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Multi-Criteria Decision Making and Artificial Intelligence for the School System: A Systematic Literature Review

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Abstract

The growing availability of data in school systems has sparked a broad academic debate [Garzón et al., 2025, Melo-López et al., 2025] on the use of advanced quantitative tools to support decision-making processes related to the planning and management of school infrastructure. This paper fits within this analytical framework, exploring, in particular, the role of Multi-Criteria Decision Making (MCDM) models and Artificial Intelligence (AI) in school systems, particularly whether these tools are used to assess the accessibility of school buildings. To this end, this paper presents a systematic review of the literature on the use of MCDM methods and AI. The analysis was conducted according to the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) guidelines [Moher et al., 2015], analyzing 416 scientific contributions in detail. The main contribution of this study is to analyze the current scientific landscape on the use of quantitative tools

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(with a focus on MCDM and AI tools) in the school system, with the aim of highlighting the main current, emerging, and potential research areas. The results show that MCDM methods and AI techniques have numerous applications in the education sector; for example, they are used to assess sustainability [Kabassi, 2021], and in information and communication technologies (ICT) [Voultsiou and Moussiades, 2025]. However, the use of MCDM and AI tools in assessing school accessibility is still limited. This latter application area may represent a potential new research field in which the use of combined tools such as MCDM methods and AI techniques can be used to classify schools based on their level of accessibility, thus further enriching the academic debate on school systems.

Keywords: Multi-criteria decision making, AI, Decision Analysis, accessibility, school. [1]

1 Introduction

In recent years, the growing availability of data within school systems [Schildkamp, 2019] has fueled a broad academic debate on the use of advanced quantitative tools to support decision-making processes related to the planning and management of school infrastructure [Barrett et al., 2019]. In this context, the topic of school building accessibility has assumed an increasingly important role in the issue of school inclusion. School accessibility represents a key dimension of modern education policies and a pillar of the 2030 Agenda for Sustainable Development [Weiland et al., 2021, UNESCO, 2020]. However, its assessment is particularly complex because it involves the integration of heterogeneous criteria and the management of a plurality of actions, including, for example, the removal of architectural barriers [Pivik, 2010], which requires extensive timescales, planning, and structural interventions that require significant financial resources [Forman et al., 2009]. Analyzing school accessibility therefore requires tools capable of combining different, often conflicting, perspectives [Cinelli et al., 2014], and processing large amounts of data from diverse fields to provide a systemic and accurate view of the phenomenon. This aspect is not only attributable to school accessibility but also to other key aspects of the school system, such as building safety and sustainability [Marinković et al., 2024], energy efficiency [Ashrafian et al., 2023], the quality of learning environments [Barrett et al., 2015], and so on. In this sense, schools are increasingly using quantitative tools that allow them to analyze decision-making problems characterized by high dimensionality (in terms of the number of alternatives and decision criteria) [Ishizaka and Nemery, 2013],

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leveraging tools related to Multi-Criteria Decision Making (MCDM) [Greco et al., 2016] and AI [Russell and Norvig, 2021]. It is well established in the literature that these tools are widely used to analyze several decision-making problems, in a variety of application fields such as manufacturing [Fattoruso et al., 2022], public tender analysis [Marcarelli and Squillante, 2020], behavioral finance [Martino et al., 2023, Ventre et al., 2024, Cruz Rambaud and González Fernández, 2018], healthcare [Esteva et al., 2019], transportation [Iyer et al., 2021] and energy management [Safari et al., 2024]. This work aims to explore how and in which contexts related to the school system, MCDM methods and AI techniques have been used, as well as to explore whether and how these tools have been used in an integrated manner. One of the focuses of this work, in the context of the use of quantitative tools in school systems, is school accessibility.

We note that the topic of school accessibility can also be considered within the Quadruple Helix Model [Carayannis and Campbell, 2010], which highlights the existing collaboration between universities, industry, government, and civil society to develop inclusive and sustainable solutions. In this model, universities have the role of providing the scientific and methodological knowledge and expertise that can be used to make school institutions more inclusive; the government can outline policies to be adopted and the available financial resources; industry can contribute by supporting universities and government; and civil society, particularly families and students, can provide concrete feedback on the current situation of school institutions and the needs that need to be addressed. This framework allows us to envision the development of tools for assessing the accessibility of schools that are not only scientifically and methodologically sound, but also effectively respond to societal needs by actively involving all stakeholders involved in decision-making processes at various levels.

The aim of this paper is to define, through a systematic literature review, the state of the art of existing literature in the school system, providing an overview of the main MCDM and AI tools used in several areas of application in the school context. This analysis aims to provide an overview that allows us to identify the main lines of research already developed by scholars but also to identify potential new areas of research that could be initiated. Studies as systematic literature review [Fattoruso, 2022, Yüksel et al., 2023, Sahoo and Goswami, 2023, Tahiru, 2021, Shams et al., 2024], allow us to understand how quantitative tools can contribute to strategic decisions by the various stakeholders involved in decision-making processes in different fields of application.

The paper is organized as follows: Section 2 describes the search strategy, the material, and the eligibility criteria; Section 3 reports the main results on the relevant scientific landscape and subsequent analysis of the main contents; Section 4 concludes the paper.

2 Search Strategy, Material and Eligibility Criteria

This systematic literature review aims to provide an overview of the use of multi-criteria decision-making methods and AI in the evaluation of the accessibility of school institutions, both individually and in combination. An exploratory approach will be applied to understand the state of the art and identify potential future directions. In selecting documents we used the PRISMA (Preferred Reporting Items for Systematic Review and Meta-Analyses) guidelines [Moher et al., 2015]. The PRISMA guidelines aim to ensure that the process of systematic literature review is carried out in a transparent manner [Moher et al., 2015, Liberati et al., 2009].

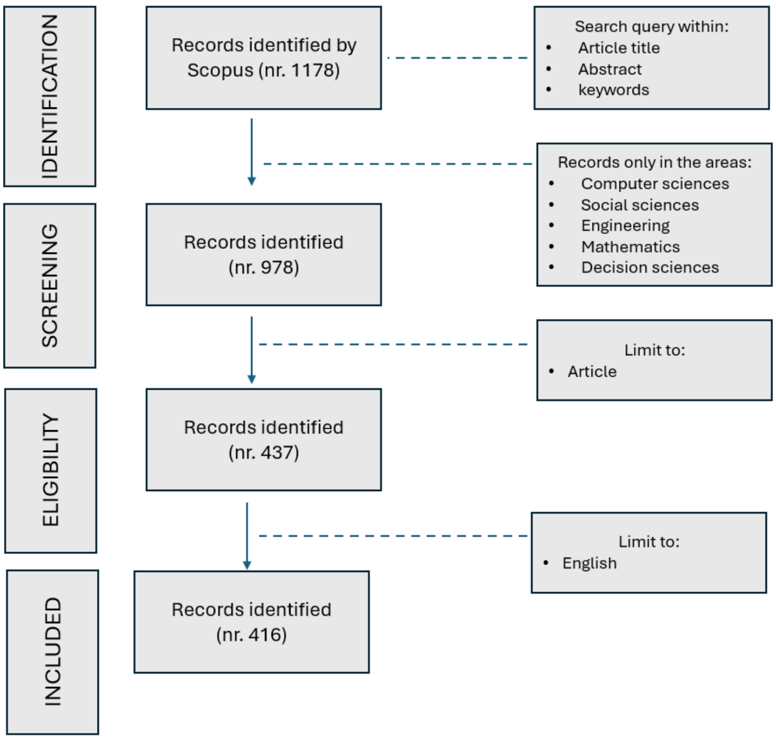

Figure 1: Stages of the systematic literature review process (database search, primary screening, and secondary screening) following PRISMA 2020 extended flow diagram (Preferred Reporting Items for Systematic Review and Meta-Analyses) flowchart. Source: our elaboration.

Academic articles from the Scopus database were analyzed, considering only English-language publications, with no temporal restrictions. A total of 416 relevant articles were found, meeting the criteria specified in the search string (“multi-criteria decision making” OR mcdm OR ”multi-criteria analysis” OR ”multiattribute decision making" OR madm OR AHP OR "analytic hierarchy process" OR "artificial intelligence" OR "ai") AND (access* OR inclus* OR "universal design" OR "barrier-free" OR "buildings") AND (school* OR "educational institution*" OR "learning environment*" OR "educational facilit*"). The search string is intentionally broad since in the first instance we tried to narrow the search by looking for applications of multi-criteria methods and AI to the accessibility of school institutions but the number of publications was null or irrelevant. For this reason we opted for a broader search that also included similar applications. The search was further refined using the keywords Artificial Intelligence, Machine Learning, Deep Learning, Decision Making, AI, Decision Support Systems, School, Machine-learning, Inclusive Education, AHP, Schools, Accessibility, Inclusion, Multicriteria Analysis, Analytical Hierarchy Process, Analytic Hierarchy Process, Natural Languages, Educational Institutions.

The identified articles were analyzed for the purpose of the ongoing literature review [Van Eck and Waltman, 2010, for History and Media, 2024]. The collected data were exported in CSV format for analysis using the VOSviewer software, to examine the relationships among article keywords, and in RIS format for further analysis using Zotero. During the export process, the following information was included: Citation information (Author(s), Document title, Year, EID, Source title, Volume, Issues, Pages, Citation count, Source and document type, Publication stage, DOI, Open access) and Abstract and keywords (Abstract, Author keywords, Indexed keywords).

3 Results

This section aims to provide a detailed and critical analysis of the current state of the literature. In the Sub-section 3.2 we provide an analysis of the current scientific landscape, highlighting the relationships between the keywords considered, identifying those that have been the main focus of the authors and examining how the scientific literature related to the research topic has evolved over time. In the Sub-section 3.3 is proposed an analysis on the main contributions in the school field to evaluate the accessibility of school institutions using multi-criteria decision-making methods and AI. More generally, an overview of the use of such methods in the education sector is provided by identifying unexplored and potential application areas.

3.1 Review Process and Selection Criteria

The review process was conducted in line with the PRISMA guidelines, which allowed us to perform a rigorous systematic literature review. The literature search was conducted using the Scopus database, which was chosen because it represents one of the main sources of scientific literature. Initially, keywords were entered into the search string and combined with Boolean operators (AND, OR) to find scientific articles related to the use of multi-criteria methods and AI in the educational setting. Subsequently, the articles were screened, considering only those in the areas of interest to our research which are computer sciences, social sciences, engineering, mathematics and decision sciences. Only articles in this type of publication are subject to methodological review, ensuring their scientific validity. Furthermore, only articles written in English were selected. Articles not related to the educational setting or concerning areas that do not concern the areas of interest we indicated were excluded. Finally, after selecting the articles, they were exported from Scopus and imported into VOSviewer. Using this software, we performed a quantitative analysis of the bibliography, identifying the main nodes and the relationships between keywords.

3.2 Current Scientific Landscape

In order to analyze the current scientific landscape, we used the VOSviewer software.
After processing the data imported from the Scopus database, the VOSviewer application allowed us to generate a map of the relationships between the keywords within the selected articles. A total of 3,519 keywords were selected. Of these, 178 met the threshold of at least 5 occurrences to be included in the visualization. Among these keywords, the following were selected: *artificial intelligence, machine learning, education, deep learning, school, decision making, learning environment, generative AI, higher education, convolutional neural networks, natural language processing, decision support systems, high school, quantitative analysis, inclusive education, AI technologies, accessibility, special education, school institutions, quality of life, AI tools, AI in education, secondary schools, AHP, analytic hierarchy process, inclusion, AI learning, multi-criteria analysis, sustainability*. The result is presented in Figure 2, which illustrates the scientific landscape of the keywords.

The cluster function is essential. The VOSviewer software uses a clustering algorithm to divide keywords into thematic groups. Keywords are placed into thematic groups based on co-occurrence, that is, how many times they appear in the same article. The logical connection between documents and keywords is precisely these co-occurrences, so if two keywords appear frequently within the article, the software connects them with a line. The distances between words on the map reflect the frequency of co-occurrences, so the closer the words, the more frequently they co-occur. Furthermore, the color of the clusters serves to visually distinguish the various groupings. It is very important to correctly interpret the map generated by the VOSviewer software because this allows us to analyze the research lines and any connections.

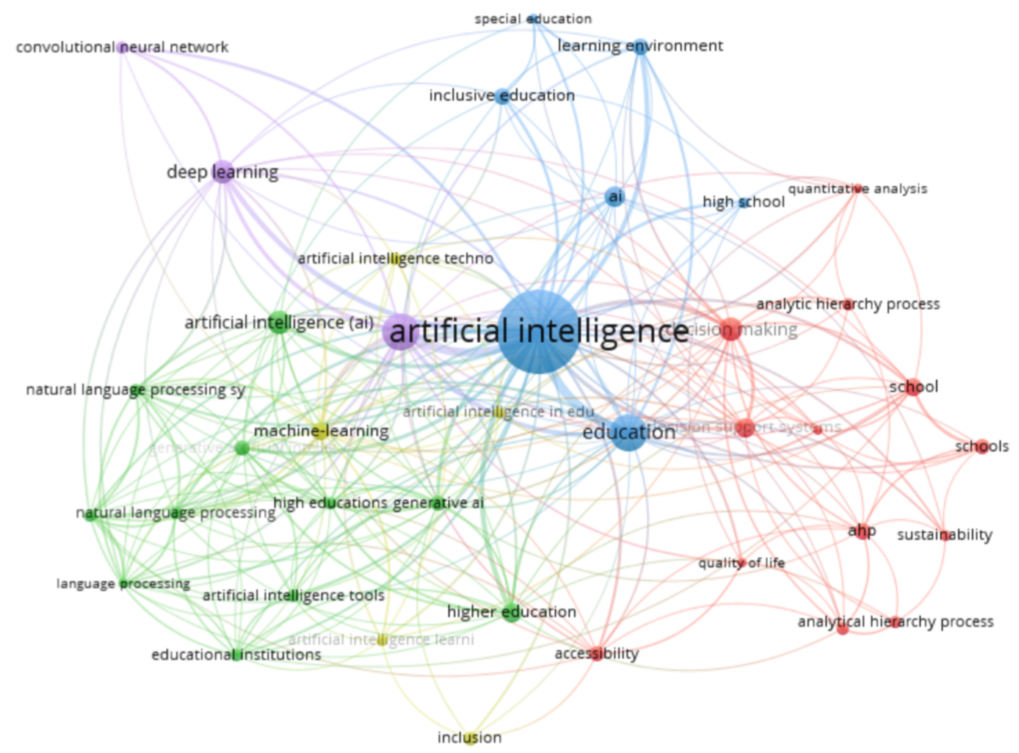

Figure 2: Scientific landscape of AI and MCDM in the school system obtained by Software VOSviewer.

As shown in the Figure 2, it is possible to identify five main clusters in which the words are classified (Cluster 1 - red, Cluster 2 - green, Cluster 3 - blue, Cluster 4 - yellow and Cluster 5 - purple). The Table 1 presents the classification of the keywords by cluster.

Table 1: Classification of keywords in five thematic clusters obtained through VOSviewer.

Cluster 1 — Red	**Cluster 2 — Green**	**Cluster 3 — Blue**	**Cluster 4 — Yellow**	**Cluster 5 — Purple**
accessibility	artificial intelligence (ai)	ai	artificial intelligence in education	convolutional neural networks
AHP	artificial intelligence tools	artificial intelligence	artificial intelligence learning	deep learning
analytic hierarchy process	educational institutions	education	artificial intelligence technologies	machine learning
analytical hierarchy process	generative ai	high school	inclusion	
decision making	generative artificial intelligence	inclusive education	machine-learning	
decision support systems	high educations	learning environment		
multicriteria analysis	higher education	special education		
quality of life	language processing			
quantitative analysis	natural language processing			
school	natural language processes			
schools	natural languages			
secondary schools				
sustainability				

In particular, analyzing the clusters specifically reported in Table 1 we can observe that:

- Cluster 1 - RED, focuses on multi-criteria decision making methods (MCDM) applied in school contexts. We can observe relationships between the elements of this cluster and artificial intelligence.

- Cluster 2 - GREEN, focuses on various AI technologies (such as natural language processing and generative AI) that have been applied in school settings, but not specifically for assessing the accessibility of school institutions.

- Cluster 3 - BLUE, focuses on school environments and inclusiveness. We also see connections between AI and education in this cluster.

- Cluster 4 - YELLOW, highlights connections between inclusion and AI.

- Cluster 5 - PURPLE, focuses exclusively on AI methods.

From the analysis of the map shown in Figure 2, we observe that the central node of the network is the keyword *"artificial intelligence"*. This represents the largest node, indicating that this keyword appears frequently in the analyzed articles and has a high number of connections with other keywords. In the analysis of the scientific landscape, we notice that there are various connections among the clusters. The themes we are focusing on MCDM, AI and school inclusion are interconnected, although the theme of inclusion appears to be more marginal. It is evident that MCDM methods and AI have been more extensively applied in other school-related areas. Looking at the map, we can observe a cohesive network, with few peripheral nodes and many connections among the keywords. From the analysis of the network, we observe strong interest in the use of AI in the school context [McMahon and Walker, 2019, Lin et al., 2024]. It is appropriate to proceed with the analysis of the Density Visualization map as well that measures the number of published documents and the strength of the relationships between keywords. The density map allows us to observe the keywords in the analyzed literature. The yellow and light green colors indicate the most relevant keywords, while the other colors correspond to lower relevance.

As can be seen in Figure 3, there is a predominance of the keyword *artificial intelligence*, indicating a strong focus. The second keyword highlighted by the density map is *education*.

Clearly, the keyword "artificial intelligence" is by far the most relevant: it is located at the center of the map and is surrounded by a bright yellow cloud. This means it appears very frequently in the analyzed articles and has many connections with other keywords. It is clearly the central thematic core of the network:

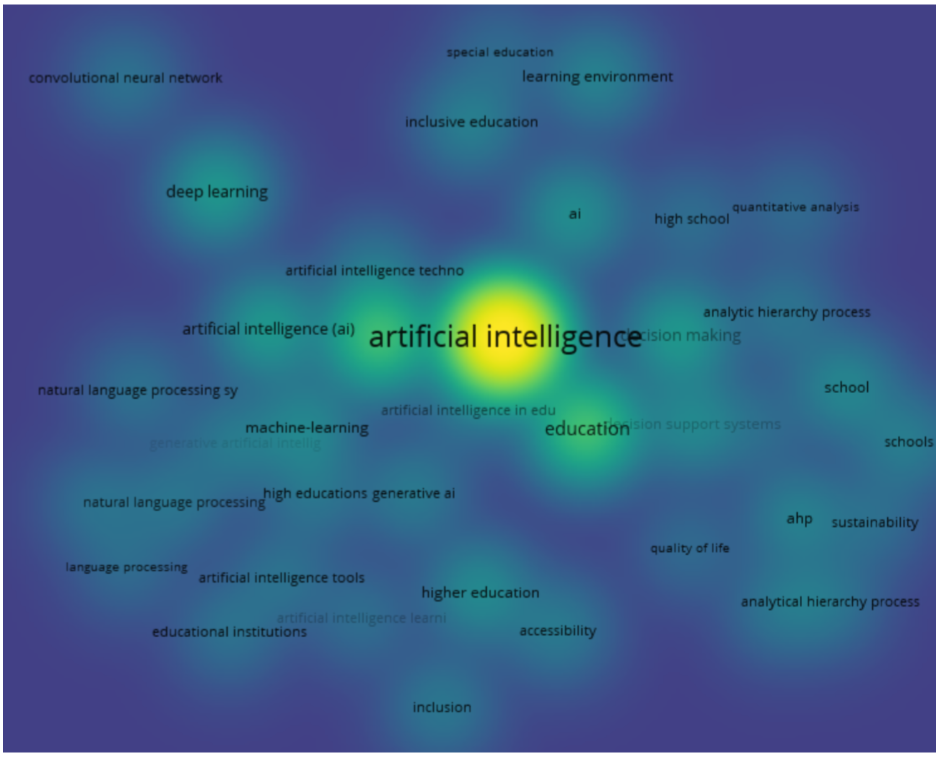

Figure 3: Density Visualization.

its central position and yellow color indicate that it occurs frequently in the documents and is highly interconnected with other terms.

Then we have education and AI, which are also surrounded by a slightly less intense but still significant yellow cloud. This suggests that AI is being used in school contexts.

The keywords machine learning, decision making, natural language processing, and higher education are located in intermediate-density areas, meaning they are fairly relevant, though not as central as those mentioned above.

Keywords such as inclusive education, learning environment, decision support systems, analytic hierarchy process, and sustainability are present with lower intensity. Therefore, they are likely less frequent in the scientific articles being analyzed.

Finally, the keywords neural network, deep learning, language processing, school institutions, inclusion, accessibility, and quality of life are the least relevant, occurring less frequently and showing fewer connections compared to the most prominent keywords. In particular, keywords such as “accessibility” and “inclusion” are low-density, which confirms that the topic of school institution accessibility in relation to AI and MCDM methods is still marginal [Gulzar et al., 2023, Chang et al., 2013].

In Figure 4, we report the graph represents the Overlay Visualization obtained by VOSviewer.

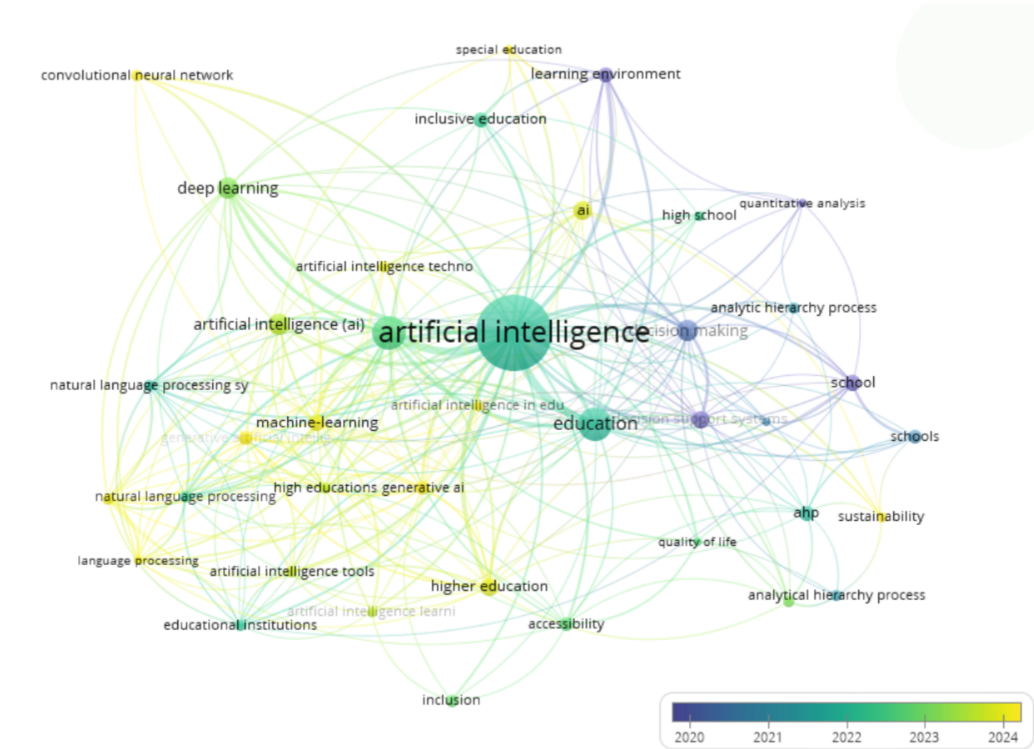

Figure 4: Overlay visualization of the keywords obtained with VOSviewer (2020–2024).

The Overlay Visualization generated by VOSviewer allows us to visualize, through a map, how the scientific literature related to the research topics has evolved over time. In this graph, the colors represent the average publication year of the scientific articles for each keyword. We distinguish three main colors:

- Blue keywords have an average publication year around 2020. Among the blue keywords, we find: decision making, analytic hierarchy process, sustainability, school, quantitative analysis. This is expected, as these are well-established topics in the scientific literature.

- Green keywords are transversal, meaning they have remained consistent across the years. Among the green keywords, we find: education, accessibility, higher education, machine learning, language processing. These are themes that have been addressed continuously over time and are still relevant today.

- Yellow keywords are those that approach 2024, indicating their recent emergence. Among the yellow keywords, we find: AI, natural language processing, artificial intelligence tools, generative AI, deep learning, convolutional neural network. These terms have recently appeared in the scientific landscape, indicating emerging areas of research.

3.3 Content Analysis

In this Section we analyze the scientific articles published in the school field with reference to the use of multi-criteria methods and AI. The focus is represented by the use of these methods individually and combined with each other to evaluate the accessibility of school institutions understood as the absence of architectural barriers that in fact limit or prevent access to people with disabilities. As we said in the section "Materials and methods", 416 articles were selected on Scopus using the keywords. We downloaded these articles in RIS format to analyze them using Zotero software.

At this point we analyzed the main contributions considering four macro-groups that concern the areas of analysis of our interest reported in Table 2.

Table 2: Keyword frequency by thematic category.

Category	**Keyword**	**Nr.**
MCDM methods	Decision Making	52
	AHP	26
	Multicriteria	11
AI	Artificial intelligence	354
	Machine learning	93
	Deep learning	43
Accessibility	Access	310
	Inclus	164
	Inclusion	75
	Accessibility	72
	Building	21
	Barrier free	4
	Universal Design	4
Educational Institutes	School	242
	Education	311

In particular, in the Table 2 we have reported for each keyword that we entered in the Zotero search field how many times it appeared (the frequency). For the MCDM methods category we note that the frequency of the keywords is low;

in fact Decision Making appears 52 times, AHP appears 26 times and Multicriteria only 11 times. As for the AI category we note that the keyword Artificial intelligence appears 354 times, Machine learning 93 times and Deep learning 43 times. This denotes that they represent the focus of the selected articles. The Accessibility category is also very relevant. In fact the keyword Access appears 310 times, Inclus 164, Inclusion 75 and Accessibility 72. We can deduce that the themes of inclusion and accessibility are very present in the articles [Abd El Karim and Awawdeh, 2020, Rogulj and Jajac, 2021]. The terms Universal Design and Barrier free are also present, even if to a lesser extent. Finally we have the School Institutes category which highlights the fact that the applications were made in a school environment. In fact, the terms Education and School are present in most of the articles; this is because our research was oriented precisely towards applications in a school environment. For the analysis of the main contributions we believe it is appropriate to provide an overview of the temporal evolution of the articles published regarding our research (see Figure 5), in order to evaluate the evolution of scientific research.

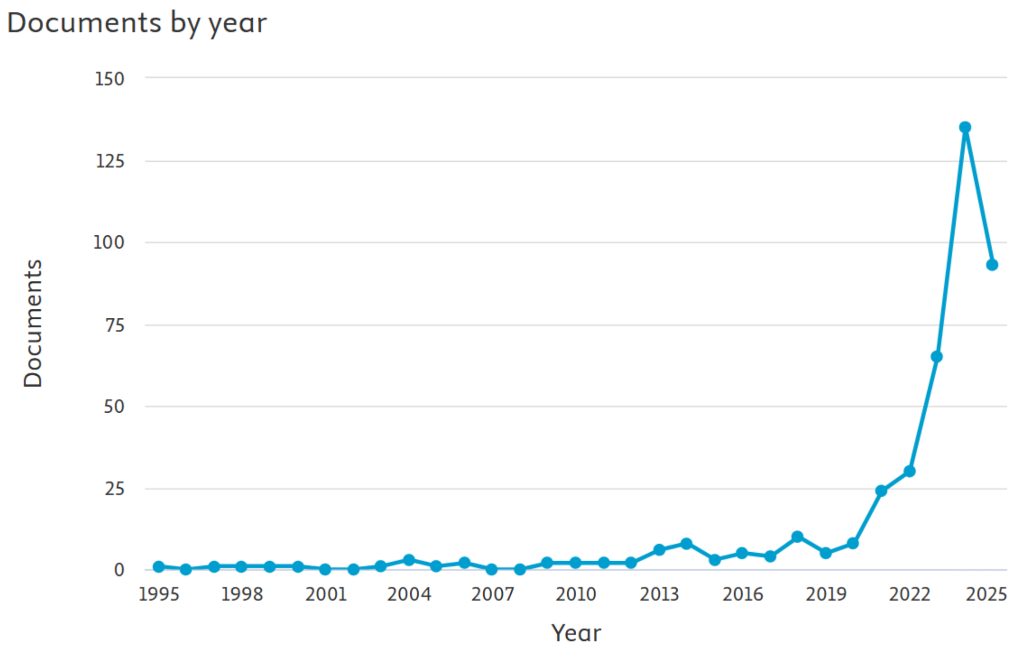

Figure 5: Documents by year. Source: Scopus.

From the analysis of the graph in Figure 5, which represents the trend of articles published over time, we can see that from 1995 to 2016 the publications related to the object of our research had few publications, demonstrating a low or non-current interest. In recent years, from 2016 to 2020 the number of published articles has gradually grown. In recent years, however, there has been an exponential growth. This is certainly due to the growing interest both at European and International level towards the issues of inclusiveness and accessibility but also to the development of multi-criteria methods and the growing applications of AI.

Analyzing the direct applications of AI on the topic of accessibility of school institutions we notice a notable discrepancy between the large scientific production related to the topic of AI in the school environment and its direct applications on the topic of school accessibility. In fact, by doing a search for the terms artificial intelligence and accessibility we obtain only 5 articles; artificial intelligence and inclusion 7 articles of which only 1 refers to the school environment. Even with the combination artificial intelligence, accessibility and education we do not obtain any results. It is clear that the applications of artificial intelligence to evaluate the accessibility of school institutions must be explored in depth.

Furthermore, analyzing the combinations between the terms multicriteria methods and accessibility of school institutions we note that the scientific applications are even more scarce. In fact, the search for the term multicriteria combined with accessibility provides only one result while the other combinations with inclusion and barrier free do not even produce one result. Even the keyword AHP (Analytic Hierarchy Process) searched in combination with accessibility or with inclusion provides only one result. No results with the combination barrier free. This highlights that multicriteria methods have currently been scarcely used to evaluate the accessibility of school institutions. Finally, we note that the combination of the keywords accessibility, multicriteria and artificial intelligence do not produce any results.

Now we can analyze (Table 3), within the scientific articles we have selected, what have been the main applications of artificial intelligence and multi-criteria methods in the school environment or in relation to issues related to accessibility.

Table 3: Overview of studies applying MCDM and AI methods in education.

Method	**Authors**	**Title**	**Purpose**
AHP	Baba et al. [2024]	A Comprehensive Framework for Assessing the Sustainability of Public Schools in Conflict Areas	*To assess sustainability of public schools in conflict zones, focusing on West Bank (Palestine).*

AHP	Shahraki et al. [2016]	Distributional planning of educational places in developing cities with case studies	*To analyze school distribution and propose redistribution using AHP and GIS.*
AHP	Chan et al. [2023]	Assessing Urban Sustainability and the Potential to Improve the Quality of Education and Gender Equality in Phnom Penh, Cambodia	*To assess sustainability and identify improvements for education and gender equality aligned with SDGs.*
AHP	Tahura et al. [2025]	Evaluating the online and offline learning effectiveness in Bangladesh using AHP	*To compare online and offline learning effectiveness using AHP.*
AHP	Teoh et al. [2022]	Analysis of ICT Implementation in Teaching and Learning using AHP	*To rank barriers to ICT implementation in rural and urban schools.*
AHP	Abd El Karim and Awawdeh [2020]	Integrating GIS and location-allocation models with MCDA for quality of life in Buraidah	*To assess accessibility to public services (schools, hospitals, etc.) as part of Vision 2030.*
AHP	Naccarella [2003]	Evaluation of the rural South Australian tri-division Adolescent Health Project	*To evaluate health project impact in rural areas on GP-adolescent relationships.*
Fuzzy AHP	Gulzar et al. [2023]	A Fuzzy AHP for Usability Requirements of Online Education Systems	*To evaluate usability of online learning platforms with fuzzy AHP.*
Fuzzy AHP	Chang et al. [2013]	Factors influencing sociability in educational MMORPGs – a fuzzy AHP approach	*To identify and rank sociability factors in educational MMORPGs.*

Fuzzy AHP	Incekara [2021]	Post-COVID-19 ergonomic school furniture design under fuzzy logic	*To design ergonomic furniture based on health, safety, and post-COVID needs.*
PROMETHEE	Rogulj and Jajac [2021]	Achieving a Construction Barrier-Free Environment: Decision Support to Policy Selection	*To support prioritization of accessibility improvements in schools.*
TOPSIS	Kumar and Banerji [2024]	Comparing inclusive education indicators: locomotor vs physical disabilities	*To rank inclusive education indicators based on disability type.*
DEA	Davies and Lewis [2020]	Children as researchers: An Appreciative Inquiry with primary-aged children	*To improve classroom communication using Appreciative Inquiry with students.*
DEA	McMahon and Walker [2019]	Leveraging emerging technology for inclusive education with UDL	*To explore how AI, robotics, and immersive tech support inclusive education.*
Generic AI	McMahon and Walker [2019]	Leveraging emerging technology to design an inclusive future with universal design for learning	*To explore opportunities and challenges from new technologies and propose an inclusive framework for experimentation.*
Automated analysis AI	Acosta-Vargas et al. [2022]	Accessibility Analysis of Worldwide COVID-19-Related Information Portals	*To evaluate accessibility of 199 official COVID-19 information websites related to medical education.*
Qualitative study AI	Castro et al. [2025]	Identifying Rural Elementary Teachers' Perception Challenges and Opportunities in Integrating Artificial Intelligence in Teaching Practices	*To investigate rural teachers' perceptions of using AI in education.*

Geospatial AI	Knoblauch et al. [2025]	Crime-associated inequality in geographical access to education: Insights from the municipality of Rio de Janeiro	*To examine the impact of crime on school access in Rio de Janeiro.*
Generic AI	Lin et al. [2024]	An Exploratory Study on the Efficacy and Inclusivity of AI Technologies in Diverse Learning Environments	*To analyze students' experiences and use of AI tools in education.*
Machine Learning, Deep Learning	Wagan et al. [2023]	Artificial Intelligence-Enabled Game-Based Learning and Quality of Experience: A Novel and Secure Framework (B-AIQoE)	*To propose a blockchain-secured AI-based game learning environment for education.*
Conceptual AI design	Xia et al. [2022]	A self-determination theory (SDT) design approach for inclusive and diverse artificial intelligence (AI) education	*To investigate how SDT-based teacher support and student attributes affect AI learning in schools.*
Generative AI	Jauhiainen and Guerra [2023]	Generative AI and ChatGPT in School Children's Education: Evidence from a School Lesson	*To examine the use of generative AI in primary school education.*
Machine Learning	Ulrich et al. [2024]	SIGNIFY: Leveraging Machine Learning and Gesture Recognition for Sign Language Teaching Through a Serious Game	*To present an ML-based educational tool for teaching sign language through a serious game.*

In the selected articles we can see that multi-criteria methods (MCDM) and AI

are used in the school environment separately. The most relevant applications regarding multi-criteria methods are the Analytic Hierarchy Process (AHP) which has been used [Baba et al., 2024] to evaluate the sustainability of public schools in conflict areas and [Shahraki et al., 2016] to analyze the distribution of school buildings in developing cities. Chan et al. [2023] also use AHP to evaluate the relationship between urban sustainability and quality of education, Tahura et al. [2025] use AHP to compare the effectiveness of online and offline learning. Teoh et al. [2022] studied the barriers to the use of ICT (Information and Communication Technologies) in urban and rural schools, while Abd El Karim and Awawdeh [2020] combined AHP with GIS (Geographic Information System) to evaluate the accessibility of school services in Saudi Arabia. The study by Naccarella [2003] used AHP in the healthcare sector.
Fuzzy AHP [Gulzar et al., 2023] was used to evaluate how the accessibility of online platforms [Chang et al., 2013] can promote socialization in educational role-playing games and [Incekara, 2021] to design ergonomic school furniture post-pandemic. The PROMETHEE (Preference Ranking Organization METHod for Enrichment Evaluations) method was used by Rogulj and Jajac [2021] in policies to eliminate architectural barriers in schools. TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) has been applied by Kumar and Banerji [2024] to classify school inclusion indicators according to the type of disability, while Davies and Lewis [2020] use DEA (Data Envelopment Analysis) to improve communication between students in participatory teaching. McMahon and Walker [2019] explored the use of technologies for inclusion purposes. As for AI, McMahon and Walker [2019] proposed an integration of AI, robotics and immersive learning for inclusion purposes, while Acosta-Vargas et al. [2022] evaluated.

4 Conclusion

This work fits within the framework of school systems by investigating the use of advanced quantitative tools to support decision-making processes related to the planning and management of school infrastructure. To do so, this article presents a systematic review of the literature on the use of MCDM methods and AI in the school system, with a focus on their use in assessing the accessibility of school buildings. From this perspective, the goal is to gain a comprehensive overview of the main current, emerging and potential research areas.
The systematic literature review was conducted following the PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) guidelines, focusing on 416 scientific contributions. The results of our investigation reveal that the academic debate focuses on the diverse areas of application of MCDM methods

and AI techniques in the school context. For example, to assess the sustainability of public schools in conflict-affected areas [Baba et al., 2024], to study the spatial distribution of school buildings in developing cities [Shahraki et al., 2016], to assess barriers to the use of ICT in rural and urban school contexts [Teoh et al., 2022] and also to measure the accessibility of school services from a geospatial perspective [Abd El Karim and Awawdeh, 2020].

Our analysis also revealed that the use of quantitative tools on issues related to school accessibility is still limited. However, given the relevance of the topic [Talen, 2001, Zhang et al., 2020, UNESCO, 2020], it can be considered that this area could represent a potential new line of research, of great social relevance, in which quantitative tools such as MCDM methods and AI can be used. It should be noted that there is a consolidated literature demonstrating the effectiveness of MCDM and AI tools in supporting decision makers in public and private strategic decisions [Saaty, 2008, Zavadskas et al., 2014].

For example, they could be used to classify schools based on various criteria, such as the presence of ramps, elevators, compliant restrooms, accessible internal routes and so on. The integrated use of AI and multi-criteria methods could also be considered to obtain even more powerful and effective tools [Cinelli et al., 2020].

We also note how the use of these tools and their integration could be incorporated into the Quadruple Helix model, which considers the joint contribution of academia, industry, government, and civil society in evaluating and improving the accessibility of school institutions [Carayannis and Campbell, 2010]. This would allow the definition and use of integrated tools that involve all stakeholders involved in decision-making processes affecting the school system. In line with this perspective, the outcomes of this study may also serve as a valuable reference for stakeholders such as families and local authorities, supporting their efforts to foster inclusive, safe and sustainable school environments.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

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The structure behind the Collatz algorithm

Michael O'Loughlin*

Abstract

The Collatz algorithm gives the operations to apply to both even and odd numbers so that starting with any natural number and applying these operations repeatedly, the number one is eventually reached. The Collatz conjecture claims that this is true no matter what starting number is used. It happens that the natural numbers can be organised into a hierarchical structure of infinite geometric sequences and the Collatz algorithm is the means of ascending through this hierarchy.
Keywords: Collatz algorithm; Collatz conjecture; geometric sequence; modulo.
2020 AMS subject classifications: 11B83, 00A05.[1]

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1 Introduction

Starting with any natural number, the Collatz algorithm [1] instructs us to divide by 2 if the number is even and to multiply by 3 and add 1 if the number is odd. Applying the algorithm to the even number 36 we get

$$18, 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Applying it to the odd number 75 produces

$$226, 113, 340, 170, 85, 256, 128, 64, 32, 16, 8, 4, 2, 1.$$

In both cases the number 1 is reached. We note that when nearing 1 the geometric sequence $1, 2, 4, 8, 16$ occurs in the first and the same sequence with a few extra terms $1, 2, 4, 8, 16, 32, 64, 128, 256$ occurs in the second case. Both occur reversed. It is no coincidence that this sequence occurs at this stage of the application of the algorithm.

The strategy we employ in this paper is to construct a hierarchical structure for the natural numbers using the inverses of the two operations used in the Collatz algorithm. Consequently, the algorithm merely becomes the means of navigating upwards through the hierarchy to reach the number 1.

We shall show that the natural numbers can be partitioned into an infinite set of infinite geometric sequences all having the same common ratio 2 and all starting with an odd number. These sequences can be organised into a hierarchical structure by using the inverse of the operation of multiplying by 3 and adding 1. Repeatedly dividing by 2 is the way to move backwards along any one of these sequences. Multiplying by 3 and adding 1 is the way to move from the initial term of any sequence to an even term of the sequence that is one level above it in the hierarchy. Using this hierarchical structure, not only can we move from any term back to the number 1 but we can also move from any number to any other number and in addition we can find a number located at any specified level within the hierarchy.

2 Partitioning the set of natural numbers

First we shall show that the set of all even numbers can be partitioned into an infinite number of infinite geometric sequences all with common ratio 2 and with distinct first terms.

Definition 2.1. *The set of natural numbers* $\mathbb{N} = \{1,\ 2,\ 3,\ \dots\}$.

In this paper the word 'number' shall mean 'natural number' unless stated otherwise.

Proposition 2.1. *The set of even numbers* $\{2n,\ n \in \mathbb{N}\} = \bigcup_{i \in \mathbb{N}} \{(2i-1)2^j,\ j \in \mathbb{N}\}$

Proof. Consider the even numbers:

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16, \ldots$$

First extract the powers of two, $\{2^k,\ k \in \mathbb{N}\}$

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \ldots$$

This leaves $6,\ 10,\ 12,\ 14,\ 18,\ 20,\ 22,\ 24,\ 26,\ 28,\ 30,\ 34,\ \ldots$
Next extract 3 times the powers of two, $\{2^k \times 3,\ k \in \mathbb{N}\}$

$$6,\ 12,\ 24,\ 48,\ \ldots$$

This leaves $10,\ 14,\ 18,\ 20,\ 22,\ 26,\ 28,\ 30,\ 34,\ \ldots$
Next extract 5 times the powers of two, $\{2^k \times 5,\ k \in \mathbb{N}\}$

$$10,\ 20,\ 40,\ 80,\ \ldots$$

This leaves $14,\ 18,\ 22,\ 26,\ 28,\ 30,\ 34,\ 36,\ 38,\ 42,\ \ldots$
By continuing in this manner we partition the even numbers into an infinite number of disjoint infinite geometric sequences:

$$2,\ 4,\ 8,\ 16,\ \ldots$$

$$6,\ 12,\ 24,\ 48,\ \ldots$$

$$10,\ 20,\ 40,\ 80,\ \ldots$$

$$\vdots$$

Therefore $\{2n,\ n \in \mathbb{N}\} = \bigcup_{i \in \mathbb{N}} \{(2i-1)2^j,\ j \in \mathbb{N}\}$ □

Next we complete the partitioning of the set of all natural numbers.

Proposition 2.2. *The set of all natural numbers can be partitioned into an infinite set of infinite geometric sequences.* $\mathbb{N} = \bigcup_{i \in \mathbb{N}} \{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$

Proof. To each of the infinite geometric sequences, $\{(2i-1)2^j,\ j \in \mathbb{N}\}$, prepend the corresponding odd number to get $\{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$.

$$\therefore\ \mathbb{N} = \bigcup_{i \in \mathbb{N}} \{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$$ □

We make the following observations regarding the elements of these sequences.

1. Every element of every sequence can be expressed in the form $2^{k-1}a$ for some $k \in \mathbb{N}$ and some a an odd number.

2. Every element which is an even number can be expressed in the form $2^k a$ for some $k \in \mathbb{N}$ and some a an odd number.

3. Where the initial term a in any sequence is congruent to $1 \pmod 3$ then the even elements in that sequence are alternately congruent to $2 \pmod 3$ and to $1 \pmod 3$.

4. Where the initial term a in any sequence is congruent to $2 \pmod 3$ then the even elements in that sequence are alternately congruent to $1 \pmod 3$ and to $2 \pmod 3$.

5. Where the initial term a in any sequence is congruent to $0 \pmod 3$, i.e., is a multiple of 3, then the even elements in that sequence are also multiples of 3.

Definition 2.2. *We shall refer to the initial term of each sequence as the **head** of the sequence.*

3 Mapping even numbers exceeding a multiple of 3 by 1, to odd numbers

It is those even elements in each sequence which are congruent to $1 \pmod 3$ that are of most interest to us. Each of these elements can be expressed in one of 2 ways:

$$\begin{cases} 2^{2k}a, k \in \mathbb{N} \quad \text{when } a \equiv 1 \pmod 3 \\ 2^{2k-1}a, k \in \mathbb{N} \quad \text{when } a \equiv 2 \pmod 3 \end{cases}$$

because an even power of 2 is congruent to $1 \pmod 3$ and an odd power to $2 \pmod 3$.

Let A be the set of all even numbers congruent to $1 \pmod 3$ and let B be the set of all odd numbers. If $x \in A$ then $(x-1)$ is odd and a multiple of 3. Hence $(x-1)/3 \in B$. We now have the following:

Proposition 3.1. *The mapping*

$$\theta : A \to B : x \to (x-1)/3 \tag{1}$$

is a bijection.

Proof. First, we show that θ is injective. Suppose that $x, y \in A$ and that $\theta(x) = \theta(y)$.
Thus, $(x-1)/3 = (y-1)/3$.
Therefore $x = y$.
Hence, each element in A maps to a distinct element in B.
Next we show that θ is surjective.
Suppose there exists $m \in B$.
Then $3m+1$ is even and congruent to 1 (mod 3).
Therefore $3m+1 \in A$.
Now $\theta(3m+1) = m$.
Hence, each element in B is the image of an element in A.
It follows that θ is a bijection. □

Because, in Section 2, we have partitioned the set of even numbers into an infinite number of geometric sequences of the form $\{2^k a, k \in \mathbb{N}\}$ where a is an odd number, the elements of set A are distributed among those sequences for which a is congruent to 1 (mod 3) or 2 (mod 3). See observations 3, 4 and, 5 in Section 2.

4 Constructing the hierarchy of sequences

1. Begin with 1 and construct the geometric sequence using $u_{n+1} = u_n \times 2$ to yield the sequence $\{1, 2, 4, 8, 16, \dots\}$

2. Every alternate even term of this sequence is congruent to 1 (mod 3) being of the form $2^{2k}, k \in \mathbb{N}$.

3. Apply the mapping θ to each of the terms congruent to 1 (mod 3) to generate an odd number which shall be the head of a new geometric sequence, i.e. $\theta(2^{2k}) = (2^{2k}-1)/3$. This produces the odd numbers $1, 5, 21, 85, 341, \dots$

4. For each of these odd numbers a — except 1 whose sequence is already generated — generate the geometric sequence $\{u_1 = a,\ u_{n+1} = u_n \times 2,\ n \in \mathbb{N}\}$ with a as head.

5. For each new sequence whose head is congruent to 1 (mod 3) or 2 (mod 3) apply the mapping θ to those terms which are even and congruent to 1 (mod 3).

6. Continue this process indefinitely.

In effect, that subset of the set A in Section 3 which is contained in each generated sequence whose head is congruent to 1 (mod 3) or 2 (mod 3) is mapped to a subset of the set of odd numbers. The set A is built up sequence by sequence.

Example 4.1. *Take the odd number* 5*, produced above by the even number* 16*. It generates the sequence* $\{5, 10, 20, 40, 80, 160, \dots\}$*. The even number* 10 *produces the odd number* 3*; the even number* 40 *produces the odd number* 13*; the even number* 160 *produces the odd number* 53 *and so on. Each of these odd numbers generates a geometric sequence with the odd number as head.*

5 Some patterns to the odd numbers generated

The odd numbers produced by any given sequence follow a recursive pattern. Let m be an even number congruent to 1 (mod 3) in a sequence. The number m produces the odd number $(m-1)/3$. The next number in the sequence congruent to 1 (mod 3) is $4m$. This produces the odd number $(4m-1)/3$. This can be rewritten in the form $4\big((m-1)/3\big)+1$ so we have the recursive pattern

$$u_{n+1} = 4u_n + 1 \tag{2}$$

for the odd numbers generated.

The pattern can be adapted to ascertain whether any given odd number is the first odd number to be generated from a sequence. If, on subtracting 1 and dividing by 4, we get another odd number then the odd number is not the first generated. If we get an even number then it is the first or if there is a remainder on dividing by 4 then it is also the first.

Example 5.1. *The odd number* 561 *is the first odd number generated from a sequence since the quotient on dividing by* 4 *is even. We can check that it is generated by the even number* 1684 *in the sequence whose head is* 421*. The odd number* 563 *is the first odd number generated from a sequence since there is a remainder on dividing by* 4*. We can check that it is generated by the even number* 1690 *in the sequence whose head is* 845*. The odd number* 565 *is not the first odd number generated from a sequence since the quotient on dividing by* 4 *is odd. We can check that it is generated by the even number* 1696 *in the sequence whose head is* 53 *and that the first odd number generated is* 35*.*

Because of the recursive pattern, the congruence of the odd numbers produced cycles through the values $0, 1, 2$ in some order depending on the congruence of the first odd number produced. Let b be the first odd number produced from some sequence. The next one is $4b+1$ and since $4b$ is congruent to b (mod 3), $4b+1$ is congruent to $(b+1)$ (mod 3). Thus, the cycle goes

$$\begin{cases} 2,0,1 & \text{when } b \equiv 2 \pmod 3 \\ 0,1,2 & \text{when } b \equiv 0 \pmod 3 \\ 1,2,0 & \text{when } b \equiv 1 \pmod 3 \end{cases}$$

Definition 5.1. *Let us define the even number congruent to* $1 \pmod 3$ *as the* ***parent*** *and the sequence that contains it as the* ***parent sequence*** *and the odd number produced by applying the mapping* θ (1) *to the parent as the* ***child*** *and the sequence which has the child as its head as the* ***child sequence****.*

The heads of the child sequences produced by the parents in the parent sequence can be pre-determined by the head of the parent sequence. Let a be the head of the parent sequence. When a is congruent to 2 (mod 3), the first parent is $2a$ and the first child is $(2a-1)/3$. Hence, the children are given recursively from (2) by:

$$u_{n+1} = 4u_n + 1, \quad u_1 = (2a-1)/3$$

Example 5.2. *Consider the parent sequence* $\{5, 10, 20, 40, \dots\}$*. Its head* 5 *is congruent to* 2 (mod 3)*. The heads of the child sequences produced by this parent sequence are* $3, 13, 53, \dots$

When a is congruent to 1 (mod 3), the first parent is $4a$ and the first child is $(4a-1)/3$. Hence, the children are given recursively by:

$$u_{n+1} = 4u_n + 1, \quad u_1 = (4a-1)/3$$

Example 5.3. *Consider the parent sequence* $\{7, 14, 28, 56, 112, \dots\}$*. Its head* 7 *is congruent to* $1 \pmod 3$*. The heads of the child sequences produced by this parent sequence are* $9, 37, 149, \dots$

We can solve the recurrence relation (2) to obtain

$$u_n = 4^{n-1}u_1 + \frac{4^{n-1}-1}{3}$$

When $a \equiv 1 \pmod 3$ then $u_1 = \dfrac{4a-1}{3}$. Inserting this and simplifying we get

$$u_n = \frac{4^n a - 1}{3} \quad \text{when } a \equiv 1 \pmod 3 \tag{3}$$

When $a \equiv 2 \pmod 3$ then $u_1 = \dfrac{2a-1}{3}$. Inserting this and simplifying we get

$$u_n = \frac{4^{n-1}.2a - 1}{3} \quad \text{when } a \equiv 2 \pmod 3 \tag{4}$$

In both Equations (3) and (4) the value of n in the index of the power of 4 tells us the position of the child among the offspring of the parent sequence with head a. Given any random odd number we can use either of (3) or (4) to determine the head of its parent sequence and its ordinal position.

Example 5.4. *Which child of its parent sequence is* 277 *and what is the head of the parent sequence?*

$$\begin{aligned} \textit{Let } 277 &= \frac{4^n a - 1}{3} \\ 832 &= 4^n a \\ &= 4^3 \times 13 \end{aligned}$$

Thus, 277 *is the third child of the parent sequence whose head is* 13*.*

We now show that every odd natural number is a child of a parent sequence.

Proposition 5.1. *Every odd natural number is a child of a parent sequence.*

Proof. Consider $\dfrac{4^n a - 1}{3}$, where a is odd and $a \equiv 1 \pmod 3$. Since $4^n \equiv 1 \pmod 3$ and $a \equiv 1 \pmod 3$ the numerator is congruent to $0 \pmod 3$. It is also an odd number. Hence, $\dfrac{4^n a - 1}{3}$ is odd. A similar argument applies to $\dfrac{4^{n-1}.2a - 1}{3}$. Let q be an odd natural number. Then q can be written in the form $\dfrac{4^n a - 1}{3}$ or $\dfrac{4^{n-1}.2a - 1}{3}$ for some value of a and of n. Hence, q is a child of the parent sequence whose head is a. $\square$

6 Levels and level numbers

We started with the sequence $\{1, 2, 4, 8, 16, \dots\}$. Let us designate this sequence as being at Level 1. Level 1 contains one infinite geometric sequence.

The infinite number of sequences generated by the odd numbers produced by those even numbers congruent to $1 \pmod 3$ contained in the first sequence we designate as being at Level 2. Level 2 contains an infinite number of infinite geometric sequences.

Those sequences which are generated from the sequences at Level 2 we designate as being at Level 3. Level 3 and each subsequent Level contain a countably infinite family of infinite geometric sequences.

Every sequence in the hierarchy is at a certain level as are the individual elements of that sequence. Thus, we have the following definition:

Definition 6.1. *The **Level** of a number x is the count of the heads of the sequences which must be traversed in order to move from x to the number 1. The count includes the number 1 and the head of the sequence containing x. The Level of a sequence is the same as the Level of any element of that sequence.*

Example 6.1. *The number 18 is at Level 7. The heads of the sequences traversed are*

$$9, 7, 11, 17, 13, 5, 1.$$

The sequence $\{9, 18, 36, \dots\}$ is also at Level 7.

Definition 6.2. *Let us define the **antecedents** of a number x as the list of the heads of the sequences which must be traversed in order to move from x to the number 1.*

Figure 1 shows the output from code written in the Python programming language to calculate the level and antecedents of a specified number. Not only can

```
To find the level and antecedents of a number:

Type any positive whole number: 1122

1122 is at level 14
Antecedents of 1122: [561, 421, 79, 119, 179, 269, 101, 19, 29, 11, 17, 13, 5, 1]
```

Figure 1: A screenshot showing the level and antecedents of a specified number.

we use the hierarchy of sequences to obtain the level and antecedents of any number but we can also use it to move from any number to any other number as shown in Figure 2. Note that in Figure 2 the path between the two numbers x and y say, contains two adjacent even numbers. These two are elements of a sequence whose head is the first common antecedent of x and y when traversing upwards. In addition we can find a number at a specified level (Figure 3).

7 Interpreting the Collatz algorithm

The Collatz algorithm tells us how to move from any given number through the hierarchy of geometric sequences to reach the number 1 at the head of the topmost sequence of the hierarchy.

```
***** Travel from any number to any other number *****

Starting number: ? 561

Finishing number: ? 563

561 is at level 14
563 is at level 14

Outbound journey from 561 to 563
[561, 421, 79, 238, 952, 317, 845, 563]

Return journey from 563 to 561
[563, 845, 317, 952, 238, 79, 421, 561]
```

Figure 2: A screenshot showing the path between two numbers.

```
***** To find a number at a given level *****

Type the level number you require ? 42

The number 3569835538891 is at level 42
```

Figure 3: A screenshot showing a number at a specified level.

The instruction to divide an even number by 2 simply means "Move backwards towards the head of the sequence containing the even number".

The instruction to multiply an odd number by 3 and add 1 means "Move up from the odd number to its parent, an even number, which lies on a sequence one level above the sequence containing the odd number".

Given any even number we can determine (a) the head of the sequence which contains it, (b) its level number, (c) its antecedents.

Given any odd number we can determine (a) its parent, (b) the head of its parent sequence, (c) its position among the children of its parent sequence, (d) its level number, (e) its antecedents.

No odd number exists in isolation. Each is connected to its parent, an even number congruent to $1 \pmod 3$, through the mapping θ (1) and the parent is connected to the head of the parent sequence through the Collatz algorithm for even numbers. Thus, we associate a triple of numbers with each odd number — (child, parent, head of the parent sequence). Consider the triple (x_1, y_1, z_1). Then, z_1 becomes the child in the triple (z_1, y_2, z_2). Next, z_2 becomes the child in (z_2, y_3, z_3). This process continues all the way up to the triple $(1, 4, 1)$ at the top level. The latter triple stops the ascent from one level to the level above it. Instead, it loops around indefinitely.

Definition 7.1. *Let us define C_e as the operation of dividing an even number by* 2 *repeatedly until we get an odd number and C_o as the operation of multiplying an odd number by* 3 *and adding* 1.

The operation C_e starts with an even number and produces an odd number and both are at the same Level ℓ. The operation C_o starts with an odd number at Level ℓ and produces an even number at Level $\ell - 1$.The compound operation $C_o \circ C_e$ starts with an even number at Level ℓ and produces an even number at Level $\ell - 1$. Likewise the compound operation $C_e \circ C_o$ starts with an odd number at Level ℓ and produces an odd number at Level $\ell - 1$. Thus, we have the following:

Proposition 7.1. *Repeated use of $C_o \circ C_e$ reduces the Level number to one. Likewise, repeated use of $C_e \circ C_o$ achieves the same result.*

Proof. Suppose we start with an even number at Level ℓ. Each application of $C_o \circ C_e$ reduces ℓ by 1. Ultimately it reaches 1.

Now, suppose we start with an odd number at some Level ℓ. As before, each application of $C_e \circ C_o$ reduces ℓ by 1 and ultimately it reaches 1. □

8 Accounting for the rise and fall of the antecedents of a number

If we consider the head of a parent sequence and map it to the heads of the child sequences generated by the parent sequence, we get a one to many mapping. But if we confine the mapping to the head of the first child sequence we get a one to one mapping. The head of the parent sequence is an odd number congruent to 1 (mod 3) or to 2 (mod 3) as described in Section 4.

Consider the odd number congruent to 2 (mod 3). It can be expressed in the form $6n - 1$. The first even number in the sequence generated by $6n - 1$ that is congruent to 1 (mod 3) is $2(6n - 1)$. This even number produces its child, the odd number $4n - 1$. Now $6n - 1 > 4n - 1 \; \forall \; n \in \mathbb{N}$ so the first child is always smaller than the head of the parent sequence. Subsequent children are all bigger due to the recursive formula $u_{n+1} = 4u_n + 1$ from Section 5.

Now consider the odd number congruent to 1 (mod 3). It can be expressed in the form $6n - 5$. The first even number in the sequence generated by $6n - 5$ that is congruent to 1 (mod 3) is $4(6n - 5)$. This even number produces its child, the odd number $8n - 7$. Now $6n - 5 < 8n - 7 \; \forall \; n > 1$ so the first child is always bigger than the head of the parent sequence for $n > 1$. Subsequent children are all bigger too.

It follows that the heads of child sequences are all bigger than the head of a parent sequence except when the head of the parent sequence is congruent to 2 (mod 3) and the child sequence is the first one. It is this that causes the antecedents of a number to fluctuate up and down. We shall illustrate with an example.

Example 8.1. *The antecedents of the number* 29 *are* $11, 17, 13, 5, 1$. *Looking at this in the opposite direction so that the heads of the parent sequences come first we have* $1, 5, 13, 17, 11$. *The head of the first parent sequence is* 1 *which is congruent to* $1 \pmod 3$ *and* 5 *is the head of its second child sequence so* $5 > 1$ *as expected. Taking* 5 *as the head of the parent sequence,* 13 *is the head of its second child sequence and* $13 > 5$ *as expected. Taking* 13 *as the head of the parent sequence,* 17 *is the head of its first child sequence so* $17 > 13$ *as expected since* 13 *is congruent to* $1 \pmod 3$. *Taking* 17 *as the head of the parent sequence,* 11 *is the head of its first child sequence so* $11 < 17$ *as expected since* 17 *is congruent to* $2 \pmod 3$. *Taking* 11 *as the head of the parent sequence,* 88 *is the second parent and* 29 *is its child so* $29 > 11$ *as expected.*

9 A related hierarchical structure

To illustrate the difficulty with the Collatz conjecture we look at a related algorithm which is equivalent to the Collatz algorithm for negative integers but applied to the natural numbers. We construct a hierarchical structure of the same geometric sequences as before, starting with $\{1, 2, 4, 8, 16, \dots\}$. Instead of making the even elements which are one more than a multiple of 3 be the parent elements we make those even elements which are one less than a multiple of 3 be the parents. Hence, it is the elements $2, 8, 32, \dots$ of the first sequence that are the parents. We generate each child by adding one to the parent and dividing by three to get $1, 3, 11, \dots$. The inverse operation to ascend the hierarchy is to multiply by 3 and subtract 1.

The properties of this other structure are very similar to those we developed already. The heads of the child sequences of a parent sequence follow a recursive pattern as before and each child can be expressed in terms of the head of the parent sequence as before. Every odd natural number can be expressed in a form similar to Proposition 5.1 so that every odd natural number is a child of a parent sequence.

Despite all these similarities the natural numbers split into 3 separate hierarchies and in only one of these can the algorithm return numbers to 1. In another, numbers can only return to cycle through the heads 5 and 7 indefinitely — more fully, through $5, 14, 7, 20, 5$. In the third hierarchy the numbers can only return to cycle through $17, 25, 37, 55, 41, 61, 91$. The reason for this is that one of the antecedents of a number turns out to be that number itself. This happens to each of the numbers listed here. Despite the result corresponding to Proposition 5.1 being true, in this instance it is no guarantee that all odd natural numbers can return to the head of the first sequence.

10 Conclusion

If we consider the even numbers $2, 4, 6, 8, 10, 12, \ldots$ we see that they are written in an orderly way. But if we write the same numbers in the same order in terms of powers of two, $2^1, 2^2, 2^1 \times 3, 2^3, 2^1 \times 5, 2^2 \times 3, \ldots$, they appear to be much less orderly. It suggests that they can be split up as follows:

$$\begin{aligned}
&2^1, 2^2, 2^3, \ldots\\
&2^1 \times 3, 2^2 \times 3, 2^3 \times 3, \ldots\\
&2^1 \times 5, 2^2 \times 5, 2^3 \times 5, \ldots
\end{aligned}$$

This is what we did in Section 2, partitioning both the even numbers and the natural numbers into infinite geometric sequences with common ratio two.

In Section 3 we set up a bijection between the set of even numbers congruent to $1 \pmod 3$ and the set of odd numbers by using the inverse of the Collatz algorithm for odd numbers. This enabled us in Section 4 to generate the heads of new child sequences from the even numbers congruent to $1 \pmod 3$ in the current sequence. The child sequences were then generated from the heads by using the inverse of the Collatz algorithm for even numbers. In this way the hierarchy of sequences was built up.

Following on from this in Section 5 we showed that the heads of the child sequences followed a recursive pattern and hence, we were able to ascertain which child was the first child of its parent sequence and also what was the position of a given child among the offspring of the parent sequence.

In Section 6 we showed how the sequences were ordered within the hierarchy by defining the level of both a sequence and its elements within the hierarchy. We demonstrated that the antecedents and level of any number can be obtained; we showed that it is possible to move from any number to any other number within the hierarchy and that, given a particular level, we can obtain a number situated at that level.

Importantly, in Section 7 we proved that both parts of the Collatz algorithm applied in succession to either an even number or an odd number reduce the Level number by one and repeated applications reduce the Level to Level 1.

In Section 8 we gave an explanation of why the antecedents of a number rise and fall in magnitude.

In this paper we have shown how the natural numbers can be partitioned into an infinite number of infinite geometric sequences and that these sequences can be organised into a hierarchy. We provided the mechanism for creating the hierarchy and for ascending from any sequence to the head of the topmost sequence. But this alone does not confirm the Collatz conjecture as the example in Section 9 illustrates. It is necessary to show that the hierarchy stays intact and does not split into two or more hierarchies as happened in the example in Section 9. To keep it intact the head of any sequence cannot appear again among its antecedents or, equivalently, among its descendants.

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Generalization of intuitionistic fuzzy n-normed spaces via ψ-extensions

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Abstract

The aim of this paper is to study the generalization of intuitionistic fuzzy normed spaces to intuitionistic fuzzy n-normed spaces. In this framework, we discuss intuitionistic fuzzy n-continuity and intuitionistic fuzzy n-boundedness. Furthermore, we introduce the intuitionistic fuzzy ψ-n-normed space, which generalizes the intuitionistic fuzzy n-normed space. Several results and properties in this new set-up are also presented and analyzed.

Keywords: Intuitionistic fuzzy n-normed space, Intuitionistic fuzzy $\psi - n$-normed space

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1 Introduction

The theory of fuzzy sets was first introduced by Zadeh (16) in 1965. Following Zadeh's pioneering work, many researchers have extended the concept into various branches of mathematics, leading to new theories such as fuzzy group theory, fuzzy differential equations, fuzzy topology, and fuzzy normed spaces (13), among others. In particular, the study of fuzzy normed spaces and their generalizations has attracted significant attention.

Atanassov (3) introduced the concept of intuitionistic fuzzy sets, which was further developed by Coker (4). Park (12) proposed the notion of intuitionistic fuzzy metric spaces, and Saadati and Park (13) introduced intuitionistic fuzzy normed spaces. In many situations where classical norms are inadequate, the framework of intuitionistic fuzzy norms proves to be more suitable. Extensive research has been conducted in this area, as seen in (7), (10), (11), (13).

More recently, M. Mursaleen (9) defined the structure of intuitionistic fuzzy 2-normed spaces and studied foundational results analogous to those in normed linear spaces. In this paper, we investigate continuity and boundedness within intuitionistic fuzzy 2-normed spaces. Additionally, T.K. Samanta and Sumit Mohinta (14) introduced intuitionistic fuzzy ψ-normed spaces and examined continuity and boundedness in that framework. Building upon these ideas, we introduce the notion of intuitionistic fuzzy ψ-n-normed spaces, which generalizes intuitionistic fuzzy n-normed spaces. This new framework provides a more flexible approach for handling situations where classical norms or n-norms may be insufficient due to inherent inexactness.

2 Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1. *(15) A binary operation* $*: [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous t-norm if it satisfies the following conditions:*

(a) $*$ *is associative and commutative;*

(b) $*$ *is continuous;*

(c) $a * 1 = a$ *for all* $a \in [0,1]$*;*

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Example 2.1. *Two typical examples of continuous t-norms are*

$$a * b = ab \ \text{ and } \ a * b = \min\{a, b\}.$$

Definition 2.2. *(15) A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*

(a) $\diamond$ *is associative and commutative;*

(b) $\diamond$ *is continuous;*

(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*

(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Example 2.2. *Two typical examples of continuous t-conorms are*

$$a \diamond b = \min\{a+b, 1\} \text{ and } a \diamond b = \max\{a, b\}.$$

Definition 2.3. *(13) The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,*

(a) $\mu(x,t) + \nu(x,t) \leq 1$*;*

(b) $\mu(x,t) > 0$*;*

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$*;*

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$*;*

(f) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$*;*

(h) $\nu(x,t) < 1$*;*

(i) $\nu(x,t) = 0$ *if and only if* $x = 0$*;*

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$*;*

(l) $\nu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous;*

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm.

Example 2.3. *Let $(V, \|\cdot\|)$ be normed space over $\mathbb{F}$. Denote $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}, \forall a, b \in [0,1]$ and let μ_0 and ν_0 be fuzzy sets on $V \times (0, \infty)$ defined as follows $\mu_0(x,t) = \dfrac{t}{t + \|x\|}$, $\nu_0(x,t) = \dfrac{\|x\|}{t+\|x\|}$, for all $t \in \mathbb{R}^+$. Then $(V, \mu_0, \nu_0, *, \diamond)$ is an intuitionistic fuzzy normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*

Definition 2.4. *(9) Let V be a real vector space of dimension d, where $n \leq d < \infty$. An n-*norm *on V is a function*

$$\|\cdot, \cdot, \dots, \cdot\| : V^n \longrightarrow \mathbb{R},$$

which satisfies, for every $x_1, x_2, \dots, x_n, y_1 \in V$ and $\alpha \in \mathbb{R}$:

(a) $\|x_1, x_2, \dots, x_n\| = 0$ *if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;*

(b) $\|x_1, x_2, \dots, x_n\|$ *is invariant under permutation of its arguments;*

(c) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \, \|x_1, x_2, \dots, x_n\|$;

(d) $\|x_1 + y_1, x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|y_1, x_2, \dots, x_n\|$.

*The pair $(V, \|\cdot, \cdot, \dots, \cdot\|)$ is then called an n-*normed space.

Definition 2.5. *(9) The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times \cdots \times V \times (0, \infty)$ satisfying the following conditions. For every $x, y, z \in V$ and $s, t > 0$,*

(a) $\mu(x,y,t) + \nu(x,y,t) \leq 1$;

(b) $\mu(x,y,t) > 0$;

(c) $\mu(x,y,t) = 1$ *if and only if x and y are linearly dependent;*

(d) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$ *for each $\alpha \neq 0$;*

(e) $\mu(x,y,t) * \mu(x,z,s) \leq \mu(x, y+z, t+s)$;

(f) $\mu(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x,y,t) = 1$ *and* $\lim_{t\to 0} \mu(x,y,t) = 0$;

(h) $\mu(x,y,t) = \mu(y,x,t)$;

(i) $\nu(x,y,t) < 1$;

(j) $\nu(x, y, t) = 0$ *if and only if* x *and* y *are linearly dependent;*

(k) $\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(l) $\nu(x, y, t) \diamond \nu(x, z, s) \geq \nu(x, y + z, t + s)$*;*

(m) $\nu(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x, y, t) = 0$ *and* $\lim_{t\to 0} \nu(x, y, t) = 1$*,*

(o) $\nu(x, y, t) = \nu(y, x, t)$*.*

In this case $(\mu, \nu)_2$ *is called an intuitionistic fuzzy 2-norm on* V*. We denote it by* $(\mu, \nu)_2$*.*

Definition 2.6. *(9) The five-tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy* n*-normed space (for short, IF* n*-NS) if* V *is a vector space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$*,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm, and* μ, ν *are fuzzy sets on* $V \times \cdots \times V \times (0, \infty)$ *satisfying the following conditions. For every* $x_1, x_2, \ldots, x_n, y, z \in V$ *and* $s, t > 0$*,*

(a) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$*;*

(b) $\mu(x_1, x_2, \ldots, x_n, t) > 0$*;*

(c) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(d) $\mu(\alpha x_1, x_2, \ldots, x_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(e) $\mu(x_1, x_2, \ldots, x_{n-1}, y, t) * \mu(x_1, x_2, \ldots, x_{n-1}, z, s)$
$\leq \mu(x_1, x_2, \ldots, x_{n-1}, y + z, t + s)$*;*

(f) $\mu(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ *and* $\lim_{t\to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$*;*

(h) $\mu(x_1, x_2, \ldots, x_n, t)$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$*;*

(i) $\nu(x_1, x_2, \ldots, x_n, t) < 1$*;*

(j) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(k) $\nu(\alpha x_1, x_2, \ldots, x_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(l) $\nu(x_1, x_2, \ldots, x_{n-1}, y, t) \diamond \nu(x_1, x_2, \ldots, x_{n-1}, z, s)$
$\geq \nu(x_1, x_2, \ldots, x_{n-1}, y + z, t + s)$*;*

(m) $\nu(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ *and* $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$,

(o) $\nu(x_1, x_2, \ldots, x_n, t)$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$.

In this case $(\mu, \nu)_n$ *is called an intuitionistic fuzzy* n*-norm on* V. *We denote it by* $(\mu, \nu)_n$.

Example 2.4. *(9) Let* $(V, \|\cdot, \cdot, \ldots, \cdot\|)$ *be an* n*-normed space over* $\mathbb{F}$ *and let* $a*b = ab$ *and* $a \diamond b = \min\{a + b, 1\}$, *for all* $a, b \in [0, 1]$. *For every* $t > 0$, *consider*

$$\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|},$$

$$\nu(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then $(V, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy* n*-normed space.*

Example 2.5. *(9) Let* $(V, \|\cdot, \cdot\|)$ *be a 2-normed space over* F *and let* $a*b = ab$ *and* $a \diamond b = \min\{a + b, 1\}$, *for all* $a, b \in [0, 1]$ *and every* $t > 0$, *consider* $\mu(x, y, t) = \dfrac{t}{t + \|x, y\|}$, $\nu(x, y, t) = \dfrac{\|x, y\|}{t + \|x, y\|}$. *Then* $(V, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy 2-normed space.*

Definition 2.7. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space and let* $r \in (0, 1)$, $t > 0$ *and* $x \in V$. *The set*

$$B(x, r, t) = \{y \in V : \mu(y - x, x_2, \ldots, x_n, t) > 1 - r,\ \ \nu(y - x, x_2, \ldots, x_n, t) < r\},$$

$\forall x_2, \ldots, x_n \in V$ *is called the open ball with center* x *and radius* r *with respect to* t.

Definition 2.8. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space. A set* $U \subset V$ *is said to be an open set if each of its points is the centre of some open ball contained in* U. *The open set in an intuitionistic fuzzy* n*-normed space* $(V, \mu, \nu, *, \diamond)$ *is denoted by* $\mathbb{U}$.

Definition 2.9. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space. A sequence* $\{x_n\}$ *in* V *is said to be Cauchy if for each* $r > 0$ *and each* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mu(x_n - x_m, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_n - x_m, x_2, \ldots, x_n, t) < r$$

for all $n, m \geq n_0$ *and for all* $x_2, \ldots, x_n \in V$.

Definition 2.10. *(9) Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. A sequence $\{x_k\}$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)_n$, if for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - L, x_2, \ldots, x_n, t) > 1 - \epsilon \quad \textit{and} \quad \nu(x_k - L, x_2, \ldots, x_n, t) < \epsilon$$

for all $k \geq k_0$ and for all $x_2, \ldots, x_n \in V$.

Definition 2.11. *(14) Let ψ be a function defined on the real field $\mathbb{R}$ into itself satisfying the following properties;*

(a) $\psi(-t) = \psi(t)$ *for all* $t \in \mathbb{R}$

(b) $\psi(1) = 1$

(c) ψ *is strictly increasing and continuous on* $(0, \infty)$

(d) $\lim_{\alpha \to 0} \psi(\alpha) = 0$ *and* $\lim_{\alpha \to \infty} \psi(\alpha) = \infty$.

Example 2.6. *(14) Consider $\psi(\alpha) = |\alpha|; \psi(\alpha) = |\alpha|^p, p \in \mathbb{R}^+$; $\psi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}^+$. The function ψ allows us to generalize fuzzy metric and normed space.*

Definition 2.12. *(14) The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy ψ-normed space if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,*

(a) $\mu(x, t) + \nu(x, t) \leq 1$*;*

(b) $\mu(x, t) > 0$*;*

(c) $\mu(x, t) = 1$ *if and only if* $x = 0$*;*

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$*;*

(e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$*;*

(f) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t \to \infty} \mu(x, t) = 1$ *and* $\lim_{t \to 0} \mu(x, t) = 0$*;*

(h) $\nu(x, t) < 1$*;*

(i) $\nu(x, t) = 0$ *if and only if* $x = 0$*;*

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$*;*

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;

(l) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) *is called an intuitionistic fuzzy* ψ*-norm.*

3 Intuitionistic fuzzy n-normed space

Theorem 3.1. *In an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$, *if* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *and* $\{y_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} y$ *then* $\{x_n+y_n\}_{n=1}^{\infty}$ *is convergent to* $x+y$. *In other words, if* $(V,\mu,\nu,*,\diamond)$ *is an intuitionistic fuzzy* n*-normed space, then the addition is continuous in* $(V,\mu,\nu,*,\diamond)$.

Theorem 3.2. *In an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$, *if* $\lambda_n, \lambda \in \mathbb{R}^+$, $\lambda_n \to \lambda$ *as* $n\to\infty$ *and* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *as* $n\to\infty$, *then* $\{\lambda_n x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} \lambda x$. *In other words, if* $(V,\mu,\nu,*,\diamond)$ *is an IF-*n*-NS, then the scalar multiplication is continuous in* $(V,\mu,\nu,*,\diamond)$.

Proof. The proofs of Theorems 3.1 and 3.2 follow directly from the definitions. □

Lemma 3.1. *Let* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *as* $n\to\infty$ *in an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$. *Then for every* $t>0$ *as* $n\to\infty$,

$$\begin{aligned} &\mu(x_n, x_2, \ldots, x_n, t) \to \mu(x, x_2, \ldots, x_n, t), \\ &\nu(x_n, x_2, \ldots, x_n, t) \to \nu(x, x_2, \ldots, x_n, t), \end{aligned} \tag{1}$$

for all $x_2,\ldots,x_n \in V$.

Proof. Let $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ as $n\to\infty$ in $(V,\mu,\nu,*,\diamond)$. Then for $t>0$, $\forall k\in\mathbb{N}^+$, we have

$$\begin{aligned} \mu(x_n, x_2, \ldots, x_n, t) &= \mu(x_n - x + x, x_2, \ldots, x_n, \tfrac{t}{k+1} + \tfrac{kt}{k+1}) \\ &\geq \mu(x_n - x, x_2, \ldots, x_n, \tfrac{t}{k+1}) * \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}) \\ &\to 1 * \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}), \qquad (n\to\infty) \\ &= \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}). \end{aligned}$$

So,

$$\underline{\lim}_{n\to\infty} \mu(x_n, x_2, \ldots, x_n, t) \geq \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}), \qquad (k=1,2,\ldots).$$

Letting $k \to +\infty$ yields

$$\underline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) \geq \mu(x, x_2, \ldots, x_n, t). \tag{2}$$

On the other hand, for all $k \in \mathbb{N}^+$,

$$\mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) \to 1 > \tfrac{k}{k+1} > 0, \qquad (n \to \infty).$$

So there exists N such that

$$\mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) > \tfrac{k}{k+1}, \qquad (\forall n > N).$$

Thus, for all $n > N$ and $t > 0$, we have

$$\begin{aligned}\mu(x_n, x_2, \ldots, x_n, t) * \tfrac{k}{k+1} &\leq \mu(x_n, x_2, \ldots, x_n, t) * \mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) \\ &\leq \mu(x, x_2, \ldots, x_n, t + \tfrac{1}{k+1}).\end{aligned}$$

Hence,

$$\overline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) * \tfrac{k}{k+1} \leq \mu(x, x_2, \ldots, x_n, t + \tfrac{1}{k+1}), \quad (k = 1, 2, \ldots).$$

Letting $k \to +\infty$ yields

$$\overline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) \leq \mu(x, x_2, \ldots, x_n, t). \tag{3}$$

From (2) and (3), we conclude that

$$\lim_{n\to\infty} \mu(x_n, x_2, \ldots, x_n, t) = \mu(x, x_2, \ldots, x_n, t).$$

Similarly, one obtains

$$\lim_{n\to\infty} \nu(x_n, x_2, \ldots, x_n, t) = \nu(x, x_2, \ldots, x_n, t).$$

The proof is completed. $\square$

Theorem 3.3. *In an intuitionistic fuzzy n-normed space $(V, \mu, \nu, *, \diamond)$, the mappings*

$$\mu, \nu : V^n \times (0, \infty) \to [0, 1]$$

are continuous.

Proof. Let $(x_1, x_2, \ldots, x_{n-1}) \in V^{n-1}$, $x \in V$ and $t > 0$ with

$$(x_n^{(j)}, x_1, x_2, \ldots, x_{n-1}, t_j) \longrightarrow (x, x_1, x_2, \ldots, x_{n-1}, t) \quad \text{as } j \to \infty$$

in $V^n \times (0, \infty)$. Then $x_n^{(j)} \to^{(\mu,\nu)_n} x$ as $j \to \infty$ in V, and $t_j \to t$ as $j \to \infty$ in $(0, \infty)$.

Thus, for every $\delta > 0$ such that $\delta < \min\{\frac{t}{2}, 1\}$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $t - \delta < t_j < t + \delta$,

$$\mu(x - x_n^{(j)}, x_1, \ldots, x_{n-1}, \delta) > 1 - \delta,$$

$$\nu(x - x_n^{(j)}, x_1, \ldots, x_{n-1}, \delta) < \delta. \tag{4}$$

Hence, for all $j \geq j_0$, we see from (4) that

$$\begin{aligned}
\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\geq \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t - \delta) \\
&= \mu(x_n^{(j)} - x + x, x_1, \ldots, x_{n-1},\ \delta + t - 2\delta) \\
&\geq \mu(x_n^{(j)} - x, x_1, \ldots, x_{n-1}, \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta) \\
&\geq (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\leq \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t - \delta) \\
&= \nu(x_n^{(j)} - x + x, x_1, \ldots, x_{n-1},\ \delta + t - 2\delta) \\
&\leq \nu(x_n^{(j)} - x, x_1, \ldots, x_{n-1}, \delta) \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta) \\
&\leq \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta).
\end{aligned}$$

Thus, for all $j \geq j_0$,

$$\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \geq\ (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta),$$

$$\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \leq\ \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta).$$

This shows that

$$\underline{\lim}_{j \to \infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \geq\ (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta), \tag{5}$$

$$\overline{\lim}_{j \to \infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \leq\ \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta). \tag{6}$$

Letting $\delta \to 0^+$ in (5) and (6), we get

$$\underline{\lim}_{j \to \infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\overline{\lim}_{j \to \infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \nu(x, x_1, \ldots, x_{n-1}, t).$$

On the other hand, applying Lemma 3.1 for IF-n-NS, we obtain

$$\begin{aligned}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\leq \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t+\delta)\\ &\longrightarrow\ \mu(x, x_1, \ldots, x_{n-1}, t+\delta),\\ \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\geq \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t-\delta)\\ &\longrightarrow\ \nu(x, x_1, \ldots, x_{n-1}, t-\delta).\end{aligned}$$

Hence,

$$\overline{\lim}_{j\to\infty}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \mu(x, x_1, \ldots, x_{n-1}, t+\delta), \tag{7}$$

$$\underline{\lim}_{j\to\infty}\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \nu(x, x_1, \ldots, x_{n-1}, t-\delta). \tag{8}$$

Letting $\delta \to 0^+$ in (7) and (8), it follows that

$$\overline{\lim}_{j\to\infty}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\underline{\lim}_{j\to\infty}\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \nu(x, x_1, \ldots, x_{n-1}, t).$$

Combining the above inequalities, we conclude that

$$\lim_{j\to\infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) = \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\lim_{j\to\infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) = \nu(x, x_1, \ldots, x_{n-1}, t).$$

Therefore, the mappings $\mu, \nu : V^n \times (0, \infty) \to [0, 1]$ are continuous. $\square$

Definition 3.1. *A linear operator*

$$T : (V, \mu, \nu, *, \diamond)\ \longrightarrow\ (V, \mu', \nu', *, \diamond)$$

is said to be intuitionistic fuzzy n-bounded (shortly, IF-n-B) if there exist constants $h, k \in \mathbb{R} \setminus \{0\}$ such that

$$\mu'(Tx_1, x_2, \ldots, x_n, t)\ \geq\ \mu(hx_1, x_2, \ldots, x_n, t)$$

and

$$\nu'(Tx_1, x_2, \ldots, x_n, t)\ \leq\ \nu(kx_1, x_2, \ldots, x_n, t)$$

for every nonzero $x_1, x_2, \ldots, x_n \in V$ and for every $t > 0$.

Theorem 3.4. *Suppose that $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *, \diamond)$ are intuitionistic fuzzy n-normed spaces over $\mathbb{F}$ with*

(a) $1 \geq a \geq c \geq 0$ *and* $1 \geq b \geq c \geq 0$ *implies* $a * b \geq c$,

(b) $0 \leq a \leq c \leq 1$ and $0 \leq b \leq c \leq 1$ implies $a \diamond b \leq c$.

If linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ are (IF-n-B) intuitionistic fuzzy n-bounded, then $T_1 + T_2$ and cT $(c \in \mathbb{F})$ are also IF-n-B.

Proof. Since the linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ are IF-n-B, there exist $k_1, k_2, h_1, h_2 \in \mathbb{R}^+$ such that

$$
\begin{aligned}
\mu^{'}(T_1x_1, x_2, \ldots, x_n, t) &\geq \mu(h_1x_1, x_2, \ldots, x_n, t),\\
\nu^{'}(T_1x_1, x_2, \ldots, x_n, t) &\leq \nu(k_1x_1, x_2, \ldots, x_n, t),\\
\mu^{'}(T_2x_1, x_2, \ldots, x_n, t) &\geq \mu(h_2x_1, x_2, \ldots, x_n, t),\\
\nu^{'}(T_2x_1, x_2, \ldots, x_n, t) &\leq \nu(k_2x_1, x_2, \ldots, x_n, t),
\end{aligned}
$$

for every nonzero $x_1, x_2, \ldots, x_n \in V$ and for every $t > 0$.

Put $h = \max\{h_1, h_2\}$ and $k = \max\{k_1, k_2\}$. Then for every $x_1, x_2, \ldots, x_n \in V$, we have

$$
\begin{aligned}
\mu^{'}((T_1 + T_2)x_1, x_2, \ldots, x_n, t) &= \mu^{'}(T_1x_1 + T_2x_1, x_2, \ldots, x_n, t)\\
&\geq \mu^{'}(T_1x_1, x_2, \ldots, x_n, \tfrac{t}{2}) * \mu^{'}(T_2x_1, x_2, \ldots, x_n, \tfrac{t}{2})\\
&\geq \mu(h_1x_1, x_2, \ldots, x_n, \tfrac{t}{2}) * \mu(h_2x_1, x_2, \ldots, x_n, \tfrac{t}{2})\\
&= \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h_1}) * \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h_2})\\
&\geq \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h}) * \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h})\\
&\geq \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{3h})\\
&= \mu(3hx_1, x_2, \ldots, x_n, t).
\end{aligned}
$$

Similarly, we obtain

$$\nu^{'}((T_1 + T_2)x_1, x_2, \ldots, x_n, t) \leq \nu(3kx_1, x_2, \ldots, x_n, t).$$

Hence $T_1 + T_2$ is IF-n-B. Analogous reasoning shows that cT is IF-n-B. □

Definition 3.2. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ is said to be* intuitionistic fuzzy n-continuous *(shortly, IF-n-C) at a point $x_1 \in V$ if $x_1^{(m)} \to^{(\mu,\nu)_n} x_1$ as $m \to \infty$ in $(V, \mu, \nu, *, \diamond)$ implies that $Tx_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} Tx_1$ as $m \to \infty$ in $(V, \mu^{'}, \nu^{'}, *, \diamond)$.*

An operator T is said to be IF-n-C if it is intuitionistic fuzzy n-continuous everywhere in V.

Definition 3.3. *A map $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ is said to be* intuitionistic fuzzy n-continuous *(shortly, IF-n-C) at a point $x_{1,0} \in V$ if, for any given*

$\epsilon > 0$ and $\alpha \in (0,1)$, there exist $\delta = \delta(\alpha,\epsilon) > 0$ and $\beta = \beta(\alpha,\epsilon) \in (0,1)$ such that for all $x_1 \in V$,

$$\mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta) > \beta \;\Rightarrow\; \mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) > \alpha,$$
$$\nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta) < 1-\beta \;\Rightarrow\; \nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < 1-\alpha.$$

Remark 3.1. *The above definitions (3.2) and (3.3) are equivalent.*

Theorem 3.5. *Suppose that $(V,\mu,\nu,*,\diamond)$ and $(V,\mu^{'},\nu^{'},*,\diamond)$ are intuitionistic fuzzy n-normed spaces over $\mathbb{F}$. If linear operators $T_1, T_2 : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ are IF-n-C, then $c_1T_1 + c_2T_2$ is IF-n-C for all scalars $c_1, c_2 \in \mathbb{F}$.*

Proof. Let $x_1^{(m)} \to^{(\mu,\nu)_n} x_1$ as $m \to \infty$ in $(V,\mu,\nu,*,\diamond)$. Since T_1 and T_2 are IF-n-C, we have

$$T_1x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} T_1x_1, \quad T_2x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} T_2x_1 \quad \text{as } m \to \infty$$

in $(V,\mu^{'},\nu^{'},*,\diamond)$. By Theorems 3.1 and 3.2 (continuity of addition and scalar multiplication in IF-n-NS), we get for all $c_1, c_2 \in \mathbb{F}$:

$$c_1T_1x_1^{(m)} + c_2T_2x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} c_1T_1x_1 + c_2T_2x_1 \quad \text{as } m \to \infty,$$

which gives

$$(c_1T_1 + c_2T_2)x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} (c_1T_1 + c_2T_2)x_1 \quad \text{as } m \to \infty.$$

Hence, $c_1T_1 + c_2T_2$ is intuitionistic fuzzy n-continuous. □

Definition 3.4. *A linear operator $T : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ is said to be* strongly intuitionistic fuzzy n-continuous *at a point $x_1 \in V$ if, for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \ge \mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$

for all $x_2, \ldots, x_n \in V$.

A linear operator T is said to be strongly IF-n-C if it is strongly IF-n-C at each point of V.

Definition 3.5. *A linear operator $T : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ is said to be* strongly intuitionistic fuzzy n-continuous *at a point $x_{1,0} \in V$ if, for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \ge \mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$

for all $x_2, \dots, x_n \in V$.

A linear operator T is said to be strongly IF-n-C if it is strongly IF-n-C at every point of V.

Theorem 3.6. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *, \diamond)$ is strongly IF-n-C, then it is IF-n-C, but the converse is not necessarily true.*

Proof. Let a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V', \mu', \nu', *, \diamond)$ be strongly IF-n-C. Let $x_{1,0} \in V$. Then for any given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x_1 \in V$ and all $x_2, \dots, x_n \in V$,

$$\mu'(Tx_1 - Tx_{1,0}, x_2, \dots, x_n, \epsilon) \geq \mu(x_1 - x_{1,0}, x_2, \dots, x_n, \delta),$$
$$\nu'(Tx_1 - Tx_{1,0}, x_2, \dots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \dots, x_n, \delta).$$

Let $\{x_1^{(m)}\}$ be a sequence in V such that $\{x_1^{(m)}\} \to^{(\mu,\nu)_n} x_{1,0}$. Then for all $x_2, \dots, x_n \in V$ and $t > 0$,

$$\lim_{m\to\infty} \mu(x_1^{(m)} - x_{1,0}, x_2, \dots, x_n, t) = 1, \quad \lim_{m\to\infty} \nu(x_1^{(m)} - x_{1,0}, x_2, \dots, x_n, t) = 0.$$

By the strong IF-n-C property, we have for all m,

$$\mu'(Tx_1^{(m)} - Tx_{1,0}, x_2, \dots, x_n, \epsilon) \geq \mu(x_1^{(m)} - x_{1,0}, x_2, \dots, x_n, \delta),$$
$$\nu'(Tx_1^{(m)} - Tx_{1,0}, x_2, \dots, x_n, \epsilon) < \nu(x_1^{(m)} - x_{1,0}, x_2, \dots, x_n, \delta),$$

which implies

$$\lim_{m\to\infty} \mu'(Tx_1^{(m)} - Tx_{1,0}, x_2, \dots, x_n, \epsilon) = 1,$$

$$\lim_{m\to\infty} \nu'(Tx_1^{(m)} - Tx_{1,0}, x_2, \dots, x_n, \epsilon) = 0.$$

Hence, $Tx_1^{(m)} \to^{(\mu',\nu')_n} Tx_{1,0}$ in $(V', \mu', \nu', *, \diamond)$. Therefore, T is IF-n-C.

Conversely, there exist operators which are IF-n-C but not strongly IF-n-C, showing that the converse is not true.

Example 3.1. *Let $(V = \mathbb{R}, \|\cdot, \dots, \cdot\|)$ be an n-normed space over $\mathbb{F}$. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$.*

Let μ, ν, μ', ν' be fuzzy sets on $V^n \times (0, \infty)$ defined by

$$\mu(x_1, \dots, x_n, t) = \frac{t}{t + \|x_1, \dots, x_n\|},$$
$$\nu(x_1, \dots, x_n, t) = \frac{\|x_1, \dots, x_n\|}{t + \|x_1, \dots, x_n\|},$$
$$\mu'(x_1, \dots, x_n, t) = \frac{t}{t + k\|x_1, \dots, x_n\|},$$
$$\nu'(x_1, \dots, x_n, t) = \frac{k\|x_1, \dots, x_n\|}{t + k\|x_1, \dots, x_n\|},$$

for all $t > 0$ and $k > 0$.

Define a mapping $T : V \to V$ by $T(Y) = \frac{Y^4}{1+Y^2}$ for all $Y \in V$. Let $Y_0 \in V$, and let $\{Y_k\}$ be a sequence in V such that $\{Y_k\} \to Y_0$ in the intuitionistic fuzzy n-norm sense, that is, for all $t > 0$,

$$\lim_{k\to\infty} \mu(Y_k - Y_0, x_2, \ldots, x_n, t) = 1 \quad \textit{and} \quad \lim_{k\to\infty} \nu(Y_k - Y_0, x_2, \ldots, x_n, t) = 0.$$

Then, for all $t > 0$,

$$\mu'(TY_k - TY_0, x_2, \ldots, x_n, t) = \frac{t}{t + k\|TY_k - TY_0, x_2, \ldots, x_n\|} \to 1,$$

and

$$\nu'(TY_k - TY_0, x_2, \ldots, x_n, t) = \frac{k\|TY_k - TY_0, x_2, \ldots, x_n\|}{t + k\|TY_k - TY_0, x_2, \ldots, x_n\|} \to 0,$$

as $k \to \infty$. Hence, T is IF-n-C.

However, to check strong IF-n-C, let $\epsilon > 0$ be given. Then

$$\mu'(TY - TY_0, x_2, \ldots, x_n, \epsilon) \geq \mu(Y - Y_0, x_2, \ldots, x_n, \delta)$$

would require

$$\delta \leq \frac{\epsilon \prod_{i=1}^{n} \|1 + Y_i^2\|}{k \prod_{i=1}^{n} \|Y_i - Y_{0,i}\| \, \|\textit{polynomial terms of } Y_i, Y_{0,i}\|}.$$

Taking the infimum over all $Y \neq Y_0$ yields $\delta_1 = 0$, which is impossible.

Hence, T is IF-n-C but not strongly IF-n-C.

□

Theorem 3.7. *Let $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *, \diamond)$ be intuitionistic fuzzy n-normed spaces (IF-n-NS), and let $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond)$ be a linear operator. Then the following conditions are equivalent:*

(a) *T is intuitionistic fuzzy n-bounded (IF-n-B).*

(b) *There exist constants $h, k \in \mathbb{R} - \{0\}$ such that*

$$\mu'(Tx_1, \ldots, Tx_n, t) \geq \mu(hx_1, \ldots, hx_n, t),$$
$$\nu'(Tx_1, \ldots, Tx_n, t) \leq \nu(kx_1, \ldots, kx_n, t),$$

for all $x_1, \ldots, x_n \in V$ (nonzero) and $t > 0$.

(c) *T is intuitionistic fuzzy n-continuous at some point $x_0 \in V$.*

(d) *T is intuitionistic fuzzy n-continuous (IF-n-C) everywhere.*

Proof. $(a) \Leftrightarrow (b)$: The result follows directly from the definition of IF-n-B.

$(c) \Leftrightarrow (d)$: Suppose T is intuitionistic fuzzy n-continuous at some point $x_0 \in V$. Let $\{x_n^{(i)}\}_{n=1}^{\infty} \to x_0$ in $(V, \mu, \nu, *, \diamond)$ for $i = 1, \ldots, n$. By the properties of IF-n-NS, we have

$$(x_n^{(1)} - x_1, \ldots, x_n^{(n)} - x_n) + x_0 \to x_0 \text{ as } n \to \infty.$$

Since T is linear, it follows that

$$T((x_n^{(1)} - x_1, \ldots, x_n^{(n)} - x_n) + x_0) = T(x_n^{(1)}, \ldots, x_n^{(n)}) \to T(x_0),$$

which shows that T is IF-n-C. The converse is obvious.

$(a) \Leftrightarrow (d)$: This follows from the generalization of results in (9) and the definitions of IF-n-B and IF-n-C. □

Definition 3.6. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. A subset D of V is said to be* compact *if any sequence in D has a subsequence converging to an element of D in the sense of $(\mu, \nu)_n$-convergence.*

Theorem 3.8. *Let $T : (V_1, \mu_1, \nu_1, *, \diamond) \to (V_2, \mu_2, \nu_2, *, \diamond)$ be a mapping and let D be a compact subset of V_1. If T is intuitionistic fuzzy n-continuous (IF-n-C) on V_1, then $T(D)$ is a compact subset of V_2.*

Proof. Let $\{y_n\}$ be a sequence in $T(D)$. Then for each n there exists $x_n \in D$ such that $T(x_n) = y_n$.

Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x_0 \in D$ such that

$$\{x_{n_k}\} \to^{(\mu_1, \nu_1)_n} x_0 \quad \text{in} \quad (V_1, \mu_1, \nu_1, *, \diamond).$$

Since T is intuitionistic fuzzy n-continuous at x_0, by Definition (3.2),

$$\{x_{n_k}\} \to^{(\mu_1, \nu_1)_n} x_0 \quad \Rightarrow \quad T\{x_{n_k}\} \to^{(\mu_2, \nu_2)_n} T(x_0).$$

Hence, the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converges to $y_0 = T(x_0) \in T(D)$, which shows that $T(D)$ is compact in V_2. □

4 Intuitionistic fuzzy ψ-n-normed space

Definition 4.1. *The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy ψ-n-normed space, if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V^n \times (0, \infty)$ satisfying the following conditions. For every $x_1, x_2, \ldots, x_n, z \in V$ and $t, s > 0$,*

(a) $\mu(x_1, \ldots, x_n, t) + \nu(x_1, \ldots, x_n, t) \leq 1$;

(b) $\mu(x_1, \ldots, x_n, t) > 0$;

(c) $\mu(x_1, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(d) $\mu(\alpha x_1, x_2, \ldots, x_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$;

(e) $\mu(x_1, \ldots, x_{n-1}, y, t) * \mu(x_1, \ldots, x_{n-1}, z, s) \leq \mu(x_1, \ldots, x_{n-1}, y+z, t+s)$;

(f) $\mu(x_1, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x_1, \ldots, x_n, t) = 1$ *and* $\lim_{t\to 0} \mu(x_1, \ldots, x_n, t) = 0$;

(h) $\mu(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) = \mu(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, t)$ *(symmetry);*

(i) $\nu(x_1, \ldots, x_n, t) < 1$;

(j) $\nu(x_1, \ldots, x_n, t) = 0$ *if and only if* $x_1, \ldots, x_n$ *are linearly dependent;*

(k) $\nu(\alpha x_1, x_2, \ldots, x_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$;

(l) $\nu(x_1, \ldots, x_{n-1}, y, t) \diamond \nu(x_1, \ldots, x_{n-1}, z, s) \geq \nu(x_1, \ldots, x_{n-1}, y+z, t+s)$;

(m) $\nu(x_1, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x_1, \ldots, x_n, t) = 0$ *and* $\lim_{t\to 0} \nu(x_1, \ldots, x_n, t) = 1$;

(o) $\nu(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) = \nu(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, t)$ *(symmetry).*

In this case $(\mu, \nu)_n$ is called an intuitionistic fuzzy ψ-n-norm on V.

Definition 4.2. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ is said to be* convergent *to $x \in V$ with respect to the intuitionistic fuzzy ψ-n-norm $(\mu, \nu)_n$ if for every $r \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$.

Definition 4.3. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ in V is said to be* Cauchy *if for every $r \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - x_m, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_k - x_m, x_2, \ldots, x_n, t) < r$$

for all $k, m \geq k_0$ and all $x_2, \ldots, x_n \in V$.

Definition 4.4. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space, $r \in (0,1)$, $t > 0$, and $x \in V$. The set*

$$B(x, r, t) = \{y \in V : \mu(y-x, x_2, \ldots, x_n, t) > 1-r,\ \nu(y-x, x_2, \ldots, x_n, t) < r\}$$

, $\forall x_2, \ldots, x_n \in V$ is called the open ball *with center x and radius r with respect to t.*

Definition 4.5. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A set $U \subset V$ is said to be* open *if each of its points is the center of some open ball contained in U. The collection of all open sets in $(V, \mu, \nu, *, \diamond)$ is denoted by $\mathbb{U}$.*

Theorem 4.1. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ converges to $x \in V$ if and only if*

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \textit{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \textit{as } k \to \infty,$$

for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_k\}$ converges to x in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. Then for a given $r \in (0,1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Thus, $1 - \mu(x_k - x, x_2, \ldots, x_n, t) < r$ and $\nu(x_k - x, x_2, \ldots, x_n, t) < r$, which implies

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \text{as } k \to \infty.$$

Conversely, if for each $t > 0$,

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \text{as } k \to \infty,$$

then for every $r \in (0,1)$, there exists an integer k_0 such that

$$1 - \mu(x_k - x, x_2, \ldots, x_n, t) < r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Hence,

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r,$$

which shows that $\{x_k\}$ converges to x in $(V, \mu, \nu, *, \diamond)$. $\square$

Theorem 4.2. *The limit is unique for a convergent sequence $\{x_k\}$ in an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$.*

Proof. Let $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} x_k = y$. Then

$$\lim_{k\to\infty} \mu(x_k - x, x_2, \ldots, x_n, t) = 1, \quad \lim_{k\to\infty} \nu(x_k - x, x_2, \ldots, x_n, t) = 0,$$

and

$$\lim_{k\to\infty} \mu(x_k - y, x_2, \ldots, x_n, t) = 1, \quad \lim_{k\to\infty} \nu(x_k - y, x_2, \ldots, x_n, t) = 0,$$

for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Now, for any $s, t > 0$,

$$\begin{aligned}\nu(x - y, x_2, \ldots, x_n, s + t) &= \nu(x - x_k + x_k - y, x_2, \ldots, x_n, s + t)\\ &\geq \nu(x - x_k, x_2, \ldots, x_n, s) \diamond \nu(x_k - y, x_2, \ldots, x_n, t).\end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$\nu(x - y, x_2, \ldots, x_n, s + t) = 0 \quad \Rightarrow \quad x = y.$$

Hence, the limit of a convergent sequence in an intuitionistic fuzzy ψ-n-normed space is unique. □

Theorem 4.3. *In an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$, every convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_k\}$ be a convergent sequence in $(V, \mu, \nu, *, \diamond)$ with $\lim_{k\to\infty} x_k = x$. Let $r \in (0, 1)$ and $s, t > 0$. Then there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x, x_2, \ldots, x_n, s) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, s) < r \quad \forall k \geq k_0.$$

For $k, p \in \mathbb{N}$, we have

$$\begin{aligned}&\mu(x_{k+p} - x_k, x_2, \ldots, x_n, s + t)\\ &= \mu(x_{k+p} - x + x - x_k, x_2, \ldots, x_n, s + t)\\ &\geq \mu(x_{k+p} - x, x_2, \ldots, x_n, s) * \mu(x - x_k, x_2, \ldots, x_n, t)\\ &= \mu(x_{k+p} - x, x_2, \ldots, x_n, s) * \mu(x_k - x, x_2, \ldots, x_n, t)\\ &> (1 - r') * (1 - r') > 1 - r', \quad \forall k \geq k_0, \text{ Since, by choosing } r' * r' > r.\end{aligned}$$

Similarly,

$$\begin{aligned}&\nu(x_{k+p} - x_k, x_2, \ldots, x_n, s + t)\\ &= \nu(x_{k+p} - x + x - x_k, x_2, \ldots, x_n, s + t)\\ &\leq \nu(x_{k+p} - x, x_2, \ldots, x_n, s) \diamond \nu(x - x_k, x_2, \ldots, x_n, t)\\ &= \nu(x_{k+p} - x, x_2, \ldots, x_n, s) \diamond \nu(x_k - x, x_2, \ldots, x_n, t)\\ &< r' \diamond r' < r, \quad \forall k \geq k_0, \text{ Since, by choosing } r' \diamond r' < r.\end{aligned}$$

Hence, $\{x_k\}$ is a Cauchy sequence in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. $\square$

Theorem 4.4. *In an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$, a sequence $\{x_k\}$ is a Cauchy sequence if and only if*

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1 \quad \textit{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0$$

as $k \to \infty$, for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_k\}$ is a Cauchy sequence in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. Then, for a given $r \in (0, 1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Thus,

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1 \text{ and } \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0 \text{ as } k \to \infty.$$

Conversely, if for each $t > 0$,

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1 \text{ and } \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0 \text{ as } k \to \infty,$$

then for every $r \in (0, 1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$1 - \mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r \text{ and } \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r \ \forall k \geq k_0.$$

Hence,

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r,$$

which shows that $\{x_k\}$ is a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$. $\square$

Definition 4.6. *An intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$ is said to be* complete *if every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ is convergent.*

Theorem 4.5. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sufficient condition for $(V, \mu, \nu, *, \diamond)$ to be complete is that every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ has a convergent subsequence.*

Proof. Let $\{x_k\}$ be a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$, and let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ that converges to $x \in V$. Let $s, t > 0$ and $r \in (0, 1)$.

Since $\{x_k\}$ is Cauchy, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x_{k_j}, x_2, \ldots, x_n, s) > 1 - r \text{ and } \nu(x_k - x_{k_j}, x_2, \ldots, x_n, s) < r, \ \forall k, j \geq k_0.$$

Also, since $\{x_{k_j}\}$ converges to x, we have

$$\mu(x_{k_j} - x, x_2, \ldots, x_n, t) > 1 - r \text{ and } \nu(x_{k_j} - x, x_2, \ldots, x_n, t) < r, \ \forall j \geq k_0.$$

Now,

$$\begin{aligned}
&\mu(x_k - x, x_2, \ldots, x_n, s + t) \\
&= \mu(x_k - x_{k_j} + x_{k_j} - x, x_2, \ldots, x_n, s + t) \\
&\geq \mu(x_k - x_{k_j}, x_2, \ldots, x_n, s) * \mu(x_{k_j} - x, x_2, \ldots, x_n, t) \\
&> (1 - r') * (1 - r') > 1 - r', \quad \forall k \geq k_0, \text{ Since, by choosing } r' * r' > r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\nu(x_k - x, x_2, \ldots, x_n, s + t) \\
&= \nu(x_k - x_{k_j} + x_{k_j} - x, x_2, \ldots, x_n, s + t) \\
&\leq \nu(x_k - x_{k_j}, x_2, \ldots, x_n, s) \diamond \nu(x_{k_j} - x, x_2, \ldots, x_n, t) \\
&< r' \diamond r' < r, \quad \forall k \geq k_0, \text{ Since, by choosing } r' \diamond r' < r.
\end{aligned}$$

Hence, $\{x_k\}$ converges to x in $(V, \mu, \nu, *, \diamond)$. Therefore, the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$ is complete. □

Remark 4.1. *Straightforwardly, we get the results* (3.1), (3.2), (3.1), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) *are also holds in IFψ-n-NS.*

Theorem 4.6. *Every intuitionistic fuzzy ψ-n-normed space is an intuitionistic fuzzy n-normed space, but the converse is not true.*

Proof. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. By the definition of a ψ-n-normed space, take $\psi(\alpha) = |\alpha|$. Then the definition of an intuitionistic fuzzy n-normed space is satisfied, so $(V, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy n-normed space.

Conversely, let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. If $\psi(\alpha) \neq |\alpha|$, then the definition of an intuitionistic fuzzy ψ-n-normed space is not satisfied. Therefore, a general intuitionistic fuzzy n-normed space is not necessarily a ψ-n-normed space. □

5 Conclusion

In this paper, we have successfully extended the concept of intuitionistic fuzzy normed spaces to the framework of intuitionistic fuzzy n-normed spaces. We have explored key properties such as intuitionistic fuzzy n-continuity and n-boundedness, providing a foundation for further analysis in these spaces. Moreover, the introduction of intuitionistic fuzzy ψ-n-normed spaces has allowed us to generalize the existing structure of n-normed spaces and examine their convergence, Cauchy sequences, and completeness. The results obtained in this study offer a comprehensive understanding of these generalized spaces and pave the way for future research in fixed point theory, functional analysis, and applications involving fuzzy and intuitionistic fuzzy structures.

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Some results on pseudo MV-algebras with n-roots

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Abstract

The paper provides a study of pseudo MV-algebras with n-roots. We outline their main properties and establish that the class of MV-algebras with n-roots forms a variety. Then, we introduce the concept of a strict n-root to classify the class of pseudo MV-algebras with n-root and demonstrate an equivalence between this variety and the class of n-divisible unital ℓ-groups. Finally, we find a relationship between strongly atomless pseudo MV-algebra and strict pseudo MV-algebras.
Keywords: Pseudo MV-algebra; Square root; n-root; n-strict MV-algebra; Boolean algebra; Strongly atomless
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1 Introduction

MV-algebras, as algebraic models for infinite-valued logic, have been pivotal in the study of root theories such as square roots and n-roots. Building on the foundational work of Chang [Chang, 1958, 1959] and Mundici [Mundici, 1986], who established their equivalence with ℓ-groups, the research has extended to pseudo MV-algebras, non-commutative generalizations that enrich the algebraic framework for non-classical logics. The study of n-roots in pseudo MV-algebras addresses unique challenges arising from non-commutative operations, offering deeper insights into algebraic structures and their connections to divisibility and atomicity.

In MV-algebras, n-roots generalize square roots, revealing decompositions into Boolean algebras and strict MV-algebras, and establishing links with n-divisible ℓ-groups. Pseudo MV-algebras, however, introduce non-commutative operations ($\oplus$ and $\odot$), necessitating a redefinition of roots to accommodate asymmetry. For instance, weak square roots in pseudo MV-algebras satisfy existence and order-compatibility conditions, while strict square roots impose additional idempotent constraints on the root of zero, linking to strongly atomless properties .

Due to the non-commutativity of operations in pseudo MV-algebras, the research methods for the nth roots of MV-algebras cannot be directly applied. As a result, issues such as the rationality of the definition of n-roots in pseudo MV-algebras, the conditions for their existence, the determination of their uniqueness, and their interaction with filters, ideals, and homomorphic mappings have not been effectively solved.

In fact, the research on the nth roots of pseudo MV-algebras holds significant theoretical value and practical significance. From a theoretical perspective, clarifying the properties of n-roots is conducive to further revealing the non-commutative structural characteristics of pseudo MV-algebras, improving the element classification theory of pseudo MV-algebras, and providing a key link for constructing a more complete non-commutative many valued logical algebra system. From an application perspective, in non-commutative fuzzy reasoning systems based on pseudo MV-algebras, n-roots can be used to describe the "detailed splitting" and "inverse operation" of proposition truth values, providing a new algebraic tool for handling the iterative calculation of truth values in complex reasoning scenarios.

These roots interact deeply with the algebraic structure:

- **Decomposition**: Similar to MV-algebras, pseudo MV-algebras with n-roots can decompose into a Boolean algebra and a strict pseudo MV-algebra, a result rooted in the categorical equivalence with ℓ-groups and the properties of normal prime ideals.

- **Divisibility**: The existence of strict n-roots corresponds to the n-divisibility of the underlying ℓ-group, ensuring unique root extraction and structural simplicity.

The main goals of this paper are to provide a study on pseudo MV-algebra with n-roots and present their main properties:

- Define n-roots and weak n-roots that in the case of MV-algebras coincide.
- Present strict n-roots on the pseudo MV-algebras.
- Decompose n-roots into two parts, one as the identity on the Boolean part and one as a strict pseudo MV-algebras.
- Study strongly atomless and find their relationship with strict pseudo MV-algebras.
- Represent n-roots by addition in unital ℓ-groups on representable symmetric pseudo MV-algebras.

2 Preliminaries

In this section, we will review several definitions and fundamental statements that will serve as the foundation for the subsequent content of this paper. For more details about pseudo MV-algebras, we refer to [Dvurečenskij, 2001a,b, 2002, Georgescu and Iorgulescu, 2001].

Definition 2.1. *[Georgescu and Iorgulescu, 2001] A pseudo MV-algebra is an algebra $(M;\oplus,-,\sim,0,1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x,y,z\in M$,*
(A1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$,
(A2) $x\oplus 0=0\oplus x=x$,
(A3) $x\oplus 1=1\oplus x=1$,
(A4) $1^-=1^\sim=0$,
(A5) $(x^-\oplus y^-)^\sim=(x^\sim\oplus y^\sim)^-$,
(A6) $x\oplus(x^\sim\odot y)=y\oplus(y^\sim\odot x)=(x\odot y^-)\oplus y=(y\odot x^-)\oplus x$,
(A7) $x\odot(x^-\oplus y)=(x\oplus y^\sim)\odot y$,

(A8) $(x^-)^\sim = x$,
where

$$x \odot y = (y^- \oplus x^-)^\sim.$$

If the operation $\oplus$ *is commutative, equivalently,* $\odot$ *is commutative, then* M *is an MV-algebra, and of course* $x^- = x^\sim$ *for each* $x \in M$. *We denote it by* $x' := x^-$. *For more about MV-algebras, see [Cignoli et al., 2013].*
It is well-known that every pseudo MV-algebra is a bounded distributive lattice [Georgescu and Iorgulescu, 2001], where axiom (A6) defines $x \vee y$ *and (A7) describes* $x \wedge y$.

An algebra $(G; \vee, \wedge, +, -, 0)$ is said to be a lattice-ordered group or ℓ-group if :(1) $(G; \vee, \wedge)$ forms a lattice, (2) $(G; +, -, 0)$ is a group, and (3) the group operation $+$ is order-preserving. In an ℓ-group G, we use G^+ to denote $\{w \in G : 0 \leq w\}$. Additionally, for any $z \in G$, we define the positive part $z^+ := z \vee 0$ and the negative part $z^- := -z \vee 0$. A strong unit of G is an ℓ-group, and it is equipped with a fixed strong unit denoted as $u \in G$.

If $(G; \vee, \wedge, +, -, 0)$ is an ℓ-group, given $0 \leq u$, we define an interval $[0, u] := \{x \in G | 0 < x < u\}$. The interval $[0, u]$ can be converted into a pseudo MV-algebra $\Gamma(G, u) = ([0, u]; \oplus, -, \sim, 0, u)$ as follows: $x \oplus y = (x + y) \wedge u,\ x \odot y = (x - u + y) \vee 0$, $x^- = u - x$ and $x^\sim = -x + u$, $x, y \in [0, u]$.

Conversely, due to the principal result [Dvurečenskij, 2002], every pseudo MV-algebra is isomorphic to $\Gamma(G, u)$ for some unital ℓ-group (G, u). The unital ℓ-group (G, u) is unique up to isomorphism of unital ℓ-groups. Moreover, the category of unital ℓ-groups is categorical equivalent to the category of pseudo MV-algebras.

Let $\mathcal{UG}$ be the category of unital ℓ-groups whose objects are unital ℓ-groups (G, u) and morphisms between objects are ℓ-group homomorphisms preserving fixed strong units. We denote by $\mathcal{PMV}$ the category of pseudo MV-algebras whose objects are pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Consider the functor $\Gamma : \mathcal{UG} \to \mathcal{PMV}$ which is defined as follows

$$\Gamma(G, u) = (\Gamma(G, u); \oplus, -, \sim, 0, u),$$

and if $h : (G, u) \to (H, v)$ is a morphism, then $\Gamma(h)$ is the restriction of h onto the interval $[0, u]$. On the other side, there is a functor from the category of $\mathcal{PMV}$ to $\mathcal{UG}$ sending a pseudo MV-algebra M to a unital ℓ-group (G, u) such that $M \cong \Gamma(G, u)$ which is denoted by $\Psi : \mathcal{PMV} \to \mathcal{UG}$. If $\varphi : M_1 \to M_2$ is a morphism of pseudo MV-algebras, then $\Psi(\phi)$ is unique extension of ϕ onto a morphism of unital ℓ-groups. It is possible to show that if $\phi : M_1 \to M_2$ is an injective (surjective) homomorphism, so is $\Psi(\phi)$.

Theorem 2.1. *[Dvurečenskij, 2002] The composite functors $\Gamma \circ \Psi$ and $\Psi \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{PMV}$ and $\mathcal{UG}$, respectively. Therefore, the categories $\mathcal{PMV}$ and $\mathcal{UG}$ are categorically equivalent.*

A pseudo MV-algebra M is said to be *symmetric* if $x^{\sim} = x^{-}$ for each $x \in M$. We underline that if M is symmetric, then $\oplus$ is not necessarily commutative. In every pseudo MV-algebra M, we can define a partial operation $+$ in the following form: $x + y$ is defined in M if $x \leq y^{-}$, equivalently $y \leq x^{\sim}$, equivalently $y \odot x = 0$, and in either case, we set $x + y := x \oplus y$, see [Dvurečenskij, 2002]. We note that if $M = \Gamma(G, u)$, then $+$ is the restriction of the group addition of the group G to the interval $[0, u]$.

For any integer $n \geq 0$ and any $x \in M$, we can define

$$0.x = 0, \ \textit{and}\ n.x = (n-1).x \oplus x, \ n \neq 1,$$

$$0x = 0, \ \textit{and}\ nx = (n-1)x + x, \ n \neq 1,$$

assuming $(n-1)x$ and $(n-1)x+x$ are defined in M. An element $x \in M$ is called a Boolean element if $x \oplus x = x$. The set $B(M)$ denotes the set of all Boolean elements of M, which is a Boolean algebra and a subalgebra of M.

In what follows, we will assume usually that M is such that $0 \neq 1$.

If that $(M; \oplus, -, \sim, 0, 1)$ is pseudo MV-algebra, for each $a \in B(M)$, the interval $([0, a]; \oplus, ^{-a}, ^{\sim a}, 0, a)$ is a pseudo MV-algebra, where $x^{-a} = x^{-} \wedge a$ and $x^{\sim a} = x^{\sim} \wedge a$, and the mapping $\varphi_a : M \to M_a := [0, a]$ defined by $\phi_a(x) = a \wedge x, \ x \in M$, is a surjective homomorphism of pseudo MV-algebras.

In each pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, we define two additional binary operations $\to$ and $\rightsquigarrow$ by

$$x \to y = x^{-} \oplus y, \ x \rightsquigarrow y := y \oplus x^{\sim}.$$

Proposition 2.1. *[Dvurečenskij, 2001a, Georgescu and Iorgulescu, 2001] In any pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, the following properties hold for all $x, y, z \in M$:*
(1) $x \odot (x \to y) = x \wedge y \leq y, \ and\ (x \rightsquigarrow y) \odot x = x \wedge y \leq y.$
(2) $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y.$

(3) $(x \odot y) \to z = y \to (x \to z)$ *and* $(x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$.
(4) $x \to (x \odot y) = (x \to 0) \vee y$ *and* $x \rightsquigarrow (y \odot x) = (x \rightsquigarrow 0) \vee y$.
(5) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.
(6)$x \odot y \leq z \Longleftrightarrow y \leq x \to z \Longleftrightarrow x \leq y \rightsquigarrow z$.
(7) $x \wedge y = x \odot (x \to y) = y \odot (y \to x) = (y \rightsquigarrow x) \odot y = (x \rightsquigarrow y) \odot x$.
(8) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ *and* $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$.
(9) $x \to (y \wedge z) = (x \to y) \odot ((x \wedge y) \to z)$ *and* $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$.
(10) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *and* $(x \wedge y) \rightsquigarrow z = ((x \wedge y) \rightsquigarrow z) \odot (x \rightsquigarrow y)$.
(11) $If\ x \leq y,\ then\ y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$.

A non-empty subset I of a pseudo MV-algebra M is called an ideal if (1) for each $y \in M,\ y \leq x \in I$ implies that $y \in I$; (2) I is closed under $\oplus$. An ideal I of M is said to be (i) prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$; (ii) normal if $x \oplus I = I \oplus x$ for any $x \in M$, where $x \oplus I := \{x \oplus i | i \in I\}$ and $I \oplus x = \{i \oplus x | i \in I\}$. We recall that an ideal I is normal if and only if given $x, y \in M$, $x \odot y^- \in I$ if and only if $y^\sim \odot x \in I$. We note that the kernel of any homomorphism of pseudo MV-algebras is a normal ideal.

A one-to-one relationship exists between congruences and normal ideals of a pseudo MV-algbera, [Georgescu and Iorgulescu, 2001]. If I is a normal of pseudo MV-algebra, then the relation $\sim_I$, defined by $x \sim_I y$ if and only if $x \odot y^-, y \odot x^- \in I$, is a congruence, and $M/I = \{x/I : x \in M\}$ and x/I is the equivalence class containing x.

$$x/I \oplus y/I = (x \oplus y)/I,\ (x/I)^- = x^-/I,\ (x/I)^\sim = x^\sim/I,\ (0/I)^\sim = 1/I.$$

Conversely, if θ is a congruence on M, then $I_\theta = \{x \in M | x \sim 0\}$ is a normal ideal such that $\sim_{I_\theta} = \sim$.

A pseudo MV-algebra M is said to be representable if M is a subdirect product of a system of linearly ordered pseudo MV-algebras. By ([Dvurečenskij, 2001b], Prop, 6.9), M is representable if and only if $a^\perp = \{x \in M | x \wedge a = 0\}$ is a normal ideal of M for each $a \in M$. Moreover, $M = \Gamma(G, u)$ is representable iff G is a representable ℓ-group, see ([Dvurečenskij, 2001b], Prop, 6.9).

Proposition 2.2. *[Cignoli R L, 2013] For each element x, y of representable pseudo MV-algebra* $(M; \oplus, -, \sim, 0, 1)$ *and each* $i, j \in \mathbb{N}$ *with the property* $i + j = n$*, we have* $x^i \odot y^j \leq x^n \vee y^n$.

Proof. Since each representable pseudo MV-algebra is a subdirect product of a family of linearly ordered MV-algebra, it suffices to prove it only for the linearly ordered case. Assume that M is a chain and $x, y \in M$. If $x \leq y$, then $x^i \odot y^j \leq y^{i+j} = y^n \leq x^n \vee y^n$ and similarly, if $y \leq x$, then $x^i \odot y^j \leq x^{i+j} = x^n \leq x^n \vee y^n$. □

3 n-roots on pseudo MV-algebras

Definition 3.1. *[Höhle, 1995] Let M be an MV-algebra and $n \geq 1$ be an integer. A unary operation $r : M \to M$ is called an n^{th} root or simply an n-root, if it fulfills (R1) and (R2) for all $z \in M$:*
(R1) $r(z)^n = \underbrace{r(z) \odot \ldots \odot r(z)}_{n} = z$,
(R2) $\omega^n \leq z$ implies that $\omega \leq r(z)$ for each $\omega \in M$.
If $n = 2$, a 2-root is simply a square root.

Definition 3.2. *Let $(M; \oplus, -, \sim, 0, 1)$ be a pseudo MV-algebra. A mapping $r : M \to M$ is said to be an n-root if it satisfies the following conditions:*
(Sq1) for all $x \in M$, $\underbrace{r(x) \odot \ldots \odot r(x)}_{n} = x$;
(Sq2) for each $x, y \in M$, $y^n \leq x$ implies $y \leq r(x)$;
(Sq3) for each $x \in M$, $r(x^-) = r(x) \to r(0)$ and $r(x\ ) = r(x) \rightsquigarrow r(0)$;
and a weak n-root if it satisfies only (Sq1) and (Sq2). If $n = 2$, a 2-root is simply a square root.

A pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$ has n-roots (weak n-roots) if there exists an n-root (weak n-root) r on M. Sometimes, one writes $x^{1/n}$ for n-roots $r(x)$.

A standard n-root is an n-root (weak n-root) r satisfying the following additional condition:
(Sq4) for all $x \in M$, $r(x) \odot r(0) = r(0) \odot r(x)$.

For each $m \in N$ and each $x \in M$, we define $r^0(x) = x$, $r^1(x) = r(x)$ and $r^{m+1}(x) = r \circ r^m(x) = r(r^m(x))$, $m \geq 1$.

Remark 3.1. *(i) Any weak n-root $r : M \to M$ is a one-to-one map. If $r(x) = r(y)$, then $x = r(x)^n = r(y)^n = y$.*
(ii) If r_1 and r_2 are two weak n-roots on a pseudo MV - algebra M, then $r_1 = r_2$. By (Sq2), for each $x \in M$, $r_1(x)^n \leq x$ implies $r_1(x) \leq r_2(x)$. In a similar way, $r_2(x) \leq r_1(x)$. That is, $r_1 = r_2$.
(iii) If r is a weak n-root on a pseudo MV-algebra M, then $r(x) \odot r(y) = r(y) \odot$

$r(x)$ implies that $x \odot y = y \odot x$. Indeed, $x \odot y = (r(x))^n \odot (r(y))^n = (r(x))^{n-1} \odot (r(x) \odot r(y)) \odot (r(y))^{n-1} = (r(x))^{n-1} \odot (r(y) \odot r(x)) \odot (r(y))^{n-1} = (r(x))^{n-2} \odot r(x) \odot (r(y) \odot r(x)) \odot r(y) \odot (r(y))^{n-2} = (r(x))^{n-2} \odot (r(y) \odot r(x)) \odot (r(y) \odot r(x)) \odot (r(y))^{n-2} = \ldots = (r(y))^n \odot (r(x))^n = y \odot x$.

Proposition 3.1. *Let r be an n-root on a pseudo MV-algebra $(M; \oplus, -, \ , 0, 1)$. For each $x, y \in M$, we have:*
(1) $x \leq x \vee r(0) \leq r(x)$ and $r(x) \odot x = x \odot r(x)$.
(2) $x \leq y$ implies that $r(x) \leq r(y)$.
(3) $x \wedge y \leq r(x) \odot r(y)$, $r(y) \odot r(x)$. Furthermore ,if b is a Boolean element of M such that $r(0) \geq b$, then $b = 0$.
(4) $x \leq r(x^n) \leq r(x)$ and $r(x \odot x)^n = r(x)^n \odot r(x)^n = x \odot x$.
(5) $(x \wedge x^-)^n \vee (x \wedge x^\sim)^n \leq r(0)$.
(6) $r(x) \wedge r(y) = r(x \wedge y)$.
(7) $r(x) \to r(y) \leq r(x \to y)$ and $r(x) \rightsquigarrow r(y) \leq r(x \rightsquigarrow y)$. Moreover, $r(x) \odot r(y) \leq r(x \odot y)$ for all $x, y \in M$ if and only if $r(x) \to r(y) = r(x \to y)$ and $r(x) \rightsquigarrow r(y) = r(x \rightsquigarrow y)$.
(8) $r(x \vee y) = r(x) \vee r(y)$.
(9) If M is representable, $r(x)^i \odot r(y)^j \leq x \vee y$ for all $i, j \in \mathbb{N}$ such that $i + j = n$ and $y' \wedge y \leq r(0)$.
(10) $r(x \odot y) \leq (r(x) \odot r(y)) \vee r(0)$, and $r(x^m) = r(x)^m \vee r(0)$ for all $m \in \mathbb{N}$, and $r(y^n) = y \vee r(0)$. Consequently, $y = r(y^n)$ for all $r(0) \leq y \in M$.
(11) $r(x \oplus y) \geq (r(x) \odot r(0)^-) \oplus r(y)$.
(12) For each $m, n \in \mathbb{N}$ with $n \leq m$, $r^m(x)^{n^k} = r^{m-k}(x)$.
(13) $r(y) = \max\{x \wedge (x^{n-1} \to y)\}$.
(14) $y = r(y)$ if and only if $r(y) \in B(M)$.
(15) For each idempotent element $a \in B(M)$, the map $r_a : [0, a] \to [0, a]$ sending x to $r(x) \odot a$ is an n-root on the pseudo MV-algebra $[0, a]$.

Proof. (1) $x \leq r(x)$ follows from (Sq1). On the other hand, $r(0)^n = 0 \leq x,$ by (Sq2), $r(0) \leq r(x)$. Hence,$x = r(x)^n \leq r(x)$. For each $x \in M$, $r(x) \odot x = r(x) \odot r(x)^n = r(x)^n \odot r(x) = x \odot r(x)$.
(2) From (Sq2) and $r(x)^n = x \leq y$, we have $r(x) \leq r(y)$.
(3) When $n = 2$, it holds.(see [Dvurecenskij and Zahiri, 2022]) By (2), $x \wedge y = r(x \wedge y)^n \leq r(x \wedge y) \odot r(x \wedge y) \leq r(x) \odot r(y)$. Choose $b \in B(M)$ such that $r(0) \geq b$. Then, by (Sq1), $b = b^n \leq r(0)^n = 0$.
(4) $x^n \leq x^n$ and (Sq2) imply that $x \leq r(x^n)$. The second part follows from (Sq1): Indeed, $r(x \odot x)^n = x \odot x = r(x)^n \odot r(x)^n$.
(5) We have $(x \wedge x^-)^n \leq (x \wedge x^-)^2 = (x \wedge x^-) \odot (x \wedge x^-) \leq x \odot x^- = 0$. Similarly, $(x \wedge x^\sim)^n \leq (x \wedge x^\sim)^2 \leq x \odot x^\sim = 0,\ n \geq 2$. Thus by (Sq2), $x \wedge x^-,\ x \wedge x^\sim \leq r(0)$.
(6) From (2) and $x \wedge y \leq x, y$, we have that $r(x \wedge y) \leq r(x) \wedge r(y)$. If $n = 2$,

$r(x \wedge y)^2 = (r(x) \odot r(x)) \wedge (r(x) \odot r(y)) \wedge (r(y) \odot r(x)) \wedge (r(y) \odot r(y)) = r(x)^2 \wedge (r(x) \odot r(y)) \wedge (r(y) \odot r(x)) \wedge r(y)^2 \leq r(x)^2 \wedge r(y)^2 = x \wedge y$, assume the formula is true for some positive integer $k-1$: $(r(x) \wedge r(y))^{k-1} \leq r(x)^{k-1} \wedge r(y)^{k-1} = x \wedge y$. Show the formula is true for $n = k$: $(r(x) \wedge r(y))^n = (r(x) \wedge r(y))^{k-1} \odot (r(x) \wedge r(y)) \leq (r(x)^{k-1} \wedge r(y)^{k-1}) \odot (r(x) \wedge r(y)) = r(y)^k \wedge (r(x)^{k-1} \odot r(y)) \wedge (r(y)^{k-1} \odot r(x)) \wedge r(x)^k \leq x \wedge y$. By mathematical induction, the formula $(r(x) \wedge r(y))^n \leq x \wedge y$ is true for all positive integers n. So by (Sq2), $r(x) \wedge r(y) \leq r(x \wedge y)$. Thus, $r(x) \wedge r(y) = r(x \wedge y)$.
(7) Let $x, y \in M$. Then

$$\begin{aligned}
(x \wedge y) \odot (r(x) \to r(y))^n &= r(x \wedge y)^n \odot (r(x) \to r(y))^n \\
&= r(x \wedge y)^{n-1} \odot r(x \wedge y) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \\
&= r(x \wedge y)^{n-1} \odot (r(x) \wedge r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \quad \text{(by Prop 2.1(1))} \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y))^{n-1} \\
&= r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-2} \\
&\leq \cdots \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\leq r(x \wedge y)^{n-1} \odot r(y) \\
&\leq r(y)^{n-1} \odot r(y) \\
&= y.
\end{aligned}$$

From Proposition 2.1 (10), it follows that $(r(x) \to r(y))^n \leq (x \wedge y) \to y \leq x \to y$ and so by (Sq1), $r(x) \to r(y) \leq r(x \to y)$.
Now, let $r(x) \odot r(y) \leq r(x \odot y)$ for all $x, y \in M$. Then $r(x) \odot r(x \to y) \leq r(x \odot (x \to y)) \leq r(y)$ which entails $r(x \to y) \leq r(x) \to r(y)$. Therefore, $r(x \to y) = r(x) \to r(y)$.
Conversely, let $r(x \to y) = r(x) \to r(y)$ for all $x, y \in M$. Then from ([Georgescu and Iorgulescu, 2001], Prop 1.13), it follows that $r(x) \to r(x \odot y) = r(x \to (x \odot y)) = r(x^- \vee y) \geq r(y)$ and so $r(x) \odot r(y) \leq r(x \odot y)$.
Similarly, we can prove the relations with $\rightsquigarrow$.

(8) By part (6) and (Sq3), we have

$$\begin{aligned}
r(x \vee y) &= r(((x \to 0) \wedge (y \to 0)) \rightsquigarrow 0) = r(x \to 0) \wedge (y \to 0)) \rightsquigarrow r(0) \\
&= (r(x \to 0) \rightsquigarrow r(0)) \vee (r(y \to 0) \rightsquigarrow r(0)) \qquad \text{(by Prop 2.1(10))} \\
&= r((x \to 0) \rightsquigarrow 0) \vee r((y \to 0) \rightsquigarrow 0) \qquad \text{(by Prop 2.1(2))} \\
&= r(x) \vee r(y).
\end{aligned}$$

(9) We know that $r(x)^i \odot r(y)^j \leq r(x)^n \vee r(y)^n = x \vee y$. In addition, $(y \wedge y')^n \leq (y \wedge y')^2 \leq y \wedge y' = 0$ and so by (Sq2), $y \wedge y' \leq r(0)$.
(10) By (6), (7) and (Sq3), we have

$$\begin{aligned}
r(x \odot y) &= r(((x \odot y)^-)^\sim) = r(((x \odot y) \to 0) \rightsquigarrow 0) \\
&= r((y \to (x \to 0)) \rightsquigarrow 0) = r(y \to (x \to 0)) \rightsquigarrow r(0) \\
&\leq (r(y) \to r(x \to 0)) \rightsquigarrow r(0) \qquad \text{(by (7) and Prop 2.1(11))} \\
&= (r(y) \to (r(x) \to r(0))) \rightsquigarrow r(0) \\
&= ((r(x) \odot r(y)) \to r(0)) \rightsquigarrow r(0) \\
&= (r(x) \odot r(y)) \vee r(0). \qquad \text{(by Prop 2.1(2))}
\end{aligned}$$

Also,

$$\begin{aligned}
r(x^m) &\leq (r(x) \odot r(x^{m-1})) \vee r(0) \\
&\leq (r(x) \odot ((r(x) \odot r(x^{m-2})) \vee r(0))) \vee r(0) \\
&= (r(x) \odot (r(x) \odot r(x^{m-2})) \vee (r(x) \odot r(0))) \vee r(0) \\
&= (r(x) \odot (r(x) \odot r(x^{m-2})) \vee r(0) \leq \ldots = r(x)^m \vee r(0).
\end{aligned}$$

For $m = n$, $r(x^n) \leq r(x)^n \vee r(0) = x \vee r(0)$. By part (4), $x,\ r(0) \leq r(x^n)$, so $x \vee r(0) \leq r(x^n)$. Therefore, $r(x^n) = x \vee r(0)$.
(11) By (7), $r(x \oplus y) = r((x^\sim)^- \oplus y) = r(x^\sim \to y) \geq r(x^\sim) \to r(y) = (r(x) \rightsquigarrow r(0)) \to r(y) = (r(0) \oplus r(x)^\sim) \oplus r(y) = (r(x) \odot r(0)^-) \oplus r(y)$.
(12) Choose $m \in \mathbb{N}$. We have $r^m(x)^n = r(r^{m-1}(x))^n = r^{m-1}(x)$. If $n = 2$, then
$r^m(x)^{n^2} = (r(r^{m-1}(x))^n = (r^{m-1}(x))^n = r^{m-2}(x)$,
$r^m(x)^{n^k} = (r^{m-1}(x))^{n^{k-1}} = \ldots = r^{m-k}(x)$, for all $1 \leq k \leq m$.
(13) Since $r(y)^n = y$, we get $r(y) \leq r(y)^{n-1} \to y$. Hence

$$r(y) = r(y) \wedge (r(y)^{n-1} \to y) \in \{x \wedge (x^{n-1} \to y) : x \in M\}.$$

For each $x \in M$,

$$(x \wedge (x^{n-1} \to y))^n = (x \wedge (x^{n-1} \to y))^{n-1} \odot x \wedge (x^{n-1} \to y) \leq x^{n-1} \odot (x^{n-1} \to y) \leq y$$

so by (Sq2), $(x^{n-1} \to y) \leq r(y)$. Therefore, $r(y) = \max\{x \wedge (x^{n-1} \to y) : x \in M\}$.

(14) If $r(y) \in B(M)$, then by (Sq1), $y = r(y)^n = r(y) \in B(M)$. Conversely, suppose that $r(y) = y$. Then $y = r(y)^n \leq r(y) = y$. $r(y) = r(y)^n$, and so $r(y) \in B(M)$.
(15) Due to ([Georgescu and Iorgulescu, 2001], Pro 4.3), we have $r_a(x) = r(x) \wedge a = a \odot r(x) = r(x) \odot a$, $x \in [0, a]$. Since $r_a(x)^n = (a \odot r(x))^n = a^n \odot r(x)^n = a \odot r(x)$, conditions (Sq1) and (Sq2) are clear. Since the mapping $\psi : M \to [0, a]$ defined by $\psi_a(x) = x \wedge a$, $x \in M$, is a homomorphism of pseudo MV-algebras, (Sq3) holds. $\square$

Proposition 3.2. *Let r be a weak n-root on a pseudo MV-algebra. The following statements hold:*
(1) x is a Boolean element if and only if $r(x) = x \oplus r(0)$ if and only if $r(x) = r(0) \oplus x$.
(2) If $a, b \in M$, $a \leq b$, then $r([a, b]) = [r(a), r(b)]$. In particular $r(M) = [r(0), 1]$.
(3) $(r(0) \to 0)^n = (r(0) \rightsquigarrow 0)^n \in B(M)$.

Proof. First, we claim that

$$r(n.y) = y \oplus r(0) = r(0) \oplus y, \ \forall y \in M.$$

Check

$$\begin{aligned} r(n.y) &= r(((y^\sim)^n)^-) = r((y^\sim)^n) \to r(0) \\ &= ((r(y^\sim)^n) \vee r(0)) \to r(0) = (y^\sim \vee r(0)) \to r(0) \qquad \text{(by (10))} \\ &= (y^\sim \to r(0)) \wedge (r(0) \to r(0)) = y^\sim \to r(0) = y \oplus r(0), \\ r(n.y) &= r(((y^-)^n)^\sim) = r((y^-)^n) \rightsquigarrow r(0) \\ &= ((r(y^-)^n) \vee r(0)) \rightsquigarrow r(0) = (y^- \vee r(0)) \rightsquigarrow r(0) \qquad \text{(by (10))} \\ &= (y^- \rightsquigarrow r(0)) \wedge (r(0) \rightsquigarrow r(0)) = y^- \rightsquigarrow r(0) = r(0) \oplus y. \end{aligned}$$

(1) If $x \in B(M)$, we can yield $r(x) = x \oplus r(0) = r(0) \oplus x$. Conversely, let $r(x) = x \oplus r(0)$, we can $r(n.x) = x \oplus r(0) = r(x)$, since r is one-to-one, $n.x = x$ and $x \in B(M)$.
(2) Let $a, b \in M$ with $a \leq b$ be given. By Proposition 3.1 (2), $r([a, b]) \subset [r(a), r(b)]$. Let $r(a) \leq y \leq r(b)$. We have $y \geq r(a) \geq r(0)$, $y = y \vee r(0)$ and by (10), $y = r(y^n)$. From $a = r(a)^n \leq y^n \leq r(b)^n = b$ it follows that $y = r(y^n) \in r([a, b])$. Consequently, $r([a, b]) = [r(a), r(b)]$. As a special case, we have $r(M) = r([0, 1]) = [r(0), r(1)] = [r(0), 1]$.
(3) We show that $(r(0) \to 0)^n \in B(M)$. By (1), it suffices to prove that $r((r(0) \to 0)^n) = (r(0) \to 0)^n \oplus r(0)$. By (10), $r((r(0) \to 0)^n) = (r(0) \to 0) \vee r(0)$. On the other hand, $(r(0) \to 0)^n \oplus r(0) = (r(0)^-)^n \oplus r(0) = r(0)^- \vee r(0)$ (see [Georgescu and Iorgulescu, 2001]). It follows that $(r(0) \to 0)^n \in B(M)$. In a similar way, we can show that $(r(0) \rightsquigarrow 0)^n \in B(M)$. $\square$

Proposition 3.3. *Let r be a weak n-root on a pseudo MV-algebra. The following statements are equivalent:*
(1) r is an n-root on M.
(2) $r(x) \odot r(x^-) \leq r(0)$.
(3) $r(x^\sim) \odot r(x) \leq r(0)$.
(4) $r(x^-) = r(x) \to r(0)$.
(5) $r(x^\sim) = r(x) \rightsquigarrow r(0)$.

Proof. Simlarly to Proposition 3.1 (7), for a weak n-root r, we always have $r(x) \to r(0) \leq r(x \to 0) = r(x^-)$ and $r(x) \rightsquigarrow r(0) \leq r(x \rightsquigarrow 0) = r(x^\sim)$.
Let $r(x) \odot r(x^-) \leq r(0)$. It implies $r(x^-) \leq r(x) \to r(0)$. Hence $r(x) \to r(0) = r(x^-)$. Conversely $r(x^-) \leq r(x) \to r(0)$ implies that $r(x) \odot r(x^-) \leq r(0)$.
This proves that (1)-(5) are equivalent. □

Proposition 3.4. *Let r be an n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. The following properties hold for $x \in M$:*
(1) $x \leq x \vee r(0) = r(x^n) \leq r(x)$,
$r(x^-)^\sim \vee r(x^\sim)^- \leq n.(r(x^-)^\sim) = x \leq x \vee r(0) = r(x^n) \leq r(x)$. *Moreover,* $x \in B(M)$ *iff* $r(x) = x \vee r(0)$.
(2) $r(x) \leq (x \oplus r(0)) \wedge (r(0) \oplus x)$.
(3) $r(x^\sim)^- \oplus r(0) = r(x) = r(0) \oplus r(x^-)^\sim$.
(4) $x \oplus r(x) = r(x) \oplus x$, $x \oplus r(0) = r(0) \oplus x$.

Proof. (1) By Proposition 3.1 (1) and (4), we have $x \leq x \vee r(0) \leq r(x)$ and $x \leq r(x^n)$. Also,by part (10) from this proposition, $r(x^n) = r(x)^n \vee r(0) = x \vee r(0)$, so we have equation (1). Hence, if $x \in M \setminus B(M)$, then $x^n \neq x$, thus $r(0) \vee x < r(x)$. Otherwise, if $r(0) \vee x = r(x)$, then $r(x) = r(x^n) = x \vee r(0)$, that is $x^n = x$. Moreover, $x \in B(M)$ if and only if $r(x) = r(0) \vee x$ (since r is a one-to-one map).
By Proposition 3.1 (1), replacing x with x^- and $x^\sim$, we have $x^- \leq r(x^-)$ and $x^\sim \leq r(x^\sim)$ which imply that $r(x^-)^\sim \vee r(x^\sim)^- \leq x$. Also, $n.(r(x^-)^\sim) = (r(x^-)^n)^\sim = (x^-)^\sim = x$. In a similar way, $x = n.(r(x^\sim)^-)$. Therefore, we have (2).
(2) According to the definition of an n-root, $r(x) = max\{z \in M | z^n \leq x\}$. We claim that $x \oplus r(0)$ is an upper bound for $r(x)$. Indeed, set $S := \{z \in M | z^n \leq 0\}$. If $z \in M$ such that $z^n \leq x$, then $z^n \odot x^- = 0$ and $x^\sim \odot z^n = 0$ which means $(z \odot x^-)^n \leq z^{n-1} \odot (z \odot x^-) = z^n \odot x^- = 0$ and $(x^\sim \odot z)^n \leq (x^\sim \odot z) \odot z^{n-1} = x^\sim \odot z^n = 0$, thus $x^\sim \odot z$, $z \odot x^- \in S$. It follows that $x^\sim \odot z$, $z \odot x^- \leq r(0)$ and hence,

$$z \leq x \vee z = x \oplus (x^\sim \odot z) \leq x \oplus r(0),$$

$$z \leq x \vee z = (z \odot x^-) \oplus x \leq r(0) \oplus x.$$

Since $r(x)$ is the greatest element of S, we conclude that (2) holds.
(3) Due to (1) and (2), $r(x^-)^\sim \leq x$ and $r(x) \leq r(0) \oplus x$. Then

$$\begin{aligned} r(0) \oplus r(x^-)^\sim &= r(0) \oplus (r(x) \to r(0))^\sim \\ &= r(0) \oplus (r(x)^- \oplus r(0))^\sim \\ &= r(0) \oplus (r(0)^\sim \odot r(x)) \\ &= r(0) \vee r(x) = r(x). \qquad \text{(by Prop 3.1(2))} \end{aligned}$$

In a similar way, $r(x) = r(x^\sim)^- \oplus r(0)$. Hence, (3) is established.
(4) (i) By (1) and (3), $x = n.(r(x)^-)^\sim = n.(r(x^\sim))^-$, $r(x) = r(x^\sim)^- \oplus r(0) = r(0) \oplus r(x^-)^\sim$. It follows that

$$\begin{aligned} x \oplus r(x) &= n.r(x^\sim)^- \oplus r(0) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus (r(x^\sim)^- \oplus r(0)) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus (r(0) \oplus r(x^-)^\sim) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus r(0) \oplus 2.r(x^-)^\sim \\ &= \dots \\ &= r(x^\sim)^- \oplus r(0) \oplus n.r(x^-)^\sim \\ &= r(x) \oplus x \end{aligned}$$

(ii) For each $x \in M$, $x \oplus r(0) = r(0) \oplus x$. $\square$

Similarly to the definition of the n-root of a pseudo MV-algebra, we can define an n-root of an individual element of a pseudo MV-algebra. Let $m \in \mathbb{N}$. We called an element $x \in M$ has an m-root if there exists $a \in M$ satisfies (Sq1) and (Sq2), that is $a^m = x$ and $y^m \leq x$ implies that $y \leq a$. The condition (Sq2) implies that if x has an m-root, then it is unique, so that, we can denote it by $\sqrt[m]{x}$ for simplicity. Clearly, if r is an m-root on M, then each element of M has an m-root, indeed, $\sqrt[m]{x} = r(x)$.

Theorem 3.1. *Suppose that M is a pseudo MV-algebra with an n-root r, and m and p are natural numbers such that $n = mp$. If $x \in M$ satisfies the condition $r(0) \leq r(x)^p$, then x has an m-root, $\sqrt[m]{x} = r(x)^p$.*

Proof. Let $mp = n$. Consider the map $v : M \to M$ defined by $v(x) = r(x)^p$ for all $x \in M$. So, $v(x)^m = r(x)^{mp} = r(x)^n = x$. In addition, if $y \in M$ such that $y^m \leq x$, then $y^n \leq x^p$, whence $y \leq r(x^p)$. By Proposition 3.1(10),

$$r(x^p) = r(x)^p \vee r(0) = v(x) \vee r(0).$$

Suppose that $r(0) \leq r(x)^p$ for some $x \in M$. Then $y \leq v(x) \vee r(0) = v(x)$, that is, v satisfies (Sq2), so $\sqrt[m]{x} = v(x)$. $\square$

Proposition 3.5. *Let $f : M_1 \to M_2$ be a homomorphism of pseudo MV-algebras and r be an n-root on M_1. Then $\tau : Im(f) \to Im(f)$, defined by $\tau(f(x)) = f(\tau(x))$ for all $x \in M_1$, is an n-root on $Im(f)$.*

Proof. (i) For each $r \in M_1$, we have $f(r(x))^n = f(r(x)^n) = f(x)$.
(ii) Let $x, y \in M_1$, be such that $f(y)^n \leq f(x)$. By Proposition 3.1(8) and (10), $r(y^n \vee x) = r(y^n) \vee r(x) = (y \vee r(0)) \vee r(x)$, whence

$$\begin{aligned}
\tau(f(x)) &= \tau(f(y)^n \vee f(x)) = \tau(f(y^n \vee x)) \\
&= f(r(y^n \vee x)) = f(r(y^n) \vee r(x)) \qquad \text{by Proposition 3.1(8)} \\
&= f((y \vee r(0)) \vee r(x)) \qquad \text{by Proposition 3.1(10)} \\
&= (f(y) \vee f(r(0))) \vee f(r(x)) \\
&= (f(y) \vee f(r(0))) \vee \tau(f(x)).
\end{aligned}$$

That is, $f(y) \leq \tau(f(x))$. Hence (Sq2) holds.
(iii) We have $\tau(f(x)) = f(r(x))^- = f(r(x)^-) = f(r(x) \to r(0)) = f(r(x)) \to f(r(0)) = \tau(f(x)) \to \tau(f(0))$. Similarly for the second equality, that is,(Sq3) holds.
(iv) We show $\tau : Im(f) \to Im(f)$ is well-defined. Indeed, if $f(x) = f(y)$, then $f(r(x))$ and $f(r(y))$ are n-roots of $f(x) = f(y)$. Parts (i) and (ii) imply that $f(r(x))^n = f(x) = f(y)$ and so $f(r(x)) \leq f(r(y))$. In similar way, $f(r(y)) \leq f(r(x))$. Thus, $\tau : Im(f) \to Im(f)$ sending $f(x)$ to $f(r(x))$ is a correctly defined n-root. □

Corolary 3.1. *Let I be a normal ideal of a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$ with an n-root r. Then:*
(1) The mapping $r_I : M/I \to M/I$ defined by $r_I(x/I) = r(x)/I$ is an n-root on M/I.
(2) For each $x, y \in M$, if $x \odot y^-$, $y \odot x^- \in I$, then $r(x) \odot r(y)^-$, $r(y) \odot r(x)^- \in I$.

Proof. The proof of part (1) follows from Proposition 3.5.
(2) Let $x, y \in M$ such that $x \odot y^-$, $y \odot x^- \in I$. Then $x/I = y/I$ and so by (i), $r(x)/I = r(y)/I$ which means $r(x) \odot r(y)^-,\ r(y) \odot r(x)^- \in I$. □

Theorem 3.2. *Let r be an n-root on a pseudo MV-algebra $(M; \vee, \wedge, \oplus, 0)$. Then M is a Boolean algebra if and only if $r(0) = 0$.*

Proof. Let $r(0) = 0$ and $x \in M$. By Proposition 3.3(2), we have $x \leq r(x^n) = r(0) \vee x \leq r(x) \leq r(0) \oplus x = x$ which implies that $r(x) = r(x^n)$ and so $x = x^n$ (since r is one-to-one). That is $M = B(M)$ is a Boolean algebra.
Conversely, let M be a Boolean algebra. Then $r(0)^n = r(0)$ and $r(0)^n = 0$. Therefore, $r(0) = 0$. □

Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra with an n-root r and I be a normal ideal of M. An ideal I of M is called r-invariant if $r(I)\subseteq I$.

Proposition 3.6. *Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra with an n-root r and I be a normal ideal of M. Then I is r-invariant if and only if I is a Boolean ideal of M.*

Proof. Suppose that I is an r-invariant normal ideal of M. By Corollary 3.1, $r_I : M/I \to M/I$ is an n-root on M/I. Since $r_I(0/I) = r(0)/I = 0/I$, Theorem 3.2 implies that M/I is a Boolean algebra and so I is a Boolean ideal of M. Conversely, let I be a Boolean ideal of M. Then M/I is a Boolean algebra and $r_I : M/I \to M/I$ is the identity map on M/I. Hence for each $x \in I$, we have $r(x)/I = r_I(x/I) = x/I = 0/I$, that is $r(x) \in I$. Therefore, I is r-invariant. $\square$

Remark 3.2. *Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra and $a \in B(M)\setminus\{0,1\}$. Set $b := a^-$. Then $[0,a]$ and $[0,b]$ are proper subsets of M. Consider the pseudo MV-algebras $([0,a];\oplus,^{-a},^{\sim a},0,a)$ and $([0,b];\oplus,^{-b},^{\sim b},0,b)$. The map $\varphi : M \to [0,a]\times[0,b]$ sending $x\in M$ to $(x\wedge a, x\wedge b)$ is an isomorphism of pseudo MV-algebras [see [Georgescu and Iorgulescu, 2001], page 23]. Hence, each pseudo MV-algebra with the non-empty set $B(M)\setminus\{0,1\}$ is not directly indecomposable. Now, let r be an n-root on M. By Proposition 3.1(15), r_a and r_b are n-roots on the pseudo MV-algebras $[0,a]$ and $[0,b]$, respectively. Therefore, in this case, M has a decomposition of pseudo MV-algebras with n-roots.*

Bělohlávek in [Bělohlávek, 2003] showed that the class of all commutative residuated lattices with square roots is a variety. Consequently, the class of MV-algebra with square roots is also a variety. The following theorem shows that the class of pseudo MV-algebra with n-roots is a variety, too.

Theorem 3.3. *The class of all pseudo MV-algebras with n-roots is a variety. The same is true for the class of pseudo MV-algebras with weak n-roots.*

Proof. Let V be the class of all pseudo MV-algebras $(M;\oplus,-,\sim,0,1)$ such that there is an n-root on M. Consider the class W of algebras of type $(M;\oplus,-,\sim,r,0,1)$ satisfying the identities (A1)-(A8) and the following additional three identities:
(1) $r(x)^n = x$;
(2) $r(y^n \vee x)\wedge y = y$;
(3) $r(x^-) = r(x) \to r(0)$ and $r(x^\sim) = r(x) \rightsquigarrow r(0)$.
First, suppose that $M \in V$. There exists an n-root r on M. We show that $(M;\oplus,-,\sim,r,0,1)$ belongs to W. We only need to show (2). For each $x,y\in M$ by Proposition 3.1(8) and 3.1(1), $r(y^n\vee x) = r(y^n)\vee r(x) \geq y \vee r(x)$ and so $r(y^n\vee x)\wedge y = y$. Hence $(M;\oplus,-,\sim,r,0,1)\in W$.

Conversely, let $(M; \oplus, -, \sim, r, 0, 1)$ belong to W. Clearly by (A1)-(A8), $(M; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra. Let $y^n \leq x$. Then by (2), $y = r(y^n \vee x) \wedge y = r(x) \wedge y$ which means $y \leq r(x)$. From (3), we get (Sq3). Hence, $(M; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra with the n-root r. That is, $(M; \oplus, -, \sim, 0, 1) \in V$. Therefore , V is a variety.
Using equations (1)-(2), we can prove that the class of pseudo MV-algebras with weak n-roots is a variety, too. □

Theorem 3.4. *(1) Let M_1 and M_2 be two pseudo MV-algebras and $r_1 : M_1 \to M_1$ and $r_2 : M_2 \to M_2$ be two n-roots. If $f : M_1 \to M_2$ is a homomorphism of pseudo MV-algebras, then f preserves n-roots if and only if $Im(f)$ is closed under r_2.*
(2) If $f : M_1 \to M_2$ is an isomorphism and r_1 is a (weak) n-root on M_1, then $r_2 = f \circ r_1 \circ f^{-1}$ is a (weak) n-root on M_2.

Proof. (1) ($\Leftarrow$) Suppose that $Im(f)$ is closed under r_2. Then $r_2|_{Im(f)} : Im(f) \to Im(f)$ is an n-root on the pseudo MV-algebra $Im(f)$. Since by Proposition 3.5, the map $\tau : Im(f) \to Im(f)$ defined by $\tau(f(x)) = f(r_1(x)), x \in M_1$, is an n-root, then $\tau = r_2$ which implies that $r_2(f(x)) = \tau(f(x)) = f(r_1(x))$ for all $x \in M$.
($\Rightarrow$) Let f preserve n-roots. For each $f(x) \in Im(f)$, we have $r_2(f(x)) = f(r_1(x)) \in Im(f)$. Therefore, $Im(f)$ is closed under r_2.
(2) (Sq1): Let $z \in M_2$. Then $r_2(z)^n = f(r_1(f^{-1}(z)))^n = f(r_1(f^{-1}(z)^n)) = f(f^{-1}(z)) = z$.
(Sq2): Let $x, y \in M_2$ and $y^n \leq x$. Then $f^{-1}(x)^n \leq f^{-1}(x)$, so that, $f^{-1}(x) \leq r_1(f^{-1}(x))$ and $y \leq f(r_1(f^{-1}(x))) = r_2(x)$.
(Sq3): Let $x \in M_2$, so $r_2(0) = f(r_1(0))$, and $r_2(x^-) = f(r_1(f^{-1}(x))) = f(r_1(f^{-1}(x)) \to f(r_1(0)) = r_2(x) \to r_2(0)$. □

Theorem 3.5. *Let $(M; \oplus, -, \sim, 0, 1)$ be a pseudo MV-algebra with an n-root r. Then the structure $([r(0), 1]; \uplus, ', *, r(0), 1)$ is a pseudo MV-algebra, where for given $r(x), r(y) \in r(M)$,*

$$r(x) \uplus r(y) = r(x \oplus y),\ r(x)' = r(x)^- \oplus r(0),\ r(x)^* = r(x)^\sim \oplus r(0).$$

Furthermore, the pseudo MV-algebras $(M; \oplus, -, \sim, 0, 1)$ and $([r(0), 1]; \uplus, ', *, r(0), 1)$ are isomorphic, and the restriction of r onto $r(M)$ is an n-root on $r(M)$.

Proof. (i) First, we note that r is one-to-one, so by Proposition 3.2(2), the operations $\uplus, '$ and $*$ are well-defined, and $[r(0), 1]$ is closed under these operations. We verify conditions (A1)-(A8). Let $x, y, z \in M$ be given.
(1)

$$\begin{aligned} r(x) \uplus (r(y) \uplus r(z)) &= r(x) \uplus r(y \oplus z) = r(x \oplus (y \oplus z)) \\ &= r((x \oplus y)) \uplus r(z) = (r(x) \uplus r(y)) \uplus r(z). \end{aligned}$$

(2) $r(x) \uplus r(0) = r(x \oplus 0) = r(x) = r(0 \oplus x) = r(0) \uplus r(x)$.
(3) $r(x) \uplus r(1) = r(x \oplus 1) = r(1) = r(1 \oplus x) = r(1) \uplus r(x)$.
(4) $1' = 1' \oplus r(0) = 0 \oplus r(0) = r(0)$. Similarly, $1^* = r(0)$.
(5) For each $x \in M$, by Proposition 3.4 (3) the following identities hold:

$$r(x)^* = r(x)^\sim \oplus r(0) = r(0) \oplus r(x)^\sim = r(x) \rightsquigarrow r(0) = r(x \rightsquigarrow 0) = r(x^\sim), \tag{a}$$

$$r(x)' = r(x)^- \oplus r(0) = r(x) \to r(0) = r(x \to 0) = r(x^-). \tag{b}$$

It follows that $(r(x)')^* = r(x^-)^* = r(x^{-\sim}) = r(x)$ and $(r(x)^*)' = r(x^\sim)' = r(x^{\sim-}) = r(x)$.
(6) From (a) and (b) deduce that

$$(r(x)' \uplus r(y)')^* = (r(x^-) \uplus r(y^-))^* = r((x^- \oplus y^-)^\sim) = r((x^\sim \oplus y^\sim)^-).$$

A similar calculus shows that $(r(x)^* \uplus r(y)^*)' = r((x^\sim \oplus y^\sim)^-)$. Hence $(r(x)^* \uplus r(y)^*)' = (r(x)' \uplus r(y)')^*$.
(7) Let $u \oslash v := (v' \uplus u')^*$ for all, $u, v \in [r(0), 1]$. Then from (a) and (b) it follows that

$$\begin{aligned} r(x) \oslash r(y) &= (r(y)' \uplus r(x)')^* = (r(y^-) \uplus r(x^-))^* \\ &= r(y^- \uplus x^-)^* = r((y^- \uplus x^-)^\sim) = r(x \odot y). \end{aligned}$$

Similarly to (6), we can prove (A6) and (A7) hold. Therefore, $([r(0), 1]; \uplus, ', *, r(0), 1)$ is a pseudo MV-algebra.
(ii) Consider the mapping $f : M \to [r(0), 1]$ defined by $f(x) = r(x)$. Clearly, $f(x) \in Im(f) = [r(0), 1]$. By (a) and (b), for each $x \in M$, $f(x)^* = r(x)^* = r(x^\sim) = f(x^\sim)$ and $f(x)' = r(x)' = r(x^-) = f(x^-)$. In addition, for each $x, y \in M$, $f(x) \uplus f(y) = r(x) \uplus r(y) = r(x \oplus y) = f(x \uplus y)$, and $f(0) = r(0)$, $f(1) = r(1) = 1$. Therefore, f is an isomorphism. Hence, M is isomorphic to $Im(r)$, and using Theorem 3.2 (2) for each $x \in M$, we have $f(r(f^{-1}(f(x)))) = f(r(x)) = r(r(x))$, is an n-root on $r(M) = [r(0), 1]$. $\square$

4 Strict n-roots and n-strict MV-algebras

In this section, we present an important class of pseudo MV-algebra with strict n-root.

Definition 4.1. *An n-root r: $M \to M$ on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$ is said to be strict if $r(0)^{n-1} = r(0)^- = r(0)^\sim$.*
A pseudo MV-algebra possessing a strict n-root is said to be an n-strict pseudo MV-algebra.

Theorem 4.1. *Let* $(M;\oplus,-,\sim,0,1)$ *be a totally ordered symmetric pseudo MV-algebra with an* n*-root* $r:M\to M$ *and let* (G,u) *be its related unital* ℓ*-group. If* $r(0)=0$*, then* $r=Id_M$*. If* $r(0)\geq 0$*, for each element* $x\in M$*, the element* $(x+(n-1)u)/n$ *exists in* M*, and* $r(x)=(x+(n-1)u)/n$ *for* $x\in M$*, where* $+$ *denotes the group addition in* G*.*

Proof. Since $M=\Gamma(G,u)$, we have G is also totally ordered, so it enjoys unique extraction of roots.
(1) If $r(0)=0$, by Theorem 3.2, M is the Boolean algebra $\{0,1\}$, and $r=Id_M$.
(2) Assume that $r(0)\neq 0$. For each $x\in M$, we have two cases:
(i) If $x\neq 0$, then $x=r(x)^n=(nr(x)-(n-1)u)\vee 0=(nr(x)-(n-1)u)$ because M is totally ordered. Then $x=nr(x)-(n-1)u$ and $x+(n-1)u=nr(x)$. The unique extraction of roots property implies that the element $(x+(n-1)u)/n$ exists in M. Hence, for each $x\in M\setminus\{0\}$, $r(x)=(x+(n-1)u)/n$.
(ii) If $x=0$, then $0=r(0)^n=(nr(0)-(n-1)u)\vee 0$ entails that $(nr(0)-(n-1)u)\leq 0$. That is $nr(0)\leq(n-1)u$, On the other hand, if $y\in M$, we have $y^n\leq 0$ if and only if $ny\leq(n-1)u$. By (Sq2), $ny\leq(n-1)u$ implies that $y\leq r(0)$. That is $r(0)=max\{z\in M: nz\leq(n-1)u\}$. Also, if $g\in G$ be such that $ng\leq(n-1)u$, then $g\in G^-$ implies that $g\leq r(0)$ and from $g\in G^+$ we get $g\leq ng\leq(n-1)u$, that is $g=[0,u]=M$ which means $g\leq r(0)$. Hence

$$r(0)=\max\{z\in G: nz\leq(n-1)u\}. \tag{c}$$

By Proposition 3.2, $(r(0)^-)^n\in B(M)$, so $(r(0)^-)^n=0$ or $(r(0)^-)^n=1$ (since M is totally ordered). If $(r(0)^-)^n=1$, then $r(0)^-=1$ entails that $r(0)=0$, but this is excluded. If $(r(0)^-)^n=0$,then $nr(x)=(n-1)u$ entails $nr(0)\geq(n-1)u$, so by (c), $nr(0)=u$. In this case, $r(0)=(0+(n-1)u)/n$. □

Theorem 4.2. *(Representation of* n*-strict MV-algebras) Let* $M=\Gamma(G,u)$ *be a representation symmetric pseudo MV-algebra with a strict* n*-root* r*, where* (G,u) *is a unital* ℓ*-group. Then* $(x+(n-1)u)/n$ *exists for each* $x\in M$*, and*

$$r(x)=\frac{x+(n-1)u}{n},\ x\in M,$$

where $+$ *denotes the group addition in* G*.*

Proof. Assume that X is the set of all normal prime ideals $P\neq M$ of M. The set X is non-empty because M is representable. We note that also G is representable. Consider the embedding $\varphi: M\to M_0:=\Pi_{P\in X}M/P$ defined by $\varphi(x)=(x/P)_{P\in X}$. We have $r(0)>0$.
(1) For each $P\in X$ we have $r(0)\notin P$(since r is strict, and $u=n.r(0)$), consequently, $r(0)/P\neq 0/P$, and $r_P(x/P)=r(x)/P$. Hence, we have

$$r_P(x/P)^{n-1}=r(x)^{n-1}/P=r(x)^-/P=(r(x)/P)^-=r_P(x/P)^-.$$

Hence, r_P is an n-root on the totally ordered pseudo MV-algebra M/P and r_P is strict.
(2) By Proposition 3.4, the homomorphism $\pi_P \circ \varphi : M \to M/P$ includes a strict n-root $t_P : M/P \to M/P$ defined by $t_P(x/P) = r(x)/P$. Let $\hat{P}$ be the ℓ-ideal (i.e. a normal convex ℓ-subgroup of G) of G generated by P. Then $\Gamma(G/\hat{P}, u/\hat{P})$ is totally ordered, symmetric pseudo MV-algebra such that $M/P \cong \Gamma(G/\hat{P}, u/\hat{P})$. Since M/P is totally ordered,(1) implies that M/P is not a Boolean algebra and so by Theorem 4.1,

$$r(x)/P = t_p(x/P) = (x/P +_P (n-1)u/P)/n,$$

where $+_P$ denotes the group addition in the unital ℓ-group $(G/\hat{P}, u/\hat{P}) = \Psi(M/P)$. If we set $t((x/P)_{P\in X}) = (t_P(x/P))_{P\in X}$, t is an n-root on M_0 and r is the restriction of t on M.
(3) Consider the unital ℓ-group $\Psi(\Pi_{P\in X}M/P) = \Pi_{P\in X}(\Psi(M/P))$. Due to Theorem 2.1, the functor $\Psi : \mathcal{PMV} \to \mathcal{UG}$ includes an ℓ-group homomorphism $\Psi(\varphi) : (G, u) \to \Psi(\Pi_{P\in X}M/P)$ preserving the strong unit that is injective. Recall that $\varphi(g) = \Psi(\varphi)(g)$ for all $g \in M$. We have $\varphi(r(x)) = (r(x)/P)_P \in X = (t_P(x/P))_{P\in X} = ((x/P +_P (n-1)u/P)/n)_{P\in X}$. Also,

$$(x/P +_P (n-1)u/P)_{P\in X} = ((x +_P (n-1)u)/P)_{P\in X}.$$

(4) By(3),

$$\varphi(nr(x)) = n\varphi(r(x)) = n\Psi\varphi(r(x)) = ((x+(n-1)u)/P)_{P\in X} = \varphi(x+(n-1)u).$$

entails that $nr(x) = (x+(n-1)u)$ and consequently, $r(x) = (x+(n-1)u)/n$. $\square$

We introduce the following notions. Let X be a non-empty subset of partially ordered set $(P, \leq)$. We defined $\uparrow X := \{a \in P | x \leq a \ for\ some\ x \in P\}$ and $\downarrow X := \{a \in P | a \leq x \ for\ some\ x \in P\}$.

Theorem 4.3. *Let $s : M \to M$ be a strict n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. The following statements hold:*
(1) For each $x \in M$, $s(x)^- \vee s(x)^\sim \leq s(0)$ and $s(0) \oplus s(x) = s(x) \oplus s(0) = 1$.
(2) For each $b \in B(M)$, $s(0) \leq b \leq 1$ implies that $b = 1$, consequently, $(\downarrow s(0) \cup \uparrow s(0)) \cap B(M) = \{0, 1\}$ and if M is not proper, then $B(M) \cap s(M) = \{1\}$.
(3) For all $x, y \in M$, $s(x) \to (s(x) \odot s(y)) = s(y) = s(x) \rightsquigarrow (s(y) \odot s(x))$.

Proof. (1) For each $x \in M$, we have $s(0) \leq s(x)$ and so $s(x)^- \leq s(0)^- = s(0)^{n-1}$ and $s(x)^\sim \leq s(0)^\sim = s(0)^{n-1}$. Due to [[Georgescu and Iorgulescu, 2001]. Prop.1.9], thus $s(x) \oplus s(0)^{n-1} = s(0)^{n-1} \oplus s(x) = 1$. Then $1 = s(x) \oplus s(0)^{n-1} \leq s(x) \oplus s(0)$, and so $s(x) \oplus s(0) = 1$.

(2) Let $b \in B(M)$ such that $s(0) \leq b \leq 1$. First we note that if $0 \neq 1$, then $s(0) < b$ (otherwise, by Proposition 3.1 (14), $0 = s(0)^n = b^n = 0$). Then $b^- < s(0)^- = s(0)^{n-1}$ and so $b^- = (b^-)^2 \leq s(0)^{n-1} \odot s(0)^{n-1} = s(0)^n \odot s(0)^{n-2} = 0$ which implies that $b = 1$. Therefore, by Proposition 3.1 (3), $(\downarrow s(0) \cup \uparrow s(0)) \cap B(M) = \{0, 1\}$. Clearly, $1 = s(1) \in B(M) \cap s(M)$. Now, let $b \in B(M) \cap s(M)$. Then by (1), $b^- \leq s(0)$, so $b = 1$.
(3) By Proposition 2.1 (4), we have

$$s(x) \to (s(x) \odot s(y)) = s(x)^- \vee s(y), s(x) \rightsquigarrow (s(y) \odot s(x)) = s(x)^\sim \vee s(y).$$

By (1), $s(x)^- \vee s(x)^\sim \leq s(0) \leq s(y)$. So we have $s(x) \to (s(x) \odot s(y)) = s(y) = s(x) \rightsquigarrow (s(y) \odot s(x))$ for all $x, y \in M$. □

Theorem 4.4. *For each n-root of a representable symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1) = \Gamma(G, u)$, the following statements hold:*
(1) $n.r(0) = r(0) \oplus r(0)$ and it is a Boolean element of M.
(2) $r(0) = 0$ or $r(0) \oplus r(0) = u$.
(3) $r(0) \geq 0$, then $r(0)^2 \neq 0$.
(4) $r(0) \geq 0$, then $r(0)^{n-1} \oplus r(0) = u$. In addition, $r(0)^{n-1} \geq 0$.
(5) $M = \{0, 1\}$ or M is n-strict.

Proof. (1) By Proposition 3.1 (2), we have

$$n.r(0) \leq r(n.r(0)) = r(0) \oplus r(0) \leq n.r(0) \text{ and } r(0) \leq r(n.r(0)).$$

It follows that $r(n.r(0)) = n.r(0) = n.r(0) \vee r(0)$. By Proposition 3.4 (1), $n.r(0) \in B(M)$. Therefore, $r(0) \oplus r(0) = 0$ if and only if $r(0) = 0$.
(2) By (1), $r(0) \oplus r(0) = 0$ or $r(0) \oplus r(0) = u$. Moreover, $r(0) \oplus r(0) = 0$ if and only if $r(0) = 0$.
(3) For $n = 2$, the proof is straightforward (refer to [Dvurečenskij and Zahiri, 2023]). Let $n \geq 3$. Since $r(0) \neq 0$, by(2), $r(0) \oplus r(0) = 1$ and so $r(0)^- \leq r(0)$.
(i) For each $y \in M \setminus \{0\}$, $0 < y = r(y)^n = (nr(y) - (n-1)u) \vee 0$ and so $y = nr(y) - (n-1)u$ entails that $nr(y) = y + (n-1)u$, whence $r(y) = (y + (n-1)u)/n$.
(ii) We claim that $r(0)^2 > 0$, otherwise, $r(0)^2 = 0$ consequently, $r(0) \leq r(0)^-$. Thus, $r(0) = r(0)^-$ or equivalently, $2r(0) = u$, that is $r(0) = u/2 \in G$. It follows that $r(u/2) = (u/2 + (n-1)u)/n = ((2n-1)u)/2n$. So, $-u/2n = ((2n-1)u)/2n - u \in G$ which implies $u/2n$, u/n, $(n-1)u/n \in G$. From $((n-1)u/n)^n = (n(n-1)u/n - (n-1)u)/n = 0$ and (Sq2) it follows that $(n-1)u/n \leq r(0) = u/2$ which is a contradiction. Therefore, $r(0)^n \neq 0$ which implies that $r(0)^2 = 2r(0) - u$.
(4) Set $y = r(0)^{n-1} \oplus r(0)$. First, we will show that $r(y) = 1$. By (2), $y \oplus r(0) =$

$r(0)^{n-1} \oplus r(0) \oplus r(0) = 1$, so $r(y) = y \oplus r(0) = 1$, and Proposition 3.2 implies that $y \in B(M)$. On the other hand M is a chain and $0 < r(0) \leq y \in B(M)$, thus $y = 1$. Hence, it suffices to prove that $r(y) = 1$ (note that $u = 1$).

$$\begin{aligned} r(y) &= r\big(r(0)^{n-1} \oplus r(0)\big) \\ &= r\big(r(0)^{n-1}\big) \oplus \big(r\big(r(0)\big) \odot r(0)^{-}\big), \quad \text{(by Theorem 3.12)} \\ &= \big(r\big(r(0)\big)^{n-1} \vee r(0)\big) \oplus \big(r\big(r(0)\big) \odot r(0)^{-}\big). \text{(by Prop3.4 (10))} \end{aligned}$$

Since $r(0) \neq 0$, by part (1) in the proof of (iii), we have

$$\begin{aligned} r\big(r(0)\big) \odot r(0)^{-} &= \left(\frac{r(0) + (n-1)u}{n} - u + \big(u - r(0)\big)\right) \vee 0 \\ &= \frac{(n-1)u - (n-1)r(0)}{n} \vee 0 \\ &= \frac{(n-1)u - (n-1)r(0)}{n}, \quad \text{(since } r(0) \leq u\text{)}, \\ r\big(r(0)\big)^{n-1} \vee r(0) &= \left(\frac{r(0) + (n-1)u}{n}\right)^{n-1} \vee r(0) \\ &= \left((n-1)\frac{r(0) + (n-1)u}{n} - (n-2)u\right) \vee 0 \vee r(0) \\ &= \left(\frac{(n-1)r(0) + (n-1)^2 u - n(n-2)u}{n}\right) \vee 0 \vee r(0) \\ &= \left(\frac{(n-1)r(0) + (n^2-2n+1)u - (n^2-2n)u}{n}\right) \vee 0 \vee r(0) \\ &= \frac{(n-1)r(0) + u}{n} \vee r(0) \\ &= \frac{(n-1)r(0) + u}{n}, \quad \text{(since } r(0) \leq u\text{)}. \end{aligned}$$

It follows that

$$\begin{aligned} r(y) &= \frac{(n-1)r(0) + u}{n} \oplus \frac{(n-1)u - (n-1)r(0)}{n} \\ &= \left(\frac{(n-1)r(0) + u}{n} + \frac{(n-1)u - (n-1)r(0)}{n}\right) \wedge u \\ &= (nu/n) \wedge u = u. \end{aligned}$$

$r(0)^{n-1} = 0$ implies that $r(0) = r(0)^{n-1} \oplus r(0) = u = r(u)$, so $0 = u$ which is absurd. It follows that, $r(0)^{n-1} > 0$.
(5) Consider $a = r(0)^{n-1} \oplus r(0)$. According to (4), $a \in M$, so $a = 0$ or $a = 1$.

From $r(0) \leq a = 0$ and Theorem we get that $r = Id_M$ and M is a Boolean algebra, so $M = \{0, 1\}$. Moreover, if $a = 1$, then $r(0)^{n-1} \geq r(0)^-$. Whence $r(0)^{n-1} = r(0)^-$ which means r is a strict n-root. Note that, in each n-root, we always have $r(0)^{n-1} \leq r(0)^-$. □

Theorem 4.5. *For each n-root of a representable symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1) = \Gamma(G, u)$ and for all $i \in \{1, 2, \ldots, n-1\}$, $r(0)^i \oplus r(0) \in B(M)$.*

Proof. If $r(0) = 0$, then by Theorem 3.2, $r = Id_M$ and $M = B(M)$. So the proof is evident. Now, let $r(0) \neq 0$. Consider the symbols and notations in the proof of Theorem 4.2. For each $P \in Y$, $r_P : M/(P \cap M) \to M/(P \cap M)$ is an n-root on a representation symmetric pseudo MV-algebra $M/(P \cap M)$, so by Theorem (4), $r_P(0/P)^{n-1} \oplus r_p(0/P) = 0$ or $r_P(0/P)^{n-1} \oplus r_P(0/P) = u/P$. In each case, $r_P(0/P)^i \oplus r_P(0/P) \in B(M)$ for all $i \in \{1, 2, \ldots, n-1\}$. Let $i \in \{1, 2, \ldots, n-1\}$ be given.

$$\begin{aligned}
\Gamma(f)(r(0)^i \oplus r(0)) &= (\frac{r(0)^i \oplus r(0)}{P \cap M})_{P \in Y} = (\frac{r(0)^i}{P \cap M})_{P \in Y} \oplus (\frac{r(0)}{P \cap M})_{P \in Y} && (1)\\
&= (\frac{r(0)}{P \cap M})^i_{P \in Y} \oplus (\frac{r(0)}{P \cap M})_{P \in Y} && (2)\\
&= (r_P(\frac{0}{P \cap M}))^i_{P \in Y} \oplus (r_P(\frac{0}{P \cap M}))_{P \in Y} && (3)\\
&= (r_P(\frac{0}{P \cap M})^i \oplus r_P(\frac{0}{P \cap M}))_{P \in Y} \in B(\prod_{P \in Y} \Gamma(G/P, u/P)). && (4)
\end{aligned}$$

Hence $r(0)^i \oplus r(0) \in B(M)$. □

Lemma 4.1. *Let r be an n-root on a represention symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. Then $r(0)^- = r(0)^\sim$, and there exists a unique idempotent element $u \in B(M)$ such that $u \vee r(0)^{n-1} = r(0)^- = r(0)^\sim$.*

Proof. Let $u := (n-1).r(0)^- \odot r(0)^-$ be an n-root. By Proposition 4.5, $u \in B(M)$ and we have

$$\begin{aligned}
u \vee r(0)^{n-1} &= ((r(0)^{n-1})^- \odot r(0)^-) \oplus r(0) \qquad \text{by Proposition 3.1(10)}\\
&= r(0)^- \vee r(0)^{n-1} = r(0)^-,
\end{aligned}$$

since $r(0)^n = 0$ which yields $r(0)^{n-1} \leq r(0)^-$. In a similar way, using Proposition 4.5, we can show that $u \vee r(0) = r(0)^\sim$.
Now, let $v \in B(M)$ such that $v \vee r(0)^{n-1} = r(0) \to 0$. Then we have

$$\begin{aligned}
(v \vee r(0))^{n-1} &= v^{n-1} \vee (v^{n-2} \odot r(0)) \vee \ldots \vee (v \odot r(0)^{n-2}) \vee r(0)^{n-1}\\
&= v^{n-1} \vee r(0)^{n-1} = v \vee r(0)^{n-1} = r(0)^-.
\end{aligned}$$

On the other hand, by Proposition 3.1 (10), $v \vee r(0) = r(v^n) = r(v)$, so $r(v)^{n-1} = r(0)^-$. In a similar way, $r(u)^{n-1} = r(0)^-$. It follows that $u = v$. □

Theorem 4.6. *Let $r : M \to M$ be an n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. Then M satisfies only one of the following statements:*
(1) The pseudo MV-algebra M is a Boolean algebra.
(2) The pseudo MV-algebra M is a strict pseudo MV-algebra.
(3) There exist MV-algebra M_1 and M_2 such that M_1 is a Boolean algebra, M_2 is an n-strict MV-algebra, and $M \cong M_1 \times M_2$. In addition, this representation for M is unique up to isomorphic.

Proof. Due to Proposition 3.2, the element $a = r(0)^{n-1} \oplus r(0)$ and $b = (n-1).r(0)^- \odot r(0)^-$ are Boolean elements, $b = a^- = a^\sim$. We have three cases.
Case 1. If $a = 0$, then $0 = r(0)$, so by Theorem 3.2, M is a Boolean algebra.
Case 2. If $a = 1$, then $r(0)^- \leq r(0)^{n-1}$. Hence, $r(0)^- = r(0)^{n-1}$, r is a strict n-root.
Case 3. If $0 < a < 1$, then consider the pseudo MV-algebras $([0, b]; \oplus, ^{-b}, ^{\sim b}, 0, b)$ and $([0, a]; \oplus, ^{-a}, ^{\sim a}, 0, a)$. By Proposition 3.1 (15), they have n-root $r_a : [0, a] \to [0, a]$ and $r_b : [0, b] \to [0, b]$, defined by $r_a(x) = r(x) \wedge a$ and $r_b(y) = r(y) \wedge b$ for all $x \in [0, a]$ and $y \in [0, b]$.

$$r_b(0) = b \wedge r(0) = b \odot r(0) = ((n-1).r(0)^-) \odot r(0)^- \odot r(0) = 0.$$

$$r_a(0) = r(0) \wedge a = r(0) \wedge (r(0)^{n-1} \oplus r(0)) = r(0).$$

So by Theorem 3.2 $[0, b]$ is a Boolean algebra, and from Lemma 4.5, we have$[0, a]$ is an n-strict pseudo MV-algebra.
In addition, by Remark 3.2, $M \cong [0, a] \times [0, b]$.
Now, we prove the second part of (3), which is the uniqueness of the direct product. Let $M \cong_\varphi M_1 \times M_2$, where M_1 is a Boolean algebra and M_2 is a strict pseudo MV-algebra. Let 0_1 and 0_2 (1_1 and 1_2) be the least (greatest) elements of M_1 and M_2, respectively. Consider the n-square root $s_i : M_i \to M_i$ $(i - 1, 2)$ (by the assumption, $s_1 : Id_{M_1}$ and s_2 is strict). Clearly, $s := (s_1, s_2) : M_1 \times M_2 \to M_1 \times M_2$, defined by $s(x_1, x_2) = (s_1(x_1), s_2(x_2))$, is an n-root on $M_1 \times M_2$. Also,

$$\begin{aligned} s(0_1, 0_2) \to (0_1, 0_2) &= (s_1(0_1), s_2(0_2)) \to (0_1, 0_2) = (0_1 \to 0_1, s_2(0) \to 0_2) \\ &= (1_1, s_2(0_2) \to 0_2) = (1_1, 0_2) \vee (0_1, s_2(0_2) \to 0_2) \\ &= (1_1, 0_2) \vee (0_1, s_2(0)^{n-1}) = (1_1, 0_2) \vee (0_1, s_2(0_2))^{n-1} \\ &= (1_1, 0_2) \vee (s_1(0_1), s_2(0_2))^{n-1} = (1_1, 0_2) \vee s(0_1, 0_2)^{n-1}. \end{aligned}$$

Hence, $(1_1, 0_2)$ satisfies the conditions of Lemma 4.5 which means $\varphi^{-1}(1_1, 0_2)$ satisfies the conditions, too. Lemma 4.5 implies that $\varphi^{-1}(1_1, 0_2) = b$ and, consequently, $[0, b] \cong [(0_1, 0_2), (1_1, 0_2)] \cong M_1$. Also,$b^\sim = \varphi^{-1}(1_1, 0_2)^\sim = \varphi^{-1}(0_1, 1_2)$, so $[0, a] \cong [(0_1, 0_2), (0_1, 1_2)] \cong M_2$. □

Theorem 4.7. *Each n-root r of a representation symmetric pseudo MV-algebra $M = \Gamma(G, u)$ satisfies the following equality:*

$$r(x) = (x \wedge b) \vee \frac{(x \wedge a) + (n-1)a}{n}, x \in M,$$

where $b = r(0)^- \odot (n-1)r(0)^-$, $a = r(0) \oplus r(0)^{n-1}$, and + is the group addition in G. Moreover, $a \neq 1$, the interal $[0, a]$ is n-divisible.

Proof. According to the proof of Theorem 4.5, $M_1 = [0, b]$ is a Boolean algebra with n-root r_1, $M_2 = [0, a]$ is an n-strict pseudo MV-algebra with an n-root r_2, where $r_1(x) = r(x) \odot b$ and $r_2(y) = r(y) \odot a$ for all $x \in M_1$ and $y \in M_2$. The mapping $f : M_1 \times M_2 \to M$, defined by $f(x, y) = x \vee y$. is an isomorphism. By Proposition 3.4, $\tau(z) = f((r_1, r_2)(x, y))$ is an n-root on M, where $f(x, y) = z$. Clearly, $x = z \wedge b$ and $y = z \wedge a$. This implies that $\tau = r$. It follows that for each $z \in M$, we have

$$\begin{aligned} r(z) = \tau(z) &= f((r_1, r_2)(z \wedge b, z \wedge a)) = f(r_1(z \wedge b), r_2(z \wedge a)) \\ &= r_1(z \wedge b) \vee r_2(z \wedge a) \\ &= (z \wedge b) \vee r_2(z \wedge a) \\ &= (z \wedge b) \vee \frac{(z \wedge a) + (n-1)a}{n}. \end{aligned}$$

Note that, $M_2 = \{z \in M : z \leq a\} = \{z \in G : 0 \leq z \leq a\} = \Gamma(G, a)$. Therefore, the formula for r can be obtained if we know $r(0)$. □

Corolary 4.1. *Suppose that r is an n-root on MV-algebra $M = (\Gamma, u)$. Then for each $m \in \mathbb{N}$ with $m|n$, the MV-algebra M has an m-root.*

Proof. Set $p = n/m$. Define v: $M \to M$ by $v(z) = r(z)^p$. According to the proof of Theorem 4.7, we have $r(z) = r_1(z \wedge b) \vee r_2(z \wedge a)^p$. Due to $a \wedge b = 0$, we have $(r_1(z \wedge b) \vee r_2(z \wedge a))^p = r_1(z \wedge b)^p \vee r_2(z \wedge a)^p$. In view of the fact r_2 is a strict n-root and Corollary 3.1, $v_2 : M_2 \to M_2$ defined by $v_2(z) = r_2(z)^p$, $z \in M_2$, is a strict m-root and $v_2(z) = (z - (m-1)u)/m$. Hence, $r_2(z \wedge a)^p = v_2(x \wedge a) = ((z \wedge a) + (m-1)a)/m$ for all $z \in M$. We have

$$\begin{aligned} r(z)^p &= (r_1(z \wedge b) \vee r_2(z \wedge a))^p = r_1(z \wedge b)^p \vee r_2(z \wedge a)^p \\ &= (z \wedge b)^p \vee r_2(z \wedge a)^p = (z \wedge b) \vee r_2(z \wedge a)^p \\ &= (z \wedge b) \vee \frac{(z \wedge a) + (m-1)a}{m}. \end{aligned}$$

The identity map is an m-root on M_1, since M_1 is a Boolean algebra. On the other hand, since M_2 is n-strict and $m|n$, by Corollary 3.1, the mapping $(x +$

$(m-1)a)/m$, $x \in M_2$, is an m-root on M_2. Similar to the proof of Theorem 4.7, $\tau(x) = (x \wedge b) \vee ((x \vee a) + (m-1)a)/m$ is an m-root on M. Clearly, $\tau = v$. Therefore, M has an m-root. $\square$

Similar to [[Belluce, 1992], Page 16], we defined a strongly atomless pseudo MV-algebra. It is a pseudo MV-algebra M such that for each non-zero element $x \in M \setminus \{0\}$, there exists a normal prime ideal P such that $x \notin P$ and x/P is not an atom of M/P. Clearly, each strongly atomless pseudo MV-algebra is representable.

Lemma 4.2. *[Dvurecenskij and Zahiri, 2022] In each representable pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, the following conditions are equivalent:*
(1) M is strongly atomless.
(2) For each $x \in M \setminus \{0\}$ there exists $y \in M$ such that $0 \leq y \leq x$ and $y \wedge (x \odot y^-) \neq 0$.
(3) For each $x \in M \setminus \{0\}$ there exists $y \in M$ such that $0 \leq y \leq x$ and $y \wedge (y^\sim \odot x) \neq 0$.

Proposition 4.1. *Let $(M; \oplus, -, \sim, 0, 1)$ be a representable pseudo MV-algebra. If M is n-strict, then it is strongly atomless.*

Proof. Assume $M = \Gamma(G, u)$, where (G, u) is a unital ℓ-group. Let $r : M \to M$ be a strict n-root. Due to Theorem 4.1, $r(x) = (x + (n-1)u)/n$ for each $x \in M$. According to Lemma 4.7, we need to prove that for each $x \in M \setminus \{0\}$, there is $y \in M$ such that $0 < y < x$ and $y \wedge (x \odot y^-) \neq 0$. If $|M| = 1$, then $M = \{0\}$ and the proof is clear. Let $|M| \geq 2$. Choose $x \in M \setminus \{0\}$.
(1) If $x = 1$, then choose $y \in M \setminus B(M)$ such that $0 < y < x$ (since M is strict, by Theorem 4.5, M is not a Boolean algebra and so we can find such an element). Then $y \wedge (x \odot y^-) = y \wedge y^- > 0$, since y is not a Boolean element (see [Georgescu and Iorgulescu, 2001]).
(2) Let $x \in M \setminus \{0, 1\}$ such that $x \in B(M)$. By Proposition 3.4, $r(x) = x \vee r(0)$ and $r(x^-) = x^- \vee r(0)$. Set $y := r(x^-)^\sim$. From relation 3.6(i), we know that $0 \leq y \leq x$. If $y = 0$, then $r(x^-) = 1 = r(1)$ and so $x = 0$, which is a contradiction. If $y = x$, in view of Proposition 3.3 (1), $x^- \vee r(0) = r(x^-) = x^-$. Thus, $r(0) \leq x^-$ and $x \leq r(0)^- = r(0)^{n-1}$ imply that

$$x = x \odot x \leq r(0)^{n-1} \odot r(0)^{n-1} = r(0)^n \odot r(0)^{n-2} = 0 \odot r(0)^{n-2} = 0.$$

This contraction to our assumption. Thus, $0 < y < x$. From $x \in B(M)$, we have

$$\begin{aligned} y \wedge (x \odot y^-) &= r(x^-)^\sim \wedge (x \wedge r(x^-)) = r(x^-)^\sim \wedge r(x^-) \\ &= (x \wedge r(0)^\sim) \wedge (x^- \vee r(0)) \\ &= (x \wedge x^- \wedge r(0)) \vee (x \wedge r(0)^\sim \wedge r(0)) \\ &= x \wedge r(0)^\sim \wedge r(0), \qquad (x \in B(M), x \wedge x^- = 0) \\ &= x \wedge r(0)^{n-1} \wedge r(0), \qquad (r \text{ is } n\text{-strict}) \\ &= x \wedge r(0)^{n-1}. \end{aligned}$$

We claim that $x \wedge r(0)^{n-1} \neq 0$. Otherwise, since $x \in B(M)$, $r(0) \leq x^-$ and so $x \leq r(0)^- = r(0)$. It follows that $x = x \odot x \leq r(0)^{n-1} \odot r(0)^{n-1} = 0$ which is absurd.
(3) If $x \in M \setminus B(M)$, then similarly to (2), we can show that $0 < r(x^-)^\sim < x$. Note that by Proposition 3.1 (1), $x \vee r(0) \leq r(x)$. Set $y := r(x^-)^\sim$. We will show that

$$0 \neq y \wedge (x \odot y^-) = r(x^-)^\sim \wedge (x \odot r(x^-)).$$

By Theorem 4.2,

$$r(x^-) = ((x^-) + (n-1)u)/n = (u - x + (n-1)u)/n = (nu - x)/n = u - x/n.$$

(i) Given $p \in G$ such that p/n exists, then $p/n + u = (p + nu)/n$. Indeed,

$$n(p/n + u) = n(p/n) + nu = p + nu,$$

so $p/n + u = (p + nu)/n$ (G enjoys unique extraction of roots). Then

$$n(p/n - u) = n(p/n) - nu = p - nu.$$

and hence $p/n - u = (p - nu)/n$.
(ii) For every $p \in G$, if p/n exists, then $(-p)/n$ exists, and $(-p)/n = -p/n$, since $n(p/n) = p$. Thus, $-p = n(-p/n)$.
(iii) Given $a \in M$, in the ℓ-group G by (i), we have

$$r(a^-)^\sim = u - ((nu - a)/n)u + (a - nu)/n = (a - nu + nu)/n = a/n.$$

That is, a/n exists and belongs to M. Also,

$$\begin{aligned} r(x^-)^\sim \wedge (x \odot r(x^-)) &= x/n \wedge ((x + r(x^-) - u) \vee 0) \\ &= x/n \wedge ((x + u - x/n - u) \vee 0) \\ &= x/n \wedge (x - x/n) = x/n \wedge (x + (-x/n)) \\ &= x/n \wedge (x + (-x)/n) = x/n \wedge (n-1)x/n = x/n. \end{aligned}$$

Since $x \neq 0$, if we choose $y = r(x^-)^\sim$, then $y \wedge (x \odot y^-) = x/n \neq 0$. □

5 Discussion and Conclusion

This paper aims to extend the concept of square roots in pseudo MV-algebras of [Dvurecenskij and Zahiri, 2022] and to achieve this goal, it is necessary to introduce the notion of n-roots. The research systematically explores the properties, classifications, and connections with related algebraic structures of n-roots in pseudo MV-algebras, providing a solid foundation for the development of non-commutative algebraic logic. First, the paper clarifies the definitions of n-roots and weak n-roots, proving that weak n-roots are injective and unique, while n-roots need to satisfy additional conditions related to inverse operations. It is revealed through the decomposition theorem that pseudo MV-algebras with n-roots can be decomposed into a Boolean algebra and a strict pseudo MV-algebra, a result based on the categorical equivalence between pseudo MV-algebras and unital ℓ-groups as well as the properties of normal prime ideals. Moreover, Boolean algebras are successfully characterized among MV-algebras by using n-roots.

In the study of strict n-roots, the paper provides a comprehensive characterization of each strict n-root. It is demonstrated that every MV-algebra with a strict n-root can be described by a unital n-divisible ℓ-group, and strict n-roots are detailedly characterized. Additionally, the paper establishes a connection between n-strict pseudo MV-algebras and strongly atomless pseudo MV-algebras, proving that in the variety of MV-algebras with n-roots, the concepts of n-strict MV-algebras and strongly atomless MV-algebras coincide. Compared with the square roots defined on pseudo MV-algebras, the n-th roots investigated in this paper exhibit a higher degree of generality. Furthermore, the paper verifies that the class of pseudo MV-algebras with n-roots forms a variety, and discusses properties such as the preservation of n-roots under homomorphic mappings and the n-root invariance of ideals. These findings not only improve the element classification theory of pseudo MV-algebras but also provide a new algebraic tool for handling the "detailed splitting" and "inverse operation" of proposition truth values in non-commutative fuzzy reasoning systems, which is of great significance for applications in fuzzy systems, quantum logic, and ordered algebraic structures.

Future research will focus on fully characterizing these n-roots, further exploring the interplay of divisibility, atomicity, and logical completeness in non-classical algebraic models, and promoting the practical application of pseudo MV-algebras with n-roots in more fields.

Declarations

Conflict of interest The authors declare no conflict of interest. No data is involved in this article.

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Robustness of Fuzzy Regression Discontinuity Designs under Imperfect Compliance

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Abstract

This paper studies the robustness of fuzzy regression discontinuity designs under imperfect compliance with clinical assignment rules. When treatment assignment based on a biomarker threshold is probabilistic, the cutoff indicator can be interpreted as an instrumental variable identifying a Local Average Treatment Effect. A Monte Carlo simulation calibrated to a cardiovascular risk setting is used to evaluate the local two stage least squares fuzzy regression discontinuity estimator under controlled violations of its identifying assumptions. We consider departures from continuity of potential outcomes at the cutoff, reductions in first stage strength, and violations of monotonicity. The results show that, under approximate continuity and sufficiently strong first stages, the estimator recovers the target local effect with limited bias and near nominal coverage. Weak first stages mainly reduce precision, whereas violations of continuity lead to severe bias even when the instrument is strong. Monotonicity violations increase variability and reduce coverage. Overall, the findings highlight that fuzzy regression discontinuity designs provide reliable local causal inference only under carefully assessed assumptions.

Keywords: fuzzy regression discontinuity, causal inference, instrumental variables, imperfect compliance, sensitivity analysis.

2020 AMS subject classifications: 62J12, 62P10, 62P20.[1]

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1 Introduction

The analysis of complex phenomena in the social, economic, and health sciences is inherently affected by multiple sources of uncertainty. In many empirical settings, uncertainty does not arise solely from randomness, but also from imprecision, vagueness, and partial observability of the mechanisms generating the data. Classical probabilistic models are well suited to represent stochastic variability, but they are often less effective in capturing situations in which concepts, rules, or assignments are not sharply defined.

The theory of fuzzy sets was introduced precisely to address this type of uncertainty, by allowing elements to belong to a set with varying degrees rather than through a crisp binary classification, see Zadeh [1965, 1968, 1975]. Since its introduction, fuzzy logic has provided a flexible conceptual and mathematical framework for representing gradual transitions, linguistic categories, and imprecise rules, and has been extensively developed both from a theoretical and an applied perspective, see Bellman and Giertz [1973], Kosko [1990].

Building on these foundations, several authors have investigated the interaction between fuzziness and randomness, leading to the notion of fuzzy random variables and hybrid probabilistic fuzzy frameworks, see Puri and Ralescu [1986]. This line of research has shown that many empirical processes cannot be adequately described by purely stochastic models or purely fuzzy representations, but require a combination of both to reflect real world complexity.

Fuzzy methods have subsequently been extended to regression analysis and statistical modeling, with the aim of handling imprecise data, vague relationships, and uncertainty in model specification, see Celmins [1987], Diamond [1988]. Further developments have emphasized the role of fuzzy probabilistic approaches for the study of statistical relationships in economic and managerial sciences, where uncertainty often arises from both incomplete information and structural vagueness, see Maturo [2016]. Related contributions have explored fuzzy regression as a tool for addressing causal complexity in social sciences, where deterministic mechanisms are rarely observed, see Maturo and Maturo [2013].

Beyond regression settings, fuzzy concepts have found applications in a wide range of domains, including the modeling of fuzzy events and fuzzy probability in economic and social systems, see Maturo and Maturo [2017], the construction of fuzzy quantities linked to classical probability distributions, see Maturo and Fortuna [2016], and the analysis of complex relational structures such as fuzzy networks and dynamic health monitoring, see Maturo et al. [2020], Porreca et al. [2025]. Together, these contributions highlight the importance of methodological

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frameworks capable of accommodating gradual rules, partial compliance, and non crisp mechanisms in applied statistical analysis.

Within this broader context, the estimation of causal effects in clinical and epidemiological research presents particularly challenging features. Causal relationships are fundamentally counterfactual in nature, since individual outcomes under both treatment and non treatment conditions cannot be simultaneously observed. Randomized Controlled Trials address this problem through exogenous treatment assignment, but their implementation is often constrained by ethical considerations, financial costs, and limited external validity. As a result, causal analyses in medical research frequently rely on observational data, where treatment assignment is potentially endogenous and subject to selection bias.

Clinical guidelines represent an important source of quasi experimental variation in such settings. Medical recommendations often prescribe treatment on the basis of a continuous biomarker X_i, where i indexes individuals, relative to a predefined threshold $c \in \mathbb{R}$. This rule generates a binary treatment indicator $D_i \in \{0, 1\}$, where $D_i = 1$ denotes treatment receipt and $D_i = 0$ denotes non treatment. In its idealized form, the guideline induces a deterministic assignment mechanism, which can be written as

$$D_i = \mathbb{1}(X_i \geq c), \tag{1}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function, equal to one if the condition in parentheses is satisfied and zero otherwise. Under this formulation, individuals with biomarker values weakly above the cutoff are assigned to treatment, while those with values below the cutoff remain untreated.

This assignment mechanism defines the Sharp Regression Discontinuity Design. Identification relies on the assumption that individuals located sufficiently close to the cutoff are comparable in both observed and unobserved characteristics, so that any discontinuity in the outcome at c can be attributed to the treatment effect, see Lee and Lemieux [2010]. The validity of this design critically depends on perfect compliance with the assignment rule, meaning that treatment status is fully determined by the threshold.

In applied clinical contexts, however, this assumption is rarely satisfied. Physicians may prescribe treatment to patients whose biomarker values fall below the recommended threshold, based on additional risk factors or clinical judgment. Conversely, some patients who exceed the cutoff may not receive treatment due to contraindications, preferences, or non adherence. As a consequence, treatment assignment becomes probabilistic rather than deterministic, and the sharp design is no longer appropriate.

These deviations from perfect compliance motivate the use of the Fuzzy Regression Discontinuity Design, in which the probability of receiving treatment exhibits a discontinuous jump at the cutoff without reaching zero or one. Formally, this setting is characterized by

$$\lim_{x \downarrow c} P(D_i = 1 \mid X_i = x) \neq \lim_{x \uparrow c} P(D_i = 1 \mid X_i = x). \quad (2)$$

The discontinuity in treatment probability allows the cutoff indicator to be interpreted as an instrumental variable, identifying a Local Average Treatment Effect for individuals whose treatment status is influenced by the guideline, see Imbens and Lemieux [2008] and Venkataramani et al. [2016].

While the identification strategy underlying fuzzy regression discontinuity designs is well established, applied analyses often rely on strong local assumptions whose empirical plausibility is rarely examined. In particular, continuity of potential outcomes at the cutoff, monotonicity of treatment assignment, and sufficient strength of the first stage are typically assumed without formal assessment. This paper focuses on the robustness of the fuzzy regression discontinuity estimator to controlled violations of these assumptions. Through a simulation framework calibrated to a cardiovascular risk setting, we evaluate how departures from ideal conditions affect bias and variability of the estimated Local Average Treatment Effect, with the aim of providing practical guidance for applied biostatistical research, see [Calonico et al., 2014].

2 Materials and Methods

This section outlines the methodological framework adopted to analyze fuzzy regression discontinuity designs under imperfect compliance. The approach follows the standard instrumental variable interpretation of fuzzy regression discontinuity designs and focuses on the identification and estimation of a local average treatment effect at the cutoff. Particular attention is devoted to the assumptions underlying identification and to their role in the subsequent robustness analysis.

2.1 Identification in Fuzzy Regression Discontinuity Designs

Let X_i denote a continuous forcing variable and c a known cutoff value determining treatment eligibility according to a clinical guideline. Treatment assignment is represented by a binary indicator $D_i \in \{0, 1\}$, and the outcome of interest by Y_i. In a fuzzy regression discontinuity design, crossing the threshold does not deterministically assign treatment, but induces a discontinuous change in the probability of treatment receipt. Formally,

$$\lim_{x \downarrow c} P(D_i = 1 \mid X_i = x) \neq \lim_{x \uparrow c} P(D_i = 1 \mid X_i = x). \quad (3)$$

Under standard assumptions, the discontinuity in treatment probability can be exploited to identify a Local Average Treatment Effect for individuals whose treatment status is influenced by the cutoff. This estimand is defined as the ratio between the discontinuity in the conditional expectation of the outcome and the discontinuity in the conditional expectation of the treatment indicator,

$$\tau_{FRDD} = \frac{\mu_Y^+ - \mu_Y^-}{\mu_D^+ - \mu_D^-}, \tag{4}$$

where μ_Y^+ and μ_Y^- denote the right and left limits of $E(Y_i \mid X_i = x)$ at $x = c$, and μ_D^+ and μ_D^- are defined analogously for D_i. This parameter corresponds to the Local Average Treatment Effect identified by the instrumental variable induced by the cutoff indicator.

2.2 Estimation via Two Stage Least Squares

In empirical applications, treatment assignment is potentially endogenous, as it may depend on unobserved clinical characteristics correlated with the outcome. Estimation is therefore conducted within a Two Stage Least Squares framework, where the cutoff indicator serves as an instrumental variable for treatment receipt.

Let $Z_i = \mathbb{1}(X_i \geq c)$ denote the instrument. The first stage models treatment assignment as

$$D_i = \gamma_0 + \gamma_1 Z_i + f(X_i - c) + \nu_i, \tag{5}$$

where $f(\cdot)$ is a smooth function capturing the relationship between the forcing variable and treatment away from the cutoff. The coefficient γ_1 measures the discontinuous change in the probability of treatment at the threshold and reflects the strength of the first stage.

The second stage relates the outcome to the predicted treatment status,

$$Y_i = \beta_0 + \tau_{FRDD} \hat{D}_i + g(X_i - c) + \epsilon_i, \tag{6}$$

where $g(\cdot)$ is a smooth function analogous to $f(\cdot)$. Under standard regularity conditions, the Two Stage Least Squares estimator of τ_{FRDD} coincides with the Wald estimator defined above.

2.3 Local Polynomial Estimation and Kernel Weighting

Identification in regression discontinuity designs is local in nature and relies on observations in a neighborhood of the cutoff. To emphasize this local structure, estimation is performed using local polynomial regression with kernel weighting.

Observations are weighted according to their distance from the cutoff using a triangular kernel,

$$K(u) = (1 - |u|)\mathbb{1}(|u| \leq 1), \tag{7}$$

which assigns greater weight to observations closer to the threshold. The choice of kernel and bandwidth reflects the standard trade off between bias and variance in local estimation and is held fixed across simulation scenarios to isolate the effects of assumption violations examined in the subsequent analysis.

3 Simulation Design

The simulation study is designed to assess the robustness of the fuzzy regression discontinuity estimator to selected violations of its identifying assumptions, using a deliberately parsimonious Monte Carlo design. Rather than exploring a large grid of parameter combinations, the data generating process focuses on a small set of representative scenarios that isolate the effects of outcome continuity, monotonicity of treatment assignment, and instrument strength. This approach allows a transparent interpretation of bias, variability, and coverage under conditions commonly encountered in applied biostatistical research.

In each Monte Carlo replication, a synthetic population of $N = 2{,}000$ individuals is generated. The forcing variable X_i, representing LDL cholesterol levels, is drawn from a normal distribution,

$$X_i \sim \mathcal{N}(160, 20),$$

and a clinical guideline defines a cutoff at $c = 160$ mg/dL.

Treatment assignment is probabilistic and depends on the position relative to the cutoff. Let p_- and p_+ denote the probabilities of receiving treatment below and above the threshold, respectively. In the baseline configuration,

$$P(D_i = 1 \mid X_i < c) = p_- = 0.15, \qquad P(D_i = 1 \mid X_i \geq c) = p_+ = 0.75,$$

which induces a discontinuity in treatment probability of magnitude $\Delta_D = 0.60$. A weak first stage scenario is obtained by reducing this gap to $\Delta_D = 0.30$, corresponding to $p_- = 0.25$ and $p_+ = 0.55$.

Departures from monotonicity are introduced by allowing a fixed fraction of defiers. In the monotonicity violation scenario, a share of 5% of the population is assigned a treatment status that moves in the opposite direction of the cutoff indicator, while preserving the same marginal probabilities p_- and p_+ as in the baseline configuration.

The outcome variable Y_i, representing systolic blood pressure, is generated according to

$$Y_i = \alpha + \beta(X_i - c) + \tau D_i + \delta \mathbb{1}(X_i \geq c) + \eta_i + \epsilon_i, \tag{8}$$

where $\alpha = 120$, $\beta = 0.1$, and $\tau = -10$ is the true treatment effect. The parameter δ introduces a direct discontinuity in untreated potential outcomes at the cutoff and is set to $\delta = 5$ in the continuity violation scenario, while $\delta = 0$ in all other configurations. Individual heterogeneity is modeled through $\eta_i \sim \mathcal{N}(0, 5)$, and $\epsilon_i \sim \mathcal{N}(0, 2)$ represents idiosyncratic noise.

Estimation is conducted using a local linear fuzzy regression discontinuity design with triangular kernel weighting and a fixed bandwidth of $h = 10$. The Local Average Treatment Effect is estimated via two stage least squares using the cutoff indicator as an instrumental variable. For each scenario, $B = 150$ Monte Carlo replications are performed, and performance is evaluated in terms of bias, standard deviation, root mean squared error, and empirical coverage of nominal 95% confidence intervals.

4 Results

The simulation study considers four configurations, namely a baseline scenario, a continuity violation scenario, a weak first stage scenario, and a monotonicity violation scenario. Table 1 reports the Monte Carlo performance of the local two stage least squares fuzzy regression discontinuity estimator and of a naive ordinary least squares benchmark. Graphical evidence supporting the main findings is provided in Figures 1, 2, 3, and 4.

4.1 First Stage Behavior and Local Sample Size

Figure 1 displays the empirical first stage in the baseline scenario, plotting binned treatment probabilities against the forcing variable. A clear discontinuity is observed at the cutoff $c = 160$, with treatment probabilities increasing from values close to 0.15 below the threshold to approximately 0.75 above it. This jump visually confirms the presence of a strong first stage induced by the guideline rule.

In the baseline configuration, the corresponding discontinuity in treatment probability is $\Delta_D = 0.60$. Consistently, the empirical first stage diagnostic is large, with mean $\widehat{F}_Z \approx 94.24$, and the average number of locally weighted observations within the bandwidth is about 766. This combination ensures reliable local identification of the causal effect.

In the weak first stage configuration, where $p_- = 0.25$ and $p_+ = 0.55$, the discontinuity decreases to $\Delta_D = 0.30$. Although the local sample size remains

Table 1: Monte Carlo performance of the fuzzy regression discontinuity estimator and naive OLS across simulation scenarios

Scenario	Method	Mean	Bias	SD	RMSE	Coverage
Baseline	FRDD local 2SLS	-10.09	-0.09	1.44	1.43	0.95
	Naive OLS	-9.97	0.03	0.29	0.29	0.95
Weak first stage	FRDD local 2SLS	-10.05	-0.05	3.07	3.06	0.97
	Naive OLS	-9.98	0.02	0.25	0.25	0.95
Monotonicity violation	FRDD local 2SLS	-9.85	0.15	1.73	1.73	0.90
	Naive OLS	-10.00	0.00	0.28	0.28	0.97
Continuity violation	FRDD local 2SLS	-1.72	8.28	1.58	8.43	0.00
	Naive OLS	-8.58	1.42	0.26	1.45	0.00

comparable, about 763, the first stage diagnostic drops markedly to a mean value of $\widehat{F}_Z \approx 17.66$. This reduction in instrument strength is not associated with a systematic change in bias but leads to a substantial loss of precision in the fuzzy regression discontinuity estimator, as documented below.

4.2 Reduced Form and Local Average Treatment Effect

Figure 2 shows the reduced form relationship between systolic blood pressure and LDL cholesterol in the baseline scenario. The outcome exhibits a smooth increasing trend in the forcing variable, together with a visible downward shift at the cutoff. This discontinuity represents the intent to treat effect, which combines the true treatment effect with the incomplete compliance induced by the guideline.

In the baseline scenario, the local fuzzy regression discontinuity estimator is centered close to the true effect $\tau = -10$. The Monte Carlo mean is $\overline{\hat{\tau}} \approx -10.09$, with bias about -0.09, standard deviation about 1.44, and empirical coverage close to the nominal level, approximately 0.95. Figure 3 summarizes this behavior by reporting the mean estimate and dispersion across scenarios, together with a reference line indicating the true parameter value.

In the weak first stage scenario, the estimator remains approximately unbiased on average, with $\overline{\hat{\tau}} \approx -10.05$ and bias about -0.05. However, dispersion increases substantially, with standard deviation about 3.07 and mean standard error about 3.20. This pattern is clearly visible in Figure 3, where the confidence band widens markedly relative to the baseline. Empirical coverage remains close to nominal, about 0.97, reflecting the larger uncertainty induced by the weaker first stage.

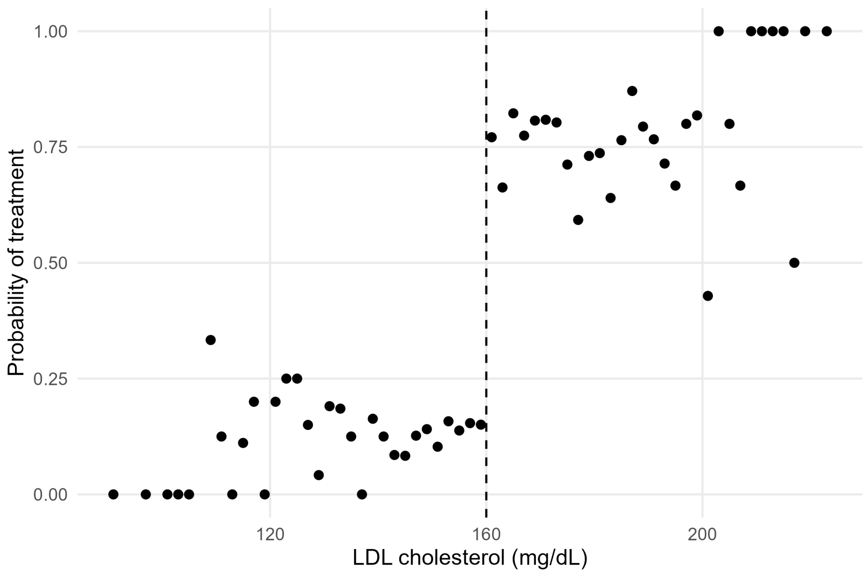

Figure 1: First stage in the baseline scenario. Binned treatment probabilities are plotted against LDL cholesterol. The vertical dashed line marks the cutoff $c = 160$.

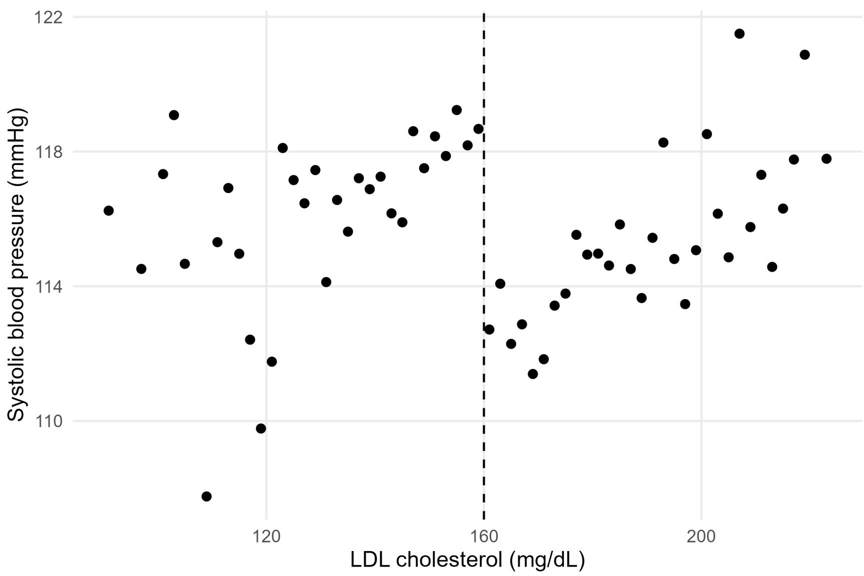

Figure 2: Reduced form in the baseline scenario. Binned averages of systolic blood pressure are plotted against LDL cholesterol. The vertical dashed line marks the cutoff $c = 160$.

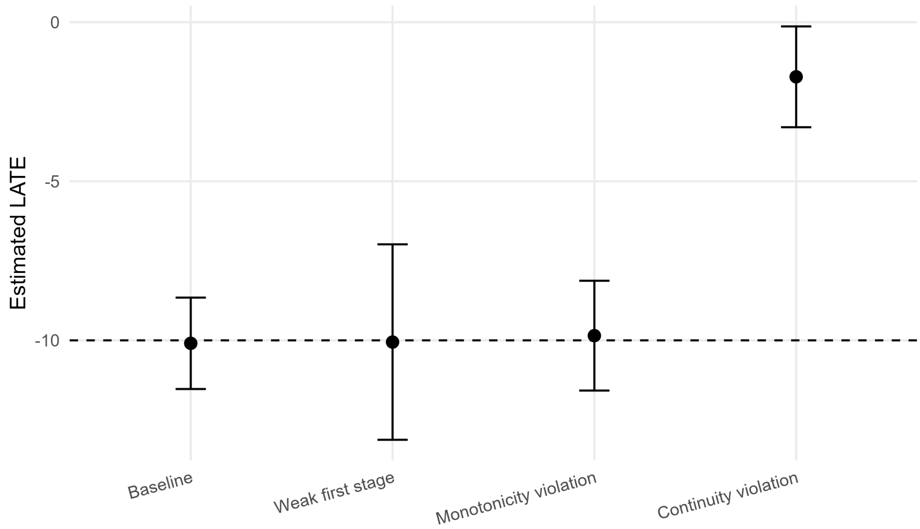

Figure 3: Monte Carlo mean and dispersion of the FRDD local 2SLS estimator across simulation scenarios. Points represent mean estimates, vertical bars represent one Monte Carlo standard deviation, the horizontal dashed line indicates the true effect $\tau = -10$.

A markedly different behavior emerges under violations of outcome continuity at the cutoff. In the continuity violation scenario, where untreated potential outcomes feature an additional discontinuity modeled through $\delta = 5$, the fuzzy regression discontinuity estimator becomes severely biased. The Monte Carlo mean is $\overline{\hat{\tau}} \approx -1.72$, with bias about 8.28, and empirical coverage collapses to zero. This failure occurs despite a very strong first stage, with mean $\widehat{F}_Z \approx 102.89$, indicating that instrument strength cannot compensate for violations of the continuity assumption. This result is clearly reflected in Figure 3, where the estimated effect lies far from the true value.

Finally, under monotonicity violations implemented through a defier share equal to 0.05, the fuzzy regression discontinuity estimator remains close to the true effect on average, with $\overline{\hat{\tau}} \approx -9.85$ and bias about 0.15. However, variability increases relative to the baseline, with standard deviation about 1.73, and empirical coverage decreases to approximately 0.90. The first stage remains strong, with mean $\widehat{F}_Z \approx 102.24$, suggesting that the observed degradation is driven by departures from monotonicity rather than by weak instrument behavior.

4.3 Coverage and First Stage Strength

Figure 4 relates empirical coverage of nominal 95% confidence intervals to the magnitude of the treatment probability discontinuity. Coverage remains close to the nominal level when the first stage gap is moderate or large, while it deteriorates sharply in the presence of continuity violations, even under strong instruments. This figure highlights that correct inference in fuzzy regression discontinuity designs depends jointly on instrument strength and on the validity of the underlying identification assumptions.

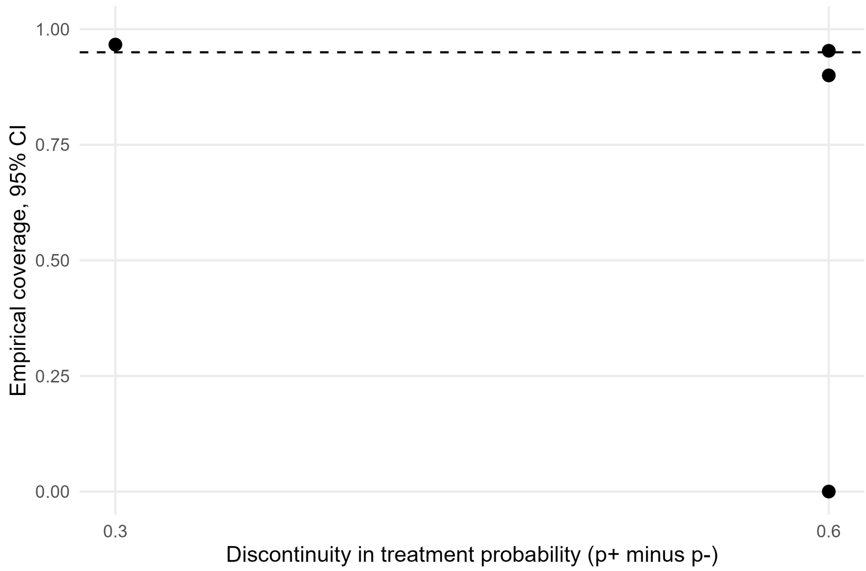

Figure 4: Empirical coverage of nominal 95% confidence intervals as a function of the discontinuity in treatment probability. The horizontal dashed line marks the nominal coverage level.

5 Discussion and Conclusions

The simulation results provide a systematic assessment of the robustness properties of fuzzy regression discontinuity designs under departures from their key identifying assumptions. The graphical evidence reported in Figures 1, 2, 3, and 4 complements the numerical results in Table 1 and helps clarify the mechanisms underlying estimator performance.

In the baseline configuration, the probability of treatment exhibits a clear discontinuity at the cutoff, as shown in Figure 1, and untreated potential outcomes are continuous. Under these conditions, the local two stage least squares estimator recovers the target Local Average Treatment Effect with negligible bias and coverage close to the nominal level. Figure 2 illustrates how the reduced form discontinuity reflects an attenuated version of the true treatment effect, which is correctly rescaled by the fuzzy regression discontinuity estimator. The corresponding point in Figure 3 lies close to the true value $\tau = -10$, confirming the validity of the design in this idealized setting.

Weakening the first stage by reducing the discontinuity in treatment probability primarily affects precision rather than bias. In the weak first stage scenario, the estimator remains centered near the true effect, but its dispersion increases markedly, as clearly visible in Figure 3. This behavior is consistent with the instrumental variable nature of the estimator, since lower compliance reduces the amount of exogenous variation available for identification. Figure 4 shows that coverage remains close to nominal despite the weaker first stage, reflecting appropriately wider confidence intervals rather than systematic distortion.

The most critical failure mode emerges under violations of continuity of potential outcomes at the cutoff. In the continuity violation scenario, the estimator becomes severely biased and empirical coverage collapses to zero, even though the first stage remains strong. This breakdown is clearly illustrated in Figure 3, where the estimated effect departs substantially from the true parameter, and in Figure 4, where coverage drops to zero despite a large treatment probability discontinuity. These results emphasize that instrument strength alone is insufficient for valid causal inference, since the local exclusion restriction implied by continuity of untreated outcomes is essential.

Violations of monotonicity generate a different pattern of sensitivity. When a nonzero share of defiers is introduced, the estimator remains approximately centered on the true effect, but variability increases and coverage deteriorates relative to the baseline. This behavior, visible in Figure 3, indicates that departures from monotonicity complicate finite sample performance and blur the interpretation of the estimated Local Average Treatment Effect, even when the first stage remains strong.

Overall, the results highlight that fuzzy regression discontinuity designs can provide reliable local causal inference in clinical and epidemiological applications only under a restricted set of conditions. Approximate continuity of potential outcomes at the cutoff and sufficient compliance are crucial, while violations of continuity lead to severe bias and loss of inferential validity. The proposed simulation framework, together with the accompanying graphical diagnostics, offers a practical tool to assess the plausibility and consequences of these assumptions in applied settings.

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Data availability statement

The dataset used in the empirical application is not publicly available. It can be provided by the author upon reasonable r ::contentReference[oaicite:0]index=0

Author Contributions

Both authors jointly contributed to the conception and design of the study, methodological development, programming and data analysis, manuscript writing and revision, responses to reviewers, and approved the final version of the manuscript.

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The authors used *AI* to improve the English language in preparing this work. They reviewed and edited the content as needed and took full responsibility for the publication's content.

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