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The construction of quantum mechanics from electromagnetism. Theory and hydrogen atom

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Abstract

We reconstruct Quantum Mechanics in a way that harmonises with classical mechanics and electromagnetism, free from mysteries or paradoxes as the *collapse of the wave-function* or *Schrödinger's cat.* The construction is inspired by de Broglie's and Schrödinger's wave mechanics while the unifying principle is Hamilton's principle of least action, which separates natural laws from particular circumstances such as initial conditions and leads to the conservation of energy for isolated systems.

In Part I we construct the Quantum Mechanics of a charged unitary entity and prescribe the form in which the entity interacts with other charged entities and matter in general. In Part II we address the quantum mechanics of the hydrogen atom, testing the correctness and accuracy of the general description. The relation between electron and proton in the atom is described systematically in a construction that is free from analogies or ad-hoc derivations and it supersedes conventional Quantum Mechanics (whose equations linked to measurements can be recovered). We briefly discuss why the concept of isolation built in Schrödinger's time evolution is not acceptable and how it immediately results in the well known measurement paradoxes of quantum mechanics. We also discuss the epistemic grounds of the development and provide a criticism of instrumentalism, the leading philosophical perspective behind conventional Quantum Mechanics.

Keywords: Wave mechanics; Schrödinger's equation; Atomic energy levels; critical epistemology; instrumentalism; consilience; unity of thought; epistemic praxis;
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Introduction

Most physicists consider their science deals with the discovery of the laws of nature. However, during the early years of the XX century a most prominent group of physicists completely dropped this position and sustained that physics was merely a collection of useful formulae (see for example (Kragh, 1990)). Both groups of physicists would however agree that the relevant product of physics was the collection of formulae or plans which allowed them to make predictions. Actually, successful predictions, and insights, boosted technological applications justifying in front of the whole society the liberties granted to scientists. In short, predictive success was the key of the game called Science since late in the XIX century.

Under this philosophical perspective, called *instrumentalism* (a form of *utilitarianism*), the process of construction of theories –including their grounding– becomes less relevant and it is usually claimed to lie outside the science (for example in (Popper, 1959)) or is substituted by a narrative that allows students to accept the desired formulae with lesser intellectual effort (sometimes called "didactic transposition"). Consider as an example the widespread misconception about absolute space being fundamental for Newtonian mechanics. However, the philosophy that developed science since (at least) Galileo Galilei and until the second industrial revolution (around 1870) was rationalist and considered the rational construction of a theory a most relevant element in judging its quality. For the very few in physics that recognise that knowledge represents the encounter of the subject and the object and as such it carries elements of both of them, the rational construction of theories remains important and it is possible to suspect that, if we turn to the old philosophy of Science, some of the utilitarian theories would show defects or imperfections that demand attention.

A long-term project in this direction has been under development during the last ten years reconstructing Mechanics (Solari & Natiello, 2018) (and particularly recovering the concept of "inertial system" much distorted in textbooks (Solari & Natiello, 2021)), making explicit the dualist epistemology and the relevance of phenomenology for science (Solari & Natiello, 2022a) and furthermore, rescuing a few rules of reason that pervade the old construction of Science (Solari & Natiello, 2023). A second constructive step consisted in developing electromagnetism in terms of relational space (Solari & Natiello, 2022b), thus reclaiming Gauss' rational project, which was left aside in favour of the idea of the ether and subsequently forgotten under the utilitarian perspective. Having recovered an Electromagnetism that is in harmony with Classical Mechanics, we present now the construction of a Quantum Mechanics that is in harmony with Classical Mechanics and Electromagnetism.

Background

Under the instrumentalist guide, the period 1870-1930 witnessed the consolidation of Electromagnetic theory (EM) and the birth of Quantum Mechanics (QM). The success of both theories as predictive tools (and thus as mediators of technological development) can hardly be challenged. The interpretative part is often modelled resorting to mechanical analogies (Mach, 2012; Boltzmann, 1974), although a conflict arose with Electromagnetism since a mechanical (Galilean) framework and the constancy of the velocity of light appeared to be incompatible. The underlying model in Bohr's atomic theory is also mechanical, with a planetary electron revolving around the atomic nucleus. A conflict between theories becomes evident since an accelerated charge is expected to radiate, thus losing energy and collapsing towards the nucleus. The solution to the conflict was to negate it: In the atomic realm radiation is not present and electronic orbits are stable. In Bohr's atomic model, stable orbits have to satisfy a quantisation condition (that the angular momentum is an integer multiple of Planck's constant, $\hbar$) and electrons can only undergo transitions among these stable orbits. Nothing is said in the model about the existence of other states. Later, the Schrödinger model will assume that the possible states of the atom are linear combinations of stationary states and stable as well. In both models all possible atomic states are stable and radiation is suppressed. However, it is not possible to ascertain which is the state of the atom since measurements make the state to "collapse" into a stationary (measurable) energy state. This dogma is often indoctrinated by resource to Schrödinger's *cat paradox*. It would appear that the laws of EM are then different for microscopic and macroscopic distances. The possibility of encompassing EM and QM in one broader theory is hopeless in their currently accepted shape since EM and QM are contradictory in their foundations.

The EM that is acceptable for atomic physics displays a number of phenomena that are very difficult to explain. Consider e.g., spontaneous emission (overwhelmingly verified in Laser physics (Carmichael, 1999; Chow *et al.*, 1994)) or the reduction postulate of QM transforming the state of the system at the moment of measurement. A natural question arises: At what distance is it mandatory to change theories? When and how EM ceases being applicable and QM is a must? Are there other experimentally observable consequences? The theories now rest on elements which are not amenable to experimental examination and moreover, there are questions that cannot be asked.

EM also suffers instrumentalist constraints on its own. The current EM approach is rooted in the heritage of Hertz, being his preferred interpretation that the EM fields are associated to space (the ether for Hertz) and that forces surround matter [2]. Thus matter and EM fields have different ontogenesis. There

[2]The expression is rather mysterious, but it can be found also in Faraday (closer to reject it than to

exists "empty space", populated by fields (and subsequently by springs, strings and branes), along with a space occupied by matter (originally impenetrable). The realm of matter and with it atomic physics is divorced from the space where fields dwell (despite the field conception being rooted in the idea of force per unit charge). The link between both realms is given mostly by the Lorentz force, which appears as an addendum to Maxwell's equations displaying the Poincaré-Lorentz symmetry (Poincaré, 1906), after repeated drift in the meaning of the symbols. The velocity, that was relational in Maxwell and in experiments, became velocity relative to the ether and later relative to an inertial frame (Assis, 1994). The last two velocities have never been measured. For example, when analysing experiments concerning Doppler effect, the standard theory refers to a velocity with respect to an inertial system, but the measured velocity is a relative one. We will not dwell in these issues, the impossibility to physically contrast Special Relativity has been discussed by (Essen, 1971, 1978), while its logical inconsistency when regarded from an epistemology that encompasses the phenomenological moment has been recently shown (Solari & Natiello, 2022a). Strictly speaking, EM floats in a realm of ideas partially divorced from observations.

The possibility of unifying EM and QM under a conception that cleaves them in two separate entities became only possible by imposing the quantification of imagined entities that populate "empty space". This is the presently accepted approach, although their original proposers finally rejected it and despite their internal inconsistencies that would force the collapse of the underlying mathematical structure (Natiello & Solari, 2015).

Unfortunately, alternative approaches have little or no room in the scientific discussion. If the outcome in formulae of two theories is practically the same; instrumentalism may be inclined to choose the option with (apparently) less complications. Coming back to Newtonian mechanics, if the outcome elaborated under the assumption of absolute space to sustain the theory is the same as (a subset of) that developed under Newton's concept of "true motion" (Newton, 1687; Solari & Natiello, 2021) instrumentalists may find no benefit in presenting the second –more abstract– theory, to the price of accepting a conflicting concept that cannot be demonstrated[3]. We promptly recognise that in the academic formation of technical professionals the difference is usually not very problematic, while in the formation of experimental philosophers (or natural philosophers), to decline the historical cognitive truth for instrumentalist reasons appears as completely inappropriate and a safe source of future conflicts [4].

accept it) and in Maxwell (closer to accept it than to reject it).

[3]Along the same lines, textbooks do not hesitate in presenting a simple but incorrect derivation of the spin-orbit interaction along with an ad-hoc correction called Thomas precession, instead of using the approach that they consider "correct" (Kastberg, 2020, p. 60).

[4]The distinction between the formation of professionals and philosophers is in (Kant, 1798).

Instrumentalism is a weakened version of the phenomenological conception presented in (Solari & Natiello, 2022a). This latter approach is rooted in the work of C. Peirce (Peirce, 1994), W. Whewell (Whewell, 1840) and J. Goethe (Goethe, 2009) developed under the influence of rational humanism and nurtured by ideas of I. Kant (Kant, 1787) and G. Hegel (Hegel, 2001).

The ideas that allow us to traverse alternative paths were presented a long time ago but their development was abandoned half way through. Faraday observed that the more familiar ideas that privilege the concept of matter before the concept of action could be confronted with the approach that recognises that only actions can be detected while matter is subsequently inferred, and as such exposed to the errors belonging in conjectured entities. In this sense, Faraday recognises that matter can only be conceived in relation to action and what we call matter is the place where action is more intense [5]. Such matter extends to infinity and is interpenetrable. Faraday's vibrating rays theory eliminates the need to conceive an ether. The relational electromagnetism initially developed by W. Weber (Assis, 1994) under Gauss influence, was later developed by the Göttingen school and by L. Lorenz (Lorenz, 1867) introducing Gauss' concept of retarded action at distance. Maxwell expressed his surprise when a theory that was based in principles very different from his could provide a body of equations in such high accordance (Maxwell, 2003). From the Hertzian point of view both theories are basically "the same". However, when experiments later demonstrated the absence (nonexistence) of a material ether (Michelson & Morley, 1887) the rational approach of Faraday-Gauss-Lorenz was already forgotten, probably because the epistemological basis of science had moved towards instrumentalism.

The development of Quantum Mechanics continued from Bohr's model into Heisenberg's matrix computations, a completely instrumental approach devoid of a priori meaning (Born, 1955). While the ideas of de Broglie (de Broglie, 1923; De Broglie, 1924) and Schrödinger (Schrödinger, 1926) revolutionised the field, at the end of the Fifth Solvay conference (1927) the statistical interpretation of quantum theory collected most of the support (Schrödinger, 1995). It is said that

[5] We read in (Faraday, 1855, p.447):

> You are aware of the speculation which I some time since uttered respecting that view of the nature of matter which considers its ultimate atoms as centres of force, and not as so many little bodies surrounded by forces, the bodies being considered in the abstract as independent of the forces and capable of existing without them. In the latter view, these little particles have a definite form and a certain limited size; in the former view such is not the case, for that which represents size may be considered as extending to any distance to which the lines of force of the particle extend: the particle indeed is supposed to exist only by these forces, and where they are it is.

only Einstein and Schrödinger were not satisfied with it. Einstein foresaw conflicts with Relativity theory, later made explicit in the Einstein-Podolsky-Rosen paper (Einstein *et al.*, 1935). Schrödinger, after several years of meditation and several changes of ideas, came to the conclusion that the statistical interpretation was forced by the, not properly justified, assumption of electrons and quantum particles being represented by mathematical points (Schrödinger, 1995). Certainly, Schrödinger knew well that no statistical perspective was introduced in his construction of ondulatory mechanics. Actually, the statistical interpretation of quantum mechanics follows strictly Hertz philosophical attitude of accepting the formulae but feeling free to produce some argument that makes them acceptable. We have called this "the student's attitude" (Solari *et al.*, 2024) since it replicates the attitude of the student that is forced to accept the course perspective and has to find a form to legitimise the mandate of the authority: the result is given and the justification (at the level of plausibility) has to be provided.

The standard QM then has to find a meaning for its formulae and such task has been called "interpretation of QM". All the interpretations that we know about, rest on the idea of point particles. An idea that allows for the immediate creation of classical images that suggest new formulae such as those entering in the atomic spectra of the Hydrogen atom. Since the accelerated electron (and proton) cannot radiate in QM by ukase[6] the radiation rule has to be set as a separated rule. The atom, according to the standard theory, can then be in an "mixed energy" state until it is measured (meaning to put it in contact with the Laboratory) which makes the state represented by the wave-function to collapse (in the simplest versions). While there are more sophisticated[7]interpretations, none of them incorporates the measurement process to QM. It remains outside QM.

The idea of isolating an atom deserves examination. One can regard as possible to set up a region of space in which EM influences from outside the region can be compensated to produce an effective zero influence. In the simplest form we think of a Faraday cage and perhaps similar cages shielding magnetic fields. In the limit, idealised following Galileo, we have an atom isolated from external influences. But this procedure is only one half of what is needed. We need the atom to be unable to influence the laboratory, including the Faraday cage. And we cannot do anything onto the atom because so doing would imply it is not isolated. This is, we must rest upon the voluntary cooperation of the atom to have it isolated. The notion of an isolated atom is thus shown to be a fantasy. If the atom, as the EM system that it is, is in the condition of producing radiation, it will produce it whether the Faraday cage is in place or not. The condition imposed by

[6]ukase: edict of the tsar

[7]We imply the double meaning as elaborated and false. The first as current use and the second implied in the etymological root "sophism" in the ancient Greek: σοφιστικός (sophistikós), latin: sofisticus.

the ukase is metaphysical (i.e., not physical, nor the limit of physical situations). Actually, if such condition is imposed to Hamilton's principle, the Schrödinger's time-evolution equation is recovered to the price that the consilience with Electromagnetism and Classical Mechanics is lost.

It was Schrödinger who dug more deeply into the epistemological problems of QM. His cat, now routinely killed in every QM course/book was a form of ridiculing the, socially accepted, statistical interpretation of QM:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.(Schrödinger, 1980)

Schrödinger understood as well that the need for a statistical interpretation was rooted in the assumption of point particles (Schrödinger, 1995).

In the present work, the conservation of energy of a quantum particle is described in the form in which Faraday conceived matter as the inferred part of the matter-action pair and it leads directly to the transition rule: $\mathcal{E}_I^f+\mathcal{E}_{EM}^f = \mathcal{E}_I^i+\mathcal{E}_{EM}^i$ with i, f meaning initial and final states while I, EM mean internal and EM energies. In the early times of QM, Bohr showed that the Balmer and Rydberg lines of the Hydrogen atom corresponded to this rule. This is to say that the observed values summarised in Rydberg's formula agree in value with the equation, yet the equation was not part of the theory because the "emission" of light was conceived as the result of external influences since the isolated atom could not decay and in such way it would not collapse with the electron falling onto the proton.

Cognitive surpass as a constructive concept

The concept of *cognitive surpass* (Piaget & García, 1989) is a guiding idea of the present work. It is a step in the development of our cognition by which what was previously viewed as diverse is recognised as a particular occurrence of a more general, abstract form. This cognitive step is dialectic in nature and displays the dialectic relation between particular and general. Cognitive surpass involves abstraction and/or generalisation, it does not suppress the preexisting theories but rather incorporates them as particular instances of a new, more general theory. It works in the direction of the unity of science(Cat, 2024) and as such, its motion opposes the specialism characteristic of the instrumentalist epistemology. Cognitive surpass is a key ingredient of the abductive process since it is *ampliative* (i.e., it enlarges the cognitive basis) as opposed to the introduction of ad-hoc hypotheses to compensate for unexplained observations.

We notice that current physics hopes for a unified theory but little to no progress has been made in such direction. This matter should not come as a surprise since the requirements for understanding were downgraded to the level of analogy when adopting instrumentalism (see (Boltzmann, 1974; Mach, 2012)). Analogies hint us for surpass opportunities connecting facts at the same level, without producing the surpass. For example, Quantum Electrodynamics superimposes quantification to electromagnetism making mechanical analogies and populating an immaterial ether, the space, with immaterial analogous of classical entities (e.g., *springs*). What is preserved in this case is the analogy with classical material bodies, the recourse to imagination/fantasy.

Plan of this work

In Part I of this work we start from the initial insight by de Broglie regarding the relation of quantum mechanics and electromagnetism as Schrödinger did, yet we soon depart from de Broglie as he believed that Special Relativity was the keystone of electromagnetism. Rather, we explore the connection with relational electromagnetism. We also depart from Schrödinger's developments since we will not explore mechanical analogies neither in meaning nor in formulae. Actually, we show that de Broglie's starting point leads directly to a rewriting of Electromagnetism producing a cognitive surpass of the Lorenz-Lorentz variational formulation (Theorem 3.2, Solari & Natiello, 2022b) which is behind Maxwell's equations as propagation in the vacuum of fields with sources (intensification) in the matter. The same variational principle produces the Lorentz Force. We show that the variational principle that produces Schrödinger's evolution equation is the complementary part of one, unifying, variational principle. The synthesis produced corresponds to the unification of the quantum entity in the terms proposed by Faraday (action-matter duality) being the entity not a mathematical point but rather extended in space in a small intensification region and reaching as far as its action reaches. We further show that the classical motion under a Lorentz force corresponds to particular variations of the action. The classical dynamical equation for the angular momentum is obtained by the same methods, as well as the conservation of charge and the total energy (that of the localised entity and its associated electromagnetic field). It is shown as well that non-radiative states correspond to those of Schrödinger's equation, matching Bohr's primitive intuition. In this construction there is no room for free interpretations; measurements correspond to interactions between the entity and the laboratory described according to the laws of electromagnetism, hence the measurement process is completely within the formulation and requires no further postulates.

In Part II we show how this conception is extended to a proton and electron interacting to form an Hydrogen atom whose energy levels calculated by a systematic computation of interactions, require no patching elements such as gyromagnetic factors or "relativistic corrections".

The QM we construct allows for a better understanding of the so-called spin-orbit interaction, which in standard QM couples the spin and the "orbit" of the electron, being then a "self-interaction" (an oxymoron) of the electron. We show that when properly and systematically considered it represents a coupling between the proton and the electron as all other couplings considered.

Part I
The theory

1 Abduction of quantum (wave) mechanics

We begin by highlighting how QM enters the scene as a requirement of consistence with electromagnetism. Consider a point-charge, q, moving at constant velocity v with respect to a system that acts as a reference. We associate with this charge the relational charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \delta(x - vt) = \begin{pmatrix} C\rho \\ j \end{pmatrix} (x, t).$$

The association of current with charge-times-velocity goes back to Lorentz, Weber and possibly further back in time.

The velocity can then be written as the quotient of the amplitude of the current divided the charge, $\beta = \dfrac{qv}{Cq}$. At the same time, Lagrangian mechanics associates the velocity with the momentum, p, of a free moving body by a relation of the form, $\beta = \dfrac{p}{Cm}$. We propose that charge and mass are constitutive attributes of microscopic systems and hence

$$\begin{pmatrix} mC \\ p \end{pmatrix} = \frac{m}{q} \begin{pmatrix} qC \\ qv \end{pmatrix}$$

will behave as the charge-current density when intervening in electromagnetic relations. [8]

If rather than a point particle we accept an extended body moving at constant velocity without deforming and represented by a charge-current density

$$\begin{pmatrix} Cq \\ qv \end{pmatrix} \rho(x,t)$$

where $\int d^3x\, \rho(x,t) = 1$, we find that the conditions posed above can be satisfied whenever

$$\rho(x,t) \propto \int d^3k\, \phi(k) \exp(i(k \cdot x - wt))$$

with $w = f(k)$. If we further associate the frequency with the (kinetic) energy in the form $E = \hbar w$, as de Broglie did (De Broglie, 1924), and write $p = \hbar k$ we have

$$E \;=\; \hbar f(\frac{p}{\hbar}).$$

Since we expect $\rho(x,t)$ to be real, it is clear that this association is not enough to rule out the unobservable negative energies, since
$\rho = \rho^* \Leftrightarrow \phi(k) \exp(i(k \cdot x - \frac{E}{\hbar}t)) = \phi^*(k) \exp(i((-k) \cdot x - (-\frac{E}{\hbar})t))$. Negative energies would be required and they would rest on an equal footing as positive

[8]A sort of myth emerges at this point. Lorentz (Lorentz, 1904) tried to fix the electromagnetism conceived by consideration of the ether. After lengthy mathematical manipulations of Maxwell's equations he arrived to the point in which the force exerted in the direction of motion held a different relation with the change of mechanical momentum than the force exerted in the perpendicular direction. He then substituted a constant velocity introduced as a change of reference frame with the time-dependent velocity of the electron with respect to the ether. This is a classical utilitarian approach, the formula is applied outside its scope although the mathematics that led to it does not hold true if the velocity is not constant. This is the origin of Lorentz' mass formula as applied for example to the Kauffman's experiment. The same utilitarian operation is used to produce the relativistic mass formulae. However, if this new mass were to be used in the derivation of Lorentz' force (Lorentz, 1892), the usual expression of the force would not hold. The story illustrates well the "advantages" of the utilitarian approach: it frees the scientist of requirements of consistency. In contrast, relational electromagnetism acknowledges that when in motion relative to the source we perceive EM fields still as EM fields but with characteristic changes as it is made evident by the Doppler effect. This allows to propose the form of the perceived fields when in motion at a constant relative velocity with respect to the source (Solari & Natiello, 2022b).

ones. Introducing the quantum-mechanical wave-function ψ satisfying

$$\begin{aligned}
\psi(x,t) &= \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k\, \Phi^*(k) \exp(\frac{i}{\hbar}(p\cdot x - Et)) \\
&= \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int d^3p\, \Phi^*(\frac{p}{\hbar}) \exp(\frac{i}{\hbar}(p\cdot x - Et)) \qquad (1) \\
\rho(x,t) &= q\, \psi(x,t)^* \psi(x,t)
\end{aligned}$$

we automatically obtain a real charge-density without the imposition of negative energies (where for free particles $E \equiv T$, the kinetic energy, whose expression will be derived below).

We retain from traditional Quantum Mechanics the idea of expressing physical observables as real valued quantities associated to an *operator* integrated over the whole entity.

We recall that the quantum entity is a unity and *not* a mathematical point. Hence, objects like $(\psi^* \, (-i\hbar\nabla\psi))$ do not necessarily have physical (measurable) meaning. Further, observables depending only on x are multiplicative operators in front of $\psi(x,t)$. For the case of the position operator x we have [9],

$$\begin{aligned}
\langle\hat{x}\rangle &= \int d^3x\, \psi^*(x,t)\, x\, \psi(x,t) \\
&= \int d^3x\, d^3p\, d^3p'\, \frac{1}{(2\pi\hbar)^3} \Phi^*(p') \exp(\frac{i}{\hbar}(p'\cdot x - Et))\, x\, \Phi(p) \exp(-\frac{i}{\hbar}(p\cdot x - Et)) \\
&= \int d^3p\, d^3p'\, \Phi^*(p')\, \left(i\hbar\nabla_p\, \Phi(p)\right) \int d^3x\, \frac{1}{(2\pi\hbar)^3} \exp(\frac{i}{\hbar}(p-p')\cdot x) \\
&= \int d^3p\, \Phi^*(p)\, (i\hbar\nabla_p \Phi(p))
\end{aligned}$$

while[10]

$$\hat{E} = i\hbar\partial_t.$$

De Broglie's proposal and eq.(1) relate x and p through a Fourier transform and hence the momentum operator becomes $\hat{p} = m\hat{v} = -i\hbar\nabla$. Thus, in the case of a free charged particle we arrive to Schrödinger's equation

$$i\hbar\partial_t\psi = \hat{E}\psi. \qquad (2)$$

[9]Integrals such as $\langle\psi^* x\psi\rangle = \int d^3x\,(\psi^* x\psi)$ or $\langle\psi^*(-i\hbar\nabla\psi)\rangle = \int d^3x\,(\psi^*(-i\hbar\nabla\psi))$ have been called *expectation values* in the traditional probability interpretation of Quantum Mechanics.

[10]Throughout this work the notations $\partial_t\psi$, $\dot{\psi}$ and $\psi_{,t}$ all denote the same concept, namely partial derivative with respect to time.

We have yet to find $f(k)$. If we consider that eq.(2) is the definition of the Hamiltonian operator $\hat{H} \equiv \hat{E}$, and recall that the operator associated to the current density is $q\hat{v} = \frac{q}{m}\hat{p}$. Hence, the conservation of charge, also known as the *continuity equation*, $\partial_t \rho + \nabla \cdot j = 0$, produces the relations

$$\begin{aligned} q\partial_t \rho &= q\left[\psi^* \partial_t \psi + \partial_t \psi^* \psi\right] = \frac{q}{i\hbar}\left[\psi^*(\hat{H}\psi) - (\hat{H}\psi)^*\psi\right] \\ &= -\nabla \cdot \frac{1}{2}\frac{q}{m}\left[\psi^* \hat{p}\psi + (\hat{p}\psi)^*\psi\right] \end{aligned}$$

and expanding the r.h.s. we can see that the only solution is:

$$\begin{aligned} \hat{p} &= -i\hbar\nabla \\ \hat{H} &= \frac{\hat{p}^2}{2m} + U(x) = -\frac{\hbar^2}{2m}\Delta + U(x) \\ j &= \frac{q}{2m}\left(\psi^*\left(-i\hbar\nabla\psi\right) + \left(-i\hbar\nabla\psi\right)^*\psi\right) = -\frac{i\hbar q}{2m}\left(\psi^*\left(\nabla\psi\right) - \left(\nabla\psi\right)^*\psi\right) \end{aligned}$$

where $U(x)$ is any scalar function of the position. We adopt $U(x) = 0$ as for a free-particle there is no justification for a potential energy. We have then found the free-particle Hamiltonian consistent with de Broglie's approach and the assumption $\rho(x,t) = |\psi(x,t)|^2$.[11]

In summary, QM results from requirements of compatibility with electromagnetism following an abduction process in which the hypothesis $E = \hbar w$ results as an extension of the photoelectric relation. As a consequence, any attempt to superimpose further formal relations with electromagnetism such as requiring invariance or equivariance of forms under the Poincaré-Lorentz group is bound to introduce more problems than solutions.

1.1 Expression of the electromagnetic potentials

1.1.1 Physical Background

In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics. Hence, we will compute Maxwell's EM interaction energy, either between the microscopic system and external EM-fields or between different fields arising within the system. We display first some classical results that are relevant for the coming computations.

[11] In his original work, de Broglie (De Broglie, 1924) assumed the relation, $E^2 = (m^2C^2 + p^2)C^2$ proposed by special relativity on an utilitarian basis as already commented. It was later Schrödinger (Schrödinger, 1926) who proposed the classical form based on the correspondence with Newton's mechanics.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2 \right]. \tag{3}$$

The classical EM potentials originating in a charge density $|\psi|^2$ and a current $-\frac{i\hbar q}{2m} \left(\psi^*(\nabla_y \psi) - (\nabla_y \psi)\psi\right)(y,t)$, as defined in the Lorenz gauge, read

$$V(x,t) \quad = \quad \frac{q}{4\pi\epsilon_0} \int d^3y \, \frac{1}{|x-y|} \, |\psi(y, t - |x-y|/C)|^2$$

for the scalar potential (where as usual $C^2 = (\mu_0 \epsilon_0)^{-1}$), while the vector potential is:

$$A(x,t) \quad = -\frac{i\hbar}{2} \quad \frac{\mu_0 q}{4\pi m} \int d^3y \, \frac{1}{|x-y|} \left(\psi^*(\nabla_y \psi) - (\nabla_y \psi)\psi\right)(y, t - |x-y|/C).$$

Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$. We recall from Maxwell (see e.g. [630] Maxwell, 1873,) that all electromagnetic quantities and the wave-function are assumed to vanish sufficiently fast at infinity. This allows for the use of partial integrations and Gauss' theorem whenever appropriate.

1.1.2 Action integral and the least action principle

Lorentz' electromagnetic action (Lorentz, 1892) is

$$\begin{aligned} \mathcal{A}_{EM} &= \int_{t_1}^{t_2} \{\mathcal{L}_{EM}\} \, ds \qquad (4) \\ \mathcal{L}_{EM} &= \frac{1}{2} \int d^3x \left\{ \frac{1}{\mu_0} B^2 - \epsilon_0 E^2 \right\} = \frac{1}{2} \int d^3x \, \{A \cdot j - V \cdot \rho\} \end{aligned}$$

Equivalence between the field and potential forms is assured by the followin

Lemma 1.1. *Up to an overall function of time and the divergence of a function vanishing sufficiently fast at infinity, the electromagnetic action satisfies*

$$\mathcal{A} \quad = \quad \frac{1}{2} \int_{t_0}^{t_1} dt \int \left(\frac{1}{\mu_0} B^2 - \epsilon_0 E^2 \right) d^3x = \frac{1}{2} \int_{t_0}^{t_1} dt \int (A \cdot j - \rho V) \, d^3x$$

See Appendix A for a proof.

Unfortunately, Lorentz worked from his expressions in a mixed form using mathematics and intuitions generated by the hypothesis regarding the existence of a material medium permeating everything: the ether. It is from these mathematical objects that he derives the electromagnetic force. Since the original derivation is marred by the ether, the force was adopted as an addition to the collection formed by Maxwell's equations and the equation of continuity.

Contrasting to Lorentz, Lorenz (Lorenz, 1867) related light to electromagnetic currents in a body, being the propagation in the form of waves as he had previously established by experimental methods (Lorenz, 1861)[12]. His idea can be expressed in terms of a variation of the action 4 with respect to the potentials where the potentials have to match the values in the region of their source. We have called this result "the Lorenz-Lorentz theorem" (Theorem 3.2 Solari & Natiello, 2022b). The variation accounts for the propagation of electromagnetic fields and imposes the condition of conservation of charge as a result ultimately produced by the observations on the propagation of the electromagnetic action. The same action 4 when varied as Lorentz proposed with respect to the charge-current reveals the Lorentz force. The details of the derivation can be found in (Natiello & Solari, 2021).

Lorentz's action accounts for the propagation of electromagnetic effects but not for the sources, in contrast Schrödinger's equation for the free charge can be derived from a variational principle

$$\mathcal{A}_{QM} = \int_{t_0}^{t_1} ds \left\langle \frac{1}{2}\left(\psi^* \left(i\hbar\dot{\psi}\right) + (i\hbar\dot{\psi})^*\psi\right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi \right\rangle$$

where $\langle\rangle$ indicates integration in all space and Δ is Laplace's operator. The Schrödinger equation is recovered when the QM-action is varied with respect to

[12]Notice that Lorenz ideas were published two years before and two years after Maxwell seminal paper (Maxwell, 1865) and at least ten years before Hertz' famous experiments of 1887-1888. The historical narrative that reaches us attributing to Maxwell and Hertz this association is historically incorrect as it is incorrect the often stated opinion (for example in Lorentz (Lorentz, 1892)) that Faraday supported the idea of the ether when actually he had warned about the risks of inventing metaphysical objects and has provided the alternative of his theory of "vibrating rays" (Faraday, 1855).

Lorenz, Faraday as much as Newton, were ahead of their times and those that followed recast their learnings in forms apt for the reader to understand (called "didactic transposition" when teaching). A first example of misreading Faraday is precisely Maxwell (V. ii, p. 177 [529] Maxwell, 1873,) who wrote about Faraday's idea: "He even speaks * of the lines of force belonging to a body as in some sense part of itself, so that in its action on distant bodies it cannot be said to act where it is not. This, however, is not a dominant idea with Faraday. I think he would rather have said that the field of space is full of lines of force, whose arrangement depends on that of the bodies in the field, and that the mechanical and electrical action on each body is determined by the lines which abut on it. * (p, 447; Faraday, 1855)" . Goethe has prevented us: "You don't have to have seen or experienced everything for yourself; but if you want to trust another person and his descriptions, remember that you are now dealing with three factors: with the matter itself and two subjects." (570, von Goethe, 1832).

ψ (or ψ^*):

Lemma 1.2. *$\mathcal{A}_{QM}$ is stationary with respect to variations of ψ^* such that $\delta\psi^*(t_0) = 0 = \delta\psi^*(t_1)$ if and only if $i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi$.*

The hope is then that by adding both actions, $\mathcal{A}_{EM}$ that almost ignores the dynamics of the sources and $\mathcal{A}_{QM}$ that ignores the associated fields we can propose a synthetic action integral

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_{EM} + \mathcal{A}_{QM} \\ &= \int_{t_0}^{t_1} ds \left\langle \frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right) + (i\hbar\dot{\psi})^*\psi\right) + \frac{\hbar^2}{2m}\psi^*\Delta\psi + \frac{1}{2}\left(\frac{1}{\mu_0}B^2 - \epsilon_0 E^2\right)\right\rangle \quad (5)\end{aligned}$$

no longer separating effects of sources, since, as stated in subsection 1.1, in the present construction all of current, charge-density and electromagnetic potentials depend of the wave-function ψ. Hamilton's variational principle applied to the action integral $\mathcal{A}$ yields all the dynamical equations in EM, and QM.

1.2 Variations of the action

The variations we are going to consider will be the product of an infinitesimal real function of time denoted ϵ when it is a scalar and $\bar{\epsilon}$ (a vector of arbitrary constant direction, $\bar{\epsilon} = \epsilon k, \epsilon r, \epsilon\Omega$) when it refers to a wave number, position or solid-angle respectively; times a function μ often of the form $\mu = \hat{O}\psi$ for some appropriate operator $\hat{O}$. The variation of the wave-function is reflected directly in variations on the charge-density, current, and fields. We will use for them the symbols $\delta\rho, \delta j, \delta E, \delta B$ for a set of variations that satisfy Maxwell's equations.

Hence, according to the previous paragraph, the variation $\delta\psi = \epsilon\mu$ is reflected in a variation of the EM quantities as follows:

1. Variation of charge-density: $\delta\rho = \epsilon q\left(\mu^*\psi + \psi^*\mu\right) \equiv \epsilon\mathfrak{X}$.

2. Variation of current: $\delta j = \epsilon\mathfrak{J}_1 + \dot{\epsilon}\mathfrak{J}_2$. We will discuss in due time the connection between $\delta\psi$ and δj.

The continuity equation then reads

$$(\delta\rho)_{,t} + \nabla\cdot\delta j = 0,$$

which discloses into

$$\begin{aligned}\mathfrak{X} + \nabla\cdot\mathfrak{J}_2 &= 0 \\ \mathfrak{X}_{,t} + \nabla\cdot\mathfrak{J}_1 &= 0\end{aligned}$$

thus demanding $\nabla\cdot\mathfrak{J}_1-\nabla\cdot\mathfrak{J}_{2,t}=0$ and suggesting $\mathfrak{J}_2=-\epsilon_0\mathfrak{E}$, with $\epsilon_0\nabla\cdot\mathfrak{E}=\mathfrak{X}$ for some electrical field that needs yet further specification.

We develop here the mathematical results of this manuscript. Proofs can be found in Appendix A.

Lemma 1.3. *(Preparatory lemma). Under the previous definitions and the conditions $\mathfrak{X}+\nabla\cdot\mathfrak{J}_2=0$ and $(\mathfrak{X})_{,t}+\nabla\cdot\mathfrak{J}_1=0$, complying with the continuity equation, a variation $\delta\psi=\epsilon\mu$ of the wave-function produces the varied action integral*

$$\begin{aligned}\delta\mathcal{A} &= \delta\int_{t_0}^{t_1} ds\left\{\left\langle\frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle+\frac{1}{2}\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle\right\}\\ &= \int_{t_0}^{t_1}\epsilon\, ds\left\langle\left(\mu^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\mu\right)+\frac{\hbar^2}{2m}\left(\mu^*\Delta\psi+\mu\Delta\psi^*\right)\right\rangle+\\ &\quad+\int_{t_0}^{t_1}\epsilon\, ds\left\langle A\cdot\mathfrak{J}_1-(A\cdot\mathfrak{J}_2)_{,t}-V\mathfrak{X}\right\rangle,\end{aligned}$$

where the fields are denoted as $B=B_m+B_r$ and $E=E_m+E_r$, with the index m corresponding to the microscopic system to be varied and r denotes the "outside world". A,V are the electromagnetic potentials.

Corollary 1.1. *Without loss of generality, we set $\mathfrak{J}_2=-\epsilon_0\mathfrak{E}$, where $\epsilon_0\nabla\cdot\mathfrak{E}=\mathfrak{X}$.*

We will address different relations in a few related results which are stated separately because of expository convenience. The details of the proofs will be displayed in Appendix A.

Theorem 1.1. *(Classical variations theorem). Under the assumptions of Lemma 1.3, the null variation of the action*

$$\mathcal{A}=\int_{t_1}^{t_2} ds\left(\left\langle\frac{1}{2}\left(\psi^*\left(i\hbar\dot{\psi}\right)+(i\hbar\dot{\psi})^*\psi\right)+\frac{\hbar^2}{2m}\psi^*\Delta\psi\right\rangle+\frac{1}{2}\left\langle\frac{1}{\mu_0}B^2-\epsilon_0E^2\right\rangle\right)$$

as a consequence of a variation of the wave-function as:

$$\delta\psi=\epsilon\psi_{,t},\ (i\epsilon k\cdot x)\,\psi,\ (\epsilon\bar{x}\cdot\nabla\psi)\,,\ (\epsilon\Omega\cdot(x\times\nabla\psi))$$

–as summarised in Table 1– results in the corresponding equations of motion displayed in Table 2.

The entries in table 1 were produced following intuitions based upon the notion of conjugated variables in classical mechanics. A displacement in time is associated with the law of conservation of energy, one in space with the law of variation of linear momentum and an angular displacement with angular momentum. The variation leading to the derivative of the position is of an abductive

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{I}_1$	$\mathfrak{I}_2$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$-2\epsilon_0\mathfrak{E}_{,t} - \mathfrak{j} = j_{,t}$	j
position	$(i\epsilon k\cdot x)\,\psi$	0	0	0
displacement	$\epsilon r\cdot\nabla\psi$	$r\cdot\nabla\rho$	$(r\cdot\nabla)\,j = \mathfrak{j}$	$-r\rho$
rotation	$\epsilon\Omega\cdot(x\times\nabla\psi)$	$\Omega\cdot(x\times\nabla\rho)$	$(\Omega\cdot(x\times\nabla))\,j - \Omega\times j = \mathfrak{j}$	$-\Omega\times(x\rho)$

Table 1: Variations

Variation/result	$\delta\psi$	Equation of motion
time/energy	$\epsilon\partial_t\psi$	$\partial_t\left\langle -\frac{\hbar^2}{2m}\psi^*\Delta\psi + \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right\rangle = 0$
position/velocity	$(i\epsilon k\cdot x)\,\psi$	$\partial_t\langle\psi^* x\psi\rangle = \dfrac{\langle j\rangle}{q}$
displacement/momentum	$\epsilon r\cdot\nabla\psi$	$\dfrac{m}{q}\partial_t\langle j\rangle = \langle j\times B + \rho E\rangle$
rotation/angular momentum	$\epsilon\Omega\cdot(x\times\nabla\psi)$	$\dfrac{m}{q}\partial_t\langle x\times j\rangle = \langle x\times(\rho E + j\times B)\rangle$

Table 2: Equations of motion.

nature since what it is really known is the cinematic relation. Hence, the variation is obtained as the answer to the question: which variation produces the cinematic relation $\langle x\rangle_{,t} = \langle j\rangle/q$?

The proposed variations do not follow a general pattern. Hence, we expect that there might be alternative forms of variations that lead to identical results which in addition present consilience. Moreover, we expect that the energy would take the place of the Lagrangian since this is one of the intuitions based upon experimental observations that lead to quantum mechanics. We present the result of this synthetic steps in the next theorem.

There are four matters that have to be put together to complete the construction of Quantum Mechanics:

1. Changing the Lagrangian presentation into a Hamiltonian one.

2. Showing that the Hamiltonian presentation has consilience and produces the particular results of Theorem 1.1 as well (it is a generalisation/correction of the particular cases shown).

3. Showing that this setting produces the usual eigenvalue-eigenvector equations of Quantum Mechanics for the stationary wave-functions.

4. Making more explicit the relation between the variation of the wave-function and the variation of the electromagnetic current.

We state these results as three Lemmas and a Corollary.

Variation	$\delta\psi$	$\mathfrak{X}$	$\mathfrak{j}$	$\epsilon_0\mathfrak{E}$	$\nabla\times\mathfrak{B}$
time	$\epsilon\partial_t\psi$	$\rho_{,t}$	$j_{,t}$	$E_{,t}$	$\nabla\times B_{,t}$
position	$(i\epsilon k\cdot x)\,\psi$	0	0	0	0
displacement	$\epsilon r\cdot\nabla\psi$	$(r\cdot\nabla)\,\rho$	$(r\cdot\nabla)\,j$	$r\rho$	$-\nabla\times(r\times j)$
rotation	$\epsilon\Omega\cdot(x\times\nabla\psi)$	$\Omega\cdot(x\times\nabla\rho)$	$(\Omega\cdot(x\times\nabla))\,j-\Omega\times j$	$\Omega\times(x\rho)$	$-\nabla\times((\Omega\times x)\times j)$

Table 3: Variations in Hamiltonian form

Lemma 1.4. *(Hamiltonian form) Let the set of functions $\mathfrak{X},\mathfrak{j},\mathfrak{E},\mathfrak{B}$ satisfy Maxwell equations with $\delta\rho=\epsilon\mathfrak{X}=\epsilon q\,(\mu^*\psi+\psi^*\mu)$ as in Lemma 1.3 . If we let the current be $\delta j=\epsilon\mathfrak{J}_1+\dot{\epsilon}\mathfrak{J}_2$, with*

$$\begin{aligned}\mathfrak{J}_2 &= -\epsilon_0\mathfrak{E}\\ \mathfrak{J}_1 &= -2\epsilon_0\mathfrak{E}_{,t}-\mathfrak{j}\end{aligned}$$

the null variation of the action integral satisfies

$$\left\langle\mu^*\left(i\hbar\dot{\psi}\right)+\left(i\hbar\dot{\psi}\right)^*\mu\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle-\left\langle\frac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle=0, \tag{6}$$

where $-\dfrac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi+\mu\Delta\psi^*\right\rangle+\left\langle\dfrac{1}{\mu_0}B\cdot\mathfrak{B}+\epsilon_0E\cdot\mathfrak{E}\right\rangle\equiv\mathcal{H}$ is recognised as the Hamiltonian of the problem.

Corollary 1.2. *The form of the variations appropriated to the Hamiltonian form corresponding to the results of Theorem 1.1 (Table 2) is the one given in Table 3.*

Lemma 1.5. *(Variation of a stationary state). In the special case when ψ corresponds to a stationary state, $\psi(x,t)=\exp(-i\omega t)\,\psi(x,0)\equiv\exp(-i\omega t)\,\psi_\omega$ and assuming that the external electromagnetic fields are time-independent, and B is homogeneous, the variation in Lemma 1.4 yields*

$$\left\langle\mu^*\left(\hbar\omega\psi_\omega\right)\right\rangle+\frac{\hbar^2}{2m}\left\langle\mu^*\Delta\psi_\omega\right\rangle-\left\langle\mu^*\hat{H}_{EM}\psi_\omega\right\rangle=0 \tag{7}$$

where

$$\hat{H}_{EM}\psi=V\psi-\left(\frac{q}{2m}\right)\left(B\cdot\left(x\times(-i\hbar\nabla)\right)\psi\right)$$

Eq.(7) *gives the Schrödinger equation for stationary states.*

The original expression by Schrödinger was developed setting $B=0$ and assuming that the electron is the quantum entity while the proton is an "external" point charge, responsible for the electrostatic potential V.

In Theorem 1.1 as well as in Corollary 1.2 we rested on phenomenological ideas (synthetic judgements) to produce the current associated to the variation. To find a general form for a set of intuitions appears as almost impossible. We proposed in the case of the rotations $\mathfrak{j} = (\Omega \cdot (x \times \nabla))\, j - \Omega \times j$ because the rotation of vectors demands to change the spatial argument as well as the orientation of the components. The confirmation of the abductive hypothesis came from the result, which has an intuitively appealing form. On the contrary, had we forgotten the term $-\Omega \times j$ we would have ended up with a result with no consilience with that of classical mechanics. We stress that these are rational arguments which are not of a deductive nature. Nevertheless, there are some mathematical relations that have to be satisfied since the continuity equation must be enforced.

Lemma 1.6. *The variations $\hat{O}\psi$ (with $\hat{O}_{,t} = 0$) that may produce a varied current $\mathfrak{j}$ fulfilling the conditions of Lemma 1.3 extend as real operators Θ acting on forms such as $(\rho,\ j,\ \nabla \cdot j)$ that can be regarded as real-valued functions of (ψ^*, ψ) as:*

$$\begin{aligned}
\Theta\rho\left(\psi^*, \psi\right) &= \rho\left(\left(\hat{O}\psi\right)^*, \psi\right) + \rho\left(\psi^*, \left(\hat{O}\psi\right)\right) \\
\Theta j\left(\psi^*, \psi, \nabla\psi^*, \nabla\psi\right) &= j\left(\left(\hat{O}\psi\right)^*, \psi, \left(\hat{O}\nabla\psi\right)^*, \nabla\psi\right) + j\left(\psi^*, \hat{O}\psi, \nabla\psi^*, \left(\hat{O}\nabla\psi\right)\right) \\
\Theta\left(\nabla \cdot j\left(\psi^*, \psi, \Delta\psi^*, \Delta\psi\right)\right) &= j\left(\left(\hat{O}\psi\right)^*, \psi, \left(\hat{O}\Delta\psi\right)^*, \Delta\psi\right) + j\left(\psi^*, \left(\hat{O}\psi\right), \Delta\psi^*, \left(\hat{O}\Delta\psi\right)\right).
\end{aligned}$$

Then, Θ preserves the continuity equation as $\Theta\left(\rho_{,t} + \nabla \cdot j\right) = 0$ and $\Theta(\nabla \cdot j) = \nabla \cdot \mathfrak{j}$ when the variation satisfies the relation

$$\psi^*\left(\left[\Delta, \hat{O}\right]\psi\right) = \psi^*\left(\left[\Delta, \hat{O}\right]^\dagger \psi\right) + \nabla \cdot K$$

(where $\dagger$ indicates Hermitian conjugation) for some vector K satisfying $\int d^3x\ (\nabla \cdot K) = \int_\Sigma K \cdot \widehat{d\Sigma} = 0$. In such a case, $\mathfrak{j} = \Theta j + K$. The operator $\hat{O}$ is then antiHermitian.

As examples for the application of the previous lemma we use the cases listed in table 3. The operators associated to energy, velocity and rotation are $\hat{O} = (\partial_t, (r \cdot \nabla), \Omega \cdot (x \times \nabla))$. All of them commute with Laplace's operator, hence $\mathfrak{j} = \Theta j$. In the case of the position $\Theta j = 0$, $\Theta\left(\nabla \cdot j\right) = 0$ and the terms including $\left[\nabla, \hat{O}\right]$ add up to zero.

1.3 Discussion

In Subsection 1.2we have shown that when the electromagnetic action outside the locus of masses, charges and currents as it appears in the Lorenz-Lorentz theorem is complemented with the action where mass, charge and currents are not necessarily null, the action proposed encodes Maxwell's electromagnetic laws, Lorentz' force and Newton's equations of motion, as well as an extension of Schrödinger equation for stationary states . Thus, electromagnetism does not question the validity of classical mechanics, nor quantum mechanics questions the validity of electromagnetism or classical mechanics. They coexist in a synthetic variational equation.

The difference between Schrodinger's eigenvalue-eigenvector equation and eq. (7) corresponds to the "static energy" which is present in Maxwell's formulation of electromagnetism ([680] p. 270 Maxwell, 1873,). The method of construction used by Maxwell is not consistent with the notion of quanta since it assumes that any charge can be divided in infinitesimal amounts, the electrostatic energy corresponding to assembling together the total charge. From a constructive point of view, such energy is not justified.

There are several possible alternatives. One of them is not to consider self-action (second alternative). A third alternative is to consider only the energy contained in transversal (rotational) non-stationary fields $(E_\perp, B)$ (where $E_\perp = E - (-\nabla V_C)$, being V_C the Coulomb potential). In the first and third cases, there will be a flow of energy associated to the EM waves each time the entity is not in a stationary state. In the second case, energy will correspond to interactions and the possibility of a quantum entity to be isolated as the limit of constructively factual situations will be doubted. One way or the other each view can be incorporated to the formalism, which in turn will provide consistency requirements that can be contrasted against experimental situations. Which alternative(s) must be discarded is to be decided experimentally. The list of alternatives will remain open until the occasion. We will see in Part II of this work that the alternatives that present "no self-actions in stationary situations" (second and third alternative) account for the energy levels of the hydrogen atom. Yet, if we want to account for the photoelectric effect or the Compton effect we need to have room for incoming or outgoing energy and momentum associated to the EM fields, thus the third alternative looks more promising.

It is worth noticing that in the statistical interpretation of quantum mechanics, the energy of the field is not considered (no reasons offered) avoiding in such way dealing with an infinite energy that corresponds to the electrostatic (self) energy of a point charge. However, self energies are not ruled out completely since they are needed (in the standard argumentation) to account for the so called spin-orbit interaction. As usual in the instrumentalist attitude, consistence is not a matter of concern.

Part II
Implementation: The hydrogen atom

2 Constructive ideas for Quantum Mechanics from Electromagnetism

We borrow some basic ingredients from Maxwell's electrodynamics and from traditional Quantum Mechanics. These elements will be combined in dynamical equations in the next Section.

1. The atom is the microscopic entity susceptible of measurements. Proton and electron within the atom do not have a sharp separable identity, they are inferences (or "shadows"). The identity of the atom consists of charge, current, magnetisation(s) and masses.

2. The experiments of Uhlenbeck & Goudsmidt and Zeeman as well as that of Stern and Gerlach suggest the existence of intrinsic magnetisations. We introduce it following the standard form relying in Pauli's seminal work (Pauli Jr, 1927).

3. The interactions within the atom and of the atom with detectors are electromagnetic.

4. Elementary entities such as the proton and the electron have no self-interactions. There is no evidence regarding the existence of self-energies (constitutive energies) at this level of description, correspondingly, they will not be included.

5. The experimental detection of properties belonging to the shadows within the atom depends on the actual possibility of coupling them with measurement devices (usually based upon electromagnetic properties). We will stay on the safe side and will not assume as measurable things like the distance from proton to electron or the probability distribution of the relative position between proton and electron. Such "measurements" remain in the domain of fantasy unless the actual procedure is offered for examination. In this respect we severely depart from textbook expositions of Quantum Mechanics.

6. The atom as an EM-entity is described with wave-functions. Any quantity associated to an atom, such as its EM fields, are represented by operators on the wave-function's space.

2.1 Expression of the electromagnetic potentials as integrated values on operators

2.1.1 Wave-functions

Two bodies that are spatially separated are idealised as represented by a *wave-function* which is the product of independent wave-functions on different coordinates. For the case of a proton and an electron we write $\Psi(x_e, x_p) = \psi_e(x_e)\psi_p(x_p)$ and call it the *electromagnetic limit*. In contrast, a single body with an internal structure is represented by the product of a wave-function associated to the body and another one representing the internal degrees of freedom. For the hydrogen atom $\Phi(x_r, x_{cm}) = \phi_r(x_r)\phi_{cm}(x_{cm})$ represents the *atomic limit*.

The election of the centre of mass, cm, to represent the position of the body will be shown later to be consistent with the form of the relational kinetic energy introduced. The variable x_r (r for *relative* or *relational*) stands for $x_e - x_p$. Our constructive postulate is that, when proton and electron are conceived as (spatially) separated entities, the laws of motion correspond to electromagnetism and when conceived as a hydrogen atom, the usual QM is recovered as a limit case[13]. In fact, the present approach is broader than the usual QM in at least two aspects: (a) The decomposition of the wave-function as a product is not imposed, but rather regarded as a limit and (b) the proton is regarded as something more than just the (point-charge) source of electrostatic potential. Hence, starting from the laws of EM, the structure of a general quantum theory for $|\Psi\rangle$ can be postulated as surpassing EM. Specialising this theory to the atomic case, the QM of the hydrogen atom is obtained.

2.1.2 Integrals on operators

We will call the value of the integral

$$\int d^3x_a \, d^3x_b \, \left(\Psi^*(x_a, x_b)(\hat{O}\Psi(x_a, x_b))\right) \equiv \left\langle \Psi^*(\hat{O}\Psi)\right\rangle \equiv \left\langle \Psi^*|\hat{O}|\Psi)\right\rangle \qquad (8)$$

the *integrated value* of the operator $\hat{O}$, where $\hat{O}$ is an operator that maps the wave-function Ψ as $\hat{O}\Psi(x_a, x_b) = \Psi'(x_a, x_b)$[14].

[13]Conventional QM also assumes whenever necessary that $\psi_{CM}(x_{CM})$ corresponds to a point-like particle at rest.

[14]We will drop the time-argument of the wave-function whenever possible to lighten the notation.

2.1.3 Physical Background

In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics, starting with the principles stated at the beginning of this Section and in Part of this work (Solari & Natiello, 2024). Hence, we will compute Maxwell's EM interaction energy, either between the atom and external EM-fields or between different fields arising within the atom. We display first some classical results that are relevant for the coming computations.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2 \right]. \tag{9}$$

As such, this result is independent of the choice of gauge for the electromagnetic potentials A, V. In the sequel, we will adopt the Coulomb gauge whenever required.

The vector and scalar potentials associated to a charge-current density $(Cq\rho, j)$ in a (electro- and magneto-) static situation are

$$\left(\frac{V}{C}, A\right)(x,t) \quad = \quad \frac{\mu_0}{4\pi} \int d^3y \, \frac{(Cq\rho, j)(y,t)}{|x-y|}.$$

The vector potential associated to a magnetisation M at the coordinate y reads:

$$A_M(x,t) \quad = \quad \frac{\mu_0}{4\pi} \nabla_x \times \int d^3y \, \frac{M(y,t)}{|x-y|}.$$

For potentials as defined in the Lorenz gauge there is a gauge transformation that brings them to Coulomb form i.e., ρ electrostatic and $\nabla \cdot A = 0$. Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$.

2.1.4 Electromagnetic static potentials

We define the density within the atom as $\rho(x_a, x_b) = |\Psi(x_a, x_b)|^2$. When appropriate, protonic and electronic densities may be defined as $\int d^3x_b \, |\Psi(x_a, x_b)|^2 \equiv \rho_a(x_a)$ for each component, $a = e, p$. In the electromagnetic limit of full separation between electron and proton ($|\Psi\rangle = |\psi_a\rangle|\psi_b\rangle$), with a wave-function for particle b normalised to unity, we recover $\rho_a(x_a) = |\psi_a(x_a)|^2$. Protonic and electronic current-densities within the atom are defined similarly for $a = e, p$ by

$$j_a(x_p, x_e) = -\frac{i\hbar q_a}{2m_a} \left(\Psi^*(x_p, x_e) \left(\nabla_a \Psi(x_p, x_e) \right) - \Psi(x_p, x_e) \left(\nabla_a \Psi^*(x_p, x_e) \right) \right)$$

with $q_e = -q_p$.

The electrostatic potential reads

$$V_a(x,t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, d^3x_b\, \frac{1}{|x-x_a|}\,|\Psi(x_a,x_b)|^2 = \frac{q_a}{4\pi\epsilon_0}\langle\Psi|\frac{1}{|x-x_a|}|\Psi\rangle.$$

Moreover,

$$V_a(x,t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{1}{|x-x_a|}\left(\int d^3x_b\,|\Psi(x_a,x_b)|^2\right) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{\rho_a(x_a)}{|x-x_a|}$$

showing that the electromagnetic structure is preserved in general, not just in the limit of full separation.

Similarly, the vector potential (Coulomb gauge) associated to an intrinsic microscopic magnetisation $M_a(x_a,x_b) = M_a(x_a)\rho(x_a,x_b)$ reads:

$$\begin{aligned} A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_b\, d^3x_a\, \frac{M_a(x_a,x_b)}{|x-x_a|} \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a(x_a)}{|x-x_a|}\rho(x_a) \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\Psi|\frac{M_a}{|x-x_a|}|\Psi\rangle \end{aligned}$$

(the index in ∇ indicates the derivation variable) where Ψ is the wave-function. In the electromagnetic limit, the vector potential reads:

$$\begin{aligned} A_{M_a}(x,t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a}{|x-x_a|}\,|\psi_a(x_a)|^2 \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\psi_a|\frac{M_a}{|x-x_a|}|\psi_a\rangle, \end{aligned}$$

which is the classical EM potential for a density $|\psi_a(x_a)|^2$.

For a static current we have

$$A_a(x,t) = \frac{1}{2}\frac{\mu_0 q_a}{4\pi m_a}\int d^3x_a\, d^3x_b\, \left(\Psi^*(x_a,x_b)\frac{1}{|x-x_a|}(-i\hbar\nabla_a)\Psi(x_a,x_b) + c.c.\right).$$

Here we may also define

$$\widetilde{j}_a(x_a) = \frac{q_a}{2m_a}\int d^3x_b\, \left(\Psi^*(x_a,x_b)(-i\hbar\nabla_a)\Psi(x_a,x_b)\right) + c.c.$$

and set

$$A_a(x,t) = \frac{\mu_0}{4\pi}\int d^3x_a\, \left(\frac{\widetilde{j}_a(x_a)}{|x-x_a|}\right).$$

2.1.5 On atomic magnetic moments

We associate an intrinsic magnetisation to proton and electron, following Pauli's original work (Pauli Jr, 1927). A general form for the intrinsic magnetisation for a microscopic constituent a reads

$$M_a = \frac{q_a}{m_a} S_a$$

where $S_a = \frac{\hbar}{2}\sigma^a$ and σ^a are the Pauli matrices for constituent a.

2.2 Variational formulation

Both Newtonian mechanics (Arnold, 1989) and Maxwell's electrodynamics (Solari & Natiello, 2022b) can be reformulated as variational theories, satisfying the *least action principle.* The action (or action integral) is the integral of the Lagrangian over time. From Part I,

$$\mathcal{L} = \int d^3y \left[\frac{1}{2}\left(\Psi^* \left(i\hbar\dot{\Psi}\right) + (i\hbar\dot{\Psi})^*\Psi\right) + \frac{\hbar^2}{2m}\Psi^* T\Psi + \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{\mu_0}B_i \cdot B_j - \epsilon_0 E_i \cdot E_j\right)\right] \tag{10}$$

with $i, j \in \{e, p, L\}$ which stand for electron, proton and Laboratory. The symbol T stands for kinetic energy and its form will be presented later. Actually "d^3y" is a symbol indicating that the integral is performed over all non-temporal arguments of the wave-function, which in the case of the hydrogen atom will be at least six.

In this Part of the work we are interested in the internal energies associated to stationary states of the microscopic entity (the Hydrogen atom). In Part I we assumed that the microscopic entity did not have an internal structure and therefore there was no "self-energy". In the present situation the atomic energy will have an internal part and an interaction part, as implied by eq.(10).

We propose therefore that the stationary states of the hydrogen atom have an associated wave-function $\Psi_\omega(x_e, x_p)$, where the indices e and p correspond to the inferred electron and proton, whose unifying entity is the atom. x represents appropriate arguments of the wave-function, satisfying

$$(\hbar\omega\Psi_\omega) - \frac{\hbar^2}{2m}T\Psi_\omega - \hat{H}\Psi_\omega = 0 \tag{11}$$

(we drop the index ω in the sequel) where

$$\left\langle \delta\Psi \hat{H}_{EM}\Psi \right\rangle = \delta_{\Psi^*}\left\langle \frac{1}{\mu_0}\left(B_L \cdot (B_e + B_p) + B_e \cdot B_p\right) + \epsilon_0\left(E_L \cdot (E_e + E_p) + E_e \cdot E_p\right)\right\rangle$$

the variation of the wave-function is $\delta\Psi$ and $\langle \cdots \rangle$ indicates integration over all coordinates as in eq.(8).

3 Contributions to the integrated value of H

3.1 Relational kinetic energy

In terms of classical mechanics, the kinetic energy associated to the hydrogen atom can be organised as follows. To fix ideas let index b correspond to the proton and a to the electron. The quantum-mechanical wave-function is $\Psi(x_a, x_b)$. The center-of-mass coordinate (relative to some external reference such as "the laboratory") and the relative coordinate (which is an invariant of the description, independent of an external reference) can be defined as usual, satisfying

$$\begin{aligned}
x_{cm} &= \frac{m_a x_a + m_b x_b}{m_a + m_b} \\
x_r &= x_a - x_b \\
x_a &= x_{cm} + x_r \frac{m_b}{m_a + m_b} \\
x_b &= x_{cm} - x_r \frac{m_a}{m_a + m_b}.
\end{aligned}$$

The wave-function and the gradient (we also make use of $p_x = -i\hbar\nabla_x$) read:

$$\begin{aligned}
\Psi(x_a, x_b) &= \Psi(x_{cm} + x_r \frac{m_b}{m_a + m_b}, x_{cm} - x_r \frac{m_a}{m_a + m_b}) = \Phi(x_{cm}, x_r) \\
\nabla_{cm} &= \nabla_a + \nabla_b \\
\nabla_r &= \frac{m_b}{m_a + m_b}\nabla_a - \frac{m_a}{m_a + m_b}\nabla_b \\
\Phi(x_{cm}, x_r) &= \phi(\frac{m_a x_a + m_b x_b}{m_a + m_b}, x_a - x_b) = \Psi(x_a, x_b) \\
\nabla_a &= \nabla_{cm}\frac{m_a}{m_a + m_b} + \nabla_r \\
\nabla_b &= \nabla_{cm}\frac{m_b}{m_a + m_b} - \nabla_r
\end{aligned}$$

Hence,

$$\begin{aligned}
T_a + T_b = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} &= -\hbar^2 \left(\nabla_{cm}^2 \frac{1}{2(m_a + m_b)} + \nabla_r^2 \frac{(m_a + m_b)}{2m_a m_b} \right) \\
&= \frac{-\hbar^2}{2(m_a + m_b)}\nabla_{cm}^2 + \frac{-\hbar^2}{2m_r}\nabla_r^2 = T_{cm} + T_r.
\end{aligned}$$

The associated integrated values read:

$$\mathcal{E}_k = \langle\Psi|T_a|\Psi\rangle + \langle\Psi|T_b|\Psi\rangle = \langle\Phi|T_{cm}|\Phi\rangle + \langle\Phi|T_r|\Phi\rangle$$

3.2 Relative electrostatic potential energy

The Coulomb electrostatic interaction energy for an atomic density $|\Psi(x_a,x_b)|^2$ reads:

$$\begin{aligned}\mathcal{E}_C^{x_a,x_b} &= \epsilon_0 \int d^3x\, E_1 \cdot E_2\, |\Psi(x_a,x_b)|^2 = -\epsilon_0 \int d^3x\, E_1 \cdot \nabla V_2\, |\Psi(x_a,x_b)|^2 \\ &= \int d^3x\, \epsilon_0 \left[-\nabla \cdot (E_1 V_2) + (\nabla \cdot E_1)\, V_2\right] |\Psi(x_a,x_b)|^2 .\end{aligned}$$

The full energy is given by averaging over the wave function Ψ. The first contribution vanishes when integrated over all space by Gauss theorem, and the second term can be recast as $-\epsilon_0(\nabla \cdot \nabla V_1)V_2$, namely

$$\begin{aligned}\mathcal{E}_C &= \int d^3x_a\, d^3x_b\ |\Psi(x_a,x_b)|^2 \int d^3x\, \epsilon_0 \left(\frac{q_a q_b}{(4\pi\epsilon_0)^2}\right)\left(-\frac{1}{|x-x_b|}\Delta\frac{1}{|x-x_a|}\right) \\ &= -\frac{e^2}{4\pi\epsilon_0}\int d^3x_e\, d^3x_p\ |\Psi(x_e,x_p)|^2\, \frac{1}{|x_e-x_p|} = -\langle\Psi|\frac{e^2}{4\pi\epsilon_0}\frac{1}{|x_e-x_p|}|\Psi\rangle\end{aligned}$$

where $q_a = e = -q_b$ and e is the electron charge (negative), i.e., a is the electron, b the proton. In the electromagnetic limit it corresponds to the potential energy between two separate charge densities $\rho_{k=}|\psi_k(x_k)|^2$ of opposite sign, corresponding to Maxwell's equation $q_k\rho_k = \epsilon_0\nabla \cdot E_k$.

3.3 Interaction of relative current with external magnetic field

For this interaction *relative current* means the currents (j_a, j_b) of both electron and proton regarding their motion relative to the (source of) external field. Recall that we associate $j = qv = \dfrac{q}{m}p = \dfrac{q}{m}(-i\hbar\nabla)$, i.e., current as a property of charge in (perceived) motion. The electromagnetic energy in the external field $B(x)$ is

$$\begin{aligned}\mathcal{E}_{jB} &= \frac{1}{\mu_0}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times(A_a+A_b)\,\Psi(x_a,x_b) \\ &= \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_a|}\frac{q_a}{m_a}p_a\right)\Psi(x_a,x_b) + c.c. \\ &+ \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_b|}\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c.\end{aligned}$$

Having in mind the (normal) Zeeman effect, we perform an explicit calculation using that the vector potential for the approximately constant external field B reads $B \equiv \nabla \times A_L = \nabla \times \big(\chi(x)\frac{1}{2}x \times B\big)$, with χ a smoothed version of the characteristic function for the experiment's region. We place the origin of coordinates inside the apparatus and assume the characteristic function to be 1 inside a macroscopically large ball K around the origin, so that

$$\int_{K\times K} dx_a\, dx_b\, |\Psi(x_a, x_b)|^2 \simeq 1$$

(i.e., the atom is inside the measurement device). In such a case,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\Big(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a+\frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\Big)\,\Psi(x_a,x_b)+c.c.\\
&= -\frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\,\frac{1}{2}B\cdot\left(x_a\times\frac{q_a}{m_a}p_a+x_b\times\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)+c.c.\\
&= -\frac{e}{2m_r}\langle B\cdot\left(\frac{m_r}{m_e+m_p}x_r\times p_{CM}+x_{CM}\times p_r+\frac{m_p-m_e}{m_p+m_e}L_r\right)\rangle\\
&\sim -\left(\frac{e}{2m_e}+O(\frac{m_e}{m_p})\right)\langle\Psi|B\cdot L_e|\Psi\rangle
\end{aligned}$$

where in the last line we have switched to CM, r coordinates and used that $\langle p_r\rangle = \langle p_{CM}\rangle = 0$ and $[x_r, \nabla_{CM}] = [x_{CM}, \nabla_r] = 0$.

Further, the argument of the above integral reads,

$$\nabla\times A_L\cdot\nabla\times A_a = \nabla\cdot(A_L\times(\nabla\times A_a)) + A_L\cdot\nabla\times(\nabla\times A_a)\,.$$

The first term vanishes by Gauss theorem. The experimental conditions assume that the atom lies well inside the region of magnetic field and that the border effects $\nabla\chi(x-x_{CM})\times\left(\frac{1}{2}x\times B\right)$ can be disregarded. Moreover, $\nabla\times(\nabla\times A_a) = \nabla(\nabla\cdot A_a) - \Delta A_a$. For the first term, we integrate over x_a, x_b, using that $\nabla_x\frac{1}{|x-x_a|} = -\nabla_{x_a}\frac{1}{|x-x_a|}$, arriving to the integral of $\frac{1}{|x-x_a|}\nabla_{x_a}\cdot\left(\Psi(x_a,x_b)^*\frac{q_a}{m_a}p_a\Psi(x_a,x_b)\right)+c.c.$, which is zero for an atom in a stationary state, since by the continuity equation it corresponds to the time-derivative of the atomic charge density. For the other term, recall that $-\Delta\frac{1}{|x-x_a|} = 4\pi\delta(x-x_a)$. Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\int d^3x\, A_L\cdot\left(\delta(x-x_a)\frac{q_a}{m_a}p_a+\delta(x-x_b)\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)\\
&\quad +c.c.\\
&= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\Big(A_L(x_a)\cdot\frac{q_a}{m_a}p_a+A_L(x_b)\cdot\frac{q_b}{m_b}p_b\Big)\,\Psi(x_a,x_b)+c.c.\\
&\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^*\Big(\frac{1}{2}x_a\times B\cdot\frac{q_a}{m_a}p_a+\frac{1}{2}x_b\times B\cdot\frac{q_b}{m_b}p_b\Big)\,\Psi(x_a,x_b)+c.c.
\end{aligned}$$

assuming that the atomic wave-function is negligibly small outside the border of the experiment's region. We now move to center-of-mass and relative coordinates, assuming further that the atom as a whole is at rest (at the centre of mass location) during the experiment (i.e., $\langle\Psi|p_{CM}|\Psi\rangle = 0$, $p_r \sim p_e$, $x_r \sim x_e$ and $\frac{m_p-m_e}{m_p+m_e} \sim 1$). Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\left\{\left\langle\Psi(x_a,x_b)^*\left(\frac{1}{2}(x_a-x_{CM})\times B\cdot\frac{q_a}{m_a}\right)\Psi(x_a,x_b)\right\rangle\right. \\
&\quad \left.+\left\langle\Psi(x_a,x_b)^*\left(\frac{1}{2}(x_b-x_{CM})\times B\cdot\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b)\right\rangle\right\}+c.c. \\
&= \frac{1}{2}\left(-\frac{e}{2(m_p+m_e)}\right)\left\langle\Psi(x_{CM},x_r)^*\,B\cdot\left(\frac{m_p}{m_e}-\frac{m_e}{m_p}\right)L_r\Psi(x_{CM},x_r)\right\rangle+c.c. \\
&= -\frac{e}{2m_r}\frac{m_p-m_e}{m_p+m_e}\langle\Psi|B\cdot L_r|\Psi\rangle \sim -\left(\frac{e}{2m_e}+O(\frac{m_e}{m_p})\right)\langle\Psi|B\cdot L_e|\Psi\rangle
\end{aligned}$$

3.4 Interaction of spin(s) with external magnetic field

We regard spin as an intrinsic magnetisation $\frac{q_a}{m_a}S_a$, leading to a vector potential

$$A_s(x,t) = \frac{\mu_0}{4\pi}\nabla\times\langle\Psi|\left(\frac{q_a}{m_a}\frac{S_a}{|x-x_a|}-\frac{q_a}{m_b}\frac{S_b}{|x-x_b|}\right)|\Psi\rangle \tag{12}$$

whose interaction energy with an external magnetic field $B(x,t)$ reads

$$\begin{aligned}
\mathcal{E}_{SB} &= \frac{1}{\mu_0}\int d^3x\, B(x,t)\cdot\nabla\times A_s(x,t) \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t)\cdot\left(\nabla\times\nabla\times\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)|\Psi\rangle \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t)\cdot\left[\nabla\left(\nabla\cdot\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)\right]|\Psi\rangle \\
&- \frac{e}{4\pi}\langle\Psi|\int d^3x\, B(x,t)\cdot\left(\Delta\left(\frac{1}{m_e}\frac{S_e}{|x-x_e|}-\frac{1}{m_p}\frac{S_p}{|x-x_p|}\right)\right)|\Psi\rangle
\end{aligned}$$

Since $\nabla\cdot B = 0$, the first contribution integrates to zero by Gauss theorem since $B\cdot\nabla\phi = \nabla\cdot(B\phi) - \phi\nabla\cdot B$. Hence,

$$\mathcal{E}_{SB} = e\langle\Psi|\left(\frac{1}{m_e}B(x_e,t)\cdot S_e-\frac{1}{m_p}B(x_p,t)\cdot S_p\right)|\Psi\rangle$$

thus completing the description of Zeeman effect. In the atomic limit the usual expression is recovered.

Stern-Gerlach effect. All forces, including the Lorentz force (Solari & Natiello, 2022b), are obtained from actions as the response of the action integral to a variation of the relative position of the interacting bodies. Displacing by δx the position of the field B one gets

$$\delta\mathcal{E}_{SB} = e\langle\Psi|\left(\frac{1}{m_e}B(x_e+\delta x,t)\cdot S_e - \frac{1}{m_p}B(x_p+\delta x,t)\cdot S_p\right)|\Psi\rangle = \delta x\cdot F$$

and then $F \sim \langle\Psi|\nabla\left(\frac{e}{m_e}B(x,t)\cdot S\right)|\Psi\rangle$. The force is nonzero only for spatially varying magnetic fields as observed in the experiment Bauer (2023).

3.5 Spin-orbit interaction

If we follow the standard discourse of Quantum physics, spin-orbit represents mainly the interaction between the electron's orbit and spin. Such heuristic approach would break one of our fundamental propositions: all the energies in the hydrogen atom are interaction energies between proton and electron, no self energy is involved. A relational view is actually forced to recognise that the orbit of the electron is a motion relative to the proton. Hence, only their relative velocity can matter. Such observation does not solve our problem, ... but let us follow its lead.

The coupling of spin and relative current has two parts, namely the interaction of proton spin $\frac{-e}{m_p}S_p$ with the relative current $\frac{ep_r}{m_r}$ as perceived by the proton and the corresponding interaction of electron spin $\frac{e}{m_e}S_e$ with relative current $\frac{-e(-p_r)}{m_r}$ as perceived by the electron. $m_r = \frac{m_e m_p}{m_e+m_p}$ stands for the reduced mass and $e\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right) = e\frac{p_r}{m_r}$. The magnetic field operator associated to the relative current as seen by the proton is hence the curl of the vector potential operator associated to that current (recall that the energy contribution is the integrated value of the operators over the wave-function), and correspondingly for the current perceived by the electron:

$$\begin{aligned}\hat{B}_{jp}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x\times\left(\frac{1}{|x-x_p|}\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)\right)|\Psi\rangle \\ &= \frac{\mu_0 e}{4\pi}\nabla_x\frac{1}{|x-x_p|}\times\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle \qquad (13)\\ \hat{B}_{je}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x\frac{1}{|x-x_e|}\times\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle.\end{aligned}$$

The magnetic field operators associated to the spin are:

$$\begin{aligned}\hat{B}_{se}(x) &= \frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_e|}\frac{S_e}{m_e}\right)\right)\\ \hat{B}_{sp}(x) &= -\frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_p|}\frac{S_p}{m_p}\right)\right)\end{aligned}$$

and the energy contribution is

$$\mathcal{E}_{SO} = \kappa\frac{1}{\mu_0}\langle\Psi|\int d^3x\ \hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) + \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x)|\Psi\rangle$$

where κ is a numerical constant that needs to be determined. We transform the spin field operators as

$$\begin{aligned}\hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot\left(\nabla\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right) - \Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\left(\nabla\cdot\left(\hat{B}_{jp}(x)\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right)\right) - \hat{B}_{jp}(x)\cdot\Delta\left(\frac{S_e}{|x-x_e|}\right)\right)\\ &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot(4\pi S_e\delta(x-x_e))\\ \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x) &= -\frac{\mu_0 e}{4\pi m_p}\hat{B}_{je}(x)\cdot(4\pi S_p\delta(x-x_p)) \qquad (14)\end{aligned}$$

using Gauss theorem and that $\nabla\cdot B = 0$. Performing the $x-$integral first, we obtain

$$\begin{aligned}\mathcal{E}_{SO} &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{-(x_p-x_e)}{|x_e-x_p|^3}\times\frac{p_r}{m_r}\right)|\Psi\rangle\\ &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{L_r}{m_r|x_e-x_p|^3}\right)|\Psi\rangle, \qquad (15)\end{aligned}$$

where $L_r = x_r\times p_r$ is the relative angular momentum operator.

Let us complete the expression determining the value of κ before we turn back to the question that opens this subsection. The expression for the electromagnetic energy (Equation 9) was introduced by Maxwell considering the kind of interactions known at his time. It is symmetric in the indexes of the two interacting systems, hence, if $\mathcal{P}(12)$ is a permutation of indexes, the energy can be written as:

$$\mathcal{E}_{SO} = \frac{1}{2}\left\{\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right] + \mathcal{P}(12)\int d^3x\left[\epsilon_0 E_1\cdot E_2 + \frac{1}{\mu_0}B_1\cdot B_2\right]\right\}.$$

We may call the first integral the way in which system one acts upon system two and the second integral is the reciprocal action. The energy can then be obtained by first establishing one action and next "symmetrising". The action of $\mathcal{P}(12)$ is merely changing the point of view in a somewhat arbitrary form, while the imposition of symmetry removes the arbitrariness since the acting group is a group of arbitrariness (see (Solari & Natiello, 2018)). The operation of taking two different view points corresponds to operating with $\mathcal{P}(ep)$ as it can be easily verified in all the previous expressions. Hence, unless $\kappa = \dfrac{1}{2}$ we would not be counting the interaction properly.

It remains to show that we are dealing with an interaction between two different entities and not an internal interaction. Consider our final expression, Eq. (15) and write it as:

$$\begin{aligned}
\mathcal{E}_{SO} &= -\frac{1}{2}\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{(x_e-x_p)\times(v_e-v_p)}{|x_e-x_p|^3}\right)|\Psi\rangle \\
&= \frac{1}{2}\int d^3x_e d^3x_p\left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{e^2}{4\pi\epsilon_0 C^2}\left(\frac{S_e\times v_e}{m_e}-\frac{S_p\times v_p}{m_p}\right)\Psi\right\} \\
&\quad -\frac{1}{2}\int d^3x_e d^3x_p\left\{\Psi^\dagger\left(\frac{(x_e-x_p)}{|x_e-x_p|^3}\right)\cdot\frac{\mu_0 e^2}{4\pi}\left(\frac{S_e\times v_p}{m_e}-\frac{S_p\times v_e}{m_p}\right)\Psi\right\}
\end{aligned}$$

($\Psi^\dagger$ stands for the row matrix that is the transpose and complex conjugate of Ψ). Consider further the case $\Psi=\psi_e\psi_p$, i.e., the case when $\Psi=\psi_e\psi_p$ and in addition the distance $|x_e-x_p|$ is macroscopic. In such a case we can consider that the variation of the relative position with respect to $\langle x_e-x_p\rangle$ is negligible in front of the macroscopic distance. The main contribution of the first term reads

$$\sim\frac{1}{2}\frac{1}{4\pi\epsilon_0 C^2}\frac{(x_p-x_e)}{|x_e-x_p|^3}\cdot\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)\int d^3x_p\left(-e|\psi_p|^2\right),$$

which represents the interaction between an electric dipole $\int d^3x_e\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$, located in the electron (now approximated as a point) and the proton (approximated as a point in front of the macroscopic distance). This view is compatible with the idea that magnetic dipoles in motion produce electric dipole fields. Up to a certain point, $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ represents a density of electric dipoles and $-e|\psi_p|^2$ a density of charge. The difference with such densities is that it is not possible to limit the interaction to "part of the electron" or "part of a proton", hence a integration over full space is always mandatory., thus the view of $\left(e\psi_e^\dagger\frac{S_e\times v_e}{m_e}\psi_e\right)$ as density of classical dipoles is only an analogy, it leaves aside the unity of the electron.

Actually, the dominant term in spin-orbit interactions corresponds to the electric dipole of the electron acting upon the proton. Thus, what was described as an interaction between the electron and its own orbit is now identified as the action of the electric dipole associated with the moving electron with the proton charge. An equivalent contribution appears with the action of $\mathcal{P}(ep)$. The last two terms are magnetic field interactions between proton and electron.

The expression 15 corresponds well with the final expression in textbook derivations. However, we have not resorted to analogy, nor have we patched this or any other expression with gyromagnetic numbers and have not further patched the expression with "relativistic corrections" (such as Thomas' correction) in the need to agree with experiments. The derivation of the spin-orbit contribution highlights the differences between the utilitarian/instrumentalist and the old style approach.

Related experiments One of the best known results of the atomic limit was the prediction of the hydrogen spectroscopic lines with Schrödinger's wave equation, providing a full theoretical expression of the Rydberg constant. The transition $n = 2$ to $n = 1$ (Kramida *et al.*, 2023) is reported as the spectroscopic line at $\lambda = 1215.6699\mathring{A}$. The line-width allows for a calculated resolution into $\lambda_1 = 1215.668237310\mathring{A}$ and $\lambda_2 = 1215.673644608\mathring{A}$ The state $n = 2$ is an octuplet, partially resolved by the spin-orbit Hamiltonian of Section 3.5 into a quadruplet and a doublet for $l = 1$ and two unresolved states with $l = 0$. The difference between the two energy levels is calculated as

$$\begin{aligned}\Delta E_{SO} &= \frac{3}{2}\hbar^2 \frac{1}{24a_0^3}\left(\frac{1}{2}\frac{\mu_0 e^2}{4\pi m_e^2}\right)\frac{m_e}{m_r} \\ &= \frac{3}{96}\frac{m_e}{m_r}\alpha^4 m_e C^2 \\ &= 7.259023470408092 \cdot 10^{-24} J\end{aligned}$$

while the calculated experimental energy difference amounts to $\Delta E_{exp} = hC(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = 7.26816814178113 \cdot 10^{-24} J$. The contribution consists of: $\frac{3}{2}\hbar^2$ coming from the level difference in $\langle L \cdot S\rangle$ between $j = 3/2$ and $j = 1/2$ (for $l = 1$ and $s = 1/2$), $\frac{1}{24a_0^3} = \langle R_{21}(r)|\frac{1}{r^3}|R_{21}(r)\rangle$ for the associated Hydrogen (n,l)-radial wave-function, $a_0 = \frac{\hbar}{\alpha m_e C}$ is the Bohr radius, m_e is the electron mass and $\alpha = \frac{\mu_0 e^2 C}{4\pi\hbar}$ is the fine-structure constant. We have disregarded the nuclear spin contribution.

3.6 Spin-spin interaction (part of hyperfine structure)

The interaction between spins within the atom uses an expression for the vector potential corresponding to that in Section 3.4 and reads

$$\begin{aligned}
\mathcal{E}_{SS} &= \frac{1}{\mu_0}\langle\Psi|\int d^3x\; B_e(x,t)\cdot B_p(x,t)|\Psi\rangle \\
B_e &= \nabla\times\left(\nabla\times\left(\frac{e}{m_e}\frac{\mu_0}{4\pi}\frac{S_e(x_e)}{|x-x_e|}\right)\right) \\
B_p &= \nabla\times\left(\nabla\times\left(\frac{-e}{m_p}\frac{\mu_0}{4\pi}\frac{S_p(x_p)}{|x-x_p|}\right)\right)
\end{aligned}$$

To proceed with the calculation we need to break the $e-p$ symmetry, namely $\mathcal{E}_{SS} = \frac{1}{2}\left(\mathcal{E}_{SS}^{e\to p} + \mathcal{E}_{SS}^{p\to e}\right)$. We pick the electron to proceed with the calculation, applying Gauss theorem again as in Section 3.4:

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{1}{\mu_0}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\; B_p(x)\cdot\left(-\Delta\frac{S_e(x_e)}{|x-x_e|}\right)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\;\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\delta(x-x_e)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\right]_{x=x_e}|\Psi\rangle
\end{aligned}$$

Through the vector calculus identities

$$\begin{aligned}
-\left(\nabla_x\times\left(\nabla_x\times\varphi(x)S_p\right)\right)\cdot S_e &= \left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\varphi(x) \\
\left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\frac{1}{|x-x_p|} &= \left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}\right]_{x=x_e}|\Psi\rangle \\
\mathcal{E}_{SS} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x - \left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_e|}\right]_{x=x_p}|\Psi\rangle
\end{aligned}$$

since both contributions are identical. The spin-spin interaction is symmetric in front of the interchange of electron and proton and then carries a factor one in front.

3.7 Interaction with an external electric field

Suppose the atom is in interaction with an external electric field, E_Z. The electromagnetic interaction is then:

$$\mathcal{E}_{AZ} = \epsilon_0 \int dx \ \left((E_A + E_A^{am}) \cdot E_Z \right)$$

where $E_A = -\nabla V$ and $E_A^{am} = -\frac{\partial A}{\partial t}$ being (V, A) the electromagnetic potentials associated to the atom:

$$\begin{aligned} A &= \frac{\mu_0}{4\pi} \nabla \times \langle \Psi | \left(\frac{e}{m_e} \frac{M_e}{|x - x_e|} - \frac{e}{m_p} \frac{M_p}{|x - x_p|} \right) | \Psi \rangle \\ V &= \frac{e}{4\pi\epsilon_0} \langle \Psi | \left(\frac{1}{|x - x_e|} - \frac{1}{|x - x_p|} \right) | \Psi \rangle \end{aligned}$$

Concluding remarks

Quantum mechanics and instrumentalism

The production of QM (a name attributed to Max Born (Born, 1955)) under the leadership of the Copenhagen school was embraced in an instrumentalist philosophical approach. Historically, the development was guided by the correspondence with classical mechanics (although it deals almost completely with electromagnetic interactions). Two of the three central productive tools in the creation of QM are: imagination (as it is illustrated by Bohr's well-known planetary atom) and analogy of forms, i.e., the proposal of mathematical relations produced in a different context and imported into QM without satisfying the conditions necessary for their meaningful production. [15]

The third element is the adoption of instrumentalism. The Copenhagen school was able to make important advances in QM introducing a matrix formalism, but as Born confessed in his Nobel lecture: "What the real significance of this formalism might be was, however, by no means clear" (Born, 1955). The situation is not a surprise since instrumentalism is focused on (predictive) success rather than significance. Its approach is more akin to technology than to science. It relies on

[15] For example: the Hamiltonian for a charged (point) particle used in classical mechanics, as can be seen in (Pauli Jr, 1927; Dirac, 1928), or Thomas' relativistic correction to the spin-orbit energy (Thomas, 1926) where, in order to patch Zeeman's formula obtained by analogy and refuted by experiments, Thomas applies Lorentz transformations relating two systems which are *not* in uniform motion with constant velocity with respect to each other (hence, the electromagnetic fields described in each system are *not* related by Lorentz transformations).

trial and error, success or failure in reproducing and collating experimental results. The process of patching contradictions or inconsistencies with ad-hoc hypotheses makes refutation almost impossible. In so doing, it puts an undue pressure over the experimental side, since it requires experiments each day more complex, scarce and in need of interpretation. Compensating refutations by introducing ad-hoc patches leads more often than not to specialism, since different patches need not be mutually consistent. Seeking for the unity of knowledge moves precisely in the opposite direction. The explosion of Science into multiple sciences, each one constantly fragmenting in a fractal design, is not a consequence of the complexity of nature but rather the consequence of sacrificing reason in the altar of productivity.

The "correspondence principle", in the context of the utilitarian development, is what remains of the idea of cognitive surpass (generalisation) where the old becomes a particular instance of the new.

In terms of the correspondence with classical mechanics, the idea of point particles is central to the statistical QM description. The concept of particle is so strongly associated to the extreme idealisation as points that, quite often, their point character is not mentioned until late in the works. If particles in classical mechanics, when possible, are idealised as mathematical points, in QM particles *are* mathematical points, the localisation in space becomes an essential property of what in conventional (textbook) QM is considered a particle.

Point particles are not an invention of the Copenhagen school, they are present as idealisations in almost every textbook of physics. In his theory of the electron, Lorentz (Lorentz, 1904) considered the electron as a point surrounded by a cloud of electricity located in the nearby ether. Further back in time, Faraday had opposed these views in his theory of vibrating rays (Faraday, 1855, p. 447) indicating that neither the material point nor the ether were needed if we considered matter as an inference arising as a consequence of the observation of EM action. Faraday recommended extreme prudence in introducing such inventions since in the past these metaphysical elements had proven to be difficult to remove.

The scene changed with the irruption of the works by de Broglie (de Broglie, 1923; De Broglie, 1924) and Schrödinger (Schrödinger, 1926) who introduced the wave formulation of QM, although both of them relied in part on point particles. Later, Schrödinger strongly opposed this view.

As mentioned in the Introduction, the EM interactions in Quantum Mechanics were at variance with electrodynamics. The expected radiation of an electron orbiting the proton in an hydrogen-like atom was suppressed by ruling that no such radiation occurs for some special orbits. In short, conventional EM did not apply to quantum phenomena. The unity of physics was broken as it was broken a few years earlier when Special Relativity and its immaterial ether (Einstein, 1924) was accepted by the guild of physics as the cornerstone of electromagnetism, thus breaking with Newtonian mechanics.

The self-appraisal of instrumentalism perceives these cognitive decisions as realism or pragmatism and not as deficits. Instrumentalism advances without retracing its steps, without reconsidering the possibility of being in error, patching "theories" and narrowing their scope as needed to avoid refutation. Actually, when the word "physicist" was coined (Yeo, 1993) the increasing specialisation and lack of unity in the advances of physics was the main concern for philosophers of Nature such as Faraday and Whewell.

The uncomfortable situation of the wave-particle duality forced by Schrödinger and de Broglie is left behind by Max Born who introduces the (now) standard interpretation of wave-functions as probability waves. As Schrödinger points out, this appears as the only form of reconciliation with point particles at hand, and it is the form in which QM has been socially transmitted to several generations of physicists. Adopting this interpretation forces us to write the wave-function in a basis of functions, $|o\rangle$, where the operator $\hat{O}$ representing the measurable property is diagonal, $\hat{O}|o\rangle = o|o\rangle,$ with $o \in \mathbb{R}$, so the "matrix" representing the density of states, $\langle o|\Psi\rangle\langle\Psi|o'\rangle$, has only positive values in the diagonal and zero otherwise. Since two non commuting matrices (operators) cannot be simultaneously diagonalised, it is not possible to obtain probabilities for their joint measurement. The theory then ruled that this impossibility is an imposition of Nature, although in deriving it we have only used our interpretation. This difficulty in separating what comes from the subject (the observer) and what from the object (the observed) has been central to the criticism of natural science by Goethe, Faraday, Whewell and Peirce. It is the attitude in front of the (possible) error what makes the most striking difference between the instrumentalist approach and the philosophical approach. The instrumentalist scientist resorts to patches, restrictions and limitations. The final outcome is posed by S. Carroll as "Physicists do not understand Quantum Mechanics. Worse, they don't seem to want to understand it" (Carroll, 2019).

In a constructivistic perspective, theories are constructs and their interpretations must be totally in accordance with the abduction process employed in the construction stage, since abduction and interpretation form the two components of the phenomenological map (Solari & Natiello, 2022a). In the constructivistic sense, to "understand a theory" implies to be able to construct it rationally starting from observations and experimental information through a process rooted in previous knowledge (which might require adjustments under the light of the new information) and using synthetic judgements which are put to test by the consequences of the theory. QM has not been reconstructed under the idea of being a statistical theory. We have then the right to ask: where was the statistics introduced when all our considerations have been about the interactions of point particles? The resolution of this conflict is simple for the instrumentalist, they tell us: "the construction is not part of science" or even more strongly: science is not about the reality but just a matter of economy of thought.

The view of inferred matter as having identity by itself with forces that abut around was the standard view used in the construction of physics in the late XIX century ([529], Maxwell, 1873) and propagates there on, without being given reconsideration. For Einstein (Einstein, 1936) the concept of material point is fundamental for Newton's mechanics (not just an idealisation). In his work Einstein criticises Lorentz' proposal of electrons extending in the space (with soft borders in (Lorentz, 1892, §33) and spherical in (Lorentz, 1904, §8)) in the form:

> The weakness of this theory lies in the fact that it tried to determine the phenomena by a combination of partial differential equations (Maxwell's field equations for empty space) and total differential equations (equations of motion of points), which procedure was obviously unnatural. The unsatisfactory part of the theory showed up externally by the necessity of assuming finite dimensions for the particles in order to prevent the electromagnetic field existing at their surfaces from becoming infinitely great. The theory failed moreover to give any explanation concerning the tremendous forces which hold the electric charges on the individual particles. (Einstein, 1936)

The criticism is faulty in several accounts. As we have shown in Part I, a single mathematical entity, Hamilton's principle, is able to produce the equations of Electromagnetism and mechanics. Einstein qualifies Lorentz' electron theory as "unsatisfactory" for using finite-size particles, since for Einstein forces (electromagnetic fields) exist in the surface of particles. Simply, he ignores Faraday, and furthermore, there seems to be a requirement for particles being mathematical points. Einstein gives no argument to justify the character of "unnatural" for Lorentz' procedure, actually, "unnatural" appears to stand for "alien to my scientific habit". In contrast, mathematical statements require proofs as we offer in Part I. The argument concerning a need for a force to held together the electron is equally faulty, it assumes that there are parts that need to be held together, that the quantum entity is a composite in which one part can act on another part. Actually, acquaintance with Maxwell's arguments, make us to realise that the electrostatic energy computed in electromagnetism requires the possibility of dividing the charge in infinitesimal amounts. Something that can be considered a reasonable limit in macroscopic physics (Maxwell's situation) but is ridiculous when it comes to the electron.

Maxwell's concept of science and our own concept differ radically from Einstein's. For Maxwell and the present work, the construction is a relevant part of the theory, for Einstein:

> There is no inductive method which could lead to the fundamental concepts of physics. Failure to understand this fact constituted the

> basic philosophical error of so many investigators of the nineteenth century. It was probably the reason why the molecular theory, and Maxwell's theory were able to establish themselves only at a relatively late date. Logical thinking is necessarily deductive; it is based upon hypothetical concepts and axioms. How can we hope to choose the latter in such a manner as to justify us in expecting success as a consequence? (Einstein, 1936).

Einstein's utterly instrumentalist statements contrast with the legend of his early reading of Kant's "Critic of pure reason" which elaborates in the opposite direction through the concept of *synthetic judgement*. The final question posed was answered by Peirce's abduction (retroduction)(Peirce, 1994) and the whole issue was considered in Whewell's philosophy of science(Whewell, 2016), also here in contrasting terms.

In contrast, Einstein accepted without criticism that "Everywhere (including the interior of ponderable bodies) the seat of the field is the empty space. (Einstein, 1936)" (referring to the EM fields) despite being conscious that the idea perpetuated the essence of the ether (Einstein, 1924).

The reason we emphasise Einstein's positions is that he has been socially instituted as the "most perfect scientist", something "known" even by the illiterate. Actually, acceptance of his statement requires the previous acceptance of the utilitarian view of science.

Concluding the historical revision, the adoption of an instrumentalist approach weakens the demands put on the construction of theories and by so doing it speeds up the process of delivering tools for the development of technologies at the price of sacrificing understanding.

The present approach

In the present work we addressed the challenge of constructing QM as a cognitive surpass of EM, in such a form that EM and QM can be regarded as two instances of the same theory. The electromagnetic formulation had to be previously unified not as an aggregation of compatible formulae but rather through a unified deduction of the formulae from a common principle. Such step was previously taken in (Solari & Natiello, 2022b) where Maxwell equations and the continuity equation all derive from a variational principle of minimal action, actually, the same principle that is used to generate Lagrangian dynamics and also the Lorentz force. In the presentation of relational electromagnetism (Solari & Natiello, 2022b) Maxwell equations represent the propagation of EM effects in regions outside its production sources. The sources of EM action are described only as Lagrangian parameters that can be identified as charge densities and current densities and satisfy the continuity equation. However, the entity of such densities inside what we intuitively call matter is not prescribed by electromagnetism. Extending EM into the material region promises then to produce the laws of QM.

In this endeavour we can continue to use the name "particle", but understanding a quantum particle as an unitary entity characterised by mass (which is the form of influencing through gravitational actions), charge, current (charge in motion) and magnetisation (spin), the latter ones being the attributes related to EM effects. Furthermore, we consider charge and mass-densities to be proportional and currents to bear the same proportion with momentum than the charge-to-mass relation. These densities cannot be thought of in analogy of the densities resulting from aggregates of matter since it is not possible to have a self-action or an interaction with only a portion of the quantum particle [16].

The wave-function in our approach will then be associated to the distribution of charge, currents and magnetisation. The need to use a wave-function and not the densities directly is taken from de Broglie's argumentation to make contact with experiments that relate frequency to energy involving Planck's constant. We must highlight that when analysing a free particle, the kinetic energy appears as the result of the above mentioned hypothesis and the conservation of charge, resulting in the usual $\frac{p^2}{2m}$, not as analogy or correspondence with classical mechanics but rather to be consistent with electromagnetism.

The wave-function is an attribute that keeps track of the state of the particle and is used primarily to determine the interaction (electromagnetic or gravitational) with other particles. In turn, the state of the particle is not just an attribute of itself but of its environment as well. Such view may result odd to physicists and chemists while it is the standard view in ecology. Thus, the problem of measurement is incorporated from the beginning since measuring means to enter in contact with a device.

Cognitive surpass means in this regard that the natural laws that govern the interaction of electron and proton as spatially separated entities must be the same as those governing their microscopic interaction, the differences corresponding to the context. When the electron and proton conform a new entity, the hydrogen atom, they act as a unity and interact with the environment as an atom. This observation suggests that the atomic wave-function may be factored as $|\Psi\rangle =$

[16] Readers familiar with Special Relativity (SR) may claim that there is a requirement of simultaneity at different points that might violate postulates of SR. However, SR displays self contradictions (Solari & Natiello, 2022a) prior to any claim. Its logical problems are usually made invisible by the instrumentalist approach that persuades us not to examine the correspondence of the construction and the interpretation. Experimental results support our view as well, for example (Aspect *et al.*, 1982).

$|\psi_r\rangle|\psi_{CM}\rangle$ while when we consider electron and proton as not entangled, it factors as $|\Psi\rangle = |\psi_p\rangle|\psi_e\rangle$, these two expressions constituting two different limits of the total wave-function. This simple argument allows us to construct, in Part II, the Hamiltonian of the hydrogen atom without resource to analogies, empirical corrections such as gyromagnetic factors or "relativistic corrections", i.e., patches inspired by SR needed to match experimental results.

We have shown that the classical laws of motion, Schrödinger's equation and the laws of Electromagnetism all derive from the same principle of minimal action. It is proper to say that they form a unit of knowledge. There is no much surprise in this fact, the principle of minimal action was introduced by Hamilton as form of collecting the system of ordinary equations of Lagrangian mechanics in one equation in partial derivatives. In turn Lagrange introduced the form today known as Lagrangian in order to incorporate in one expression those force laws given by their effects (usually called constraints of motion) and those given as interactions (the original realm of Newton's theory). Maxwell constructed Electromagnetism guided by the least action principle[17].

The particle is indissolubly associated to its localisation when it is though as: "the particle is that thing that lies there". In such regard, the vibration of Faraday's rays becomes dissociated from the particle. It is not an action performed by nature but rather by our way of knowing. In contrast, the "problem" of measurement in QM is dissolved in our construction, actually, all what is needed is a quantum mechanical description of the measurement device and its (usually) electromagnetic form of interacting with the system of interest. The universe cannot be suppressed in an idealisation, its presence will be always in action, to the very least in the form of time. If in addition, we allow ourselves to separate the inferred (matter) from what is sensed (action) as it is done in the pair QM and EM, we simply create a fantasy, an illusion, where poor Schrödinger's cat is both dead and alive until we measure. Actually, under the present view, the quantum mechanical entity will have associated time-dependent electromagnetic fields in as much it has not reached a stationary state. A state initially described as a superposition of stationary states will evolve into an apparent stationary state if we agree to ignore the relation with the electromagnetic energy whose space locus has not longer an overlap with the space locus of the material entity. Notice that in order to make this statement we have to include the localisation between the attributes of the entity creating a second entity, the photon, associated to the EM radiated energy.

[17]Actually, this is the mathematics about which Hertz said: "Many a man has thrown himself with zeal into the study of Maxwell's work, and, even when he has not stumbled upon unwonted mathematical difficulties, has nevertheless been compelled to abandon the hope of forming for himself an altogether consistent conception of Maxwell's ideas. I have fared no better myself." Hertz proceeds then to "understand" Maxwell's theory, actually to accept Maxwell's formulae resorting to the ether. See (p.20, Hertz, 1893)

In the present work we have advanced a unified action integral inspired in de Broglie and Schrödinger intuitions as well as Maxwell's electrodynamics (Lemma 1.2 to Theorem 1.1). Through Hamilton's principle we obtained the equations of motion ruling microscopic systems in accordance with classical mechanics and electromagnetism. Further, we reformulated Hamilton's principle in terms of the (conserved) energy, obtaining a slightly more general equation of Schrödinger type (Lemma 1.4to Lemma 1.5). Finally we showed the connection between the possible form of the variations and Hermitian operators (Lemma 1.6). Hamilton's variational principle appears as central to the development, since it separates circumstantial issues from "laws of nature". The law is what remains invariant regardless of initial or final conditions (or condition at any time) [18]. The occurrence of Hamilton's principle is the result of successive cognitive surpasses: from Newton equations to Lagrange equations and next to Hamilton's principle in Mechanics. Hamilton's structure was used by Maxwell to organise Electromagnetism (Maxwell, 1873) and by Lorentz to reorganise Maxwell's electromagnetism in terms of the electromagnetic Lagrangian (Lorentz, 1892) introducing his EM-force from the application of this principle. It was later elevated to the level of fundamental principle by Helmholtz because it systematically produces the conservation of energy, and finally Poincaré (Poincaré, 1913) considered it a fundamental principle as well. Hence, Hamilton's principle was born as cognitive surpass and evolved as a guide for the construction of physics, a history that situates it as the primary choice for an attempt to unify theories.

In Part II, the quantum mechanics that emerges from electromagnetism has been tested in the construction of the internal energies for the Hydrogen atom producing outstanding results, superior to those of standard quantum mechanics. Equal in precision but definitely superior in consistence and consilience. Another insight gained is that the moving spin reveals its role as the carrier of *electric* polarisation in the spin-orbit interaction.

The path followed in this construction is blocked for the standard presentation of QM as extension from mechanics by the early assumption of point particles. The usual form used to incorporate the Lorentz force in the case of point particles was the constructive basis for several authors such as Pauli (Pauli Jr, 1927) and Dirac (Dirac, 1928). This led to the identification of the momentum p with the operator $-i\hbar\nabla$ (in all situations), a compensating error sustained in Bohr's principle of correspondence. In turn, this chain of decisions made impossible to associate the operator with the current and prevented the production of EM interactions in QM except by resource to analogy.

[18] The original concept of variation by Hamilton reads "...any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data..."(Hamilton, 1834)

In contrast, in this work the unity of concepts is sought in the fundamentals. The Lorentz force was integrated to Maxwell equations in (Solari & Natiello, 2022b) (see Part I as well), without restrictions to point particles, using Lorentz' Lagrangian formulation. The integration with mechanics was then performed by incorporating the kinetic part of the action integral in the form suggested by the QM of the isolated particle, thus reuniting the *material* and *action* sides of the entity. After this synthetic effort was completed, Hamilton's principle showed the extraordinary power of consilience which was its initial justification (Hamilton, 1834). It is worth mentioning that so far there has been no need to request the quantification of the EM field. All results are consistent with Planck's view (Brush, 2002) who recognised that the exchange of energy in absorption and emission processes was quantified yet, this fact does not imply that the field itself needs to be quantified a priori, being the latter an idea usually attributed to Einstein (Einstein, 1905).

With the present work we have achieved the unification of the foundations of classical mechanics, electromagnetism and quantum physics in a sequence of works. This view is incompatible with the statistical interpretation of QM where measuring is a sort of magic act in which the laws of QM do not apply. As Carroll puts it "Quantum Mechanics, as it is enshrined in textbooks, seems to require separate rules for how quantum objects behave when we're not looking at them, and how they behave when they are being observed" (Carroll, 2019). Furthermore, a condition for a theory to be considered rational consists in the possibility of "filling the gap", which means that the details left without specification can be provided in accordance to the theory. The current QM is unable to specify the details of the measurement, they remain in an abstract sphere where apparently, for example, the position of the electron relative to the proton in the hydrogen atom can be determined with as much precision as desired, yet not a word is said on how such measurement would be performed. It fails then a requisite of rationality as discussed in general form in (Solari & Natiello, 2023).

We have shown that quantum mechanics can be conceived in a form that does not oppose classical mechanics but coexists with it and complements electromagnetism. The consilience of the present formulation not only contrasts with the statistical interpretation of quantum mechanics, it contrasts with what is implied in the notion of interpretation, i.e., that equations are first herded together as needed in terms of their usefulness without regard to their meaning, and a posteriori meaning is attributed in the "interpretation" to promote acceptance and facilitate the use of the equations. Such procedure is not needed, it is not forced by the limitations of our mind and much less by Nature. Any claim regarding the Truth in such collections is ludicrous. However, students in sciences are indoctrinated constantly in terms of the Truth value of such "theories". It is the social power of the guild of physicists what prevents progress towards understanding.

There are two obstacles that have to be overcome to grasp our presentation, the first is the entrenched belief in the representation of fundamental quantum particles as points. Schrödinger was, in our limited knowledge, the only one to point out this problem, yet he did not notice that this view was threaded with Special Relativity (SR). The second obstacle is the belief that Special Relativity (SR) is the cornerstone of electromagnetism.

Einstein's instrumentalism is declared in:

> Physics constitutes a logical system of thought which is in a state of evolution, and whose basis cannot be obtained through distillation by any inductive method from the experiences lived through, but which can only be attained by free invention. The justification (truth content) of the system rests in the proof of usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively. (Einstein, 1936, p. 381)

SR closed the debate around the ether at the price of elitism (Essen, 1978; Lerner, 2010), and dropping the advances made by Lorentz in terms of a Lagrangian formulation of Maxwell's theory. Only Lorentz' law of forces was kept, a law that was herded into electromagnetism probably to hide that it comes from Galilean transformations, hence, its deduction by Lorentz casts doubts on Einstein's alleged role of Lorentz transformations (Einstein, 1907) relating to inertial systems. We cannot avoid wondering what "inertial system" means for Einstein, who believes that Newton's mechanics rests upon absolute space (Einstein, 1924). But the problem is not Einstein, all of us are limited and to show him limited only proves what is not disputed: Einstein was human. The force driving the social construction of idols is the rent obtained by the clergy. In situations like this, the philosophical debate is obscured and impeded by the social struggle. Any one who has objected the idea of SR as the cornerstone of EM has been threatened by the guild of physics. We came to know only the testimony of those strong enough to (socially) survive the attacks. Let us name them: Essen (Essen, 1971, 1978) who wrote "The theory is so rigidly held that young scientists dare not openly express their doubts" as abstract of his 1978 intervention. Essen provided evidence that SR does not make sense in experimental terms. Dingle, warned us about the epistemological problems of SR (Dingle, 1960b), the relevance of the Doppler effect by itself (Dingle, 1960a) and gave testimony of the attacks he endured (Dingle, 1972). Phipps insisted from both theoretical and experimental points of view in the relational view of physics (Phipps Jr, 2006; Phipps, 2014).

If we go back in time, the men in charge of producing theories about nature were philosophers-mathematicians. For Galileo, mathematics provided the language to understand the universe by the philosopher, who was understood as a master at reasoning. When the production of scientific results was "industrialised" a new social character was born: the scientist (in particular, the physicist). The professional scientist, produced at the universities, was declared free from the control of philosophy (i.e., reason), each science had its method and forms of validations, it was proclaimed. The guild of the profession was then left in charge of the quality control. The immediate consequence of this social change in physics was a new form of explanation that left behind natural philosophy and empowered the "second physicist" (Jungnickel & McCormmach, 2017) with the social responsibility of producing theories. The technical success of Electromagnetism contrasted with the failure of the new epistemology that had made of the ether the weapon for attacking the remains of the old epistemology, for example, e.g. Göttingen's electromagnetism (Solari *et al.*, 2024). When it became clear that the ether was a no-end route, the task of saving the theoretical physicist became urgent. The merit of SR is this social rescue and is also the ultimate reason for not being challenged.

The present work, constructed much under the old epistemological view shows that physics depends on epistemology, and through epistemology it depends on social interests that have nothing to do with Nature. The immediate utilitarian consequences appear as unchanged but the foundations for advancing understanding of nature appear as dubious if the instrumentalist perspective is adopted.

We believe we have shown by example that when we adopt a different *epistemic praxis* such as we have done in this work and in our reconstruction of EM (Solari & Natiello, 2022b), we reach a different theoretical understanding, actually a deeper understanding when the epistemic praxis is more demanding, as it is the relation between our dualist phenomenology (Solari & Natiello, 2022a) and instrumentalism. Thus, physics depends not only on the observable natural phenomena but it depends as well on our philosophical disposition. Yet, if the concept of theory is weakened so as to be reduced to the equations (cf. (p. 21 Hertz, 1893)), the distinctions between theories mostly fade out, the phenomenological link disappears and interpretation crops up to guide the use of the formulae after sacrificing the unity of thought, critical and phenomenological actions.

In summary, there is no compelling reason to believe scientific results that have been achieved by weakening the old scientific demand for Truth into the current demand of usefulness. Newtonian physics is not incompatible with EM, as it is usually preached; and both of them can live in harmony with quantum mechanics. There is no reason for abandoning (relational) Cartesian geometry or to consider time as an "odd" spatial coordinate. There is no license to use formulae outside their range of validity as it is too often done.

Too often in the course of this investigation we have encountered misrepresentations of the thoughts and writings of the natural philosophers that developed physics until the middle of the XIX century, when the *scientist* was born.

Abridged representations of their ideas and writings creep in as soon as the creators fade out. Absolute space is attributed to Newton's mechanics and "true motion" disappears (Solari & Natiello, 2021), Faraday becomes a supporter of the ether rather than the careful philosopher he was, absolutely inclined to entertain doubts as much as possible avoiding to precipitate into simplifying inventions. Faraday writes in a letter to R Taylor:

> But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgment [...]

Maxwell's plead to consider the hypothesis of the ether as worth of research (p. 493 Maxwell, 1873) was transformed as well into a belief when the story was told. Yielding to the needs of the new social position of science (and its emerging epistemic praxis), Maxwell's theory was deprived of its mathematical construction lowering the acceptance standard from correct into plausible.

In our own experience, we have spent more time undoing promissory hunches than constructing correct reasoning. It is worth noticing that such hunches trigger exploratory actions and as such are important. It is believing, instead of doubting them, what makes them prejudice. What appears to us as correct is made of the debris of our errors.

The great minds that constructed Special Relativity and Quantum Theory were passengers of their epoch, a time when "conquering nature" (the old imperialist dream of Francis Bacon) had finally taken prevalence over the "understanding nature," sought by his contemporary Galileo Galilei. Our own work cannot escape the rule, even though we are not "great minds". We live in a time where Nature reminds us that we must understand ki [19] and, consequently, love ki. Science must then rescue the teaching of the old masters, the natural philosophers, and their practice of critical thinking.

[19] The word ki has been proposed as a new and respectful pronoun for Nature and all what is part of Nature. The word comes from Anishinaabe's language but relates as well to the Japanese and Chinese Ki and to the old English kin (like in kinship).

A Proof of variational results

A.1 Proof of Lemma 1.1:

Proof. We use standard vector calculus operations on the integrand, namely

$$\begin{aligned}
A\cdot j-\rho V &= A\cdot\left(\frac{1}{\mu_0}\nabla\times B-\epsilon_0\frac{\partial E}{\partial t}\right)-\epsilon_0 V\nabla\cdot E\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)+\epsilon_0 E\cdot\nabla V\\
&= \frac{1}{\mu_0}\left(|B|^2-\nabla\cdot(A\times B)\right)-\epsilon_0 A\cdot\frac{\partial E}{\partial t}-\epsilon_0\nabla\cdot(VE)+\epsilon_0 E\cdot\left(-E-\frac{\partial A}{\partial t}\right)\\
&= \frac{1}{\mu_0}|B|^2-\epsilon_0|E|^2-\nabla\cdot\left(\frac{1}{\mu_0}A\times B+\epsilon_0 VE\right)-\epsilon_0\frac{\partial}{\partial t}\left(A\cdot E\right)
\end{aligned}$$

□

A.2 Proof of Lemma 1.2:

Proof. Since the wave-function admits the independent variations $\psi\mapsto\psi+\delta\psi$ and $\psi\mapsto\psi+i\delta\psi$ we have that

$$\begin{aligned}
\psi\frac{\delta\mathcal{A}}{\delta\psi}\delta\psi+\frac{\delta\mathcal{A}}{\delta\psi^*}\delta\psi^* &= 0\\
\frac{\delta\mathcal{A}}{\delta\psi}\left(i\delta\psi\right)+\frac{\delta\mathcal{A}}{\delta\psi^*}\left(-i\delta\psi^*\right) &= 0
\end{aligned}$$

where $\delta\psi^*=(\delta\psi)^*$, and hence, it is sufficient to consider the variations of ψ and ψ^* as independent variations that must result in a null change of $\mathcal{A}$. By partial integration we obtain $-i\hbar\dot{\psi}^*\psi=-i\hbar\left(\psi^*\psi\right)_{,t}+i\hbar\psi^*\dot{\psi}$ and the stationary action corresponds to

$$\delta\mathcal{A}_{QM}=\int_{t_0}^{t_1}ds\left\langle\delta\psi^*\left[i\hbar\dot{\psi}+\frac{\hbar^2}{2m}\Delta\psi\right]\right\rangle=0.$$

□

A.3 Proof of Lemma 1.3

Proof. We act the variations piecewise, recalling that $\delta\psi = \epsilon\mu$ gives $\delta\dot{\psi} = (\epsilon\mu)_{,t} = \dot{\epsilon}\mu + \epsilon\dot{\mu}$,

$$\begin{aligned}
\mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^*\dot{\psi} - \dot{\psi}^*\psi \right\rangle \\
\delta\mathcal{A}_1 &= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle + \frac{i\hbar}{2}\int_{t_0}^{t_1} ds \left\langle \psi^*\left(\epsilon\mu\right)_{,t} - \left(\epsilon\mu\right)_{,t}{}^*\psi \right\rangle \\
&= \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\dot{\psi} - \dot{\psi}^*\mu \right\rangle - \frac{i\hbar}{2}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \dot{\psi}^*\mu - \mu^*\dot{\psi} \right\rangle \\
&= \int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\mu \right\rangle
\end{aligned}$$

after a partial integration in time of the $\dot{\epsilon}$ contribution (in the second line) that does not modify the variation. Partial integrations in space using Gauss' theorem give

$$\begin{aligned}
\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} ds \left\langle \psi^*\Delta\psi \right\rangle \\
\delta\mathcal{A}_2 &= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\Delta\psi + \psi^*\Delta\mu \right\rangle \\
&= \frac{\hbar^2}{2m}\int_{t_0}^{t_1} \epsilon\, ds \left\langle \mu^*\Delta\psi + \mu\Delta\psi^* \right\rangle .
\end{aligned}$$

Since the variation complies with Maxwell equations, after some partial integrations, we have:

$$\begin{aligned}
\delta\frac{1}{2}\int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}B^2 - \epsilon_0 E^2 \right\rangle &= \int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}\delta B_m \cdot B - \epsilon_0 \delta E_m \cdot E \right\rangle \\
&= \int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0} A\cdot(\nabla\times\delta B_m) - V\delta(\epsilon_0\nabla\cdot E_m) - \epsilon_0 A\cdot\delta E_{m,t} \right\rangle \\
&= \int_{t_0}^{t_1} ds \left\langle A\cdot\delta j_m - V\cdot\delta\rho_m \right\rangle
\end{aligned}$$

where $\delta j_m = \epsilon\mathfrak{J}_1 + \dot{\epsilon}\mathfrak{J}_2$ and $\delta\rho_m = \epsilon\mathfrak{X}$. Consequently,

$$\begin{aligned}
\delta\frac{1}{2}\int_{t_0}^{t_1} ds \left\langle \frac{1}{\mu_0}B^2 - \epsilon_0 E^2 \right\rangle &= \int_{t_0}^{t_1} ds \left\langle A\cdot\epsilon\mathfrak{J}_1 + \dot{\epsilon}A\cdot\mathfrak{J}_2 - V\epsilon\mathfrak{X} \right\rangle \\
&= \int_{t_0}^{t_1} \epsilon\, ds \left\langle A\cdot\mathfrak{J}_1 - \left(A\cdot\mathfrak{J}_2\right)_{,t} - V\mathfrak{X} \right\rangle
\end{aligned}$$

after another partial integration in time. □

A.4 Proof of Corollary 1.1

Proof. $\mathfrak{J}_2 = -\epsilon_0 \mathfrak{E}$ satisfies the assumption $\mathfrak{X} + \nabla \cdot \mathfrak{J}_2 = 0$ of Lemma 1.3, as well as $\mathfrak{J}_2 + \nabla \times W$ does it, for any suitable W. However, setting $\mathfrak{J}_1 := \mathfrak{J}_1 + \nabla \times W_{,t}$, the vector W does not appear in any of the equations of the Lemma. Note that the precise form of $\mathfrak{J}_1$ as a function of μ has not been prescribed so far. □

A.5 Proof of Theorem 1.1

Proof. Note that the column $\mathfrak{X}$ in the table corresponds to propagating $\delta\psi$ on the quantum charge density $\rho = \psi^*\psi$.

For the "time/energy" entry, the variation of the action corresponds to setting $\mu \equiv \dot{\psi}$ in Lemma 1.3. Since ϵ is any possible variation function, the only way to ensure a stationary action is to let the space integral be identically zero, namely:

$$\left\langle \mu^* \left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* \mu \right\rangle + \frac{\hbar^2}{2m} \left\langle \mu^* \Delta\psi + \mu \Delta\psi^* \right\rangle + \left\langle A \cdot \mathfrak{J}_1 - \left(A \cdot \mathfrak{J}_2\right)_{,t} - qV\mathfrak{X} \right\rangle = 0.$$

It follows that $\dot{\psi}^* \left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* \dot{\psi} = 0$ and $\dot{\psi}^* \Delta\psi + \dot{\psi}\Delta\psi^* = (\psi^*\Delta\psi)_{,t}$, by repeated application of Gauss' theorem. Letting $\mathfrak{J}_1 = -2\epsilon_0 \mathfrak{E}_{,t} - \mathfrak{j}$, we propose $\mathfrak{j} = j_{,t}$ so that $\mathfrak{X}_{,t} + \nabla \cdot \mathfrak{j} = 0$. Since $\mathfrak{X} = (\psi^*\psi)_{,t}$, we have that $\mathfrak{J}_2 = -\epsilon_0 \mathfrak{E} = j$ and $\mathfrak{J}_1 = j_{,t}$. The conditions of Lemma 1.3 are satisfied and

$$\begin{aligned}
A \cdot \delta j - V\delta\rho &= \epsilon \left[A \cdot \mathfrak{J}_1 - \left(A \cdot \mathfrak{J}_2\right)_{,t} - V\mathfrak{X}\right] \\
&= \epsilon \left[A \cdot \left(\mathfrak{J}_1 - \mathfrak{J}_{2,t}\right) - A_{,t} \cdot \epsilon_0 \mathfrak{E} - V\mathfrak{X}\right] \\
&= -\epsilon \left[-\left(A_{,t} + \nabla V\right) \cdot \epsilon_0 \mathfrak{E}\right] \\
&= -\epsilon \left[E \cdot j\right]
\end{aligned}$$

The latter expression is the time-derivative of the electromagnetic energy:

$$\begin{aligned}
\frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)_{,t} &= E \cdot \epsilon_0 E_{,t} + \frac{1}{\mu_0} B \cdot B_{,t} \\
&= E \cdot \left(\frac{1}{\mu_0} \nabla \times B - j\right) - \frac{1}{\mu_0} B \cdot (\nabla \times E) \\
&= -E \cdot j + \frac{1}{\mu_0} \nabla \cdot (B \times E)
\end{aligned}$$

By Gauss' theorem, the divergence vanishes when integrated and we finally obtain the equation

$$\left\langle \partial_t \left[-\frac{\hbar^2}{2m} \left(\psi^* \Delta\psi\right) + \frac{1}{2} \left(\frac{1}{\mu_0} B^2 + \epsilon_0 E^2\right)\right] \right\rangle = 0$$

expressing the conservation of energy.

For the "position/velocity" entry in the table, in order to obtain a stationary action integral, a similar argument about the independence and arbitrariness of the real-valued time function ϵ forces the following vector valued integral to be zero:

$$\begin{aligned}\left\langle -i\psi^* (k\cdot x)\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* i(k\cdot x)\psi\right\rangle &+ \frac{\hbar^2}{2m}\langle -i\psi^*(k\cdot x)\Delta\psi + i(k\cdot x)\psi\Delta\psi^*\rangle \\ &+ \left\langle A\cdot\mathfrak{J}_1 + (A\cdot\mathfrak{L}\epsilon_0)_{,t} - V\mathfrak{X}\right\rangle\end{aligned}$$

where $\mathfrak{X} = (ix\psi)^*\psi + \psi^*(ix\psi) = 0$. Hence, we have $-\mathfrak{J}_2 = \epsilon_0\mathfrak{L} = 0$. The continuity equation is satisfied by taking $\mathfrak{J}_1 = 0$. The electromagnetic integral is identically zero and we obtain

$$\left\langle k\cdot\left(-i\psi^* x\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^* ix\psi\right)\right\rangle + \frac{\hbar^2}{2m}\langle k\cdot(-i\psi^* x\Delta\psi + ix\psi\Delta\psi^*)\rangle = 0.$$

The first term gives $k\cdot(\hbar\partial_t\langle\psi^* x\psi\rangle)$ while the second reduces to $\frac{i\hbar^2}{2m}k\cdot\langle\psi^*(\nabla\psi) - (\nabla\psi)^*\psi\rangle$ after repeated application of Gauss' theorem. Finally,

$$0 = \partial_t\langle\psi^* x\psi\rangle + \frac{i\hbar}{2m}\langle\psi^*(\nabla\psi) - (\nabla\psi)^*\psi\rangle = \partial_t\langle\psi^* x\psi\rangle - \frac{1}{q}\langle j\rangle.$$

For the "displacement/momentum" case, we compute $\mathfrak{X} = r\cdot\nabla\rho$, set $\mathfrak{j} = (r\cdot\nabla)j$ and $\mathfrak{J}_2 = -\epsilon_0\mathfrak{L} = -r\rho$. Then $\mathfrak{J}_1 - \mathfrak{J}_{2,t} = \epsilon_0\mathfrak{L}_{,t} + \mathfrak{j}$, which computes to $\mathfrak{J}_1 - \mathfrak{J}_{2,t} = r\rho_{,t} + (r\cdot\nabla)j = -r(\nabla\cdot j) + (r\cdot\nabla)j$ and further $-r(\nabla\cdot j) + (r\cdot\nabla)j = -\nabla\times(r\times j)$. Hence, up to a global divergence vanishing by Gauss' theorem,

$$\begin{aligned}A\cdot\left(\mathfrak{J}_1 - \mathfrak{J}_{2,t}\right) - A_t\cdot\mathfrak{J}_2 - V\mathfrak{X} &= A\cdot(-\nabla\times(r\times j)) + (A_t + \nabla V)\cdot(r\rho) \\ &= -A\cdot\nabla\times(r\times j) - E\cdot(r\rho) \\ &= -(r\times j)\cdot(\nabla\times A) - r\cdot\rho E \\ &= -r\cdot(\rho E + j\times B).\end{aligned}$$

Finally, the arbitrariness of ϵ demands

$$r\cdot\left\langle(\nabla\psi)^*\left(i\hbar\dot{\psi}\right) + \left(i\hbar\dot{\psi}\right)^*\nabla\psi\right\rangle + \frac{\hbar^2}{2m}r\cdot\langle(\nabla\psi^*)\Delta\psi + (\nabla\psi)\Delta\psi^*\rangle - r\cdot\langle\rho E + j\times B\rangle = 0.$$

By repeated use of Gauss' theorem we obtain $\langle(r\cdot\nabla\psi)^*\Delta\psi + (r\cdot\nabla\psi)\Delta\psi^*\rangle = 0$. Since the directions r are arbitrary, we have

$$i\hbar\left\langle(\nabla\psi)^*\dot{\psi} - \dot{\psi}^*\nabla\psi\right\rangle = \frac{m}{q}\partial_t\langle j\rangle = \langle(j\times B) + \rho E\rangle.$$

For the "rotation/angular momentum" entry $\mathfrak{X} = \Omega\cdot(x\times\nabla\rho)$, $\mathfrak{J}_1 = \mathfrak{j} = (\Omega\cdot(x\times\nabla))j - \Omega\times j$ and we set $\mathfrak{J}_2 = -(\Omega\times x\rho)$ since $\nabla\cdot\mathfrak{J}_2 = -\Omega\cdot(x\times\nabla\rho) = -\nabla\cdot(\Omega\times x\rho)$.

The proposed variation $\delta\psi$ enters in the space integrals, which specifically become

$$\Omega\cdot\left\langle i\hbar\left(\left(x\times\nabla\psi^*\right)\dot{\psi}-\left(x\times\nabla\psi\right)\dot{\psi}^*\right)+\frac{\hbar^2}{2m}\left(\left(x\times\nabla\psi^*\right)\Delta\psi+\left(x\times\nabla\psi\right)\Delta\psi^*\right)\right\rangle .$$

The second term vanishes as in the previous case, while the first term reads

$$\Omega\cdot\left\langle x\times\left(i\hbar\left\langle\left(\nabla\psi\right)^*\dot{\psi}-\dot{\psi}^*\nabla\psi\right\rangle\right)\right\rangle \quad = \quad \frac{m}{q}\Omega\cdot\partial_t\left\langle x\times j\right\rangle$$

The electromagnetic part reads

$$\begin{aligned} A\cdot\left(\mathfrak{J}_1-\mathfrak{J}_{2,t}\right)-A_t\cdot\mathfrak{J}_2-V\mathfrak{X} &= A\cdot\left[\Omega\cdot\left(x\times\nabla\right)\right]j-\Omega\cdot\left(j\times A\right)\\ &+ \Omega\cdot\left(\left(x\rho_{,t}\right)\times A\right)-\Omega\cdot\left[\left(x\rho\right)\times E\right]\\ &= -\Omega\cdot\left(x\times\left(j\times B\right)\right)-\Omega\cdot\left[x\times\rho E\right]\\ &= -\Omega\cdot\left(x\times\left(\rho E+j\times B\right)\right). \end{aligned}$$

Being Ω a vector of arbitrary direction, equating the variation to zero we get

$$\frac{m}{q}\partial_t\left\langle x\times j\right\rangle-\left\langle x\times\left(\rho E+j\times B\right)\right\rangle=0$$

which is the evolution of the angular momentum under the action of the Lorentz force. □

A.6 Proof of Lemma 1.4

Proof. We only need to check that $\left(\delta\rho\right)_{,t}+\nabla\cdot\delta j=0$ which follows from computing

$$\mathfrak{J}_1-\mathfrak{J}_{2,t}=-\epsilon_0\mathfrak{E}_{,t}-\mathfrak{j}=-\frac{1}{\mu_0}\nabla\times\mathfrak{B}$$

according to Maxwell's equations. Substituting in the result of the preparatory lemma and using the arbitrary $\epsilon(t)$ argument, the results follow. □

A.7 Proof of Lemma 1.5

Proof. We begin from Eq. 6 of Lemma 1.4. We write $\mathfrak{X}=\left(\mu^*\psi+\mu\psi^*\right)$ and set $\mathfrak{j}=-\frac{i\hbar q}{2m}\left(\mu^*\nabla\psi-\psi\nabla\mu^*+\psi^*\nabla\mu-\nabla\psi^*\mu\right)$. According to Lemma 1.3 the variation of the electromagnetic contribution to the action reads:

$$\int_{t_0}^{t_1}\epsilon\,ds\left\langle A\cdot\mathfrak{J}_1-\left(A\cdot\mathfrak{J}_2\right)_{,t}-V\mathfrak{X}\right\rangle$$

which according to Lemma 1.4 is:

$$\int_{t_0}^{t_1} \epsilon \, ds \left\langle -\frac{1}{\mu_0} A \cdot (\nabla \times \mathfrak{B}) - \epsilon_0 E \cdot \mathfrak{E} \right\rangle$$

The variation of the action becomes:

$$\int (\epsilon(s)\, ds) \left\langle \mu^* \left(i\hbar\dot{\psi} \right) + \left(i\hbar\dot{\psi} \right)^* \mu \right\rangle + \frac{\hbar^2}{2m} \left\langle \mu^* \Delta\psi + \mu\Delta\psi^* \right\rangle - \left\langle -\frac{1}{\mu_0} A \cdot (\nabla \times \mathfrak{B}) - \epsilon_0 E \cdot \mathfrak{E} \right\rangle .$$

Letting $\epsilon(t)$ to be constant, integrating in $-T \le s \le T$, and letting $T \to \infty$, projects μ into its ω Fourier component, μ_ω, while the electromagnetic terms integrate to $A \cdot \mathfrak{j}_\omega - V\mathfrak{X}_\omega$ projecting in the same form the variation of density and current, with $\mathfrak{X}_\omega = (\mu_\omega^* \psi_\omega + \mu_\omega \psi_\omega^*)$ and $\mathfrak{j}_\omega = -\frac{i\hbar q}{2m} (\mu_\omega^* \nabla\psi_\omega - \psi_\omega \nabla\mu_\omega^* + \psi_\omega^* \nabla\mu_\omega - \nabla\psi_\omega^* \mu_\omega)$. Since μ is arbitrary, its Fourier components are arbitrary and the null variation of the action results in the requisite:

$$\langle \mu_\omega^* \left(\hbar\omega\psi_\omega \right) \rangle + \frac{\hbar^2}{2m} \langle \mu_\omega^* \Delta\psi_\omega \rangle + \left\langle A \cdot \left(\frac{i\hbar q}{2m} \right) (\mu_\omega^* \nabla\psi_\omega - \psi_\omega \nabla\mu_\omega^*) - V\nu_\omega^* \psi_\omega \right\rangle = 0.$$

Partial integration gives $A \cdot (\nu^* \nabla\psi_\omega - \psi_\omega \nabla\nu^*) = A \cdot (2\nu^* \nabla\psi_\omega - \nabla(\psi_\omega \nu^*))$ and by Gauss' theorem $\langle -A \cdot \nabla(\nu^* \psi_w) \rangle = \langle \nu^* \psi_w \nabla \cdot A \rangle$. For an homogeneous magnetic field we have $A = \frac{1}{2} x \times B$ (and hence $\nabla \cdot A = 0$) and further $A \cdot \left(\frac{i\hbar q}{2m} \right) 2\nu^* \nabla\psi_\omega = \nu^* \left(\frac{i\hbar q}{2m} \right) (x \times B) \cdot \nabla\psi_w$, which finally leads to

$$\left\langle \nu^* \left(\hbar\omega\psi_\omega + \frac{\hbar^2}{2m} \Delta\psi_\omega - \frac{i\hbar q}{2m} B \cdot (x \times \nabla\psi_\omega) - qV\psi_\omega \right) \right\rangle = 0.$$

It follows that $\hbar\omega\psi \equiv \mathcal{E}\psi = -\frac{\hbar^2}{2m} \Delta\psi - \frac{q}{2m} B \cdot (x \times (-i\hbar\nabla)\psi) + qV\psi.$ □

A.8 Proof of Lemma 1.6

Proof. Letting $\delta\psi = \epsilon\hat{O}\psi$, the operator $\hat{O}$ must be linear as a consequence of the infinitesimal character of variations. Acting with Θ on the continuity equation we get

$$\begin{aligned} \Theta\left(\rho_{,t} + \nabla \cdot j\right) &= 0 \\ \left(\Theta\rho\right)_{,t} + \Theta\left(\nabla \cdot j\right) &= 0 \end{aligned}$$

provided $\hat{O}_{,t} = 0$. Since

$$\Theta\left(\nabla \cdot j\right) = \nabla \cdot (\Theta j) - \frac{i\hbar q}{2m} \left(\psi^* \left(\left[\nabla, \hat{O}\right] \cdot \nabla \right) \psi - \psi \left(\left[\nabla, \hat{O}\right] \cdot \nabla\psi \right)^* + \left(\left[\nabla, \hat{O}\right] \psi \right)^* \cdot \nabla\psi - \nabla\psi^* \cdot \left(\left[\nabla, \hat{O}\right] \psi \right) \right)$$

It is then possible to propose a suitable current $\mathfrak{j}$ if and only if

$$-\frac{i\hbar q}{2m}\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)=\nabla\cdot K \quad (16)$$

for some vector K. In such a case, $\mathfrak{j}=\Theta j+K$, satisfies $(\Theta\rho)_{,t}+\nabla\cdot\mathfrak{j}=0$.

Let us further investigate condition eq.(16). Clearly, it implies

$$-\frac{i\hbar q}{2m}\left\langle\left(\psi^*\left(\left[\nabla,\hat{O}\right]\cdot\nabla\right)\psi-\psi\left(\left[\nabla,\hat{O}\right]\cdot\nabla\psi\right)^*+\left(\left[\nabla,\hat{O}\right]\psi\right)^*\cdot\nabla\psi-\nabla\psi^*\cdot\left(\left[\nabla,\hat{O}\right]\psi\right)\right)\right\rangle=\langle\nabla\cdot K\rangle$$

Introducing the Hermitian adjoint, the lhs can be rewritten as follows:

$$\frac{i\hbar q}{2m}\left\langle\psi^*\left(\left[\Delta,\hat{O}\right]-\left[\Delta,\hat{O}\right]^\dagger\right)\psi\right\rangle \quad = \quad \langle\nabla\cdot K\rangle\,.$$

Hence, if $\left[\Delta,\hat{O}\right]$ is Hermitian (or equivalently if $\hat{O}^\dagger=-\hat{O}$, since Δ is itself Hermitian), eq.(16) ensures that an adequate K exists with $\langle\nabla\cdot K\rangle=0$. Conversely, if a vector field K exists such that

$$\int d^3x\ (\nabla\cdot K)=\int_\Sigma K\cdot\widehat{d\Sigma}=0$$

where the volume integral is taken as limit over all space and Σ is the oriented surface bounding the volume of integration, then $\hat{O}$ is anti-Hermitian. □

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Declaration of Interest

The authors declare that there exists no actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, their work.

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