Science & Philosophy Volume 10(1), 2022

Can the entanglement be considered a basic concept of quantum mechanics?

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Abstract

Recently, several experimental tests devoted to quantitatively establish complementarity relations in quantum systems have been reported. Starting from the results of fully quantum single-particle self-interference experiments, we critically review the concept of entanglement, arguing that this quantity is a peculiar trait of composite quantum systems, and thus it can be looked as a basic concept of quantum mechanics.
Keywords: Entanglement; Particle-wave duality; Basic concepts in quantum mechanics [1]

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[1]Received on January 12th, 2021. Accepted on May 12th, 2021. Published on June 30th, 2021. doi: 10.23756/sp.v10i1.739. ISSN 2282-7757; eISSN 2282-7765.

1 Introduction

Quantum mechanics (QM) developed over many decades of last century, providing theoretical explanations of experiments that the classical physics, namely classical mechanics, thermodynamics, and electromagnetism, could not explain [Guerra et al., 2020]. Many scientists contributed to the foundation of this theory, guessing rationalisations, and supplying satisfactory description of classically inexplicable phenomena [Guerra and Mercaldo, 2020]. Hence, QM gradually gained acceptance, also due to the experimental verification of new concepts. Fascinatingly, QM introduced new revolutionary principles, i. e. quantised properties of dynamical quantities and the ultimate nature of light and/or matter, that cannot find any analogous within the classical physics realm [Figliolia et al., 2020].

Concerning the properties of the light and the matter it is well-known that light can sometimes behave as a particle, even though a plethora of experiments shows that it behaves as a wave. On the other hand, matter can also act as a wave, albeit many experiments suggest that matter exists as a collection of particles. The concept of wave-particle duality was introduced by Planck at the beginning of 20th century, by which it was possible to explain the black body radiation, marking the start of QM. Reinforced by Young experiment of double slit and photoelectric effect it was argued that the light could be considered simultaneous wave and particle [Avitabile and Nigro, 2020].

It was 1924 when de Broglie for the first time spoke about wave-like behaviour referring to the matter. Aware of the light duality, de Broglie wondered if also the electron could show a wave-like behaviour. Thanks to his knowledge of special relativity and the newborn Planck constant, it was able to assign to each particle (corpuscular entity) a plane wave property characterised by a wavelength later called “de Broglie wavelength”. A corpuscular entity can be understood as any kind of macroscopic and microscopic objects. It is not possible, of course, to observe the wave nature of a macroscopic object, and a description based on classical mechanics is quite sufficient. Instead, with a microscopic particle like an electron, the de Broglie wavelength is comparable to the x-ray wavelength. Therefore, in this case one can consider and study the dual nature of electrons by using a QM description. This is the reason why quantum duality turns out to be one of QM basic concepts. Moreover, this hypothesis persuaded Bohr to state that the particle and wave descriptions of light and of matter are both necessary even though they are logically incompatible with each other. Thus, they must be regarded as being “complementary” to each other like different sides of a coin. This proposal led Bohr to formulate what is presently called the complementarity principle (CP) [Bohr, 1928a,b]: *The wave and particle models are both required for a complete description of matter and of electromagnetic radiation. Since these two models are mutually exclusive, they cannot be used simultaneously. Each exper-*

iment, or the experimenter who designs the experiment, selects one or the other description as the proper description for that experiment.

It was then realised that some physical objects exhibit multiple properties whose nature depends on what type of specific detecting devices is used. One well-known example is the wave-particle duality considered for a single particle in a two-way interferometer [Feynman et al., 1963]. One can choose to observe the wave-like or particle-like behaviours of the particle through different detection arrangements. Interference fringes have been observed for massive particles, such as neutrons [Summhammer et al., 1983], electrons [Tonomura et al., 1989], atoms [Carnal and Mlynek, 1991, Keith et al., 1991], and molecules [Arndt et al., 1999]. Remarkably, these entities were all previously thought to only be particle-like. In the case of light, both the anti-bunching effect and interference fringes, which are associated with the particle-like and wavelike properties, respectively, have been previously demonstrated [Grangier et al., 1986, Braig et al., 2003, Jacques et al., 2005].

Let us now talk about another extraordinary quantum property: the entanglement. Quantum entanglement is one of the most important and counter-intuitive phenomena of QM, initially contrasted by a part of the scientific community and currently assumed as a basic ingredient of most advanced forms of quantum-based technology such as quantum computers and quantum cryptography [Dowling and Milburn, 2003]. Erwin Schrödinger referring to the entanglement said [Schrödinger, 1935]: *Then one can say that the entanglement consists in that one and only one observable (or set of commuting observables) of one system is uniquely determined by a definite observable (or set of commuting observables) of the other system.*

To understand the sense and value of this statement is necessary to recognize the essence of this phenomenon, so that we will use an example to provide a first intuitive explanation. The entanglement can be seen as the correlation between two people that is established at the time when they are linked by marriage. If tragically one of the two spouses was to immediately fail the other would pass from the status of husband/wife to that of widower/widow; what happens to one partner inevitably changes the status of the other. However, if the quantum entanglement would be reduced to this simple analogy there would be no way to justify the words of Schrödinger and nothing would be so absurd and unusual in the phenomenon more than there is in the marriage between two individuals.

In a more formal way, a system of several particles is defined entangled when the total wave function of the system cannot be written as product of the individual wave functions of the constituent particles. A simple example of correlated systems is the helium atom: two electrons present in the atomic region interacting by Coulomb repulsion make the helium atom Hamiltonian not separable and the solutions of Schrödinger equation, because not factorizable, describes entangled states of the system.

Generally, it is possible to obtain entangled systems simply by making interact with each other different particles, or by making these particles be correlated with each other as happens in the decay reactions. In this regard, a prototype example is the decay process of the neutral pion, π^0 at rest in a positron, e^+, and an electron, e^-, [Bohm, 1951]:

$$\pi^0 \to e^- + e^+ \,. \tag{1}$$

Since the pion has spin S=0, as consequence of the conservation of angular momentum, the entangled system electron-positron must be in the singlet configuration for which if a particle is in one of the two possible spin states, the other must necessarily be in the opposite configuration:

$$\frac{1}{\sqrt{2}}\Big(\mid \uparrow>_{e^-} \mid \downarrow>_{e^+} - \mid \downarrow>_{e^-} \mid \uparrow>_{e^+}\Big)\,, \tag{2}$$

where we have indicated the spin-up $\mid \uparrow>$ and spin-down $\mid \downarrow>$ states of the two particles, respectively. Therefore, if the electron has spin up, the positron must necessarily have spin down, and vice versa. The spin of the two particles is said, therefore, correlated and the whole system is entangled. If we do not make a measurement, the two particles do not have defined spin: there is equal probability to find the system in one of the two spin configurations of the particles. It is the act of measuring [Ferraioli and Noce, 2019] the spin of one of these to ensure, not only, that it assumes a specific value of spin but that the other, as it is related to it, assumes instantaneously a configuration with opposite spin, thus obtaining the collapse of the total wave function of the entangled system.

This paper is devoted to clarifying the role played by the entanglement within the frame of basic concepts of QM, and it is organised as follows. In the next Section we critically review the wave-particle duality and the CP; in Section 3 we describe the Wooters-Zurek experiment while Section 4 is devoted to the Qian *et al.* experiment. Finally, the last Section contains a critical discussion of the experiments presented as well as final remarks.

2 Wave-particle duality, complementarity principle and quantum mechanics

It is important to understand that by accepting the wave–particle duality as a fact of nature, light and electrons (or other objects) encompass potentially the properties of both particles and waves, until they are observed. At this stage, they behave as if they are either one or the other, depending upon the experiment and the experimenter choice. This is a profound statement, for it meant that what we observe in our experiments is not the way nature *really is* when we are not observing it. Indeed, nature does not favour any specific model when we are not observing it; rather, it is a mixture of the many possibilities that it could be until we finally do observe it. By setting up an experiment, we select the model that nature will exhibit, and we decide how photons and electrons, for example, are going to behave either as particles or as waves. In other words, according to Bohr, the experimenter becomes part of the experiment since he interacts with nature, so that we can never observe all aspects of nature *as it really is* by itself. That sentence, while so appealing, has no operational meaning. Instead, we should say that we can know only the part of nature revealed by our experiments. The consequence of this fact, for events at the quantum level, said Bohr, is the uncertainty principle, which states a quantitative limitation upon what we can learn about nature in any given interaction. As a consequence of this limitation we must accept the a-priori probability interpretation of individual quantum processes, according to the orthodox approach to QM [Figliolia et al., 2020].

To explain the Bohr CP, we need to describe the experiment of the double-slit, a Young-type experiment. To adapt it within quantum realm in 1961, it was used an electron source and a properly modified new experimental set-up [Jönsson, 1961]. To recreate the exact conditions of Young experiment, more sophisticated detector slabs were used: electrons were accelerated through an appropriate potential, the slits separation and the reciprocal distance were adapted to the electron wavelength. The first images on the screen showed isolated spots in correspondence of the two slits, hence a typical corpuscular behaviour, gradually as soon as the number of electrons increased, the spots were replaced by the same interference pattern of light.

In the *which way* experiment [Durr et al., 1998], an attempt was made to understand exactly in which one of the two slits the electron went through by using a detector besides the two slits. Unexpectedly, it was clear that there could not see an interference pattern but a simple particle distribution, the electrons lost their wave nature. Obviously, this result arises from the fact that with this kind of configuration we are studying the electron corpuscular nature, the wave one vanishing.

Summarising, the Bohr CP is well explained in this case since the wave and corpuscular behaviour are mutually exclusive and complementary aspects of a quantum system. Einstein argued against such a principle and proposed a thought experiment that he claimed showing both the wave and particle nature in the same experiment. But Bohr was able to point out a flaw in Einstein argument and the CP stood its ground. It was recognised later that the wave and particle natures are not mutually exclusive in the strict sense of the word. It is possible to get partial information on which slit a quanton passes through and observe an interference pattern that is not sharp. We note that the quanton is here defined as a quantum entity that may be characterised by a wave and a particle nature [Bunge, 1973]. Therefore, the wave-particle duality in Bohr CP can be stated not just in terms of mutual exclusivity of purely particle and purely wave natures, but in terms of quantitative measures of these properties [Qureshi, 2016].

3 Wooters-Zurek conceptual experiment

Gradually, the CP was deeply studied as soon as the theory grew and improved the physicist ability about the interpretation of the microscopic new quantum world. For this reason, we will now discuss the Wooters and Zurek experimentally testable complementarity relation [Wootters and Zurek, 1979] and how it influenced the Bohr theory.

At the Solvay congress, Einstein suggested a Gedankenexperiment based on the well-known double slit experiment to reject the CP. According to Einstein, it was possible to evaluate the momentum of a double slit placed on a mobile support free to move up and down. Photons are deflected toward a given slit in a way that they impart a characteristic momentum to the mobile support. It was Einstein intention to determine the path of each photon without disturbing the interference pattern. The simultaneous knowledge of the path and the interference automatically involved the violation of the Bohr CP confirming a flaw in the theory. Bohr reinterpreted the Gedankenexperiment using the orthodox QM theory, used the Heisenberg uncertainty principle. In this manner one considered the slit as microscopic particles

At the end of the 70's, Wooters and Zurek studied the Einstein version of the double slit experiment, and choosing to face again the complementarity theme after the Bohr-Einstein debate. Wooters and Zurek were the first to overlook the Bohr principle as an aut-aut of the traditional quantum mindset, but rather as a principle able to link the two natures of the quantum entity at stake [Wootters and Zurek, 1979]. From this point of view, the wave behavior and the corpuscular one are not mutually exclusive but they have a complementary attitude adapt to fully describe the quanton.

From now on, it is possible to describe the Bohr principle by using the two characteristic properties of a quanton: the coherence, that is the ability to interfere, and the position, useful to describe corpuscular properties. In this regard, the duality relation for pure state was born, in it appears the interference visibility V, the quantity linked to the wave nature, and the distinguishability D, linked, instead, to the corpuscular nature, such as:

$$V^2 + D^2 \leq 1 \,. \tag{3}$$

It was possible to obtain partial information about the crossed slit and to observe a low-defined interference pattern [Qureshi, 2016]. But nowadays, can this relation be seen as still valid considering the new scientific discoveries? In the following Section, we will analyse the modern aspects connected to the concept of quantum duality and the entanglement.

4 Qian *et al.* experiment

In January 2020, in the Physical Review Research appeared an article titled *Turning off quantum duality* [Qian et al., 2020] in which it is again addressed the problem of duality wave-particle. Significantly, for the first time it is tested the concept of duality, expressed in Eq. (3), modifying a Young-type experiment where, apart from usual single pure-photon states, states describing photons interfering among them are included. More precisely, in this experiment were realised seven photon states, as reported in Table I. The states 1-3 are single-photon states whereas states 4-7 have been realised with a Mach-Zehnder interferometer able to produce single photon states that interact with each other. Moreover, the entanglement measure appears in the form of the concurrence C.

	V	D	C	$V^2 + D^2$	$V^2 + D^2 + C^2$
1	0.992	0.009	0.003	0.985	0.985±0.014
2	0.719	0.680	0.012	0.980	0.980±0.054
3	0.068	0.994	0.008	0.992	0.992±0.060
4	0.048	0.708	0.703	0.503	0.998±0.084
5	0.058	0.011	0.991	0.004	0.986±0.040
6	0.720	0.011	0.691	0.518	0.996±0.070
7	0.587	0.568	0.570	0.667	0.992±0.070

Table 1: Measured values of interference visibility V, the distinguishability D and of the concurrence C. The states are enumerated from 1 to 7. Data taken from Ref. [Qian et al., 2020].

Testing Eq. (3), they were able to confirm that the duality relation well describes the cases of one single photon characterised only by corpuscular properties when D=1, or only characterised by wave properties when V=1. Indeed, for the states 1-3 the column $V^2 + D^2$, that exemplifies Eq. (3), correctly describes experimental data giving a value near to 1. Nevertheless, for other states the duality seems to be weakened, and in one case, i. e. the state 5, it seems to be fully turned off giving a value of Eq. (3) practically vanishing. Hence, if we analyse the collected data we can infer that the duality relation is sufficient to describe single pure-photon states (1-3) but no states that interferes with themselves when the so-called entangled configurations (4-7) are involved.

We may now ask: how can we guarantee the validity of the duality relation for entangled states considering that Eq. (3) is not able to describe experimental data? The authors of the mentioned paper decided to encompass a quantity connected to the entanglement: the concurrence. Without contradicting what Wooters-Zurek reported, they improved Eq. (3) to entangled states preserving the basic concepts described in the previous Section. They converted the duality relation Eq. (3) in a triality identity where, in addition to interference visibility and distinguishability, appears the concurrence C:

$$V^2 + D^2 + C^2 = 1\,. \tag{4}$$

It is remarkable how in one relation we find two of the most counter-intuitive concepts of quantum mechanics, specifically the entanglement and the wave-particle duality. Considering this new point of view, it is now necessary to reconsider the experimental data.

Let us try to further comment on Table I focusing on last column which collects the values of Eq. (4) for all single considered states. Relation Eq. (4) confirms Eq. (3) for single photon pure-state where the entanglement is totally absent, as it can be inferred comparing the last column with the column $V^2 + D^2$ for states 1-3. Remarkably, Eq. (4) gives back a value equal to 1 even in those critical cases where duality is attenuated or turned off. Specifically, case 4, where the duality seems to be attenuated, corresponds to a weak entangled state, case 5, where the duality is completely turned off, corresponds to a fully entangled state. In this latter case, the measured value of concurrence is higher than the value of interference visibility and distinguishability. Therefore, due to the introduction of the entanglement, a consistency with experimental data has been recovered. These results show how the entanglement concept requires a modification of quantum duality, becoming a basic aspect similarly to quantum duality and CP.

5 Discussion and final remarks

It would be reasonable to ask for the following question: the entanglement must be seen in the same way as the basic principles of QM? We think that the entanglement, besides playing a fundamental role in QM, can be assumed as one of the basic concepts belonging to key principles of QM. The results presented and discussed in the previous Section have shown that a relationship constituted only by visibility and distinguishability is incomplete since the concurrence, and then the entanglement, is the missing piece for a complete formula, establishing the important role of the entanglement in the context of interference of quantum states. Therefore, the entanglement redefines the constraints related to quantum duality, and within this scheme, the state of a single particle would be seen as a sub-case of entangled state, as the one for which the state results separable.

Moreover, the development of communication and quantum computation are considering the entanglement as a real and powerful resource to be exploited [Ste]. There are indeed numerous applications, especially in the computer science field, as for example for quantum cryptography, where transmission protocols based on the presence of entangled states are implemented, since it is much more efficient and secure than classical protocols.

Furthermore, it has been shown that there exists a noteworthy connection between the uncertainty principle and the entanglement. The uncertainty principle mainly elucidates the correlation between the simultaneous measurement of two non-commuting observables, whereas the entanglement describes the correlations between simultaneous measurement of two or more commuting observables selected within a complete set of commuting observables for a given system. To this end, a suitable criterion of entanglement has proposed, which may be viewed as an extension of the uncertainty principle to many-particle or multi-degree of freedom system, further corroborating the relevance of the entanglement in QM frame [Zeng et al., 2013].

Recently, special attention has been also devoted to quantify the entanglement in real quantum systems. Within the condensed matter physics realm, spin chains have been largely investigated [Arnesen et al., 2001], due to their possible application to quantum computation. Concerning the experimental results, spectroscopic evidence for the development of entangled macroscopic quantum states has been provided in biased Josephson-junction qubits coupled to a capacitor [Berkley et al., 2003]. A signature of the entanglement is the identification of the so-called entanglement witnesses, which are observables having positive expectation values for separable states and negative ones for entangled states [Horodecki et al., 2001, Bourennane et al., 2004, Wu et al., 2005]. Using this tool, entanglement witnesses can be detected and quantified by means of measurement of magnetic susceptibility [Wiesniak, 2005, Brukner et al., 2006] and neutron diffraction scattering [Rappoport et al., 2007].

Looking at the entanglement involving spin and orbital degrees of freedom in transition metal oxides [Gotfryd et al., 2020], the spin-orbital entanglement manifests when a quantum many-body system with interacting spin and orbital degrees of freedom is split into the subsystems with separated degrees of freedom, and one is attempting to write interacting spin and orbital wave functions separately. Interestingly, the concept of spin-orbital entanglement could be connected to the so-called Goodenough-Kanamori rules [Goodenough, 1963, Kanamori, 1959]. It was indeed realised that this entanglement is crucial to understand the systems where spin and orbital variables are intertwined [Ole].

Therefore, from these remarks and keeping in mind the experimental results discussed in the previous Section IV, we can conclude that the answer to the title question definitely is: *YES, the entanglement is a novel, innovative, **basic** concept of QM.*

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