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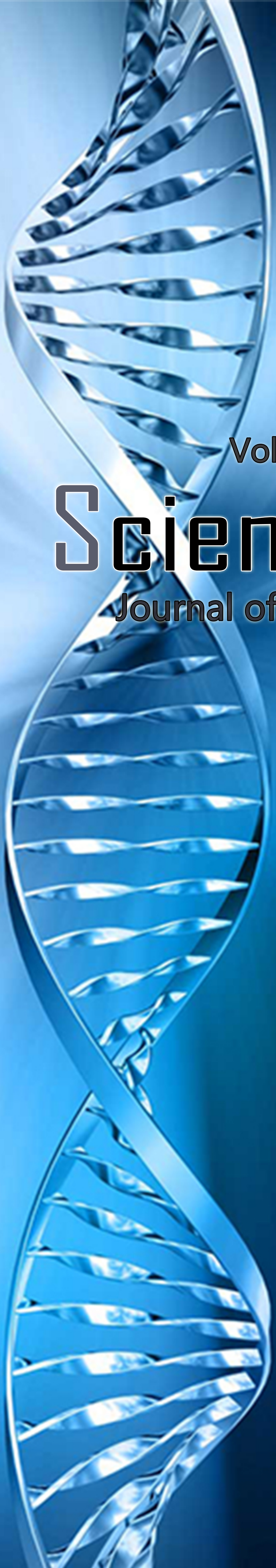

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Social sciences on stage: a theatrical scientific dissemination project

Davide Costa*

Abstract

One of the biggest challenges of contemporary science is to develop innovative approach to excite society about science and scientific topics. One of the attempts to find new ways to communicate with the public has been to use artistic language to explore scientific topics. Based on these premises, a project, titled *Social sciences on stage*, is presented that create a creative link between social science and theater. The aim is to encourage the participation of scientists in the creation and expression of theater and to reflect on the social context of science. It will generate a reflection on the social context of science. According with this paper the theatre is not an end in itself, but a useful resource for educational practice. With my project, theater is considered a real educational tool for human development. It favors the increase of individual sensitivity, channeling the various emotions of the spectators which often remain unexpressed.

Keywords: Social sciences on stage; Theatrical anthropology; Theatre; Public engagement with science.†

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1. Introduction

Modern society is deeply influenced by science and technology in many aspects such as political stability, economic growth, national prosperity, health, social welfare, education and citizenship (Bubela et al., 2009). There is a significant gap between society and science (Rédey, 2006). This lack of communication does not allow for a valuable and complete understanding between these two worlds and undermines the building of relationships of trust and appreciation (European Commission, 2007).

Science communication is pivotal to addressing these problems by bringing science closer to the public. The main objective of science communication initiatives is to enable researchers and the public to learn from each other through a process of interaction and collaboration (Davies, 2008). Using artistic language to explore scientific topics is a creative way to communicate and engage the public (Riesch, 2014).

Theater is an art and is a way of dealing with a specific theme, namely science and technology and its political, social, and ethical issues (Shepherd-Barr, 2006). One of the hallmarks of the growing importance of science in society is an indicator of the growing importance of science in society is its presence in artistic productions.

In the last 30 years, it has been one of the themes of increasing interest in theater (Zehelein, 2009). Science theater has covered a wide range of disciplines and scientific ideas, including biology, genetics, neuroscience, mathematics, physics, and astronomy.

At this regard "The appropriation of science by playwrights may take many forms. Some plays analyze the social and ethical impacts of scientific progress—some examples are 'The Burning Glass' that focuses on the invention and use of the atomic bomb and Wendy Lill's 'Chimera' that focuses on genetic engineering. Other address questions about the history of science, the daily life of scientists or the scientific process. There are numerous examples, particularly in the last decades, such as 'Copenhagen' which focuses on a secret meeting between two renowned physicists during the second World War, or Timberlake Wertenbaker's 'After Darwin' which explores Charles Darwin's voyage on the Beagle, or (…) Anna Ziegler's 'Photograph 51' about Rosalind Franklin and the discovery of the structure of the DNA. There are many other examples that use science as a subject and scientists as protagonists. The history of science in theatre is long, multidisciplinary and proves that many scientific themes succeed in theatre" (Amaral et al., 2017, p.2).

Based on these premises, a project, titled *Social sciences on stage* is presented that create a creative link between science and theater. The aim is to encourage the participation of scientists in the creation and expression of theater and to reflect on the social context of science. It will generate a reflection on the social context of science. In doing so, the initiative has the potential to reduce two gaps between science and society.

These are demystifying the image of scientists and science itself between science and society (Haynes, 2014) and addressing the relationship between social science and 'laypeople', i.e. those who do not have sufficient knowledge in the field.

2. The scientific bases of Social Sciences on stage

Social sciences on stage is a project that has scientific bases deriving from theatrical anthropology and the sociological dramaturgical approach,and the main characteristics of which will be presented.

Theatrical anthropology or anthropology of theatre

The anthropology of theater studies the expressive behavior of artists, shaping different genres and traditions. In organizational performance, the artist's physical and mental being is modeled differently from everyday life, forming the basis of technology. Techniques can be conscious or unconscious, yet implicit in landscape practices. The principle of repetition creates physical tension, generating new energy and captivating the viewer. This "before" stage constitutes the basic level of organization, inseparable from the audience and performance. Analyzing this level allows for knowledge expansion in practical, professional, and historical areas with immediate results (Barba, 2005). The anthropology of theater combines three levels:

1) the artist's personality, sensitivity, artistic intelligence, social personality: the characteristics that make an artist unique;

2) the historical and cultural context in which the characteristics of the natural tradition and the unique personality of the artist emerged;

3) the use of the body mind according to the extra daily technique, in which the principle of ultra-cultural repetition can be found.

These levels are defined by theatrical anthropology as the field of preliminary representation (Barba, 2005). In particular, “The first two aspects determine the transition from pre-expressivity to performing. The third is the idem, that which does not vary; it underlies the various personal, stylistic and cultural differences. It is the level of the scenic bios, the ‘biological’ level of performance, upon which the various techniques and the particular uses of the performer’s scenic presence and dynamism are founded. The only affinity connecting Theatre Anthropology to the methods and fields of study of cultural anthropology is the awareness that what belongs to our tradition and appears obvious to us can instead reveal itself to be a knot of unexplored problems. This implies a displacement, a journey, a détour strategy which makes it possible for us to single out that which is ‘ours’ through confrontation with what we experience as ‘other’. Displacement educates our way of seeing and renders it both participatory and detached. Thus a new light is thrown on our own professional ‘country’”(https://ista-online.org/about-ista/theatre-anthropology/).

Specifically, theater anthropology is a useful science when experts can understand creativity and increase the power of those involved in creativity (Barba, 2005). As a sub-discipline it falls more broadly within the discipline known as anthropology of art. It has focused on material objects. Less attention has been given to physical performance (music, dance, theatre) despite pioneering work by anthropologists such as Victor Turner (1979) in recent decades.

There are different anthropological approaches to work from work and workplace studies. Anthropologists examine the relationship between society and the resulting theatrical and artistic performances. Human approaches to work reflect a range of human concerns: social structure, function, meaning, identity, and experience. Performing arts research is akin to ritualistic practice, a well-documented field in which functional elements do not always receive the recognition they deserve. The study of anthropology helps teach students about cultural diversity and the role of different theoretical perspectives in learning. Anthropology is a relatively new field of study, so there is no easy introduction or general reading (Islam et al., 2024).

Sociology and theater

Another trend that has a strong connection with the world of theater and social sciences is certainly dramaturgy.

Dramaturgy is a perspective of sociology that analyzes micro-sociological descriptions of everyday social interactions through analogies of performativity and theatrical dramaturgy, dividing such interactions between "actors", "audiences", and various "frontstages" and "backstages". The term was first adopted into sociology from theater by Erving Goffman (1959), who developed most of the related terminology and ideas in his book *The Presentation of Self in Everyday Life*.

Dramaturgy sociology asserts that elements of human interaction are time, place and audience dependent. In other words, for Goffman, the self is a sense of who one is and the dramatic effect that results from the presented immediate scene. Goffman forms a theatrical metaphor by defining the way one person presents themselves to another based on cultural values, norms, and beliefs (Macionis and Gerber, 2010). Although the performances can have glitches (the actors are aware of this), most are successful. The goal of this self-presentation is to be accepted by the audience through a carefully executed performance. If the actor is successful, the audience will see the actor as he wants to be seen.

The theatrical metaphor can be seen in the origin of the word person, which comes from the Latin persona, meaning mask worn by an actor. A person acts differently because plays different roles, in front of different people, i.e. the audience. A person chooses her clothes (costume) to match the image she wants to convey. With the help of friends, caterers, and decorators (fellow actors and stage crew), a person successfully "stages" a dinner with parents, a friend's birthday party, etc. If they need to adjust the outfit or say something unpleasant about one of the guests, they do so discreetly (behind the scenes) where others cannot see (Brissett Edgley, 1990). The way we present ourselves to others is called dramaturgy. The dramaturgical perspective is one of several sociological paradigms that differ from other sociological theories and theoretical frameworks because it does not investigate the causes of human behavior, but rather analyzes context. However, this is controversial within sociology (Edgley, 2013).

For Goffman "What is important is the sense he (a person or actor) provides them (the others or audience) through his dealing with them of what sort of person he is behind the role he is in"(1974, p.298). Because it relies on this agreement to define the social situation, this perspective argues that there is no concrete meaning to interactions that cannot be redefined. Dramaturgy emphasizes expressiveness as a key element of interaction i.e. a fully two-sided view of human interaction (Goffman, 1959).

Dramaturgy theory assumes that a person's identity is not a stable, independent psychological entity, but is constantly reshaped as the person interacts with others. The dramaturgy model analyzes social interactions through the way people live their lives, like actors on a stage. This analysis gives insight into the concept of status, which is similar to theatrical roles, and roles that act as scripts and provide dialogue and action for the characters. Just as on stage, in everyday life, people manage their environment, clothing, words, non-verbal behavior, etc. to convey a certain impression to others. Goffman(1959) described each individual's "performance" as a representation of the self. A person's effort to leave a certain impression on the minds of others. This process is sometimes called impression management. Goffman(1959) distinguishes between onstage behavior, that is, behavior that is part of the performance visible to the audience, and backstage behavior, that is, behavior that people do when there is no audience. For example, a professor will probably behave a certain way in front of students in class, but will be much more informal in his home. He may also do things in class that would be considered inappropriate in front of students or colleagues.

Before people interact with others, they prepare the roles or impressions they want to give to the other person. These roles are subject to what is known in theater as breaking character (Brissett, Edgley, 1990). Inconvenient interruptions can occur if they are interrupted by people who are not supposed to be watching the performance backstage (Goffman, 1959). In addition, there are examples that show how the audience plays a role in determining the direction of an individual's performance, for example, how people ignorantly ignore many performance mistakes, such as someone tripping or spitting while speaking(Goffman, 1959). In dramaturgy analysis, a team is a group of people who work together to share a party line. Since mistakes affect everyone, team members need to share information. Team members also have inside information and are not fooled by each other's achievements (Goffman, 1959).

The scientific functions of theater

Regarding the role of theater in general, Vieites (2014) identified three functions of theater:1) education, where theatrical practice is aligned with the purpose of the process;

2) theater as a comprehensive means for developing social skills and competencies; 3) education through theater, linking it with content and skills from other disciplines.

Therefore, theater provides an opportunity for public participation and performances in which the audience is invited to think, participate and decide, as a place for personal and social change. Without going into the history of theater and analyzing its various genres, a first mention is Augusto Boal's Theater of the Oppressed (Sajnani et al., 2021) inspired by the ideas of Paulo Freire (Coudray, 2016), this method allows not only to talk, but also to analyze, to give answers, to act, to interact: the story that is presented is not which came from outside the author, but from the actors from the audience, so he can think about his social situation, identify vulnerable situations, and understand problems to contribute to the creation of solutions, to integrate various representations of reality and to explore the possibilities of change in creative and social situations. By providing methods and tools for reflection and analysis, Theatre of the Oppressed aims to foster an informed and active citizenry (Gigli, Tolomelli, & Zanchettin, 2008; Santos, 2018; Aglieri & Aprigliano, 2019).

One of the main techniques is interactive theater, where people can hold objects and move from one side of the stage to another, becoming characters in the show, guided by group vote, and give suggestions for the continuation of the conversation. Developments such as the Augusto Boal Theater (Sajnani et al., 2021) Interactive theater is not only for entertainment, like TEDs (Corazza and Macauda, 2022), but is often used for debates on social or political issues in real life: the audience can participate in what is being said in it. at the center of the debate. The activities related to these scientific topics can achieve the goal of creating the desired relationship between science and society and increase the need for a new "scientific thinking" (Murtonen, Balloo, 2019).

The theatre is not an end in itself, but a useful resource for educational practice. Theater is considered a real educational tool for human development. It favors the increase of individual sensitivity, channeling the various emotions of the spectators which often remain unexpressed. The theater also allows you to express emotions while watching a show. What is observed in practical terms has a significant impact on those who take part in a performance (Gómez, 2006). Furthermore, according to Llamas (2013) the theater and its educational function allow a real application of Gardner's theory (Gardner, Hatch, 1989) on multiple intelligences, and its impact on skills and holistic education.

The ease of attrition in public performance spaces is effective for theater spaces. Compared to static art experiences in art galleries or other art experiences such as watching movies, the coexistence of actors and audience is a unique feature of this live performance art (Sun et al., 2023). The actors breathe in the same space as the audience and communicate with them verbally and non-verbally (Karmakar, 2013). Theater is a place where three types of dyadic interaction exist: actor-performer, actor-audience and audience-audience (Sun et al., 2023). It is important to hightlited that these three combinations influence each other, whichis why theater audiences are as interested as scientists in the types of interactions that take place in the theater space (Karmakar, 2013). Also important are the physical and mental experiences of the actors and the audience.

Social interaction in the theater is choreographed and repeatable. Theater performances are probably the closest thing to real-world social interaction, but they can be controlled and manipulated experimentally. During a performance, actors do not necessarily feel the same way every time, but the plot of the play is usually the same throughout the performance. Theater allows for adaptations that are both reproducible and considered valid (Sun et al., 2023).

Therefore, theater favors the development of intelligence in all its manifestations, in particular favoring the overall development of the individual through self-awareness and personal self-construction. In this way, theater comes to favor two processes: 1) mediate creativity and innovation; 2) favor contents and notions in a more engaging and dynamic way. In this way, through the theatre, spectators develop a more critical, questioning and transformative vision. Therefore, theater is a powerful pedagogical resource (Serrani et al., 2023).

3. The project “Social sciences on stage” in detail

It is on the basis of all these scientific assumptions that the Social sciences on stage project was structured.

The project has several objectives. First, to communicate the elements that characterize the social sciences (especially anthropology, criminology and sociology) through theatre, which is understood to be one of the best tools for acquiring knowledge;

also to address contemporary social problems; and above all, to link the experience of scientific research with the world of performing arts.

The research carried out by the author of this contribution is presented and therefore attributable to topics such as: gender issues, healthcare, criminology, cannibalism, chronic diseases and much more.

The meaning of the logo

It is known that the identity of any entity firstly requires symbols to assert itself. Precisely for these reasons, a logo was created that was able to incorporate the many "dualities" that underlie this project (Figure 1).

Figure 1. The logo

First of all, the colours: black and white, as a reference both to the light and shadow areas of human action, but also related to the contrast between the backstage and the foreground, but more generally they refer to the various forms of ambivalence that accompany the man. The symbols: on the left there are the classic masks that represent the theatre, but they also have another meaning, namely the one proposed by Jung (1967) who borrows the term from the Latin *persōna persōnam*, or the mask that actors used to wear during stage performances . The person was only a reflection of the image of the character played by the actor, it took up his features, characterized him, inserted him into a role. On the right, however, we find a network, and therefore a symbolism that recalls the human ability to enter into relationships with others.

The structure of social sciences on stage

The project has a special structure. First, it opens with a video with background music. This stage has a twofold role. To create an intense atmosphere to attract attention and at the same time to be a real visual stage set.

Then the theatrical phase begins, where not only a passage from my research is interpreted, but all the characters are analyzed. This is an important moment as it serves to put the audience 'inside' the topic under discussion. The aim is to show how interviews are conducted from a social science perspective, i.e. to make the audience understand how complex it is to remain unted about a topic. At the same time, it helps to start the final stage - the analysis of everything brought to the stage - with simple, understandable language that conveys a form of knowledge that is useful not only for the purpose of the event, but also for everyday life.

Other peculiarities of the project

Social sciences on stage, intersects social sciences not only with acting, but also with art. In fact, a central element of this project is both the creation of photographs together with the presence of artistic installations on stage in line with the theme covered.

The project is totally autonomous and self-financed, because only in autonomy can a research product be provided free from any form of political, economic or other influence.

The ultimate goal would be to launch the figure of the *theatrical popularizer*: that is, a social scientist capable of transmitting his knowledge, his experience in the field, his scientific products in the theatrical context.

4. Conclusions

The project presented, based on a series of scientific references, constitutes a form of intersection between science and art, fitting into the context of scientific dissemination in a theatrical key. The intersection between these different activities such as creativity, aesthetics, it could allow communities to be brought closer to the contents and research typical of the social sciences. It is also important to highlight the advantages regarding the potential of the project to involve scientists in scientific communication activities, leading to staff development and professional skills, as already described in other studies on science communication (Clark et al., 2016). Many researches show how the difficulty of science the communicative task is usually associated with a perceived negative image and therefore with its removal scientists from these types of initiatives (Jensen et al., 2008; Davies, 2008; Dowell and Weitkamp, 2011; Borrow and Russo, 2015).This perspective had already been described in other similar science communication projects, including the cabaret initiatives the British “Brightclub” (Bultitude, 2011), “Cientistas de Pé”, the Portuguese projects by Pinto et al.(2013) and by Amaral et al. (2017).

In order to evaluate the impact that the project has on people, after each event an anonymous questionnaire is sent via email containing both personal data questions and a series of open-ended questions such as: "what do you think of the event?"; "what struck you the most?"; etc.

At this regards common adjectives in the responses were: exciting, exciting, inspiring, pleasant. Here are some of the answers: It helps you think about what you are doing; You feel like you are inside; Yes, because people want people you have to work with; Talk directly to others face to face and observe what each person is doing; But it works in a context where people come to all events and I'll stop here; I don't know if things like this can easily attract people's attention; Dedicated only to acting; It is more confusing to express yourself verbally and draw your hand to show a card to enter because something, but far from the crowd; etc.

Theater, then, is also a source of expression of stories and truths, which serves as a basis for the progress and connection of social studies. The results of this approach are not only entertaining, but also informative, educational, or educationally stimulating for social change. In this way, the theater can take various steps to analyze the socio-cultural reality presented through the performances (Veselková, 2024).

Among the most relevant aspects is the recognition of a dual need: to save people engage by closing the distance between the "scientist" and the audience responding to the topic by working on a verbal level.

The project's ability to show singularities is noteworthy of the scientific world and the daily life of researchers, approaching science and society and demystifying it the complex image of scientists.

By lifting the veil of the scientific world, theater has the ability to show concepts and discoveries useful to everyone, and not just to the scientific community.

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Between objectivism and perspectivism: reasons for a philosophy and science encounter

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Abstract

In this study we explore the encounters emerging from the restructured categorizations of subjectivity in the context of an open system that constantly confronts the structures that create it. On the one hand, science formulates valid propositions through the ratio that ensures and confirms the acquisition of knowledge, but on the other hand, the truth of its propositions is related to those mental processes which seek the truth of deeper values than those discovered on the surface of reality. The question that arises in this context is which criteria allow us to resolve these antinomies (objectivism/perspectivism) in order to develop the interdisciplinary terms between the humanitarian field of science and sciences in a spirit of unification. In this way, we approach the potential of scientific reformulation in terms of the above interpretive perspectives in order to project the overcoming of divisions under the principles of diversity and critical thinking both in the field of philosophy and science as well.

Keywords: objectivism, perspectivism, differentiation, epistemology.†

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1. Introduction

In this study we attempt to examine epistemological issues in the fields of interdisciplinary research in the light of a discourse between philosophy and science. We refer to the above discourse as a critical elevation of a condition that defines the truth of existence to the extent that it (the existence) transcends its ruptures due to the applications of artificial language. Under this premise it liberates itself from the representations of the truth and expands the perspective of experiential reality as a revolutionary rupture against the dominance of rational thought. In other words, if the cognitive desire concerns the controlling disposition of the subject in order to dominate over the nature or the society, then we enlarge upon the area of the divisions of science and the philosophical thinking in order for the above divisions to serve their asymmetries (Rotry, 1982).

On the other hand, when Snow states the different perspectives concerning the two cultures (1995), those of the sciences and those of literature and the arts to point out their misconnection, he highlights an epistemological gap that is further exacerbated even in the science itself in which a crisis is observed within it when, for example, "in the framework of the Copenhagen interpretation and with regards to the fundamental problem of elementary particles and their interactions, there was no consensus regarding the epistemology of the research programs aiming at solving it" (Evangelopoulos, 2023, p. 94).

In this way, the question we are dealing with concerns the criteria according to which it is possible to identify the lost universal concepts that scientific practice and positivist epistemology have deconstructed, being unable to integrate them into individual social events. In other words, we claim that the way in which the circuit of the unity of historico-critical forms (of a hyper-empirical dialectical epistemology) can remain open passing through the reading of those routes of subjectivity that moves in the direction of the questioning discourse and the criticism towards the legitimizations immersed in mine of their certainties.

How can one judge historical consciousness and human values by reflecting on the suffering caused by the two world wars under the blessing of scientific progress and ideologies of the time in force? With what perspective would one be able to read the truth as the ultimate truth and meaning for people's lives during the tragic periods of fascism and Stalinism? What accusation could one possibly cast at Husserl just before the Nazi atrocity spread, when he pointed out that positivism is naive because it interrupted the course of the European spirit from its historicity (Husserl, 2012)? Approaches to the above questions presuppose a cross-section of human actions in order to locate living according to the legitimations of divisions and dualistic arrangements of the world.

2. Towards a divisive consciousness

The modern world’s lack of certainty stems from rational thought's divisive practices that dissects the discourse by stripping it of any existential reflection. Both Foucault's remarks on the standardization of subjectivity (Foucault, 2017) and Castoriadis's denunciations of identifying-holistic rationality (1999) place the ratio in a state of division, since the dualistic regulations of the world have promoted the "incongruously flows of energy in the flow of the world" (Deligiorgi, 2007, p. 47).

The human, being excluded within the indisputable walls of rational thought, experiences the "time of a clock that skips moments, and is content to show dates with no meaning beyond the sum of events..." (Deligiorgi, 2007, p. 256) ensuring in this way the establishment of the Cartesian cogito. This movement of the hands of the clock transforms historicity and duration of time into entrenched structures of self-referentiality in order not to shake the shaped structure from any person's projections and actions. Therefore, scientific practice is important, serving a world of similarities in order for the circuit of the structure to remain firmly closed.

An exclusive discourse is therefore produced which entrenches persons within a passivity in order to make them objects of a controlling rationality excluding any deviation from their study as historical subjects. Thus, the control of people through knowledge occupies more and more the field of the rationality of things, rising a rationality that obeys the positivist trajectories of science. In other words, the way reality is organized passes through its most rational belief according to its interweaving with goals and arrangements, creating divisive subjects who are categorized in dualistic forms of alienation from the truth of themselves.

However, beyond the above *desire* found in the dominance of positivist clarification over its assembly against the order of a positivist cognitive model, there is also identified the desire to know which goes beyond formal validity by delving into those processes which involve a continuous opening towards the truth. But where are these processes located?

Gaston Bachelard (1884-1962) in his epistemological approaches (2000) already points out a new scientific spirit at the basis of the anti-Cartesian revolution and in the field of electromagnetic physics of mathematics and also non-Euclidean geometries (Deligiorgi, 2007). In other words, what is being attempted is the search for the possible through the search for truth beyond the circuit of what has already been done. Thus, the issue is not the imposition of certainties but rather the unraveling of connections between scientific categories and the broader historical and social realities. But how is this transformation achieved?

3. Towards an open subjectivity

In order to highlight the above transformation it is necessary to survey what happens beyond the fields of rational symbols, representations and calculations. The conventional boundaries that define and separate science from philosophy, validating the refutation or certainty of their propositions, are re-engraved under the experimental expressions of a philosophical discourse that moves in the spectrum of open subjectivities. There, where "the venturous formulations" of Logginos "sweep all the rest away with their flow" (Logginos, 1990, p. 143).

What is derived from the above approach relating to an open subjectivity in order to transcend the finite borders of dominant self-referential discourses by projecting new formulations through an expanded reflection. Against the objectivist utterances of speech produced by the artificial language of science, open subjectivity maintains its relationship with reality by shaping those terms produced by its experiential and epistemological processes. The way in which the experiences of the subject are inserted between the subject and science but also philosophy expresses the infinity of views composing concepts and ideas. As Deligiorgi also notes:

"It is necessary that we open up to the unity of reality that includes us and makes us partakers of the interactions and differentiations overlooking its objectifying approaches and become its receivers again, this time consciously and with intention..." (2002, p. 392).

However, when we refer to open subjectivity we mean the testing of conceptualizations with experiential experience in order not to absolutize the fields of science and philosophy into dead-end antinomies. So what is perspectivism about but “the negation of objectivity, that is, of an ‘intrinsic’ value of knowledge, independent of the particular context and conditionings within which this knowledge is acquired”? (Agazzi, 2016, p. 350). In other words, it is not recorded a single perspective that is shown in order to exclude another, but a differentiation that specifies the manifestations of the above fields within relevant interactions. What is expressed is a journey into the world of subjectivity to the extent that the perspective elevation of judgment is achieved.

In other words, the issue is not to solve an antinomy but to verify the propositions that make up their entanglements. And this path goes through the extent to which we can emerge in our relationship with the world in order to form a new view towards the world of science. In other words, our relationship with the world reveals our traces in history through our experiences and actions. As Berghofer state “experiences present their objects in a certain way that at least partly depends on subjective factors such as previous experiences, background beliefs, etc” (2020, p. 9). And this relationship is unique and unprecedented projecting its readings from science and philosophy in order to interpret the unity of the world.

It is not registered a dualistic rationality that legitimizes the authoritative aspirations of an absolutist proposition by recycling the terms of rational thought, but a dialectic that favors broader perspectives of interdisciplinary approaches. When an interdisciplinary relationship is established, a new form of thought emerges which constantly exceeds its limits. It also denies an absolute authoritative power. In other words, an epistemological perspectivism is produced that is interpreted as a network of relationships making our thinking more active since the doctrinal conditions of uncritical thinking are exceeded. As Massimi states “It is also, first and foremost, a forward-looking commitment to engage across scientific perspectives and retain knowledge claims that have served us well across multiple perspectives” (2021, p. 6139). When, for example, Heisenberg (1901-1976) adopts a kind of skepticism towards idealized reality in scientific validation, he turns his gaze to another reality that passes through the true face of being (1971).

Because in what way is the dismemberment of the dualisms produced by rational thought (the world from man, philosophy from science, Being from truth) overcome than with the free spirit of education, deviating from the traditional forms of domination that produce a closed system of unalterable principles? In other words, the way in which the philosophical wondering poses uncertainty regarding the rational judgments of science reduces the critical power of the mind to a form of clarification on new ways of saying. Therefore, from the dominant mathematical language that acts "as the most effective weapon for the domination of nature but also of human behavior" (Deligiorgi, 2008, p. 61) we move to where the gap between the historicity of existence and the consciousness is reduced.

Therefore, a dynamic is established, in the circuit of which the ego emerges as an existence that seeks those openings that lead it to the horizon seen under the plottings of its function with the real. One could argue that existence constantly claims the non-definite, joining one's voice with Johann Gottlieb Fichte (1762-1814) who notes that "he sees nothing definite, he expects more than he can conceive in this life , of what it will ever be able to conceive until the end of time" (1931, p. 176).

As the subject conquers the world through a reflective critique, it widens the perspective point of discourse of science by revealing the traces of open subjectivity in history. Thus, against a timeless historicity of postmodernism that reduces "the new moment" to a key dimension of its relativistic spirit, a science that is in line with philosophy is juxtaposed in order to survey the subterranean routes that constitute the being/ontological relations.

At the point where philosophy and science meet, we can shed light on political and moral issues guarded by rationality in the entire spectrum of our lives. We should look for those remnants of the path taken by rationalism. We will identify what was not leveled by the nihilism produced by History.

4. Conclusion

According to the above remarks, we would argue with regards to the formed structures of rationality that what is examined concerns the Law and the operation of the phenomena but not the becoming and the meaning that emerges from the reason. A deterministic logic dominates, which detects the surface of reality on the basis of an axiomatic logic, adding grist to the mill of a closed system that ultimately becomes an ideology deviating from the truth. Thus, an ahistorical consciousness is formed that persists in its solipsistic routs by recycling the present as an absolute condition that delimits its certainties in order to exist within the current discourse.

In these closed structures, then, scientism is guarded in order to observe the cognitive process that ensures valid scientific achievements and in this way becomes consistent in its idealized reference structure. In this way, however, the real is idealized, making the human experience a measurable calculation that ultimately cancels any attempt to reduce it to reflective imagination.

Because if we can finally go beyond the closed circuits of the discourse in force that reduces its validity to the meaning instituted by the value of its rules, we need to question the way in which science checks the validity of its power by necessitating its questioning of reflective discourse in order to deepens into the clarifying processes of knowing.

Under these conditions, science and philosophy are not subject to Cartesian dualistic readings of a distorted reality that resents finality in order to deseive the division of ratio, but to a real world where the experiences of the subject are true within the everyday biota giving meaning to its existence. In this way, Frankenstein restores the human sciences as a need of existence to experience the sensitivity of others while simultaneously discovering his own.

In this context, the discourse of philosophy constitutes a morality that criticizes both the historical conditions that have been formed to what extend and what constitutes our existence. Finally, the views of the circuits concern the open character of reality by deeping into the roots of the Laws in order to break the chains of the authority in force and looking at the flows of the river the cycles of History that do not identify with the beginning of their previous point.

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Matematica e Retorica a Roma: una lezione di geometria piana nell'*Institutio oratoria* di Quintiliano

(Mathematics and Rhetoric in Rome: A Lesson in Plane Geometry in Quintilian's Institutio Oratoria)

Mariacarolina Santoro[1]

Sunto

Prendendo in esame quanto il celebre maestro di retorica Marco Fabio Quintiliano (35 d.C. ca - 100 d.C. ca) scrive in età flavia nella sua *Institutio oratoria* a proposito dell'importanza dello studio della Matematica nella formazione di base del futuro perfetto oratore romano, si intende approfondire in particolare una porzione del lungo passo presente nel I libro (I 10, 34-49), nello specifico i §§ 39-45. In essi l'autore latino, partendo dall'affermazione che la geometria, non meno dell'aritmetica, con il suo procedimento razionale smaschera ciò che, pur apparendo verisimile, in realtà è falso, inserisce in funzione esemplificativa una vera e propria lezione di geometria piana sul problema dell'isoperimetria. Lo scopo dell'autore è di fornire una prova dell'utilità della geometria per educare a un uso corretto dell'*ars dicendi*.

Keywords: Quintiliano; retorica; isoperimetria.[2]

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Mariacarolina Santoro

Abstract

Examining what the famous rhetoric teacher Marcus Fabius Quintilian (ca. 35 CE – ca. 100 CE) writes during the Flavian period in his Institutio Oratoria about the importance of studying mathematics as part of the foundational education for the future perfect Roman orator, this paper focuses on a specific section of the long passage found in Book I (I 10, 34-49), particularly §§ 39-45. In this section, the Latin author, starting from the assertion that geometry, no less than arithmetic, with its rational method, reveals what may appear plausible but is actually false, includes a true lesson in plane geometry on the problem of isoperimetry. The author's goal is to demonstrate the utility of geometry in educating one to make proper use of the ars dicendi (art of speaking).
Keywords: Quintilian; rhetoric; isoperimetry.

1. Introduzione

Il presente contributo è una parziale ripresa e prosecuzione di un lavoro realizzato in occasione del Workshop "Matematica e Latino nella scuola secondaria di secondo grado", svoltosi a dicembre del 2023 presso Sapienza Università di Roma[3]. In quell'occasione la mia attenzione si è rivolta a un lungo passo del libro I dell'*Institutio oratoria* di Quintiliano in cui l'autore latino argomenta a favore dello studio della matematica per la formazione completa del futuro oratore ideale[4]. In questa sede è mia intenzione approfondire di quel passo una porzione specifica, dove troviamo, in forma di digressione, una lezione elementare del problema isoperimetrico utilizzata dall'autore in quel contesto allo scopo di dimostrare come la geometria possa smascherare il falso che ha parvenza di vero e, quindi, come possa educare a un corretto e onesto uso dell'*ars dicendi*.

[3] "Workshop Matematica e Latino nella scuola secondaria di secondo grado", organizzato dal Dipartimento di Matematica "Guido Castelnuovo" e dal Dipartimento di Scienze dell'Antichità, Sapienza Università di Roma, 15-16 dicembre 2023.

[4] *L'importanza dello studio della matematica nella formazione del perfetto oratore secondo Quintiliano:* vir bonus "geometriae" peritus, in Atti del Workshop Matematica e Latino nella scuola secondaria di secondo grado (in corso di stampa su «Res Publica Litterarum», III serie).

2. Lo studio della matematica nel percorso formativo dell'antica Roma

Nel sistema d'istruzione scolastico dell'antica Roma,[5] dalla tarda repubblica all'impero, lo studio della matematica e, in generale, delle discipline scientifiche pare non avere rivestito molta importanza, relegato sostanzialmente al solo livello elementare e quindi fortemente svalutato nel percorso formativo del *civis Romanus*, a vantaggio di altre discipline afferenti all'area umanistico-letteraria, prime fra tutte la retorica.

Se, infatti, gli studi elementari presso il *litterator* (detto anche *primus magister* o *ludi magister*) prevedevano non solo l'acquisizione delle abilità di lettura e scrittura, ma anche, presso il *calculator*[6], quella di fare conti, gli studi di secondo livello alla scuola del *grammaticus* vertevano essenzialmente su materie letterarie. Nello specifico, coloro che vi accedevano studiavano lingue e letterature greca e latina attraverso la lettura dei poeti, da cui si ricavavano nozioni di storia, geografia, mitologia e pochi rudimenti di fisica e astronomia; pertanto, quel poco di conoscenze scientifiche che venivano acquisite in questa fase erano «in margine ai classici»[7]. Presso la scuola del *rhetor*, poi, ovvero al livello più alto d'istruzione, i giovani più fortunati approfondivano lo studio dei testi dei grandi prosatori, memorizzando e componendo essi stessi testi di diversa tipologia e difficoltà, con l'obiettivo di raggiungere il pieno possesso dell'arte oratoria.

Da questa ridotta considerazione dei Romani nei confronti delle discipline scientifiche è nata l'opinione diffusa secondo la quale a Roma la matematica non abbia mai attecchito[8]. Tuttavia, tale giudizio è inficiato da un confronto

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[5]Sul percorso scolastico a Roma, ispirato a quello greco delle scuole ellenistiche, v. [14] pp. 353-83; e [16] cap. 14.

[6] La figura del *calculator*, che purtroppo ci è poco nota, secondo H.I. Marrou [14], pp. 360 e 554 nota 13, sarebbe una specie di professore d'insegnamento tecnico. Secondo D.A. Russel, [18], p. 214, il *calculator* era «the expert in shuttling around the *calculi*, pebbles, used for counting on the abacus, and in reckoning large number on his fingers, according to the very complex conventions in use ».

[7] V. [14], p. 371sg.

[8] È noto il giudizio *tranchant* dello storico del pensiero matematico Morris Kline [11], p. 208s., secondo il quale "Il primo disastro fu l'avvento dei Romani, il cui ruolo nella storia complessiva della matematica fu quello di agenti distruttori"; scrive che "la matematica romana merita a malapena di essere menzionata" visto anche che in undici secoli di esistenza della civiltà latina "non vi fu un solo matematico romano" e "questo solo fatto dice virtualmente tutto". Entrando poi nello specifico della disciplina, lo studioso definisce "rozza" l'aritmetica e "approssimate" le formule geometriche romane, inoltre l'applicazione tanto dell'una quanto delle altre era eminentemente pratica, finalizzata "all'agrimensura, alla fissazione dei confini delle città e alla determinazione degli appezzamenti di terreno su cui dovevano essere costruite le case e i templi". Non molto lusinghiere pure le parole di William H. Stahl [20], il quale

implicito con lo sviluppo delle scienze nella Grecia classica, il quale però non risulta ben posto: non è infatti corretto mettere a confronto la cultura latina con quella dei Greci del V-IV secolo a.C. Se è vero che a Roma è presente, sin dall'età repubblicana e poi per tutti i secoli dell'impero, una netta preferenza per gli studi letterari, è anche vero che essa è dovuta all'influsso della contemporanea cultura ellenistica, nella quale pure, dopo secoli di straordinaria fioritura degli studi scientifico-matematici, si registra lo stesso fenomeno limitatamente all'ambito scolastico.[9] A partire dall'età ellenistica, infatti, anche in Grecia lo studio delle scienze cede il posto sempre di più alle discipline letterarie nell'insegnamento secondario ed è riservato agli studi superiori per specialisti e filosofi[10].

3. La matematica nella formazione del perfetto oratore

Se poi prendiamo in considerazione quanto scrive in epoca flavia il celebre *rhetor* Marco Fabio Quintiliano (35 d.C. ca - 100 d.C. ca) in un passo della sua *Institutio oratoria* (90 d.C. ca), a proposito dell'importanza dello studio della matematica nella preparazione di base del futuro perfetto *orator* romano, si potrebbero individuare altre ragioni per attenuare posizioni così fortemente negative sul rapporto tra matematica e cultura latina[11], in special modo all'interno di un discorso di carattere pedagogico. L'autore, dotato di una sensibilità di educatore per molti versi vicina alla nostra, ha posizioni originali e non consuete per il suo tempo e per il mondo antico in generale, in fatto sia di pratica didattica che di teoria pedagogica. Anche a proposito degli studi matematici sembra allontanarsi dalle opinioni più diffuse tra i suoi contemporanei, influenzati verosimilmente dalle posizioni svalutative ciceroniane[12], e mostra di rifarsi piuttosto a una più antica tradizione

scrive che la scienza romana "si mantenne sempre su di un piano decisamente mediocre" (p. 5) e che, addirittura, nel II secolo d.C. questa "creaturina malformata nata in un mondo inospitale, cresciuta rachitica per carenza di nutrimento adeguato, era precipitata in uno stato di decadenza senza avere mai dato prova di essere in grado di diventare adulta" (p. 161).

[9] Cf. [14] p. 372.

[10] Cf. [14] pp. 250sg.

[11] Il sapere scientifico e tecnico dei Romani di recente è stato oggetto di una rivalutazione in seguito a un lavoro di revisione storiografica che ha superato la tesi della stagnazione tecnologica, ossia la valutazione negativa nei confronti delle conoscenze scientifiche nell'antichità e soprattutto nel mondo romano. Su questo v. G. Di Pasquale [8] pp. 169-172.

[12] Cicerone, che per Quintiliano è la principale fonte di teoria oratoria oltre che modello di stile, ha posizioni molto negative sull'importanza dello studio della matematica: nei primi capitoli del trattato *De oratore* (I 3-6, 11-16) leggiamo opinioni chiaramente contrarie alla padronanza di discipline scientifiche poiché la matematica sarebbe una scienza astrusa e

pedagogica greca[13] risalente a Platone, che, com'è noto, assegnava grande rilievo alla formazione scientifica[14].

Il passo che ci interessa chiude il capitolo decimo del primo libro (I 10, 34-49) e arriva dopo che l'autore ha già affrontato una lunga serie di questioni preliminari sull'educazione dei bambini a partire dalla prima infanzia fino all'età scolare. Nel capitolo 10, in particolare, discute la questione se sia indispensabile a un oratore una cultura generale (*orbis doctrinae*), quella che i Greci definivano "enciclopedica" (*encyclion paedian* I 10, 1) ovvero una formazione che comprendesse conoscenze in tutte le discipline (*artes*)[15], anche quelle compiute e perfette in sé indipendentemente dall'eloquenza, ma che da sole non sono sufficienti a formare un oratore. Se dunque l'istruzione a Roma, benché fondata sul concetto greco della *enkyklios paideia,* che comprende tanto le discipline letterarie quanto quelle matematiche, era, come già evidenziato, fortemente sbilanciata a favore delle prime[16], le posizioni di Quintiliano, come vedremo, sembrano molto più vicine a quelle dell'autentica *paideia* greca.

Dopo aver trattato della necessità dello studio della grammatica in tutti i suoi aspetti (I 4-9), non senza aver preso in considerazione le possibili obiezioni alla sua tesi a favore di una cultura ampia e aver spiegato le ragioni delle sue posizioni (I 10, 1-8), Quintiliano indica le altre materie a suo giudizio necessarie per la preparazione culturale di base del futuro oratore ideale, ovvero la musica (I 10, 9-33) e la *geometria* (I 10, 34-49), termine attraverso cui si riferisce sia all'aritmetica che alla geometria stessa.[17] A chi si chiede a cosa servano musica e matematica a un oratore (I 10, 3), l'autore con tono garbatamente polemico risponde che non sono certo queste discipline a fare un

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oscura, complicata ed esatta; visto che sono molti coloro che in essa hanno raggiunto la perfezione, se ne deduce, sorprendentemente, che chiunque vi si dedichi possa padroneggiarla con successo. Insomma, la matematica sarebbe una materia alla portata di tutti. In altri passi dello stesso dialogo Cicerone mostra di apprezzare un approccio interdisciplinare per quanto concerne la formazione dell'oratore, ma non si allontana quasi mai dall'ambito umanistico: tra le discipline in qualche modo connesse con l'oratoria figurano, infatti, filosofia, storiografia, musica, pittura, diritto civile, studio dell'animo umano (psicologia) e della natura, astronomia, arte militare, arboricoltura, navigazione, architettura, poesia. Su questo vd. F. Boldrer, [2], pp. 281-87.

[13] Sulla formazione scientifica in Grecia dall'età classica all'epoca ellenistica v. [14], pp. 241-253.

[14] Sull'importanza della matematica per Platone v. [9]. Una raccolta completa dei passi matematici nelle opere di Platone è nel volume di M. Cavallaro, [4]. Il rilievo che il filosofo assegnò agli studi matematici si evince in particolare dal VII libro della *Repubblica* (521 c - 527 d), dove afferma che i futuri dirigenti dello stato devono possedere un'istruzione accurata in matematica, geometria e astronomia. Cf. [18], pp. 210-225.

[15] Il Russel [18] p. 211s., discutendo la problematicità dell'espressione greca *enkyklios paideia,* pensa che si possa trattare di un "prototype of tha later trivium and quadrivium".

[16] Cf. [14] p. 371s.

[17] M. Kline [11] p. 210, spiega che il termine "matematica" cadde in disgrazia perché gli astrologi erano chiamati *mathematici* e l'astrologia era condannata dagli imperatori romani.

oratore, la loro conoscenza tuttavia gli gioverà e contribuirà a renderlo completo (I 10, 6), giacché l'eloquenza necessita di una molteplicità di discipline le quali, anche se non si mostreranno in piena evidenza nel parlare, forniscono però una forza segreta (*vim occultam*) e vengono percepite anche se *tacitae*, "silenziose" (I 10, 7).

Partendo da questa prospettiva ampia, che coniuga l'ambito umanistico con quello scientifico, la trattazione di Quintiliano si articola in tre fasi: viene sottolineata in primo luogo la funzione formativa della materia, si passa poi alla sua ricaduta immediata, in termini di utilità pratica, nell'attività oratoria, infine, allargando ulteriormente lo sguardo, si lascia intravedere la sua valenza in campo etico.

In questa sede si intende approfondire del lungo e interessantissimo passo dedicato alle ragioni dell'importanza dello studio della matematica nella formazione oratoria, solo i §§ 39-45, in cui Quintiliano si allontana dall'argomento principale per inserire una vera e propria lezione elementare di geometria piana. Tuttavia, il discorso non può prescindere dal contesto in cui si trova, che tenterò di riassumere brevemente[18].

Nel primo paragrafo (I 10, 34), dopo aver sottolineato i benefici della matematica, noti ai più (tenere viva la mente, rendere acuto l'ingegno e procurare velocità nell'apprendimento ai piccoli discenti), l'autore aggiunge anche che essa giova non tanto, come le altre discipline, una volta che se ne siano acquisiti i contenuti, ma nel processo stesso dell'acquisizione di essi, cioè nel processo di apprendimento e costruzione del sapere. L'autore è quindi consapevole dei benefici *in itinere* degli studi matematici sulle giovani menti in formazione, i cui effetti vanno al di là del possesso nozionistico dei contenuti disciplinari. Questo non è, tuttavia, frutto originale del suo pensiero, ma si tratta, come precisa lui stesso (§ 35), della *"vulgaris opinio*", che proviene da posizioni[19] evidentemente già diffuse, utilizzate qui solo come punto di avvio della parte più personale della trattazione.

Quintiliano, specificati i due settori in cui si divide la "*geometria*" ("*numeros*", ovvero l'aritmetica, e "*formas*" ossia la geometria), passa a esplicitarne i rispettivi vantaggi pratici. La conoscenza della prima ("*numerorum notitia*") occorre al futuro *orator* nei processi per non apparire "*indoctus*" allorquando in pubblico, nel pieno di un'orazione, dimostri incertezza nel fare somme e appaia goffo nell'uso delle mani[20], mostrando con

[18] Il passo si trova inserito tra i §§ 34-38 e §§ 46-49.

[19] In queste affermazioni Quintiliano sembra riecheggiare concetti platonici. Su questo cf. [10] pp. 114sg. e 122.

[20] L'indigitazione o *digitorum flexio* era un complesso sistema di numerazione strumentale che si serviva delle dita della mano, usato soprattutto dai Romani, ma universalmente noto agli antichi popoli del bacino mediterraneo, e di lì trasmesso agli Arabi e ai Berberi, presso i quali ancora si riscontra. Le regole dell'indigitazione ci sono note grazie ai trattati medievali, come il *Liber de loquela per gestum digitorum* di Beda e la *Summa* di Luca Pacioli (1494).

le posizioni delle dita numeri non rispondenti ai calcoli espressi a voce[21]. Subito dopo (§ 36) ricorda come anche la conoscenza della geometria ("*linearis ratio*") sia spesso utile nei processi civili, quando si trattano contese su confini e relative misurazioni di terreni.

Il contributo della geometria, però, non si riduce a questi pur rilevanti vantaggi pratici, giacché essa presenta un legame più significativo con l'oratoria (§ 37): si tratta dell'ordine ("*ordo*"), importante punto di contatto tra le due materie, in quanto requisito necessario a entrambe. E c'è ancora dell'altro, ossia il rigoroso metodo logico-deduttivo: sia la geometria che il discorso, muovendo da premesse, provano le conclusioni e dimostrano l'incerto partendo dal certo[22]; inoltre, entrambe si servono dei sillogismi (su cui si fondano le soluzioni dei problemi geometrici) i quali, se è vero che sono propri più della dialettica che della retorica, tuttavia nei discorsi possono talvolta risultare necessari e assumere anche una particolare forma, quella dell'entimema, il sillogismo imperfetto (§ 38).

A questo punto (§ 39) inizia la parte per noi più significativa, in cui l'autore individua un ulteriore elemento di contatto tra retorica e geometria: l'orazione ha come suo scopo più importante quello di provare la veridicità di qualcosa, ovvero dimostrarla con prove efficaci, proprio come quelle usate nelle dimostrazioni di geometria piana. A tale proposito Quintiliano aggiunge che la geometria, non meno dell'aritmetica, con il suo procedimento razionale, smaschera ciò che, pur apparendo verisimile, in realtà è falso. Questa importante affermazione, la quale sottintende che smascherare gli inganni a favore del vero è quanto pure l'orazione deve fare in molte situazioni, è seguita da un'esposizione di argomento geometrico in funzione esemplificativa, in cui sono trattate questioni relative a estensione di superfici di diversa forma e a misurazioni di perimetri e aree.

4. Una lezione elementare sul problema isoperimetrico

La trattazione si presenta come una vera e propria lezione di geometria piana in forma di digressione di lunghezza alquanto sproporzionata rispetto all'insieme (§§ 40-45). Sembra quasi che l'autore abbia voluto dare un saggio della sua preparazione in materia, la stessa che ogni buon oratore dovrebbe arrivare a possedere. Ma probabilmente lo scopo principale che egli intende perseguire è risultare convincente ai suoi lettori con esempi chiari su una questione che gli sta particolarmente a cuore.

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[21] Su questo v. [11] p. 209.
[22] Russel [18] p. 212, richiama a questo proposito Euclide.

Riporto di seguito il passo in questione con relativa traduzione.[23]

39. *Falsa quoque veris similia geometria ratione deprendit. Fit hoc et in numeris per quasdam quas pseudographias vocant, quibus pueri ludere solebamus. Sed alia maiora sunt: nam quis non ita proponenti credat : «Quorum locorum extremae lineae eandem mensuram colligunt, eorum spatium quoque, quod iis lineis continetur, par sit necesse est?* ***40.*** *At id falsum est : nam plurimum refert cuius sit formae ille circumitus, reprehensique a geometris sunt historici, qui magnitudinem insularum satis significari navigationis ambitu crediderunt.*

Nam ut quaeque forma perfectissima, ita capacissima est. ***41.*** *Ideoque illa circumcurrens linea, si efficiet orbem, quae forma est in planis maxime perfecta, amplius spatium complectetur quam si quadratum paribus oris efficiat, rursus quadrata triangulis, triangula ipsa plus aequis lateribus quam inaequalibus.* ***42.*** *Sed alia forsitan obscuriora ; nos facillimum etiam imperitis sequamur experimentum. Iugeri mensuram ducentos et quadraginta longitudinis pedes esse dimidioque in latitudinem patere non fere quisquam est qui ignoret, et qui sit circumitus et quantum campi cludat colligere expeditum.* ***43.*** *At centeni et octogeni in quamque partem pedes idem spatium extremitatis, sed multo amplius clusae quattuor lineis areae faciunt. Id si computare quem piget, brevioribus numeris idem discat. Nam deni in quadram pedes quadraginta per oram, intra centum erunt. At si quini deni per latera, quini in fronte sint, ex illo quod amplectuntur quartam deducent eodem circumductu.* ***44.*** *Si vero porrecti utrimque undeviceni singulis distent, non plures intus quadratos habebunt, quam per quot longitudo ducetur; quae circumibit autem linea, eiusdem spatii erit, cuius ea, quae centum continet.*

Ita quidquid formae quadrati detraxeris, amplitudini quoque peribit. ***45.*** *Ergo etiam id fieri potest ut maiore circumitu minor loci amplitudo cludatur. Haec in planis; nam in collibus vallibusque etiam imperito patet plus soli esse quam caeli.*

39. La geometria con il suo procedere razionale smaschera anche ciò che è falso ma assomiglia al vero. Questo accade pure nell’aritmetica, attraverso quelle che chiamano false figure, con cui eravamo soliti giocare da piccoli. Ma ci sono applicazioni più importanti: chi infatti non crederebbe alla proposizione «Figure con perimetri di identica misura, non possono che occupare, dentro quei perimetri, identico spazio»? **40.** Eppure è affermazione falsa: infatti a contare più di tutto è la forma di quel perimetro, e i geometri hanno biasimato gli storici convinti che l’area di un’isola venisse indicata con sufficiente esattezza dal suo periplo.

[23] Il testo latino è tratto dall’edizione per la Collection Budé curata da J. Cousin, Le Belles Lettres, Paris 1975-1980. La traduzione è quella di S. Corsi [6], pp. 239-241.

In effetti, quanto più una figura è regolare, tanto più copre una superficie estesa. **41.** Perciò, descrivendo un cerchio, ovvero la figura piana più regolare, un perimetro comprenderà un'area maggiore che, a parità di lunghezza, descrivendo un quadrato; lo stesso vale fra quadrati e triangoli, e fra i triangoli medesimi hanno superficie più grande gli equilateri rispetto agli scaleni. **42.** Ma ci sono altre questioni forse più oscure; seguiamo un esperimento di estrema semplicità anche per chi non se ne intenda. Al mondo quasi nessuno ignora che la misura dello iugero è di duecentoquaranta piedi di lunghezza e centoventi di larghezza, sicché si fa presto a calcolarne il perimetro e l'area. **43.** Ebbene: centottanta piedi per ogni lato danno un perimetro identico, ma racchiudono un quadrilatero di area assai più vasta. E se qualcuno non ha voglia di effettuare questo conto, provi a capacitarsene su numeri più piccoli. Infatti, dieci piedi di lato per costruire un quadrato danno quaranta di perimetro, e cento come superficie interna. Ma se i piedi di lunghezza saranno quindici, e cinque quelli di larghezza, essi da quell'area, pur restando identico il perimetro, perderanno la quarta parte. **44.** Se, invece, due lati di diciannove piedi distano la larghezza di un piede solo, all'interno conterranno un'area di piedi quadrati non superiore alla loro lunghezza, mentre il perimetro sarà identico a quello del quadrato con superficie di cento.

Così, tutto quanto si toglierà alla figura del quadrato, verrà meno anche alla sua area. **45.** E può dunque perfino accadere che un perimetro maggiore contenga uno spazio minore. Questo per quanto concerne le figure piane; infatti, anche a un inesperto appare chiaro che valli e colline racchiudono più suolo del cielo che le copre.

L'esposizione parte da un'affermazione che solo in apparenza è veritiera, ovvero che "figure con perimetri di identica misura non possono che occupare, dentro quei perimetri, identico spazio". L'affermazione, infatti, è falsa poiché a contare più di tutto è la forma di quel perimetro. Ad essere caduti in tale errore, dice Quintiliano, sono stati gli storici, spesso rimproverati dai geometri per la loro convinzione che l'area di un'isola venisse indicata con sufficiente precisione dal suo periplo[24]. L'estensione di una superficie, spiega l'autore, dipende dalla sua forma e non dal suo perimetro e quanto più una figura è regolare, tanto più estesa sarà la sua superficie.

Dunque, un cerchio, che è chiaramente la figura piana più regolare di tutte, non può che comprendere un'area maggiore di un quadrato a parità di perimetro (fig. 1); e lo stesso vale se mettiamo a confronto quadrati e triangoli o anche triangoli stessi di uguale perimetro ma di diversa forma, giacché gli equilateri risulteranno di superficie maggiore rispetto agli scaleni.

[24] Secondo P. Maroscia [13] p. 384, Quintiliano potrebbe alludere a un passo di Tucidide.

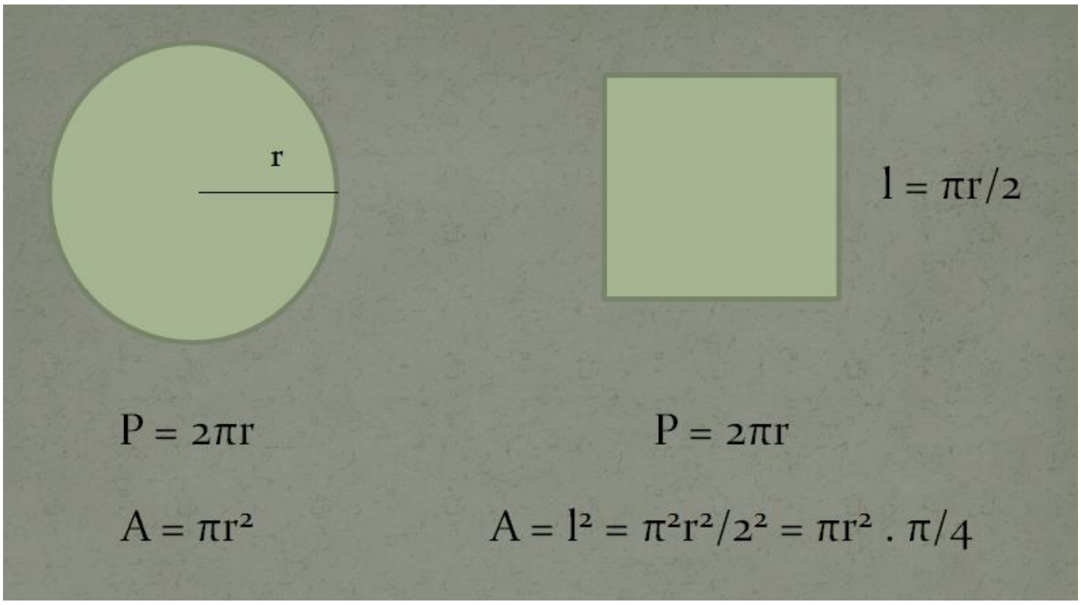

Figura 1.

Se però questo sembra di facile comprensione e risulta in modo evidente, più complesse potrebbero essere altre questioni che vengono affrontate dal maestro subito dopo attraverso la proposta di un esperimento semplice rivolto ai più inesperti.

Questa volta il discorso prende in considerazione lo iugero (fig. 2), unità di misura di superficie usata nell'antica Roma, equivalente a un rettangolo di 240 × 120 piedi romani[25] (ossia circa 2500 metri quadrati), del quale si calcola facilmente perimetro (720 piedi) e area (28800 piedi quadrati). Lo stesso perimetro di 720 piedi è dato anche da un quadrato di 180 piedi per lato, il quale però racchiude un'area molto più vasta (32400 piedi quadrati).

[25] Il *pes* romano, che era la base delle misure di lunghezza, equivaleva a circa 30 centimetri.

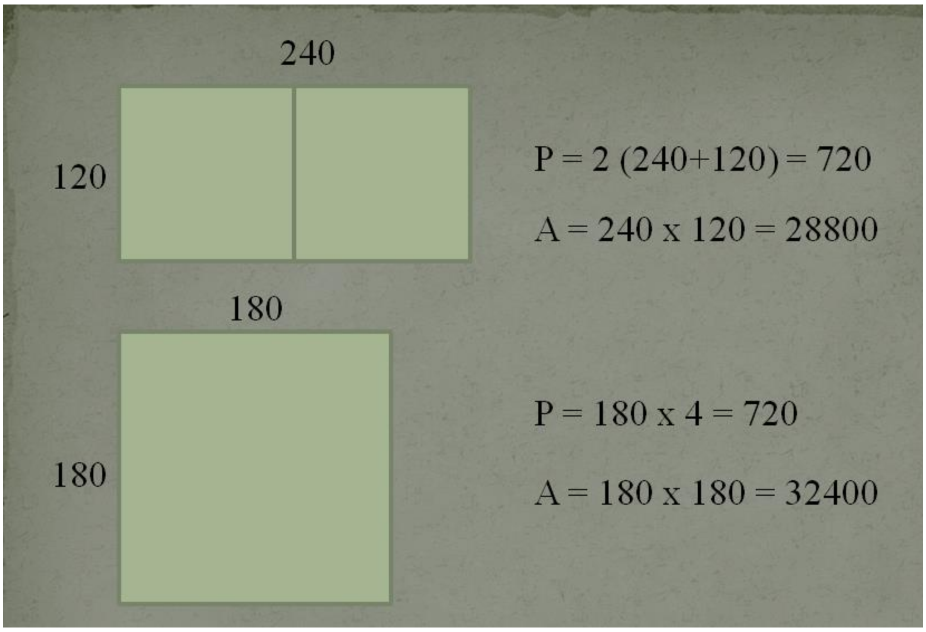

Figura 2

Per chi non abbia voglia di fare calcoli così grandi, Quintiliano, da buon docente, propone lo stesso esperimento con numeri più piccoli: un quadrato di 10 piedi per lato (fig. 3) avrà un perimetro di 40 piedi e 100 piedi quadrati come superficie interna; se invece misuriamo un rettangolo di 15 piedi di lunghezza e di 5 piedi di larghezza (fig. 3), avremo un perimetro identico al quadrato suddetto (40 piedi), ma un'area minore (75 piedi quadrati).

Se poi prendiamo in considerazione un rettangolo della dimensione di piedi 19 × 1, avremo ancora una volta un perimetro di 40 piedi identico al quadrato con superficie di 100 piedi quadrati, ma un'area di molto più piccola, non maggiore della lunghezza del rettangolo stesso (19 piedi quadrati). Perciò, tutto quanto è sottratto alla figura del quadrato, verrà meno anche alla sua area; pertanto, conclude l'autore, può addirittura accadere che un perimetro maggiore contenga uno spazio minore.

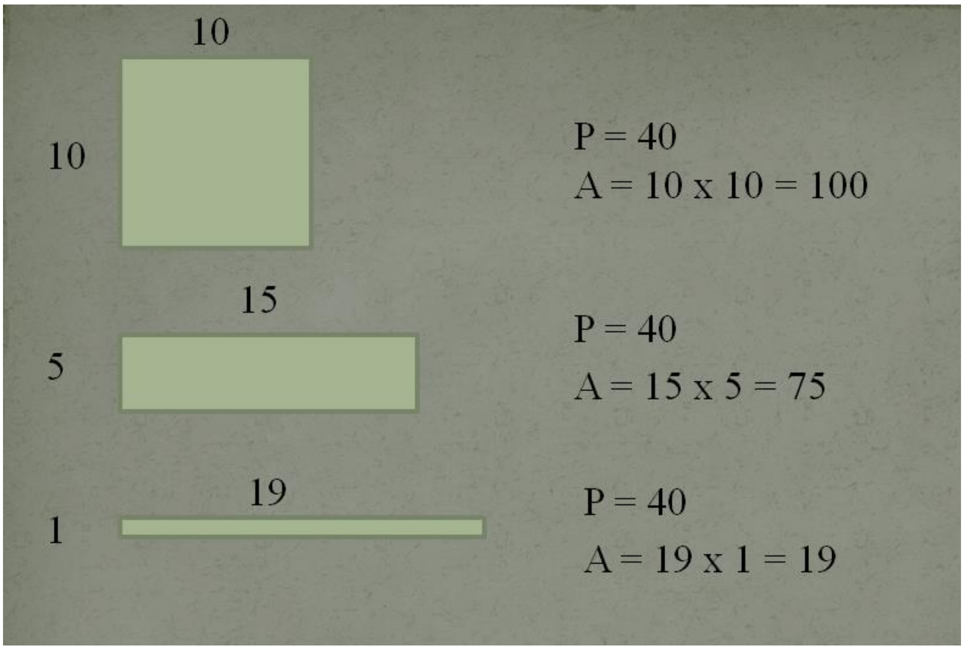

Figura 3

Alla fine troviamo anche un breve accenno a superfici non piane (come colline e valli), le quali avranno ovviamente un'estensione maggiore del cielo che le ricopre, come è evidente anche ai più inesperti.

Come si vede, si tratta di una lezione[26], con tanto di esemplificazioni, che richiama, pur senza enunciarlo, il problema classico dell'isoperimetria[27]: *"Tra tutte le figure piane aventi lo stesso perimetro, determinare quelle aventi area massima.*" La cui soluzione è: *"Il cerchio ha l'area maggiore di qualsiasi poligono regolare di uguale perimetro."*

Tale problema, noto anche come "problema di Didone"[28], ha origini antichissime che affondano le radici addirittura nel mito[29]. Nella matematica

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[26] Questa sezione del passo è definita "introduzione elementare al classico problema isoperimetrico" da P. Maroscia [13] p. 375. A p. 382 lo studioso, apprezzando la "sensibilità e la finezza psicologica di Quintiliano nel trattare argomenti di matematica", afferma che "l'esposizione in questione potrebbe venir presa a modello per una lezione di geometria, anche oggi".

[27] Sull'argomento in generale v. [1] e [15].

[28] Su questo v. [3] pp. 41-43.

[29] La fonte letteraria più antica del mito di fondazione di Cartagine da parte della regina fenicia Didone, fuggita da Tiro e giunta nel territorio del re dei Getuli Iarba, è un passo dell'*Eneide* virgiliana (I vv. 335-368), in cui si accenna all'astuzia della donna nell'aggirare l'ostacolo posto da Iarba: il re, infatti, le aveva concesso di occupare tanto territorio quanto ne può delimitare una pelle di bue; la regina allora fece tagliare la pelle in strisce sottilissime, le fece legare alle estremità in modo da formare una lunghissima corda e con essa fece circondare una porzione di terreno lungo la costa a forma di semicerchio per poter racchiudere un'area il più possibile ampia.

greca antica fu probabilmente formulato per la prima volta intorno al IX secolo a.C., non abbiamo però notizie dei matematici che si occuparono dell'argomento prima del periodo ellenistico. Il primo di cui possiamo dire con certezza che studiò il problema fu Zenodoro[30], matematico greco vissuto nel II sec. a.C., autore di un perduto trattato sullo studio delle figure isoperimetriche a noi noto attraverso l'opera *Mathematicae collectiones* di Pappo e gli scritti di Teone (entrambi attivi, a poca distanza l'uno dall'altro, come docenti alla scuola di Alessandria nel IV sec. d.C.). Zenodoro per primo tentò di fornire una dimostrazione della proprietà isoperimetrica dei poligoni regolari e del cerchio che, per quanto corretta, conteneva delle lacune[31]. Il suo teorema può essere così enunciato: "*Tra i poligoni aventi uguale perimetro, quelli regolari hanno area massima e il cerchio ha l'area maggiore di ogni poligono di uguale perimetro".* Possiamo ipotizzare che la sua opera fu la fonte da cui Quintiliano attinse materiale per il contenuto del nostro passo, come si può ricavare attraverso il confronto tra il suo testo e alcune delle proposizioni fondamentali della dimostrazione del teorema isoperimetrico del matematico greco ricostruite attraverso i suddetti scritti[32].

Si tratta di uno dei problemi più affascinanti e longevi della storia della Matematica. Pur essendo la soluzione intuitivamente nota sin dall'antichità, sono trascorsi molti secoli prima che si arrivasse a una sua dimostrazione completa e rigorosa. Molti matematici, infatti, hanno lavorato a tale scopo a partire dalla fine del Seicento, riuscendoci parzialmente solo dopo la prima metà del 1800. Tra questi ricordiamo in primo luogo Jacob Steiner, matematico svizzero della prima metà del XIX secolo, le cui dimostrazioni furono però ritenute incomplete dal matematico tedesco Dirichlet; poi, il tedesco Weierstrass, che fornì la prima vera dimostrazione rigorosa, di tipo analitico e non geometrico, applicabile solo a una classe di insiemi regolari. Bisogna arrivare al XX secolo per una risoluzione completa del problema fornita dal matematico italiano Ennio De Giorgi[33].

5. Conclusioni

Finita l'esposizione geometrica, Quintiliano ritorna all'argomento principale e introduce un ulteriore straordinario vantaggio degli studi

[30] Sulla figura di questo matematico alessandrino v. [21].
[31] Sul problema isoperimetrico in Zenodoro v. il contributo di G.P. Leonardi, [12].
[32] In particolare, nel testo quintilianeo ritroviamo concetti simili a quelli contenuti nella seconda e nella decima proposizione individuata da G.P. Leonardi, [12] pp. 6s., laddove lo studioso elenca le dieci proposizioni che compongono la dimostrazione del teorema sulla base dell'ordine seguito da Pappo. *Proposizione 2: un cerchio ha area maggiore di qualunque poligono regolare di egual perimetro; Proposizione 10: tra tutti i poligoni di ugual perimetro e numero di lati, quelli regolari (equilateri ed equiangoli) hanno area maggiore.*
[33] Sulla storia del problema classico dell'isoperimetria v. [5] e [7].

matematici (§ 46), affermando che la *geometria* è quella disciplina che si eleva fino a studiare razionalmente il cosmo e quindi a spiegare con calcoli numerici le orbite regolari e fisse degli astri, grazie ai quali impariamo che nulla nell'universo avviene a caso. L'autore, riferendosi evidentemente all'astronomia, individua nelle conoscenze scientifico-matematiche applicate ai fenomeni naturali quella funzione di antidoto contro le paure e le superstizioni popolari che attanagliano l'uomo e gli impediscono di vivere serenamente, funzione tradizionalmente riconosciuta alla filosofia[34].

Tutta la trattazione, che volge ormai alla conclusione, culmina nella seguente riflessione (§ 49): se è vero che attraverso le dimostrazioni di geometria piana si possono risolvere molti problemi altrimenti difficilmente comprensibili (come la tecnica della divisione, la sezione all'infinito e la velocità di crescita), e se è vero pure che l'oratore deve saper parlare di tutti gli argomenti, allora egli non può in nessun modo ignorare la matematica ("*ut, si est oratori ... de omnibus rebus dicendum, nullo modo sine geometria esse possit*").

Secondo Marrou[35], Quintiliano, quando conclude la sua trattazione dicendo che l'oratore non può in nessun modo essere privo di *geometria*, usa una "bella formula, degna in sé di Platone", ma di portata pratica molto attenuata, giacché l'autore stesso, poco più avanti, alla fine del primo libro (I 12, 12-14) afferma che una volta avviata l'attività creativa più impegnativa, il futuro oratore dovrà dedicare alla matematica e alle altre discipline solo i ritagli di tempo. In realtà, quest'ultima affermazione non sminuisce l'utilità e il valore formativo che l'autore assegna agli studi matematici, ma riguarda unicamente il tempo ridotto da dedicare ad essi una volta che la formazione del giovane sia giunta ad un punto di svolta, quando cioè sia arrivato il momento di dedicarsi il più possibile alla composizione scritta, attività che richiede impegno, tempo ed energie. Al contrario, l'invito, collocato sempre alla fine del libro primo, rivolto

[34] Quintiliano non manca di sottolineare anche come conoscere le leggi matematiche che regolano i fenomeni naturali risulti utile all'uomo politico e all'oratore in certe circostanze particolarmente difficili o imprevedibili. A dimostrazione di ciò (§§ 47-48) riporta alcuni episodi famosi che vedono come protagonisti celebri uomini politici Greci e Romani, come quello in cui Pericle tranquillizzò gli Ateniesi spaventati da un'eclissi di sole e quello in cui, similmente, Sulpicio Gallo allontanò la paura e la superstizione dall'animo dei soldati dell'esercito di Lucio Emilio Paolo (console e vincitore contro i Macedoni nel 168 a.C. a Pidna nella terza guerra macedonica), spaventati da un'eclissi di luna. In entrambi i casi i due uomini politici, assumendo il ruolo di oratori e utilizzando le loro conoscenze scientifiche in discorsi rivolti a un ampio uditorio popolare e incolto, in un momento di particolare pericolo, riuscirono ad allontanare paure irrazionali, derivanti da ignoranza in *geometria*, che avrebbero potuto avere conseguenze disastrose. Lo stesso non avvenne invece in un'altra circostanza, pure riferita dall'autore per dimostrare i pericoli a cui va incontro chi non possiede una formazione dello stesso tipo: nel 415 in Sicilia, Nicia, all'oscuro di conoscenze scientifiche, mentre era a capo dell'esercito ateniese, fu spaventato da un'eclissi e andò incontro alla rovina.

[35] V. [14] p. 372.

al giovane a trascorrere il tempo libero in compagnia di un maestro di matematica o di musica, piuttosto che fra divertimenti volgari (*ineruditae voluptates*), come spettacoli, giochi, dormite, mangiate e inutili chiacchiere, è una prova del fatto che l'autore ritiene quegli studi sempre utili e educativi, attività dignitose (*honesta*), oltre che piacevoli, donate agli uomini dalla Provvidenza[36].

Da quanto fin qui esposto, è possibile ricavare che nella visione quintilianea alla base dell'importanza dello studio della matematica nella formazione oratoria vi sono contemporaneamente ragioni pedagogiche, professionali ed etiche. In tutto ciò verisimilmente possiamo riconoscere le motivazioni di quella *vis occulta* (I 10, 7) a cui l'autore ha fatto riferimento prima dell'avvio della sua lunga trattazione volta a spiegare e sostenere le sue posizioni.

Quintiliano, esponendo i punti di contatto tra matematica e retorica, vuole implicitamente farci comprendere cosa intende per *ars dicendi*: non quella tecnica pericolosa che, eliminando le argomentazioni razionali a favore delle tecniche capaci di suscitare emozioni, mira a persuadere l'uditorio a prescindere dalla correttezza del ragionamento sotteso ai contenuti e quindi dalla veridicità degli stessi, magari anche a scopo di lucro[37], bensì quell'arte nobilissima e utilissima per i cittadini e per lo stato che ha il fine di dimostrare con rigoroso metodo logico, attraverso un corretto uso delle parole, ciò che è giusto e vero, smascherando al contempo ciò che è falso e ingannevole. In questo modo l'autore, superando ogni contrapposizione tra le due discipline, dimostra come nella retorica, correttamente intesa, la matematica, con i suoi metodi e le sue caratteristiche, sia non solo utile ma addirittura indispensabile. Di qui l'importanza dello studio della *geometria* nella formazione dell'oratore ideale, *vir bonus dicendi peritus*[38], ovvero un uomo che sappia usare in modo onesto e, per così dire, "scientifico" la propria abilità tecnica nel parlare.

Tirando le somme, possiamo affermare che Quintiliano, dall'alto della sua lunga esperienza didattica e della sua attenta riflessione teorica, con una visione moderna, ampia e interdisciplinare, ci dimostra una delle possibili e utilissime interconnessioni tra discipline appartenenti alle cosiddette "due culture"[39], quella delle scienze umane e quella delle scienze esatte, indicandoci

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[36] Quint., *Inst. or.*, I 12, 18.

[37] Cf. Quint., *Inst. or.* I 12, 16; e XII 1, 32. Già Platone distingueva la retorica buona (cf. *Phaedr*. 276e-277c) da quella cattiva (cf. *Gorg*. 450e-451d).

[38] Questa famosa definizione, già coniata da Catone il Censore e poi ripresa da Cicerone, si trova nel primo e nell'ultimo libro del trattato (Quint., *Instit. or*., I 1, 1; XII 1, 1).

[39] L'espressione "Le due Culture", com'è noto, fu usata per la prima volta da Charles Percy Snow in una *Lectio magistralis* all'Università di Cambridge nel Maggio del 1959, poi pubblicata nel volume [19]. Le due culture hanno differenze intrinseche su cui si è insistito a lungo e solo in tempi relativamente recenti si è compreso che esse, anche se oggettivamente differenti, hanno in realtà più punti di contatto di quanto si pensi comunemente. Anche nel

una strada per superare visioni riduttive e limitanti della *scientia* umana da perseguire sia in ambito scolastico che al di fuori di esso. Finché le scienze non dialogheranno tra loro integrandosi senza diffidenze e pregiudizi, non ci potrà essere un progresso culturale vero, profondo e completo: è questo un tema cruciale dal quale partire per affrontare le sfide più ardue della nostra complessa società.

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mondo antico non mancava una forma di separazione tra le due culture nella polemica tra l'*umanista* Isocrate e il *matematico* Platone, per la quale vd. D. Pesce, [17].

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From Archimedes and the oxen problem to Pell's equation

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Abstract

As part of the laboratory activities designed for the students of the second year of the Mathematical High School by the research group in Mathematics Education of the Department of Mathematics of the University of Salerno, the theme "From Archimedes and the oxen problem to Pell's equation" has been designed. The course has not yet been delivered to students and in this paper the didactic path to be developed in the classroom will be described. The students initially will retrace the various solution approaches proposed over the centuries. Subsequently, with a heuristic approach, they will tackle the resolution of the proposed problem by exploiting properties related to the theory of numbers. With the help of some schedules, provided by the teachers-tutors, they will conjecture and argue by answering questions aimed at stimulating didactic reflection. This workshop path involves processes such as exploring and conjecturing through the articulation of different ways of working (individual and in groups), manual and digital.

Keywords: Mathematical High School; technology advanced learning; history of mathematics; Number Theory.‡

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1. Introduction

> What does mathematics really consist of?
> Axioms (such as the parallel postulate)?
> Theorems (such as the fundamental theorem of algebra)?
> Proofs (such as Godel's proof of undecidability)?
> Concepts (such as sets and classes)?
> Definitions (such as the Menger definition of dimension)?
> Theories (such as category theory)?
> Formulas (such as Cauchy's integral formula)?
> Methods (such as the method of successive approximations)?
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions.
>
> Paul Halmos

In this article, a proposal for a didactic path designed for the solution of a problem that spans two millennia of history through various solving processes of famous mathematicians will be illustrated and developed. It is the problem of Archimedes' oxen. Eratosthenes' epigram describes in common language a seemingly simple mathematical problem: it deals with "small" numbers and relates concrete objects. Its solution leads to algebraic challenges that are within the reach of students in the two years of upper secondary school, but with numbers not usually used in school problems. This activity makes the students reflect on the use of language, mathematical formalization, the problem-posing process and problem-solving processes on a problem that possesses an intrinsic complexity.

In fact, it requires finding integer solutions to a system of Diophantine equations, which are polynomial equations with integer coefficients and required solutions that are also integers. These types of equations are known for their complexity, making the problem a considerable challenge even for mathematicians of various eras.

In 1880, the German mathematician Hermann H. Amthor was the first to make significant progress. He did not solve the problem in the strict sense, but he showed that his solution required the solution of a gigantic quadratic Diophantine equation. Amthor published a detailed analysis that clarified the intractable nature of the problem, estimating that the solution would require very huge numbers.

In 1965, mathematician Hermann H. Hitz and his team at the *University of Illinois* finally solved the problem with the use of an IBM 7090 computer. This computer, one of the most powerful at the time, made it possible to perform the complex calculations necessary to find the solution.

Hitz used advanced algorithms to calculate the solution, which required integers of an order of magnitude that defied the known one. The result was that the solution involved numbers with hundreds of digits, impossible to manage without the help of machinery.

Dealing with a classic problem in the classroom that has an important mathematical history behind it, allows you to develop a didactic discourse starting from the point of view of those who over time have built solution processes to then face an activity of exploration, problem-posing and problem-solving.

Referring to Polya, problem-solving means finding a way out of a difficulty, to reach a goal that is not immediately attainable, solving problems is a specific undertaking of intelligence, and intelligence is the specific gift of humankind.

In fact, solving problems is a highly formative activity, whose value goes beyond Mathematics itself (hard skill), involves personal skills or transversal skills (soft skills) and, as Vygotsky states, thanks to problem-solving activities, the student can significantly exceed the level of actual development to enter the zone of proximal development. Duly consolidated, this process leads to raising the level of effective development.

Schoenfeld deals extensively with thinking, teaching, and learning. His book, Mathematical Problem-solving, describes the importance of research in solving mathematical problems by specifying what it means to think mathematically.

For Pellerey, problem-solving and metacognition are closely interconnected: the acquisition of new knowledge passes through the awareness achieved by the learner (and who reflects on his or her cultural background). All metacognitive activity therefore belongs to the zone of proximal development.

The decision to propose in the classroom an educational workshop such as the one presented in the article is aimed at developing significant mathematical skills in students built on curricular knowledge and acquired skills (in particular in the manipulation of algebraic objects). In fact, the activity is in line with the curriculum for 12th-grade students (4th high school year in Italy). This workshop, to be held at the Mathematical High School Project (MHS), aims to delve into mathematical topics to foster technological proficiency and bolster critical thinking skills and mathematical skills. The approach taken in this laboratory resonates with the overarching methodology employed in the diverse educational programs offered by MHS, as evidenced by studies conducted (for example: Bimonte et al., 2021, 2022, 2023, Tortoriello & Veronesi, 2021, and Aramo et al., 2023).

2. The problem of Archimedes' oxen

At MHS, the evolution of teaching methods in a laboratory setting aligns with Vygotsky's constructivist approach to education and learning. This educational model emphasizes the integration of new knowledge through collaborative interactions among students, teachers, and researchers. In this educational environment, educators play a crucial role as mediators of meaning, guiding students toward a deeper understanding of the theme. To cultivate this development, educators should embrace a research-action methodology that prioritizes hands-on learning, teamwork, critical thinking, and the introduction of engaging content to spark students' curiosity. The project described in the paper utilizes a peer-collaboration action-research technique within small group settings, where students delve into diverse discourses and identify the fundamental framework of the hypothetical deductive method.

The progression of the activity can be delineated into multiple distinct phases, each playing a crucial role in shaping its final form:

- From the text to the problem: transformation from the first part of the epigram into a system of linear equations
- The system "gets complicated": from a mathematical point of view, the approach does not change, but the numbers involved are "larger"
- Archimedes describes a quadratic Diophantine problem
- Pell's Equation as a solution tool
- Let's find a possible solution
- Let's look for help in technology.

2.1 *Part 1. Introduction to the problem*

"*Reckon, <u>o stranger</u>, the number of the oxen of the Sun,*
working with care, if you possess any wisdom;
Calculate how many of them once grazed on the plains
of the Sicilian Island of Trinacria, distributed in 4 groups
of various colours: one milky white,
the second shining black,
the third is golden brown, the fourth mottled.
In each flock <u>the bulls</u> were distributed in considerable numbers,
In the following reports: Consider whites
as equal to the middle of the third
part of all blacks and browns;
the blacks then equal to the fourth part
And to the fifth of the mottled and all swarthy,

From an algebraic point of view, the first part is a multilinear Diophantine problem with four variables and three equations. This first part of the problem describes the proportions of bulls of the four different colors. Let W be the number of white bulls, B the number of black bulls, D the number of piebald bulls, and Y the number of yellow bulls. So, algebraically, this part of the problem can be formalized like this:

$$W = \left(\frac{1}{2}+\frac{1}{3}\right)B + Y = \frac{5}{6}B + Y$$

$$B = \left(\frac{1}{4}+\frac{1}{5}\right)D + Y = \frac{9}{20}D + Y$$

$$D = \left(\frac{1}{6}+\frac{1}{7}\right)W + Y = \frac{13}{42}W + Y$$

These three linear equations are easily solved.

$$W = \frac{5}{6}B + Y = \frac{5}{6}\left(\frac{9}{20}D + Y\right) + Y = \frac{5}{6}\left(\frac{9}{20}\left(\frac{13}{42}W + Y\right) + Y\right) + Y.$$

Therefore

$$\left(1 - \frac{5}{6}\cdot\frac{9}{20}\cdot\frac{13}{42}\right)W = \left(\frac{5}{6}\cdot\frac{9}{20} + \frac{5}{6} + 1\right)Y.$$

After some calculations, we get that 4455W=11130Y.

To obtain a general solution, it is sufficient to assume

W=11130t and Y=4455t,

from which it follows that

D=7900t and B=8010t.

These 4 coefficients are all divisible by 5, so we can reparametrize the solution as follows:

W=2226t, Y=891t, D=1580t, B=1602t

where t is a positive integer.

2.2 ***Part 2 Cows and Oxen... The problem is taking shape.***

The next part of the problem introduces four more variables and four more linear equations. In principle, it shouldn't be more difficult to solve than in part 1, but the numbers are much larger, so it's much easier to make miscalculations.

"These were the proportions of the cows: The white cows were exactly equal to the third part and a quarter of the whole herd of the black cows; while the blacks were once again equal to the fourth part of the piebald, and with it a fifth part, when all, including the bulls, went to graze together. Now the piebald in four parts were equal in number to one-fifth and one-sixth of the yellow flock. Finally, the yellows were equal in number to one-sixth and one-seventh of the white herd. If thou couldst tell precisely, O stranger, the number of the Sun's cattle, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be considered inexperienced or ignorant of numbers, but yet thou shalt not be numbered among the wise."

We indicate with w the number of white cows, b the number of black cows, d the number of piebald cows, and y the number of yellow cows. So, algebraically, this part can be formalized like this:

$$w = \left(\frac{1}{3} + \frac{1}{4}\right)(B + b) = \frac{7}{12}(B + b)$$

$$b = \left(\frac{1}{4} + \frac{1}{5}\right)(D + d) = \frac{9}{20}(D + d)$$

$$d = \left(\frac{1}{5} + \frac{1}{6}\right)(Y + y) = \frac{11}{30}(Y + y)$$

$$y = \left(\frac{1}{6} + \frac{1}{7}\right)(W + w) = \frac{13}{42}(W + w)$$

These four linear equations should be easily solved for the four new variables $w, b, d\ e\ y$.

So we have:

$$w = \frac{7}{12}(B + b) = \frac{7}{12}\left(B + \frac{9}{20}(D + d)\right)$$

$$= \tfrac{7}{12}\left(B + \tfrac{9}{20}\left(D + \tfrac{11}{30}(Y + y)\right)\right)$$

$$= \tfrac{7}{12}\left(B + \tfrac{9}{20}\left(D + \tfrac{11}{30}\left(Y + \tfrac{13}{42}(W + w)\right)\right)\right)$$

Therefore

$$\left(1 - \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}\right)w = \frac{7}{12}B + \frac{7}{12}\cdot\frac{9}{20}D + \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}Y + \frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}W$$

Simplifying the denominators, we get:

$$\begin{aligned}(12\cdot 20\cdot 30\cdot 42 - 7\cdot 9\cdot 1\cdot 13)w = {} & 7\cdot 20\cdot 30\cdot 42\, B \\ & + 7\cdot 9\cdot 30\cdot 42\, D \\ & + 7\cdot 9\cdot 11\cdot 42\, Y \\ & + 7\cdot 9\cdot 11\cdot 13\, W\end{aligned}$$

So you have,

$$293391\, w = 176400\, B + 79380\, D + 29106\, Y + 9009\, W$$

Putting them together with the previously found values we find that:

$$W = 2226t, Y = 891t, D = 1580t, e\ B = 1602t,$$

$$293391w = 454000680t.$$

Therefore

$$w = \frac{454000680}{293391}t$$

There are essentially two ways to find y, d and b. One way is to express y in terms of w, then d in terms of y and then b in terms of w. Another is to look for them directly. Using this other method, we find:

$$y = \frac{13}{42\left(W + \frac{7}{12\left(B + \frac{9}{20\left(D + \frac{11}{30(Y+y)}\right)}\right)}\right)}$$

$$d = \frac{11}{30\left(Y + \frac{13}{42\left(W + \frac{7}{12\left(B + \frac{9}{20(D+d)}\right)}\right)}\right)}$$

$$b = \frac{9}{20\left(D + \frac{11}{30\left(Y + \frac{12}{42\left(W + \frac{7}{12(B+b)}\right)}\right)}\right)}$$

$$\left(1 - \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}\right)y = \frac{13}{42}W + \frac{13}{42}\cdot\frac{7}{12}B + \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}D + \frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\cdot\frac{11}{30}Y$$

$$\left(1 - \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}\right)y = \frac{11}{30}Y + \frac{11}{30}\cdot\frac{13}{42}W + \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}B + \frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\cdot\frac{9}{20}D$$

$$\left(1-\frac{9}{20}\cdot\frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\right)y=\frac{9}{20}\ D+\frac{9}{20}\cdot\frac{11}{30}\ Y+\frac{9}{20}\ \cdot\frac{11}{30}\cdot\frac{13}{42}W+\frac{9}{20}\ \cdot\frac{11}{30}\cdot\frac{13}{42}\cdot\frac{7}{12}\ B$$

As before, simplifying the denominators, after a few steps you get:

$$293391y = 93600\ B + 54600\ D + 24570\ Y + 9008\ W\ = 342670419t$$

$$293391d = 110880\ Y + 34320\ W20020\ B + 9009\ D\ = 221496660t$$

$$293391b = 136080\ W + 49896\ B + 15444D + 9009\ Y = 308274498t$$

Ultimately, the general solution has and t.

$$w=\frac{434000680}{293391}t, y=\frac{342670419}{293391}t, d=\frac{221496660}{293391}t\ b=\frac{308274498}{293391}$$

The numerator and denominator are divisible by 63, so we can rewrite the solution more simply:

$$w=\frac{7206360}{4657}t, y=\frac{5439213}{4657}t, d=\frac{3515820}{4657}t \text{ and } b=\frac{4893246}{4657}t.$$

4657 is a prime number so we can't reduce the fraction again. So, *t* must be divisible by 4657 by the number of cows of the 4 types of colors that are integers. We have the solution from part 2.$t = 4657s,$

$w = 7206360\ s$	$W\ = 2226t = 10366482s$
$y = 5439213\ s$	$Y = 891t = 4149387s$
$d = 3515820\ s$	$D\ = 1580t = 7358060s$
$b = 4893246\ s$	$B\ = 1602t = 7460514s$

where s is a positive integer.

2.3 *Part 3. The last part of Archimedes' Problem.*

In the last part, Archimedes describes a quadratic Diophantine problem, which we report here:

"But come, understand also all these conditions concerning the cattle of the Sun. When the white bulls mingled with the black bulls, they stood still, equal in depth and breadth, and the plains of Trinacria, which stretched in every direction, were filled with their multitude. Moreover, when the yellow bulls and the piebald

bulls were gathered together in a herd, they stood in such a way that their number, beginning with one, slowly increased until they completed a triangular figure, there being neither bulls of other colors nor any in the middle. of them deficient. If thou art able, O stranger, to discover all these things, and to gather them together in thy mind, expounding all their relations, thou shalt depart crowned with glory, and knowing that thou hast been judged perfect in this kind of wisdom."

This part asserts that W + B *is a square number, while $Y + D$ is a triangular number.*

$W + B = 2226t + 1602t = 3828t = 3828 \cdot 4657\ s$, while $Y + D = 891t + 1580t = 2471t = 2471 \cdot 4657\ s$. We are looking for a value for S such that $3280\ t = 3828 \cdot 4657\ s$ is a square number, that is, of the form n^2, at the same time $2471 \cdot 4657\ s$ must be a triangular number, that is, of the form $\frac{k(k+1)}{2}$.

$3828 = 957 \cdot 2^2$, so $3828\ t = 3828 \cdot 4657\ s$ has to be a square number. It is sufficient to show that $957 \cdot 4657\ s$ is a square number. And, for our need, $s = 957 \cdot 4657n$ for a positive integer n.

Triangular numbers are more difficult to manage. We need $2471 \cdot 4657s = 2471 \cdot 957 \cdot 4657^2 n^2 = \frac{k(k+1)}{2}$ for a few whole k. The whole problem comes down to solve the quadratic Diophantine equation for k and n.

$$2471 \cdot 957 \cdot 4657^2 n^2 = \frac{k(k+1)}{2}$$

We can complete the square of the left member of the equation by writing:

$$\frac{k^2+k}{2} = \frac{\left(k+\frac{1}{2}\right)^2 - \frac{1}{4}}{2} = \frac{1}{8}((2k+1)^2 - 1).$$

We then multiply the above Diophantine equation by 8 to bring it into the following form:

$$8 \cdot 2471 \cdot 957 \cdot 4657^2 n^2 = (2k+1)^2 - 1,$$

By introducing a new variable $m = 2k + 1$, we have

$$8 \cdot 2471 \cdot 957 \cdot 4657^2 n^2 = m^2 - 1$$

Or

$$m^2 - Cn^2 = 1$$

Where $C = 410286423278424 = 8 \cdot 2471 \cdot 957 \cdot 4657^2 = 2 \cdot 2471 \cdot 957 \cdot (2 \cdot 2471)^2$

This equation is a classic example of what goes by the name of Pell's equation.

We'll solve it in two steps. In the first, we will solve Pell's equation of the form $m^2 - C'n^2 = 1$ where $C' = 4729494$. In the second step of the solutions to this equation, we will find one in which n is a multiple of 2942.

2.4 Part 4 The Pell Equation and Continuous Fractions.

Pell's equation $m^2 - Cn^2 = 1$ has solutions that are the rational approximation of $\sqrt{C}$. Let's take two examples: C=19 and C=13. How can we find the expansion of the continuous fraction for $C = 19$ which is about 4.3588989435407? $\sqrt{19}$ is a little more than 4. If we make it so that a is reciprocal of this something then $\sqrt{19} = 4 + \frac{1}{a}$ where $a = \frac{1}{0.3588989435} = 2.7862996478$.

Continuing, $a = 2 + \frac{1}{b}$ with $b = \frac{1}{0.7862996478} = 1.2717797887$. Successively $b = 1 + \frac{1}{c}$ with $c = \frac{1}{0.2717797887} = 3.6794494718$.

Then, $c = 3 + \frac{1}{d}$ with $d = \frac{1}{0.6794494718} = 1.4717797887$. Again, $d = 1 + \frac{1}{e}$ where $e = \frac{1}{0.4717797887} = 2.1196329812$. Finally, $e = 2 + \frac{1}{f}$ with $f = \frac{1}{0.1196329812} = 8.3588989435$. It may be noted that $f - 8 = 0.3588989435 = a - 2$, then $f = 8 + \frac{1}{b}$.

This process leads us to the continuous fraction expansion for $\sqrt{19}$ which is represented in the following way:

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \cdots}}}}}}$$

To save space, this expression can be written

$$\sqrt{19} = 4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{2}{1+}\frac{1}{8+\cdots}$$

Or

$$\sqrt{19} = [4,2,1,3,1,2,8,\ldots]$$

At this point, the expansion of the continuous fraction repeats, and usually the repetition is indicated by a line on the repeating part, namely:

$$\sqrt{19} = [4,2,1,3,1,2,8,2,1,3,1,2,8,\ldots] = \left[4,\overline{2,1,3,1,2,8}\right]$$

This continuous fraction expansion gives a close approximation of $\sqrt{19}$

Its first approximations are:

$$4 = 4$$

$$4 + \frac{1}{2} = \frac{9}{2}$$

$$4 + \frac{1}{2+}\frac{1}{1} = \frac{7}{2}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3} = \frac{11}{3}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1} = \frac{18}{5}$$

$$4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2} = \frac{119}{33}$$

These approximations are easy to calculate recursively, as in the table below, using the sequence $\left[4,\overline{2,1,3,1,2,8}\right]$. To create the table, you have to write the first two lines of 0 and 1 as shown, then on the left write the sequence of numbers of the continuous fraction. It calculates each new row in the table from the previous two rows so that each entry in that new row is found by adding the entry two rows above it and the factor on the left multiplied by the entry directly above it. (see Fig.1).

	m	n	m^2	$19n^2$	$m^2 - 19n^2$
	0	1			
	1	0			
4	4	1	16	19	-3
2	9	2	81	76	5
1	13	3	169	171	-2
3	48	11	2304	2299	5
1	61	14	3721	3724	-3
2	170	39	28900	28899	1
8	1421	326	2019241	2019244	-3

Fig.1 The table shows the approximations of the continuous fraction

The left-hand entry m in the row is the numerator of a rational approximation while the right-hand entry n is the denominator.

The interesting thing about these rational approximations of $\sqrt{19}$ is that the square of the numerator m is alternately less than and greater than 19 times the square of the denominator n. The above table has columns for m, $19n^2$ and $m^2 - 19n^2$. We can observe that when $(m, n) = (170{,}39)$, we have $170^2 - 19 \cdot 38^2 = 1$, which, solves Pell's equation $m^2 - 19n^2 = 1$.

This represents the smallest solution of Pell's equation.

In addition, all the solutions of Pell's equation can be found starting from this smallest equation $(m, n) = (170{,}39)$.

The equation $m^2 - 19n^2 = 1$ can be interpreted as follows:

$$(m + n\sqrt{19})(m - n\sqrt{19})$$

And from this, it follows that $(m + n\sqrt{19})^k \left(m - n\sqrt{19}\right)^k = 1$ for every k. This means, for example, that

$$(170 + 39\sqrt{19})\left(170 - 39\sqrt{19}\right) = 1$$

implies, with k=2, that

$$(170 + 39\sqrt{19})^2 \left(170 - 39\sqrt{19}\right)^2 = 1$$

From

$$\left(170 + 39\sqrt{19}\right)^2 = (170^2 + 19 \cdot 39^2) + (2 \cdot 170 \cdot 39)\sqrt{19}$$
$$= 57799 + 13260\sqrt{19}$$

We have a second solution, $(m, n) = (57799{,}13260)$.

Graphically, Pell's equation can be represented as a particular elliptic curve of equation $m^2 = 19n^2 + 1$ (see Fig.2)

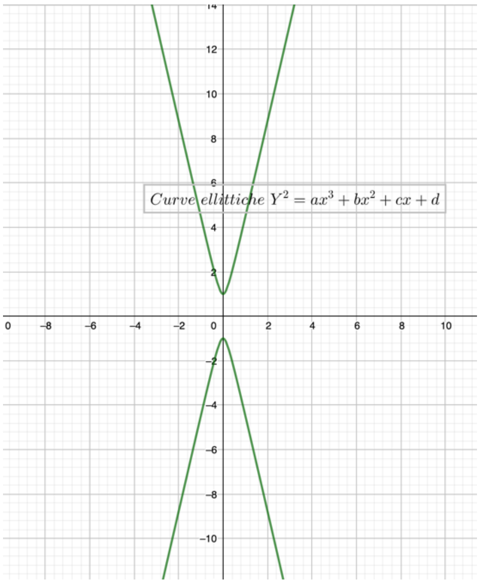

Fig. 2 Representation of the Pell equation as a particular elliptic curve

2.5 Part 5. Back to Archimedes' Problem and possible solution.

Let's go back to Archimedes' problem. We have two stages to complete. Initially, we have to solve Pell's equation of the form $m^2 - C'n^2 = 1$ with $C' = 4729494$. Second, among the solutions of this equation, we need to find one where is n a multiple of 2942 .

All we need is the continuous fraction expansion for $C' = 4729494$, which is about 2174.7399844579. At this point, we show the calculations to find it.

2.174.7399844579=2174+1/1.35137973413856

1.35137973413856=1+1/2.84592394735536

2.84592394735536=2+1/1.18213936740570

1.18213936740570=1+1/5.49030126898696

5.49030126898696=5+1/2.03956233290230

2.03956233290230=2 + 1/25.2765805450719

25.2765805450719=25+1/3.61574659004589

3.61574659004589=3 + 1/1.62404472256270

1.62404472256270=1 + 1/1.60244925378649

1.60244925378649= 1 + 1/1.65989084344423

1.65989084344423= 1 + 1/1.51540214557397

1.51540214557397= 1 + 1/1.94023251278156

1.94023251278156= 1 + 1/1.06356670972973

1.06356670972973= 1 + 1/15.7315048120601

15.7315048120601= 15 + 1/1.36704500573790

Then

$$\sqrt{4729494} = [2174,\overline{1,2,1,5,2,25,3,1,1,1,1,1,1,15,1,2,16,1,2,1,1,8,6,1,21,1,1,3,1,1,1,2,2,6,}$$
$$\overline{1,1,5,1,17,1,1,47,3,1,1,6,1,1,3,47,1,1,17,1,5,1,1,6,2,2,1,1,1,3,1,1,21,1,6,8,1,1,2,1,16,2,1,15,1,}$$
$$\overline{1,1,1,1,1,3,25,2,5,1,2,1,4348}].$$

We then calculate the first approximations of $\sqrt{C}$:

	m	n
	0	1
	1	0
2174	2174	1
1	2175	1
2	6524	3
1	8699	4
5	50019	23

Fig. 3 The table shows the first approximations of $\sqrt{C}$

The smallest solution of $m^2 - Cn^2 = 1$ is in the row $m = 8102213$ and $n = 4$ that gives $81022113^2 - 410286423278424 \cdot 4 = 1$

3. Use of technologies

The presented didactic path highlights a considerable intense computational activity with even high numbers that make the solution systems rather complex. This could lead one to think that the development of an activity of this type requires a lot of time and careful proximity of the teacher to the students, risking impoverishing the exploratory aspect of the research due to an excessive attention to the numerical component. To avoid this possible drift of the activity, it can be extremely effective to choose to deal with the modelling and construction of relationships between the data presented in the text, then delegate the algebraic solution to a technological tool.

In this regard, we propose approach to the solution with two technological tools currently present among teachers and their didactic choices in the classroom, namely the use of graphing calculators and a numerical approach programming in Python. The first tools, graphing calculators, has long been recognized for its utility in enhancing mathematical instruction and promoting problem-solving skills among students. With their ability to graph equations, perform complex calculations, and visualize mathematical concepts, graphing calculators have become an indispensable resource for educators seeking to facilitate deeper understanding and engagement in the learning process.

In addition to graphing calculators, the integration of numerical programming in Python presents a promising approach for enhancing the pedagogical experience in the classroom. Python, a versatile and widely-used programming language, offers educators the opportunity to introduce students to the world of computational thinking and algorithmic problem-solving in a user-friendly manner and to apply numerical methods and algorithms to solve mathematical problems.

3.1 Solving the problem with graphing calculators for a heuristic approach to mathematical exploration

The choice of graphing calculators is linked to the fact that in Italy since 2017 graphing calculators have been authorized for the final mathematics test of the final exam of studies and therefore many students have them and many schools develop paths and laboratory activities to help students learn their effective and conscious use. Below we see the system resolution menus with the Casio FxCG50 calculator.

Fig. 4 Select the linear systems menu

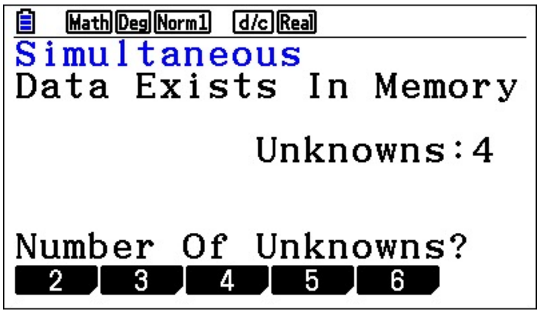

Fig. 5 Select the number of unknown

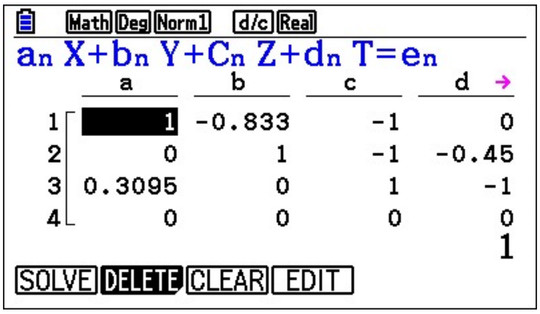

Fig. 6 Enter numerical coefficients (in fractional form)

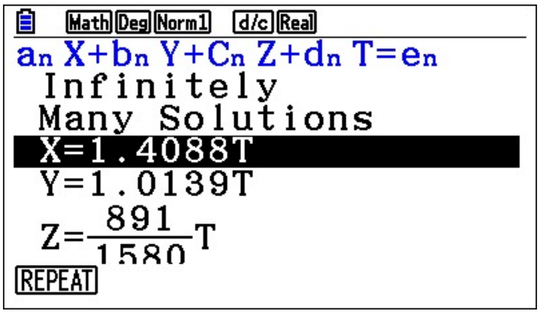

Fig. 7 The solution is expressed in parametric form

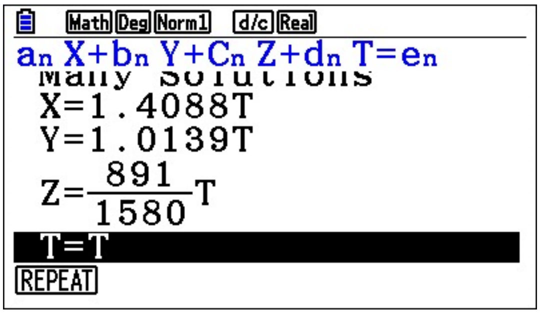

Fig. 8 The parameter is the last unknown factor you entered

It is interesting to note that the solutions are expressed in parametric form and can then be used in the numerical analysis that has been done in our article.

3.2 Solving the problem in Python with a numerical approach suggested by the very nature of the question.

The Archimedean oxen problem, as we have seen, requires solving a system of Diophantine equations involving large integers.

One version is to solve Pell's equation as follows:

$$m^2 - 410286423278424\, n^2 = 1$$

The solution of this equation involves finding a continuous fraction for the square root of the coefficient D=410286423278424 and determining the period of the continuous fraction to obtain the fundamental solutions.

Based on a conjecture taken from "*An Experimental Introduction to number theory, Benjamin Huts 2010"* which states that the fundamental solution of Pell's Equation is given by C_{K-1} for $K = r$ or $2r$ where r is the period of expansion of the continuous fraction of $\sqrt{D}$. In addition, if C_t is the fundamental solution, all solutions are given by C_{t+mr} with m ∈ to the natural numbers if r is even and m ∈ to 2 naturals if r is odd.

Below we list the steps to Solve Archimedes' Oxen Problem with the Numerical Approach in Python.

- Calculation of the continuous fraction for $\sqrt{D}$: This involves implementing the algorithm to find the terms of the continuous fraction.

```
import math

def continued_fraction_sqrt(D, max_terms=1000):
    """Calculates the continued fraction representation of the square root of D.

    Args:
        D (int): The integer to find the square root of.
        max_terms (int): The maximum number of terms to calculate.

    Returns:
        list: A list of the terms in the continued fraction representation of sqr
    """
    if D < 0:
        raise ValueError("D must be a non-negative integer.")

    m = 0
    d = 1
    a0 = a = int(math.sqrt(D))
```

Fig.9 Python Implementation for Continuous Fraction Computation

```
    if a * a == D:
        return [a0]

    terms = [a0]
    for _ in range(max_terms - 1):
        m = d * a - m
        d = (D - m * m) // d
        a = (a0 + m) // d
        terms.append(a)

        # Periodicity check
        if d == 1:
            break

    return terms
```

Fig.10 Python Implementation for Continuous Fraction Computation

- Calculation of convergences from the continuous fraction:

```
def continued_fraction_convergents(terms):
    """Calculates the convergents of a continued fraction given its terms.

    Args:
        terms (list): A list of terms in the continued fraction.

    Returns:
        list: A list of tuples representing the convergents (numerator, denomina
    """
    convergents = []
    n0, d0 = 1, 0
    n1, d1 = terms[0], 1

    convergents.append((n1, d1))

    for term in terms[1:]:
        n = term * n1 + n0
        d = term * d1 + d0
        convergents.append((n, d))

        n0, d0 = n1, d1
        n1, d1 = n, d

    return convergents
```

Fig.11 Python calculates the convergents of the continuous fraction.

- Finding the Period of the Continuous Fraction:

The period helps to determine the fundamental solution of Pell's equation as evidenced by the conjecture.

```python
import math

def continued_fraction_sqrt_period(D):
    """Calculates the continued fraction representation of the square root of D

    Args:
        D (int): The integer to find the square root of.

    Returns:
        list: A list of the terms in the continued fraction representation of sq
        list: The period of the continued fraction.
    """
    if D < 0:
        raise ValueError("D must be a non-negative integer.")

    m = 0
    d = 1
    a0 = a = int(math.sqrt(D))

    if a * a == D:
        return [a0], []  # No period for perfect squares

    terms = [a0]
    seen = {}

    while (m, d, a) not in seen:
        seen[(m, d, a)] = len(terms)

        m = d * a - m
        d = (D - m * m) // d
        a = (a0 + m) // d
        terms.append(a)

    period_start = seen[(m, d, a)]
    period = terms[period_start:]

    return terms, period
```

Fig.12 Program that calculates the period of the continuous fraction

Solving this problem with Python allows students to integrate mathematical knowledge with programming skills, showing how mathematics can be used to develop algorithms and solve complex problems efficiently. For example, in a secondary school, students can implement an algorithmic solution by analyzing the problem and breaking it down into simpler parts. This process develops critical thinking and problem-solving skills, which are fundamental skills in many scientific and engineering disciplines. Python also allows for more interactive and engaging teaching than traditional methods. Students can immediately see the result of their efforts, make changes, and better understand abstract concepts through visualization and experimentation.

4. Discussion and Conclusions

In the activity presented, doing mathematics involves various levels of manipulation from the abstract to the concrete: manipulating language and transforming it into symbols, manipulating symbols, and manipulating processes. Mathematics learning therefore takes place in situations where students can develop heuristic processes and build mathematical skills through problem-posing and problem-solving.

The practical experiment offers students a unique chance to improve their understanding of how technology can be used effectively to address classical mathematical problems found in scholarly literature. Students are equipped with the essential ability to collaborate efficiently within a group, optimize the strengths of their peers, and articulate their thoughts to accomplish a shared goal. These proficiencies are fundamental not only within an academic environment but also in professional settings, where the synergy of teamwork and cooperation is paramount for achieving desired outcomes.

At the end of the classroom laboratory, a comprehensive analysis of the results will be conducted and subsequently disseminated in an upcoming publication.

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Il latino della scienza Galilei e i fratelli Bernoulli in uno studio interdisciplinare sulla brachistocrona

(Latin in Science, Galilei and the Bernoulli brothers in interdisciplinary study over the Brachistochrone) *

Francesca Coppa†
Piera Filippi‡

Abstract

In this paper, we describe an interdisciplinary study between Latin and Mathematics, where Latin serves as a tool for inquiry, facilitating access to and reflection on the new mathematical concepts that emerged during the Scientific Revolution of the 17th and 18th centuries. By examining original texts in Latin, we trace the genesis of these concepts and their development through the creation of a new Latin lexicon tailored to express them.

We analyze texts by Galileo and the Bernoulli brothers concerning the brachistochrone problem, which marks the transition from classical mathematics to the new infinitesimal calculus that laid the groundwork for the Scientific Revolution. From an epistemological perspective, solving this problem—which requires the new tools of differential calculus—offers a significant example of the limitations of classical mathematics. The analysis of primary sources allows us to reconstruct the original thinking that led scientists to propose solutions, while also encountering an epistemological

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obstacle: the concept of the limit. Furthermore, analyzing the Latin texts reveals the construction and evolution of scientific language as it developed alongside the new mathematics.
Keywords: Brachistochrone, Latin, Galilei, Bernoulli, History of Mathematics.

Sunto

In questo lavoro descriviamo uno studio interdisciplinare tra Latino e Matematica in cui il latino diventa strumento di indagine che facilita l'accesso e la riflessione sui nuovi concetti matematici che si sviluppano durante la rivoluzione scientifica tra Sei e Settecento. Tramite la lettura dei testi originali in Latino, è possibile seguire la genesi dei nuovi concetti anche tracciandone lo sviluppo in un nuovo lessico latino necessario alla loro espressione. Vengono analizzati alcuni testi di Galileo e dei fratelli Bernoulli relativi al problema della brachistocrona, nei quali si realizza il passaggio tra la matematica classica e il nuovo calcolo infinitesimale, che creerà i presupposti per la rivoluzione scientifica. Da un punto di vista epistemiologico la risoluzione di tale problema, che richiede i nuovi strumenti del calcolo differenziale, rappresenta un esempio significativo dei limiti della matematica classica. L'analisi dalle fonti dirette consente di ricostruire il pensiero originale, che ha permesso agli scienziati di formulare le proposte risolutive, e di incontrare un esempio di ostacolo epistemiologico che è rappresentato dal concetto di limite. Inoltre, l'analisi del testo in latino permette di apprezzare realmente la costruzione e l'evoluzione del linguaggio scientifico che si viene a creare di pari passo con la nuova matematica.
Parole chiave: Brachistocrona, Latino, Galilei, Bernoulli, Storia della Matematica.

1. Introduzione

Nel ‘700 il latino era la lingua franca della nuova Scienza in Europa. In quel periodo storico, sono state scritte le pagine fondamentali per lo sviluppo della matematica e della fisica ed inevitabilmente il nuovo linguaggio scientifico è stato formalizzato nella lingua latina.

L’accesso ai testi originali è una grande occasione per ripercorrere, ed in qualche misura rivivere, la genesi di concetti matematici fondamentali e complessi, che prendono vita tramite la lingua nella quale sono ststi descritti.

Abbiamo considerato un problema classico, quello della brachistocrona, la cui risoluzione mette in luce la crisi della matematica classica, necessitando dei nuovi strumenti introdotti da Leibniz e Newton a metà del ‘600. Ne abbiamo analizzato le risoluzioni proposte da Galilei e dai fratelli Bernoulli, a partire dai testi originali.

Lo studio delle fonti originali ha richiesto, in una interazione continua, da un lato l’analisi in chiave diacronica e filologica del latino e dall’altra l’interpretazione del pensiero scientifico del tempo, diventando così i testi il *boundary object* [1] per il latinista ed il matematico§ ovvero un oggetto analizzato con sensibilità differenti da comunità diverse. Già Calvino aveva individuato tale confine valicabile in una linea ideale Dante-Galilei- Leopardi: ‘Non è scandaloso accostare Galilei a Dante: l’opera letteraria come mappa del mondo e dello scibile, la scrittura mossa da una spinta conoscitiva. La scienza e la filosofia naturale sono la vocazione della letteratura italiana, che coinvolge altri grandi autori come L. Ariosto e G. Leopardi’ [6].

Ci soffermiamo ora su alcuni passi scelti con l’obiettivo di mostrare come l’analisi della fonte primaria in lingua latina possa contribuire ad acquisire una comprensione profonda dell’argomento. Inevitabilmente, ogni traduzione e/o compendio in linguaggio a noi più familiare, se da un lato rende più fruibile il contenuto dall’altro fa perdere di vista allo studioso il pensiero che ha portato gli scenziati del tempo a quel tipo di dimostrazione, ma anche il lungo lavoro di definizione ed evoluzione del nuovo linguaggio scientifico.

2. Il linguaggio di Galilei

Galileo ha utilizzato il latino in alcune delle sue opere più importanti. Questa scelta era comune tra gli studiosi dell’epoca, poiché il latino era la lingua ufficiale della comunicazione tra dotti e sapienti.

Per analizzare il latino di Galilei è opportuno conoscere quali forme sintattiche, lessicali, quali particolarità ortografiche, morfologiche, quali modelli lo influenzassero in un contesto storico e culturale lontano ormai da quello precedente, umanistico-rinascimentale.

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§ *“I am conscious of myself and become myself only while revealing myself for another, through another, and with the help of another. . . [E]very internal experience ends up on the boundary.”Michail M. Bachtin (1984, p. 287),***(***Akkerman, S. F., & Bakker, A., Review of Educational Research, 2011).*

Nella scrittura in latino delle sue opere scientifiche e filosofiche, si è avvalso e ha attinto da fonti classiche in latino. Egli, infatti, era immerso profondamente nella letteratura classica, in testi di autori come Virgilio, Ovidio, Cicerone e Seneca, i quali hanno influenzato il suo uso del latino. Era anche ben versato nei testi scientifici e filosofici dell'antichità, come quelli di Aristotele, Euclide, Tolomeo e Archimede. Sebbene le sue teorie e scoperte scientifiche si discostassero in gran parte da quelle dei suoi predecessori antichi, il linguaggio scientifico e filosofico dell'epoca era ancora influenzato dai testi classici. Infine, essendo nato e vissuto in un periodo di significativa influenza della chiesa cattolica, era familiare anche con il latino utilizzato nei testi religiosi e teologici della tradizione ecclesiastica.

Queste fonti classiche gli hanno fornito una solida base linguistica e stilistica per le sue opere in latino. Tuttavia, è importante notare che egli era un innovatore nel suo campo e talvolta adattava il linguaggio latino esistente per descrivere concetti scientifici innovativi e rivoluzionari. Nonostante lo studio delle lingue classiche (imparò anche il greco) e l'uso del latino in alcune sue opere, preferì certamente esprimersi nella lingua fiorentina anche per poter raggiungere un più largo pubblico di lettori, che comprendesse chiunque fosse dotato di 'bon snaturale' (intelligenza), come egli scriveva in una lettera a Paolo Gualdo del 16 giugno 1612, in cui spiegava la scelta dell'italiano e delineava il suo 'lettore modello': non quelli che andavano all'Università perché sapevano il latino e magari erano 'inettissimi', ma quelli che non lo sapevano e «non potendo vedere le cose scritte in baos», benché intelligenti, si dedicavano ad altro. Galilei scriveva in 'vulgare' perché voleva che si convincessero che la natura «come gl'ha dati gl'occhi per veder l'opere sue», così ha dato loro «il cervello da poterle intendere e capire» [5].

Riprendendo le parole del Migliorini [18], si può affermare che «Con il Seicento siamo alla svolta decisiva: nascono le nuove scienze, svincolandosi dallo sterile metodo peripatetico, e questo spirito nuovo ha bisogno di un'espressione adeguata. Sarà il vecchio otre della scolastica, già malconcio per i colpi che gli hanno dato gli umanisti?

O sarà l'elegante vaso antico dissotterrato e polito dal classicismo? Il latino, sia quello di tipo scolastico, sia quello di tipo umanistico, avrebbe certo il vantaggio di rivolgersi ai dotti di tutta l'Europa. Ma Galileo non ha solo dei concetti da esprimere: egli ha in sé un'energia, un entusiasmo, un furore di proselitismo che vuol esercitare non sulla corporazione dei dotti, catafratta di citazioni aristoteliche e di sillogismi pseudo-scientifici, ma sugli uomini che hanno esperienza di vita e ingegno aperto. A costoro non bisogna parlare in baos, ma servirsi di una lingua viva, che sia insieme capace d'arte e di scienza».

3. I Discorsi e dimostrazioni

Tra le opere di Galilei scritte in latino si includono anche i *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (da ora in avanti *Discorsi e dimostrazioni*)

Sebbene gran parte del testo sia scritto in volgare italiano e solo alcune sezioni in latino, il quale risulta chiaro e lineare, adatto per comunicare le sue scoperte scientifiche anche a un pubblico non prettamente accademico. Pur rispettando le convenzioni linguistiche e grammaticali del latino classico, egli ha talvolta adattato il vocabolario esistente per descrivere concetti scientifici emergenti o introdotto nuovi termini, scelta che rifletteva la sua audacia in campo scientifico e la sua volontà di esplorare nuovi orizzonti. L'uso del latino era anche una scelta socio-culturale, volta a raggiungere un pubblico più ampio e sottolineare l'importanza delle sue scoperte nel contesto culturale dell'epoca. Poste tali considerazioni, è interessante riflettere su alcuni termini e ricercarne il significato traslato in ambito matematico da un'accezione più letterale.

I *Discorsi e dimostrazioni* [11] rappresentano la nascita della scienza moderna. Nonostante fosse stato proibito a Galilei, dopo la condanna e l'abiura, di occuparsi di questioni cosmologiche, e fosse stata vietata dall'Inquisizione la pubblicazione di qualsiasi suo scritto, la sua incessante curiosità, tuttavia, non lo fece desistere dalla ricerca, cui diede un nuovo oggetto: la fisica terrestre anziché la fisica celeste. Decise pertanto di far stampare il libro in Leida in Olanda, in cui i veti dell'Inquisizione non furono rispettati. Il testo cerca di sistematizzare le sue teorie affrontando campi diversi della fisica che vanno dalle proprietà della materia, connesse alla sua resistenza a rompersi, ai fenomeni del moto, argomentazioni arricchite da molte digressioni sui più svariati effetti naturali. Il libro non parve destare particolari reazioni di sdegno o apprensione [14].

Ci soffermeremo adesso su alcuni passi fondamentali della terza giornata[**], in cui Galilei pone le fondamenta della cinematica del moto accelarato.

Interessante è leggervi, nel paragrafo introduttivo del *De motu* <u>*naturaliter accelerato*</u>, che:

[.] tamen, quandoquidem quadam accelerationis specie gravium descendentium utitur natura, eorundem speculari passiones decrevimus, si eam, quam allaturi sumus de nostro motu accelerato definitionem, cum essentia motus naturaliter accelerati congruere contigerit[††].

La natura si serve di una certa forma di accelerazione nei gravi discendenti. Galilei, dunque, intende studiarne le proprietà e fornire una definizione del moto naturalmente accelerato, che corrisponda a quello di cui si serve la natura che 'suole far uso dei mezzi più immediati', grazie alla formulazione e l'introduzione di un linguaggio specialistico della fisica moderna che si lega ad un sostrato grammaticale classico. Qualche riga più avanti si legge che:

[**] Per lo studio del testo si è fatto riferimento a [11; 12; 13; 14; 15]

[††] Ossia, 'Tuttavia dal momento che la natura si serve di una certa forma di accelerazione nei gravi discendenti, abbiamo stabilito di studiarne le proprietà posto che la definizione che daremo del nostro moto accelerato abbia a corrispondere con l'essenza del moto naturalmente accelerato'.

[.]sicut enim motus aequabilitas et uniformitas per temporum spatiorumque aequabilitates definitur ac concipitur (lationem, enim, tunc aequabilem appellamus, cum temporibus aequalibus aequalia conficiuntur spatia) ita per easdem aequalites partium temporis, incrementa celeritatis simpliciter facta percipere possumus; mente concipientes, motum illum uniformiter eodemque modo continue acceleratum esse, dum temporibus quisbuscumque aequalibus aequalia ei superaddantur celeritatis additimenta[‡‡].

Compaiono quindi rispettivamente l'espressione *naturaliter accelerato*, nel primo paragrafo riportato, che indica l'osservazione della natura mentre nel secondo paragrafo troviamo *uniformiter accelerato* (attestato in Arnobio), legato alle uguali porzioni di velocità in porzioni uguali di tempo e che, come si legge nel brano, spiega rifacendosi alla suddivisione dei tempi nel moto uniforme.

Egli usa spesso l'espressione *gradus velocitatis*, di provenienza medievale, attestato nell'opera di Nicola Oresme, come sinonimo di *momentum velocitatis* per indicare la velocità istantanea. Con il termine *momentum* intende spesso la quantità di moto ma a volte anche il concetto di energia. L'*impeto* fa riferimento spesso all'energia cinetica, altre volte indica quella che oggi denominiamo quantità di moto. Il termine gravità può essere interpretato come la forza peso [8].

Aequabilis diventa uniforme; *latio,-onis* (latum da fero), presente ad esempio nel corollario II del teorema II, giornata terza, *de motu naturaliter accelerato*, è inteso come moto da un originario 'atto di portare, proporre', già attestato in Cicerone, Livio e Gellio; così *passio,-onis*, che significa 'sofferenza, patimento', indica la proprietà conseguente al moto stesso, e ugualmente si assimila al significato di "proprietà, caratteristica" il termine *symptoma-atis* propriamente 'sintomo', già ricorrente nel libro I, proposizioni 11-13, de Le coniche, testo di Apollonio di Perga [2], di cui la prima edizione, in greco e latino, fu pubblicata a Bologna da Federico Commandino nel 1566. Entrambi i termini figurano ancora nel primo paragrafo del *De motu naturaliter accelerato:*

[.] et consequentes eius passiones contemplari, non sit inconveniens (ita, enim, qui helicas aut conchoides lineas ex motibus quibusdam exortas, licet talibus non utatur natura, sibi finxerunt, earunt symptomata ex suppositione demonstrarunt cum laude)[§§] *[.].*

[‡‡] Ossia, 'Come infatti l'uguaglianza e uniformità del moto si definisce e si concepisce attraverso l'uguaglianza dei tempi e degli spazi (chiamiamo, infatti, moto uniforme quello per cui in tempi uguali si percorrono spazi uguali), così possiamo percepire gli incrementi della velocità, semplicemente compiuti, attraverso la stessa uguaglianza delle parti del tempo; concependo mentalmente che quel moto è continuamente accelerato in modo uniforme, mentre in qualsiasi tempo uguale si aggiungono incrementi uguali alla velocità'.

[§§] Ossia, 'e considerare le sue conseguenze non è fuori luogo (infatti, coloro che hanno immaginato curve come le eliche o le conchoidi, nate da determinati moti, sebbene la natura non usi tali curve, ne hanno dimostrato le proprietà, secondo una supposizione, con lode)'.

Dai teoremi e dalle proposizioni analizzati l'attenzione si pone, in ambito lessicale, su alcune espressioni particolarmente interessanti: *velocitatis gradus, subduplus,* presente ad esempio nel *Theorema I - Propositio I,* giornata terza, *De motu naturaliter accelerato,* qui di seguito riportato.

Tempus in quo aliquod spatium a mobili conficitur latione ex quiete uniformiter accelerata, est aequale tempori in quo idem spatium conficeretur ab eodem mobili motu aequabili delato, cuius velocitatis gradus subduplus sit ad summum et ultimum gradum velocitatis prioris motus uniformiter accelerati[***].

Nel Theorema IV- Propositio IV, di seguito riportato, compare l'espressione *subdupla ratione*.

Tempora lationum super planis aequalibus, sed inaequaliter inclinatis, sunt inter se in subdupla ratione elevationum eorundem planorum permutatim accepta[†††].

Entrambi gli aggettivi *subduplus* e *subdupla* sono composti dal prefisso *sub-*, sotto, inferiore, che indica, nel primo, la metà rispetto al *duplus*, al doppio, nel secondo, una ragione di inferiorità rispetto al quadrato, *duplicatus*, quindi la radice quadrata. Dal latino imperiale proviene l'aggettivo *subduplus*, rilevato nell'opera di Boezio, e si attesta poi verso il 1377 nell'opera di Nicola Oresme, filosofo, matematico, fisico, astronomo. Se ne hanno altre attestazioni nel 1500 fino all'uso prettamente scientifico che ne fa Galilei.

Dal punto di vista linguistico, destano interesse alcuni termini, presenti nell'enunciato del Theorema XXII - Propositio XXXVI, di seguito riportato:

Si in circulo ad horizontem erecto ab imo puncto elevetur planum non maiorem subtendens circumferentiam quadrante, a terminis cuius duo alia plania ad quodlibet circunferentiae punctum inflectantur, quam in solo priori plano elevato, vel quam in altero tantum ex illis duobus, nempe in inferiori.[‡‡‡]

e in molti altri presenti nel testo quali, ad esempio, *horizontem,* corrispondente a linea di base, derivante da un significato di linea passante per il centro del quadrante solare, come

[***] Ossia, 'Il tempo in cui uno spazio dato è percorso da un mobile con moto uniformemente accelerato a partire dalla quiete, è eguale al tempo in cui quel medesimo spazio sarebbe percorso dal medesimo mobile mosso di moto equabile, il cui grado di velocità sia sudduplo (metà) del grado di velocità ultimo e massimo'.

[†††] Ossia, 'I tempi dei moti su piani di eguale lunghezza, ma di diversa inclinazione, stanno tra di loro in sudduplicata proporzione delle elevazioni dei medesimi piani permutatamente prese (cioè in un rapporto pari alla radice quadrata del rapporto inverso tra le elevazioni)'.

[‡‡‡] Ossia, 'Se in un cerchio, eretto sull'orizzonte, dal suo punto più basso si innalza un piano inclinato, il quale sottenda un arco non maggiore di un quadrante, e se dagli estremi di tale piano si conducono due altri piani inclinati a un qualsiasi punto dell'arco, la discesa lungo [il sistema di] questi due ultimi piani inclinati si compirà in minor tempo che lungo il solo primo piano inclinato, o che lungo uno soltanto di questi due ultimi piani, e precisamente l'inferiore'.

attestato in Vitruvio e *subtendens*, che sottende (attestazione in Catone) e che continuiamo a trovare nel nostro linguaggio della geometria euclidea.

3.1 Il problema della brachistocrona

Dopo aver dunque presentato le assunzioni e diversi teoremi relativi alla cinematica del moto accelerato, Galilei pone il problema della brachistrocrona mostrando come essa non coincida con la geodetica (quindi con un percorso rettilineo trattandosi di geometria piana).
Nello *scholium* del Theorema XXII - Propositio XXXVI giornata terza, affronta il problema e, introducendo un procedimento iterativo, suppone che più punti intermedi si

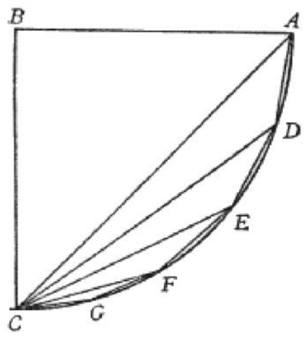

Fig. 1 Scholium teorema XXII, giornata terza

aggiungono lungo la traiettoria poligonale, più diminuisce il tempo necessario a percorrerla: il movimento piú veloce per andare da A a C «*non per brevissimam lineam, nempe per rectam, sed per circuli portionem* [.]. *Quo igitur per iscriptos poligono magis ad circumferentiam accedimus, eo citius absolvitur motus inter duos terminos signatos A, CV*».

Arriva dunque ad individuare correttamente, e con una brillante dimostrazione, che la circonferenza è una curva migliore rispetto al piano inclinato, relativamente a tale problema, ma non può dimostrare che è la migliore, ciò perché la matematica nota fino ad allora, che non gestisce procedimenti di limite, non può fornire risoluzioni per problemi di questo tipo. Gli strumenti con i quali cerca di risolvere il problema sono quelli della geometria euclidea: similitudine, teorema di Pitagora e teoria delle proporzioni. C'è un'idea implicita di passaggio al limite, che in qualche modo ricorda il metodo di esaustione. Il ragionamento iterativo che propone Galilei poggia sulla dimostrazione da lui realizzata (nel Theorema XXII - Propositio XXXVI) che il tempo per percorrere un sistema di due piani inclinati consecutivi (AD e DC in fig. 1) sia minore di quello richiesto per percorrere un tratto con gli stessi estremi (AC in fig. 1). Tuttavia egli non tiene conto del fatto che, già nella prima iterazione, quando si parte dal punto D per sostituire il tratto DC con il sistema DE+EC, la condizione iniziale di velocità non è la stessa [7; 16]. La lettura del testo originale permette di evidenziare e discutere questi nodi concettuali, ostacoli epistemiologici, e apprezzare la novità nelle soluzioni che la matematica moderna fornirà.

La sua grande innovazione risiede nel cercare di utilizzare strumenti che ricordano il metodo di esaustione per risolvere nuovi problemi di cinematica e che anticipano problemi di tipo differenziale. La soluzione da lui proposta, l'arco di circonferenza, risulta effettivamente più veloce di qualunque poligonale in essa inscritta ma non è la brachistocrona, ovvero la linea più breve che, come mostreremo nel seguito, i fratelli Bernoulli dimostreranno essere un arco di cicloide ordinaria.

4. Il latino dei fratelli Bernoulli e dei matematici del tempo

Negli *Acta Eruditorum* del 1697 compaiono le dimostrazioni dei due fratelli Bernoulli e di altri matematici del tempo. Dall'analisi dei testi, secondo un punto di vista linguistico- lessicale e metodologico, appare evidente il passaggio da una struttura di sintassi ancora influenzata dal latino classico e umanistico- rinascimentale di Galilei a un formalismo sistematico nell'uso di espressioni morfosintattiche molto semplificate e di termini ormai divenuti specifici in un linguaggio propriamente matematico, tra i quali si annoverano *diaphanus*, *linea recta*, *curva*, *sinus*, *linea verticalis*, *abscissa*, *applicata*, *differentialem* [7; 19]. Questi ultimi compaiono già nel *Nova methodus* di Leibnz [17].

Il testo della dimostrazione di Johann: *Curvatura radii in Diaphanis non uniformibus, Solutioque Problematis a se* in *Actis* 1696, p. 269 [4] è scritto in una lingua latina più semplificata. Infatti, Johann riprende la lezione di Galilei, sperimentando però strumenti diversi, che egli rende in una lingua latina ormai lontana dai canoni classici o umanistici e divenuta funzionale all'espressione della matematica.

Johann si distanzia dalla lezione galileiana per giungere alla soluzione con strumenti diversi, spiegati in un latino ormai duttile alle esigenze del linguaggio scientifico, come, ad esempio, si può apprezzare nella descrizione della analogia ottica-meccanica che egli utilizza come espediente per aiutare il lettore a comprendere la dimostrazione:

Si nunc concipiamus medium non uniformiter densum, sed velut per <u>infinitas lamellas</u> horizontaliter interjecta distinctum [.]; manifestum est, radium, quem ut globulum cinsideramus, non emanaturm in liena recta, sed in curva quadam (notante id iam &ipso hugenio in eodem Tractatu de Lumine, sed ipsam curvai naturam minime determinante) quae eius sit naturae, ut gloobulus per illam decurrens celeritate continue aucta vel *diminuta, pro ratione graduum raritatis, brevissimo tempore perveniat a punto ad punctum.* [§§§] [.].

Nella brillante dimostrazione, Johann adotta, come si è appena detto, un procedimento introdotto attraverso una analogia ottica-meccanica tra il problema di minimo tempo di un raggio di luce che si propaga in un mezzo a densità variabile ed il minimo tempo di percorrenza di un grave vincolato su una curva e soggetto all'azione della gravità; entrambi sono fenomeni a velocità variabili per i quali viene richiesto il tempo minimo di percorrenza (principio di minor tempo di Fermat[****].) Considera dunque un mezzo che non sia omogeneamente denso ma che sia composto di strati orizzontali con densità crescente o decrescente (*infinitas lamellas*).

Fa dunque un esperimento concettuale, se esistesse un mezzo fatto da un'infinità di lamelle a densità crescente o decrescente, un raggio non si propagherà in linea retta ma curva, analogamente ad una palla sotto l'azione della sola gravità e che percorre il cammino secondo una velocità sempre aumentata o diminuta (il raggio si comporta come la palla se si avesse una materia con tante lamelle a varie densità, con una opportuna legge di variazione).

Per un tale sistema a velocità variabile la richiesta di tempo minimo ha come soluzione la legge di Snell. La legge della rifrazione di Snell[††††] si può ricavare dal principio di Fermat.

Tale legge vale per qualunque sistema a velocità non uniforme e non solo in campo ottico e rende minimo il tempo di percorrenza su un percorso fatto di due spezzoni finiti di traiettoria. Essendo però la ricerca della brachistocrona un problema di meccanica, Johann, per determinare la soluzione, applica la fondamentale legge di caduta di Galilei.

[§§§] Ossia, 'è manifesto che un raggio che noi consideriamo come una particella non si propagherà lungo una linea retta, ma lungo una curva (come già lo stesso Huygens aveva notato sul suo Trattato sulla Luce ma senza aver determinato la natura della curva) che è di tale natura, che una palla che la percorre con una velocità continuamente aumentata o diminuita, secondo il grado di densità, può giungere da un punto all'altro nel più breve tempo' [9].

[****] Principio del minimo tempo impiegato per il raggio nella rifrazione, formalizzato intorno al 1629: di tutti i possibili cammini che un raggio di luce può percorrere per andare da un punto a un altro, esso segue il cammino che richiede il tempo più breve.

[††††] La luce che viaggia dal punto A in un mezzo 1 al punto B in un mezzo 2 può intraprendere infiniti percorsi possibili costituiti da segmenti di linea in ciascun mezzo. Ogni percorso attraversa l'interfaccia in una certa posizione x, e quindi ne crea gli angoli θ_1e θ_2 rispetto alla normale all'interfaccia. Per la dimostrazione di tale legge si sfrutta il principio fisico secondo cui il tempo varia in modo proporzionale alla distanza percorsa e in modo inversamente proporzionale alla velocità.

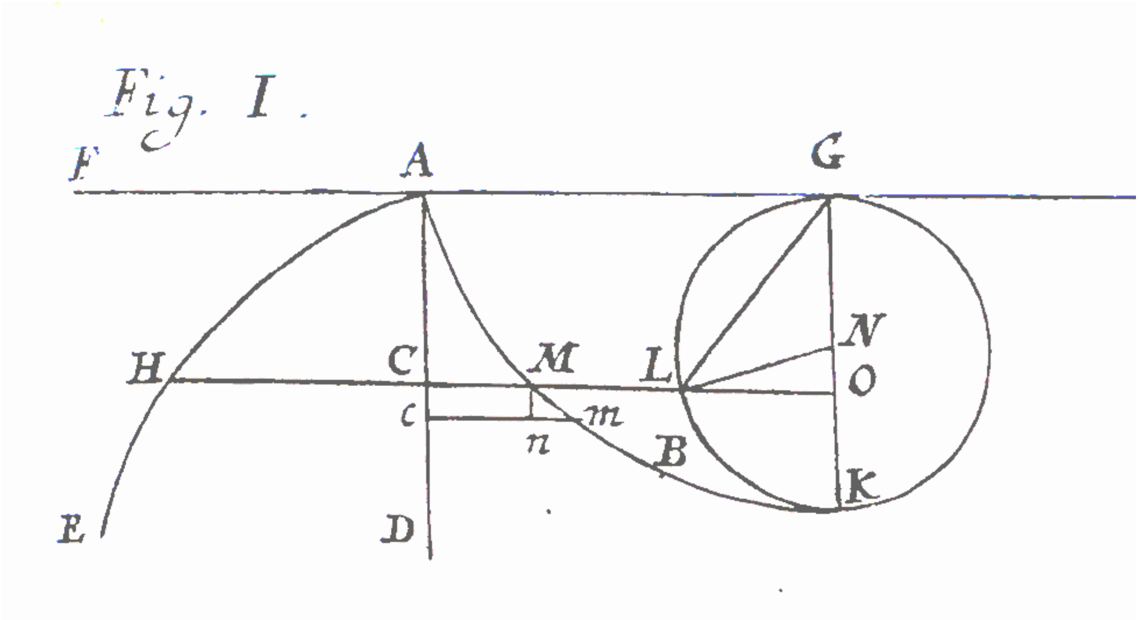

fig 2. *Dimostrazione di Johann*

Nella fig.2, il punto *A* è nella parte meno densa mentre *B* sta nella parte più densa; ogni applicata ordinata CH di AC indica la densità media all'altezza AC, oppure le velocità del raggio, o della palla, in M (sulla curva AMB). Mediante l'uso dei differenziali si perviene alla forma differenziale della cicloide ordinaria [7, 9, 10]

L'articolo di Jakob, fratello maggiore di Johann, è strutturato in maniera simile a quello del fratello: *Solutio problematum fraternorum* [.] [3]. Anche in questo caso il lessico presenta una struttura latina estremamente funzionale alla matematica. Interessante si rivela l'etimologia del termine oligochrona, derivante da όλίγος e κρόνος, quale sinonimo di brachistocrona: «[.] *qua quidem proprietate isochronam illam Hugenianam, nunc et oligochronam futuram, tritam nempe notamque Geometris Cycloidem gaudere deprehendo*».

La soluzione presentata da Jakob è di tipo geometrico e si basa su un lemma, assicurante che, se per una curva assegnata la proprietà di essere suscettibile a massimizzazione o minimizzazione vale globalmente, allora questa proprietà vale pure localmente, cioè per un arco qualsiasi di questa curva. E' egli stesso che afferma:

adeo ut problema ad puram Geometriam redactum huc redeat, ut inveniatur curva, quae elementa sua habeat composita ex elementis abscissarum directe, & radicibus applicatorum inverse: qua quidem proprietate Isochronam illam Hgenianam, nunc & Oligochronoma futuram, tritam nempe notamque Geometris Cycloidem, gaudere deprehendo; [.] [‡‡‡‡].

‡‡‡‡ Ossia, 'Il problema, dunque ridotto alla pura geometria, si riduce a questo: trovare una curva i cui elementi siano composti da quelli delle ascisse in modo diretto e dalle radici delle ordinate in modo inverso. Con questa proprietà, mi accorgo che la famosa cicloide, ben nota ai geometri, gode di essere quell'isocrona

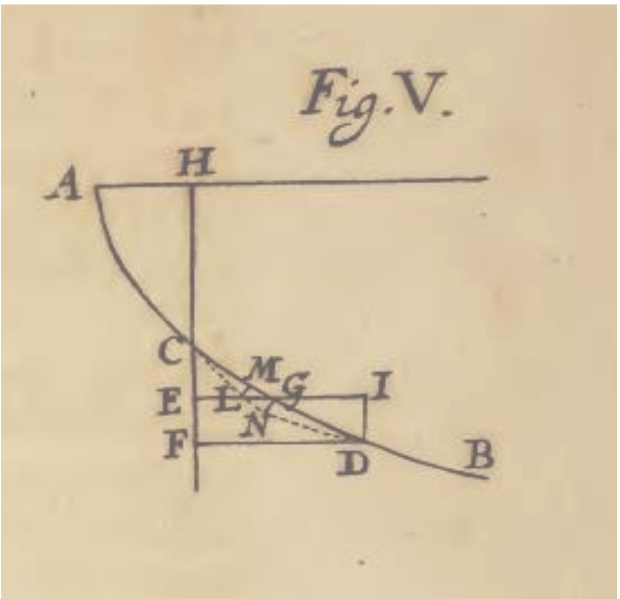

Fig. 3 Dimostrazione di Jakob

Jakob, come il fratello, utilizza la legge di caduta di Galilei mostrandone anch'egli l'importanza nello sviluppo della fisica moderna. L'uso dei differenziali permette anche a lui di arrivare alla forma differenziale della cicloide. Nel complesso, questa dimostrazione risulta più tecnica e fa un uso massiccio di proporzioni e la scrittura formale matematica pare più rudimentale di quella del fratello. Utilizza materiali già da lui elaborati e il suo approccio influenzerà i matematici futuri, fino a prina dell'introduzione del nuovo metodo da parte di Lagrange [16].
Ci piace chiudere evidenziando il seguente passo:

Ubi in transitu considerandum proponimus Celeberrimo Domino Nieuwentiit usum differerentio – differentialium [quae ipse immerito explodit [.]

in cui Jakob probabilmente manifesta la presa di posizione a favore di Leibniz nella diatriba sulla paternità del calcolo differenziale.

Le diatribe e le invettive sono sempre attuali, qualunque sia il periodo storico d'indagine.

di Huygens, ora anche chiamata oligochrona. Questa proprietà appartiene all'isocrona di Huygens, che è quindi anche l'oligocrona cioè la cicloide ben nota ai Geometri [.]'.

4. Conclusioni

L'analisi condotta sui testi ha mostrato l'esigenza già di Galilei di introdurre un lessico in supporto all'espressione dei nuovi concetti. Tale lessico, recepito nel linguaggio scientifico, ritorna negli articoli dei fratelli Bernoulli, e più in generale nei lavori dei matematici del 1700, ampliato e sostenuto dalle conoscenze dei lavori di Leibniz e Newton.

Lo studio dell'evoluzione del pensiero scientifico e il ragionamento matematico sono intimamente uniti alla semantica del linguaggio, in quanto codice aperto a cambiamenti continui. In questo caso la lingua latina, con cui sono esposti i testi analizzati, diventa strumento di analisi scientifica anch'essa, a dimostrazione della complementarietà di ambiti di studio e di indagine affini e connessi, nient'affatto distanti.

Dal punto di vista della comprensione dei concetti matematici, la lettura dei testi originali in lingua permette di cogliere nel profondo l'essenza dei nodi concettuali riconoscendone gli ostacoli epistemiologici insiti e apprezzandone le strade che hanno portano al loro superamento.

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The construction of quantum mechanics from electromagnetism. Theory and hydrogen atom

Hernán Gustavo Solari* and Mario Alberto Natiello†

Abstract

We reconstruct Quantum Mechanics in a way that harmonises with classical mechanics and electromagnetism, free from mysteries or paradoxes as the *collapse of the wave-function* or *Schrödinger's cat.* The construction is inspired by de Broglie's and Schrödinger's wave mechanics while the unifying principle is Hamilton's principle of least action, which separates natural laws from particular circumstances such as initial conditions and leads to the conservation of energy for isolated systems.

In Part I we construct the Quantum Mechanics of a charged unitary entity and prescribe the form in which the entity interacts with other charged entities and matter in general. In Part II we address the quantum mechanics of the hydrogen atom, testing the correctness and accuracy of the general description. The relation between electron and proton in the atom is described systematically in a construction that is free from analogies or ad-hoc derivations and it supersedes conventional Quantum Mechanics (whose equations linked to measurements can be recovered). We briefly discuss why the concept of isolation built in Schrödinger's time evolution is not acceptable and how it immediately results in the well known measurement paradoxes of quantum mechanics. We also discuss the epistemic grounds of the development and provide a criticism of instrumentalism, the leading philosophical perspective behind conventional Quantum Mechanics.

Keywords: Wave mechanics; Schrödinger's equation; Atomic energy levels; critical epistemology; instrumentalism; consilience; unity of thought; epistemic praxis; **AMS subject classification:** 81P05, 81Q05, 81V45[1]

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Introduction

Most physicists consider their science deals with the discovery of the laws of nature. However, during the early years of the XX century a most prominent group of physicists completely dropped this position and sustained that physics was merely a collection of useful formulae (see for example (Kragh, 1990)). Both groups of physicists would however agree that the relevant product of physics was the collection of formulae or plans which allowed them to make predictions. Actually, successful predictions, and insights, boosted technological applications justifying in front of the whole society the liberties granted to scientists. In short, predictive success was the key of the game called Science since late in the XIX century.

Under this philosophical perspective, called *instrumentalism* (a form of *utilitarianism*), the process of construction of theories –including their grounding– becomes less relevant and it is usually claimed to lie outside the science (for example in (Popper, 1959)) or is substituted by a narrative that allows students to accept the desired formulae with lesser intellectual effort (sometimes called "didactic transposition"). Consider as an example the widespread misconception about absolute space being fundamental for Newtonian mechanics. However, the philosophy that developed science since (at least) Galileo Galilei and until the second industrial revolution (around 1870) was rationalist and considered the rational construction of a theory a most relevant element in judging its quality. For the very few in physics that recognise that knowledge represents the encounter of the subject and the object and as such it carries elements of both of them, the rational construction of theories remains important and it is possible to suspect that, if we turn to the old philosophy of Science, some of the utilitarian theories would show defects or imperfections that demand attention.

A long-term project in this direction has been under development during the last ten years reconstructing Mechanics (Solari & Natiello, 2018) (and particularly recovering the concept of "inertial system" much distorted in textbooks (Solari & Natiello, 2021)), making explicit the dualist epistemology and the relevance of phenomenology for science (Solari & Natiello, 2022a) and furthermore, rescuing a few rules of reason that pervade the old construction of Science (Solari & Natiello, 2023). A second constructive step consisted in developing electromagnetism in terms of relational space (Solari & Natiello, 2022b), thus reclaiming Gauss' rational project, which was left aside in favour of the idea of the ether and subsequently forgotten under the utilitarian perspective. Having recovered an Electromagnetism that is in harmony with Classical Mechanics, we present now the construction of a Quantum Mechanics that is in harmony with Classical Mechanics and Electromagnetism.

Background

Under the instrumentalist guide, the period 1870-1930 witnessed the consolidation of Electromagnetic theory (EM) and the birth of Quantum Mechanics (QM). The success of both theories as predictive tools (and thus as mediators of technological development) can hardly be challenged. The interpretative part is often modelled resorting to mechanical analogies (Mach, 2012; Boltzmann, 1974), although a conflict arose with Electromagnetism since a mechanical (Galilean) framework and the constancy of the velocity of light appeared to be incompatible. The underlying model in Bohr's atomic theory is also mechanical, with a planetary electron revolving around the atomic nucleus. A conflict between theories becomes evident since an accelerated charge is expected to radiate, thus losing energy and collapsing towards the nucleus. The solution to the conflict was to negate it: In the atomic realm radiation is not present and electronic orbits are stable. In Bohr's atomic model, stable orbits have to satisfy a quantisation condition (that the angular momentum is an integer multiple of Planck's constant, $\hbar$) and electrons can only undergo transitions among these stable orbits. Nothing is said in the model about the existence of other states. Later, the Schrödinger model will assume that the possible states of the atom are linear combinations of stationary states and stable as well. In both models all possible atomic states are stable and radiation is suppressed. However, it is not possible to ascertain which is the state of the atom since measurements make the state to "collapse" into a stationary (measurable) energy state. This dogma is often indoctrinated by resource to Schrödinger's *cat paradox*. It would appear that the laws of EM are then different for microscopic and macroscopic distances. The possibility of encompassing EM and QM in one broader theory is hopeless in their currently accepted shape since EM and QM are contradictory in their foundations.

The EM that is acceptable for atomic physics displays a number of phenomena that are very difficult to explain. Consider e.g., spontaneous emission (overwhelmingly verified in Laser physics (Carmichael, 1999; Chow *et al.*, 1994)) or the reduction postulate of QM transforming the state of the system at the moment of measurement. A natural question arises: At what distance is it mandatory to change theories? When and how EM ceases being applicable and QM is a must? Are there other experimentally observable consequences? The theories now rest on elements which are not amenable to experimental examination and moreover, there are questions that cannot be asked.

EM also suffers instrumentalist constraints on its own. The current EM approach is rooted in the heritage of Hertz, being his preferred interpretation that the EM fields are associated to space (the ether for Hertz) and that forces surround matter [2]. Thus matter and EM fields have different ontogenesis. There

[2] The expression is rather mysterious, but it can be found also in Faraday (closer to reject it than to

exists “empty space”, populated by fields (and subsequently by springs, strings and branes), along with a space occupied by matter (originally impenetrable). The realm of matter and with it atomic physics is divorced from the space where fields dwell (despite the field conception being rooted in the idea of force per unit charge). The link between both realms is given mostly by the Lorentz force, which appears as an addendum to Maxwell’s equations displaying the Poincaré-Lorentz symmetry (Poincaré, 1906), after repeated drift in the meaning of the symbols. The velocity, that was relational in Maxwell and in experiments, became velocity relative to the ether and later relative to an inertial frame (Assis, 1994). The last two velocities have never been measured. For example, when analysing experiments concerning Doppler effect, the standard theory refers to a velocity with respect to an inertial system, but the measured velocity is a relative one. We will not dwell in these issues, the impossibility to physically contrast Special Relativity has been discussed by (Essen, 1971, 1978), while its logical inconsistency when regarded from an epistemology that encompasses the phenomenological moment has been recently shown (Solari & Natiello, 2022a). Strictly speaking, EM floats in a realm of ideas partially divorced from observations.

The possibility of unifying EM and QM under a conception that cleaves them in two separate entities became only possible by imposing the quantification of imagined entities that populate “empty space”. This is the presently accepted approach, although their original proposers finally rejected it and despite their internal inconsistencies that would force the collapse of the underlying mathematical structure (Natiello & Solari, 2015).

Unfortunately, alternative approaches have little or no room in the scientific discussion. If the outcome in formulae of two theories is practically the same; instrumentalism may be inclined to choose the option with (apparently) less complications. Coming back to Newtonian mechanics, if the outcome elaborated under the assumption of absolute space to sustain the theory is the same as (a subset of) that developed under Newton’s concept of “true motion” (Newton, 1687; Solari & Natiello, 2021) instrumentalists may find no benefit in presenting the second –more abstract– theory, to the price of accepting a conflicting concept that cannot be demonstrated[3]. We promptly recognise that in the academic formation of technical professionals the difference is usually not very problematic, while in the formation of experimental philosophers (or natural philosophers), to decline the historical cognitive truth for instrumentalist reasons appears as completely inappropriate and a safe source of future conflicts [4].

accept it) and in Maxwell (closer to accept it than to reject it).

[3]Along the same lines, textbooks do not hesitate in presenting a simple but incorrect derivation of the spin-orbit interaction along with an ad-hoc correction called Thomas precession, instead of using the approach that they consider “correct” (Kastberg, 2020, p. 60).

[4]The distinction between the formation of professionals and philosophers is in (Kant, 1798).

Instrumentalism is a weakened version of the phenomenological conception presented in (Solari & Natiello, 2022a). This latter approach is rooted in the work of C. Peirce (Peirce, 1994), W. Whewell (Whewell, 1840) and J. Goethe (Goethe, 2009) developed under the influence of rational humanism and nurtured by ideas of I. Kant (Kant, 1787) and G. Hegel (Hegel, 2001).

The ideas that allow us to traverse alternative paths were presented a long time ago but their development was abandoned half way through. Faraday observed that the more familiar ideas that privilege the concept of matter before the concept of action could be confronted with the approach that recognises that only actions can be detected while matter is subsequently inferred, and as such exposed to the errors belonging in conjectured entities. In this sense, Faraday recognises that matter can only be conceived in relation to action and what we call matter is the place where action is more intense [5]. Such matter extends to infinity and is interpenetrable. Faraday's vibrating rays theory eliminates the need to conceive an ether. The relational electromagnetism initially developed by W. Weber (Assis, 1994) under Gauss influence, was later developed by the Göttingen school and by L. Lorenz (Lorenz, 1867) introducing Gauss' concept of retarded action at distance. Maxwell expressed his surprise when a theory that was based in principles very different from his could provide a body of equations in such high accordance (Maxwell, 2003). From the Hertzian point of view both theories are basically "the same". However, when experiments later demonstrated the absence (nonexistence) of a material ether (Michelson & Morley, 1887) the rational approach of Faraday-Gauss-Lorenz was already forgotten, probably because the epistemological basis of science had moved towards instrumentalism.

The development of Quantum Mechanics continued from Bohr's model into Heisenberg's matrix computations, a completely instrumental approach devoid of a priori meaning (Born, 1955). While the ideas of de Broglie (de Broglie, 1923; De Broglie, 1924) and Schrödinger (Schrödinger, 1926) revolutionised the field, at the end of the Fifth Solvay conference (1927) the statistical interpretation of quantum theory collected most of the support (Schrödinger, 1995). It is said that

[5] We read in (Faraday, 1855, p.447):

> You are aware of the speculation which I some time since uttered respecting that view of the nature of matter which considers its ultimate atoms as centres of force, and not as so many little bodies surrounded by forces, the bodies being considered in the abstract as independent of the forces and capable of existing without them. In the latter view, these little particles have a definite form and a certain limited size; in the former view such is not the case, for that which represents size may be considered as extending to any distance to which the lines of force of the particle extend: the particle indeed is supposed to exist only by these forces, and where they are it is.

only Einstein and Schrödinger were not satisfied with it. Einstein foresaw conflicts with Relativity theory, later made explicit in the Einstein-Podolsky-Rosen paper (Einstein *et al.*, 1935). Schrödinger, after several years of meditation and several changes of ideas, came to the conclusion that the statistical interpretation was forced by the, not properly justified, assumption of electrons and quantum particles being represented by mathematical points (Schrödinger, 1995). Certainly, Schrödinger knew well that no statistical perspective was introduced in his construction of ondulatory mechanics. Actually, the statistical interpretation of quantum mechanics follows strictly Hertz philosophical attitude of accepting the formulae but feeling free to produce some argument that makes them acceptable. We have called this "the student's attitude" (Solari *et al.*, 2024) since it replicates the attitude of the student that is forced to accept the course perspective and has to find a form to legitimise the mandate of the authority: the result is given and the justification (at the level of plausibility) has to be provided.

The standard QM then has to find a meaning for its formulae and such task has been called "interpretation of QM". All the interpretations that we know about, rest on the idea of point particles. An idea that allows for the immediate creation of classical images that suggest new formulae such as those entering in the atomic spectra of the Hydrogen atom. Since the accelerated electron (and proton) cannot radiate in QM by ukase[6] the radiation rule has to be set as a separated rule. The atom, according to the standard theory, can then be in an "mixed energy" state until it is measured (meaning to put it in contact with the Laboratory) which makes the state represented by the wave-function to collapse (in the simplest versions). While there are more sophisticated[7]interpretations, none of them incorporates the measurement process to QM. It remains outside QM.

The idea of isolating an atom deserves examination. One can regard as possible to set up a region of space in which EM influences from outside the region can be compensated to produce an effective zero influence. In the simplest form we think of a Faraday cage and perhaps similar cages shielding magnetic fields. In the limit, idealised following Galileo, we have an atom isolated from external influences. But this procedure is only one half of what is needed. We need the atom to be unable to influence the laboratory, including the Faraday cage. And we cannot do anything onto the atom because so doing would imply it is not isolated. This is, we must rest upon the voluntary cooperation of the atom to have it isolated. The notion of an isolated atom is thus shown to be a fantasy. If the atom, as the EM system that it is, is in the condition of producing radiation, it will produce it whether the Faraday cage is in place or not. The condition imposed by

[6]ukase: edict of the tsar

[7]We imply the double meaning as elaborated and false. The first as current use and the second implied in the etymological root "sophism" in the ancient Greek: σοφιστικός (sophistikós), latin: sofisticus.

the ukase is metaphysical (i.e., not physical, nor the limit of physical situations). Actually, if such condition is imposed to Hamilton's principle, the Schrödinger's time-evolution equation is recovered to the price that the consilience with Electromagnetism and Classical Mechanics is lost.

It was Schrödinger who dug more deeply into the epistemological problems of QM. His cat, now routinely killed in every QM course/book was a form of ridiculing the, socially accepted, statistical interpretation of QM:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.(Schrödinger, 1980)

Schrödinger understood as well that the need for a statistical interpretation was rooted in the assumption of point particles (Schrödinger, 1995).

In the present work, the conservation of energy of a quantum particle is described in the form in which Faraday conceived matter as the inferred part of the matter-action pair and it leads directly to the transition rule: $\mathcal{E}_I^f + \mathcal{E}_{EM}^f = \mathcal{E}_I^i + \mathcal{E}_{EM}^i$ with i, f meaning initial and final states while I, EM mean internal and EM energies. In the early times of QM, Bohr showed that the Balmer and Rydberg lines of the Hydrogen atom corresponded to this rule. This is to say that the observed values summarised in Rydberg's formula agree in value with the equation, yet the equation was not part of the theory because the "emission" of light was conceived as the result of external influences since the isolated atom could not decay and in such way it would not collapse with the electron falling onto the proton.

Cognitive surpass as a constructive concept

The concept of *cognitive surpass* (Piaget & García, 1989) is a guiding idea of the present work. It is a step in the development of our cognition by which what was previously viewed as diverse is recognised as a particular occurrence of a more general, abstract form. This cognitive step is dialectic in nature and displays the dialectic relation between particular and general. Cognitive surpass involves abstraction and/or generalisation, it does not suppress the preexisting theories but rather incorporates them as particular instances of a new, more general theory. It works in the direction of the unity of science(Cat, 2024) and as such, its motion opposes the specialism characteristic of the instrumentalist epistemology. Cognitive surpass is a key ingredient of the abductive process since it is *ampliative* (i.e., it enlarges the cognitive basis) as opposed to the introduction of ad-hoc hypotheses to compensate for unexplained observations.

We notice that current physics hopes for a unified theory but little to no progress has been made in such direction. This matter should not come as a surprise since the requirements for understanding were downgraded to the level of analogy when adopting instrumentalism (see (Boltzmann, 1974; Mach, 2012)). Analogies hint us for surpass opportunities connecting facts at the same level, without producing the surpass. For example, Quantum Electrodynamics superimposes quantification to electromagnetism making mechanical analogies and populating an immaterial ether, the space, with immaterial analogous of classical entities (e.g., *springs*). What is preserved in this case is the analogy with classical material bodies, the recourse to imagination/fantasy.

Plan of this work

In Part I of this work we start from the initial insight by de Broglie regarding the relation of quantum mechanics and electromagnetism as Schrödinger did, yet we soon depart from de Broglie as he believed that Special Relativity was the keystone of electromagnetism. Rather, we explore the connection with relational electromagnetism. We also depart from Schrödinger's developments since we will not explore mechanical analogies neither in meaning nor in formulae. Actually, we show that de Broglie's starting point leads directly to a rewriting of Electromagnetism producing a cognitive surpass of the Lorenz-Lorentz variational formulation (Theorem 3.2, Solari & Natiello, 2022b) which is behind Maxwell's equations as propagation in the vacuum of fields with sources (intensification) in the matter. The same variational principle produces the Lorentz Force. We show that the variational principle that produces Schrödinger's evolution equation is the complementary part of one, unifying, variational principle. The synthesis produced corresponds to the unification of the quantum entity in the terms proposed by Faraday (action-matter duality) being the entity not a mathematical point but rather extended in space in a small intensification region and reaching as far as its action reaches. We further show that the classical motion under a Lorentz force corresponds to particular variations of the action. The classical dynamical equation for the angular momentum is obtained by the same methods, as well as the conservation of charge and the total energy (that of the localised entity and its associated electromagnetic field). It is shown as well that non-radiative states correspond to those of Schrödinger's equation, matching Bohr's primitive intuition. In this construction there is no room for free interpretations; measurements correspond to interactions between the entity and the laboratory described according to the laws of electromagnetism, hence the measurement process is completely within the formulation and requires no further postulates.

In Part II we show how this conception is extended to a proton and electron interacting to form an Hydrogen atom whose energy levels calculated by a systematic computation of interactions, require no patching elements such as gyromagnetic factors or "relativistic corrections".

The QM we construct allows for a better understanding of the so-called spin-orbit interaction, which in standard QM couples the spin and the "orbit" of the electron, being then a "self-interaction" (an oxymoron) of the electron. We show that when properly and systematically considered it represents a coupling between the proton and the electron as all other couplings considered.

Part I
The theory

1 Abduction of quantum (wave) mechanics

We begin by highlighting how QM enters the scene as a requirement of consistence with electromagnetism. Consider a point-charge, q, moving at constant velocity v with respect to a system that acts as a reference. We associate with this charge the relational charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \delta(x - vt) = \begin{pmatrix} C\rho \\ j \end{pmatrix} (x, t).$$

The association of current with charge-times-velocity goes back to Lorentz, Weber and possibly further back in time.

The velocity can then be written as the quotient of the amplitude of the current divided the charge, $\beta = \dfrac{qv}{Cq}$. At the same time, Lagrangian mechanics associates the velocity with the momentum, p, of a free moving body by a relation of the form, $\beta = \dfrac{p}{Cm}$. We propose that charge and mass are constitutive attributes of microscopic systems and hence

$$\begin{pmatrix} mC \\ p \end{pmatrix} = \frac{m}{q} \begin{pmatrix} qC \\ qv \end{pmatrix}$$

will behave as the charge-current density when intervening in electromagnetic relations. [8]

If rather than a point particle we accept an extended body moving at constant velocity without deforming and represented by a charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \rho(x,t)$$

where $\int d^3x\, \rho(x,t) = 1$, we find that the conditions posed above can be satisfied whenever

$$\rho(x,t) \propto \int d^3k\, \phi(k) \exp(i(k \cdot x - wt))$$

with $w = f(k)$. If we further associate the frequency with the (kinetic) energy in the form $E = \hbar w$, as de Broglie did (De Broglie, 1924), and write $p = \hbar k$ we have

$$E \;=\; \hbar f(\frac{p}{\hbar}).$$

Since we expect $\rho(x,t)$ to be real, it is clear that this association is not enough to rule out the unobservable negative energies, since
$\rho = \rho^* \Leftrightarrow \phi(k) \exp(i(k \cdot x - \frac{E}{\hbar}t)) = \phi^*(k) \exp(i((-k) \cdot x - (-\frac{E}{\hbar})t))$. Negative energies would be required and they would rest on an equal footing as positive

[8] A sort of myth emerges at this point. Lorentz (Lorentz, 1904) tried to fix the electromagnetism conceived by consideration of the ether. After lengthy mathematical manipulations of Maxwell's equations he arrived to the point in which the force exerted in the direction of motion held a different relation with the change of mechanical momentum than the force exerted in the perpendicular direction. He then substituted a constant velocity introduced as a change of reference frame with the time-dependent velocity of the electron with respect to the ether. This is a classical utilitarian approach, the formula is applied outside its scope although the mathematics that led to it does not hold true if the velocity is not constant. This is the origin of Lorentz' mass formula as applied for example to the Kauffman's experiment. The same utilitarian operation is used to produce the relativistic mass formulae. However, if this new mass were to be used in the derivation of Lorentz' force (Lorentz, 1892), the usual expression of the force would not hold. The story illustrates well the "advantages" of the utilitarian approach: it frees the scientist of requirements of consistency. In contrast, relational electromagnetism acknowledges that when in motion relative to the source we perceive EM fields still as EM fields but with characteristic changes as it is made evident by the Doppler effect. This allows to propose the form of the perceived fields when in motion at a constant relative velocity with respect to the source (Solari & Natiello, 2022b).

ones. Introducing the quantum-mechanical wave-function ψ satisfying

$$\begin{aligned}
\psi(x,t) &= \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k \, \Phi^*(k) \exp(\frac{i}{\hbar}(p \cdot x - Et)) \\
&= \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int d^3p \, \Phi^*(\frac{p}{\hbar}) \exp(\frac{i}{\hbar}(p \cdot x - Et)) \\
\rho(x,t) &= q\, \psi(x,t)^* \psi(x,t)
\end{aligned} \tag{1}$$

we automatically obtain a real charge-density without the imposition of negative energies (where for free particles $E \equiv T$, the kinetic energy, whose expression will be derived below).

We retain from traditional Quantum Mechanics the idea of expressing physical observables as real valued quantities associated to an *operator* integrated over the whole entity.

We recall that the quantum entity is a unity and *not* a mathematical point. Hence, objects like $(\psi^* \, (-i\hbar\nabla\psi))$ do not necessarily have physical (measurable) meaning. Further, observables depending only on x are multiplicative operators in front of $\psi(x,t)$. For the case of the position operator x we have [9],

$$\begin{aligned}
\langle \hat{x} \rangle &= \int d^3x \, \psi^*(x,t) \, x \, \psi(x,t) \\
&= \int d^3x \, d^3p \, d^3p' \, \frac{1}{(2\pi\hbar)^3} \Phi^*(p') \exp(\frac{i}{\hbar}(p' \cdot x - Et)) \, x \, \Phi(p) \exp(-\frac{i}{\hbar}(p \cdot x - Et)) \\
&= \int d^3p \, d^3p' \, \Phi^*(p') \, \left(i\hbar\nabla_p \, \Phi(p)\right) \int d^3x \, \frac{1}{(2\pi\hbar)^3} \exp(\frac{i}{\hbar}(p - p') \cdot x) \\
&= \int d^3p \, \Phi^*(p) \, (i\hbar\nabla_p \Phi(p))
\end{aligned}$$

while[10]

$$\hat{E} = i\hbar\partial_t .$$

De Broglie's proposal and eq.(1) relate x and p through a Fourier transform and hence the momentum operator becomes $\hat{p} = m\hat{v} = -i\hbar\nabla$. Thus, in the case of a free charged particle we arrive to Schrödinger's equation

$$i\hbar\partial_t\psi = \hat{E}\psi. \tag{2}$$

[9] Integrals such as $\langle \psi^* x \psi \rangle = \int d^3x \, (\psi^* x \psi)$ or $\langle \psi^*(-i\hbar\nabla\psi) \rangle = \int d^3x \, (\psi^*(-i\hbar\nabla\psi))$ have been called *expectation values* in the traditional probability interpretation of Quantum Mechanics.

[10] Throughout this work the notations $\partial_t\psi$, $\dot{\psi}$ and $\psi_{,t}$ all denote the same concept, namely partial derivative with respect to time.

We have yet to find $f(k)$. If we consider that eq.(2) is the definition of the Hamiltonian operator $\hat{H} \equiv \hat{E}$, and recall that the operator associated to the current density is $q\hat{v} = \frac{q}{m}\hat{p}$. Hence, the conservation of charge, also known as the *continuity equation*, $\partial_t \rho + \nabla \cdot j = 0$, produces the relations

$$\begin{aligned} q\partial_t \rho &= q\left[\psi^* \partial_t \psi + \partial_t \psi^* \psi\right] = \frac{q}{i\hbar}\left[\psi^*(\hat{H}\psi) - (\hat{H}\psi)^*\psi\right] \\ &= -\nabla \cdot \frac{1}{2}\frac{q}{m}\left[\psi^* \hat{p}\psi + (\hat{p}\psi)^*\psi\right] \end{aligned}$$

and expanding the r.h.s. we can see that the only solution is:

$$\begin{aligned} \hat{p} &= -i\hbar\nabla \\ \hat{H} &= \frac{\hat{p}^2}{2m} + U(x) = -\frac{\hbar^2}{2m}\Delta + U(x) \\ j &= \frac{q}{2m}\left(\psi^*\left(-i\hbar\nabla\psi\right) + \left(-i\hbar\nabla\psi\right)^*\psi\right) = -\frac{i\hbar q}{2m}\left(\psi^*\left(\nabla\psi\right) - \left(\nabla\psi\right)^*\psi\right) \end{aligned}$$

where $U(x)$ is any scalar function of the position. We adopt $U(x) = 0$ as for a free-particle there is no justification for a potential energy. We have then found the free-particle Hamiltonian consistent with de Broglie's approach and the assumption $\rho(x,t) = |\psi(x,t)|^2$.[11]

In summary, QM results from requirements of compatibility with electromagnetism following an abduction process in which the hypothesis $E = \hbar w$ results as an extension of the photoelectric relation. As a consequence, any attempt to superimpose further formal relations with electromagnetism such as requiring invariance or equivariance of forms under the Poincaré-Lorentz group is bound to introduce more problems than solutions.

1.1 Expression of the electromagnetic potentials

1.1.1 Physical Background

In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics. Hence, we will compute Maxwell's EM interaction energy, either between the microscopic system and external EM-fields or between different fields arising within the system. We display first some classical results that are relevant for the coming computations.

[11] In his original work, de Broglie (De Broglie, 1924) assumed the relation, $E^2 = (m^2C^2 + p^2)C^2$ proposed by special relativity on an utilitarian basis as already commented. It was later Schrödinger (Schrödinger, 1926) who proposed the classical form based on the correspondence with Newton's mechanics.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2 \right]. \tag{3}$$

The classical EM potentials originating in a charge density $|\psi|^2$ and a current $-\frac{i\hbar q}{2m} \left(\psi^*(\nabla_y \psi) - (\nabla_y \psi)\psi \right)(y,t)$, as defined in the Lorenz gauge, read

$$V(x,t) \;=\; \frac{q}{4\pi\epsilon_0} \int d^3y \, \frac{1}{|x-y|} \, |\psi(y, t - |x-y|/C)|^2$$

for the scalar potential (where as usual $C^2 = (\mu_0 \epsilon_0)^{-1}$), while the vector potential is:

$$A(x,t) \;= -\frac{i\hbar}{2} \;\; \frac{\mu_0 q}{4\pi m} \int d^3y \, \frac{1}{|x-y|} \left(\psi^*(\nabla_y \psi) - (\nabla_y \psi)\psi \right)(y, t - |x-y|/C).$$

Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$. We recall from Maxwell (see e.g. [630] Maxwell, 1873,) that all electromagnetic quantities and the wave-function are assumed to vanish sufficiently fast at infinity. This allows for the use of partial integrations and Gauss' theorem whenever appropriate.

1.1.2 Action integral and the least action principle

Lorentz' electromagnetic action (Lorentz, 1892) is

$$\begin{aligned} \mathcal{A}_{EM} \;&=\; \int_{t_1}^{t_2} \{\mathcal{L}_{EM}\} \, ds \\ \mathcal{L}_{EM} \;&=\; \frac{1}{2} \int d^3x \left\{ \frac{1}{\mu_0} B^2 - \epsilon_0 E^2 \right\} = \frac{1}{2} \int d^3x \, \{ A \cdot j - V \cdot \rho \} \end{aligned} \tag{4}$$

Equivalence between the field and potential forms is assured by the followin

Lemma 1.1. *Up to an overall function of time and the divergence of a function vanishing sufficiently fast at infinity, the electromagnetic action satisfies*

$$\mathcal{A} \;=\; \frac{1}{2} \int_{t_0}^{t_1} dt \int \left(\frac{1}{\mu_0} B^2 - \epsilon_0 E^2 \right) d^3x = \frac{1}{2} \int_{t_0}^{t_1} dt \int (A \cdot j - \rho V) \, d^3x$$

See Appendix A for a proof.

Unfortunately, Lorentz worked from his expressions in a mixed form using mathematics and intuitions generated by the hypothesis regarding the existence of a material medium permeating everything: the ether. It is from these mathematical objects that he derives the electromagnetic force. Since the original derivation is marred by the ether, the force was adopted as an addition to the collection formed by Maxwell's equations and the equation of continuity.

Contrasting to Lorentz, Lorenz (Lorenz, 1867) related light to electromagnetic currents in a body, being the propagation in the form of waves as he had previously established by experimental methods (Lorenz, 1861)[12]. His idea can be expressed in terms of a variation of the action 4 with respect to the potentials where the potentials have to match the values in the region of their source. We have called this result "the Lorenz-Lorentz theorem" (Theorem 3.2 Solari & Natiello, 2022b). The variation accounts for the propagation of electromagnetic fields and imposes the condition of conservation of charge as a result ultimately produced by the observations on the propagation of the electromagnetic action. The same action 4 when varied as Lorentz proposed with respect to the charge-current reveals the Lorentz force. The details of the derivation can be found in (Natiello & Solari, 2021).

Lorentz's action accounts for the propagation of electromagnetic effects but not for the sources, in contrast Schrödinger's equation for the free charge can be derived from a variational principle

$$\mathcal{A}_{QM} = \int_{t_0}^{t_1} ds \left\langle \frac{1}{2}\left(\psi^* \left(i\hbar\dot{\psi}\right) + (i\hbar\dot{\psi})^*\psi\right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi \right\rangle$$

where $\langle\rangle$ indicates integration in all space and Δ is Laplace's operator. The Schrödinger equation is recovered when the QM-action is varied with respect to

[12]Notice that Lorenz ideas were published two years before and two years after Maxwell seminal paper (Maxwell, 1865) and at least ten years before Hertz' famous experiments of 1887-1888. The historical narrative that reaches us attributing to Maxwell and Hertz this association is historically incorrect as it is incorrect the often stated opinion (for example in Lorentz (Lorentz, 1892)) that Faraday supported the idea of the ether when actually he had warned about the risks of inventing metaphysical objects and has provided the alternative of his theory of "vibrating rays" (Faraday, 1855).

Lorenz, Faraday as much as Newton, were ahead of their times and those that followed recast their learnings in forms apt for the reader to understand (called "didactic transposition" when teaching). A first example of misreading Faraday is precisely Maxwell (V. ii, p. 177 [529] Maxwell, 1873,) who wrote about Faraday's idea: "He even speaks * of the lines of force belonging to a body as in some sense part of itself, so that in its action on distant bodies it cannot be said to act where it is not. This, however, is not a dominant idea with Faraday. I think he would rather have said that the field of space is full of lines of force, whose arrangement depends on that of the bodies in the field, and that the mechanical and electrical action on each body is determined by the lines which abut on it. * (p, 447; Faraday, 1855)" . Goethe has prevented us: "You don't have to have seen or experienced everything for yourself; but if you want to trust another person and his descriptions, remember that you are now dealing with three factors: with the matter itself and two subjects." (570, von Goethe, 1832).

ψ (or ψ^*):

Lemma 1.2. *$\mathcal{A}_{QM}$ is stationary with respect to variations of ψ^* such that $\delta\psi^*(t_0) = 0 = \delta\psi^*(t_1)$ if and only if $i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi$.*

The hope is then that by adding both actions, $\mathcal{A}_{EM}$ that almost ignores the dynamics of the sources and $\mathcal{A}_{QM}$ that ignores the associated fields we can propose a synthetic action integral

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_{EM} + \mathcal{A}_{QM} \\ &= \int_{t_0}^{t_1} ds \left\langle \frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right) + (i\hbar\dot{\psi})^*\psi\right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi + \frac{1}{2}\left(\frac{1}{\mu_0}B^2 - \epsilon_0 E^2\right)\right\rangle \end{aligned} \quad (5)$$

no longer separating effects of sources, since, as stated in subsection 1.1, in the present construction all of current, charge-density and electromagnetic potentials depend of the wave-function ψ. Hamilton's variational principle applied to the action integral $\mathcal{A}$ yields all the dynamical equations in EM, and QM.

1.2 Variations of the action

The variations we are going to consider will be the product of an infinitesimal real function of time denoted ϵ when it is a scalar and $\bar{\epsilon}$ (a vector of arbitrary constant direction, $\bar{\epsilon} = \epsilon k, \epsilon r, \epsilon\Omega$) when it refers to a wave number, position or solid-angle respectively; times a function μ often of the form $\mu = \hat{O}\psi$ for some appropriate operator $\hat{O}$. The variation of the wave-function is reflected directly in variations on the charge-density, current, and fields. We will use for them the symbols $\delta\rho, \delta j, \delta E, \delta B$ for a set of variations that satisfy Maxwell's equations.

Hence, according to the previous paragraph, the variation $\delta\psi = \epsilon\mu$ is reflected in a variation of the EM quantities as follows:

1. Variation of charge-density: $\delta\rho = \epsilon q\left(\mu^*\psi + \psi^*\mu\right) \equiv \epsilon\boldsymbol{\mathfrak{X}}$.
2. Variation of current: $\delta j = \epsilon\boldsymbol{\mathfrak{I}}_1 + \dot{\epsilon}\boldsymbol{\mathfrak{I}}_2$. We will discuss in due time the connection between $\delta\psi$ and δj.

The continuity equation then reads

$$(\delta\rho)_{,t} + \nabla\cdot\delta j = 0,$$

which discloses into

$$\begin{aligned} \boldsymbol{\mathfrak{X}} + \nabla\cdot\boldsymbol{\mathfrak{I}}_2 &= 0 \\ \boldsymbol{\mathfrak{X}}_{,t} + \nabla\cdot\boldsymbol{\mathfrak{I}}_1 &= 0 \end{aligned}$$

thus demanding $\nabla\cdot\mathfrak{J}_1-\nabla\cdot\mathfrak{J}_{2,t}=0$ and suggesting $\mathfrak{J}_2=-\epsilon_0\mathfrak{E}$, with $\epsilon_0\nabla\cdot\mathfrak{E}=\mathfrak{X}$ for some electrical field that needs yet further specification.

We develop here the mathematical results of this manuscript. Proofs can be found in Appendix A.

Lemma 1.3. *(Preparatory lemma). Under the previous definitions and the conditions $\mathfrak{X}+\nabla\cdot\mathfrak{J}_2=0$ and $(\mathfrak{X})_{,t}+\nabla\cdot\mathfrak{J}_1=0$, complying with the continuity equation, a variation $\delta\psi=\epsilon\mu$ of the wave-function produces the varied action integral*

$$\begin{aligned}\delta\mathcal{A} &= \delta\int_{t_0}^{t_1} ds\left\{\left\langle\frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle+\frac{1}{2}\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle\right\}\\ &= \int_{t_0}^{t_1}\epsilon\, ds\left\langle\left(\mu^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\mu\right)+\frac{\hbar^2}{2m}\left(\mu^*\Delta\psi+\mu\Delta\psi^*\right)\right\rangle+\\ &\quad+\int_{t_0}^{t_1}\epsilon\, ds\left\langle A\cdot\mathfrak{J}_1-(A\cdot\mathfrak{J}_2)_{,t}-V\mathfrak{X}\right\rangle,\end{aligned}$$

where the fields are denoted as $B=B_m+B_r$ and $E=E_m+E_r$, with the index m corresponding to the microscopic system to be varied and r denotes the "outside world". A,V are the electromagnetic potentials.

Corollary 1.1. *Without loss of generality, we set $\mathfrak{J}_2=-\epsilon_0\mathfrak{E}$, where $\epsilon_0\nabla\cdot\mathfrak{E}=\mathfrak{X}$.*

We will address different relations in a few related results which are stated separately because of expository convenience. The details of the proofs will be displayed in Appendix A.

Theorem 1.1. *(Classical variations theorem). Under the assumptions of Lemma 1.3, the null variation of the action*

$$\mathcal{A}=\int_{t_1}^{t_2} ds\left(\left\langle\frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle+\frac{1}{2}\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle\right)$$

as a consequence of a variation of the wave-function as:

$$\delta\psi=\epsilon\psi_{,t},\ (i\epsilon k\cdot x)\,\psi,\ (\epsilon\bar{x}\cdot\nabla\psi)\,,\ (\epsilon\Omega\cdot(x\times\nabla\psi))$$

–as summarised in Table 1– results in the corresponding equations of motion displayed in Table 2.

The entries in table 1 were produced following intuitions based upon the notion of conjugated variables in classical mechanics. A displacement in time is associated with the law of conservation of energy, one in space with the law of variation of linear momentum and an angular displacement with angular momentum. The variation leading to the derivative of the position is of an abductive

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{I}_1$	$\mathfrak{I}_2$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$-2\epsilon_0\mathfrak{E}_{,t} - \mathfrak{j} = j_{,t}$	j
position	$(i\epsilon k \cdot x)\,\psi$	0	0	0
displacement	$\epsilon r \cdot \nabla\psi$	$r \cdot \nabla\rho$	$(r \cdot \nabla)\, j = \mathfrak{j}$	$-r\rho$
rotation	$\epsilon\Omega \cdot (x \times \nabla\psi)$	$\Omega \cdot (x \times \nabla\rho)$	$(\Omega \cdot (x \times \nabla))\, j - \Omega \times j = \mathfrak{j}$	$-\Omega \times (x\rho)$

Table 1: Variations

Variation/result	$\delta\psi$	Equation of motion
time/energy	$\epsilon\partial_t\psi$	$\partial_t \left\langle -\frac{\hbar^2}{2m}\psi^*\Delta\psi + \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right\rangle = 0$
position/velocity	$(i\epsilon k \cdot x)\,\psi$	$\partial_t \langle \psi^* x \psi\rangle = \dfrac{\langle j\rangle}{q}$
displacement/momentum	$\epsilon r \cdot \nabla\psi$	$\dfrac{m}{q}\partial_t \langle j\rangle = \langle j \times B + \rho E\rangle$
rotation/angular momentum	$\epsilon\Omega \cdot (x \times \nabla\psi)$	$\dfrac{m}{q}\partial_t \langle x \times j\rangle = \langle x \times (\rho E + j \times B)\rangle$

Table 2: Equations of motion.

nature since what it is really known is the cinematic relation. Hence, the variation is obtained as the answer to the question: which variation produces the cinematic relation $\langle x\rangle_{,t} = \langle j\rangle / q$?

The proposed variations do not follow a general pattern. Hence, we expect that there might be alternative forms of variations that lead to identical results which in addition present consilience. Moreover, we expect that the energy would take the place of the Lagrangian since this is one of the intuitions based upon experimental observations that lead to quantum mechanics. We present the result of this synthetic steps in the next theorem.

There are four matters that have to be put together to complete the construction of Quantum Mechanics:

1. Changing the Lagrangian presentation into a Hamiltonian one.

2. Showing that the Hamiltonian presentation has consilience and produces the particular results of Theorem 1.1 as well (it is a generalisation/correction of the particular cases shown).

3. Showing that this setting produces the usual eigenvalue-eigenvector equations of Quantum Mechanics for the stationary wave-functions.

4. Making more explicit the relation between the variation of the wave-function and the variation of the electromagnetic current.

We state these results as three Lemmas and a Corollary.

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{j}$	$\epsilon_0\mathfrak{E}$	$\nabla\times\mathfrak{B}$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$j_{,t}$	$E_{,t}$	$\nabla\times B_{,t}$
position	$(i\epsilon k\cdot x)\,\psi$	0	0	0	0
displacement	$\epsilon r\cdot\nabla\psi$	$(r\cdot\nabla)\,\rho$	$(r\cdot\nabla)\,j$	$r\rho$	$-\nabla\times(r\times j)$
rotation	$\epsilon\Omega\cdot(x\times\nabla\psi)$	$\Omega\cdot(x\times\nabla\rho)$	$(\Omega\cdot(x\times\nabla))\,j-\Omega\times j$	$\Omega\times(x\rho)$	$-\nabla\times((\Omega\times x)\times j)$

Table 3: Variations in Hamiltonian form

Lemma 1.4. *(Hamiltonian form) Let the set of functions $\mathfrak{X},\mathfrak{j},\mathfrak{E},\mathfrak{B}$ satisfy Maxwell equations with $\delta\rho=\epsilon\mathfrak{X}=\epsilon q\,(\mu^*\psi+\psi^*\mu)$ as in Lemma 1.3 . If we let the current be $\delta j=\epsilon\mathfrak{J}_1+\dot\epsilon\mathfrak{J}_2$, with*

$$\begin{aligned}\mathfrak{J}_2 &= -\epsilon_0\mathfrak{E}\\ \mathfrak{J}_1 &= -2\epsilon_0\mathfrak{E}_{,t}-\mathfrak{j}\end{aligned}$$

the null variation of the action integral satisfies

$$\left\langle\mu^*\left(i\hbar\dot\psi\right)+\left(i\hbar\dot\psi\right)^*\mu\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle-\left\langle\frac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle=0,\tag{6}$$

where $-\dfrac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle+\left\langle\dfrac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle\equiv\mathcal{H}$ is recognised as the Hamiltonian of the problem.

Corollary 1.2. *The form of the variations appropriated to the Hamiltonian form corresponding to the results of Theorem 1.1 (Table 2) is the one given in Table 3.*

Lemma 1.5. *(Variation of a stationary state). In the special case when ψ corresponds to a stationary state, $\psi(x,t)=\exp(-i\omega t)\,\psi(x,0)\equiv\exp(-i\omega t)\,\psi_\omega$ and assuming that the external electromagnetic fields are time-independent, and B is homogeneous, the variation in Lemma 1.4 yields*

$$\left\langle\mu^*\left(\hbar\omega\psi_\omega\right)\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi_\omega\right\rangle-\left\langle\mu^*\hat H_{EM}\psi_\omega\right\rangle=0\tag{7}$$

where

$$\hat H_{EM}\psi=V\psi-\left(\frac{q}{2m}\right)\left(B\cdot\left(x\times(-i\hbar\nabla)\right)\psi\right)$$

Eq.(7) gives the Schrödinger equation for stationary states.

The original expression by Schrödinger was developed setting $B=0$ and assuming that the electron is the quantum entity while the proton is an "external" point charge, responsible for the electrostatic potential V.

In Theorem 1.1 as well as in Corollary 1.2 we rested on phenomenological ideas (synthetic judgements) to produce the current associated to the variation. To find a general form for a set of intuitions appears as almost impossible. We proposed in the case of the rotations $\mathfrak{j} = (\Omega \cdot (x \times \nabla))\, j - \Omega \times j$ because the rotation of vectors demands to change the spatial argument as well as the orientation of the components. The confirmation of the abductive hypothesis came from the result, which has an intuitively appealing form. On the contrary, had we forgotten the term $-\Omega \times j$ we would have ended up with a result with no consilience with that of classical mechanics. We stress that these are rational arguments which are not of a deductive nature. Nevertheless, there are some mathematical relations that have to be satisfied since the continuity equation must be enforced.

Lemma 1.6. *The variations $\hat{O}\psi$ (with $\hat{O}_{,t} = 0$) that may produce a varied current $\mathfrak{j}$ fulfilling the conditions of Lemma 1.3 extend as real operators Θ acting on forms such as $(\rho,\ j,\ \nabla \cdot j)$ that can be regarded as real-valued functions of (ψ^*, ψ) as:*

$$\begin{aligned}
\Theta\rho\,(\psi^*, \psi) &= \rho\left(\left(\hat{O}\psi\right)^*, \psi\right) + \rho\left(\psi^*, \left(\hat{O}\psi\right)\right) \\
\Theta j\,(\psi^*, \psi, \nabla\psi^*, \nabla\psi) &= j\left(\left(\hat{O}\psi\right)^*, \psi, \left(\hat{O}\nabla\psi\right)^*, \nabla\psi\right) + j\left(\psi^*, \hat{O}\psi, \nabla\psi^*, \left(\hat{O}\nabla\psi\right)\right) \\
\Theta\left(\nabla \cdot j\,(\psi^*, \psi, \Delta\psi^*, \Delta\psi)\right) &= j\left(\left(\hat{O}\psi\right)^*, \psi, \left(\hat{O}\Delta\psi\right)^*, \Delta\psi\right) + j\left(\psi^*, \left(\hat{O}\psi\right), \Delta\psi^*, \left(\hat{O}\Delta\psi\right)\right).
\end{aligned}$$

Then, Θ preserves the continuity equation as $\Theta\left(\rho_{,t} + \nabla \cdot j\right) = 0$ and $\Theta(\nabla \cdot j) = \nabla \cdot \mathfrak{j}$ when the variation satisfies the relation

$$\psi^*\left(\left[\Delta, \hat{O}\right]\psi\right) = \psi^*\left(\left[\Delta, \hat{O}\right]^\dagger \psi\right) + \nabla \cdot K$$

(where $\dagger$ indicates Hermitian conjugation) for some vector K satisfying $\int d^3x\ (\nabla \cdot K) = \int_\Sigma K \cdot \widehat{d\Sigma} = 0$. In such a case, $\mathfrak{j} = \Theta j + K$. The operator $\hat{O}$ is then antiHermitian.

As examples for the application of the previous lemma we use the cases listed in table 3. The operators associated to energy, velocity and rotation are $\hat{O} = (\partial_t, (r \cdot \nabla), \Omega \cdot (x \times \nabla))$. All of them commute with Laplace's operator, hence $\mathfrak{j} = \Theta j$. In the case of the position $\Theta j = 0$, $\Theta\left(\nabla \cdot j\right) = 0$ and the terms including $\left[\nabla, \hat{O}\right]$ add up to zero.

1.3 Discussion

In Subsection 1.2we have shown that when the electromagnetic action outside the locus of masses, charges and currents as it appears in the Lorenz-Lorentz theorem is complemented with the action where mass, charge and currents are not necessarily null, the action proposed encodes Maxwell's electromagnetic laws, Lorentz' force and Newton's equations of motion, as well as an extension of Schrödinger equation for stationary states . Thus, electromagnetism does not question the validity of classical mechanics, nor quantum mechanics questions the validity of electromagnetism or classical mechanics. They coexist in a synthetic variational equation.

The difference between Schrodinger's eigenvalue-eigenvector equation and eq. (7) corresponds to the "static energy" which is present in Maxwell's formulation of electromagnetism ([680] p. 270 Maxwell, 1873,). The method of construction used by Maxwell is not consistent with the notion of quanta since it assumes that any charge can be divided in infinitesimal amounts, the electrostatic energy corresponding to assembling together the total charge. From a constructive point of view, such energy is not justified.

There are several possible alternatives. One of them is not to consider self-action (second alternative). A third alternative is to consider only the energy contained in transversal (rotational) non-stationary fields $(E_\perp, B)$ (where $E_\perp = E - (-\nabla V_C)$, being V_C the Coulomb potential). In the first and third cases, there will be a flow of energy associated to the EM waves each time the entity is not in a stationary state. In the second case, energy will correspond to interactions and the possibility of a quantum entity to be isolated as the limit of constructively factual situations will be doubted. One way or the other each view can be incorporated to the formalism, which in turn will provide consistency requirements that can be contrasted against experimental situations. Which alternative(s) must be discarded is to be decided experimentally. The list of alternatives will remain open until the occasion. We will see in Part II of this work that the alternatives that present "no self-actions in stationary situations" (second and third alternative) account for the energy levels of the hydrogen atom. Yet, if we want to account for the photoelectric effect or the Compton effect we need to have room for incoming or outgoing energy and momentum associated to the EM fields, thus the third alternative looks more promising.

It is worth noticing that in the statistical interpretation of quantum mechanics, the energy of the field is not considered (no reasons offered) avoiding in such way dealing with an infinite energy that corresponds to the electrostatic (self) energy of a point charge. However, self energies are not ruled out completely since they are needed (in the standard argumentation) to account for the so called spin-orbit interaction. As usual in the instrumentalist attitude, consistence is not a matter of concern.

Part II
Implementation: The hydrogen atom

2 Constructive ideas for Quantum Mechanics from Electromagnetism

We borrow some basic ingredients from Maxwell's electrodynamics and from traditional Quantum Mechanics. These elements will be combined in dynamical equations in the next Section.

1. The atom is the microscopic entity susceptible of measurements. Proton and electron within the atom do not have a sharp separable identity, they are inferences (or "shadows"). The identity of the atom consists of charge, current, magnetisation(s) and masses.

2. The experiments of Uhlenbeck & Goudsmidt and Zeeman as well as that of Stern and Gerlach suggest the existence of intrinsic magnetisations. We introduce it following the standard form relying in Pauli's seminal work (Pauli Jr, 1927).

3. The interactions within the atom and of the atom with detectors are electromagnetic.

4. Elementary entities such as the proton and the electron have no self-interactions. There is no evidence regarding the existence of self-energies (constitutive energies) at this level of description, correspondingly, they will not be included.

5. The experimental detection of properties belonging to the shadows within the atom depends on the actual possibility of coupling them with measurement devices (usually based upon electromagnetic properties). We will stay on the safe side and will not assume as measurable things like the distance from proton to electron or the probability distribution of the relative position between proton and electron. Such "measurements" remain in the domain of fantasy unless the actual procedure is offered for examination. In this respect we severely depart from textbook expositions of Quantum Mechanics.

6. The atom as an EM-entity is described with wave-functions. Any quantity associated to an atom, such as its EM fields, are represented by operators on the wave-function's space.

2.1 Expression of the electromagnetic potentials as integrated values on operators

2.1.1 Wave-functions

Two bodies that are spatially separated are idealised as represented by a *wave-function* which is the product of independent wave-functions on different coordinates. For the case of a proton and an electron we write $\Psi(x_e, x_p) = \psi_e(x_e)\psi_p(x_p)$ and call it the *electromagnetic limit*. In contrast, a single body with an internal structure is represented by the product of a wave-function associated to the body and another one representing the internal degrees of freedom. For the hydrogen atom $\Phi(x_r, x_{cm}) = \phi_r(x_r)\phi_{cm}(x_{cm})$ represents the *atomic limit*.

The election of the centre of mass, cm, to represent the position of the body will be shown later to be consistent with the form of the relational kinetic energy introduced. The variable x_r (r for *relative* or *relational*) stands for $x_e - x_p$. Our constructive postulate is that, when proton and electron are conceived as (spatially) separated entities, the laws of motion correspond to electromagnetism and when conceived as a hydrogen atom, the usual QM is recovered as a limit case[13]. In fact, the present approach is broader than the usual QM in at least two aspects: (a) The decomposition of the wave-function as a product is not imposed, but rather regarded as a limit and (b) the proton is regarded as something more than just the (point-charge) source of electrostatic potential. Hence, starting from the laws of EM, the structure of a general quantum theory for $|\Psi\rangle$ can be postulated as surpassing EM. Specialising this theory to the atomic case, the QM of the hydrogen atom is obtained.

2.1.2 Integrals on operators

We will call the value of the integral

$$\int d^3x_a\, d^3x_b\, \left(\Psi^*(x_a, x_b)(\hat{O}\Psi(x_a, x_b))\right) \equiv \left\langle \Psi^*(\hat{O}\Psi)\right\rangle \equiv \left\langle \Psi^*|\hat{O}|\Psi)\right\rangle \tag{8}$$

the *integrated value* of the operator $\hat{O}$, where $\hat{O}$ is an operator that maps the wave-function Ψ as $\hat{O}\Psi(x_a, x_b) = \Psi'(x_a, x_b)$[14].

[13] Conventional QM also assumes whenever necessary that $\psi_{CM}(x_{CM})$ corresponds to a point-like particle at rest.

[14] We will drop the time-argument of the wave-function whenever possible to lighten the notation.

2.1.3 Physical Background

In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics, starting with the principles stated at the beginning of this Section and in Part of this work (Solari & Natiello, 2024). Hence, we will compute Maxwell's EM interaction energy, either between the atom and external EM-fields or between different fields arising within the atom. We display first some classical results that are relevant for the coming computations.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2 \right]. \tag{9}$$

As such, this result is independent of the choice of gauge for the electromagnetic potentials A, V. In the sequel, we will adopt the Coulomb gauge whenever required.

The vector and scalar potentials associated to a charge-current density $(Cq\rho, j)$ in a (electro- and magneto-) static situation are

$$\left(\frac{V}{C}, A \right)(x,t) \;\; = \;\; \frac{\mu_0}{4\pi} \int d^3y \, \frac{(Cq\rho, j)(y,t)}{|x-y|}.$$

The vector potential associated to a magnetisation M at the coordinate y reads:

$$A_M(x,t) \;\; = \;\; \frac{\mu_0}{4\pi} \nabla_x \times \int d^3y \, \frac{M(y,t)}{|x-y|}.$$

For potentials as defined in the Lorenz gauge there is a gauge transformation that brings them to Coulomb form i.e., ρ electrostatic and $\nabla \cdot A = 0$. Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$.

2.1.4 Electromagnetic static potentials

We define the density within the atom as $\rho(x_a, x_b) = |\Psi(x_a, x_b)|^2$. When appropriate, protonic and electronic densities may be defined as $\int d^3x_b \, |\Psi(x_a, x_b)|^2 \equiv \rho_a(x_a)$ for each component, $a = e, p$. In the electromagnetic limit of full separation between electron and proton ($|\Psi\rangle = |\psi_a\rangle|\psi_b\rangle$), with a wave-function for particle b normalised to unity, we recover $\rho_a(x_a) = |\psi_a(x_a)|^2$. Protonic and electronic current-densities within the atom are defined similarly for $a = e, p$ by

$$j_a(x_p, x_e) = -\frac{i\hbar q_a}{2m_a} \left(\Psi^*(x_p, x_e) \left(\nabla_a \Psi(x_p, x_e) \right) - \Psi(x_p, x_e) \left(\nabla_a \Psi^*(x_p, x_e) \right) \right)$$

with $q_e = -q_p$.

The electrostatic potential reads

$$V_a(x,t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, d^3x_b\, \frac{1}{|x-x_a|}\, |\Psi(x_a,x_b)|^2 = \frac{q_a}{4\pi\epsilon_0}\langle\Psi|\frac{1}{|x-x_a|}|\Psi\rangle.$$

Moreover,

$$V_a(x,t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{1}{|x-x_a|}\left(\int d^3x_b\, |\Psi(x_a,x_b)|^2\right) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{\rho_a(x_a)}{|x-x_a|}$$

showing that the electromagnetic structure is preserved in general, not just in the limit of full separation.

Similarly, the vector potential (Coulomb gauge) associated to an intrinsic microscopic magnetisation $M_a(x_a,x_b) = M_a(x_a)\rho(x_a,x_b)$ reads:

$$\begin{aligned} A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_b\, d^3x_a\, \frac{M_a(x_a,x_b)}{|x-x_a|} \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a(x_a)}{|x-x_a|}\rho(x_a) \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\Psi|\frac{M_a}{|x-x_a|}|\Psi\rangle \end{aligned}$$

(the index in ∇ indicates the derivation variable) where Ψ is the wave-function. In the electromagnetic limit, the vector potential reads:

$$\begin{aligned} A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a}{|x-x_a|}\, |\psi_a(x_a)|^2 \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\psi_a|\frac{M_a}{|x-x_a|}|\psi_a\rangle, \end{aligned}$$

which is the classical EM potential for a density $|\psi_a(x_a)|^2$.

For a static current we have

$$A_a(x,t) = \frac{1}{2}\frac{\mu_0 q_a}{4\pi m_a}\int d^3x_a\, d^3x_b\, \left(\Psi^*(x_a,x_b)\frac{1}{|x-x_a|}(-i\hbar\nabla_a)\Psi(x_a,x_b) + c.c.\right).$$

Here we may also define

$$\widetilde{j}_a(x_a) = \frac{q_a}{2m_a}\int d^3x_b\, \left(\Psi^*(x_a,x_b)(-i\hbar\nabla_a)\Psi(x_a,x_b)\right) + c.c.$$

and set

$$A_a(x,t) = \frac{\mu_0}{4\pi}\int d^3x_a\, \left(\frac{\widetilde{j}_a(x_a)}{|x-x_a|}\right).$$

2.1.5 On atomic magnetic moments

We associate an intrinsic magnetisation to proton and electron, following Pauli's original work (Pauli Jr, 1927). A general form for the intrinsic magnetisation for a microscopic constituent a reads

$$M_a = \frac{q_a}{m_a} S_a$$

where $S_a = \frac{\hbar}{2}\sigma^a$ and σ^a are the Pauli matrices for constituent a.

2.2 Variational formulation

Both Newtonian mechanics (Arnold, 1989) and Maxwell's electrodynamics (Solari & Natiello, 2022b) can be reformulated as variational theories, satisfying the *least action principle.* The action (or action integral) is the integral of the Lagrangian over time. From Part I,

$$\mathcal{L} = \int d^3y \left[\frac{1}{2}\left(\Psi^*\left(i\hbar\dot{\Psi}\right) + (i\hbar\dot{\Psi})^*\Psi\right) + \frac{\hbar^2}{2m}\Psi^* T\Psi + \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{\mu_0}B_i\cdot B_j - \epsilon_0 E_i \cdot E_j\right)\right] \tag{10}$$

with $i, j \in \{e, p, L\}$ which stand for electron, proton and Laboratory. The symbol T stands for kinetic energy and its form will be presented later. Actually "d^3y" is a symbol indicating that the integral is performed over all non-temporal arguments of the wave-function, which in the case of the hydrogen atom will be at least six.

In this Part of the work we are interested in the internal energies associated to stationary states of the microscopic entity (the Hydrogen atom). In Part I we assumed that the microscopic entity did not have an internal structure and therefore there was no "self-energy". In the present situation the atomic energy will have an internal part and an interaction part, as implied by eq.(10).

We propose therefore that the stationary states of the hydrogen atom have an associated wave-function $\Psi_\omega(x_e, x_p)$, where the indices e and p correspond to the inferred electron and proton, whose unifying entity is the atom. x represents appropriate arguments of the wave-function, satisfying

$$(\hbar\omega\Psi_\omega) - \frac{\hbar^2}{2m}T\Psi_\omega - \hat{H}\Psi_\omega = 0 \tag{11}$$

(we drop the index ω in the sequel) where

$$\left\langle \delta\Psi \hat{H}_{EM}\Psi\right\rangle = \delta_{\Psi^*}\left\langle \frac{1}{\mu_0}\left(B_L\cdot(B_e + B_p) + B_e\cdot B_p\right) + \epsilon_0\left(E_L\cdot(E_e+E_p) + E_e\cdot E_p\right)\right\rangle$$

the variation of the wave-function is $\delta\Psi$ and $\langle\cdots\rangle$ indicates integration over all coordinates as in eq.(8).

3 Contributions to the integrated value of H

3.1 Relational kinetic energy

In terms of classical mechanics, the kinetic energy associated to the hydrogen atom can be organised as follows. To fix ideas let index b correspond to the proton and a to the electron. The quantum-mechanical wave-function is $\Psi(x_a, x_b)$. The center-of-mass coordinate (relative to some external reference such as "the laboratory") and the relative coordinate (which is an invariant of the description, independent of an external reference) can be defined as usual, satisfying

$$\begin{aligned}
x_{cm} &= \frac{m_a x_a + m_b x_b}{m_a + m_b} \\
x_r &= x_a - x_b \\
x_a &= x_{cm} + x_r \frac{m_b}{m_a + m_b} \\
x_b &= x_{cm} - x_r \frac{m_a}{m_a + m_b}.
\end{aligned}$$

The wave-function and the gradient (we also make use of $p_x = -i\hbar\nabla_x$) read:

$$\begin{aligned}
\Psi(x_a, x_b) &= \Psi(x_{cm} + x_r \frac{m_b}{m_a + m_b}, x_{cm} - x_r \frac{m_a}{m_a + m_b}) = \Phi(x_{cm}, x_r) \\
\nabla_{cm} &= \nabla_a + \nabla_b \\
\nabla_r &= \frac{m_b}{m_a + m_b}\nabla_a - \frac{m_a}{m_a + m_b}\nabla_b \\
\Phi(x_{cm}, x_r) &= \phi(\frac{m_a x_a + m_b x_b}{m_a + m_b}, x_a - x_b) = \Psi(x_a, x_b) \\
\nabla_a &= \nabla_{cm}\frac{m_a}{m_a + m_b} + \nabla_r \\
\nabla_b &= \nabla_{cm}\frac{m_b}{m_a + m_b} - \nabla_r
\end{aligned}$$

Hence,

$$\begin{aligned}
T_a + T_b = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} &= -\hbar^2 \left(\nabla_{cm}^2 \frac{1}{2(m_a + m_b)} + \nabla_r^2 \frac{(m_a + m_b)}{2m_a m_b} \right) \\
&= \frac{-\hbar^2}{2(m_a + m_b)}\nabla_{cm}^2 + \frac{-\hbar^2}{2m_r}\nabla_r^2 = T_{cm} + T_r.
\end{aligned}$$

The associated integrated values read:

$$\mathcal{E}_k = \langle\Psi|T_a|\Psi\rangle + \langle\Psi|T_b|\Psi\rangle = \langle\Phi|T_{cm}|\Phi\rangle + \langle\Phi|T_r|\Phi\rangle$$

3.2 Relative electrostatic potential energy

The Coulomb electrostatic interaction energy for an atomic density $|\Psi(x_a, x_b)|^2$ reads:

$$\begin{aligned}\mathcal{E}_C^{x_a,x_b} &= \epsilon_0 \int d^3x\, E_1 \cdot E_2\, |\Psi(x_a, x_b)|^2 = -\epsilon_0 \int d^3x\, E_1 \cdot \nabla V_2\, |\Psi(x_a, x_b)|^2 \\ &= \int d^3x\, \epsilon_0 \left[-\nabla \cdot (E_1 V_2) + (\nabla \cdot E_1)\, V_2\right] |\Psi(x_a, x_b)|^2 .\end{aligned}$$

The full energy is given by averaging over the wave function Ψ. The first contribution vanishes when integrated over all space by Gauss theorem, and the second term can be recast as $-\epsilon_0(\nabla \cdot \nabla V_1)V_2$, namely

$$\begin{aligned}\mathcal{E}_C &= \int d^3x_a\, d^3x_b\, |\Psi(x_a, x_b)|^2 \int d^3x\, \epsilon_0 \left(\frac{q_a q_b}{(4\pi\epsilon_0)^2}\right)\left(-\frac{1}{|x - x_b|}\Delta\frac{1}{|x - x_a|}\right) \\ &= -\frac{e^2}{4\pi\epsilon_0}\int d^3x_e\, d^3x_p\, |\Psi(x_e, x_p)|^2\, \frac{1}{|x_e - x_p|} = -\langle\Psi|\frac{e^2}{4\pi\epsilon_0}\frac{1}{|x_e - x_p|}|\Psi\rangle\end{aligned}$$

where $q_a = e = -q_b$ and e is the electron charge (negative), i.e., a is the electron, b the proton. In the electromagnetic limit it corresponds to the potential energy between two separate charge densities $\rho_{k=}|\psi_k(x_k)|^2$ of opposite sign, corresponding to Maxwell's equation $q_k\rho_k = \epsilon_0\nabla \cdot E_k$.

3.3 Interaction of relative current with external magnetic field

For this interaction *relative current* means the currents (j_a, j_b) of both electron and proton regarding their motion relative to the (source of) external field. Recall that we associate $j = qv = \dfrac{q}{m}p = \dfrac{q}{m}(-i\hbar\nabla)$, i.e., current as a property of charge in (perceived) motion. The electromagnetic energy in the external field $B(x)$ is

$$\begin{aligned}\mathcal{E}_{jB} &= \frac{1}{\mu_0}\int d^3x_a\, d^3x_b\, \Psi(x_a, x_b)^* \int d^3x\, B(x) \cdot \nabla \times (A_a + A_b)\, \Psi(x_a, x_b) \\ &= \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a, x_b)^* \int d^3x\, B(x) \cdot \nabla \times \left(\frac{1}{|x - x_a|}\frac{q_a}{m_a}p_a\right)\Psi(x_a, x_b) + c.c. \\ &+ \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a, x_b)^* \int d^3x\, B(x) \cdot \nabla \times \left(\frac{1}{|x - x_b|}\frac{q_b}{m_b}p_b\right)\Psi(x_a, x_b) + c.c.\end{aligned}$$

Having in mind the (normal) Zeeman effect, we perform an explicit calculation using that the vector potential for the approximately constant external field B reads $B \equiv \nabla \times A_L = \nabla \times \big(\chi(x)\tfrac{1}{2}x \times B\big)$, with χ a smoothed version of the characteristic function for the experiment's region. We place the origin of coordinates inside the apparatus and assume the characteristic function to be 1 inside a macroscopically large ball K around the origin, so that

$$\int_{K\times K} dx_a\, dx_b\, |\Psi(x_a,x_b)|^2 \simeq 1$$

(i.e., the atom is inside the measurement device). In such a case,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\left(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a+\frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.\\
&= -\frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\,\frac{1}{2}B\cdot\left(x_a\times\frac{q_a}{m_a}p_a+x_b\times\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.\\
&= -\frac{e}{2m_r}\langle B\cdot\left(\frac{m_r}{m_e+m_p}x_r\times p_{CM}+x_{CM}\times p_r+\frac{m_p-m_e}{m_p+m_e}L_r\right)\rangle\\
&\sim -\left(\frac{e}{2m_e}+O(\frac{m_e}{m_p})\right)\langle\Psi|B\cdot L_e|\Psi\rangle
\end{aligned}$$

where in the last line we have switched to CM, r coordinates and used that $\langle p_r\rangle = \langle p_{CM}\rangle = 0$ and $[x_r,\nabla_{CM}] = [x_{CM},\nabla_r] = 0$.

Further, the argument of the above integral reads,

$$\nabla\times A_L\cdot\nabla\times A_a = \nabla\cdot(A_L\times(\nabla\times A_a)) + A_L\cdot\nabla\times(\nabla\times A_a)\,.$$

The first term vanishes by Gauss theorem. The experimental conditions assume that the atom lies well inside the region of magnetic field and that the border effects $\nabla\chi(x-x_{CM})\times\left(\frac{1}{2}x\times B\right)$ can be disregarded. Moreover, $\nabla\times(\nabla\times A_a) = \nabla(\nabla\cdot A_a)-\Delta A_a$. For the first term, we integrate over x_a, x_b, using that $\nabla_x\frac{1}{|x-x_a|} = -\nabla_{x_a}\frac{1}{|x-x_a|}$, arriving to the integral of $\frac{1}{|x-x_a|}\nabla_{x_a}\cdot\left(\Psi(x_a,x_b)^*\frac{q_a}{m_a}p_a\Psi(x_a,x_b)\right)+c.c.$, which is zero for an atom in a stationary state, since by the continuity equation it corresponds to the time-derivative of the atomic charge density. For the other term, recall that $-\Delta\frac{1}{|x-x_a|} = 4\pi\delta(x-x_a)$. Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\int d^3x\, A_L\cdot\left(\delta(x-x_a)\frac{q_a}{m_a}p_a+\delta(x-x_b)\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)\\
&\quad +c.c.\\
&= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\left(A_L(x_a)\cdot\frac{q_a}{m_a}p_a+A_L(x_b)\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.\\
&\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\left(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a+\frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.
\end{aligned}$$

assuming that the atomic wave-function is negligibly small outside the border of the experiment's region. We now move to center-of-mass and relative coordinates, assuming further that the atom as a whole is at rest (at the centre of mass location) during the experiment (i.e., $\langle\Psi|p_{CM}|\Psi\rangle = 0$, $p_r \sim p_e$, $x_r \sim x_e$ and $\frac{m_p - m_e}{m_p + m_e} \sim 1$). Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\left\{\left\langle \Psi(x_a, x_b)^* \left(\frac{1}{2}(x_a - x_{CM}) \times B \cdot \frac{q_a}{m_a}\right)\Psi(x_a, x_b)\right\rangle\right. \\
&\quad \left. + \left\langle \Psi(x_a, x_b)^* \left(\frac{1}{2}(x_b - x_{CM}) \times B \cdot \frac{q_b}{m_b} p_b\right)\Psi(x_a, x_b)\right\rangle\right\} + c.c. \\
&= \frac{1}{2}\left(-\frac{e}{2(m_p + m_e)}\right)\left\langle \Psi(x_{CM}, x_r)^* B \cdot \left(\frac{m_p}{m_e} - \frac{m_e}{m_p}\right) L_r \Psi(x_{CM}, x_r)\right\rangle + c.c. \\
&= -\frac{e}{2m_r}\frac{m_p - m_e}{m_p + m_e}\langle\Psi|B \cdot L_r|\Psi\rangle \sim -\left(\frac{e}{2m_e} + O(\frac{m_e}{m_p})\right)\langle\Psi|B \cdot L_e|\Psi\rangle
\end{aligned}$$

3.4 Interaction of spin(s) with external magnetic field

We regard spin as an intrinsic magnetisation $\frac{q_a}{m_a} S_a$, leading to a vector potential

$$A_s(x,t) = \frac{\mu_0}{4\pi}\nabla \times \langle\Psi|\left(\frac{q_a}{m_a}\frac{S_a}{|x - x_a|} - \frac{q_a}{m_b}\frac{S_b}{|x - x_b|}\right)|\Psi\rangle \tag{12}$$

whose interaction energy with an external magnetic field $B(x,t)$ reads

$$\begin{aligned}
\mathcal{E}_{SB} &= \frac{1}{\mu_0}\int d^3x\, B(x,t) \cdot \nabla \times A_s(x,t) \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t) \cdot \left(\nabla \times \nabla \times \left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)|\Psi\rangle \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t) \cdot \left[\nabla\left(\nabla \cdot \left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)\right]|\Psi\rangle \\
&- \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t) \cdot \left(\Delta\left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)|\Psi\rangle
\end{aligned}$$

Since $\nabla \cdot B = 0$, the first contribution integrates to zero by Gauss theorem since $B \cdot \nabla\phi = \nabla \cdot (B\phi) - \phi\nabla \cdot B$. Hence,

$$\mathcal{E}_{SB} = e\langle\Psi|\left(\frac{1}{m_e}B(x_e,t) \cdot S_e - \frac{1}{m_p}B(x_p,t) \cdot S_p\right)|\Psi\rangle$$

thus completing the description of Zeeman effect. In the atomic limit the usual expression is recovered.

Stern-Gerlach effect. All forces, including the Lorentz force (Solari & Natiello, 2022b), are obtained from actions as the response of the action integral to a variation of the relative position of the interacting bodies. Displacing by δx the position of the field B one gets

$$\delta\mathcal{E}_{SB} \;=\; e\langle\Psi|\left(\frac{1}{m_e}B(x_e+\delta x,t)\cdot S_e - \frac{1}{m_p}B(x_p+\delta x,t)\cdot S_p\right)|\Psi\rangle = \delta x\cdot F$$

and then $F \sim \langle\Psi|\nabla\left(\frac{e}{m_e}B(x,t)\cdot S\right)|\Psi\rangle$. The force is nonzero only for spatially varying magnetic fields as observed in the experiment Bauer (2023).

3.5 Spin-orbit interaction

If we follow the standard discourse of Quantum physics, spin-orbit represents mainly the interaction between the electron's orbit and spin. Such heuristic approach would break one of our fundamental propositions: all the energies in the hydrogen atom are interaction energies between proton and electron, no self energy is involved. A relational view is actually forced to recognise that the orbit of the electron is a motion relative to the proton. Hence, only their relative velocity can matter. Such observation does not solve our problem, ... but let us follow its lead.

The coupling of spin and relative current has two parts, namely the interaction of proton spin $\frac{-e}{m_p}S_p$ with the relative current $\frac{ep_r}{m_r}$ as perceived by the proton and the corresponding interaction of electron spin $\frac{e}{m_e}S_e$ with relative current $\frac{-e(-p_r)}{m_r}$ as perceived by the electron. $m_r = \frac{m_e m_p}{m_e+m_p}$ stands for the reduced mass and $e\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right) = e\frac{p_r}{m_r}$. The magnetic field operator associated to the relative current as seen by the proton is hence the curl of the vector potential operator associated to that current (recall that the energy contribution is the integrated value of the operators over the wave-function), and correspondingly for the current perceived by the electron:

$$\begin{aligned}
\hat{B}_{jp}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x\times\left(\frac{1}{|x-x_p|}\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)\right)|\Psi\rangle \\
&= \frac{\mu_0 e}{4\pi}\nabla_x\frac{1}{|x-x_p|}\times\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle \qquad (13)\\
\hat{B}_{je}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x\frac{1}{|x-x_e|}\times\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle.
\end{aligned}$$

The magnetic field operators associated to the spin are:

$$\begin{aligned}\hat{B}_{se}(x) &= \frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_e|}\frac{S_e}{m_e}\right)\right)\\ \hat{B}_{sp}(x) &= -\frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_p|}\frac{S_p}{m_p}\right)\right)\end{aligned}$$

and the energy contribution is

$$\mathcal{E}_{SO} = \kappa\frac{1}{\mu_0}\langle\Psi|\int d^3x\ \hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) + \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x)|\Psi\rangle$$

where κ is a numerical constant that needs to be determined. We transform the spin field operators as

$$\begin{aligned}\hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot\left(\nabla\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right) - \Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\left(\nabla\cdot\left(\hat{B}_{jp}(x)\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right)\right) - \hat{B}_{jp}(x)\cdot\Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot\left(4\pi S_e\delta(x-x_e)\right)\\ \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x) &= -\frac{\mu_0 e}{4\pi m_p}\hat{B}_{je}(x)\cdot\left(4\pi S_p\delta(x-x_p)\right)\end{aligned}\tag{14}$$

using Gauss theorem and that $\nabla\cdot B = 0$. Performing the $x-$integral first, we obtain

$$\begin{aligned}\mathcal{E}_{SO} &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{-(x_p-x_e)}{|x_e-x_p|^3}\times\frac{p_r}{m_r}\right)|\Psi\rangle\\ &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{L_r}{m_r|x_e-x_p|^3}\right)|\Psi\rangle,\end{aligned}\tag{15}$$

where $L_r = x_r\times p_r$ is the relative angular momentum operator.

Let us complete the expression determining the value of κ before we turn back to the question that opens this subsection. The expression for the electromagnetic energy (Equation 9) was introduced by Maxwell considering the kind of interactions known at his time. It is symmetric in the indexes of the two interacting systems, hence, if $\mathcal{P}(12)$ is a permutation of indexes, the energy can be written as:

$$\mathcal{E}_{SO} = \frac{1}{2}\left\{\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right] + \mathcal{P}(12)\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right]\right\}.$$

We may call the first integral the way in which system one acts upon system two and the second integral is the reciprocal action. The energy can then be obtained by first establishing one action and next "symmetrising". The action of $\mathcal{P}(12)$ is merely changing the point of view in a somewhat arbitrary form, while the imposition of symmetry removes the arbitrariness since the acting group is a group of arbitrariness (see (Solari & Natiello, 2018)). The operation of taking two different view points corresponds to operating with $\mathcal{P}(ep)$ as it can be easily verified in all the previous expressions. Hence, unless $\kappa = \dfrac{1}{2}$ we would not be counting the interaction properly.

It remains to show that we are dealing with an interaction between two different entities and not an internal interaction. Consider our final expression, Eq. (15) and write it as:

$$\begin{aligned}
\mathcal{E}_{SO} &= -\frac{1}{2}\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{(x_e-x_p)\times(v_e-v_p)}{|x_e-x_p|^3}\right)|\Psi\rangle \\
&= \frac{1}{2}\int d^3x_e d^3x_p \left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{e^2}{4\pi\epsilon_0 C^2}\left(\frac{S_e\times v_e}{m_e}-\frac{S_p\times v_p}{m_p}\right)\Psi\right\} \\
&\quad -\frac{1}{2}\int d^3x_e d^3x_p \left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{\mu_0 e^2}{4\pi}\left(\frac{S_e\times v_p}{m_e}-\frac{S_p\times v_e}{m_p}\right)\Psi\right\}
\end{aligned}$$

($\Psi^\dagger$ stands for the row matrix that is the transpose and complex conjugate of Ψ). Consider further the case $\Psi = \psi_e\psi_p$, i.e., the case when $\Psi = \psi_e\psi_p$ and in addition the distance $|x_e - x_p|$ is macroscopic. In such a case we can consider that the variation of the relative position with respect to $\langle x_e - x_p\rangle$ is negligible in front of the macroscopic distance. The main contribution of the first term reads

$$\sim \frac{1}{2}\frac{1}{4\pi\epsilon_0 C^2}\frac{(x_p-x_e)}{|x_e-x_p|^3}\cdot\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)\int d^3x_p\left(-e|\psi_p|^2\right),$$

which represents the interaction between an electric dipole $\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$, located in the electron (now approximated as a point) and the proton (approximated as a point in front of the macroscopic distance). This view is compatible with the idea that magnetic dipoles in motion produce electric dipole fields. Up to a certain point, $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ represents a density of electric dipoles and $-e|\psi_p|^2$ a density of charge. The difference with such densities is that it is not possible to limit the interaction to "part of the electron" or "part of a proton", hence a integration over full space is always mandatory., thus the view of $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ as density of classical dipoles is only an analogy, it leaves aside the unity of the electron.

Actually, the dominant term in spin-orbit interactions corresponds to the electric dipole of the electron acting upon the proton. Thus, what was described as an interaction between the electron and its own orbit is now identified as the action of the electric dipole associated with the moving electron with the proton charge. An equivalent contribution appears with the action of $\mathcal{P}(ep)$. The last two terms are magnetic field interactions between proton and electron.

The expression 15 corresponds well with the final expression in textbook derivations. However, we have not resorted to analogy, nor have we patched this or any other expression with gyromagnetic numbers and have not further patched the expression with "relativistic corrections" (such as Thomas' correction) in the need to agree with experiments. The derivation of the spin-orbit contribution highlights the differences between the utilitarian/instrumentalist and the old style approach.

Related experiments One of the best known results of the atomic limit was the prediction of the hydrogen spectroscopic lines with Schrödinger's wave equation, providing a full theoretical expression of the Rydberg constant. The transition $n = 2$ to $n = 1$ (Kramida *et al.*, 2023) is reported as the spectroscopic line at $\lambda = 1215.6699\mathring{A}$. The line-width allows for a calculated resolution into $\lambda_1 = 1215.668237310\mathring{A}$ and $\lambda_2 = 1215.673644608\mathring{A}$ The state $n = 2$ is an octuplet, partially resolved by the spin-orbit Hamiltonian of Section 3.5 into a quadruplet and a doublet for $l = 1$ and two unresolved states with $l = 0$. The difference between the two energy levels is calculated as

$$\begin{aligned} \Delta E_{SO} &= \frac{3}{2}\hbar^2 \frac{1}{24a_0^3}\left(\frac{1}{2}\frac{\mu_0 e^2}{4\pi m_e^2}\right)\frac{m_e}{m_r} \\ &= \frac{3}{96}\frac{m_e}{m_r}\alpha^4 m_e C^2 \\ &= 7.259023470408092 \cdot 10^{-24} J \end{aligned}$$

while the calculated experimental energy difference amounts to $\Delta E_{exp} = hC(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = 7.26816814178113 \cdot 10^{-24} J$. The contribution consists of: $\frac{3}{2}\hbar^2$ coming from the level difference in $\langle L \cdot S\rangle$ between $j = 3/2$ and $j = 1/2$ (for $l = 1$ and $s = 1/2$), $\frac{1}{24a_0^3} = \langle R_{21}(r)|\frac{1}{r^3}|R_{21}(r)\rangle$ for the associated Hydrogen (n, l)-radial wave-function, $a_0 = \frac{\hbar}{\alpha m_e C}$ is the Bohr radius, m_e is the electron mass and $\alpha = \frac{\mu_0 e^2 C}{4\pi\hbar}$ is the fine-structure constant. We have disregarded the nuclear spin contribution.

3.6 Spin-spin interaction (part of hyperfine structure)

The interaction between spins within the atom uses an expression for the vector potential corresponding to that in Section 3.4 and reads

$$\begin{aligned}
\mathcal{E}_{SS} &= \frac{1}{\mu_0}\langle\Psi|\int d^3x\; B_e(x,t)\cdot B_p(x,t)|\Psi\rangle \\
B_e &= \nabla\times\left(\nabla\times\left(\frac{e}{m_e}\frac{\mu_0}{4\pi}\frac{S_e(x_e)}{|x-x_e|}\right)\right) \\
B_p &= \nabla\times\left(\nabla\times\left(\frac{-e}{m_p}\frac{\mu_0}{4\pi}\frac{S_p(x_p)}{|x-x_p|}\right)\right)
\end{aligned}$$

To proceed with the calculation we need to break the $e-p$ symmetry, namely $\mathcal{E}_{SS} = \frac{1}{2}\left(\mathcal{E}_{SS}^{e\to p} + \mathcal{E}_{SS}^{p\to e}\right)$. We pick the electron to proceed with the calculation, applying Gauss theorem again as in Section 3.4:

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{1}{\mu_0}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\; B_p(x)\cdot\left(-\Delta\frac{S_e(x_e)}{|x-x_e|}\right)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\;\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\delta(x-x_e)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\right]_{x=x_e}|\Psi\rangle
\end{aligned}$$

Through the vector calculus identities

$$\begin{aligned}
-\left(\nabla_x\times\left(\nabla_x\times\varphi(x)S_p\right)\right)\cdot S_e &= \left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\varphi(x) \\
\left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\frac{1}{|x-x_p|} &= \left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}\right]_{x=x_e}|\Psi\rangle \\
\mathcal{E}_{SS} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_e|}\right]_{x=x_p}|\Psi\rangle
\end{aligned}$$

since both contributions are identical. The spin-spin interaction is symmetric in front of the interchange of electron and proton and then carries a factor one in front.

3.7 Interaction with an external electric field

Suppose the atom is in interaction with an external electric field, E_Z. The electromagnetic interaction is then:

$$\mathcal{E}_{AZ} = \epsilon_0 \int dx \ \left((E_A + E_A^{am}) \cdot E_Z \right)$$

where $E_A = -\nabla V$ and $E_A^{am} = -\frac{\partial A}{\partial t}$ being (V, A) the electromagnetic potentials associated to the atom:

$$\begin{aligned} A &= \frac{\mu_0}{4\pi} \nabla \times \langle \Psi | \left(\frac{e}{m_e} \frac{M_e}{|x - x_e|} - \frac{e}{m_p} \frac{M_p}{|x - x_p|} \right) | \Psi \rangle \\ V &= \frac{e}{4\pi\epsilon_0} \langle \Psi | \left(\frac{1}{|x - x_e|} - \frac{1}{|x - x_p|} \right) | \Psi \rangle \end{aligned}$$

Concluding remarks

Quantum mechanics and instrumentalism

The production of QM (a name attributed to Max Born (Born, 1955)) under the leadership of the Copenhagen school was embraced in an instrumentalist philosophical approach. Historically, the development was guided by the correspondence with classical mechanics (although it deals almost completely with electromagnetic interactions). Two of the three central productive tools in the creation of QM are: imagination (as it is illustrated by Bohr's well-known planetary atom) and analogy of forms, i.e., the proposal of mathematical relations produced in a different context and imported into QM without satisfying the conditions necessary for their meaningful production. [15]

The third element is the adoption of instrumentalism. The Copenhagen school was able to make important advances in QM introducing a matrix formalism, but as Born confessed in his Nobel lecture: "What the real significance of this formalism might be was, however, by no means clear" (Born, 1955). The situation is not a surprise since instrumentalism is focused on (predictive) success rather than significance. Its approach is more akin to technology than to science. It relies on

[15] For example: the Hamiltonian for a charged (point) particle used in classical mechanics, as can be seen in (Pauli Jr, 1927; Dirac, 1928), or Thomas' relativistic correction to the spin-orbit energy (Thomas, 1926) where, in order to patch Zeeman's formula obtained by analogy and refuted by experiments, Thomas applies Lorentz transformations relating two systems which are *not* in uniform motion with constant velocity with respect to each other (hence, the electromagnetic fields described in each system are *not* related by Lorentz transformations).

trial and error, success or failure in reproducing and collating experimental results. The process of patching contradictions or inconsistencies with ad-hoc hypotheses makes refutation almost impossible. In so doing, it puts an undue pressure over the experimental side, since it requires experiments each day more complex, scarce and in need of interpretation. Compensating refutations by introducing ad-hoc patches leads more often than not to specialism, since different patches need not be mutually consistent. Seeking for the unity of knowledge moves precisely in the opposite direction. The explosion of Science into multiple sciences, each one constantly fragmenting in a fractal design, is not a consequence of the complexity of nature but rather the consequence of sacrificing reason in the altar of productivity.

The "correspondence principle", in the context of the utilitarian development, is what remains of the idea of cognitive surpass (generalisation) where the old becomes a particular instance of the new.

In terms of the correspondence with classical mechanics, the idea of point particles is central to the statistical QM description. The concept of particle is so strongly associated to the extreme idealisation as points that, quite often, their point character is not mentioned until late in the works. If particles in classical mechanics, when possible, are idealised as mathematical points, in QM particles *are* mathematical points, the localisation in space becomes an essential property of what in conventional (textbook) QM is considered a particle.

Point particles are not an invention of the Copenhagen school, they are present as idealisations in almost every textbook of physics. In his theory of the electron, Lorentz (Lorentz, 1904) considered the electron as a point surrounded by a cloud of electricity located in the nearby ether. Further back in time, Faraday had opposed these views in his theory of vibrating rays (Faraday, 1855, p. 447) indicating that neither the material point nor the ether were needed if we considered matter as an inference arising as a consequence of the observation of EM action. Faraday recommended extreme prudence in introducing such inventions since in the past these metaphysical elements had proven to be difficult to remove.

The scene changed with the irruption of the works by de Broglie (de Broglie, 1923; De Broglie, 1924) and Schrödinger (Schrödinger, 1926) who introduced the wave formulation of QM, although both of them relied in part on point particles. Later, Schrödinger strongly opposed this view.

As mentioned in the Introduction, the EM interactions in Quantum Mechanics were at variance with electrodynamics. The expected radiation of an electron orbiting the proton in an hydrogen-like atom was suppressed by ruling that no such radiation occurs for some special orbits. In short, conventional EM did not apply to quantum phenomena. The unity of physics was broken as it was broken a few years earlier when Special Relativity and its immaterial ether (Einstein, 1924) was accepted by the guild of physics as the cornerstone of electromagnetism, thus breaking with Newtonian mechanics.

The self-appraisal of instrumentalism perceives these cognitive decisions as realism or pragmatism and not as deficits. Instrumentalism advances without retracing its steps, without reconsidering the possibility of being in error, patching "theories" and narrowing their scope as needed to avoid refutation. Actually, when the word "physicist" was coined (Yeo, 1993) the increasing specialisation and lack of unity in the advances of physics was the main concern for philosophers of Nature such as Faraday and Whewell.

The uncomfortable situation of the wave-particle duality forced by Schrödinger and de Broglie is left behind by Max Born who introduces the (now) standard interpretation of wave-functions as probability waves. As Schrödinger points out, this appears as the only form of reconciliation with point particles at hand, and it is the form in which QM has been socially transmitted to several generations of physicists. Adopting this interpretation forces us to write the wave-function in a basis of functions, $|o\rangle$, where the operator $\hat{O}$ representing the measurable property is diagonal, $\hat{O}|o\rangle = o|o\rangle,$ with $o \in \mathbb{R}$, so the "matrix" representing the density of states, $\langle o|\Psi\rangle\langle\Psi|o'\rangle$, has only positive values in the diagonal and zero otherwise. Since two non commuting matrices (operators) cannot be simultaneously diagonalised, it is not possible to obtain probabilities for their joint measurement. The theory then ruled that this impossibility is an imposition of Nature, although in deriving it we have only used our interpretation. This difficulty in separating what comes from the subject (the observer) and what from the object (the observed) has been central to the criticism of natural science by Goethe, Faraday, Whewell and Peirce. It is the attitude in front of the (possible) error what makes the most striking difference between the instrumentalist approach and the philosophical approach. The instrumentalist scientist resorts to patches, restrictions and limitations. The final outcome is posed by S. Carroll as "Physicists do not understand Quantum Mechanics. Worse, they don't seem to want to understand it" (Carroll, 2019).

In a constructivistic perspective, theories are constructs and their interpretations must be totally in accordance with the abduction process employed in the construction stage, since abduction and interpretation form the two components of the phenomenological map (Solari & Natiello, 2022a). In the constructivistic sense, to "understand a theory" implies to be able to construct it rationally starting from observations and experimental information through a process rooted in previous knowledge (which might require adjustments under the light of the new information) and using synthetic judgements which are put to test by the consequences of the theory. QM has not been reconstructed under the idea of being a statistical theory. We have then the right to ask: where was the statistics introduced when all our considerations have been about the interactions of point particles? The resolution of this conflict is simple for the instrumentalist, they tell us: "the construction is not part of science" or even more strongly: science is not about the reality but just a matter of economy of thought.

The view of inferred matter as having identity by itself with forces that abut around was the standard view used in the construction of physics in the late XIX century ([529], Maxwell, 1873) and propagates there on, without being given reconsideration. For Einstein (Einstein, 1936) the concept of material point is fundamental for Newton's mechanics (not just an idealisation). In his work Einstein criticises Lorentz' proposal of electrons extending in the space (with soft borders in (Lorentz, 1892, §33) and spherical in (Lorentz, 1904, §8)) in the form:

> The weakness of this theory lies in the fact that it tried to determine the phenomena by a combination of partial differential equations (Maxwell's field equations for empty space) and total differential equations (equations of motion of points), which procedure was obviously unnatural. The unsatisfactory part of the theory showed up externally by the necessity of assuming finite dimensions for the particles in order to prevent the electromagnetic field existing at their surfaces from becoming infinitely great. The theory failed moreover to give any explanation concerning the tremendous forces which hold the electric charges on the individual particles. (Einstein, 1936)

The criticism is faulty in several accounts. As we have shown in Part I, a single mathematical entity, Hamilton's principle, is able to produce the equations of Electromagnetism and mechanics. Einstein qualifies Lorentz' electron theory as "unsatisfactory" for using finite-size particles, since for Einstein forces (electromagnetic fields) exist in the surface of particles. Simply, he ignores Faraday, and furthermore, there seems to be a requirement for particles being mathematical points. Einstein gives no argument to justify the character of "unnatural" for Lorentz' procedure, actually, "unnatural" appears to stand for "alien to my scientific habit". In contrast, mathematical statements require proofs as we offer in Part I. The argument concerning a need for a force to held together the electron is equally faulty, it assumes that there are parts that need to be held together, that the quantum entity is a composite in which one part can act on another part. Actually, acquaintance with Maxwell's arguments, make us to realise that the electrostatic energy computed in electromagnetism requires the possibility of dividing the charge in infinitesimal amounts. Something that can be considered a reasonable limit in macroscopic physics (Maxwell's situation) but is ridiculous when it comes to the electron.

Maxwell's concept of science and our own concept differ radically from Einstein's. For Maxwell and the present work, the construction is a relevant part of the theory, for Einstein:

> There is no inductive method which could lead to the fundamental concepts of physics. Failure to understand this fact constituted the

> basic philosophical error of so many investigators of the nineteenth century. It was probably the reason why the molecular theory, and Maxwell's theory were able to establish themselves only at a relatively late date. Logical thinking is necessarily deductive; it is based upon hypothetical concepts and axioms. How can we hope to choose the latter in such a manner as to justify us in expecting success as a consequence? (Einstein, 1936).

Einstein's utterly instrumentalist statements contrast with the legend of his early reading of Kant's "Critic of pure reason" which elaborates in the opposite direction through the concept of *synthetic judgement*. The final question posed was answered by Peirce's abduction (retroduction)(Peirce, 1994) and the whole issue was considered in Whewell's philosophy of science(Whewell, 2016), also here in contrasting terms.

In contrast, Einstein accepted without criticism that "Everywhere (including the interior of ponderable bodies) the seat of the field is the empty space. (Einstein, 1936)" (referring to the EM fields) despite being conscious that the idea perpetuated the essence of the ether (Einstein, 1924).

The reason we emphasise Einstein's positions is that he has been socially instituted as the "most perfect scientist", something "known" even by the illiterate. Actually, acceptance of his statement requires the previous acceptance of the utilitarian view of science.

Concluding the historical revision, the adoption of an instrumentalist approach weakens the demands put on the construction of theories and by so doing it speeds up the process of delivering tools for the development of technologies at the price of sacrificing understanding.

The present approach

In the present work we addressed the challenge of constructing QM as a cognitive surpass of EM, in such a form that EM and QM can be regarded as two instances of the same theory. The electromagnetic formulation had to be previously unified not as an aggregation of compatible formulae but rather through a unified deduction of the formulae from a common principle. Such step was previously taken in (Solari & Natiello, 2022b) where Maxwell equations and the continuity equation all derive from a variational principle of minimal action, actually, the same principle that is used to generate Lagrangian dynamics and also the Lorentz force. In the presentation of relational electromagnetism (Solari & Natiello, 2022b) Maxwell equations represent the propagation of EM effects in regions outside its production sources. The sources of EM action are described only as Lagrangian parameters that can be identified as charge densities and current densities and satisfy the continuity equation. However, the entity of such densities inside what we intuitively call matter is not prescribed by electromagnetism. Extending EM into the material region promises then to produce the laws of QM.

In this endeavour we can continue to use the name “particle”, but understanding a quantum particle as an unitary entity characterised by mass (which is the form of influencing through gravitational actions), charge, current (charge in motion) and magnetisation (spin), the latter ones being the attributes related to EM effects. Furthermore, we consider charge and mass-densities to be proportional and currents to bear the same proportion with momentum than the charge-to-mass relation. These densities cannot be thought of in analogy of the densities resulting from aggregates of matter since it is not possible to have a self-action or an interaction with only a portion of the quantum particle [16].

The wave-function in our approach will then be associated to the distribution of charge, currents and magnetisation. The need to use a wave-function and not the densities directly is taken from de Broglie’s argumentation to make contact with experiments that relate frequency to energy involving Planck’s constant. We must highlight that when analysing a free particle, the kinetic energy appears as the result of the above mentioned hypothesis and the conservation of charge, resulting in the usual $\frac{p^2}{2m}$, not as analogy or correspondence with classical mechanics but rather to be consistent with electromagnetism.

The wave-function is an attribute that keeps track of the state of the particle and is used primarily to determine the interaction (electromagnetic or gravitational) with other particles. In turn, the state of the particle is not just an attribute of itself but of its environment as well. Such view may result odd to physicists and chemists while it is the standard view in ecology. Thus, the problem of measurement is incorporated from the beginning since measuring means to enter in contact with a device.

Cognitive surpass means in this regard that the natural laws that govern the interaction of electron and proton as spatially separated entities must be the same as those governing their microscopic interaction, the differences corresponding to the context. When the electron and proton conform a new entity, the hydrogen atom, they act as a unity and interact with the environment as an atom. This observation suggests that the atomic wave-function may be factored as $|\Psi\rangle =$

[16]Readers familiar with Special Relativity (SR) may claim that there is a requirement of simultaneity at different points that might violate postulates of SR. However, SR displays self contradictions (Solari & Natiello, 2022a) prior to any claim. Its logical problems are usually made invisible by the instrumentalist approach that persuades us not to examine the correspondence of the construction and the interpretation. Experimental results support our view as well, for example (Aspect *et al.*, 1982).

$|\psi_r\rangle|\psi_{CM}\rangle$ while when we consider electron and proton as not entangled, it factors as $|\Psi\rangle = |\psi_p\rangle|\psi_e\rangle$, these two expressions constituting two different limits of the total wave-function. This simple argument allows us to construct, in Part II, the Hamiltonian of the hydrogen atom without resource to analogies, empirical corrections such as gyromagnetic factors or "relativistic corrections", i.e., patches inspired by SR needed to match experimental results.

We have shown that the classical laws of motion, Schrödinger's equation and the laws of Electromagnetism all derive from the same principle of minimal action. It is proper to say that they form a unit of knowledge. There is no much surprise in this fact, the principle of minimal action was introduced by Hamilton as form of collecting the system of ordinary equations of Lagrangian mechanics in one equation in partial derivatives. In turn Lagrange introduced the form today known as Lagrangian in order to incorporate in one expression those force laws given by their effects (usually called constraints of motion) and those given as interactions (the original realm of Newton's theory). Maxwell constructed Electromagnetism guided by the least action principle[17].

The particle is indissolubly associated to its localisation when it is though as: "the particle is that thing that lies there". In such regard, the vibration of Faraday's rays becomes dissociated from the particle. It is not an action performed by nature but rather by our way of knowing. In contrast, the "problem" of measurement in QM is dissolved in our construction, actually, all what is needed is a quantum mechanical description of the measurement device and its (usually) electromagnetic form of interacting with the system of interest. The universe cannot be suppressed in an idealisation, its presence will be always in action, to the very least in the form of time. If in addition, we allow ourselves to separate the inferred (matter) from what is sensed (action) as it is done in the pair QM and EM, we simply create a fantasy, an illusion, where poor Schrödinger's cat is both dead and alive until we measure. Actually, under the present view, the quantum mechanical entity will have associated time-dependent electromagnetic fields in as much it has not reached a stationary state. A state initially described as a superposition of stationary states will evolve into an apparent stationary state if we agree to ignore the relation with the electromagnetic energy whose space locus has not longer an overlap with the space locus of the material entity. Notice that in order to make this statement we have to include the localisation between the attributes of the entity creating a second entity, the photon, associated to the EM radiated energy.

[17] Actually, this is the mathematics about which Hertz said: "Many a man has thrown himself with zeal into the study of Maxwell's work, and, even when he has not stumbled upon unwonted mathematical difficulties, has nevertheless been compelled to abandon the hope of forming for himself an altogether consistent conception of Maxwell's ideas. I have fared no better myself." Hertz proceeds then to "understand" Maxwell's theory, actually to accept Maxwell's formulae resorting to the ether. See (p.20, Hertz, 1893)

In the present work we have advanced a unified action integral inspired in de Broglie and Schrödinger intuitions as well as Maxwell's electrodynamics (Lemma 1.2 to Theorem 1.1). Through Hamilton's principle we obtained the equations of motion ruling microscopic systems in accordance with classical mechanics and electromagnetism. Further, we reformulated Hamilton's principle in terms of the (conserved) energy, obtaining a slightly more general equation of Schrödinger type (Lemma 1.4to Lemma 1.5). Finally we showed the connection between the possible form of the variations and Hermitian operators (Lemma 1.6). Hamilton's variational principle appears as central to the development, since it separates circumstantial issues from "laws of nature". The law is what remains invariant regardless of initial or final conditions (or condition at any time) [18]. The occurrence of Hamilton's principle is the result of successive cognitive surpasses: from Newton equations to Lagrange equations and next to Hamilton's principle in Mechanics. Hamilton's structure was used by Maxwell to organise Electromagnetism (Maxwell, 1873) and by Lorentz to reorganise Maxwell's electromagnetism in terms of the electromagnetic Lagrangian (Lorentz, 1892) introducing his EM-force from the application of this principle. It was later elevated to the level of fundamental principle by Helmholtz because it systematically produces the conservation of energy, and finally Poincaré (Poincaré, 1913) considered it a fundamental principle as well. Hence, Hamilton's principle was born as cognitive surpass and evolved as a guide for the construction of physics, a history that situates it as the primary choice for an attempt to unify theories.

In Part II, the quantum mechanics that emerges from electromagnetism has been tested in the construction of the internal energies for the Hydrogen atom producing outstanding results, superior to those of standard quantum mechanics. Equal in precision but definitely superior in consistence and consilience. Another insight gained is that the moving spin reveals its role as the carrier of *electric* polarisation in the spin-orbit interaction.

The path followed in this construction is blocked for the standard presentation of QM as extension from mechanics by the early assumption of point particles. The usual form used to incorporate the Lorentz force in the case of point particles was the constructive basis for several authors such as Pauli (Pauli Jr, 1927) and Dirac (Dirac, 1928). This led to the identification of the momentum p with the operator $-i\hbar\nabla$ (in all situations), a compensating error sustained in Bohr's principle of correspondence. In turn, this chain of decisions made impossible to associate the operator with the current and prevented the production of EM interactions in QM except by resource to analogy.

[18]The original concept of variation by Hamilton reads "...any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data..."(Hamilton, 1834)

In contrast, in this work the unity of concepts is sought in the fundamentals. The Lorentz force was integrated to Maxwell equations in (Solari & Natiello, 2022b) (see Part I as well), without restrictions to point particles, using Lorentz' Lagrangian formulation. The integration with mechanics was then performed by incorporating the kinetic part of the action integral in the form suggested by the QM of the isolated particle, thus reuniting the *material* and *action* sides of the entity. After this synthetic effort was completed, Hamilton's principle showed the extraordinary power of consilience which was its initial justification (Hamilton, 1834). It is worth mentioning that so far there has been no need to request the quantification of the EM field. All results are consistent with Planck's view (Brush, 2002) who recognised that the exchange of energy in absorption and emission processes was quantified yet, this fact does not imply that the field itself needs to be quantified a priori, being the latter an idea usually attributed to Einstein (Einstein, 1905).

With the present work we have achieved the unification of the foundations of classical mechanics, electromagnetism and quantum physics in a sequence of works. This view is incompatible with the statistical interpretation of QM where measuring is a sort of magic act in which the laws of QM do not apply. As Carroll puts it "Quantum Mechanics, as it is enshrined in textbooks, seems to require separate rules for how quantum objects behave when we're not looking at them, and how they behave when they are being observed" (Carroll, 2019). Furthermore, a condition for a theory to be considered rational consists in the possibility of "filling the gap", which means that the details left without specification can be provided in accordance to the theory. The current QM is unable to specify the details of the measurement, they remain in an abstract sphere where apparently, for example, the position of the electron relative to the proton in the hydrogen atom can be determined with as much precision as desired, yet not a word is said on how such measurement would be performed. It fails then a requisite of rationality as discussed in general form in (Solari & Natiello, 2023).

We have shown that quantum mechanics can be conceived in a form that does not oppose classical mechanics but coexists with it and complements electromagnetism. The consilience of the present formulation not only contrasts with the statistical interpretation of quantum mechanics, it contrasts with what is implied in the notion of interpretation, i.e., that equations are first herded together as needed in terms of their usefulness without regard to their meaning, and a posteriori meaning is attributed in the "interpretation" to promote acceptance and facilitate the use of the equations. Such procedure is not needed, it is not forced by the limitations of our mind and much less by Nature. Any claim regarding the Truth in such collections is ludicrous. However, students in sciences are indoctrinated constantly in terms of the Truth value of such "theories". It is the social power of the guild of physicists what prevents progress towards understanding.

There are two obstacles that have to be overcome to grasp our presentation, the first is the entrenched belief in the representation of fundamental quantum particles as points. Schrödinger was, in our limited knowledge, the only one to point out this problem, yet he did not notice that this view was threaded with Special Relativity (SR). The second obstacle is the belief that Special Relativity (SR) is the cornerstone of electromagnetism.

Einstein's instrumentalism is declared in:

> Physics constitutes a logical system of thought which is in a state of evolution, and whose basis cannot be obtained through distillation by any inductive method from the experiences lived through, but which can only be attained by free invention. The justification (truth content) of the system rests in the proof of usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively. (Einstein, 1936, p. 381)

SR closed the debate around the ether at the price of elitism (Essen, 1978; Lerner, 2010), and dropping the advances made by Lorentz in terms of a Lagrangian formulation of Maxwell's theory. Only Lorentz' law of forces was kept, a law that was herded into electromagnetism probably to hide that it comes from Galilean transformations, hence, its deduction by Lorentz casts doubts on Einstein's alleged role of Lorentz transformations (Einstein, 1907) relating to inertial systems. We cannot avoid wondering what "inertial system" means for Einstein, who believes that Newton's mechanics rests upon absolute space (Einstein, 1924). But the problem is not Einstein, all of us are limited and to show him limited only proves what is not disputed: Einstein was human. The force driving the social construction of idols is the rent obtained by the clergy. In situations like this, the philosophical debate is obscured and impeded by the social struggle. Any one who has objected the idea of SR as the cornerstone of EM has been threatened by the guild of physics. We came to know only the testimony of those strong enough to (socially) survive the attacks. Let us name them: Essen (Essen, 1971, 1978) who wrote "The theory is so rigidly held that young scientists dare not openly express their doubts" as abstract of his 1978 intervention. Essen provided evidence that SR does not make sense in experimental terms. Dingle, warned us about the epistemological problems of SR (Dingle, 1960b), the relevance of the Doppler effect by itself (Dingle, 1960a) and gave testimony of the attacks he endured (Dingle, 1972). Phipps insisted from both theoretical and experimental points of view in the relational view of physics (Phipps Jr, 2006; Phipps, 2014).

If we go back in time, the men in charge of producing theories about nature were philosophers-mathematicians. For Galileo, mathematics provided the language to understand the universe by the philosopher, who was understood as a master at reasoning. When the production of scientific results was "industrialised" a new social character was born: the scientist (in particular, the physicist). The professional scientist, produced at the universities, was declared free from the control of philosophy (i.e., reason), each science had its method and forms of validations, it was proclaimed. The guild of the profession was then left in charge of the quality control. The immediate consequence of this social change in physics was a new form of explanation that left behind natural philosophy and empowered the "second physicist" (Jungnickel & McCormmach, 2017) with the social responsibility of producing theories. The technical success of Electromagnetism contrasted with the failure of the new epistemology that had made of the ether the weapon for attacking the remains of the old epistemology, for example, e.g. Göttingen's electromagnetism (Solari *et al.*, 2024). When it became clear that the ether was a no-end route, the task of saving the theoretical physicist became urgent. The merit of SR is this social rescue and is also the ultimate reason for not being challenged.

The present work, constructed much under the old epistemological view shows that physics depends on epistemology, and through epistemology it depends on social interests that have nothing to do with Nature. The immediate utilitarian consequences appear as unchanged but the foundations for advancing understanding of nature appear as dubious if the instrumentalist perspective is adopted.

We believe we have shown by example that when we adopt a different *epistemic praxis* such as we have done in this work and in our reconstruction of EM (Solari & Natiello, 2022b), we reach a different theoretical understanding, actually a deeper understanding when the epistemic praxis is more demanding, as it is the relation between our dualist phenomenology (Solari & Natiello, 2022a) and instrumentalism. Thus, physics depends not only on the observable natural phenomena but it depends as well on our philosophical disposition. Yet, if the concept of theory is weakened so as to be reduced to the equations (cf. (p. 21 Hertz, 1893)), the distinctions between theories mostly fade out, the phenomenological link disappears and interpretation crops up to guide the use of the formulae after sacrificing the unity of thought, critical and phenomenological actions.

In summary, there is no compelling reason to believe scientific results that have been achieved by weakening the old scientific demand for Truth into the current demand of usefulness. Newtonian physics is not incompatible with EM, as it is usually preached; and both of them can live in harmony with quantum mechanics. There is no reason for abandoning (relational) Cartesian geometry or to consider time as an "odd" spatial coordinate. There is no license to use formulae outside their range of validity as it is too often done.

Too often in the course of this investigation we have encountered misrepresentations of the thoughts and writings of the natural philosophers that developed physics until the middle of the XIX century, when the *scientist* was born.

Abridged representations of their ideas and writings creep in as soon as the creators fade out. Absolute space is attributed to Newton's mechanics and "true motion" disappears (Solari & Natiello, 2021), Faraday becomes a supporter of the ether rather than the careful philosopher he was, absolutely inclined to entertain doubts as much as possible avoiding to precipitate into simplifying inventions. Faraday writes in a letter to R Taylor:

> But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgment [...]

Maxwell's plead to consider the hypothesis of the ether as worth of research (p. 493 Maxwell, 1873) was transformed as well into a belief when the story was told. Yielding to the needs of the new social position of science (and its emerging epistemic praxis), Maxwell's theory was deprived of its mathematical construction lowering the acceptance standard from correct into plausible.

In our own experience, we have spent more time undoing promissory hunches than constructing correct reasoning. It is worth noticing that such hunches trigger exploratory actions and as such are important. It is believing, instead of doubting them, what makes them prejudice. What appears to us as correct is made of the debris of our errors.

The great minds that constructed Special Relativity and Quantum Theory were passengers of their epoch, a time when "conquering nature" (the old imperialist dream of Francis Bacon) had finally taken prevalence over the "understanding nature," sought by his contemporary Galileo Galilei. Our own work cannot escape the rule, even though we are not "great minds". We live in a time where Nature reminds us that we must understand ki [19] and, consequently, love ki. Science must then rescue the teaching of the old masters, the natural philosophers, and their practice of critical thinking.

[19]The word ki has been proposed as a new and respectful pronoun for Nature and all what is part of Nature. The word comes from Anishinaabe's language but relates as well to the Japanese and Chinese Ki and to the old English kin (like in kinship).

A Proof of variational results

A.1 Proof of Lemma 1.1:

Proof. We use standard vector calculus operations on the integrand, namely

$$\begin{aligned}
A\cdot j-\rho V &= A\cdot\left(\frac{1}{\mu_0}\nabla\times B-\epsilon_0\frac{\partial E}{\partial t}\right)-\epsilon_0 V\nabla\cdot E\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)+\epsilon_0 E\cdot\nabla V\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)+\epsilon_0 E\cdot\left(-E-\frac{\partial A}{\partial t}\right)\\
&= \frac{1}{\mu_0}|B|^2-\epsilon_0|E|^2-\nabla\cdot\left(\frac{1}{\mu_0}A\times B+\epsilon_0 VE\right)-\epsilon_0\frac{\partial}{\partial t}\left(A\cdot E\right)
\end{aligned}$$

□

A.2 Proof of Lemma 1.2:

Proof. Since the wave-function admits the independent variations $\psi\mapsto\psi+\delta\psi$ and $\psi\mapsto\psi+i\delta\psi$ we have that

$$\begin{aligned}
\psi\frac{\delta\mathcal{A}}{\delta\psi}\delta\psi+\frac{\delta\mathcal{A}}{\delta\psi^*}\delta\psi^* &= 0\\
\frac{\delta\mathcal{A}}{\delta\psi}\left(i\delta\psi\right)+\frac{\delta\mathcal{A}}{\delta\psi^*}\left(-i\delta\psi^*\right) &= 0
\end{aligned}$$

where $\delta\psi^*=(\delta\psi)^*$, and hence, it is sufficient to consider the variations of ψ and ψ^* as independent variations that must result in a null change of $\mathcal{A}$. By partial integration we obtain $-i\hbar\dot{\psi}^*\psi=-i\hbar\left(\psi^*\psi\right)_{,t}+i\hbar\psi^*\dot{\psi}$ and the stationary action corresponds to

$$\delta\mathcal{A}_{QM}=\int_{t_0}^{t_1}ds\left\langle\delta\psi^*\left[i\hbar\dot{\psi}+\frac{\hbar^2}{2m}\Delta\psi\right]\right\rangle=0.$$

□

A.3 Proof of Lemma 1.3

Proof. We act the variations piecewise, recalling that $\delta\psi = \epsilon\mu$ gives $\delta\dot{\psi} = (\epsilon\mu)_{,t} = \dot{\epsilon}\mu + \epsilon\dot{\mu}$,

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^*\dot{\psi} - \dot{\psi}^*\psi \right\rangle \\
\delta\mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle + \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^* (\epsilon\mu)_{,t} - (\epsilon\mu)_{,t}{}^*\psi \right\rangle \\
&= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle - \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \dot{\psi}^*\mu - \mu^*\dot{\psi} \right\rangle \\
&= \int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^* \left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\mu \right\rangle
\end{aligned}
$$

after a partial integration in time of the $\dot{\epsilon}$ contribution (in the second line) that does not modify the variation. Partial integrations in space using Gauss' theorem give

$$
\begin{aligned}
\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} ds \left\langle \psi^*\Delta\psi \right\rangle \\
\delta\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\Delta\psi + \psi^*\Delta\mu \right\rangle \\
&= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\Delta\psi + \mu\Delta\psi^* \right\rangle .
\end{aligned}
$$

Since the variation complies with Maxwell equations, after some partial integrations, we have:

$$
\begin{aligned}
\delta\frac{1}{2}\int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}B^2 - \epsilon_0 E^2 \right\rangle &= \int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}\delta B_m \cdot B - \epsilon_0 \delta E_m \cdot E \right\rangle \\
&= \int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0} A \cdot (\nabla \times \delta B_m) - V\delta(\epsilon_0 \nabla \cdot E_m) - \epsilon_0 A \cdot \delta E_{m,t} \right\rangle \\
&= \int_{t_0}^{t_1} ds \left\langle A \cdot \delta j_m - V \cdot \delta\rho_m \right\rangle
\end{aligned}
$$

where $\delta j_m = \epsilon\boldsymbol{\mathfrak{J}}_1 + \dot{\epsilon}\boldsymbol{\mathfrak{J}}_2$ and $\delta\rho_m = \epsilon\boldsymbol{\mathfrak{X}}$. Consequently,

$$
\begin{aligned}
\delta\frac{1}{2}\int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}B^2 - \epsilon_0 E^2 \right\rangle &= \int_{t_0}^{t_1} ds \left\langle A \cdot \epsilon\boldsymbol{\mathfrak{J}}_1 + \dot{\epsilon} A \cdot \boldsymbol{\mathfrak{J}}_2 - V\epsilon\boldsymbol{\mathfrak{X}} \right\rangle \\
&= \int_{t_0}^{t_1} \epsilon\, ds \left\langle A \cdot \boldsymbol{\mathfrak{J}}_1 - (A \cdot \boldsymbol{\mathfrak{J}}_2)_{,t} - V\boldsymbol{\mathfrak{X}} \right\rangle
\end{aligned}
$$

after another partial integration in time. □

A.4 Proof of Corollary 1.1

Proof. $\mathfrak{I}_2 = -\epsilon_0\boldsymbol{\mathfrak{E}}$ satisfies the assumption $\boldsymbol{\mathfrak{X}} + \nabla\cdot\mathfrak{I}_2 = 0$ of Lemma 1.3, as well as $\mathfrak{I}_2 + \nabla\times W$ does it, for any suitable W. However, setting $\mathfrak{I}_1 := \mathfrak{I}_1 + \nabla\times W_{,t}$, the vector W does not appear in any of the equations of the Lemma. Note that the precise form of $\mathfrak{I}_1$ as a function of μ has not been prescribed so far. □

A.5 Proof of Theorem 1.1

Proof. Note that the column $\boldsymbol{\mathfrak{X}}$ in the table corresponds to propagating $\delta\psi$ on the quantum charge density $\rho = \psi^*\psi$.

For the "time/energy" entry, the variation of the action corresponds to setting $\mu \equiv \dot{\psi}$ in Lemma 1.3. Since ϵ is any possible variation function, the only way to ensure a stationary action is to let the space integral be identically zero, namely:

$$\left\langle \mu^* \left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* \mu \right\rangle + \frac{\hbar^2}{2m}\left\langle \mu^*\Delta\psi + \mu\Delta\psi^* \right\rangle + \left\langle A\cdot\mathfrak{I}_1 - \left(A\cdot\mathfrak{I}_2\right)_{,t} - qV\boldsymbol{\mathfrak{X}} \right\rangle = 0.$$

It follows that $\dot{\psi}^*\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\dot{\psi} = 0$ and $\dot{\psi}^*\Delta\psi + \dot{\psi}\Delta\psi^* = (\psi^*\Delta\psi)_{,t}$, by repeated application of Gauss' theorem. Letting $\mathfrak{I}_1 = -2\epsilon_0\boldsymbol{\mathfrak{E}}_{,t} - \mathfrak{j}$, we propose $\mathfrak{j} = j_{,t}$ so that $\boldsymbol{\mathfrak{X}}_{,t} + \nabla\cdot\mathfrak{j} = 0$. Since $\boldsymbol{\mathfrak{X}} = (\psi^*\psi)_{,t}$, we have that $\mathfrak{I}_2 = -\epsilon_0\boldsymbol{\mathfrak{E}} = j$ and $\mathfrak{I}_1 = j_{,t}$. The conditions of Lemma 1.3 are satisfied and

$$\begin{aligned}
A\cdot\delta j - V\delta\rho &= \epsilon\left[A\cdot\mathfrak{I}_1 - \left(A\cdot\mathfrak{I}_2\right)_{,t} - V\boldsymbol{\mathfrak{X}}\right] \\
&= \epsilon\left[A\cdot\left(\mathfrak{I}_1 - \mathfrak{I}_{2,t}\right) - A_{,t}\cdot\epsilon_0\boldsymbol{\mathfrak{E}} - V\boldsymbol{\mathfrak{X}}\right] \\
&= -\epsilon\left[-\left(A_{,t} + \nabla V\right)\cdot\epsilon_0\boldsymbol{\mathfrak{E}}\right] \\
&= -\epsilon\left[E\cdot j\right]
\end{aligned}$$

The latter expression is the time-derivative of the electromagnetic energy:

$$\begin{aligned}
\frac{1}{2}(\epsilon_0E^2 + \frac{1}{\mu_0}B^2)_{,t} &= E\cdot\epsilon_0E_{,t} + \frac{1}{\mu_0}B\cdot B_{,t} \\
&= E\cdot\left(\frac{1}{\mu_0}\nabla\times B - j\right) - \frac{1}{\mu_0}B\cdot(\nabla\times E) \\
&= -E\cdot j + \frac{1}{\mu_0}\nabla\cdot(B\times E)
\end{aligned}$$

By Gauss' theorem, the divergence vanishes when integrated and we finally obtain the equation

$$\left\langle \partial_t\left[-\frac{\hbar^2}{2m}\left(\psi^*\Delta\psi\right) + \frac{1}{2}\left(\frac{1}{\mu_0}B^2 + \epsilon_0E^2\right)\right]\right\rangle = 0$$

expressing the conservation of energy.

For the "position/velocity" entry in the table, in order to obtain a stationary action integral, a similar argument about the independence and arbitrariness of the real-valued time function ϵ forces the following vector valued integral to be zero:

$$\begin{aligned} \left\langle -i\psi^* (k\cdot x)\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* i(k\cdot x)\psi\right\rangle &+ \frac{\hbar^2}{2m}\left\langle -i\psi^*(k\cdot x)\Delta\psi + i(k\cdot x)\psi\Delta\psi^*\right\rangle \\ &+ \left\langle A\cdot\mathfrak{J}_1 + (A\cdot\mathfrak{L}\epsilon_0)_{,t} - V\mathfrak{X}\right\rangle \end{aligned}$$

where $\mathfrak{X} = (ix\psi)^*\psi + \psi^*(ix\psi) = 0$. Hence, we have $-\mathfrak{J}_2 = \epsilon_0\mathfrak{L} = 0$. The continuity equation is satisfied by taking $\mathfrak{J}_1 = 0$. The electromagnetic integral is identically zero and we obtain

$$\left\langle k\cdot\left(-i\psi^* x\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* ix\psi\right)\right\rangle + \frac{\hbar^2}{2m}\left\langle k\cdot(-i\psi^* x\Delta\psi + ix\psi\Delta\psi^*)\right\rangle = 0.$$

The first term gives $k\cdot(\hbar\partial_t\langle\psi^* x\psi\rangle)$ while the second reduces to $\frac{i\hbar^2}{2m}k\cdot\langle\psi^*(\nabla\psi) - (\nabla\psi)^*\psi\rangle$ after repeated application of Gauss' theorem. Finally,

$$0 = \partial_t\langle\psi^* x\psi\rangle + \frac{i\hbar}{2m}\langle\psi^*(\nabla\psi) - (\nabla\psi)^*\psi\rangle = \partial_t\langle\psi^* x\psi\rangle - \frac{1}{q}\langle j\rangle.$$

For the "displacement/momentum" case, we compute $\mathfrak{X} = r\cdot\nabla\rho$, set $\mathfrak{j} = (r\cdot\nabla)j$ and $\mathfrak{J}_2 = -\epsilon_0\mathfrak{L} = -r\rho$. Then $\mathfrak{J}_1 - \mathfrak{J}_{2,t} = \epsilon_0\mathfrak{L}_{,t} + \mathfrak{j}$, which computes to $\mathfrak{J}_1 - \mathfrak{J}_{2,t} = r\rho_{,t} + (r\cdot\nabla)j = -r(\nabla\cdot j) + (r\cdot\nabla)j$ and further $-r(\nabla\cdot j) + (r\cdot\nabla)j = -\nabla\times(r\times j)$. Hence, up to a global divergence vanishing by Gauss' theorem,

$$\begin{aligned} A\cdot\left(\mathfrak{J}_1 - \mathfrak{J}_{2,t}\right) - A_t\cdot\mathfrak{J}_2 - V\mathfrak{X} &= A\cdot(-\nabla\times(r\times j)) + (A_t + \nabla V)\cdot(r\rho) \\ &= -A\cdot\nabla\times(r\times j) - E\cdot(r\rho) \\ &= -(r\times j)\cdot(\nabla\times A) - r\cdot\rho E \\ &= -r\cdot(\rho E + j\times B). \end{aligned}$$

Finally, the arbitrariness of ϵ demands

$$r\cdot\left\langle(\nabla\psi)^*\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\nabla\psi\right\rangle + \frac{\hbar^2}{2m}r\cdot\langle(\nabla\psi^*)\Delta\psi + (\nabla\psi)\Delta\psi^*\rangle - r\cdot\langle\rho E + j\times B\rangle = 0.$$

By repeated use of Gauss' theorem we obtain $\left\langle(r\cdot\nabla\psi)^*\Delta\psi + (r\cdot\nabla\psi)\Delta\psi^*\right\rangle = 0$. Since the directions r are arbitrary, we have

$$i\hbar\left\langle(\nabla\psi)^*\dot{\psi} - \dot{\psi}^*\nabla\psi\right\rangle = \frac{m}{q}\partial_t\langle j\rangle = \langle(j\times B) + \rho E\rangle.$$

For the "rotation/angular momentum" entry $\mathfrak{X} = \Omega\cdot(x\times\nabla\rho)$, $\mathfrak{J}_1 = \mathfrak{j} = (\Omega\cdot(x\times\nabla))j - \Omega\times j$ and we set $\mathfrak{J}_2 = -(\Omega\times x\rho)$ since $\nabla\cdot\mathfrak{J}_2 = -\Omega\cdot(x\times\nabla\rho) = -\nabla\cdot(\Omega\times x\rho)$.

The proposed variation $\delta\psi$ enters in the space integrals, which specifically become

$$\Omega\cdot\left\langle i\hbar\left(\left(x\times\nabla\psi^{*}\right)\dot{\psi}-\left(x\times\nabla\psi\right)\dot{\psi}^{*}\right)+\frac{\hbar^{2}}{2m}\left(\left(x\times\nabla\psi^{*}\right)\Delta\psi+\left(x\times\nabla\psi\right)\Delta\psi^{*}\right)\right\rangle .$$

The second term vanishes as in the previous case, while the first term reads

$$\Omega\cdot\left\langle x\times\left(i\hbar\left\langle\left(\nabla\psi\right)^{*}\dot{\psi}-\dot{\psi}^{*}\nabla\psi\right\rangle\right)\right\rangle \;=\; \frac{m}{q}\Omega\cdot\partial_{t}\left\langle x\times j\right\rangle$$

The electromagnetic part reads

$$\begin{aligned} A\cdot\left(\mathfrak{J}_{1}-\mathfrak{J}_{2,t}\right)-A_{t}\cdot\mathfrak{J}_{2}-V\mathfrak{X} &= A\cdot\left[\Omega\cdot\left(x\times\nabla\right)\right]j-\Omega\cdot\left(j\times A\right)\\ &+ \Omega\cdot\left(\left(x\rho_{,t}\right)\times A\right)-\Omega\cdot\left[\left(x\rho\right)\times E\right]\\ &= -\Omega\cdot\left(x\times\left(j\times B\right)\right)-\Omega\cdot\left[x\times\rho E\right]\\ &= -\Omega\cdot\left(x\times\left(\rho E+j\times B\right)\right). \end{aligned}$$

Being Ω a vector of arbitrary direction, equating the variation to zero we get

$$\frac{m}{q}\partial_{t}\left\langle x\times j\right\rangle-\left\langle x\times\left(\rho E+j\times B\right)\right\rangle=0$$

which is the evolution of the angular momentum under the action of the Lorentz force. □

A.6 Proof of Lemma 1.4

Proof. We only need to check that $\left(\delta\rho\right)_{,t}+\nabla\cdot\delta j=0$ which follows from computing

$$\mathfrak{J}_{1}-\mathfrak{J}_{2,t}=-\epsilon_{0}\mathfrak{E}_{,t}-\mathfrak{j}=-\frac{1}{\mu_{0}}\nabla\times\mathfrak{B}$$

according to Maxwell's equations. Substituting in the result of the preparatory lemma and using the arbitrary $\epsilon(t)$ argument, the results follow. □

A.7 Proof of Lemma 1.5

Proof. We begin from Eq. 6 of Lemma 1.4. We write $\mathfrak{X}=\left(\mu^{*}\psi+\mu\psi^{*}\right)$ and set $\mathfrak{j}=-\frac{i\hbar q}{2m}\left(\mu^{*}\nabla\psi-\psi\nabla\mu^{*}+\psi^{*}\nabla\mu-\nabla\psi^{*}\mu\right)$. According to Lemma 1.3 the variation of the electromagnetic contribution to the action reads:

$$\int_{t_{0}}^{t_{1}}\epsilon\,ds\left\langle A\cdot\mathfrak{J}_{1}-\left(A\cdot\mathfrak{J}_{2}\right)_{,t}-V\mathfrak{X}\right\rangle$$

which according to Lemma 1.4 is:

$$\int_{t_0}^{t_1} \epsilon\, ds \left\langle -\frac{1}{\mu_0} A \cdot (\nabla \times \mathfrak{B}) - \epsilon_0 E \cdot \mathfrak{E} \right\rangle$$

The variation of the action becomes:

$$\int (\epsilon(s)\, ds) \left\langle \mu^* \left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* \mu \right\rangle + \frac{\hbar^2}{2m} \left\langle \mu^* \Delta\psi + \mu\Delta\psi^* \right\rangle - \left\langle -\frac{1}{\mu_0} A \cdot (\nabla \times \mathfrak{B}) - \epsilon_0 E \cdot \mathfrak{E} \right\rangle .$$

Letting $\epsilon(t)$ to be constant, integrating in $-T \leq s \leq T$, and letting $T \to \infty$, projects μ into its ω Fourier component, μ_ω, while the electromagnetic terms integrate to $A \cdot \mathfrak{j}_\omega - V\mathfrak{X}_\omega$ projecting in the same form the variation of density and current, with $\mathfrak{X}_\omega = (\mu^*_\omega \psi_\omega + \mu_\omega \psi^*_\omega)$ and $\mathfrak{j}_\omega = -\frac{i\hbar q}{2m} (\mu^*_\omega \nabla\psi_\omega - \psi_\omega \nabla\mu^*_\omega + \psi^*_\omega \nabla\mu_\omega - \nabla\psi^*_\omega \mu_\omega)$. Since μ is arbitrary, its Fourier components are arbitrary and the null variation of the action results in the requisite:

$$\langle \mu^*_\omega (\hbar\omega\psi_\omega)\rangle + \frac{\hbar^2}{2m} \langle \mu^*_\omega \Delta\psi_\omega \rangle + \left\langle A \cdot \left(\frac{i\hbar q}{2m}\right) (\mu^*_\omega \nabla\psi_\omega - \psi_\omega \nabla\mu^*_\omega) - V\nu^*_\omega \psi_\omega \right\rangle = 0.$$

Partial integration gives $A \cdot (\nu^* \nabla\psi_\omega - \psi_\omega \nabla\nu^*) = A \cdot (2\nu^* \nabla\psi_\omega - \nabla(\psi_\omega \nu^*))$ and by Gauss' theorem $\langle -A \cdot \nabla(\nu^* \psi_w)\rangle = \langle \nu^* \psi_w \nabla \cdot A\rangle$. For an homogeneous magnetic field we have $A = \frac{1}{2} x \times B$ (and hence $\nabla \cdot A = 0$) and further $A \cdot \left(\frac{i\hbar q}{2m}\right) 2\nu^* \nabla\psi_\omega = \nu^* \left(\frac{i\hbar q}{2m}\right) (x \times B) \cdot \nabla\psi_w$, which finally leads to

$$\left\langle \nu^* \left(\hbar\omega\psi_\omega + \frac{\hbar^2}{2m} \Delta\psi_\omega - \frac{i\hbar q}{2m} B \cdot (x \times \nabla\psi_\omega) - qV\psi_\omega \right) \right\rangle = 0.$$

It follows that $\hbar\omega\psi \equiv \mathcal{E}\psi = -\frac{\hbar^2}{2m}\Delta\psi - \frac{q}{2m} B \cdot (x \times (-i\hbar\nabla)\psi) + qV\psi.$ □

A.8 Proof of Lemma 1.6

Proof. Letting $\delta\psi = \epsilon\hat{O}\psi$, the operator $\hat{O}$ must be linear as a consequence of the infinitesimal character of variations. Acting with Θ on the continuity equation we get

$$\begin{aligned} \Theta\left(\rho_{,t} + \nabla \cdot j\right) &= 0 \\ (\Theta\rho)_{,t} + \Theta\left(\nabla \cdot j\right) &= 0 \end{aligned}$$

provided $\hat{O}_{,t} = 0$. Since

$$\Theta(\nabla \cdot j) = \nabla \cdot (\Theta j) - \frac{i\hbar q}{2m} \left(\psi^* \left(\left[\nabla, \hat{O}\right] \cdot \nabla \right) \psi - \psi \left(\left[\nabla, \hat{O}\right] \cdot \nabla\psi \right)^* + \left(\left[\nabla, \hat{O}\right] \psi \right)^* \cdot \nabla\psi - \nabla\psi^* \cdot \left(\left[\nabla, \hat{O}\right] \psi \right) \right)$$

It is then possible to propose a suitable current $\mathfrak{j}$ if and only if

$$-\frac{i\hbar q}{2m}\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)=\nabla\cdot K \quad (16)$$

for some vector K. In such a case, $\mathfrak{j}=\Theta j+K$, satisfies $(\Theta\rho)_{,t}+\nabla\cdot\mathfrak{j}=0$.

Let us further investigate condition eq.(16). Clearly, it implies

$$-\frac{i\hbar q}{2m}\left\langle\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)\right\rangle=\langle\nabla\cdot K\rangle$$

Introducing the Hermitian adjoint, the lhs can be rewritten as follows:

$$\frac{i\hbar q}{2m}\left\langle\psi^*\left(\left[\Delta,\hat{O}\right]-\left[\Delta,\hat{O}\right]^\dagger\right)\psi\right\rangle \quad = \quad \langle\nabla\cdot K\rangle\,.$$

Hence, if $\left[\Delta,\hat{O}\right]$ is Hermitian (or equivalently if $\hat{O}^\dagger=-\hat{O}$, since Δ is itself Hermitian), eq.(16) ensures that an adequate K exists with $\langle\nabla\cdot K\rangle=0$. Conversely, if a vector field K exists such that

$$\int d^3x\;(\nabla\cdot K)=\int_\Sigma K\cdot\widehat{d\Sigma}=0$$

where the volume integral is taken as limit over all space and Σ is the oriented surface bounding the volume of integration, then $\hat{O}$ is anti-Hermitian. □

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Declaration of Interest

The authors declare that there exists no actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, their work.

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I Beni archeologici: un equilibrio tra competenze scientifiche e saperi umanistici
(Archaeological heritage: a balance between scientific skills and humanistic knowledge)

Giuliana Soppelsa[1]

Sunto

Il presente lavoro intende mettere in risalto i rapporti matematici nella realizzazione del Tempio di Hera nel santuario meridionale della città magnogreca di Posidonia-Paestum. Progettato intorno alla metà del VI sec. a. C. e portato a termine nell'ultimo decennio del secolo, il primo tempio dedicato a Hera, più noto come Basilica, fu eretto in stile dorico in base ad un ordine planimetrico basato sul rapporto 1:2, evidente già nella peristasi (con numero di colonne pari a 9x18 sullo stilobate). Il volume interno definisce una composizione perfettamente simmetrica rispetto ad un asse definito dalla colonna mediana di ciascuna fronte e dal colonnato assiale che divide la cella in due navate uguali; la larghezza di ciascuna navata interna corrisponde inoltre esattamente a due intercolumni, generando così una ripartizione armonicamente equilibrata e proporzionata in quattro parti uguali di tutto lo spazio racchiuso nel peristilio. La tematica, per il suo peculiare intreccio tra saperi scientifici e umanistici, si presta ad essere utilizzata come esempio di didattica interidsciplinare.

Keywords: Tempio; Posidonia-Paestum; simmetria. [2]

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Giuliana Soppelsa

Abstract

This work aims to highlight the mathematical relationships in the design of the Temple of Hera in the southern sanctuary of the Magna Graecia city of Posidonia-Paestum. Designed around the mid-6th century B.C. and completed in the last decade of the century, the first temple dedicated to Hera, known as the Basilica, was built in Doric style based on a planimetric order following the 1:2 ratio, which is already evident in the peristasis (with a number of columns equal to 9 x 18) on the stylobate. The internal volume defines a perfectly symmetrical composition with respect to an axis defined by the median column of each front and by the axial colonnade that divides the cell into two equal naves; the width of each internal nave also corresponds exactly to two intercolumns, generating a harmoniously balanced and proportional division of the entire space enclosed in the peristyle into four equal parts. The topic, due to its peculiar intertwining of scientific and humanistic knowledge, lends itself to being used as an example of interdisciplinary teaching.

Keywords: Temple; Posidonia-Paestum; symmetry.

1. Introduzione

Questo contributo intende ampliare e approfondire la tematica presentata con il poster dal titolo "Simmetrie e rapporti numerici nel Tempio di Hera a Posidonia-Paestum" esposto al Primo Convegno MatEleatica "La Matematica e la Cultura classica"[3]. In quella occasione sono stati sottolineati gli evidenti rapporti matematici che sono alla base della progettazione dei templi posidoniati ed in particolare nel Tempio di Hera, come chiaro esempio dell'importanza dei saperi scientifici nella determinazione dell'assetto monumentale e urbanistico della *polis* magnogreca, mentre in questa sede si intende approfondire il tema e soprattutto proporlo anche come contenuto centrale per un esempio di didattica pluridisciplinare, nella convinzione che determinati argomenti, se opportunamente sottolineati, possono mostrare agli studenti l'origine greco-romana della cultura europea e occidentale.

La pedagogia del patrimonio, infatti, se considerata nell'ottica del connubio tra scienze esatte e cultura classica, si qualifica come ambiente di apprendimento privilegiato ove si intrecciano in maniera indissolubile saperi di diversi ambiti disciplinari. I beni archeologici e storico-artistici costituiscono dunque un ambito di indagine che coniuga un'approfondita preparazione umanistica e storico-artistica con la sistematicità degli studi di carattere scientifico. Proprio una conoscenza integrata dell'antico, con particolare attenzione ai beni culturali, può sviluppare una formazione equilibrata, in cui il rigore del linguaggio scientifico funga da irrinunciabile base per comprendere la complessità della

[3] Primo Convegno MatEleatica- La Matematica e la Cultura classica, Velia-Ascea, 17-19 Maggio 2024, promosso dall' Associazione Math&Phys-Salerno, in collaborazione con l'Associazione Napoletana di Filosofia e Scienze Umane "Renato Caccioppoli", l'Accademia Piceno Aprutina dei Velati in Teramo, e l'Associazione Mathesis "Aldo Morelli".

cultura umanistica. Il tema dei beni culturali, tradizionalmente associato agli studi umanistici, si presta molto bene in realtà anche alle necessità formative previste dall'asse matematico e scientifico- tecnologico, proprio per l'evidente relazione sostanziale tra arte e scienza. In particolare, l'archeologia si avvale di operazioni concettuali quali la selezione, la catalogazione, la comparazione, il calcolo delle percentuali, la composizione di tabelle, l'analisi statistica dei dati; necessita inoltre di disegni tecnici di oggetti, di piante, di sezioni e di assonometrie: sono tutte abilità che si acquisiscono proprio con lo studio delle discipline scientifiche e del disegno, cioè competenze che tradizionalmente vengono sviluppate nel curricolo caratterizzato da discipline scientifiche. In più, bisogna sottolineare che l'archeologo, lo storico e lo storico dell'arte si avvalgono del prezioso apporto delle materie scientifiche per numerose applicazioni, quali l'analisi chimico-fisica dei materiali di cui si compongono i reperti, oppure la datazione delle evidenze, tra cui quella sui materiali di origine organica, attraverso la misura della concentrazione dell'isotopo 14 del carbonio, per citare solo alcuni degli importanti contributi dell'archeometria.

Per dare un esempio concreto delle connessioni interdisciplinari pertinenti ai beni culturali, si propone in questa sede una riflessione sulle simmetrie e sui rapporti numerici che regolano la costruzione del Tempio di Hera nella colonia magnogreca di Posidonia[4], poi denominata Paestum in età romana (comune di Capaccio, Salerno). Come è noto, i templi pestani rappresentano una delle espressioni più potenti del vasto patrimonio culturale della provincia di Salerno[5], la cui conoscenza unisce da un lato storia, archeologia, arte e pedagogia del patrimonio, dall'altro temi propri alle discipline scientifiche, applicate allo studio del monumento e del territorio.

2. Il Tempio di Hera a Posidonia-Paestum

Il tempio di Hera[6], il più antico dei tre grandi edifici sacri della *polis* di Posidonia, fu progettato intorno alla metà del VI sec. a. C. e portato a termine circa nel 530 a. C. (Figura 1). Appartiene alla prima generazione dei grandi templi in pietra, e dunque ad un periodo cruciale per la formazione dell'architettura greca. Manca la parte alta costituita dal tetto e dai frontoni, e ciò, unito alla considerazione che l'impianto planimetrico non è quello considerato canonico per il tempio dorico, ha fatto sì che per molto tempo la sua destinazione non fosse chiara e che si supponesse una funzione civile; nasce da qui la denominazione di "basilica" attribuita all'edificio dagli eruditi e dai viaggiatori che nel Settecento riscoprirono la città antica e ne diffusero la conoscenza attraverso una serie di disegni e rilievi. Ancora oggi viene chiamato "Basilica", anche se è ormai scientificamente provato che si tratta di un edificio di culto, che nel corso degli scavi

[4] Per una descrizione dell'edificio templare e in generale di tutti monumenti della *polis,* v. [9].
[5] Per lo studio dell'architettura templare posidoniate, v. [1], pp. 203-214.
[6] Sull'analisi puntuale del tempio in tutti i suoi elementi strutturali v. [6], pp. 230- 237.

stratigrafici ha restituito oggetti votivi ed iscrizioni riferibili al culto di Hera, protettrice degli Achei, qui probabilmente associata al suo sposo, Zeus.

Figura 1. Posidonia. Tempio di Hera.

Il maestoso tempio dorico è situato nel santuario meridionale della città (Figura 2), insieme al cosiddetto Tempio di Nettuno, un po' più recente, dedicato in realtà non a Poseidon-Nettuno ma ad un'altra divinità maschile, identificata con Zeus, in base al confronto con il tempio di Zeus ad Olimpia, oppure con Apollo[7]. Insieme al Tempio di Atena, più noto come Tempio di Cerere, situato nel santuario settentrionale della città, essi costituiscono la più grandiosa manifestazione della potenza economica raggiunta dalla colonia greca già alla fine del VI sec. a. C., ad un secolo della sua fondazione.

Figura 2. Posidonia. Santuario meridionale.

La città nasce come sub-colonia ad opera di Sibari, a sua volta fondata dagli achei sullo Ionio, colonia presto assurta ad un tale grado di ricchezza e potere (le fonti parlano di un "impero sibarita") che, per disporre di uno sbocco territoriale e marittimo anche nel

[7] Cf. [4], pp. 75-78.

Mar Tirreno, diede vita appunto alla città di Posidonia. Questa fu impiantata sulle sponde dell'importante percorso fluviale del Silaro (oggi Sele); il fiume alla fine del VII sec. a. C. era navigabile ed era solcato dalle numerose navi greche e fenicie che attraversavano il Mar Tirreno, portando sulle coste dell'Italia meridionale e dell'Etruria merci raffinate, genti, idee e tecniche (Figura. 3)[8].

Figura 3. Posidonia. Veduta della città.

Il Tempio di Hera, che fu eretto in stile dorico, presenta uno stilobate che misura 26 x 55.70 metri su cui si erge una peristasi di nove colonne sul lato corto e diciotto sul lato lungo. La cella vera e propria è divisa in due navate da una fila di colonne centrali (come accade nelle antiche architetture in legno); la prima colonna interna non è addossata al muro trasversale e contro la parete di fondo non si trovava una colonna, ma un'anta, il che rende la divisione in due metà ancora più accentuata (Figura 4). Alcuni studiosi attribuiscono la divisione in sole due navate, con un'unica fila centrale di colonne, all'arcaicità dell'edificio sacro; altri invece, ritenendo tale ipotesi non sostenibile alla luce delle conquiste dell'architettura greca di quel periodo, ritengono che essa sia dovuta al fatto che veniva qui venerata un'altra divinità maschile oltre a Hera.

[8] Sulla colonizzazione greca in Occidente, v. [2].

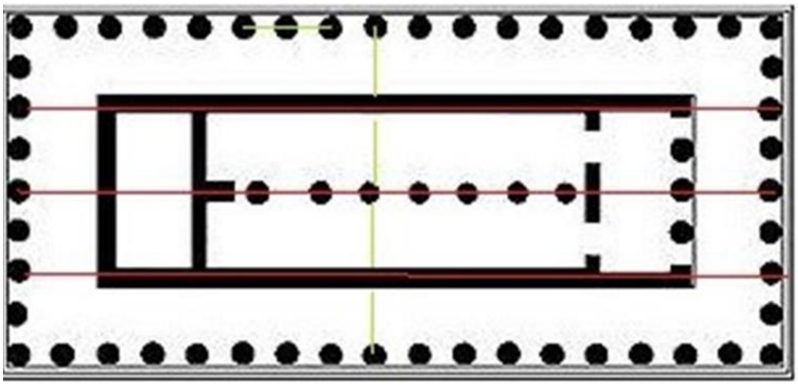

Figura 4. Posidonia. Pianta del Tempio di Hera con ripartizione dello spazio basata su due intercolumni.

La cella è preceduta da un pronao con tre colonne e seguita da un vano simmetrico, originariamente concepito come opistodomo (un ambiente cioè simile al pronao, aperto, con tre colonne nella facciata), ma poi sostituito da un ambiente chiuso (*adyton*), inaccessibile dall'esterno, che era probabilmente adibito a deposito per il tesoro del tempio. Per quel che riguarda l'alzato, le colonne in calcare hanno venti scanalature poco profonde su un fusto sagomato da una raffinata *entasis* (restringimento) fino al sommo scalpo, da cui inizia una profonda gola a foglie che penetra all'interno del profilo dell'echino. Al di sopra delle colonne, si conserva anche l'architrave e la fascia interna del fregio, che doveva comprendere metope e triglifi. È stato possibile ricostruire anche la parte mancante dell'epistilio in base ad alcuni frammenti recuperati durante gli scavi stratigrafici, fino alle terrecotte architettoniche che costituivano l'orlo del tetto, con frontoni triangolari sui lati brevi (Figura 5).

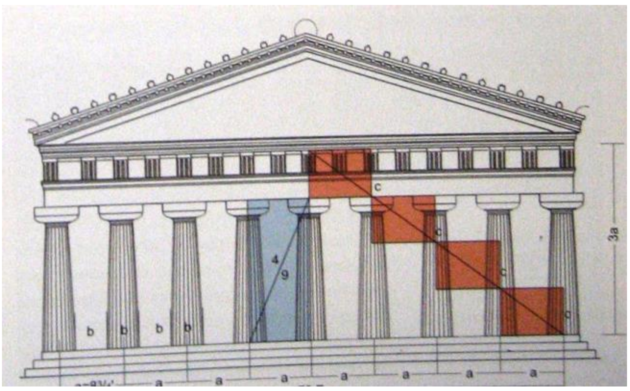

Figura 5. Posidonia. Tempio di Hera, ricostruzione dell'alzato con schema delle proporzioni elementari.

L'edificio templare diventa oggetto di specifica attenzione proprio per le sue peculiari caratteristiche planimetriche, modulate su precisi rapporti numerici e su evidenti simmetrie: la pianta è basata infatti sul rapporto 1:2, evidente già nella peristasi. Tale rapporto tra lunghezza e larghezza si riduce nella cella a 1:3. Il volume interno, come sopra già accennato, si compone di due parti simmetriche rispetto all'asse centrale che è definito dal colonnato interno della cella e si origina e si conlude con la colonna mediana di ciascuna fronte, in maniera tale che ogni navata misuri precisamente due intrcolumni. Infatti, proprio la necessità di creare un rapporto tra le divisioni interne dell'edificio e il ritmo dei colonnati esterni determina la scelta progettuale basata su un modulo costituito dall'intercolumnio. I muri della cella si trovano dunque in asse rispettivamente con la terza e la settima colonna dei lati brevi della peristasi e quindi ciascuna delle gallerie laterali del peristilio corrisponde anch'essa a due intercolumni della facciata, dando vita ad una divisione perfettamente simmetrica e regolare di tutto lo spazio in quattro parti uguali[9]. Gli evidenti rapporti modulari che sono alla base della progettazione dei templi posidoniati, in particolare quelli evidenziati nel Tempio di Hera, sono un chiaro esempio dell'importanza dei saperi scientifici nella determinazione dell'assetto monumentale e urbanistico della *polis* magnogreca; rimarcati in maniera adeguata, possono ricordare in tal modo la matrice greco-romana della cultura occidentale, considerata nella sua integrità.

3. Connessioni interdisciplinari

L'edificio templare, collocato nel suo contesto storico e archeologico, funge da idea centrale sulla quale si costruisce un percorso interdisciplinare che coinvolge ampiamente da un lato materie scientifiche (Fisica e Scienze), dall'altro l'intero blocco delle discipline umanistiche, nonché tematiche di cittadinanza democratica (Figura 6).

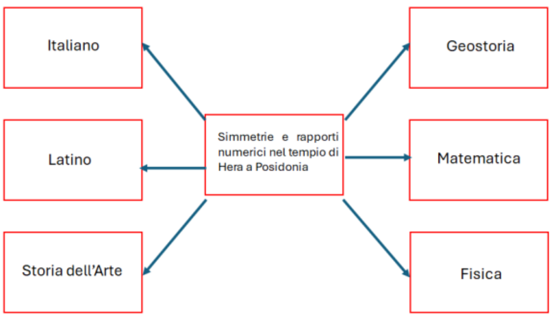

Figura 6. Mappa concettuale dell'Unità didattica di Apprendimento.

[9] Per l'importanza dell'intercolumio come modulo dell'intero monumento, v. [11], pp. 139-149.

Aedium compositio constat ex symmetria, cuius rationem diligentissime architecti tenere debent. Ea autem paritur a proportione, quae graece analogia dicitur. Proportio est ratae partis membrorum in omni opere totoque commodulatio, ex qua ratio efficitur symmetriarum. Namque non potest aedis ulla sine symmetria atque proportione rationem habere compositionis, nisi uti [ad] hominis bene figurati membrorum habuerint exactam rationem[10].

La composizione dell'edificio è basata sulla simmetria, che gli architetti devono considerare attentamente. Essa però nasce dalla proporzione, che in greco si chiama analogia. La proporzione è il rapporto *di ogni singolo elemento rispetto all'opera nella sua interezza, dalla quale si genera il rispetto delle simmetrie. Infatti nessun tempio può avere un ordine di composizione senza simmetria e proporzione, come se uomini ben formati non avessero un armoniosa misura delle membra*[11].

Il Tempio di Hera rientra nel più vasto tema della città greca in generale, intesa come organismo urbanistico, sociale e politico, e della colonizzazione greca in Occidente e in Sicilia, fenomeno storico entro il quale si colloca la fondazione della città di Posidonia-Paestum. Per ricostruire le giuste coordinate spazio-temporali della città antica, è opportuno ricordare, oltre alla fase greca, anche le trasformazioni successive, dal periodo lucano fino alla città romana, nonché la fase dell'abbandono in età medievale e della "riscoperta" nel Settecento[12].

Inoltre, partendo dalle divinità a cui sono dedicati i templi pestani, si possono approfondire alcuni aspetti peculiari della religiosità nel mondo antico[13]. La divinità principale è Hera, sposa di Zeus, a cui è dedicato il santuario meridionale urbano e l'importante santuario extraurbano di Foce Sele, posto a rimarcare il territorio (*chora*) occupato dai coloni greci. La dea a Posidonia viene venerata in una pluralità di funzioni: come divinità armata protettrice della guerra, raffigurata nell'atto di impugnare una lancia, oppure come *kourotrofos* o protettrice della maternità, come indicano le statuette che la ritraggono in atto di allattare un bambino, ed infine, in età classica, come protettrice della fecondità della terra, simboleggiata dal melograno che regge nella mano destra nelle statue che la rappresentano seduta in trono, nel tipo che in letteratura archeologica si definisce appunto "Hera pestana". Accanto alla dea, come sopra accennato, vi era il suo paredro maschile, Zeus, di cui è stata rinvenuta una statua accanto all'altare posto di fronte al tempio, e che probabilmente è il nume titolare anche del secondo tempio del santuario meridionale. Nella fase della fondazione della città doveva essere ritenuto di grande importanza anche Poseidon, visto che il nome della colonia deriva proprio dal dio del mare. L'altro santuario urbano, posto nella parte settentrionale della città, è invece dedicato ad Atena: i recenti studi sui materiali (offerte votive e armi), provenienti dagli

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[10] Vitruvio, *De Arch., Liber* III, IV, I.1, v. [5], p. 124.
[11] La traduzione è di chi scrive.
[12] Sulle vicende della *polis* posidoniate in seguito all'abbandono, v. [8].
[13] J. P. Vernant ha dato un contributo fodamentale alla conosceza della religione greca antica, v. [12].

scavi effettuati negli anni '20 e '30, hanno consentito di delineare con maggiore precisione l'importanza del culto poliadico e le funzioni attribuite alla Atena di Posidonia: a giudicare dalla coroplastica votiva e dalle molteplici offerte di armi, ella è venerata innanzi tutto come *Athena Promachos*, legata al mondo della guerra, ma oltre a questo ruolo rivela competenze molteplici, tra cui la protezione degli *erga* (lavori) e della transizione dei giovani all'età adulta[14].

Passando ora all'asse scientifico-tecnologico, la descrizione del Tempio di Hera può implicare cenni alle leggi fisiche della Statica, ossia della descrizione delle forze che si bilanciano in maniera tale da assicurare la staticità del monumento, ove i blocchi di travertino sono giustapposti senza l'ausilio di malta. Altre leggi della fisica che possono essere chiamate in causa per comprendere il progetto inventivo dell'architetto ideatore del tempio sono quelle che spiegano le "correzioni ottiche" adottate nella costruzione dell'edificio sacro[15]: è presente infatti l'*entasis* delle colonne, cioè il rigonfiamento del fusto a circa un terzo dell'altezza (utilizzato per eliminare l'illusione ottica che, a distanza, fa apparire la parte centrale della colonna più stretta rispetto alle estremità)[16].

Un'altra possibile connessione interdisciplinare prevede un approfondimento della conformazione geomorfologica del sito[17], in particolare in relazione alla disponibilità del calcare, che è stato largamente impiegato come materiale da costruzione per i templi e per gli altri monumenti posidoniati.

Infine, la conoscenza dei monumenti e dell'archeologia dell'antica *polis* comportano lo sviluppo di un percorso di cittadinanza democratica basato sulla conoscenza, la tutela e la fruizione del patrimonio culturale, inteso come legame con il proprio territorio, scoperta delle proprie radici e consolidamento dell'identità.

4. Conclusioni

I beni culturali rappresentano un contesto in cui appare pressoché irrilevante la consueta divisione (più apparente che reale) tra le discipline scientifiche e quelle

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[14] Per un'attenta disamina delle funzioni e degli ambiti posti sotto la protezione della dea nella *polis* di Posidonia – Paestum, v. [3].

[15] Le correzioni ottiche sono modifiche di alcuni dettagli della costruzione che correggono determinati effetti di curvatura delle linee o di sproporzione dovuti a punti di vista o a condizioni di luci particolari. Oltre all'*entasis*, nel tempio greco vengono spesso adottate anche altri tipi di correzione ottica. Una delle più note è la curvatura delle orizzontali: poiché le linee orizzontali del basamento, della trabeazione e della base del frontone del tetto viste da lontano sembrano inarcarsi, vengono curvate leggermenteverso l'alto, in modo da non apparire concave alla distanza ma perfettamente rettilinee, mentre nei lati lunghi della peristasi venivano posizionate colonne più massicce, con un diametro maggiore. Un'altra soluzione frequentemente adottata riguarda l'aumento dell'interasse delle colonne: per correggere le distorsioni dovute sempre alla luce del sole, l'intercolumnio tra le colonne in corrispondenza della cella è maggiore di quello tra le colonne laterali, in modo da ottenere l'effetto di uniformità.

[16] Particolarmente efficace per un utilizzo didattico risulta la scheda didattica 15, curata da G. Grassi, consultabile online nel sito Alipes. Arte e cultura nella pubblicità, v. [13].

[17] Per la geomorfologia dell'area pestana, v. [7].

umanistiche, e che rivela invece la complessità del sapere. Un progetto formativo che voglia essere efficace e capace di raccogliere le sfide che la società odierna, così mutevole e in rapido cambiamento, pone continuamente ai giovani e agli educatori, deve necessariamente risultare equilibrata e offrire una visione organica della conoscenza, ed è per tale motivo che il settore dei beni culturali, in particolar modo dell'archeologia, si pone sempre più frequentemente al centro di proposte educative che abbiano valore di orientamento (scoperta delle proprie risorse e delle proprie attitudini) e di preparazione al mondo del lavoro. I parchi archeologici, gli oggetti d'arte o di cultura materiali di cui sono pieni i nostri musei, proprio perché provocano una indiscussa carica emozionale e perché suscitano una profonda percezione dell'antico e dell'avvicendarsi dei tempi, rappresentano dunque un patrimonio inesauribile di conoscenza e formazione per le nuove generazioni che, se guidate ad una corretta interpretazione, sono in grado di avvertire il senso della storia e del passato, su cui costruire il proprio futuro.

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The Existence of Chance

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Abstract

This paper aims to study a particular case to provide a rational basis for recognizing the existence of chance by finding its real trace, that is, specifying its practical meaning, which consists of the following statement known as the Statistical Law of Large Numbers:
If an event E has a constant probability p of occurrence on any one trial, and has occurred m times in n trials, then, if the relative frequency of E, m/n, approaches the value of a limit point l and the accuracy of the approximation increases as the number of trials increases, we have l = p.
The argument we propose is based on the concepts of "event" and "trial", formulated by the author himself, and their direct implications.

Keywords: Chance; Gambling System; Random Event; Statistical Law of Large Numbers; Trial.[1]

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1. Introduction

Is chance real, true? Everyday life and some formal analyses seem to lead us to a negative answer, suggesting the idea that chance, or randomness, are relative and not absolute notions; indeed, it seems that the same phenomenon can be considered random based on some evidence and no longer be so in the presence of a stronger, more refined evidence[2].

It is therefore necessary to decide what exactly is to be understood by "chance", that is, whether it denotes an inescapable attribute of reality, as indeterminism suggests, or whether, as determinism and necessitarianism hold, it denotes our non-exhaustive knowledge of the exact causes underlying and figuring out various phenomena.

If, instead, our knowledge/evidence is such as to allow us to prove by itself the impossibility of predicting a certain phenomenon, we are in the presence of a random event. Examples of events of this last type are easily found in dice games, meteorology, etc., that is, in all those situations in which minimal or imperceptible variations of the starting conditions – so small as to be impossible for us to record or to exclude them – produce macroscopic differences crucial for the event to occur or not[3]. An event is therefore random if it, because of its instability, its complexity, or any other relevant property, refers in a total and exclusive way to a cognitive state, sufficient to affirm its absolute imponderability.

The objectivity that characterizes this notion of randomness, however, is such in relation to the living being, to the world in which it lives and therefore to what in this world is a cognitive patrimony common to all. Note, then, that those who support this position would seem to have to refrain, perforce, from taking a decision for or against indeterminism necessarily. In fact, they could neither deny determinism nor exclude that, also in the context of determinism, unpredictable events can be found, which are such because of the set of conditions/knowledge to which they refer.

But is it possible to dispel this doubt? In other words, is it possible that the concept of randomness is marked by truthfulness, that is, is characterized by objectivity in the strict sense, of an ontological type, and not only by what is subjectively universal – i.e., valid for all men –?

Yes, so it is, if we take some interpretations within post-quantum sciences, in which "real" is only what can be measured and where a distinction

[2] By the term *evidence* we refer here to all the knowledge we possess when we are preparing to argue about a given phenomenon. When the very broad meaning just outlined is attributed to evidence, it is impossible to speak of randomness in the absence of evidence. The randomness of a phenomenon will therefore always be related to evidence.

[3] See M.C. GALAVOTTI, *Probabilità*, La Nuova Italia, Milano, 2000, pp. 70-79.

between gnoseological unpredictability and ontological indeterminacy cannot exist: in other terms, pre-assumption of a certain ontological reality – beyond those possible experimental measures – would signify the fall into metaphysics.

Yes, it is. Provided that we reject the traditional distinction between a truthful reality and a subjective one, we reject the firm belief that the world is divided into a truthful development in space-time and a consciousness that merely sees and thinks about this evolution. And it is precisely this last point of view that we wish to propose here.

Reality is no longer something independent of us and of our knowledge, something that can go ahead autonomously without our contribution. It is not the set of objects, material substances, of what is outside of us, to which to oppose the internal sphere of subjectivity. Rather, the real thing[4] becomes the objective result of the necessary and constant interaction between something that is seen and what is known about this something, the experience of their fusion, which will take on different forms and degrees in relation to the different contexts that are treated.

Chance can therefore still be considered a name for our ignorance, specifying however that by "our ignorance" we are not referring to our lack of information on reality, but to a precise state of reality itself, in which we are inevitably involved and which changes because of the variation of our knowledge on what we see about it. Changing our knowledge about reality is in fact equivalent to changing it.

At this point it becomes necessary to give to this position a coherent logical structure and an operational sense. And that is exactly what we will try to do in the next two paragraphs.

2. Event and Trial

In close connection with what explained in the introduction, we argue in the following our denunciation of a truthful randomness, focusing on three key concepts which we will be discussing at once and that are closely related: *event and trial*.

The notion of event that we adopt here can be summarized in the following two basic points:

ε1) By *event*, we mean a thing that refers totally and exclusively to a well-defined set (or class) of conditions, such that, when this set of conditions is realized – i.e., when all the conditions that constitute it are realized –, and

[4] For *real thing* we mean here simply "what is (appears)".

only in this case, the thing gives rise to an ascertainable fact by which we can always determine whether or not the event has occurred(presented) [5].

ε2) When we affirm that some event A is *random*, we mean that the set of conditions C, or more briefly set C or only C, to which event A refers, holds the whole class of reasons necessary and sufficient for there to be no immanent way of predicting whether A will or will not occur when C. The event A is instead *certain* (*impossible*) with respect to C, if the set C includes the necessary and sufficient conditions for A to occur (not to occur) when C[6] (See [8] pp. 21-24). It follows that every event of C is necessarily random, certain, or impossible[7].

The single realization of the set of conditions C, called *trial of C,* can understandably take place, at least mentally, more than once – in this case we

[5] The event, realized the class conditions to which it relates, gives rise to a verifiable fact by which it can always be inferred whether the event occurred. By *fact* we mean, therefore, a happened that can be described by a boolean proposition of the type «the thing or entity x has the property y» that makes sense, that is, for which a criterion is given by which it is technically possible to attribute it the truth value: *true* (See [1] pp.13-14). Since, as is obvious, the meaning and truth value of a proposition depend on some aspects of the context in which it is inserted – e.g., the phrase «there is a pen on that table» makes sense and is a proposition because it is placed, even implicitly, in environments where one knows perfectly well what «table» and «pen» are and how one recognizes their presence; in some of these areas, it could thus be (judged) true, in others false etc. –, one can speak of a fact (as of an event) only in relation to a set of conditions.

[6] When we talk about randomness, certainty, or impossibility of any event, we always mean randomness, certainty, or impossibility with respect to the set of conditions to which the event refers.

By use of the three known symbols of logical implication, introduced as follow: "$\Leftrightarrow$" as short for "co-implies", or "is logically equivalent to", "$\not\Rightarrow$" as short for "not imply" and "$\Rightarrow$"as short for "implies", we can summarize the concepts of randomness and certainty (impossibility) of an event, respectively, by the following relations:

$$\{C \Leftrightarrow [«C \not\Rightarrow A» \text{ and } «C \not\Rightarrow \text{not } A»]\}\ ,\ \ C \Leftrightarrow\ «C \Rightarrow A \text{ (not A)}»$$

where, for the sake of brevity, "C" stands for "C is realized," "A" for "A occurs," and "not A" for "A does not occur."

[7] Let us first show that $S_1 = (C\Rightarrow A) \Leftrightarrow S_2 = [C\Rightarrow(C\Rightarrow A)]$, where A is an event of C.

Recall in this regard that, from a logical point of view, the two relations $C\Rightarrow A$ e $C\not\Rightarrow A$ are necessary and mutually exclusive, and that each of these two options implies C; in fact neither $C\Rightarrow A$ nor $C\not\Rightarrow A$ would make sense without the realization of C, since, by ε1, A presupposes C.

Now, $S_2 \Rightarrow S_1$ is a self-evident truth. Conversely, if the realization of C is a sufficient condition for the occurrence of A, i.e., if $C\Rightarrow A$, then, if S_2 were false (i.e., if $C\not\Rightarrow(C\Rightarrow A)$, we would have at least one classification, one example in which C is realized and A does not occur. But this would contradict the assumption that $C\Rightarrow A$. Consequently, it must be $C\Rightarrow (C\Rightarrow A)$. Finally, we prove that

$$S_3 = [C \not\Rightarrow (C \Rightarrow A)] \Leftrightarrow S_4 = [C\Rightarrow (C\not\Rightarrow A)].$$

If S_3 were false, that is, if $C\Rightarrow(C \Rightarrow A)$, then, evidently, $C\Rightarrow(C\not\Rightarrow A)$ could not be true; therefore, $S_4 \Rightarrow S_3$.

Vice versa, suppose that $S_3 = C\not\Rightarrow(C\Rightarrow A)$ is true. Since $S_1 \Leftrightarrow S_2$, we get $C\not\Rightarrow A$, necessarily – it cannot occur $(C\Rightarrow A)$, if C is true; otherwise, we would have $C\Rightarrow(C\Rightarrow A)$ against the hypothesis: S_3 true –. It must therefore be $C\Rightarrow(C\not\Rightarrow A)$. Hence, $S_3\Rightarrow S_4$. It follows that $S_3 \Leftrightarrow S_4$, and so that $(C\not\Rightarrow A)\Leftrightarrow[C\Rightarrow(C\not\Rightarrow A)]$.

We thus proved that [$C\Rightarrow$ «$C\not\Rightarrow$ ($\Rightarrow$) "A" "not A"»] $\Leftrightarrow$ «$C\not\Rightarrow$($\Rightarrow$) "A" "not A"». The statement that every speak of *repeatable trial of C* – or only once, when it coincides with a totality of repeatable trials. The various trials, being exactly individual realizations of the same set of conditions, can only be identical and mutually independent, in the sense that the outcome of each of them does not influence and is not influenced in any way by that of another or the others. The set of all physical or conceptual trials of C – i.e., of all thinkable ones – will be written down with H_C. H_C is hence a singleton[8] or an infinite countable set.

Regarding each single trial, certainty, randomness, and impossibility are qualities that are not restricted to events, but also extend to facts; so, they are properties applicable not only to propositions concerning future events, but also to propositions that describe past or present events. This can be easily proved as follows:

Without loss of generality, suppose we do not know which of the random events A and -A[9], of the set of conditions C, is the one that, by happening, caused the occurrence of a certain circumstance B. Since B inevitably occurred together with one and only one of the two events A and -A, we have (with respect to B) the following and conflicting hypotheses:

H_1: «Event A has occurred» H_2: « Event -A has occurred».

Now, due to ε1, it must be possible to carry out at least one T-test – even if it were only a direct verification – that allows one to find whether A has occurred, and therefore to check which of the two hypotheses H_1 and H_2 is proven true, and which is refuted. Two complementary events are so defined:

$\hat{H}$: "the hypothesis H_1 is confirmed" -$\hat{H}$: "the hypothesis H_2 is confirmed",

both inherent to the set of conditions $\hat{C}$, which consists of the condition "C has been fulfilled" and all the other conditions that, together, define the properties of the T-test and the technical modalities of its execution.

Also note that the event $\hat{H}$ (-$\hat{H}$) occurs in any trial of $\hat{C}$ only if the event A (-A) has already occurred in the corresponding trial of C. Hence, if the premises $\hat{C}$ are true, the conclusion $\hat{H}$ or -$\hat{H}$ does not necessarily follow,

event of C must be certain, random, or impossible is thus, with good reason, a theorem.

[8] Recall that by singleton we mean a set that has exactly a single element.

[9] Let -A be the complementary event of A. We remember that two events are said to be complementary if they are incompatible and necessary, that is when, in the same realization of the set of conditions to which they refer, the occurrence of the one excludes the occurrence of the second – incompatible – but one of the two must necessarily occur – necessary –. For example, consider the single roll of a dice. As it is easy to guess, the following two cases may occur: either an even number comes out or an odd number comes out. The two events are complementary since one of the two will necessarily occur and one excludes the other.

because A(-A) is a random event of C and its occurrence, since it predates the T-test building, is independent of all T characteristics and specificities.

So, both $\hat{H}$ and $-\hat{H}$ are random events with respect to $\hat{C}$ [10].

On the other hand, as can easily be seen, the element "C has been realized", which constitutes the singleton $\hat{C}_H$, is a necessary and sufficient condition not only for the hypotheses H_1 and H_2 to make sense, for them to be, but also for the impossibility of foreseeing their correctness or falsity when C has been realized, or, what it is the same, whenever $\hat{C}_H$ is realized [11]. Both the hypotheses H_1 and H_2 satisfies thus the two requirements ε1 and ε2 characterizing the notion of event; therefore, H_1 and H_2 are two random events with respect to the set of conditions $\hat{C}_H$.

Hence, the property of being a random event can apply as well to that which has happened and which we can discover to be a fact. But no fact can be a random event; a fact is something whose truth it is impossible to doubt. It would seem to be facing a contradiction. It seems, as argued by Pierre-Simon de Laplace, that chance should be thought of negatively, not as a concrete reality, but simply as an epistemic fact, that is, as a deprivation of knowledge. It would then seem that chance/contingency cannot be truthfully characterized, that is, marked by ontological objectivity but must depend solely on the fact that we have "partial" knowledge of the data necessary for certain prediction. It seems to be, but it is not.

We explain ourselves better by proving the following important result (*Mister Tanatò's Paradox*):

R_1: Let A be an event with respect to a well-defined and realizable set of conditions C. If A is a certain or random event of C, then A occurs in at least one trial of C and vice versa.

Proof: The property is clear if A is a certain or impossible event of C. Suppose then that Mister Tanatò, a man sentenced to death, could avoid the death penalty in a way completely free from his will. Specifically, a coin M will be flipped (condition set C): if HEAD gets out (random event A), Mister Tanatò will have won his life challenge and will continue his existence; if

[10] If it were $\hat{C} \Rightarrow \hat{H}(-\hat{H})$, given that $\hat{H}(-\hat{H}) \Rightarrow A(-A)$, we would have $\hat{C} \Rightarrow A(-A)$ and thus $C \Rightarrow A(-A)$, being the occurrence of A(-A) completely independent of the T-test. But this would be absurd since A and -A are two random events of C and therefore $C \not\Rightarrow A(-A)$.
Finally, it is obvious that $\hat{C} \Leftrightarrow$ [«$\hat{C} \not\Rightarrow \hat{H}$» and «$\hat{C} \not\Rightarrow -\hat{H}$»], since $[C \not\Rightarrow A(-A)] \Leftrightarrow \{C \Rightarrow [C \not\Rightarrow A(-A)]\}$ (See footnote 7), $[C \not\Rightarrow A(-A)] \Rightarrow [\hat{C} \not\Rightarrow \hat{H}(-\hat{H})]$ and $\hat{C} \Rightarrow C$.
[11] If the conditions that form the set $\hat{C}$ yield that it is impossible to predict whether the event $\hat{H}$ occurs in any trial of $\hat{C}$, the conditions of $\hat{C}_H$, a fortiori, are necessary and sufficient to make it impossible to decide whether the hypothesis H_1 is true or false. In fact, $\hat{C}_H$ is a subset of $\hat{C}$, the realization of $\hat{C}_H$ is compatible with that of $\hat{C}$ and the occurrence of $\hat{H}$ $(-\hat{H})$ co-implies the truth of H_1 (H_2).

instead TAILS (random event -A) comes out, he will be promptly administered intravenously a lethal substance that will kill him instantly.

Since A is a random event with respect to the realizable set C, we cannot exclude that, in the immediately following instant of time τ to the toss of the coin M, the sentenced Tanatò has won the said bet and, hence, he has managed to survive, escaping his executioner.

At the time τ, Mister Tanatò might then find himself as a member of a special circle of the appearing (of reality), thus, still forced to keep going towards to one or more possibilities that will be ineluctably offered or imposed on him, including that of his own death. So, simultaneously, there must be a set of realized conditions, let us call this Č, respect to which the existence of Mister Tanatò is a random event.

On the other hand, at the same time τ, to affirm the impossibility of the existential fact of Mister Tanatò, of finding him still alive, it would be equivalent to supplying notification of his death – which therefore happened at least one moment before τ – to public. In other words, at the instant τ, this would mean that Mister Tanatò never existed, or as they say that he has already passed away in all possible worlds. But if so, one would have to admit that the set of conditions $Č_1$: {"The event -A occurred in every possible (thinkable) toss of the coin M"} has realized.

Hence, to avoid facing an antinomy[12], given the arbitrariness in the choice of Mister Tanatò and the experiment (launch of M) in which he is an interested spectator, we must recognize the veracity of the $\mathbf{R_1}$-statement, which is thus a theorem.

Remark 2.1: Let C be any realizable set of conditions. Let A be any certain or random event of C.

Because of the definition of an event, the occurrence of A in any trial of C is necessarily predictable – if A is certain – or always unpredictable – if A (and therefore -A) is random –. It follows, for $\mathbf{R_1}$ result, that the randomness of A co-implies the truth of the proposition $\mathbf{r_1}$: « “A occurs in a trial of C” is sometimes true». For these considerations, it seems that the thesis of indeterminism, which admits the ontological reality of chance/contingency, can be argued, since experience shows that $\mathbf{r_1}$ is true in at least one instance – consider, for example, the outcomes of repeatedly tossing of a fair coin –.

The validity of remark 2.1 assumes that randomness is a truthful characteristic of reality – of the type of the mass of the universe – which

[12] That is, of coming across the concomitant presence of two sets of conditions Č $\supseteq${“C has been realized”} and $Č_1$, both realized at time τ, the union of which is such as make the nonexistence of Mr. Tanató (in τ) a fact – it is unequivocally a fact if $Č_1$ is realized –whose truth turns out to be foreseeable and unpredictable at the same time.

involves the knowledge/evidence of the one who uses it, since it may otherwise also involve the happened, the latent fact – what has happened may not yet have happened –. Evidence thus sensitively affects what is real. Reality, therefore, can no longer be thought of as something independent of the knowledge of the one who thematizes it – who would then have only "degrees of confidence" about the manifestation of the phenomena by which it is affected – but what appears to us is what is, because to experience what directly appears to us is exactly to experience reality. So that conditions which for some observers are indeterminate, i.e., which are known, which appear to them (and thus are real) in the form of unknown variables, may then, paradoxically, appear to others (and thus belong to reality) as constants, i.e., may, for the latter, be precisely specified. Variables and constants therefore have the same dignity as real things and can be objects of experience and knowledge.

What are the boundaries within which the case acts? What quantitative limits does it have to respect? The next section will deal with this last issue.

3. Probability and relative frequency of an event

The percentage of repeatable trials, in which a given random event occurs, cannot experimentally be found, because we never have all the possible achievements of the class of conditions that define an event. If we think about the "last night it snowed" event, it is technically impossible to repeat independently «last night» an arbitrary number of times. We could do several numerical simulations on the computer (compatible with what is known about the weather situation of yesterday evening) and record the percentage of cases in which the simulation gives snow. In doing so, however, we would only get a correct approximation of our situation. The repeatability of the complex of «last night» conditions is indeed only theoretically possible if the circumstances of «last night» could be reconfigured in the same way as they were configured. To deal with such idealizations, it is necessary to involve probabilities. Probabilities replace uncertainty with something more usable and consistent.

Probability theory uses statements of the type «$P(E) = \lambda$», where E is any event and λ a numeral.

The expression «$P(E) = \lambda$» should read as follows: «the probability of the event E is λ». We now introduce another event F. This allows us two new combinations of symbols: $E \vee F$ and $E \& F$ each of which is another event. The event $E \vee F$ corresponds to the occurrence of three distinct cases: E, or F, or E and F, while the event E&F corresponds to the occurrence of the only case: E and F. We complete our presentation of probabilities with four basic rules:

1) $P(E) \geq 0$.

2) $P(-E) = 1-P(E)$.

3) If E and F are incompatible events, then $P(E \vee F) = P(E) + P(F)$.

4) $P(E\&F) = P(F\&E) = P(F)*P(E|F)$, where $P(E|F)$ is «the probability of event E, supposedly verified the event F».

All the other rules of the calculation of probabilities can be deduced from these through an exclusively logical-mathematical reasoning[13].

There are different points of view on what the statements of probability mean and how their veracity can be found. Limited to the needs of this discussion, we will consider probability as the numerical measure of the possibility of an event taking place and therefore as something truthful, independent of any human judgment and constant from person to person.

Our interpretation of the probability of an event is based on the following result:

R2: Let A be an event related to a well-defined set of conditions C. Suppose that A has a constant probability p of occurring in each trial of C and that it occurs μ times in ν trials of C, with $\nu > 1$. Let H_C is the set of all (physical or conceptual) trials of C.

Then, the relative frequency with which A occurs, μ/ν, converges to the probability p of A, i.e., it stabilizes around the p-value, and the approximation improves as the number ν of trials of C increases.

Proof: A theorem in probability theory, called the «Strong law of large numbers»[14], implies that, if the probability of event A occurring in each trial of C is constant and equal to p, then, with a probability of 1, the relative frequency of A, μ/ν, in the set H_C of all independent trials of C converges to p. The set H_C is thus, by definition, an infinite countable set, having assumed that $\nu > 1$. Worth noting is also that a probability of 1 does not necessarily mean that the event will happen, although it is clearly possible.
Now, if it were admissible that the relative frequency of A (in H_C) does not converge to p, then even "the relative frequency of A does not converge to p" would be a possible outcome of class H_C – understood as a single trial – and

[13] See M. D. RESNIK, *Choices*, University of Minnesota Press, Minneapolis, 1987, trad it. *Scelte*, franco muzzio & c. editore spa, 2003, pp. 76-89.
[14] See M. SPIEGEL – A. SRINIVASAN – J. SCHILLER, *Schaum's Outline of Theory and Problems of PROBABILITY AND STATISTICS*, The Mc Graw-Hill Companies Inc., New York 2000, pp. 80-88.

thus a random event with respect to that set of conditions, let us call it C', whose single realization consists in performing all the trials of C[15].

If it did, by $\mathbf{R_1}$, the relative frequency of the event A of C would simultaneously have to converge and not converge to p (in H_C), since a unique set, H_C, consisting of all trials of C is conceivable. A contradiction has been reached. The $\mathbf{R_2}$ claim is then true.

In summary: in a «long» trial sequence of C, in which the probability p of the event A of C always be the same, the relative frequency of A is about the probability p, the approximation improving as the number of trials (of C) increases. In other terms, if A and -A are two complementary events, then both A and -A are present in n_A and $n_{\bar{A}}$ "meta-real worlds", respectively, such that the ratio $n_A/(n_A + n_{\bar{A}})$ [$n_{\bar{A}}/(n_A + n_{\bar{A}})$] is equal to the probability p [1-p] of the event A [-A]. In addition, the greater the number of trials of C that we record as having concretely occurred, which we subtract from the sum $n_A + n_{\bar{A}}$, the smaller ordinarily will be the absolute deviation between the relative frequency of A in these trials and its probability p of occurrence.

The $\mathbf{R_2}$ result, known as the statistical law of large numbers, summarizes our conception of probability - i.e., it stands for the meaning we attach to probability assessments - and thus allows us to apply probability calculus to practical cases. An interesting example of this can be found in the following:

Problem (of the certain winner). Let C be the set of conditions that define the following experiment: «Two players X and Y alternately roll a pair of unrigged dice. The first player to throw a sum of 7 wins and ends the game. We assume that the game is a priori unlimited». Show that:

i) A: "the game is not won by either player" is an impossible event with respect to C.

ii) The methodical choice a priori of infinite subsequences of repeatable trials of C does not change the probability of a specific event of C (*impossibility of a gambling system*).

Solution: Let $\hat{C}$ be the class of conditions defining the following experiment:

"X or Y throws the pair of unrigged dice from experiment C". The probability P(B) of the event B: "X or Y gets a seven in a double throw with unrigged dice" of $\hat{C}$, as it is easy to calculate, is equal to 1/6. If player X is the

[15] Note that the expression «the relative frequency of A does not converge to p», by construction, makes sense and it is likely to have a truth value if and only if it refers to the set of conditions C'. This expression, therefore, describes here an event that can legitimately be assumed not to be impossible, even though, by the "strong law of large numbers", it must be assigned a probability value of zero – zero probability, however, does not necessarily mean impossibility –.

first to throw, then X will win in one of the mutually exclusive cases listed below with their respective odds:

- X wins on the first toss. Probability = 1/6.
- X loses on the first roll, then Y loses, then X wins. Probability = =(5/6)*(5/6)*(1/6) = 25/216.
- X loses on the first toss, then Y loses, then X loses, then Y loses, then X wins.
  Probability = (5/6)*(5/6)*(5/6)*(5/6)*(1/6) = 625/7776.
- ………………………………………………………………………… ;
- ………………………………………………………………………… .

The probability that X wins the game is then:

(1/6) +(5/6)*(5/6)*(1/6) +(5/6)*(5/6)*(5/6)*(5/6)*(1/6) +…= (1/6)*[1+ +$(5/6)^2$ +$(5/6)^4$ +…+$(5/6)^{2n}$ + +…] = (1/6)/{1-$(5/6)^2$} = 6/11, being the expression in square brackets, as known from the Mathematical Analysis, the development of a geometric series of common ratio q = $(5/6)^2$.

In like manner, assuming that Y is the second to throw, the probability that Y wins the game is equal to:

(5/6)*(1/6) +[$(5/6)^2$]*(5/6)*(1/6) +... = (5/6)*(1/6)*[1 +$(5/6)^2$ +$(5/6)^4$ +… + +$(5/6)^{2n}$ +…] = (5/6)*(6/11) = 5/11.

It is worth noting that the probability P(A) of a tie is equal to 1-[(6/11) + + (5/11)] = 0, but this data is not sufficient to say that event A (draw) is impossible, because events with a probability of zero are not necessarily impossible. Probability theory does not seem to be able to solve the problem of the certain winner placed above. It does not seem, but it is.

In this regard, we note that each trial of C, by construction, consists of a sequence of equiprobable trials of Ĉ[16]. Specifically, if X or Y were the winner of the game, there would be at least one realization of C consisting of a finite sequence of trials Ĉ – of at least one term if X (the first to throw) were the winner; or two if he had won Y – in which the event B has occurred once.

Moreover, if it were possible that no one between X and Y wins the game, then of course B might not occur for an entire infinite sequence of trials of Ĉ, that is, for a sequence whose terms form an equipotent set[17] to the set of all the trials of Ĉ. In this latter hypothesis, one could then not rule out the possibility that the event -B occurs in every single trial of Ĉ. From all this,

[16] That is to say that there is no reason, either factual or in principle, to believe that any of these trials (of Ĉ) is more likely than any other to be a constituent of a fixed trial of C. It follows that the randomness or certainty of event B is a sufficient condition for it to be possible for the game to end with a winner.

[17] Remember that two sets are said to be equipotent when their elements can be put in one-to-one correspondence, that is, if each element of the first can be associated with one and only one element of the second and vice versa.

because of **R_1**, it easily follows that the game always has a winner, always a limit, if and only if B has occurred in at least one trial of $\hat{C}$.

So, by **R_1**, if A were a certain or random event of C, then none of the possible (thinkable) repeatable trials of $\hat{C}$ could have resulted in a 7, i.e., in none of these trials could event B have occurred.

On the other hand, since $P(B) = 1/6 > 0$, there must be, by **R_2**, at least ν trials of $\hat{C}$ in which event B occurred μ times, with $\mu/\nu = (1/6)$. Hence, if A were a random or certain event of C, there would be a clear contradiction.

A is thus an impossible event of C (thesis ***i***).

Similarly, if it were possible that the relative frequency of B in a whole infinite subsequence of trials of $\hat{C}$, selected regardless of their outcomes, gives a value greater or less than 1/6, then it would be possible that
$F(B) \neq P(B) = 1/6$, where F(B) is the relative frequency of the event B in each sequence of all the trials of $\hat{C}$. But this would contradict the **R_2** result, according to which it must be $F(B) = P(B) = 1/6$.

So, it is impossible to find any pre-established function or rule that can select a priori, among all the repeatable trials of $\hat{C}$, an infinite subsequence in which, as regards the total number of its elements, the frequency of B is not equal to its probability $P(B) = 1/6$ (thesis ***ii***).[18]

4 Conclusions

We have achieved our goal, which was to show how randomness can be truthfully characterized, that is, marked by an ontological objectivity. By recognizing chance in the reality, we have also proved the rationality of the statistical law of large numbers, thereby giving an example of how determinism can appear from underlying indeterminism.

Our ignorance of things, from which the chance arises, therefore consists not so much in the unknowability of such things in their entirety as in the delineation of a precise state of reality constituted by the conjunction between everything that is seen and the wealth of knowledge that one owns on what is seen. It follows that there cannot be any kind of partial knowledge, that there is no reality without knowledge, that the complete annihilation of our knowledge entails (our) true non-existence. If, on the other hand, we were to gradually eliminate all phenomena that we study or could study – cease to be occupied with that matter, cease to hear that sound…–, everything that is or could be the object of our knowledge, with the exeption of itself, we would at some point arrive at a theoretical state of appearing, at that which is not but

[18] See R. D'AMICO, *Chance and The Statistical Law of Large Numbers,* in Journal of Mathematical Economics and Finance, [S.l.], v. 7, n. 2, p. 41-53, dec. 2021. ISSN 2458-0813. https://doi.org/10.14505/jmef.v7.2(13).03.

legitimately aims to be, at the mediator between the non-real and the real: the possible, the certain or random event.

Reality therefore depends on our knowledge. The fact that mathematical knowledge is the certain and rigorous form of knowledge par excellence would thus seem to explain why mathematics is effective in physics, why its precise language is the most suitable for describing reality. Artificial intelligence (AI), by collecting and analysing data and identifying patterns, can help us improve our knowledge, but it cannot replace us.

Hence, the randomness of phenomena is a feature of reality and probability is an intrinsic aspect to reality itself and it is no longer an instrument at the service of an imperfect knowledge. Possibility and probability thus become two notions both endowed with that empirical as well as truthful character that is claimed by the natural sciences.

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Contents

Social sciences on stage: a theatrical scientific dissemination project *Davide Costa*	7-18
Between objectivism and perspectivism: reasons for a philosophy and science encounter *Konstantinos Mantzanaris*	19-25
Matematica e Retorica a Roma: una lezione di geometria piana nell'Institutio oratoria di Quintiliano (Mathematics and Rhetoric in Rome: A Lesson in Plane Geometry in Quintilian's Institutio Oratoria) *Mariacarolina Santoro*	26-42
From Archimedes and the oxen problem to Pell's equation *Ilaria Veronesi, Antonio Caserta*	43-65
Il latino della scienza Galilei e i fratelli Bernoulli in uno studio interdisciplinare sulla brachistocrona (Latin in Science, Galilei and the Bernoulli brothers in interdisciplinary study over the Brachistochrone) *Francesca Coppa, Piera Filippi*	66-79
The construction of quantum mechanics from electromagnetism. Theory and hydrogen atom *Hernán Gustavo Solari, Mario Alberto Natiello*	80-138
I Beni archeologici: un equilibrio tra competenze scientifiche e saperi umanistici (Archaeological heritage: a balance between scientific skills and humanistic knowledge) *Giuliana Soppelsa*	139-149
The existence of chance *Rosario D'Amico*	150-163

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