C}_{n-1}(X) \subset \underset{x\in X}{\cup} \mathcal{C}_{n-1}(x)$, for $n \in \mathbb{N}^*$, and consider $y \in \mathcal{C}_n(X)$. This means that there exists $Q \in P(H)$ such that $y \in Q$ and $Q \cap \mathcal{C}_{n-1}(X) \neq \emptyset$. Then, by hypothesis, $Q \cap \mathcal{C}_{n-1}(x) \neq \emptyset$, for some $x \in X$. Therefore $Q \subset \mathcal{C}_n(x)$, that is $y \in \left(\underset{x\in X}{\cup} \mathcal{C}_n(x)\right)$, which proves that the converse inclusion holds.

(6) We proceed once again by induction on $n \in \mathbb{N}$.

For $n = 0$ the relation (6) is valid because $\mathcal{C}_0(\mathcal{C}_m(X)) = \mathcal{C}_m(X)$.

Suppose now that $\mathcal{C}_{n-1}(\mathcal{C}_m(X)) = \mathcal{C}_{m+n-1}(X)$, where $n \in \mathbb{N}^*$. Then

$\mathcal{C}_n(\mathcal{C}_m(X)) = \cup\{Q \in P(H) \mid Q \cap \mathcal{C}_{n-1}(\mathcal{C}_m(X)) \neq \emptyset\} = \cup\{Q \in P(H) \mid Q \cap \mathcal{C}_{m+n-1}(X) \neq \emptyset\} = \mathcal{C}_{n+m}(X)$.

(7) In order to prove that $\mathcal{C}(X) \subset \bigcup_{n\in\mathbb{N}} \mathcal{C}_n(X)$ it is sufficient to establish that $A = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n(X)$ is a complete part of H. Let $Q \in P(H)$ such that $Q \cap A \neq \emptyset$. Then $Q \cap \mathcal{C}_n(X) \neq \emptyset$, for some $n \in \mathbb{N}$. Consequently $Q \subset \mathcal{C}_{n+1}(X) \subset A$ and thus $\mathcal{C}(X) \subset \bigcup_{n\in\mathbb{N}} \mathcal{C}_n(X)$. On the orther hand, from i) and ii) , by induction on n , we get that $\mathcal{C}_n(X) \subset \mathcal{C}(X)$, for every $n \in \mathbb{N}$. Hence $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n(X) \subset \mathcal{C}(X)$. ∎

Note that we also have the properties below :

(8) If $Q \in P(H)$ then $\mathcal{C}(Q) = \mathcal{C}(x)$, for any $x \in Q$.

(9) $\mathcal{C}_n(x)y \subset \mathcal{C}_n(xy) \supset x\mathcal{C}_n(y)$, for any x and y in H.

(10) $\mathcal{C}(x)y \subset \mathcal{C}(xy) \supset x\mathcal{C}(y)$, for any x and y in H.

The connexion between the relation β^* and the complete parts of a hypergroupoid is given by the following result.

2.2. PROPOSITION. *Let x and y be two elements of a hypergroupoid H. Then $x\beta^*y$ if and only if $\mathcal{C}(x) = \mathcal{C}(y)$.*

PROOF :

In order to prove the required equivalence suppose first that $x\beta^*y$ and show that $\mathcal{C}(x) = \mathcal{C}(y)$. It suffices to establish that for the couples (x, y) of elements of H satisfying $x\beta y$. If $x\beta y$ then there exists $Q \in P(H)$ which contains both x and y. Therefore, according to (8), $\mathcal{C}(x) = \mathcal{C}(Q) = \mathcal{C}(y)$. Consequently, if $x\beta^*y$ then $\mathcal{C}(x) = \mathcal{C}(y)$.

Assume now that $\mathcal{C}(x) = \mathcal{C}(y)$. Then $x \in \mathcal{C}_n(y)$, for some $n \in \mathbb{N}$. We prove by induction on n that $x\beta^*y$. For $n = 0$ this is true because $x = y$. Assume this is also true for any integer $k < n$ and prove that the corresponding assertion for n is true, too. As $x \in \mathcal{C}_n(y)$ there exists $Q \in P(H)$ such that $x \in Q$ and $Q \cap \mathcal{C}_{n-1}(y) \neq \emptyset$. Let $z \in Q \cap \mathcal{C}_{n-1}(y)$. From $x \in Q$ we get $\{x\}\overline{\overline{\beta^*}}Q$, whence $x\beta^*z$. On the other hand, as $z \in \mathcal{C}_{n-1}(y)$, by inductive hypothesis, $z\beta^*y$. Hence $x\beta^*y$. ■

If R is a relation on the hypergroupoid H we define on $\mathcal{P}^*(H) = \{X \subset H \mid X \neq \emptyset\}$ two others relations R and $\overline{\overline{R}}$ by :

$$A\overline{R}B \text{ iff } \begin{cases} \forall a \in A, \exists b \in B \text{ such that } aRb \\ \forall b \in B, \exists a \in A \text{ such that } aRb \end{cases}$$

$$A\ \overline{\overline{R}}\ B \text{ iff } (\forall a \in A, \forall b \in B \text{ we have } aRb).$$

The relation $\overline{\overline{\beta}}$ which intervine in the previous proof is obtained in this manner.

Using the previous results we obtain the following characterization of the transitivity of the realtion β in hypergroupoids.

2.3. THEOREM. *The relation β is transitive in a hypergroupoid H if and only if*

$$(*) \quad \mathcal{C}(x) = \mathcal{C}_1(x), \ \textit{for any} \quad x \in H.$$

PROOF :

Suppose β is transitive. In order to prove that $\mathcal{C}(x) = \mathcal{C}_1(x)$, for any $x \in H$, it suffices to establish that $\mathcal{C}_1(x)$ is a complete part of H. Let $Q \in P(H)$ such that $Q \cap \mathcal{C}_1(x) \neq \emptyset$ and let $y \in Q \cap \mathcal{C}_1(x)$. We have to show that $Q \subset \mathcal{C}_1(x)$.

It is obvious that $x\beta y$. As $y\beta z$, for $z \in Q$, we obtain that $x\beta z$. Consequently there exists Q' in $P(H)$ containing both x and z, whence $z \in \mathcal{C}_1(x)$.

Conversely, suppose that $(*)$ holds. Consider x, y, z elements of H such that $x\beta y$ and $y\beta z$. Then $x\beta^* z$ and thus $\mathcal{C}(x) = \mathcal{C}(z)$. It follows that $z \in \mathcal{C}_1(x)$. Hence, there exists Q in $P(H)$ which contains both x and z, that is $x\beta z$. ∎

Several examples of semihypergroups for which the relation β is not transitive are presented in [7]. However there is no known example of quasihypergroup for which the associated relation β is not transitive.

Conjecture. If H is a quasihypergroup then β is transitive on H.

If we deal with quasihypergroups H such that $H/_{\beta^*}$ is a quasigroup we can give a necessary and sufficient condition for the transitivity of β, more simple than the previous condition $(*)$.

2.4. THEOREM. *Let H be a quasihypergroup such that $H/_{\beta^*}$ is a quasigroup. Then the relation β is transitive in H if and only if*

(**) *there exists x in H for which $\mathcal{C}(x) = \mathcal{C}_1(x)$.*

PROOF :

The result to prove is a direct consequence of the following.

2.5. LEMMA. *Let H be a quasihypergroup such that $H/_{\beta^*}$ is a quasigroup. Then the following assertions are valid :*

a) *$\mathcal{C}(xy) = \mathcal{C}(x)y$, for every x and y in H.*

b) *If there exist x in H and $n \in \mathbb{N}$ such that $\mathcal{C}(x) = \mathcal{C}_n(x)$ then $\mathcal{C}(y) = \mathcal{C}_n(y)$, for every $y \in H$.*

PROOF :

a) According to (10), $\mathcal{C}(x)y \subset \mathcal{C}(xy)$. In order to prove the converse inclusion let $t \in \mathcal{C}(xy)$. As $t \in H = Hy$ we get that $t \in uy$, for some $u \in H$. Therefore $\beta^*(t) = \beta^*(x)\beta^*(y) = \beta^*(u)\beta^*(y)$, whence $\beta^*(t) = \beta^*(u)$. Thus $t \in \mathcal{C}(u)y = \mathcal{C}(x)y$, that is $\mathcal{C}(xy) \subset \mathcal{C}(x)y$.

b) Let $y \in H$. Then there exists $u \in H$ such that $y \in xu$. Hence $\mathcal{C}(y) = \mathcal{C}(xu) = \mathcal{C}(x)u = \mathcal{C}_n(x)u \subset \mathcal{C}_n(xu) = \mathcal{C}_n(y)$.

It folllows that $\mathcal{C}(y) = \mathcal{C}_n(y)$, for every $y \in H$. ∎

2.6. COROLLARY. *Let H be a finite quasihypergroup. Then the relation β is transitive in H if and only if H verifies the condition (**).*

2.7. REMARK. If H is a infinite quasihypergroup then $H/_{\beta^*}$ is not necessarily a quasigroup. This follows from the example below.

2.8. EXAMPLE. Let G be a groupoid and $(A_x)_{x\in G}$ be a family of nonempty disjoint sets. On $H = \bigcup_{x\in G} A_x$ we define a hyperoperation by $a \cdot b = A_{xy}$, where $a \in A_x, b \in A_y, x \in G$ and $y \in G$. Then $(H, \cdot)$ is a hypergroupoid for which $\beta^* = \beta = \beta_2$ and $H/_{\beta^*} \simeq G$.

If we take $G = (\mathbb{N}, *)$, where $x * y = |x - y|$, by the above construction we obtain a quasihypergroup for which $H/_{\beta^*} \simeq (\mathbb{N}, *)$ is not a quasigroup.

3. CLASSES OF QUASIHYPERGROUPS IN WHICH THE RELATION β IS TRANSITIVE

In this section, we present some of the most important classes of quasihypergroups in which we can prove that the relation β is transitive. We use Theorems 2.3 and 2.4, established in the previous section.

3.1. THE CASE OF HYPERGROUPS

3.1.1. PROPOSITION. *Let H be a hypergroup.*

i) *If $a \in H$ and $Q \in P_n(H)$, where $n \in \mathbb{N}^*$, then there exists $Q' \in P_n(H)$ such that $Q \subset Q'a$.*

ii) *If Q and Q' are two elements of $P(H)$ such that*

$Q \cap Q' \neq \emptyset$ and $a \in Q'$ then $(Q \cup \{a\}) \subset Q''$, for some $Q'' \in P(H)$.

iii) $\mathcal{C}(x) = \mathcal{C}_1(x)$, *for any* $x \in H$.

PROOF :

i) Consider $x_1, \ldots, x_n$ in H such that $Q = x_1 \ldots x_n$. As $Ha = H$ there exists $x'_n \in H$ such that $x_n \in x'_n a$. Then, taking $Q' = x_1 \ldots x_{n-1} x'_n$ we get $Q \subset Q'a$.

ii) Let $b \in Q \cap Q'$. Then, because of the reproductibility, then exists $c \in H$ such that $a \in bc$. According to i) we have $Q \subset Q_1 a$, for some $Q_1 \in P(H)$. Therefore $Q \subset Q_1 a \subset Q_1 bc \subset Q_1 Q' c$ and $a \in bc \subset Qc \subset Q_1 ac \subset Q_1 Q' c$. Hence for $Q'' = Q_1 Q' c$ we obtain that $(Q \cup \{a\}) \subset Q''$ and $Q'' \in P(H)$.

iii) It suffices to show that $\mathcal{C}_1(x)$ is a complete part of H, for every $x \in H$. Let $Q \in P(H)$ such that $Q \cap \mathcal{C}_1(x) \neq \emptyset$. According to the definition of $\mathcal{C}_1(x)$ there exists $Q' \in P(H)$ such that $x \in Q'$ and $Q \cap Q' \neq \emptyset$. From ii), we get $(Q \cup \{x\}) \subset Q''$ for some $Q'' \in P(H)$, where $Q \subset Q'' \subset \mathcal{C}_1(x)$. ∎

Using this above proposition and Theorem 2.3 we have :

3.1.2. PROPOSITION. *The relation β is transitive in any hypergroup.* ∎

3.1.3. REMARK. Another proof for this previous result have been given by D. Freni in [4] using the notion of heart of a hypergroup.

We present now an interesting consequence of Proposition 3.1.1 for finite hypergroups.

3.1.4. PROPOSITION. *Let H be a finite hypergroup and Q, Q' be in $P(H)$ such that $Q \cap Q' \neq \emptyset$. Then there exists $Q'' \in P(H)$ such that $Q \cup Q' \subset Q''$.* ■

3.2. THE CASE OF QUASIHYPERGROUPS WITH IDENTITY

Recall that an element e of a hypergroupoid H is an identity if $x \in (ex \cap xe)$, for every $x \in H$.

We get the following result.

3.2.1. LEMMA. *Let H be a quasihypergroup in which there exists at least one identity e. Then $C(e) = C_1(e)$.*

PROOF :

It suffices to show that $C_1(e)$ is a complete part of H. Let $Q \in P(H)$ such that $Q \cap C_1(e) \neq \emptyset$. Hence there exists $Q' \in P(H)$ such that $e \in Q'$ and $Q \cap Q' \neq \emptyset$. Consider $x \in Q \cap Q'$. Then $e \in x'x$, for some $x' \in H$. Taking $Q'' = (x'Q')Q \in P(H)$ we have $e \in x'x \subset (x'e)x \subset (x'Q')Q = Q''$ and $Q \subset eQ \subset (x'x)Q \subset (x'Q')Q = Q''$. Hence $Q \subset C_1(e)$. ■

Using the previous Lemma and Theorem 2.4 we get :

3.2.2. THEOREM. *Let H be a quasihypergroup such that $H/_{\beta^*}$ is a quasigroup. If H contains at least one identity then the relation β is transitive in H.* ■

Here are now two remarquable consequences of Theorem 3.2.2.

3.2.3. PROPOSITION. *Let H be a weakly associative quasihypergroup (i.e., according to* [10], *a H_V-group). If H has at least one identity then the relation β is transitive in H.*

PROOF :

It follows from Theorem 3.2.2 and because $H/_{\beta^*}$ is a group whenever H is a H_V-group. ∎

We mention that another proof of Proposition 3.2.3 has been given by D.Freni in Corollary 2.4 of [5].

3.3. THE CASE OF QUASIHYPERGROUPS CONTAINING SINGLE ELEMENTS

An element x of a hypergroupoid H is *single* if $\mathcal{C}(x) = \{x\}$ (see [12]). The following result holds.

3.3.1. THEOREM. *Let H be a quasihypergroup such that $H/_{\beta^*}$ is a quasigroup. If there exists $x \in H$ such that $\mathcal{C}(x) \in P(H)$ then $\beta = \beta^*$.*

PROOF :

If $\mathcal{C}(x) = Q \in P(H)$ then $\mathcal{C}_1(x) = Q = \mathcal{C}(x)$ and, according to Theorem 2.4, $\beta = \beta^*$. ∎

An immediate consequence of Theorem 3.3.1 is :

3.3.2. PROPOSITION. *Let H be a quasihypergroup such that $H/_{\beta^*}$ is a quasigroup. If H contains a single element then β is transitive.* ∎

A similar result for H_V-groups has been obtained by T. Vougiouklis (see [12], Theorem 1.3.3).

3.4. THE CASE OF QUASIHYPERGROUPS HAVING UNDERLYING GROUPS

Let $(H,)$ be a quasihypergroup which has an underlying group $(H,\cdot)$, that is $x \cdot y \subset xy$, for every x and y in H. Then $(H,)$ is a weakly associative quasihypergroup, also called a H_b-*group* (see T. Vougiouklis, [12]).

In the following we show how to determine the relation β^* for this kind of quasihypergroups. Notice that according to Theorem 3.2.2 we have $\beta^* = \beta$.

Consider the set $A = \cup\{(x \cdot y)^{-1}(xy) \mid x, y \in H\}$ and let N be the normal closure of A (i.e. N is the least normal divisor of the group $(H,\cdot)$ containing the set A).

Consider the canonical projections $q : H \to H/N$, respectively $\pi : H \to H/_{\beta^*}$.

Then $q(xy) = \{t \cdot N \mid t \in xy\}$ and $q(x) \cdot q(y) = (x \cdot y) \cdot N$, whence $q(xy) = q(x) \cdot q(y)$, for every x and y in H. Therefore the heart ω_H of H is included in N. On the other hand, as $1 = x^{-1} \cdot x \subset x^{-1}x$, for every $x \in H$, it follows that $\pi(x^{-1}) = \pi(x)^{-1}$. Using this equality we have that $x^{-1} \cdot y \in \omega_H$ and also $a^{-1} \cdot x \cdot a \in \omega_H$, whenever x and y are in ω_H and $a \in H$. Hence ω_H is a normal subgroup of H. More than that, as $\pi((xy)^{-1}(xy)) = \pi(1)$, it follows that $A \subset \omega_H$. Therefore $N \subset \omega_H$. Hence $\omega_H = N$. It results that $H/_{\beta^*}$ coincides with H/N. We get that $x\beta^* y$ if and only if $x\beta y$, that is, if and only if $xN = yN$.

A direct consequence of these remarks are the following results.

3.4.1. THEOREM. *Let $(H,)$ be a quasihypergroup having an underlying group $(H,\cdot)$. Then the heart ω_H of $(H,)$ is the normal closure of the set $\cup\{(x\cdot y)^{-1}(xy)\ |x,y\in H\}$ and the relations β and β^* coicide in $(H,)$.* ∎

3.4.2. PROPOSITION. *Let H be a H_V-group having only one proper hyperproduct. Then the relations β and β^* coicide in H.*

PROOF :

According to [8], H is either a hypergroup or a H_b-group. Hence $\beta=\beta^*$. ∎

We mention that in [12] some particular case of Proposition 3.4.1 are studied (see Examples 1.2.3, 1.2.4 and 1.2.5).

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