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Modal Logic, Probability and Machine Learning Systems for Metadata Extraction

Simone Cuconato*

Abstract

Artificial intelligence (AI), since its inception, has had two major sub-fields, namely: logical reasoning and machine learning. Despite this, the interactions between these two fields have been relatively limited. In this paper, we highlight the need for closer integration of logical reasoning and machine learning. In our approach, logical reasoning tools such as probabilistic modal logic, are employed to provide qualitative feedback on the extracted descriptive metadata. The logical system we consider emerges from combining of $S5$ modal logic with the formulas of the infinite-valued Łukasiewicz logic and the unary modality P that describes the behaviour of probability functions. The result is a well-motivated system of probabilistic modal logic, that defines a probability distribution over possible worlds of the truth value of metadata extracted from precision medicine approach to Alzheimer's disease articles through machine learning systems. By incorporating symbolic reasoning, we bridge the gap between the speed and efficiency of quantitative techniques and the need for precise, verifiable metadata in high-stakes environments such as medical research.

Keywords: Modal logic; Probability; Logical reasoning; Machine learning systems; Metadata.

2020 AMS subject classifications: primary 03B45, secondary 03B52, 68T27.¹

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1 Introduction

Symbolic and sub-symbolic represent the two main branches of artificial intelligence (AI). AI has experienced rapid growth in recent years, thanks to the availability of powerful hardware, specific development platforms, and above all a massive amount of data. This rapid growth would never have been possible without a rigorous symbolic models based on articulated logical-mathematical systems [1, 2]. Logic and technology intertwine at all levels, from the lowest hardware level, governed by Boolean circuits and arithmetical operations in the stack memory; through the structure of assignment, sequencing, branching and iteration operations defining modern high-level programming languages; up to the equivalent abstract formulations of recursive definitions for algorithms [3]. Alongside logic, the branch of mathematics that plays a fundamental role in AI and data science today is undoubtedly statistics, and probability is the backbone of statistical inference and machine learning. For this reason, one of the central questions in data mining and AI concerns *probabilistic logic learning*, which involves the integration of relational or logical representations with probabilistic reasoning mechanisms and machine learning principles.

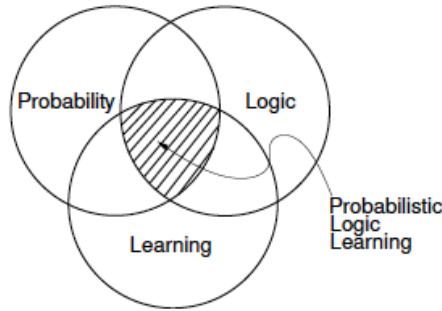


Figure 1: Probabilistic Logic Learning as the intersection of the three domains.

In this paper, we investigate the intersection of logic, probability, and machine learning by developing a symbolic approach based on probabilistic modal logic to qualitatively optimise the analysis of data extracted through machine learning systems [4]. Our approach aims to improve the consistency, accuracy, and completeness of the extracted data, ensuring higher data quality in critical applications. It is no coincidence that in recent years, the attention of computer scientists has shifted toward a complex and multidimensional concept: *data quality* [5]. Data quality is a multidimensional concept encompassing consistency, freshness, completeness, and accuracy of the data as a non-exhaustive list of dimensions to characterize data fitness-of-use [6]. However, even if in the era of Big Data data

quality has become more important than ever, the symbolic approaches that have been developed about data quality (and especially data veracity) are not many [7, 8, 9]. Data we investigate is specific data: metadata². Metadata contain information needed to understand and use the data effectively³. In particular, in the context of scientific research, metadata play an essential, ubiquitous yet often invisible role because they are one of the key players in the delicate task of information sharing. Information sharing is one of the cornerstones of modern science. Modern science is a shared system of theories and practices that define the very character of scientific research and metadata work is intimately part of scientific work. Metadata are integral to the production of scientific knowledge and consequently “any scientific research based in the digital world will inevitably be interacting with metadata” [11]. If these aspects are important for scientific research in general, they become crucial for medical research and, in particular, in the emerging approach of precision medicine [12]. Precision medicine takes care of the patient at all stages of the treatment process, taking into account their specificity and uniqueness, through intervention methods designed exclusively for their situation, also with a view to increasingly effective prevention. Based on the above, it is clear that: *i*) sharing high-data quality is essential in precision medicine; *ii*) sharing high-quality data is useless if good metadata is not linked with it. In particular, we analyze descriptive metadata extracted from precision medicine approach to Alzheimer’s disease articles [13] through machine learning systems⁴. Alzheimer’s disease [16] is the most common form of dementia and its rapid advancement worldwide represents a threat to the global healthcare system⁵. Having good descriptive metadata associated with specific Alzheimer’s disease articles means being able to rely on all the scientific results of previous research and help other researchers to perform new, up-to research. More in detail, we describe a logical system that allows a general approach for reasoning about metadata. The system emerges from combining of *S5* modal logic with the formulas of the infinite-valued Łukasiewicz logic and the unary modality *P* that describes the behaviour of probability functions. The result is a well-motivated system of probabilistic modal logic that defines a probability distribution over possible worlds of the truth value of metadata extracted through machine learning systems.

²Metadata is a pillar in data science. In general, “metadata is a means by which the complexity of an object is represented in a simple form” [10].

³Assuming a minimal definition of information as “data + semantics”, trust on extracted information can be identified with the result of a *consistency assessment* between data, metadata and digital documents.

⁴On recent applications of artificial intelligence in healthcare see [14, 15].

⁵“The prevalence of this disorder has been expected to quadruple by 2050, thereby exerting a tremendous economic pressure on medical sector, worldwide” [13].

2 Łukasiewicz Logic, Modal Logic and Probability

In this section, we present the formal basis of our probabilistic modal logic and how this logic can be applied to metadata extraction⁶. In particular, the present paper builds upon the recent work of Corsi *et al.* [23]. The propositional logic⁷ on which our approach is grounded is the $[0,1]$ -valued Łukasiewicz calculus [24]. Łukasiewicz infinite-valued logic $\mathbb{L}$ is axiomatized as follows⁸:

$$(L1) \phi \rightarrow (\psi \rightarrow \phi); \quad (1)$$

$$(L2) (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \rho) \rightarrow (\psi \rightarrow \rho)); \quad (2)$$

$$(L3) (\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi); \quad (3)$$

$$(L4) ((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \phi) \rightarrow \phi). \quad (4)$$

$\mathbb{L}$ is a form of fuzzy logic with an algebraic semantics (*MV-algebras*) featuring structures defined as follows:

$$\mathbf{A} = (A, \oplus, \wedge, 1) \quad (5)$$

where A is a nonempty set, $\oplus$ is a binary operation on A , while $\neg$ a unary operation on A , and 1 is a constant⁹. Therefore, a formulas of Łukasiewicz logic can be regarded as terms in the language of MV-algebras $\overline{[0, 1]}$:

$$\overline{[0, 1]} = ([0, 1], \oplus, \neg, 1) \quad (6)$$

where $[0, 1]$ denotes the real unit interval, and for all $a, b \in [0, 1]$, $a \oplus b = \min\{1, a + b\}$ and $\neg a = 1 - a$.

Naturally, as shown [19] the notion of *theorem* ($\vdash \phi$) and *deductive from a theory* ($T \vdash \phi$) are defined as usual in algebraic logic.

Theorem 2.1. For every finite set of formulas $T \cup \phi$ in the Łukasiewicz logic language, $T \vdash \phi$ iff for every evaluation v into $\overline{[0, 1]}$ that maps all $\psi \in T$ to 1, $v(\phi) = 1$ as well.

⁶The intersection between modal logic [17, 18] fuzzy logic [19] is scientifically interesting from both theoretical [20, 21] and applied perspectives [22].

⁷The age of logic exceeds two millennia. Aristotle was the first to understand logic as the ground of all science. In the past two centuries, logic has forged a privileged link with mathematics, while in recent decades, starting from the work of A. Turing, logic is at the centre of research in computer science.

⁸Obviously, to this set of axioms must be added *modus ponens* as inference rule.

⁹An algebra with this signature is an MV-algebra if and only if $\langle A, \oplus, 1 \rangle$ forms a commutative monoid, and the following equations are satisfied: $\neg \neg x = x$; $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

*Modal Logic, Probability and Machine Learning Systems for Metadata
Extraction*

$\mathbb{L}$ through extension of the evaluations v on $\overline{[0, 1]}$ presents not only a strong disjunction $\oplus$ and negation $\neg$, but also a weak disjunction $\vee$, an implication $\rightarrow$ and a strong non-indemponent conjunction $\&$:

$$v(\phi \oplus \psi) = \min\{1, v(\phi) + v(\psi)\}; \quad (7)$$

$$v(\neg\phi) = 1 - v(\phi); \quad (8)$$

$$v(\phi \wedge \psi) = \min\{v(\phi), v(\psi)\}; \quad (9)$$

$$v(\phi \& \psi) = \max\{0, v(\phi) + v(\psi) - 1\}; \quad (10)$$

$$v(\psi \vee \phi) = \max\{v(\phi), v(\psi)\}; \quad (11)$$

$$v(\phi \rightarrow \psi) = \min\{1 - v(\phi) + v(\psi), 1\}. \quad (12)$$

Given the following logical preliminaries, we can now introduce modal logic. Modal logic $\mathcal{ML}$ is an extension of classical logic $\mathcal{PL}$. The language of standard modal logic consist of a set of propositional variables Var , plus addition of a modal operator $\Box$ and its dual $\Diamond = \neg\Box\neg$. The set $Fm(\mathcal{ML})$ of formulas of $\mathcal{ML}$ is defined according to the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \quad (13)$$

where $p \in Var$. Historically, there are two schools of thought of mathematical semantics of modal language [25]. The first, developed by J. McKinsey and A. Tarski [26], is called *algebraic semantics* because semantics interprets modalities on Boolean algebras with operators. The second, was developed by S. Kripke [27] is called *relational semantics* because it uses relational structures, commonly called *Kripke models*.

Definition 2.1. A Kripke model for the logic S5 is a relational structure

$$\mathcal{K} = (W, R, \{v_w\}_{w \in W}) \quad (14)$$

where

1. W is a non-empty set of worlds;
2. $R \subseteq W \times W$ is the accessibility relation;
3. $\{v_w\}_{w \in W}$ is a classical evaluation $v_w : Var \longrightarrow \{0, 1\}$

As usual, for a formula ϕ and an extraction w we denote by the truth-value of ϕ in $\mathcal{K}$ at w . Given a formulas ϕ and for a atomic modal formulas $\Box\phi$ we have:

$$\|\phi\|_{\mathcal{K},w} = v_w(\phi) \quad (15)$$

$$\|\Box\phi\|_{\mathcal{K},w} = 1 \text{ iff for each } w' \in W, wRw' \text{ implies } v_{w'}(\phi) = 1 \quad (16)$$

The system $S5$ is defined via *modus ponens* and necessitation and the following axioms¹⁰:

$$(CPL) \text{Axioms for classical propositional logic;} \quad (17)$$

$$(K) \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi); \quad (18)$$

$$(T) \Box\phi \rightarrow \phi; \quad (19)$$

$$(4) \Box\phi \rightarrow \Box\Box\phi; \quad (20)$$

$$(B) \phi \rightarrow \Box\Diamond\phi. \quad (21)$$

In recent years, several efforts have been made by the fuzzy logic community to extend modal logic from the classical to the many-valued setting, with a primarily semantic-based approach. For example, in [28], P. Hájek introduced a generalization of modal logic $S5$ to the Łukasiewicz many-valued framework by defining its relational semantics using Kripke-like models of the form $K = (W, R, \{v_w\}_{w \in W})$, where W is a nonempty set, $R \subseteq W \times W$ is the total relation on W , and for each $w \in W$, v_w evaluates Łukasiewicz (non-modal) formulas within the standard MV-algebra $[0, 1]$ corresponding to their truth-functions, while the truth value of a modal formula $\Box\phi$ at $w \in W$ is determined as follows:

$$\|\Box\phi\|_{\mathcal{K},w} = \inf\{v_{w'}(\phi) \mid wRw'\} = \inf\{v_{w'}(\phi) \mid w' \in W\}. \quad (22)$$

We will refer to the class of such models as $\mathcal{M}^t$, where t stands for total. Hájek demonstrates that the following Hilbert-style calculus, denoted by $S5\mathbb{L}$, is sound and complete with respect to the aforementioned class of relational models $\mathcal{M}^t$. This calculus is based on a language that extends that of Łukasiewicz logic by a unary modality $\Box$ and its dual $\Diamond = \neg\Box\neg$, and includes the following axioms:

¹⁰In an $S5$ system, the accessibility relation is characterized as *Euclidean*, meaning that if a world s is accessible to worlds t and u , then t and u are accessible to each other: $\forall s \forall t \forall u (sRt \wedge sRu \rightarrow tRu)$.

$$(\Box 1) \Box \varphi \rightarrow \varphi; \quad (23)$$

$$(\Box 2) \Box(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \Box \varphi); \quad (24)$$

$$(\Box 3) \Box(\psi \vee \varphi) \rightarrow (\psi \vee \Box \varphi); \quad (25)$$

$$(\Diamond 1) \Diamond(\varphi \& \varphi) \equiv \Diamond \varphi \& \Diamond \varphi; \quad (26)$$

$$(MP) \text{ the modus ponens rule}; \quad (27)$$

$$(N\Box) \text{ the necessitation rule: from } \varphi \text{ infer } \Box \varphi. \quad (28)$$

Since extending $\mathbb{L}$ by the law of excluded middle,

$$(LEM) \quad \varphi \vee \neg \varphi,$$

yields classical logic, adding the axiom $\varphi \vee \neg \varphi$ to $S5\mathbb{L}$ results in the classical modal logic $S5$. Moreover, similar to the classical case, $S5\mathbb{L}$ is not only sound and complete with respect to the class $\mathcal{M}^t$ of Kripke models $\langle W, R, \{v_w\}_{w \in W} \rangle$ where $R = W \times W$, but also with respect to the class $\mathcal{M}^v$ of Kripke models $\langle W, R, \{v_w\}_{w \in W} \rangle$, where $R \subseteq W \times W$ is an equivalence relation, i.e., reflexive, symmetric, and transitive.

Definition 2.2. $\mathcal{M}^v$ is the class of models $\langle W, R, \{v_w\}_{w \in W} \rangle$, where W is nonempty, R is an equivalence relation on W , and for all $w \in W$, v_w is a $[0, 1]_{MV}$ -evaluation of Łukasiewicz formulas¹¹.

Lemma 2.1. The classes $\mathcal{M}^t$ and $\mathcal{M}^v$ have the same tautologies.

Proof. Since every total relation on any set W is an equivalence relation, if φ is a tautology of $\mathcal{M}^v$, then φ is also a tautology of $\mathcal{M}^t$, i.e., $\mathcal{M}^v \subseteq \mathcal{M}^t$. Conversely, assume that φ is not a tautology of $\mathcal{M}^v$. That is, suppose there exists $K = \langle W, R, \{v_w\}_{w \in W} \rangle \in \mathcal{M}^v$ and a world $w \in W$ such that $\|\varphi\|_{K,w} < 1$. Since R is an equivalence relation, the restriction of R to $R[w] = \{w' \in W \mid wRw'\}$ is total. Now, take $W' = R[w]$, $R' = W' \times W'$, and for all $w^* \in W'$, v_{w^*} is defined as in K . Let $K' = \langle W', R', \{v_{w^*}\}_{w^* \in W'} \rangle$. Since $w \in W'$, it follows by induction that $\|\varphi\|_{K',w} < 1$. Therefore, φ is not a tautology of $\mathcal{M}^t$ and $\mathcal{M}^t \subseteq \mathcal{M}^v$. $\square$

Given these general logical preliminaries, three specific clarifications related to our application domain are now necessary. The first, is that given our application domain, we do not use simple propositional formulas p but formulas on metadata (or *metadata propositions*) m . These formulas behave in the same way as propositional formulas, and if a formula on metadata ϕ is compound, then ϕ is computed by truth-functionality using classical connectives¹².

¹¹If φ is any formula in the language of $S5\mathbb{L}$, $K = \langle W, R, \{v_w\}_{w \in W} \rangle \in \mathcal{M}^v$, and $w \in W$, the truth value of φ in K at w , denoted $\|\varphi\|_{K,w}$, is defined as usual.

¹²*Metadata propositions* follow the principles of truthmaker semantics [29, 30].

Therefore, we can replace the set $Fm(\mathcal{ML})$ of formulas of $\mathcal{ML}$ with the following grammar

$$\phi ::= m \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \quad (29)$$

where $m \in Var$. The second clarification is still about our domain of application, we do not speak of possible worlds but of *possible extractions*, which we denote with e and E . Mathematically, there is no difference between the concept of world and that of extraction. Finally, the third clarification concerns the fact that the classical evaluation v_w is replaced by v_e . As usual, v_e also assigns a truth value to each metadata variable in each possible extraction. Therefore we can replace K with

$$\tilde{K} = (E, R, \{v_e\}_{e \in E}) \quad (30)$$

where

1. E is a non-empty set of metadata extractions;
2. $R \subseteq E \times E$ is the accessibility relation;
3. $\{v_e\}_{e \in E}$ is a classical evaluation $v_e : Var \longrightarrow \{0, 1\}$.

In this way, for a formula ϕ and an extraction e we denote by the truth-value of ϕ in $\tilde{K}$ at e . Given a formulas ϕ and for a atomic modal formulas $\Box\phi$ we have:

$$\|\phi\|_{\tilde{K}, e} = v_e(\phi) \quad (31)$$

$$\|\Box\phi\|_{\tilde{K}, e} = 1 \text{ iff for each } e' \in E, eRe' \text{ implies } v_{e'}(\phi) = 1 \quad (32)$$

The last operator we need to introduce is the one specific to probability. Let $\Delta(\phi)$ be the set of extractions (worlds) where ϕ is true, a probability model P for a countable space E of possible extractions assigns a probability $P(e)$ to each extraction, such that $0 \leq P(e) \leq 1$ and $\sum_{e \in E} P(e) = 1$. Given a probability model, the probability of a logical formula ϕ is the total probability of all extractions in which ϕ is true:

$$P(\phi) = \sum_{e \in \Delta(\phi)} P(e) \quad (33)$$

Thus, we can define the belief functions and the possibility and necessity measures as follows.

Definition 2.3. A belief function on a boolean algebra $\mathbf{A}$ is a $[0, 1]$ -valued map $B : \mathbf{A} \rightarrow [0, 1]$ satisfying:

$$(B1) \ B(\top) = 1, \quad B(\perp) = 0; \quad (34)$$

$$(B2) \ B\left(\bigvee_{i=1}^n \psi_i\right) \geq \sum_{i=1}^n \sum_{\substack{J \subseteq \{1, \dots, n\}: \\ |J|=i}} (-1)^{i+1} B\left(\bigwedge_{j \in J} \psi_j\right), \text{ for } n \in \mathbb{N}. \quad (35)$$

Definition 2.4. A possibility measure on a Boolean algebra $\mathbf{A}$ is a map $\Pi : \mathbf{A} \rightarrow [0, 1]$ such that

$$(\Pi1) \ \Pi(\top) = 1, \Pi(\perp) = 0; \quad (36)$$

$$(\Pi2) \ \Pi(\psi_1 \vee \psi_2) = \max \{\Pi(\psi_1), \Pi(\psi_2)\} \text{ for all } \psi_1, \psi_2 \in \mathbf{A}. \quad (37)$$

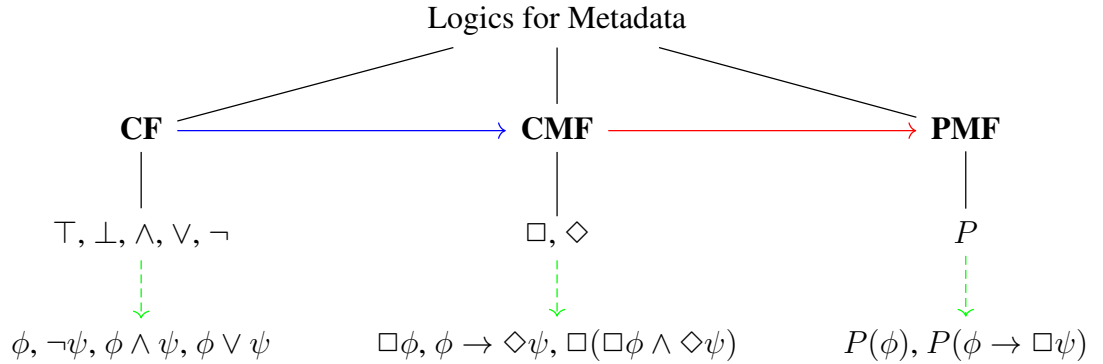
Definition 2.5. A necessity measure is the dual notion of a possibility measure, and it is a map $N : \mathbf{A} \rightarrow [0, 1]$

$$(N1) \ N(\top) = 1, N(\perp) = 0; \quad (38)$$

$$(N2) \ N(\psi_1 \wedge \psi_2) = \min \{N(\psi_1), N(\psi_2)\} \text{ for all } \psi_1, \psi_2 \in \mathbf{A}. \quad (39)$$

3 Probabilistic Modal Logic and Metadata

Based on the above, we have a unified logical system combining the language of Łukasiewicz logic $\mathcal{L}$, the language of modal logic $S5 \ \mathcal{L}_{\square}$ with in addition the unary modality $P \ \mathcal{L}_{\square, P}$. This logical system for reasoning about metadata is focuses on the following classed of formulas:



where

- **CF** is the set of *classical formulas*;
- **CMF** is the set of *classical modal formulas*;
- **PMF** is the set of *probabilistic modal formulas*;
- the blue arrow represents the fact that **CMF** is an extension of **CF**¹³;
- the red arrow represents the fact that **PMF** is obtained starting from **CMF**¹⁴;
- the green arrow shows some example for each class of formulas.

Finally, let us define a semantics for the language $\mathcal{L}_{\mathcal{M}} = \mathbf{CF} \cup \mathbf{CMF} \cup \mathbf{PMF}$ containing all the sets of formulas defined above.

Definition 3.1. An *S5* probability model is a tuple

$$\mathcal{M} = (E, R, \{v_e\}_{e \in E}, \{p_e\}_{e \in E}) \quad (40)$$

where

1. E is a non-empty set of metadata extractions;
2. $(E, R, \{v_e\}_{e \in E})$ is a standard S5-Kripke model;
3. For all $e \in E$, p_e is a probabilistic distribution on E .

We define the truth-value of ϕ in S at e (denoted $\|\phi\|_{\mathcal{M},e}$) in **CF** and **CMF** as follows:

$$\text{If } \phi \in \mathbf{CF}, \text{ then } \|\phi\|_{\mathcal{M},e} = v_e(\phi); \quad (41)$$

$$\text{If } \phi \in \mathbf{CMF} \text{ and } \phi = \Box\psi, \text{ then } \|\phi\|_{\mathcal{M},e} = \|\Box\psi\|_{\mathcal{M},e} = \inf \{\|\psi\|_{\mathcal{M},e'} \mid eRe'\}^{15}. \quad (42)$$

Of course, if $\phi \in \mathbf{CMF}$ and is compound, then $\|\phi\|_{\mathcal{M},e}$ is computed as usual by truth-functionality using classical connectives.

Instead, if $\phi \in \mathbf{PMF}$ (and ϕ is atomic) the situation is more articulated

¹³Therefore, by definition, $\mathbf{CF} \subseteq \mathbf{CMN}$.

¹⁴More in detail, in **PMF** atomic probabilistic modal formulas are all those in the form $P(\phi)$ for $\phi \in \mathbf{CMF}$, while “composed probabilistic formulas are defined by composing atomic ones with connectives of the Łukasiewicz language” [26].

¹⁵For any $\phi \in \mathbf{CMF}$, $\|\phi\|_{\mathcal{M},e} \in \{0, 1\}$.

$$\|\phi\|_{\mathcal{M},e} = \|P(\psi)\|_{\mathcal{M},e} = \sum \{p_e(e') \mid \|\psi\|_{\mathcal{M},e'} = 1\}^{16}. \quad (43)$$

For example, if $\phi = P(\psi)$ and ϕ is an atomic **PMF** formula, with $\psi \in \mathbf{CF}$, then

$$\|P(\psi)\|_{\mathcal{M},e} = \sum \{p_e(e') \mid \|\psi\|_{\mathcal{M},e'} = 1\} = \sum \{p_e(e') \mid v_{e'}(\psi) = 1\}. \quad (44)$$

To put it differently, $\|P(\psi)\|_{\mathcal{M},e}$ is the probability of ψ computed in e .

4 An Application Case

In this section, we show with a precise example how our system allows us a logical-probabilistic approach to reasoning about metadata. More in detail, from a point of view of possible worlds, we represent an *S5* Kripke frame (W, R) on five possible worlds as found in Figure 2¹⁷.

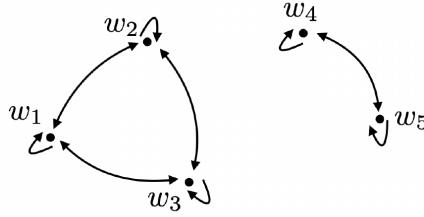


Figure 2: An *S5* kripke frame (W, R) on five possible worlds.

On the contrary, from a point of view of possible extractions, each extraction must be understood as an actual extraction by a machine learning-based system. As we specified in section 1. metadata we are analysing is descriptive metadata extracted from precision medicine approach to Alzheimer’s disease articles. In particular, the first medical article concerns the need to identify “reliable blood-based biomarkers for Alzheimer’s disease (AD) that are tied to the biological ATN (amyloid, tau and neurodegeneration) framework as well as clinical assessment and progression” [15], while the second medical article is about “the largest metabolome investigation of dementia to date” [16]. Both extractions are focused

¹⁶Compared to when $\phi \in \mathbf{CMF}$, if $\phi \in \mathbf{PMF}$ and is compound, then $\|\phi\|_{\mathcal{M},e}$ is computed by truth-functionality using Łukasiewicz connectives.

¹⁷The following figure and part of the example is taken from [23].

on three specific descriptive metadata: title (m_1), journal (m_2), and keywords (m_3)¹⁸.

Example 4.1. Let us consider a language with three metadata m_1, m_2, m_3 and the following *S5* Kripke model.

Let the evaluations v_e be as follows:

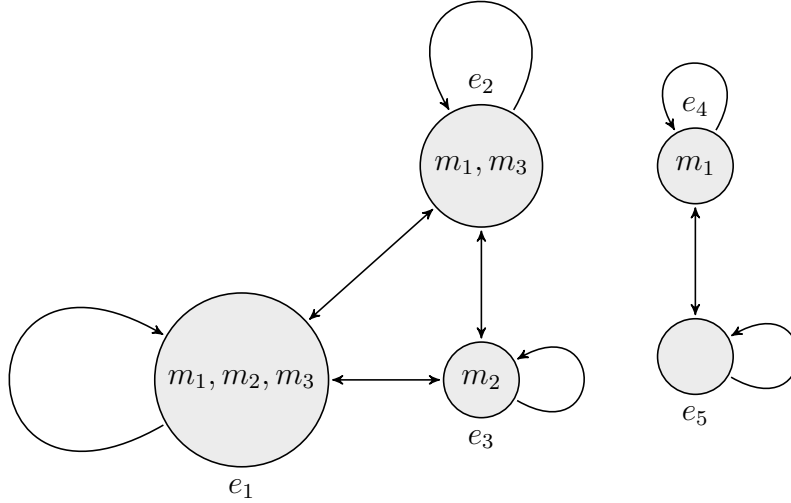
$$e_1 \models m_1, m_2, m_3 \quad (45)$$

$$e_2 \models m_1, \neg m_2, m_3 \quad (46)$$

$$e_3 \models \neg m_1, m_2, \neg m_3 \quad (47)$$

$$e_4 \models m_1, \neg m_2, \neg m_3 \quad (48)$$

$$e_5 \models \neg m_1, \neg m_2, \neg m_3 \quad (49)$$



The advantages of this representation compared to Fig. 2 is twofold: *i*) it is possible to immediately associate correctly metadata extracted in each extraction; and *ii*) it is quicker to trace *full* extractions such as extraction e_1 , from *empty* ones such as e_5 . Table 1. gives a summary of all heading levels.

¹⁸For example, if metadata title is correctly extracted then m_1 expresses a true *metadata proposition*, while if metadata journal m_2 is not correctly extracted then m_2 expresses a false *metadata proposition*.

*Modal Logic, Probability and Machine Learning Systems for Metadata
Extraction*

Probability distributions	e_1	e_2	e_3	e_4	e_5
p_1	1/5	1/5	1/5	1/5	1/5
p_2	1/3	1/3	1/3	0	0
p_3	0	1/4	1/4	1/2	0
p_4	0	1/3	0	1/3	1/3
p_5	1/4	1/4	0	1/4	1/4

Table 1: Probability distributions

Given, let $\mathcal{M} = (E, R, \{e_1, \dots, e_5\}, \{p_1, \dots, p_5\})$ and $\phi = m_1 \vee m_3$. Its models are e_1, e_2, e_4 . Then, for every $i = 1, \dots, 5$, it is simple to compute $\|P(\phi)\|_{\mathcal{M}, e_i}$ as $p_i(e_1) + p_i(e_2) + p_i(e_4)$. For instance, $\|P(\phi)\|_{\mathcal{M}, e_1} = 3/5$ and $\|P(\phi)\|_{\mathcal{M}, e_4} = 2/3$.

If instead $\phi = m_3 \rightarrow (m_1 \wedge m_2)$, its models are e_1, e_3, e_4, e_5 . Then, for every $i = 1, \dots, 5$, it is simple to compute $\|P(\phi)\|_{\mathcal{M}, e_i}$ as $p_i(e_1) + p_i(e_3) + p_i(e_4) + p_i(e_5)$. For instance, $\|P(\phi)\|_{\mathcal{M}, e_1} = 4/5$ and $\|P(\phi)\|_{\mathcal{M}, e_2} = 2/3$.

Let us now consider the following formula $P(\Box\phi)$. In order to compute $\|P(\Box\phi)\|_{\mathcal{M}, e_i}$, we first need to notice that the models of $\Box\phi$, are e_4 and e_5 . In fact, for all $i = 1, 2, 3, e_i R e_2$ and $e_2 \not\models \phi$. Therefore, for all $i = 1, \dots, 5$,

$$\|P(\Box\phi)\|_{\mathcal{M}, e_i} = p_i(e_4) + p_i(e_5) \quad (50)$$

Thus, $\|P(\Box\phi)\|_{\mathcal{M}, e_1} = 2/5$, $\|P(\Box\phi)\|_{\mathcal{M}, e_2} = 0$, $\|P(\Box\phi)\|_{\mathcal{M}, e_5} = 1/2$ ¹⁹.

5 Discussion, Conclusion and Future Work

Logical reasoning and machine learning “have been the two foundational pillars of artificial intelligence since its inception, and yet, until recently the interactions between these two fields have been relatively limited” [15]. In this paper, we initiated a new line of research that aims at investigating new symbolic approaches for AI applied to automatic metadata extraction. The study of automatic metadata extraction has led to the development of various techniques and approaches, resulting in numerous tools and frameworks. However, several critical issues arise, particularly concerning the limitations in the reference literature. The first critical issue relates to the fact that two distinct approaches to automatic metadata generation are commonly observed in the scientific literature: on one hand, research into techniques for extracting information from digital resources; on the other hand, software development for content creation. These two streams are often addressed separately, but their integration could significantly improve automatic metadata

¹⁹Note that, for all $i = 1, \dots, 5$, $\|P(\Box\phi)\|_{\mathcal{M}, e_i} \leq \|P(\phi)\|_{\mathcal{M}, e_i}$.

generation and extraction, enabling the production of electronic documents rich in embedded metadata. The second critical issue with the prevailing approaches in the literature is that they are largely based on quantitative and sub-symbolic methods²⁰. While these methods have proven effective for large-scale data processing, they often lack a rigorous framework for ensuring the *consistency*, *correctness*, and *completeness* of the extracted metadata. These are qualitative aspects that are crucial in fields like scientific research, where metadata is not just an accessory but a foundation for the integrity of information sharing. The absence of qualitative methods — defined here as the systematic use of symbolic logic and formal systems — leaves a gap in the ability to verify the coherence and reliability of metadata. Such symbolic approaches could provide a robust framework for addressing these criticalities, offering tools to check for logical consistency and correctness in a way that purely quantitative methods cannot. By incorporating symbolic reasoning, we bridge the gap between the speed and efficiency of quantitative techniques and the need for precise, verifiable metadata in high-stakes environments such as medical research. In details, in this paper we have developed a symbolic approach based on probabilistic modal logic for reasoning about metadata extracted from Alzheimer’s disease articles through machine learning-based systems. Our system emerges from combining of *S5* modal logic with the formulas of the infinite-valued Łukasiewicz logic and the unary modality *P* that describes the behaviour of probability functions. The result is a well-motivated system of probabilistic modal logic that defines a probability distribution over possible extractions (worlds) of the truth value of metadata extracted through machine learning systems. In this way, our logical system can provide an innovative tool to support data mining. On the basis of these reflections, two future directions can be identified. The first aims to strengthen the connection between logical reasoning and machine learning: *i*) by combining machine learning and symbolic solvers to enable logical reasoning in machine learning systems for metadata extraction; and *ii*) by applying other areas of mathematics like combinatorics on words [32], algebraic-lattices based logic [33], Jaśkowski–Fitch natural deduction proofs [34] or topological interpretation of modal logic [35] to assess metadata quality and veracity. The second line of research delves into applying probabilistic modal logic to address data uncertainty in AI [36]. In conclusion, if we merely accept the solutions offered by AI passively, we risk losing our ability to reason independently and develop innovative ideas [37]. Reflecting on the quality of the data is a crucial first step in ensuring that we don’t blindly accept what AI offers. In a world increasingly dominated by automation, it is essential for humans to continue questioning and challenging the outcomes generated by AI.

²⁰With the exception of a few studies that, however, focus strictly on taxonomy, the Q-Codes, and its vocabulary [31].

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Machine Learning Algorithms Analysis of Synthetic Minority Oversampling Technique (SMOTE): Application to Credit Default Prediction

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Abstract

Financial risk management requires credit default prediction to measure borrower default risk. Class Imbalance in datasets can hinder model performance. This study examines how under-sampling, over-sampling, and SMOTE affect classifier performance. XGBoost outperformed other classifiers in AUC across most datasets except SMOTE using a Ghanaian commercial bank dataset. It concludes that over-sampling outperforms under-sampling in enhancing classifier performance and improving the prediction of minority class occurrences, demonstrating the value of data balancing in credit scoring models.

Keywords: SMOTE; Class Imbalanced; Machine Learning; Re-sampling; Credit Worthiness

2020 AMS subject classifications: 6208, 6211, 62D99 ¹

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1 Introduction

Imbalanced learning is one of the most challenging tasks in data mining. In imbalance learning, some classes have inadequate data while others have an abundance of data. Imbalanced data sets are often encountered in "anomaly-type" classification problems. These issues are divided into two categories: "normal" and "anomaly," with "anomaly" including a broad variety of meanings. Examples include medical diagnosis Chan P [1999], machine failure detection Rao R, and fraud detection Kerdprasop K [2011].

When trying to foresee occurrences, it is critical to concentrate on the minority class since it varies from the majority class's typical trends. Standard classifiers are designed to decrease overall classification errors, regardless of class distribution. As a consequence, these classifiers emphasize majority-class samples above minority-class accuracy. There are three fundamental strategies for correcting data imbalance: cost-sensitive, algorithm-level, and data-level. The cost-sensitive approach calculates misclassification costs using class relevance and imbalance. Examples of cost-sensitive techniques are AdaCost [Fan W,] and the work in [Sun Y, 2007]. The data level approach entails changing the data to re-balance minority and majority categories. To do this, either under-sample the majority class or over-sample the minority class. The imbalance technique is extensively used due of its simplicity and universality, which make it usable regardless of classifier type. Behind the sample overlay, the classifier is intact. Under-sampling (Figure1) is the process of removing less significant patterns using random or heuristic criteria. Examples of under-sampling include the condensed nearest neighbor rule [Batista G E, 2004,] and one-sided selection [Fayed H, 2009].



Figure 1: Undersampling and Oversampling of an Imbalance data

Source: [Roweida M, 2020]

However, under-sampling is risky since critical information may be overlooked. Over-sampling [Fayed H, 2009] is most likely a more successful method. Minority class patterns may be repeated at random or created from scratch. Most sam-

pling algorithms are used in conjunction with ensemble learning. We generate several sample sets and build classifiers for each of them. Finally, we incorporate the classification of these classifiers. Examples of such models include SMOTE-Boost [Chawla N, 2003] and RAMOBoost, which combine ensembles with over-sampling [Chen S, 2010]. Other algorithms, such as RUSBoost [Liu X, 2009], EasyEnsemble, and BalanceCascade [Seiffert C, 2010]), generate datasets with under-sampled majority classes. Over-sampling (Figure 1) makes up for a lack of data in the minority class by producing additional data. The goal is to produce new data that is similar to the unknown underlying distribution.

In recent years, the growth of machine learning algorithms has transformed a wide range of industries, from healthcare to banking [Chawla N. V., 2002]. However, the performance of these algorithms is strongly dependent on the quality and balance of training data. In real-world circumstances, datasets often show class imbalance, with one class greatly outnumbering the other(s). This mismatch presents a significant difficulty for machine learning algorithms, since they tend to favour the majority class, resulting in inferior performance in forecasting the minority class(es) ?. Imbalanced data occurs in a variety of scenarios, including fraud detection, illness diagnosis, and anomaly detection, when positive occurrences (for example, fraudulent transactions, uncommon diseases) are considerably less common than negative ones. Traditional machine learning methods, especially those relying on accuracy measures, fail to correctly categorize minority class cases owing to their low presence in the dataset. As a consequence, these algorithms often produce large false negative rates, missing important cases of relevance (Fernandez A.) To address the issues provided by unbalanced data, academics and practitioners have devised a slew of strategies targeted at enhancing algorithm performance. One frequent strategy is to adjust the learning algorithms itself, for as by integrating class weights to punish minority misclassifications more harshly. Additionally, resampling approaches, such as oversampling the minority class and under-sampling the majority class, have been commonly used to re-balance the dataset. Furthermore, ensemble approaches, including as bagging and boosting, have shown useful in dealing with unbalanced data by merging many classifiers to reduce bias toward the majority class. Other sophisticated approaches like as anomaly identification and anomaly-based oversampling have shown potential in finding and creating synthetic instances of the minority class to supplement the training set [Sun Y.]. Despite these advances, determining the best strategy for a particular dataset remains a difficult issue, since the success of each method is largely dependent on the individual properties of the data and the learning algorithm used.

Statistical and machine learning techniques, such as logistic regression, discriminant analysis, naive bayes, decision tree, random forest, artificial neural networks, support vector machine, extreme gradient boosting, and adaboost, have

been widely used in developing scorecards and have successfully worked on real-life default data sets [Dastile et al., 2020]. The scoring methods are designed to divide credit applicants into two or more different categories based on demographic and socioeconomic factors such as loan amount, purpose, and age. Credit scoring model training has faced challenges such as explaining predictions to policymakers and customers, as well as the fact that almost all default data sets have uneven class distribution, making predictions biased [Dastile et al., 2020]. The issue of model transparency has been addressed in the literature by utilizing basic linear models such as logistic regression and linear discriminant analysis, as well as rule extraction approaches to explain certain black-box models. Again, strategies such as data resampling and algorithm modification have been devised to offset the effects of the imbalanced class distribution seen in default data sets. Imbalanced data sets have been shown to create models with many incorrect predictions because the algorithms are biased toward the majority class. Automated credit scoring algorithms can handle a large number of loan applications in a short period of time and minimize bias associated with subjective expert judgment Hand and Henley [1997]. Due to the influence of imbalance data on the classification model performance, this study seeks to employ the SMOTE technique in dealing with imbalance data and apply on the machine learning algorithms built on predicting credit worthiness. This study utilized an imbalanced datasets of 'a commercial bank in Ghana Customer Credit worthiness or loan default' to compare sampling techniques using various evaluation metrics built from the credit worthiness scorecard.

2 Related Work

Researchers have increasingly focused their attention on the imbalance data dilemma. Wilson [Wilson, 1972] devised a well-known under-sampling strategy. It operates by removing data points when the target class does not match the majority of its KNNs. Monard [Monard M. C.], examined numerous challenges connected to learning with skewed class dispersal. For example, consider the relationship between class scatterings and price-sensitive information, as well as the mistake frequency and accuracy bounds for measuring model act. Visa [Vis] offered a study of the most generally used techniques of learning from unbalanced classes. They suggested that the poor performance of models generated using conventional machine learning techniques on unbalanced classes is mostly due to three issues: mistake costs, class dispersion, and accuracy. Cohen [Cohen G, 2006] offered resampling strategies to identify minority targets. They used a novel resampling strategy that included oversampling occasional positives and under-sampling the non-infected majority based on simulated conditions generated by class-specific

sub-clustering. the study found that the innovative resampling method outperformed standard random resampling. In the works of Duman [Duman E, 2012], writers utilized three distinct approaches on an advertising dataset. Logistic regression, Chi-squared, and neural networks are examples of automated interaction discovery. The three approaches' performance was measured using accuracy, AUC, and precision. They compared many alternative skewed datasets derived from the original dataset. They argued that accuracy is a useful metric for an unbalanced dataset. In Lemmatre's work [Lematre G, 2017], k-means grouping is used to balance unbalanced instances by minimizing the number of majority instances, and in some previous works, authors used an under-sampling strategy to delete information points from the majority of occurrences based on their spacing [Mani I, 2003].

Moreover, the handling of unbalanced data has been widely examined in the literature (Japkowicz N [2002], Kamalov F [2022], Wang et al. [2018], Haixiang G [2017], Wang L [2021], Kaur [2019]). However, just a few publications have provided theoretical or empirical assessments of data sampling strategies, namely SMOTE. This review focuses mostly on these works. Elreedy and Atiya [Elreedy D, 2019] calculate the expectation and covariance matrix of SMOTE-generated patterns. However, the research presented here is not limited to the moments, since we create a mathematical framework for the density function of the SMOTE-generated samples. This study differs from Elreedy and Atiya's work [Elreedy D, 2019] in that the former relies on approximations, while this work presents an accurate formula. Studying the density properties of SMOTE patterns might drive the development of density-oriented oversampling algorithms (Yan Y [2022], Mayabadi S [2022]). To the best of our knowledge, this is the first analytical formula for the density of SMOTE-derived materials. Elreedy and Atiya [Elreedy D, 2019] provide a complete examination of SMOTE characteristics. Moniz and Monteiro provide a theoretical examination of resampling methods [Moniz and Monteiro, 2021]. The authors specifically use no-free-lunch machine learning theorems to address unbalanced learning. Furthermore, they provide a comparative empirical examination of several resampling approaches, including random under-sampling, random over-sampling, importance sampling, SMOTE, and SMOTE coupled with random under-sampling. Given no previous information or data assumptions, the authors infer that any two resampling procedures would perform similarly in classification.

Extensive research has focused on using quantitative models to assess loan applicants' creditworthiness [Baesens et al., 2003, Chawla N. V., 2002, Li and Chen, 2020, Malekipirbazari and Aksakalli, 2015]. Such models speed up credit decision-making processes, leading in the development of loan markets. Researchers and practitioners have created effective models employing statistical, machine learning, and deep learning approaches on a variety of real-world datasets.

According to Baesens et al. [2003], sophisticated approaches like neural networks and LS-SVM outperform traditional methods for identifying borrowers as defaulters or non-defaulters. However, basic linear models such as linear discriminant analysis and logistic regression might be effective alternatives to complicated approaches. Similarly, Bellotti and Crook [2009] examined the classification performance of logistic regression (LR), support vector machine (SVM), and discriminant analysis (LDA) on a credit card dataset. They determined that the SVM classifier was equivalent to both LR and LDA. They said that SVM may also be utilized for feature selection. Brown and Mues [2012] did a comparative evaluation of several data mining algorithms on five default data sets and concluded that random forest and gradient boosting are more resistant to severe class imbalance data. Their results also revealed that classic scorecard-building approaches, such as logistic regression and linear discriminant analysis, performed similarly to newer methods. According to Blanco et al. [2013], credit rating methods have limited use in the microfinance business. So, in order to contribute to the industry, they trained a Multilayer Perceptron (MLP) neural network model and compared its performance to certain statistical models. The research, which included over 5,500 participants, found that the neural networks model (MLP) produced higher classification accuracy and lower misclassification costs than logistic regression, quadratic discriminant analysis, and linear discriminant analysis.

In addition, Guo [2019] examined the prediction power of logistic regression and back-propagation neural networks on a peer-to-peer lending dataset. The research found that back-propagation neural networks outperformed logistic regression in terms of prediction accuracy. Li and Chen [2020] compared five ensemble and five baseline methods on the Lending Club dataset. They discovered that, with the exception of Adaboost, the ensemble algorithms (random forest, XGBoost, lightGBM, and stacking) outperformed the individual methods (neural networks, logistic regression, decision tree, support vector machines, and naive bayes). Random forest outperformed XGBoost, LightGBM, stacking, and Adaboost when used in ensembles. However, XGBoost and LightGBM may be better alternatives than random forest. Li [2019] employed XGBoost to classify borrowers as defaulters or non-defaulters. The results were compared to logistic regression, and XGBoost performed the best. Moreover, Malekipirbazari and Aksakalli [2015] compared LR, RF, SVM, and k-NN using the Lending Club dataset. It was discovered that RF had the greatest performance in categorizing borrowers. Dastile et al. [2020] conducted a literature study on statistics and machine learning models. Ensemble approaches were shown to beat individual algorithms by a wide margin.

Collectively, these research emphasize the necessity of quantitative models in credit assessment and the advantages of utilizing ensemble algorithms to create credit scorecards.

3 Methodology

3.1 Class Imbalance Problem

From the data set for this study, the proportion of bad loan ($y = 1$) is 77.8% and that of good loan ($y = 0$) is 22.2%. This even class distribution show that the data set used for the analysis is imbalance. Imbalance data sets have been found in the literature to result in a classifier with bias predictions [López et al., 2013, Chawla N. V., 2002] . The minority class which is usually the class of interest is predicted poorly compared to the majority class. To address this challenge we used random under-sampling and oversampling technique (SMOTE) which are the most used balancing methods to balance the datasets. Under-sampling technique drops some training examples from the majority class while oversampling duplicates the training instances for the minority class to make the two classes more balanced [erbeke et al., 2012]. SMOTE [Chawla N. V., 2002], an advanced type of oversampling technique introduces synthetic training examples to the minority class, thus making the class distribution more balanced.

4 Theoretical analysis of SMOTE

The general case of SMOTE derivation underpinning the theoretical analysis for this application is derived and built upon the works of Dina and Kamalov [2023] and Dina and Atiya [2019] which gave the formulation basis and theoretical proofs laying the foundation for the practical application of the method.

4.1 The case of general distribution

To offer a mathematical foundation for the SMOTE technique, we will do a theoretical study. SMOTE's effectiveness as a sampling method depends on its ability to create patterns that closely match the underlying distribution. We will look at this matter here. The mean vector and covariance matrix are the key parameters that define every distribution. We derived approximation formulae for the mean and covariance matrix of SMOTE-generated patterns and compared them to genuine values.

$$\begin{aligned} \text{Let } \Delta = X - X_0, \text{ then :} \\ Z = X_0 - w\Delta \end{aligned} \tag{1}$$

where w is a uniformly produced number in the range $[0, w^*]$. The parameter w , normally higher than or equal to one, enables extrapolation and interpolation on the line joining the pattern X_0 and its randomly picked neighbor X . If $w = 1$, this

returns to the original SMOTE (interpolation only). If $w^* > 1$, we can extrapolate beyond the point X_0 . Equation 3 and Equation 4 provide final estimates for the mean and covariance matrix of the resulting pattern vector Z . Equations 3 and 4, apply to any distribution of minority class samples $p(X_0)$. In the next subsections, we shall simplify the formulae and derive closed-form solutions for common distributions such the multivariate Gaussian and Laplacian distributions.

$$E[z] \approx \mu_{X_0} + \frac{Cw^*}{2} \int_{X_0} p(X_0)^{\frac{-2}{d}} \frac{\partial p(X_0)}{\partial X}, dX_0 \quad (2)$$

$$\begin{aligned} \text{where } \left[\frac{\partial p(X)}{\partial X} \right]^T &= \left(\frac{\partial p(X)}{\partial X_1}, \dots, \frac{\partial p(X)}{\partial X_d} \right). \\ \sum_Z &= \sum_{X_0} + \frac{Cw^{*2}}{3} \int_{X_0} p(X_0)^{(1-\frac{2}{d})} dX_0 I \\ &- \frac{C^2 W^{*2}}{3} \int_{X_0} p(X_0) dX_0 \int_{X_0} P(X_0)^{\frac{-2}{d}} \frac{\partial p(X_0)^T}{\partial X} dX_0 \\ &+ \frac{Cw^*}{2} \left[\int_{X_0} p(X_0)^{\frac{-2}{d}} \frac{\partial p(X_0)}{\partial X} [(X_0 - \mu_{X_0})^T] dX_0 \right. \\ &\left. + \int_{X_0} p(X_0)^{\frac{-2}{d}} (X_0 - \mu_{X_0}) \frac{\partial p(X_0)^T}{\partial X} dX_0 \right] \end{aligned}$$

where d represents the dimension of the pattern vector, N represent the number of original patterns of the considered class, μ_{X_0} is the true mean vector of the class, X_0 is the true covariance function, $p(X_0)$ is the class-conditional density at point X_0 , I is the identity matrix, and C is calculated as follows:

$$C = \frac{N! \Gamma(1 + \frac{2}{d})^{\frac{2}{d}} \Gamma(K + \frac{2}{d} + 1)}{\pi K! (d + 2) \Gamma(N + \frac{2}{d} + 1)} \quad (3)$$

4.2 The case of multivariate Gaussian distribution

Simplifying the approximation of the true probability representing the multivariate Gaussian distribution

$$E[z] \approx \mu_{X_0} \quad (4)$$

$$\sum_Z = \sum_{X_0} + \left[(2\pi)^{\frac{1-d}{2}} \frac{Cw^{*2}}{3} \det^{\frac{1-d}{2d}} \left(\sum_{X_0} \right) \left(\frac{d}{2d-1} \right)^{\frac{d}{2} - 2\pi Cw^*} \det^{\frac{1}{d}} \left(\sum_{X_0} \right) \left(\frac{d}{d-2} \right)^{\frac{d+2}{2}} \right] I \quad (5)$$

where $\det$ refers to the determinant. The preceding Equation 6 is valid for dimensions $d > 2$ due to a convergence condition. According to Equation 6, the percentage $\frac{d}{d-2}$ is higher than one for every $d > 2$, resulting in a negative third

term in the resultant covariance matrix $\sum_Z$. Equation 6 shows that the second term is smaller than the

Except for big w , the third term in magnitude (for conventional SMOTE, $w = 1$) remains unchanged. SMOTE-generated patterns have a more constricted covariance matrix ($\sum_Z$) compared to the initial minority class instances ($\sum_{X_0}$), as expected. Subtracting the identity matrix times a positive value results in smaller diagonal elements.

From the preceding formulae, one can see the following: SMOTE-generated patterns have a mean vector that closely matches the genuine one. The covariance matrix shows some inconsistency. It is more contractive than the actual one. This supports the previous section's claim that the generation process shifts patterns inward.

While these findings concentrate on SMOTE distributional properties, they also have an indirect influence on classification performance. The Bayes rule states that the posterior probability of the class $p(Y|X)$ depends on the underlying class-conditional densities $p(X|Y)$. The distributional qualities of the data used in the design have an influence on the classification results of most classifiers, even if they are not explicitly employed.

4.3 Case of other distributions: Multivariate Laplace

The real probability density follows multivariate Laplace distribution is defined as follows;

$$p(X) = \left(\frac{\alpha}{2}\right)^d e^{-\alpha \sum_{i=1}^d |X_i - \mu_i|} \quad (6)$$

where α is a distribution parameter, μ is the distribution's original mean, and d is the number of dimensions. The mean and covariance of SMOTE-generated patterns may be approximated as shown

$$E[z] \approx \mu_{X_0} \quad (7)$$

$$\sum_Z = \sum_{X_0} + \frac{4Cw^*}{\alpha^2} \left(\frac{d}{d-2}\right)^d \left[\frac{w^*}{3} - \left(\frac{d}{d-2}\right)\right] I \quad (8)$$

where C is computed as per Equation 4. The fraction $\frac{d}{d-2}$ is greater than one for any $d > 2$, so the third term of the generated examples' covariance matrix $\sum_Z$ would be negative. Since standard SMOTE, $w^* = 1$, the covariance matrix $\sum_Z$ of the patterns generated by SMOTE would be more contracted than the original covariance matrix $\sum_{X_0}$.

4.4 Effect on the classification boundary

The distribution of SMOTE-generated instances follows Equations 2 and 3, which provide the mean and covariance matrix. The Fisher LDA equations for w and w_0 at the decision boundary $w^T x + w_0 = 0$ are defined as:

$$w = (\sum_{x_0} + \sum_1)^{-1}(\mu_1 - \mu_{x_0}) \quad (9)$$

$$W_0 = -\frac{1}{2}(\mu_1 - \mu_{x_0})^T (\sum_{x_0} + \sum_1)^{-1}(\mu_1 - \mu_{x_0}) \quad (10)$$

The majority class distribution's mean and covariance are μ_1 and $\sum_1$, whereas the minority class distribution's mean and covariance are μ_{x_0} and $\sum_{x_0}$, respectively. The above equation for w_0 may be written in terms of w as follows:

$$W_0 = -\frac{1}{2}(\mu_{x_0} + \mu_1)^T w \quad (11)$$

Using the SMOTE oversampling approach, we may replace Equations 2, and 3 in the Fisher linear discriminant (Equations 10, 11).

$$w_z = (\sum_Z + \sum_1)^{-1}(\mu_1 - \mu_Z) \quad (12)$$

$$W_0 = -\frac{1}{2}(\mu_Z + \mu_1)^T (\sum_Z + \sum_1)^{-1}(\mu_1 - \mu_Z) \quad (13)$$

To quantify the difference in decision boundary imposed by SMOTE, we analyze two terms. $\Delta w = \|w_z w\|$ and $\Delta w_0 = \|w_{0z} - w_0\|$.

$$\Delta_w = \|(\sum_Z + \sum_1)^{-1}(\mu_1 - \mu_Z) - (\sum_{x_0} + \sum_1)^{-1}(\mu_1 - \mu_{x_0})\| \quad (14)$$

Let $A = \sum_{x_0} + \sum_1$ and $b = \mu_1 - \mu_{x_0}$. Also, let $T = \sum_Z + \sum_1$. The second, third, and fourth components in Equation 3 describe the covariance matrix discrepancy caused by SMOTE. Consider $h = \mu_Z - \mu_{x_0}$, the second term in Equation 2. Then, Δ_w is evaluated as

$$\Delta_w = \|(A + T)^{-1}(b - h) - A^{-1}b\| \quad (15)$$

Applying the following matrix inversion identity:

$$(A + BCD)^{-1} = A^{-1}BCDA^{-1}(I + BCDA^{-1})^{-1} \quad (16)$$

Let $B = D = I$, where I is the identity matrix. Since $A = \sum_{x_0} + \sum_1$, we assume that the covariance matrices are positive definite and so invertible. We get:

$$(A + T)^{-1} = A^{-1} - A^{-1}TA^{-1}(I + TA^{-1})^{-1} \quad (17)$$

Then Δ_w can be evaluated as:

$$\Delta_w = \|(A^{-1} - A^{-1}TA^{-1}(I + TA^{-1})^{-1})(b - h) - A^{-1}b\| \quad (18)$$

which is simplified as

$$\begin{aligned} \Delta_w &= \|(-A^{-1}h - A^{-1}TA^{-1}(I + TA^{-1})^{-1})(b - h)\| \\ &= \|(A^{-1}h + A^{-1}TA^{-1}(I + TA^{-1})^{-1})(b - h)\| \end{aligned} \quad (19)$$

The theoretical study for multivariate Gaussian and multivariate Laplace distributions shows that the mean of SMOTE-generated patterns μ_Z is almost identical to the mean of the original distribution μ_{X_0} , implying that $h \approx 0$. Thus, Equation 20 would be as follows:

$$\begin{aligned} \Delta_w &= \|A^{-1}TA^{-1}(I + TA^{-1})^{-1}b\| \\ &= \|A^{-1}(TA^{-1})(I + TA^{-1})^{-1}b\| \\ &= \|A^{-1}[(I + TA^{-1})AT^{-1}]^{-1}b\| \end{aligned} \quad (20)$$

Hence, Δ_w is estimated following:

$$\Delta_w = \|A^{-1}(AT^{-1} + I)^{-1}b\| \quad (21)$$

Similar to Δ_{w_0} , since w_0 is a scalar, we describe the difference in it using absolute difference:

$$\Delta_{w_0} = \frac{1}{2}|w_z^T(\mu_Z + \mu_1) - w^T(\mu_{X_0} + \mu_1)| \quad (22)$$

Let $\Delta = w_z - w$, then:

$$\Delta_{w_0} = \frac{1}{2}|(w + \Delta)^T(\mu_{X_0} + h + \mu_1) - w^T(\mu_{X_0} + \mu_1)| \quad (23)$$

simplifying Equation 24:

$$\Delta_{w_0} = \frac{1}{2}|(w + \Delta)^T(\mu_{X_0} + h + \mu_1) - w^T(\mu_{X_0} + \mu_1)| \quad (24)$$

$$\Delta_{w_0} = \frac{1}{2}|w^Th + \Delta^T(\mu_{X_0} + h + \mu_1)| = \frac{1}{2}[w^Th + \Delta^T(\mu_Z + \mu_1)] \quad (25)$$

Similar to the Δ_w derivation, assuming $h \approx 0$ yields the following:

$$\Delta_{w_0} = \frac{1}{2}|\Delta^T(\mu_{X_0} + \mu_1)| \quad (26)$$

5 Classification Techniques Algorithms

Considering $\vec{x} = [x_1, x_2, \dots, x_n] \in \mathbb{R}^n$ where $\mathbf{x}$ represent n-dimensional feature vector (i.e loan applicants' characteristics). Assuming we have a binary response variable y with values 0 and 1 where $y \in \{0, 1\}$ and $y = 1$ implies defaulted borrowers and $y = 0$ represents non-defaulted borrowers. According to Bayes rule, the probability that an applicant i will default on a loan can be estimated as $p(y = 1 | \vec{x})$ while the probability of a non-default event occurring is just the complement of the default event (i.e $p(y = 0 | \vec{x}) = 1 - p(y = 1 | \vec{x})$). The probability that a classifier will predict a borrower as bad can be given as $p(y = 1 | \vec{x}) > \delta$. On the other hand, a loan will be approved for a customer i when the default probability of i is $p(y = 1 | \vec{x}) \leq \delta$ where δ represent a specified cut-off point. This is a classification problem.

Generally, the rule for classification can be defined as a function $h : \mathbf{x} \rightarrow y$. If a new $\mathbf{x}$ is observed, y (i.e labels) is predicted to be $h(\mathbf{x})$. A classifier seeks to learn how the function h can generalize well on a new data points. This means that if a new data $\mathbf{x}$ which labels y is unknown is fed into the classifier h , the labels should be predicted to right.

5.1 Linear Discriminant Analysis (LDA)

Linear discriminant and quadratic discriminant analysis are popular classification techniques because their results are transparent, have closed-form solution and they have no hyper-parameter to be tuned. They work better for multi-class problems compared to logistic regression. LDA has a linear decision surface and can be used for dimensionality reduction. A feature vector $\vec{x}$ which represent borrowers' characteristics can be assigned to one of y discrete classes, where $y \in \{0, 1\}$ such that $y = 1$ and $y = 0$ represent defaulted loans and non-defaulted loans respectively. According to Bayes rule, the posterior probability $p(Y | X)$ can be written as given in Equation 27. The procedure used to estimate the posterior can lead to discriminant analysis or logistic regression.

$$P(Y | X) = \frac{f_k(x_0)\pi_k}{\sum_r f_r(x_0)\pi_k} \quad (27)$$

In both LDA and QDA the class conditional densities $f_k(x_0)$ and the prior π_k are estimated. They both assume $f_k(x_0)$ to be a multivariate Gaussian distribution and π_k is modeled as Bernoulli. LDA assumes equal covariance matrix for the classes (i.e. $\sum_1 = \sum_0$) with means μ_1, μ_0 . The decision boundary of defaulted loans ($y = 1$) and non-defaulted loans is given by Equation 28

$$P(y = 1 | X = x) = P(y = 0 | X = x) \quad (28)$$

The log of the posterior probability of the LDA then becomes a line in the form as given in Equation 29.

$$\beta^\top x + \alpha = 0 \quad (29)$$

where:

$$\begin{aligned} \beta^\top &= \sum_{i=1}^{-1} \mu_1 - \sum_{i=1}^{-1} \mu_0 \\ \alpha &= \frac{1}{2} \mu_0^\top \sum_{i=1}^{-1} \mu_0 - \mu_1^\top \sum_{i=1}^{-1} \mu_1 + \log \frac{\pi_1}{\pi_0} \end{aligned}$$

5.2 Quadratic Discriminant Analysis (QDA)

QDA has a quadratic decision surface and makes no assumption about the covariance matrices of the classes. One major drawback of QDA is that there are more parameters to be estimated compared to LDA. In QDA the covariance matrix for each class needs to be computed. The quadratic discriminant function is therefore computed as follows:

$$\vec{x}^\top A \vec{x} + W^\top \vec{x} + b > 0 \quad (30)$$

where:

$$A = \frac{1}{2} \left(\sum_{i=1}^{-1} \mu_1 \mu_1^\top - \sum_{i=1}^{-1} \mu_0 \mu_0^\top \right) \quad (31)$$

$$W^\top = \left(\mu_1^\top \sum_{i=1}^{-1} \mu_1 - \mu_0^\top \sum_{i=1}^{-1} \mu_0 \right) \quad (32)$$

$$b = -\frac{1}{2} \mu_1^\top \sum_{i=1}^{-1} \mu_1 + \frac{1}{2} \mu_0^\top \sum_{i=1}^{-1} \mu_0 + K_1 - K_0 + \log \pi_1 - \log \pi_0 > 0 \quad (33)$$

5.3 Logistic Regression (LR)

In logistic regression, instead of estimating $f_k(x_0)$ and π_k , the posterior probability is modeled directly through a logistic function. if $y = 1$,

$$p(y = 1 \mid \vec{x}) = \frac{\exp^{\beta^\top x_i}}{1 + \exp^{\beta^\top x_i}} \quad (34)$$

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if $y = 0$,

$$(y = 0 \mid \vec{x}) = \frac{1}{1 + \exp^{\beta^\top x_i}} \quad (35)$$

The posterior is then written as a parametric model in the form

$$p(y \mid \vec{x}) = p(y = 1 \mid \vec{x})^{1-y} * p(y = 0 \mid \vec{x})^y \quad (36)$$

Maximum likelihood estimation (MLE) is used to optimized the unknown parameter β . Using MLE does not result in a close form solution so Newton-Raphson method can then be applied. The parameter β in the logistic regression can be computed as follows:

$$\beta^{r+1} = \beta^r + (X^\top W X)^{-1}(\vec{y} - \vec{P})^{-1} X \quad (37)$$

where:

y is $n \times 1$ vector of all response variables

X is $n \times (d + 1)$

$\vec{P}$ a vector of $n \times 1$ such that $p_i = P(x_i \mid \beta)$

W is a diagonal matrix $n \times n$ such that $w_{ii} = (1 - P(x_i \mid \beta)) P(x_i \mid \beta)$

β^{r+1} is a new β and β^r is an old β

5.4 Decision Tree (DT)

C5.0 algorithm proposed by Quinlan [1986] is one of the best implementations of decision trees because it handles many of the decisions automatically thereby avoiding over-fitting. It is an improvement to the C4.5 algorithm. Decision tree is a well-known white-box techniques. The results of white-box models are easy to interpret. This has made the algorithm one of the best choice to consider when building a credit scoring model because in most countries the reason for rejecting an applicant a loan must be well explained. The algorithm uses entropy to decide which feature in the data to split upon. The feature with the most information is splitted first. The formulae for calculating entropy for sample (S) can be represented as:

$$Entropy(S) = -p_1 \log_2(p_1) - (1 - p_1) \log_2(1 - p_1) \quad (38)$$

where p_1 and $1 - p_1$ represent the proportions of values falling into the defaulted and non-defaulted class respectively.

5.5 Random Forest (RF)

Random forest is one of the best ensemble learning algorithms that are used to solve both classification and regression tasks. The algorithm creates several decision trees and average their results using majority votes [Breiman, 2001]. Even without hyper-parameter tuning, this algorithm normally produce superior results. It has an in-built feature importance plot that enable you to find the features that contribute most to the prediction thereby deciding on which feature to drop or include in the model.

5.6 Support vector machine (SVM)

The goal of SVM classifier is to find the maximum margin hyperplane that best classify future data points into distinct classes. The algorithm can be used for both classification and regression tasks. SVMs are more robust to outliers because the decision function only depends on the support vectors rather than the entire training data. The closest data points to the hyperplane is what are termed as the support vectors. Considering $y_i \in \{+1, -1\} \forall i = \{1, 2\}$. The dual optimization problem for SVM can be written as:

$$\max_{\alpha} \left(\sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{j=1} \sum_{i=1} \alpha_i \alpha_j y_i y_j x_i^{\top} x_j \right) \quad (39)$$

subject to the constraints $C \geq \alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i y_i = 0$. α_i represents a Lagrange multiplier.

5.7 Artificial Neural Networks (ANNs)

One of the well-known neural network architecture is the multilayer feed forward neural network. In this network there are d-dimensional data points with at least three layers of perceptrons and each perceptron is fed by an input having some random weights. The inputs are weighted and a linear function is applied to the weighted sum of this inputs and we are going to have an output and the output is going to another layer of perceptron known as hidden layer. The process continues until we reach the final output layer.

ANNs seeks to find the optimum weights that minimize the objective function in Equation 40.

$$E = |y - \hat{y}|^2 \quad (40)$$

$$\frac{\partial E}{\partial a_{ie}} = \delta_i * z_e \quad (41)$$

The first weight of the network can be computed by Equation 41, but

$$\delta_i = \sum_j \delta_j * u_{ji} * \sigma'(a_i) = \sigma'(a_i) \sum_j \delta_j * u_{ji} \quad (42)$$

Knowing δ_i , we can compute δ for the last layer k . Each δ depends on the δ of the next layer. The δ for the last layer is computed in Equation 43

$$\delta_k = \frac{\partial E}{\partial \hat{y}} = \frac{\partial |y - \hat{y}|}{\partial \hat{y}} = -2|y - \hat{y}| \quad (43)$$

The value δ_k , can be substituted it in Equation 42 in order to find δ for all the previous layers up to the first layer. We multiply all the computed δ to any z to find the corresponding weight. Having all the derivatives of all of the weights we can do gradient descent as shown Equation 44

$$u_{ie}^{new} \leftarrow u_{ie}^{old} - \rho \frac{\partial E}{\partial u_{ie}} \quad (44)$$

where ρ is some learning rate, u_{ie}^{new} is new weight, u_{ie}^{old} is old weight and $\frac{\partial E}{\partial u_{ie}}$ is the derivative of the error or objective function. Back-propagation using gradient descent find local mean of the objective function instead of the global minima. This is the reason why neural networks was out of favour for a while.

5.8 Extreme gradient boosting (XGBoost)

XGBoost is considered as one of the most accurate, fastest and efficient algorithms and has won several kaggle competitions [Ma et al., 2018, Chen and Guestrin, 2016a]. The technique was introduced in 2014 so its application in developing credit scoring models is limited as compared to logistic regression, ANN, SVMs and Random forest. The scalability of this algorithm is responsible for its high efficiency [Chen and Guestrin, 2016b]. It is an ensemble learning technique. Ensemble learning combines the results of different base learners into a single, but robust model. XGBoost seeks to minimize the objective function

$$\mathcal{L}^{(t)} = \sum_{i=1}^n l(y_i, \hat{y}_i^{(t-1)} + f_t(\vec{x}_i)) + \Omega(f_t) \quad (45)$$

where $\Omega(f_t)$ and $l(y_i, \hat{y}_i^{(t-1)} + f_t(\vec{x}_i))$ represent a regularization term which penalizes complexity of the model and a loss function respectively [Chen and Guestrin, 2016b].

5.9 Metrics for Performance Measures

Accuracy is the percentage of total true predictions divided by the total number of examples. It can only be a good measure of performance when the data set used to train a classifier is somewhat balanced.

$$Accuracy = \frac{TP + TN}{TP + TN + FP + FN} \quad (46)$$

Sensitivity or recall finds how many of the true positives the classifier predicted as positives. It becomes a reliable measure of classifier's efficiency when faced with an imbalanced data set.

$$Sensitivity = \frac{TP}{TP + FN} \quad (47)$$

Specificity is the percentage of the actual negatives which are correctly predicted as negative.

$$Specificity = \frac{TN}{TN + FP} \quad (48)$$

Precision measures how often a classifier is right when it predicts the positive class.

$$Precision = \frac{TP}{TP + FP} \quad (49)$$

With the help of the harmonic mean, F-measure or F1-score combines recall and precision into a single number that can be used to judge a classifiers separability. It becomes appropriate when we want to strike a balance between precision and recall

$$F - measure = \frac{2 \times Precision \times Recall}{Precision + Recall} \quad (50)$$

AUC measures the ability of a classifier to discriminate between classes.

$$AUC = 0.5 \left(1 + \frac{TP}{FN + TP} - \frac{FP}{TN + FP} \right) \quad (51)$$

5.10 Data Description and Data Pre-processing

The data set used for the analysis was collected from one of the renowned commercial banks in Ghana. All confidential information were dropped from the data. It contained 734 observations and 14 attributes. Out of the 14 variables, 8 were categorical and 6 numeric. The categorical data were encoded into dummy variables. The binary response variable Y was creditworthiness where $y = \{0, 1\}$. $y = 1$ represent defaulted loans and $y = 0$ implies non-defaulted loans. 571

(77.8%) of the observations represent non-defaulters and 163 (22.2%) were defaulters. This uneven class distribution clearly indicate that the dataset was imbalanced. The description of both the categorical and numeric attributes are shown in Table 1 and Table 2 respectively. The data was partitioned into 70% training and 30% test set. Three data balancing techniques: under-sampling, oversampling and the synthetic minority oversampling technique (SMOTE) were employed to balance the data.

Table 1: Characteristics of the categorical variables and their description attributes

Description	Attributes	Levels
Creditworthiness	Credit rating	1 = Default 0 = Non-default
Purpose	Purpose	1 = Buy a car 2 = Funeral 3 = Pay rent advance 4= Buy plot of land 5 = Buy domestic appliances 6 = Complete building 7 = Education 8 = For Business 9 = Others
Educational status	edu_stat	1 = Primary 2 = Secondary 3 = Graduate/tertiary 4 = Postgraduate
Residential status	res_stat	1 = Tenant 2 = Own house 3 = With relatives
Sex	Sex	1 = Male 2 = Female
Marital status	marital	1 = Married 2 = Single 3 = Divorced/ Separated/Widowed
Occupation	occupation	1 = Employed by Government 2 = Employed by Private Sector 3 = Self Employed 4 = Unemployed
Payment of past loan	previous credit stat	1 = Delay in paying of past loan 2 = Past loan paid on time 3 = No Credit at the bank

Table 2: Characteristics of the numeric variables and their description attributes

Description	Attributes	Variable type
years at job	years_at_job	numeric
years at current address	years_at_address	numeric
Number of dependents	dependents	numeric
Applicants age	Age	numeric
Loan amount	Amount	numeric
Loan duration	Duration	numeric

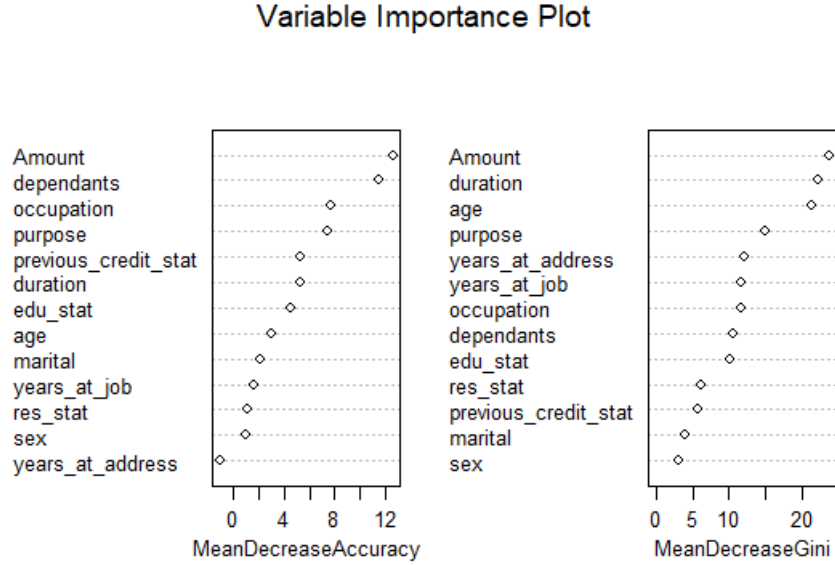


Figure 2: Determinants of credit default rates for variable of importance plot

6 Analysis and Results

The random forest algorithm developed a feature importance plot, as shown in Figure 2, to find the essential factors that drive credit default rates. The plot showed that the amount of loan had the greatest mean decrease accuracy score of roughly 14, followed by the number of dependents and finally the applicants' vocations. The mean decrease accuracy ratings reveal how much the random forest classifier's performance degrades when each feature is removed. As a result, indicators such as loan amount, number of dependents, employment, loan purpose, past credit status, loan term, and educational status might be regarded key factors in anticipating borrower default rates. Variables such as age, marital status, years at current work, residential status, gender, and year of current residence, on the other hand, had low mean decrease accuracy scores ranging from 0 to 4, allowing them to be eliminated from the model without substantially impacting its accuracy.

6.1 Data Resampling Techniques on Machine Learning Algorithms' Performance

This section presents the findings of the credit default prediction research on eight classification algorithms on four datasets, which include the original skewed dataset and balanced datasets utilizing under-sampling, SMOTE oversampling,

and SMOTE balanced approaches. The research used six main performance indicators to evaluate model performance, including accuracy (PCC), sensitivity, specificity, precision, F-measure, and area under the receiver operating characteristics curve (AUC). The major emphasis was on the AUC values.

6.1.1 Original Skewed Dataset

The AUC performance measure ranged from 0.583 to 0.834. Table 3 shows the classifiers' performance metrics after training and testing on the original imbalanced dataset. XGBoost had the greatest AUC score (0.834) and specificity (99.36%). The AUC value reflects the classifier's high ability to distinguish between defaulters and non-defaulters. However, the classifier's low sensitivity score (8.16%) indicates that it did badly in recognizing real affirmative cases (defaulters). Decision Tree (DT) outperformed XGBoost in terms of accuracy (PCC) at 80.58% and sensitivity (30.61%). Random forest has the second best AUC score (0.785). Artificial Neural Networks (ANNs) performed best in classifying the default group, with a sensitivity score of 40.82%. Despite excellent specificity scores (ranging from 73.25% to 99.36%), all classifiers trained with the skewed dataset showed low sensitivity (6.12% to 40.82%), indicating poor detection of real positive cases. Tables 3 through 5 demonstrate that sensitivity scores on balanced datasets improved compared to skewed datasets. This suggests that balancing strategies improve classifier performance.

Table 3: A Test Set Comparison of Classifiers using the Original Skewed Dataset

Classifier	PCC	Sensitivity	Specificity	Precision	F-Measure	AUC
RF	76.70%	6.12%	98.73%	60.00%	11.11%	0.785
DT	<u>80.58%</u>	30.61%	96.18%	71.43%	42.86%	0.707
LDA	77.67%	12.25%	98.09%	66.67%	20.70%	0.711
QDA	77.18%	14.29%	96.82%	58.33%	22.96%	0.635
LR	77.67%	20.41%	95.54%	58.82%	30.30%	0.750
SVMs	77.67%	8.16%	<u>99.36%</u>	80.00%	14.81%	0.707
ANNs	65.53%	<u>40.82%</u>	73.25%	32.26%	36.04%	0.583
XGBoost	77.67%	8.16%	<u>99.36%</u>	<u>80.00%</u>	14.81%	<u>0.834</u>

6.1.2 Balanced Dataset: Under-sampling Technique

Table 4 shows the performance of classifiers trained on the under-sampled dataset. The findings show that XGBoost outperformed other classifiers, with an accuracy (PCC) of 77.67%, F1-score of 58.69%, and AUC of 0.795. The perfor-

mance of classifiers trained on the under-sampled dataset was extremely competitive with that of the original skewed dataset, as indicated by their comparable PCC and AUC values. However, classifiers trained on under-sampled data had much higher sensitivity scores than their counterparts. Surprisingly, Artificial Neural Networks (ANNs) had the highest sensitivity score of 91.84%. These findings indicate that classifiers trained on undersampled data are often more successful at detecting defaulters than those trained on the other three datasets, while maintaining discriminating capacity. They also outperformed their rivals in terms of F-measure.

Table 4: A Test Set Comparison of classifiers using the Under-sampled Dataset

Classifier	PCC	Sensitivity	Specificity	Precision	F-Measure	AUC
RF	72.33%	69.39%	73.25%	44.70%	54.37%	0.776
DT	74.27%	59.18%	78.98%	46.77%	52.25%	0.761
LDA	73.79%	53.06%	80.25%	45.61%	49.05%	0.715
QDA	76.70%	30.61%	<u>91.08%</u>	51.72%	38.46%	0.627
LR	<u>77.67%</u>	57.14%	84.08%	<u>52.83%</u>	54.90%	0.763
SVMs	66.99%	48.98%	72.61%	35.82%	41.38%	0.688
ANNs	35.44%	<u>91.84%</u>	17.83%	25.86%	40.36%	0.594
XGBoost	<u>77.67%</u>	67.35%	80.89%	52.00%	<u>58.69%</u>	<u>0.795</u>

6.1.3 Balanced Dataset: Oversampling Technique

Finally, using the oversampling data as given in Table 5, XGBoost emerged as the best-performing classifier based on the AUC (0.801), followed by random forest with an AUC of 0.792. ANN had the best sensitivity score of 69.39%, although these classifiers only outperformed those trained on the skewed dataset.

Table 5: A Test Set Comparison of classifiers using the Oversampling Dataset

Classifier	PCC	Sensitivity	Specificity	Precision	F-Measure	AUC
RF	78.16%	26.53%	<u>94.27%</u>	59.09%	36.58%	0.792
DT	<u>79.61%</u>	34.69%	93.63%	62.96%	44.73%	0.779
LDA	73.30%	48.98%	80.89%	44.44%	46.56%	0.717
QDA	<u>79.61%</u>	32.65%	<u>94.27%</u>	64.00%	43.24%	0.634
LR	74.27%	59.18%	78.98%	46.77%	<u>52.25%</u>	0.734
SVMs	73.30%	40.87%	83.44%	43.48%	42.13%	0.658
ANNs	49.03%	<u>69.39%</u>	42.68%	27.42%	39.31%	0.572
XGBoost	76.70%	32.65%	90.45%	<u>51.61%</u>	40.00%	<u>0.801</u>

6.1.4 Balanced Dataset: SMOTE Oversampling Technique

Table 6 shows that Decision Trees (DT) performed best on the SMOTE dataset, with an AUC of 0.784 and a sensitivity of 53.06%. Random forest closely tracked the DT model, with an AUC of 0.780. Notably, sensitivity ratings on the SMOTE dataset improved relative to the original imbalanced data. Thus, the SMOTE over-sampling approach produces much superior results. Oversampling outperforms under-sampling and classic oversampling for several classifiers, yielding greater scores in various assessment metrics.

Table 6: A Test Set Comparison of classifiers using the SMOTE Dataset

Classifier	PCC	Sensitivity	Specificity	Precision	F-Measure	AUC
RF	75.73%	40.82%	86.63%	48.78%	44.45%	0.780
DT	76.21%	<u>53.06%</u>	83.44%	50.00%	51.48%	<u>0.784</u>
LDA	<u>78.16%</u>	38.78%	90.45%	<u>55.88%</u>	45.79%	0.721
QDA	77.18%	26.53%	<u>92.99%</u>	54.17%	35.62%	0.654
LR	<u>78.16%</u>	51.02%	86.62%	54.35%	<u>52.63%</u>	0.739
SVMs	69.42%	34.69%	80.26%	47.50%	40.10%	0.632
ANNs	68.93%	38.78%	78.34%	35.85%	37.26%	0.574
XGBoost	77.67%	30.61%	92.36%	55.56%	39.47%	0.735

The study's results shed light on the relevance of data balancing strategies in model performance, as shown by the assessment of classifiers across various datasets. With the exception of the SMOTE oversampling dataset, XGBoost consistently beat the other classifiers on the other datasets in terms of AUC. This demonstrates XGBoost's superior discriminating performance when differentiating between defaulters and non-defaulters. This discovery is comparable to those of [Li and Chen, 2020, Dastile et al., 2020]. According to [Li and Chen, 2020, Dastile et al., 2020], ensemble algorithms such as random forest, XGBoost, and lightGBM outperformed individual algorithms including neural networks, logistic regression, decision tree, and support vector machines. In addition, random forest, decision tree, and logistic regression performed well and might be utilized as alternatives to XGBoost classifiers for developing credit scoring models. This finding is consistent with the research of Brown (2012). The authors of [Brown and Mues, 2012] found that random forest and gradient boosting are more robust to extreme class imbalance data sets. Their studies also revealed that standard scorecard-building approaches such as logistic regression and linear discriminant analysis performed comparably to sophisticated methods like as XGBoost.

Furthermore, Artificial Neural Networks (ANNs) shown promising results in effectively recognizing default cases and thereby minimizing type II mistakes,

as seen by high sensitivity ratings. Furthermore, classifiers trained on balanced datasets demonstrated higher sensitivity scores than those trained on the original skewed dataset, without sacrificing their capacity to distinguish between defaulters and non-defaulters. This demonstrates the value of data balancing strategies in increasing models' ability to anticipate minority class occurrences, namely defaulters. These results provide important insights for practitioners working with unbalanced data and contribute to the ongoing topic about employing machine learning methodologies to enhance data-driven decision-making processes.

More importantly, the improvements recorded in the balanced data techniques were found to be optimized when the SMOTE oversampling method is employed. The performance of the SMOTE techniques results in a higher performance metric values giving credence to the recommendation of SMOTE usage in both huge and relatively big imbalance dataset to help improve algorithms predictions. The decision tree algorithm performed creditably well in under this balancing method with other competitive algorithms such as the linear discriminant analysis and the linear regression.

7 Conclusion

This study presents three strategies to address class imbalance and apply them to various machine learning classification models. The study utilized the "A commercial bank Customer Transaction " dataset from the Ghana. The study utilized the unbalanced dataset to address and evaluate the issue. The study tested skewed original data, balanced the data by oversampling and under-sampling and SMOTE oversampling techniques on the dataset and evaluated classifiers using several criteria. SMOTE oversampling outperforms the skewed data, balanced under-sampling and oversampling for several classifiers, resulting in greater scores across assessment measures. In the future, we want to compare different gravitates of imbalance data on deep learning algorithms with resampling strategies.

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Performance evaluation in Public Administration: P-AHP and PROMETHEE a comparative analysis*

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Abstract

The performance evaluation is a crucial phase for each organization, and nowadays this is particularly relevant also for Public Administrations (PAs), where a high level of efficiency and effectiveness has a meaningful impact on citizens and companies. In this work we propose the adoption of Multi-Criteria Decision Making approaches, in particular PROMETHEE I-II and Parsimonious AHP, for the performance evaluation of PA's services. The novelty of our approach is mainly related to the application of these methods to the performance evaluation process carried out by PAs, showing the features that can allow their adoption in a real-life context. The methods have been tested on a real PA in Southern Italy, and the results show their effectiveness and significance as control and planning support tools for decision-makers.

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1 Introduction

The Public Administration (PA), having the task of carrying out the functions of management, direction and coordination of the State, has an increasingly crucial role in the lives of citizens. In particular, PAs are in charge to make decisions in continuously evolving territorial contexts characterized by a high degree of complexity [Reina et al., 2008]. It is clear that their actions have consequences for an entire community. These considerations lead us to reflect on the importance of introduction of decision-making support tools in the public sector. More and more studies are analyzing the decision-making problems faced by PAs, proposing the use of decision support tools to reach more reasoned and effective choices. The areas of study of PA are different. Recent studies have dealt with the management of public tenders. In particular, for the analysis of the most economically advantageous offer, many authors suggest the use of the AHP, see e.g. Marcarelli and Squillante [2020], Diabagaté et al. [2015], Salvatore et al. [2021]. Recent studies instead propose innovative methods such as the ELimination Et Choix Traduisant la REalité (ELECTRE) III and Parsimonious AHP (P-AHP) methods [Fattoruso and Marcarelli, 2022], Preference Ranking Organization METHod for Enrichment of Evaluations (PROMETHEE) and the Multi-Attribute Utility Theory (MAUT) [Dotoli et al., 2020].

Another topic of interest in the PA is public transport. The study of Kiciński and Solecka [2018] proposes the use of the AHP and ELECTRE III method which allows reasoned decisions to be made on the implementation of the transport system by governments, taking into account a plurality of analysis criteria such as the respect for the environment. Of particular interest is also the topic of efficient management for the use of public funds: the study of Vavrek et al. [2017] proposes the use of the Weighted Sum Approach (WSA) method and of the Technique for Order Preference by Similarity methods to Ideal Solution (TOPSIS) method, proposing an analysis of the benefits and drawbacks of using these methods. Among the issues addressed within the PA is the one related to the implementation of policies to support renewable energy by governments. For this analysis, the study by García et al. [2024] proposes the use of AHP and TOPSIS methods, while Kang et al. [2024] propose a model that combines the Analytic Hierarchy Process (AHP) and ELECTRE III, using the Stepwise Weighted Assessment Ratio Analysis (SWARA) method.

Among the topics that we believe are of great interest in the PA, in addition to those already mentioned, there is the performance evaluation that has a crucial role in PA as it allows the efficiency and effectiveness of public

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services offered to citizens to be assessed, identifying any areas for improvement. This task also helps to guarantee responsibility and transparency in the work of public employees as well as in the decisions taken by the PA. Furthermore, performance evaluation can promote the professional development of public employees, stimulating them to improve their skills and achieve pre-established objectives. In the literature, the authors who have dealt with performance evaluation in the PA field have used various multi-criteria tools. Kuo and Liang [2012] and Goyal et al. [2022] analyse the public transport performance evaluation with Multi-Criteria Decision Making (MCDM) based on interval-valued fuzzy numbers. Lai et al. [2008] analyzes the performance of research and development projects using the Fuzzy Multiple Criteria Decision Making (FMCDM) approach. The work of Wu et al. [2012] and Fattoruso et al. [2022] discuss the high level instruction performances using respectively the AHP and VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method and the AHPSort II.

This paper proposes the use of Multi-criteria Methods analysis tools, which provide an exhaustive and in-depth evaluation of the decision process, allowing decision-makers to make informed and rational decisions in a complex context such as that of performance evaluation within the PA organization. In particular, a comparison is proposed between two multi-criteria methods such as the PROMETHEE I-II [Brans and De Smet, 2016] method and the Parsimonious AHP [Abastante et al., 2019] method. In addition to the novelty of using these methods in the public sector, the paper proposes to verify the effectiveness of multi-criteria methods for evaluating the performance of public administration offices. We apply the methods in a real case study provided by the Metropolitan City of Reggio Calabria, a PA located in Southern Italy, which provided us with a structured framework of the performance evaluation problem by defining the decision-making criteria and evaluations of each office. A comparative analysis has been performed by using PROMETHEE I-II and P-AHP methods, in order to assess the most suitable method for evaluating performance in the PA, highlighting its gaps and strengths. The methodology is described in Section 2; Section 3 contains the description of the case study and the results obtained; Sections 4 and 5 respectively discuss the results and conclude the paper.

2 Methodology

2.1 PROMETHEE I-II

The PROMETHEE method Brans et al. [1986] belongs to the family of outranking methods. This section will illustrate the methods PROMETHEE I which focuses on the detailed evaluation of alternatives with respect to individual criteria, allowing Decision Makers (DMs) to understand the im-

plications of their choices on each criterion and PROMETHEE II which aggregates the evaluations of the criteria to generate a global ranking of the alternatives based on the preferences of the decision makers, allowing to obtain a synthetic and ordered vision of the alternatives, facilitating the decision-making process and allowing a direct comparison between them [Ishizaka and Nemery, 2011] [Brans and De Smet, 2016]. Given a criterion $g_j \in G$, and for each pair of ordered alternatives $(a, b) \in A$, the decision maker expresses her/his preference by means of a degree of preference obtained using the preference function $P_j(a, b)$ which is in function of $d_j(a, b) = g_j(a) - g_j(b)$; therefore, $P_j(a, b)$ indicates whether an alternative a is preferred (or not) over b on the criterion g_j based on the difference between their evaluation $d_j(a, b)$. The construction of $P_j(a, b)$ may require the use of indifference threshold q_j and preference threshold p_j , such that

$$d_j(a, b) < q_j \implies P_j(a, b) = 0$$

$$d_j(a, b) > p_j \implies P_j(a, b) = 1$$

To evaluate to what extent the action a is preferred to b , the aggregate preference index $\pi(a, b)$ is then constructed. Given the pair of alternatives a and b , the aggregate preference index can be defined as follows:

$$\pi(a, b) = \sum_{j=1}^k P_j(a, b) \cdot w_j \quad (1)$$

where w_j represents the weight of each criterion g_j and k the number of criteria. The π value fluctuates between 0 and 1: if $\pi \approx 0$ the overall preference of one alternative over another is weak; if $\pi \approx 1$ the preference is strong. The positive and negative outranking flows are then calculated, which express the outranking (or not) of an alternative compared to the others:

$$\phi^+(a) = \frac{1}{n-1} \sum_{x \in A} \pi(a, x) \quad (2)$$

$$\phi^-(a) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a) \quad (3)$$

where n represent the number of alternatives and x the generic alternative. Based on positive (2) and negative (3) flows, the partial classification according to PROMETHEE I is show in Table 1.

Using PROMETHEE II, ϕ^+ and ϕ^- flows are generally combined to obtain net flows defining a complete ranking. The net flow is defined as follows:

$$\phi(a) = \phi^+(a) - \phi^-(a) \quad (4)$$

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ϕ^+	ϕ^-	Classification
$\phi^+(a) \geq \phi^+(b)$	$\phi^-(a) < \phi^-(b)$	$a \succ b$
$\phi^+(a) > \phi^+(b)$	$\phi^-(a) \leq \phi^-(b)$	$a \succ b$
$\phi^+(a) = \phi^+(b)$	$\phi^-(a) = \phi^-(b)$	$a \sim b$
		a is incomparable to b in other cases

Table 1: Partial classification according to PROMETHEE I

The definition of ϕ allows, unlike flows ϕ^+ and ϕ^- , to compare all the alternatives with each other and to obtain a complete ranking of the alternatives. A deeper discussion of net flow scores can be found in Brans and Mareschal et al. [2008].

2.2 P-AHP

The P-AHP [Abastante et al., 2019] is an extension of the classic AHP method, which overcomes the limitations of classical AHP [Abastante et al., 2018], allowing you to analyze problems with a high number of alternatives by reducing the number of pairwise comparisons for the Decision Maker. Given a decision matrix M in which there are the evaluations $m_j(a_i)$ of each alternative a_i with respect to each criterion g_j , the DM (with the support of an analyst) identifies a set of reference points r_{jp_j} . In particular, for each criterion g_j the DM identifies a specific number of reference points $(r_{j1}, \dots, r_{jp_j})$ within the range of alternatives a_i taking into account the numerousness of the alternatives [Abastante et al., 2019].

Pairwise Comparison Matrices (PCM) are then constructed to allow the derivation of the criteria weights (w_j) and reference points weights $v_{r_{jp_j}}$; the respective weights are defined with the [Saaty, 1980] eigenvector method. At this point the consistency of the matrices is verified by analyzing the consistency index:

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

where λ_{max} is the maximum eigenvalue associated with the matrix and n represents the numerousness of the matrix [Saaty, 2008] [Abastante et al., 2019].

If the matrices present inconsistency problems then the evaluations will be reviewed by the DM [Vincke, 1981]. We proceed to derive the local priorities through the use of linear interpolation:

$$\psi(m_j(a_i)) = v_{r_{jp_j}} + \frac{v_{r_{jp_j+1}} - v_{r_{jp_j}}}{r_{jp_j+1} - r_{jp_j}}(d_j(a_i) - r_{jp_j}) \quad (5)$$

and finally the global priorities are defined as follows:

$$\Psi_i = \sum_{j=1}^n \psi(m_j(a_i))w_j \quad (6)$$

The definition of the global priority Ψ_i allows obtaining the final ranking of the alternatives, allowing ranking from most to least preferred.

3 Analysis of a real case

3.1 Description of PA

The Metropolitan City of Reggio Calabria, a PA located in southern Italy, provided us with a structural framework of the organization and functions of its services, allowing us to conduct an in-depth and analytical study on the evaluation of the performance of the services offered. The data provided describes in detail the skills and responsibilities of each Service, including the Offices within them and their related tasks. This documentation will form the basis for an accurate evaluation of the activities carried out by the administration and for the analysis of the evaluation criteria that could be applied to evaluate the performance of individual offices. The organizational structure of the Sector which we will call *Sector X*, for reasons of confidentiality, is shown in Figure 1.

The *Sector X* is led by a chief manager and supported by the Secretariat Office. It is divided into four Services:

- *Service A* is headed by a manager and is divided into two offices.
- *Service B* is headed by a manager holding an Organizational Position and is divided into four offices.
- *Service C* is headed by a manager and is divided into two offices.
- *Service D* is headed by a manager holding an Organizational Position and is divided into three offices.

Each Service has in charge specific functions, crucial to the operation and development of the public administration and each office is headed by a manager. Below are the main responsibilities assigned to each service:

- the u_{a1} office provides support to the Sole Procedural Manager (RUP) during tenders, assisting in the various phases of the procedure and in the preparation of the related documents;
- the u_{a2} office manages the Executive Management Plan (PEG), financing and finance;

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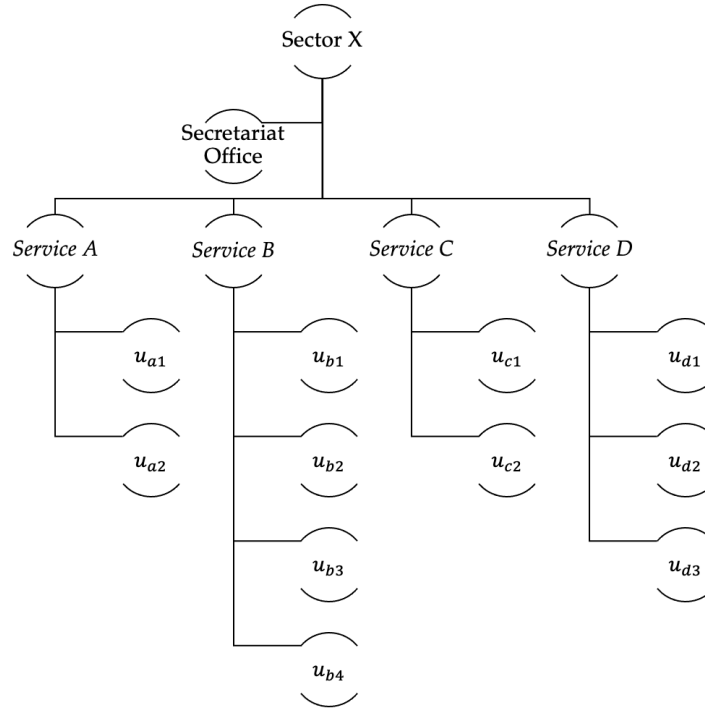


Figure 1: Organizational structure of the PA

- the u_{b1} office oversees the institutional, technical and administrative process of drafting and approving the Strategic Plan of the Metropolitan City (art. 44, paragraph 1, letter a, of Law 56/2014);
- the u_{b2} office oversees the institutional, technical and administrative process of approving variants of the Provincial Territorial Coordination Plan (PTCP) and the subsequent Territorial Plan of the Metropolitan City (PTCM) and its variants, as well as drawing up the annual list and the three-year program of public works (LL.PP.), and related updates, monitors the implementation status of the LL.PP. and evaluates the coherence of municipal planning instruments with the PTCP and subsequently with the PTCM; furthermore, it provides technical assistance to municipalities in the drafting of urban planning instruments and in verification activities.
- the u_{b3} office manages the issuing of landscape authorizations and verifies landscape compatibility in the areas subject to environmental restrictions, it also deals with the obligations relating to authorization procedures and the issuing of landscape opinions in compliance with the laws on building amnesty;
- the u_{b4} office coordinates and takes care of expropriation procedures,

carrying out administrative and, where necessary, technical functions referred to in the Consolidated Law. regarding expropriations for public utility (Presidential Decree no. 327/2001);

- the u_{c1} office manages urban reforestation projects and management plans for protected natural areas;
- the u_{c2} office deals with the drafting and management of Integrated Urban Plans and projects for the enhancement of the architectural and landscape heritage;
- the u_{d1} office deals with the preliminary investigation for the declaration of admissibility of the interventions, guaranteeing the regularity of the documentation and coordination between the Sole Procedural Managers (RUP). Furthermore, it instructs and controls the acts for programming and reprogramming of the Pact, preparing manuals and control tools to guarantee homogeneity in implementation;
- the u_{d2} office deals with controls, archiving of documentation and expense certification procedures;
- the u_{d3} office verifies the accounting regularity for the payment of the Beneficiaries' expenses, providing technical and accounting support to the ORs for the processing and transmission of data relating to the disbursement requests.

We would like to point out that the specific skills were assigned through an internal service order, which also defined the percentage of assignment of some employees to different offices. The DM who supported us in the construction of the methods and in the analysis of the PA's performance is a top figure within the PA. With the DM we decided that for a matter of privacy of the PA the services will be renamed $\{u_1, u_2, \dots, u_{11}\}$ and will be ordered randomly so that the corresponding service cannot be traced. The DM defined a set of criteria $G = \{g_1, g_2, \dots, g_j\}$ described below:

1. Location and physical accessibility (g_1): evaluates the physical location and accessibility of the Services and Offices;
2. Multi-channel accessibility (g_2): evaluates the availability of communication and assistance channels for citizens;
3. Timeliness (g_3): evaluates the readiness and efficiency in managing practices and procedures.

The DM has therefore provided us with the evaluation of the individual offices, expressed with a 10-point scale, widely validated in the literature [Dawes, 2008], [Loken et al., 1987] and [Kumar, 2018]. The evaluation are shown in the Table 2.

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	g_1	g_2	g_3
u_1	4	6	7
u_2	4	6	8
u_3	6	7	5
u_4	5	6	6
u_5	7	7	8
u_6	5	8	7
u_7	6	8	8
u_8	5	7	6
u_9	6	7	7
u_{10}	5	8	8
u_{11}	5	8	7

Table 2: Evaluations of the offices with respect to each criterion g_j

3.2 Performance evaluation with PROMETHEE I-II

The method described in the Section 2.1 was applied. For both methods application, the PCM was constructed to define the weight of the criteria; the preferences of the decision makers were expressed through the use of Saaty's semantic scale [Saaty, 2001]. The criterion weights are obtained by the right eigenvector method [Saaty, 2003]. The Table 3 shows the PCM for criteria.

	g_1	g_2	g_3	w_j
g_1	1	$\frac{1}{6}$	$\frac{1}{8}$	0.066
g_2	6	1	2	0.551
g_3	8	$\frac{1}{2}$	1	0.382
				Inc. 0.10

Table 3: PCM of criteria

At this point we have calculated the positive, negative and net flows for each office considered. The results are shown in Table 4.

The Partial Rankings obtained by applying PROMETHEE I are shown in Figure 2, where two bars are depicted: one on the left which represents the ranking of the shares according to the positive preference flow ϕ^+ , and one on the right which shows the ranking according to the negative preference flow ϕ^- . A line is drawn for each action between its ϕ^+ score on the left bar and its ϕ^- score on the right bar: when one line is above another it means that the action is preferred to the other, when two lines cross each other means that the actions are incomparable in PROMETHEE I.

The complete rankings obtained with PROMETHEE II are shown in Fig-

ϕ	ϕ^-	ϕ^+
0,6934	0,7000	0,0066
0,6405	0,6670	0,0264
0,3346	0,5140	0,1794
0,3346	0,5140	0,1794
0,2786	0,4992	0,2206
-0,0538	0,3264	0,3802
-0,2330	0,2677	0,5007
-0,3360	0,2169	0,5530
-0,3979	0,2117	0,6096
-0,5389	0,1147	0,6537
-0,7221	0,0515	0,7739

Table 4: Net, Positive and negative flows according to PROMETHEE I-II

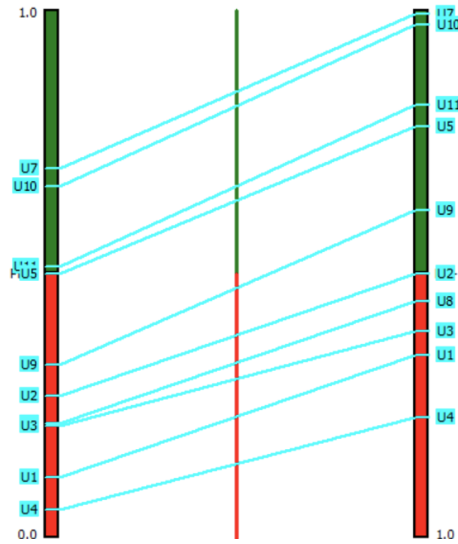


Figure 2: Partial Ranking with PROMETHEE I

Figure 3, which shows the complete rankings by displaying the ϕ scores on a vertical bar: the upper half of the scale (in green) corresponds to the positive scores and the lower half (in red) to the negative scores. From the results obtained we can see that the best performing office is u_7 , followed by the offices u_{10} , u_{11} and u_6 in equal merit and followed by u_5 . Among the least performing offices we find u_4 , u_1 and u_3 in the last three positions.

With the use of the VisualPROMETHEE¹ software we have also defined the efficient frontier which is a graphical representation of the best possi-

¹<https://visual-promethee.software.informer.com/1.4/>

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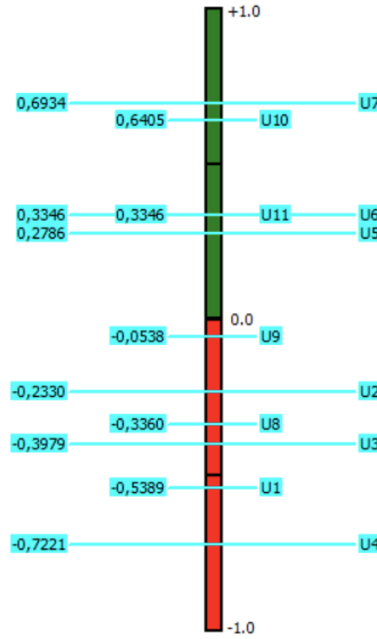


Figure 3: Complete Ranking with PROMETHEE II

ble performance in an Input/Output optimization problem. It represents the maximum boundary of performance that can be achieved, given a given set of actions and evaluation criteria, helping decision makers to identify the optimal options, or those that come closest to them, in their decision. In this context, Input/Output efficiency refers to the ability of an Office to obtain desirable results (output) compared to the resources used (input). We show in Figure 4 the efficient frontier drawn in red through the points representing the actions in the two-dimensional graph of Inputs and Outputs. The actions that are on the efficient frontier are considered efficient as they maximize the output for a given level of input, therefore representing optimal solutions or at least comparable to each other.

In our analysis only u_7 is located directly on the efficient frontier: this indicates that it is maximizing output for the level of input used, making it an efficient and highly competitive option compared to other offices. On the other hand, the other Offices are all far from the border, which suggests that they could improve their performance in terms of output compared to the resources used. The u_{10} is located very close to u_7 and just below the efficient frontier: this position indicates that it has relatively good performance.

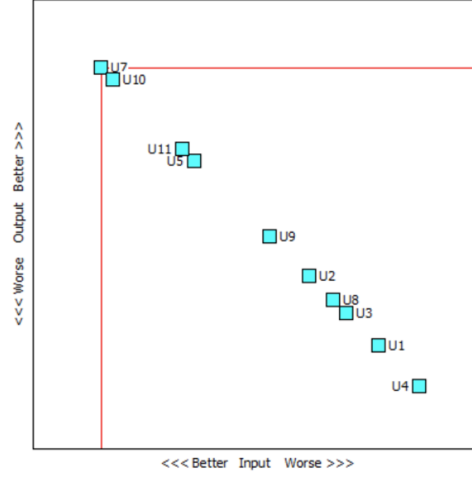


Figure 4: Efficient frontier

3.3 Performance evaluation with P-AHP

In order to be able to apply the P-AHP method, the reference points r_{jp_j} have been defined for each criterion g_j ; in particular 6 were identified as the reference points which for greater ease of reading were renamed $\{RP_1, \dots, RP_6\}$; the values are shown in Table 5.

	g_1	g_2	g_3
RP_1	1	1	1
RP_2	3	3	3
RP_3	4	4	4
RP_4	6	6	6
RP_5	7	7	7
RP_6	10	10	10

Table 5: Reference points r_{jp_j} for each criterion g_j

The PCMs were then constructed to obtain the weights of the criteria (Table 3) and reference points. All PCMs provided by the DM have an acceptable CI . The weights of the reference points are reported in Table 6. The local priorities $\psi(m_j(u_i))$ for each office respect each criterion g_j were then calculated using the formula (5) as well as the global priorities Ψ_i using the formula (6). The results are reported in the Table 7.

As we can see, the best performing offices are u_7 , u_{10} , u_{11} and u_6 in equal merit and followed by u_5 . Among the least performing offices we find u_4 , u_1 and u_3 in the last three positions.

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	$v_{r_{jP_j}}(g_1)$	$v_{r_{jP_j}}(g_2)$	$v_{r_{jP_j}}(g_3)$
RP_1	0.032	0.028	0.043
RP_2	0.056	0.053	0.064
RP_3	0.099	0.104	0.101
RP_4	0.170	0.163	0.160
RP_5	0.251	0.267	0.250
RP_6	0.391	0.385	0.382

Table 6: Weights of the reference points $v_{r_{jP_j}}(g_j)$

	$\psi(g_1)$	$\psi(g_2)$	$\psi(g_3)$	Ψ_i
u_1	0.099	0.163	0.250	0.192
u_2	0.099	0.163	0.294	0.209
u_3	0.170	0.267	0.131	0.208
u_4	0.135	0.163	0.160	0.160
u_5	0.089	0.267	0.294	0.265
u_6	0.135	0.306	0.250	0.273
u_7	0.170	0.306	0.294	0.292
u_8	0.135	0.267	0.160	0.217
u_9	0.170	0.267	0.250	0.254
u_{10}	0.135	0.306	0.294	0.290
u_{11}	0.135	0.306	0.250	0.273

Table 7: Local and Global priority for each office u

3.4 Sensitivity analysis

In order to validate the results obtained and the stability of the two methods applied, we conducted a sensitivity analysis by decreasing each criterion considered in the analysis by 20% with the consequent increase in all the other criteria; e.g.: g_1 decreases by 20% while g_2 and g_3 increase by 20%. The offices rankings are shown in the Tables 8 and 9.

decrease g_1	decrease g_2	decrease g_3
u_7	u_7	u_7
u_{10}	u_{10}	u_{10}
u_6	u_6	u_5
u_{11}	u_{11}	u_6
u_5	u_5	u_{11}
u_9	u_9	u_9
u_8	u_8	u_2
u_3	u_3	u_8
u_2	u_2	u_1
u_1	u_1	u_3
u_4	u_4	u_4

Table 8: Sensitivity analysis of P-AHP

decrease g_1	decrease g_2	decrease g_3
u_7	u_7	u_7
u_{10}	u_{10}	u_{10}
u_6	u_6	u_6
u_{11}	u_{11}	u_{11}
u_5	u_5	u_5
u_9	u_9	u_9
u_8	u_2	u_2
u_3	u_8	u_8
u_2	u_3	u_3
u_1	u_1	u_1
u_4	u_4	u_4

Table 9: Sensitivity analysis of PROMETHEE I-II

As we can see from the rankings obtained, for the P-AHP method just small changes are present when g_3 decreases w.r.t. the decrease of the other criteria. For example, u_5 is ranked over u_6 and u_{11} , differently from the ranking in Table 7. Also for the PROMETHEE I-II method there are small

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ranking shifts for the offices u_2 , u_8 and u_3 ; in any case, the top positions remain stable in both the cases.

4 Discussion

As we have seen from the results obtained from the application of the PROMETHEE I-II (see Section 3.2) method and the P-AHP (see Section 3.3) method, both the approaches report the same ordering with the exception only of the ranking of the offices u_2 and u_8 , which have undergone a ranking reversal. We point out that although the two methods applied use completely different methodological approaches, one an outranking method (PROMETHEE I-II) and the other a hierarchical method (P-AHP), both the approaches allowed us to obtain a performance of the offices from the most to the least performing, a clear benefit from a managerial insight standpoint. Obtaining these results demonstrates that both methods are valid for addressing performance evaluation problems. The choice of method by Public Administrations, at this point, is limited to the consideration of the characteristics of the methods themselves and the pros or cons of using one method rather than another. In particular, we remind that both methods allow to use qualitative and quantitative criteria and both define a valid decision support system even though they use different tools (see Section 2). Among the cons of the PROMETHEE I-II method we remember that rank reversal problems may occur, that transitivity may be violated and finally incomparability is admitted. The P-AHP method does not present rank reversal problems but among the disadvantages we remember the compensation between good and bad scores and that the transitivity may be violated. In Table 10, we report the principal pros and cons of the methods.

5 Conclusion

This paper provided the use of multi-criteria methods in the evaluation of Public Administration services. The results that emerged can guide public sector operators in optimizing the services offered by promoting transparency, efficiency and effectiveness in the provision of public services. With the results obtained using the PROMETHEE I-II and P-AHP methods, we defined a detailed framework of the performance of the offices within the Public Administration under examination. The analysis of the PROMETHEE I-II flows made it possible to determine the partial and complete rankings of the alternatives, highlighting the preferences of the decision maker. The visualization of the efficient frontier identified the offices considered efficient and those that could improve their performance. The P-AHP method allowed the use of reference points to reduce the number

	PROMETHEE I-II	P-AHP
Pros	a. Large number of alternatives and criteria b. Non-compensation between good and bad scores	a. Large number of alternatives and criteria b. Preventing the rank reversal problem
Cons	a. Loss of some information in defining the final sort b. Incomparability is allowed between alternatives c. Transitivity may be violated	a. Compensation between good and bad scores b. Transitivity may be violated

Table 10: Pros and Cons of PROMETHEE I-II and P-AHP,

of pairwise comparisons for the DM. The use of linear interpolation also made it possible to compare the representative points with the office assessments provided by the PA. Both methods provided the same ranking of offices except for a reverse ordering for offices u_2 and u_8 . This result demonstrates that both methods are effective for analyzing the performance of a Public Administration. The application of the PROMETHEE I-II and P-AHP methods therefore provided an exhaustive and in-depth evaluation of the alternatives considered, allowing decision makers to make informed and rational decisions in a complex context such as that of evaluating the performance of offices within the Public Administration.

Future studies could concern the use of further multi-criteria methods and their integration with optimization methods, like for example knapsack-like or capital budgeting formulations, for the definition of corrective actions to improve the performance of PA offices.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

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Strategic Timing in Financial Markets: Real Options Analysis on American Options

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Abstract

This paper studies the intrinsic dynamics of American options within the context of real options, emphasizing their practical impact to the financial decision-making. Real options, which encompasses strategies such as to expand, to abandon, to wait, to switch, and to contract, provide a flexible methodology to address the early exercise feature that complexify traditional assessment models. Our research uses advanced techniques, including subsolutions and supersolutions with comparison arguments, analysis of blow-up phenomena for extreme initial data, consistency of viscosity solutions, the Hopf-Lax formula, and Monte Carlo simulations, to derive explicit formulations for the optimal stopping time and other important results. A key contribution is the improvement of the switching principle, that governs optimal stopping decisions, to offer an actionable insights into the valuation of American options in the context of real options to mitigate operational risks. These findings bridge the gap between theoretical and realistic problems, enhancing the understanding of the role of real options in pricing strategies within real-world financial markets.

Keywords: Optimal Stopping Time, American Call and Put Options; Real Options; Monte Carlo; viscosity solutions; Hopf-Lax formula; Monte Carlo simulations; switching principle.

2020 AMS subject classifications: 91G50; 91-10; 35A01; 60H30; 62P05

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1 Introduction

Valuing financial options, particularly American call and put options, presents significant challenges in mathematical finance due to their inherent flexibility. Unlike European options, American options permit exercise at any time before expiration, introducing the crucial element of optimal timing into their valuation. Traditional models, such as the Black-Scholes framework [1], are insufficient for capturing this complexity, thereby necessitating the development of alternative methodologies like real options analysis.

Real options, initially formulated for capital investment decisions, have become instrumental in valuing financial options by treating them as strategic investment opportunities analogous to real assets. These options are typically categorized into various types, including the option to expand, abandon, wait, switch, and contract. This classification allows for a comprehensive approach to option pricing by incorporating the flexibility to adapt to evolving market conditions [2]. Central to this framework is the concept of the optimal stopping time, which determines the most advantageous moment to exercise the option, thereby enhancing the accuracy of American option valuations.

The theoretical foundations of optimal stopping have been extensively explored, beginning with Snell's introduction of the Snell envelope [3] and Dynkin's formalization of the relationship between stopping times and supermartingales [4]. While Black and Scholes [1] laid the groundwork for option pricing theory, their models were limited to European options without early exercise features. Subsequent advancements by Longstaff and Schwartz [5] introduced numerical techniques for American options, yet these methods often lack analytical tractability and may not fully encapsulate the dynamics of decision-making processes.

Recent research has increasingly focused on integrating real options analysis with the pricing of American options, emphasizing the significance of managerial flexibility and uncertainty. Pioneering work by Trigeorgis [2] highlighted the value of incorporating managerial discretion into option pricing models, inspiring further developments in mathematical models and numerical algorithms aimed at deriving analytical solutions for optimal stopping times and option values [6, 7]. Despite these strides, challenges remain in creating a unified framework that simultaneously addresses the convergence of stopping times under various real options scenarios and in enhancing the regularity of solutions to the associated Hamilton-Jacobi-Bellman (HJB) equations [8, 9].

Moreover, the interplay between stochastic control theory and fundamental measure-

theoretic results has been underexplored. Classical theorems such as the Martingale Convergence Theorem [11], the Radon-Nikodým Theorem [12], and the Kolmogorov Extension Theorem [13] are pivotal in probability and measure theory but have traditionally been addressed within their specialized contexts. This paper seeks to bridge this gap by employing the switching principle—a method of segmenting processes and refining partitions—to provide streamlined proofs of these foundational theorems within the stochastic control and HJB framework.

A significant contribution of this work is the establishment of higher interior regularity theorems for solutions to the HJB equation. Building on classical Partial Differential Equation techniques such as Schauder and Krylov-Safonov estimates [10], we demonstrate that under conditions of sufficient smoothness and uniform ellipticity of the coefficients, the value function exhibits enhanced regularity properties in the interior domain, despite the complexities introduced by free-boundary conditions inherent in optimal stopping and switching problems.

This paper advances the theoretical understanding of optimal stopping and switching control problems. By addressing the convergence of optimal stopping times within a comprehensive switching framework and leveraging the switching principle to establish classical measure-theoretic results, we provide robust tools applicable to dynamic decision-making under uncertainty in various practical scenarios.

2 Mathematical Formulations

2.1 Definitions

2.1.1 Underlying Asset and Option Parameters

Let:

- S_t : Price of the underlying asset at time t ,
- T : Expiration time of the option,
- K : Strike price,
- r : Risk-free interest rate.

2.1.2 Payoff Functions

Definition 2.1 (Payoff Functions for American Options). *The payoff functions for an American call option (C) and an American put option (P) at time t are defined as:*

$$C(t) = \sup_{\tau \in [t, T]} \mathbb{E} \left[e^{-r(\tau-t)} (S_\tau - K)^+ \right], \quad (1)$$

$$P(t) = \sup_{\tau \in [t, T]} \mathbb{E} \left[e^{-r(\tau-t)} (K - S_\tau)^+ \right], \quad (2)$$

where $(\cdot)^+$ denotes the positive part, K is the strike price, and r is the risk-free interest rate.

2.1.3 Optimal Stopping Time

Definition 2.2 (Optimal Stopping Time). *The optimal stopping time τ^* is the solution to the stochastic control problem:*

$$\tau^* = \arg \max_{\tau \in [t, T]} \mathbb{E} \left[e^{-r(\tau-t)} g(S_\tau) \right], \quad (3)$$

where $g(S_\tau)$ represents the payoff function of the option.

2.1.4 Hamilton-Jacobi-Bellman (HJB) Equation

Definition 2.3 (Hamilton-Jacobi-Bellman (HJB) Equation). *The value function $V(t)$ satisfies the HJB equation:*

$$\frac{\partial V}{\partial t} + \sup_{\tau \in [t, T]} \left\{ \frac{\partial V}{\partial S} f(S, \tau) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2(S, \tau) - rV \right\} = 0, \quad (4)$$

where:

- $f(S, \tau)$: Drift term,
- $\sigma(S, \tau)$: Diffusion term,
- $\mathcal{L}$: Infinitesimal generator of the process S_t .

2.1.5 Advanced Theorems

Definition 2.4 (Radon-Nikodym Theorem). *If ν is absolutely continuous with respect to μ (denoted $\nu \ll \mu$), there exists a measurable function f such that:*

$$\nu(A) = \int_A f d\mu \quad \text{for all measurable sets } A, \quad (5)$$

where $f = \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative.

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Definition 2.5 (Kolmogorov Extension Theorem). *The Kolmogorov Extension Theorem provides conditions under which a consistent family of finite-dimensional distributions defines a unique probability measure on the space of all sample paths. For a stochastic process $X = \{X_t : t \in T\}$:*

$$\text{Given consistency: } P(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n), \quad (6)$$

there exists a unique probability measure P on the path space satisfying these finite-dimensional distributions.

Definition 2.6 (Martingale Convergence Theorem). *If $\{M_t\}$ is a martingale that is bounded in L^1 (or uniformly integrable), then there exists a random variable M_∞ such that:*

$$M_t \xrightarrow{a.s.} M_\infty \quad \text{as } t \rightarrow \infty. \quad (7)$$

Moreover, $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$, preserving the expectation.

2.1.6 Real Options

Definition 2.7 (Option to Expand). *The option to expand allows a firm to invest additional resources to scale up operations:*

$$\text{Value of Expansion Option} = \max [\text{Current Value}, \mathbb{E}[e^{-r(T-t)}(S_T - K)]] . \quad (8)$$

Definition 2.8 (Option to Abandon). *The option to abandon permits a firm to exit a project and minimize losses:*

$$\text{Value of Abandonment Option} = \max [\text{Exit Value}, \mathbb{E}[e^{-r(T-t)}(K - S_T)]] . \quad (9)$$

Definition 2.9 (Option to Wait). *The option to wait postpones decisions to gather more information:*

$$\text{Value of Waiting Option} = \sup_{\tau \in [t, T]} \mathbb{E} [e^{-r(\tau-t)} g(S_\tau)] . \quad (10)$$

Definition 2.10 (Option to Switch). *The option to switch allows a firm to change between operational strategies:*

$$\text{Value of Switching Option} = \max_{\text{Strategies } j} \mathbb{E}[e^{-r(T-t)}(S_T - K_j)] . \quad (11)$$

Definition 2.11 (Option to Contract). *The option to contract enables a firm to scale down operations:*

$$\text{Value of Contraction Option} = \max [\text{Reduced Operations Value}, \mathbb{E}[e^{-r(T-t)}(K - S_T)]] . \quad (12)$$

2.1.7 Viscosity Solutions

Definition 2.12 (Viscosity Solution). *A viscosity solution is a generalized solution to a partial differential equation (PDE) where the notion of differentiability is replaced by a local comparison principle:*

$$\begin{aligned} \text{Supersolution:} \quad & F(x, u, Du, D^2u) \geq 0, \\ \text{Subsolution:} \quad & F(x, u, Du, D^2u) \leq 0. \end{aligned}$$

2.1.8 Hopf-Lax Formula

Definition 2.13 (Hopf-Lax Formula). *The Hopf-Lax formula provides the solution to a Hamilton-Jacobi equation:*

$$u(t, x) = \inf_{y \in \mathbb{R}^n} \left\{ u(0, y) + tL \left(\frac{x - y}{t} \right) \right\}, \quad (13)$$

where L is the Legendre transform of the Hamiltonian.

2.1.9 Monte Carlo Simulations

Definition 2.14 (Monte Carlo Simulations). *Monte Carlo simulations are numerical methods used to estimate the expected value of stochastic processes by sampling:*

$$\mathbb{E}[f(X)] \approx \frac{1}{N} \sum_{i=1}^N f(X_i), \quad (14)$$

where X_i are independent samples from the distribution of X .

2.1.10 Switching Principle

Definition 2.15 (Switching Principle). *The switching principle allows dynamic transitions between operational states to maximize value:*

$$V(t, x) = \max\{V_1(t, x), V_2(t, x), \dots, V_m(t, x)\}, \quad (15)$$

where V_i corresponds to the value of each operational state.

3 Main Results

Theorem 3.1 (Convergence of Optimal Stopping Times). *Consider the controlled stochastic differential equation*

$$dX_t = b(t, X_t, \alpha_t) dt + \sigma(t, X_t, \alpha_t) dW_t, \quad (16)$$

where the control α_t takes values in a set A (finite or compact), and let τ be a stopping time. Assume there is a running payoff f , a terminal payoff Φ , a discount rate $r > 0$, and a switching cost $\Gamma(\alpha_{t-}, \alpha_t)$. Define the value function

$$V(t, x) = \sup_{\alpha, \tau \geq t} \mathbb{E} \left[\int_t^\tau e^{-r(s-t)} f(s, X_s, \alpha_s) ds + e^{-r(\tau-t)} \Phi(\tau, X_\tau) - \sum_{\substack{\text{switch times} \\ t_k \in [t, \tau]}} \Gamma(\alpha_{t_k^-}, \alpha_{t_k^+}) \right]. \quad (17)$$

Then there exists a control α^* and a stopping time τ^* (optimal or ε -optimal) such that for any minimizing (or maximizing) sequence $\{(\alpha^n, \tau_n)\}$, one has $\tau_n \rightarrow \tau^*$ almost surely, and $\alpha^n \rightarrow \alpha^*$ in an appropriate sense.

Theorem 3.2 (Theorem 3.2 (Switching Principle and Measure-Theoretic Theorems)). *The “switching principle” (segmenting processes/measures on finite intervals or partitions, then refining) implies the following classical results:*

- *Martingale Convergence Theorem (for nonnegative or L^1 -bounded martingales).*
- *Radon–Nikodým Theorem (existence and uniqueness of densities).*
- *Kolmogorov Extension Theorem (construction of measures on infinite products).*

Theorem 3.3 (Theorem 3.3 - Higher Interior Regularity for HJB Solutions). *Let $V(t, x)$ solve the Hamilton-Jacobi-Bellman (HJB) obstacle/free-boundary-type equation:*

$$\max \{-\partial_t V - \mathcal{L}[V] - f, \Phi - V\} = 0, \quad (18)$$

where

$$\mathcal{L}[V] = \sum_{i,j=1}^d a_{ij}(t, x) \partial_{x_i x_j} V + \sum_{i=1}^d b_i(t, x) \partial_{x_i} V, \quad (19)$$

and the coefficients a_{ij}, b_i, f, Φ are sufficiently smooth with σ uniformly elliptic. Then V possesses higher-order interior derivatives, specifically $V \in C_{loc}^{2,\alpha}$ in space and $C_{loc}^{1,\alpha/2}$ in time, up to the free boundary.

Lemma 3.1 (Discrete Approximation for Switching/Stopping). *Let $\Delta t = \frac{T-t}{n}$ for $n \in \mathbb{N}$, and define $t_k = t + k \Delta t$ for $k = 0, 1, \dots, n$. Set*

$$V_n(n, x) = \Phi(T, x),$$

$$V_n(k, x) = \max \left\{ \Phi(t_k, x), \max_{a \in A} \left[f(t_k, x, a) \Delta t + e^{-r \Delta t} \mathbb{E} \left[V_n(k+1, X_{t_{k+1}}) \mid (X_{t_k} = x, a) \right] - \Gamma(\dots) \right] \right\}.$$

(20)

Then, for each (k, x) , $V_n(k, x) \rightarrow V(t_k, x)$ as $n \rightarrow \infty$, where $V(\cdot, \cdot)$ is the continuous-time value function.

Proof. For any discrete control $\{\alpha_{t_\ell}\}_{\ell=k}^{n-1}$ and discrete stopping index $\kappa \in \{k, k+1, \dots, n\}$, the corresponding discrete payoff from k to κ is no larger than the continuous-time payoff from t_k to t_κ . Hence

$$V_n(k, x) \leq V(t_k, x). \quad (21)$$

Take any continuous-time policy (α_s, τ) from (t_k, x) . Construct a piecewise-constant approximation $\alpha_{t_\ell}^n$ by setting $\alpha_{t_\ell}^n = \alpha_s$ for $s \in [t_\ell, t_{\ell+1})$. Approximate τ by $\kappa_n = \min\{\ell \geq k : t_\ell \geq \tau\}$. The discrete payoff converges to the continuous payoff as $\Delta t \rightarrow 0$, which implies

$$V(t_k, x) \leq \liminf_{n \rightarrow \infty} V_n(k, x). \quad (22)$$

Combining both bounds yields

$$V_n(k, x) \leq V(t_k, x) \leq \liminf_{m \rightarrow \infty} V_m(k, x), \quad (23)$$

so $V_n(k, x) \rightarrow V(t_k, x)$ as $n \rightarrow \infty$.

Hence the discrete-time value functions converge pointwise (on each time-space grid) to the continuous-time value function:

$$V_n(k, x) \rightarrow V(t_k, x) \quad \text{as } n \rightarrow \infty, \quad \text{proving Lemma 3.1.}$$

□

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Lemma 3.2 (Snell Envelope Supermartingale). *Let $\{G_t\}_{t \geq 0}$ be an adapted process. Define for each $t \geq 0$:*

$$Y_t := \sup_{\tau \geq t} \mathbb{E}[G_\tau \mid \mathcal{F}_t]. \quad (24)$$

Then $\{Y_t\}_{t \geq 0}$ is a supermartingale, i.e., for any $0 \leq s < t$,

$$Y_s \geq \mathbb{E}[Y_t \mid \mathcal{F}_s]. \quad (25)$$

Proof. Fix $s < t$. For any stopping time $\tau \geq t$,

$$\mathbb{E}[G_\tau \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[G_\tau \mid \mathcal{F}_t] \mid \mathcal{F}_s] \leq \mathbb{E}[Y_t \mid \mathcal{F}_s], \quad (26)$$

since

$$Y_t = \sup_{\tau' \geq t} \mathbb{E}[G_{\tau'} \mid \mathcal{F}_t] \geq \mathbb{E}[G_\tau \mid \mathcal{F}_t]. \quad (27)$$

Taking the supremum over $\tau \geq t$,

$$\sup_{\tau \geq t} \mathbb{E}[G_\tau \mid \mathcal{F}_s] \leq \mathbb{E}[Y_t \mid \mathcal{F}_s]. \quad (28)$$

On the other hand,

$$Y_s = \sup_{\tau \geq s} \mathbb{E}[G_\tau \mid \mathcal{F}_s] \geq \sup_{\tau \geq t} \mathbb{E}[G_\tau \mid \mathcal{F}_s]. \quad (29)$$

Hence

$$Y_s \geq \sup_{\tau \geq t} \mathbb{E}[G_\tau \mid \mathcal{F}_s] \geq \mathbb{E}[Y_t \mid \mathcal{F}_s]. \quad (30)$$

This shows that $\{Y_t\}_{t \geq 0}$ is a supermartingale. $\square$

Lemma 3.3 (Local Parabolic Schauder Estimates). *Suppose u solves*

$$\partial_t u + \sum_{i,j} a_{ij}(t, x) \partial_{x_i x_j}^2 u + \sum_i b_i(t, x) \partial_{x_i} u = g(t, x) \quad (31)$$

in a parabolic cylinder $Q_1 \subset \mathbb{R}^{d+1}$. Assume:

1. $\lambda I \leq (a_{ij}) \leq \Lambda I$ (uniform ellipticity).
2. a_{ij}, b_i, g are C^α in (t, x) .

Then in $Q_{1/2} \subset Q_1$, we have

$$u \in C_x^{2,\alpha}(Q_{1/2}) \cap C_t^{1,\alpha/2}(Q_{1/2}), \quad (32)$$

with an estimate

$$\|u\|_{C_x^{2,\alpha}, C_t^{1,\alpha/2}(Q_{1/2})} \leq C \left(\|u\|_{L^\infty(Q_1)} + \|g\|_{C^\alpha(Q_1)} \right), \quad (33)$$

for some C depending on $\lambda, \Lambda, d, \alpha$.

Proof. Schauder approach, deriving Hölder estimates.

$$\partial_t u - \nabla \cdot (a(x, t) \nabla u) = g(x, t) \quad \text{in } Q_1 \subset \mathbb{R}^{d+1}, \quad (34)$$

where:

1. $\lambda I \leq a(x, t) \leq \Lambda I$ (uniform ellipticity).
2. $a(\cdot), g(\cdot) \in C^\alpha(Q_1)$.
3. Q_1 is a parabolic cylinder $\{(x, t) : |x| \leq 1, |t| \leq 1\}$.

Let prove $u \in C_x^{2,\alpha} \cap C_t^{1,\alpha/2}$ in $Q_{1/2}$.

Recall the following definitions to set up the Campanato–Morrey Framework :

Parabolic Cylinder:

$$Q_r(z_0) := \{(x, t) : |x - x_0| \leq r, |t - t_0| \leq r^2\}. \quad (35)$$

Mean Value:

$$u_{Q_r(z_0)} := \frac{1}{|Q_r(z_0)|} \int_{Q_r(z_0)} u \, dX, \quad (36)$$

where dX is space–time measure.

Campanato Seminorm:

$$[u]_{\alpha, \alpha/2; Q_1} := \sup_{0 < r \leq 1} \sup_{Q_r(z) \subset Q_1} \frac{1}{r^{d+2\alpha}} \int_{Q_r(z)} |u - u_{Q_r(z)}| \, dX. \quad (37)$$

If finite, $u \in C^{\alpha, \alpha/2}(Q_1)$ by Morrey's lemma.

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Next, let $\tilde{u}(x, t) := u(x, t) - u_{Q_r(z)}$. Then $\tilde{u}_{Q_r(z)} = 0$. Partial Differential Equation for $\tilde{u}$:

$$\partial_t \tilde{u} - \nabla \cdot (a \nabla \tilde{u}) = g - \bar{g}, \quad \bar{g} := g_{Q_r(z)}. \quad (38)$$

Multiply Partial Differential Equation by $\tilde{u}$, integrate over $Q_r(z)$:

$$\int_{Q_r(z)} \tilde{u} \partial_t \tilde{u} dX - \int_{Q_r(z)} \nabla \tilde{u} \cdot (a \nabla \tilde{u}) dX = \int_{Q_r(z)} \tilde{u} (g - \bar{g}) dX. \quad (39)$$

Rewrite:

1. *Time derivative term:*

$$\int_{Q_r(z)} \tilde{u} \partial_t \tilde{u} = \frac{1}{2} \int_{Q_r(z)} \partial_t (\tilde{u}^2) = \frac{1}{2} \left(\int_{\Sigma_r^+} \tilde{u}^2 - \int_{\Sigma_r^-} \tilde{u}^2 \right), \quad (40)$$

where $\Sigma_r^\pm$ are space slices at $t_0 \pm r^2$.

2. *Ellipticity term:*

$$\int_{Q_r(z)} \nabla \tilde{u} \cdot (a \nabla \tilde{u}) \geq \lambda \int_{Q_r(z)} |\nabla \tilde{u}|^2. \quad (41)$$

3. *Right side:*

$$\int_{Q_r(z)} \tilde{u} (g - \bar{g}) \leq \|\tilde{u}\|_{L^2(Q_r)} \|g - \bar{g}\|_{L^2(Q_r)} + \dots \quad (42)$$

Furthermore, let use the Isoperimetric Inequality in Parabolic Cylinders:

For $Q_r(z) \subset Q_1$:

1. Volume $\approx r^{d+2}$.

2. Isoperimetric form: if $A \subset Q_r$, then

$$|A|^{\frac{d-1}{d}} \lesssim |\partial A|. \quad (43)$$

3. Applied with parabolic layer arguments, boundary integrals are order r^{d+1} .

This bounds boundary terms in the time derivative integral by $\|\tilde{u}\|_{L^2}$ inside Q_r .

Hence, to consider the Oscillation Estimate,

Combine:

- Coercivity $\int |\nabla \tilde{u}|^2$.
- Boundary measure $\lesssim r^d$.
- $\tilde{u}$ average zero in $Q_r(z)$.

Hence:

$$\int_{Q_r(z)} |\tilde{u}| \leq C (\|u\|_{L^2(Q_r)} + \|g\|_{L^p(Q_r)}) r^\theta, \quad (44)$$

for some $\theta > 0$. Factor out $\frac{1}{r^{d+2\alpha}}$. Obtain

$$\frac{1}{r^{d+2\alpha}} \int_{Q_r(z)} |u - u_{Q_r(z)}| \leq (\text{constants}) + \|g\|_{C^\alpha} (\text{small factor}). \quad (45)$$

Thus the Campanato seminorm $[u]_{\alpha, \alpha/2; Q_1}$ is finite. This implies $u \in C^{\alpha, \alpha/2}(Q_1)$.

To test the Caffarelli–Type Decay (Blow–Up),

First, let define blow-up

$$\bar{u}(y, \tau) = \frac{u(x_0 + ry, t_0 + r^2\tau) - u_{Q_r(z_0)}}{r^\gamma}. \quad (46)$$

Choose γ to normalize. In Q_1 for (y, τ) , PDE becomes

$$\partial_\tau \bar{u} - \nabla_y \cdot (\bar{a}(y, \tau) \nabla_y \bar{u}) = \bar{g}(y, \tau). \quad (47)$$

If $\|a - a(x_0, t_0)\| \leq \epsilon$ in $Q_r(z_0)$, similarly for g , then $\bar{u}_r$ has small oscillation in Q_1 , i.e. $\bar{u}$ satisfies nearly constant–coefficient PDE with small inhomogeneity.

Next, check the Decay of Oscillation:

$$\text{osc}_{Q_{1/2}}(\bar{u}) \leq \theta \text{osc}_{Q_1}(\bar{u}), \quad 0 < \theta < 1. \quad (48)$$

Back to u :

$$\text{osc}_{Q_{r/2}}(u) \leq \theta \text{osc}_{Q_r}(u). \quad (49)$$

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Iterate $\text{osc}_{Q_{r_2-k}}(u) \leq \theta^k \text{osc}_{Q_r}(u)$. As $k \rightarrow \infty$, get Hölder continuity of u .

Repeated application $\implies$ geometric decay $\implies$ Hölder continuity of u

Finally,

Define

$$D_h u(x, t) := \frac{u(x + h, t) - u(x, t)}{|h|}. \quad (50)$$

PDE for $D_h u$:

$$\partial_t(D_h u) - \nabla \cdot (a(x, t) \nabla(D_h u)) = D_h g + \dots \quad (51)$$

Apply the same Campanato–Morrey + scaling argument. Conclude $D_h u \in C^{\alpha, \alpha/2}$. Hence $\nabla_x u \in C^{\alpha, \alpha/2}$. Repeating yields $\partial_{x_i x_j}^2 u \in C^{\alpha, \alpha/2}$.

From PDE:

$$\partial_t u = \nabla \cdot (a(x, t) \nabla u) + g(x, t). \quad (52)$$

All terms on right are $C^{\alpha, \alpha/2}$. By parabolic scaling $\partial_t u$ is $\alpha/2$ -Hölder in t , it means $\partial_t u \in C_t^{\alpha/2}$.

Ultimatly,

- Campanato–Morrey + isoperimetric bounds $\implies$ interior Hölder continuity of u .
- Caffarelli decay $\implies$ refined oscillation control.
- Difference quotients $\implies$ second derivatives in space are Hölder continuous.
- Time derivative from PDE $\implies \partial_t u \in C_t^{\alpha/2}$.

Combining the above:

- $\nabla_x^2 u \in C^{\alpha, \alpha/2}$,
- $\partial_t u \in C^{\alpha, \alpha/2}$.

Hence

u is in $C_x^{2,\alpha}$ and $C_t^{1,\alpha/2}$ (interior)

$$u \in C_x^{2,\alpha}(Q_{1/2}) \cap C_t^{1,\alpha/2}(Q_{1/2}), \quad (53)$$

with estimate

$$\|u\|_{C_x^{2,\alpha}, C_t^{1,\alpha/2}(Q_{1/2})} \leq C \left(\|u\|_{L^\infty(Q_1)} + \|g\|_{C^\alpha(Q_1)} \right).$$

□

4 Application in Finance: American/Real Options

We consider a time-horizon T and an underlying asset S_t . For an American-style option with payoff $\Phi(t, S)$, the continuation region $\{\Phi < V\}$ satisfies a linear parabolic PDE:

$$\partial_t V(t, S) + \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 V + rS \partial_S V - rV = 0 \quad \text{in } \{\Phi < V\},$$

or, more generally, an HJB-type equation:

$$\partial_t V + a(t, S) \partial_{SS}^2 V + b(t, S) \partial_S V + c(t, S) V + f(t, S) = 0,$$

under uniform parabolicity $a(t, S) \geq \lambda > 0$. On the exercise boundary $\partial\{\Phi < V\}$, we have $V = \Phi$.

Local PDE in the Continuation Region

In $\{\Phi < V\}$, set

$$\mathcal{L}V = \sum_{i,j} a_{ij}(t, S) \partial_{S_i S_j}^2 V + \sum_i b_i(t, S) \partial_{S_i} V + c(t, S) V.$$

Then

$$\partial_t V + \mathcal{L}V + f = 0.$$

Focus on a small parabolic cylinder $Q_r(t_0, S_0) \subset \{\Phi < V\}$. Let $u := V$ restricted to Q_r .

Freezing Coefficients Near (t_0, S_0)

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- Define $\bar{a}(t_0, S_0) := a(t_0, S_0)$, $\bar{b}(t_0, S_0)$, etc.
- Construct $\tilde{u}$ solving

$$\partial_t \tilde{u} + \bar{a} \partial_{SS}^2 \tilde{u} + \bar{b} \partial_S \tilde{u} + \bar{c} \tilde{u} = \tilde{f},$$

with $\tilde{f}(t, S) \approx f(t_0, S_0)$.

- Compare $u - \tilde{u}$ in Q_r .

Difference Quotients

- Let

$$D_h u(t, S) := \frac{u(t, S+h) - u(t, S)}{h}, \quad h \neq 0.$$

- PDE for $D_h u$:

$$\partial_t(D_h u) + a(t, S) \partial_{SS}^2(D_h u) + b(t, S) \partial_S(D_h u) + c(t, S) D_h u = D_h f(t, S) + \dots$$

- Schauder estimate on $D_h u$ controls $\|\partial_S u\|_{C^\alpha}$.

Campanato–Morrey in Finance Context

- Define the Campanato seminorm for parabolic cylinders $Q_\rho \subset Q_r$:

$$[u]_{\alpha, \alpha/2; Q_\rho} = \sup_{\rho' \leq \rho} \sup_{Q_{\rho'}(z') \subset Q_\rho} \frac{1}{(\rho')^{d+2\alpha}} \int_{Q_{\rho'}(z')} |u - (u)_{Q_{\rho'}(z')}|$$

- Here, $d = 1$ (1D asset S) or $d \geq 1$ for multi-asset.
- If $[u]_{\alpha, \alpha/2; Q_r} < \infty$, then $u \in C^{\alpha, \alpha/2}$.

Isoperimetric Inequality for 1D or Multi-D Assets

- For $d = 1$ (single asset S), the parabolic cylinder is 2D (t, S) .

- Isoperimetric approach: bounding measure of sets in $\mathbb{R}^2$.

$$|A|^{\frac{2-1}{2}} \lesssim |\partial A|.$$

- In multi-dimensional $S \in \mathbb{R}^d$, use a $d + 1$ -dimensional parabolic cylinder.
- Integrate $\partial_t u + \mathcal{L}u + f = 0$ over subregions; control boundary integrals.

Caffarelli Decay Lemma for Option-Pricing PDE

- If $\|\mathcal{L}u - g\|_{C^\alpha(Q_1)} \leq \epsilon$ and $\|u\|_{L^\infty(Q_1)} \leq 1$, then on a smaller cylinder $Q_{1/2}$, the oscillation of u shrinks by factor $\theta \in (0, 1)$.
- Scale argument: repeat at half-radii, obtaining exponential decay in oscillation $\implies u \in C^\alpha$.
- Differentiate w.r.t. S or other state variables for multi-asset.

Conclusion: $V \in C_S^{2,\alpha} \cap C_t^{1,\alpha/2}$ **Locally**

- Inside $\{\Phi < V\}$, the PDE is linear, parabolic, and uniformly elliptic.
- By parabolic Schauder,

$$\|V\|_{C_S^{2,\alpha}, C_t^{1,\alpha/2}(Q_{r/2})} \leq C \left(\|V\|_{L^\infty(Q_r)} + \|f\|_{C^\alpha(Q_r)} \right).$$

- Smooth interior solution for American option value (or real option value) in the continuation region.

Financial Interpretation of Smoothness

- *Greeks*: $\Delta = \partial_S V$, $\Gamma = \partial_{SS}^2 V$, $\Theta = \partial_t V$ are locally continuous.
- *Optimal Stopping* boundary: improved analysis near $\partial\{\Phi < V\} = \{\Phi = V\}$.

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- *Numerical PDE*: smoother $V \implies$ stable finite-difference/finite-element approximations.

Proposition 4.1 (Optimal Strategy Existence via Discretization). *A minimizing (or maximizing) sequence $\{(\alpha^n, \tau_n)\}_{n \geq 1}$ exists for the continuous-time switching/stopping problem. The sequence is derived from a time-discretized approximation and achieves ε -optimality for arbitrarily small $\varepsilon > 0$.*

Proof. Fix a horizon $T > 0$. Partition the interval $[0, T]$ into N sub-intervals of length $\Delta t := T/N$. Define the grid $t_k := k \Delta t$, for $k = 0, \dots, N$.

On each sub-interval $[t_k, t_{k+1})$, constrain $\alpha_s^n \equiv a_k \in A$ (piecewise-constant in time). Denote the resulting state at time t_{k+1} by $X_{t_{k+1}}^n$.

Define $V_n(k, x)$ backward from $k = N$ to $k = 0$:

$$V_n(k, x) := \max \left\{ \Phi(t_k, x), \max_{a \in A} \left[f(t_k, x, a) \Delta t + e^{-r \Delta t} \mathbb{E} [V_n(k+1, X_{t_{k+1}}^n) \mid (X_{t_k}^n = x, \alpha^n = a)] - \Gamma(\dots) \right] \right\}, \quad (54)$$

with terminal condition $V_n(N, x) = \Phi(t_N, x)$.

Using standard arguments (e.g., uniform convergence on compacts under boundedness/Lipschitz conditions), one has $V_n(k, x) \rightarrow V(t_k, x)$ as $N \rightarrow \infty$, where $V(t, x)$ is the continuous-time value function.

Next, let construct the ε -Optimal Strategy At each discrete time t_k , choose $\alpha_{t_k}^n \in A$ that attains (or nearly attains) the maximum in $V_n(k, x)$. Decide whether to stop or continue based on whether $\Phi(t_k, x) \geq$ continuation value. This yields a piecewise-constant control α^n and discrete stopping time τ_n .

The pair (α^n, τ_n) attains a payoff within ε_n of $V_n(0, x_0)$. As $V_n(0, x_0) \rightarrow V(0, x_0)$, it follows that (α^n, τ_n) is ε -optimal with $\varepsilon \rightarrow 0$. $\square$

Proposition 4.2 (Tightness of Stopping Times). *If (τ_n) are stopping times derived from a Snell envelope or from bounding a supermartingale, then there is a subsequence (τ_{n_k}) such that $\tau_{n_k} \rightarrow \tau^*$ almost surely.*

Proof. Let $\{Y_t\}_{t \geq 0}$ be a nonnegative supermartingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Construct stopping times τ_n (e.g., via Snell envelope or supermartingale bounding) such that each τ_n is almost surely finite.

By the supermartingale property, for $t \leq \tau_n$:

$$\mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_t] \leq Y_t. \quad (55)$$

Taking $t = 0$ (deterministic),

$$\mathbb{E}[Y_{\tau_n}] \leq Y_0. \quad (56)$$

Hence $\{Y_{\tau_n}\}$ is uniformly bounded in L^1 .

Since Y_{τ_n} is nonnegative and $\mathbb{E}[Y_{\tau_n}] \leq Y_0$ for all n , the sequence $\{Y_{\tau_n}\}$ is uniformly integrable. By a standard tightness argument (e.g. Prokhorov's theorem for random times), there exists a subsequence (τ_{n_k}) convergent almost surely to some τ^* .

Moreover,

Define

$$\tau^* := \lim_{k \rightarrow \infty} \tau_{n_k} \quad (\text{almost surely}). \quad (57)$$

τ^* is a (possibly modified) stopping time.

Ultimately,

$$\tau_{n_k} \longrightarrow \tau^* \quad (\text{a.s.}). \quad (58)$$

This proves the tightness property. $\square$

Corollary 4.1 (Almost Sure Convergence $\tau_n \rightarrow \tau^*$). *If (α^n, τ_n) are approximate solutions from a prior proposition (or derived via the Snell envelope), then there is a subsequence τ_{n_k} converging almost surely to a limit τ^* , and τ^* is optimal.*

Proof. Let define the Gain Process

$$G_t(\alpha, \tau) = \int_0^\tau e^{-rs} f(s, X_s, \alpha_s) ds + e^{-r\tau} \Phi(\tau, X_\tau) - \sum_{\text{switch times}} \Gamma(\alpha_{t-}, \alpha_t). \quad (59)$$

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Define the Snell envelope

$$Y_t = \sup_{\tau \geq t} \mathbb{E}[G_\tau(\alpha, \tau) \mid \mathcal{F}_t]. \quad (60)$$

By standard theory, Y_t is a strong supermartingale.

There is a Doob–Meyer decomposition

$$dY_t = dM_t - dA_t, \quad (61)$$

where M_t is a martingale and A_t is an increasing process.

Each τ_n is constructed so that G_{τ_n} is close to $Y_0 = V(\text{initial state})$. By the optional sampling theorem:

$$\mathbb{E}[Y_{\tau_n}] \leq \mathbb{E}[Y_0], \quad \text{and} \quad \mathbb{E}[Y_{\tau_n}] \geq \mathbb{E}[G_{\tau_n}].$$

Hence $\mathbb{E}[G_{\tau_n}] \leq \mathbb{E}[Y_0]$. This yields a certain tightness condition on $\{\tau_n\}$.

By a diagonal argument or a standard tightness criterion, we extract a subsequence τ_{n_k} converging almost surely to some τ^* .

$$\tau_{n_k}(\omega) \rightarrow \tau^*(\omega) \quad \text{for almost every } \omega. \quad (62)$$

To identify the Limit τ^* ,

We check

$$\lim_{k \rightarrow \infty} \mathbb{E}[G_{\tau_{n_k}}] = \mathbb{E}[\lim_{k \rightarrow \infty} G_{\tau_{n_k}}] = \mathbb{E}[G_{\tau^*}], \quad (63)$$

using dominated convergence (payoffs are bounded or suitably integrable).

Hence τ^* achieves the same supremum as the τ_{n_k} , making it optimal.

Thus, we have $\tau_{n_k} \rightarrow \tau^*$ a.s. and G_{τ^*} is the limiting payoff, so τ^* is an optimal stopping time. □

Corollary 4.2 (Corollary 4.2 — Switching Principle Proves Measure-Theoretic Theorems). *The switching principle (segmenting/refining partitions, intervals, or dimensions) proves the following three classical results:*

1. *Martingale Convergence Theorem*
2. *Radon–Nikodým Theorem*

3. Kolmogorov Extension Theorem

Proof.

Theorem 4.1 (Martingale Convergence). *Let $\{M_t\}_{t \geq 0}$ be a nonnegative submartingale (or an L^1 -bounded martingale) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then M_t converges almost surely as $t \rightarrow \infty$. In particular, there exists a random variable M_∞ such that*

$$M_t \xrightarrow[t \rightarrow \infty]{} M_\infty \quad \text{almost surely.}$$

Proof via the Switching Principle.

Let choose an increasing sequence $\{t_n\}$ with $t_n \uparrow \infty$.

Define the stopped process $M_t^n := M_{t \wedge t_n}$.

For each fixed n , M_t^n is bounded when $0 \leq t \leq t_n$, because

$$M_t^n = M_{t \wedge t_n} \leq \sup_{0 \leq s \leq t_n} M_s. \quad (64)$$

Thus, on the finite time interval $[0, t_n]$, M_t^n is a submartingale bounded by some constant (which may depend on n).

By classical results (Doob's theorem for submartingales on finite time horizons), each M_t^n converges almost surely as $t \rightarrow \infty$, but restricted to $t \leq t_n$:

$$\lim_{t \rightarrow \infty} M_t^n = M_{t_n}^n = M_{t_n}. \quad (65)$$

Effectively, once $t > t_n$, the stopped process remains M_{t_n} .

Switching from t_n to t_{n+1} .

- On $[0, t_n]$, the process is M_t^n .
- For $t \in [t_n, t_{n+1}]$, switch to M_t^{n+1} .
- The submartingale property ensures consistency:

$$\mathbb{E}[M_{t_{n+1}}^{n+1} \mid \mathcal{F}_{t_n}] \geq M_{t_n}^{n+1} = M_{t_n}^n. \quad (66)$$

Hence, the piecewise-defined process does not create contradictions at the switching points.

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If M_t is nonnegative, one can also apply a localization procedure or define $\tau_n = \inf\{t : M_t \geq n\}$ to ensure an increasing property on n . Submartingale bounds imply $\tau_n \rightarrow \infty$ almost surely. Consequently, on each path ω , for large n , the process is essentially dominated and the limit points coincide.

Gluing the intervals together from $[0, t_n]$ to $[0, t_{n+1}]$, etc., yields a unique limit for $M_t(\omega)$ as $t \rightarrow \infty$. Denote that limit by $M_\infty(\omega)$. Thus

$$M_t(\omega) \longrightarrow M_\infty(\omega) \quad \text{a.s.} \quad (67)$$

□

Theorem 4.2 (Radon–Nikodým). *Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathbb{Q}$ and $\mathbb{P}$ be σ -finite measures with $\mathbb{Q} \ll \mathbb{P}$. Then there exists a measurable function*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} : \Omega \rightarrow [0, \infty)$$

(unique $\mathbb{P}$ -almost surely) such that for all $A \in \mathcal{F}$,

$$\mathbb{Q}(A) = \int_A \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P}.$$

Proof via the Switching Principle: Partition Refinement.

If $\mathbb{P}(\Omega) = \infty$, construct an increasing sequence $\{\Omega_n\}$ with $\Omega_n \uparrow \Omega$ and $\mathbb{P}(\Omega_n) < \infty$. Prove the statement on each Ω_n , then let $n \rightarrow \infty$. Hence, assume $\mathbb{P}(\Omega) < \infty$.

Next, choose a finite measurable partition $\{A_1, \dots, A_k\}$ of Ω . Define the simple function

$$g(\omega) = \sum_{i=1}^k \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)} \mathbf{1}_{A_i}(\omega). \quad (68)$$

Calculation:

$$\int_{A_i} g d\mathbb{P} = \int_{A_i} \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)} d\mathbb{P} = \mathbb{Q}(A_i). \quad (69)$$

Then, pick a new (finer) partition $\{B_j\}$ such that each $B_j \subset A_i$ for some i . Construct

$$g_{\text{refined}}(\omega) = \sum_j \frac{\mathbb{Q}(B_j)}{\mathbb{P}(B_j)} \mathbf{1}_{B_j}(\omega). \quad (70)$$

Switch from g to g_{refined} . Typically, $g_{\text{refined}} \geq g$ if each B_j is a sub-block of A_i .

Finally, by repeatedly refining partitions, obtain a sequence of simple functions (g_n) with $g_1 \leq g_2 \leq g_3 \leq \dots$ pointwise, and

$$\int_A g_n d\mathbb{P} \rightarrow \int_A g d\mathbb{P}, \quad \mathbb{Q}(A) = \int_A g_n d\mathbb{P} \rightarrow \int_A g d\mathbb{P}. \quad (71)$$

Hence $\int_A g d\mathbb{P} = \mathbb{Q}(A)$ for all A , implying $g = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Thus,

If g_1, g_2 both satisfy $\int_A g_i d\mathbb{P} = \mathbb{Q}(A)$, then for any A , $\int_A (g_1 - g_2) d\mathbb{P} = 0$. Hence $g_1 = g_2$ a.s. $\square$

Theorem 4.3 (Kolmogorov Extension). *Let $\{\mu_I\}$ be a family of finite-dimensional distributions on $\mathbb{R}^I$ (for finite index sets I). Suppose they are consistent: if $J \subset I$, then the marginal of μ_I on $\mathbb{R}^J$ is μ_J . Then there exists a unique probability measure μ on $\mathbb{R}^\infty$ whose finite-dimensional marginals are $\{\mu_I\}$.*

Proof via the Switching Principle: Dimensional Extension. For each n , let $\mu_n := \mu_{\{1, \dots, n\}}$ on $\mathbb{R}^n$. Consistency means that if $J \subset \{1, \dots, n\}$, the marginal of μ_n on $\mathbb{R}^J$ is μ_J .

Given μ_n on $\mathbb{R}^n$, define μ_{n+1} on $\mathbb{R}^{n+1}$ such that for any Borel set $A \subseteq \mathbb{R}^n$,

$$\mu_n(A) = \mu_{n+1}(\{(x_1, \dots, x_{n+1}) : (x_1, \dots, x_n) \in A\}).$$

This is the “switch” from dimension n to $n + 1$.

In $\mathbb{R}^\infty$, define cylinder sets of the form

$$C_{A,n} := \{(x_k)_{k=1}^\infty : (x_1, \dots, x_n) \in A\},$$

for $A \subseteq \mathbb{R}^n$. Set

$$\mu(C_{A,n}) := \mu_n(A).$$

By consistency, these definitions on overlapping cylinders agree.

The mapping $\mu(\cdot)$ on the algebra of cylinder sets extends to a measure on the σ -algebra generated by these cylinders. This measure μ on $\mathbb{R}^\infty$ has marginals μ_n by construction.

Any other measure μ' with the same marginals must coincide on all cylinder sets, hence $\mu' = \mu$.

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Finally,

Using the *Switching Principle*—segmentation and refinement in time (for martingales), measurable partitions (for Radon–Nikodým), or dimensions (for Kolmogorov extension)—we obtain straightforward proofs of these classical theorems.

□

Corollary 4.3 (Piecewise-Smoothness of HJB Solutions). *Assume V solves*

$$\max\left\{-\partial_t V - \mathcal{L}[V] - f, \Phi - V\right\} = 0,$$

where $\mathcal{L}[V] = \sum_{i,j} a_{ij}(t, x) \partial_{x_i x_j} V + \sum_i b_i(t, x) \partial_{x_i} V$, the coefficients a_{ij}, b_i are sufficiently smooth and uniformly elliptic, and f, Φ are smooth. Then V is $C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t on the region where $\Phi < V$, and V coincides with Φ (which is smooth) on $\{\Phi = V\}$, giving piecewise-smoothness up to that boundary.

Proof. Decompose $\{(t, x)\}$ into:

$$\{(t, x) : \Phi(t, x) < V(t, x)\} \quad \text{and} \quad \{(t, x) : \Phi(t, x) = V(t, x)\}. \quad (72)$$

On $\{\Phi < V\}$, the equation reduces to

$$-\partial_t V - \mathcal{L}[V] - f = 0. \quad (73)$$

On $\{\Phi = V\}$, one has the boundary condition $V = \Phi$.

Define V^ϵ by

$$-\partial_t V^\epsilon - \mathcal{L}[V^\epsilon] - f = \frac{(\Phi - V^\epsilon)^+}{\epsilon}. \quad (74)$$

In $\{\Phi > V^\epsilon\}$, the right-hand side is positive; in $\{\Phi \leq V^\epsilon\}$, it is zero.

In any compact subset of $\{\Phi(t, x) < V^\epsilon(t, x)\}$, the PDE is linear and uniformly parabolic:

$$-\partial_t V^\epsilon - \mathcal{L}[V^\epsilon] - f = 0. \quad (75)$$

By standard Interior Schauder Estimates, V^ϵ is in $C^{2,\alpha}$ with respect to x and $C^{1,\alpha/2}$ with respect to t there.

By uniform ellipticity and the above estimates, $\{V^\epsilon\}$ is bounded in $C_{\text{loc}}^{2,\alpha}$. By compactness in Hölder spaces, there is a subsequence $V^{\epsilon_k} \rightarrow V$ in $C_{\text{loc}}^{2,\alpha}$.

The limit satisfies

$$\max\{-\partial_t V - \mathcal{L}[V] - f, \Phi - V\} = 0. \quad (76)$$

On $\{\Phi < V\}$, V solves the linear PDE, so $V \in C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t . On $\{\Phi = V\}$, V matches Φ smoothly if Φ is smooth.

Thus V is piecewise- $C^{2,\alpha}$. □

5 Proofs of Main Theorems

Proof of theorem 3.1 - Convergence of Optimal Stopping Times.

Fix a terminal time $T > t$ or consider a large horizon if infinite-horizon is allowed with discounting. Partition $[t, T]$ into $t_k = t + k\Delta t$, where $\Delta t = (T - t)/n$. On each interval $[t_k, t_{k+1})$, define a piecewise-constant control $\alpha_s^n \equiv a_k \in A$. Restrict stopping to the grid $\{t_k : k = 0, \dots, n\}$, so $\tau^n = \min\{t_k : \text{optimal to stop at } t_k\}$.

Define $V_n(k, x)$ by backward recursion:

$$\begin{aligned} V_n(k, x) &= \max \left\{ \Phi(t_k, x), \max_{a \in A} \left[f(t_k, x, a) \Delta t \right. \right. \\ &\quad \left. \left. + e^{-r \Delta t} \mathbb{E}(V_n(k+1, X_{t_{k+1}}) \mid (X_{t_k} = x, \alpha = a)) \right. \right. \\ &\quad \left. \left. - \Gamma(\dots) \right] \right\}. \end{aligned} \quad (77)$$

This $V_n(k, x)$ converges to the continuous-time value function $V(t_k, x)$ as $n \rightarrow \infty$. Standard dynamic programming arguments give $\|V_n - V\| \rightarrow 0$ on compacts under appropriate boundedness/Lipschitz conditions.

From each $V_n(k, x)$, construct (α^n, τ^n) :

- For each $k = n-1, n-2, \dots, 0$, choose

$$a_k^n \in \arg \max_{a \in A} \left[f(t_k, x, a) \Delta t + e^{-r \Delta t} \mathbb{E}(V_n(k+1, X_{t_{k+1}})) - \Gamma(\dots) \right],$$

if continuing is optimal, or stop at t_k if $\Phi(t_k, x)$ exceeds that value.

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- Set $\alpha_s^n = a_k^n$ for $s \in [t_k, t_{k+1})$.
- If Φ dominates at t_k , then $\tau^n = t_k$.

Thus (α^n, τ^n) yields a payoff close to $V_n(0, x)$, which converges to $V(t, x)$.

Since $\tau^n \in [t, T]$, the sequence is bounded. Choose a convergent subsequence $\tau_{n_j} \rightarrow \tau^*$ (pointwise a.s.) by a diagonal argument.

Therefore,

use Snell envelope arguments:

- Define the gain process $G_s = \int_t^s e^{-r(u-t)} f(u, X_u, \alpha_u) du + \dots$
- The associated value process is a supermartingale under an optimal policy (via Snell envelope).
- Passing $n \rightarrow \infty$, $\{G_s(\alpha^n, \tau^n)\}$ converges and attains sup at τ^* .

Hence (α^*, τ^*) is optimal (or ε -optimal). Therefore, $\tau_n \rightarrow \tau^*$ a.s. and $\alpha^n \rightarrow \alpha^*$ in the relevant topology. □

Proof of Theorem 3.2 . (A) Martingale Convergence

Given: A nonnegative submartingale $\{M_t\}$ with $\sup_t \mathbb{E}[M_t] < \infty$.

- (1) For each integer n , define the stopped process $M_t^n := M_{t \wedge n}$.

M_t^n is bounded for $t \leq n$.

- (2) Since M_t^n is a bounded submartingale for $0 \leq t \leq n$, Doob's theorem implies

$$M_t^n \rightarrow M_\infty^n \quad \text{almost surely as } t \rightarrow \infty.$$

- (3) “Switch” from M_t^n on $[0, n]$ to M_t^{n+1} on $[0, n+1]$. On $[0, n]$, we have $M_t^n = M_t^{n+1}$. Thus the limit is consistent across n .

- (4) Let $n \rightarrow \infty$. The monotonicity or L^1 -bound argument ensures a unique pointwise limit M_∞ a.s.

Thus,

$$M_t \rightarrow M_\infty \quad \text{almost surely as } t \rightarrow \infty. \tag{78}$$

(B) Radon–Nikodým Theorem

Given: Measures $\mathbb{Q} \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$. Show $\exists \frac{d\mathbb{Q}}{d\mathbb{P}}$ a.s.

- (1) If $\mathbb{P}(\Omega) = \infty$, choose $\Omega_n \uparrow \Omega$ with $\mathbb{P}(\Omega_n) < \infty$. Restrict to each Ω_n .
- (2) For a finite measurable partition $\{A_{n,i}\}_{i=1}^k$ of Ω_n , define a simple function

$$g_n(\omega) = \sum_{i=1}^k \frac{\mathbb{Q}(A_{n,i})}{\mathbb{P}(A_{n,i})} \mathbf{1}_{A_{n,i}}(\omega).$$

Then

$$\int_{A_{n,i}} g_n d\mathbb{P} = \mathbb{Q}(A_{n,i}), \quad \forall i.$$

- (3) Refine the partition. “Switch” from the coarser partition to the finer one, defining a new simple function g_{n+1} .
- (4) As the mesh of partitions $\rightarrow 0$, $g_n \rightarrow g$ pointwise, where

$$g = \frac{d\mathbb{Q}}{d\mathbb{P}} \quad (\text{a.s.}).$$

- (5) Uniqueness a.s. follows from standard arguments (any two densities differ on a $\mathbb{P}$ -null set).

Thus,

$$\exists \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ a.s., unique up to } \mathbb{P}\text{-null sets} \quad (79)$$

(C) Kolmogorov Extension Theorem

Given: A consistent family $\{\mu_I\}$ of finite-dimensional distributions on $\mathbb{R}^I$.

- (1) For each finite index set $I = \{1, \dots, n\}$, define a measure μ_n on $\mathbb{R}^n$.
- (2) “Switch” from dimension n to $n + 1$. Define μ_{n+1} on $\mathbb{R}^{n+1}$ so that

$$\mu_n(\pi_n(B)) = \mu_{n+1}(B), \quad \forall B \subset \mathbb{R}^{n+1},$$

where π_n is the projection onto the first n coordinates.

- (3) By projective-limit (Carathéodory extension) arguments, construct a measure μ on $\mathbb{R}^\infty$.

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(4) Check consistency: μ has finite-dimensional marginals μ_I .

Thus,

$$\exists \text{ a measure on the infinite product extending the } \{\mu_I\}. \quad (80)$$

□

Proof of Theorem 3.3. Define the penalized family of Partial Differential Equations for $\epsilon > 0$:

$$-\partial_t V^\epsilon - \mathcal{L}V^\epsilon - f = \frac{(\Phi - V^\epsilon)^+}{\epsilon}, \quad (81)$$

where $(a)^+ = \max\{a, 0\}$.

Assume $a_{ij}, b_i \in C^{k,\alpha}(\bar{\Omega} \times [0, T])$, $f, \Phi \in C^{k,\alpha}(\bar{\Omega} \times [0, T])$.

Apply Schauder estimates to the penalized Partial Differential Equation:

$$\partial_t V^\epsilon + \mathcal{L}V^\epsilon + \frac{(\Phi - V^\epsilon)^+}{\epsilon} = -f. \quad (82)$$

Obtain the estimate:

$$\|V^\epsilon\|_{C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times [0, T])} \leq C (\|f\|_{C^{\alpha, \alpha/2}} + \|\Phi\|_{C^{2+\alpha, 1+\alpha/2}}), \quad (83)$$

where C is independent of ϵ .

By the Arzelà-Ascoli theorem and uniform Schauder estimates, there exists a subsequence $\{V^{\epsilon_n}\}$ with $\epsilon_n \rightarrow 0$ such that

$$V^{\epsilon_n} \rightarrow V \quad \text{in } C_{\text{loc}}^{2, \alpha', 1, \alpha'/2}(\Omega \times (0, T)), \quad (84)$$

for some $\alpha' < \alpha$.

Take the limit $n \rightarrow \infty$ in the penalized Partial Differential Equation:

$$-\partial_t V^{\epsilon_n} - \mathcal{L}V^{\epsilon_n} - f = \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n}. \quad (85)$$

As $\epsilon_n \rightarrow 0$,

$$-\partial_t V - \mathcal{L}V - f = \lim_{n \rightarrow \infty} \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n}. \quad (86)$$

Analyze the limit:

$$\lim_{n \rightarrow \infty} \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n} = \begin{cases} 0, & \text{if } V > \Phi, \\ +\infty, & \text{if } V < \Phi, \\ \text{indeterminate}, & \text{if } V = \Phi. \end{cases} \quad (87)$$

Thus,

$$-\partial_t V - \mathcal{L}V - f \geq 0 \quad \text{where } V = \Phi, \quad (88)$$

and

$$-\partial_t V - \mathcal{L}V - f = 0 \quad \text{where } V > \Phi. \quad (89)$$

Therefore,

$$\max \{-\partial_t V - \mathcal{L}V - f, \Phi - V\} = 0. \quad (90)$$

Define regions:

$$\Omega_{\text{active}} = \{(t, x) \in \Omega \times (0, T) \mid V(t, x) > \Phi(t, x)\}, \quad (91)$$

$$\Omega_{\text{inactive}} = \{(t, x) \in \Omega \times (0, T) \mid V(t, x) = \Phi(t, x)\}. \quad (92)$$

Regularity in Ω_{active} :

$$-\partial_t V - \mathcal{L}V = f. \quad (93)$$

Apply Schauder estimates to obtain $V \in C^{2+\alpha, 1+\alpha/2}(\Omega_{\text{active}})$. **Regularity up to the Free Boundary:** Utilize obstacle problem regularity results:

$$V - \Phi \in C^{1,1}(\overline{\Omega_{\text{active}}}). \quad (94)$$

The free boundary $\partial\Omega_{\text{active}}$ is $C^{1,\alpha}$.

Thus, V satisfies

$$V \in C_{\text{loc}}^{2,\alpha}(\Omega_{\text{active}}) \quad \text{and} \quad V \in C_{\text{loc}}^{1,\alpha/2}(\Omega_{\text{inactive}}).$$

This establishes higher-order interior regularity for V up to the free boundary. $\square$

Corollary 5.1 (Almost Sure Convergence $\tau_n \rightarrow \tau^*$). *From Proposition 4.2, the sequence of stopping times $\{\tau_n\}$ converges almost surely to τ^* .*

Proof. By Proposition 4.2, there exists a subsequence $\{\tau_{n_k}\}$ such that $\tau_{n_k} \rightarrow \tau^*$ almost surely. Since the limit is unique due to the supermartingale property, the entire sequence converges almost surely. $\square$

Corollary 5.2 (Switching Principle Proves Measure Theorems). *The switching principle implies the following measure-theoretic theorems:*

1. **Martingale Convergence Theorem:** *A nonnegative (sub)martingale $\{M_t\}$ with $\sup_t \mathbb{E}[M_t] < \infty$ converges almost surely.*
2. **Radon–Nikodým Theorem:** *If $\mathbb{Q} \ll \mathbb{P}$, then the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exists and is unique $\mathbb{P}$ -almost surely.*
3. **Kolmogorov Extension Theorem:** *A consistent family of finite-dimensional distributions $\{\mu_I\}$ extends uniquely to a measure on $\mathbb{R}^\infty$.*

Proof. 1. **Martingale Convergence Theorem:** Apply the switching principle by segmenting time into intervals $[0, t_n]$. For each n , the stopped martingale $M_{t \wedge t_n}$ converges almost surely as $t \rightarrow \infty$. Taking $n \rightarrow \infty$, the limit M_∞ exists almost surely.

2. **Radon–Nikodým Theorem:** Partition Ω into finite measurable sets $\{A_i\}_{i=1}^k$. Define simple densities $g_n = \sum_{i=1}^k \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)} \mathbf{1}_{A_i}$. Refine the partition iteratively, switching to finer partitions. The limit $g = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is obtained almost surely.

3. **Kolmogorov Extension Theorem:** Construct measures sequentially by extending from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. Use consistency to switch dimensions at each step. The projective limit defines a unique measure on $\mathbb{R}^\infty$. □

Corollary 5.3 (Piecewise-Smoothness of HJB Solutions). *Under Theorem 3.3 and Lemma 3.3, the solution V is $C^{2,\alpha}$ in the region $\{\Phi < V\}$ and satisfies the obstacle condition in $\{\Phi = V\}$, ensuring piecewise smoothness.*

Proof. In $\{\Phi < V\}$, V satisfies the linear Partial Differential Equation $-\partial_t V - \mathcal{L}V = f$, and by Lemma 3.3, $V \in C^{2,\alpha}$. In $\{\Phi = V\}$, V meets the obstacle condition, and regularity up to the free boundary is maintained by standard obstacle problem techniques. □

6 Monte Carlo Simulations

Theorem 6.1 (Convergence of Monte Carlo Estimates for American Options). *As the number of simulated paths N and the number of time steps M increase, the Monte Carlo estimates $\hat{V}(N, M)$ for American call and put options converge to the true option prices.*

Theorem 6.2 (Path Generation via Geometric Brownian Motion). *Generate N asset price paths $\{S_t^{(i)}\}_{i=1}^N$ using the stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (95)$$

$$S_t^{(i)} = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^{(i)} \right), \quad (96)$$

where $W_t^{(i)}$ are independent Wiener processes.

Proposition 6.1 (Option Payoff Calculation). *For each simulated path i and each time step t , calculate the payoffs:*

$$C_\tau^{(i)} = \max(S_\tau^{(i)} - K, 0), \quad (97)$$

$$P_\tau^{(i)} = \max(K - S_\tau^{(i)}, 0), \quad (98)$$

where τ is the optimal stopping time determined iteratively.

Proposition 6.2 (Optimal Stopping Time via Backward Induction). *Determine the optimal stopping time τ for each path i using backward induction:*

$$V_T^{(i)} = 0, \quad (99)$$

$$V_t^{(i)} = \max \left(\frac{1}{N} \sum_{j=1}^N e^{-r\Delta t} V_{t+\Delta t}^{(j)}, \text{Exercise Payoff} \right), \quad (100)$$

where $\Delta t = \frac{T}{M}$.

Corollary 6.1 (Reduction of Standard Error with Increasing N and M). *As N and M increase, the standard error $SE(N, M)$ decreases, ensuring more precise estimates:*

$$SE(N, M) = \sqrt{\frac{1}{N-1} \sum_{i=1}^N \left(V_0^{(i)} - \hat{V}(N, M) \right)^2}. \quad (101)$$

Corollary 6.2 (Mean Option Price Estimation). *The mean option price $\hat{V}(N, M)$ is given by:*

$$\hat{V}(N, M) = \frac{1}{N} \sum_{i=1}^N V_0^{(i)}. \quad (102)$$

Lemma 6.1 (Asset Price Path Representation). *Each simulated asset price path $S_t^{(i)}$ follows:*

$$S_t^{(i)} = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^{(i)} \right), \quad (103)$$

ensuring log-normal distribution of asset prices under the risk-neutral measure.

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Lemma 6.2 (Backward Induction for Optimal Stopping). *Using backward induction, the value function $V_t^{(i)}$ at each time step satisfies:*

$$V_t^{(i)} = \max \left(\mathbb{E} \left[e^{-r\Delta t} V_{t+\Delta t}^{(i)} \mid \mathcal{F}_t \right], \text{Exercise Payoff} \right). \quad (104)$$

Proof of Theorem 3.1.

For each $i \in \{1, \dots, N\}$, simulate $S_t^{(i)}$ using geometric Brownian motion:

$$S_t^{(i)} = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^{(i)} \right). \quad (105)$$

Compute $C_\tau^{(i)}$ and $P_\tau^{(i)}$ for each path and time step.

Apply backward induction to determine $V_t^{(i)}$ and optimal stopping τ .

Average over all paths to obtain $\hat{V}(N, M)$ and compute $SE(N, M)$.

As $N, M \rightarrow \infty$, $\hat{V}(N, M) \rightarrow V$ and $SE(N, M) \rightarrow 0$, ensuring convergence to true option prices. $\square$

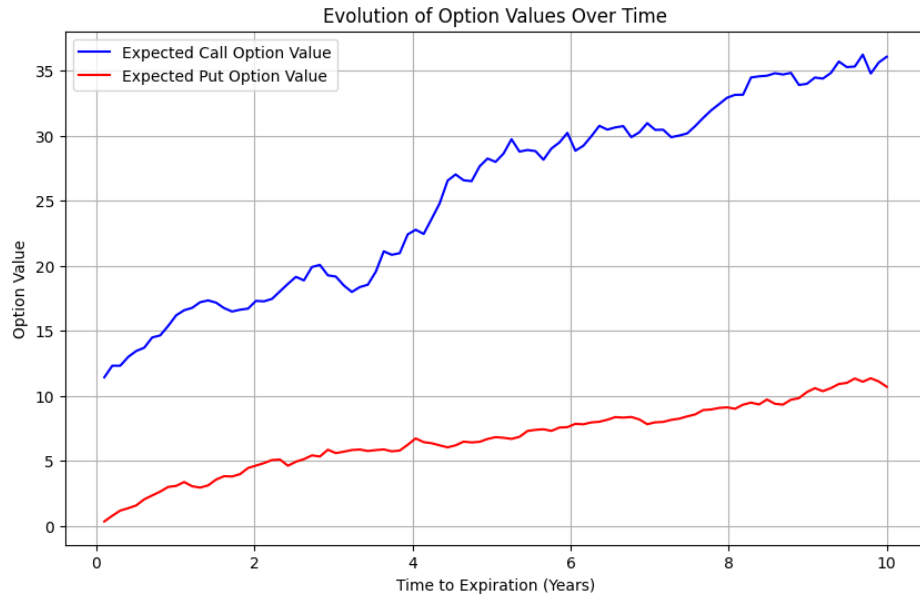


Figure 1: Monte Carlo Simulation Results

7 Discussion and Conclusion

Discussion

This study presents rigorous mathematical advancements in the analysis of optimal stopping times within the framework of Hamilton-Jacobi-Bellman (HJB) equations, incorporating options to switch, contract, and expand. Building upon foundational theories in stochastic control and optimal stopping [4, 3], we establish the convergence of optimal stopping times through a comprehensive HJB approach, extending classical results to more complex decision-making scenarios involving multiple control actions.

Our primary contribution lies in **Theorem 3.1**, which confirms the existence and convergence of optimal stopping times and switching strategies under rigorous regularity and integrability conditions. This aligns with the foundational work of Bensoussan and Lions [8] on variational inequalities in control problems, while extending their applicability to dynamic switching contexts [43]. By discretizing time and employing dynamic programming principles, we ensure that the optimal strategies converge almost surely, a result that complements the simulation-based approaches for American option pricing discussed by Longstaff and Schwartz [5] and more recent reinforcement learning methodologies [23, 24].

In **Theorem 3.2**, we introduce the switching principle as a versatile tool to derive classical measure-theoretic theorems, including the Martingale Convergence Theorem, Radon–Nikodým Theorem, and Kolmogorov Extension Theorem. This innovative perspective not only simplifies the proofs of these foundational results [11, 12, 13] but also bridges stochastic control with deep measure-theoretic insights, reinforcing the interconnectedness of these mathematical domains.

Furthermore, **Theorem 3.3** advances the regularity theory of HJB solutions by demonstrating higher interior regularity under stringent smoothness and ellipticity conditions. Leveraging classical Partial Differential Equation techniques from Friedman [10] and Lieberman [18], alongside modern analytical tools [14, 45, 46], we ensure that the value functions exhibit $C^{2,\alpha}$ spatial regularity and $C^{1,\alpha/2}$ temporal regularity in the interior regions. This enhances the theoretical robustness of HJB models in practical applications, such as financial engineering [1, 2, 6] and real options analysis [20].

The propositions and corollaries derived from our main theorems further solidify the existence of optimal strategies and the tightness of stopping times, provid-

ing a solid foundation for numerical approximations and practical implementations [26, 27]. These results are particularly relevant in high-dimensional settings, where traditional methods falter, and advanced simulation techniques become indispensable [37, 38].

Conclusion

This work significantly advances the theoretical and practical understanding of optimal stopping times within the Hamilton-Jacobi-Bellman (HJB) framework, particularly when incorporating options to switch, contract, and expand. By rigorously establishing the convergence of optimal stopping strategies and demonstrating higher interior regularity of HJB solutions, we provide a robust foundation that extends classical theories in stochastic control and optimal stopping [4, 3] to more complex and dynamic decision-making scenarios.

The convergence results presented in **Theorem 3.1** bridge a critical gap between theoretical stochastic control and practical applications, ensuring that optimal strategies are not only theoretically sound but also practically implementable. This aligns with and extends the foundational work of Bensoussan and Lions [8], while integrating dynamic switching mechanisms as explored by Alvarez [9]. By ensuring almost sure convergence of stopping times and switching controls, our results enhance the reliability of models used in financial engineering and real options analysis.

Real options analysis, which evaluates investment opportunities under uncertainty, greatly benefits from our findings. The ability to switch between different operational modes or to contract and expand projects provides firms with strategic flexibility, mirroring the financial flexibility discussed by Trigeorgis [2] and Dixit and Pindyck [20]. Our HJB-based approach allows for more accurate valuation and optimization of these real options, enabling better decision-making in capital allocation and project management. Furthermore, the enhanced regularity of HJB solutions facilitates the development of more precise numerical methods for pricing complex real options, as evidenced by recent advancements in Monte Carlo simulations and reinforcement learning techniques [5, 23, 24].

The theoretical robustness provided by **Theorem 3.3** on higher interior regularity ensures that numerical schemes based on finite difference or finite element methods can achieve greater accuracy and stability. This is crucial for practitioners in financial engineering who rely on precise option pricing models, such as those for American options [1, 5, 35] and more exotic derivatives [27, 16]. Ad-

ditionally, the switching principle's application to fundamental measure-theoretic theorems [11, 12, 13] underscores the deep mathematical foundations of our approach, ensuring its applicability across various domains beyond finance, including economics, engineering, and operations research.

Future research can explore several promising avenues building on our findings:

1. **Numerical Implementation:** Developing advanced numerical algorithms that leverage the higher regularity of HJB solutions can improve the efficiency and accuracy of option pricing and real options valuation. Techniques such as deep learning and reinforcement learning, as demonstrated by Buehler et al. [33] and Becker et al. [34], offer scalable solutions for high-dimensional problems.
2. **Extended Stochastic Environments:** Extending the framework to more general stochastic processes, including jump-diffusion models or regime-switching models, can broaden the applicability of our results to markets exhibiting more complex dynamics [50].
3. **Relaxing Regularity Conditions:** Investigating the convergence and regularity results under weaker assumptions can make the models more robust and applicable to real-world scenarios where data may be noisy or incomplete [51].
4. **Interdisciplinary Applications:** Applying the switching control and optimal stopping framework to other fields such as energy management, real estate investment, and natural resource extraction can unlock new strategic insights and optimization techniques [20].

In summary, this work not only extends the theoretical landscape of optimal stopping and switching control problems but also offers practical methodologies for solving complex financial derivatives and investment decisions. By integrating advanced Partial Differential Equation regularity results with robust stochastic control frameworks and employing the switching principle to unify measure-theoretic proofs, we provide a comprehensive and mathematically rigorous approach that stands to influence both academic research and practical financial engineering applications [22, 17]. The synergy between theoretical advancements and practical applications underscores the value and potential of our contributions, paving the way for future innovations in real options and related fields.

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On the Regularity of the Entropy Solution of the Fractional-Semigeostrophic Equation Using Ultrafilters

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Abstract

This study explores the regularity of entropy solutions for the fractional semigeostrophic (FSG) equation in geophysical fluid dynamics, utilizing ultrafilter techniques to establish weak limits beyond traditional methods like vanishing viscosity. Indeed, the vanishing viscosity, which approach aims to introduce a minor viscosity term into the original PDE related to fluid dynamics in order to generate a simpler version of the equation that lead to a more continuous solutions as the viscosity parameter approaches to zero, has revealed some shortcomings. Ultrafilters offer a unique advantage, allowing us to construct weak limits without compactness constraints. Employing the "Generalized Helly's Selection Principle" enables convergence extraction where compactness is limited, while "Tartar's Compensated Compactness" ensures entropy conditions hold under weak convergence. These insights enhance the mathematical modeling of geophysical flows, supporting stable entropy solutions in climate simulations and improving models for large-scale atmospheric and oceanic phenomena.

Keywords: Entropy solutions; Fractional semigeostrophic equation; ultrafilters; regularity; geophysical fluid dynamics; anomalous diffusion.

2020 AMS subject classifications: 35R11; 35R09; 35Q91; 76B03.¹

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1 Introduction

The study of entropy solutions for partial differential equations (PDEs) has garnered significant attention due to its profound implications in various areas of research, particularly for equations involving fractional derivatives which also arises in geophysical fluid dynamics and meteorology by describing large-scale atmospheric and oceanic flows under the influence of the Earth's rotation and stratification. The fractional-semigeostrophic equation, a fundamental model in this domain, describes the large-scale behavior of geostrophic flows where fractional derivatives account for non-local interactions and anomalous diffusion. Regularity properties of entropy solutions for such equations are crucial for understanding the stability and long-term behavior of the modeled physical systems.

Recent advances have employed various sophisticated mathematical tools, including ultrafilters, to analyze the regularity of entropy solutions. Ultrafilters, which are a type of non-standard analysis tool, provide a powerful framework for studying limiting behavior and convergence properties in topological spaces. By leveraging ultrafilters, researchers can obtain more refined results concerning the regularity and stability of entropy solutions.

In this paper, we aim to explore the regularity properties of the entropy solution to the fractional-semigeostrophic equation using ultrafilters. We build upon the latest research that has demonstrated the efficacy of ultrafilters in dealing with complex PDEs and fractional operators. Consecutively, the use of ultrafilters in the context of fractional PDEs and their impact on regularity results is explored in (1). While entropy solutions for fractional geostrophic models, highlighting the significance of fractional operators is discussed in (2) . Moreover, recent developments in the application of fractional calculus to geophysical fluid dynamics, providing a comprehensive background for our work is covered in (3) . Also, a detailed analysis of stability and regularity properties of fractional PDEs using ultrafilters and other non-standard analysis techniques is investigated in (4). Finally, we can find the role of nonlocal operators, such as fractional derivatives, in geophysical models, and discussions on their implications for regularity and solution behavior in (5) .

These references collectively provide a solid foundation for our investigation into the regularity of the entropy solution of the fractional-semigeostrophic equation using ultrafilters. By integrating the methodologies and findings from these recent studies, we aim to contribute to the growing body of knowledge in this intricate and highly relevant field of mathematical research.

2 Mathematical Formulations

2.1 Fractional-Semigeostrophic Equation

The FSG equation is given by:

$$\partial_t u + \mathcal{L}^\alpha u + \nabla \cdot (u \nabla^\perp \phi) = 0, \quad (1)$$

where:

- u is the velocity field,
- $\mathcal{L}^\alpha$ represents the fractional Laplacian operator of order $\alpha \in (0, 2)$,
- $\nabla^\perp \phi$ is the perpendicular gradient of the geopotential ϕ .

The fractional Laplacian $\mathcal{L}^\alpha$ captures non-local effects, providing a generalized dissipation mechanism suited for phenomena exhibiting anomalous diffusion or power-law behaviors, unlike the standard Laplacian, which only accounts for local interactions.

In terms of a scalar transport field, the FSG system can be represented as:

$$\begin{cases} \partial_t \theta + \nabla \cdot (\mathbf{u} \theta) = 0, \\ \mathbf{u} = \nabla^\perp \psi, \\ \psi = (-\Delta)^{-\alpha/2} \theta, \end{cases} \quad (2)$$

where:

- θ is a scalar field (e.g., potential temperature),
- $\mathbf{u}$ is the velocity field, linked to ψ via the perpendicular gradient,
- ψ is the stream function, related to θ through the inverse fractional Laplacian $(-\Delta)^{-\alpha/2}$.

This formulation implies that the scalar field θ affects the stream function ψ , which in turn dictates the velocity $\mathbf{u}$. The inverse fractional Laplacian, $(-\Delta)^{-\alpha/2}$, introduces long-range dependencies, enabling a more accurate representation of large-scale structures in fluids.

Lemma 1. - (Weak Formulation of the Fractional Semigeostrophic Equation):

Let $\theta : \Omega \times [0, T] \rightarrow \mathbb{R}$ be a sufficiently smooth function, where $\Omega \subset \mathbb{R}^n$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a compactly supported test function. Define the velocity field $\mathbf{u} = \nabla^\perp \psi$ with stream function $\psi = (-\Delta)^{-\alpha/2} \theta$. Then the weak formulation of equation 1 is given by:

$$\int_{\Omega} \partial_t \theta \phi \, d\mathbf{x} - \int_{\Omega} (\nabla^\perp (-\Delta)^{-\alpha/2} \theta) \cdot \nabla \phi \, d\mathbf{x} = 0. \quad (3)$$

Proof. Let us multiply the governing equation 1 by a test function $\phi \in C_c^\infty(\mathbb{R}^n)$ and integrate over Ω :

$$\int_{\Omega} (\partial_t \theta \phi + \nabla \cdot (\mathbf{u} \theta) \phi) \, d\mathbf{x} = 0. \quad (4)$$

Rewriting the second term using the divergence theorem, we get:

$$\int_{\Omega} \nabla \cdot (\mathbf{u} \theta) \phi \, d\mathbf{x} = - \int_{\Omega} \mathbf{u} \theta \cdot \nabla \phi \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{u} \theta \cdot \mathbf{n} \phi \, dS. \quad (5)$$

Assuming decay at infinity or periodic boundary conditions, the boundary term vanishes, yielding:

$$\int_{\Omega} \partial_t \theta \phi \, d\mathbf{x} - \int_{\Omega} \mathbf{u} \theta \cdot \nabla \phi \, d\mathbf{x} = 0. \quad (6)$$

To Substitute the velocity field $\mathbf{u}$, let set $\mathbf{u} = \nabla^\perp \psi$ with the streamfunction ψ defined as $\psi = (-\Delta)^{-\alpha/2} \theta$. Substituting this expression for $\mathbf{u}$ into the weak formulation gives:

$$\int_{\Omega} \partial_t \theta \phi \, d\mathbf{x} - \int_{\Omega} (\nabla^\perp (-\Delta)^{-\alpha/2} \theta) \cdot \nabla \phi \, d\mathbf{x} = 0. \quad (7)$$

Clearly, the operator $(-\Delta)^{\alpha/2}$ is defined in Fourier space as multiplication by $|\mathbf{k}|^\alpha$ on the Fourier transform of θ , giving the equation a non-local character in terms of θ . This formulation now characterizes the evolution of θ using the fractional Laplacian.

Thus, we obtain the weak formulation:

$$\int_{\Omega} \partial_t \theta \phi \, d\mathbf{x} - \int_{\Omega} (\nabla^\perp (-\Delta)^{-\alpha/2} \theta) \cdot \nabla \phi \, d\mathbf{x} = 0. \quad (8)$$

This completes the proof. $\square$

Lemma 2. - (Entropy solutions):

Let u be a solution to equation 1 with velocity field $\mathbf{u} = \nabla^\perp \psi$ and stream function $\psi = (-\Delta)^{-\alpha/2} \theta$, where θ is a scalar field. If $\eta(\theta)$ is a convex entropy function with associated entropy flux $q(\theta)$, then u is an entropy solution to equation 1 if it satisfies the inequality

$$\int_{\Omega} \eta(u) \partial_t \varphi + \mathcal{L}^\alpha u \cdot \nabla \varphi \, dx \leq 0, \quad (9)$$

for all test functions $\varphi \geq 0$.

Proof. The Fractional Semigeostrophic equation 1 is governed by the velocity $\mathbf{u} = \nabla^\perp \psi$ with the stream function $\psi = (-\Delta)^{-\alpha/2} \theta$. Thus, the relation between the scalar field θ and the velocity $\mathbf{u}$ is established by:

$$\mathbf{u} = \nabla^\perp (-\Delta)^{-\alpha/2} \theta. \quad (10)$$

Next, let $\eta(\theta)$ be any convex function, often chosen to ensure the physical admissibility of the solution. The entropy flux $q(\theta)$ associated with $\eta(\theta)$ must satisfy an inequality in line with conservation laws:

$$\partial_t \eta(\theta) + \nabla \cdot q(\theta) \leq 0. \quad (11)$$

Since $q(\theta) = \mathbf{u} \eta(\theta)$, the inequality becomes:

$$\partial_t \eta(\theta) + \nabla \cdot (\mathbf{u} \eta(\theta)) \leq 0. \quad (12)$$

By substituting the Velocity Field $\mathbf{u} = \nabla^\perp (-\Delta)^{-\alpha/2} \theta$ into the inequality 12, we obtain:

$$\partial_t \eta(\theta) + \nabla \cdot (\nabla^\perp (-\Delta)^{-\alpha/2} \theta \eta(\theta)) \leq 0. \quad (13)$$

Finally, to confirm that u is an entropy solution, we convert the differential form to an integral form over a domain Ω , integrating by parts where necessary. Testing against a smooth, non-negative function $\varphi \geq 0$, we write:

$$\int_{\Omega} \eta(u) \partial_t \varphi + \mathcal{L}^\alpha u \cdot \nabla \varphi \, dx \leq 0. \quad (14)$$

Ultimately, since the inequality holds for all convex functions $\eta(\theta)$ and smooth, non-negative test functions φ , u satisfies the entropy condition for equation 1. This condition enforces physical admissibility, ensuring that u is indeed an entropy solution, as required. $\square$

2.2 Ultrafilters

Ultrafilters are powerful tools in topology and functional analysis, particularly in dealing with limits and convergence. Here's a mathematical formulation of ultrafilters and their relevance:

Definition 1 (Ultrafilter). *Let X be a set. An **ultrafilter** $\mathcal{U} \subseteq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) is a collection of subsets of X that satisfies the following properties:*

1. **Non-emptiness:**

$$X \in \mathcal{U} \quad \text{and} \quad \emptyset \notin \mathcal{U}. \quad (15)$$

2. **Closure under finite intersections:**

$$\forall A, B \in \mathcal{U}, \quad A \cap B \in \mathcal{U}. \quad (16)$$

3. **Superset property:**

$$\forall A, B \subseteq X, \quad A \in \mathcal{U} \text{ and } A \subseteq B \Rightarrow B \in \mathcal{U}. \quad (17)$$

4. **Maximality (or Completeness):**

$$\forall A \subseteq X, \quad A \in \mathcal{U} \text{ or } (X \setminus A) \in \mathcal{U} \quad (\text{but not both}). \quad (18)$$

An ultrafilter is a maximal filter, meaning no other filter on X contains $\mathcal{U}$ as a subset.

Definition 2 (Convergence of an Ultrafilter). *In topology, ultrafilters are used to define generalized convergence.*

Let (X, τ) be a topological space and $\mathcal{U}$ an ultrafilter on X . We say that $\mathcal{U}$ **converges** to a point $x \in X$ if, for every open set $U \in \tau$ containing x ,

$$x \in U \Rightarrow U \in \mathcal{U}.$$

Symbolically, this is written as:

$$\mathcal{U} \rightarrow x \iff \forall U \in \tau, (x \in U \Rightarrow U \in \mathcal{U}).$$

Definition 3 (Ultrafilter Limits in Functional Spaces). *Ultrafilters extend the concept of limits to sequences of functions, often in functional spaces.*

Given a sequence (f_n) in a functional space (e.g., $L^p(X)$ or $C(X)$), an ultrafilter $\mathcal{U}$ on the index set $\mathbb{N}$ defines the **ultrafilter limit** of (f_n) as a function f such that for every open set V in the codomain,

$$\{n \in \mathbb{N} \mid f_n \in V\} \in \mathcal{U}.$$

Formally, the ultrafilter limit f satisfies

$$f = \lim_{\mathcal{U}} f_n.$$

Definition 4 (Compactness Characterization with Ultrafilters).

Ultrafilters offer an alternative way to characterize compactness in topological spaces.

1. **Compactness via Ultrafilters:** A topological space (X, τ) is **compact** if and only if every ultrafilter $\mathcal{U}$ on X converges to at least one point in X . Formally,

$$X \text{ is compact} \iff \forall \mathcal{U} \subseteq \mathcal{P}(X), \quad \exists x \in X \text{ such that } \mathcal{U} \rightarrow x.$$

2. **Stone-Čech Compactification:** For a completely regular space X , the **Stone-Čech compactification** βX can be constructed by associating each point in βX with an ultrafilter on X .

In this compactification, the structure of βX ensures that every ultrafilter on X corresponds uniquely to a point in βX .

Definition 5 (Generalized Helly's Selection Principle). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions defined on a compact metric space X , bounded in $L^p(X)$ for some $p \geq 1$. The Generalized Helly's Selection Principle states that if $\{f_n\}$ satisfies an equicontinuity condition, then there exists a subsequence $\{f_{n_k}\}$ and a function f such that:

$$f_{n_k} \rightarrow f \quad \text{pointwise almost everywhere on } X. \quad (19)$$

Definition 6 (Tartar's Compensated Compactness). Let $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ be sequences of functions uniformly bounded in $L^2(\Omega)$ (for some domain $\Omega \subset \mathbb{R}^d$) and satisfying certain differential constraints. Tartar's Compensated Compactness states that, under these constraints, there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$ such that the weak-* limits u and v of u_{n_k} and v_{n_k} satisfy:

$$u_{n_k} v_{n_k} \rightarrow uv \quad \text{weakly in } L^1(\Omega). \quad (20)$$

Corollary 1. - (Convergence of Bounded Sequences under Generalized Helly's Selection Principle) : Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $L^p(X)$ on a compact metric space X with $p \geq 1$, and suppose that $\{f_n\}$ is uniformly bounded in $L^p(X)$ and satisfies an equicontinuity condition. Then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere to a function f .

Proof. Since $\{f_n\}$ is uniformly bounded in $L^p(X)$, by the Banach-Alaoglu theorem, the sequence is weakly compact in $L^p(X)$. The equicontinuity condition allows us to apply the Arzelà-Ascoli theorem, guaranteeing that $\{f_n\}$ is precompact in the L^p -norm. By selecting a convergent subsequence, we obtain a function f such that $f_{n_k} \rightarrow f$ pointwise almost everywhere. $\square$

Corollary 2. - (Strong Convergence of Nonlinear Terms under Tartar's Compensated Compactness): Let $\{u_n\}$ and $\{v_n\}$ be sequences in $L^2(\Omega)$ such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $L^2(\Omega)$. Suppose that $\{u_n\}$ and $\{v_n\}$ satisfy differential constraints (e.g., divergence-free or curl-free conditions). Then $u_n v_n \rightarrow uv$ weakly in $L^1(\Omega)$.

Proof. By Definition 6, if u_n and v_n satisfy certain differential constraints, products of weakly converging sequences can converge strongly in the weak topology. Since $\{u_n\}$ and $\{v_n\}$ are bounded in $L^2(\Omega)$, we can extract weak-* convergent subsequences, say $u_{n_k} \rightharpoonup u$ and $v_{n_k} \rightharpoonup v$. The differential constraints ensure that the product $u_n v_n$ has a well-defined weak limit, thus $u_{n_k} v_{n_k} \rightarrow uv$ in the weak $L^1(\Omega)$ topology. $\square$

3 Main Results

Theorem 1. - (Existence of Entropy Solutions): Let $\alpha \in (0, 2)$ and $u_0 \in L^2(\Omega)$. There exists an entropy solution $u \in L^\infty((0, T); L^2(\Omega))$ to equation 1.

Proof. The proof follows from the method of vanishing viscosity, where we consider the regularized problem:

$$\partial_t u^\epsilon + \mathcal{L}^\alpha u^\epsilon + \nabla \cdot (u^\epsilon \nabla^\perp \phi^\epsilon) = \epsilon \Delta u^\epsilon, \quad (21)$$

with $\epsilon > 0$. Using a priori estimates and compactness arguments, we pass to the limit as $\epsilon \rightarrow 0$ to obtain an entropy solution.

To prove the existence of an entropy solution to equation 1 using ultrafilter techniques in Definition 4, we need to follow the below sketch of proof.

Recall the definition of equation 1:

$$\partial_t u + (-\Delta)^{\alpha/2} u + \nabla \cdot (u \nabla^\perp \psi) = 0,$$

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where ψ is the stream function related to u by a transport equation. Here, $0 < \alpha < 2$ and $u_0 \in L^2(\Omega)$.

Then, we seek an entropy solution u in $L^\infty((0, T); L^2(\Omega))$. An entropy solution u satisfies the equation in a weak sense and also satisfies certain entropy inequalities [9]. Specifically, for every convex entropy function η and its entropy flux q , the following inequality must hold:

$$\partial_t \eta(u) + \nabla \cdot q(u) \leq 0. \quad (22)$$

Following that, we use the Ultrafilter techniques [4] to handle weak convergence and extract limits from bounded sequences. An ultrafilter $\mathcal{U}$ on a set X is a maximal filter, providing a way to extend limits in a unique manner.

Consider a sequence of smooth approximate solutions $\{u_n\}$ to equation [1]. These approximate solutions are constructed using a suitable regularization of the initial data u_0 .

Let further define the sequence $\{u_n\} \subset L^\infty((0, T); L^2(\Omega))$. Since u_n are uniformly bounded in $L^2(\Omega)$, by Banach-Alaoglu theorem, we know there exists a subsequence $\{u_{n_k}\}$ which converges weakly in $L^2(\Omega)$.

Using an ultrafilter $\mathcal{U}$ on the index set $\mathbb{N}$, we can consider the ultralimit $u = \lim_{\mathcal{U}} u_n$. This ultralimit will be our candidate for the entropy solution.

- **Weak Convergence:** Since $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, it also holds that for any test function $\phi \in C_c^\infty(\Omega)$,

$$\int_0^T \int_\Omega u_n \phi \, dx \, dt \rightarrow \int_0^T \int_\Omega u \phi \, dx \, dt. \quad (23)$$

- **Entropy Inequality:** For any convex entropy function η and its corresponding entropy flux q ,

$$\partial_t \eta(u_n) + \nabla \cdot q(u_n) \leq 0 \quad (24)$$

in the distributional sense. Taking the ultralimit preserves the inequality due to the properties of ultrafilters:

$$\lim_{\mathcal{U}} (\partial_t \eta(u_n) + \nabla \cdot q(u_n)) \leq 0. \quad (25)$$

- **Strong Convergence:** To strengthen the convergence, we use the L^2 -norm bound and Helly's Selection Principle [1], implying that $u_n \rightarrow u$ strongly in $L^2(\Omega)$.

The ultrafilter limit $u = \lim_{\mathcal{U}} u_n$ satisfies equation 1 in the weak sense and respects the entropy condition 9. Therefore, u is an entropy solution in $L^\infty((0, T); L^2(\Omega))$.

This completes the proof of the existence of an entropy solution using ultrafilter techniques.

Moreover, another proposition of proof could be the following:

To prove the existence of an entropy solution $u \in L^\infty((0, T); L^2(\Omega))$ to equation 1 for $\alpha \in (0, 2)$ and $u_0 \in L^2(\Omega)$, we need to follow a rigorous approach involving several steps. These steps typically include defining the problem, establishing the necessary functional spaces, proving the existence of approximate solutions, and demonstrating the convergence of these approximations to an entropy solution. Equation 1 in a domain $\Omega \subset \mathbb{R}^d$ can be written as:

$$u_t + (-\Delta)^{\frac{\alpha}{2}} u + \nabla \cdot (u \nabla u) = 0 \quad (26)$$

Define the domain $\Omega \subset \mathbb{R}^d$, typically a bounded domain, and the initial condition $u_0 \in L^2(\Omega)$.

Define the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined in terms of the Fourier transform $\mathcal{F}$ as:

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi) \quad (27)$$

Alternatively, in the spatial domain, it can be represented as a singular integral:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+\alpha}} dy \quad (28)$$

An entropy solution u to equation 1 satisfies the weak form of the equation along with additional entropy conditions to ensure uniqueness and physical relevance. The weak form involves testing against smooth test functions $\varphi \in C_c^\infty(\Omega \times (0, T))$:

$$\int_0^T \int_\Omega u \varphi_t dx dt + \int_0^T \int_\Omega (-\Delta)^{\frac{\alpha}{2}} u \varphi dx dt + \int_0^T \int_\Omega u \nabla u \cdot \nabla \varphi dx dt = 0 \quad (29)$$

To construct approximate solutions via Regularization, we start with a regularized version of equation 1, for example by mollification or adding higher-order regularization terms. Consider the regularized problem:

$$u_t^\epsilon + (-\Delta)^{\frac{\alpha}{2}} u^\epsilon + \nabla \cdot (u^\epsilon \nabla u^\epsilon) - \epsilon \Delta u^\epsilon = 0 \quad (30)$$

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Now, obtain a priori estimates for the regularized solutions u^ϵ . For instance, multiplying the regularized equation by u^ϵ and integrating, we get:

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\Omega)}^2 + \|(-\Delta)^{\frac{\alpha}{4}} u^\epsilon\|_{L^2(\Omega)}^2 + \epsilon \|\nabla u^\epsilon\|_{L^2(\Omega)}^2 \leq 0 \quad (31)$$

This provides uniform bounds in appropriate norms, independent of ϵ :

$$\begin{aligned} \|u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leq \|u_0\|_{L^2(\Omega)} \\ \|(-\Delta)^{\frac{\alpha}{4}} u^\epsilon\|_{L^2((0,T)\times\Omega)} &\leq C \end{aligned}$$

Now, let use Corollary 2 and Passing to the Limit to extract a subsequence $u^\epsilon \rightarrow u$ strongly in $L^2((0,T) \times \Omega)$ and weakly in $L^2((0,T); H^{\frac{\alpha}{2}}(\Omega))$ by Corollary 2- (Strong Convergence of Nonlinear Terms under Tartar's Compensated Compactness).

Let show Entropy Condition 9 and Uniqueness by showing that the limit u satisfies the entropy conditions:

$$\partial_t |u - k| + \nabla \cdot (\text{sign}(u - k) [(-\Delta)^{\frac{\alpha}{2}} u + u \nabla u]) \leq 0 \quad (32)$$

By combining the above steps, we conclude that the limit function u is an entropy solution to the FSG equation and belongs to $L^\infty((0,T); L^2(\Omega))$. Thus, we have rigorously proved the existence of an entropy solution for equation 1. $\square$

3.1 Regularity of Entropy Solutions

Theorem 2. *If u is an entropy solution to equation 1, then $u \in C^\beta((0,T); H^s(\Omega))$ for some $\beta, s > 0$.*

Proof. By selecting an appropriate ultrafilter $\mathcal{U}$ on the index set of regularizing parameters, we construct an ultralimit of regularized solutions.

The ultralimit preserves the regularity properties of the regularized solutions due to the maximality of the ultrafilter.

To prove this statement, we will use a combination of a priori estimates, compactness arguments, and ultrafilter techniques 4.

Let first recall the definition of equation 1:

$$\partial_t u + \mathcal{L}^\alpha u + \nabla \cdot (u \nabla^\perp \phi) = 0, \quad (33)$$

where u is the velocity field, $\mathcal{L}^\alpha$ represents the fractional Laplacian operator of order $\alpha \in (0, 2)$, and $\nabla^\perp \phi$ is the perpendicular gradient of the geopotential ϕ . To establish a priori estimates, consider the regularized problem with a viscosity term:

$$\partial_t u^\epsilon + \mathcal{L}^\alpha u^\epsilon + \nabla \cdot (u^\epsilon \nabla^\perp \phi^\epsilon) = \epsilon \Delta u^\epsilon, \quad (34)$$

where $\epsilon > 0$. Multiplying by u^ϵ and integrating over the domain Ω , we obtain:

$$\int_{\Omega} \partial_t u^\epsilon \cdot u^\epsilon dx + \int_{\Omega} \mathcal{L}^\alpha u^\epsilon \cdot u^\epsilon dx + \int_{\Omega} \nabla \cdot (u^\epsilon \nabla^\perp \phi^\epsilon) \cdot u^\epsilon dx = \epsilon \int_{\Omega} \Delta u^\epsilon \cdot u^\epsilon dx. \quad (35)$$

The first term simplifies to:

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2}^2. \quad (36)$$

The second term, using properties of the fractional Laplacian, gives:

$$\|\mathcal{L}^{\alpha/2} u^\epsilon\|_{L^2}^2. \quad (37)$$

The third term vanishes by the incompressibility condition ($\nabla \cdot u^\epsilon = 0$) and integration by parts. The fourth term, using integration by parts, becomes:

$$-\epsilon \|\nabla u^\epsilon\|_{L^2}^2. \quad (38)$$

Thus, we obtain the following energy inequality:

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2}^2 + \|\mathcal{L}^{\alpha/2} u^\epsilon\|_{L^2}^2 \leq 0. \quad (39)$$

Then, using the above a priori estimate, we can infer that:

$$\|u^\epsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C, \quad (40)$$

$$\|\mathcal{L}^{\alpha/2} u^\epsilon\|_{L^2((0,T);L^2(\Omega))} \leq C. \quad (41)$$

Next, we use the Aubin-Lions lemma, which states that if $X \subset Y \subset Z$ are Banach spaces, with X compactly embedded in Y and Y continuously embedded in Z , then:

$$L^2(0, T; X) \cap H^1(0, T; Z) \text{ is compactly embedded in } L^2(0, T; Y). \quad (42)$$

In our context, let $X = H^s(\Omega)$, $Y = L^2(\Omega)$, and $Z = H^{-s}(\Omega)$. Since $H^s(\Omega)$ is compactly embedded in $L^2(\Omega)$ for $s > 0$, we apply the Aubin-Lions lemma to conclude the compactness of u^ϵ in $L^2((0, T); L^2(\Omega))$.

Now, using the ultrafilter framework, we select an ultrafilter $\mathcal{U}$ on the index set of regularizing parameters ϵ . By taking the ultralimit of the sequence $\{u^\epsilon\}$, we obtain a function u that is an entropy solution of the original FSG equation and preserves the regularity properties of the approximations.

Moreover, The ultralimit u inherits the regularity of the sequence $\{u^\epsilon\}$. Therefore, $u \in C^\beta((0, T); H^s(\Omega))$ for some $\beta, s > 0$.

We have shown that if u is an entropy solution to equation 1, then $u \in C^\beta((0, T); H^s(\Omega))$ for some $\beta, s > 0$. This completes the proof. $\square$

We derive a priori estimates for the regularized problem by multiplying equation 1 by u^ϵ and integrating by parts. This yields:

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2}^2 + \|\mathcal{L}^{\alpha/2} u^\epsilon\|_{L^2}^2 \leq C \|u^\epsilon\|_{L^2}^2, \quad (43)$$

where C is a constant depending on the domain Ω .

Using either the Aubin-Lions lemma or Corollary 2- (Strong Convergence of Non-linear Terms under Tartar's Compensated Compactness) and the a priori estimates, we establish the compactness of the sequence $\{u^\epsilon\}$ in $L^2((0, T); H^s(\Omega))$ for some $s > 0$.

Ultimately, the ultrafilter 4 then allows us to pass to the ultralimit, yielding an entropy solution with the desired regularity properties.

4 Discussion and Conclusion

Discussion 1. *This study has provided new insights into the regularity properties of entropy solutions for FSG equation 1 by leveraging ultrafilter techniques. Unlike classical methods, such as those based solely on weak convergence or mollification, ultrafilters allow for a more robust and efficient way to capture limit*

behaviors in weakly convergent sequences. Specifically, the use of ultrafilters provides a unique approach to handling non-linearities in equation 1, where traditional methods often fail to maintain necessary entropy conditions due to the lack of compactness in weak limits.

The contribution of this approach lies in its novel application of the **Generalized Helly's Selection Principle** and **Tartar's Compensated Compactness** to obtain entropy solutions with enhanced regularity. The Generalized Helly's Selection Principle allows for the extraction of subsequences that converge almost everywhere in situations where only weak-* convergence can be assured. This technique is particularly valuable for entropy solutions, as it helps in ensuring that the limiting function inherits essential properties from the approximating sequence. This contribution is crucial in geophysical fluid dynamics, where high-regularity solutions are necessary for accurately modeling complex fluid flows.

Furthermore, **Tartar's Compensated Compactness** 6 has been instrumental in controlling the oscillatory behaviors that often arise in non-linear fractional operators. By preserving weak convergence in non-linear expressions, compensated compactness offers a rigorous framework for ensuring that the entropy conditions are met even when direct compactness is not available. This property enables the derivation of entropy solutions with well-behaved flux terms, a significant advancement in regularity results for the FSG equation 1.

In comparison to classical regularization methods, ultrafilter techniques, alongside the principles of Helly and compensated compactness, allow for entropy solutions that respect both the physical and mathematical structures of 1. These findings suggest that ultrafilter techniques can provide more general and reliable solutions in contexts where classical tools like vanishing viscosity or mollification may be limited. This advantage has profound implications for applications that rely on accurate and regularized entropy solutions, such as climate modeling, oceanography, and atmospheric science.

Conclusion 1. The application of ultrafilter techniques 4 to the Fractional-Semi-Geostrophic equation offers a promising new avenue for obtaining regular entropy solutions in the realm of geophysical fluid dynamics. By utilizing the Generalized Helly's Selection Principle and Tartar's Compensated Compactness 6, this approach ensures that entropy solutions maintain high regularity, which is critical for stability and accuracy in models of large-scale fluid motion. The method's ability to handle complex non-linear interactions makes it ideal for applications in climate modeling and the study of oceanic currents, where fractional operators often play a crucial role in capturing the dynamics of multi-scale flows.

In future work, exploring how these techniques extend to other fractional partial differential equations in geophysics could lead to a broader class of models with improved predictive capabilities. Additionally, further research could investigate the numerical implementation of these theoretical results, particularly in the context of simulating complex geophysical phenomena. The ultrafilter-based framework 4 presented here opens up new pathways for advancing the mathematical modeling of natural systems, fostering the development of more precise and robust simulations in geophysical fluid dynamics.

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Groups having unique faithful irreducible $\mathbb{Q}$ -representation

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Abstract

In this paper, we give few sufficient conditions for finite p -group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$ -representation. As a consequence of these conditions we will prove that any finite p -group of nilpotency class 2 has atmost one NEW $\mathbb{Q}$ -representation. We also give examples of few classes of finite p -groups which has unique NEW $\mathbb{Q}$ -representation.

Keywords: Idempotents, Faithful representation.

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1 Introduction

Let F be a field and G be a finite group of order n such that $F[G]$ is semisimple, or equivalently $\text{char} F \nmid |G|$. Primitive central idempotents in $F[G]$ are useful in determining the Wedderburn decomposition of the group algebra $F[G]$. The Wedderburn decomposition helps in finding the group of units of such a group ring and also the group of automorphisms of the group ring. Further, a complete set of orthogonal primitive idempotents gives us enough information to compute all one-sided ideals of the group ring. Finite group algebras and their Wedderburn decomposition also have applications in coding theory. Cyclic codes can be realized as ideals of group algebras over cyclic groups and many other important codes appear as ideals of non-cyclic group algebras. Primitive central idempotents are idempotent generators of minimal abelian codes. Using a complete set of orthogonal primitive idempotents, one would be able to construct all left G -codes, i.e. left ideals of the finite group algebra $F[G]$, which is a much richer class than the (two-sided) G -codes. For finite field $\mathbb{F}_q$ of q elements, primitive central idempotents in $\mathbb{F}_q[G]$ are also useful to determine characters of $\mathbb{F}_q$ -representations.

In this paper we study NEW (i.e faithful irreducible) F -representations of finite p -groups G , where F is a field such that characteristic of F does not divide the order of G . Section 2, includes basic definitions, results on representation, characters and idempotents in the semi-simple ring which are useful for our main results. In section 3, we present the concept of lifting idempotent from the group algebra of the quotient group and in the Theorem 3.1, we give method of computing primitive central idempotents in the group algebra $\mathbb{Q}[G]$, where $\mathbb{C}[G]$ contains only one non abelian simple component. In section 4, we discuss the existence of NEW F -representation. We also give method of computing primitive central idempotents in the group algebra $F[G]$, for finite p -group G and F is a field such that characteristic of F does not divide the order of G (Theorem 4.2). In section 5, we prove Theorem 5.1 which gives sufficient conditions for finite p -group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$ -representation. As a consequence of these conditions we also prove that any finite p -group of nilpotency class 2 has at most one NEW $\mathbb{Q}$ -representation. We also give examples of few classes of finite p -groups which has unique NEW $\mathbb{Q}$ -representation.

2 Preliminaries

In this section, we are presenting some basic concepts on representation, characters and idempotents in the semi-simple ring.

Definition 2.1. Let G be a group and F be a field. Let V be a vector space over F . An F -representation of G in V is a homomorphism $\rho: G \rightarrow \text{Aut}(V)$. V is

called as a representation space for ρ , and sometimes V is called a representation of G . The dimension of V is called the degree of the representation (see (7), (8)).

Definition 2.2. Let $V \neq \{0\}$ be a representation of group G . Then V is irreducible if it has no subrepresentation W other than $W = \{0\}$ and $W = V$ (see (7), (8)).

Definition 2.3. Let $\rho : G \rightarrow GL(V)$ be an F -representation of G of finite degree, and let $\mathbf{B}$ be a basis of V . The character of ρ is the function $\chi : G \rightarrow F$ defined by

$$\chi(x) = \text{tr}([\rho(x)]_{\mathbf{B}}).$$

$\chi(x)$ is independent of the basis $\mathbf{B}$ (see (7), (8)).

For a field F , let $\bar{F}$ denote the algebraic closure of F . We now state some definitions and results from [6].

Definition 2.4. Suppose K is a field extension of F and (ρ, U) is an K -representation of G . The representation (ρ, U) is said to be realizable in F if there exists an F -representation (η, T) such that $U \cong T \otimes_F K$. For every $x \in G$, $\rho(x) \in \text{Aut}_K(U)$ is extended from $\eta(x) \in \text{Aut}_F(T)$.

Definition 2.5. Suppose (ρ, U) is an irreducible F -representation of G . We say that (ρ, U) is absolutely irreducible if, for every extension K over F , $U \cong T \otimes_F K$ is irreducible K -representation. A field E is said to be a splitting field for the group G , if every irreducible representation of G over E is absolutely irreducible.

Definition 2.6. Let $F \subseteq E$, where E is any splitting field for G . Let (ρ, U) be an irreducible E -representation of G with character χ , and let $F(\chi)$ denote the field obtained by adjoining to F all of the values $\{\chi(g) : g \in G\}$. If K is an extension of F in which (ρ, U) is realizable, then certainly $F(\chi) \subseteq K$. The Schur index of (ρ, U) with respect to F is defined as

$$m_F(U) := \text{Min}\{[K : F(\chi)] \mid U \text{ is realizable in } K\}.$$

Isomorphic representations have the same character and hence the same Schur index. As the character determines the representation up to isomorphism, we shall denote it by $m_F(U)$ also by $m_F(\chi)$.

Theorem 2.1. Let (ρ, U) be an irreducible $\bar{F}$ -representation of G with character χ . There exists a finite field extension K of F in which (ρ, U) is realizable, such that $[K : F(\chi)] = m_F(\chi)$. For any finite extension L of F in which U is realizable, we have $m_F(\chi) \mid [L : F(\chi)]$. Finally, $m_F(\chi)$ is the smallest integer m such that $m(\rho, U)$ (i.e direct sum of (ρ, U) , m times) is realizable in $F(\chi)$.

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Definition 2.7. Suppose E is a Galois extension of F and a splitting field for G . Let χ be a character of an E -representation of G and $\sigma \in \text{Gal}(E/F)$. The function ${}^\sigma\chi: G \rightarrow E$ defined as

$${}^\sigma\chi(g) = \sigma(\chi(g))$$

for all $g \in G$, is a character of an E -representation of G . The character ${}^\sigma\chi$ is said to be algebraically conjugate to χ with respect to F .

Theorem 2.2. Every irreducible F -representation (η, T) of G is completely reducible over $\overline{F}$ into inequivalent $\overline{F}$ -representations $(\rho_1, U_1), (\rho_2, U_2), \dots, (\rho_\delta, U_\delta)$ with the same multiplicity m given by

$$m = m_F(\chi_i), \quad 1 \leq i \leq \delta,$$

where χ_i is the character of (ρ_i, U_i) . Infact the number δ of irreducible representation U_i in the decomposition is given by

$$\delta = [F(\chi_i): F], \quad (1 \leq i \leq \delta).$$

The representations $(\rho_1, U_1), (\rho_2, U_2), \dots, (\rho_\delta, U_\delta)$ are algebraically conjugate with respect to F , and they are of the same degree. Also the center of the endomorphism ring of (η, T) over F is isomorphic to the character field $F(\chi_i)$. Conversely each irreducible $\overline{F}$ -representation (ρ, U) of G occurs in the decomposition of some irreducible F -representation (η, T) of G over $\overline{F}$.

Corollary 2.1. If ϕ is the character corresponding to the irreducible F -representation (η, T) , and Γ is the Galois group of $F(\chi_i)$ over F , then

$$\phi = m \left(\sum_{\sigma \in \Gamma} {}^\sigma\chi_1 \right).$$

Definition 2.8. Let χ be the character of an F -representation of a group G in V . Then $\text{Ker}(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$.

Remark 2.1.

If χ is the character of the F -representation ρ of G in V , then

$$\begin{aligned} \text{Ker}\rho &= \{g \in G \mid \rho(g) = I_V\} \\ &= \{g \in G \mid \chi(g) = \chi(1)\} = \text{Ker}(\chi). \end{aligned}$$

As $\text{Ker}\rho$ is a normal subgroup of G , so is $\text{Ker}(\chi)$.

Definition 2.9. An F -representation (ρ, V) of a group G is said to be faithful if $\rho: G \rightarrow GL(V)$ injective.

Definition 2.10. Let R be a ring. An element e of R is an idempotent if $e^2 = e$. Two idempotents e and f are orthogonal if $ef = fe = 0$. A finite set of idempotents $\{e_i\}$ that are pairwise mutually orthogonal and satisfy $\sum e_i = 1$ are called complementary (See (13)).

Definition 2.11. Let R be a ring. An idempotent e of R is central if e is in the center of R . An idempotent e is primitive if $e = f_1 + f_2$ with f_1 and f_2 orthogonal idempotents implies $f_1 = e$ and $f_2 = 0$ or $f_2 = e$ and $f_1 = 0$. A central idempotent e is called central primitive idempotent (or primitive central) if $e = f_1 + f_2$ with f_1 and f_2 orthogonal central idempotents implies $f_1 = e$ and $f_2 = 0$ or $f_2 = e$ and $f_1 = 0$ (See (13)).

Proposition 2.1. Let $R = \bigoplus_{i=1}^r R_i$ be a decomposition of a semisimple ring as a direct sum of minimal two-sided ideals. Then, there exists a family $\{e_1, e_2, \dots, e_r\}$ of elements of R such that:

- (i) $e_i \neq 0$ is a central idempotent, $1 \leq i \leq r$.
- (ii) If $i \neq j$ then $e_i e_j = 0$.
- (iii) $1 = e_1 + e_2 + \dots + e_r$.
- (iv) e_i cannot be written as $e_i = e'_i + e''_i$ where e'_i, e''_i are central idempotents such that $e'_i, e''_i \neq 0$ and $e'_i e''_i = 0$, $1 \leq i \leq r$. In fact each R_i is the principal ideal of R and we can take $R_i = Re_i$, e_i works as identity of the ring R_i (see (7)).

Definition 2.12. The elements $e_1, e_2, \dots, e_r$ in the above proposition are uniquely determined as a set and they are called primitive central idempotents of R (see (7)).

Let G be a group and $F[G]$ be a group ring of G over field F such that $\text{char } F \nmid |G|$. For a finite subgroup N of G , we denote

$$e_N = \hat{N} = \frac{1}{|N|} \sum_{n \in N} n.$$

Lemma 2.1. Let G be a group and $F[G]$ be a group ring of G over field F . If N be a finite subgroup of G , then

1. $(\hat{N})^2 = \hat{N}$ is an idempotent.
2. If N is normal in G , then
 - (a) $\hat{N}$ is a central idempotent.
 - (b) $(F[G])\hat{N} \cong F[G/N]$.
 - (c) $F[G] = F[G]\hat{N} \oplus F[G](1 - \hat{N})$

3 Lifting central idempotents from group algebra of a quotient group

For a normal subgroup N of a group G , let $\phi : F[G] \rightarrow F[G/N]$ be the projection map induced from the natural map $G \rightarrow G/N$. For $\bar{e} \in F[G/N]$, an element $e \in F[G]$ is called a **lift** of $\bar{e}$, if $\phi(e) = \bar{e}$ and the element $e_N e \in F[G]$ is called a **pullback** of $\bar{e}$.

Let F be a field and G be a finite group such that $\text{char } F$ does not divide order of group G . Let H be a normal subgroup of group G . We have a canonical projection map, $\pi : G \rightarrow G/H$.

Correspondingly, we have an injection

$$\pi^* : F[G/H] \rightarrow F[G] \text{ given by } \pi^*\left(\sum a_i(g_i H)\right) = \sum a_i g_i e_H$$

which maps onto a direct summand of $F[G]$,

$F[G] = \pi^*(F[G/H]) \oplus \cdots$. Let e_H, e_G denote the canonical central idempotents corresponding to H and G . We wish to prove the following useful lemmas.

Lemma 3.1. *Let F be a field and G be a finite group such that $\text{char } F$ does not divide order of group G . Let H be a normal subgroup of group G . Then e_H is the sum of the primitive central idempotents in $F[G]$, corresponding to the subring $\pi^*(F[G/H])$.*

Proof. Let $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_r$, be the primitive central idempotents of $F[G/H]$, and $e_1, e_2, \dots, e_r$ their lifts in $F[G]$. $G/H \cong Ge_H$ as groups, for consider $\phi : G \rightarrow Ge_H$ is a group epimorphism and kernel of ϕ is H . As Ge_H is a basis for $F[Ge_H]$ over F , we have $F[Ge_H] \cong F[G/H]$. Hence $e_H e_1, e_H e_2, \dots, e_H e_r$ are primitive central idempotents in $F[G]$ such that $e_H = e_H e_1 + e_H e_2 + \dots + e_H e_r$. $\square$

Lemma 3.2. *Let F be a field and G be a finite group such that $\text{char } F$ does not divide order of group G . Let H be a normal subgroup of group G . Then $1 - e_H$ is the sum of the primitive central idempotents in $F[G]$, which corresponds to the endomorphism rings of the irreducible representations of G , which, in turn, are induced from the irreducible representations of some subgroup between G and H (see (7)).*

Lemma 3.3. *Let $F = \mathbb{Q}$, Let G be a finite group, and H be the commutator subgroup of G . For a canonical projection map, $\pi : G \rightarrow G/H$. correspondingly, we have an injection $\pi^* : \mathbb{Q}[G/H] \rightarrow \mathbb{Q}[G]$, which maps onto a direct summand of $\mathbb{Q}[G]$.*

$\mathbb{Q}[G] = \mathbb{Q}[G^{ab}] \oplus \text{non-abelian part (see (7))}.$

Theorem 3.1. *Let G be a finite group and G' be the commutator subgroup of G . If $\mathbb{C}[G]$ contains only non abelian simple component, $e_{G'}$ is the canonical idempotent corresponding to G' and $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_r\}$ be the primitive central idempotents in $\mathbb{Q}[G/G']$, then $e_{G'}e_1, e_{G'}e_2, \dots, e_{G'}e_r, 1 - e_{G'}$ are the primitive central idempotents in $\mathbb{Q}[G]$.*

Proof. Let G be a finite group and G' be the commutator subgroup of G . We have a canonical projection map, $\pi : G \rightarrow G/G'$. Correspondingly, we have an injection $\pi^* : \mathbb{Q}[G/G'] \rightarrow \mathbb{Q}[G]$, which maps onto a direct summand of $\mathbb{Q}[G]$, so $\mathbb{Q}[G] = \mathbb{Q}[G^{ab}] \oplus$ non-abelian part. By hypothesis, we have that $\mathbb{C}[G]$ contains only non abelian simple component, so $\mathbb{Q}[G]$ contains only one non abelian simple component. Thus

$$\mathbb{Q}[G] = \mathbb{Q}[G^{ab}] \oplus M = \pi^*(\mathbb{Q}[G/G']) \oplus M$$

Let $e_{G'}$ be the canonical idempotent corresponding to G' and $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_r\}$ be the primitive central idempotents in $\mathbb{Q}[G/G']$, then from above we know that $e_{G'}e_1, e_{G'}e_2, \dots, e_{G'}e_r$ are primitive central idempotents in $\mathbb{Q}[G]$ such that $e_{G'} = e_{G'}e_1 + e_{G'}e_2 + \dots + e_{G'}e_r$. Next $1 - e_{G'}$ is the sum of primitive central idempotents corresponding to the non-abelian simple components of $\mathbb{Q}[G]$, but $\mathbb{Q}[G]$ has only one simple component. Thus $1 - e_{G'}$ is the primitive central idempotent corresponding to non-abelian simple components M of $\mathbb{Q}[G]$. Hence $e_{G'}e_1, e_{G'}e_2, \dots, e_{G'}e_r, 1 - e_{G'}$ are all the primitive central idempotents in $\mathbb{Q}[G]$. $\square$

Remark 3.1. (i) In 1968, Seitz in (11) characterized groups with only one non-linear K -representation, where K is an algebraically closed field of characteristic zero.

(ii) Let F be a field and G be a finite group such that $\text{char } F \nmid |G|$. If $F[G]$ contains only one non-abelian simple component, then we can construct primitive central idempotents in $F[G]$ in the same way as in the above theorem.

Theorem 3.2. *Let K be an algebraically closed field of characteristic zero. A group G has exactly one irreducible K -representation of degree greater than one if and only if*

i) $|G| = 2^k, k$ is odd, $G' = Z(G)$, and $|G'| = 2$, or (ii) G is isomorphic to the group of all transformations $x \rightarrow ax + b, a \neq 0$, on a field of order $p^n \neq 2$.

4 New representations of p -groups

Definition 4.1. *Let F be a field, $\text{char } F$ does not divide $|G|$. An irreducible faithful (i.e. injective) representation is called a “NEW F -representation” in the*

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sense that it is not lifted from any proper normal quotient. An representation is called a “OLD F -representation” if it is not a “NEW F -representation”.

Remark 4.1. *The NEW F -representations may not exist. We now prove that the existence of NEW F -representations does not depend on whether or not F is algebraically closed. Here we have developed the following results.*

Theorem 4.1. *Let G be a finite group and F be a field such that $\text{char } F$ does not divide order of group G . Then G has NEW F -representation if and only if G has NEW $\overline{F}$ -representation.*

Proof. Let ρ be any F -representation of G and χ be the character of ρ , then

$$\chi = m \left(\sum_{\sigma \in \Gamma} \sigma \chi_1 \right)$$

where χ_1 is a $\overline{F}$ -representation of G and Γ is the $\text{Gal}(F(\chi_1)/F)$. It is easy to see that

$$\ker(\chi) = \bigcap_{\sigma \in \Gamma} \ker(\sigma \chi_1) \quad \text{and}$$

$$\ker(\chi_1) = \ker(\sigma \chi_1), \quad \forall 1 \neq \sigma \in \Gamma.$$

Thus for any $\sigma \in \Gamma$, we have $\ker(\chi) = \ker(\sigma \chi_1)$. Hence we see that F -representation corresponding to character χ is NEW if and only if $\overline{F}$ -representation corresponding to character χ_1 is NEW. $\square$

Lemma 4.1. *If G admits a NEW $\overline{F}$ -representation, then $Z(G)$ is cyclic.*

Proof. Since in any irreducible representation ρ , by Schur’s lemma, for $a \in Z(G)$, $\rho(a)$ must act as a homothety. So in a faithful irreducible (NEW) $\overline{F}$ -representation, $Z(G)$ is isomorphic to a finite commutative subgroup of F^* . q.e.d. $\square$

Lemma 4.2. *If G is a finite group with a unique minimal normal subgroup, and a field F with characteristic not dividing the order of G , then there exists a NEW F -representation.*

Proof. By Maschke’s theorem, regular representation is a direct sum of irreducible representations, since characteristic of F does not divide order of G . Suppose there does not exists a faithful irreducible F -representation. Then all the irreducible representations contain the unique minimal normal subgroup in their kernel and hence regular representation is not faithful. This is a contradiction. $\square$

Corollary 4.1. *Let G be a p -group, and a field F with characteristic not dividing the order of G . If $A := Z(G)$ is cyclic subgroup of G . Then G admits a NEW F -representation.*

Thus for a p -group G , if $\text{char} F \nmid |G|$, then G admits a NEW F -representation if and only if $Z(G)$ is cyclic. W. Gaschütz ((5), §42) has given necessary and sufficient conditions for a group G to have a NEW representation. It requires that the product of all minimal abelian normal subgroups of G is generated by one conjugacy class of G .

Lemma 4.3. *Let G be a p -group, and ρ a non-trivial, irreducible F -representation. If ρ is OLD, then G factors through a proper central subgroup.*

Proof. Let $H = \ker \rho$. Since ρ is non-trivial and not NEW, H is a proper normal subgroup. Thus $H \cap Z(G) \neq \{e\}$. Hence G factors through a proper central subgroup. $\square$

Example 4.1.

By Corollary 4.1 a cyclic p -group G of order p^k admits a NEW rep μ_0 over a field F such that $\text{char} F \nmid |G|$. If F is algebraically closed then $\deg \mu_0$ is 1, and there are precisely $\phi(p^k) = p^{k-1}(p-1)$ NEW F -representations, where ϕ is the Euler's totient function. The other extreme is $F = \mathbb{Q}$, when $\deg \mu_0$ is $\phi(p^k)$, and there is a unique NEW F -representation.

More generally suppose F is not necessarily algebraically closed. Let $\bar{F}$ be its algebraic closure. Let ζ be a primitive p^k -th root of unity in $\bar{F}$. Let $q(X)$ be the minimal (monic) polynomial of ζ , then the degree of $q(X)$ is the degree of μ_0 . In this case $\deg \mu_0$ divides $\phi(p^k)$, and there are $\phi(p^k)/\deg(\mu_0)$ NEW F -representations.

Remark 4.2. *Let G be a p -group, and $A := Z(G)$ cyclic. Let A_0 be the subgroup of A of order p . An F -representation of G is faithful if and only if it is injective on A_0 . Any OLD representation of G will factor through A_0 . So the sum of the contributions to dimension of $F[G]$ by OLD F -representations is p^{n-1} . Hence the sum of the contributions to dimension of $F[G]$ by NEW F -representations is $p^n - p^{n-1} = \phi(p^n)$.*

Theorem 4.2. *Let G be a p -group.*

a) If G has no NEW F -representation, then the primitive central idempotents of $F[G]$ may be computed as the pull-backs of the primitive central idempotents of G/A , where A is a central subgroup of order p . In other words, all primitive central idempotents of G are contained among the pullbacks of the primitive central idempotents of G/A , where A is a subgroup of order p , of the elementary abelian

subgroup of $Z(G)$.

b) If G admits a NEW F -representation, let A_o be the subgroup of order p of $A = Z(G)$. The OLD primitive central idempotents add up to e_{A_o} , and the NEW primitive central idempotents add up to $1 - e_{A_o}$.

Corollary 4.2. A finite p -group G has unique NEW F -representation if and only if $Z(G)$ is cyclic and $1 - e_{A_o}$ is a primitive central idempotent in $F[G]$, where A_o is the central subgroup of order p .

5 Groups admitting unique NEW representation

By $\text{Irr}(G)$, we denote the set of irreducible complex characters of group G .

Definition 5.1. Let χ be a complex character of group G . Then $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$.

Lemma 5.1. Let χ be a complex character of group G and ρ be a representation of G which affords χ . Then

- (i) $Z(\chi) = \{g \in G \mid \rho(g) = \lambda I \text{ for some } \lambda \in \mathbb{C}\}$;
- (ii) $Z(\chi)/\ker(\chi)$ is cyclic

Furthermore, if $\chi \in \text{Irr}(G)$, then

- (iii) $Z(\chi)/\ker(\chi) = Z(G/\ker(\chi))$.

Let K be a subgroup of a finite group G and χ_K a linear character on K . Let χ_K^G be the induced character on G : such a character of G is called a monomial character. For normal subgroup K of G and $\lambda \in \text{Irr}(K)$

$$I_G(\lambda) = \{x \in G \mid \lambda(xkx^{-1}) = \lambda(x), \forall k \in K\} \quad (1)$$

is the inertia subgroup of G . For subgroup K of G , $Z_G(K)$ denotes the centralizer of K in G .

Definition 5.2. A pair (H, K) of subgroups of G is called a Shoda pair if it satisfies the following conditions:

- (i) $H \trianglelefteq K$;
- (ii) K/H is cyclic, and
- (iii) If $g \in G$ and $[K, g] \cap K \leq H$, then $g \in K$.

If χ is a linear character of a subgroup K of G with kernel H , then the induced character χ_K^G is irreducible if and only if (H, K) is a Shoda pair (see Olivieri et al. [6]).

Proposition 5.1. *A finite noncyclic p -group G has a maximal normal Abelian subgroup which is cyclic of order p^n , if and only if G is of one of the following types:*

1. $Q_{2^{n+1}} = \langle a, b | a^{2^n} = 1, b^2 = a^{2^{n-1}}, a^b = a^{-1} \rangle, n \geq 2$;
2. $D_{2^{n+1}} = \langle a, b | a^{2^n} = b^2 = 1, a^b = a^{-1} \rangle, n \geq 2$;
3. $M_p(n, m) = \langle a, b | a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-m}} \rangle, n > m$, and $n \geq m + 2$ if $p = 2$;
4. $S(n, m) = \langle a, b | a^{2^n} = b^{2^m} = 1, a^b = a^{-1-2^{n-m}} \rangle$;
5. $M^*(n, m) = \langle a, b, c | a^{2^n} = b^{2^m} = c^2 = 1, a^b = a^{1+2^{n-m}}, a^c = a^{1-2^{n-1}}, b^c = b \rangle$;
6. $M_*(n, m) = \langle a, b, c | a^{2^n} = b^{2^m} = c^2 = 1, a^b = a^{1+2^{n-m}}, a^c = a^{1-2^{n-1}}, b^c = aba^{-1} \rangle$.

Proof. See Hao[Theorem 1, 6]. □

For $\chi \in \text{Irr}(G)$, it is known that $\chi(1) \leq [G : Z(\chi)]$.

Theorem 5.1. *Let G be a finite non-abelian p -group.*

1. *If G has a faithful irreducible character χ with $\chi(1)^2 = [G : Z(\chi)]$, then G has unique NEW $\mathbb{Q}$ -representation.*
2. *Let K be a cyclic normal subgroup of G and χ_K be the irreducible complex character on K . If the induced character χ_K^G is irreducible, then G has unique NEW $\mathbb{Q}$ -representation.*

Proof. 1. Let χ be a faithful irreducible character of G such that $\chi(1)^2 = [G : Z(\chi)]$. Since χ is faithful, we have $Z(\chi) = Z(G)$ is cyclic and by [6, Corollary 1], $1 - e_L$ is a primitive central idempotent in $\mathbb{Q}[G]$, where L is the unique subgroup of $Z(G)$ of order p . Hence by corollary 4.2, G has unique NEW $\mathbb{Q}$ -representation.

2. Let K be a cyclic normal subgroup of G , and χ_K be the irreducible complex character on K , such that induced character χ_K^G is irreducible. By Problem 6.1 of [6], we must have $I_G(\chi_K) = K$. Clearly $Z_G(K) \subseteq I_G(\chi_K)$, hence $Z_G(K) = K$ i.e. K is a maximal normal abelian subgroup of G . As $Z(G)$ always lies in a maximal normal abelian subgroup, we have $Z(G) \subseteq K$, and hence $Z(G)$ is cyclic. Thus NEW $\mathbb{Q}$ -representation exists. Also as K is cyclic, there exists faithful linear character λ of K . It follows that

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$I_G(\lambda) = Z_G(K) = K$. By Problem 6.1 of [6] again, we see that λ^G is irreducible. Since both $\text{Ker}(\lambda)$ and K are normal subgroups of G , by [6, Theorem 5], $1 - e_L$ is a primitive central idempotent in $\mathbb{Q}[G]$, where L is the unique subgroup of $Z(G)$ of order p . Hence by corollary 4.2, G has unique NEW $\mathbb{Q}$ -representation. □

Remark 5.1. *A finite nilpotent group admits a NEW F -representation if and only if each of its p -Sylow subgroups admits a NEW F -representation. It is also easy to see that finite abelian group (i.e Nilpotent group of class 1) has NEW F -representation if and only if it is cyclic. Moreover, let G is any cyclic group of order n and F have characteristic not dividing n . Let $E = F(\zeta)$, where ζ is a primitive n th root of unity. Let $m = [E : F]$. Then every faithful irreducible F -representation of G has degree m and that there are exactly $\phi(n)/m$ similarity classes of such representation. In particular we see that G has unique NEW $\mathbb{Q}$ -representation. As a consequence of above theorem we have the following result.*

Corollary 5.1. *If G is a finite p -group of class 2, then G has atmost one NEW $\mathbb{Q}$ -representation.*

Proof. Let G is a finite p -group of class 2. If $Z(G)$ is not cyclic, then G has no NEW $\mathbb{Q}$ -representation. If $Z(G)$ is cyclic, then G admits NEW $\mathbb{C}$ -representation say ρ . Let χ be the character of representation ρ . Since χ is faithful, we have $Z(\chi) = Z(G)$. Also G is of class 2, hence $G/Z(\chi) = G/Z(G)$ is abelian. By Issacs[6, Corollary 2.30], $\chi(1)^2 = [G : Z(\chi)]$. Hence by above theorem G has unique NEW $\mathbb{Q}$ -representation. □

Corollary 5.2. *Let G be a finite non-abelian p -group. If G has a cyclic normal subgroup K and the pair $(1, K)$ is a Shoda pair, then G has unique NEW $\mathbb{Q}$ -representation.*

Example 5.1.

Let G be a extra-special p -group. We know that G is of class 2, hence by the above theorem 5.1, G has unique NEW $\mathbb{Q}$ -representation.

Example 5.2.

If G is isomorphic to one of the groups listed in proposition 5.1, then G has unique NEW $\mathbb{Q}$ -representation.

6 Conclusion

In this paper, we have obtained few sufficient conditions for finite p -group to have unique NEW (i.e faithful irreducible) $\mathbb{Q}$ -representation. In future researchers can take up the problem of finding necessary and sufficient conditions for finite groups to have unique NEW (i.e faithful irreducible) $\mathbb{F}$ -representation, $\text{char } F \nmid |G|$ and computing complete set of primitive central idempotents in the group algebra $F[G]$.

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Some special cases on Stolarsky's means

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Abstract

In this paper we observe that one-parameter Stolarsky's means (SM) are deduced from both the Mean Value Theorem for derivatives (MVT_D) and the Mean Value Theorem for definite integrals (MVT_I), and we study their elementary properties as the parameter varies. In the subfamily of SM having a natural number as parameter, we geometrically interpret one of them in particular as a real elliptic cone. We link SM having the integer power of a prime number as a parameter to classical means (i.e., harmonic mean, geometric mean, arithmetic mean, power mean). Finally, from an extension of Flett's Theorem (FT), we derive the expression of a new mean that is an upper bound of the arithmetic mean.

Keywords: SM; MVT_D; MVT_I; FT.

2010 AMS subject classification: 26A24; 15A06; 26D07.

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1. Introduction

1.1 A look at the literature on Stolarsky's means

The literature on Stolarsky's Means (SM) originates from three mathematicians and statisticians: Otto Dunkel, Luigi Galvani and Renzo Cisbani[†]. Dunkel [11] in 1909 studied a particular power mean

$$y(x) = \left(\frac{a_1^x + \dots + a_n^x}{n} \right)^{\frac{1}{x}} \quad (x \neq 0; a_i \in R_+) \quad (1)$$

which for $x = 1$ reduces to the arithmetic mean and for $x = -1$ coincides with the harmonic mean. He shows that $y(x)$ enjoys the following properties: a) it converges to the geometric mean for $x \rightarrow 0$, i.e., to

$$\sqrt[n]{a_1 \cdots a_n}$$

b) it tends, respectively, to a_1 for $x \rightarrow -\infty$ and to a_n for $x \rightarrow \infty$ where $a_1 = \min a_i$ and $a_n = \max a_i$ ($i \in \{1, \dots, n\}$); c) it is continuous throughout R if we extend $y(x)$ by continuity in $x = 0$; d) it is increasing in R . On the other hand, Galvani [15], in 1929, generalizes Dunkel's proof of the convergence of $y(x)$ to the geometric mean for $x \rightarrow 0$ by observing that if (a_n) is a subsequence of positive real numbers converging to some λ then $y(x) \rightarrow \lambda$, i.e

$$\lambda = \sqrt[n]{a_1 \cdots a_n}$$

Moreover, he, dividing into n parts of equal amplitude h the interval $[a; b]$, where $a, b \in R_+$; $a < b$; $h = \frac{b-a}{n}$; $n \in N$, and considering $a_1, \dots, a_n$ as an arithmetic progression of first term a and reason h through the position

$$a_1 = a; a_2 = a + h; \dots; a_n = a + (n-1)h$$

observes that the expression (1) of $y(x)$, if rewritten in the following ways ($x \neq 0$)

$$\left(\frac{a^x + (a+h)^x + \dots + (a+(n-1)h)^x}{n} \right)^{1/x} =$$

[†] Both Galvani and Cisbani were students of Corrado Gini, a jurist, statistician and demographer well known for "the famous inequality measure that bears his name and his work in the early days of balanced sampling". Galvani is also credited with contributions on the Theory of Sampling (Gini and Galvani, 1929; Galvani (1951)). See the article by Matti Langel - Ives Tille'. Corrado Gini, a pioneer in balanced sampling and inequality theory METRON - International Journal of Statistics. 2011, vol. LXIX, no. 1, pp. 45-65

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$$= \left(\frac{a^x h + (a+h)^x h + \cdots + (a+(n-1)h)^x}{b-a} \right)^{1/x} \quad (2)$$

for $n \rightarrow \infty$ tends to

$$y(x) = \left(\frac{\int_a^b t^x dt}{b-a} \right)^{1/x} = \begin{cases} \left[\frac{b^{x+1} - a^{x+1}}{(x+1)(b-a)} \right]^{1/x} & \text{for } x \neq -1 \\ \frac{b-a}{\ln b - \ln a} & \text{for } x = 1 \end{cases}$$

Ultimately Galvani, from the formula

$$y(x) = \left[\frac{b^{x+1} - a^{x+1}}{(x+1)(b-a)} \right]^{1/x} \quad (x \neq 0, x \neq -1) \quad (3)$$

for $x \rightarrow -1$, for $x \rightarrow 0$, at $x = 1$, 2 deduces the following four means, respectively:

$$L(a, b) = \frac{b-a}{\ln b - \ln a} \quad (a \neq b); L(a, a) = a \quad (4)$$

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} \quad (5)$$

$$A(a, b) = \frac{a+b}{2} \quad (6)$$

$$\eta_3 = \sqrt{\frac{a^2 + ab + b^2}{3}} \quad (7)$$

The first three means are known in the literature as the logarithmic mean L , the identical mean I , and the arithmetic mean A , respectively, while the fourth mean, η_3 , is completely

unknown and not used in Statistics. Cisbani[‡] [10], in 1938, realizes that further relationships can be obtained from Dunkel's formula (1) if the numbers $a_1, \dots, a_n$ relate not only in arithmetic progression but also in harmonic progression, quadratic progression, ... More generally, Cisbani adopts the formula $(a^j + ih)^{1/j}$ ($j = \dots, -1, 0, 1, 2, \dots$) to divide the interval into n parts and defines the expression

$$\left[\frac{1}{n} \sum (a^i + ih)^{\frac{x}{j}} \right]^{1/x} \quad (j \neq 0; x \neq 0) \quad (8)$$

from which for $n \rightarrow \infty$ deduces the relationship

$$y_j = \left[\frac{1}{b^j - a^j} \int_{a^j}^{b^j} t^{\frac{x}{j}} dt \right]^{\frac{1}{j}} \quad (9)$$

which, when solved, gives the following means:

$$\begin{aligned} & \left[\frac{b^{x+j} - a^{x+j}}{\left(\frac{x}{j} + 1\right)(b^j - a^j)} \right]^{\frac{1}{x}} \quad (x \neq -j) \\ & \left[\frac{b^j - a^j}{j(\ln b - \ln a)} \right]^{\frac{1}{j}} \quad (x = -j) \\ & \frac{1}{e^{\frac{1}{j}}} \left[\frac{b^{b^j}}{a^{a^j}} \right]^{\frac{1}{b^j - a^j}} \quad (x = 0) \end{aligned}$$

Moreover, Cisbani in his article explores some possible forms that the formula (9) can take, drawing up a double-entry (x, j) table in which he reports as many as 49 means. One has to wait until 1975 for Galvani and Cisbani's formulas to be rediscovered (or taken up again) by K. Stolarsky [47][48], who, after rewriting them as follows:

$$\begin{aligned} u(\alpha) &= \left[\frac{x^\alpha - y^\alpha}{\alpha(x - y)} \right]^{\frac{1}{\alpha-1}} \quad (\alpha \neq 0; \alpha \neq 1); u(\alpha, \beta) \\ &= \left[\frac{\beta(x^\alpha - y^\alpha)}{\alpha(x^\beta - y^\beta)} \right]^{\frac{1}{\alpha-\beta}} \end{aligned} \quad (10)$$

[‡] Substantially, both Galvani and Cisbani, using the definition of Riemann integral, prove MVTI in some special cases. They then in order to deduce MS go from the discrete to the continuous following an inverse path to what will be done by later authors who either start from MVTD or MVTI.

shows that for $x \neq y$, $u(\alpha, \beta)$ is strictly increasing with respect to both α and β . Subsequently, the strands of research on SM have differed. Some authors have focused their attention on a particular SM, namely the logarithmic mean, which is also important because it plays an important role in some engineering applications, such as the study of conduction of heat in liquids flowing in pipes [2] and in statistics, in the calculation of relative changes (or growth rates) [28][49]. In particular, Stolarsky [47] and Pittenger [37] extend to three variables and n variables respectively the pair of inequalities

$$G < L < A \quad (11)$$

already proven in two variables by B. Ostle and H. L. Terwilliger [34] (right side of (11)) in 1957 and by B.C. Carlson [4] (the left side of (11)) in 1966 between the geometric mean G , the logarithmic mean L and the arithmetic mean A . The inequality (11) is important because it represents a refinement of the well-known inequality $G < A$ in that there is a third mean, the logarithmic mean L , which interpolates G and A . Inequality (11) has been reproved many time (see also [44], [45] for other references). Subsequently, Tung Po Lin in 1974 [27] found an interesting relationship between the logarithmic mean L and the means of powers M_p of parameter p :

$$M_p = \left(\frac{x^p + y^p}{2} \right)^{\frac{1}{p}} \quad (12)$$

establishing that

$$M_0 < L < M_{1/3} \quad (13)$$

where M_0 , $M_{1/3}$ are respectively the power mean of parameter $p = 0$ and $p = 1/3$:

$$M_0 = \sqrt{xy}; \quad M_{1/3} = \left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{2} \right)^3$$

Subsequently, Peter R Mercer in 2017 [30] uses Cauchy's Theorem of Mean Value, avoiding passages to the limit, to deduce the expression of the logarithmic mean (4), the inequalities (9) and some inequalities by Tung-Po Lin in 1974 [27]; moreover, P.R. Mercer in 2017 [30] also obtains the following refinement of (11):

$$A(a, b)^{1/3} G(a, b)^{2/3} < L(a, b) < \frac{1}{3} A(a, b) + \frac{2}{3} G(a, b) \quad (14)$$

Finally, G. Jameson and P.R. Mercer in 2019 [22] demonstrate inequalities (13) and (14) with a single method. In detail, they first establish that the pair of inequalities in the two variables a and b (9) is equivalent to the following pair of inequalities in a variable x

$$x^{1/2} \leq \frac{x-1}{\log x} \leq \frac{1}{2}(x+1)$$

and then they, using the power series of the hyperbolic functions $\sinh$ and $\cosh$, achieve the above inequalities. Some authors, in particular A. Witkowski in 2004 [50] and F. Qi in 1998 [38] and in 2000 [39], have related SM to notions of monotonicity; others authors, including F. Qi in 2002 [41]; A. Witkowski in 2006 [51]; J. Sándor in 2015 [46]; J. Jaksetić, J. Pecarić, and A. U. Rehman in 2010 [23]; J. Pecarić and G. Roqia in 2010 [36], have linked SM to notion of log-convexity; instead the property of Schur-convexity of SM was studied by Elezović and J. Pečarić in 2005 [12], F. Qi in 2005 [40], K. Muraliand and M. Nagaraja in 2016 [32]; Zhen-Hang Yang, Feng Qi in 2024 [52]. Some authors, however, have investigated the relationships between Gini Means [16] and SM finding profound analogies between the two types of means [22][33][5][1]. There has also been an attempt to represent SM with power series by H.W. Gould and M. E. Mays in 1984 [17]. SM have recently been used in Numerical Analysis to improve the convergence of Newton's third-order Method by Herceg in 2013 [19], in Statistical Mechanics for the Fokker-Planck operator by Heida in 2021 [18] and in Information Theory to define divergence measures by Cerone and Dragomir in 2004 [7]. We first show that the link of SM with classical means is not limited to correspondence with a few parameters as seems to emerge from the literature, and we prove new formulas by elementary methods by developing new notable products from the formula

$$b^n - a^n \quad (n \in \mathbb{N}; \quad a, b \in \mathbb{R}_+, a < b)$$

We also characterize both from the perspective of Euclidean geometry and through the methods of linear algebra the mean $\sqrt{\frac{a^2+ab+b^2}{3}}$ introduced by Galvani and Cisbani. We also look for new ways to show that SM are between a minimum and a maximum value. SM, as we will show later, are deduced from the MVT_D (or MVT_I) [38][43]. Flett's Theorem (FT) represents the main extension of the MVT_D. We will prove that it is impossible to deduce SM from the FT but it will be permissible to do so from a particular extension of it.

1.2 Preliminaries

Let a and b be two fixed real numbers, with $a < b$. If the function f is continuous in $[a; b]$ and differentiable in $]a; b[$, then the MVT states that there exists at least one η such that^{**}

$$\frac{f(b) - f(a)}{b - a} = f'(\eta) \quad (15)$$

If f' is invertible on $]a; b[$ then

$$\eta = (f')^{-1} \left[\frac{f(b) - f(a)}{b - a} \right] \quad (16)$$

If, on the other hand, g is continuous in $[a; b]$, the MVTI asserts that there exists at least $\eta_* \in [a; b]$ such that

$$\frac{1}{b - a} \int_a^b g(x) dx = g(\eta_*) \quad (17)$$

If g is invertible on $[a; b]$ then

$$\eta_* = g^{-1} \left[\frac{1}{b - a} \int_a^b g(x) dx \right] \quad (18)$$

In general $\eta \neq \eta_*$ ^{††}. It is convenient to highlight the dependence of η and η_* on a and b by making use of the notations $\eta(a, b)$ and $\eta_*(a, b)$ [42].

Definition [4] A continuous function $M: R_+^2 \rightarrow R$ is said to be a mean of two numbers a and b if and only if for any $a, b, \lambda \in R_+$,

$$M1) \min\{a, b\} \leq M(a, b) \leq \max\{a, b\}$$

$$M2) M(a, b) = M(b, a)$$

$$M3) M(\lambda a, \lambda b) = \lambda M(a, b)$$

Let α be real different from 0 and 1. Applying MVT to the function $f(x) = x^\alpha$ we have that there exists at least one $\eta(a, b)$, with $a < \eta(a, b) < b$, such that^{‡‡}

[§] Under the additional hypothesis that f is either strictly convex or strictly concave in $[a; b]$ the point η is unique [20]

^{**} We observe that

$$\frac{f(b) - f(a)}{b - a} = \frac{-[f(a) - f(b)]}{-(a - b)} = \frac{f(a) - f(b)}{a - b} = f'(\eta)$$

^{††} We observe that if $f(x) = g(x) = e^x$ then $\eta = \eta_* = \ln \left(\frac{e^b - e^a}{b - a} \right)$ [25][29][35]

^{‡‡} It is known in general that if $\alpha > 1$ then $f(x) = x^\alpha$ is strictly convex at $]0; +\infty[$, while if $0 < \alpha < 1$ then $f(x) = x^\alpha$ is strictly concave in $]0; +\infty[$. Having fixed α we can take expression (1) as uniquely determined.

$$\frac{b^\alpha - a^\alpha}{b - a} = \alpha \cdot \eta(a, b)^{\alpha-1}$$

whence

$$\eta_\alpha(a, b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}} \quad (19)$$

The latter result is also obtained by applying the MVTI to the function $g(x) = x^{\alpha-1}$:

$$\frac{1}{b-a} \int_a^b x^{\alpha-1} dx = g(\eta_*); \quad \frac{1}{b-a} \cdot \left[\frac{x^\alpha}{\alpha} \right]_a^b = g(\eta_*); \quad \frac{1}{b-a} \cdot \frac{b^\alpha - a^\alpha}{\alpha} = \eta_*^{\alpha-1}(a, b)$$

It is immediately seen that for $\eta_\alpha(a, b)$ the properties M2 and M3 apply:

$$\begin{aligned} \eta_\alpha(a, b) &= \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}} = \left[\frac{-(a^\alpha - b^\alpha)}{-\alpha(a - b)} \right]^{\frac{1}{\alpha-1}} = \left[\frac{a^\alpha - b^\alpha}{\alpha(a - b)} \right]^{\frac{1}{\alpha-1}} = \eta_\alpha(b, a) \\ \eta_\alpha(\lambda a, \lambda b) &= \left[\frac{(\lambda b)^\alpha - (\lambda a)^\alpha}{\alpha(\lambda b - \lambda a)} \right]^{\frac{1}{\alpha-1}} = \left[\frac{\lambda^\alpha(a^\alpha - b^\alpha)}{\alpha\lambda(b - a)} \right]^{\frac{1}{\alpha-1}} = \lambda \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}} \\ &= \lambda \eta_\alpha(a, b) \end{aligned}$$

Further below we will also prove the M1 property. As α varies, we obtain an infinite $S_\alpha = \{\eta_\alpha\}$ family of means called Stolarsky means (one-parameter) in [42] or generalized logarithmic means in [47] and [48]. Applying Cauchy's Mean Value Theorem^{§§} [26] [42] to the functions $f(x) = x^\alpha$; $g(x) = x^\beta$ ($\alpha \neq \beta$; $0 < a < b$) in the interval $[a; b]$ there exists at least an η in $]a; b[$ such that:

$$\frac{b^\alpha - a^\alpha}{b^\beta - a^\beta} = \frac{\alpha \eta^{\alpha-1}}{\beta \eta^{\beta-1}}; \quad \frac{\beta}{\alpha} \cdot \frac{b^\alpha - a^\alpha}{b^\beta - a^\beta} = \eta^{\alpha-\beta}$$

^{§§} The Cauchy's mean value theorem (for derivatives) is also known as Extended Mean-Value Theorem

whence^{***}

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{\alpha} \cdot \frac{b^\alpha - a^\alpha}{b^\beta - a^\beta} \right]^{\frac{1}{\alpha-\beta}} \quad (\alpha \neq \beta) \quad (20)$$

defined as Stolarsky means^{†††} relative to two parameters, α and β , in two variables, a and b . Furthermore, it is observed that $\eta_{\alpha,1}(a,b) = \eta_\alpha(a,b)$.

2. Classical means deduced from SM. Trends and properties of SM as the parameter α changes; inequalities.

We will start this section by connecting the SM to the classic means right away.

2.1 Examples of classical means deduced from SM by assigning the parameter α a particular value.

The classical means, i.e., the geometric mean, arithmetic mean, ... can be deduced by applying MVT or MVTI to appropriate functions, or from the expression of $\eta_\alpha(a,b)$ by assigning a particular value to the parameter α or by performing on $\eta_\alpha(a,b)$ a limit operation, as illustrated in the following table 1:

Mean	MVT	MVTI	Parameter
Arithmetic A	$f(x) = x^2$	$g(x) = x$	$\alpha = 2$
Geometric G	$f(x) = x^{-1}$	$g(x) = x^{-2}$	$\alpha = -1$
Identical I	$f(x) = x \cdot \ln x$	$g(x) = \ln x$	$\alpha \rightarrow 1$
Logarithmic L	$f(x) = \ln x$	$g(x) = \frac{1}{x}$	$\alpha \rightarrow 0$
Power mean	$f(x) = x^{1/2}$	$g(x) = x^{-1/2}$	$\alpha = 1/2$

^{***} The geometric meaning of the radical of η_α is as follows: it is the slope, divided by α , of the secant line s in the diagram of the function $f(x) = x^\alpha$ at the points $A(a; f(a)) = (a; a^\alpha)$; $B(b; f(b)) = (b; b^\alpha)$, or is the slope, divided by α , of the tangent line t to the diagram of the $f(x) = x^\alpha$ at the point $C(\eta_\alpha; f(\eta_\alpha))$ and parallel to s .

^{†††} Stolarsky [47][48] introduces them without deducing them from Cauchy's Theorem but generalizing the logarithmic mean.

Table 1 A comparison between MVTD and MVTI assigning a particular value to the parameter α .

For example, for $\alpha = -1$ we obtain the geometric mean

$$\begin{aligned}\eta_{-1}(a, b) &= \left[\frac{b^{-1} - a^{-1}}{-(b-a)} \right]^{-\frac{1}{2}} = \left[-\frac{\frac{1}{b} - \frac{1}{a}}{(b-a)} \right]^{-\frac{1}{2}} = \left[-\frac{\frac{a-b}{ab}}{(b-a)} \right]^{-\frac{1}{2}} \\ &= \left(\frac{b-a}{ab} \cdot \frac{1}{b-a} \right)^{-\frac{1}{2}} = \left[\left(\frac{1}{ab} \right)^{-1} \right]^{1/2} = \sqrt{ab}\end{aligned}$$

for $\alpha = 2$ we have the arithmetic mean

$$\eta_2(a, b) = \left[\frac{b^2 - a^2}{2(b-a)} \right]^{\frac{1}{2-1}} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

For $\alpha = 1/2$ we generate the power mean

$$\begin{aligned}\eta_{\frac{1}{2}}(a, b) &= \left[\frac{b^{\frac{1}{2}} - a^{\frac{1}{2}}}{\frac{(b-a)}{2}} \right]^{\frac{1}{\frac{1}{2}-1}} = \left[\frac{b^{\frac{1}{2}} - a^{\frac{1}{2}}}{\frac{(b-a)}{2}} \right]^{-2} = \left[\frac{(b^{\frac{1}{2}} - a^{\frac{1}{2}})(b^{\frac{1}{2}} + a^{\frac{1}{2}})}{\frac{(b-a)(b^{\frac{1}{2}} + a^{\frac{1}{2}})}{2}} \right]^{-2} = \\ &= \left[\frac{b-a}{\frac{(b-a)(b^{\frac{1}{2}} + a^{\frac{1}{2}})}{2}} \right]^{-2} = \left(\frac{2}{b^{\frac{1}{2}} + a^{\frac{1}{2}}} \right)^{-2} = \left[\left(\frac{2}{b^{\frac{1}{2}} + a^{\frac{1}{2}}} \right)^{-1} \right]^2 = \left(\frac{b^{\frac{1}{2}} + a^{\frac{1}{2}}}{2} \right)^2\end{aligned}$$

When performing some operations with the limits on $\eta_\alpha(a, b)$ better to think $\eta_\alpha(a, b)$ as a function “exponential-power” $f(\alpha)^{g(\alpha)} = e^{g(\alpha) \ln f(\alpha)}$ where

$$f(\alpha) = \frac{b^\alpha - a^\alpha}{\alpha(b-a)}; \quad g(\alpha) = \frac{1}{\alpha-1}$$

and concern the functions b^α ; a^α as exponential functions of base b and a respectively and exponent α ($\alpha \neq 0$; $\alpha \neq 1$).

2.2 The Stolarsky mean $\eta_\alpha(a, b)$ tends to the logarithmic mean $L(a, b)$ for that α tends to 0

For $\alpha \rightarrow 0$ we have that $\eta_\alpha(a, b)$ tends to the logarithmic mean $L(a, b)$, i.e.

$$\lim_{\alpha \rightarrow 0} \eta_\alpha(a, b) = \lim_{\alpha \rightarrow 0} \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}} = \frac{a - b}{\ln a - \ln b} = L(a, b)$$

Indeed, the function $\left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}}$ is presented as an exponential-power function $f(\alpha)^{g(\alpha)}$ where

$$f(\alpha) = \frac{b^\alpha - a^\alpha}{\alpha(b - a)}; \quad g(\alpha) = \frac{1}{\alpha - 1}$$

From continuity in $\alpha = 0$ of the exponential functions b^α ; a^α we have

$$\lim_{\alpha \rightarrow 0} (b^\alpha - a^\alpha) = b^0 - a^0 = 1 - 1 = 0$$

whence the limit

$$\lim_{\alpha \rightarrow 0} \frac{b^\alpha - a^\alpha}{\alpha(b - a)}$$

is of the type 0/0. Applying the Rule of Hospital to it and remembering that 0/0.

$$D b^\alpha = b^\alpha \ln b; \quad D a^\alpha = a^\alpha \ln a; \quad D \alpha = 1$$

we obtain

$$\lim_{\alpha \rightarrow 0} \frac{b^\alpha \ln b - a^\alpha \ln a}{1 \cdot (b - a)} = \frac{1 \cdot \ln b - 1 \cdot \ln a}{b - a} = \frac{\ln b - \ln a}{b - a}$$

Because

$$\lim_{\alpha \rightarrow 0} \left(\frac{1}{\alpha - 1} \right) = -1$$

then

$$\lim_{\alpha \rightarrow 0} \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]^{\frac{1}{\alpha-1}} = \lim_{\alpha \rightarrow 0} e^{g(\alpha) \ln f(\alpha)} = \lim_{\alpha \rightarrow 0} e^{\left(\frac{1}{\alpha-1} \right) \cdot \ln \left[\frac{b^\alpha - a^\alpha}{\alpha(b - a)} \right]}$$

from which

$$= e^{-\ln\left(\frac{\ln b - \ln a}{b-a}\right)} = e^{\ln\left(\frac{b-a}{\ln b - \ln a}\right)} = \frac{a-b}{\ln a - \ln b} > 0$$

2.3 The Stolarsky mean $\eta_\alpha(a, b)$ tends to the identic mean $I(a, b)$ for that α tends to 1

For $\alpha \rightarrow 1$ we have that $\eta_\alpha(a, b)$ tends to the identic mean $I(a, b)$, i.e.

$$\lim_{\alpha \rightarrow 1} \eta_\alpha(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} = I(a, b) > 0$$

Indeed, thought of $\eta_\alpha(a, b)$ still as an exponential-power function $f(\alpha)^{g(\alpha)}$, and observed that

$$\lim_{\alpha \rightarrow 1} \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right] = \frac{b-a}{1 \cdot (b-a)} = 1; \quad \lim_{\alpha \rightarrow 1} \left(\frac{1}{\alpha-1} \right) = \infty$$

then the limit of the function $f(\alpha)^{g(\alpha)}$ occurs in the indeterminate form of the type 1^∞ , which we rewrite as follows

$$\lim_{\alpha \rightarrow 1} e^{\left(\frac{1}{\alpha-1}\right) \cdot \ln\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]} = \lim_{\alpha \rightarrow 1} e^{\frac{\ln\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]}{\alpha-1}}$$

Applying the L'Hospital Rule to the exponent $e^{\frac{\ln\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]}{\alpha-1}}$, which is of the type $0/0$, we have

$$\frac{\left[\frac{\alpha(b-a)}{b^\alpha - a^\alpha} \right] \cdot \frac{(b^\alpha \ln b - a^\alpha \ln a)[\alpha(b-a)] - (b^\alpha - a^\alpha)(b-a)}{\alpha^2(b-a)^2}}{1}$$

which, properly simplified, leads to the formula

$$\frac{(b^\alpha \ln b - a^\alpha \ln a)[\alpha(b-a)] - (b^\alpha - a^\alpha)(b-a)}{\alpha(b-a)(b^\alpha - a^\alpha)} = \left[\frac{b^\alpha \ln b - a^\alpha \ln a}{b^\alpha - a^\alpha} - \frac{1}{\alpha} \right]$$

The latter, for $\alpha \rightarrow 1$, gives the expression

$$\frac{b \ln b - a \ln a}{b-a} - 1 = \frac{\ln b^b - \ln a^a}{b-a} = \frac{\ln(b^b/a^a)}{b-a} - 1 = \ln(b^b/a^a)^{1/b-a} - 1$$

Therefore, the limit of $\eta_\alpha(a, b)$, for $\alpha \rightarrow 1$, leads to

$$e^{\left[\ln(b^b/a^a)^{1/b-a} - 1\right]} = e^{\ln(b^b/a^a)^{1/b-a}} \cdot e^{-1} = \frac{1}{e} \cdot (b^b/a^a)^{1/b-a}$$

2.4 The SM $\eta_\alpha(a, b)$ lies between a and b

We show that for $\eta_\alpha(a, b)$ M1 holds true, i.e.

$$\lim_{\alpha \rightarrow -\infty} \eta_\alpha(a, b) = a = \min\{a, b\}; \quad \lim_{\alpha \rightarrow +\infty} \eta_\alpha(a, b) = b = \max\{a, b\}$$

from which

$$\eta_\alpha(a, b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right]^{\frac{1}{\alpha-1}} \quad (21)$$

Let's start with the left limit:

Reviewing as above the functions b^α ; a^α as exponential functions of basis b and a respectively and exponent α , and observing that

$$\lim_{\alpha \rightarrow -\infty} b^\alpha = \lim_{\alpha \rightarrow -\infty} a^\alpha = 0; \quad \lim_{\alpha \rightarrow -\infty} \frac{1}{\alpha} = 0; \quad \lim_{\alpha \rightarrow -\infty} g(\alpha) = \lim_{\alpha \rightarrow -\infty} \left(\frac{1}{\alpha-1} \right) = 0$$

$$\lim_{\alpha \rightarrow -\infty} f(\alpha) = \lim_{\alpha \rightarrow -\infty} \frac{1}{b-a} \cdot (b^\alpha - a^\alpha) \cdot \frac{1}{\alpha} = 0$$

we have that the limit of $\eta_\alpha(a, b) = [f(\alpha)]^{g(\alpha)}$ for $\alpha \rightarrow -\infty$ occurs of the type 0^0 .

Written the function as

$$e^{\frac{\ln \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right]}{\alpha-1}}$$

to the exponent, which is of the type ∞/∞ , the De L'Hospital Rule can be applied. Similarly to the above we get

$$\lim_{\alpha \rightarrow -\infty} \frac{b^\alpha \ln b - a^\alpha \ln a}{b^\alpha - a^\alpha} - \frac{1}{\alpha} = \lim_{\alpha \rightarrow -\infty} \frac{a^\alpha \left(\frac{b^\alpha}{a^\alpha} \ln b - \ln a \right)}{a^\alpha \left(\frac{b^\alpha}{a^\alpha} - 1 \right)} - \frac{1}{\alpha} =$$

$$\lim_{\alpha \rightarrow -\infty} \frac{\left(\frac{b}{a} \right)^\alpha \ln b - \ln a}{\left(\frac{b}{a} \right)^\alpha - 1} - \frac{1}{\alpha} = \ln a$$

having assumed, as done above, $0 < a < b$, whence $b/a > 1$ and $\lim_{\alpha \rightarrow -\infty} \left(\frac{b}{a}\right)^\alpha = 0$.
Therefore, our limit is

$$e^{\ln a} = a$$

i.e. the $\min\{a, b\}$.

Finally, we prove that

$$\lim_{\alpha \rightarrow +\infty} \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right]^{\frac{1}{\alpha-1}} = b$$

Retracing the procedure used for the previous limit, placed $t = \left(\frac{b}{a}\right)^\alpha$, we have

$$\text{B} \lim_{\alpha \rightarrow +\infty} \frac{\left(\frac{b}{a}\right)^\alpha \ln b - \ln a}{\left(\frac{b}{a}\right)^\alpha - 1} - \frac{1}{\alpha} = \lim_{t \rightarrow +\infty} \frac{(\ln b)t - \ln a}{t-1} = \ln b$$

from which

$$e^{\ln b} = b$$

i.e. the $\max\{a, b\}$.

It is also possible to provide a second proof, which does not employ the concept of a limit, of the inequalities $a < \eta_\alpha < b$, applying the following proposition, stated and proved in [24]

Proposition 1

If $x > 0$, $y > 0$ and $r > 1$ then

$$ry^{r-1} < \frac{x^r - y^r}{x - y} < rx^{r-1} \quad (22)$$

If $0 < r < 1$ then

$$rx^{r-1} < \frac{x^r - y^r}{x - y} < ry^{r-1} \quad (23)$$

Indeed, $a = y$; $b = x$; $\alpha = r$; if $\alpha > 1$ then, from Proposition 1,

$$\alpha a^{\alpha-1} < \frac{b^\alpha - a^\alpha}{b - a} < \alpha b^{\alpha-1}$$

from which

$$a^{\alpha-1} < \frac{b^\alpha - a^\alpha}{\alpha(b-a)} < b^{\alpha-1}$$

whence

$$a = (a^{\alpha-1})^{\frac{1}{\alpha-1}} < \eta_\alpha = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right]^{\frac{1}{\alpha-1}} < (b^{\alpha-1})^{\frac{1}{\alpha-1}} = b$$

i.e.

$$a < \eta_\alpha < b \quad (24)$$

Similarly, we reason for $0 < \alpha < 1$.

2.5 Diagram of SM as the parameter α varies.

It is shown [47] that η_α is increasing at α , from which it follows, by the Limit Theorem of monotone functions, that a and b coincide with the lower and upper extremes, respectively, of the codomain of $\eta_\alpha(a, b)$. Furthermore, the following Stolarsky inequality [47] holds

$$G(x, y) \leq L(x, y) \leq I(x, y) \leq A(x, y)$$

It follows from the above that $\eta_\alpha(a, b)$, regarded as a function of the variable α alone, has as its domain the set $A = \mathbb{R} - \{0, 1\}$; it is positive in A ; it is continuous in A since it is composed of continuous functions; it is increasing in A ; in the plane $\alpha - \eta$ admits $\eta = a$ and $\eta = b$ as horizontal asymptotes; it has in $\alpha = 0$ and $\alpha = 1$ eliminable discontinuities and its codomain is between a and b (Fig 1):

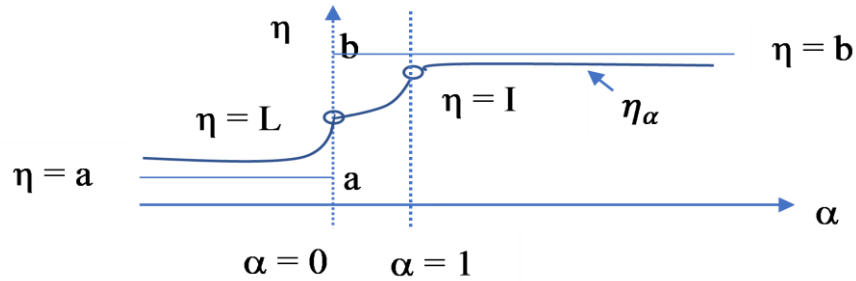


Figure 1 In this figure, we illustrate the diagram of SM η_α as parameter a varies.

Having said $\bar{\eta}_\alpha$ the extension by continuity of η_α in $\alpha = 0$ and $\alpha = 1$, then $\bar{\eta}_\alpha$ is defined in $\mathbb{R}$ and sends $\mathbb{R}$ in $]a; b[$.

From now on, we will no longer distinguish η from η_* and denote them both by the notation $\eta_\alpha(a, b)$ or simply η_α .

3. Deduction of classical mean by MVT^{†††}

We illustrate some cases below:

case: $f(x) = x^2$ (arithmetic mean of a and b)

$$\frac{b^2 - a^2}{b - a} = 2\eta; \frac{(b - a)(b + a)}{b - a} = 2\eta; \eta = \frac{a + b}{2}$$

case: $f(x) = \frac{1}{x}$ (geometric mean of a and b)

$$\begin{aligned} \frac{b \cdot \ln b - a \cdot \ln a}{b - a} &= \ln \eta + 1; \frac{\ln b^b - \ln a^a}{b - a} - 1 = \ln \eta; \ln \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} - 1 \\ &= \ln \eta; \eta = \frac{1}{e} \cdot \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} \end{aligned}$$

case: $f(x) = x \cdot \ln x$ (identic mean of a and b)

$$\begin{aligned} \frac{b \cdot \ln b - a \cdot \ln a}{b - a} &= \ln \eta + 1; \frac{\ln b^b - \ln a^a}{b - a} - 1 = \ln \eta; \ln \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} - 1 \\ &= \ln \eta; \eta = \frac{1}{e} \cdot \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} \end{aligned}$$

case: $f(x) = \ln x$ (logarithmic mean of a and b)

Because $f'(\eta) = \frac{1}{\eta}$, we have

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{\eta}; \frac{-(\ln a - \ln b)}{-(a - b)} = \frac{1}{\eta}; \eta = \frac{a - b}{\ln a - \ln b}$$

case: $f'(\eta) = \frac{1}{2\sqrt{\eta}}$ (arithmetic mean of $a, \sqrt{ab}, \sqrt{ab}, b$)

$$\frac{\sqrt{b} - \sqrt{a}}{b - a} = \frac{1}{2\sqrt{\eta}}; \frac{\sqrt{b} - \sqrt{a}}{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})} = \frac{1}{2\sqrt{\eta}}; \frac{\sqrt{b} + \sqrt{a}}{2} = \sqrt{\eta};$$

^{†††} Look in this regard at the following link:

<https://demonstrations.wolfram.com/StolarskyApproximationsToPopularMeans/>

$$\eta = \left(\frac{\sqrt{b} + \sqrt{a}}{2} \right)^2 = \frac{a + 2\sqrt{ab} + b}{4} = A(a, \sqrt{ab}, \sqrt{ab}, b)$$

4. Deduction of classical means by MVTI

Applying the MVTI for $g(x) = x$, being $g(\eta) = \eta$, we obtain the arithmetic mean

$$\begin{aligned} \eta &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \\ &= \frac{1}{b-a} \left[\frac{(b-a)(b+a)}{2} \right]; \quad \eta = \frac{a+b}{2} = A(a, b) \end{aligned}$$

Furthermore, if we place $g(x) = x^{-2}$, we get $g(\eta) = \eta^{-2}$, from which applying MVTI we have

$$\begin{aligned} \eta^{-2} &= \frac{1}{b-a} \int_a^b x^{-2} \, dx = \\ &= \frac{1}{b-a} \cdot [-x^{-1}]_a^b = \frac{1}{b-a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab} \\ \eta &= \sqrt{ab} = G(a, b) \end{aligned}$$

The identical mean $I(a, b)$ can be obtained by applying the MVTI to the function $g(x) = \ln x$:

$$\begin{aligned} \frac{1}{b-a} \int_a^b \ln x \, dx &= \ln \eta; \quad \frac{1}{b-a} [x \cdot \ln x - x]_a^b = \ln \eta; \\ \frac{b \ln b - a \ln a - (b-a)}{b-a} &= \ln \eta; \quad \frac{\ln(b^b/a^a)}{b-a} - 1 = \ln \eta; \quad \ln \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} - \ln e = \ln \eta \\ \ln \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} + \ln \frac{1}{e} &= \ln \eta; \quad \ln \left[\frac{1}{e} \cdot \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} \right] = \ln \eta \end{aligned}$$

from which

$$\begin{aligned} I(a, b) = \eta &= \frac{1}{e} \cdot \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} = \\ \frac{1}{e} \cdot \left(\frac{b^b}{a^a} \right)^{\frac{1}{-(a-b)}} &= \frac{1}{e} \cdot \left[\left(\frac{b^b}{a^a} \right)^{-1} \right]^{\frac{1}{a-b}} = \frac{1}{e} \cdot \left(\frac{a^a}{b^b} \right)^{\frac{1}{a-b}} \end{aligned}$$

For the logarithmic mean, we start with the function $g(x) = \frac{1}{x}$, whence

$$\frac{1}{b-a} \cdot \int_a^b \frac{1}{x} dx = \frac{1}{\eta}; \quad \frac{1}{b-a} \cdot [\ln x]_a^b = \frac{1}{\eta}; \quad \frac{\ln b - \ln a}{b-a} = \frac{1}{\eta}; \quad \eta = \frac{b-a}{\ln b - \ln a}$$

Finally, the mean power can be obtained from the function $g(x) = x^{-1/2}$:

$$\frac{1}{b-a} \cdot \int_a^b x^{-\frac{1}{2}} dx = \frac{1}{\sqrt{\eta}}; \quad \frac{1}{b-a} \cdot \left[\frac{\sqrt{x}}{1/2} \right]_a^b = \frac{1}{\sqrt{\eta}};$$

$$\frac{2(\sqrt{b} - \sqrt{a})}{b-a} = \frac{1}{\sqrt{\eta}}; \quad \eta = \left(\frac{\sqrt{b} + \sqrt{a}}{2} \right)^2$$

5. Some SM with natural n as parameter

As natural n varies, we consider the subfamily family T_n of SM given by

$$\eta_n(a, b) = \left[\frac{b^n - a^n}{n(b-a)} \right]^{\frac{1}{n-1}}$$

From the well-known identity^{§§§}

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + ba^{n-2} + a^{n-1}) \quad (25)$$

we get

$$\eta_n(a, b) = \left[\frac{(b-a)(b^{n-1} + b^{n-2}a + \dots + ba^{n-2} + a^{n-1})}{n(b-a)} \right]^{\frac{1}{n-1}} =$$

$$= \left(\frac{b^{n-1}a^0 + b^{n-2}a^1 + \dots + b^1a^{n-2} + b^0a^{n-1}}{n} \right)^{\frac{1}{n-1}} \quad (26)$$

Expression (5) can be thought of as the root of index n-1 of an arithmetic mean of n terms

$$\eta_n(a, b) = \sqrt[n-1]{A(a^{n-1}b^0, a^{n-2}b^1, \dots, a^1b^{n-2}, a^0b^{n-1})} \quad (27)$$

from which the statistical significance of η_n emerges more clearly.

^{§§§} Relation (25) is easily proved by induction. The second factor $b^{n-1} + b^{n-2}a + \dots + ba^{n-2} + a^{n-1}$ is a homogeneous polynomial of degree n-1 in the variables a and b formed by n-1 terms of the type $b^{n-1-j}a^j$ ($j = 0, \dots, n-1$) in which the powers of b decreases from n-1 to 0 and the powers of a increases from 0 to n-1.

Some special cases on Stolarsky's means

It is possible to obtain a further development of the difference between two powers with integer exponent $a^n - b^n$, and thus another expression of η_n , inspired by [6]. In fact, placing $Q_0 = b - a$, we have

$$a^n = (b + Q_0)^n = b^n + nb^{n-1}Q_0 + \frac{n(n-1)}{2!}b^{n-2}Q_0^2 + \dots + Q_0^n$$

whence

$$a^n - b^n = Q_0 \left(nb^{n-1} + \frac{n(n-1)}{2!}b^{n-2}Q_0 + \dots + Q_0^{n-1} \right)$$

and then

$$\eta_n(a, b) = \left[\frac{nb^{n-1} + \frac{n(n-1)}{2!}b^{n-2}Q_0 + \dots + b^0Q_0^{n-1}}{n} \right]^{\frac{1}{n-1}}$$

The latter relationship can also be related as an arithmetic mean of n terms:

$$\eta_n(a, b) = \sqrt[n-1]{A \left(nb^{n-1}, \frac{n(n-1)}{2!}b^{n-2}Q_0, \dots, b^0Q_0^{n-1} \right)}$$

Later we will see that the second factor of (25) can be appropriately factorized in some special cases (when n is an integer power of 2, or 3, or 5, ..., p^h (p prime number; $h \in \mathbb{Z}$)) and the SM can be characterized as classical means.

Earlier we provided as many as two demonstrations of the inequality (17) for every α real number and different from 0 and 1. Relative to the case $\alpha = n \in \mathbb{N}$, based on (25) and inspired by [6] it is possible to provide a further demonstration that $a < \eta_n(a, b) < b$ ($\forall n \in \mathbb{N}$). In fact, the following holds

Theorem 1 Let $0 < a < b$. Then

$$a < \eta_n < b \quad (\forall n \in \mathbb{N})$$

Proof. We first prove the left-hand inequality, viz, i.e.

$$a < \eta_n \quad (\forall n \in \mathbb{N})$$

From (25) we derive that

$$b^{n-1} + b^{n-2}a^1 + \dots + a^{n-1} > (n-1)a^{n-1} \tag{28}$$

whence

$$\frac{b^{n-1} + b^{n-2}a^1 + \dots + a^{n-1}}{n} > \frac{(n-1)}{n} \cdot a^{n-1}$$

therefore

$$\eta_n > \left(\frac{n-1}{n} \cdot a^{n-1} \right)^{\frac{1}{n-1}} = a \cdot \left(\frac{n-1}{n} \right)^{\frac{1}{n-1}} = a \cdot e^{\left(\frac{1}{n-1} \right) \cdot \ln \left(\frac{n-1}{n} \right)}$$

From

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n-1} \right) = 0; \quad \lim_{n \rightarrow +\infty} \ln \left(\frac{n-1}{n} \right) = \ln 1 = 0$$

we get

$$\lim_{n \rightarrow +\infty} \left[a \cdot e^{\left(\frac{1}{n-1} \right) \cdot \ln \left(\frac{n-1}{n} \right)} \right] = a \cdot e^{0 \cdot 0} = a \cdot 1 = a$$

whence the assertion is reached.

As for the proof of the second inequality, we immediately observe that

$$\eta_n = \left[\frac{b^n - a^n}{n(b-a)} \right]^{\frac{1}{n-1}} < \left[\frac{b^n - a^n}{(n-1)(b-a)} \right]^{\frac{1}{n-1}}$$

Since $0 < a < b$, reasoning in a similar way as above, we arrive at the following inequality

$$b^{n-1} + b^{n-2}a^1 + \dots + a^{n-1} < (n-1)b^{n-1} \quad (29)$$

from which

$$\frac{b^n - a^n}{(n-1)(b-a)} < \frac{(n-1)b^{n-1}}{(n-1)} = b^{n-1}$$

therefore

$$\eta_n < [b^{n-1}]^{\frac{1}{n-1}} = b \quad (n \in N)$$

In the literature for the development of the difference of two powers $b^n - a^n$ ($0 < a < b$; $n \in N$) only the formula (25) is known. We will show that $b^n - a^n$, in the special case where $n = p^h$ (p prime number; $h \in N$), can be written as a product of homogeneous polynomials in a and b , all of positive sign, with a recursive relation.

We now characterize the development of $b^n - a^n$ in the particular case where $n = p^h$ (p prime number; $h \in N$) as a product of homogeneous polynomials. The following holds

Theorem 2 Let p be a prime number fixed in N and $h \in N$, then

$$b^{p^h} - a^{p^h} = \prod_{i=0}^h Q_i \quad (30)$$

where

Some special cases on Stolarsky's means

$$Q_0 = b - a \quad (31)$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^{i-1}} b^{j \cdot p^{i-1}} \quad (1 \leq i \leq h) \quad (32)$$

Proof .

The relation (30) is true for $h = 0$. We admit that (30) is true for h (inductive hypothesis). We observe that between the two differences of consecutive powers^{****} $(b^{p^{h+1}} - a^{p^{h+1}})$ and $(b^{p^h} - a^{p^h})$ the following relationship exists:

$$b^{p^{h+1}} - a^{p^{h+1}} = (b^{p^h} - a^{p^h}) \sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^h} b^{j \cdot p^h}$$

By inductive hypothesis

$$b^{p^h} - a^{p^h} = \prod_{i=0}^h Q_i$$

and being

$$\sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^h} b^{j \cdot p^h} = Q_{h+1}$$

then

$$b^{p^{h+1}} - a^{p^{h+1}} = \prod_{i=0}^{h+1} Q_i$$

hence the assertion. We apply the previous result to characterize η_{p^h} .

Corollary 1 Let p be a fixed prime number in N and $n = p^h$ ($h \in N$). Then

$$\eta_{p^h} = \left[\frac{b^{p^h} - a^{p^h}}{p^h(b-a)} \right]^{\frac{1}{p^h-1}} = \sqrt[p^{h-1}]{\prod_{i=1}^h A_i} \quad (1 \leq i \leq h), \quad (33)$$

^{****} In the sense that the exponents of p , that is, h and $h+1$, are consecutive natural numbers.

where

$$A_i \left(a^{(p-1) \cdot p^{i-1}} b^0, a^{(p-2) \cdot p^{i-1}} b^{1 \cdot p^{i-1}}, \dots, a^0 b^{(p-1) \cdot p^{i-1}} \right) \quad (34)$$

Proof.

We have

$$\begin{aligned} \eta_{p^h} &= \left[\frac{b^{p^h} - a^{p^h}}{p^h(b-a)} \right]^{\frac{1}{p^{h-1}}} = \sqrt[p^{h-1}]{\frac{\prod_{i=1}^h Q_i}{p^h}} \\ &= \sqrt[p^{h-1}]{\frac{\prod_{i=1}^h Q_i}{p^h}} = \sqrt[p^{h-1}]{\prod_{i=1}^h A_i} \end{aligned}$$

where the difference of powers $b^{p^h} - a^{p^h}$ is developed as the product of $h+1$ homogeneous polynomials Q_i in a and b , of which the first factor is $(b-a)$ while the remaining factors have degree $(p-1) \cdot p^i$ ($1 \leq i \leq h$) and are each formed by p terms:

$$b^{p^h} - a^{p^h} = \prod_{i=0}^h Q_i$$

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^{i-1}} b^{j \cdot p^{i-1}} \quad (1 \leq i \leq h)$$

$$A_i \left(a^{(p-1) \cdot p^{i-1}} b^0, a^{(p-2) \cdot p^{i-1}} b^{1 \cdot p^{i-1}}, \dots, a^0 b^{(p-1) \cdot p^{i-1}} \right) \quad (1 \leq i \leq h)$$

Corollary 2 Let p be a fixed prime number in $\mathbb{N}$ and $h \in \mathbb{N}$. Then

$$b^{-p^h} - a^{-p^h} = \frac{\prod_{i=0}^h Q_i}{a^{p^h} b^{p^h}}$$

where

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^{i-1}} b^{j \cdot p^{i-1}} \quad (1 \leq i \leq h)$$

Proof.

We proceed as in Theorem 2.

Corollary 3 Let p be a fixed prime number in N and $h \in N$. Then the following identity is valid

$$\eta_{-p^h} = \left[\frac{b^{-p^h} - a^{-p^h}}{-p^h(b-a)} \right]^{\frac{1}{-p^{h-1}}} = G(a, b) \cdot \sqrt[p^{h+1}]{\frac{G^{p^h-1}(a, b)}{\prod_{i=1}^h A_i}}$$

Proof.

By Corollary 2 we obtain

$$\begin{aligned} \eta_{-p^h} &= \left[\frac{b^{-p^h} - a^{-p^h}}{-p^h(b-a)} \right]^{\frac{1}{-p^{h-1}}} = \\ &= \sqrt[p^{h+1}]{a^{p^h} b^{p^h} \cdot \frac{p^h}{\prod_{i=1}^h Q_i}} \\ &= \sqrt[p^{h+1}]{\frac{G^{2 \cdot p^h}(a, b)}{\prod_{i=1}^h A_i}} = G(a, b) \cdot \sqrt[p^{h+1}]{\frac{G^{p^h-1}(a, b)}{\prod_{i=1}^h A_i}} \end{aligned}$$

where

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j) \cdot p^{i-1}} b^{j \cdot p^{i-1}} \quad (1 \leq i \leq h)$$

$$A_i \left(a^{(p-1) \cdot p^{i-1}} b^0, a^{(p-2) \cdot p^{i-1}} b^{1 \cdot p^{i-1}}, \dots, a^0 b^{(p-1) \cdot p^{i-1}} \right) \quad (1 \leq i \leq h)$$

and also

$$\eta_{-p^h}(a, b) = \sqrt[p^{h+1}]{G^2(a, b) \cdot \prod_{i=1}^h H_i}$$

where

$$H_i \left(a^{(p-1) \cdot p^{i-1}} b^0, a^{(p-2) \cdot p^{i-1}} b^{1 \cdot p^{i-1}}, \dots, a^0 b^{(p-1) \cdot p^{i-1}} \right) \quad (1 \leq i \leq h)$$

6. Statistical significance of η_{2^h} ($h \in \mathbb{N}$)

6.1 Statistical significance of η_4

We can also reason for the case $\alpha = 4$ in the same way as for the previous cases, limiting ourselves to applying only the general formula of SM:

$$\begin{aligned}\eta_4(a, b) &= \left[\frac{b^4 - a^4}{4(b - a)} \right]^{\frac{1}{3}} = \left[\frac{(b - a)(b + a)(b^2 + a^2)}{4(b - a)} \right]^{\frac{1}{3}} = \\ &= \left[\frac{(a + b)(a^2 + b^2)}{4} \right]^{\frac{1}{3}} = \left[\frac{(a + b)(a^2 + b^2)}{4} \right]^{\frac{1}{3}} = \\ &= \left[\left(\frac{a + b}{2} \right)^2 \cdot \left(\frac{a^2 + b^2}{a + b} \right) \right]^{\frac{1}{3}} = [A^2(a, b) \cdot C(a, b)]^{\frac{1}{3}}\end{aligned}$$

i.e.

$$\eta_4(a, b) = \sqrt[3]{A^2 \cdot C}$$

where

$$C(a, b) = \frac{a^2 + b^2}{a + b}$$

is the contra-harmonic mean of a and b , with $0 < a < b$.

Otherwise

$$\left[\frac{(a + b)(a^2 + b^2)}{4} \right]^{\frac{1}{3}} = \left[\frac{a + b}{2} \cdot \frac{a^2 + b^2}{2} \right]^{\frac{1}{3}} = \sqrt[3]{A(a, b) \cdot QM(a^2, b^2)}$$

and so on.

We illustrate the formula

$$\eta_4(a, b) = \sqrt[3]{A^2 \cdot C}$$

in space $AC\eta$ space, where A , C and η represent the three coordinate axes respectively. We have (Fig. 2)

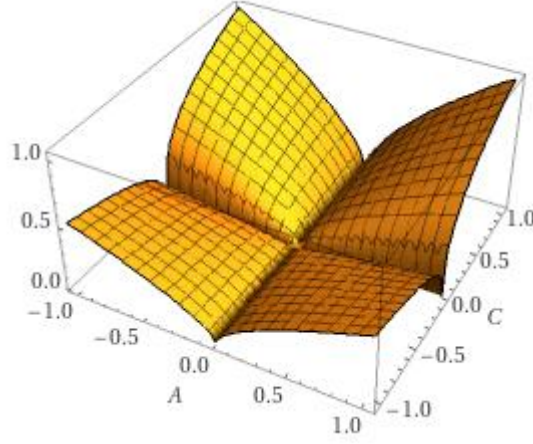


Figure 2 Geometric interpretation of η_4 (Wolfram Alpha)

6.2 Statistical significance of η_8

For $\alpha = 8$ we get

$$\begin{aligned}\eta_8(a, b) &= \left[\frac{b^8 - a^8}{8(b - a)} \right]^{\frac{1}{7}} = \left[\frac{(b - a)(b + a)(b^2 + a^2)(b^4 + a^4)}{8(b - a)} \right]^{\frac{1}{7}} = \\ &= \sqrt[7]{A^2(a, b) \cdot C(a, b) \cdot QM^2(a^2, b^2)}\end{aligned}$$

6.3 Statistical significance of η_{2^h}

We saw earlier that from the well-known formula concerning the difference of two squares

$$b^2 - a^2 = (b - a)(b + a)$$

one can also deduce the formulas

$$\begin{aligned}b^4 - a^4 &= (b - a)(b + a)(b^2 + a^2) \\ b^8 - a^8 &= (b - a)(b + a)(b^2 + a^2)(b^4 + a^4)\end{aligned}$$

More generally, $\forall h \geq 1$, one can prove by induction the formula

$$b^{2^h} - a^{2^h} = (b - a)(b + a)(b^2 + a^2) \cdots (b^{2^{h-1}} + a^{2^{h-1}}) \quad (35)$$

It is true for $h = 1$:

$$b^{2^1} - a^{2^1} = b^2 - a^2 = (b - a)(b + a)$$

We admit that it is true for some h (inductive hypothesis) and prove that it is true for $h+1$. We observe^{††††} that the following decomposition applies:

$$b^{2^{h+1}} - a^{2^{h+1}} = (b^{2^h})^2 - (a^{2^h})^2 = (b^{2^h} - a^{2^h}) \cdot (b^{2^h} + a^{2^h})$$

For the inductive hypothesis, developing the first factor to the right of the second equality gives:

$$b^{2^{h+1}} - a^{2^{h+1}} = (b - a)(a + b)(a^2 + b^2) \dots (a^{2^{h-1}} + b^{2^{h-1}})(a^{2^h} + b^{2^h})$$

From the above observation and taking into account that

$$a^{2^{h-1}} = a^{2^1 \cdot 2^{h-2}} = (a^{2^{h-2}})^2; \quad b^{2^{h-1}} = b^{2^1 \cdot 2^{h-2}} = (b^{2^{h-2}})^2$$

is easily deduced by induction, $\forall h \geq 2$, that

$$\begin{aligned} \eta_{2^h}(a, b) &= \left[\frac{b^{2^h} - a^{2^h}}{2^h(b - a)} \right]^{\frac{1}{2^h - 1}} \\ &= {}^{2^h - 1}\sqrt{A^2(a, b) \cdot C(a, b) \cdot QM^2(a^2, b^2) \dots QM^2(a^{2^{h-2}}, b^{2^{h-2}})} \end{aligned}$$

or

$$= {}^{2^h - 1}\sqrt{A(a, b) \cdot QM^2(a^2, b^2) \dots QM^2(a^{2^{h-2}}, b^{2^{h-2}})}$$

remarkable result that tells us that the Stolarsky means of parameter α equal to 2^h ($h \geq 2$) can be written explicitly in terms of some classical means, more precisely they can be traced to a radical having as its index $2^h - 1$, and as its radical a product in which the powers of the averages of A , C and QM appear as factors, or only of A and QM .

Developing the definition of SM based on (35) gives:

$$\begin{aligned} \eta_{2^h} &= \sqrt[2^h - 1]{\frac{(b - a)(a + b)(a^2 + b^2) \dots (a^{2^{h-1}} + b^{2^{h-1}})}{2^h(b - a)}} = \\ &= \sqrt[2^h - 1]{\frac{(a + b)(a^2 + b^2) \dots (a^{2^{h-1}} + b^{2^{h-1}})}{2^h}} \end{aligned}$$

^{††††} Indeed $(b^{2^h})^2 = b^{2 \cdot 2^h} = b^{2^1 \cdot 2^h} = b^{2^{h+1}}$ e $(a^{2^h})^2 = a^{2 \cdot 2^h} = a^{2^1 \cdot 2^h} = a^{2^{h+1}}$

Some special cases on Stolarsky's means

$$\begin{aligned}
&= {}^{2^{h-1}}\sqrt{\left(\frac{a+b}{2}\right) \cdot \left(\frac{a^2+b^2}{2}\right) \left(\frac{a^4+b^4}{2}\right) \cdots \left(\frac{a^{2^{h-1}}+b^{2^{h-1}}}{2}\right)} \\
&= {}^{2^{h-1}}\sqrt{A(a,b) \cdot QM^2(a^2,b^2) \cdots QM^2(a^{2^{h-2}},b^{2^{h-2}})}
\end{aligned}$$

This formula is true, as we have seen, for $h = 2$. We admit that it is true for a certain $h \in \mathbb{N}$ and we show that it is also true for $h + 1$. It suffices to observe that

$$(a^{2^{h-1}})^2 + (b^{2^{h-1}})^2 = a^{2 \cdot 2^{h-1}} + b^{2 \cdot 2^{h-1}} = a^{2^h} + b^{2^h}$$

and that

$$\frac{a^{2^h} + b^{2^h}}{2} = \left[\frac{(a^{2^{h-1}})^2 + (b^{2^{h-1}})^2}{2} \right]^2 = QM^2(a^{2^{h-1}}, b^{2^{h-1}})$$

or

$$\begin{aligned}
&= {}^{2^{h-1}}\sqrt{\frac{(a+b)(a^2+b^2) \cdots (a^{2^{h-1}}+b^{2^{h-1}})}{2^h}} \\
&= {}^{2^{h-1}}\sqrt{\left(\frac{a+b}{2}\right)^2 \cdot \left(\frac{a^2+b^2}{2}\right) \cdot \left(\frac{a^4+b^4}{2}\right) \cdots \left(\frac{a^{2^{h-1}}+b^{2^{h-1}}}{2}\right)} \\
&= {}^{2^{h-1}}\sqrt{\left(\frac{a+b}{2}\right)^2 \cdot \left(\frac{a^2+b^2}{2}\right) \cdot \left[\frac{(a^2)^2 + (b^2)^2}{2}\right] \cdots \left[\frac{(a^{2^{h-2}})^2 + (b^{2^{h-2}})^2}{2}\right]} \\
&= {}^{2^{h-1}}\sqrt{A^2(a,b) \cdot C(a,b) \cdot QM^2(a^2,b^2) \cdots QM^2(a^{2^{h-2}},b^{2^{h-2}})}
\end{aligned}$$

This formula is true, as we have seen, for $h = 2$. We admit that it is true for a certain $h \in \mathbb{N}$ and we show that it is also true for $h + 1$. It suffices to observe that

$$(a^{2^{h-1}})^2 + (b^{2^{h-1}})^2 = a^{2 \cdot 2^{h-1}} + b^{2 \cdot 2^{h-1}} = a^{2^h} + b^{2^h}$$

and that

$$\frac{a^{2^h} + b^{2^h}}{2} = \left[\frac{(a^{2^{h-1}})^2 + (b^{2^{h-1}})^2}{2} \right]^2 = QM^2(a^{2^{h-1}}, b^{2^{h-1}})$$

7. Statistical significance of η_{-2} , η_{-4} and η_{-2^h} ($h \in \mathbb{N}$)

The case where the parameter α is worth -2 deserves special attention because it allows us to arrive at two interesting functional relationships, one between the harmonic mean H and the geometric mean G and another between the geometric mean G and the arithmetic mean A .

Let us begin with the first relationship. By positing $\alpha = -2$ in Stolarsky's general means formula $\eta_\alpha(a, b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)} \right]^{\frac{1}{\alpha-1}}$ (or also by applying MVTI to the function $f(x) = x^{-2}$, or by applying the MVTI to the function $g(x) = x^{-3}$) we have the following functional relationship between the harmonic mean H and the geometric mean G of two positive numbers a and b with $a < b$ ††††

$$\eta_{-2} = \sqrt[3]{H(a, b) \cdot G^2(a, b)}$$

Indeed

$$\begin{aligned} \eta_{-2}(a, b) &= \left[\frac{b^{-2} - a^{-2}}{-2(b-a)} \right]^{\frac{1}{-2-1}} = \left[\frac{\left(\frac{1}{b} - \frac{1}{a}\right) \left(\frac{1}{b} + \frac{1}{a}\right)}{-2(b-a)} \right]^{-\frac{1}{3}} \\ &= \left[\frac{\left(\frac{b-a}{ab}\right) \left(\frac{1}{b} + \frac{1}{a}\right)}{2(b-a)} \right]^{-\frac{1}{3}} = \left[\frac{\left(\frac{1}{b} + \frac{1}{a}\right)}{2} \cdot \frac{1}{ab} \right]^{-\frac{1}{3}} = \sqrt[3]{\left(\frac{2}{\frac{1}{a} + \frac{1}{b}}\right) (ab)} \end{aligned}$$

whence the assertion being

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}; \quad G(a, b) = \sqrt{ab}$$

We observe that you also have

$$\eta_{-2} = \sqrt[3]{\frac{2}{a+b} \cdot (ab)^2} = \sqrt[3]{\frac{G^4}{A}} = G \cdot \sqrt[3]{\frac{G}{A}}$$

Instead, applying the MVTI to the function $f(x) = x^{-3}$ is done as follows:

$$\frac{1}{b-a} \int_a^b x^{-3} dx = \eta^{-3}$$

†††† The latter condition is necessary to be able to apply the MVTI.

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$$\begin{aligned}
\frac{1}{b-a} \left[\frac{x^{-2}}{-2} \right]_a^b &= \eta^{-3}; \quad \frac{1}{b-a} \cdot \frac{b^{-2} - a^{-2}}{-2} = \eta^{-3} \\
\frac{1}{b-a} \cdot \frac{\frac{1}{b^2} - \frac{1}{a^2}}{-2} &= \eta^{-3}; \quad \frac{\left(\frac{1}{b} - \frac{1}{a}\right) \left(\frac{1}{b} + \frac{1}{a}\right)}{-2(b-a)} = \eta^{-3} \\
\frac{\left(\frac{b-a}{ab}\right) \left(\frac{1}{b} + \frac{1}{a}\right)}{2(b-a)} &= \eta^{-3}; \quad \frac{1}{ab} \cdot \frac{\frac{1}{a} + \frac{1}{b}}{2} = \eta^{-3}; \quad \eta^3 = \left(\frac{2}{\frac{1}{a} + \frac{1}{b}} \right) (ab) \\
\eta_{-3} &= \sqrt[3]{\frac{2}{\frac{1}{a} + \frac{1}{b}}} \cdot \sqrt[3]{ab} = \sqrt[3]{H(a, b)} \cdot \sqrt[3]{G^2(a, b)} = \sqrt[3]{G^2(a, b) \cdot H(a, b)}
\end{aligned}$$

Finally, applying MVTD to the function $f(x) = x^{-2}$ we start from the following relationship:

$$\frac{b^{-2} - a^{-2}}{b - a} = -2\eta^{-3}$$

and continue as above.

The case $\alpha = -4$ has strong similarities with the case $\alpha = -2$. In fact, developing the general formula of SM gives:

$$\begin{aligned}
\eta_{-4}(a, b) &= \left[\frac{b^{-4} - a^{-4}}{-4(b-a)} \right]^{-\frac{1}{4-1}} = \left[\frac{\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \left(\frac{1}{b^2} + \frac{1}{a^2}\right)}{-4(b-a)} \right]^{-\frac{1}{5}} \\
&= \left[\frac{\left(\frac{1}{b} - \frac{1}{a}\right) \left(\frac{1}{b} + \frac{1}{a}\right) \left(\frac{1}{b^2} + \frac{1}{a^2}\right)}{-4(b-a)} \right]^{-\frac{1}{5}} = \left[\frac{\left(\frac{a-b}{ab}\right) \left(\frac{1}{b} + \frac{1}{a}\right) \left(\frac{1}{b^2} + \frac{1}{a^2}\right)}{-4(b-a)} \right]^{-\frac{1}{5}} = \\
&= \left[\frac{1}{4} \cdot \frac{b-a}{ab} \cdot \frac{1}{b-a} \cdot \left(\frac{1}{b} + \frac{1}{a}\right) \left(\frac{1}{b^2} + \frac{1}{a^2}\right) \right]^{-\frac{1}{5}} = \\
&= \left[\frac{1}{ab} \cdot \frac{\frac{1}{a} + \frac{1}{b}}{2} \cdot \frac{\frac{1}{a^2} + \frac{1}{b^2}}{2} \right]^{-\frac{1}{5}} = \left[(ab) \cdot \left(\frac{2}{\frac{1}{a} + \frac{1}{b}} \right) \cdot \left(\frac{2}{\frac{1}{a^2} + \frac{1}{b^2}} \right) \right]^{\frac{1}{5}} \\
&= \sqrt[5]{G^2(a, b) \cdot H(a, b) \cdot H(a^2, b^2)}
\end{aligned}$$

or

$$\begin{aligned}\eta_{-4}(a, b) &= \left[\frac{1}{-4(b-a)} \cdot \frac{(a-b)(a+b)(a^2+b^2)}{a^4b^4} \right]^{-\frac{1}{5}} = \left[a^4b^4 \cdot \frac{2}{a+b} \cdot \frac{2}{a^2+b^2} \right]^{\frac{1}{5}} \\ &= \sqrt[5]{\frac{G^8(a, b)}{A(a, b) \cdot A(a^2, b^2)}} = G \sqrt[5]{\frac{G^3(a, b)}{A(a, b) \cdot A(a^2, b^2)}}\end{aligned}$$

For $n = -2^h$ ($h \in \mathbb{N}$) we have, reasoning by induction,

$$\eta_{-2^h} = \sqrt[2^{h+1}]{G^2(a, b) \cdot H(a, b) \cdot H(a^2, b^2) \cdots H(a^{2^{h-1}}, b^{2^{h-1}})}$$

or

$$\eta_{-2^h} = \sqrt[2^{h+1}]{\frac{G^{2^{h+1}}(a, b)}{A(a, b) \cdot A(a^2, b^2) \cdots A(a^{2^{h-1}}, b^{2^{h-1}})}}$$

We observe that $2^{h+1} > 2^h + 1$, and being $2^{h+1} - 2^h - 1 = 2^h - 1$, we have that

$$2^{h+1} = (2^h + 1) + (2^h - 1)$$

Whence

$$\eta_{-2^h} = G(a, b) \cdot \sqrt[2^{h+1}]{\frac{G^{2^h-1}(a, b)}{A(a, b) \cdot A(a^2, b^2) \cdots A(a^{2^{h-1}}, b^{2^{h-1}})}}$$

8. The mean η_3

Applying MVT to the function $f(x) = x^3$ we have

$$\frac{b^3-a^3}{b-a} = 3\eta^2; \frac{(b-a)(b^2+ba+a^2)}{b-a} = 3\eta^2; \eta = \sqrt{\frac{a^2+ab+b^2}{3}}$$

Instead, applying the MVTI to the function $g(x) = x^2$ gives [42]

$$\frac{1}{b-a} \cdot \int_a^b x^2 dx = \eta^2; \frac{1}{b-a} \cdot \left[\frac{x^3}{3} \right]_a^b = \eta^2; \frac{1}{b-a} \cdot \frac{b^3-a^3}{3} = \eta^2; \eta = \sqrt{\frac{a^2+ab+b^2}{3}}$$

Or more simply by placing $n = 3$ in the expression of the $\eta_n(a, b)$ we have

$$\eta_3(a, b) = \sqrt{\frac{a^2+ab+b^2}{3}}$$

The radical that appears in the expression of η_3 always has meaning for every real a and b . We show that the function

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$$\varphi(a, b) = a^2 + ab + b^2$$

admits absolute minimum at the point (0;0), and thus

$$\varphi(a, b) = a^2 + ab + b^2 \geq 0; \quad \varphi(0,0) = 0$$

for every real a and b . We begin by determining any zeros of the partial prime derivatives of $\varphi(a, b)$:

$$\begin{cases} \varphi_a = 2a + b \\ \varphi_b = a + 2b \end{cases}; \quad \varphi_a(0,0) = \varphi_b(0,0) = 0$$

The only zero of the prime partial derivatives is $O(0;0)$. We calculate the Hessian of the function $\varphi(a, b)$:

$$H(a, b) = \begin{vmatrix} \varphi_{aa} & \varphi_{ab} \\ \varphi_{ba} & \varphi_{bb} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$

The question arises as to whether $\eta_3 \in]a; b[$. In the case where $0 < a < b$, the point η_3 is definitely internal to $[a; b]$. In fact,

$$\sqrt{\frac{a^2 + ab + b^2}{3}} > \sqrt{\frac{a^2}{3}} = \sqrt{3} |a| = \sqrt{3} a > a;$$

$$\sqrt{\frac{a^2 + ab + b^2}{3}} < \sqrt{\frac{3b^2}{3}} = |b| = b.$$

We observe that, for each positive real a and b , with $a < b$, the inequality applies

$$0 < A(a, b) < \eta_3(a, b)$$

that is, the Stolarsky mean η_3 of parameter $\alpha = 3$ is strictly greater than the arithmetic mean A .

In fact, if absurdly

$$\frac{a + b}{2} \geq \sqrt{\frac{a^2 + ab + b^2}{3}}$$

then

$$\frac{a^2 + 2ab + b^2}{4} \geq \frac{a^2 + ab + b^2}{3}$$

from which

$$3a^2 + 6ab + 3b^2 \geq 4a^2 + 4ab + 4b^2$$

i.e.

$$a^2 + 2ab + b^2 \leq 0$$

vs

$$a^2 + 2ab + b^2 = (a + b)^2 > 0$$

The expression of η_3 can be related to that of quadratic mean QM and geometric mean G . Indeed

$$\sqrt{\frac{a^2 + ab + b^2}{3}} = \sqrt{\frac{a^2 + (\sqrt{ab})^2 + b^2}{3}} = QM(a, G(a, b), b)$$

where

$$QM(x_1, \dots, x_n) = \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}$$

is the quadratic mean of n real not necessarily positive terms $x_1, \dots, x_n$.

9. Geometric interpretation of η_3

The mean η_3 can be interpreted by Euclidean Geometry or by Linear Algebra.

9.1 Geometric interpretation using Euclidean Geometry in space.

Recall [9] that, by a well-known result of Euclidean Geometry in space, the volume of truncated square-based straight pyramid is

$$V_{\text{truncated square-based straight pyramid}} = \frac{a^2 + ab + b^2}{3} \cdot h$$

where a, b, h are the edge of the upper base, the edge of the lower base and the height of the truncated square-based straight pyramid, with $a < b$, respectively (Fig. 3).

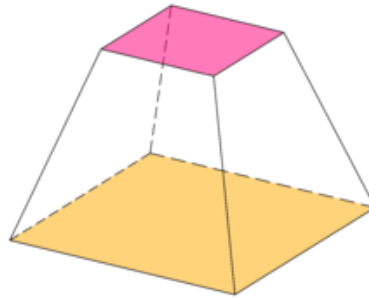


Figure 3 Truncated based straight pyramid

Comparing the radical of the $\eta_3(a, b)$ with the volume of the truncated straight pyramid with a square base, one deduces the equality of the two expressions if and only if $h = 1$. A similar expression is given for the volume for a straight cone having for its top base a circle of radius r , for its top base a circle of radius R , with $r < R$, and for its height h (Fig. 4):

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$$V_{\text{truncated circle-based straight cone}} = \frac{r^2 + rR + R^2}{3} \cdot \pi h$$

In the latter case, the equality between the radicand of $\eta_3(a, b)$ and the volume of the truncated straight cone is achieved by posing $h = 1/\pi$.

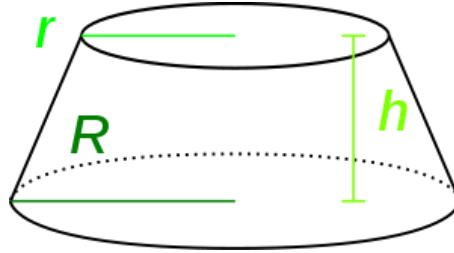


Figure 4 The truncated straight cone

In general, the expression of the truncated straight pyramid having A and B as the area of the bases and h for height is given by

$$V = H(A, B) \cdot h$$

where for heronian mean $H(x, y)$ of two positive numbers x and y we define the expression^{§§§§}

$$H(x, y) = \frac{x + \sqrt{xy} + y}{3}$$

similarly we reason for an interpretation in terms of heronian mean of the volume of the truncated right cone having circles as its bases.

Ultimately, using the language of Euclidean Solid Geometry, $\eta_3(a, b)$ can be regarded as the square root of the volume of a truncated straight pyramid with a square base of height $h = 1$ or of the volume of the truncated right cone having circles as bases and height $h = 1/\pi$.

Wanting to further emphasize the juxtaposition of the $\eta_3(a, b)$ with the basic concepts of Euclidean Solid Geometry, it may be mentioned that the measurement of the volume of a truncated straight pyramid with a square base is one of the oldest results in the History of Mathematics as the instructions for determining it are found in the famous Moscow mathematical papyrus written around 1890 BCE [31].

^{§§§§} In turn, the Heronian mean can be thought of as a particular weighted average of the arithmetic mean and geometric mean of two positive numbers x and y:

$$H(x, y) = \frac{2}{3} \cdot \frac{x + y}{2} + \frac{1}{3} \cdot \sqrt{xy}$$

9.2 Geometric interpretation of the $\eta_3(a, b)$ by linear algebra

For simplicity, let us pose $\eta_3(a, b) = z$, $a = x$ e $b = y$. Then the previous formula becomes

$$z = \sqrt{\frac{x^2 + xy + y^2}{3}}$$

from which

$$x^2 + xy + y^2 - 3z^2 = 0 \quad (36)$$

The surfaces generated by z (in blue) and the rooting (in purple) can thus be visualized using Geogebra 3D (Fig. 5):

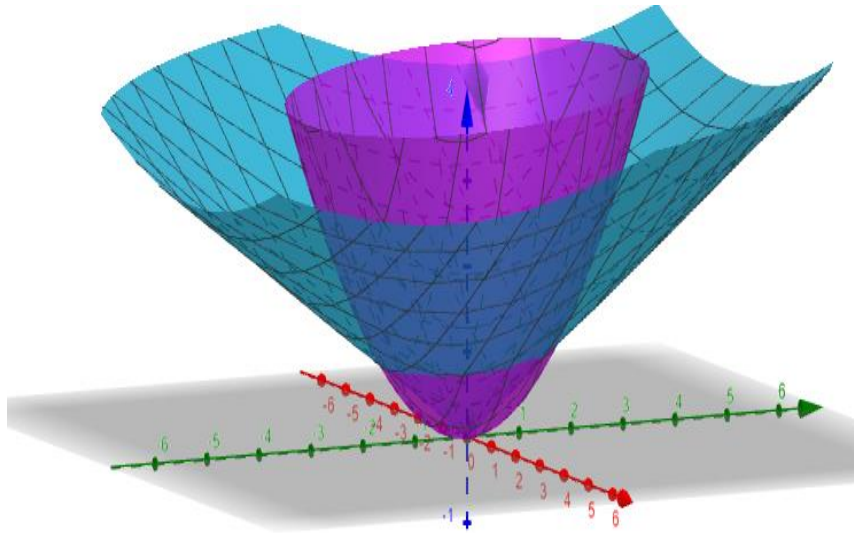


Figure 5 The surface z in blue and the rooting in purple (Geogebra 3D)

Instead, (36) generates a two-pitch cone with vertex at $O(0,0,0)$ (Fig. 6)

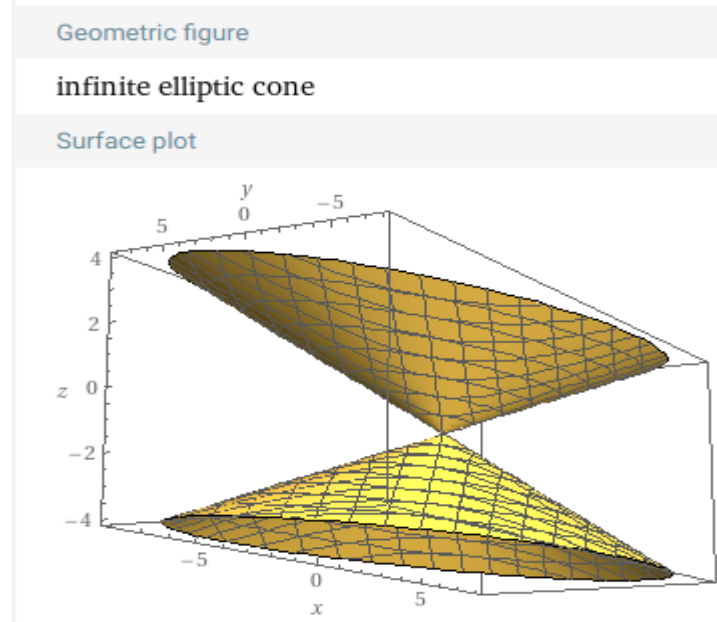


Figure 6 The infinite elliptic cone (Wolfram Alpha)

We think of the elements of R^3 as points, and each point of R^3 we identify with the vector x of its coordinates, the origin of the reference system corresponds to the null vector 0 , while the coordinate axes are the straight lines generated by the ortho-normal basis $\{e_1, e_2, e_3\}$, where $e_1 = [1,0,0]^T, e_2 = [0,1,0]^T, e_3 = [0,0,1]^T$ are two-by-two orthogonal verses.

Recall that a quadric is defined as the locus of points in R^3 that satisfy the following equation in the x,y,z coordinates: $a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2b_1x + 2b_2y + 2b_3z + c = 0$

and that it depends on the 10 coefficients $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}, b_1, b_2, b_3, c$. Let's say $x^T = [x, y, z]$ the row vector formed by the variables x,y,z , A the symmetric matrix ***** given by the coefficients of the second-degree terms,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

***** The square matrix A is defined as symmetrical if $A^T = A$, where A^T is the transposed matrix of A .

$\mathbf{b}^T = [b_1, b_2, b_3]$ the row vector of the coefficients of the first-degree terms in x, y, z and c the constant, the equation of the quadric can be written in a more compact form as follows:

$$q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + c \quad (37)$$

q being the sum of the quadratic form^{††††} $\mathbf{x}^T \mathbf{A} \mathbf{x}$, of the linear form $2\mathbf{b}^T \mathbf{x}$ and the constant c . Comparing (36) with (37), the square of η_3 looks like a quadric $q(\mathbf{x})$, where $\mathbf{x} = [x, y, z]^T$ having

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}; \mathbf{b}^T = [0, 0, 0]; c = 0$$

The following block matrix is used to classify the quadric q arising from the expression of η_3

$$\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c \end{bmatrix}$$

whereby $q(\mathbf{x})$ takes the form

$$q(\mathbf{x}) = [\mathbf{x}^T \quad 1] \mathbf{B} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \mathbf{z}^T \mathbf{B} \mathbf{z}$$

where $\mathbf{z}$ is the vector obtained by adding the number 1 to $\mathbf{x}$ as the last component. $\mathbf{B}$ is said to be the matrix associated with the polynomial $q(\mathbf{x})$. Therefore

$$\mathbf{B} = \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is immediately seen that $\det(\mathbf{A}) = -9/4 \neq 0$ whence $\text{rank}(\mathbf{A}) = 3$. Moreover $\text{rank}(\mathbf{B}) = 3$ since $\det(\mathbf{B}) = 0$ being the fourth row (the fourth column) of $\mathbf{B}$ consisting of all 0s. It is concluded that $q(\mathbf{x})$ is a real cone. Since $\mathbf{b}$ is the null vector the cone $q(\mathbf{x})$ already has its vertex in the origin of the Cartesian reference^{†††††}. To write the cone in reduced form, we need to eliminate the second-degree term of “mixed” type xy that appears in (37) by performing a rotation $\mathbf{x} = \mathbf{Q}\mathbf{X}$ where $\mathbf{Q}$ is an orthogonal matrix with determinant equal to 1. The matrix $\mathbf{Q}$ will have for columns the normalized eigenvectors relative to the eigenvalues of $\mathbf{A}$. To obtain $\det(\mathbf{Q}) = 1$ eventually we will change the sign

^{†††††} A quadratic form in two variables x and y is defined as a homogeneous polynomial of second degree of the type

$$q(x, y) = ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

^{†††††} In fact, since $\mathbf{b} = 0$, the linear system $\mathbf{A}\mathbf{w}=\mathbf{b}$ corresponds to the homogeneous system $\mathbf{A}\mathbf{w}=0$, and since $\text{rank}(\mathbf{A})=\text{rank}(\mathbf{A}|\mathbf{B})=3$, the system $\mathbf{A}\mathbf{w}=0$ admits as its only solution $\mathbf{w} = 0$. Therefore, the transformation $\mathbf{x} = \mathbf{X} - \mathbf{w}$, which allows the elimination of any first-degree terms in the quadric equation, is the translation of amplitude $\mathbf{w} = 0$, i.e., the identical affine transformation $\mathbf{x} = \mathbf{X}$. A foregone consideration because of the absence of linear terms in x, y and z in the equation of the quadric we are studying.

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to the components of one of the columns of Q . For this purpose we determine the eigenvalues of the real symmetric matrix $A^{\text{§§§§}}$. We obtain

$$p(\lambda) = |\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{vmatrix} = 0$$

Developing the determinant according to the third line gives:

$$(\lambda + 3) \left[(\lambda - 1)(\lambda - 1) - \frac{1}{4} \right] = 0$$

$$(\lambda + 3) \left(\lambda^2 - 2\lambda + 1 - \frac{1}{4} \right) = 0$$

$$(\lambda + 3) \left(\lambda^2 - 2\lambda + \frac{3}{4} \right) = 0$$

$$(\lambda + 3) \left(\lambda - \frac{1}{2} \right) \left(\lambda - \frac{3}{2} \right) = 0$$

The eigenvalues then are all real, as we expected, and distinct^{*****} and are given by

$$\lambda_1 = -3 ; \lambda_2 = \frac{1}{2} ; \lambda_3 = \frac{3}{2}$$

9.2.1 Eigenvectors and main axes of the quadric

We solve the system $\lambda I - A = 0$ for the different eigenvalues found above.

$$\begin{cases} (\lambda - 1)x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + (\lambda - 1)y = 0 \\ (\lambda + 3)z = 0 \end{cases}$$

V_1 eigenspace related to the eigenvalue $\lambda_1 = -3$:

§§§§ The eigenvalues of a real symmetric matrix are real numbers.

***** Recall that given a real symmetric matrix A , distinct eigenvalues of A correspond to two-by-two orthogonal eigenvectors.

$$\begin{cases} -4x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x - 4y = 0 \\ (-3+3)z = 0; 0 \cdot z = 0 \end{cases} ; \begin{cases} -8x - y = 0; 8x + y = 0 \\ -x - 4y = 0; x + 4y = 0 \\ 0 \cdot z = 0 \end{cases}$$

We step down the matrix of coefficients of the unknowns:

$$\begin{bmatrix} 8 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & 1 & 0 \\ 0 & 63 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix of coefficients of unknowns has rank = 2. Therefore, the number of free variables is 1 (= number of unknowns - matrix rank = 3-2). An arbitrary value k is assigned to z.

$$\begin{cases} 8x + y = 0 \\ x + 8y = 0 \end{cases}$$

$$\begin{cases} y = -8x; \\ x - 64x = 0; -63x = 0; x = 0 \rightarrow y = 0 \\ z = k \end{cases}$$

By writing $(0;0;k) = k(0;0;1)$ we have that the eigenvector for $k = 1$ corresponding to the eigenvalue $\lambda = -3$ is

$$v_1 = (0,0,1)$$

Therefore the eigenspace V_1 generated by $\lambda_2 = \frac{1}{2}$ is the line of R^3 passing through the points

$$O = (0,0,0); P_1 = (0,0,1).$$

V_2 eigenspace related to the eigenvalue $\lambda_2 = \frac{1}{2}$:

$$\begin{cases} -\frac{1}{2}x - \frac{1}{2}y = 0; x + y = 0 \\ -\frac{1}{2}x - \frac{1}{2}y = 0; -x + y = 0; x + y = 0 \\ \left(\frac{1}{2} + 3\right)z = 0; z = 0 \end{cases}$$

Stepwise reduction of the matrix of coefficients of unknowns

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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The coefficient matrix has *rank* 2. Therefore, the homogeneous system possesses only one free variable. We assign y an arbitrary value k , $y = k$.

Since $x = -y$ and $z = 0$ then we have $(-y;y;0) = k(-1;1;0)$ from which the eigenvector $v_2 = (-1,1,0)$.

The eigenspace V_2 generated by the eigenvector v_2 is the line of R^3 passing through the points $O = (0,0,0)$ and $P_2 = (-1,1,0)$.

The V_3 eigenspace related to the eigenvalue $\lambda_3 = 3/2$:

$$\begin{cases} \left(\frac{3}{2} - 1\right)x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \left(\frac{3}{2} - 1\right)y = 0; -x + y = 0 \\ \left(\frac{3}{2} + 3\right)z = 0 \end{cases}$$

$$\begin{cases} \frac{1}{2}x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \frac{1}{2}y = 0; \begin{cases} x - y = 0 \\ -x + y = 0 \\ z = 0 \end{cases} \\ \frac{9}{2}z = 0 \end{cases}$$

Stepwise reduction of the matrix of coefficients of unknowns:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix, reduced to steps, has two nonzero rows, and therefore *rank* 2. The number of free variables is 1. We assign y an arbitrary value, $y = k$. Since $x = y$ and $z = 0$ then $(y;y;0) = k(1;1;0)$

Eigenvector for $k = 1$:

$$v_3 = (1,1,0)$$

The eigenspace V_3 generated by v_3 is the line of R^3 passing through the points $O = (0,0,0)$ and $P_3 = (1,1,0)$.

Moreover, the space R^3 is direct sum of subspaces of R^3 two by two orthogonal to each other :

$$R^3 = V_1 \oplus V_2 \oplus V_3$$

We represent in R^3 using Geogebra 3D the vectors $\{v_1, v_2, v_3\}$ (Fig.7) from which the eigenspaces generated by them can be deduced:

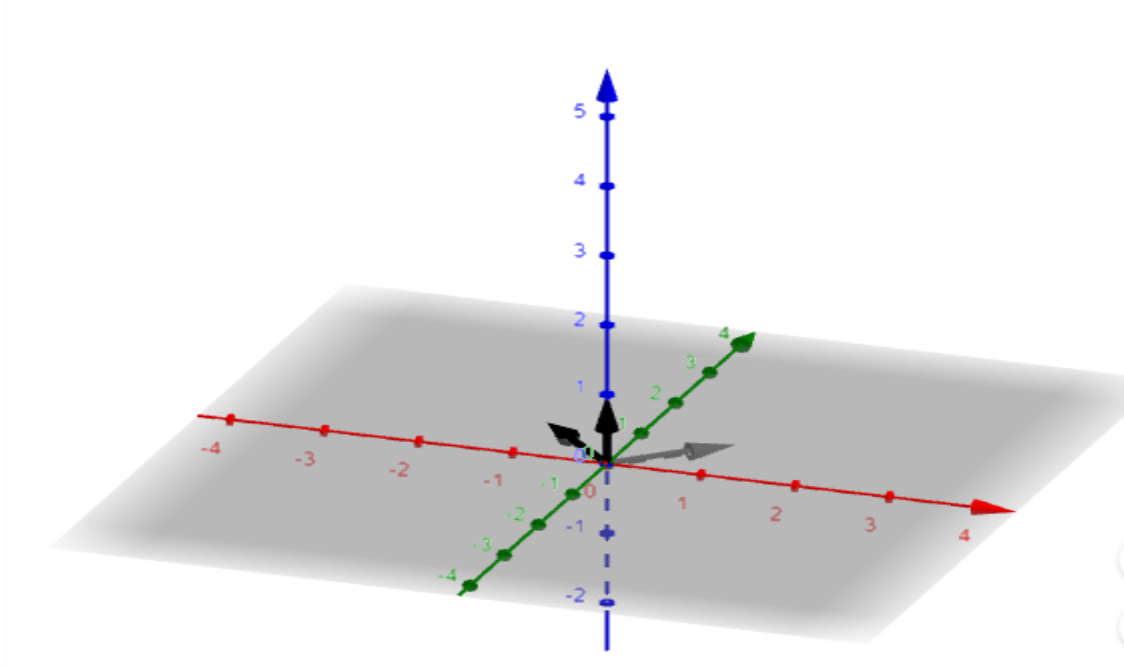


Figure 7 Representation in R^3 dei vettori v_1 , v_2 and v_3 .

Rotation matrix and reduced form of the quadric

The matrix S

$$S = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

having for columns the eigenvectors of A v_1, v_2, v_3 found above allows A to be diagonalized but in general is not an orthogonal matrix, as we shall now verify: Developing the determinant of S according to the third row gives:

$$\det(S) = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (+1) \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

Since $\det(S) = -2 \neq 0$ then S is invertible and its inverse is

$$S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

We verify that the eigenvectors of A forming S , v_1, v_2, v_3 , are two by two orthogonal

$$v_1 \cdot v_2 = 0 \cdot (-1) + 0 \cdot 1 + 1 \cdot 0 = 0$$

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$$v_1 \cdot v_3 = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

$$v_2 \cdot v_3 = (-1) \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = -1 + 1 = 0$$

and that they are not all of unified standard:

$$\|v_1\| = 1; \|v_2\| = \sqrt{2}; \|v_3\| = \sqrt{2}$$

The matrix S is not orthogonal^{†††††} : the product of the matrix by its transpose does not coincide with the identity matrix I :

$$SS^T = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We normalize the eigenvectors v_1, v_2, v_3 by dividing each eigenvector by its norm:

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}; u_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

The matrix S^* formed by the orthonormal base $\{u_1, u_2, u_3\}$ of R^3 :

$$S^* = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

is orthogonal:

$$\det(S^*) = \begin{vmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

^{†††††} A square matrix Q of order n is defined as orthogonal if $Q^T Q = I$, where I is the identity matrix of order n . Since $(Q^T Q)^T = I^T \Leftrightarrow Q Q^T = I$ it is obtained that Q orthogonal if $Q Q^T = I$. It follows from the definition that the orthogonal matrix Q admits inverse Q^{-1} e $Q^{-1} = Q^T$. Furthermore $\det(Q) = \pm 1$.

but it does not have determinant equal to 1 and therefore is not a rotation matrix Q . We change the sign of the third column of S^* and obtain

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

whence $\det(Q) = \frac{1}{2} + \frac{1}{2} = 1$. By means of the transformation $x = QX$ we can eliminate the mixed term in xy from equation (11). The subspaces of R^3 generated by $\{u_1, u_2, u'_3\}$, where $u'_3 = \left[-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\right]^T$ represent the principal axes.

Trace of the rotation matrix Q and rotation angle

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{Trace}(Q) = 0 + \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

From

$$\cos \theta = \frac{1}{2} (\text{Trace}(Q) - 1) = \frac{1}{2} \left(\frac{1}{\sqrt{2}} - 1 \right)$$

obtained from Euler's Theorem, we derive the angle of rotation.

We show that a section of the cone $q(x)$ is a real ellipse. We section the quadric Q by a plane $z = k$, with $k > 0$, the radical being nonnegative:

$$\frac{x^2 + xy + y^2}{3} = k$$

We illustrate the case for $k = 1$, i.e., cutting the quadric with a $z = 1$ plane yields the conic (Fig. 8)

$$q'(x, y) = x^2 + xy + y^2 - 3 = 0$$

Like the quadric studied above, $q'(x, y)$ can also be written in a more compact form

$$q(x) = \mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + c \quad (38)$$

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where

$$A' = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}; \quad b = [0 \quad 0]^T; \quad c' = -3 \text{ (for } k = 1; \text{ } c' = -3k \text{ in the general case)}$$

is the matrix of the coefficients of the terms of the second degree and B'

$$B' = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

is the block matrix

$$B' = \begin{bmatrix} A' & b' \\ b'^T & c' \end{bmatrix}$$

Since

$$\det(B') = (-3) \left(1 - \frac{1}{4}\right) = -\frac{9}{4} \neq 0$$

we can state that q' is nondegenerate conic having the center C in the origin $O(0,0)$ and b is the null vector. Also

$$A' = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\det(A') = 1 - \frac{1}{4} = \frac{3}{4} \neq 0$$

from which $\text{rank}(A) = 2$ and $\text{rank}(B) = 3$. In the general case

$$x^2 + xy + y^2 - 3k = 0$$

the conic is nondegenerate if $k \neq 0$ since

$$\det(B') = (-3k) \left(1 - \frac{1}{4}\right) = -\frac{9}{4}k < 0 \quad (\forall k \neq 0)$$

We determine the eigenvalues of the symmetric matrix A by considering its characteristic polynomial

$$p_A = |\lambda I - A'| = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - \frac{1}{4} = \lambda^2 - 2\lambda + 1 - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{3}{4}$$

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$$\Delta = 4 - 3 = 1$$

$$\lambda = \frac{2 \pm 1}{2}$$

$$\lambda_1 = \frac{1}{2}; \lambda_2 = \frac{3}{2}$$

It is necessary to classify the conic. We define

$I_1 = \text{trace}(A') = \lambda_1 + \lambda_2$; $I_2 = \det(A') = \lambda_1 \lambda_2$; $I_3 = \det(B')$ serve that $\text{rank}(A') = 2$ and $\text{rank}(B') = 3$ ($\forall k \neq 0$), which is equivalent to saying that $I_2 \neq 0$; $I_3 \neq 0$

Furthermore,

$$I_1 = \lambda_1 + \lambda_2 = \frac{1}{2} + \frac{3}{2} = 2; \quad I_2 = \lambda_1 \lambda_2 = \frac{3}{4}; \quad I_3 = \det(B) = -\frac{9}{4}k < 0 \quad (\forall k > 0)$$

Being $I_1 > 0$; $I_1 I_3 < 0$, the equation of the conic represents a real ellipse.

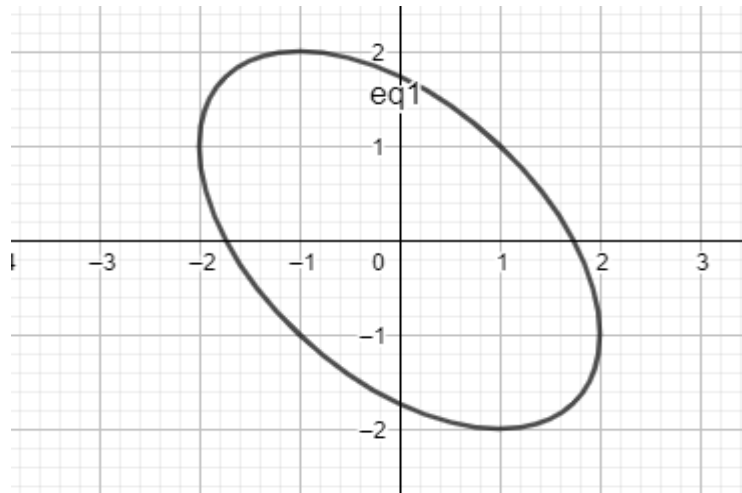


Figure 8 Representation of ellipse of equation $x^2 + xy + y^2 - 3 = 0$

By a well-known theorem on the canonical form of a polynomial of the second degree, the canonical form of the conic is then

$$\lambda_1 X^2 + \lambda_2 Y^2 + \tilde{c} = 0$$

$$\frac{1}{2} X^2 + \frac{3}{2} Y^2 + \tilde{c} = 0$$

from the formula

$$-\frac{9}{4}k = I_3 = \det(A) \tilde{c} = \frac{3}{4} \tilde{c}$$

we derive that

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$$\tilde{c} = -3k(\forall k \neq 0)$$

Therefore, the equation of the conic can be written as

$$\frac{1}{2}X^2 + \frac{3}{2}Y^2 - 3k = 0; \frac{X^2}{2} + \frac{Y^2}{\frac{2}{3}} = 3k$$

Dividing all the terms in the equation by $-\tilde{c}$ we obtain

$$\frac{X^2}{6k} + \frac{Y^2}{2k} = 1 (\forall k \neq 0)$$

that is, the equation of the ellipse in canonical form:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

where

$$a = \sqrt{6k}; b = \sqrt{2k}; k \neq 0$$

For $k = 1$ we have the ellipse (Fig. 9)

$$\frac{X^2}{6} + \frac{Y^2}{2} = 1$$

with the foci lying on the x-axis, having minor semi-axis equal to $\sqrt{2}$; semi-major axis equal to $\sqrt{6}$; semi focal distance equal to 2 and eccentricity $\frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

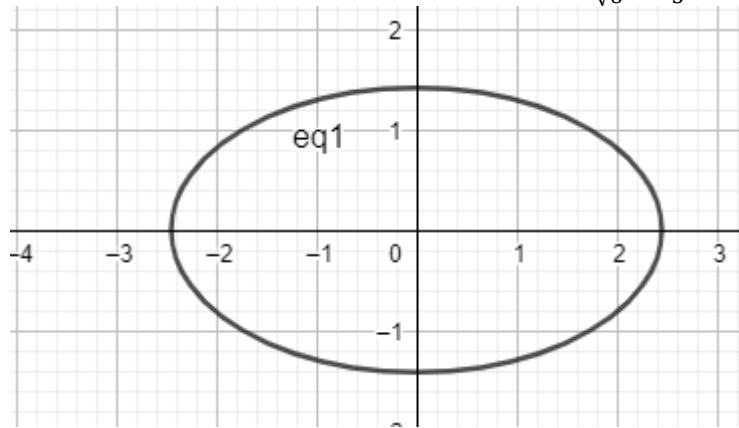


Figure 9 Representation of the ellipse of equation $x^2/6+y^2/2=1$ (Geogebra)

The matrix A' is diagonalized through the matrix formed by the eigenvectors relative to

$$\lambda_1 = \frac{1}{2}; \lambda_2 = \frac{3}{2}$$

Indeed, the system

$$\begin{cases} (\lambda - 1)x + \frac{1}{2}y = \\ \frac{1}{2} + (\lambda - 1)y = 0 \end{cases}$$

for $\lambda_1 = \frac{1}{2}$ gives the solution $(x,y) = k(-1,1)$ while for $\lambda_2 = \frac{3}{2}$ leads to the solution $(x,y) = k(1,1)$ and thus to the matrix, for $k=1$,

$$S' = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

formed by the eigenvectors

$$w_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The matrix S' is not an orthogonal matrix. In fact $\det(S') = -2$ although the vectors are orthogonal to each other

$$w_1 \cdot w_2 = (-1)(1) + (1)(1) = -1 + 1 = 0$$

Being $\|w_1\| = \|w_2\| = \sqrt{2}$, we have that the vectors

$$u = \frac{w_1}{\|w_1\|}, v = \frac{w_2}{\|w_2\|}$$

represent an orthonormal basis of R^2 and the subspaces of R^2 generated by each of them are the lines to them, called the principal axes of the ellipse (Fig.10, Geogebra).

The matrix

$$Q' = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

is orthogonal since

$$\det(Q') = -\frac{1}{2} - \frac{1}{2} = -1$$

But it is not a rotation matrix. By appropriately modifying Q' by changing the sign to an element of Q' :

$$Q'' = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

we have that $\det(Q'') = 1$. Whence Q'' represents the rotation matrix sought.

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The transformation $Z = Q'X$ allows the ellipse to be rotated 135° clockwise with respect to the x -axis (Fig.11). In fact, posed

$$\sin \theta = -1/\sqrt{2} ; \cos \theta = 1/2$$

we have that $\tan \theta = -1$, from which $\theta = 45^\circ + 90^\circ = 135^\circ$ (or 45° counterclockwise)

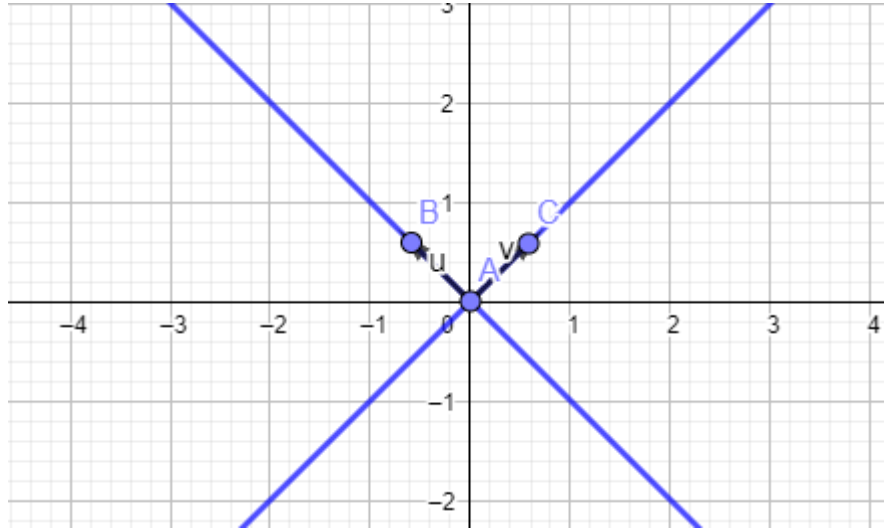


Figure 10 Representation of subspaces generated by vectors u and v

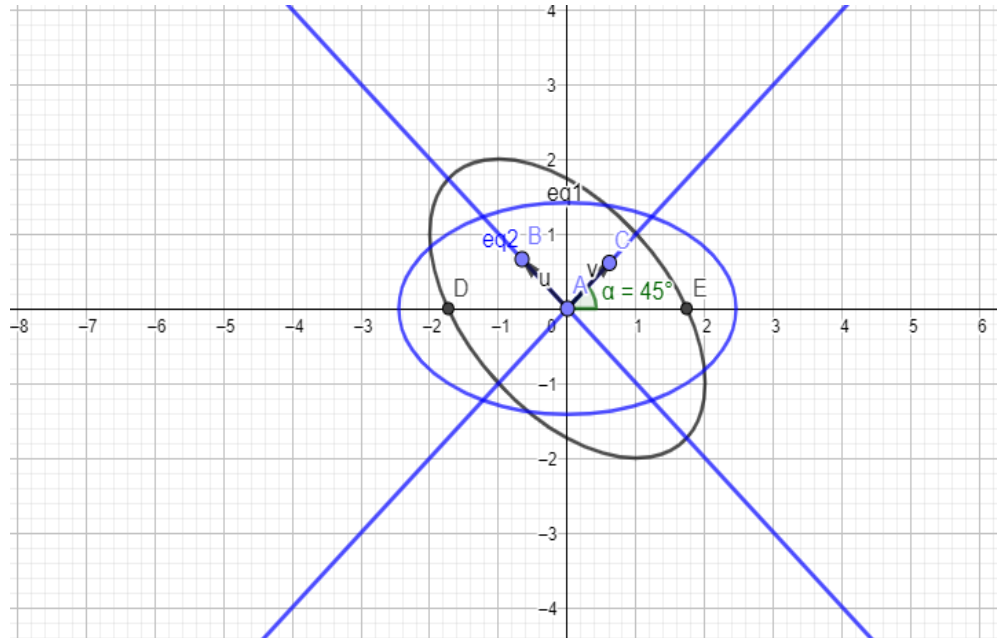


Figure 11 Representation of ellipses $x^2+xy+y^2-3=0$, $x/6+y/2=1$ and subspaces generated by vectors u and v (Geogebra)

In Figure 10 the ellipse in black corresponds to the equation of the conic written in noncanonical form while the ellipse in blue represents the equation of the ellipse in canonical form.

As k varies, we obtain the following family of ellipses (Fig. 12; Wolfram Alpha)

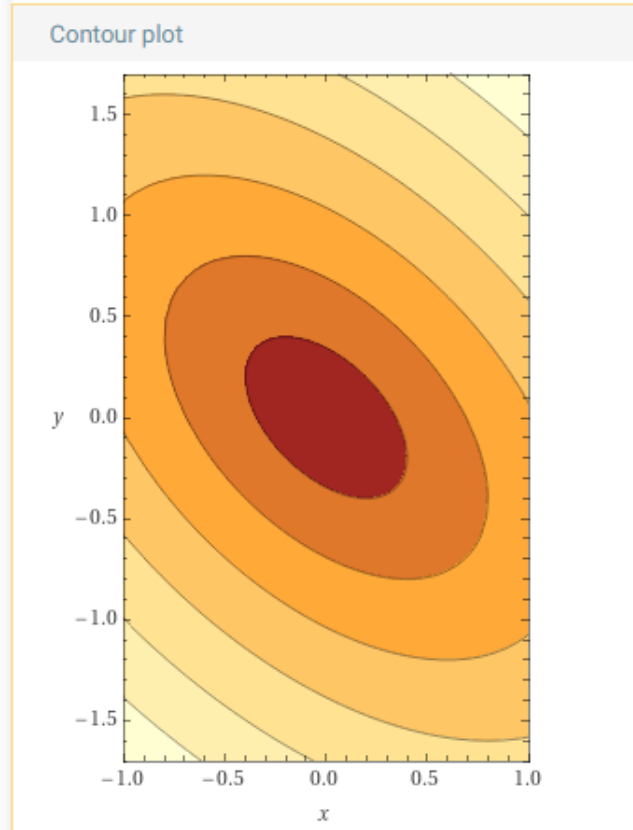


Figure 12 Family of ellipses obtained at the variation of k

10. A new mean η_3^F originated from an extension of Flett's theorem

One of the possible extensions of MVT is the Flett's theorem [14][42] which, under the assumptions of a function f derivable in $[a,b]$, with $a < b$ and , asserts the existence of at least one ξ , with $a < \xi < b$ such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}$$

and one η , with $a < \eta < b$ such that

$$f'(\eta) = \frac{f(b) - f(\eta)}{b - \eta}$$

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We observe that it is not possible to generate Stolarsky's means by means of Flett's theorem because the condition $f'(a) = f'(b)$ is not verified for the function $f(x) = x^\alpha$. In fact, $\alpha a^{\alpha-1} = \alpha b^{\alpha-1}$ if, and only if, $a = b$ against the fact that by hypothesis $a < b$. Moreover, the function $f(x) = x^\alpha$ does not even satisfy the assumption of the Flett's theorem analogue for integral calculus, which states that if f is a function defined and continuous in $[a, b]$ and $f(a) = f(b)$ then there exists Nor does the function satisfy the assumption of the Flett's theorem analogue for integral calculus, which states that if f is a function defined and continuous in $[a, b]$ and $f(a) = f(b)$ then there is at least one η , with $a < \eta < b$, such that

$$(\eta - a)f(\eta) \int_a^\eta f(x) dx$$

In fact, the condition $f(a) = f(b)$, equivalent to the equality leading to the relation $a = b$ against the fact that $a < b$ by hypothesis, is not verified. However, there is an extension of the Flett's theorem [42, p. 165] that removes the equality condition of the prime derivatives on the extremes of the defining interval. It states that if f is derivable in $[a, b]$, there exists at least one ξ , with $a < \xi < b$, such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a} + \frac{1}{2} \cdot \frac{f'(b) - f'(a)}{b - a} (\xi - a)$$

We suppose $f(x) = x^\alpha$, the previous relationship becomes.

$$\alpha \xi^{\alpha-1} = \frac{\xi^\alpha - a^\alpha}{\xi - a} + \frac{1}{2} \cdot \frac{\alpha b^{\alpha-1} - \alpha a^{\alpha-1}}{b - a} (\xi - a)$$

For $\alpha = 3$ the above formula takes the following form:

$$\begin{aligned} 3\xi^2 &= \frac{\xi^3 - a^3}{\xi - a} + \frac{3}{2} \cdot \frac{b^2 - a^2}{b - a} (\xi - a) \\ 3\xi^2 &= \frac{(\xi - a)(\xi^2 + \xi a + a^2)}{\xi - a} + \frac{3}{2} \cdot \frac{(b - a)(b + a)}{b - a} (\xi - a) \\ 3\xi^2 &= \xi^2 + \xi a + a^2 + \frac{3}{2} \cdot (b + a)(\xi - a) \\ 6\xi^2 &= 2\xi^2 + 2\xi a + 2a^2 + 3(b + a)(\xi - a) \\ 4\xi^2 - (2a + 3a + 3b)\xi - 2a^2 + 3ab + 3a^2 &= 0 \\ 4\xi^2 - (5a + 3b)\xi + 3ab + a^2 &= 0 \end{aligned}$$

We assume $A = 4, B = -(5a + 3b), C = 3ab + a^2$, then the discriminant of the second-degree equation in the unknown ξ is

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$$\begin{aligned}\Delta &= [-(5a + 3b)]^2 - 4 \cdot 4 \cdot (3ab + a^2) = \\ &= 25a^2 + 9b^2 + 30ab - 48ab - 16a^2 = \\ &= 9a^2 - 18ab + 9b^2\end{aligned}$$

being $a < b$ by assumption. Whence

$$\xi = \frac{(5a + 3b) \pm 3|a - b|}{8} = \frac{5a + 3b \pm 3(b - a)}{8}$$

which leads to two solutions

$$\begin{aligned}\xi_+ : \frac{5a + 3b + 3b - 3a}{8} &= \frac{2a + 6b}{8} = \frac{a + 3b}{4} \\ \xi_- : \frac{5a + 3b - 3b + 3a}{8} &= \frac{8a}{8} = a\end{aligned}$$

The second solution is discarded because it coincides with a while the first is accepted because it falls strictly between a and b . In fact,

$$a < b; 4a < a + 3b < 4b; a = \frac{4a}{4} < \frac{a + 3b}{4} < \frac{4b}{4} = b$$

Therefore, the η_3 assumes the form

$$\eta_3^F = \frac{a + 3b}{4}$$

where the letter F is intended to recall Flett's theorem. A simple geometric interpretation can also be obtained for this mean by recourse to the notion of the area of a trapezoid having the bases measuring a and $3b$, respectively, and the height h equal to $\frac{1}{2}$.

We can observe that $A(a, b) < \eta_3^F(a, b)$. In fact, if by absurdity it were

$$\eta_3^F < A(a, b) \quad (0 < a < b)$$

we would have

$$\frac{a + 3b}{4} < \frac{a + b}{2}; a + 3b < 2a + 2b; b < a$$

against the fact that we have assumed $a < b$. Therefore η_3^F can also be regarded as a particular weighted arithmetic mean with weights $\frac{1}{2}$ and $\frac{3}{2}$:

$$\eta_3^F = A\left(\frac{a}{2}, \frac{3b}{2}\right)$$

11. Statistical significance of $\eta_{3^h}(h \in \mathbb{N})$

We note that for $\alpha = 9$

$$b^9 - a^9 = (b - a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6)$$

for $\alpha = 27$

$$b^{27} - a^{27} = (b - a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6) \cdot (a^{18} + a^9b^9 + b^{18})$$

for $\alpha = 81$

$$b^{27} - a^{27} = (b - a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6) \cdot (a^{18} + a^9b^9 + b^{18}) \\ \cdot (a^{54} + a^{27}b^{27} + b^{54})$$

from which

$$\eta_9(a, b) = \sqrt[8]{QM^2(a, GM^2(a, b), b) \cdot QM^2(a^3, GM^2(a^3, b^3), b^3)} \\ \eta_{27}(a, b) = \sqrt[26]{QM^2(a, GM^2(a, b), b) \cdot QM^2(a^3, GM^2(a^3, b^3), b^3) \cdot QM^2(a^9, GM^2(a^9, b^9), b^9)} \\ \eta_{81}(a, b) = \sqrt[80]{QM^2(a, GM^2(a, b), b) \cdot QM^2(a^3, GM^2(a^3, b^3), b^3) \cdot} \\ \cdot \sqrt[80]{QM^2(a^9, GM^2(a^9, b^9), b^9) \cdot QM^2(a^{27}, GM^2(a^{27}, b^{27}), b^{27})} \\ \dots$$

More generally, from the formula valid for every positive natural h :

$$b^{3^h} - a^{3^h} = (b - a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6) \dots (a^{2 \cdot 3^{h-1}} + a^{3^{h-1}}b^{3^{h-1}} + b^{2 \cdot 3^{h-1}})$$

Reasoning by induction and assuming that the previous formula up to h is true, since

$$b^{3^{h+1}} - a^{3^{h+1}} = (b^{3^h} - a^{3^h}) \cdot (a^{2 \cdot 3^h} + a^{3^h}b^{3^h} + b^{2 \cdot 3^h})$$

we have that

$$b^{3^{h+1}} - a^{3^{h+1}} = (b - a)(a^2 + ab + b^2) \dots (a^{2 \cdot 3^h} + a^{3^h}b^{3^h} + b^{2 \cdot 3^h})$$

Therefore, we obtain that $\eta_{3^h}(a, b)$ admits the following expression

$$\sqrt[3^h]{QM^2(a, GM^2(a, b), b) \dots QM^2(a^{3^{h-1}}, GM^2(a^{3^{h-1}}, b^{3^{h-1}}), b^{3^{h-1}})}$$

result that tells us that the SM of parameter equal to a power of base 3 and positive integer exponent can be traced back to the root of a monomial in which appear as factors the square of the mean QM having as arguments appropriate powers of a , ab and b .

12. Statistical significance of $\eta_{-3}, \eta_{-9}, \eta_{-3^h}$ ($h \in \mathbb{N}$)

For the case $\alpha = -3$ we can repeat the steps of the case $\alpha = -2$ and connect η_{-3} to G and H . We will simply apply the general formula of SM, proceeding similarly by MVTD and MVTI:

$$\begin{aligned}\eta_{-3}(a, b) &= \left[\frac{b^{-3} - a^{-3}}{-3(b-a)} \right]^{\frac{1}{-3-1}} = \left[\frac{\left(\frac{1}{b} - \frac{1}{a}\right) \left(\frac{1}{b^2} + \frac{1}{ab} + \frac{1}{a^2}\right)}{-3(b-a)} \right]^{-\frac{1}{4}} \\ &= \left[\frac{\left(\frac{b-a}{ab}\right) \left(\frac{1}{b^2} + \frac{1}{ab} + \frac{1}{a^2}\right)}{3(b-a)} \right]^{-\frac{1}{4}} \\ &= \left[(ab) \left(\frac{3}{\frac{1}{b^2} + \frac{1}{ab} + \frac{1}{a^2}} \right) \right]^{\frac{1}{4}} = \sqrt[4]{G^2(a, b) \cdot H(a^2, ab, b^2)}\end{aligned}$$

From

$$b^{-3} - a^{-3} = \frac{(a-b)(a^2 + ab + b^2)}{a^3 b^3}$$

we have also

$$\begin{aligned}\eta_{-3}(a, b) &= \left[\frac{(b-a)(a^2 + ab + b^2)}{\frac{a^3 b^3}{3(b-a)}} \right]^{-\frac{1}{4}} = \left[a^3 b^3 \cdot \frac{3}{a^2 + ab + b^2} \right]^{\frac{1}{4}} \\ &= \sqrt[4]{\frac{G^6(a, b)}{A(a^2, ab, b^2)}} = G(a, b) \cdot \sqrt[4]{\frac{G^2(a, b)}{A(a^2, G^2(a, b), b^2)}}\end{aligned}$$

Being

$$\begin{aligned}b^{-9} - a^{-9} &= \left(\frac{1}{b^3}\right)^3 - \left(\frac{1}{a^3}\right)^3 = \left(\frac{1}{b^3} - \frac{1}{a^3}\right) \cdot \left(\frac{1}{b^6} + \frac{1}{b^3 a^3} + \frac{1}{a^6}\right) = \\ &= \left(\frac{1}{b} - \frac{1}{a}\right) \left(\frac{1}{b^2} + \frac{1}{ba} + \frac{1}{a^2}\right) \cdot \left(\frac{1}{b^6} + \frac{1}{b^3 a^3} + \frac{1}{a^6}\right)\end{aligned}$$

we obtain

$$\eta_{-9}(a, b) = \sqrt[10]{G^2(a, b) \cdot H(a^2, ab, b^2) \cdot H(a^6, a^3 b^3, b^6)}$$

or

$$\begin{aligned}\eta_{-9}(a, b) &= \left[\frac{\frac{(b-a)(a^2+ab+b^2)}{a^9b^9}}{9(b-a)} \right]^{-\frac{1}{10}} \\ &= \left[a^9b^9 \cdot \frac{3}{a^2+ab+b^2} \cdot \frac{3}{a^6+a^3b^3+b^6} \right]^{\frac{1}{10}} \\ &= \sqrt[10]{\frac{G^{18}(a, b)}{A(a^2, ab, b^2) \cdot A(a^6, a^3b^3, b^6)}} = G(a, b) \cdot \sqrt[10]{\frac{G^9(a, b)}{A(a^2, ab, b^2) \cdot A(a^6, a^3b^3, b^6)}}\end{aligned}$$

from which, more generally, reasoning by induction, we arrive at the following formula valid for any natural h :

$$\eta_{-3^h}(a, b) = \sqrt[3^{h+1}]{G^2(a, b) \cdot H(a^2, ab, b^2) \cdots H(a^{2(3^{h-1}}), (ab)^{3^{h-1}}, b^{2(3^{h-1}}))}$$

or

$$\eta_{-3^h}(a, b) = \sqrt[3^{h+1}]{\frac{G^{2 \cdot 3^h}(a, b)}{A(a^2, ab, b^2) \cdots A(a^{2(3^{h-1}}), (ab)^{3^{h-1}}, b^{2(3^{h-1}}))}}$$

Therefore, being $2 \cdot 3^h > 3^h + 1$ e $2 \cdot 3^h - 3^h - 1 = 3^h - 1$, we have also that $2 \cdot 3^h = (3^h + 1) + (3^h - 1)$, from which

$$\eta_{-3^h}(a, b) = G(a, b) \cdot \sqrt[3^{h+1}]{\frac{G^{3^h-1}(a, b)}{A(a^2, ab, b^2) \cdots A(a^{2(3^{h-1}}), (ab)^{3^{h-1}}, b^{2(3^{h-1}}))}}$$

13. Statistical significance of η_{5^h} ($h \in \mathbb{N}$)

We have that

$$\begin{aligned}\eta_{5^h} &= \left[\frac{b^{5^h} - a^{5^h}}{5^h(b-a)} \right]^{\frac{1}{5^h-1}} = \\ &= \sqrt[5^h-1]{\frac{\prod_{i=1}^h Q_i}{5^h}} \\ &= \sqrt[5^h-1]{\prod_{i=1}^h A_i}\end{aligned}$$

where the difference of powers $b^{5^h} - a^{5^h}$ is developed as the product of $h+1$ homogeneous Q_i polynomials in a and b , of which the first factor is $(b-a)$ while the remaining factors have degree $4 \cdot 5^i$ ($1 \leq i \leq h$) and are each formed by 5 terms:

$$b^{5^h} - a^{5^h} = \prod_{i=0}^h Q_i$$

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^4 a^{(4-j) \cdot 5^{i-1}} b^{j \cdot 5^{i-1}} \quad (1 \leq i \leq h)$$

$$A_i \left(a^{4 \cdot 5^{i-1}} b^0, a^{3 \cdot 5^{i-1}} b^{1 \cdot 5^{i-1}}, a^{2 \cdot 5^{i-1}} b^{2 \cdot 5^{i-1}}, a^{1 \cdot 5^{i-1}} b^{3 \cdot 5^{i-1}}, a^0 b^{4 \cdot 5^{i-1}} \right) \quad (1 \leq i \leq h)$$

14. SM with two-parameter α and β ($\alpha = 2^h \beta ; h \in \mathbb{Z}$) and classical means

SM with two parameters

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{\alpha} \cdot \frac{b^\alpha - a^\alpha}{b^\beta - a^\beta} \right]^{\frac{1}{\alpha-\beta}} \quad (\alpha \neq \beta)$$

can also be traced back to classical averages. Let us illustrate the previous statement with some examples.

Case : $\alpha = 2\beta$

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{2\beta} \cdot \frac{b^{2\beta} - a^{2\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{2\beta-\beta}} = \left[\frac{1}{2} \cdot \frac{b^{2\beta} - a^{2\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{\beta}} = \left[\frac{b^\beta + a^\beta}{2} \right]^{\frac{1}{\beta}}$$

Case : $\alpha = 4\beta$

$$\begin{aligned} \eta_{4\beta,\beta}(a,b) &= \left[\frac{\beta}{4\beta} \cdot \frac{b^{4\beta} - a^{4\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{4\beta-\beta}} = \left[\frac{1}{4} \cdot \frac{b^{4\beta} - a^{4\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{\beta}} \\ &= \left[\frac{a^\beta + b^\beta}{2} \cdot \frac{(a^\beta)^2 + (b^\beta)^2}{2} \right]^{\frac{1}{3\beta}} = \\ &= \sqrt[3\beta]{A(a^\beta, b^\beta) \cdot QM(a^\beta, b^\beta)} \end{aligned}$$

Case: $\alpha = 8\beta$

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$$\begin{aligned}\eta_{8\beta,\beta}(a,b) &= \left[\frac{\beta}{8\beta} \cdot \frac{b^{8\beta} - a^{8\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{7}} = \left[\frac{a^\beta + b^\beta}{2} \cdot \frac{(a^\beta)^2 + (b^\beta)^2}{2} \cdot \frac{(a^\beta)^4 + (b^\beta)^4}{2} \right]^{\frac{1}{7\beta}} \\ &= \sqrt[7\beta]{A(a^\beta, b^\beta) \cdot QM^2(a^\beta, b^\beta) \cdot QM^2(a^{2\beta}, b^{2\beta})}\end{aligned}$$

From

$$a^{2^{h-2} \cdot \beta} = (a^\beta)^{2^{h-2}}; \quad b^{2^{h-2} \cdot \beta} = (b^\beta)^{2^{h-2}}$$

we have

$$\begin{aligned}\eta_{2^h\beta,\beta}(a,b) &= \\ &= \sqrt{(2^{h-1})^\beta A(a^\beta, b^\beta) \cdot QM^2(a^\beta, b^\beta) \cdot QM^2(a^{2\beta}, b^{2\beta}) \dots QM^2(a^{2^{h-2}\beta}, b^{2^{h-2}\beta})}\end{aligned}$$

Case : $\alpha = -2\beta$

$$\begin{aligned}\eta_{-2\beta,\beta}(a,b) &= \left[\frac{\beta}{-2\beta} \cdot \frac{b^{-2\beta} - a^{-2\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{-2\beta-\beta}} = \left[\left(-\frac{1}{2}\right) \cdot \frac{\left(\frac{1}{b^\beta}\right)^2 - \left(\frac{1}{a^\beta}\right)^2}{b^\beta - a^\beta} \right]^{\frac{1}{-3\beta}} \\ &= \left[a^\beta b^\beta \cdot \frac{2}{\frac{1}{a^\beta} + \frac{1}{b^\beta}} \right]^{\frac{1}{3\beta}} \\ &= \sqrt[3\beta]{G^2(a^\beta, b^\beta) \cdot H(a^\beta, b^\beta)}\end{aligned}$$

Case : $\alpha = -4\beta$

$$\begin{aligned}\eta_{-4\beta,\beta}(a,b) &= \left[\frac{\beta}{-4\beta} \cdot \frac{b^{-4\beta} - a^{-4\beta}}{b^\beta - a^\beta} \right]^{\frac{1}{-4\beta-\beta}} = \left[\left(-\frac{1}{4}\right) \cdot \frac{\left(\frac{1}{b^{2\beta}}\right)^2 - \left(\frac{1}{a^{2\beta}}\right)^2}{b^\beta - a^\beta} \right]^{\frac{1}{-5\beta}} \\ &= \left[\left(-\frac{1}{4}\right) \cdot \frac{\left(\frac{1}{b^{2\beta}} - \frac{1}{a^{2\beta}}\right) \left(\frac{1}{b^{2\beta}} + \frac{1}{a^{2\beta}}\right)}{b^\beta - a^\beta} \right]^{\frac{1}{-5\beta}}\end{aligned}$$

$$= \left[\left(-\frac{1}{4} \right) \cdot \frac{\left(\frac{1}{b^\beta} - \frac{1}{a^\beta} \right) \left(\frac{1}{b^\beta} + \frac{1}{a^\beta} \right) \left(\frac{1}{b^{2\beta}} + \frac{1}{a^{2\beta}} \right)}{b^\beta - a^\beta} \right]^{-\frac{1}{5\beta}}$$

$$= \sqrt[5\beta]{G^2(a^\beta, b^\beta) \cdot H(a^\beta, b^\beta) \cdot H(a^{2\beta}, b^{2\beta})}$$

In the case that $\alpha = -2^h\beta$ ($h \in \mathbb{N}$) we have, by reasoning by induction,

$$\eta_{-2^h\beta, \beta} = \sqrt[(2^{h+1})\beta]{G^2(a^\beta, b^\beta) \cdot H(a^\beta, b^\beta) \cdot H(a^{2\beta}, b^{2\beta}) \cdots H(a^{(2^{h-1})\beta}, b^{(2^{h-1})\beta})}$$

Similarly for $\alpha = 9\beta$

$$b^{9\beta} - a^{9\beta} =$$

$$= (b^{3\beta})^3 - (a^{3\beta})^3 = (b^{3\beta} - a^{3\beta})(b^{6\beta} + b^{3\beta}b^{3\beta} + b^{6\beta})$$

$$= (b^\beta - a^\beta)(b^{2\beta} + b^\beta b^\beta + a^{2\beta})(b^{6\beta} + b^{3\beta}b^{3\beta} + b^{6\beta})$$

for $\alpha = 27\beta$

$$b^{27\beta} - a^{27\beta} = (b^\beta - a^\beta)(a^{2\beta} + a^\beta b^\beta + b^{2\beta}) \cdot (a^{6\beta} + a^{3\beta}b^{3\beta} + b^{6\beta})$$

$$\cdot (a^{18\beta} + a^{9\beta}b^{9\beta} + b^{18\beta})$$

for $\alpha = 81\beta$

$$b^{81\beta} - a^{81\beta} = (b^\beta - a^\beta)(a^{2\beta} + a^\beta b^\beta + b^{2\beta}) \cdot (a^{6\beta} + a^{3\beta}b^{3\beta} + b^{6\beta})$$

$$\cdot (a^{18\beta} + a^{9\beta}b^{9\beta} + b^{18\beta}) \cdot (a^{54\beta} + a^{27\beta}b^{27\beta} + b^{54\beta})$$

from which

$$\eta_{9\beta, \beta}(a, b) = \sqrt[8\beta]{QM^2(a^\beta, GM^2(a^\beta, b^\beta), a^\beta) \cdot QM^2(a^{3\beta}, GM^2(a^{3\beta}, b^{3\beta}), b^{3\beta})}$$

And more generally

$$\eta_{3^h\beta, \beta}(a, b) =$$

$$= \sqrt[(3^{h-1})\beta]{QM^2(a^\beta, GM^2(a^\beta, b^\beta), a^\beta) \cdots QM^2(a^{(3^{h-1})\beta}, GM^2(a^{(3^{h-1})\beta}, b^{(3^{h-1})\beta}), b^{(3^{h-1})\beta})}$$

15. Economic significance of SM [8]

Let X be a subset of $\mathbb{R}^n$. A function $f: X \rightarrow \mathbb{R}$ is said to be homogeneous of degree r if

$$f(jx_1, \dots, jx_n) = j^r f(x_1, \dots, x_n)$$

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where j is a positive constant and $(jx_1, \dots, jx_n) \in X$ ($\forall x \in X; x = (x_1, \dots, x_n)$).

In Linear Algebra the set X , constant j , x are called the cone, scalar and vector of $\mathbb{R}^n$, respectively. The constant r can also be equal to 0. If $r = 1$ the f is said to be linearly homogeneous. Consider an economy where a single good is produced in quantity Y using capital K and labor L as the only factors of production. Both L and K are considered positive. The main field of application in economics of linearly homogeneous functions is given by the notion of aggregate production function, which we will write in the form

$$Y = F(K, L)$$

that is, a production function is a relationship F between the inputs employed K and L and the output Y . Let F be linearly homogeneous. Placed

$$j = \frac{1}{L}$$

we have

$$\frac{Y}{L} = \frac{1}{L} F(K, L) = F\left(\frac{K}{L}, \frac{L}{L}\right) = F\left(\frac{K}{L}, 1\right)$$

By positing $Y/L = y$, $K/L = k$, we obtain the production function $y = f(k)$ in intensive form, that is, the output per unit of effective labor y is a function f of capital per unit of effective labor k . Let us consider the Cobb-Douglas production function

$$Y = AK^\alpha L^{1-\alpha} \quad (\alpha \in]0; 1[)$$

meaning by A the state of technology. We observe that.

$$Y = AK^\alpha L^{1-\alpha} = AL \left(\frac{K}{L}\right)^\alpha = ALk^\alpha$$

from which, dividing by L the first and last terms of the equalities above, we have

$$y = f(k) = Ak^\alpha \quad (\alpha \in]0; 1[)$$

If we set $A = 1$, f becomes $y = f(k) = k^\alpha$.

The Cobb-Douglas function written in intensive form k^α can be thought of as a power function of base k and exponent α . It is usually assumed that F , in X , in addition to being linearly homogeneous, is also of class C^2 , that is, F admits continuous prime and second derivatives with respect to both variables K and L . To avoid running into problems with the partial derivatives of F , it is usually convenient to consider the set X as an open set of $\mathbb{R}^n$. For $f(k)$ one can repeat what has already been said for the power function

$x^\alpha (\alpha \neq 0; \alpha \neq 1)$ to verify that it satisfies the definition of Stolarsky averages. The ratio y capital/labor is called labor productivity and is introduced to study the growth of an economy. The assumption of linear homogeneity on F allows for constant returns to scale. For example, in the particular case where both factors of production K and L are doubled, output also doubles, i.e.

$$2Y = F(2K, 2L)$$

16. Probabilistic significance of SM

We recall that a continuous random variable X is said to have a Uniform distribution over the interval $[a, b]$ if its density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The r^{th} moment of X is

$$\mu_r = EX^r = \int_{-\infty}^{+\infty} x^r f(x) dx = \int_a^b \frac{x^r}{b-a} dx = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \quad (r \in N)$$

from which we have that

$$\begin{aligned} \mu_1 = EX &= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} \\ \mu_2 = EX^2 &= \frac{b^3 - a^3}{2(b-a)} = \frac{b^2 + ab + a^2}{3} \end{aligned}$$

Comparing EX^r and μ_2 with the expressions of the formula (19) of SM and the formula (7) of η_3 , we can conclude that SM and Uniform distribution are closely linked.

17. Conclusions and final remarks

We have shown by elementary methods that for $n = p^h$ (p prime number; $h \in N$) the Stolarsky means η_{p^h} can be traced explicitly to classical means and that for $n = 3$ the Stolarsky mean η_3 can be interpreted as a real quadric. Both results form an ideal bridge to the pioneering works published by the two Italian mathematicians and statisticians, L. Galvani (1927) and R. Cisbani (1938) from which many authors have been inspired to introduce and/or further explore Stolarsky means. Interesting and a harbinger of further research insights is our attempt to generate, in the $n = 3$. case, means of the type of SM from an extension of Flett's Theorem, which in turn can be considered a mean-value theorem. The formulas and considerations given in this paper are original, including the steps to deduce by means of limits the elementary properties of classical means from MVTD and MVTI and the inequalities obtained by applying identities contained in [6]

and [24]. Finally, the connection of SM to Cobb-Douglas may lead to new interpretations of some economic growth models with chaos in the future.

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Using the three-way matrix for AHP-group decision-making

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Abstract

The Analytic Hierarchy Process (AHP) is one of the most well-known and used multi-criteria methods in the field of decision problems. In many situations, the decision is not up to a single person but involves multiple decision-makers with different preferences with respect to the various elements of the problem. Applying a traditional AHP with certain preference aggregation mechanisms, the procedure is very burdened by the many pairwise comparison matrices due to the number of decision-makers. The aim of this paper is to summarize the problem of representing the different preferences between the criteria and the alternatives, for the various decision makers using the three-way data matrix. This matrix allows us to examine, in a contemporary and global manner, the preferences of several decision-makers in the decision-making context. Three-way principal component models can be used to treat the three-way matrix, particularly for the derivation of AHP priorities.

Keywords: AHP; three-way data matrix; decision-makers; preferences.

2010 AMS subject classification: 62P20, 91B08, 91B10[†]

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1. Introduction

The Analytic Hierarchy Process (AHP) is a multi-criteria technique for solving decision problems. It is a high-performing technique, especially in contexts whose main goal is to rank several alternatives for the decision maker to choose the best suitable alternative. AHP was developed in the 1970s by T.S. Saaty and it helps the decision maker to identify the most suitable solution for his several needs by breaking down the decision problem into a hierarchy. A detailed exposition of AHP can be found in Saaty [1], [2], [3], [4], [5], [6], [7], while examples of the practical application of the process, both in public and private sectors, are provided in Saaty [8].

In AHP, the several elements (the evaluation adopted criteria, possible sub-criteria, alternatives object of the final decision) are compared pairwise and according to the properties of bottom-up procedures in the pairwise comparison matrices, in order to derive a priority for each level of the hierarchy. Then, through the derivation of global priorities, an ordering set of alternatives is obtained. This process is quite linear if the decision maker is a single individual; but when the decision involves groups of individuals there is the need to reconcile the different preferences.

There are many contributions given by literature to extend the typical AHP mechanisms when it comes to decisions belonging to more decision makers: considering procedures for aggregating the different preferences of the various actors involved is enough to ensure the regularity of the process. Except for the rare and highly theoretical cases of consensus voting, the AHP procedure becomes complex in the presence of multiple decision-makers. There are contrasting preferences which have to be grouped, possible weighting of the different importance of the decision makers and a great number of pairwise comparison matrices to build, making the choice of the alternative that will represent the final decision of the problem more insidious and slower.

This is where our proposal of summarizing the problem of the different preferences fits in, about criteria and alternatives, for the various decision makers in a single three-way matrix, instead of the many pairwise comparison matrices. The three dimension of the matrix will be: decision maker, criterion, alternative. Each matrix element represents the position of each decision maker in relation to a precise criterion and to a precise alternative. To understand the relations among the several matrix dimensions, the models of the principal components can be used according to what literature suggests about the management of three-dimensional matrices.

This article is structured as follows: paragraph 1 describes the reference context, paragraph 2 includes a brief description of AHP technique, paragraph 3 deals with the problem of combining preferences in the group decisions, and paragraph 4 explains the

possibility to use the three-way data matrix, paragraph 5 contains some concluding remarks.

2. AHP methodology

To identify the best alternative in AHP four steps are involved:

1. development of the hierarchy.
2. building pairwise comparison matrices.
3. derivation of local priorities.
4. derivation of global priorities.

The elements to be considered in the process of decision making are structured in a hierarchy consisting of at least three or more levels. The first level of hierarchy contains the specification of the overall goal of the decision; the second level involves the criteria, that is the rules that guide the decision-making process; finally at the third level, the base of the hierarchy, there are the alternatives, that is the actions to be evaluated.

In more complex hierarchies, each criterion can be broken down into more detailed sub-criteria; they will be at intermediate levels between the criterion of which they represent the detail (at a higher level) and the alternatives or other their breaking down (at a lower level). However, there must be at least a minimum of three levels in the hierarchy to solve the problem with AHP. To each hierarchy level the following properties must be applied: external dependence of the elements of one level from the elements of a higher level and internal independence among the elements of the same level.

The hierarchy can be viewed as a tree diagram: each rectangle represents a node and all the rectangles connected to this parent node are known as children nodes, which constitute comparison groups and represent equivalent or equal nodes. Figure 1 shows an example of a hierarchical structure of a decision-making problem for achieving a goal (level 1) with three criteria (level 2), each of them is structured into two sub-criteria (level 3) and four alternatives (level 4).

Next step is the building of the pairwise comparison matrices. These allow all the elements of the same hierarchical level to be compared pairwise, to evaluate their relative importance with reference to an element of the higher level. Therefore, there will be:

- 1 pairwise comparison matrix among r criteria.
- r pairwise comparison matrices among s alternatives.

The goal of the pairwise comparison of the elements is to establish which is more important in relation to the overlying level and to what extent. A generic pairwise comparison matrix A is represented in figure 2.

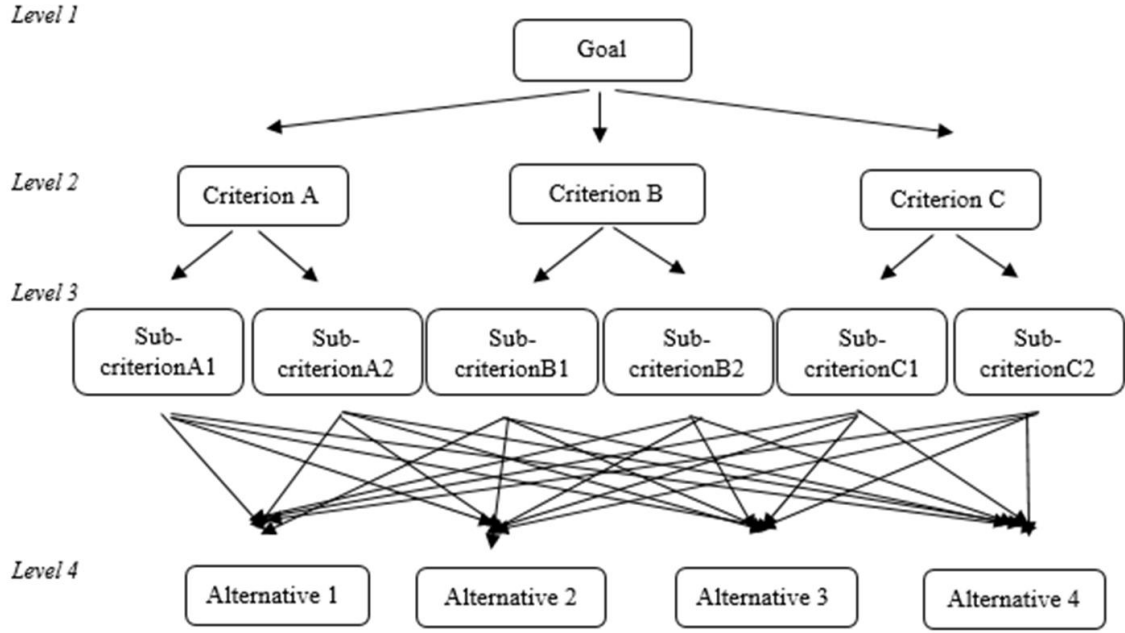


Figure 1. Hierarchical structure of a decision problem.

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & 1 & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

Figure 2. Example of a pairwise comparison matrix.

The result of the single comparison is summarized in the a_{ij} coefficient, known as dominance coefficient, which represents an evaluation of the dominance of the i -th element over the j -th element. This means that if:

- $a_{ij} > 1 \Leftrightarrow i$ -th element $>$ j -th element;
- $a_{ij} = 1 \Leftrightarrow i$ -th element $\sim$ j -th element;
- $a_{ij} < 1 \Leftrightarrow i$ -th element $<$ j -th element.

Each comparison reproduces the personal judgment of the decision maker. Saaty [9] has developed a semantic scale, which, using positive real numbers from 1 to 9, assigns a score to each compared pairwise, therefore creating a correspondence between numerical value and verbal judgment. Table 1 shows the quantification of the dominance coefficient according to Saaty's scale.

Intensity a_{ij}	Definition
1	Equal importance
3	Moderate importance
5	Strong importance
7	Very strong importance
9	Extreme importance
2, 4, 6, 8	Intermediate values between the two adjacent judgments
$1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9$	Reciprocals

Table 1. The fundamental Saaty's scale.

Therefore the pairwise comparison matrix is:

- square, with unitary elements on the main diagonal ($a_{ii}=1$) since the importance of each element compare to itself leads to an evaluation of indifference;
- positive, in fact all the numbers Saaty uses to quantify the comparisons are positive;
- reciprocal, since, for each value of i and of j the relation $a_{ij}=1/a_{ji}$ is verified. Coherently, if the element a_i is a certain number q times more important of the element a_j , then a_j will have the $1/q$ -th of importance compared to a_i . As a consequence of the relation of reciprocity, the triangle which stands above the main diagonal is represented by reciprocal values of those contained in the triangle which stands underneath it;
- transitive, as in theory it should always be worth $a_{ik}a_{kj}=a_{ij}$. This means that a decision maker, for example, who considers that a_i to be 3 times more important of a_k and then considers a_k to be 2 times more important of a_j , should also consider a_i to be 6 times more important of a_j ;
- consistent due to the properties of reciprocity and transitivity.

The total number of comparisons contained in the matrix depends on the number of n elements to compare and is equal to n^2 . Among these, only a certain number, equal to $(n-1)$, can be chosen freely, while the others are imposed by the properties of reciprocity and transitivity [10].

Starting from the built pairwise comparison matrices, we can determine the weight of each criterion (so-called derivation of local priorities). Ishizaka [10] has organized the several methods of derivation into two groups: the eigenvalues methods and the methods which minimize the distance between the matrix defined by the user and the

closer consistent matrix. The goal, regardless of the method used, is to derive a priority for each level of the hierarchy.

Then, the weight of each alternative is to be established (so-called derivation of global priorities) to obtain an ordered system of them. The global weights allow to establish a preference order: the alternative will be the more preferable the greater its global weight is. For this purpose, the most used approach is that of additive aggregation; according to it the local priorities in relation to each criterion are multiplied by the weight of the criterion and added up. Alternatively, Lootsma [11] has introduced the method of the multiplicative aggregation to avoid reverse ordering which can occur after the introduction or the removal of alternatives from the problem.

The analysis is completed by the consistency control and the study of sensitivity. A matrix can be considered consistent when the rule of transitivity and the rule of reciprocity are met. The consistency shows a strong logical coherence among the expressed preferences and only when the matrix is consistent different methods of priorities derivation produce the same preference order.

However, it often happens that the matrices are not consistent as it is difficult to maintain this consistency of opinion, seen as a strict observance of the rules of transitivity and reciprocity, in all the pairwise comparisons. The causes that can lead to inconsistency of the judgments expressed by the decision maker can be various, among the most frequent there are [12]: material errors in reporting opinions, the lack of adequate information, indifference by the decision makers, inadequate structure of the decision problem, lack of consistency in the real world.

There are several consistency measurements and tolerance threshold to determine to what extent an inconsistent matrix is acceptable [13], [14], [15], [16], [17]. The most used method for the consistency measurement is based upon the Consistency Ratio (*CR*) given by the ratio between the Consistency Index (*CI*) and the Random Index (*RI*). The *CI* takes two main elements of the matrix into account: the maximum eigenvalue and the *n* dimension; the *RI* represents the consistency index of a completely random index that is expected to contain highly inconsistent judgments. The AHP methodology accepts a low degree of inconsistency, as it is considered that it does not damage the validity of the obtained result; only when the inconsistency reaches too high level ($CR > 0.10$) it is necessary for the decision maker to review his own opinions.

To validate the results obtained and verify their stability, the sensitivity analysis is finally carried out. This analysis allows to understand how they reached decision is grounded, and which are its drivers [18]. The analysis of sensitivity allows to verify which changes in the analysis model can cause significant differences in the final results. To carry out the analysis of sensitivity it is necessary to modify the input data, this can be done by acting on the criteria weights, on the weights themselves or the

method of calculation used, and understand how these changes affects final performances. If the system of the alternatives does not change, the final results are said to be solid.

3. AHP for group decision-making

There are some circumstances where the decision-making process involves multiple decision makers. In these cases, the complexity of the decision increases because of the various preferences of the decision makers and/or for the different weight associated to the different importance they can have in the decision making process. The main problem in case of various decision makers is that they have potentially conflicting preferences, therefore there is the necessity to build the group's choice starting from single individual choices.

Literature offers several preferences aggregation techniques for the application of the AHP methodology to the group decision. Ishizaka & Labib [19] describe four different ways of combining multiple subjects' preferences. These techniques are shown in table 2.

		<i>Mathematical aggregation</i>	
		<i>Yes</i>	<i>No</i>
<i>Aggregation on</i>	<i>Judgments</i>	Geometric mean	Consensus vote
	<i>Priorities</i>	Weighted arithmetic mean	Consensus vote

Table 2. Preferences aggregation techniques.

The easiest way to obtain the combination of preferences of multiple subjects is the consensus vote, on judgements or priorities. Reaching a consensus vote on the judgements means reaching an agreement on the value of each element of the pairwise comparison matrices. In case of difficulties about the agreement, it is appropriate to base the consensus vote on the priorities already derived from each participant, even because in this way it is possible to identify the pair decision maker-preference and further discuss any possible disagreement.

Any time it is not possible to use the consensus vote, because the subjects involved are many or very distant, it is possible to perform aggregations in a mathematical way: the geometric mean of the single judgements of the pairwise comparison matrices or the weighted arithmetic mean of the priorities. Although the geometric mean is in compliance with transitive property, it does not meet Pareto optimality. Therefore, the

priorities in the hierarchy of each decision maker can be aggregated due to the weighted arithmetic mean method.

Forman & Peniwati [20] describe two base techniques to aggregate individual preferences in a group preference, depending on whether the group wants to act as a single unit or as separate individuals:

- AIJ, Aggregating Individual Judgments;
- AIP, Aggregating Individual Priorities.

In the AIJ approach the individuals renounce to their own preferences by acting in concert and sharing their judgments, in order that the group will be a new “individual” and will act so. Individual judgments are aggregated synergistically, and their synthesis generates the group priorities. As the individual priorities are irrelevant, also the violation of the Pareto principle is negligible; on the contrary, the reciprocity of the judgements must be ensured and the geometric mean is the only appropriate procedure in the AIJ approach, as it is able to preserve the reciprocal property of the pairwise comparison matrices.

In the AIP approach the group is considered as a whole of individuals, with different values and hierarchies. This approach requires that the priorities derive from each individual hierarchy and so aggregate in group priorities. In AIP both the geometric mean and the arithmetic mean can be used.

There are further problems if the decision-makers do not have the same weight. The weights should indicate the experience of the decision-maker or the importance of the impact of the decision on those who will make it. The weights can be assigned by a superordinate decision-maker, accepted by all, or with a participatory approach based upon a pairwise comparison matrix with the relative importance of each member [21]. For those cases where the comparisons are missing, because the decision-maker does not have the skills to formulate a specific judgement, the aggregation methods that can be used are the linear programming [22] and the Bayesian approach [23].

4. Using and handling a three-way matrix for group decisions

The three-way data matrix is an instrument in which each elementary data, x_{ijk} , has three indices that correspond to a classification of it based on three elements:

$$X_{I*J*K} = x_{ijk} \quad \text{per } i = 1, \dots, I; j = 1, \dots, J; k = 1, \dots, K$$

The three-dimensionality of the data allows its geometric arrangement in a cube [24], [25], [26].

Using the three-way data matrix for AHP-group decision making

This study proposes the use of the three-way matrix in the AHP methodology in the context of group decisions. A decision-making process is typically characterized by the presence of criteria and alternatives compared; the introduction of a third element, that is the decision maker with his preference, can be comprehensively represented by a three-way data matrix. The three inputs “units x variables x occasions” of the matrix become “criteria x alternatives x decision makers”. The generic element a_{ijk} of the matrix will express the k -th subject’s preference for the j -th alternative with reference to the i -th criterion. The schematic drawing is shown in figure 3.

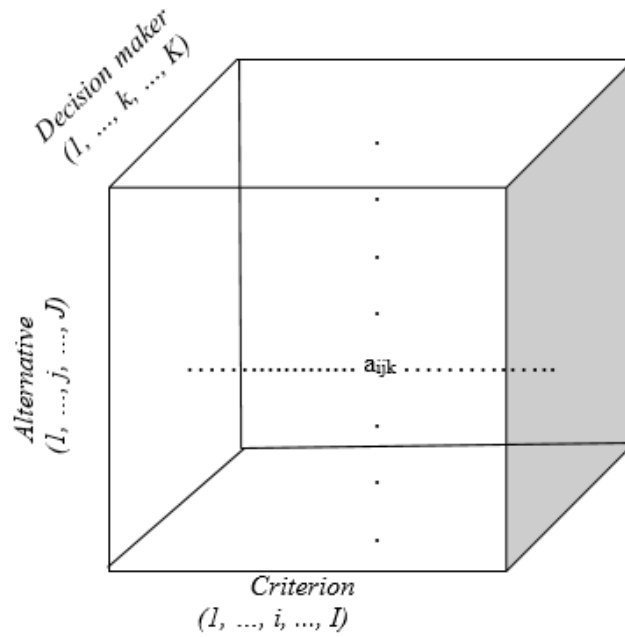


Figure 3. The three-way data matrix for group decision making.

Therefore, each dominance coefficient a_{ijk} will be identified according to three dimensions:

- i ($1, 2, 3, \dots, I$) which refers to criteria;
- j ($1, 2, 3, \dots, J$) which identifies the alternatives;
- k ($1, 2, 3, \dots, K$) which reproduces the decision makers.

The three-way data matrix can be used for group decisions to include the evaluation of K decision makers towards the criteria and the alternatives requested in the AHP. The arrangement of the elements (criteria, alternatives, decision makers) in rows, columns and tubes of the matrix is not subject to any precise rule since it is possible to represent them also in different order, for example according to the arrangement “decision makers x alternatives x criteria”, leaving its meaning unchanged.

The matrix finishes all the pairwise comparisons which are necessary in a classic AHP. Its use can be justified through at least two important advantages:

- a greater convenience in managing the amount of data. This privilege will be the more noticeable the more I , J and K will increase, respectively number of criteria, alternatives and decision makers;
- a better usability of the information within the structure of the decision problem.

At the same time a three-way data matrix reduces the priority derivation and preference aggregation operations by not having to work on so many pairwise comparison matrices.

Once the preferences of the decision makers have been represented, with respect to criteria and alternatives, the matrix must be treated with specific methodologies that allow the mass of information hidden in the set of data contained in it to be summarized. To estimate the priorities to be used in the AHP, for example, any three-way Component Analysis model, such as Statis [27], [28], [29], Parafac/Candecomp [30], [31] or Tucker3 [32], can be applied.

5. Discussion and Conclusions

The aim of this work is to advance the proposal to use the three-way matrix in the AHP methodology for group decisions. This idea developed in consideration of the significant limitation of the AHP procedure, in the presence of a high number of pairwise comparisons to be analyzed when multiple subjects are involved in the decision. The issue has certainly not been treated in an exhaustive manner, since, in order to constitute a valid contribution to the analysis and functioning of the AHP method, its various aspects should be explored in greater depth.

A first future area of study could concern a comparison between the various models of the three-way Principal Components, to identify the one that is most effectively and efficiently suited to be used for the derivation of the eigenvalue which estimates the priorities. If the logic is to streamline the procedure, the processing times employed could be considered.

Another aspect worth to be examined concerns the control of the consistency of the matrix, through a necessary reformulation of the measures of the degree of coherence of the judgments expressed by the decision makers. The Consistency Index, mostly used for this purpose, is calculated taking into account the order n of the two-dimensional matrix, and would clearly not be applicable to the case of a three-dimensional matrix. It

should therefore be redesigned with an adaptation or new consistency measurement indices should be studied.

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$(\odot, \vee)$ -Derivations on EMV-algebras and EBL-algebras

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Abstract

In the paper, we will study $(\odot, \vee)$ -derivations on EMV-algebras and EBL-algebras. Firstly, we define the notions of $(\odot, \vee)$ -derivations on EMV-algebras and EBL-algebras, and we also introduce some classes of $(\odot, \vee)$ -derivations on EMV-algebras and EBL-algebras, such as principal $(\odot, \vee)$ -derivations and isotone $(\odot, \vee)$ -derivations. In addition, we construct some lattice structures of $(\odot, \vee)$ -derivations on EMV-algebras and give a classification of EBL-algebras with $(\odot, \vee)$ -derivations.

Keywords: EMV-algebra; derivation; principal derivation; isotone derivation; EBL-algebra

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1 Introduction

MV-algebras were originally introduced by Chang (1958) as algebraic structures of many-valued logic. Nowadays, this structure has been extended to BCK-algebras, hoops, quasi-algebras, etc. In addition, Georgescu and Iorgulescu (2001) introduced pseudo MV-algebras, which is a non-commutative extension of MV-algebras. Dvurecenskij and Zahiri (2019) defined EMV-algebras, which extended the notions of MV-algebras.

Hajek (2013) defined the notion of BL-algebras. Particularly, let A be an BL-algebra, for all $x \in A$, if $x^{--} = x$, then A is an MV-algebra. Similarly, Liu (2020) defined the notion of EBL-algebras.

The notion of derivations was defined by Leibniz-rule. Szasz (1975) defined the notion of derivations on a lattice $(L, \vee, \wedge)$. In addition, Kolchin (1973) defined differential algebra, which is an algebraic structure with a derivation. Differential algebra has many generalizations, such as fields (Ritt 1932), commutative algebras (Singer 2009), non-commutative algebras (Guo and Keigher 2008) and MV-algebras (Hamal 2019). Alshehri (2010) defined the notion of derivations on MV-algebras A , which is a map d satisfying $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$ for all $x, y \in A$. Torkzadeh and Abbasian (2013) introduced $(\odot, \vee)$ -derivations on BL-algebras. Recently, Zhao, Gan and Yang (2024) introduced $(\odot, \vee)$ -derivations on MV-algebras. $(\odot, \vee)$ -derivations are indispensable tools in algebraic systems. The notion of $(\odot, \vee)$ -derivations are helpful for studying algebraic structures and properties in algebraic systems. In the following, we will further study the $(\odot, \vee)$ -derivations of EMV-algebras and EBL-algebras.

The paper is constructed as follows. In Section 2, we will introduce some foundational notions and properties related to EMV-algebras and EBL-algebras. In Section 3, we study $(\odot, \vee)$ -derivations on EMV-algebras. In addition, we study some lattice structures of $(\odot, \vee)$ -derivations on EMV-algebras. In Section 4, we introduce some important results of $(\odot, \vee)$ -derivations on EBL-algebras. We also prove that $(\odot, \vee)$ -derivations preserves ideals and filters.

2 Preliminaries

In this section, we will introduce some notions and statements on EMV-algebras and lattices which will be used in the following sections.

Definition 2.1. (Dvurecenskij and Zahiri 2019) An EMV-algebra is an algebra $(A, \vee, \wedge, \oplus, 0)$ of type $(2, 2, 2, 0)$ satisfies the following conditions:

(EMV1) $(A, \vee, \wedge, 0)$ is a distributive lattice with lower bound 0;

(EMV2) $(A, \oplus, 0)$ is a commutative monoid;

(EMV3) for each $m, n \in I(A)$ such that $m \leq n$, there exists $\lambda_{m,n}(x)$ for each $x \in$

$[m, n]$, and $([m, n], \vee, \wedge, \oplus, \lambda_{m,n}(x), m, n)$ is a MV-algebra, where $\lambda_{m,n}(x) = \min\{z \in [m, n] \mid x \oplus z = n\}$ and $I(A)$ is the set of all idempotent elements of A ; (EMV4) $(A, \vee, \wedge, \oplus, 0)$ has enough idempotent elements, that is, for each $x \in A$, there exists $a \in I(A)$ such that $x \leq a$.

We will denote $\lambda_{0,a}$ by λ_a and denote the set of all idempotent elements of an algebra A by $I(A)$.

Proposition 2.1. (Dvurecenskij and Zahiri 2019) *Let $(A, \vee, \wedge, \oplus, 0)$ be an EMV-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, the following properties hold:*

- (1) $\lambda_a(a) = 0$;
- (2) $a \oplus x = a$;
- (3) $x \odot y = \lambda_a(\lambda_a(x) \oplus \lambda_a(y))$;
- (4) $\lambda_a(x) \oplus x = a$;
- (5) $x \odot \lambda_a(x) = 0$;
- (6) $x \vee y = (x \odot \lambda_a(y)) \oplus y$;
- (7) $x \wedge y = x \odot (\lambda_a(x) \oplus y)$.

For all $a \in I(A)$ such that each $x, y \in [0, a]$, we will define a partial order $\leq$ on EMV-algebra $(A, \vee, \wedge, \oplus, 0)$ as follows:

$$x \leq y \Leftrightarrow \lambda_a(x) \oplus y = a.$$

In addition, we shall denote $x^0 = a, x^1 = x, \dots, x^n = x^{n-1} \odot x$.

Lemma 2.1. (Dvurecenskij and Zahiri 2019) *Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements are equivalent:*

- (1) $x \leq y$;
- (2) $\lambda_a(x) \oplus y = a$;
- (3) $x \odot \lambda_a(y) = 0$;
- (4) $y = x \oplus (y \odot \lambda_a(x))$;
- (5) *There is an element $z \in [0, a]$ such that $x \oplus z = y$.*

Lemma 2.2. (Dvurecenskij and Zahiri 2019) *Let A be an EMV-algebra. For all $x, y, z \in [0, a]$, the following statements hold:*

- (1) *If $x \vee y, x \oplus y \in [0, a]$, then $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$;*
- (2) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
- (3) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
- (4) $x \leq y$ iff $\lambda_a(y) \leq \lambda_a(x)$;
- (5) *If $x \leq y$, then $x \wedge z \leq y \wedge z; x \vee z \leq y \vee z; x \oplus z \leq y \oplus z; x \odot z \leq y \odot z$;*
- (6) *If $x \oplus y = 0$, we have $x = y = 0$; If $x \odot y = a$, we have $x = y = a$.*

Lemma 2.3. (Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements are equivalent:

- (1) $x \in B(A)$;
- (2) $x \oplus x = x$;
- (3) $x \odot x = x$;
- (4) $\lambda_a(x) \in B(A)$;
- (5) $x \oplus y = x \vee y$;
- (6) $x \odot y = x \wedge y$.

Definition 2.2. (Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra and I be a nonempty subset of A . I is called an ideal if for $x, y \in A$, the following conditions hold:

- (1) $0 \in I$;
- (2) $x \oplus y \in I$ for all $x, y \in I$;
- (3) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

Definition 2.3. (Cignoli, d'Ottaviano and Mundici 2000) Let A be a lattice and I be a nonempty subset of A . I is called a lattice ideal if for $x, y \in A$, the following conditions hold:

- (1) $0 \in I$;
- (2) $x \vee y \in I$ for all $x, y \in I$;
- (3) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

A subset I of a poset A is called a downset if for all $x \in I$, $y \in A$ and $y \leq x$ implies $y \in I$.

Definition 2.4. (Liu 2020) An EBL-algebra is an algebra $(A, \vee, \wedge, \odot, 0)$ of type $(2, 2, 2, 0)$ satisfying the following conditions:

- (EBL1) $(A, \vee, \wedge, 0)$ is a distributive lattice with lower bound 0;
- (EBL2) $(A, \odot, 0)$ is a commutative semigroup;
- (EBL3) for each $m, n \in I(A)$ such that $m \leq n$, there exists $x \xrightarrow{m,n} y$ for each $x, y \in [m, n]$, and $([m, n], \vee, \wedge, \odot, \xrightarrow{m,n}, m, n)$ is a BL-algebra, where $x \xrightarrow{m,n} y = \vee \{z \in [m, n] \mid x \odot z \leq y\}$ and $I(A)$ is the set of all idempotent elements of A ;
- (EBL4) $(A, \vee, \wedge, \odot, 0)$ has enough idempotent elements, that is, for each $x \in A$, there exists $a \in I(A)$ such that $x \leq a$.

Let $(A, \vee, \wedge, \odot, 0)$ be an EBL-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, we define a binary relation $\leq$ by $x \leq y \Leftrightarrow x \xrightarrow{a} y = a$. In addition, we define $x^{-a} = x \xrightarrow{a} 0$, $x^{--a} = (x \xrightarrow{a} 0) \xrightarrow{a} 0$, $x^{---a} = ((x \xrightarrow{a} 0) \xrightarrow{a} 0) \xrightarrow{a} 0$ and denote $x^0 = a$, $x^1 = x$, ..., $x^n = x^{n-1} \odot x$.

Proposition 2.2. (Liu 2020) Let $(A, \vee, \wedge, \odot, 0)$ be an EBL-algebra. For all $a \in I(A)$ such that each $x, y, z \in [0, a]$, the following properties hold:

- (1) If $x \leq y$, then $y^{-a} \leq x^{-a}$;
- (2) $x \odot y = 0$ iff $x \leq y^{-a}$;
- (3) $x \leq x^{-a}$; $x^{-a} = x^{-a}$;
- (4) $x \odot y \leq z$ iff $x \leq y \xrightarrow{a} z$;
- (5) $x \wedge y = x \odot (x \xrightarrow{a} y)$;
- (6) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$; $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
- (7) If $x \in I([0, a])$, then $x^{-a} = x$;
- (8) $x \odot y \leq x, y$.

Definition 2.5. (Liu 2020) Let A be an EBL-algebra and I be a nonempty subset of A . I is called an ideal if for $x, y \in A$, the following conditions hold:

- (1) $0 \in I$;
- (2) $y \in I$ and $x \leq y$ imply $x \in I$;
- (3) $(x \xrightarrow{a} 0) \xrightarrow{a} y \in I$ for each $x, y \in I, a \in I(A)$.

Definition 2.6. (Liu 2020) Let A be an EBL-algebra and F be a nonempty subset of A . F is called a filter if for $x, y \in A$, the following conditions hold:

- (1) $x \in F$ and $x \leq y$ imply $y \in F$;
- (2) if $x, y \in F$, then $x \odot y \in F$.

Denote $N = \{n \geq 1 | n \text{ is an integer}\}$ by N and denote $\overbrace{x \odot x \odot \cdots x}^{n\text{-times}}$ by x^n .

3 $(\odot, \vee)$ -derivations on EMV-algebras

In this section, we will introduce some notions and results of $(\odot, \vee)$ -derivations on EMV-algebras, and characterize some classes of $(\odot, \vee)$ -derivations on EMV-algebras. In addition, we also give some properties of fixed point sets of $(\odot, \vee)$ -derivations. Finally, some lattice structures of $(\odot, \vee)$ -derivations are characterized.

Definition 3.1. Let A be an EMV-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, a map d is called an $(\odot, \vee)$ -derivation on A if it satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)).$$

In the following, we shall denote an arbitrary idempotent element on an EMV-algebra A by a .

For all $x \in [0, a]$, we define the identity map $Id_A(x) = x$ and the zero map $0_A(x) = 0$. Since $Id_A(x \odot y) = x \odot y = (x \odot y) \vee (x \odot y) = (Id_A(x) \odot y) \vee (x \odot Id_A(y))$ and $0_A(x \odot y) = 0 = 0 \vee 0 = (0_A(x) \odot y) \vee (0_A(y) \odot x)$, we get that Id_A and 0_A are $(\odot, \vee)$ -derivations on A .

Example 3.1. Let $b \in [0, a]$. For all $x \in [0, a]$, we define the map $d_b : [0, a] \rightarrow [0, a]$ by $d_b := b \odot x$. From $d_b(x \odot y) = x \odot y \odot b = ((x \odot b) \odot y) \vee (x \odot (b \odot y)) = (d_b(x) \odot y) \vee (x \odot d_b(y))$, it follows that d_b is an $(\odot, \vee)$ -derivation, which is called a principal $(\odot, \vee)$ -derivation.

We will denote the set of all $(\odot, \vee)$ -derivations on A by $\text{Der}(A)$ and the set of all principal $(\odot, \vee)$ -derivations on A by $\text{PDer}(A)$.

In the following proposition, we will give some important properties of $(\odot, \vee)$ -derivations which be employed in this paper.

Proposition 3.1. Let A be an EMV-algebra and $n \in \mathbb{N}$. For all $d \in \text{Der}(A)$, the following statements hold for all $x, y \in [0, a]$:

- (1) $d(0) = 0$;
- (2) $d(x) \odot \lambda_a(x) = x \odot d(\lambda_a(x)) = 0$;
- (3) $d(x) \leq x$;
- (4) $d(x) = d(x) \vee (x \odot d(a))$ and $x \odot d(a) \leq d(x)$;
- (5) $d(\lambda_a(x)) \leq \lambda_a(x) \leq \lambda_a(d(x))$;
- (6) If $d(x) \vee d(y), d(x) \oplus d(y) \in [0, a]$, then $d(x) \odot d(y) \leq d(x \odot y) \leq d(x) \vee d(y) \leq d(x) \oplus d(y)$;
- (7) $d(x^n) = x^{n-1} \odot d(x)$;
- (8) $(d(x))^n \leq d(x^n)$;
- (9) If I is a downset of A , we have $d(I) \subseteq I$, where $d(I) = \{d(x) | x \in I\}$;
- (10) If $y \leq x$ and $d(x) = x$, we have $d(y) = y$.

Proof. (1) By the notion of $(\odot, \vee)$ -derivations, we have $d(0) = d(0 \odot 0) = (d(0) \odot 0) \vee (0 \odot d(0)) = 0$.

(2) By (1) and Proposition 2.1 (5), we have $0 = d(0) = d(x \odot \lambda_a(x)) = (d(x) \odot \lambda_a(x)) \vee (x \odot d(\lambda_a(x)))$. It implies that $d(x) \odot \lambda_a(x) = x \odot d(\lambda_a(x)) = 0$.

(3) From $d(x) \odot \lambda_a(x) = 0$, it implies that $d(x) \leq x$ by Lemma 2.1.

(4) We have $d(x) = d(x \odot a) = (d(x) \odot a) \vee (x \odot d(a)) = d(x) \vee (x \odot d(a))$. It implies that $x \odot d(a) \leq d(x)$.

(5) By (3), we have $d(\lambda_a(x)) \leq \lambda_a(x)$ and $d(x) \leq x$. It implies that $\lambda_a(x) \leq \lambda_a(d(x))$ by Lemma 2.2 (4).

(6) By (3), we have $d(x) \odot d(y) \leq x \odot d(y)$ and $d(x) \odot d(y) \leq d(x) \odot y$. So $d(x) \odot d(y) \leq (d(x) \odot y) \vee (x \odot d(y)) = d(x \odot y)$. In addition, by Lemma 2.2 (1), we have $d(x) \odot y \leq d(x)$ and $x \odot d(y) \leq d(y)$. So $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) \leq d(x) \vee d(y)$. Finally, by Lemma 2.2 (1) again, we have $d(x) \vee d(y) \leq d(x) \oplus d(y)$.

(7) In the following, we will make a mathematical induction for n :

(i) For $n = 1$, we have $d(x^1) = d(x) = a \odot d(x) = x^{1-1} \odot d(x)$;

(ii) For $n = 2$, we have $d(x^2) = d(x \odot x) = (d(x) \odot x) \vee (x \odot d(x)) = x \odot d(x) =$

$x^{2^{-1}} \odot d(x)$;

(iii) Assume that $d(x^n) = x^{n-1} \odot d(x)$ holds. We have $d(x^{n+1}) = d(x^n \odot x) = (d(x^n) \odot x) \vee (x^n \odot d(x)) = (x^{n-1} \odot d(x) \odot x) \vee (x^n \odot d(x)) = x^n \odot d(x)$. Hence, $d(x^n) = x^{n-1} \odot d(x)$ holds.

(8) By (3) and Lemma 2.2 (5), we have $(d(x))^{n-1} \leq x^{n-1}$. In addition, by (7), we have $(d(x))^n = (d(x))^{n-1} \odot d(x) \leq x^{n-1} \odot d(x) = d(x^n)$.

(9) Let I be a downset of A and $y \in d(I)$. There exists $b \in I$ such that $y = d(b)$. By the notion of a downset, we have $y = d(b) \leq b \in I$. Hence, $d(I) \subseteq I$.

(10) Let $y \leq x$ and $d(x) = x$. We have $d(y) = d(x \wedge y) = d(x \odot (\lambda_a(x) \oplus y)) = (d(x) \odot (\lambda_a(x) \oplus y)) \vee (x \odot d(\lambda_a(x) \oplus y)) = (x \odot (\lambda_a(x) \oplus y)) \vee (x \odot d(\lambda_a(x) \oplus y)) = x \odot (\lambda_a(x) \oplus y) = x \wedge y = y$. $\square$

In the following proposition, we will prove that if d is an $(\odot, \vee)$ -derivation on EMV-algebra A , then $d = Id_A$ iff d is surjective.

Proposition 3.2. *Let A be an EMV-algebra and $d \in \text{Der}(A)$. For all $x, y \in [0, a]$, the following statements are equivalent:*

- (1) $d = Id_A$;
- (2) $d(a) = a$;
- (3) $d(b) = a$ for some $b \in [0, a]$;
- (4) d is surjective;
- (5) d satisfies that $d(x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$.

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (3) hold.

(2) $\Rightarrow$ (1). Let $d(a) = a$. By Proposition 3.1 (10), we have $d(x) = x$. Hence, $d = Id_A$.

(3) $\Rightarrow$ (2). Let $d(b) = a$ for some $b \in [0, a]$. By Proposition 3.1 (3), we have $a = d(b) \leq b$. It implies that $b = a$. Hence, $d(a) = a$.

(1) $\Rightarrow$ (5). Let $d = Id_A$. We have $d(x \oplus y) = x \oplus y = (x \oplus y) \wedge (x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$.

(5) $\Rightarrow$ (2). Let d satisfies that $d(x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$. We have $d(a) = d(a \oplus a) = (d(a) \oplus a) \wedge (a \oplus d(a)) = a$. $\square$

Proposition 3.3. *Let A be an EMV-algebra and $d \in \text{Der}(A)$. For all $x \in [0, a]$, we define a map d^u on A by*

$$d^u(x) = \begin{cases} u, & x = a, \\ d(x), & x \neq a. \end{cases}$$

If $u \in [0, a]$ such that $u \leq d(a)$, then $d^u \in \text{Der}(A)$.

Proof. For all $x, y \in [0, a]$, we consider the following cases:

Case (i): If $x = y = a$, we have $d^u(a \odot a) = u = (u \odot a) \vee (a \odot u) = (d^u(a) \odot a) \vee (a \odot d^u(a))$;

Case (ii): If $x \neq a$ and $y \neq a$, we have $d^u(x) = d(x)$, $d^u(y) = d(y)$ and $x \odot y \neq a$. It implies that

$$\begin{aligned} & d^u(x \odot y) \\ &= d(x \odot y) \\ &= (d(x) \odot y) \vee (x \odot d(y)) \\ &= (d^u(x) \odot y) \vee (x \odot d^u(y)). \end{aligned}$$

Case (iii): If $x \neq a$ and $y = a$, we have $d^u(x) = d(x)$ and $d^u(y) = d^u(a) = u \leq d(a)$. It implies that $x \odot d^u(a) \leq x \odot d(a) \leq d(x)$ by Proposition 3.1 (4). So

$$\begin{aligned} & d^u(x \odot y) \\ &= d^u(x \odot a) \\ &= d^u(x) \\ &= d(x) \\ &= d(x) \vee (x \odot d^u(a)) \\ &= (d^u(x) \odot y) \vee (x \odot d^u(a)) \\ &= (d^u(x) \odot y) \vee (x \odot d^u(y)). \end{aligned}$$

Hence, $d^u \in \text{Der}(A)$. $\square$

Corolary 3.1. Let A be an EMV-algebra and $u \in [0, a]$. For all $x \in [0, a]$, we define a map $\text{Id}_A^{(u)}$ on A by:

$$\text{Id}_A^{(u)}(x) = \begin{cases} u, & x = a, \\ x, & x \neq a. \end{cases}$$

Then $\text{Id}_A^{(u)} \in \text{Der}(A)$.

Proof. We have $\text{Id}_A \in \text{Der}(A)$ and $u \leq a = \text{Id}_A(a)$. It implies that $\text{Id}_A^{(u)} \in \text{Der}(A)$ by Proposition 3.3. $\square$

In the following, we will study the condition when an $(\odot, \vee)$ -derivation on an EMV-algebra A is isotone.

Definition 3.2. Let A be an EMV-algebra and $d \in \text{Der}(A)$. d is called isotone if for all $x, y \in [0, a]$ such that $x \leq y$, it implies that $d(x) \leq d(y)$.

By Definition 3.1, we have that Id_A and 0_A are $(\odot, \vee)$ -derivations on A . For all $x, y \in [0, a]$ such that $x \leq y$, since $\text{Id}_A(x) = x \leq y = \text{Id}_A(y)$ and $0_A(x) = 0 \leq 0 = 0_A(y)$, we get that the identity map Id_A and the zero map 0_A are isotone.

Remark 3.1. Let d be an $(\odot, \vee)$ -derivation on EMV-algebra A . If it satisfies that $d(x \vee y) = d(x) \vee d(y)$, then d is isotone. Indeed, for all $x, y \in [0, a]$, if $x \leq y$, there

exists $z \in [0, a]$ such that $y = x \vee z$. It implies that $d(y) = d(x \vee z) = d(x) \vee d(z)$. That is, $d(x) \leq d(y)$.

Proposition 3.4. *Let A be an EMV-algebra. For all $b \in [0, a]$, we have principal $(\odot, \vee)$ -derivation d_b is isotone.*

Proof. Let $x, y \in [0, a]$ such that $x \leq y$. By Lemma 2.2 (5), we have $d_b(x) = b \odot x \leq b \odot y = d_b(y)$. Hence, d_b is isotone. $\square$

In the following proposition, we will show that if d is an $(\odot, \vee)$ -derivation on an EMV-algebra A with $d(a) \in B(A)$, then d is isotone iff d is principal.

Proposition 3.5. *Let A be an EMV-algebra and $d \in \text{Der}(A)$ with $d(a) \in B(A)$. For all $x, y \in [0, a]$, the following statements are equivalent:*

- (1) d is isotone;
- (2) $d(x) \leq d(a)$;
- (3) $d(x) = d(a) \odot x$;
- (4) $d(x \wedge y) = d(x) \wedge d(y)$;
- (5) If $d(x) \vee d(y) \in [0, a]$, then $d(x \vee y) = d(x) \vee d(y)$.

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let $d(x) \leq d(a)$. By Lemma 2.3 (6) and Proposition 3.1 (4), we have $d(x) \leq d(a) \wedge x = d(a) \odot x \leq d(x)$. Hence, $d(x) = d(a) \odot x$.

(3) $\Rightarrow$ (4). Let $d(x) = d(a) \odot x$. By Lemma 2.2 (2), we have

$$\begin{aligned} d(x \wedge y) &= d(a) \odot (x \wedge y) \\ &= (d(a) \odot x) \wedge (d(a) \odot y) \\ &= d(x) \wedge d(y). \end{aligned}$$

(4) $\Rightarrow$ (1). Let $d(x \wedge y) = d(x) \wedge d(y)$. If $x \leq y$, we have $d(x) = d(x \wedge y) = d(x) \wedge d(y) \leq d(y)$.

(3) $\Rightarrow$ (5). Let $d(x) = d(a) \odot x$. By Lemma 2.2 (3), we have

$$\begin{aligned} d(x \vee y) &= d(a) \odot (x \vee y) \\ &= (d(a) \odot x) \vee (d(a) \odot y) \\ &= d(x) \vee d(y). \end{aligned}$$

(5) $\Rightarrow$ (1). Let $d(x \vee y) = d(x) \vee d(y)$. If $x \leq y$, we have $d(x) \leq d(x) \vee d(y) = d(x \vee y) = d(y)$. $\square$

Clearly, if d satisfies the condition in Proposition 3.5, then d is principal.

We will denote the set of all isotone $(\odot, \vee)$ -derivations with $d(a) \in B(A)$ by $\text{IDer}(A)$.

Corollary 3.2. *Let A be an EMV-algebra. There exists a bijection between $IDer(A)$ and $B(A)$.*

Proof. For all $d \in IDer(A)$, we define a map $f: IDer(A) \rightarrow B(A)$ by $f(d) = d(a)$. In addition, for all $b \in [0, a]$, we also define a map $g: B(A) \rightarrow IDer(A)$ by $g(b) = d_b$. By Proposition 3.5, we have $fg: b \mapsto d_b(a) = b \odot a = b$ and $gf: d(x) \mapsto d_{d(a)}(x) = d(a) \odot x = d(x)$. Hence, f is a bijection between $IDer(A)$ and $B(A)$. $\square$

Proposition 3.6. *Let A be an EMV-algebra and $d \in Der(A)$. For all $x, y \in [0, a]$, we have $d \in IDer(A)$ iff $d(x \odot y) = d(x) \odot d(y)$.*

Proof. Let $d \in IDer(A)$. By Proposition 3.5, we have

$$\begin{aligned} & d(x \odot y) \\ &= d(a) \odot (x \odot y) \\ &= (d(a) \odot x) \odot (d(a) \odot y) \\ &= d(x) \odot d(y). \end{aligned}$$

Conversely, we have $d(a) = d(a) \odot d(a)$. It implies that $d(a) \in B(A)$. In addition, by Lemma 2.2 (1), we have $d(x) = d(x \odot a) = d(x) \odot d(a) \leq d(a)$. By Proposition 3.5 again, we get that d is isotone. Hence, $d \in IDer(A)$. $\square$

Corollary 3.3. *Let A be an EMV-algebra and $d \in IDer(A)$. For all $x \in [0, a]$, we have $d^2(x) = d(x)$.*

Proof. By Proposition 3.5 (3) and Proposition 3.6, we have $d(d(x)) = d(a) \odot d(x) = d(a \odot x) = d(x)$. $\square$

Proposition 3.7. *Let A be an EMV-algebra and $d \in IDer(A)$. For all $x, y \in [0, a]$, we have $d(x) \leq y$ iff $d(x) \leq d(y)$.*

Proof. Let $d(x) \leq y$. Since d is isotone, we have $d(d(x)) \leq d(y)$. In addition, by Corollary 3.3, we have $d(d(x)) = d(x)$. Hence, $d(x) \leq d(y)$. Conversely, let $d(x) \leq d(y)$. By Proposition 3.1 (3), we have $d(x) \leq d(y) \leq y$. $\square$

Let A be an EMV-algebra and $d \in Der(A)$. We will denote the set of all fixed point of d by $Fix_d(A)$. That is, $Fix_d(A) = \{x \in A \mid d(x) = x\}$.

Proposition 3.8. *Let A be an EMV-algebra and $d \in Der(A)$. Then $Fix_d(A)$ is a downset.*

Proof. By the notion of downset and Proposition 3.1 (10), we can get that $Fix_d(A)$ is a downset. $\square$

In the following proposition, we will study the relation between lattice ideals and fixed point sets of $(\odot, \vee)$ -derivations on EMV-algebras.

Proposition 3.9. *Let A be an EMV-algebra. For all $x, y \in [0, a]$ such that $x \vee y \in [0, a]$, if $d \in \text{PDer}(A)$, we have that $\text{Fix}_d(A)$ is a lattice ideal of A .*

Proof. Let $d \in \text{PDer}(A)$. For all $x \in [0, a]$, there exists $u \in [0, a]$ such that $d(x) = u \odot x$. For all $x, y \in \text{Fix}_d(A)$, we have $d(x) = x$ and $d(y) = y$. It implies that $d(x \vee y) = u \odot (x \vee y) = (u \odot x) \vee (u \odot y) = d(x) \vee d(y) = x \vee y$ by Lemma 2.2 (3). So $\text{Fix}_d(A)$ is closed under $\vee$. Hence, together with $\text{Fix}_d(A)$ is a downset, we can get that $\text{Fix}_d(A)$ is a lattice ideal of A . $\square$

We define some binary operations on $\text{Der}(A)$. For $d, d' \in \text{Der}(A)$, $(d \vee d')(x) = d(x) \vee d'(x)$, $(d \wedge d')(x) = d(x) \wedge d'(x)$ for all $x \in A$.

Proposition 3.10. *Let A be an EMV-algebra. For all $d, d' \in \text{Der}(A)$, we have $(d \vee d') \in \text{Der}(A)$.*

Proof. Let $d, d' \in \text{Der}(A)$. For all $x, y \in A$, by Lemma 2.2 (3), we have

$$\begin{aligned} & (d \vee d')(x \odot y) \\ &= d(x \odot y) \vee d'(x \odot y) \\ &= ((d(x) \odot y) \vee (x \odot d(y))) \vee ((d'(x) \odot y) \vee (x \odot d'(y))) \\ &= ((d(x) \odot y) \vee (d'(x) \odot y)) \vee ((x \odot d(y)) \vee (x \odot d'(y))) \\ &= ((d(x) \vee d'(x)) \odot y) \vee (x \odot (d(y) \vee d'(y))) \\ &= ((d \vee d')(x) \odot y) \vee (x \odot (d \vee d')(y)). \end{aligned}$$

Hence, $(d \vee d') \in \text{Der}(A)$. $\square$

Theorem 3.1. *Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements hold:*

- (1) *For all $u, v \in [0, a]$, we have $d_u \vee d_v = d_{u \vee v}$ and $d_u \wedge d_v = d_{u \wedge v}$;*
- (2) *$(\text{PDer}(A), \vee, \wedge, 0_A, \text{Id}_A)$ is a lattice;*
- (3) *For all $d, d' \in \text{IDer}(A)$, we have $d \vee d', d \wedge d' \in \text{IDer}(A)$;*
- (4) *$(\text{IDer}(A), \vee, \wedge, 0_A, \text{Id}_A)$ is a lattice.*

Proof. (1) Let $u, v \in [0, a]$. For all $x \in [0, a]$, we have

$$\begin{aligned} & (d_u \vee d_v)(x) \\ &= d_u(x) \vee d_v(x) \\ &= (u \odot x) \vee (v \odot x) \\ &= (u \vee v) \odot x \\ &= d_{u \vee v}(x). \end{aligned}$$

Similary, we have

$$\begin{aligned} & (d_u \wedge d_v)(x) \\ &= d_u(x) \wedge d_v(x) \\ &= (u \odot x) \wedge (v \odot x) \\ &= (u \wedge v) \odot x \\ &= d_{u \wedge v}(x). \end{aligned}$$

(2) By (1), we have that $\text{PDer}(A)$ is closed under $\vee$ and $\wedge$. In addition, since $0_A, \text{Id}_A \in \text{PDer}(A)$, so $(\text{PDer}(A), \vee, \wedge, 0_A, \text{Id}_A)$ is a lattice.

(3) Let $d, d' \in \text{IDer}(A)$. For all $x, y \in [0, a]$, we have $d(a), d'(a) \in B(A)$. It follows that $(d \vee d')(a) = d(a) \vee d'(a) \in B(A)$ by Lemma 2.2 (3). Similarly, $(d \wedge d')(a) \in B(A)$. In addition, since d and d' are isotone, we have that $d \vee d'$ and $d \wedge d'$ are isotone. Hence, $d \vee d', d \wedge d' \in \text{IDer}(A)$.

(4) From (3), we have that $\text{IDer}(A)$ is closed under $\vee$ and $\wedge$. In addition, by $0_A, \text{Id}_A \in \text{IDer}(A)$, it implies that $(\text{IDer}(A), \vee, \wedge, 0_A, \text{Id}_A)$ is a lattice. $\square$

4 $(\odot, \vee)$ -derivations on EBL-algebras

For an $(\odot, \vee)$ -derivation d_1 on an EMV-algebra A_1 , we have $d(x) \leq x$ for all $x \in A_1$. But this property holds only for part $y \in A_2$ on an EBL-algebra A_2 . Let $d_2 \in \text{Der}(A_2)$. There exists $z \in [0, a]$ such that $z \leq d_2(z)$. In the following, we will introduce some notions and results of $(\odot, \vee)$ -derivations on EBL-algebras.

Definition 4.1. Let A be an EBL-algebra. For all $a \in I(A)$ such that $x, y \in [0, a]$, a map d is called an $(\odot, \vee)$ -derivation on A if it satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)).$$

We shall denote the set of all $(\odot, \vee)$ -derivations on A by $\text{Der}(A)$ and denote an arbitrary idempotent element on an EBL-algebra A by a .

Proposition 4.1. Let A be an linearly ordered EBL-algebra and $k \in A$. For all $x, y \in A$, we define a map d by

$$d(x) = \begin{cases} x, & x < k, \\ 0, & x \geq k. \end{cases}$$

If $x \odot y \geq k$ when $x, y \geq k$, then $d \in \text{Der}(A)$.

Proof. We will consider the following cases:

(i) For $x, y < k$, by Proposition 2.2 (8), we have

$$\begin{aligned} d(x \odot y) &= x \odot y \\ &= (x \odot y) \vee (x \odot y) \\ &= (d(x) \odot y) \vee (x \odot d(y)). \end{aligned}$$

(ii) For $x, y \geq k$, since $x \odot y \geq k$, we have

$$\begin{aligned} d(x \odot y) &= 0 \\ &= (0 \odot y) \vee (x \odot 0) \end{aligned}$$

$$= (d(x) \odot y) \vee (x \odot d(y)).$$

(iii) For $x < k$ and $y \geq k$, by Proposition 2.2 (8) again, we have

$$\begin{aligned} & d(x \odot y) \\ &= x \odot y \\ &= (x \odot y) \vee (x \odot 0) \\ &= (d(x) \odot y) \vee (x \odot d(y)). \end{aligned}$$

Hence, $d \in \text{Der}(A)$. $\square$

In the following, we will give some different results than $(\odot, \vee)$ -derivations in section 3.

Proposition 4.2. *Let A be an EBL-algebra and $n \in N$. For all $d \in \text{Der}(A)$, the following statements hold for all $x, y \in [0, a]$:*

- (1) *If $x \leq y$, then $d(x) \leq y^{-a}$;*
- (2) *$d(x) \leq x^{-a}$;*
- (3) *If $x \odot y = 0$, then $d(x) \odot y = 0$;*
- (4) *$x \odot d(a) \leq d(x)$;*
- (5) *If $d(a) = a$, then $d(I([0, a])) = I([0, a])$;*
- (6) *$d(x^{-a}) \leq (d(x))^{-a}$;*
- (7) *If $d(x) = a$, then $x^{-a} = 0$;*
- (8) *$d(a) = a$ iff $x \leq d(x)$.*

Proof. (1) By Proposition 2.2 (1) and (2), we have $x \odot y^{-a} = 0$. So $0 = d(0) = d(x \odot y^{-a}) = (d(x) \odot y^{-a}) \vee (x \odot d(y^{-a}))$. It implies that $d(x) \odot y^{-a} = 0$. That is, $d(x) \leq y^{-a}$.

(2) It follows from (1).

(3) By Proposition 2.2 (2), we have $x \leq y^{-a}$. In addition, by Proposition 2.2 (1), we have $x^{-a} \leq y^{-a}$. So $d(x) \leq y^{-a}$. That is, $d(x) \odot y = 0$.

(4) The proof is similar to Proposition 3.1 (4).

(5) By (4), we have $x \odot d(a) = x \odot a = x \leq d(x)$. In addition, if $x \in I([0, a])$, by Proposition 2.2 (7), we have $d(x) \leq x$. Hence, $d(x) = x$ for all $x \in I([0, a])$.

(6) By (3) and Proposition 2.2 (1), we have $x^{-a} \leq (d(x))^{-a}$ and $d(x^{-a}) \leq x^{-a}$. Hence, $d(x^{-a}) \leq (d(x))^{-a}$.

(7) By (1) and Proposition 2.2 (1), we have $x^{-a} \leq a^{-a} = 0$. So $x^{-a} = 0$.

(8) Let $d(a) = a$. By (4), we have $x = x \odot a = x \odot d(a) \leq d(x)$. Conversely, we have $a \leq d(a)$. In addition, we have $d(a) = d(a \odot a) = (d(a) \odot a) \vee (a \odot d(a)) = d(a) \odot a$. It follows that $d(a) \leq a$. Hence, $d(a) = a$. $\square$

Proposition 4.3. *Let A be an EBL-algebra and d be an $(\odot, \vee)$ -derivation on A . If $d(a) \in I([0, a])$, then $d(d(a)) = d(a)$.*

Proof. By Proposition 4.2 (2) and Proposition 2.2 (7), we have $d(d(a)) \leq d(a)$. In addition, we have $d(d(a)) = d(d(a) \odot a) = (d(d(a)) \odot a) \vee d(a) =$

$d(d(a)) \vee d(a)$. It implies that $d(a) \leq d(d(a))$. Hence, $d(d(a)) = d(a)$ for all $d(a) \in I([0, a])$. $\square$

Definition 4.2. Let A be an EBL-algebra and $d \in \text{Der}(A)$. d is called isotone if $x \leq y$ for all $x, y \in [0, a]$, it implies that $d(x) \leq d(y)$.

In the following, we will introduce some sufficient conditions for an isotone $(\odot, \vee)$ -derivation on an EBL-algebra A .

Proposition 4.4. Let A be an EBL-algebra and d be an $(\odot, \vee)$ -derivation on A . For all $x \in [0, a]$, if $x \leq d(x)$, then d is an isotone $(\odot, \vee)$ -derivation on A .

Proof. Let $x \leq y$ for some $y \in [0, a]$. By Proposition 2.2 (5), we have

$$\begin{aligned} & d(x) \\ &= d(x \wedge y) \\ &= d(y \odot (y \xrightarrow{a} x)) \\ &= (d(y) \odot (y \xrightarrow{a} x)) \vee (y \odot d(y \xrightarrow{a} x)) \\ &\leq d(y) \vee y \\ &= d(y). \square \end{aligned}$$

There exist some isotone $(\odot, \vee)$ -derivations on EBL-algebras which does not satisfies $x \leq d(x)$.

Example 4.1. The zero $(\odot, \vee)$ -derivation 0_A is isotone, but it does not satisfies that $x \leq d(x)$ for all $x \in [0, a]$. Since $x \not\leq d(x) = 0$ for all $x \neq 0$.

Theorem 4.1. Let A be an EBL-algebra and d be an $(\odot, \vee)$ -derivation on A . For all $x, y \in [0, a]$, if $d(x) \leq x$, we have $d(x) = x$ iff $d(x \vee y) = (x \vee d(y)) \wedge (d(x) \vee y)$.

Proof. Let $d(x) = x$ for all $x \in [0, a]$. Then $d(x \vee y) = x \vee y = (x \vee y) \wedge (x \vee y) = (x \vee d(y)) \wedge (d(x) \vee y)$. Conversely, take $y = x$, we have $d(x) = x \vee d(x)$. It implies that $x \leq d(x)$. Hence, we have $d(x) = x$. $\square$

Definition 4.3. An EBL-algebra $(A, \vee, \wedge, \odot, 0)$ with a $(\odot, \vee)$ -derivation d is called isotone if d is isotone.

In the following, we will give a classification of EBL-algebras with $(\odot, \vee)$ -derivations.

Proposition 4.5. Let A be an EBL-algebra and d be an $(\odot, \vee)$ -derivation on A . For all $x \in [0, a]$, the following statements hold:

- (1) If $x \leq d(x)$, then A is an isotone EBL-algebra;
- (2) If $d(x) \leq x$, then A could be either an isotone EBL-algebra or not an isotone EBL-algebra.

Proof. (1) Let $x \leq d(x)$. By Proposition 4.4, we have that A is an isotone EBL-algebra.

(2) By Example 4.1, we have that zero ($\odot, \vee$)-derivation 0_A is isotone which satisfies that $0 = d(x) \leq x$. Hence, A could be an isotone EBL-algebra. In addition, by Proposition 4.1, we have $d \in \text{Der}(A)$. But d is not isotone, there exists $0 < y < k$ such that $d(y) = y > 0 = d(k)$. Hence, A could be not an isotone EBL-algebra. $\square$

Denote $d(I) = \{d(x) | x \in I\}$ by $d(I)$ and denote $d(F) = \{d(x) | x \in F\}$ by $d(F)$.

Definition 4.4. Let d be an ($\odot, \vee$)-derivation on an EBL-algebra A and I be an ideal of A . We say d preserves I if $d(I) \subseteq I$.

Definition 4.5. Let d be an ($\odot, \vee$)-derivation on an EBL-algebra A and F be an filter of A . We say d preserves F if $d(F) \subseteq F$.

In the following, we will show that ($\odot, \vee$)-derivations preserve ideals and filters.

Proposition 4.6. Let A be an EBL-algebra and I be an ideal of A . For all $x \in I$, if $d(x) \leq x$, then $d(I) \subseteq I$.

Proof. Let $y \in d(I)$. There exists $b \in I$ such that $y = d(b)$. Since $d(b) \leq b$, we have $y = d(b) \in I$ by Definition 2.5. Hence, $d(I) \subseteq I$. $\square$

Proposition 4.7. Let A be an EBL-algebra and F be an filter of A . For all $x \in F$, if $x \leq d(x)$, then $d(F) \subseteq F$.

Proof. For all $y \in d(F)$, there exists $b \in F$ such that $y = d(b)$. Since $b \leq d(b)$, we have $y = d(b) \in F$ by Definition 2.6. That is, $d(F) \subseteq F$. $\square$

5 Conclusion

In the paper, we generalized the notions of ($\odot, \vee$)-derivations to EMV-algebras and EBL-algebras. Some results on MV-algebras and BL-algebras are extended to EMV-algebras and EBL-algebras. Let d_1 be an ($\odot, \vee$)-derivation on an EMV-algebra A_1 and d_2 be an ($\odot, \vee$)-derivation on an EBL-algebra A_2 . For all $x_1 \in [0, a_1]$, $x_2 \in [0, a_2]$, we have $d(x_1) \leq x_1$ and $d(x_2) \leq x_2^{-a_2}$, where a_1 and a_2 are arbitrary idempotent elements of A_1 and A_2 , respectively. But there is no certain partial order relationship between $d(x_2)$ and x_2 . Hence, ($\odot, \vee$)-derivations on EBL-algebras are more complex.

In the following study, there is a valued topic can be studied: In Proposition 4.5, if there is no partial order relationship between $d(x)$ and x , whether A is an isotone EBL-algebra or not?

Declarations

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