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Special Issue:

LIE-ISOTOPIC REPRESENTATION OF NUCLEAR DATA

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Foreword for isotopic generalizations of quantum theories

Jeremy Dunning-Davies *

Abstract

Here it is attempted to give a brief introduction to the work of Ruggero Santilli via a slightly more detailed discussion of two areas in which he has made major contributions. The motivation for his work is also discussed throughout and an attempt is made also to highlight some of the problems which have inspired him over the years. The two examples taken are, firstly, a fascination with the early work of Rutherford in viewing a neutron, which was undiscovered at the time, as a proton and an electron. This model was discarded because it wasn't allowed by conventional quantum mechanics. The second was a deep interest in the paradox introduced into physics by the appearance of the now famous article by Einstein, Podolsky and Rosen in 1935. Both these contributed to Santilli's realization that, for further progress to be made in physics, a new approach had to be adopted which would involve the use of as yet unused, or unknown, mathematical methods. Over the years, a huge body of new mathematics has emerged through the efforts of Santilli, helped to some extent by various other workers, but the main body has been due to him. This relatively short foreword has as its aim to promote this lifetime's work of Santilli and hopefully encourage others to be motivated to study it with truly open minds and, where necessary, carry out actual experiments to test independently some of the predictions made – especially the one relating to a possible method for the safe, rapid disposal of nuclear waste.

Keywords: nuclear physics; radioactive waste; hadronic mechanics

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1 Introduction

In the very last paragraph of his well-known book on quantum mechanics [1], the Nobel Prize Winner Paul Dirac states that:

“It would seem that we have followed as far as possible the path of logical development of the ideas of quantum mechanics as they are at present understood. The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr’s orbit theory to the present quantum mechanics.”

This is a powerful statement by an eminent, highly respected theoretician but echoes accurately concerns which have existed about quantum mechanics since the subject was born. This is not to decry its enormous achievements in the intervening years but merely to draw attention to the fact that it, like all other theories, cannot be accepted as the final answer; again like all other theories, it is not complete and depends crucially on any assumptions made in its beginnings.

Very often the queries about quantum mechanics have revolved around the role of the observer and over whether or not quantum mechanics is an objective theory. One man who has considered these points is Karl Popper, one of the best known philosophers of science. Contrary to the so-called Copenhagen Interpretation, he expresses the view that the observer, or as he prefers to call him, the experimentalist, plays exactly the same role in quantum mechanics as he does in classical physics – that is, he is there to test the theory.

As has been noted elsewhere, a great many eminent physicists have switched allegiance away from the pro-Copenhagen camp over the years. However, where does Popper fit into anything to do with Hadronic Mechanics? Quite simply, the answer lies in the fact that it was in his 1982 book [2] that he, Karl Popper, drew attention to the thoughts and ideas of Ruggero Santilli. In the ‘Introductory Comments’ to his book, Popper reflects on, amongst other things, Chadwick’s model of a neutron. He notes that it could be viewed and indeed was interpreted originally as being composed of a proton and an electron. However, again as he notes, orthodox quantum mechanics offered no viable explanation for such a composition. Hence, in time, it became accepted as a new particle. Popper then notes that, around his (Popper’s) time of writing, Santilli had produced an article in which the “first structure model of the neutron” was being revived by “resolving the technical difficulties which had led, historically, to the abandonment of the model”. It is noted that Santilli felt the difficulties were all associated with the assumption that quantum mechanics applied within the neutron and disappeared when a generalised mechanics is used. Later, at the end of section IV of his ‘Introductory Comments’, Popper makes the following assertion:

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“I should like to say that he (Santilli) – one who belongs to a new generation - seems to me to move on a different path. Far be it from me to belittle the giants who founded quantum mechanics under the leadership of Planck, Einstein, Bohr, Born, Heisenberg, de Broglie, Schrödinger, and Dirac. Santilli too makes it very clear how greatly he appreciates the work of these men. But in his approach he distinguishes the region of the arena of incontrovertible applicability of quantum mechanics (he calls it atomic mechanics) from nuclear mechanics and hadronics, and his most fascinating arguments in support of the view that quantum mechanics should not, without new tests, be regarded as valid in nuclear and hadronic mechanics, seem to me to augur a return to sanity: to that realism and objectivism for which Einstein stood, and which had been abandoned by those two very great physicists, Heisenberg and Bohr”.

Obviously, these comments of Popper will not be too well-received by some but, at the very least, they provide much food for thought and, considering his own well-deserved reputation, should convince people to assess Santilli’s contributions with open minds at the very least.

2 Limitations of quantum mechanics?

As stated above, in more recent times, one man who has worried about the extent of the claims for much of conventional theory is Ruggero Santilli. He has devoted his life to studying and attempting to extend the theory to cover situations to which it was not, in its usually accepted form, truly applicable. The fact that it is, at the very least, not applicable in certain cases is something which is hidden from the public and from most students and Santilli’s investigations have placed him squarely in opposition to the ‘godfathers’ of ‘conventional wisdom’.

All this has put him at a grave disadvantage in the scientific world where questioning the currently accepted views on basic theory is still a perilous route to follow just as it was in the days of Waterston. It might be remembered that Lord Kelvin opined that, ignoring Waterston’s work on the kinetic theory of gases, had probably delayed advances in the field by a great many years. This scientific blunder is well documented in Brush’s two volume work ‘The Kind of Motion we call Heat’. [3]. As already indicated, Ruggero Santilli has dedicated his life to examining the bases of not just quantum mechanics but relativity as well, feeling both theories to be incomplete.

3 The need for new clean nuclear power

Santilli's investigations have led, in recent years, to possibilities for new clean energies and it is this which is now so important to consider, especially at this time when the world is so troubled by the depletion of energy stocks and worries about environmental effects of the energy sources presently being utilised so widely. This whole problem of future energy supplies is probably far more serious than usually imagined. Present demand is increasing but, when countries such as those of both the Indian sub-continent and of Africa come on line fully and require as much energy as the countries of the present west, that demand will escalate enormously. Given the present state of orthodox fundamental knowledge, the only realistic solution to this problem is presented by nuclear power. To many, this is not an acceptable option.

Alternatives such as solar power, wind power, geothermal energy, wave energy, and others are all put forward but, in truth, these in total would come nowhere near satisfying the probable future demands for energy. No; as has been pointed out on more than one occasion, the only realistic answer at the world's disposal at present is nuclear power. [4] However, nuclear power is felt to pose two major problems and both are concerned with safety. The safety of the actual power stations is, not unreasonably, a tremendous worry for many. This is accentuated by incidents such as the Three Mile Island problem in the U.S.A. and, more recently, the disaster at Chernobyl. However, it is only the latter case that proved a true disaster; the first was fundamentally contained by the safety systems in place.

There is little doubt that, provided adequate funds are made available, nuclear power plants can be made extremely safe, although, as with all man-made structures, no-one can guarantee complete safety of anything and, whether those in authority like to admit it or not, genuine accidents will, and do, occur. Therefore, there can be no room for complacency but, if a sensible number of safety measures are incorporated into the plant, nuclear power stations should be safe.

4 The need for recycling nuclear waste

The disposal of nuclear waste is another matter, as has been highlighted by all the problems being faced in the U.S.A. over its proposed storage facility in Nevada. This brings the story back to Santilli for another outcome of his work has been the emergence of a possibility for the safe disposal of nuclear waste in-house; by which is meant, the safe disposal of the waste without any need for transportation. [5] The idea is still only at the theoretical stage and, as Santilli has been requesting for some time now, requires the performance of about three experiments to see if the theory actually works in practice. Such experiments would not be cheap to

perform but, considering the enormous sums spent on some elementary particle work, the cost would not be too great and, if successful, the ensuing benefit for mankind would truly be out of all proportion to that cost!

Most will ask at this point why these experiments haven't been performed. This is a difficult, if not impossible, question to answer, but it may be noted that, on the one hand, the theory behind all this does not conform to 'conventional wisdom' and does, in fact, raise questions about the range of validity (at least) of the widely accepted theories of relativity and quantum mechanics, while, on the other hand, the theory has led already to the production of the new clean fuel, 'magne-gas'! Hence, although the theory may be abstruse, may contain elements which some feel unacceptable, and may conflict with 'conventional wisdom', nevertheless something concrete has been produced already which can be, and has been, used. The theory definitely appears to have had a readily identifiable success already.

On the other hand, enormous profits are being made by people in the business of disposing of nuclear waste using the current somewhat crude and unsatisfactory methods. So the question arises as to whether, in some sense, 'conventional wisdom' and 'big business' have combined to prevent the performance of these experiments which, if successful, could have such a dramatic effect on both. Santilli has extended his work to cover a huge number of seemingly disparate fields but, as far as much of his basic work is concerned, he derived a large amount of inspiration from a relatively small number of sources.

5 Rutherford's synthesis of the neutron in a star

From the point of view of physics, it seems that Santilli obtained inspiration from early ideas of Rutherford. It was in 1920 [6] that Rutherford postulated the existence of a new particle, which was, in essence a 'compressed hydrogen atom'; that is, it was composed of an electron compressed entirely within the proton. This he called a neutron. Presumably Rutherford thought that, when a hydrogen atom is compressed, for example, in the core of a star, the high pressures involved could result in it being reduced in size to that of a proton, with an electrically neutral particle emerging finally. Twelve years later, Chadwick [7] established the existence of the neutron experimentally. However, Rutherford's original conception of this particle was dismissed by many of the founders of quantum mechanics for a variety of seemingly good reasons at the time: - the model would require a positive binding energy; both constituents possess spin $\frac{1}{2}$ and so, the resulting particle would not be permitted to have spin $\frac{1}{2}$ by normal quantum mechanics; orthodox quantum mechanics would also not allow the correct magnetic moment to follow in this model. Hence, the rejection of Rutherford's model of a neutron and this

heralded a change in the direction of physics' research.

Up to that time, physics had been based on the notion that the constituents of so-called bound states have to be capable of being isolated and identified in laboratories. The rejection of Rutherford's conception appears to have altered this view. This then was the spur for Santilli and, having devised some totally new mathematical techniques, he first succeeded in producing a consistent model of the meson, π^0 , as a bound state of an electron and a positron. This model is not possible in conventional quantum mechanics for a number of reasons, one of which concerns binding energy. Quantum bound states possess negative binding energies and this implies a total mass less than the sum of the constituent masses. For a π^0 meson, this would imply a rest energy appreciably less than its actual rest energy of 135 *Mev*.

This problem, as are all others, is resolved by hadronic mechanics or, at least, that is the claim with all the evidence clearly available for examination by those with a mind so to do. The model Santilli proposes does, in fact, explain all the characteristics of the said particle – zero spin, electrically neutral, null magnetic moment, a rest energy of 135 *Mev*, a mean-life of approximately 10^{-16} *sec.*, a charge radius of about 1 fm (that is, 10^{-15} *m*), decay according to $\pi^0 \rightarrow e^- + e^+$ - and this model of the smallest of hadrons has now been extended successfully to all mesons. Further, although the theory does not view quarks as actual physical particles, but rather as mathematical objects with a composite structure, this new model for hadrons does prove compatible with the current quark theories, always assuming that quarks have a composite structure. For those interested, further details of this model may be found in a variety of publications but especially in volume 4 of the *Journal of New Energy*. [8] In fact this reference is a veritable goldmine of information on this general topic of hadronic mechanics and its consequences both for physics itself and probably for mankind as a whole through its consideration of the possibilities offered by the theory for alternative new clean energies.

6 The need to generalise Heisenberg's uncertainty principle

Central to all of this was the generalization of Heisenberg's uncertainty principle of hadronic mechanics (for details of this refer to the summary in reference below [9]) because in its absence none of this would have been possible and the above mentioned new method for the recycling of nuclear waste could never have been contemplated.. Hence, it is his success in using the new hadronic mechanics to resurrect the Rutherford model for the structure of the neutron successfully

which could conceivably turn out to be Santilli's most important achievement. This model recognises a neutron as being composed of a bound state of a proton and an electron at a distance of 1fm; that is, at a distance of 10-15 m. As mentioned earlier, such a model is prohibited by conventional quantum mechanics, so, if Santilli's ideas are valid, what are the consequences for physics? The answer is, quite simply, enormous! The abandonment of the original approach to the structure of physical particles will have had a profound and far-reaching effect on research in the area of particle physics obviously.

However, it is the possible ecological implications which are staggering and of so much direct relevance to absolutely everyone. The orthodox approach has conceivably prevented the study of the neutron as a major source of clean energy and actually seems to have obstructed the study of new forms of clean nuclear energy.

7 Recycling nuclear waste via stimulated decay

As for the actual proposal for a safe method of disposal of nuclear waste, that has been treated in a number of articles and more details may be found in these. The basic idea revolves around the fact that the nuclei concerned are large and naturally unstable. One idea is to expose the highly radioactive nuclear waste to an intense, coherent flow of photons with the required resonating frequency. It is felt that this may be achieved via a synchrotron of about three meters diameter; - a size which could be accommodated in nuclear power plants. A typical example is provided by uranium (92-U-238) which has a life-time of the order of 109 *years*. A double stimulated transmutation of this element could change it into Plutonium (94-Pu-238).

Again, this is an unstable quantity and has harmful emissions as well, but its life-time is a mere 86 days and it could well be retained under suitable shields for that period of time. It may be superfluous to draw extra attention to this point, but it is worth noting the different life-times involved here – 86 days as against 109 *years*!

The phenomenal advantage of this stimulated transmutation is immediately evident. Will it work? The theory certainly suggests that it should, but only experimentation will give the actual answer to that question. Possibly the bigger, more relevant, question to ask at this time is whether or not the scientific community and national governments are prepared to finance the experiments necessary to test this thesis? There is little doubt powerful forces, both within the scientific establishment and in big business, will violently oppose the performance of these but can the possibility of the existence of such a prize be ignored any longer? As has been stated above, Santilli derived much inspiration for much of his work

from the earlier ideas of Rutherford but that was merely one source of inspiration for the work that has occupied his entire working life.

8 The Einstein, Podolsky and Rosen paradox

Another important source was supplied by the well-known article by Einstein, Podolsky and Rosen [10] which appeared in 1935 and raised important questions concerning the completeness of quantum mechanics. As far as the Einstein-Podolsky-Rosen, or EPR, ideas are concerned, it is worth noting that questions concerning the completeness of quantum mechanics as a physical theory have been discussed at length ever since that famous article first appeared. Many experiments have been carried out in attempts to both prove and disprove the assertions it contains and much deep thought has gone into the theoretical investigations of such as Bell. In fact all the references to this work may be found in the collection of Bell's papers on quantum philosophy which may be found in 'Speakable and Unspeakable in Quantum Mechanics' [11] as published by Cambridge University Press in 1993. Less well-known is the resolution of the paradox advanced by Santilli in 1998 [12] and it is the lack of publicity for this work which poses a significant question for the scientific community as a whole.

However, when you read even the abstract for this paper, some possible answers become apparent. With talk of such concepts as nonlinear, nonlocal, non-canonical, axiom-preserving isotopies and spin-isospin symmetry and iso-spaces, some will be put off by the implied effort to understand properly what follows in the body of the article, while others will dismiss the work out-of-hand because it depends on concepts unfamiliar to them. Hence, the question facing the scientific community is 'when do we agree to examine openly and without prejudice radical new proposals aimed at producing solutions to age-old problems?'

It seems there was little concerted effort to deny investigating results emanating from the use of the methods of Riemannian geometry – and, at the time, these would have been a total mystery to many practicing physicists – as well as the uncertainties introduced into physics and chemistry by the advent of quantum mechanics some one hundred years ago, so why not afford the same respect to Hadronic Mechanics in this present time or are the basic tenets of quantum mechanics to remain sacrosanct even when they fail to answer some of the important questions facing the World today?

These are important questions generally but are particularly apposite when considering the so-called EPR paradox and work related to it. The EPR criticisms of quantum mechanics, together with those by others, led to the establishment of the notion of so-called local realism which is formulated via conventional mathematical ideas.

Santilli introduced a generalisation of this under the name iso-local realism which is based on his own covering theory of iso-mathematics. Basically, the EPR claim is that quantum mechanics is an incomplete theory because its description of physical reality does not include all elements of reality, while every element of physical reality should be precisely represented in a complete theory. Santilli's new approach has important consequences as far as the EPR argument is concerned. Traditionally, commuting quantities are believed to be independent but, in the so-called iso-topic completion of quantum mechanics, iso-commuting quantities can be mutually interacting, although it should be understood that such interactions are structurally different from those of action-at-a-distance/potential type. Fundamentally, quantum mechanics may be considered an incomplete theory in that it does not contain the element of reality given by the nonlocal structure of interactions expected from the mutual wave overlapping. Hadronic mechanics overcomes this problem.

However, as Santilli himself points out, hadronic mechanics is not intended to represent all elements of reality. It is not meant to be a final theory. Physics is, after all, a discipline which will never admit final theories. Hadronic mechanics simply provides one type of completion of quantum mechanics – that of axiom preserving type.

9 The limitations of von Neumann's, Bell's and other theorems

It should be noted at this point that Santilli has shown via his new mathematics that von Neumann's theorem on hidden variables is inapplicable under isotopies – note, not violated but inapplicable! He has established also that the oft quoted Bell's inequality is not valid universally but holds specifically for the conventional form of quantum mechanics. Of course, more recently the matter has resurfaced with the announcement of experimental results supporting the EPR assertions at Basel. [13] The team there noted that the phenomenon dated back to a thought experiment of 1935 and that it allowed measurement results to be predicted precisely. However, it is always important to remember that thought experiments are just that – thought experiments - and, as such, are very difficult to interpret due to the assumptions made not always being totally clear, possibly not even to the originators themselves. The essence of a good practical experiment is that it should be readily repeatable. It is relatively easy to see how this could be true but, equally, could be untrue of any thought experiment. It follows that important results derived via thought experiments should be viewed with extreme care. Nevertheless, as far as the thought experiment leading to the EPR paradox is concerned, it is

one which has been viewed and examined over many years and has always led to a genuine paradox in physics. Basically, via a thought experiment, Einstein, Podolsky and Rosen showed that precise predictions are possible theoretically in certain circumstances. Such a notion may be explained briefly as follows;- they considered two systems in

an entangled state in which their properties are strongly correlated. In this case, the results of measurements on one system may be used to predict the results of corresponding measurements on the second system with arbitrary precision in principle. It was also the case that the two systems could be separated spatially. The resulting paradox is that an observer may use measurements on the first system to make more precise statements about the second system than an observer who has direct access to that second system but not the first.

The Basel team used lasers to cool atoms to a small fraction of a degree above the absolute zero of temperature. At such low temperatures, the atoms are thought to behave completely according to the rules of quantum mechanics and form a Bose-Einstein condensate. In this ultra-cold cloud, the atoms collide with one another constantly, causing their spins to become entangled. The researchers involved then took measurements of the spin in spatially separated regions of the condensate. By using high-resolution imaging, they were able to measure the spin correlations between the separate regions directly and simultaneously localise the atoms in precisely defined positions.

Hence, in this experiment, the researchers seem to have succeeded in using measurements in a given region to predict precisely the results for another region. This writer is unaware of any objections being raised as yet about this work by the Basel team. If such have or, indeed, do in the future then the argument over the EPR paradox will, no doubt, rumble on. If that does not happen then it is conceivable that a new era might be opening up for physics since, if that does happen, it is likely that applications will follow which it is hoped will be of benefit to mankind rather than the opposite.

The whole issue of the EPR paradox was discussed at length recently at a teleconference held in Florida [14] and the proceedings of that conference provide an excellent source of references and information for the present situation concerning it. Also, a detailed summing up of the situation is to be found in a fairly recent review article. [15] Both of these publications draw attention to the pioneering work of Santilli and his numerous contributions.

10 Concluding remarks

This introduction has focussed on just two of the various issues that provided inspiration to Ruggero Santilli for all that has followed over the years. The aban-

doned original Rutherford model for a neutron has been resurrected and examination of the new proposals indicates that, if the theory is correct, great benefits by way of the safe disposal of nuclear waste and the possibility of new clean energies could be at hand. As for the suggested theoretical resolution of the well-known EPR paradox, that too could conceivably lead to advances which could benefit mankind although such may not be as immediately obvious as in the case of the resurrection of the Rutherford model of the neutron. Further details on both these topics as well as on other issues addressed by Ruggero Santilli may be found in the general texts listed below. [16] where references to the theoretical background may be found also. In short it seems now is the time for an unbiased examination by the science community at large of this huge body of work. For too long have ideas been summarily dismissed purely because they conflict with ideas of presently accepted ‘conventional wisdom’.

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Problematic aspects of generalized quantum theories and their apparent resolution via hadronic mechanics

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Abstract

We show that nonlinear and nonlocal completions of quantum mechanics on conventional spaces over conventional numeric fields are generally afflicted by a number of problematic aspects such as the general loss of observability, superposition principle and causality as well as the general inability to predict the same numeric values under the same conditions at different times. We show that said problematic aspects are resolved by hadronic mechanics in view of its embedding of all nonlinear and nonlocal interactions in the new operator $\hat{S} > 0$ of the axiom-preserving product $A \hat{\times} B = A \hat{S} B$ of the enveloping algebra, resulting in the representation of new potential-free interactions between extended particles, while reconstructing linearity, locality and potentiality on isospaces over numeric isofields. We prove, apparently for the first time, a theorem according to which hadronic mechanics admits as particular incomplete cases all possible nontrivial generalizations of quantum mechanics, by therefore unifying seemingly disparate experimental tests. In particular, we show that available experiments on the violation of Bell's inequalities, or on departures from quantum mechanical predictions, constitute direct experimental verifications of hadronic mechanics.

Keywords: quantum mechanics; nonunitary transforms; hadronic mechanics¹

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1 Introduction

As it is well known, the 1935 historical argument by A. Einstein, B. Podolsky and N. Rosen that "*Quantum mechanics is not a complete theory*" [1] (see also related works [2]-[5]) has stimulated a large number of completions/generalizations of quantum mechanics as well as of its basic laws, such as: 1) Nonlinear theories [6]- [8]. 2) Nonlocal theories [9]-[10]. 3) Theories with nonpotential interactions [11] [12]. 4) Generalizations of Heisenberg's uncertainty principle [13]-[15]. 5) Generalizations of Bell's inequalities [16]-[18]. 6) q-deformations [19]-[21]. 7) Operator deformations [22]- [26]. 8) Squeezed states theories [27]. 9) String theories [28], and others.

Let us recall the majestic axiomatic structure of quantum mechanics for the conditions of its original conception and experimental verification (point-like particles in vacuum under electromagnetic interactions), such as the capability of predicting *the same numeric values under the same conditions at different times*. This important feature is due to the unitary character of the time evolution of a quantum mechanical observable A ,

$$U(t)U^\dagger(t) = U^\dagger(t)U(t) = 1, \quad (1)$$

e.g. in the finite Heisenbergs representation

$$A(t) = e^{Ht}A(0)e^{-itH} = U(t)A(0)U^\dagger(t), \quad (2)$$

with ensuing verification of causality laws and preservation over time of the numeric value of the (multiplicative) unit of quantum mechanics

$$\hbar = 1 \rightarrow 1' = U1U^\dagger \equiv 1. \quad (3)$$

In turn, the preservation of the unit over time implies the preservation over time of the numeric field $\mathcal{R}$ in which measurements belong, and the preservation over time of Hermiticity/observability on a Hilbert space $\mathcal{H}$ with states ψ over the field of complex numbers $\mathcal{C}$

$$\langle \psi | U(t)HU^\dagger(t) | \psi \rangle \equiv \langle \psi | [U(t)HU^\dagger(t)]^\dagger | \psi \rangle \in \mathcal{R}. \quad (4)$$

resulting in unquestionable experimental verifications that have justly made quantum mechanics part of history.

In this paper we note that, notwithstanding their clear historical character, generalized theories 1-9 generally depart from the above axiomatic structure and, consequently, are afflicted by a number of problematic aspects, such as [29]-[36]:

1.1. Loss of unitary time evolution. A necessary and sufficient condition for a generalization of quantum mechanics to be nontrivial is to exit from the infinite

class of unitary theories (1), which is possible if and only if the time evolution of the generalized theory is *nonunitary*

$$A(t) = W(t)A(0)W^\dagger(t), \quad W(t)W^\dagger(t) \neq 1, \quad (5)$$

with ensuing: i) lack of preservation over time of the basic unit

$$\hbar = 1 \rightarrow 1' = W(t)1W^\dagger(t) \neq 1, ; \quad (6)$$

ii) consequential loss over time of the numeric field $\mathcal{R}$; and iii) lack of consistent experimental verifications.

1.2 . General loss of Hermiticity/observability. Even though condition (7) holds at time $t = t_0$ for generalized theories 1-9,

$$\langle \psi | [W(t_0)HW^\dagger(t_0)] | \psi \rangle \equiv \langle \psi | [W(t_0)HW^\dagger(t_0)]^\dagger | \psi \rangle \in \mathcal{R}. \quad (7)$$

the same property cannot be consistently defined at a later time with ensuing loss of the necessary conditions for the validity of essential self-adjointness. Also, the time evolution of a number of generalized theories has the structure

$$A(t) = W(t)A(0)W(t)^{-1} \quad (8)$$

in which case Hermiticity condition (7) no longer holds, with consequential loss of observability at all times (see [36] for a detailed study).

1.3. General loss of the superposition principle, i.e., the wave function Ψ of a bound state of two particles with relative coordinate r

$$[\Sigma_{k=1,2} \frac{1}{2m} p_k^2 + V(r)]\Psi(r) = E\Psi(r), \quad (9)$$

cannot be generally reduced to the sum of the wave functions of the individual particle constituents

$$\Psi(r) \neq \Sigma_{k=1,2} \psi_k(r), \quad (10)$$

with consequential inability to characterize the constituents of generalized bound states.

1.4. General violation of causality laws, e.g., the admission of possible solutions in which the effect precedes the cause (see the violation of causality for half- odd-integer eigenvalues of the angular momentum in Ref. [37]).

1.5. General loss of associative character of the enveloping algebras, with consequential, inapplicability of Lie's theory loss of all space-time symmetries and ensuing lack of preservation over time of numeric predictions (see Refs. [31] [34]).

In this paper, we show the apparent resolution of insufficiencies 1.1-1.5 for theories constructed via the mathematical and physical laws of *hadronic mechanics* which was proposed in 1978 by R. M. Santilli [11] [12] at Harvard University under DOE support and developed by several scholars (see [38] for a classification of hadronic mechanics, [39]-[41] for a review, monographs [42] [43] [44] for a comprehensive treatment and Refs. [45]-[54] for independent studies).

In this paper we assume a knowledge of the preceding papers of this Special Issue, with particular reference to: the insufficiencies of quantum mechanics in nuclear physics [55]; Sect. 2 (axiomatic formulation of quantum mechanics), Sect. 3 (outline of isomathematics) and Sect. 4 (outline of isomechanics) of [56]; Sect. 4 (representation of nuclear stability) and Sect. 6 (recycling of nuclear waste) of [57].

2 Apparent resolution of axiomatic insufficiencies

2.1. The fundamental assumptions of hadronic mechanics.

The first fundamental condition by R. M. Santilli [11] [12] for the completion of quantum mechanics into hadronic mechanics is the *preservation of the associative character of the quantum mechanical enveloping algebra*,

$$\begin{aligned}\xi &= \{A, B, \dots; AB = A \times B; 1\}, \\ A \times (B \times C) &= (A \times B) \times C,\end{aligned}\tag{11}$$

as the only known way to avoid the mathematical and physical inconsistencies of nonassociative envelopes (see Ref. [34] for a technical treatment).

Santilli's [11] [12] second fundamental condition is the use of an enveloping algebra characterized by a generalized, yet axiom-preserving (thus isotopic) associative product, called isoproduct (First introduced in Charts 5.2, 5.3, 5.4, p. 154 on of [12] as part of the step-by-step isotopies of Lie's theory)

$$\begin{aligned}\hat{\xi} &= \{A, B, \dots; A \hat{\times} B; \hat{1}\}, \\ A \hat{\times} B &= A \hat{S} B, \quad \hat{S} > 0, \\ A \hat{\times} (B \hat{\times} C) &= (A \hat{\times} B) \hat{\times} C,\end{aligned}\tag{12}$$

where $\hat{S}$, called the Santillian (see, e.g., [3]), with ensuing *necessary* generalization of the quantum mechanical unit $\hbar = 1$ into the *isounit* of hadronic mechanics

$$\hbar = 1 \rightarrow \hat{1} = 1/\hat{S} > 0,\tag{13}$$

and resulting *nonunitary* character of the generalized theory (Sect. 4 of [56]).

Santilli's [42] [43] third fundamental condition is the generalization of the entire 20-th century applied mathematics, with no known exceptions, into *isomathematics* (Sect. 3 of [56]) which is based on isoproduct (12) and related isounit (13), resulting in the novel isonumbers, isofunctions, Lie-isotopic theory, isospaces, isogeometries, isosymmetries, isorelativities, etc. The Lie-isotopic branch of hadronic mechanics is the axiom-preserving completion of quantum mechanics based on isomathematics (see Sect. 3 of [56] for a review and references).

The above third condition is important for the resolution of the mathematical and physical inconsistencies of nonunitary generalizations of quantum mechanics treated in Sect. 1. This important point can best be seen as follows. Note that the Santillian is positive-definite $\hat{S} > 0$ like the unit $1 > 0$. Therefore, at the abstract, realization-free level, the isoproduct $A \hat{\times} B$ coincides with the conventional product $A \times B$ and the isounit $\hat{I}$ coincides with the conventional unit 1. It then follows that the Lie-isotopic branch of hadronic mechanics [38] coincides at the abstract level with quantum mechanics to such an extent that they can be expressed with the same symbols, only subject to different realizations.

This occurrence guarantees that *hadronic mechanics preserves all properties of quantum mechanics* to such an extent that, to the authors best knowledge, any possible preservation by hadronic mechanics of inconsistencies 1.1-1.5 is due to insufficient knowledge or use of isomathematics.

The reader should keep in mind that Santilli's proposed "hadronic" mechanics is intended for the *time invariant* (hence the need for isosymmetries) representation of the numeric value of the generally ignored actual dimension, shape and density of nucleons (spheroids with 0.78 fm radius), as well as for the representation of nuclei as generalized bound states of extended, thus deformable nucleons in conditions of partial mutual penetration [56] with conventional, action-at-a-distance, variationally selfadjoint (SA) interactions represented with the Hamiltonian H plus contact, zero-range, variationally nonselfadjoint (NSA) interactions represented with the Santillian $\hat{S}$ [11] [12], under the condition of recovering quantum mechanics identically, uniquely and unambiguously for all distances sufficiently bigger than the range of the strong interactions (about 1 fm) [42] [43].

Recall also that the Santillian is solely subjected to the condition of being positive-definite but otherwise possesses an unrestricted functional dependence for the representation of the indicated, generally ignored, nuclear structure, e.g., a dependence on: time t , relative coordinates r , relative momentum p , acceleration a , energy E , charge radius R , nucleon density d , pressure π , temperature τ , wave function $\psi(r)$, their derivatives $\partial_r \psi(r)$ and any other needed local quantity,

$$\hat{S} = 1/\hat{I} = \hat{S}(t, r, p, a, E, R, d, \pi, \tau, \psi(r), \partial_r \psi(r), \dots) > 0, \quad (14)$$

with explicit realization for a hadronic two-body system given by Eq. (15) of Ref. [56], i.e.,

$$\begin{aligned} \hat{S} = \hat{S}_{4 \times 4} = \Pi_{\alpha=1,2,3,4} \begin{pmatrix} \frac{1}{n_{1,\alpha}^2} & 0 & 0 & 0 \\ 0 & \frac{1}{n_{2,\alpha}^2} & 0 & 0 \\ 0 & 0 & \frac{1}{n_{3,\alpha}^2} & 0 \\ 0 & 0 & 0 & \frac{1}{n_{4,\alpha}^2} \end{pmatrix} \times \\ \times e^{-\Gamma(r,p,a,R,d,\psi,\hat{\psi}(r),\partial_r\psi(r)\dots)} > 0, \\ n_{\mu,\alpha} > 0, \quad \Gamma > 0, \end{aligned} \quad (15)$$

where the exponential structure (which is crucial for the results of this paper) originates from the axiom-preserving restriction on the isodifferential (for spherical nucleons)[58]

$$\hat{d}\hat{r} = \hat{S}d(r\hat{1}) = \hat{S}d[r\hat{I}(r, \dots)] = dr - rd\Gamma(r, \dots). \quad (16)$$

We see in this way the emergence of the new term $-rd\Gamma$ responsible for the numerically exact and time invariant representation of nuclear data [55]-[57], which term is completely absent in quantum mechanics.

2.2. Isounitariness.

Recall that the *projection* of hadronic mechanics on conventional spaces over conventional fields can be easily constructed via a nonunitary transformation representing the size, shape, density of nucleons and their interactions

$$UU^\dagger = U1U^\dagger = \hat{1}, \quad (17)$$

under the condition of subjecting the *totality* of 20th century applied mathematics to said transformation, such as:

$$\begin{aligned} A \times B &\rightarrow A' \hat{\times} B' = U(AB)U^\dagger = \\ &= (UAB)(UU^\dagger)^{-1}(UBU^\dagger) = A'\hat{S}B', \end{aligned} \quad (18)$$

$$\hat{S} = (UU^\dagger)^{-1} > 0.$$

To prevent a rather frequent misrepresentation, it should be stressed that Eqs. (17)-(18) *do not* belong to hadronic mechanics because they are not formulated according to isomathematics, as one can see from the fact that all products are

conventional. In fact, said equations belong to the class of *nonunitary generalizations of quantum mechanics* with ensuing physical inconsistencies identified in Sect. 1.

Most insidious for physical applications, is the lack of invariance of the Santillian under an additional nonunitary transformation $WW^\dagger \neq 1$,

$$\begin{aligned} W(A'\hat{S}B')W^\dagger &= A''\hat{\times}\hat{S}'\hat{\times}B'', \\ \hat{S}' &= W\hat{S}W^\dagger \neq \hat{S}, \end{aligned} \quad (19)$$

with consequential lack of invariance over time of nonpotential/NSA interactions.

The above insufficiency was resolved in the 1998 paper [59] with the reformulation of all possible nonunitary transformations in terms of the *isounitary transformations*

$$U = \hat{U}\hat{S}^{1/2}, \quad (20)$$

$$UU^\dagger = \hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{1},$$

under which we have indeed the isoinvariance of the isounit and of the Santillian, provided that all quantities are formulated on isospaces over isofields (Sect. 3.12 of [56])

$$\begin{aligned} \hat{I} &\rightarrow \hat{U}\hat{\times}\hat{I}\hat{\times}\hat{U}^\dagger = \hat{I}' \equiv \hat{I}, \\ \hat{A}\hat{\times}\hat{B} &\rightarrow \hat{U}\hat{\times}(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{U}^\dagger = \hat{A}'\hat{S}'\hat{B}', \\ \hat{S}' &\equiv \hat{S}. \end{aligned} \quad (21)$$

The resolution of the mathematical and physical inconsistencies indicated in Sect. 1 crucially depends on the above isotopic reformulation of nonunitary theories.

The notion of isounitariness can be best expressed via the *iso-Heisenberg equation* (first achieved in p.153 of [12] and developed in Chap. 4, p. 127 of [43]) also called the *Heisenberg-Santilli isoequation*, here presented for the particular case with $\hat{t} = t$, $I_t = 1$ in the infinitesimal form

$$\begin{aligned} i\frac{d\hat{A}}{dt} &= [\hat{A}, \hat{H}] = \hat{A}\hat{\times}\hat{H} - \hat{H}\hat{\times}\hat{A} = \\ &= \hat{A}\hat{S}(e, p, \psi, \dots)\hat{H} - \hat{H}\hat{S}(r, p, \psi, \dots)\hat{A}; \end{aligned} \quad (22)$$

with integrated form showing its clean isounitary structure

$$\begin{aligned} \hat{A}(t) &= e^{H\hat{S}t} \hat{A}(0) e^{-it\hat{S}H} = \hat{e}^{Ht} \hat{\times} \hat{A}(t) \hat{\times} \hat{e}^{-itH} = \\ &= \hat{U}(t) \hat{\times} \hat{A}(0) \hat{\times} \hat{U}^\dagger(t), \quad \hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{1}, \end{aligned} \quad (23)$$

where we have used the isoexponential of an operator X characterized by the Poincaré-Birkhoff-Witt-Santilli isothem of the isoenvlope $\hat{\xi}$ (Chart 5.2, p. 154 on of [12])

$$\hat{e}^X = (e^{X\hat{S}})\hat{1} = \hat{1}(e^{\hat{S}X}). \quad (24)$$

Note that hadronic mechanics requires *two* operators for the representation of systems: the Hamiltonian $\hat{H}$ for the representation of Hamiltonian interactions and the Santillian $\hat{S}$ for the representation of non-Hamiltonian interactions;

2.3. Isolinearity.

Consider the Schrödinger-Santilli isoequation for a two-body hadronic bound state on the iso-Hilbert space $\hat{\mathcal{H}}$ with isostates $|\hat{\Psi}(\hat{r})\rangle$ over the isofield $\hat{\mathcal{C}}$ with relative Isocoordinates $\hat{r} = UrU^\dagger = r\hat{I}$ (first introduced in Eq. (15), p. 259 of [12]), developed in Chapter 5, p. 182 of [43] and reviewed in Sect. 4 of [56])

$$\begin{aligned} \hat{H} \hat{\times} |\hat{\Psi}(\hat{r})\rangle &= \hat{H} \hat{S}(\hat{r}, \hat{p}, \hat{\psi}, \dots) |\hat{\Psi}(\hat{r})\rangle = \\ &= [\Sigma_{k=1,2} \frac{1}{2m_k} \hat{\times} \hat{p}_k \hat{\times} \hat{p}_k + \hat{V}(\hat{r})] \hat{\times} |\hat{\Psi}(\hat{r})\rangle = \\ &= \hat{E} \hat{\times} |\hat{\Psi}(\hat{r})\rangle = E |\hat{\Psi}(\hat{r})\rangle. \end{aligned} \quad (25)$$

Since all nonlinear terms are embedded in the Santillian $\hat{S}$ (or, equivalently, in the isounit $\hat{1}$), the above equation is clearly *linear* in the isowave function when formulated on isospaces over isofields, a property called *isolinearity* (Append. 4.C.1 and Definition 4.C.1, p. 166 of [43]), although highly nonlinear when *projected* on conventional spaces over conventional fields.

In a language accessible to nuclear physicists, we introduce the following important property:

THEOREM 2.3.1 [43]: *Any regular, algebraic or partial differential equation which is nonlinear on the three-dimensional Euclidean space $E(r, \delta, 1)$ over $\mathcal{R}$ always admits an identical isolinear reformulation on the isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{1})$ over $\hat{\mathcal{R}}$*

In fact, for any given nonlinear equation, there always exists at least one Santillian $\hat{S}$ for the representation of all nonlinear terms with ensuing isotopy of the product (12) and/or of the differential calculus (16) [58].

Th above setting has stimulated the following problem that has remained open to this day:

QUESTION 2.3.1 [43]: *Is an elementary solution of an isolinear equation on an isospace over an isofield a solution of the nonlinear projection of said equation on a conventional space over a conventional field?*

In the event the answer to the above question is positive, isotopies can provide new means to identify at least one solution of nonlinear equations.

2.4. Isolocality

In view of the abstract identity between quantum and hadronic formulations (beginning with the abstract identity between coordinates r and isocoordinates $\hat{r}$), it is evident that iso-Schrödinger equation (25) is *local* on $\hat{\mathcal{H}}$ over $\mathcal{C}$ yet it is *nonlocal* when projected on $\mathcal{H}$ over $\mathcal{C}$, resulting in a property called *isolocality* (Append. 4.C.1 and Definition 4.C.1 p. 166 on of [43]). In fact, the isocoordinate $\hat{r}$ of an isolated nucleon is a *point* on an isospace over an isofield, yet its projection in an ordinary space over an ordinary numeric field represents the *volume* occupied by the charge distribution of the considered nucleon

$$\hat{r} = r\hat{1} = r \text{Diag.}(n_1^2, n_2^2, n_3^2, n_4^2). \quad (26)$$

Consequently, Santilli's isodifferential calculus [58] (see S. Georgiev's extended studies [54]) is a generalization of the Newton-Leibnitz differential calculus from points to volumes, as needed to represent *extended* particles.

When we have two nucleons under strong nuclear interactions, the Santillian (15) represents the experimentally established *overlapping* of the charge distribution of the two nucleons, resulting in the contact, zero range NSA interactions that are at the foundation of the exact representations of nuclear data [56] and stability [57].

Since the wave packet of a particle is identically null only at infinity, the volume characterized by Santillian (15) for the EPR entanglement of particles is generally infinite.

2.5. Isouncertainties.

As indicated earlier, a necessary condition for a generalization of quantum mechanics to be nontrivial is that its time evolution must be *nonunitary*, Eqs. (23). In turn, a nonunitary time evolution implies a necessary generalization of Heisenberg's uncertainty principle first identified by R. M. Santilli [60] in 1981 for strong interactions, extended in 1994 to black holes [61] and finalized in 2019 [62] (see also studies [3] [40] and the possible experimental verification [62]) resulting in the *isouncertainty principle* of hadronic mechanics, also called *Einstein's isode-*

terminism, which can be written for spherical particles

$$\begin{aligned}
 \Delta r \Delta p &\approx \frac{1}{2} |U\{\langle \psi r | [r, p] | \psi(r) \rangle\} U^\dagger| = \\
 &= \frac{1}{2} |\langle \hat{\psi}(\hat{r}) | \hat{\times} [\hat{r}, \hat{p}] \hat{\times} | \hat{\psi}(\hat{r}) \rangle| = \\
 &= \frac{1}{2} |\langle \hat{\psi}(\hat{r}) | \hat{S} [\hat{r}, \hat{p}] \hat{S} | \hat{\psi}(\hat{r}) \rangle| = \\
 &= \frac{1}{2} \hbar e^{-\Gamma(t, r, p, a, E, R, d, \pi, \tau, \psi(r), \partial_r \psi(r), \dots)} \approx \\
 &\approx \frac{1}{2} \hbar [1 - \Gamma(t, r, p, a, E, R, d, \pi, \tau, \psi(r), \partial_r \psi(r), \dots)] \ll \frac{1}{2} \hbar,
 \end{aligned} \tag{27}$$

As one can see, the above principle includes as particular cases all axiomatically incomplete generalizations of the uncertainty principle known to the authors (see Refs. [13]-[15] and papers quoted therein). The significance of reformulation (27) is given by the resolution of the inconsistencies identified in Sect. 1 plus the isoinvariance for the prediction of the same numeric values under the same conditions at different times.

As an illustration, Heisenberg's uncertainty principle prohibits a quantitative representation of the synthesis of the neutron from the proton and the electron in the core of stars, because the standard deviation Δr_e for the coordinate of the electron is much bigger than the size of the neutron and the standard deviation Δp_e of the momentum implies a kinetic energy of the electron bigger than the rest energy of the neutron [55],

$$\begin{aligned}
 \Delta r_e &> R_n = 0.87 \times 10^{-13} \text{ cm}, \\
 \Delta v_e &> \frac{\hbar}{\Delta r_e \times m_e} > 10^{10} \text{ m/s}, \\
 \Delta K_e &= \frac{1}{2m_e} \times (\Delta p_e)^2 > m_n = 939.56 \text{ MeV}/c^2.
 \end{aligned} \tag{28}$$

By comparison, studies of the neutron synthesis via hadronic mechanics under isouncertainty principle (27), imply standard isodeviations for which Eqs. (28) become [56] [62]

$$\begin{aligned}
 \hat{\Delta} r_e &= \hat{S} \Delta r_e \leq R_n = 0.87 \times 10^{-13} \text{ cm}, \\
 \hat{\Delta} v &= \hat{S} \Delta v_e \ll 10^{10} \text{ m/s}, \\
 \hat{\Delta} K_e &= \hat{S} \Delta K_e \ll m_n = 939.56 \text{ MeV}/c^2,
 \end{aligned} \tag{29}$$

by therefore allowing advances that cannot be formulated via quantum mechanics, let alone treated, such as:

2.5.1) Numerically exact and time invariant representation of *all* Deuteron data in its true ground state, namely, that without orbital contributions [56].

2.5.2) Numerically exact and time invariant representations of *all* characteristics of the neutron in its synthesis from the hydrogen at the nonrelativistic and relativistic levels [64] [57] (see [65] for experimental verifications).

2.5.3) Apparent elimination of the Coulomb barrier opposing nuclear fusions via the synthesis of *negatively charged* pseudonuclei [66] .

2.5.4) The possible return to the continuous creation of matter in the universe to explain the 0.782 *MeV* missing in the neutron synthesis [65].

2.5.5) Apparent confirmation of the original conception of all matter in the universe as being composed by protons and electrons [64] [57].

2.6. Isosuperposition

It is evident that the regaining of linearity (Sect. 2.3) implies the recovering of the superposition principle on isospaces over isofields. This property can be illustrated via the simple nonunitary transformation of linear equations (9)

$$\begin{aligned} U\{\Sigma_{k=1,2}\frac{1}{2m}p_k^2 + V(r)]\Psi(r)\}U^\dagger = \\ = U[E_1|\psi_1(r)\rangle + E_2|\psi_2(r)\rangle]U^\dagger = E_1|\hat{\psi}_1(\hat{r})\rangle + E_2|\hat{\psi}_2(\hat{r})\rangle, \end{aligned} \quad (30)$$

where one should note the preservation of the eigenvalues under isotopies since nonpotential/NSA forces carry no energy [56].

Isosuperposition principle (30) has permitted the characterization of the individual nuclear constituents under nonlinear internal interactions [56] [57].

3 Experimental verification of hadronic mechanics via the violations of Bell's inequality.

We are now sufficiently equipped to state, prove and illustrate the following:

THEOREM 3.1: Particle experiments measuring deviations from numerical predictions of quantum mechanics constitute experimental verifications of hadronic mechanics.

PROOF: The sole possibility of achieving theoretical predictions numerically different than those predicted by quantum mechanics is via nonunitary transformations of quantum predictions that, as such, are within the class of isounitary equivalence of hadronic mechanics. Q.E.D.

Let us illustrate the above theorem with the following representative examples:

3.1) Quantum electrodynamics predicts the following value for the giromagnetic factor (g-2) of the muon [67] [68]

$$g^{QED} = 2.00233183620, \quad (31)$$

which value is invariant under the infinite class of unitary equivalence of quantum mechanics (Sect. 1). Consequently, the anomalous experimental value of the g-2 factor [69]

$$g^{EXP} = 2.00233184122, \quad (32)$$

can solely be achieved by subjecting the derivation of the theoretical value g^{QED} to a step-by-step nonunitary transformation that, as such, constitutes experimental verification of hadronic mechanics [72] [71]. The expectation of additional deviation caused by the reversibility of quantum mechanics versus the irreversibility of spontaneous decays should be noted [70].

3.2) Under the use of the tabulated values of the magnetic moments of the proton and of the neutron in vacuum [67] [68]

$$\mu_p = +2.79285 \mu_N, \quad \mu_n = -1.91304 \mu_N, \quad (33)$$

(where μ_N represents the *nuclear magneton*) quantum mechanics (qm) predicts that the magnetic moment of the Deuteron is given by

$$\mu_D^{qm} = (2.79285 - 1.91304) \mu_N = 0.87981 \mu_N, \quad (34)$$

which is about 3% off the experimentally measured value [67] [68]

$$\mu_D^{exp} = 0.85647 \mu_N. \quad (35)$$

By remembering the invariance of quantum mechanical predictions under unitary transformations, the experimental value μ_D^{exp} can be solely achieved via nonunitary transformations of the quantum mechanical treatment that, as such, belong to hadronic mechanics (Sect. 2), as confirmed in Refs. [73] [74].

3,3) For various other cases, see the analysis of Refs. [44] [56] [57] and independent studies [45].

In 1964, J. S. Bell [75] published a theorem essentially stating that, under the assumption of quantum mechanics per axiomatic structure (12), (13), including the representation of the spin 1/2 of particles via Pauli's matrices, and other assumptions, *a system of two point-like particles with spin 1/2 does not admit a classical counterpart*. The theorem was proved by showing that a certain expression D^{Bell} (whose explicit value depends on the assumed conditions of the two particles) is always *smaller* than the corresponding classical value D^{Clas} ,

$$D^{Bell} < D^{Clas}, \quad (36)$$

for all possible values of D^{Bell} . Consequently, Bell's theorem confirms the impossibility for quantum mechanics to recover Einstein determinism [1].

In 1998, R. M. Santilli [76] published the value $\hat{D}^{Bell}$ for *two extended nucleons under strong interactions according to hadronic mechanics* via the *Pauli-Santilli iso-matrices* on an isospace $\mathcal{H}$ over the isofield $\mathcal{C}$ [40], resulting in the value (Eq. (5.8), p. 189 of [76])

$$\hat{D}^{Bell} = \frac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) D^{Bell}, \quad (37)$$

where λ_1 and λ_2 are Bohm's hidden variables [77].

Experiments [78] [79] on the polarization of entangled photons have measured numeric values of a generalized Bell's inequality which violates inequality (36). Since the latter deviations can be solely achieved via nonunitary transformations of Bell's derivation, experiments [78] [79] constitute direct experimental verifications of hadronic mechanics, and the same holds for the additional experiments [80] [81] [82] whose papers quote the EPR argument in their titles (see Sect. 1 of [55] for a comprehensive list of experimental verifications of hadronic mechanics).

4 Concludig remarks

As shown in the preceding sections, all generalizations of quantum mechanics, thus including theories 1 to 9 of Sect. 1, must have a *nonunitary time evolution* as a necessary condition to be nontrivial. Consequently, all generalized theories imply isotopies (12) with the addition of nonlinear, nonlocal and nonpotential/NSA interactions represented by the Santillian $\hat{S}$. It then follows that all generalized theories 1 to 9 are incomplete particular cases of the infinite class of isounitary equivalence of hadronic mechanics.

The universality of hadronic mechanics can be more formally expressed by the following:

THEOREM 4.1: *Hadronic mechanics is universal for the representation of all infinitely possible, regular, linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions (universality) directly in the frame of the experimentalist (direct universality).*

PROOF: *Under the assumption that the Hamiltonian is conserved (closed systems, e.g., stable nuclei), the Hamilltonian H of the iso-Schrödinger equation (25) represents all infinitely possible linear, local and potential/SA interactions, while the Santillian $\hat{S}$ represents all infinitely possible nonlinear, nonlocal and nonpotential/NSA interactions. When the Hamiltonian is not conserved (open*

systems, e.g., nuclear fusions), the bimodular structure of Eqs. (25) is ordered to the right (left) to represent motion forward (backward) in time for the additional representation of irreversibility over time [83] [84]. Q.E.D.

It is hoped that the unification of seemingly different generalized theories presented in this paper may remove the need of a number of experimental verifications for different generalized theories and their restriction to only one test, as it is the case for generalized uncertainties.

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Lie-isotopic representation of stable nuclei

I: Apparent insufficiencies of quantum mechanics in nuclear physics

Ruggero Maria Santilli*

Abstract

In this paper, we recall the majestic axiomatic consistency of quantum mechanics for *point-like particles and electromagnetic waves in vacuum*. By following the 1935 historical argument by A. Einstein, B. Podolsky and N. Rosen that *quantum mechanics is not a complete theory*, we identify a number of apparent insufficiencies of quantum mechanics in nuclear physics with particular reference to the lack of numerically exact representation in one century of nuclear data, the prohibition by Heisenberg's uncertainty principle to represent the neutron synthesis from the electron and the proton in the core of stars despite their extremely big Coulomb attraction and the ensuing inability to represent the nuclear stability. We then point out that the axiomatic origin of the indicated insufficiencies appears to be due to the representation of nuclear constituents as *dimensionless* particles, compared to the experimentally measured *extended* character of the charge distribution of protons and neutrons in conditions of partial mutual penetration within a nuclear structure, with consequential strong interactions of nonlinear, non-local and nonpotential. In the second paper, we attempt a resolution of the indicated insufficiencies with ensuing exact and invariant representation of the Deuteron data. In the third paper, we present a consequential representation of nuclear stability with ensuing new means of recycling nuclear waste by nuclear power plants and other advances.

Keywords: nuclear physics 81V35, EPR argument, hadronic mechanics, nuclear data.¹

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1 Introduction

This author has expressed various times in his writings the view that *non-relativistic and relativistic quantum mechanics, thus including Galilean and special relativities, have a majestic axiomatic structure* and an impressive amount of experimental verifications within the conditions of their original conception, those of *dimensionless particles in vacuum under linear, local and potential interactions*.

Contrary to historical discoveries in atomic physics, quantum mechanics has been unable to achieve in one century under billions of dollars of research funds: exact representations of nuclear data; a structural (rather than kinematical) representation of the synthesis of the neutron from the hydrogen in the core of stars; the representation of nuclear stability despite the natural instability of the neutron as well as the strongly repulsive protonic Coulomb forces; industrially viable recycling of radioactive nuclear waste; sustainable and controllable nuclear fusions; and the lack of other advances of societal need.

Consequently, for societal as well as scientific accountability, we need to reinspect the *exact* validity of quantum mechanics in nuclear physics, with the understanding that its *approximate* validity in nuclear physics is beyond scientific doubts.

Among supporting views, we mention the widely forgotten (and often opposed) 1935 historical view by A. Einstein, B. Podolsky and N. Rosen (EPR) that "*quantum mechanics is not a complete theory*" or that "*the wave function of quantum mechanics does not provide a complete description of physical reality*", herein called the EPR argument [1], which was based on the experimental evidence on *particle entanglements*, namely, on the evidence that particles initially bonded together and then separated, can continuously and instantaneously influence each other at arbitrary mutual distances (see the recent paper [2] for entanglements at the classical level).

By noting that electromagnetic and gravitational interactions are ignorable at large mutual distances and superluminal interactions would violate special relativity, Ref. [1] argued that particle entanglements constitute experimental evidence on the lack of universal validity of quantum mechanics, thus suggesting the study of its completion.

On theoretical grounds, R. M. Santilli [3] has recently proved Einstein's rejection of the very name "quantum entanglement" on grounds that quantum mechanics can solely represent interactions derivable from a potential, and consequently, can solely represent entangled particle as being free due to the lack of any conceivable potential interactions at large mutual distances, by therefore suggesting the study of a suitable completion of quantum mechanics.

On experimental grounds, the lack of exact character of quantum mechanics in nature has been established by direct experiments [4] [5] [6] citing the EPR ar-

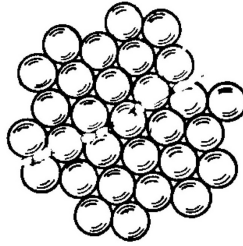


Figure 1: *In this figure, we illustrate experimental data on nuclear volumes compared to the volume of individual nucleons [23]-[27], according to which nuclei are composed by extended nucleons in conditions of partial mutual penetration of their dense charge distributions, with ensuing non-linear, non-local and non-potential interactions which have been technically studied via the conditions of variational self-adjointness in monograph [30].*

gument in their titles, as well as measurements in virtually all branches of physics, including: electrodynamics [7] [8] [9] [10]; large ion physics [11]; particle physics [12] [13]; Bose-Einstein correlation [14] [15]; propagation of light within physical media [16]; cosmology [17] [18]; and in other fields.

Following decades of studies in the field initiated in the late 1970's at Harvard University under support from the U. S. Department of Energy, the protracted general ignorance in nuclear physics of opposing experimental evidence [4]-[18] appears to be a primary reason for the failed achievement to date of an industrially viable controlled nuclear fusion despite the use of trillions of dollars of public funds, with known implications for our decaying environment.

In fact, the engineering realization of reactors for nuclear fusions may well be inadequate to achieve sustainability because: 1) Quantum mechanics has been experimentally established not to be exact for complex physical processes [4]-[18]. 2) Quantum mechanics has been unable to achieve an exact representation of nuclear data and stability. 3) The engineering components of nuclear reactors are based on the theoretical assumption of the validity for nuclear structures of Heisenberg's uncertainty principle, contrary to the recently achieved, progressive recovering of Einstein's determinism in the structure of hadrons, nuclei and stars, and its full recovering at the limit of gravitational collapse [19] [20] [21].

In the author's view and experience, a primary reason for the century old opposition by mainstream physicists against the completion of relativistic quantum mechanics in nuclear physics, as well as for the oblivion of experimental deviations [4]-[18], is a widespread resilience against the consequential completion of *special relativity* for point-like particles in vacuum into a covering relativity, known as *isorelativity*, for extended, therefore deformable and dense particles [22].

In this paper, we illustrate the validity of the EPR argument in nuclear physics via the following primary aspects:

1.1. The experimental evidence [23]-[27] that nuclei are composed by *extended* protons and neutrons (collectively called nucleons) in conditions of partial mutual penetration of their *dense* charge distribution, with ensuing expectation that, in addition to conventional linear, local and potential terms, nuclear forces contain additional terms that are *non-linear* (in the sense of depending on powers of the wave-function) as first studied by W. Heisenberg [28], *non-local* (in the sense of occurring in volumes) as first studied by L. de Broglie and D. Bohm [29], and *non-potential* (in the sense of being of contact, thus zero-range type not derivable from a potential), as first studied by R. M. Santilli in monographs [30] [31].

1.2. The consequential lack of exact representation in one century of nuclear data, such as nuclear spins, nuclear magnetic moments and nuclear stability.

1.3. The time reversal invariance of quantum mechanics (in view of the invariance of Heisenberg's equations under anti-Hermiticity), compared to the time irreversibility of all physical, chemical and biological energy releasing processes.

By using a language specifically intended for nuclear physicists, in Sect. 2 of this paper we identify a number of apparent insufficiencies of quantum mechanics in nuclear physics. In Sect. 3, we identify their apparent axiomatic origin as a necessary pre-requisite for their possible resolution.

In the second paper [32] (hereon indicated with the prefix II) we attempt a resolution of the insufficiencies indicated in this paper via the representation of the *extended* character of nucleons and their contact, thus *non-potential interactions*, with ensuing exact and invariant representation of the Deuteron data.

In the third paper [33] (hereon referred with the prefix III), we use the methods of paper II for a quantitative representation of the synthesis of the neutron from an electron and a proton in the core of stars with a consequent representation of nuclear stability despite the natural *instability* of the neutron and despite the strongly *repulsive* protonic Coulomb forces. We then apply the results to the recycling of radioactive nuclear waste and other problems.

2 Apparent insufficiencies of quantum mechanics in nuclear physics

It is conceivably possible that above insufficiencies 1.1, 1.2 and 1.3 may be at least partially responsible for the lack of achievement to date of controlled nuclear fusions, thus deserving their closer inspection.

2.1. Insufficient representation of strong nuclear forces

As it is well known, the achievement of a full representation of nuclear forces has remained elusive to this day despite the use in one century of a large number of potentials, including the Yukawa potential [34], the Woods-Saxon potential [35], the Reid potential [36], magnetic resonance technique [37], solutions of the Klein-Gordon equation [38], meson exchange formulations and other methods [39] (see Refs. [40]-[41] for historical accounts). This insufficiency suggests the study of possible additional terms in the nuclear force of non-linear, non-local and non-potential type which have been technically identified as being *variationally non-selfadjoint* (NSA) [30]. An explicit and concrete representation of strong nuclear forces is presented in Sect. II-3.4 and applied in Sect. II-5, III-3, III.4.

2.2. Insufficient representation of nuclear stability due to strongly repulsive protonic forces

Let us recall that protons repel each other with a Coulomb force, which at the mutual nuclear distance of 10^{-13} cm , is extremely big for particle standards, since it is of the order of hundreds of Newtons. As an example, the two protons of the Helium experience the *repulsive* Coulomb force of 230 Newton,

$$\begin{aligned} F &= +\frac{e^2}{r^2} = \\ &= +(8.99 \times 10^9) \frac{(1.60 \times 10^{-19})^2}{(10^{-15})^2} = +230N. \end{aligned} \tag{1}$$

It is possible that the Yukawa, Woods-Saxon, Reid and other attractive nuclear potentials may overcome such a big repulsive force, but it is unlikely whether there exists a residual attractive force sufficient for a quantitative representation of the stability of the Helium and other nuclei. A representation of nuclear stability despite the strongly repulsive protonic Coulomb forces is presented in Sect. III-4.2.

2.3. Inability to represent the synthesis of the neutron from the Hydrogen atom in the core of stars

Stars begin their lives as an aggregate of hydrogen atoms that increases in time via the accretion of hydrogen from intergalactic spaces. When the temperature (and pressure) in the core of stars reaches values estimated to be in the order of 10^8 K , the electron is "compressed" inside the proton according to E. Rutherford [42] resulting in a new particle called the *neutron* $p^+ + e^- \rightarrow n$.

W. Pauli noted that Rutherford's synthesis does not conserve the spin, by therefore violating special relativity, and suggested for their preservation the emission of a hypothetical massless particle with spin 1/2 called by E. Fermi the *neutrino* [40] [41]

$$p^+ + e^- \rightarrow n + \nu. \tag{2}$$

It appears that W. Pauli did succeed in maintaining the validity of special relativity for the neutron synthesis, but he jointly failed to maintain the validity of quantum mechanics. In fact, despite the existence of an extremely big *attractive* Coulomb force of 230 Newtons between the (negatively charged) electron and the (positively charged) proton, Eq. (1), quantum mechanics prohibits the synthesis of the neutron from the hydrogen for various reasons identified by R. M. Santilli in the 1978 Harvard University memoir [43].

The first insufficiency is caused by the fact that the validity of Heisenberg's uncertainty principle under strong interactions implies that the maximal value of the standard deviation Δr_e for the coordinate r_e of an electron inside a nucleus may be bigger than the radius of the neutron, with consequential speed such that the kinetic energy of the electron K_e may be bigger than the mass m_n of the neutron,

$$\Delta r_e > R_n = 0.87 \times 10^{-13} \text{ cm},$$

$$\Delta v_e = \frac{\hbar}{\Delta r_e \times m_e} > 10^{10} \text{ m/s}, \quad (3)$$

$$\Delta K_e = \frac{1}{2m_e} \times (\Delta p_e)^2 > m_n = 939.56 \text{ MeV}/c^2,$$

by therefore prohibiting any meaningful synthesis of the neutron, and suggesting the completion of Heisenberg's uncertainty principle for strong interactions presented in Ref. [21] (Sect. II-4.7), and applied to the resolution of insufficiencies (3) in Sect. III-3.3.1.

Additionally, Ref. [43] noted that the mass of the neutron is 0.782 *MeV bigger* than the sum of the masses of the proton and of the electron

$$E_p = 938.272 \text{ MeV}, \quad E_e = 0.511 \text{ MeV},$$

$$E_n = 939.565 \text{ MeV}, \quad (4)$$

$$\Delta E = E_n - (E_p + E_e) = 0.782 \text{ MeV} > 0,$$

by therefore requiring a *positive binding energy* and resulting in a *mass excess* for which the Schrödinger equation *for a bound state with two constituents* admits no physically meaningful solution.

Ref. [43] finally noted that, while being fully valid for the representation for the Hydrogen atom, Dirac's equation fails to provide any representation of its "compression" into the neutron (for extensive theoretical studies in the neutron synthesis from the Hydrogen see the theoretical [44], experimental studies Ref. [45] and Sect. III-3).

It should be noted that the above insufficiency has been generally dismissed by 20th century nuclear physics on grounds that the missing energy of 0.782 *MeV* is

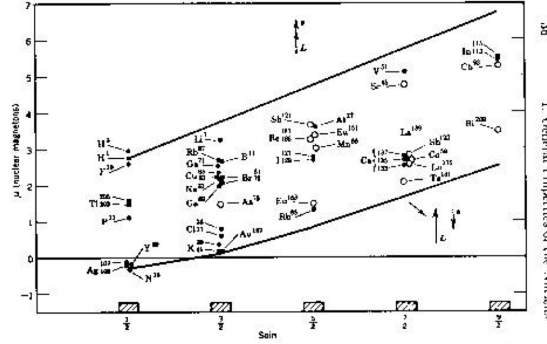


Figure 2: In this figure, we illustrate the well known Schmidt limits in nuclear physics (see, e.g., Ref. [41]) establishing deviations between the predictions of quantum mechanics for nuclear magnetic moments and their experimental values.

carried by the neutrino of reaction (3). However, in the event that is the case, the synthesized neutron n in reaction (3) could not exist because of insufficient energy for its synthesis. Also, the missing energy of 0.782 MeV cannot be provided by the $e - p$ -relative kinetic energy because the $e - p$ cross section at $1M \text{ MeV}$ is virtually null, thus preventing any synthesis. In the event said missing energy is provided by the $e - p$ -relative energy, stars would never produce light because they synthesize about 10^{40} neutrons per second, by therefore requiring about 10^{40} MeV/s . Additionally, scientific accountability prohibits the dismissal of a *generalized structure equation* for the $e - p$ bound state just because it does not verify quantum mechanics in view of its implications for controlled nuclear fusions and the recycling of radioactive nuclear waste [44] [45].

A review and upgrade of half a century of studies in the representation of *all* characteristics of the neutron in its synthesis from the Hydrogen in the coire of stars via hadronic mechanics is presented in Sect. III.3.

2.4. Insufficient representation of nuclear stability despite the instability of the neutron

As it is also well known, neutrons are naturally *unstable* (when isolated) with a mean life $\tau = 880 \text{ s}$ and spontaneous decay from Eq.(3) $n \rightarrow p^+ + e^- + \bar{\nu}$. Consequently, the stability of the neutron under strong nuclear forces deserves indeed a study. A representation of nuclear stability despite the natural instability of the neutron is presented in Sect. III-4.1.

Mainstream nuclear physicists generally dismiss the above problem (formulated in this paper apparently for the first time) on grounds that the evident nuclear stability dismisses the existence of the problem itself.

2.5. Insufficient representation of nuclear spins

According to quantum mechanical basic axioms, the only stable bound state of two particles with spin 1/2 (such as the proton and the neutron) is the singlet coupling. Consequently, quantum mechanics predicts that the Deuteron D has the structure

$$D = (p_{\uparrow}, n_{\downarrow})_{qm}, \quad (5)$$

for which the total angular momentum is null, $J_D = 0$, contrary to the experimental value of the spin of the Deuteron $J_D = 1$.

As a result of this insufficiency, quantum mechanics represents the spin of the Deuteron via such a *collection of orbital contributions* with value $L_D = 1$ (see, e.g., Ref. [41]). However, the spin $J_D = 1$ has been measured for the Deuteron *in its true ground state*, i.e., the experimentally detected state for which $L_D \equiv 0$. Consequently, quantum mechanics does not allow a consistent representation of the spin of the Deuteron in its true ground state, with similar insufficiencies for heavier nuclei (Sect. III-4.1).

Mainstream nuclear physicists generally dismissed the above insufficiency on grounds that the spin of the Deuteron is fully recovered at the level of nuclear isospins. However, the nuclear isospin symmetry is a purely mathematical symmetry solely definable in a complex valued two-dimensional carrier space that, as such, cannot possibly have any sound connection with the spin of the Deuteron in a state that, by definition, has no excited orbital states. Other attempts at salvaging quantum mechanics via hypothetical symmetric and antisymmetric states of the Deuteron are afflicted by the same shortcoming.

Since spin is a fundamental nuclear characteristic with direct implications for nuclear fusions, recycling of nuclear waste and other unsolved problems of direct societal relevance, scientific accountability requires at least the search for a consistent representation of the Deuteron spin *without* orbital contributions, which study is done in Sect. II-33.3.2.

2.6. Insufficient representation of nuclear magnetic moments

Under the use of the tabulated values of the magnetic moments of the proton and of the neutron in vacuum (see, e.g., Ref. [23])

$$\mu_p = +2.79285 \mu_N, \quad \mu_n = -1.91304 \mu_N, \quad (6)$$

(where μ_N represents the *nuclear magneton*) quantum mechanics (qm) predicts that the magnetic moment of the Deuteron is given by

$$\mu_D^{qm} = (2.79285 - 1.91304) \mu_N = 0.87981 \mu_N, \quad (7)$$

while the experimentally measured value is given by

$$\mu_D^{ex} = 0.85647 \mu_N, \quad (8)$$

resulting in a deviation for about 3% of the quantum mechanical prediction μ_D^{qm} , Eq. (7), from the experimental value μ_D^{ex} , Eq. (8), with bigger deviations for heavier nuclei (Figs. 1, 2). A numerically exact and time invariant representation of the Deuteron magnetic moment in its true ground state (that without orbital contributions) is presented in Sect. II.3.3.

The above insufficiency of quantum mechanics is generally dismissed by mainstream nuclear physicists on grounds that an exact representation of the deuteron magnetic moment can be achieved via the angular contribution to its spin. The credibility of the dismissal is however in question due to the vast experimental evidence for which the *Deuteron spin* $S_D = 1$ holds in the ground state, that is, the experimentally detected state without orbital contributions by its very definition.

2.7. Insufficient representation of nuclear irreversible processes

As it is well known, quantum mechanics is a theory invariant under time reversal in view of the invariance of Heisenberg's time evolution of an observable A under anti-Hermiticity

$$i\frac{dA}{dt} - [A, H] \equiv -\{i\frac{dA}{dt} - [A, H]\}^\dagger. \quad (9)$$

Consequently, under the assumption that the total energy $H(r, p)$ is an observable (rather than a complex-valued, thus non-observable quantity), the same Schrödinger equation in the relative coordinate r has to be applied for time irreversible energy producing processes, such as the controlled fusion of two nuclei into a third, moving forward in time (fw), as well as for their image under motion backward in time (bw), such as the disintegration of the synthesized third nucleus,

$$H = H^\dagger,$$

$$H|\psi(r)\rangle = E^{fw}|\psi(r)\rangle, \quad \langle\psi(r)|H = \langle\psi(r)|E^{bw}, \quad (10)$$

$$|\psi(r)\rangle \equiv (\langle\psi(r)|)^\dagger, \quad E^{fw} \equiv E^{bw},$$

by therefore lacking an axiomatically consistent representation of the irreversibility of energy releasing processes at large and for controlled nuclear fusions in particular.

Additionally, quantum mechanics can solely represent the conservation of the energy due to the known antisymmetric character of Lie's brackets,

$$i\frac{dH}{dt} = [H, H] = HH - HH \equiv 0. \quad (11)$$

Hence, quantum mechanics additionally lacks an axiomatically consistent representation of the *energy production* of controlled nuclear fusions (see Ref. [46] for details).

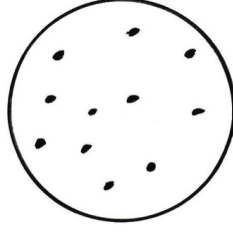


Figure 3: In this figure, we illustrate the conception of nuclei originating from the locality of quantum mechanics [1] as ideal spheres with isolated dimensionless particles in their interior. Such a view is disproved by actual value 0.87×10^{-13} cm of the charge distribution (rather than of the wave packet) of protons and neutron [27].

For the apparent intent of maintaining quantum mechanics under physical conditions beyond those of its original conception, the limitation caused by time reversibility has been generally dismissed by mainstream nuclear physicists on grounds that the direction of time is set by thermodynamical laws. The scientific character of the dismissal is however in question due to the notorious incompatibility of quantum mechanics with thermodynamics, as well as by known opposition against the EPR completion of quantum mechanics with non-potential interactions to achieve compatibility between mechanics and thermodynamics.

3 Apparent axiomatic origin of quantum mechanical insufficiencies

In this section, we attempt to identify the axiomatic origin of the insufficiencies of quantum mechanics studied in the preceding section as a necessary pre-requisite for their possible resolution studied in paper II [32], Sects, II-3 and II-4.

3.1. Is the linearity of quantum mechanics an axiomatic insufficiency?

As it is well known from the axiomatic structure of *linear* operators on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ [41], *quantum mechanics is linear in the wave function*, while non-linear interactions are expected in nuclear structures. Various nuclear models with interactions non-linear in the wave functions have been studied (see the first study by W. Heisenberg [28]), but all of them have been formulated via the quantum mechanical *Hamiltonian* structure

$$H(r, p, \psi, \dots)|\Psi(r)\rangle = E\psi(r)|\Psi(r)\rangle, \quad (12)$$

in which case: A) The Hamiltonian does not generally represent the total energy. B) There is the general violation of the superposition principle for the decomposi-

tion of a state $|\Psi(r)\rangle$ representing a nucleus with A nucleons into states $|\psi_k(r_k)\rangle$ representing the individual nucleons

$$|\Psi(r)\rangle \neq \prod_{k=1,2,\dots,A} |\psi_k(r_k)\rangle, \quad (13)$$

with consequential inability to characterize individual constituents of a nucleus. C) Being additive to the kinetic energy in a Hamiltonian, non-linear interactions are generally interpreted as having a potential energy against their expected non-potential character (Fig. 1). A possible resolution of the above axiomatic insufficiency is presented in Sect. II-3.4 and applied in Sect. II-5, III-3, III.4.

3.2. Is the locality of quantum mechanics an axiomatic insufficiency?

This axiomatic insufficiency is due to the fact that, as voiced by A. Einstein, B. Podolsky and N. Rosen [1], the wave function $\psi(r)$, the potential $V(r)$, the Laplacian Δ_r and the Newton-Leibnitz differential calculus of the Schrödinger equation

$$H(r, p)\psi(r) = \left[-\frac{\hbar^2}{2m}\Delta_r + V(r)\right]\psi(r) = E\psi(r), \quad (14)$$

can only be defined at a finite number of isolated points r , resulting in an excessively simple conception of nuclei as ideal spheres with isolated massive points in their interior (Fig. 3). In reality, nuclei are composed by extended, therefore deformable and dense nucleons with a charge radius of about $0.87 \times 10^{-13} \text{ cm}$ [27] in conditions of partial mutual penetration, with ensuing non-linear, non-local and non-potential interactions (Sect. 1). It appears evident that these complex conditions cannot be represented in a final form by the linear, local and potential structure of quantum mechanics.

It should be noted that, for the apparent intent of maintaining quantum mechanics under non-local conditions beyond its representational capability, 20th century nuclear physics dismissed the insufficiency on the dimensionless character of nuclear constituents on grounds that the dimension of protons and neutrons are represented by their wave packets. In reality, such a dismissal does not appear to have solid physical grounds because:

1) Wave packets fill up the entire universe, while protons and neutrons have the measured dimension of $0.87 \times 10^{-13} \text{ cm}$.

2) Assuming that wave packets can be used for the representation of the size of particles, their interactions are non-local (because occurring in volumes not reducible to points), thus being outside the representational capabilities of quantum mechanics.

3) The dismissal of dimensionless constituents is in disrespect of A. Einstein, B. Podolsky and N. Rosen because their concluding statement is that ... *the wavepackets of quantum mechanics cannot represent the entire physical reality* [1].

For the apparent intent of opposing the representation of the actual, measured, dimension of nucleons achieved in Paper II, mainstream nuclear physicists generally dismiss the extended character of nucleons on grounds that their size is represented by the standard deviations Δr of Heisenberg's uncertainty principle. This additional dismissal does not appear to have solid physical grounds because:

i) The validity of Heisenberg's uncertainty for nuclear constituents is a personal opinion by individual physicists because said principle has never been directly tested under strong nuclear interactions.

ii) To have physical value, the dismissal should prove in refereed publications that, while being a constituent of a nucleus such as the Zirconium, a proton has a standard deviations Δr , Δp *exactly equal* to that when the same proton is the nucleus of the hydrogen atom.

iii) To remain within the boundaries of science, the indicated dismissal should first disprove the *iso-uncertainty principle under strong interactions* of Refs. [19] [20] [21].

A possible resolution of the above axiomatic insufficiency is presented in Sect. I-3.4 and applied in Sect. I-5, II-3, II.4.

3.3. Is the potentiality of quantum mechanics an axiomatic insufficiency?

Let us recall the following technical treatment of potential forces by R. M. Santilli in the 1978 Springer-Verlag monograph [30]:

DEFINITION 3.1: A force is said to be 'variationally self-adjoint' (SA) when it verifies all necessary and sufficient conditions for its derivation as an additive potential in a Hamiltonian.

However, as recalled in Sect. 1.1 and Fig. 1, experimental nuclear data [23]-[27] suggest the presence of new terms in nuclear force that can be technically identified via the following:

DEFINITION 3.2 [30]: A force is said to be 'variationally non-self-adjoint' (NSA) when it violates the necessary and sufficient conditions for its derivation from an additive potential in a Hamiltonian.

NSA forces are known to exist since Newton's time, they are represented by the external terms in Lagrange's and Hamilton's equations, and their operator counterpart cannot be ignored in view of the *No-Reduction Theorem* studied by R. M. Santilli in the 1983 Springer-Verlag monograph [31] whose understanding requires a knowledge of the progressive recovering of Einstein's determinism studied in Refs. [19] [20] [21].

THEOREM 3.1: A macroscopic system with SA and NSA internal forces cannot be consistently decomposed into a finite number of quantum mechanical par-

ticles all with sole SA forces, and vice versa, a quantum mechanical system with sole SA internal forces cannot represent a classical system with SA and NSA forces under the correspondence or other principles.

As an example, the resistive force experienced by a spaceship during re-entry in our atmosphere is a NSA force, that analytically, can be solely represented via the external terms in Lagrange's and Hamilton's equations. Theorem 3.1 establishes that said spaceship cannot be consistently decomposed into a collection of quantum mechanical particles thus suggesting the EPR completion of quantum mechanics via the inclusion of NSA forces.

In any case, following decades of studies by various scholars along the EPR argument, quantum mechanics has been completed into the covering *hadronic mechanics* [47] [48] [49] [50] which has achieved a numerically exact and time invariant representation of nuclear data [52] thanks to the characterization of strong interactions as the most general conceivable superposition of SA and NSA forces [53] by therefore establishing that *irreversibility originates at the most elementary possible level in nature*, e.g., in meson's spontaneous decays. A possible resolution of the above axiomatic insufficiency is presented in Sect. I-3.4 and applied in Sect. I-5, II-3, II.4.

3.4. Is the modularity of quantum mechanics an axiomatic insufficiency?

We are here referring to the conventional associative action (called modular) of a Hamiltonian on a state $|\psi\rangle$ of the Hilbert space verifying conditions (10), which modular action is evidently responsible for possible causality violation by quantum mechanical treatments of controlled nuclear fusions. Axiomatic insufficiency 3.4 implies the expectation that quantum mechanics cannot represent consistently time irreversible systems, such as nuclear fusions as well as high energy particle collisions. We can therefore say that the causal treatment of controlled nuclear fusions suggests the construction of a suitable completion of the modular structure of quantum mechanics to be studied in subsequent papers.

3.5. Is the single-valuedness of quantum mechanics an axiomatic insufficiency?

We are here referring to the quantum mechanical property for which the product of two quantities (such as numbers, matrices, operators, etc.) yields one single result. By contrast, the correlation of two atoms in a DNA may yield a full organ, such as a liver, with about 10^{30} atoms. The mathematical representation of the correlation is given by the multiplication. Hence, axiomatic limitation 3.5 suggests the need for an additional completion of quantum mechanics of *multi-valued character* in which the product of two quantities can yield an ordered, but otherwise arbitrary number of results in which case, there appears to be hope to initiate a quantitative representation of life [54].

4 Concluding remarks

In preparation of our search for new radiation-free controlled nuclear fusions of light elements preliminarily indicated in report [46], in this paper we recalled the majestic axiomatic consistency of non-relativistic and relativistic quantum mechanics with corresponding Galilean and special relativities, as well as their impressive record of experimental verifications for the conditions of their original conception, those of *point-like particles in vacuum under electromagnetic interactions*.

As an illustration of the 1935 historical argument by A. Einstein, B. Podolsky and N. Rosen that *quantum mechanics is not a complete theory* [1], we have identified a number of apparent insufficiencies of quantum mechanics for *extended protons and neutrons under strong interactions*, with particular reference to the inability of achieving in one century under billions of dollars of research funds: exact representations of nuclear data; a structural (rather than kinematical) representation of the synthesis of the neutron from the hydrogen in the core of stars; the representation of nuclear stability despite the natural instability of the neutron as well as the strongly repulsive protonic Coulomb forces; the recycling of radioactive nuclear waste; sustainable and controllable nuclear fusions; and the lack of other advances of societal needs.

In the author's view, the insufficiency at the foundation of the preceding ones is the impossibility by quantum mechanics to achieve a representation via a structure equation (rather than "ad hoc" kinematic arguments) of the synthesis of the neutron from an electron and a proton in the core of stars despite their extremely big Coulomb attraction (1), because Heisenberg's uncertainty principle implies a maximal standard deviation for the electron coordinate bigger than the size of the neutron and independently, a maximal standard deviation of the momentum yielding a kinetic energy of the electron bigger than the mass of the neutron (3), by therefore prohibiting any possible neutron synthesis.

This rather serious insufficiency confirms the need for a suitable EPR completion of quantum mechanics including, most importantly, a completion for strong interactions of Heisenberg uncertainty principle studied in Sect. II-4.7, which has been assumed for one century to be valid in nuclear physics without any direct or indirect experimental verification.

We have then identified the apparent axiomatic origin of the indicated insufficiencies in the *point-like* approximation of nuclear constituents, compared to the experimentally established *extended* character of nucleons, with consequential, expected new terms in the nuclear force of *nonlinear* type initiated by W. Heisenberg [28], *nonlocal* type initiated by L. de Broglie and D. Bohm [29] and *nonpotential* type initiated by R. M. Santilli [30] [31].

In paper II [32], we present in a language specifically intended for nuclear

physicists an apparent resolution of the indicated insufficiencies and apply the results to achieve exact and invariant representations of the Deuteron data.

In paper III [33], we apply the same methods to achieve an exact and invariant representation of *all* characteristics of the neutron in its synthesis from the Hydrogen in the core of stars thanks to the completion of Heisenberg's uncertainty principle into the *isouncertainty principle for strong interactions* [21] (Sect. II-4-7), and apply the results for new means of recycling radioactive nuclear waste by nuclear power plants and other applications.

The methods underlying our studies are given by the *axiom-preserving EPR completion*— of quantum mechanics into *hadronic mechanics* [47] [48] [49] [50] according to the following branches with increasing complexity for the representation of progressively more complex nuclear conditions (see Ref. [51] for a summary classification of hadronic mechanics):

4.1. Lie-isotopic branch of hadronic mechanics [31] [48] comprising a time reversal invariant completion of quantum mechanics for the representation of stable nuclei as a collection of extended and dense nucleons in conditions of partial mutual penetration with linear and non-linear, local and non-local and potential, as well as non-potential internal forces (Fig. 1).

4.2. Lie-admissible branch of hadronic mechanics [48] [46] comprising a time irreversible completion of the Lie-isotopic branch for the causal representation of the controlled nuclear fusion, recycling of radioactive nuclear waste and other problems of societal relevance.

4.3. Hyperstructural branch of hadronic mechanics [48] [54] comprising a multi-valued completion of the irreversible Lie-admissible branch in the hope of initiating a quantitative representation of life intended as the difference between organic and inorganic molecules.

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Lie-isotopic representation of stable nuclei II: Exact and time invariant representation of the Deuteron data

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Abstract

In the preceding paper, we have presented apparent insufficiencies of quantum mechanics in nuclear physics. To attempt a resolution of the indicated insufficiencies, in this paper we present a systematic and upgraded outline of the axiom-preserving, time reversal invariant, *Lie-isotopic branch of hadronic mechanics* for the characterization of *stable* nuclei via the representation of the dimension, shape and density of protons and neutrons in the experimentally measured conditions of partial mutual penetration in a nuclear structure, with ensuing potential-Hamiltonian and contact non-Hamiltonian terms in nuclear force. We show that the Lie-isotopic methods allow a numerically exact and time invariant representations of all Deuteron data in the ground state without orbital contributions. We finally show that said representations are primarily due to the violation by Lie-isotopic methods of Bell's inequalities with explicit and concrete realizations of Bohm's hidden variables, as well as to the completion of Heisenberg's uncertainty principle for point-like particles in vacuum under electromagnetic interactions into the *isouncertainty principle* of hadronic mechanics for extended nucleons in conditions of partial mutual penetration allowing a progressive recovering of Einstein's determinism under strong interactions up to its full recovery at the limit of Schwarzschild's horizon.

Keywords: nuclear physics 81V35, EPR argument, hadronic mechanics, nuclear data.¹

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1 Introduction

1.1. Experimental foundations

Inspired by the 1935 historical argument by A. Einstein, B. Podolsky and N. Rosen (EPR) that *Quantum mechanics is not a complete theory* [1] (see also recent studies [2] [3] [4]), in the preceding paper [5] (herein referred to with the prefix I), we have presented apparent insufficiencies of quantum mechanics in nuclear physics with particular reference to the prohibition by Heisenberg's uncertainty principle for point-like particles in vacuum to achieve a quantitative representation of the neutron synthesis from the Hydrogen in the core of stars, Eq. I-3) and of other nuclear data (Sect. I-2).

In this paper, we attempt a resolution of the indicated insufficiencies based on the rather dramatic physical differences between atomic and nuclear structure. In fact, atomic structures can be well approximated as being composed by point-like constituents in vacuum under sole action at a distance, thus potential interactions and ensuing exact validity of quantum mechanics.

By comparison, experimental evidence [6]-[?] establishes that nuclei are composed by *extended* protons and neutrons (collectively called nucleons) in conditions of *partial mutual penetration* when they are members of a nuclear structure. As an example, the charge radius of the Helium-4 $D_{He} = 1.67 \text{ fm}$ [9] is 0.07 fm smaller than the diameter of the proton $D_N = 1.74 \text{ fm}$ [11]. Consequently, the charge distributions of the four nucleons composing the Helium-4 are in conditions of 0.07 fm mutual penetration of their dense structure, with generally increasing values for heavier nuclei.

The latter conditions imply the expectation that nuclei are composed by extended constituents under conventional action at a distance, thus Hamiltonian interactions, plus basically new contact, thus zero range and non-Hamiltonian interactions outside any possible representation by quantum mechanics with ensuing need for its suitable completion.

1.2. Hamiltonian interactions

We are collectively referring to interactions that are linear, local and derivable from a potential, thus being fully representable by the conventional Hamiltonian of quantum mechanics (Sect. 2).

1.3. Non-Hamiltonian interactions

We are collectively referring to interactions that are: *Nonlinear* (in the wave function) as pioneering by Werner Heisenberg [13]; *Nonlocal* (distributed in a volume not reducible to points) as pioneered by Louis de Broglie and David Bohm [14]; *Nonpotential* (of contact zero-range type) as pioneered by R. M. Santilli in the 1978 Harvard University monograph [15] via the *conditions of variational self-*

adjointness according to which Hamiltonian interactions are variationally selfadjoint (SA), while non-Hamiltonian interactions are variationally nonselfadjoint (NSA) (Sect. I-3).

1.4. Hadronic mechanics

In this paper, we shall outline and update the EPR completion of quantum into *hadronic mechanics*, first introduced by R. M. Santilli in the 1983 second volume [16] for the representation of extended nucleons under Hamiltonian/SA and non-Hamiltonian/NSA interactions in such a way to preserve the basic axioms of quantum mechanics at their abstract level, according to the following two primary branches:

TIME REVERSIBLE LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS studied in this paper, which was introduced in Charts 5.2, 5.3 and 5.4, p. 165 on of [16] and comprises the axiom-preserving *isomathematics* (Sect. 3) and *isomechanics* (Sect. 4) for the representation of *stable* (thus, time-reversal invariant) systems of *extended* particles at short mutual distances under Hamiltonian/SA and non-Hamiltonian/NSA interactions. The Lie-isotopic methods are based on the generalization of the conventional associative product $AB = A \times B$ of generic quantities into the associativity-preserving form $A \hat{\times} B = A \hat{T} B, \hat{T} > 0$ (first presented in Eq. (5), p. 71 of [16]) where T , called the isotopic element (and at times the Santillian) is positive-definite but possesses otherwise an unrestricted functional dependence on all needed local variables, with ensuing *Lie-isotopic generalization of Heisenberg equation* $idA/dt = A \hat{\times} H - H \hat{\times} A = A \hat{T} H - H \hat{T} A$ (Eq. (18a), p. 153 of [16]) where the Hamiltonian H represents all SA interactions and the Santillian $\hat{T}$ represents the extended character of particles and their NSA interactions.

TIME IRREVERSIBLE LIE-ADMISSIBLE BRANCH OF HADRONIC MECHANICS introduced in Chart 5.1, p. 148 on of [16] which comprises *genomathematics* and *genomechanics*. Lie-admissible formulations are intended to achieve an axiomatization of irreversibility by restricting all associative products $AB = A \times B$ to be ordered to the right $A > B = A \hat{S} B, \hat{S} > 0$ (left $A < B = A \hat{R} B, \hat{R} > 0$) for motion forward (backward) in time. Irreversibility is then assured for $\hat{R} \neq \hat{S}$ (for technical details, see the 1981 paper [17] from Harvard's Department of Mathematics). The basic dynamical equation is given by the *Lie-admissible generalization of Heisenberg's equations* (Eqs.(19a), p. 153 of [16]) $idA/dt = (A, H) = A < H - H > A = A \hat{R} H - H \hat{S} A$ which admits the Lie-isotopic case for $\hat{R} = \hat{S} = \hat{T} > 0$ and, for the particular case $\hat{R} = 1, \hat{S} = 1 - F/H$, assumes the form $idA/dt = (AH - HA) + FA$, by therefore providing the first known operator realization of Lagrange's and Hamilton's external terms F for the representation of irreversibility in their celebrated analytic equations. Lie-

admissible methods are suggested for the axiomatically consistent representation of controlled nuclear fusions and other irreversible processes to be studied in subsequent papers.

For advances on hadronic mechanics that occurred during the past decades, the interested reader can inspect: the final classification of hadronic mechanics (including the additional *hyperstructural branch* for biological structures and the *isodual branch* for antiparticles) [18]; the collection of papers [19]; the list of early workshops and conferences [20]; independent studies [21]-[29]; and the general presentation [30] [31] [32].

1.5. EPR entanglement

Experimental evidence known since Einstein's times establishes that particles, which are initially bounded together and then separated, can influence each other continuously and instantaneously at arbitrary distances [33]. Albert Einstein strongly objected against the terms "quantum entanglement" on grounds that the sole possible representation of particle entanglements via the Copenhagen interpretation of quantum mechanics would require superluminal communications that violate special relativity.

For the intent of honoring the generally forgotten Einsteins view, R. M. Santilli [34] proved that the sole possible representation of particle entanglement by the Copenhagen interpretation of quantum mechanics is that *the particles are free*, evidently because the sole possible interactions admitted by said interpretation are those derivable from a potential which is identically null for particles at large mutual distances.

By recalling that the wave packet of particles is identically null solely at infinite distance, Ref. [34] then pointed out that the sole interactions that are continuous, instantaneous and at arbitrary distances are given by the mutual penetration of wave packets of particles which, being NSA [15], are beyond any hope of treatment via quantum mechanics.

Thanks to the prior development of isomathematics for the representation of nonlinear, nonlocal and NSA interactions [26] [29] [30], Santilli [34] proposed the axiom-preserving completion of quantum into hadronic entanglement under the suggested name of *EPR entanglement* which does indeed provide a *quantitative* representation of particle entanglements without superluminal speeds via contact, nonlinear, nonlocal and non-Hamiltonian, thus NSA interactions between the wave packets of entangled particles (Fig. 1).

Einstein's entanglement has intriguing implications for the studies presented in this paper, such as:

1.5.A) A new conception of nuclei as hadronic bound states of extended nucleons in condition of deep entanglement that persists following nuclear fission or

fusion.

1.5.B) A new conception of computers based on isomathematics, which has been suggested under the name of *EPR computers* [35], with: increased energy efficiency because NSA interactions carry no potential energy; better cybersecurity (Sect. 3.5) and faster computations (Sect. 4.6).

1.5.C) A new conception of continuous and instantaneous interactions that, in view of the recent detection of entanglement at classical distances [36], is here suggested as a possible realization of the *fifth interaction* (see also Ref. [37]). Its lack of identification to date is easily explained by the fact that the EPR entanglement cannot be conceived, let alone treated via quantum mechanics.

1.6. Bell's inequalities

In the 1950's, J. S. Bell [38] proved a number of quantum inequalities, the first one of which essentially states that systems of (point like) particles with spin $1/2$ represented via quantum mechanics do not admit classical counterparts. This view was assumed by mainstream physicists for over half a century to be the final disproof of the EPR argument, and to establish the validity of Heisenberg's uncertainty principle for all possible particle conditions existing in the universe.

Again thanks to the prior development of isomathematics, R. M. Santilli [39] proved in 1998 a number of hadronic inequalities essentially stating that *systems of extended particles with spin $1/2$ represented via the Lie-isotopic branch of hadronic mechanics do indeed admit classical counterparts*, while providing explicit examples. Santilli's hadronic inequalities are confirmed by experiments [40] [41] [42] establishing the existence in nature of particle conditions which violate Bell's inequalities (see also experiments [7]-[19] of Paper I).

A deeper understanding of this paper requires the knowledge that the theoretical [39] and experimental works [40] [41] [42] disproving Bell's inequalities imply the expectation that *Heisenberg's uncertainties principle is correspondingly violated by strong interactions between extended nucleons in conditions of mutual penetration*.

1.7. Einstein's isodeterminism

Soon after joining the faculty of Harvard University under DOE support in late 1977, R. M. Santilli expressed doubts on the exact validity for strong interactions of Heisenberg's uncertainty principle and other basic quantum mechanical laws, as one can see from the titles of the 1978 introductory memoir on hadronic mechanics [43] (see also the subsequent papers [44] [45]). The argument underlying such a conviction is that the standard deviations for coordinates Δr , momenta Δp and their product are certainly valid for the conditions of their original conception, i.e., for *point-like charged particles under electromagnetic interactions*, because a point-like particle can move within a star by solely sensing action-at-a-distance

interactions due to their dimensionless character.

The situation is conceptually, mathematically, theoretically and experimentally different when considering *extended* nucleons in conditions of *mutual penetration* because, in view of their 'strength', their strong interactions imply the emergence of a *pressure* created on a given nucleon by its surrounding nucleons (Fig. 3) according to a view pioneered by L. de Broglie and D. Bohm with their nonlocal theory [14]. It is then evident that the standard deviations for the indicated nucleon Δr and Δp *cannot* be the same as the corresponding standard deviation for an electron in vacuum, thus implying the need for a suitable completion of Heisenberg's uncertainty principle for strong interactions.

Thanks to works written at Harvard University in the late 1970's [43] [44] and the recent works [45] [34] [39], R. M. Santilli [46] finally achieved the axiom-preserving, EPR completion of Heisenberg's uncertainty principle for point-like particle under electromagnetic interactions into the *isouncertainty principle for extended nucleons under electromagnetic, weak and strong interactions* collectively called *Einstein's isouncertainties* (see Sect. 4.7 for its formulation), which new principle essentially states that Einstein's determinism [1] is progressively recovered with the increase of the density in the interior of hadrons, nuclei and stars and fully recovered at the limit of Schwarzschild's horizon.

As known to experts, despite one century of studies under large public funds, nuclear physics has been unable to achieve exact representations of nuclear data, the synthesis of the neutron from the Hydrogen atom in the core of stars, nuclear stability despite the natural instability of the neutron and extremely repulsive protonic Coulomb forces and other open problems.

A main point which is attempted to convey in this and in the next paper is that the indicated open nuclear problems appear to be due to the *theoretical assumption* that Heisenberg's uncertainty principle for point-like particles under electromagnetic interactions is also valid for extended nucleons under strong interactions. However, in view of the lack of any direct experimental verification, said assumption is a personal view by the individual physicist, while the indicated open problems deal with clear physical evidence, by therefore suggesting the need for collegial mathematical, theoretical and experimental studies on the uncertainties of nucleons in a nuclear structure as a necessary premise for advances in the recycling of radioactive nuclear waste, controlled nuclear fusions and other clear societal needs.

1.8. Bohm hidden variables

From the preceding outline one can see that Bohm's hidden variables [48] [49] are *hidden in the associative axiom of quantum mechanics* with generic realization in terms of the Santillian $\lambda = T$, $A \hat{\times} B = A \lambda B$, $A \hat{\times} (B \hat{\times} C) = (A \hat{\times} B) \hat{\times} C$.

It should be noted that, despite its apparent elementary character, the quantita-

tive study of the indicated realization of Bohm's hidden variables required nonlocal lifting of 20-th century applied mathematics, including the Newton-Leibnitz differential calculus [50] (see also studies [29]). A preliminary realization of Bohm's hidden variables has been presented in Sect. 4.C.3, p. 170 on and Sect. 6.8, p. 254 on of [31] (see Sect. 5.5 for details). Additional explicit and concrete realizations of Bohm's hidden variables have been presented in Refs. [51] [52].

The deterministic interpretation of Bohm's hidden variables resulted to be progressive [46] because dependent on the density of a composite strongly interacting system represented by the isotopic element $\hat{T}$ which becomes null for gravitational collapse under which nucleons are essentially frozen in their position with identically null standard deviations $\Delta r \equiv 0$, $\Delta p \equiv 0$ due to the immense surrounding pressures. (see Sect. 4.7 for details).

1.9. Simple construction of hadronic models

On pedagogical grounds, the main objectives of this paper are:

1.A) Show that quantum mechanical models for point-like nucleons with sole Hamiltonian interactions can be mapped via simple *nonunitary transformations* into covering hadronic models for extended, deformable and hyperdense nucleons under Hamiltonian and non-Hamiltonian interactions.

1.1.B) Show that, when formulated on quantum mechanical spaces over conventional fields, said nonunitary models are afflicted by serious consistency problems, such as the lack of conservation of the Hermiticity/observability, lack of prediction of the same numerical values at different times, loss of causality and others (Sect. 3.11).

1.C) Show that, the indicated inconsistencies are resolved when said quantum mechanical nonunitary models are reformulated into hadronic models on isospaces over isofields (3.12).

In this paper, we shall: review the main axiomatic structure of quantum mechanics; outline and upgrade the foundations of the axiom-preserving isomathematics and isomechanics; and then show that the resulting Lie-isotopic methods, including above all Einstein's isodeterminism presented in Sect. 4.7 [46], allow the first known numerically exact and time invariant representation of the Deuteron experimental data in a form expectedly extendable to other stable nuclei.

2 Axiomatic foundations of quantum mechanics

In this section, we recall a few axiomatic aspects of quantum mechanics that are relevant later on for comparison with their corresponding axiom-preserving completion.

As it is well known (see, e.g., Refs. [53] [54] [55], nuclear physics has

been based since its inception on the *universal enveloping associative algebra* $\xi : \{A, B, \dots; AB, = A \times B; 1\}$ of Hermitean operators $A, B, \dots$ on a Hilbert space $\mathcal{H}$ with states $|\psi(r)\rangle$ over the field of complex numbers $\mathcal{C}$ with conventional associative product

$$AB = A \times B, \quad (1)$$

and unit

$$1 \times A = A \times 1 = A \quad \forall A \in \xi. \quad (2)$$

In fact, the above basic axioms are sufficient to characterize, uniquely and unambiguously, all quantum mechanical methods used in nuclear physics including:

2.1) The representation of stable nuclei via a primitive, k -dimensional space-time or internal Lie symmetry with anti-symmetric Lie algebra attached to the enveloping algebra $L \equiv \xi^-$ with commutation rules for Hermitean generators $X_k = X_k^\dagger$, $k = 1, 2, \dots, N$

$$[X_i, X_j] = X_i X_j - X_j X_i = C_{ij}^k X_k, \quad (3)$$

including the rotational symmetry $SO(3)$, the spin symmetry $SU(2)$, the Lorentz symmetry $SO(3,1)$, the Lorentz-Poincaré symmetry $P(3,1)$, the spinorial covering of the Poincaré symmetry $\mathcal{P}(3,1)$, and others.

2.2) The infinite class of equivalent nuclear models under the unitary law

$$UU^\dagger = U^\dagger U = I = \text{Diag}_{k \times k}(1, 1, \dots, 1). \quad (4)$$

2.3) *Heisenberg's representation* of the time evolution of a Hermitean operator A in terms of the Hamiltonian H here expressed in the infinitesimal and its unitary finite form

$$i \frac{dA}{dt} = [A, H] = A \times H - H \times A,$$

$$A(t) = e^{H \times t \times i} \times A(0) \times e^{-i \times t \times H} = W(t) A(0) W^\dagger(t), \quad (5)$$

$$WW^\dagger = W^\dagger W = I.$$

2.4) The unitarily equivalent *Schrödinger representation* of the time evolution of A on a Hilbert space $\mathcal{H}$ with states $|\psi(r)\rangle$ over the field of complex numbers $\mathcal{C}$

$$\langle \psi(r) | \psi(r) \rangle = I,$$

$$< \langle \psi(r) | H | \psi(r) \rangle = < \langle \psi(r) | H | \psi(r) \rangle^*,$$

$$H \times |\psi(r)\rangle = \left[\frac{1}{2m} \delta^{ij} p_i \times p_j + V(r) \right] \times |\psi(r)\rangle = E \times |\psi(r)\rangle, \quad (6)$$

$$p \times |\psi(r)\rangle = -i \partial_r |\psi(r)\rangle,$$

$$[r_i, p_j] \times |\psi(r)\rangle = -i \delta_{ij}, \quad [r_i, r_j] \times |\psi(r)\rangle = [p_i, p_j] \times |\psi(r)\rangle = 0.$$

2.5) The important prediction of the same numerical values under the same conditions at different times (hereon called time invariance), which is evidently due to the unitary structure of time evolutions (5) (6) with ensuing invariant of numeric values, e.g.,

$$1 \rightarrow 1' = U1U \equiv 1. \quad (7)$$

2.6) The *perturbation theory* based, e.g., on power series

$$A(t) = A(0) + w \times \frac{[A, H]}{1!} + w^2 \times \frac{[[A, H], H]}{2!} + \dots \quad (8)$$

2.7) The remaining well known methods used for a century in the quantum mechanical study of nuclear physics.

3 Elements of isomathematics

3.1. Historical notes

On September 8, 1977, R. M. Santilli joined the Lyman Laboratory of Physics of Harvard University with a research contract from the Department of Energy (DOE) to search for possible new clean forms of nuclear energies. To conduct this task, Santilli requested authorization from the DOE to initiate the research with the study of the most fundamental fusion in nature, that of the neutron from the Hydrogen atom in the core of stars [56], (the inability by quantum mechanics to provide any quantitative representation of the neutron synthesis despite the extremely big $e - p$ Coulomb attraction) was identified as part of this initial study.

The origin of the insufficiency was identified with the need to represent the charge distributions of the proton as being *extended*. The admission of the extended character of particles then implied Insufficiency I.2.1 (the expectation at short mutual distances of non-Hamiltonian interactions 1.3, Fig. I.1), which expectation is essentially implied by Rutherford's "compression" [56] of the electron within the hyperdense proton. The analysis essentially confirmed the EPR argument [1] because a quantitative representation of the neutron synthesis was beyond any hope of achievement via quantum mechanics, thus leaving no other possibility than that of constructing a suitable completion.

Since the task was of primary *mathematical* nature, Santilli was moved in 1979 from the Lyman Laboratory of Physics to the Department of Mathematics of Harvard University (see, e.g., paper [17]) where he proposed the completion of quantum mechanics into *hadronic mechanics* [16] (to indicate its intended use for strong interaction), under the challenging intent of maintaining the abstract axioms of quantum mechanics despite the admission of non-Hamiltonian interactions.

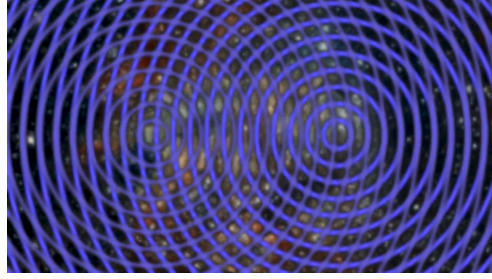


Figure 1: *In this figure we illustrate the notion at the foundation of the studies presented in these papers, which is given by the "Einstein-Podolsky-Rosen (EPR) entanglement of particles" when interpreted as being due to the continuous and instantaneous overlapping of the wave packets of particles at arbitrary mutual distance [34], with resulting nonlinear, nonlocal and nonpotential interactions whose representation required the construction of the new isomathematics, isomechanics and isosymmetries (see also the overview [3]).*

3.2. The birth of isomathematics

As it is well known, Lie's theory (see, e.g., Ref. [57] and works quoted therein) has acquired a justly historical value for the representation of *point-like particles under sole Hamiltonian interactions*, such as the representation of the spin $S = 1/2$ of atomic electrons via Pauli's matrices, and the consequential exact representation of the magnetic moment of the electron via the giromagnetic factor, $\mu = gS$.

By contrast, Lie's theory appears to be excessively restrictive in nuclear physics due to its inability to represent the *extended* character of nucleons and their expected *non-Hamiltonian* interactions, with consequential insufficiencies studied in Paper I. At the same time, the numeric value $1/2$ of the spin of nucleons is beyond scientific doubt. Hence, the problem addressed by Santilli in the late 1970's was that of identifying a completion of Lie's theory for the representation of extended particles under Hamiltonian and non-Hamiltonian interactions while preserving conventional spin values.

To achieve the intended needs, Santilli proposed in Eq. (5) p. 71 of the 1983 Springer-Verlag monograph [16] the completion (also called lifting) of the universal enveloping associative algebra $\xi\{A, B, \dots; AB = A \times B; I\}$ (Sect. 2) into the *associativity-preserving*, thus isotopic envelope $\hat{\xi}$ characterized by the new product called the *isoproduct* (first introduced in Eq. (5), p. 71, and then studied in details from Sect. 5.1, p. 148 and Sect. 5.2 of [16])

$$A \times B \rightarrow A \hat{\times} B = A \hat{T} B, \quad \hat{T} > 0, \quad (9)$$

with related new unit, called the *isounit*

$$\hbar = 1 \rightarrow \hat{I} = 1/\hat{T}, \quad \hat{I} \hat{\times} A = A \hat{\times} I = A, \quad \forall A \in \hat{\xi}, \quad (10)$$

where the isotopic element $\hat{T}$, at times called the Santillian, is solely restricted by the condition of being positive-definite $\hat{T} > 0$, while admitting an arbitrary functional dependence on the relative coordinate r , momentum p , acceleration a , energy E , charge radius R , density d , pressure π , temperature τ wave function ψ , and any other needed local variable (Fig. 3),

$$\hat{T} = 1/\hat{I} = \hat{T}(r, p, a, E, R, d, \psi, \pi, \tau, \psi, \dots) > 0, \quad (11)$$

which dependence is hereon tacitly assumed.

The reader should be unformed that the generalization of the basic unit of quantum mechanics, $\hbar = 1$, into a nonlinear, nonlocal and nonpotential operator $\hat{I} = 1/\hat{T}$ intends to represent the possible ack of quantized energy exchanges within a nuclear structure in favor of rather complex integro-differential energy exchanges, as expected for extended nucleons in conditions of mutual penetration (see Sect. 3.12 for more comments, experimental works [47] and papers quoted therein).

3.3. The Lie-isotopic theory

Despite its simplicity, isotopic completion (9) of associative product (1) has required, for consistency, a compatible isotopic completion of the totality of 20th century applied mathematics (with no exception known to the author) into the novel *isomathematics* [15] [16] (see monographs for physicists [28] [30] and those for mathematicians [26] [29]) whose knowledge is essential to avoid structural inconsistencies in isotopic nuclear models that may remain unknown to non-experts in the field.

To begin a brief outline of the isomathematics implied by basic axiomatic assumptions (9) and (10), in Chapter 5, p. 148 on of [16] Santilli constructed the step-by-step isotopies of the various branches of Lie's theory, including the isotopies of the universal enveloping associative algebras (Sect. 5.2), Lie algebras and Lie's Theorems (Sect. 5.3), Lie groups and the Lie transformation theory (Sect. 5.4), resulting in the *Lie-isotopic theory*, today known as the *Lie-Santilli isothory* [22] [58] [59], with generic *isocommutation rules* for a k -dimensional isoalgebra with iso-Hermitean generators

$$\begin{aligned} [J_i, J_j] &= J_i \hat{\times} J_j - J_j \hat{\times} J_i = J_i \hat{T} J_j - J_j \hat{T} J_i = \\ &= C_{ijk}(r, p, a, E, R, d, \psi, \pi, \tau, \psi, \dots) J_k. \end{aligned} \quad (12)$$

Lie-Santilli isoalgebras are nowadays classified into [60]: *regular isoalgebras* when the structure quantities C_{ij}^k are constant, and *irregular isoalgebras* when

said structure quantities are functions of local variables. As we shall see in Sect. 3.11, regular isoalgebras can be easily constructed via *nonunitary transformations* of the original Lie algebras, while irregular isoalgebras cannot, by therefore constituting new realizations of Lie's axioms which remain vastly unexplored to this day.

It should be noted since these introductory words that the above assumptions have permitted various explicit and concrete realizations of Bohm's *hidden variables* λ [48] (Sect. 1.8). Such a realization is known to be impossible for the Copenhagen interpretation of quantum mechanics, but it becomes possible for isomathematics because *Bohm's variable are hidden in the axiom of the associative product of quantum mechanics* [31]

$$\lambda = \hat{T}, \quad (13)$$

$$A \hat{\times} (B \hat{\times} C) = A \lambda (B \lambda C) = (A \hat{\times} B) \hat{\times} C = (A \lambda B) \lambda C.$$

The time invariant and numerically exact representation of nuclear data presented in Sect. 5 is a direct consequence of the capability of isomathematics (and, therefore, of isomechanics) to represent Bohm's hidden variables.

It should also be noted that isoproduct (9) assures that hadronic mechanics is outside the infinite class of unitary equivalence of quantum mechanics (because the new product $A \hat{T} B$ cannot be obtained via a unitary transformation of the original product AB). Despite that, isoproduct (9) does preserve the "majestic axiomatic structure of quantum mechanics" indicated in Sect. I.1.

3.4. Representation of extended particles and their non-Hamiltonian interactions

The main objective of the isotoopic element $\hat{T} > 0$ is that of resolving axiomatic insufficiencies I-3.1, I-3.2 and I-3.3 of quantum mechanics in nuclear physics, (namely, its linear, local and potential character) via its representation of nonlinear, nonlocal and nonpotential/NSA interactions among extended nucleons.

The representation was initiated by Newton's historical conception of velocity-dependent resistive, thus nonpotential forces which were then represented by Lagrange and Hamilton via the external terms in their analytic equations.

As indicated in Sect. 1, particle interactions nonlinear in the wave function but represented with a Hamiltonian were initiated by W. Heisenberg [13]; interactions occurring in volumes not reducible to points but also represented with a Hamiltonian were initiated by L. de Broglie and D. Bohm [14]; nonlinear and nonlocal interactions not representable with a Hamiltonian were initiated by R. M. Santilli in the 1978 Harvard University monograph [15] via their technical characterization as being variationally non selfadjoint (NSA).

Exact Lie-isotopic representation of the Deuteron data

The first known application of NSA interactions in particle physics occurred in Santilli's 1978 memoir [43] via an isotopic element of the simple type

$$\hat{T} = e^{-\frac{\hat{F}(r)}{F(r)}}, \quad \hat{F}(r) > 0, \quad F(r) > 0, \quad (14)$$

where $\hat{F}(r)$ ($F(r)$) are positive-definite, generally nonlinear and nonlocal functions of relative coordinates r representing NSA (SA) forces in the *Schrödinger-Santilli isoequation* $H \hat{\times} |\hat{\psi}\rangle = H \hat{T} |\hat{\psi}\rangle = E |\hat{\psi}\rangle$ (see Sect. 4.3 for the full representation).

Despite its simplicity, isotopic element (14) allowed the first known representation of *all* characteristics of the π^0 meson, the $\mu^\pm$ lepton and other particles (including the mechanism for their spontaneous decay and the representation of their mean lives) via hadronic structure equations of physical constituents, electrons and positrons, produced free in the spontaneous decay with the lowest modes (Sect. 5.1, p. 827 on of Ref. [43]).

The isotopic element for the joint representation the extended character of particles and their non-Hamiltonian/NSA interactions adopted in these papers is given by the (contravariant) form [3]

$$\hat{T} = \hat{T}_{4 \times 4} = \Pi_{\alpha=1,2,3,4} \begin{pmatrix} \frac{1}{n_{1,\alpha}^2} & 0 & 0 & 0 \\ 0 & \frac{1}{n_{2,\alpha}^2} & 0 & 0 \\ 0 & 0 & \frac{1}{n_{3,\alpha}^2} & 0 \\ 0 & 0 & 0 & \frac{1}{n_{4,\alpha}^2} \end{pmatrix} \times e^{-\Gamma(r,p,a,R,d,\psi,\hat{\psi},\dots)} > 0, \\ n_{\mu,\alpha} > 0, \quad \Gamma > 0, \quad (15)$$

where

3.4.A) The representation of the dimension and shape of the particle is done via semi-axes $n_{k,\alpha}^2$, $k = 1, 2, 3$ (with n_3 parallel to the spin) with normalization for the vacuum $n_{k,\alpha}^2 = 1$ (Fig. 2).

3.4.B) The representation of the density is done via the characteristic quantity $n_{4,\alpha}^2$ per individual nucleons with normalization for the vacuum $n_{4,\alpha}^2 = 1$.

3.4.C) The representation of the nonlinear, nonlocal and nonpotential interactions between extended particle is done via the exponential term $e^{\Gamma(\psi,\dots)}$ generally realized in terms of volume integrals, as we shall see in Sects. 3 and 4.

3.4.D) When representing nucleons and their NSA interactions, the dimension of the ellipsoids is restricted not to surpass the range of strong interactions $R = 1 \text{ fm} = 10^{-13} \text{ cm}$.

3.4.E) When considering fifth interactions (Sect. 1.5.C), the dimension of the particle is generally assumed to be infinite so that isointegrals [30] [29] yield measurable predictions.

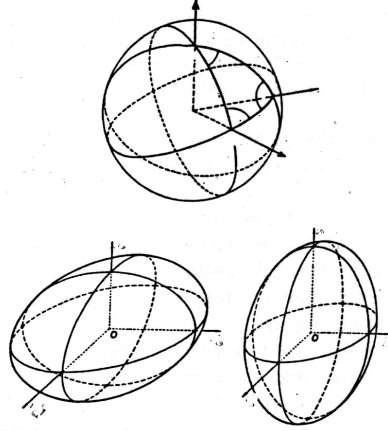


Figure 2: In this figure, we illustrate the second fundamental notion of these studies, the implementation of the nonlocality advocated by Einstein-Podolsky-Rosen [1] via the representation of protons and neutrons (nucleons) as being extended, thus deformable charge distribution that, as such, can be prolate or oblate ellipsoids [3].

The reader should keep in mind that the isotopic element for *covariant* nuclear models is the *inverse* of contravariant isotopic element (15), thus have a positive value in the exponent.

3.5. Isonumber theory

The formulation of experimental values of isotopic nuclear models over the conventional field of real numbers $\mathcal{R}(n, \times, 1)$ implies the loss of their invariance over time because the conventional unit 1 of the theory is no longer invariant under the nonunitary transformations of hadronic mechanics, $1 \rightarrow 1' = W1W^\dagger \neq 1$, with the consequential loss of the base field over time and the invalidation of experimental measurements.

In order to maintain the crucial time invariance of isotopic numeric predictions, it has been necessary to complete the conventional number theory into a form applicable for arbitrary, positive-definite isounits $\hat{I} > 0$. This completion was achieved by Santilli in the 1993 memoir [61] (see independent studies [25] [62]) via the notion of *isofields* $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$ for which the infinite family of rings with elements $\hat{n}$, called *isoreal*, *isocomplex* or *isoquaternionic isonumbers*, characterized by the ordinary product of conventional numbers n times a given isounit $\hat{I}$

$$\hat{n} = n \times \hat{I}, \quad (16)$$

when equipped with isoproduct (9)

$$\hat{n} \hat{\times} \hat{m} = (n\hat{I})\hat{T}(m\hat{I}) = (nm)\hat{I} \in \hat{F}, \quad (17)$$

Exact Lie-isotopic representation of the Deuteron data

verify all axioms of a numeric field, thus being fully acceptable for experimental values.

Predictably, *all* conventional operations with numbers should be replaced, for consistency, with the compatible operations for isonumbers, such as (see [30] for details)

$$\hat{n}^2 = \hat{n} \hat{\times} \hat{n} = (n^2) \hat{I}, \quad \hat{n} / \hat{m} = \hat{n} (/ \hat{I}) \hat{m}, \quad \hat{n}^{1/2} = n^{1/2} \hat{I}, \text{ etc.} \quad (18)$$

In essence, Santilli discovered that *the abstract axioms of a numeric field do not necessarily require that the multiplicative unit be the millenary unit 1* since said axioms merely require that the multiplicative unit be invertible and it is here assumed to be positive definite for isotopic theories.

It should be noted that the primary difficulty in using isonumbers in nuclear physics is of *conceptual*, rather than of mathematical nature because from realizations (15), *isonumbers represent volumes*, as expected for the needed representation of extended nucleons at all levels of study, thus including the number theory. For example, the isoreal isonumbers for the nonrelativistic representation of one nucleon are given by the following *volumes* on isospaces over isofields which, when projected into the Euclidean space, acquire the meaning of volumes (Fig. 2)

$$\hat{n} = n \times \text{Diag.}(n_1^2, n_2^2, n_3^2), \quad (19)$$

although, on grounds of abstract axioms, there is no difference between n and $\hat{n}$ due to the positive-definiteness of the isounit.

The isonumbers for a nucleus with A nucleons are given by the following collection of overlapping volumes

$$\hat{n} = n \times \Pi_{\alpha=1,2,\dots,A} \text{Diag.}(n_{1,\alpha}^2, n_{2,\alpha}^2, n_{3,\alpha}^2). \quad (20)$$

It should also be noted that the implications of Santilli's isonumber theory are not trivial. As an illustration, consider the isoreal isofield $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$ for which $\hat{I} = 3 \in F$. In this case $2 \hat{\times} 3 = 2$ and

$$\hat{1} = 3, \quad \hat{2} = 6, \quad \hat{3} = 9, \quad \hat{5} = 15, \quad \hat{7} = 21, \quad \text{etc.}, \quad (21)$$

by therefore suggesting that *the numeric values of prime numbers depend on the selected multiplicative unit*.

The above reformulation of prime numbers is not trivial because it has permitted the development and use of *isocryptograms*, (Appendix 2C, p. 84 of Ref. [30]) consisting of conventional cryptograms reformulated via isomathematics with a change of the unit such to require infinite time for the solution by hackers (see also Ref. [63]).

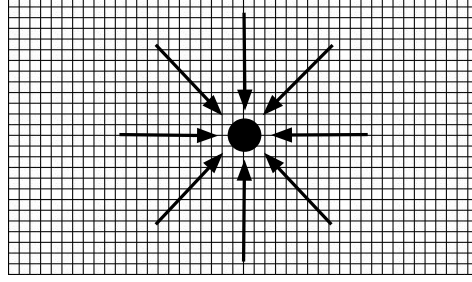


Figure 3: In this figure, we illustrate a third basic notion of these studies, which is given by the representation in the nuclear force of pressure, density, temperature and other measurable quantities for nuclei conceived as physical media (generally called 'hadronic media') [31].

3.6. Isofunction analysis

In quantum mechanics, all scalar quantities must have values in an ordinary field F . Corresponding quantities of isotopic theories must be *isoscalar*, that is, they must have values in the isofield $\hat{F}$. This means that all functions used in 20th century physics must be replaced by suitable *isofunctions* whose study was initiated with the *isoexponential isofunction* (Eq. (55), p. 171, Ref.[16])

$$\hat{e}^X = \hat{I} + X + X \hat{\times} X / \hat{1}! + \dots = (e^{X\hat{T}})\hat{I} = \hat{I}(e^{\hat{T}X}). \quad (22)$$

Consequently from definition (16), ordinary coordinates $r = (x, r, z)$ representing a point of the 3-dimensional Euclidean space are mapped into *iso-coordinates* representing volumes

$$\hat{r} = r \times \hat{I} = r \times \text{Diag}(n_1^2, n_2^2, n_3^2). \quad (23)$$

This implies that Newtonian massive points r are replaced by *massive volumes* $\hat{r}$ (Fig. 2).

Similarly, functions $f(r)$ of local coordinates r must be completed into isofunctions $\hat{f}$ of isocoordinates $\hat{r}$, and in addition, have values in $\hat{F}$, thus suggesting the structure

$$\hat{f}(\hat{r}) = [f(\hat{r})]\hat{I}. \quad (24)$$

Interested readers should be aware that the elaboration of isotopic nuclear models studied in these papers requires the use of *isotrigonometric*, *isohyperbolic* and other *isofunctions* that cannot be reviewed here for brevity (see monographs [30] [31] and Refs. [64]-[66]).



Figure 4: This picture illustrates the isodifferential calculus [50] with birds flying in close formation without wing interferences. This flying formation can be best understood by assuming that birds conceive themselves as a volume encompassing their wings, rather than a mass concentrated in their center of gravity as it would be requested by the Newton-Leibnitz differential calculus [50]

3.7. Isodifferential calculus

Following the construction of all possible isotopies of 20th century mathematics reviewed in the preceding sections, in the early 1990's hadronic mechanics was still missing a consistent isotopic formulation of the linear momentum, since the sole available formulation at that time was the familiar form from Eq. (6) $p|\psi\rangle = -i\partial_r|\psi\rangle$. This insufficiency implied the inability, at that time, of formulating the isotopies of the Schrödinger equations in a way compatible with all other aspects of isomathematics, with ensuing inability to formulate meaningful experiments (Fig. 4).

Studies of this impasse revealed that its ultimate origin rested in the conventional differential calculus due to its sole definition at isolated points, while the representation of the extended character of particles was needed for consistency at all levels, including the differential calculus.

The above insufficiencies left no other option than that of constructing the completion of the Newton-Leibnitz differential calculus, from its historical definition at isolated points r , to a formulation defined over *volumes* $\hat{r} = r\hat{I}$, which completion was finally achieved by Santilli in the 1996 special issue of Springer Nature *Rendiconti* [50] via the following:

Isoaxiom 3.7.1: *The isodifferential $\hat{d}$ of an isocoordinate $\hat{r} = r\hat{I}$ includes all possible completions $\hat{d}\hat{r}$ of the conventional differential dr under the sole condition of recovering the latter at the limit when the isounit $\hat{I}$ recovers the conventional unit 1*

$$\text{Lim}_{\hat{I}=1} \hat{d}\hat{r} = dr, \quad (25)$$

with solution for the isodifferential [50]

$$\hat{d}\hat{r} = \hat{T}d\hat{r} = \hat{T}d[r\hat{I}(r, \dots)] = dr + r\hat{T}d\hat{I}, \quad (26)$$

and consequential solution for the isoderivative

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \quad (27)$$

The discovery of the isodifferential calculus finally allowed the formulation of the isolinear isomomentum with the consequential consistent isotopies of Schrödinger equations presented in next section (for systematic studies on the isodifferential calculus, see the monumental works by the mathematician, S. Georgiev [29]).

3.8. Isospaces

Let $E(r^k, \delta_{ij}, I)$, $i, j = 1, 2, 3$ be the three-dimensional Euclidean spaces in the (contravariant) coordinates r^k , over the field of real numbers $\mathcal{R}$ with invariant $r^2 = r^i \delta_{ij} r^j = r_1^2 + r_2^2 + r_3^2$. The infinite family of isotopic images of $E(r^k, \delta_{ij}, I)$ are given by the *iso-Euclidean isospaces* [26] [31]) $\hat{E}(\hat{r}^k, \hat{\Delta}_{ij}, \hat{I})$ characterized by the contravariant isotopic element

$$\hat{T} = \text{Diag.}(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}), \quad (28)$$

in which the exponential term of Eq. (15) has been imbedded in the n -characteristic quantities, with isocoordinates $\hat{r}^k = r^k \hat{I}$, *isometrics*

$$\hat{\Delta}^{ij} = \hat{\delta}^{ij} \hat{I}_{cr} = (\hat{T}_k^i \delta^{kj}) \hat{I}, \quad (29)$$

and *isoinvariant*

$$\hat{r}^2 = \hat{r}^i \hat{\Delta}_{ij} \hat{r}^j = (r^i \delta_{ij} r^j) \hat{I}_{cr} = (\frac{r_1^2}{n_1^2} + \frac{r_2^2}{n_2^2} + \frac{r_3^2}{n_3^2}) \hat{I}, \quad (30)$$

where one should note that $\hat{\Delta} = \hat{\delta} \hat{I}$ is an *iso-metric*, namely, a metric whose elements are iso-numbers. Note also that the covariant isotopic element is the inverse of the contravariant expression (28).

Let $M(x, \eta, I)$ be the conventional Minkowski space over the field of real numbers $\mathcal{R}$ with contravariant spacetime coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = \text{Diag.}(1, 1, 1, -1)$, unit $I = \text{Diag.}(1, 1, 1, 1)$ and invariants $x^2 = x^\mu \eta_{\mu\nu} x^\nu$ and $p^2 = p_\mu \eta^{\mu\nu} p_\nu$. The infinite family of *iso-Minkowskian isospaces* [21] [31] $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ are characterized by the infinite family of isotopic elements

$$\hat{T} = 1/\hat{I} = \text{Diag.}(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2}), \quad (31)$$

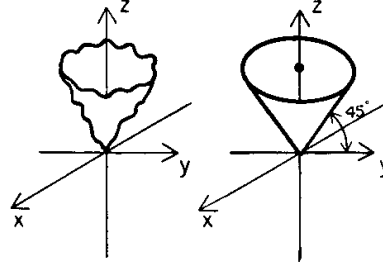


Figure 5: In this figure, we illustrate a basic notion of the isotopies of the Lorentz symmetry (Sect. 3.10.1), that of providing the invariance of Lorentz's locally varying speed of light $C = c/n$ within physical media with density $n = n_4$ (left view), while reconstructing the universal invariance of the constant speed c on isospaces over isofields (right view) [3].

isocoordinates $\hat{x} = x\hat{I}$, isometrics

$$\hat{\Gamma} = \hat{\eta}\hat{I} = (\hat{T}\eta)\hat{I}, \quad (32)$$

and iso-invariants

$$\hat{x}^2 = \hat{x}^\mu \hat{\times} \hat{\Gamma}_{\mu\nu} \hat{\times} \hat{x}^\nu = \left(\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - \frac{x_4^2}{n_4^2} \right) \hat{I}. \quad (33)$$

3.9. Isogeometries

Euclidean, Minkowskian, Riemannian, symplectic and other 20th century geometries are axiomatically based on the conventional associative product and their values are formulated over numeric fields. It is then evident to see that the lifting of the associative product into its isotopic form (9), with compatible lifting of number theory and functional analysis, implies non-trivial isotopies of 20th century geometries we cannot possibly review here (see Refs. [30] [31], independent studies [24], the recent monograph in iso-Euclidean geometry [67] and papers quoted therein).

For future needs, we limit ourselves to recall that the isometric (32) of the Iso-Minkowskian isospace $\hat{M}$ coincides with the Riemannian metric $\omega(x)$, as a result of which:

3.9.1) The iso-Minkowskian geometry admits the complete machinery of the Riemannian geometry although expressed in terms of the isodifferential calculus [68].

3.9.2) The Riemannian metric $\omega(x)$ admits the factorization into the gravitational isotopic element (Santillian) here illustrated with the particular case of the

Schwarzschild's isotopic element [69]

$$\begin{aligned}\omega(x)_{Riem} &= \hat{T}_{Sant} \times \eta_{Mink}, \quad \hat{T}_{Sant} > 0, \\ \hat{T}_{Schwar} &= \text{Diag.}(\frac{1}{(1-\frac{2M}{r})}, \frac{1}{(1-\frac{2M}{r})}, \frac{1}{(1-\frac{2M}{r})}, 1 - \frac{2M}{r}),\end{aligned}\tag{34}$$

where the positive definiteness $T_{Sant} > 0$ is guaranteed by the factorization of the Minkowski metric.

3.9.3) Despite its dependence on iso-coordinates, the iso-Minkowskian geometry is *isoflat*, that is, with null curvature when formulated on isospaces over isofields, because the lifting of the Minkowski metric $\eta \rightarrow \omega(x) = T(x) \times \eta$ is compensated by the *inverse* lifting of the Riemannian unit $I_{4 \times 4} \rightarrow \hat{I}_{Sant} = 1/\hat{T}_{Sant}$.

3.9.4) Relativistic isomechanics characterized by the isounit $\hat{I} = 1/\hat{T}_{Sant}$ provides an axiomatically consistent unification of Einstein's field equations and quantum mechanical axioms. [70].

3.9.5) In view of the absence of curvature, the above identical reformulation of Einstein's field equations provides an axiomatically consistent unification of gravitation and electroweak interactions [70] studied in detail in the Springer Nature monograph [71].

3.10. Isosymmetries

The main methods used in *nonrelativistic hadronic mechanics* are given by the *isotopies of the Galilean symmetry and relativity* first presented in Chapter 6, p. 199 on of monograph [16] with various applications in pages 253-267. Said isotopies were then studied in the 1991 monographs [72] [73] (see also notes [21] from Santilli's 1991 lectures at the ICTP, Trieste, Italy, and review [28]).

The main conceptual and technical difference between the conventional Galileo symmetry and its isotopic image is that the former symmetry is based on the Newtonian notion of *massive points*, as a consequence of which no resistance can be admitted during the free fall in our atmosphere. By contrast, the iso-Galilean symmetry represents the dynamics of *massive volumes* moving within physical media with ensuing resistive and other NSA forces represented via the isounit of the theory in such a way to recover the conventional Galilean symmetry at the limit $\hat{I} \rightarrow I$.

Regrettably, we cannot review these Galilean studies to prevent excessive length of this paper and refer the interested reader to monographs [72] [73] and independent reviews [21] [28].

The main methods used for *relativistic hadronic mechanics* are given by the *isotopies of spacetime symmetries* which include: 1) The isotopies $\hat{SO}(3)$ of the rotational symmetry [74]-[76]; 2) The isotopies $\hat{SU}(2)$ of the spin symmetry

$SU(2)$ [77]; 3) The isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ in classical [78] and operator [79] formulation; 4) The isotopies $\hat{P}(3.1)$ of the Lorentz-Poincaré symmetry $P(3.1)$ [80]; 5) The isotopies $\hat{\mathcal{P}}(3.1)$ of the spinorial symmetry $\hat{P}(3.1)$ [81]; 6) The isotopies of the various discrete spacetime symmetries [31]; 7) The isotopies $\hat{M}(3.1)$ of the Minkowskian geometry $M(3.1)$ [82].

To render this paper minimally self-sufficient, let us outline the isotopies of the Lorentz and Poincaré symmetries which are needed for the representation of the Deuteron data of Section 5.

3.10.1. Iso-Lorentz symmetry. As it is well known to historians, H. Lorentz attempted in 1904 the construction of the invariance of the speed of light within physical media $C = c/n$ where n is the local density of the medium considered, but failed to do so and had to restrict his study to the invariance of the constant speed of light *in vacuum* c . The latter symmetry was characterized by the unitary irreducible representations of the Lie group $SO(3.1)$, which are the justly celebrated *Lorentz transformations*. It is unfortunate for science that Lorentz's crucial words *in vacuum* were generally ignored in the 20th century physics, resulting in the widespread belief that the speed of light is a universal constant c throughout all possible conditions existing in the universe. A list of inconsistencies for such an assumption is available in Sect. 8.4.4., p. 134 of the overview [3] and a decade of experiments in the U.S.A. and in Europe establishing Lorentz's local value $C = c/n$ of the speed of light in a physical medium, such as our atmosphere, are presented in Ref. [?] and papers quoted therein.

Santilli pointed out in the 1983 Springer-Verlag monograph that the restriction of the invariance to the constant speed c was due to the Hamiltonian character of Lie's theory, while Lorentz's problem was highly non-Hamiltonian due to the complex functional dependence (11) of the index of refraction n . For this reason, Santilli constructed in Ref. [16] the Lie-isotopic completion of Lie's theory for non-Hamiltonian systems, and achieved in the 1983 paper [78] the universal invariance of Lorentz's speed of light (34), which is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli isosymmetry $\hat{SO}(3.1)$ of isoinvariant (33), today known as the *Lorentz-Santilli isosymmetry*. It's isoalgebra $\hat{so}(3.1)$ is characterized by the isogenerators (Eq. (10), p. 550 of [78] expressed in terms of the characteristic quantities $b_\mu = 1/n_\mu$)

$$\begin{aligned} \hat{J}_1 &= n_2 n_3 J_1, \quad \hat{J}_2 = n_1 n_3 J_2, \quad \hat{J}_3 = n_1 n_2 J_3, \\ \hat{M}_k &= n_k M_k, \end{aligned} \tag{35}$$

where J_k, M_k are the conventional generators of $so(3.1)$, with isocommutation

rules (Eq. (12), p. 551 of [78])

$$\begin{aligned} [\hat{J}_i, \hat{J}_j] &= \epsilon_{ijk} \hat{J}_k, \quad [\hat{M}_i, \hat{M}_j] = C^2 \epsilon_{ijk} \hat{J}_k, \\ [\hat{J}_i, \hat{M}_j] &= -\epsilon_{ijk} \hat{M}_k, \end{aligned} \quad (36)$$

and *iso-Casimir isoinvariant* (Eq. (13), p. 551 of Ref. [78])

$$\hat{C}_1 = \hat{J}^2 - \frac{1}{C} \hat{M}^2 = -3\hat{I}, \quad \hat{C}_2 = \hat{\mathbf{J}} \hat{\times} \hat{\mathbf{M}} = 0. \quad (37)$$

A main feature of the Lorentz-Santilli isosymmetry at the abstract level is that of being identical to the conventional Lorentz transformations when formulated on an iso-Minkowskian isospace $\hat{M}(\hat{x}, \hat{\Delta}, \hat{I})$ over the isoreal isofield $\hat{\mathcal{R}}$, in which case the speed of light is indeed the universal constant c (Fig. 5). The invariance of the Lorentz speed $C = c/n$ occurs in the *projection* of the isosymmetry on the conventional Minkowski space $M(x, \eta, I)$ over the field of real numbers $\mathcal{C}$, e.g., for the case of the *isoboosts* (Eq. (15), p. 551 of Ref. [78] for isotransformations in the $(3, 4)$ -plane, and Chapter 8, p.329 on of Ref. [31] for a general treatment)

$$\begin{aligned} x^{1'} &= x^1, \quad x^{2'} = x^2, \\ x^{3'} &= \hat{\gamma}(x^3 - \hat{\beta}_{\frac{n_3}{n_4}} x^4), \quad x^{4'} = \hat{\gamma}(x^4 - \hat{\beta}_{\frac{n_4}{n_3}} x^3), \\ C &= \frac{c}{n_4} = \text{invariant}, \end{aligned} \quad (38)$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \quad \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}. \quad (39)$$

It should be indicated that the Lorentz-Santilli isosymmetry leaves invariant the local speed of light $C = c/n_4$, with ensuing wiggly-shaped light cone (left of Fig. 5), only when the isosymmetry is *projected* into the conventional spacetime over the conventional field of real numbers, as it is the case for isotransformations (38). By contrast, the isosymmetry $\hat{SO}(3.1)$ restores the perfect light cone in the isospacetime over the isoreal isofield (right of Fig. 5). We can therefore say that the speed of light c is indeed a *universal constant*, but only on isospacetime over the isoreal isofield, while it is a local variable in the physical reality.

It should be indicated that *isomathematics introduces no restrictions on the density of the medium* besides that of positive-definiteness, $n_4 > 0$ which, as such, can be smaller or greater than one. Consequently, the Lorentz-Santilli isosymmetry confirms Lorentz's local speed of light for transparent media of low density for which $C = c/n_4 < c$, $n_4 > 1$ and predicts superluminal speeds for physical

particles within hyperdense media in the interior of hadronic, nuclei and stars for which $C = c/n_4 > c$, $n_4 < 1$ [83] [84] [85], a prediction that has been confirmed by all exact and time invariant representations to date of all characteristics of unstable particles with physical constituents (Sect. 5).

Intriguingly, the Lorentz-Santilli isosymmetry has essentially proved that *the abstract axioms of special relativity do indeed admit arbitrary speeds*, with a consequential considerable increase of its representational capability [86].

Note finally that the Lorentz-Santilli isosymmetry $\hat{SO}(3.1)$ is *irregular* because the structure quantities are isofunctions (Sect. 3.3), thus constituting a new realization of Lie's axioms.

As it is the case for the symmetries, the axioms of special relativity and its isotopic extension, called *isospecial relativity*, coincide at the abstract level, since physical differences solely occur in specific realizations. Regrettably, we are unable to review this additional profile for brevity and have to refer the interested reader to monographs [72] [73] [30] [31], the 2021 upgrade [3] and independent reviews [21] [28].

3.10.2. Iso-Poincaré symmetry. The second basic isosymmetry of relativistic hadronic mechanics is given by the isotopies $\hat{P}(3.1)$ of the lorentz-Poincaré symmetry $P(3.1)$ introduced by Santilli in the 1993 paper [80] [81] (see also Sect. 8.3, p. 342 on of Ref. [16] and independent studies [21]), today called the *Lorentz-Poincaré-Santilli isosymmetry*. It's isoalgebra with Hermitean generators ($J_{\mu\nu} = (J_k, M_k)$) is characterized by the isocommutation rules where

$$\begin{aligned} [J_{\mu\nu}, \hat{J}_{\alpha\beta}] &= i(\hat{\eta}_{\nu\alpha}J_{\beta\mu} - \hat{\eta}_{\mu\alpha}J_{\beta\nu} - \hat{\eta}_{\nu\beta}J_{\alpha\mu} + \hat{\eta}_{\mu\beta}J_{\alpha\nu}), \\ [J_{\mu\nu}, \hat{P}_\alpha] &= i(\hat{\eta}_{\mu\alpha}P_\nu - \hat{\eta}_{\nu\alpha}P_\mu), \quad [P_\mu, \hat{P}_\nu] = 0, \end{aligned} \quad (40)$$

and *iso-Casimir-isoinvariants*

$$\begin{aligned} \hat{C}_1 &= \hat{I}(t, r, p, E, \mu, \tau, \psi, \partial\psi, \dots) > 0, \\ \hat{C}_2 &= \hat{P}^2 = \hat{P}_\mu \hat{\times} \hat{P}^\mu = (\hat{\eta}_{\mu\nu} P^\mu \hat{T} P^\nu) \hat{I}, \\ \hat{C}_3 &= \hat{W}^2 = \hat{W}_\mu \hat{\times} \hat{W}^\mu, \quad \hat{W} = W \hat{I}, \quad \hat{W}_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \star J^{\alpha\beta} \hat{\times} P^\rho. \end{aligned} \quad (41)$$

We should note that the isounit of isoinvariant $\hat{C}_2$ is the inverse as that of the Lorentz-Santilli isosymmetry due to the *covariant* nature of the considers iso-Casimir invariant. We should also note that isomomenta isocommute on isospaces over isofields, but they do not commute when they are projected on conventional spaces over conventional field, thus confirming the nonlinearity of the Lie-Santilli isothory.

As it is well known, a *particle* can be assumed to be physical according to 20th century physics (that is, existing and experimentally detectable in our spacetime vacuum) if and only if it is represented by a unitary irreducible representation of the Poincaré symmetry $P(3.1)$.

An important feature identified since the 1978 Harvard University memoir [43], and confirmed in subsequent works [3], is that in the transition from motion in vacuum to motion within a hadronic medium (such as a hadron, a nucleus or a star), *an extended particle generally experiences alterations of its intrinsic characteristics* originally called *mutations* and nowadays referred to as the characteristics of *isoparticles* [87].

We shall therefore say that a particle within a hadronic medium is considered to be extended in these papers if and only if it is an isoirreducible isounitary isorepresentation of the Lorentz-Poincaré-Santilli isosymmetry $\hat{P}(3.1)$. As we shall see, the mutations of the intrinsic characteristics of particles appears to be dependent on the density of the hadronic medium, with minimal mutations (if any) in the interior of mesons and maximal mutations in the core of stars or in the interior of gravitational collapse. Intriguingly, the indicated mutations appear to be deeply linked with, or complementary to the expected deviations from Heisenberg's uncertainty principle for extended particles in the interior of hadronic media [46].

Note that the above isosymmetries are sufficient for the representation of the Deuteron data studied in this paper, but they need a spinorial extension for the studies of the subsequent paper.

3.10.3. Isorelativities. In these papers we gave assumed since the Introduction of Paper I that the Galilean and the special relativity are exactly valid for point-like particles in vacuum under Hamiltonian (thus including electromagnetic and weak) interactions, although they need a suitable completions for extended particles within physical media under Hamiltonian and non-Hamiltonian (thus including strong) interactions. Their study was initiated in 1983 [16] (see Chapter 6), continued in monographs [72] [73] [30] [31] and upgraded in the 2021 work [3] (see independent studies [21] [24] [28]).

By recalling from the preceding sections the isomorphism between conventional space time symmetries and their isotopic formulation, the primary intent for the EPR completion of the Galilean, special and general relativities, collectively known as *isorelativities*, is that of *preserving the original basic axioms and merely introducing broader realizations*, by therefore enlarging their representational capabilities.

3.10.4. Experimental verifications. Let us indicated the generally ignored experimental verifications of isorelativities for:

1) The invariant representation of the the Bose-Einstein correlation [88] [89] with physical characteristics of the proton-antiproton fireball (dimension, shape and density) rather than with unknown "chaoticity parameters".

2) The representation of all characteristics, including their anomalies of muons [35] [90] [91] [92], kaons [93] [94] [95], and unstable baryons [96] via hadronic structure models with physical constituents generally produced free in the spontaneous decays with the lowest modes.

3) The representation of all characteristics of the neutron in its synthesis from the electron and the proton in the core of stars at the non-relativistic and relativistic levels [97] and its experimental verification [98] and generic cases in generic cases in classical mechanics [3] (see the subsequent paper).

3.11. Simple construction of isotopic theories

As indicated in more details in Sect. 4, in practical applications of isotopic methods in nuclear physics, *the Hamiltonian $H(r, p) > 0$ represents all possible linear, local and potential interactions, while the Santillian $\hat{T} = 1/\hat{I} > 0$, represents all possible nonlinear, nonlocal and nonpotential interactions expected from the extended character of nucleons in conditions of partial mutual penetration* (Fig. I.1).

It then follows that, despite their apparent mathematical complexity, all aspects of regular (Sect. 3.3) isotopic formulations, thus including isomathematics, isomechanics and all their applications, can be constructed via the following simple *nonunitary transformation of the quantum mechanical unit $1 = \hbar$ defined on conventional spaces over conventional fields* [99]

$$UU^\dagger = \hat{I} > 0, \quad \hat{T} = 1/\hat{I} = (UU^\dagger)^{-1} > 0, \quad (42)$$

which transformation merely *completes* quantum mechanical models with the representation of the extended character of nucleons and their non-Hamiltonian interactions, provided that said transformation is applied to the *totality* of quantum mechanical quantities and all their operations without any known exception.

In fact, the above procedure maps the quantum mechanical unit, product, Lie algebras, etc., into their isotopic images, according to the simple rules ($1 = \hbar$)

$$\hbar = 1 \rightarrow U1U^\dagger = \hat{I} \neq 1,$$

$$AB \rightarrow U(AB)U^\dagger = (UAU^\dagger)(UU^\dagger)^{-1}(UBU^\dagger) = \hat{A}\hat{T}\hat{B}, \quad (43)$$

$$[X_i, X_j] \rightarrow U[X_i, X_j]U^\dagger = [\hat{X}_i, \hat{X}_j] = \hat{X}_i\hat{T}\hat{X}_j - \hat{X}_j\hat{T}\hat{X}_i = \hat{C}_{ij}^k \hat{\times} \hat{X}_k,$$

where one should note that the transformations are defined on conventional spaces over conventional fields and all products are conventional. Therefore Eqs. (43) characterize *quantum mechanical (rather than hadronic) transformations*.

Note the necessity of abandoning the century-old notion of quantized energy exchanged for point-like electrons in vacuum when dealing with extended nucleons in conditions of mutual penetration (see the experimental works [47] suggesting integro-differential energy exchanges for complex/extended structures and applications in Sect. 5).

Note also that, once quantum mechanical models have been completed via the above simple procedure, when remaining within the quantum mechanical formalism, the resulting new models are afflicted by a number of inconsistencies which are typical of all nonunitary theories on conventional spaces over conventional fields, such as:

3.11.1) Lack of invariance of the observables under additional nonunitary transformations $WW^\dagger \neq I$, including the lack of invariance of the isounit, isoproduct, etc.

$$\begin{aligned}\hat{I} &\rightarrow W\hat{I}W^\dagger = \hat{I}' \neq I, \\ \hat{A}\hat{T}\hat{B} &\rightarrow W(\hat{A}\hat{T}\hat{B})W^\dagger = \hat{A}'\hat{T}'\hat{B}', \quad \hat{T}' \neq \hat{T},\end{aligned}\tag{44}$$

with consequential lack of invariance of numeric predictions.

3.11.2) Consequential general lack of conservation of Hermiticity/observability, Eqs. (6), $U(\langle\psi|H|\psi\rangle)U^\dagger \neq [U(\langle\psi|H\psi\rangle)U^\dagger]$.

3.11.3) General loss of causality, including the possible admission of solutions in which the effect precedes the cause (see also the analysis by L. Biedenharn [100] on the violation of causality for nonunitarily achieved, half-odd-integer eigenvalues of the conventional $SO(3)$ orbital symmetry). Additional problematic aspects of nonunitary theories formulated over a conventional numeric field are addressed in the so-called *Theorems of catastrophic inconsistencies of nonunitary theories* (see p. 451 on of [101]).

The axiomatic origin of the above inconsistencies is that their basic product is the *conventional* associative product (1), rather than isoproduct (9), with ensuing nonconserved basic unit 1, loss under time evolution of the basic numeric field and other insufficiencies.

3.12. Invariance of isotopic formulations

Problems 3.11.1-3.11.3 were resolved in 1998 by Santilli [99] via the identification of the infinite class of equivalence of the dynamical equations of isotopic theories (Sect. 4) which is given by the completion of unitary law (4) into the *isounitary law*

$$\hat{W} \hat{\times} \hat{W}^\dagger = \hat{W}^\dagger \hat{\times} \hat{W} = \hat{I},\tag{45}$$

completed by the identical reformulation of transformations (43) into the isouni-

tary form

$$\begin{aligned}
 U &= \hat{W}T^{1/2}, \\
 UU^\dagger &= \hat{I} \rightarrow \hat{W} \hat{\times} \hat{W}^\dagger = \hat{W}^\dagger \hat{\times} \hat{W} = \hat{I}, \\
 \hat{I} &\rightarrow \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^\dagger = \hat{I}' \equiv \hat{I}, \\
 \hat{A} \hat{\times} \hat{B} &\rightarrow \hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^\dagger = \\
 &= (\hat{W} \hat{\times} \hat{A} \hat{\times} \hat{W}^\dagger) \hat{\times} \hat{W}^{\dagger, -1} \hat{\times} \hat{W}^{-1} \hat{\times} (\hat{W}^\dagger \hat{\times} \hat{B} \hat{\times} \hat{W}^\dagger) = \\
 &= \hat{A}' \hat{\times} \hat{B}' = \hat{A}' \hat{T} \hat{B}', (UU^\dagger)^{-1},
 \end{aligned} \tag{46}$$

(where we have used the isotopoic inverse W^{-1} [30]) with consequential invariance of isotopic formulations.

It should be indicated that isoinvariance (46) [99] embodies the very essence of hadronic mechanics as an *axiom-preserving image of quantum mechanics* because:

3.12A) Quantum mechanical *unitary* transformations on $\mathcal{H}$ over $\mathcal{C}$ leave the product $\times$ numerically invariant, i.e., $U \times (A \times B) \times U^\dagger = A' \times B'$, $U \times U^\dagger = 1$.

3.12B) Quantum mechanical *nonunitary* transformations on $\mathcal{H}$ over $\mathcal{C}$ change the numeric value of the product $\times$, Eq. (43), i.e., $W \times (A \times B) \times W^\dagger = A' \hat{T} B'$, $\times \rightarrow \hat{T} = (W \times W^\dagger)^{-1}$, $W \times W^\dagger \neq 1$.

3.12C) Isomechanical *isounitary* transformations on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ leave numerically invariant the issoproduct $\hat{\times}$, Eq. (46), i.e., $\hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^\dagger = \hat{W} \hat{\times} (\hat{A} \hat{T} \hat{B}) \hat{\times} \hat{W}^\dagger = \hat{A}' \hat{\times} \hat{B}' = \hat{A}' \hat{T} \hat{B}'$, $\hat{W} = \hat{W} \hat{T}^{-1/2}$, $\hat{W} \hat{\times} \hat{W}^\dagger = \hat{W}^\dagger \hat{\times} \hat{W} = \hat{1}$.

Note that the isotopies of spaces and numeric fields. thus of the entire 20th century applied mathematics, are *necessary* to achieve isoinvariance (46).

The axiomatically and physically important significance of transformations 3.12A and 3.12C is that they coincide at the abstract, realization-free level to such an extent that *quantum mechanics and the Lie-isotopic branch of hadronic mechanics can be expressed via the same equations, only subjected to different realizations of the associative product* to such an extent that the lack of formal identity of quantum and isotopoic expressions is due to the insufficient knowledge and/or use of isomathematics.

It is easy to see that the above reformulation resolves insufficiencies 3.11.1-3.11.3 because: expressions (46) imply the *numerical— invariance of the isounit $\hat{I}$ and of the Santillian $\hat{T}$* , with ensuing invariance of the numerical predictions of the theory; such an invariance implies that all observables of quantum mechanics remain observable in hadronic mechanics, e.g., $H^\dagger \equiv H^{\hat{\dagger}}$ and isotopic formulations

verify the causality laws of quantum mechanics (see the fully causal isoorbital angular momentum with half-odd-integer isoeigenvalues of Rutherford's electron compressed inside the proton in the core of stars of Sect. 5.3) [30] [31].

On mathematical grounds, the invariance of numeric predictions, the conservation of Hermiticity under unitary transformations and the verification of causality laws are ultimately reducible to the abstract associative enveloping algebra, thus having an assured validity for conventional as well as isotopic realization of said envelope.

On pedagogical grounds, all applications of isotopic formulations in nuclear physics to date (including those of Sect. 5) have been derived via conventional nonunitary transformations (42) (43) and then their reformulation in terms of isomathematics to prevent the indicated inconsistency problems.

Alternatively we can say that applications of isotopic theories in physics can be initially done via simple rules (42) (43) because they represent the *projection* on conventional spaces over conventional fields of covering formulations (45) (46) on isospaces over isofields.

It should be indicated that this author has received various claims of inconsistency of hadronic mechanics via arguments based on product (1) with basic unit 1, which claims are essentially equivalent to the early 20th century claims of inconsistency of quantum mechanics based on Newtonian mechanics.

3.13. Inconsistencies of deformed Lorentz transformations

It should be indicated that decades following the 1983 isotopies of the Lorentz symmetries [78], iso-Lorentz transformations (38) have been identically used under the name of 'deformed Lorentz transformations' but formulated over a conventional numeric field' (see books [102] [103] and literature quoted therein). As an example, the 2005 deformed Lorentz transformations (4.9), p. 23 of [102] coincide with the 1983 isotransformations (15), p. 351 of [78] and they coincide with Eqs. (38) with $b_\mu = 1/n_\mu$, although without the quotation, specifically, of their origination [78], contrary to admission of said knowledge in preceding works [89] [93] [94] [104]. Similarly, one can see that the 2005 principles of deformed special relativity [102] coincide with the 1991 isoaxioms of special isorelativity [73].

Readers should be aware that deformed Lorentz transformations and special relativity formulated on conventional numeric fields verify the *Theorems of catastrophic inconsistencies* of Ref. [101]) with particular reference to the lack of invariance over time of the numerical predictions and consequential lack of meaningful application to experiments.

This occurrence illustrates the importance of formulating nonunitary completions of quantum mechanics over isospaces and isofields due to their reconstruction of unitarity, Eq. (45).

Quantum Mechanics

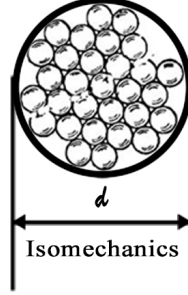


Figure 6: In this figure, we illustrate that the application of hadronic mechanics (*hm*) in nuclear physics is restricted to hold in the interior of a sphere of radius $R_{hm} = 10^{-13}$ cm, called the 'nuclear hadronic horizon' in representation of strong nuclear forces, while uniquely and unambiguously recovering quantum mechanics outside said range. The understanding is that, in general, the continuous and instantaneous fifth interactions represented by Santillian (15) hold at arbitrary distances in view of the EPR entanglement [34].

4 Elements of isomechanics

In order to render this and the following papers minimally self-sufficient, in this section we review, update and specialize to nuclear physics the *Lie-isotopic branch of hadronic mechanics*, also known as *isomechanics*, first proposed in the 1978-1983 Springer-Verlag monographs [15] [16] and subsequently studied by various authors (see the 1995 general treatment [30] [31], monographs [21] [29] and papers quoted therein).

4.1. Iso-Newtonian mechanics

It is important for these papers to indicate that the assumed realization of strong nuclear forces as the most general possible combination of potential/SA (Definition 3.1) and nonpotential/NSA forces (Definition 3.2) originates at the purely Newtonian level, carries over at the operator level and permits the numerically exact and time invariant representation of the experimental data of the Deuteron while preserving the abstract axioms of quantum mechanics (Sect. 2, p. 27 on of Ref. [50]).

Consider the historical Newton's equations for N "massive points" (in Newton's language) on a conventional 3-dimensional Euclidean space $E(r, \delta, I)$

$$m_a \frac{dv_{ak}}{dt} - F_{ak}(t, r, v, \dots) = 0, \quad k = 1, 2, 3, \quad a = 1, 2, \dots, N, \quad (47)$$

which, under the conditions of variational selfadjointness [15] can be written

$$m_a \frac{dv_{ak}}{dt} - F_{ak}^{SA}(r, v) - F_{ak}^{NSA}(t, r, v, \dots) = 0. \quad (48)$$

It was generally believed for centuries that the above equations can solely represent open nonconservative systems generally illustrated via Newton's nonpotential velocity-dependent forces of the type $F^{NSA} = k_1 v + k_2 v^2, \dots$. Eqs. (6.3.6), p. 236 of Ref. [16] introduced the new notion of *closed NS systems*, which are given by systems (39) for $N \geq 2$ verifying all ten conservation laws of Galileo relativity under the conditions

$$\sum_{ak} F_{ak}^{NSA} = 0, \quad \sum_{ak} \mathbf{r} \cdot \mathbf{F}^{NSA} = 0, \quad \sum_{ak} \mathbf{r} \wedge \mathbf{F}^{NSA} = 0. \quad (49)$$

The main equations of isomechanics are given by the formulation of Newton's equations (39) on iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ (Sect. 3.6) over isoreal isonumbers $\hat{\mathcal{R}}$ (Sect. 3.4) first achieved in Eqs. (2.5), p. 31 of Ref. [50] following the discovery of the isodifferential calculus (Sect. 3.7), and nowadays called *Newton-Santilli isoequations* [21] [28],

$$\hat{m}_a \hat{\times} \frac{\hat{d}\hat{v}_{ak}}{\hat{d}\hat{t}} - F_{ak}^{SA}(\hat{r}, \hat{v}) = 0. \quad (50)$$

The main features of Eqs. (41), hereon considered for the particular case $\hat{t} = t$, $\hat{I}_t = 1$, are the following:

1) By recalling that the isodifferential calculus coincides at the abstract level with the conventional differential calculus, iso-Newton's equation (41) coincide at the abstract level with the Newton's equations (39) with $F^{NSA} = 0$.

2) Eqs. (41) provide the first known, axiomatically consistent representation of the actual shapes and dimensions of the particles considered thanks to their representation via the isodifferential calculus with realization of the isotopic element of type (15).

3) Eqs. (41) are *directly universal* for NSA systems [31] in the sense of representing all possible NSA systems directly in the r -coordinate system of the experimentalist, thus without any needs for the transformation theory. In fact, by writing explicitly the isoderivative $\hat{d}\hat{v}_{ak}/\hat{d}\hat{t}$, we reach the representation of NSA forces in terms of isotopic element (15)

$$F^{NSA}(t, r, v, \dots) = mv\hat{T} \frac{d\hat{I}}{dt}, \quad (51)$$

which always admit an isotopic element T for any given (nonsingular) NSA force.

4.2. Iso-Lagrange and iso-Hamilton mechanics

A primary motivation for developing the isodifferential calculus has been the re-formulation of NSA Newton's equation in such a way to admit the following *iso-variational principle* (Sect. 2.4, p. 43 on of Ref. [50]), here written for simplicity for the case of one particle with evident extension to many particles

$$\begin{aligned}\hat{\delta}A &= \hat{\delta}\hat{\int}(\hat{p}\hat{\times}\hat{d}\hat{r} - \hat{H}\hat{\times}\hat{d}\hat{t}) = \\ &= \hat{\delta}\hat{\int}[(p\hat{I}_r)\hat{T}_r\hat{d}\hat{r} - (H\hat{I}_t)\hat{T}_t\hat{d}\hat{t}] = \hat{\delta}\hat{\int}(pT_r\hat{d}r - HT_t\hat{d}t) = 0,\end{aligned}\tag{52}$$

because important for operator images.

The above principle has permitted the formulation of the following main equations of *classical isomechanics* [50]:

i) The *iso-Hamilton equations*

$$\frac{\hat{d}\hat{r}}{\hat{d}\hat{t}} = \frac{\partial H(\hat{r}, \hat{p})}{\partial \hat{p}}, \quad \frac{\hat{d}\hat{p}}{\hat{d}\hat{t}} = -\frac{\partial H(\hat{r}, \hat{p})}{\partial \hat{r}},\tag{53}$$

(where we have assumed $\hat{H} = H\hat{I}_t \equiv H$) with time evolution for a quantity Q

$$\frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [Q, H] = \frac{\partial Q}{\partial \hat{r}} \frac{\partial H}{\partial \hat{p}} - \frac{\partial H}{\partial \hat{p}} \frac{\partial Q}{\partial \hat{r}},\tag{54}$$

where the brackets $[Q, H]$ constitute a classical realization of Lie-Santilli isoalgebras [16] [22].

ii) The *iso-Lagrange isoequations*

$$\frac{\hat{d}}{\hat{d}\hat{t}} \frac{\partial \hat{L}(\hat{r}, \hat{v})}{\partial \hat{v}} - \frac{\partial \hat{L}(\hat{r}, \hat{v})}{\partial \hat{r}} = 0.\tag{55}$$

iii) The *iso-Hamilton-Jacobi equations* also called *Hamilton-Jacobi-Santilli isoequations*

$$\frac{\partial \hat{A}}{\partial \hat{t}} + H = 0 \quad (47a), \quad \frac{\partial \hat{A}}{\partial \hat{r}} - \hat{p} = 0 \quad (47b), \quad \frac{\partial \hat{A}}{\partial \hat{p}} = 0 \quad (47c).\tag{56}$$

It should be indicated that the above analytic equations represent the *external terms* of the historical Hamilton's and Lagrange's equations via the embedding of said terms in the isodifferential calculus, as one can see for the simple case with $\hat{t} = t$, $\hat{I}_t = 1$, $\hat{r} = r$, $\hat{I}_r = 1$ and $\hat{I}_v$ representing the external terms via equations of type (50).

4.3. Iso-Schrödinger representation

The isotopic lifting of the conventional naive quantization [105] characterizes the following map of the (classical) iso-Hamiltonian mechanics into the (operator) isomechanics

$$\hat{A} = \int (\hat{p} \hat{\times} \hat{d}\hat{r} - \hat{H} \hat{\times} \hat{d}\hat{t}) \rightarrow -\hat{i} \hat{\times} \hat{\log} \hat{\psi}(\hat{t}, \hat{r}) = -i \hat{I} \log |\psi(\hat{t}, \hat{r})\rangle, \quad (57)$$

where we use the notion of *isolog* $\hat{\log} \hat{\psi} = \hat{I} \log \psi$ [30]. Note that map (57) is restricted in nuclear physics to hold within a radius $R_{hm} = 10^{-13} \text{ cm}$ called the *nuclear hadronic horizon* in representation of strong nuclear forces, while uniquely and unambiguously recovering quantum mechanics outside said range (Fig. 6).

The resulting *iso-Schrödinger representation*, also known as the *Schrödinger-Santilli isorepresentation*, is characterized by:

1) The *iso-Schrödinger equation* here derived from the iso-Hamilton-Jacobi Eq. (47a) (see Chapter 5, p. 182 of Ref. [31] for a general treatment)

$$\begin{aligned} -\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{t}} |\hat{\psi}(\hat{t}, \hat{r})\rangle &= \hat{H} \hat{\times} |\hat{\psi}(\hat{t}, \hat{r})\rangle = \\ &= [\frac{1}{2m} \hat{p} \hat{\times} \hat{p} + \hat{V}(r)] \hat{T}(\hat{\psi}, \dots) |\hat{\psi}(\hat{t}, \hat{r})\rangle = \hat{E} \hat{\times} |\hat{\psi}(\hat{t}, \hat{r})\rangle = E \times |\hat{\psi}(\hat{t}, \hat{r})\rangle, \end{aligned} \quad (58)$$

where we should recall our assumption $\hat{H} = H \hat{I}_t \equiv H$ for our assumption that, in first approximation, $\hat{t} = t$, $\hat{I}_t = 1$, the last identity trivially follows from the value $\hat{E} = E \hat{I}$, and in case the isostates $|\hat{\psi}(\hat{t}, \hat{r})\rangle$ are k -dimensional, the isotopic element $\hat{t}$ must be a $k \times k$ -isomatrix to avoid evident inconsistencies.

2) The *isolinear isomomentum* from Eq. (47b)

$$\hat{p} \hat{\times} \hat{\psi}(\hat{r}) = -\hat{i} \hat{\times} \hat{\partial}_r \hat{\psi}(\hat{r}) = -i \hat{I} \partial_r \hat{\psi}(\hat{r}), \quad (59)$$

with ensuing *isocanonical isocommutation rules*

$$[\hat{r}_i, \hat{p}_j] \hat{\times} |\hat{\psi}\rangle = -\hat{i} \hat{\times} \hat{\delta}_{i,j} \hat{\times} |\hat{\psi}\rangle = -i \delta_{i,j} |\hat{\psi}\rangle, [\hat{r}_i, \hat{r}_j] |\hat{\psi}\rangle = [\hat{p}_i, \hat{p}_j] |\hat{\psi}\rangle = 0. \quad (60)$$

3) The independence of the isowavefunction from the isolinear isomomentum from Eq. (47c)

$$\hat{\partial}_{\hat{p}} \hat{\psi} = 0. \quad (61)$$

The above naive isoquantization is formulated on the *Hilbert-Myung-Santilli isospace* [106] on an isocomplex isofield $\hat{\mathcal{C}}$ (Sect. 3.4) with: *isostates* $|\hat{\psi}\rangle$, *isonormalization*

$$\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle = \langle \hat{\psi} | \hat{T} | \hat{\psi} \rangle = \hat{I}, \quad (62)$$

(where different isonormalizations per different applications should be kept in mind); *isoexpectation isovalues* of an iso-Hermitean operator $\hat{A}$,

$$\langle \hat{A} \rangle = \langle \hat{\psi} | \hat{\times} \hat{A} \hat{\times} | \hat{\psi} \rangle, \quad (63)$$

and *isidentity*

$$\langle \hat{I} \rangle = \langle \hat{\psi} | \hat{\times} \hat{I} \hat{\times} | \hat{\psi} \rangle = \hat{I}. \quad (64)$$

It should be recalled from Ref. [106] that the condition of iso-Hermiticity coincides with the conventional Hermiticity. Therefore, *all quantities that are observable in quantum mechanics remain observable in isomechanics* [31].

Note also that the characterization of extended particles via isomechanics requires *two* quantities, the conventional Hamiltonian H for the representation of SA interactions and the isotopic element $\hat{T}$ for the representation of NSA interactions.

We should finally note that the quantum mechanical representations of nonlinear interactions via the Hamiltonian violate the superposition principle, by therefore preventing the study of individual nucleons under nonlinear nuclear forces. This insufficiency has been resolved by isomathematics and isomechanics because the representation of all nonlinear interactions via the isounit restores linearity at the abstract, realization-free level called *isolinearity* (see Sect. 4.2, p. 128 on Ref. [30]), by therefore allowing the decomposition of an isostate $|\Psi\rangle$ (representing a nucleus) into the isostates $|\psi_k\rangle$ (representing the individual nucleons)

$$|\Psi(r)\rangle = \Pi_{k=1,2,\dots,N} |\psi_k(r_k)\rangle, \quad (65)$$

with consequential capability of quantitative representations of individual nuclear constituents.

4.4. Iso-Heisenberg representation

Nonrelativistic isomechanics is additionally based on the *iso-Heisenberg isoequations*, also called *Heisenberg-Santilli isoequations* (first formulated in Eq. (18a), p. 153, Ref. [16] (see Sect. 3.1, p. 80 on Ref. [31] for a detailed treatment), here written in the isoinfinitesimal form for the time evolution of an iso-Hermitean operator $\hat{Q}$ in terms of the Hamiltonian H representing all linear, local and potential interactions and the Santillian $\hat{T}$ representing all all nonlinear, nonlocal and nonpotential interactions

$$\begin{aligned} \hat{i} \hat{\times} \frac{d\hat{Q}}{dt} &= [\hat{Q}, \hat{H}] = \hat{Q} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{Q} = \\ &= \hat{Q} \hat{T}(\psi, \dots) H(r, p) - H(r, p) \hat{T}(\psi, \dots) \hat{Q}, \end{aligned} \quad (66)$$

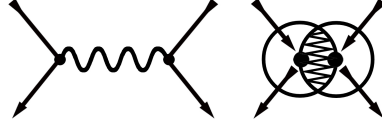


Figure 7: An illustration on the left of the conventional Feynman's diagrams representing an elastic scattering of two point-like electrons via the exchange of a point-like photon, and an illustration on the right of the elastic scattering of two extended hadrons mediated by their EPR entanglement (Figure 1) without particle exchange due to the density of the scattering region (Ch. 12 of [31]).

and in their isoexponentiated form

$$\begin{aligned} Q(t) &= e^{Htti} \hat{\times} Q(0) \hat{\times} e^{-itH} = \hat{U}(t) \hat{\times} Q(0) \hat{\times} U(t)^\dagger = \\ &= e^{H\hat{T}ti} Q(0) e^{-it\hat{T}H}, \end{aligned} \quad (67)$$

where we have used isoexponentiation (22), showing the reconstruction by the Lie-Santilli isogroups of unitarity on isospaces over isofields, called *isounitariness* (see Sect. 4.2, p. 128 on Ref. [30])

$$\hat{U}(t) \hat{\times} \hat{U}(t)^\dagger = \hat{U}(t)^\dagger \hat{\times} \hat{U}(t) = \hat{I}. \quad (68)$$

Note that *the iso-Heisenberg equations (65) verify the invariance under anti-Hermiticity of the conventional equations (5), by therefore restricting the sole consistent application of the Lie-isotopic formulations to stable nuclei.*

The author contacted Prof. Werner Heisenberg in the early 1970's on the expected nonlinear interactions in nuclear physics, with particular reference to the inability for the Hamiltonian representation to characterize individual nucleons. Prof. Heisenberg responded very nicely to a young physicist (then an Assistant Professor of physics at Boston University), but essentially indicated no knowledge available at the time of including nonlinear interactions while preserving the superposition principle. The isotopic solution of the problem via the incorporation of all nonlinear interactions in the isounit of the theory, Eqs. (49), was the outcome of years of trials and errors by the author stimulated by his correspondence with Prof. Heisenberg.

4.5. Relativistic isoequations

The relativistic isoequations of hadronic mechanics are characterized by the isosymmetry *Poincaré-Santilli isosymmetry* $\hat{P}(3.1)$ (Sect. 3.8.2) of the covariant formulation of the iso-Minkowski isospacetime $\hat{M}(\hat{p}, \hat{\Omega}, \hat{I})$ over the isoreal isofield $\hat{\mathcal{R}}$,

with four-dimensional isomomentum

$$\hat{p}_\mu \hat{\times} |\hat{\psi}\rangle(\hat{x}) = -i \hat{\times} \hat{\partial}_{\hat{x}_\mu} |\hat{\psi}\rangle(\hat{x}), \quad \mu = 1, 2, 3, 4, \quad \hat{x}^4 = \frac{x^4}{n_4}, \quad (69)$$

resulting in the local speed of light within the considered physical medium, Eq. (34), with density $n = n_4$ (Figure 4). Consequently, the isometric $\hat{\Omega} = (\hat{\eta})\hat{I}$ is now the inverse of isometric (29).

The second order relativistic isoequation, called *Klein-Gordon-Santilli isoequation* [107] [108] (see Chapter 10, p.477 on Ref. [31] for a detailed study and review [28]), is characterized by the second order iso-Casimir invariant of $\hat{\mathcal{P}}$ (3.1) and can be written

$$\begin{aligned} \hat{p}^2 |\hat{\psi}\rangle(\hat{x}) &= \hat{\Omega}^{\mu\nu} \hat{\times} \hat{p}_\mu \hat{\times} \hat{p}_\nu \hat{\times} |\hat{\psi}\rangle(\hat{x}) = \\ &= \hat{\eta}^{\mu\nu} (-i\hat{I}\partial_\mu)(-i\hat{I}\partial_\nu) |\hat{\psi}\rangle(\hat{x}) = \hat{I}(mC)^2 |\hat{\psi}\rangle(\hat{x}). \end{aligned} \quad (70)$$

The first order isorelativistic equation of isomechanics, called the iso-Dirac equation, will be introduced in Section 5.5.

4.6. Isoconvergence

In all applications to date, the isotopic element (15) resulted to have a numeric value much smaller than one, $\|\hat{T}\| \ll 1$. Since this numeric value has to be sandwiched between *all* products of isomechanics, isomathematics may eliminate quantum mechanical divergencies, besides allowing much faster data elaboration via perturbative and other methods (see Chapter 11, p. 500 on Ref. [31]).

To illustrate this important feature, consider the following divergent quantum mechanical series $A(w) = A(0) + (AH - HA)/1! + \dots \rightarrow \infty$, $w > 1$. But the value of the isotopic element is much smaller than the parameter w . Therefore, the isotopic completion of the above series

$$A(w) = A(0) + (A\hat{T}H - H\hat{T}A)/1! + \dots \rightarrow N < \infty, \quad \hat{T} \ll w, \quad (71)$$

is always strongly convergent.

More specifically, the divergencies of quantum mechanics originate from the singularity existing at the origin of the *Dirac delta distribution* which divergence, in turn, originates from the point-like approximation of particles.

Another important feature of isomechanics is that of avoiding these singularities as illustrated by the isotopic image of Dirac's delta distribution, known as *Dirac-Myung-Santilli isodelta isofunction* first introduced in Ref. [106] (see also Nishioka's studies [109]-[112])

$$\hat{\delta}(\hat{r}) = \int \hat{e}^{\hat{k} \star \hat{r}} \star d\hat{k} = \int e^{\hat{k} \hat{T} \hat{r}} d\hat{k}, \quad (72)$$

where we have used isoexponentials and isointegrals [30]. In fact, the appearance of the isotopic element in the *exponent* of the integrant may change a sharp singularity at the origin $r = 0$ into a bell-shaped function.

The author visited P. A. M. Dirac, in 1981 at the Department of Physics of the University of Florida, Tallahassee, in connection with his suggestion to remove the divergencies of quantum mechanics so as to avoid the removal of infinities via infinities. During a brief encounter, the author indicated to Prof. Dirac isotopic product (9) with ensuing strongly convergent perturbative series (64) without divergencies. Following a long minute of complete silence, Prof. Dirac simply stated "*Please send me the papers.*" Following his return to Harvard University, the author mailed the day after to Prof. Dirac: paper [106] that he was writing with the mathematician H. C. Myung, a copy of monograph [15] and the draft of [16]. It was unfortunate for science that, quite likely, Dirac never saw these works because he became ill and died a few years later.

4.7. Einstein isodeterminism

Recall the physical reality that stars initiate their lives as an aggregate of Hydrogen, synthesize the neutron from the electron and the proton and then synthesize all natural elements. Recall the prohibition by the 1927 Heisenberg's indeterminacy (also called uncertainty) principle to represent the synthesis of the neutron from an electron and a proton in the core of stars despite their extremely big Coulomb attraction at 1 *fm* mutual distances which can be computed to be of the order of 230 *N*.

The above occurrences suggested the need to generalize Heisenberg's uncertainty principle for *extended* nucleons in conditions of *mutual penetration* whose study was initiated by Santilli in the late 1970s at Harvard University under DOE support (see the title of memoir [43]) which was first formulated in the 1981 paper "Generalization of Heisenberg's uncertainty principle for strong interactions" [44], followed by the 1994 paper for gravitational collapse [45], and applied in the 1998 paper [39] to set the foundations for the recovering of Einstein's determinism by showing that a system of *extended* spin 1/2 particles with potential and non-potential interactions violate Bell's inequalities [38] thus admitting classical counterparts.

The considerable lapse of time between papers [44] - [45] - [39] is due to the need for corresponding prior advances in isomathematics. In fact, starting with the isotopies of Lie's theory [16], the 1981 paper [44] established the need that the formulations had to be defined over numeric isofields that were discovered in 1993 [61]. The 1994 paper [45] established the need for the extension of the Newton-Leibnitz differential calculus to *volumes*, that was achieved in 1996 [50]. The 1998 paper [45] identified the final needs to predict the same numeric values under the same conditions at different times, such as the invariance under isounitary

transformations reviewed in Sect. 3.12.

Santilli's final (isotopic) form of the generalized uncertainty principle for strong interactions was presented in the 2019 publication at *Acta Mathematica* [46] with an extended 2020 presentation also published by *Acta Mathematica* [113].

Nowadays, the EPR completion of Heisenberg' indeterminacy principle which is needed for the synthesis of the neutron (studied in the next section) and the representation of nuclear stability (studied in paper III) can be derived via a truly elementary, *quantum nonunitary* transformation of the conventional principle on conventional spaces and fields (Sect. 3.11) reformulated in terms of *hadronic isounitary transformations* on isospaces over isofields to avoid the inconsistencies of quantum nonunitary theories (Sect. 3.12) and can be reviewed as follows.

For applications in nuclear physics it is sufficient to assume the isonormalization necessary for constant $\hat{T}$

$$\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle = \hat{T}, \quad (73)$$

and isocommutation rules (60), under which a simple nonunitary-isounitary transformation of the conventional principle yields the *isoindeterminacy (or isouncertainty) principle of hadronic mechanics* whose projection on our spacetime (as needed for experiments) is given by

$$\begin{aligned} \Delta r \Delta p &\approx \frac{1}{2} | \langle \hat{\psi}(\hat{r}) | \star [\hat{r}, \hat{p}] \star | \hat{\psi}(\hat{r}) \rangle = \\ &= \frac{1}{2} | \langle \hat{\psi}(\hat{r}) | \hat{T} [\hat{r}, \hat{p}] \hat{T} | \hat{\psi}(\hat{r}) \rangle = \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r} \approx \frac{1}{2} \bar{h} \hat{T} = \\ &\quad \frac{1}{2} \bar{h} e^{-\Gamma(r,p,a,R,d,\psi,\hat{\psi},\dots)} \approx \\ &\approx \frac{1}{2} \bar{h} [1 - \Gamma(r,p,a,R,d,\psi,\hat{\psi},\dots) + \dots] \ll 1, \end{aligned} \quad (74)$$

where the Santillian $\hat{T}$ has been derived from realization (15) to represent the non-potential-NSA component of strong interactions.

It should be noted that isoprinciple (77) contains as particular cases all generalizations of Heisenberg's uncertainty principle known to the author (see, e.g., Refs. [114] [115] [116] and papers quoted therein).

Note in isoprinciple (77) that the quantum expression $\Delta r \Delta p \geq (12)\hbar$ (also denoted $\sigma_r \sigma_p \geq (12)\hbar$) is turned into the hadronic form $\Delta r \Delta p \approx (1/2\hat{T}\hbar$ because of the impossibility to maintain quantum mechanical statistical averages of point-like particles in vacuum for extended protons and neutrons under the contact-nonlocal effects cause by the surrounding particles (Fig. 3).

The connection between the above isoindeterminacy principle and Einstein's determinism [1] is intriguing indeed. Recall from Sect. 1:

4.7.1) The prohibition of Einstein's determinism by Bell's inequalities because they prohibit the existence of classical images for quantum mechanical systems of point-like spin 1/2 particles in vacuum [38];

4.7.2) The violation of Bell's inequalities by extended spin 1/2 particles in conditions of mutual penetration with the ensuing existence of classical counterparts [39] ;

4.7.3) Bohm's attempt of achieving Einstein's determinism via the hypothesis of hidden variables [48]-[49];

4.7.4) Santilli's explicit and concrete realization of Bohm's variables as being hidden in the axiom of associativity of quantum mechanics [31] [51].

4.7.5) Santilli [46] [113] has shown that isomathematics and isomechanics uniquely and unambiguously characterize the *explicit form of the standard isodeviations* for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$ which, again in their projection on conventional spaces over conventional fields, can be written

$$\begin{aligned}\Delta r &= \sqrt{\hat{T} < \hat{\psi}(\hat{r}) | [\hat{r} - < \hat{\psi}(\hat{r}) | \star \hat{r} \star | \hat{\psi}(\hat{r}) >]^2 | \hat{\psi}(\hat{r}) >}, \\ \Delta p &= \sqrt{\hat{T} < \hat{\psi}(\hat{p}) | [\hat{p} - < \hat{\psi}(\hat{p}) | \star \hat{p} \star | \hat{\psi}(\hat{p}) >]^2 | \hat{\psi}(\hat{p}) >}.\end{aligned}\tag{75}$$

In view of the lack of energy by contact zero-range NSA interactions and the ensuing very low value of $\hat{T}$ for all known physical applications, the above isodeviations, *individually* tend to zero with the increase of the density of the hadronic medium up to the limit identity of $\hat{T}$ with Schwarzschild's horizon, Eq. (34),

$$\lim_{\hat{T}=0} \Delta r = 0, \quad \lim_{\hat{T}=0} \Delta p = 0,\tag{76}$$

resulting in the Lemma (p. 127 of [113])

LEMMA 4.7.1 (EINSTEIN'S ISODETERMINISM): *The standard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$ progressively and individually tend to be zero for extended particles in the interior of hadrons, nuclei, and stars, and fully achieve Einstein's determinism at the limit of gravitational collapse.*

The first significant implication of the above Lemma is a confirmation of the isoconvergence of Sect. 4.6, which can be expressed via the following Corollary (p. 128 of [113]):

COROLLARY 4.7.1. *Einstein's isodeterminism according to Lemma 4.7.1 implies the removal of quantum mechanical divergencies in hadronic mechanics.*

As we shall see, Einstein's isodeterminism plays a fundamental role for first known, numerically exact and time invariant representation of the Deuteron data

(Sect. 5), as well as of the neutron synthesis from the Hydrogen and, consequently, of nuclear stability (paper III).

4.8. Isoscattering theory without divergencies

As indicated in Section I.1, we have assumed in our study the exact validity of quantum mechanics for point-like particles and electromagnetic waves *in vacuum*. Consequently, we can safely assume that *quantum mechanics is exactly valid for particles in accelerators*.

However, the mathematical, theoretical and experimental evidence presented in these papers imply that *quantum mechanics is not valid for high energy scattering regions*, for numerous reasons, such as the lack of representation of the time irreversibility of nonelastic scattering processes (Insufficiency I-2.7), unavoidable nonlinear, nonlocal and nonpotential effects due to the impact of *extended* protons on *dense* targets, and other reasons (Fig. 7).

Studies on the EPR completion of the conventional scattering theory into a more realistic theory representing the collision of extended particles on dense targets have been initiated by R. Mignani [117]-[118], A. K. Aringazin and D. A. Kirukhin [119], A. O. E. Animalu and R. M. Santilli [120] and others.

These studies have established the foundations of the scattering theories of hadronic mechanics known under the names of *isoscattering and genoscattering theories* and collectively, as *hadronic scattering theories* (Figure 7). Regrettably, we cannot review these studies to avoid a prohibitive length. We limit ourselves to the following comments (see Chapter 12, p. 507 on of Ref. [31] for details):

4.8.1. In view of the time-reversal invariance of the Lie-Santilli theory, the isoscattering theory can solely represent the *elastic* scattering of extended particles/wavepackets on a dense target. The causal and axiomatically consistent representation of *inelastic* scatterings is done via the broader genoscattering theory with a Lie-admissible algebraic structure [120].

4.8.2. Evidently, hadronic scattering theories cannot change experimentally measured values, such as scattering angles, cross sections, etc. However, *hadronic scattering theories do require a revision of the theoretical interpretations derived via the conventional scattering theory*.

4.8.3. Recall that rather vast studies have been conducted on *nonunitary scattering theories* (see, e.g., Ref. [121] and papers quoted therein). These studies had to be abandoned because nonunitary theories violate causality laws (Sect. 3.10). By contrast, hadronic scattering theories reconstruct unitarity on generalized space over generalized fields, thus restoring causality [30].

4.9. Connection between Birkhoffian mechanics and isomechanics

Following the technical characterization of NSA forces in the first volume of the 1978 Springer-Verlag monographs [15], the primary aim of the 1983 second vol-

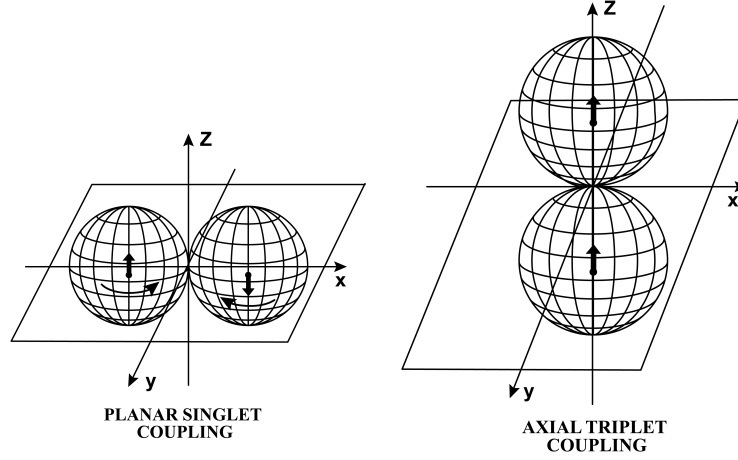


Figure 8: In this figure, we illustrate the two stable bound states of particles with spin predicted by hadronic mechanics, which are given by the 'planar singlet coupling' on the left and the 'axiom triplet coupling' on the right.

ume [16] was the achievement, via the use of conventional mathematics, of direct universality for the representation of NSA Newtonian systems in terms of a generalized variational principle so as to allow apparently for the first time the optimization, for instance, of the actual shape of a wing moving in atmosphere (see representative papers [122]-[126]).

This aim was achieved via the formulation of Birkhoffian mechanics in the 6-dimensional phase space with local coordinates $a^\mu = (r, p)$, $\mu = 1, 2, \dots, 6$, and the assumption of the most general possible action principle in the indicated phase space

$$\delta \mathcal{A} \delta \int [R_\mu(t, a) da^\mu - B(t, a) dt] = 0, \quad (77)$$

with consequential *Birkhoff's equations* (Sect. 4.2, p. 30 of [16])

$$\left[\frac{\partial R_\mu(t, a)}{\partial a^\nu} - \frac{\partial R_\nu(t, a)}{\partial a^\mu} \right] \frac{da^\nu}{dt} - \left[\frac{\partial B(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial t} \right] = 0, \quad (78)$$

where $B(t, a)$ is the *Birkhoffian* (which does not generally represent the energy) with the most general possible realization of the Lie-Santilli isothory in classical mechanics in terms of brackets between observables while admitting as particular case the conventional Hamilton's equation in phase space (those without external

terms) for $R = (0, p)$

$$\begin{aligned}
 [A, B] &= \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu}, \\
 \omega_{\mu\nu} \frac{da^\nu}{dt} - \frac{\partial H(t, a)}{\partial a^\mu} &= 0, \\
 [A, B] &= \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r} \frac{\partial B}{\partial p} - \frac{\partial B}{\partial r} \frac{\partial A}{\partial p}.
 \end{aligned} \tag{79}$$

Note that the conventional Hamilton's equations can only represent SA Newtonian systems while, by comparison, Birkhoff's equations are directly universal for NSA systems (Sect. 4.5, p. 54 of Ref. [16]).

A crucial point occurred in these studies with the identification of the fact that the *Birkhoffian form of the Hamilton-Jacobi equations* (p. 205 on of Ref. [16]) no longer verifies the crucial property (47c) on the independence of the action from the linear momentum. As a consequence, *the Birkhoffian generalization of Schrödinger equation* (p. 211 of Ref. [16]) *requires a generalization of quantum mechanical axioms* because wave functions would also depend on momenta $\psi = \psi(t, r, p)$.

In order to maintain the validity of quantum axioms for NSA interactions, R. M. Santilli proposed in Eq. (15), p. 259 of Ref. [16] the decomposition of the Birkhoffian eigenvalue equation into the iso-Hamiltonian

$$B(t, a) \times |\psi(t, a)\rangle = H \hat{\times} |\psi(t, r)\rangle = H \hat{T} |\psi(t, r)\rangle = E |\psi(t, r)\rangle, \tag{80}$$

which is possible thanks to the embedding of all NSA terms in the isounit of the theory.

By looking in retrospect, Birkhoffian mechanics has seen various applications for the classical treatment of NSA systems (see representative papers [122]-[126]) while the isotopic formulation of NSA systems has been preferred over the operator Birkhoffian form in view of its clear preservation of quantum mechanical axioms.

5 Representation of the Deuteron data

5.1. Foreword

Thanks to studies initiated in 1978 [43] and continued thereafter (see Refs. [127]-[131]), in this section we review and upgrade the numerically exact and time invariant representation permitted by the Lie-isotopic mechanics of the following experimental data of the Deuteron [6]-[?] in the ground state used for experimental measurements, that with null orbital contributions:

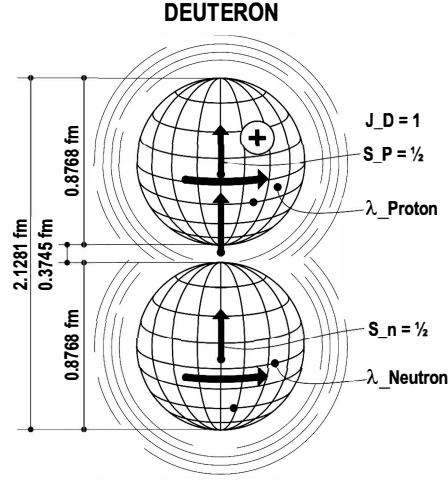


Figure 9: In this figure, we reproduce known experimental data on the dimensions of the Deuteron [8] and its constituent proton and neutron [11], as well as their interpretation as a hadronic bound state in axial triplet coupling (Fig. 1), thus representing for the first time the spin of the Deuteron $S_D = 1$ in its ground state, that with null angular contributions $L_D = 0$ [51].

5.1.1. The spin $S_D = 1$.

5.1.2. The magnetic moment $\mu_D^{exp} = 0.85647 \mu_N$.

5.1.3. The total energy $E_D^{tot} = M_p + M_n - BE = 1875.7 \text{ MeV}$.

5.1.4. The charge radius $R_D = \mu_D^{exp} = 0.85647 \mu_N$.

5.1.5. The mean life $\tau_D = \infty$.

In order to apply isotopic formulations, in this section we assume in first approximation that the Deuteron is a hadronic bound state of a proton and a *permanently stable neutron*. The representation of the stability of the Deuteron despite the natural instability of the neutron and the Deuteron stability despite strongly repulsive protonic forces, are studied in the following paper.

5.2. Representation of the Deuteron spin

Recall that the sole stable bound state predicted by quantum mechanics between two *point-like* particles with spin $S = 1/2$ is the singlet for which $S_D = 0$. The stable bound states predicted by hadronic mechanics for two *extended* particles with spin $S = 1/2$ are the *planar singlet coupling* and the *axial triplet coupling* illustrated in Figure 8. It then follows that, according to the Lie-isotopic branch of hadronic mechanics, *the Deuteron in its true ground state is composed by a proton*

and a neutron in axial triplet coupling (Fig. 9) [51]

$$D = \begin{pmatrix} p^\uparrow \\ n^\uparrow \end{pmatrix}, \quad (81)$$

resulting in a structure illustrated in Figure 9 with dimensions from Refs. [8] [11]. In the subsequent Figure 10, we illustrate the consequential structure of the Helium-4 (Fig. 10).

It should be noted that the above representation of the Deuteron spin is highly preliminary because a deeper study of the spin of the Deuteron requires the prior representation of the synthesis of the neutron in the core of stars, which representation is studied in the next paper.

5.3. Hadronic angular momentum

As it is well known, the quantum mechanical angular momentum is given by the familiar expression on an Euclidean space over the field of complex numbers $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}|b\rangle$ acting on a state of the Hilbert space $\mathcal{H}$. The rotational symmetry $so(3)$ represents the conservation of the angular momentum.

The angular momentum for extended nucleons under strong interactions, called *hadronic angular momentum* (first presented in the 1985-1991 papers [76]-[78] and then studied in detail in Chapter 6, p. 209-285 of Ref. [31]) is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli isogroup $\hat{SO}(2)$ characterized by the *isorotations* on the iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I})$ (Sect. 3.6)

$$\begin{aligned} \hat{r}' &= \hat{R}(\hat{\theta}) \hat{\times} \hat{r}, \quad \hat{R}(\hat{\theta}) \hat{\times} \hat{R}(\hat{\theta})^{-1} = \hat{R}(\hat{\theta})^{-1} \hat{\times} \hat{R}(\hat{\theta}) = \hat{I}, \\ \hat{R}(\hat{\theta}) \hat{\times} \hat{R}(\hat{\theta}') &= \hat{R}(\hat{\theta}') \hat{\times} \hat{R}(\hat{\theta}) = \hat{R}(\hat{\theta} + \hat{\theta}'), \quad \hat{R}(\hat{\theta}) \hat{\times} \hat{R}(-\hat{\theta}) = \hat{R}(0) = \hat{I} = 1\hat{T}. \end{aligned} \quad (82)$$

The isoalgebra $\hat{so}(2)$ can be realized on an iso-Hilbert isospace over an isofield via a simple isotopy of the conventional angular momentum (Section 3.6)

$$\begin{aligned} \hat{\mathbf{L}} \hat{\times} |\hat{\mathbf{b}}\rangle &= (\hat{\mathbf{r}} \hat{\wedge} \hat{\mathbf{p}}) \hat{\times} |\hat{\mathbf{b}}\rangle, \\ \hat{\mathbf{r}} &= \mathbf{r}\hat{\mathbf{I}}, \quad \hat{\mathbf{p}} \hat{\times} |\hat{\mathbf{b}}\rangle = -i\hat{\Delta}\hat{\mathbf{r}}|\hat{\mathbf{b}}\rangle = -i\hat{\mathbf{I}}\partial_{\hat{\mathbf{r}}}|\hat{\mathbf{b}}\rangle. \end{aligned} \quad (83)$$

As expected, the $\hat{so}(3)$ isosymmetry characterizes the conservation of the hadronic angular momentum, which conservation is evidently verified for stable nuclei.

The simplest possible realization of the isoalgebra $\hat{so}(3)$ can be constructed via the isotransformations of the conventional generators J_k , $k = 1, 2, 3$ (Sect. 3.8)

$$\begin{aligned} U J_k U^\dagger &= \hat{J}_k, \\ U U^\dagger &= \text{Diag.}(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}) = (b_1^2, b_2^2, b_3^2). \end{aligned} \quad (84)$$

It is easy to see that the above isoalgebra preserves the conventional structure constants of $so(3)$

$$[\hat{J}_i, \hat{J}_j] \hat{\times} |\hat{b}\rangle = (\hat{J}_i \hat{T} \hat{J}_j - \hat{J}_j \hat{T} \hat{J}_i) \hat{\times} |\hat{b}\rangle = \epsilon_{ijk} \hat{J}_k \hat{\times} |\hat{b}\rangle, \quad (85)$$

as well as conventional quantum mechanical eigenvalues

$$\begin{aligned} \hat{J}_3 \hat{\times} |\hat{b}\rangle &= m |\hat{b}\rangle, \quad m = J, j-1, \dots, -J, \quad j = 0, 1, 2, \dots \\ \hat{J}^2 \hat{\times} |\hat{b}\rangle &= (\hat{J}_1 \hat{T} \hat{J}_1 + \hat{J}_2 \hat{T} \hat{J}_2 + \hat{J}_3 \hat{T} \hat{J}_3) \hat{T} |\hat{b}\rangle = j(j+1) |\hat{b}\rangle, \end{aligned} \quad (86)$$

by therefore assuring the isomorphic $\hat{so}(3) \approx SO(3)$.

It should be indicated that the above simplest possible isorealization of $\hat{so}(3)$, called *standard* (p. 212 of Ref. [31]), is sufficient for the representation of the Deuteron magnetic moment done in the next section. Nevertheless, the representation of the conserved angular momentum of the neutron in its synthesis from the Hydrogen atom (studied in the next paper) will require the broader *regular isorepresentation* (p. 213 of Ref. [31]) while the representation of *nonconserved* angular momenta in nuclear fusions requires the more general *irregular isorepresentations* [58] [59] [60].

5.4. Representation of the Deuteron magnetic moment via the hadronic angular momentum

Let us recall the historical view by E. Fermi [53], V. F. Weisskopf [54], and other founders of nuclear physics according to which the inability by quantum mechanics to represent nuclear magnetic moments may be due to a deformation of the charge distribution of nucleons under strong nuclear forces, a prediction hereon referred to as the *Fermi-Weisskopf hypothesis*. In fact, p. 31 of Ref. [54] presents the statement *it is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon*.

We are here referring to the fact that a rotating charge distribution creates a magnetic field along its rotation axis. When said charge distribution is deformed into a prolate (oblate) ellipsoid, the magnetic field decreases (increases). Since the quantum mechanical prediction for the magnetic moment of the Deuteron is in *excess* by about 3% of the experimental value $\mu_D^{exp} = 0.85647 \mu_N$, the deformation of the charge distribution of the proton and the neutron is expected to be of *prolate* type (bottom right view of Fig. 2).

The representation of the Deuteron magnetic moment via the Fermi-Weisskopf hypothesis achieved for the first time in the 1994 paper [127] (see also monograph [128] and independent studies [129]-[131]) is based on the representation of the deformation of nucleons via isotopic element (28) which reconstructs the perfect sphere on the iso-Euclidean isospace over an isofield (top of figure 2). The

Exact Lie-isotopic representation of the Deuteron data

reformulation (isotopy) via hadronic mechanics (hm) of the conventional connection between magnetic moment and spin of nucleons [55] [7] (here omitted for brevity) implies the following *mutation of the magnetic moments of the protons and the neutron* [127]

$$\mu_D^{hm} = \frac{n_4}{n_3}(g_p S + g_n S) = \frac{n_4}{n_3}\mu_D^{qm}. \quad (87)$$

By recalling the experimental value $\mu_D^{exp} = 0.8565\mu_N$, and the quantum mechanical (qm) value $\mu_D^{qm} = 0.8798\mu_N$, we reach the numeric value

$$\frac{\hat{\mu}_D^{exp}}{\mu_D^{qm}} = \frac{n_4}{n_3} = \frac{0.8565}{0.8798} = 0.9735. \quad (88)$$

In the above expressions, n_4 represents the density of individual nucleons whose currently available best value is given by the density $n_4 = 605$ of the $p - \bar{p}$ fireball of the Bose-Einstein correlation [88] [89], resulting in the numeric values

$$n_3 = \frac{0.605}{0.9735} = 0.6215, \quad n_3^2 = 0.3862. \quad (89)$$

To obtain the value of the remaining characteristic quantities, we assume that the mutation of the magnetic moment does not alter the volume of nucleons. By assuming a normalization of the type

$$n_1^2 + n_2^2 + n_3^2 = 3, \quad (90)$$

we obtain the numeric values of the semiaxes of the spheroids here expressed for the characteristic quantities n_μ of this paper as well as for their inverse $b_\mu = 1/n_\mu$ used in paper [127]

$$\begin{aligned} n_1^2 = n_2^2 = 1.3069, \quad n_3^2 = 0.3862, \quad n_4^2 = 0.3660, \\ b_1^2 = b_2^2 = \frac{1}{n_1^2} = \frac{1}{n_2^2} = 0.7652, \quad \frac{1}{n_3^2} = 2.5893, \quad \frac{1}{n_4^2} = 1.7668, \\ n_1 = n_2 = 1.1432, \quad n_3 = 0.6215, \quad n_4 = 0.605, \end{aligned} \quad (91)$$

$$b_1 = b_2 = \frac{1}{n_1} = \frac{1}{n_2} = 0.8709, \quad b_3 = \frac{1}{n_3} = 1.6098, \quad b_4 = \frac{1}{n_4} = 1.653,$$

which confirm their prolate character because $n_3^2 < n_1^2 = n_2^2$.

It should be noted that Ref. [127] studied the isotopic representation of a possible 1% deformation of the neutron under nuclear interactions. By using the same nucleon density $b_4 = 1/n_4 = 1.653$ of Refs. [88] used above [89], Ref.[127] reached the numeric value $b_3 = 1/n_3 = 1.662$ (Eqs. (3.5), p. 124, of Ref.

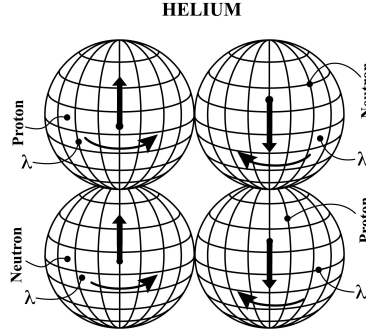


Figure 10: In this figure, we illustrate the structure of the Helium as a hadronic bound state of two Deuterons composed by protons and neutrons in axial triplet coupling [51].

[127]) which is rather close to the value $b_3 = 1/n_3 = 1.6098$ of Eq. (90) above confirming the prolate character of the deformation, The $b_1 = b_2$ value of Ref. [127] different than that of Eqs. (90) is due to a normalization of the characteristic quantities different than used above.

The possible application of the above model to a numeric representation of the magnetic moments of light stable nuclei with N nucleons can be done via the following isotopy of standard representation of nuclear magnetic moments [54] [?] first presented in Eq. (3.8), p. 125 of Ref. [127]

$$\mu^{hm} = \sum_{k=1,2,\dots,N} \left(\frac{n_{4k}}{n_{3k}} g_k^L L_{k3} + \frac{n_{4k}}{n_{3k}} g_k^S S_{k3} \right), \quad (92)$$

where g_k^L and (g_k^S) are tabulated, orbital (spin) giromagnetic factors of the k -nucleon, n_{4k} is the density of the k -nucleon and n_{3k} is its semiaxis along the spin direction.

By assuming again in first approximation that nuclei are composed by protons and neutrons, it is easy to see that isotopic model (91) provides a numeric representation of the magnetic moment of the Helium and other light stable nuclei with potentially significant information on the deformation of the individual nucleons.

5.5. Hadronic spin

The spin of extended nucleons under strong interactions, called *hadronic spin* (first presented in Ref. [77] is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli $\hat{su}(2)$ isoalgebra whose general case is studied in Chapter 6, pages 209-285 of monograph [31], jointly with the hadronic angular momentum of the preceding section and the isotopies of spin-orbit couplings. The hadronic spin was then used for the 1998 verification of the EPR argument [39] and in other applications [32] (Fig. 8).

Exact Lie-isotopic representation of the Deuteron data

We here study the particular case of the hadronic spin with conventional quantum value $S = 1/2$, yet with new degrees of freedom representing Bohm's hidden variables [51]. Its formulation requires the isolinearization of the second order isoinvariant $\hat{C}_2$ (41) of the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ (Sect. 4.8.2) done in Ref. [82] (see [31], Chapter 8, p. 329, particularly Sect. 8.5, p. 353). Said isolinearization required:

- 1) A four-dimensional real valued isospacetime $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ for the characterization of the orbital motion (orb).
- 2) A two-dimensional complex valued isospace $E(\hat{z}, \hat{\delta}, \hat{I}_{spin})$ for the representation of the spin (spin).
- 3) A six-dimensional total (tot) isospace $\hat{M} \hat{\times} \hat{E}$.

From Sect. 3.9, we introduce the corresponding nonunitary transformations

$$\begin{aligned} U_{orb} U_{orb}^\dagger &= \hat{I}_{orb} = \text{Diag.}(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2}), \\ U_{spin} U_{spin}^\dagger &= \text{Diag.}(\lambda^{-1}, \lambda), \\ (U_{orb} \times U_{spin}) \times (U_{orb} \times U_{spin})^\dagger &= \hat{I}_{tot}, \end{aligned} \tag{93}$$

where λ represents Bohm's hidden variable [48] [49].

We now introduce the 4×4 -isomatrices $\hat{\Gamma}$ verifying the conditions

$$\begin{aligned} \hat{\Gamma}_\mu &= \hat{\gamma}_\mu \hat{I}_{orb}. \\ \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} &= \hat{\gamma}_\mu \hat{T}_{orb} \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{T}_{orb} \hat{\gamma}_\mu = 2\hat{\eta}_{\mu\nu} \hat{I}_{orb}, \end{aligned} \tag{94}$$

The second order isoinvariant of $\hat{P}(3.1)$ is then decomposed into the isoproduct (Eq. (6.1a), p. 190, of Ref. [82])

$$\begin{aligned} \hat{\eta}^{\mu\nu} \hat{p}_\mu \hat{T}_{orb} \hat{p}_\nu + \bar{m}^2 C^2 &= \\ &= (\hat{\eta}^{\mu\nu} \hat{\gamma}_\mu \hat{T}_{tot} \hat{p}_\nu + i\hat{m}C) \hat{T}_{tot} (\hat{\eta}^{\mu\nu} \hat{\gamma}_\mu \hat{T}_{tot} \hat{p}_\nu + i\hat{m}C), \quad C = c/n_4, \end{aligned} \tag{95}$$

where the local speed of light $C = c/n_4$ within a hadronic medium with density n_4 originates from the Lorentz-Santilli isosymmetry $\hat{S}\hat{O}(3.1)$ [78] (Sect. 3.8.1), resulting in the *Dirac-Santilli isoequation* [107] (see also Refs. [50] [31]), here written according to isomathematics and in its projection on the conventional Minkowski space

$$\begin{aligned} [\hat{\Omega}^{\mu\nu} \hat{\times} \hat{\Gamma}_\mu \hat{\times} \hat{P}_\nu + \hat{M} \hat{\times} \hat{C}] \hat{\times} |\hat{\psi}(\hat{x})\rangle &= \\ &= (-i\hat{I} \hat{\eta}^{\mu\nu} \hat{\gamma}_\mu \partial_\nu + mC) |\hat{\psi}(\hat{x})\rangle = 0, \end{aligned} \tag{96}$$

where the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\begin{aligned}\hat{\gamma}_k &= \frac{1}{n_k} \begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 &= \frac{i}{n_4} \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix},\end{aligned}\quad (97)$$

and the *Pauli-Santilli isomatrices* (Eq. (6.8.20), page 254 of Ref. [31] and Ref. [39])

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \quad (98)$$

It is easy to see that the Pauli-Santilli isomatrices verify the iso-commutation rules with the same structure constants of Pauli's matrices

$$\begin{aligned}[\hat{\sigma}_i, \hat{\sigma}_j] &= \hat{\sigma}_i \hat{\times} \hat{\sigma}_j - \hat{\sigma}_j \hat{\times} \hat{\sigma}_i = \\ &= \hat{\sigma}_i \hat{T}_{spin} \hat{\sigma}_j - \hat{\sigma}_j \hat{T}_{spin} \hat{\sigma}_i = i2\epsilon_{ijk} \hat{\sigma}_k,\end{aligned}\quad (99)$$

showing the iso-morphism $\hat{su}(2) \approx su(2)$.

The representation of the spin 1/2 of nucleons is given by the isoeigenvalues on an iso-state $|\hat{b}\rangle$ of the *Hilbert-Myung-Santilli iso-space* $\hat{\mathcal{H}}$ [106] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [61]

$$\begin{aligned}\hat{S}_k &= \frac{1}{2} \hat{\times} \hat{\sigma}_k = \frac{1}{2} \hat{\sigma}_k, \\ \hat{\sigma}_3 \hat{\times} |\hat{b}\rangle &= \hat{\sigma}_3 \hat{T} |\hat{b}\rangle = \pm |\hat{b}\rangle,\end{aligned}\quad (100)$$

$$\hat{\sigma}^2 \hat{\times} |\hat{b}\rangle = (\hat{\sigma}_1 \hat{T} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{T} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}\rangle = 3 |\hat{b}\rangle.$$

It should be noted that the above hadronic spin, called *standard*, is sufficient for our needs for this section, but it is insufficient for the synthesis of the neutron studied in the next paper in favor of the broader notions of *regular and irregular hadronic spins* [31].

5.6. Representations of the Deuteron magnetic moment via the hadronic spin

Let us recall the factorization of the Pauli-Santilli isomatrices in Eqs. (6.8.18), p. 254 of Ref. [31]

$$\hat{\sigma}_k = \sigma_k \hat{I} = \hat{T} \sigma_k, \quad \hat{I} = 1/\hat{T} = \text{Diag.}(\lambda^{-1}, \lambda), \quad \text{Det.} \hat{I} = \text{Det} \hat{T} = 1, \quad (101)$$

Exact Lie-isotopic representation of the Deuteron data

and assume the realization of the hidden variable [51] [52]

$$\lambda = e^\phi \geq 0,$$

$$\hat{I} = \cosh \phi + \sigma_3 \sinh \phi = e^{\phi \sigma_3}, \quad (102)$$

$$\hat{T} = \cosh \phi - \sigma_3 \sinh \phi = e^{-\phi \sigma_3}.$$

By recalling that σ_3 characterizes the nucleon spin $S = 1/2$, we have

$$\hat{\sigma}_3 |\hat{b}\rangle = \sigma_3 \hat{I} |\hat{b}\rangle = \sigma_3 e^{\phi \sigma_3} |\hat{b}\rangle, \quad (103)$$

with ensuing characterization of the magnetic moment

$$\mu_{hm} |\hat{b}\rangle = K g S |\hat{b}\rangle, \quad (104)$$

where K is an iso-renormalization constant of the gyromagnetic factor g created by Bohm's hidden variables in the hadronic spin. By using the above property, we reach the relation

$$\mu_{hm} |\hat{b}\rangle = e^{\phi \sigma_3} \mu_{qm} |\hat{b}\rangle = e^{\phi \sigma_3} g S |\hat{b}\rangle. \quad (105)$$

Note that the proton and the neutron have the same spin $1/2$, they have essentially the same mass (thus being characterized by the same λ) and they have the same eigenvalue equation

$$\sigma_3 |\hat{b}\rangle = -|\hat{b}\rangle. \quad (106)$$

We can, therefore, write the expression per each nucleon

$$\mu_{hm,k} \approx (1 + \phi \sigma_3) \mu_{qm,k} = (1 - \phi) \mu_{qm,k}, \quad k = p, n, \quad (107)$$

from which we obtain the isorenormalized value of the magnetic moment of the proton and of the neutron

$$\hat{\mu}_p = +(1 - \phi) 2.79285 \mu_N, \quad \hat{\mu}_n = -(1 - \phi) 1.91304 \mu_N, \quad (108)$$

with corresponding value for the magnetic moment of the Deuteron

$$\begin{aligned} \mu_D^{hm} &= (1 - \phi) 2.79285 - (1 - \phi) 1.91304 \mu_N = \\ &= (1 - \phi) 0.87981 \mu_N = \mu_D^{ex} = 0.85647 \mu_N, \end{aligned} \quad (109)$$

from which we obtain the numeric value

$$\phi = 1 - 0.87981/0.85647 = 1 - 0.02334 = 0.97666, \quad (110)$$

with corresponding *numeric value of Bohm's hidden variable for the Deuteron* [51]

$$\lambda = e^\phi = e^{0.97666} = 2.65557. \quad (111)$$

We can, therefore, state that isomathematics and isomechanics do allow a numerically exact representation of the Deuteron magnetic moment in terms of Bohm hidden variable λ which is invariant over time due to the invariance of the Dirac-Santilli isoequation under the isosymmetry $\hat{\mathcal{P}}(3.1)$ [82].

5.7. Representation of the Deuteron mass, stability and charge radius

In accordance with our primary assumptions of Sect. 1, the terms *hadronic structure equations* are referred to iso-Galilean isoinvariant isoequation (57) representing stable bound state of extended nucleons in conditions of partial mutual penetration. The conventional linear, local and potential interactions are represented by the Hamiltonian H , while nonlinear, nonlocal and nonpotential interactions are represented by the isotopic element $\hat{T}$ under the condition of rapidly recovering the conventional unit, $\hat{T} \rightarrow I$ at mutual distances bigger than the range of strong interactions $R = b^{-1} = 10^{-13} \text{ cm}$.

The first hadronic structure equation was proposed by Santilli in Sect. 5.1, p. 827 on of the 1978 Harvard University memoir [43] for the representation of *all* characteristics of the pion π^0 as a bound state of an electron and a positron at $R = 10^{-13} \text{ cm}$ mutual distance, so as to have a physical origin of the great instability of the particle as well as of the spontaneous decay with the lowest mode $\pi^0 \rightarrow 2 \gamma$.

The isotopic element was selected to be of type (14), with the function $\hat{F}(r)$ given by the Hulthén potential in representation of nonpotential interactions, and the function $F(r)$ representing the Coulomb attraction between the electron and the positron

$$\hat{T} = e^{-\frac{\hat{F}(r)}{F(r)}} \approx 1 - \frac{\hat{F}(r)}{F(r)} = 1 - \frac{\frac{e^{-br}}{1-e^{-br}}}{\frac{e^2}{r}}, \quad (112)$$

resulting in the iso-Galilean invariant [72] [73] [21] iso-Schrödinger equation

$$\left[-\frac{1}{\bar{m}} \Delta_r - \frac{e^2}{r} - N \frac{e^{-br}}{1-e^{-br}} \right] |\psi(r)\rangle = E |\psi\rangle, \quad (113)$$

where N is a normalization constant, E is the binding energy and the reduced mass

$$m = m_1 m_2 / (m_1 + m_2), \quad (114)$$

is isorenormalized to the form on isospaces over isofields $\hat{m} = m/n_4$ originating from the Lorentz-Santilli isosymmetry (Sect. 3.8.1), with value for the π^0 $\bar{m} = m_e/2n_4$.

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The resulting *hadronic structure equations of the π^0* in their radial form are given by (Eqs. (5.1.14), p. 835 of Ref. [43])

$$\left[\frac{1}{r^2} \left(\frac{d}{dr} r^2 \frac{d}{dr} \right) + \bar{m} \left(E + \frac{e^2}{r} + N \frac{e^{-br}}{1-e^{-br}} \right) \right] = 0,$$

$$E_{\pi^0} = E_{e^-} + E_{e^+} - E = 135 \text{ MeV},$$

$$\tau_{\pi^0} = 2\pi\lambda^2 |\hat{\psi}(0)|^2 \frac{\alpha^2 E_e}{\hbar} = 10^{-16} s,$$

$$R_{\pi^0} = b^{-1} = 10^{-13} \text{ cm} = 1 \text{ fm}.$$
(115)

A detailed study of the analytic solution of the radial *differential* equation with the inclusion of boundary conditions (p. 836-841 of Ref. [43]), reduced the structure model of the π^0 to the solution of the following *algebraic* equations in the parameters k_1, k_2 (Eqs. (5.1.32a) and (5.1.32b), p. 840 of Ref. [43])

$$\tau_{\pi^0} = \frac{48 \times (137)^2}{4\pi b c} \frac{k_1}{(k_2 - 1)^3} = 0,$$
(116)

$$E_{\pi^0} = k_1 [1 - (k_2 - 1)2] \frac{2\hbar c}{b} = 135 \text{ MeV},$$

with energy spectrum

$$E = \frac{1}{4R^2 \bar{m}} \left(k_2 \frac{1}{n} - n \right)^2, \quad n = 1, 2, 3, \dots,$$
(117)

numeric values for the π^0

$$k_1 = 0.34, \quad k_2 = 1 + 4.27 \times 10^{-2},$$
(118)

and consequential approximate value of the Binding Energy $E \approx 0$.

Following the above 1978 proposal, the hadronic structure model of the π^0 has been applied to the structure of other particles, in each case providing a numerically exact and time invariant representation of *all* characteristics of the particle considered [91] [35].

The application of the hadronic structure model of the π^0 to the Deuteron as a bound state of a proton and a neutron has been studied extensively in the 1998 monograph [128] resulting in the iso-Galilean isoinvariant *hadronic structure equations of the Deuteron* (we cannot review to prevent a prohibitive length)

$$\left[\frac{1}{r^2} \left(\frac{d}{dr} r^2 \frac{d}{dr} \right) + \bar{m} \left(E + N \frac{e^{-br}}{1-e^{-br}} \right) \right] = 0,$$

$$E_D = E_p + E_n - E = 1875.7 \text{ MeV},$$
(119)

$$\tau_D = 2\pi\lambda^2 |\hat{\psi}(0)|^2 \frac{\alpha^2 E_e}{\hbar} = \infty,$$

$$R_D = b^{-1} = 10^{-13} \text{ cm} = 1 \text{ fm},$$

with values for the k -parameters

$$k_1 = 2.5 \quad k_2 = 1. \quad (120)$$

The following comments may be helpful for identifying some essential aspects of structure model (119):

5.7.1. As expected, *non-Hamiltonian interactions generate no binding energy*, because for $k_2 = 1$ the finite spectrum of the Hulten potential (116) solely admits the solution $E = 0$. Consequently, *the binding energy and related missing mass in nuclear structures is created by potential interactions*.

5.7.2. Model (114) contains the clearly identified Coulomb potential between the electron and the positron with ensuing well identified binding energy E . By contrast, the author could identify no physical, independently verifiable potential between the proton and the neutron. Consequently, *the representation of the total energy of the Deuteron in model (118) occurs on isospaces over isofields via the isorenormalized mass of the constituents*

$$E_{tot} = \bar{m}_1 + \bar{m}_2 = \left(\frac{m_1 + m_2}{n_4} \right), \quad (121)$$

where n_4 is a geometrization of the energy density of the Deuteron.

5.7.3. The author has no words to express the highly approximate character of model (118) and therefore, of the Deuteron conceived as a bound state of a proton and a neutron, thus suggesting the reinspection of model (118) with the synthesis of the neutron done in the next paper.

6 Concluding remarks

In accordance with the Einstein-Podolsky-Rosen argument that *quantum mechanics is not a complete theory* [1], the main aim of these papers is to present the Lie-isotopic methods as a recommendable ore-requisite for the broader Lie-admissible methods for the search (to be conducted in future papers) of new radiation-free controlled nuclear fusions of light element, which have been preliminarily presented in report [132] (see also the video on a reactor built with funds from China [133]).

Along the above lines in this paper we have reviewed and upgraded the EPR completion of quantum mechanics into *hadronic mechanics* as originally proposed in 1978 by R. M. Santilli [43] [15] [16] at Harvard University under DOE support comprising:

6.1) The technical characterization of nonpotential interactions via the conditions of variational self-adjointness presented in the 1978 monograph [15] and

their classification into Self-Adjoint/Hamiltonian (SA) interactions and Non-Self-Adjoint/non-Hamiltonian (NSA) interactions.

6.2) The proposal presented in Charts 5.2-5.4, p. 154 on of the 1983 monograph [16] of representing *stable*, thus time reversal invariant systems, via the axiom-preserving time-reversal invariant *Lie-isotopic formulations* representing SA interactions via the conventional Hamiltonian H and representing NSA interactions via the Santillian $\hat{T} > 0$. The representation is based on the completion of the universal enveloping associative algebra of quantum mechanics with generic product $AB = A \times B$ into the associativity-preserving form with product $A \hat{\times} B = A \hat{T} B$, $\hat{T} > 0$.

6.3) We have then reviewed, upgraded and specialized to nuclear physics studies conducted on hadronic mechanics by various scholars since the late 1970's [30]- [32] [19] [18] [21]-[29].

We have shown that isotopic formulations are based on:

6.i) The novel EPR entanglement characterized by the non-Hamiltonian interactions of the wave packets of particles [34].

6.ii) The violation of Bell's inequalities [38] by a system of extended spin $1/2$ particles under NSA/non- Hamiltonian interactions with ensuing existence of classical counterpart [39].

6.iii) The completion of Heisenberg's uncertainty principle for point-like particles under electromagnetic interactions into the *isouncertainrty principle* for extended particles under electromagnetic and strong interactions (Sect. 4.7) [46] which implies a progressive recovering of Einstein's determinism [1] with the increase of the density of hadronic media (decrease of the value of $\hat{T}$) in the interior of hadrons, nuclei and stars and its full recovering at the limit of gravitational collapse due to the identity of the isotopic element with Schwartzschild's horizon, Eq. (34).

6.iv) Explicit and concrete realizations [51] [52] of Bohm's hidden variables λ [48] [49] in terms of the Santillian $\hat{T} = \lambda$, as being hidden in the axiom of isoassociativity of hadronic mechanics $sA \hat{\times} (B \hat{\times} C) = (A \hat{\times} B) \hat{\times} C$, $A \hat{\times} B = A \lambda B$.

We have shown that the time reversal invariant isotopic formulations per the above outline permit the representation of *stable*, thus time reversal invariant nuclei as a collection of extended protons and neutrons in conditions of partial mutual penetration with Hamiltonian and Santillian interactions, with the consequential numerically exact and time invariant representation of the experimental data of the Deuteron conceived as a hadronic bound state of a proton and a neutron in its ground state as experimentally measured without orbital contributions.

In the hope that the methods may be applied to strong interactions at large, , we have shown that quantum mechanical nuclear models for *point-like nucleons* with sole Hamiltonian interactions and insufficient representation of nuclear data

can be uniquely and unambiguously mapped into hadronic models for *extended nucleons* with Hamiltonian and non-Hamiltonian interactions with ensuing exact representation of nuclear data via the simple *quantum nonunitary transformation* on conventional spaces over conventional fields $U \times U^\dagger = \hat{I} = 1/\hat{T}$ (Sect. 3.11), provided that the resulting model is reformulated in terms of the *hadronic isounitary transformations* on isospaces over isofields $\hat{U} \hat{\times} \hat{U}^\dagger = \hat{I} = 1/\hat{T}$ to avoid insidious inconsistencies (Sect. 3.12).

We have finally shown that, despite the indicated advances, the conception of the Deuteron as a hadronic bound state of a proton and a neutron does not allow a quantitative representation of the Deuteron stability in view of the natural instability of the neutron or of the strongly repulsive protonic Coulomb forces. These insufficiencies, combined with unsolved nuclear problems of societal relevance (such as the recycling of radioactive nuclear waste and the controlled nuclear fusion) suggests the conduction of deeper studies on nuclear structures which are presented in the subsequent paper.

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Lie-isotopic representation of stable nuclei III: Exact and time invariant representation of nuclear stability

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Abstract

In the preceding two papers, we presented exact and time invariant representations of stable nuclei via the Lie-isotopic branch of hadronic mechanics and showed the necessity for the results of lifting Heisenberg's uncertainty principle for point-like particles into the isouncertainty principle of hadronic mechanics, also called *Einstein's isodeterminism*, for extended nucleons in condition of mutual penetration. In this paper we present apparently for the first time: the treatment of protons and neutrons as *isoparticles* characterized by the isosymmetries and isorelativities of hadronic mechanics; the representation of *all* characteristics of the neutron at the nonrelativistic and relativistic levels as a hadronic bound state of an isoelectron and an isoproton; the identification of the density of the neutron in a way compatible with other experimental data; and the representation of nuclear stability despite the natural instability of the neutron and despite the extremely repulsive protonic Coulomb forces. The main implications of the above studies are: 1) The prediction of novel means for the recycling of radioactive nuclear waste by nuclear power plants via new stimulated decays. 2) The possible return to the continuous creation of matter in the universe as a consistent way to explain the 0.782 MeV missing in the neutron synthesis. 3) The apparent confirmation of the historical reduction of all matter in the universe to protons and electrons.

Keywords: nuclear physics 81V35, EPR argument, hadronic mechanics, nuclear data.¹

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1 Introduction

Inspired by the Einstein-Podolsky-Rosen view that *Quantum mechanics is not a complete theory* [1], in the preceding first paper [2] (herein referred to with the letter I) we have presented apparent insufficiencies of quantum mechanics in nuclear physics. In the preceding second paper [3] (herein referred to with the letter II) we have presented a systematic review and upgrade, specialized for the first time to nuclear physics, of the axiom-preserving time reversal invariant *Lie-isotopic branch of hadronic mechanics* [4]- [6] comprising the novel *isomathematics* (Sect. II-3), *isomechanics* (Sect. II-4) and *isosymmetries* (Sect. II-3.10) which are based on the axiom-preserving completion of the universal enveloping associative algebra of quantum mechanical Hermitean operators $\hat{\xi} : \{A, B, \dots; A \times B = AB; I\}$ into the isoassociative enveloping algebra of hadronic mechanics $\xi : \{A, B, \dots; A \hat{\times} B = A\hat{T}B; \hat{I} = 1/\hat{T}\}$ with consequential iso-Heisenberg's equation (Sect. II-5.4) [5] $idA/dt = A \hat{\times} H - \hat{\times} A = A\hat{T}H - \hat{T}A$ in which potential interactions are represented by the Hamiltonian H and non-potential interactions are represented by the isotopic element $\hat{T}$ at times called the Santillian.

We have shown in paper [2] that the Lie-isotopic mechanics implies:

1.1) The first known representation invariant over time of the dimension, shape and density of particles via the isotopic element $\hat{T}$ per realization (II-15) [5].

1.2) The first known quantitative representation of subluminal, continuous and instantaneous interactions of entangled particles at arbitrary distances characterized by the non-Hamiltonian interactions of the wave packets of particles [7].

1.3) The first known explicit and concrete realizations of Bohm's hidden variables λ [8] [9] in terms of the Santillian $\lambda = \hat{T}$, as being hidden in the axiom of associativity of quantum mechanics [10] [11].

1.4) The violation of Bell's inequalities [12] by a system of extended particles with spin 1/2 under non-Hamiltonian interactions with ensuing existence of classical counterpart [13].

1.5) The EPR completion of Heisenberg's uncertainty principle for point-like particles under electromagnetic interactions into the *isouncertainrty principle* of hadronic mechanics, also called *Einstein's isodeterminism*, for extended particles under electromagnetic and strong interactions (Sect. 4.7) [14] (see [15] for an extended derivation), which principle implies a progressive recovering of Einstein's determinism [1] with the increase of the density of hadronic media (decrease of the value of $\hat{T}$) in the interior of hadrons, nuclei and stars and its full recovering at the limit of gravitational collapse due to the identity of the isotopic element with Schwartzschild's horizon, Eq. (II-34).

It should be indicated that none of the results of paper I or of this paper are possible under the validity for nuclear structures of Heisenberg's uncertainty prin-

ciple for electromagnetic interactions, while the results are all possible for the completion of said principle into Einstein's isodeterminism for strong interactions (Sect. 3.3.1).

We have then shown in the preceding paper I that the Lie-isotopic branch of hadronic mechanics has permitted the achievement of the first known, numerically exact and time invariant representation of the experimental data of the Deuteron conceived as a hadronic bound state of a proton and a neutron in its ground state without orbital contributions used in experimental measurements.

Based on tabulated fundamental characteristics of electrons, protons and neutrons [16]-[23], in this paper we study, apparently for the first time, nuclear stability despite the neutron natural instability and the strongly repulsive protonic Coulomb forces via the following primary steps:

1.A) The EPR completion of the 20th century notion of *particle*, as a unitary irreducible representation of the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1) = SL(2.C) \times \mathcal{T}$ into that of *isoparticle*, as an isounitary, isoirreducible isorepresentation of the isospinorial isosymmetry $\hat{\mathcal{P}}(3.1) = \hat{SL}(2.\hat{C}) \hat{\times} \hat{\mathcal{T}}$ [24] [25] [26] which appears to be necessary under non-Hamiltonian interactions between extended particles due to an apparent mutation of the *intrinsic characteristics* of 20th century particles predicted in the 1978 memoir [27].

1.B) A review of extensive studies in the synthesis of the neutron from the Hydrogen atom in the core of stars as a hadronic bound state of an isoelectron $\hat{e}^-$ totally compressed and entangled inside the dense isoproton $\hat{p}^+$.

1.C) The representation of nuclear stability despite the natural instability of the neutron via the *decoupling* of the neutron into its original, *permanently stable* constituents, the isoproton and the isoelectron, resulting in a three-body structure of the Deuteron.

1.D) A representation of the nuclear stability despite the very big, protonic Coulomb *repulsion* via isoprotons.

1.E) The prediction of means for the recycling of radioactive nuclear waste by nuclear power plants via *new* stimulated decays.

1.F) The possible return to the continuous creation of matter in the universe as a consistent way to explain the 0.782 MeV missing in the neutron synthesis.

1.G) The apparent confirmation of the historical reduction of all matter in the universe to protons and electrons.

As we shall see, the primary applications of the above representation of nuclear stability are given by:

1.i) The prediction of means for the recycling of radioactive nuclear waste by nuclear power plants via new stimulated decays.

1.ii) The possible return to the continuous creation of matter in the universe a consistent way to explain 0.782 MeV missing in the neutron synthesis.

1.iii) The apparent confirmation of the historical reduction of all matter in the universe to protons and electrons.

the prediction of basically new means for the recycling of radioactive nuclear waste via their stimulated decay by the nuclear power plants themselves, the apparent reduction of all matter in the universe to protons and electrons in conditions of increasing complexity and the apparent *origin* (rather than the sole description) of gravitation in the nuclear structure.

The understanding of the main lines of this paper can be achieved via a knowledge of Ref. [2], although a technical understanding of this paper can be best achieved via a full knowledge of hadronic mechanics [4]-[6].

2 Isoparticles and isorelativities

To our best knowledge, the notion of *mutation* of elementary particles was first introduced in Sect. 5, p. 819 of the 1978 Harvard University memoir [27] to denote the expected alteration of the *intrinsic* characteristics of particles under strong interactions. To upgrade the notion of mutation with the advances occurred in the meantime, let us recall also that, according to 20th century physics, a spin $1/2$ *particle* [28] is a unitary irreducible representation of the *spinorial* covering $\mathcal{P}(3.1) = SL(2.C) \times \mathcal{T}(3.1)$ of the Poincaré symmetry $P(3.1) = SO(3.1) \times \mathcal{T}(3.1)$ (Sect. II-3.10).

The impossibility of representing the indicated mutation with conventional spacetime symmetries and relativities has suggested the following generalized notion of particles introduced at the nonrelativistic level in the 1991 paper [29] and at the relativistic level in Sect. 7.7, p. 312 on of the 1995 monograph [5].

DEFINITION 2.1: A '*relativistic isoparticle*' is an *isounitary isoirreducible isorepresentation of the spinorial covering of the Poincaré symmetry*

$$\hat{\mathcal{P}}(3.1) = \hat{S}L(2.\hat{C}) \hat{\times} \hat{\mathcal{T}}(3.1), \quad (1)$$

formulated on the Hilbert-Myung-Santilli isospace over an isofield (Sect. II-4.3).

The Galilean isorelativity for the non-relativistic dynamics of extended objects within resistive media has been proposed in Refs. [30] [31] and cannot be reviewed for brevity (see the excellent general review [32]). The isotopies of special relativity have been first introduced in the 1983 paper [24], then developed in papers [25] [26], in the 1991 monographs [30] [31] and upgraded in the 2021 overview [33] (see independent reviews [32] [34] [35]). A conceptual review the isorelativities appears recommendable in view of their applications in subsequent sections.

DEFINITION 2.2: The term 'isorelativities' denotes the infinite family of axiom-preserving isotopies of special relativity characterized by the infinite family of isotopic elements and related isounits of (contravariant) type (II-15), represented in terms of the characteristic quantities $n_\mu > 0$ of paper [2] as well as in terms of the characteristic quantities $b_\mu = 1/n_\mu > 0$ used in the literature of the 1990's

$$\hat{T} = 1/\hat{I} = \text{Diag.}(n_1^2, b_2^2, b_3^2, b_4^2) \equiv \text{Diag.}(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2}), \quad (2)$$

where the exponential term of Eq. (II-15) has been incorporated in the characteristics quantities and can be factorized whenever needed.

All isorelativities are formulated on an iso-Minkowski isospace $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I}$ over the isoreal isofield $\hat{\mathcal{R}}$ (Sect. II-3.5) with isocoordinates $\hat{x} = x\hat{I} = (\hat{x} - \mu)\hat{I} = (x^k, x^4 = ct)\hat{I}$, isometric $\hat{\Omega} = \hat{\eta}\hat{I}$, $\hat{\eta} = \hat{T}\eta$, where $\eta = \text{Diag.}(1, 1, 1, 1)$, isoinvariant in isospace

$$(\hat{x} - \hat{y})^{\hat{2}} = (\hat{x} - \hat{y})^\mu \hat{\times} \hat{\Omega}_{\mu\nu} \hat{\times} (\hat{x} - \hat{y})^\nu \quad (3)$$

and its projection in our spacetime $M(x, \eta, I)$ over the field of real numbers $\hat{\mathcal{R}}$

$$\begin{aligned} & \left[\frac{(x^1 - y^1)^2}{n_1^2} + \frac{(x^2 - y^2)^2}{n_2^2} + \frac{(x^3 - y^3)^2}{n_3^2} - \frac{(x^4 - y^4)^2}{n_4^2} \right] \hat{I} = \\ & = [b_1^2(x^1 - y^1)^2 + b_2^2(x^2 - y^2)^2 + b_3^2(x^3 - y^3)^2 - b_4^2(x^4 - y^4)^2] \hat{I}, \end{aligned} \quad (4)$$

under the universal isosymmetry (1), with ensuing unique and unambiguous characterization of relativistic isolaws (Sect. 8.4.4, p. 134 on of [33]).

Besides the admission of special relativity for the particular case $\mu = 1$, $\mu = 1, 2, 3, 4$, the most important isorelativities are given by:

2.1. Special isorelativity, which occurs for the particular case $\mu = \text{constant} \neq 1$, $\mu = 1, 2, 3, 4$, for the characterization of extended-spheroidal particles with semiaxes n_1^2, n_2^2, n_3^2 moving within a hadronic medium with density $n_4 = 1/b_4$ (see the isoaxioms of special isorelativity, p. 138 of [33] and Refs. [24]-[35]).

2.2. Exterior isogravitation, which occurs for the factorization of the isotopic element $\hat{T}_{gr}$ from any nonsingular Riemannian metric $g(x)$ for the exterior treatment of gravitation and its formulation in terms of the isounit $\hat{I} = 1/\hat{T}$ of relativistic hadronic mechanics (first presented at the 1992 *Marcel Grossmann Meeting in Gravitation*, [36] [37])

$$\begin{aligned} g(x) &= \hat{T}_{gr}(x)\eta, \\ \hat{I}_{gr}(x) &= 1/\hat{T}_{gr}(x), \end{aligned} \quad (5)$$

(whose positive-definite character is guaranteed by the non-singularity of the metric $g(x)$) under which realization the isoaxioms of special isorelativity, p. 138 of [33] remain valid for $n_\mu = n_\mu(x) > 0$, with particular realization given by the first Schwartzchild's paper on the exterior gravitational problem [38]

$$\hat{T}_{kk} = \frac{1}{1 - \frac{2M}{r}}, \quad \hat{T}_{44} = 1 - \frac{2M}{r}. \quad (6)$$

2.3. Interior isogravitation, holding for general nonsingular, symmetric, 3+1-dimensional isometrics $\Omega_{\mu\nu}$ with an unrestricted functional dependence on all needed local variables of type (II-11), when represented via the isodifferential calculus (Sect. II-3.7) and related iso-Minkowskian geometry [39] for the study of the *origin* (rather than the sole exterior description) of gravitation in the nuclear structure [40] [45] under which assumptions the isoaxioms of p. 138, Ref. [33], remain valid for an arbitrary functional dependence of the characteristic quantities $n_\nu > 0$.

These studies are along the second Schwartzchild's paper on the interior gravitational problem [42], which paper has been vastly ignored because incompatible with general relativity, but compatible with interior isogravitation.

Note that realization (6) of the Santillian is a particular case of the nuclear analysis of [2], by therefore indicating the possible primary source of gravitation in the *electromagnetic* structure of nuclei according to the 1905 historical hypothesis by H. Poincaré [43] and its apparent realization by R. M. Santilli in the mid 1970's when at MIT [44] under the *necessary* resolution of the insufficiencies of 20-th gravitation into a form compatible with all century sciences [45].

A realization of the isoalgebra of isosymmetry (1) is given by [26]

$$\hat{R}_k = \frac{1}{2}\epsilon_{kij}\hat{\Gamma}_i \hat{\times} \hat{\Gamma}_j, \quad \hat{B}_k = \frac{1}{2}\hat{\Gamma}_k \hat{\times} \hat{\Gamma}_4, \quad k = 1, 2, 3, \quad (7)$$

where $\hat{R}_k$ represents the generators of *isorotations*, $\hat{B}_k$ represents the generators of the *isoboosts* and $\hat{\Gamma}_\mu = \hat{\gamma}_\mu \hat{I}$ represent the Dirac-Santilli isomatrices (II-96), while a realization of the isotranslations $\hat{\mathcal{T}}(3.1)$ is given by 4-isomomentum (II-68) $\hat{P}_\mu$, $\mu = 1, 2, 3, 4$.

The isocommutation rules of isosymmetry (1) with generators $\hat{J}_{\mu\nu} = \{\hat{R}_k, \hat{S}_k\}$ are then given by the spinorial covering of Eqs. (II-40)

$$\begin{aligned} \left[\hat{J}_{\mu\nu}, \hat{J}_{\alpha\beta} \right] &= \hat{\eta}_{\nu\alpha} \hat{J}_{\beta\mu} - \hat{\eta}_{\mu\alpha} \hat{J}_{\beta\nu} - \hat{\eta}_{\nu\beta} \hat{J}_{\alpha\mu} + \hat{\eta}_{\mu\beta} \hat{J}_{\alpha\nu}, \\ \left[\hat{J}_{\mu\nu}, \hat{P}_\alpha \right] &= i\hat{\eta}^{\mu\alpha} \hat{P}_\nu - \hat{\eta}_{\nu\alpha} \hat{P}_\mu, \quad \left[\hat{P}_\mu, \hat{P}_\nu \right] = 0, \\ \hat{\eta}_{\mu\nu} &= \hat{T} \eta = (\hat{T}_\rho^\mu \eta^{\rho\nu}), \end{aligned} \quad (8)$$

while the iso-Casimir invariants are given by the spinorial covering of isoinvariants (II-41)

$$\begin{aligned}\hat{C}_1 = \hat{I} > 0, \quad \hat{C}_2 = \hat{P}^2 = \Sigma_{k=1,2,3}(b_k^2 \hat{P}_k^2 - m^2 c^2 b_4^2), \\ \hat{C}_3 = \hat{W}^2 = \hat{W}_\mu \hat{\times} \hat{W}^\mu, \quad \hat{W}_\mu = \epsilon_{\mu\alpha\beta\rho} \hat{J}_{\alpha\beta} \hat{\times} \hat{P}_\rho.\end{aligned}\tag{9}$$

In this paper, we shall also use the explicit form of the *hadronic angular momentum* $\hat{O}(3)$ as an isosubgroup of isosymmetry (1) (Eqs. (6.4a) (6.4b) of Ref. [26] and Sect. II-5.3) and related isoeigenvalues

$$\hat{L}_k = \epsilon_{ijk} \hat{r}_i \hat{\times} \hat{p}_j, \quad [\hat{L}_1, \hat{L}_j] = \epsilon_{ijk} n_k^2 \hat{L}_k = \epsilon_{ijk} b_k^2 \hat{L}_k,\tag{10}$$

$$\hat{L}_3 \hat{\times} |\psi\rangle = \pm \frac{1}{2} b_1^{-1} b_2^{-1} |\psi\rangle, \quad \hat{L}^2 \hat{\times} |\psi\rangle = (b_1^2 b_2^2 + b_1^2 b_3^2 + b_3^2 b_1^2) |\psi\rangle,$$

as well as the realization of the *hadronic spin* and related isoeigenvalues (see Eqs. (6.4c) (6.4d), p. 190 of Ref. [26] and Sect. II-5.5)

$$\hat{S}_k = \frac{1}{2} \epsilon_{ijk} \hat{\gamma}_i \hat{\times} \hat{\gamma}_j, \quad [\hat{S}_i, \hat{S}_j] = \epsilon_{ijk} b_k^{-2} \hat{S}_k,\tag{11}$$

$$\hat{S}_3 \hat{\times} |\psi\rangle = \pm \frac{1}{2} b_1 b_2 |\psi\rangle, \quad \hat{S}^2 \hat{\times} |\psi\rangle = \frac{1}{4} (b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2) |\psi\rangle.$$

Some of the main features occurring for all isorelativities are given by [33]:

2.A) The *invariance* of events described by all isorelativities under isosymmetry (1) which is fundamental for the prediction of the same numeric values under the same conditions but at different times (Sect. II-3-12). By contrast, covariant theories (such as general relativity) violate this fundamental condition for experimental verifications that will pass the test of time.

2.B) The admission of arbitrary speeds of light within hadronic media with subluminal (superluminal) speeds, i.e., for media with low (high) density,

$$C = c/n_4,\tag{12}$$

$$C < c, \quad n_4 > 1, \quad C > c \quad n_4 < 1.$$

2.C) Under the assumption of perfect sphericity, the isorenormalization of the energy of extended particles within a hadronic medium called the *mass-energy isoequivalence principle* (see again the isoaxioms of special isorelativity, p. 138 of [33], particularly Eq. (133), p. 139)

$$E = mc^2 \rightarrow \tilde{E} = mC^2 = \frac{n_k^2}{n_4^2} E = \frac{b_4^2}{b_k^2} E,\tag{13}$$



Figure 1: In this figure, we illustrate some of the hadronic reactors built by Thunder Energies Corporation (now Hadronic Technologies Corporation) for the synthesis of neutrons from a commercially available Hydrogen gas.

where $n_k^2 = 1/b_k^2$ is the semiaxis of the considered particle in the direction of its spin, and the difference $\tilde{E} - E$ is due to the *nonlocal effects* at the foundation of hadronic mechanics, namely, the effects in the mass-energy equivalence on a given mass from its hadronic medium under a state of deep EPR entanglement [7].

As we shall see in the next section, the above realization of isosymmetry (1) and related notion of isoparticles is sufficient for a consistent representation of the synthesis of the neutron in a star and for the consequential study of nuclear stability, with the understanding that progressively broader realizations of isoparticles are needed for progressively more complex physical conditions.

3 Representation of the neutron synthesis from the Hydrogen

3.1. Historical notes. As it is well known, stars initiate their lives as an aggregate of Hydrogen that grows by accretion during travel in interstellar spaces. When the temperature in the core of the aggregate reaches a value of the order of $10^8 K$, E. Rutherford [46] suggested in 1920 that the Hydrogen atom is "compressed" into a new neutral particle which he called the *neutron* [16]-[23]

$$e^- + p^+ \rightarrow n. \quad (14)$$

The existence of the neutron was experimentally established in 1932, by J. Chadwick [47]. In 1933, W. Pauli [48] pointed out that synthesis (11) violates the

conservation of the angular momentum. Therefore, E. Fermi [49] submitted in 1935 the hypothesis that the synthesis of the neutron occurs with the joint *emission* of a neutral and massless particle ν with spin $1/2$ which he called the *neutrino* (meaning "little neutron" in Italian)

$$e^{-} + p^{+} \rightarrow n + \nu, \quad (15)$$

with the neutron decay

$$n \rightarrow p^{+} + e^{-} + \bar{\nu}, \quad (16)$$

where $\bar{\nu}$ is the antineutrino.

Subsequent tests [22] established that *the neutron is naturally unstable* with a mean life of $\tau = 877$ and spontaneous decay

$$n \rightarrow e^{-} + p^{+} + \bar{\nu}, \quad (17)$$

where $\bar{\nu}$ is the *antineutrino*.

Predictably, the synthesis of the neutron from the Hydrogen attracted attention soon following Chadwick's confirmation. According to the historical account [50] Ernest J. Sternglass conducted in 1951 the first test for the laboratory synthesis of the neutron from the Hydrogen, followed by tests in 1952 by E. Trownson and others, although none of these initial tests were reported in published papers in view of the incompatibility of the neutron synthesis with quantum mechanics and for other reasons.

To the author's best knowledge, the first published tests on the laboratory synthesis of the neutron from the Hydrogen were done in the 1960's by the Italian priest-physicist Don Carlo Borghi and his associates [51]. In essence the experimentalists constructed a cylindrical metal chamber (called *klystron*) which was filled up with a Hydrogen gas at a fraction of 1 *bar* pressure kept mostly ionized by an electric arc with about 500 *V* and 10 *mA*. Additionally, the gas was traversed by microwaves with the frequency of 10^{-10} *s*. The experimentalists then placed in the exterior of the Klystron various materials suitable to be activated when exposed to a neutron flux (such as gold or silver).

Following exposures over several weeks, the experimentalists reported clear and reproducible nuclear transmutations that can only be due to a neutral hadron emitted from the Klystron. Due to insufficient evidence on neutron emission, the experimentalists conjectured that the detected nuclear transmutations were due to a new neutral particle with the mass of the neutron but spin different than $1/2$ that they called the *neutroid*.

3.2. Recent studies. In view of its fundamental character in nuclear physics, R. M. Santilli and his associates have conducted over the past four decades mathematical, theoretical, experimental and industrial research on the synthesis of the

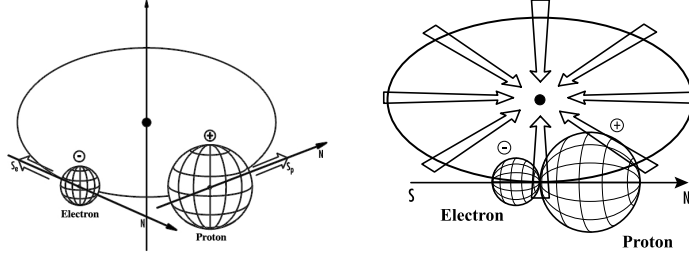


Figure 2: In this figure, we illustrate the mechanism used by hadronic reactors (Fig. 1) for the synthesis of neutroids and neutrons via an especially designed DC electric arc between Carbon electrodes submerged within a Hydrogen gas causing the ionization of Hydrogen atoms into electrons and protons by the activation of the arc (left view) and the compression of the electron within the dense proton by the de-activation of the arc (right view).

neutron from the Hydrogen atom in the core of stars as well as in laboratory (see theoretical studies [52]-[58], experimental works [59]-[65] and independent studies [66]-[71]).

The studies were initiated in the late 1970's at Harvard University under DOE support with the inapplicability of quantum mechanics for the neutron synthesis [27], followed by the proposal to construct *hadronic mechanics* in the 1978-1983 Springer-Verlag monographs [72] [73].

By far the biggest difficulty of the above studies was the representation of the spin of the neutron $S_n = 1/2$ from two particles each having spin $1/2$ as originally conceived by Rutherford [46]. This problem stimulated the construction of the Lie-Santilli isothory (Sect. 4.4, p. 173 on of Ref. [73] and Sect. II-3.2), followed by systematic studies on the isotopies of spacetime symmetry reported in Sect. II-3.10, with particular reference on the isotopies $\hat{\mathcal{SO}}(3)$ of the angular momentum symmetry $\mathcal{SO}(3)$ at the classical and operator levels [74] [75].

As a result of these preparatory studies, Santilli was able to achieve a numerically exact and time invariant representation of *all* characteristics of the neutron at the nonrelativistic level in the 1990 paper [52] and at the relativistic level in the 1995 paper [56] with a more extensive study in monograph [57].

Following, and only following, the achievement of a consistent representation of the neutron synthesis via the Lie-isotopic branch of hadronic mechanics, Santilli initiated in 2007 experimental tests on the laboratory synthesis of the neutron from the Hydrogen [59]-[63]. According to these studies, the neutron synthesis from the Hydrogen can be generated by *hadronic reactors* (Fig. 1) consisting of a metal vessel containing in its interior a commercially available Hydrogen gas at pressure and a pair of submerged carbon electrodes powered by a specially

Exact Lie-isotopic representation of nuclear stability

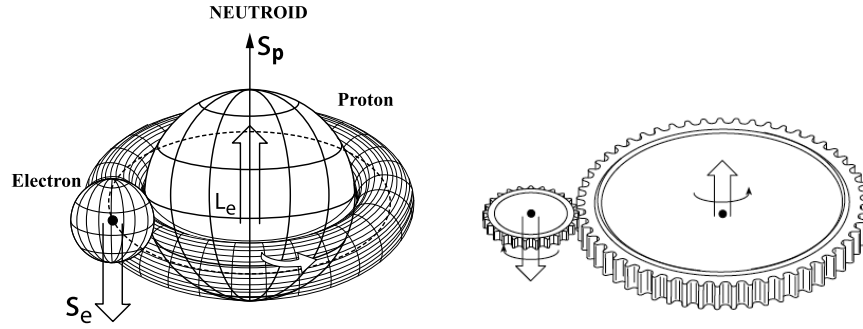


Figure 3: In the left of this figure, we illustrate the predicted structure of the neutroid in its ground state as a hadronic bond of electrons and protons in singlet couplings with null eigenvalues of the angular momentum and of the spin under their very big Coulomb attraction. In the right of this figure, we present a conceptual gear equivalent of the left view to illustrate the reason for the half life of neutroids as being about 10% that of neutrons, i.e., of about 8 s (which is extremely big for particle standards).

designed, external DC source via a gap controllable from the outside. During operations, the DC arc is continuously connected and disconnected because of the consumption of the carbon electrodes. During its activation (left of Fig. 2), the special form of the DC arc ionizes the Hydrogen gas by creating a plasma mostly composed by protons and electrons in its cylindrical surroundings, while during its deactivation (right of Fig. 2), the DC arc compresses the ionized gas from all directions toward its symmetry axis, by realizing in laboratory Rutherford compression of the electron inside the proton.

Tests [59]- [65] have confirmed: 1) The production of Don Borghi's neutroids (Fig. 3) for DC power of the order of 5 kw, gas pressure of 5 psi and electrode gap of 2 mm. 2) The production of neutrons (Fig. 4) for powers of the order of 50 kw, gas pressure from 10 psi on and electrode gap of 5 mm. In particular, the synthesis of neutroids resulted to be an unavoidable step prior to the synthesis of the neutron.

Following, and only following, sufficient experimental evidence on the laboratory synthesis of the neutron from a Hydrogen gas, Santilli founded in 2012 the U. S. publicly traded company *Thunder Energies Corporation* (now the privately held *Hadronic Technologies Corporation*, www.hadronictechnologies.com) for the production and sale of *Directional Neutron Source* (DNS), namely, equipment producing a beam of low energy neutrons (with energy less than 100 eV) in a preferred direction and in the desired intensity for the detection of fissionable material that may be hidden in baggage (lower left view of Fig. 1).

Interested readers should be aware that commercially available AC-DC power

units to create electric arcs between carbon electrodes submerged within a Hydrogen gas may synthesize neutroids (Fig. 3) and other unstable states under the big electron-proton Coulomb attraction (Eq. I.1), but said power units are not designed to synthesize neutrons via the compression of electron inside the proton.

3.3. Nonrelativistic representation of the neutron synthesis from the Hydrogen. Studies [52]-[56] were conducted under the assumptions (dating back to the 1978 memoir [27]) that:

3.3.A) The angular momentum of the electron compressed inside the proton is *constrained* to be equal to the spin of the proton to prevent big resistive forces caused by the motion of its extended wave packet against the dense medium in the interior of the proton.

3.3.B) When compressed inside the dense proton, the electron e is mutated into a new particle called the *eleton* in Sect. of [27] and hereon called the *isoelectron* $\hat{e}^-$, while the proton is mutated into the *isoproton*, although in first approximation the proton can be assumed to be unmutated, $\hat{p}^+ \approx p^+$ since it is about 1800 times heavier than the electron.

3.3.C) While the numeric value of the mean life of the resulting hadronic state may be subject to scientific debates, the existence of a bound state between the electron and the proton at short distance should be beyond scientific doubt in view of their extremely big Coulomb attraction.

The above assumptions imply the following *structure model of the neutron according to hadronic mechanics* (hm) which can be constructed via isounitary transformations (Sect. 3.12) of the quantum mechanical (qm) model of the Hydrogen atom

$$n = (\hat{e}_\downarrow^-, \hat{p}_\uparrow^+)_{hm} = \hat{U} \hat{\times} (e_\downarrow^-, p_\uparrow^+)_{qm} \hat{\times} \hat{U}^\dagger, \quad (18)$$

under the Coulomb *attraction* in the macroscopic value of 230 Newton,

$$F = -\frac{e^2}{r^2} = -(8.99 \times 10^9) \frac{(1.60 \times 10^{-19})^2}{(10^{-15})^2} = -230 \text{ Newtons}. \quad (19)$$

3.3.1. The fundamental role of Einstein's isodeterminism. The most serious insufficiency of quantum mechanics in nuclear physics identified in [2] has been the prohibition by Heisenberg's uncertainty principle to represent the synthesis of the neutron from the hydrogen in the core of stars, because when said principle is applied to the coordinate r_e and momentum p_e of an electron in the core of stars,

$$\Delta r \Delta p = \frac{1}{2} | \langle \psi(r_e) | \times [r_e, \hat{p}_e] \times | \psi(r_r) \rangle \geq \frac{1}{2} \hbar, \quad (20)$$

it implies standard deviations of the coordinate bigger than the neutron radius with consequential kinetic energy of the electron bigger than the neutron mass,

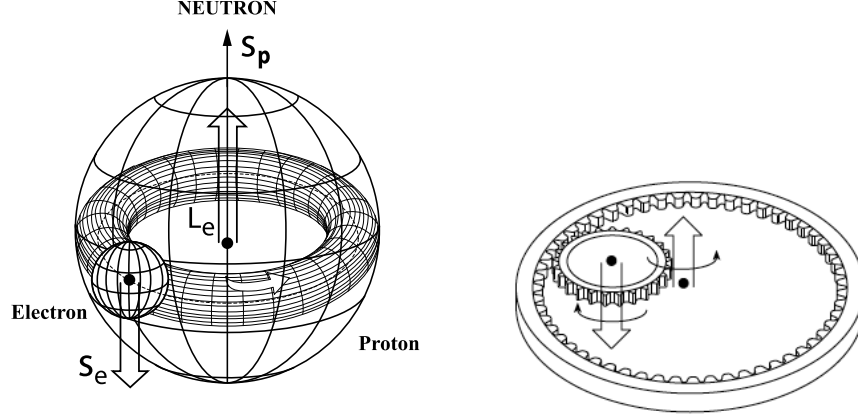


Figure 4: In the left of this figure, we illustrate the compression of the neutroid (Fig. 3) into the neutron (Fig. 4), resulting in a constrained hadronic angular momentum of the electron within the dense medium inside the proton that, to avoid extreme resistive forces, has to be equal to the proton spin with ensuing total angular momentum $1/2$.

Eq. (I-3), i.e.,

$$\Delta r_e > R_n = 0.87 \times 10^{-13} \text{ cm},$$

$$\Delta v_e > \frac{\hbar}{\Delta r_e \times m_e} > 10^{10} \text{ m/s}, \quad (21)$$

$$\Delta K_e = \frac{1}{2m_e} \times (\Delta p_e)^2 > m_n = 939.56 \text{ MeV}/c^2.$$

By recalling that the validity of the uncertainty principle for point-like particles under electromagnetic interactions (as occurring in the atomic structure) is beyond doubt, its validity for nuclear constituents is a personal opinion by the individual physicist due to the lack of any direct experimental verification denounced beginning with the title of the 1978 Harvard memoir [27] while, by comparison, the synthesis of the neutron from the electron and the proton in the core of stars is a physical reality.

It is easy to see that the *isouncertainty principle* of hadronic mechanics (Sect. II-4.7) [1] [14] [15]

$$\Delta r \Delta p = \frac{1}{2} | \langle \hat{\psi}(\hat{r}) | \star [\hat{r}, \hat{p}] \star | \hat{\psi}(\hat{r}) \rangle | < \frac{1}{2} \hbar \hat{T} \ll 1, \quad (22)$$

resolves the above impasse, because in view of very small value of $\hat{T}$, Eq. (II-15),

the isotopic image of values (51)

$$\begin{aligned}\Delta r_e &\leq R_n = 0.87 \times 10^{-13} \text{ cm}, \\ \Delta v_e &\ll 10^{10} \text{ m/s},\end{aligned}\tag{23}$$

$$\Delta K_e = \frac{1}{2m_e} \times (\Delta p_e)^2 \ll m_n = 939.56 \text{ MeV}/c^2,$$

implies standard deviations of the coordinates of the order of the neutron radius and electron kinetic energy much smaller than the neutron mass.

3.3.2. Representation of the neutron spin. Assumption 3.3.A requires that the hadronic angular momentum of the isoelectron be equal to the spin of the proton, thus having value $\hat{L}_{3,\hat{e}} = 1/2$. The study of this assumption was initiated in the 1984 papers on the isotopies of the rotational symmetry [74] [75] and presented in the 1990 paper [52] via the isotrigonometric isofunctions (see also p. 304 on of Ref. [4]), as well as the use of the Lie-Santilli isoalgebra $\hat{so}(3)$. We here study, apparently for the first time, the nonrelativistic representation of the hadronic angular momentum $\hat{L}_3 = 1/2$ of the isoelectron in model (18) under the assumptions in first approximation that *the orbit of the isoelectron is contained in a plane and it is a perfect circle with radius $R = 10^{-13} \text{ cm}$* . In fact, deviations from the above assumptions imply instabilities generally preventing a representation of the neutron mean life of 887 s. Under the above assumptions, the acting isosymmetry is the two-dimensional Lie-Santilli isogroup $\hat{O}(2)$ [74] [75] (see also Sect. 6.4, p. 233 on of [5]).

Consider the conventional $O(2)$ symmetry which is classically formulated on the two-dimensional Euclidean space $E(z, \delta, I)$, and quantum mechanically treated on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$. In accordance with the simple rule of Sect. II-3.9 (for the construction of isotopic images of quantum models to add the representation of extended nucleons and their non-Hamiltonian interactions), we map the entire classical and quantum mechanical formulation of $O(2)$ under the nonunitary transformation

$$\begin{aligned}UU^\dagger &= \hat{I} = 1/\hat{T} == \text{Diag.}(n_1^2, n_2^2), = \\ &= \text{Diag.}(b_1^{-2}, b_2^{-2}), \quad b_k = 1/n_k > 0, \quad k = 1, 2,\end{aligned}\tag{24}$$

and introduce Bohm's hidden variable [8]

$$\frac{1}{n_1} = \frac{1}{n_2} = b_1 = b_2 = \lambda > 0.\tag{25}$$

Therefore, the isorepresentation occurs in the two-dimensional iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I})$ over the isofield $\hat{\mathcal{C}}$ with isocoordinates

$$\hat{r} = r\hat{I} = \{x, y\}\lambda^2 I_{2 \times 2},\tag{26}$$

Exact Lie-isotopic representation of nuclear stability

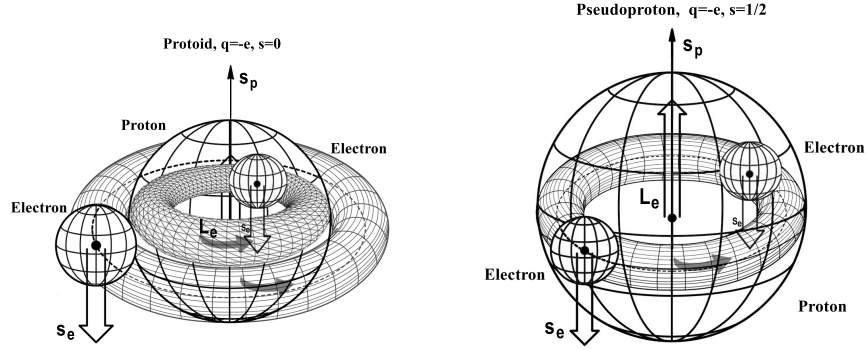


Figure 5: In this figure, we illustrate the prediction by hadronic mechanics of the synthesis in the core of stars of negatively charged strongly interacting particles characterized by a hadronic singlet bond between an electron and a neutron, with null total spin, as well as the pseudoproton on the right with total spin 1/2, and other hadronic states [101].

isometric

$$\hat{\Delta} = (\hat{\delta})\hat{I}, \hat{\delta} = \hat{T}\delta = \lambda^2\delta, \quad (27)$$

isoinvariant

$$\hat{r}^2 = \lambda^2 r^2, \quad (28)$$

and isotrigonometric representation (App. 5C, p. 300 on of Ref. [4])

$$x = r\lambda^{-1}\cos\hat{\phi}, \quad y = r\lambda^{-1}\sin\hat{\phi}, \quad (29)$$

$$\hat{\phi} = T_\phi\phi = n_1n_2\phi = \lambda^{-2}\phi \quad \hat{T}_\phi = b_1 - 1b_2^{-2} = \lambda^{-2}.$$

The isounitary isoirreducible isorepresentations of $\hat{O}(2)$ is defined on a Hilbert-Myung-Santilli isospace $\mathcal{H}$ over the iso-complex isofield $\mathcal{C}$ with iso-states $|\hat{\psi}(\hat{r})\rangle$, isonormalization

$$\langle \hat{\psi}(\hat{r}) | \hat{\times} | \hat{\psi}(\hat{r}) \rangle = \hat{I}, \quad (30)$$

isoeigenvalues

$$\begin{aligned} \hat{R}(\hat{\phi}) \hat{\times} | \hat{\psi} \rangle &= \hat{e}^{iM\hat{\phi}} \hat{\times} | \hat{\psi} \rangle = (e^{i\hat{M}\hat{\phi}}) \hat{I}_\psi \hat{\times} | \hat{\psi} \rangle = (e^{i\lambda^2 M\hat{\phi}}) | \hat{\psi} \rangle, \\ \hat{M} &= b_1 b_2 M = \lambda^2 M, \end{aligned} \quad (31)$$

(where $\hat{e}$ is, this time, the isoexponentiation in the ϕ -plane) with Lie-Santilli isogroup laws

$$\begin{aligned} \hat{R}(\hat{\phi}) \hat{\times} \hat{R}(\hat{\phi}') \hat{\times} | \hat{\psi} \rangle &= \hat{R}(\hat{\phi}') \hat{\times} \hat{R}(\hat{\phi}) \hat{\times} | \hat{\psi} \rangle = \hat{R}(\hat{\phi} + \hat{\phi}') \hat{\times} | \hat{\psi} \rangle, \\ \hat{R}(\phi) \hat{\times} \hat{R}(-\phi) \hat{\times} | \hat{\psi} \rangle &= \hat{R}(0) \hat{\times} | \hat{\psi} \rangle = | \hat{\psi} \rangle. \end{aligned} \quad (32)$$

The isoeigenvalue of the hadronic angular momentum $\hat{L}$ is given by

$$\hat{L} \times |\hat{\psi}\rangle = \hat{M} |\hat{\psi}\rangle = \lambda^2 M |\hat{\psi}\rangle. \quad (33)$$

But isotopies preserve original numeric values. Therefore,

$$\hat{M} = \lambda^2 M = 0, 1, 2, 3, \dots \quad (34)$$

Consequently, the angular momentum measured by the experimentalist is given by

$$M = \frac{\hat{M}}{\lambda^2}, \quad (35)$$

and can represent the *constrained* angular momentum of the isoelectron inside the proton for the value

$$M = \frac{\hat{M}}{\lambda^2} = \frac{1}{2}, \quad \lambda = \sqrt{2} = 1.4142. \quad (36)$$

The total spin of the neutron is then given by

$$S_n = s_p + s_e + L_e = \frac{1}{2} - \frac{1}{2} + \frac{1}{2}, \quad (37)$$

and therefore, *the spin of the neutron coincides with the spin of the proton as expected.*

For brevity, we leave to the interested reader the representation of the spin 1/2 of the isoelectron via the isosymmetry $\hat{SO}(2)$, which can be derived from the above treatment with the $\hat{O}(2)$ symmetry with Santillian

$$\hat{T} = \text{Diag.} \left(\frac{1}{n_1}, \frac{1}{n_2} \right) = \text{Diag.}(b_1, b_2) = \text{Diag.}(\lambda, \lambda^{-1}), \quad \text{Det.} \hat{T} = I, \quad (38)$$

and the use of suitable isostates to separate positive and negative eigenvalues.

An important rule for the isorepresentation of the neutron synthesis from the Hydrogen atom is that *the total angular momentum of the isoelectron compressed inside the proton is identically null.* As we shall see, in case confirmed, the above rule appears to have basic implications for models of nuclear structures according to hadronic mechanics.

The spin of the neutroid according to Fig. 3 is characterized by the following value for the hadronic angular momentum of the isoelectron

$$M = \frac{\hat{M}}{\lambda} = 0, \quad \hat{M} = 0, \quad \lambda > 0. \quad (39)$$

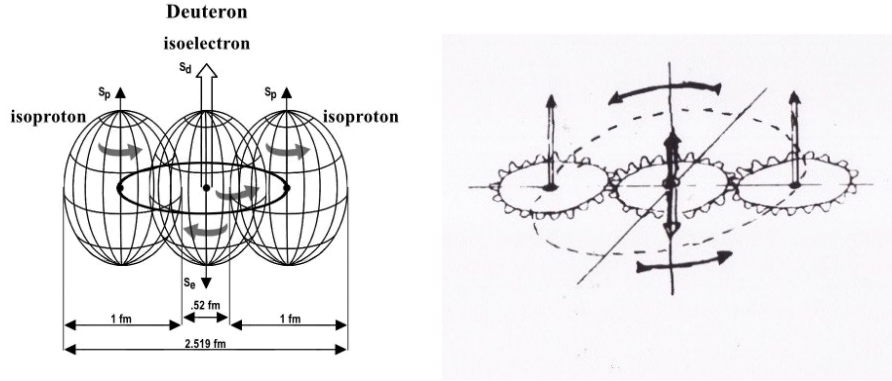


Figure 6: In the left of this figure, we illustrate the first three-body structure model of the Deuteron permitted by the neutron synthesis from the Hydrogen [57] consisting of two protons in triplet coupling with an electron acting as their "gluon." Despite its stability illustrated in the right of this figure, this model had to be abandoned because of its inability to represent: 1) The spin $S_D = 1$ of the Deuteron as a bound state of a proton and a neutron in their ground state [2]; 2) The constrained angular momentum of the electron being equal to the proton spin with ensuing null total value $S_e^{tot} = 0$; and 3) The stability of the Deuteron under the natural instability of the neutron.

Consequently, the spin of the neutroid according to Figure 3 is predicted to be zero, by therefore explaining the reason for their lack of detection via commercially available neutron detectors.

3.3.3. Representation of the neutron magnetic moment. The anomalous magnetic moment of the neutron according to model (18) has been first represented in the original proposal via the following *three* contributions [52]

$$\mu_n = \mu_p + \mu_e^{spin} + \mu_{\hat{e}}^{orb}. \quad (40)$$

The biggest difficulty of the above representation is that the magnetic moment of the electron μ_e^{spin} is too big for nuclear standard to prevent a quantum mechanical model of the neutron synthesis as well as to prevent that electrons can be members of nuclear structures. In fact, we have the value

$$\begin{aligned} \mu_e^{spin} &= -9.284764 \times 10^{-24} \text{ J/T} = \\ &= -9.284764 \times 10^{-24} \times 1.9798907610^{26} \mu_N = \\ &= -928.4784 \times 1.979890 \mu_N = 1838.2851 \mu_N, \end{aligned} \quad (41)$$

which is 961-times *bigger* than the neutron magnetic moment.

Under the assumption that the proton is a point-like particle, there is no known possibility to reduce magnetic value (41) down to the magnitude needed for the neutron magnetic moment. The above impasse is here resolved, apparently for the first time for value (41), via the magnetic moment of the *orbital* motion of the isoelectron $\mu_{\hat{e}}^{orb}$ which is *opposite* that of the isoelectron (Fig. 4) and its value is predicted to be [52]

$$\mu_{\hat{e}}^{orb} = 1833.580 \mu_N. \quad (42)$$

By recalling the known values of the magnetic moments of the proton and the neutron [76] $\mu_p = 2.792 \mu_N$, $\mu_n = -1.913 \mu_N$, we have the *numerically exact and time invariant representation of the anomalous magnetic moment of the neutron* [52]

$$\begin{aligned} \mu_n &= \mu_p + \mu_e^{spin} + \mu_{\hat{e}}^{orb} = \\ &= 2.792 \mu_N - 1838.285 \mu_N + 1833.5801 \mu_N = -1.913 \mu_N. \end{aligned} \quad (43)$$

It should be noted that the assumption of the above orbital contribution from the isoelectron not only allows a representation of the *numeric value* of the anomalous magnetic moment of the neutron, but also of its *negative value*.

3.3.4. Representation of the neutron rest energy, mean life and charge radius.

In this section we shall identify, apparently for the first time, the numeric value of *neutron density* $n_4 = 1/b_4$ from isoequivalence principle (13) (the isoaxiom IV of special isorelativity, p. 139 of [33]) in remarkable agreement with the density of the proton-antiproton fireball of the Bose-Einstein correlation [77] [78].

By assuming the iso-Galilean invariant hadronic structure model of Sect. II-5.7, first presented in Eqs. (5.1.14), p. 835 of Ref. [27] (see the review in Sect. 2.5.4 of Ref. [79]) we have the following *hadronic structure equations of the neutron*

$$\begin{aligned} \left[\frac{1}{r^2} \left(\frac{d}{dr} r^2 \frac{d}{dr} \right) + \bar{m} \left(E + e^{\frac{2}{r}} + N \frac{e^{-br}}{1-e^{-br}} \right) \right] &= 0, \\ E_n^{tot} = E_p + E_{\hat{e}} - E &= 939.565 \text{ MeV}, \end{aligned} \quad (44)$$

$$\tau^{-1} = 2\pi\lambda^2 |\hat{\psi}(0)|^2 \frac{\alpha^2 E_e}{\hbar} = 877 \text{ s},$$

$$R = b^{-1} = 10^{-13} \text{ cm} = 1 \text{ fm},$$

which can be reduced to the *algebraic* equations in the parameters k_1 , k_2 (Eqs. (5.1.32a) and (5.1.32b), p. 840 of Ref. [27])

$$E = k_1 [1 - (k_2 - 1)2] \frac{2\hbar c}{b} = 939.565 \text{ MeV}, \quad (45)$$

$$\tau = \frac{48 \times (137)^2}{4\pi b c} \frac{k_1}{(k_2 - 1)^3} = 877 \text{ s}, \quad \xi = \tilde{n},$$

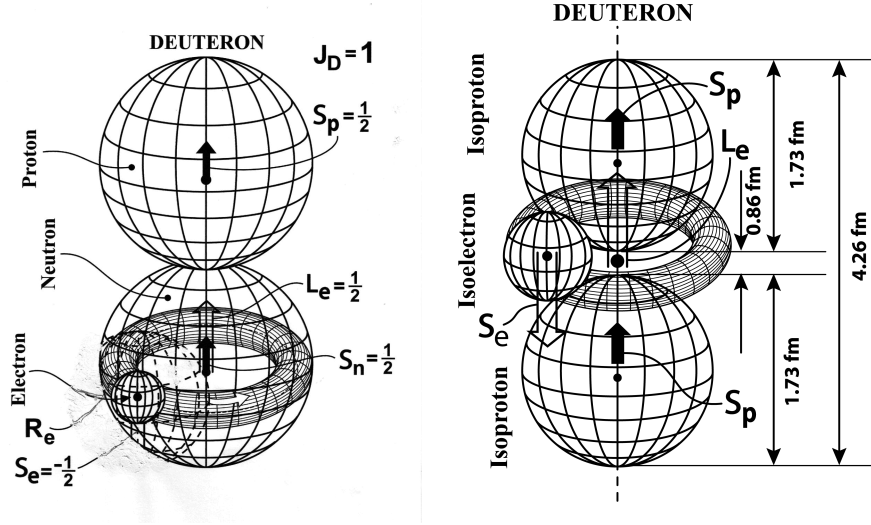


Figure 7: In the left of this figure, we illustrate the structure of the Deuteron as an axial triplet coupling (Fig. II-8) of a proton and a neutron which permits the representation of the stability of the Deuteron (illustrated in the right of this figure) via the decoupling of the neutron into its permanently stable constituents, the proton and the electron. An important point for the consistency of the representation is that the electron in the indicated axial triplet configuration preserves its null total angular momentum due to the persistence of its total immersion within the proton which is not possible for the planar singlet coupling (Fig. II-8) of Fig. 6.

energy spectrum

$$E = -\frac{1}{4K_h k_2} \left(\frac{k_2}{N} - N \right)^2 = 0, \quad k_2 = K_h \frac{\tilde{E}_{\bar{e}}}{\hbar^2 b^2} = 1, \quad (46)$$

and numeric solutions[52] (see also Refs. [63])

$$k_1 = 2.6, \quad k_2 = 1, \quad (47)$$

which numeric values should be compared with those for the meson and baryon octets of Refs.[27] [79].

The primary difference between the above equations and Eqs. (II-118) is the absence of a potential in the latter while the former exhibits indeed a potential, evidently given by the Coulomb potential between the electron and the proton.

We now compute the rest energy of the neutron via the isoequivalence principle (13) under the approximation that all particles are perfectly spherical, thus having $n_d = 1/b_d = 1$, $d = 1, 2, 3$ (see next section for the spheroid case). We shall then compare the results with other experimental particle data. By recalling

from Sect. II-5.7 that *the Hulthen potential does not generate a binding energy*, the total energy of the neutron in model (18) is given by

$$E_n = \tilde{E}_{\tilde{e}} + E_p - E, \quad (48)$$

where, from principle (13) for the assumed values $n_d = 1/b_d = 1$, $d = 1, 2, 3$,

$$\tilde{E}_{\tilde{e}} = \frac{E_e}{n_4^2} = \frac{0.511}{n_4^2}. \quad (49)$$

By keeping in mind that we cannot use quantum mechanical methods for the computation of the binding energy of neutron due to its *excess* energy of 0.782 MeV , we can assume in first approximation that the binding energy E of the neutron according to structure model (18) is of the same order of magnitude as that of the positronium in the $1s$ state because the charge structures and radii of the neutron and the positronium are essentially the same, [80]

$$E = -0.948 \text{ eV}. \approx 0. \quad (50)$$

Hence, we can assume in first approximation that

$$\tilde{E}_{\tilde{e}} = E_n - E_p = 0.511 + 0.762 \text{ MeV} = 1.293 \text{ MeV} \quad (51)$$

resulting in the *best available estimate of the density of the proton* based on the isoequivalence law (13)

$$\begin{aligned} \tilde{E}_{\tilde{e}} &= \frac{E_e}{n_4^2} = 1.293 \text{ MeV}/c^2, \\ n_4^2 &= \frac{0.511}{1.293} = 0.395, \\ n_4 &= 0.628, \end{aligned} \quad (52)$$

resulting in the value of the neutron density $n_4 = 1/b_4 = 0.610$ which is in remarkable agreement with the numeric value of $n_4^2 = 0.429$, $n_4 = 0.654$ for the proton-antiproton fireball of the Bose-Einstein correlation, Eqs. (10.28b), p. 127 of [77] (see also Ref. [78]). Therefore, from Eqs. (44), (46) and (50), the rest energy of the neutron is given by the rest energy of the proton plus the isorenormalized rest energy of the isoelectron, $E_n = E_p + \tilde{E}_{\tilde{e}}$.

Note that values (52) imply the following *superluminal tangential speed of the electron within the proton structure*

$$C = \frac{c}{n_4} = 1.592 \text{ c}, \quad (53)$$

which is fully possible for hadronic mechanics because contact nonpotential interactions carry no energy [13] [81].

Intriguingly, values (52) confirm the nonlocal effects at the foundation of the isoequivalence principle (13), the second digit difference between the value $n_4 = 0.628$ of Eqs. (48) and the value $n_4 = 0.654$ of Ref. [77] being due to the assumption of perfect sphericity of the particle in isoprinciple (13) compared to the spheroid shape of the particles in Ref. [77].

3.4. Relativistic representation of the neutron synthesis. Recall that the Dirac equation has acquired a justly historical stature because of its exact and time invariant relativistic representation of the electron under the Hamiltonian interaction of the *point-like* proton in the structure of the Hydrogen atom.

The Dirac-Santilli isoequation (I-95) has been constructed to attempt the exact and time invariant representation of the isoelectron under the Hamiltonian and non-Hamiltonian interactions of the *extended* proton in the structure of the neutron according to Fig. 3, thus requiring its characterization via the isotopies of the spinorial covering of the Poincaré symmetry $\hat{\mathcal{P}}(3.1) = \hat{S}L(2, \hat{C}) \hat{\times} \hat{T}(3.1)$, Eq. (1) first introduced in the 1995 paper [26] jointly with the Dirac-Santilli isoequation and the first relativistic representation of the neutron synthesis.

For consistency, model (18) requires that the absolute value of the hadronic angular momentum of the isoelectron $\hat{L}_3$ is equal to its spin $\hat{S}_3$, thus requiring that

$$\hat{L}_3 = \hat{S}_3, \quad \hat{L}^2 = \hat{S}^2. \quad (54)$$

From Eqs. (8) and (10) of the isospinorial isosymmetry $\hat{\mathcal{P}}(3.1)$, we therefore obtain the conditions on the characteristic quantities of the isotopic element (2)

$$b_1^{-1}b_2^{-1} = \frac{1}{2}b_1b_2, \quad (55)$$

$$b_1^{-2}b_2^{-2} + b_2^{-2}b_3^{-2} + b_3^{-2}b_1^{-2} = \frac{1}{4}(b_1^2b_2^{-2} + b_2^2b_3^{-2} + b_3^2b_1^{-2}),$$

with numeric value for the case of spherical symmetry [26]

$$b_1^2 = b_2^2 = \frac{1}{n_1^2} = \frac{1}{n_2^2} = \sqrt{2} = 1.415, \quad b_1 = b_2 = \frac{1}{n_1} = \frac{1}{n_2} = 1.189. \quad (56)$$

The value for the semi-axis $b_3^2 = 1/n_3^2$ of the isoelectron can be found by assuming the volume preserving normalization $n_1^2 + n_2^2 + n_3^2 = 3$ used for neutron, Eq. (I-89), resulting in the value [26]

$$n_1^2 = n_2^2 = 0.707, \quad n_3^2 = 1.587, \quad (57)$$

by therefore suggesting that the wave packet of the isoelectron is *oblate* (because $n_3^2 < n_1^2 = n_2^2$).

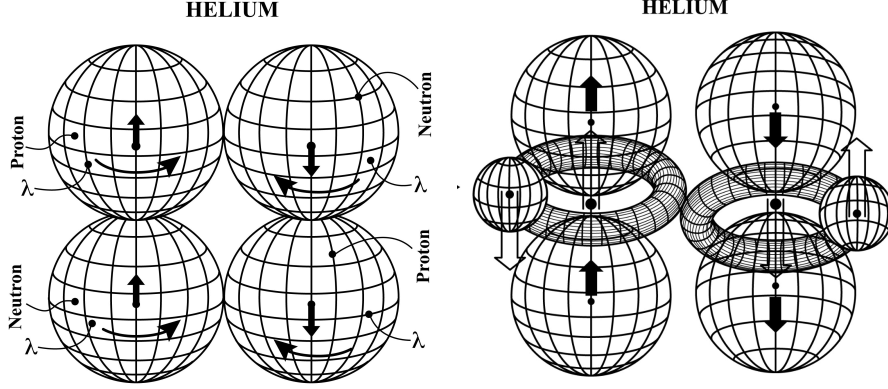


Figure 8: In this figure we compare the structure of the Helium according to ref. [2] (left side) and that according to the analysis of paper (right view) to illustrate the achievement of stability as well as the identification of an internal attractive force in the transition from the former to the latter.

To compute the rest energy of the neutron, we now assume the full form of the isoequivalence principle (13) from which, instead of Eq. (48), we have the following expression for the energy $\tilde{E}_e$ of the isoelectron from the value (52) of the shape of the electron wave packet

$$\begin{aligned}\tilde{E}_{\hat{n}} &= \frac{n_2^2}{n_4^2} E_e = 1.37 \text{ MeV}/c^2, \\ n_4^2 &= \frac{1.587}{1.37} 0.511 \text{ MeV}/c^2 = 0.592, \\ n_4 &= 0.789,\end{aligned}\tag{58}$$

which remains close to the value $n_4 = 0.6$ for the density of the fireball of the Bose-Einstein correlation of [77] [78].

The relativistic representations of neutron magnetic moment is essentially the same as that of the preceding non-relativistic analysis (see Eq. (7.4), p. 182 of [26] for an alternative derivation). As it is well known, the radial equation of Dirac's equation for the Hydrogen atom is essentially the same as the corresponding nonrelativistic equation, and the same holds for the Dirac-Santilli isoequation. Consequently, the relativistic representation of the mean life and charge radius of the neutron per model (19) is essentially the same as that of Sect. 3.3.3. of Ref. [26] and, as such, it is omitted for brevity.

4 Representation of nuclear stability

4.1. Representation of nuclear stability despite the neutron instability. We shall now search for a mechanism according to which the naturally unstable neutron (when isolated) becomes permanently stable under strong nuclear forces.

Let us recall our initial structure model of the Deuteron as a three-body hadronic bound state of two protons in plane triplet alignment with spins $1/2$ and one isoelectron with total angular momentum $J = 0$ (Fig. 6) [52] [57]. This model had to be abandoned despite the representation of the Deuteron spin $S_D = 1$ and its great stability (illustrated with the 'gear model' in the right of Fig. 6), because of the impossibility to represent proton-neutron exchange forces (due to the inability that the isoelectron could be consistently transferred from the neutron to the proton), the inability to represent the stability of the Deuteron despite the neutron natural instability, and for other reasons.

In the hope of resolving the above insufficiencies, we have then assumed model (I-81) on the Deuteron structure as a two-body hadronic bound state of a isoproton and an isoneutron in axial triplet couplings with spins $1/2$. We here submit the hypothesis that, when it is a member of the Deuteron structure, *the isoelectron of model (18) decouples from the neutron to acquire a position intermediate between the two protons* (right of Fig. 7).

$$D = \begin{pmatrix} \hat{p}^\uparrow \\ \hat{n}^\uparrow \end{pmatrix} = \begin{pmatrix} p^\uparrow \\ \hat{e} \\ p^\uparrow \end{pmatrix}, \quad (59)$$

in which the isoelectron remains immersed within dense hadronic matter as a necessary condition to activate hadronic mechanics with ensuing restricted angular momentum equal to the proton spin and null total angular momentum (Sect. 3.3.1 and Fig. 4)

$$J_{\hat{e}}^{tot} = S^{spin} + L^{orb} = \frac{1}{2} - \frac{1}{2} = 0. \quad (60)$$

Consequently, the decoupled isoelectron assumes an intermediate position between the two protons by occupying the distance between their charge distributions of $0.3745 \times 10^{-13} \text{ cm}$ indicated in Figure II-9 [82].

Note that the above hypothesis is supported by null values (46) and (50) of the neutron binding energies for the SA and NSA components of strong interactions, according to which the isoelectron is *quasi-free*, and indeed decouples from the neutron under the strong Coulomb attraction by the second proton.

It appears that, unlike the case for the planar configuration (Fig. 6), the axial triplet coupling of the isoproton and the isoneutron in the Deuteron structure (left of Fig. 7) allows the representation of: all experimental data of the Deuteron in its true ground state; the representation of proton-neutron exchange forces via the

mere exchange of the isoelectron in between the two protons; and the representation of the deuteron stability despite the natural instability of the neutron.

Additionally, the decoupling of the isoelectron from the neutron indicates the apparent existence of a strongly *attractive* Coulomb force $e - p$ in the Deuteron structure which is absent in quantum mechanical treatments, by therefore contributing to the stability of the Deuteron.

4.2. Representation of nuclear stability despite repulsive protonic forces. Let us recall that, according to quantum mechanics, protons *repel* each other (in view of their equal charge) with a Coulomb force of such an extreme value not to be counter balanced by potential nuclear forces.

We here indicate, apparently for the first time, that hadronic mechanics can represent this additional problem of nuclear stability via a mechanism similar to the achievement by *hadronic chemistry* of a strongly *attractive* force between the *identical* electrons of valence couplings [83] [84] [85].

The proposed isotopic mechanism is essentially that of counter-balancing repulsive nuclear force via the completion of a conventional repulsive quantum mechanical model with non-Hamiltonian forces generated by a suitably selected nonunitary transform (Sect. II-3.11).

Various measurements [86] have established that the Helium $He(2, 4, 0)$ [87] has a charge radius of $1.678 \times 10^{-13} \text{ cm}$, against the radius of two protons and two neutrons each having the value of $0.841 \times 10^{-13} \text{ cm}$ [20] with a total radius of $1.682 \times 10^{-13} \text{ cm}$.

The above measurements confirm the primary assumption of hadronic mechanics, according to which nuclei are composed by extended nucleons in conditions of partial mutual penetration of their dense charge distributions, with ensuing non-Hamiltonian interactions. Consequently, the indicated measurements justify the presence of non-Hamiltonian interactions in the structure of the Helium.

Let us assume in first approximation that the Helium is a quantum mechanical (qm) bound state of two Deuterons with anti-parallel spins (Fig. 8)

$$He(2, 4, 0) = [D(1, 2, 1_{\uparrow}), D(1, 2, 1_{\downarrow})]_{qm}, \quad (61)$$

under the sole action of the Coulomb potential between the two protons,

$$V_c(r) = +\frac{e^2}{r}, \quad (62)$$

where, as usual, the positive sign denotes *repulsion*, with the understanding that conventional, attractive, nuclear potentials will be added later on.

In order to add non-Hamiltonian interactions in model (61), we subject its

Schrödinger equation to the nonunitary transformation (Sect. II-3.7)

$$\begin{aligned}
 UU^\dagger = \hat{I} = 1/\hat{T} &= e^{\frac{V_h(r)}{V_c(r)}} = e^{\frac{K \frac{e^{-br}}{1-e^{-br}}}{\frac{e^2}{r}}}, \\
 \hat{T} &\approx 1 - \frac{V_h(r)}{V_c(r)} = 1 - \frac{K \frac{e^{-br}}{1-e^{-br}}}{\frac{e^2}{r}},
 \end{aligned} \tag{63}$$

where N is a normalization constant, resulting in the iso-Schrödinger equation (Sect. II-5.7 and Sect. 3 of Ref. [79])

$$\begin{aligned}
 U[\frac{1}{2m}\delta^{ij}p_i p_j + V_c(r)]|\psi(r)\rangle U^\dagger &= \\
 = U(\frac{1}{2m}\delta^{ij}p_i p_j)U^\dagger \hat{T}|\hat{\psi}(r)\rangle + UV_c(r)U^\dagger \hat{T}|\hat{\psi}(r)\rangle &\approx \\
 \approx \{-\frac{1}{m}\hat{\partial}_r \hat{\partial}_r + V_c(\hat{r})[1 - \frac{V_h(r)}{V_c(r)}]\}|\hat{\psi}(r)\rangle &= \\
 = [-\frac{1}{m}\Delta_r + \frac{e^2}{r} - K\frac{e^{br}}{1-e^{br}}]|\hat{\psi}(r)\rangle,
 \end{aligned} \tag{64}$$

where K is a normalization constant. But the Hulthen potential behaves like the Coulomb potential at short distances (Eq. (5.1.6), p. 833, of the 1978 memoir [27]), by therefore *absorbing* the Coulomb potential via a mere redefinition of its normalization constant from K to K'

$$V_h(r) \approx V_c(r), \quad V_c + V_h = +\frac{e^2}{r} - K\frac{e^{br}}{1-e^{br}} \approx -K'\frac{e^{br}}{1-e^{br}}, \tag{65}$$

resulting in the radial equation without any presence of repulsive forces

$$\left[\frac{1}{r^2} \left(\frac{d}{dr} + m(E + K'\frac{e^{-br}}{1-e^{-br}}) \right) \right] |\hat{\psi}\rangle = 0. \tag{66}$$

Consequently, it appears that *nonlinear, nonlocal and nonpotential nuclear interactions due to the mutual penetration of nucleons can be so strongly attractive to overcome repulsive Coulomb force between protons.*

For brevity, we leave to the interested reader the completion of the above isotopic model with some of the attractive nuclear potentials such as the *Yukawa potential* [88], the *Woods-Saxon potential* [89] and other potentials or their combination [90].

4.3. Open problems in nuclear stability. It should be indicated that the representation of the nuclear stability of Sect. 4.1 via the decoupling of the neutron into its original, permanently stable constituents appears to be solved on both mathematical and physical grounds.

By contrast, the representation of nuclear stability under repulsive proton-proton Coulomb forces of Sect. 4.2 is purely *mathematical*, by therefore creating the fundamental open problem on the *physical* origin of the attraction.

In the author's view, the above open problem can be best studied via the completion of the quantum mechanical notion of *point-like charge* into the nonlocal notion of *isocharge* of isoparticles (Definition 2.1). In turn a nonlocal formulation of the Coulomb law requires a *structure model of the charge*, e.g., along the author's undergraduate thesis [91] [92] [93] (see also the review in Sects. 3.2 and 3.3 of [35]).

5 Implications of the neutron synthesis

5.1. Stimulated neutron decay. In the author's view, an intriguing implications of the synthesis of the neutron from the Hydrogen is the possible existence of mechanisms suitable to trigger the neutron decay, of course, under the strict validity of all nuclear conservation laws.

As it is well known, according to quantum mechanics the electron is not a constituent of the neutron, which therefore with consequential prohibition to stimulate the neutron decay. Nevertheless, the representation by hadronic mechanics of *all* characteristics of the neutron at the *nonrelativistic and relativistic levels* as a hadronic bound state of an isoproton and an isoelectron admits indeed the stimulated neutron decay. In fact, the decoupling of the neutron for nuclear stability predicts a possible stimulated decay of the neutron via the irradiation with resonating photons γ_{res} with the frequency [57]

$$E_{res} = E_{\hat{e}} = 1,293 \text{ MeV}. \quad (67)$$

Note that the stimulated neutron decay is generally *not* admitted by stable nuclei, except for double beta decays of the type

$$\begin{aligned} \gamma_{res}(0,0,1) + Mo(100,42,0) &\rightarrow Tc(100,43,1) + \beta^-(0,-1,0), \\ Tc(100,43,1) &\rightarrow Ru(100,44,0) + \beta^-(0,-1,1), \end{aligned} \quad (68)$$

which transmutation has been tentatively verified by Santilli [6] [67]) and by the experimental team [94] (see Fig. 9 and review [67]) via the following simple and inexpensive experimental set up:

5.1.1) Purchase a small sample of the commercially available radioisotope Europe-152 (emitting photons with E_γ) and of the pure isotope $Mo(100,42,0)$.

5.1.2) Place the Europe-152 sample in contact with the sample of $Mo(100,42,0)$.

5.1.3) The emission of easily detectable electrons from the $Mo(100,42,0)$ sample, or the possible detection of traces of $Ru(100,44,0)$ in the originally pure

sample of $Mo(100, 42, 0)$, would establish the existence of the stimulated double beta decays (75) beyond scientific doubt.

5.2. The etherino hypothesis. It should be indicated that, despite the best possible efforts conducted over decades, *the author has been unable to identify any possibility of maintaining the Pauli-Fermi neutrino hypothesis in a structural (rather than kinematical) representation of the neutron synthesis in a star.*

This impasse has left no other option than that, in line with Rutherford's original conception (18), of assuming that *the neutron is a generalized bound state of the permanently stable electron and proton without any additional constituent.* Different models are encouraged, provided that they interpret the totality of the characteristics of the neutron during its synthesis from the electron and the proton as occurring in nature.

To begin, recall that bound states of two particles were assumed throughout the 20-th century to have *negative* binding energies, resulting in a *mass defect* which is at the foundation of nuclear fusions. By comparison, the neutron synthesis misses 0.872 MeV which can be provided, quite simply, by the kinetic energy of the electron (Sect. 3.3.4). However, the author has located no plausible source of the additional energy needed to accomodate the neutrino.

Similarly, the only known structural representation of the neutron synthesis is Rutherford's compression of the electron inside the dense proton in the core of stars. In turn, such a compression mandates that, to prevent extreme resistive forces, the angular momentum of the internal electron must be equal to the proton spin, thus implying Rutherford's original conception (18) rather than Pauli-Fermi conception (15). Despite years of attempts, the author has found no possibility of accommodating the neutrino spin $1/2$ in the neutron synthesis.

To illustrate the intriguing character of the problem, let us also indicate that the studies reviewed so far on the neutron synthesis have left open the fundamental problem of the *origin* of the missing energy of 0.782 MeV (Sect. I-1). Contrary to a rather general view, said missing energy cannot be provided by the relative kinetic energy between the electron and the proton in the core of stars, because at that energy, the electron-proton cross section is essentially null, thus prohibiting any synthesis.

Similarly, said missing energy cannot be provided by the Sun, because the Sun synthesizes about 10^{38} neutrinos *per second* [95], that would require about

$$\Delta E = 10^{38} \text{ MeV per second}, \quad (69)$$

by therefore implying such an energy loss for which the Sun would cool down and never produce light.

For these and other reasons, Santilli proposed in 2007 the hypothesis that *the missing energy in the neutron synthesis is provided by the ether conceived as a*

universal substratum with extremely big energy density, and that the energy of 0.782 MeV is transferred from the ether to the neutron by a massless, chargeless and spinless *longitudinal impulse* called *etherino* (denoted with the letter a from the Latin *aether*) [96] according to the synthesis

$$\hat{e}^- + a + \hat{p}^+ \rightarrow n. \quad (70)$$

Note that, contrary to the case of the neutrino in the Pauli-Fermi synthesis (15), the etherino is on the *left* of synthesis (14) as a condition to supply the missing energy. It should also be recalled that (see the original 1956 study [91] and Chapter 3 of the review [35]):

5.2.1) The ether as a universal substratum is necessary for a coherent conception of electromagnetic *waves*.

5.2.2) The ether as a universal substratum is not in conflict with special relativity because of the inability by us to identify a reference frame at rest with the ether due to inertia.

5.2.3) The *aethereal wind*, that has been historically used to maintain the validity of special relativity, is eliminated by admitting that, as it is the case for electromagnetic waves, the electron and other truly elementary particles are oscillations of points of the ether.

It should be also indicated that Santilli proposed the etherino hypothesis to *assure the gravitational stability of the Sun*, in view of the fact that the Sun releases into light $2.3 \times 10^{38} \text{ MeV/s}$ [97] (see also [98]), corresponding to about $4.3 \times 10^6 \text{ t/s}$. Since in a Gregorian year there are 10^7 seconds, the loss of mass by the Sun ΔM_S per year due to light emission is given by

$$\Delta M_S = 10^{23} \text{ metric tons per year.} \quad (71)$$

The above loss of mass is of such a magnitude to cause a change of planetary orbits detectable in astrophysical laboratories, which change is contrary to centuries of measurements on the stability of the Sun gravitational field. However, the energy needed for the star to synthesize neutrons, Eq. (50), is essentially equal to the energy loss into light, by therefore allowing a representation of the gravitational stability of the star, with the understanding that it implies a *return to cosmology based on the continuous creation of matter in the universe* [99], with intriguing implications, e.g., for a realistic representation of the energy released in supernova explosions, since it is of such a dimension and of such instantaneous character not to be realistically representable via the sole use of nuclear fusions.

By keeping in mind that (as it is the case for quarks) neutrinos cannot be directly detected, we should finally indicate for interested readers the existence of unpublished indications that available experiments on the *predictions* of the neutrino hypothesis [100] appear to be numerically representable via the corresponding *predictions* of the etherino hypothesis [96].

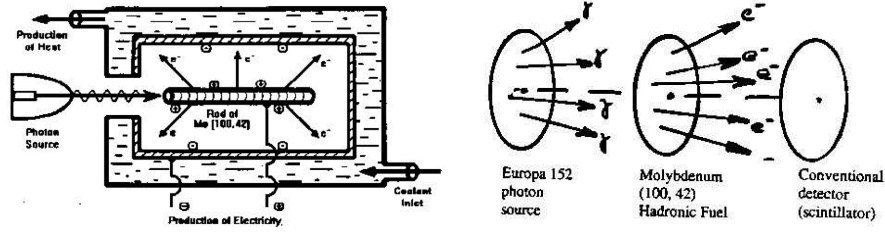


Figure 9: In the left of this figure, we reproduce the original drawing of paper [105] showing (from the left) a beam of resonating photons irradiating a cylinder of $\text{Mo}(100, 42, 0)$ which emits electrons easily trapped by a metal casing with the production of a clean DC electric current of nuclear origin (for which reason the set up is called a 'nuclear battery'), plus clean heat utilized by a heat exchanger (because the metal casing heats up while trapping electrons with 0.782 MeV and the transmutation is esoeenergetic). In the right of this figure, we reproduce the original figure of paper [105] illustrating the simplicity as well as the low cost of experimental verifications [6] [94].

5.3. The pseudoproton hypothesis. Following the synthesis of the neutron via the compression of an electron within the dense proton, hadronic reactors (Fig. 1) progressively synthesizes, in statistical smaller amounts, *various negatively charged, strongly interacting particles* [101] such as:

5.3.1) The *protoid* $\tilde{p}_1^-$ with spin 0, mass essentially that of the neutron and mean-life predicted to be of about 7 s which is predicted to be a planar singlet coupling of an isoelectron and a neutron (left of Fig. 5).

5.3.2) The *pseudo-proton* $\tilde{p}_2^-$ with spin 1/2, mass equal to that of the neutron and mean-life of the order of 5 s, which is predicted from the compression of the isoelectron, this time, inside the hyperdense neutron (right of Fig. 5).

5.3.3) Additional states with decreasing mean life [101] (see also Ref. [108]). The protoid and pseudoproton are representable with the unified notation

$$\tilde{p}^- = (\hat{e}^-, n)_{hm}, \quad (72)$$

where, as it is the case for the neutron synthesis, the isoelectron has total angular momentum $J^{tot} = S^{spin} + L^{orb} = 1/2$ for the exterior orbital configuration and $J^{tot} = S^{spin} + L^{orb} = 0$ for the constrained interior configuration of Fig. 5.

The existence of the above hadronic states is predicted from the fact that *non-Hamiltonian interactions caused by deep EPR entanglement of particles are insensitive to charges*, thus being somewhat reminiscent of the charge independence of strong interactions.

A reason currently stimulating mathematical, theoretical, experimental and industrial interest is that being negatively charged and strongly interacting, *pseudo-protons are attracted by nuclei* here represented in standard notation $N(Z, A, J, u)$

[87] with resulting unavoidable transmutation into *new nuclei called nucloids* [101]

$$\tilde{p}_s^- + N(Z, A, J) \rightarrow \tilde{N}(Z - 1, A + 1, J + s, u + m_{\tilde{p}}), \quad (73)$$

whose possible significance for the recycling nuclear waste, medical treatments and other fields is indicated in Sect. 6.

It should be finally noted that the existence of the pseudo-proton, let alone its long mean life for particle standards, is impossible without the EPR completion of quantum into hadronic mechanics.

6 Recycling radioactive nuclear waste

6.1. The societal problem of recycling radioactive nuclear waste. As it is well known [102], in view of the popular opposition against the transportation of radioactive nuclear waste through populated areas to reach an assigned general depository, the radioactive nuclear waste generated by nuclear power plants has been stored on their site since start up, by reaching current storage levels such to constitute a serious danger for local residents as well as to the environment.

Amidst a social problem of such a dimension, it appears recommendable to study *all* possible forms of recycling of nuclear waste irrespective of whether they are aligned with preferred theories. The problem is compounded by the failure to date of all forms of recycling based on 20-th century theories which have been attempted under billions of dollars of public funds.

The problem is compounded by the fact that radioactive nuclear waste can solely be handled by governmental or para-governmental conduits, by therefore preventing the open participation by the industry at large. Societal accountability then suggests the continuation of the ongoing research based on the assumption that quantum mechanics is exactly valid for nuclear waste, as well as the initiation of *governmental* research based on new vistas, such as the Einstein-Podolsky-Rosen argument that suggesting a suitable completion of quantum mechanics for nuclear structures as a recommendable premise for the successful recycling of nuclear waste or the development of radiation-free controlled nuclear fusions.

With the understanding that the standard model has produced a valid *classification* of particles, the recycling of nuclear waste is additionally precluded by the assumption that the hypothetical quarks are the actual physical constituents of protons and neutrons without the dismissal in refereed publications of related inconsistencies, such as the belief that the permanently stable electron and proton *disappear* from the universe at the time of the neutron synthesis to be replaced by the hypothetical quarks, and at the time of the neutron decay, the hypothetical quarks disappear to be replaced by the original electron and proton [103]. For brevity, we refer the interested reader to the inconsistencies of quark conjectures

in Sect. 8.2.2 of [33] and the *return to sanity* advocated by Sir Karl Popper [104] for the structure model of unstable particles with physical constituents produced free in the spontaneous decays as it is the case for the neutron and all unstable particles, of course, according to hadronic rather than quantum mechanics [27] [79]. It is hoped that governmental agencies do not expose themselves to a judgment by posterity for keeping the recycling of nuclear waste hostage to hypothetical quarks that are incompatible with the physical reality of the neutron synthesis, admit a rigorously proved finite probability of tunnel effect and have not been detected free despite hadron collisions at extremely high energies.

By keeping in mind the prohibition of transporting radioactive waste through populated areas, R. M. Santilli has proposed the *recycling of radioactive nuclear waste by the nuclear plants themselves via their stimulated decay reducing their very long mean lives down to minutes or seconds* [105]-[107] (see also [108][33] [109]). The main issue is that the stimulated decays predicted by quantum mechanics are insufficient for the indicated reduction of mean lives, while hadronic mechanics does indeed predict a number of effective stimulated decay deserving due scientific process for proper societal accountability.

6.2. Recycling radioactive nuclear waste based on the neutron synthesis. In the author's view, the most important societal implication of the studies on the synthesis of the neutron from the proton and the electron [52]-[71] is the prediction of innovative means for the recycling of radioactive nuclear waste via nuclear transmutations.

The first possibility is offered by the representation of nuclear stability via the decoupling of the isoelectron $\hat{e}$ from the neutron (Sect. 4) since it implies that *neutrons can be stimulated to decay via irradiation with photons γ_r with the resonating frequency or energy $E_r = E_{\hat{e}}$ which, for the case of the Deuteron, is given by $E_{\hat{e}} = 1.293 \text{ MeV}$, according to the reaction first proposed in the 1994 paper [105]*

$$\gamma_r^\uparrow + n^\downarrow \rightarrow +p^\downarrow + e^\uparrow + \nu, \quad (74)$$

with possible application to the transmutation of $Mo(100, 42, 0)$ into the stable $Ru(100, 44, 0)$ preliminarily verified by Santilli [6] as well as by the experimental team [94] (see also Refs. [66] [71])

$$\gamma_r(0, 0, 1) + Mo(100, 42, 0) \rightarrow Tc(100, 43, 1) + \beta^-(0, -1, 0), \quad (75)$$

$$Tc(100, 43, 1) \rightarrow Ru(100, 44, 0) + \beta^-(0, -1, 1).$$

Note that, in the event confirmed, the above transmutation would provide a dual source of nuclear energy without the emission of harmful radiation or the release of radioactive waste (Fig. 9), which are given by the production of a DC current (since one resonating gamma would produce two easily trappable electrons)

as well as of heat since the transmutation is isoenergetic with $\Delta E = 1.828 \text{ MeV}$,

$$\begin{aligned}\Delta E &= E(100, 42, 0) - E(100, 44, 0) - E(\gamma) - 2E(e) = \\ &= (3.034 - 0.184 - 1.022)\text{MeV} = 1.828\text{MeV}.\end{aligned}\tag{76}$$

A second possible recycling of radioactive nuclear waste via their stimulated decay is given by the irradiation of waste pellets with a beam of low energy neutrons produced by the *Directional Neutron Source* of Hadronic Technologies Corporation (see Sect. 3.3 and Fig. 1). By remembering that radioactive nuclear waste can solely be handled by governmental agencies, the possibility can be illustrated via the following neutron irradiation of the unstable $42 - Mo - 100$ with half life of 10^{19} years into the isotope $42 - Mo - 102$ with half life of one minute,

$$2n + Mo(42, 100, 0) \rightarrow Tc(42, 102, 1).\tag{77}$$

For brevity, we leave to interested colleagues the extension of the above results to radioactive nuclear waste.

6.3. Recycling radioactive nuclear waste via pseudoproton irradiation. It should be mentioned that the confirmation and separation of the pseudoproton and/or the protoid (Sect. 3.6 and Ref. [101]) would imply the possible recycling of radioactive nuclear waste via their stimulated decay, as illustrated by the predicted transmutations of the type

$$2\bar{p}^-(-1, 1, 0) + Mo(42, 100, 0) \rightarrow Zr(40, 102, 0),\tag{78}$$

in which case the 10^{19} years main-life of $42 - Mo - 100$ is reduced to the of 2.9 seconds mean-life of $40 - Zr - 102$.

It should be noted that, being negatively charged and having low energy (estimated up to 100 eV), all protoids or pseudoprotons are expected to be absorbed by nuclei of an irradiated waste pellet. It should also be noted that protoids and pseudoprotons are evidently *unstable*. Nevertheless, their half life is predicted to be of the order of seconds (about 10% the half life of the neutron), thus being suitable for industrial applications.

In the author's view, the above application of protoids and pseudoprotons should not be dismissed on grounds of its incompatibility with the *antiprotons* allegedly produced at various particle laboratories, in view of their unsettled character as true antiparticles [101]. It is hoped that particle physics laboratories may develop reactors producing a beam of antiprotons/pseudoprotons to test possible recycling of radioactive nuclear waste.

In view of these and other unsettled aspects, Santilli has also proposed the test of the gravity of true antiparticles such as the positrons in a horizontal vacuum

tube [110] (see also A . P. Mills [111], V. de Haan [112] and the Springer Nature monograph [113]), and test the gravity of the apparent anti-hydrogen atom only following the verification that its nucleus is indeed a true antiparticle.

7 Concluding remarks

In the hope that the studies presented in these papers can be applied to strong interactions at large, in the preceding [2] we have presented a systematic and updated study, specialized to nuclear physics of the axiom-preserving time reversal invariant *Lie-isotopic branch of hadronic mechanics*, whose knowledge is recommendable prior to the study of the time irreversible *Lie-admissible branch of hadronic mechanics* for the axiomatically consistent treatment of controlled nuclear fusions [114]- [116].

We have then presented numerically exact and time invariant representations of the experimental data of the Deuteron conceived as a bound state of a proton and a neutron via simple *quantum mechanical* nonunitary transformations of available models (Sect. I-3-11) under the condition of being reformulated in terms of isomathematics and isomechanics (Sect. I-3.12) to avoid known inconsistencies of quantum nonunitary theories.

In this paper, we have presented apparently for the first time:

7.1) The relativistic completion of the quantum mechanical notion of *particle* into that of *isoparticles* via special isorelativity and related Lorentz-Poincaré-Santilli isosymmetry for extended nucleons under potential and nonpotential interactions.

7.2) The representation of *all* characteristics of the neutron at the nonrelativistic and relativistic levels in its synthesis from the isoelectron and the isoproton in the core of stars under their big *attractive* Coulomb force.

7.3) The identification of the numerical value of the density of the neutron in a way compatible with other experimental data.

7.4) The representation of nuclear stability despite the natural instability of the neutron via the *decoupling* of the electron from the neutron in the Deuteron due to the big Coulomb attraction from the second proton.

7.5) The representation of nuclear stability despite the big Coulomb *repulsion* between protons thanks to the admission of nonpotential forces between extended nucleons.

The main implications of the above advances are:

7.A) The prediction of means for the recycling of radioactive nuclear waste by the nuclear power plants themselves via new stimulated decays permitted by hadronic mechanics which apparently reduce very large mean lives down to seconds (Sect. 6).

7.B) The possible return to the historical cosmology with continuous creation of matter in the universe because the sole consistent possibility of explaining the missing 0.782MeV in the neutron synthesis is that it originates from the ether as a universal substratum (Sect. 3).

7.C) The return to the first conception of the neutron and of nuclei at large as being bound states of the original constituents of stars, the proton and the electron. Said conception was prohibited by the theoretical assumption that Heisenberg's indeterminacy principle for point-like particles in vacuum is also valid inside nuclei. However, the conception of the neutron, and therefore, of all matter in the universe as being bound states of protons and electrons is fully valid under Einstein's isodeterminism (Sects. 3 and 4.7) [1] [14] [15] for extended particles within hadronic matter.

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Lie-Santilli admissible hyper-structures, from numbers to H_v -numbers

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Abstract

The class of H_v -structures defined on a set is very big and admits a partial order. For this reason, it has a numerous of applications in mathematics and other sciences as physics, biology, linguistics, to mention but a few. Here, we focus on the Lie-Santilli's admissible case, where the hyper-numbers, called H_v -numbers, are used. In order to verify all needed axioms for Lie-Santilli's admissibility, as the irreversibility and uniqueness of living organisms and time, on the one side and small results on the other side, we use the very-thin H_v -fields. Therefore, we take rings and we enlarge only one result by adding only one element in order to obtain an H_v -field. This means that, we use only the associativity on the product and we transfer this to the weak-associativity on the hyper-product. Thus, from a semigroup on the product, we construct an H_v -group on the hyper-product.

Keywords: Lie-Santilli iso-theory, weak axioms, H_v -fields.

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1. Weak Hyper-structures

The largest class of hyper-structures is the class of H_V -structures which was introduced in 1990 [16]. These satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Definitions 1.1 *Hyper-structure* is any set H equipped with, at least one, *hyper-operation*:

$$\cdot : H \times H \rightarrow P(H) - \{\emptyset\}.$$

A hyper-operation satisfy the *weak associativity*, if $(xy)z \cap x(yz) \neq \emptyset$, $\forall x, y, z \in H$ and satisfy the *weak commutativity*, if $xy \cap yx \neq \emptyset$, $\forall x, y \in H$.

A hyper-structure $(H, \cdot)$ is an H_V -semigroup if it is weak associative and it is an H_V -group if it is reproductive H_V -semigroup: $xH = Hx = H$, $\forall x \in H$.

In the classical theory, the quotient of a group with respect to an invariant subgroup is a group. In hyper-structures, the quotient of a group with respect to any subgroup is a hypergroup. Finally, Vougiouklis introduced and proved in 1990, that the quotient of a group with respect to any partition is an H_V -group.

Definitions 1.2 The $(R, +, \cdot)$ is an H_V -ring if both $(+)$ and $(\cdot)$ are weak associative, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset, \quad \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be H_V -ring, $(M, +)$ be a weak commutative H_V -group and there exists an external hyper-operation

$$\cdot : R \times M \rightarrow P(M): (a, x) \rightarrow ax$$

such that, $\forall a, b \in R$ and $\forall x, y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset,$$

then M is an H_V -module over F . In the case of an H_V -field, which is defined later, F instead of an H_V -ring R , then the H_V -vector space is defined.

For more definitions and applications on H_V -structures one can see the books and papers as the: [1], [2], [5], [7], [16], [17], [18], [28], [29].

Definition 1.3 The *fundamental relations* β^* , γ^* and ε^* , are defined, in H_V -groups, H_V -rings and H_V -vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [5], [16], [19], [23], [27].

The way to find the fundamental classes is given by analogous to the following theorems:

Let $(H, \cdot)$ be H_V -group, U the set of all finite products of elements of H . Define the relation β in H by setting $x\beta y$ if, and only if, $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β .

Let $(R, +, \cdot)$ be H_v -ring, U the set of all finite polynomials of elements of R . Define the relation γ in R by: $xy\gamma$ if, and only if, $\{x, y\} \subset u$ where $u \in U$. Then γ^* is the transitive closure of γ .

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions:

Definition 1.4 An H_v -ring $(R, +, \cdot)$ is called H_v -field if R/γ^* is a field. The elements of an H_v -field are called *hyper-numbers* or H_v -numbers.

Definition 1.5 Let $(H, \cdot)$, $(H, *)$ be H_v -semigroups defined on the same set H . $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, if, and only if, there exists an automorphism

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x^*y), \forall x, y \in H.$$

Then we say that $(H, *)$ contains $(H, \cdot)$. If $(H, *)$ contains a group, then is called H_b -group.

The Little Theorem. Greater hyper-operations than the ones which are weak associative or weak commutative are, as well, weak associative or weak commutative, respectively.

The Little Theorem leads to a partial order, *posets*, on H_v -structures and introduce several new classes of hyper-structures [16], [27].

Definition 1.6 An H_v -structure is called *very-thin* if all hyper-operations are operations except one, which has all hyper-products singletons except only one, which is a subset of cardinality more than one.

Definition 1.7 Let $(G, \cdot)$ be groupoid, then for all $P \subset G$, $P \neq \emptyset$ we define the following P -hyper-operation:

$$\underline{P}: x\underline{P}y = (xP)y \cup x(Py), \quad \forall x, y \in G.$$

If $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$, so $(G, \underline{P})$ is a semi-hypergroup.

A generalization of P -hyper-operations is the following [4], [5], [24]:

Let $(G, \cdot)$ be abelian group and P , subset of G . We define the hyper-operation $\times_P$, by

$$\begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\}, & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y, & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this P_e -hyper-operation and the hyper-structure $(G, \times_P)$ is an abelian H_v -group.

The general definition of an H_v -Lie algebra was given as follows [5], [15]:

Definition 1.8 Let $(L, +)$ be an H_v -vector space over $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$ the canonical map, $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is zero of F/γ^* . Let ω_L the core of $\varphi': L \rightarrow L/\varepsilon^*$ and denote 0 the zero of L/ε^* . Consider the *bracket hyper-operation*

$$[,]: L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is an H_v -Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hyperoperation is bilinear:

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \quad \forall x, x_1, x_2, y, y_1, y_2 \in L \text{ and } \forall \lambda_1, \lambda_2 \in F$$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L.$

2. H_v -fields and hyper-matrix representations

The enlargements and the reductions of hyper-structures are examined in the sense that extra elements appear in results or we take out elements. In both directions most interesting cases in the research are those H_v -structures when the fundamental structures remain the same [16], [19], [23], [27].

Theorem 2.1 Let $(G, \cdot)$ be semigroup and $v \notin G$ be an extra element appearing in a product ab , where $a, b \in G$, thus the result becomes a hyper-product $a \otimes b = \{ab, v\}$. Then the minimal hyper-operation $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G , and such that $(G', \otimes)$ is a minimal H_v -semigroup which has fundamental structure isomorphic to $(G, \cdot)$, is defined by

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy, \quad \forall (x, y) \in G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \text{ and } v \otimes x = abx, \quad \forall x \in G.$$

$(G', \otimes)$ is very-thin H_v -semigroup. If $(G, \cdot)$ is commutative then we obtain that $(G', \otimes)$ is strongly commutative.

Now we present some ‘small’ H_v -fields obtained mainly by finite ordinary rings, by enlarging only one result by adding only one element from the underline ring. These H_v -fields are useful since the results are as small as possible. Moreover, they obey in axioms needed in applications in hadronic mechanics [15], [23], [26].

Constructions 2.2 On the rings $(Z_4, +, \cdot)$ and $(Z_6, +, \cdot)$ we will define all the multiplicative H_v -fields which have non-degenerate fundamental field and, moreover they are,

(a) very-thin minimal, (b) weak commutative, (c) have 0 and 1, scalars.

(I) On $(Z_4, +, \cdot)$ we have the isomorphic cases: $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$. The fundamental classes are $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ and we have $(Z_4, +, \otimes) / \gamma^* \cong (Z_2, +, \cdot)$. In this H_v -group there is only one unit and every element have a unique double inverse.

(II) On $(Z_6, +, \cdot)$, we have the only one hyperproduct,

(i) $2 \otimes 3 = \{0, 3\}$, $2 \otimes 4 = \{2, 5\}$, $3 \otimes 4 = \{0, 3\}$, $3 \otimes 5 = \{0, 3\}$, $4 \otimes 5 = \{2, 5\}$. Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ and we have $(Z_6, +, \otimes) / \gamma^* \cong (Z_3, +, \cdot)$.

- (ii) $2 \otimes 3 = \{0, 2\}$ or $2 \otimes 3 = \{0, 4\}$, $2 \otimes 4 = \{0, 2\}$ or $\{2, 4\}$, $2 \otimes 5 = \{0, 4\}$ or $2 \otimes 5 = \{2, 4\}$,
 $3 \otimes 4 = \{0, 2\}$ or $\{0, 4\}$, $3 \otimes 5 = \{3, 5\}$, $4 \otimes 5 = \{0, 2\}$ or $\{2, 4\}$. Fundamental classes:
 $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ and we have $(Z_6, +, \otimes) / \gamma^* \cong (Z_2, +, \cdot)$.

Representations of H_v -groups, can be considered either by H_v -matrices [5], [14], [16], [17], [18] or by generalized permutations [16]. Here we focus on representations by H_v -matrices because they are used in Santilli's admissibility.

Definition 2.3 H_v -matrix is called a matrix with entries from an H_v -ring or H_v -field. The hyper-product of H_v -matrices $A = (a_{ij})$ and $B = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v -matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) \mid c_{ij} \in \oplus \Sigma a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ is the n -ary circle hyper-operation, which means the union of all possible patterns of parentheses put for elements, on the hyper-addition.

The representation problem by H_v -matrices is the following:

Definition 2.4 Let $(H, \cdot)$ be an H_v -group, $(R, +, \cdot)$ be an H_v -ring and take a set of H_v -matrices $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then any

$$T: H \rightarrow M_R: h \mapsto T(h) \text{ with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H,$$

is called *H_v -matrix representation*. If $T(h_1 h_2) \subset T(h_1)T(h_2)$, then T is called *inclusion representation*, if $T(h_1 h_2) = T(h_1)T(h_2)$, then T is called *good representation*, in this case an induced representation T^* for the H_v -group algebra, is obtained. If T is one to one and good, then it is a *faithful representation*.

Theorem 2.5 A necessary condition in order to have an inclusion representation T of an H_v -group $(H, \cdot)$ by $n \times n$ H_v -matrices over the H_v -ring $(R, +, \cdot)$ is the following:

For all $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}.$$

Thus, every inclusion representation $T: H \rightarrow M_R: a \mapsto T(a) = (a_{ij})$ induces a homomorphic representation T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

Several classes of H_v -structures can give special representations. Some of those classes are [16], [20], [26]:

Definition 2.6 Let $M = M_{m \times n}$, the set of $m \times n$ matrices on R and take $P = \{P_i: i \in I\} \subseteq M$. We define, a kind of, P -hyper-operation $\underline{P}$ on M as follows

$$\underline{P}: M \times M \rightarrow P(M): (A, B) \underline{A} \underline{P} B = \{A P_i^t B: i \in I\} \subseteq M$$

where P^t denotes the transpose of P . $\underline{P}$ is bilinear Rees' like operation where, instead of one sandwich matrix, a set is used. $\underline{P}$ is strong associative and the inclusion distributive to addition is valid:

$$\underline{A}\underline{P}(\underline{B}+\underline{C}) \subseteq \underline{A}\underline{P}\underline{B}+\underline{A}\underline{P}\underline{C}, \forall \underline{A},\underline{B},\underline{C} \in M$$

So $(M, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices.

Let $M = M_{m \times n}$ be module of $m \times n$ matrices on R and take the sets

$$S = \{s_k : k \in K\} \subseteq R, \quad Q = \{Q_i : i \in J\} \subseteq M, \quad P = \{P_i : i \in I\} \subseteq M.$$

Define three hyperoperations as follows

$$\underline{S} : R \times M \rightarrow P(M) : (r, A) \rightarrow r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq M$$

$$\underline{Q}_+ : M \times M \rightarrow P(M) : (A, B) \rightarrow A\underline{Q}_+B = \{A + Q_i + B : i \in J\} \subseteq M$$

$$\underline{P} : M \times M \rightarrow P(M) : (A, B) \rightarrow A\underline{P}B = \{A P_i B : i \in I\} \subseteq M$$

Then, $(M, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra on R the *general matrix P-hyperalgebra*.

3. The Lie-Santilli H_v -admissibility

Hyper-structures have applications in other sciences, which range from hadronic physics, leptons, Santilli's iso-theory, bio-mathematics to mention but a few. The hyper-structure theory is related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too [1], [2], [3], [5], [6], [7], [10], [11], [12], [13], [16], [28], [29].

In [21], with '*The Santilli's theory 'invasion' in hyperstructures*', there is a description how Santilli's theories effect in hyper-structures and how new theories in Mathematics appeared by Santilli's pioneer research. In 1996 Santilli & Vougiouklis [14], [21], point out that in physics the interesting hyper-structures are the e-hyperstructures. These hyper-structures contain a unique left and right scalar unit, which is an important tool in Lie-Santilli theory. See books and related papers for definitions and results related topics: [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [20], [21], [22], [24], [25].

Definition 3.1 A hyper-structure $(F, +, \cdot)$, where $(+)$ is operation and $(\cdot)$ hyper-operation, is called *e-hyperfield* if the following are valid: $(F, +)$ is abelian group with additive unit 0, $(\cdot)$ is weak associative, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$, there exist a product scalar unit 1: $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there is a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If $1 = x \cdot x^{-1} = x^{-1} \cdot x$, then we say that we have a *strong e-hyperfield*.

Definition 3.2 *Main e-Construction*. Given a group $(G, \cdot)$, where e the unit, we define in G , a huge number of hyper-operations $(\otimes)$ by

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \quad \forall x, y \in G - \{e\}, \text{ where } g_1, g_2, \dots \in G - \{e\}.$$

$(G, \otimes)$ is e-hypergroup, in fact is an H_b -group because contains $(G, \cdot)$. Moreover, if $\forall x, y$ such that $xy = e$, so $x \otimes y = xy$, then $(G, \otimes)$ becomes strong e-hypergroup.

The proof is immediate since we enlarge results of the group by putting elements from G applying the Little Theorem. Moreover, e is unique scalar and $\forall x \in G$, there is a unique inverse x^{-1} , such that $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Finally, if the last condition of this definition is valid, then $e = x \cdot x^{-1} = x^{-1} \cdot x$, so $(G, \otimes)$ is strong e -hypergroup.

The useful e -hypergroups are if only one product is enlarged by one element.

An important topic in hyper-mathematics comes from Santilli's admissibility. We may transfer this theory to representations in two ways: using either, ordinary matrices with a hyper-operation on them, or hyper-matrices with ordinary operations [5], [9], [26].

Definition 3.3 Let L be H_v -vector space over the H_v -field $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of F/γ^* . Let ω_L be the core of $\varphi': L \rightarrow L/\varepsilon^*$ and denote again by 0 the zero of L/ε^* . Take two subsets $R, S \subseteq L$ then a *Lie-Santilli admissible hyperalgebra* is obtained by taking the Lie bracket hyper-operation

$$[\cdot, \cdot]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = xRy - ySx = \{xry - ysx \mid r \in R, s \in S\}$$

We can define admissibility on non square matrices, as well:

Definition 3.4 Let $(L = M_{m \times n}, +)$ be the H_v -vector space of $m \times n$ hyper-matrices on the H_v -field $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, canonical map, $\omega_F = \{x \in F: \varphi(x) = 0\}$. Similarly, let ω_L the core of $\varphi': L \rightarrow L/\varepsilon^*$. Take any two subsets $R, S \subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hyper-operation

$$[\cdot, \cdot]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x, y]_{RS} = xR^t y - yS^t x = \{x r^t y - y s^t x \mid r \in R \text{ and } s \in S\}$

Definition 3.5 According to Santilli's iso-theory, on a field $F = (F, +, \cdot)$, a *general isofield* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$, is defined, to be a field with elements $\hat{a} = a \times \hat{1}$, called *isonumbers*, where $a \in F$, and $\hat{1}$ is a positive-defined element outside F , equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0 , and $\hat{\times}$ is a new product

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \quad \forall \hat{a}, \hat{b} \in \hat{F} \quad (i)$$

called *iso-product*, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \times \hat{1} = \hat{a}, \quad \forall \hat{a} \in \hat{F} \quad (ii)$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory to hyper-structures, we generalize only the new product $\hat{\times}$ from (i), by replacing with a hyper-operation including the old one. There are two general constructions on this direction as follows:

Construction 3.6 *General enlargement.* On a field $F = (F, +, \cdot)$ and isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ we replace in the results of the iso-product

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}$ where $\hat{x}_1, \hat{x}_2, \dots \in \hat{F}$, containing $\hat{T}$, for all $\hat{a} \hat{\times} \hat{b}$ for which $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$ and $\hat{x}_1, \hat{x}_2, \dots \in \hat{F} - \{\hat{0}, \hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore, the new iso-hyper-operation is

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\} \times \hat{b}, \quad \forall \hat{a}, \hat{b} \in \hat{F} \quad (\text{iii})$$

$\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is *iso- H_v -field*. The elements of F are called *iso- H_v -numbers* or *iso-numbers*.

Important hopes, of this construction, are those where only for few ordered pairs $(\hat{a}, \hat{b})$ the result is enlarged, even more, the extra element $\hat{x}_i$, is exactly one. Thus, there exists only one pair $(\hat{a}, \hat{b})$ for which

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \quad \forall \hat{a}, \hat{b} \in \hat{F}$$

and the rest are ordinary, then we have a *very-thin iso- H_v -field*.

The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, where $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the iso- H_v -field has scalar absorbing $\hat{0}$, scalar $\hat{1}$, and every $\hat{a} \in \hat{F}$, has one inverse.

Construction 3.7 *The P-hyper-operation.* Consider an isofield $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ with $\hat{a}=a \times \hat{1}$, isonumbers, where $a \in F$, and $\hat{1}$ is positive-defined outside F , with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-product

$$\hat{a} \hat{\times} \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \quad \forall \hat{a}, \hat{b} \in \hat{F}$$

Take a $\hat{P}=\{\hat{T}, \hat{p}_1, \dots, \hat{p}_s\}$, with $\hat{p}_1, \dots, \hat{p}_s \in \hat{F} - \{\hat{0}, \hat{1}\}$, define the *isoP- H_v -field*, $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times}_P)$, where the hyper-operation $\hat{\times}_P$ is defined as follows:

$$\hat{a} \hat{\times}_P \hat{b} := \begin{cases} \hat{a} \times \hat{P} \times \hat{b} = \{\hat{a} \times \hat{h} \times \hat{b} \mid \hat{h} \in \hat{P}\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \times \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases} \quad (\text{iv})$$

The elements of $\hat{F}$ are called *isoP- H_v -numbers*.

If $\hat{P} = \{\hat{T}, \hat{p}\}$, the inverses in isoP- H_v -fields, are not necessarily unique.

Example 3.8 The generalized P-construction can be applied on rings to obtain H_v -fields. Thus for, $\hat{\mathbf{Z}}_{10} = \mathbf{Z}_{10}(\hat{\underline{a}}, \hat{+}, \hat{\times})$, and if we take $\hat{P}=\{\hat{\underline{2}}, \hat{\underline{7}}\}$, then we have the table

$\hat{\times}$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$	$\hat{9}$
$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$
$\hat{1}$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$	$\hat{9}$
$\hat{2}$	$\hat{0}$	$\hat{2}$	$\hat{8}$	$\hat{2}$	$\hat{6}$	$\hat{0}$	$\hat{4}$	$\hat{8}$	$\hat{2}$	$\hat{6}$
$\hat{3}$	$\hat{0}$	$\hat{3}$	$\hat{2}$	$\hat{3}, \hat{8}$	$\hat{4}$	$\hat{0}, \hat{5}$	$\hat{6}$	$\hat{2}, \hat{7}$	$\hat{8}$	$\hat{4}, \hat{9}$
$\hat{4}$	$\hat{0}$	$\hat{4}$	$\hat{6}$	$\hat{4}$	$\hat{2}$	$\hat{0}$	$\hat{8}$	$\hat{6}$	$\hat{4}$	$\hat{2}$
$\hat{5}$	$\hat{0}$	$\hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$
$\hat{6}$	$\hat{0}$	$\hat{6}$	$\hat{4}$	$\hat{6}$	$\hat{8}$	$\hat{0}$	$\hat{2}$	$\hat{4}$	$\hat{6}$	$\hat{8}$
$\hat{7}$	$\hat{0}$	$\hat{7}$	$\hat{8}$	$\hat{2}, \hat{7}$	$\hat{6}$	$\hat{0}, \hat{5}$	$\hat{4}$	$\hat{3}, \hat{8}$	$\hat{2}$	$\hat{1}, \hat{6}$
$\hat{8}$	$\hat{0}$	$\hat{8}$	$\hat{2}$	$\hat{8}$	$\hat{4}$	$\hat{0}$	$\hat{6}$	$\hat{2}$	$\hat{8}$	$\hat{4}$
$\hat{9}$	$\hat{0}$	$\hat{9}$	$\hat{6}$	$\hat{4}, \hat{9}$	$\hat{2}$	$\hat{0}, \hat{5}$	$\hat{8}$	$\hat{1}, \hat{6}$	$\hat{4}$	$\hat{2}, \hat{7}$

Then the fundamental classes are

$$(0)=\{\hat{0}, \hat{5}\}, (1)=\{\hat{1}, \hat{6}\}, (2)=\{\hat{2}, \hat{7}\}, (3)=\{\hat{3}, \hat{8}\}, (4)=\{\hat{4}, \hat{9}\},$$

and the multiplicative table is the following

$\times$	(0)	(1)	(2)	(3)	(4)
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{\mathbf{Z}}_{10} = \mathbf{Z}_{10}(\hat{\mathbf{a}}, \hat{+}, \hat{\times})$, is an H_v -field.

Example 3.9 Consider the $\hat{\mathbf{Z}}_{14} = \mathbf{Z}_{14}(\hat{\mathbf{a}}, \hat{+}, \hat{\times})$, and take $\hat{P}=\{\hat{2}, \hat{9}\}$, then we have the table

$\otimes$	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	<u>10</u>	<u>11</u>	<u>12</u>	<u>13</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	<u>10</u>	<u>11</u>	<u>12</u>	<u>13</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>2</u>	<u>12</u>	<u>2</u>	<u>6</u>	<u>10</u>	<u>0</u>	<u>4</u>	<u>8</u>	<u>12</u>	<u>2</u>	<u>6</u>	<u>10</u>
<u>3</u>	<u>0</u>	<u>3</u>	<u>12</u>	<u>4,11</u>	<u>10</u>	<u>2,9</u>	<u>8</u>	<u>0,7</u>	<u>6</u>	<u>5,12</u>	<u>4</u>	<u>3,10</u>	<u>2</u>	<u>1,8</u>
<u>4</u>	<u>0</u>	<u>4</u>	<u>2</u>	<u>10</u>	<u>4</u>	<u>12</u>	<u>6</u>	<u>0</u>	<u>8</u>	<u>2</u>	<u>10</u>	<u>4</u>	<u>12</u>	<u>6</u>
<u>5</u>	<u>0</u>	<u>5</u>	<u>6</u>	<u>2,9</u>	<u>12</u>	<u>1,8</u>	<u>4</u>	<u>0,7</u>	<u>10</u>	<u>6,13</u>	<u>2</u>	<u>5,12</u>	<u>8</u>	<u>4,11</u>
<u>6</u>	<u>0</u>	<u>6</u>	<u>10</u>	<u>8</u>	<u>6</u>	<u>4</u>	<u>2</u>	<u>0</u>	<u>12</u>	<u>10</u>	<u>8</u>	<u>6</u>	<u>4</u>	<u>2</u>
<u>7</u>	<u>0</u>	<u>7</u>	<u>0</u>	<u>0,7</u>	<u>0</u>	<u>0,7</u>	<u>0</u>	<u>0,7</u>	<u>0</u>	<u>0,7</u>	<u>0</u>	<u>0,7</u>	<u>0</u>	<u>0,7</u>
<u>8</u>	<u>0</u>	<u>8</u>	<u>4</u>	<u>6</u>	<u>8</u>	<u>10</u>	<u>12</u>	<u>0</u>	<u>2</u>	<u>4</u>	<u>6</u>	<u>8</u>	<u>10</u>	<u>12</u>
<u>9</u>	<u>0</u>	<u>9</u>	<u>8</u>	<u>5,12</u>	<u>2</u>	<u>6,13</u>	<u>10</u>	<u>0,7</u>	<u>4</u>	<u>1,8</u>	<u>12</u>	<u>2,9</u>	<u>6</u>	<u>3,10</u>
<u>10</u>	<u>0</u>	<u>10</u>	<u>12</u>	<u>4</u>	<u>10</u>	<u>2</u>	<u>8</u>	<u>0</u>	<u>6</u>	<u>12</u>	<u>4</u>	<u>10</u>	<u>2</u>	<u>8</u>
<u>11</u>	<u>0</u>	<u>11</u>	<u>2</u>	<u>3,10</u>	<u>4</u>	<u>5,12</u>	<u>6</u>	<u>0,7</u>	<u>8</u>	<u>2,9</u>	<u>10</u>	<u>4,11</u>	<u>6</u>	<u>3,10</u>
<u>12</u>	<u>0</u>	<u>12</u>	<u>6</u>	<u>2</u>	<u>12</u>	<u>8</u>	<u>4</u>	<u>0</u>	<u>10</u>	<u>6</u>	<u>2</u>	<u>6</u>	<u>8</u>	<u>4</u>
<u>13</u>	<u>0</u>	<u>13</u>	<u>10</u>	<u>1,8</u>	<u>6</u>	<u>4,11</u>	<u>2</u>	<u>0,7</u>	<u>12</u>	<u>3,10</u>	<u>8</u>	<u>3,10</u>	<u>4</u>	<u>2,9</u>

Then the fundamental classes are

$$(0)=\{\underline{0},\underline{7}\}, (1)=\{\underline{1},\underline{8}\}, (2)=\{\underline{2},\underline{9}\}, (3)=\{\underline{3},\underline{10}\}, (4)=\{\underline{4},\underline{11}\}, (5)=\{\underline{5},\underline{12}\}, (6)=\{\underline{6},\underline{13}\},$$

and the multiplicative table is the following

$\times$	(0)	(1)	(2)	(3)	(4)	(5)	(6)
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(6)	(1),(4)	(3),(5)	(5),(6)
(2)	(0)	(2),(4)	(1),(2)	(5)	(2)	(6)	(3)
(3)	(0)	(3),(6)	(5)	(4)	(3)	(2)	(1)
(4)	(0)	(1),(4)	(2)	(3)	(4)	(5),(6)	(3),(6)
(5)	(0)	(3),(5)	(6)	(2)	(5),(6)	(1)	(4)
(6)	(0)	(5),(6)	(3)	(1)	(3),(6)	(4)	(2)

Consequently, $\hat{\mathbf{Z}}_{14} = \mathbf{Z}_{14}(\hat{\mathbf{a}}, \hat{+}, \hat{\otimes})$, is an H_V -field.

4 Conclusions

The very big number of hyper-structures, especially of H_v -structures, defined on a set gives the opportunity to work in applications in an opposite direction. This means that we ask from applied sciences to give more axioms in order to reduce the number of possible H_v -structures to express the problem by a mathematical model. This is the problem of Lie-Santilli's admissibility which can be faced by defining the appropriate hyper-numbers or H_v -numbers, needed to express the irreversibility and uniqueness of living organisms and time.

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Problematic aspects of 20-th century antiparticles and their apparent resolution via isodual mathematics

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Abstract

In this paper, we recall Dirac's *negative energy* antiparticles, their compatibility with particle-antiparticle annihilation into light and their lack of compatibility with special relativity as well as causality laws. We then recall the 20th century *positive energy* antiparticles, their compatibility with special relativity and causality laws but their incompatibility with annihilation into light, with ensuing problematic aspects for a true antimatter character of antiprotons, anti-Hydrogen atoms and related gravity tests. We then review the *isodual branch of hadronic mechanics* whose *isodual theory of antimatter*: 1) Represents Dirac's negative energy antiparticles without causality problems. 2) Admits special and general relativities due to the invariance of quantum axioms under the isodual map. 3) Implies matter-antimatter anti-gravity at all levels. 4) Predicts the existence of the negative energy *antiphoton*. 5) Is compatible with existing experimental evidence on antiparticles. We suggest the conduction of resolatory tests on the gravity of well established antiparticles, such as the positrons in horizontal flight in a supercooled vacuum tube. We conclude with the indication of intriguing open problems in antimatter, such as the possible *expulsion* of antiphotons by black holes following internal particle-antiparticle creation and annihilation.

Keywords: antiprotons; pseudoprotons; antimatter; hadronic mechanics ¹

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1 Dirac's negative energy antiparticles

As it is well known, the first quantitative formulation of antimatter was achieved in 1928 by P. A. M. Dirac [1] via his celebrated equation on the Minkowski space-time $M = M(x, \eta, I)$ with coordinates $x = (x^\mu) = (x^1, x^2, x^3, x^4 = ct)$, $\mu = 1, 2, 3, 4$, metric $\eta = \text{Diag.}(1, 1, 1, -1)$ and unit $I = \text{Diag.}(1, 1, 1, 1)$ over the field of real numbers $\mathcal{R}$

$$(i\eta^{\mu\nu}\gamma_\mu\partial_\nu - mc)|\psi(x)\rangle = 0, \quad (1)$$

where the Dirac's matrices γ_μ , $\mu = 1, 2, 3, 4$ are given by

$$\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = i \begin{pmatrix} +I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}, \quad (2)$$

and Pauli's matrices σ_k , $k = 1, 2, 3$ represent the spin $S = 1/2$ of the particles.

According to Dirac's conception, the electron e^- is represented by the top 2 components of Eq. (1) with unit $+I_{2 \times 2}$, thus having *positive energy* $m_e > 0$, while the "antielectron" $\bar{e}^- = e^+$ (now called the positron), is represented by the bottom 2×2 component of Eq. (1) with unit $-I_{2 \times 2}$, thus having *negative energy* $\bar{m}_e = -m_e < 0$.

An important property of Dirac's conception of antimatter is that, in view of the opposite sign of the masses of the two particles, Eq. (1) allows a consistent representation of the electron-positron annihilation into light [2]

$$e^- + e^+ \rightarrow \gamma + \gamma. \quad (3)$$

More specifically, Dirac established the foundations of the notion of particle-antiparticle *annihilation*, which according to the very meaning of the word, deals with the elimination of the particle masses and ensuing transition to light.

Unfortunately, Dirac's conception of negative energy antiparticles is incompatible with special and general relativities on various counts and is afflicted by a number of problems of physical consistency that were identified by Dirac himself.

For instance, negative energies implies the possibility that the (positive energy) electron can emit a photon and pass to a negative energy state. The repetition of the process would then imply the lack of conservation of the energy. Similarly, the scattering between a positive and a negative energy particle may imply effects preceding their causes, by therefore violating causality.

Dirac attempted to resolve the above problems via his *hole theory* [3] which assumed the existence of an infinite sea of negative state positrons with occasional "holes" occupied by positive energy electrons, which theory provided a conceptual rendering of electron-positron annihilation and other events.

2 20th century positive energy antiparticles

In 1930, to achieve compatibility with special relativity (which was an understandable task for the time), J. R. Oppenheimer [4] assumed Dirac's equation as the fundamental equation of relativistic quantum mechanics, but under the theoretical assumption that *antiparticles have the same positive mass of particles* $\bar{m}_{e^+} = m_{e^-} > 0$, which assumption remains in full effect to this day.

The above view was implemented in the particle-antiparticle conjugation via the *PTC* theorem [5]. As an illustration, consider the wave packet of a free particle $\psi(t, r) = \exp[i(p \times r - E \times t)]$ on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with state $|p, J, \psi\rangle$ where $p = mv$, J is the spin of the particle along the third axis and $E = v^2/2m$. Then, the *charge conjugation* of the particle is given by the anti-Hermitean transformation

$$C\psi(t, r) = -\psi^\dagger(t, r), \quad (4)$$

and the *particle-antiparticle conjugation* is given by

$$\begin{aligned} PTC|p, J, \psi(t, r)\rangle &= PT|p, J, -\psi^\dagger(t, r)\rangle = \\ &= P| -p, -J, -\psi^\dagger(-t, r)\rangle = |p, -J, -\psi^\dagger(t, r)\rangle, \end{aligned} \quad (5)$$

according to which antiparticles have the same energy E and momentum $p = mv$ of antiparticles and are defined on the same coordinates t, r , the same Hilbert space $\mathcal{H}$ of particles over the same numeric field $\mathcal{C}$.

Recall that a particle with spin 1/2 is technically defined as a unitary irreducible representation of the spinorial covering of the Poincare' symmetry $\mathcal{P}(3.1) = SL(2, \mathcal{C}) \times \mathcal{T}(3.1)$ [6]. In order to render conjugation (7) compatible with the $\mathcal{P}(3.1)$ symmetry while considering Eq. (1) the fundamental equation of relativistic quantum mechanics, the physicists of the early 20th century assumed the realization of the generators of $\mathcal{P}(3.1)$ in terms of the 4×4 gamma matrices (2) [6] [7] [8] (rather than as the Kronecker product of the symmetries of the two distinct 2×2 -components of Dirac's equation as done in the next section)

$$\mathcal{P}(3.1) = SL(2, \mathcal{C}) \times \mathcal{T}(3.1), \quad (6)$$

$$\mathcal{SL}(2, \mathcal{C}) : S_k = \frac{1}{2}\gamma_k \times \gamma_4, \quad R_k = \frac{1}{2}\epsilon_{ij}^k \gamma_i \times \gamma_j, \quad \mathcal{T}(3.1) : P_\mu,$$

where S_k , R_k and T_μ are the generators of spin, boosts and linear momentum, respectively, with commutation rules for $J_{\mu, \nu} = \{S_k, R_k\}$

$$\begin{aligned} [J_{\mu\nu}, J_{\alpha\beta}] &= i(\eta_{\nu\alpha}J_{\beta\mu} - \eta_{\mu\alpha}J_{\beta\nu} - \eta_{\nu\beta}J_{\alpha\mu} + \eta_{\mu\beta}J_{\alpha\nu}), \\ [J_{\mu\nu}, P_\alpha] &= i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \quad [P_\mu, P_\nu] = 0, \end{aligned} \quad (7)$$

and Casimir invariants

$$\begin{aligned} C_1 &= I, \\ C_2 &= P^2 = P_\mu P^\mu = (\eta^{\mu\nu} P_\mu P_\nu), \\ C_3 &= W^2 = W_\mu W^\mu, \quad W_\mu = \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} P^\rho. \end{aligned} \tag{8}$$

Unfortunately, the conception of antiparticles as having positive masses with conjugation (5) implies the departure from Dirac's particle-antiparticle *annihilation* into light, since positive mass particles and antiparticles can only experience a *scattering* with the production of a shower of positive mass particles, as experimentally established in the Bose-Einstein correlation [9].

The important distinction between Dirac's annihilation and 20th century particle scattering has been technically expressed in the preceding paper [10] (see also the recent comments [11]) via the following:

THEOREM 2.1: Under the spinorial Poincaré symmetry (6), particle-antiparticle conjugation (5) is incompatible with the experimental evidence of electron-positron annihilation into light (3).

PROOF: Within the class of unitary equivalence of relativistic quantum mechanics, there exists no possibility of rendering null the sum of two positive masses in the Casimir invariants of the $\mathcal{P}(3.1)$ symmetry, Eq. (8).

The above theorem casts shadows on the true antimatter character of the 20th century notion of antiparticles characterized by conjugation (5), because said theorem prohibits their annihilation into light, as predicted by basic axioms, available preliminary evidence (Sect. 3.5 and Fig. 1) and the following comments:

2.1. Under the condition of being represented via Dirac's equation (1), we shall call *cosmic antiprotons* the antiparticles expected to be the origin of the bright flashes detected by astronauts and cosmonauts in the dark side of our upper atmosphere [12], or detected in cosmic rays [13]. We shall then call *laboratory antiprotons* the particles $\bar{p}^-$ produced in laboratory via collision of high energy protons p^+ against a matter target t (e. g., an iridium rod) according to the reaction [15]

$$p^+ + t \rightarrow (p^+, \bar{p}^-) + p^+ + t, \tag{9}$$

for which a proton-antiproton pair is expected to be produced by *pure kinetic energy*. Under these conditions, it is natural to expect that protons and true antiprotons should annihilate into pure electromagnetic radiations (here symbolically denoted γ_k , $k = 1, 2, \dots$) without massive residues

$$p^+ + \bar{p}^- \rightarrow \Sigma_k \gamma_k. \tag{10}$$

The apparent lack of true antimatter character of laboratory antiprotons $\bar{p}^-$ is then expected from the experimental evidence that proton-antiproton collisions in the Bose-Einstein correlation produce instead a shower of bosons $B_k = 1, 2, 3, \dots$ [9]

$$p^+ + \bar{p}^- \rightarrow \Sigma_k B_k. \quad (11)$$

2.2. The synthesis of the neutron from an electron and a proton in the core of stars [16]

$$e^- + p^+ \rightarrow n + \nu, \quad (12)$$

has received significant theoretical [17] as well as experimental [18] verifications. The same studies have identified the subsequent apparent synthesis in the Sun and in laboratory of an electron, and this time, a neutron, into the *negatively charged* pseudoproton [19]

$$e^- + n \rightarrow \tilde{p}^- + \nu, \quad (13)$$

which is essentially a positive energy proton with a negative charge. It is therefore possible that high energy collisions of protons against a matter-target (thus, initially against the *electron clouds* of the target, rather than their nuclei) may first synthesize neutrons (11) and subsequently pseudoprotons (12), rather than true antiprotons (see [11] for additional comments).

2.3. Conjugation (5) does not allow a study of the rather general view that the antimatter world is a complete dual of the matter world [20] [21]. In fact, the representation of *particle-antiparticle* conjugation with *charge* conjugation (4) is insufficient to study as to whether the photon, the π^0 meson and other particles do have an antimatter counterpart as predictable by the extension of Dirac's negative energy antiparticles to the photon and all other particles.

Needless to say, doubts on the true antimatter character of laboratory antiprotons cast shadows on the entire antimatter character of the anti-Hydrogen atom [22], and consequently, on its claimed conventional gravity [23] (see Sect. 4 for specific comments on antigravity tests).

3 Isodual theory of antiparticles

3.1. Isodual branch of hadronic mechanics

From the analysis of the preceding sections, it appears that, whenever treated with *20th century applied mathematics*, the notion of antiparticles is not entirely compatible with physical evidence irrespective of whether the antiparticle masses are negative (Sect. 1) or positive (Sect. 2).

This author has indicated various times in his writings that the lack of resolution of century old *physical* problems is generally due to the use of insufficient

mathematics, as illustrated by the advent of Hilbert spaces for the Hydrogen structure, Riemannian geometry for gravitation, and other historical cases.

By noting that charge conjugation (4) is anti-Hermitean, it is plausible to expect that the same conjugation applies to all other characteristics of antiparticles. This expectation lead to the construction of the *isodual branch of hadronic mechanics* [24]-[28] which can be constructed via the anti-Hermitean conjugation, called *isoduality* (and denoted with an upper or lower letter d) applied to the *totality* of quantities Q , all possible local variables $t, r, p, E, \psi, \dots$ and all operations of 20th century particle physics

$$\begin{aligned} Q(t, r, p, E, \psi, \dots) &\rightarrow Q^d(t^d, r^d, p^d, E^d, \psi^d, \dots) = \\ &= -Q^\dagger(-t^\dagger, -r^\dagger, -p^\dagger, -E^\dagger, -\psi^\dagger, \dots) = -Q^\dagger(-t, -r, -p, -E, -\psi^\dagger, \dots). \end{aligned} \quad (14)$$

A quantity Q is said to be *isoselfdual* when it coincides with its isodual, $Q \equiv Q^d$.

In this paper, we study *point-like antiparticles in vacuum* (see Refs. [24]-[26] for extended antiparticles) treated via the *isodual mathematics* with fundamental, *negative-definite* multiplicative unit (called *isodual unit*)

$$1^d = -1^\dagger = -1, \quad (15)$$

compatible multiplication (called the *isodual product*)

$$\begin{aligned} s^d &= (1^d)^{-1}, \\ 1^d \times^d n^d &= 1^d s^d n^d = n^d s^d 1^d \equiv n^d, \end{aligned} \quad (16)$$

and compatible isodual images of applied mathematics, including: new numbers with negative (multiplicative) unit called *isodual numbers* [29]; *isodual spaces* [30]; *isodual Lie's theory* [24]-[26]; *isodual space-time symmetries* [31]-[34]; *isodual Minkowskian and Riemannian geometries* [35]; and other aspects of isodual mathematics.

To prevent a prohibitive length, we limit ourselves to recall:

3.1) *Isodual fields* $\mathcal{F}^d(n^d, \times^d, 1^d)$ of *isodual real, complex and quaternionic numbers* $n^d = -n^\dagger$ [29] with isodual multiplication

$$n^d \times^d m^d = k^d \forall n^d, m^d, k^d \in F^d. \quad (17)$$

Recall that functions $f(r)$ have values on their base field. Similarly, *isodual functions* $f^d(r^d)$ have value in their isodual isofield, e.g., the isodual of e^r is given by $e_d^{r^d} = -e^{-r}$.

Problematic aspects of 20-th century antiparticles

3.2) *Isodual space-time* [24] [25] [26] $M^d = M^d(x, \Omega^d, I^d)$ over $\mathcal{R}^d$ with *isodual coordinates*

$$\begin{aligned} x^d = (x_\mu^d) &= (-x^{d,1}, -x^{d,2}, -x^{d,3}, -x^{d,4} = \\ &= -c^d \times^d t^d = ct) \in R^d, \end{aligned} \quad (18)$$

isodual metric

$$\begin{aligned} \Omega^d &= (1^d)^{-1} \eta^d = (1^d)^{-1} \text{Diag.}(1^d, 1^d, 1^d, -1^d) = \\ &= \text{Diag}(-1, -1, -1, +1), \end{aligned} \quad (19)$$

where the isoduality $\eta^d = (1^d, 1^d, 1^d, -1^d)$ is needed for the metric η^d to be an *isodual matrix*, namely, a matrix whose elements are isodual numbers [24].

3.3) *Isodual differential calculus* of an isodual coordinate $r^d \in R^d$ [30] (see also [27])

$$d^d r^d = s^d d(r^d), \quad (20)$$

$$\frac{\partial^d f^d(r^d)}{\partial^d r^d} = 1^d \frac{\partial f^d(r^d)}{\partial r^d}.$$

Note that isodual mathematics is somehow "hidden" in the abstract axioms of 20th century mathematics because of the isoselfduality of the space-time invariant and of the differential

$$\begin{aligned} x^{d2d} &\equiv x^2, \quad d^d r^d \equiv dr, \\ d^d F^d(r^d) &\equiv dF(r^d), \end{aligned} \quad (21)$$

as well as of other axioms, thus explaining its lack of detection until the 1993 paper [29].

3.2. Isodual theory of antiparticles

Following the achievement of maturity in the isodual mathematics, Santilli [36] proposed in 1994 the *isodual theory of antimatter* which can be defined as the isodual image of 20th century particle physics, by therefore being elaborated via isodual mathematics.

In this section, we shall attempt to verify the primary aims of proposal [36]:

1) Represent Dirac's negative energy antiparticles without its causality problems.

2) Achieve an axiomatic distinction between Dirac's particle and negative-energy antiparticle "annihilation" into light and the 20th century "scattering" of particles and positive-energy antiparticles.

3) Represent all known experimental data in antiparticles.



Figure 1: We present a picture of the 1908 Tunguska explosion [52] that: A) Had the equivalent of hundreds of Hiroshima atomic bombs; B) Ionized the entire Earth atmosphere for days; and C) Produced on the ground no crater or solid debris. These features can only be quantitatively explained as due to an antimatter asteroid annihilating into sole electromagnetic radiations at contact with Earth's upper atmosphere, Eq. (10), thus constituting evidence of Dirac's particle-antiparticle annihilation into light (Sect. 1), while casting shadow on: the particle-antiparticle scattering into a shower of particles of the Bose-Einstein correlation [50] [51]; the true antimatter character of the nucleus of the anti-Hydrogen atom [22]; and its recent gravity test [23](Sects. 2, 3). The number of large explosions in the sky without a crater or solid debris in the ground is too large to be listed here (see, e.g. reports [53] [54]). The astrophysical origin of cosmic ray antiprotons [12] [13] and antimatter asteroids is under study for reporting in a subsequent paper.

Problematic aspects of 20-th century antiparticles

Isodual antiparticles are defined in the isodual Minkowski space $M^d(x^d, \Omega^d, 1^d)$ over the isodual numeric field $\mathcal{R}^d$ which space is independent but co-existing with our space-time $M(x, \eta, 1)$ thanks to identities (21) [37].

The application of isodual map (14) to charge conjugation (4) yields the *isodual map from particles to antiparticles* [26]

$$\psi(t, r) \rightarrow \psi^d(t^d, r^d) = -\psi^\dagger(-t, -r), \quad (22)$$

without any need for the *isodual parity* P^d (that would map negative into positive coordinates) and *isodual time inversions* T^d (that would map motion backward into forward in time).

The isodual theory of antiparticles resolves the problematic aspects of Dirac's negative energy antiparticles indicated in Sect 1. In fact, it is easy to see that the isodual image $\mathcal{P}^d(3.1)$ [35] of the spinorial covering of the Poincare' symmetry $\mathcal{P}(3.1, \text{Eq. (6)})$, has negative-valued isodual Casimir invariants (8), by therefore bypassing Theorem II.1 and allowing an axiomatically consistent annihilation of particle-antiparticle into light.

The causality problems of Dirac's antiparticles are equally resolved by isodual mathematics because numeric values of ordinary particles are measured in terms of units (e.g., *gr, sec, cm* etc.) that are elements of the basic field $\mathcal{R}$. Consequently, under isoduality numeric values of isodual antiparticles must be measured in terms of units that are elements of the isodual field $\mathcal{R}^d$, thus having *negative* values $gr^d = -gr, sec^d = -sec, cm^d = -cm$, etc.

Consequently, negative energy antiparticles moving backward in time referred to negative units of mass and time are as causal as ordinary particles with positive mass moving forward in time referred to positive units of mass and time.

The problem of the conservation of the energy in particle-antiparticle scattering appears to be resolved by the accurate implementation of isodual mathematics. In fact, an antiparticle is represented in *isodual quantum mechanics* [25] [26] via the isodual Hamiltonian $H^d(r^d, p^d)$ acting on the isodual Hilbert space $\mathcal{H}^d$ with isodual states $|\psi^d(r^d)\rangle$, resulting in isodual eigenvalues E^d defined on the isodual field $\mathcal{R}^d$

$$H^d(r^d, p^d) \times^d |\psi^d(r^d)\rangle = E^d \times^d |\psi^d(r^d)\rangle, \quad E^d < 0_{R^d}, \quad (23)$$

with consequential quantitative representation of Dirac's antimatter with negative energy E^d . When projected in our Hilbert space $\mathcal{H}$ with state $|\psi(r)\rangle$ over the ordinary field $\mathcal{R}$, the same antiparticle of Eq. (23) acquires *positive eigenvalues* on $\mathcal{R}$

$$H^d(r^d, p^d) \times^d |\psi(r)\rangle = E \times |\psi(r)\rangle, \quad E > 0_R. \quad (24)$$

Similarly, a quantum mechanical particle characterized by a Hamiltonian $H(r, p)$ on $\mathcal{H}$ over $\mathcal{R}$ with *positive energy* E

$$H(r, p) \times |\psi(r)\rangle = E \times |\psi(r)\rangle, \quad E > 0_R, \quad (25)$$

when projected in the isodual space $\mathcal{H}^d$ over $\mathcal{R}^d$ acquires *negative energies* when inspected from the isodual world

$$H(r, p) \times |\psi^d(r^d)\rangle = E^d \times^d |\psi^d(r^d)\rangle, \quad E^d < 0_{R^d}. \quad (26)$$

Consequently, the total energy of the Positronium for quantum mechanics (qm) $\mathcal{P} = (e^-, e^+)_{qm}$ is positive in our space-time and negative in the isodual space-time (for brevity see Sect. 2.3.14, p. 131 of [26]).

Note that the isodual image of the wave function of free particles $\psi(t, lr) = \exp[i(p \times r - E \times t)]$ considered in Sect. 1 is given by

$$\begin{aligned} \psi^d(t^d, r^d) &= [e^{i(p \times r - E \times t)}]^d = \\ &= e_d^{i^d \times^d (p^d \times^d r^d - E^d \times^d t^d)} = -e^{-i(p \times r - E \times t)}, \end{aligned} \quad (27)$$

and the same rules apply for the isoduality of generic wave functions.

It should be finally indicated that the isodual map is equivalent to charge conjugation when projected in our world because it is anti-Hermitean, Eq. (14.) like charge conjugation, Eq. (4.) and admits positive energy eigenvalues, Eq. (24). Consequently, to our best knowledge, the isodual theory of antimatter verifies all existing experimental evidence on antiparticles merely subjected to the indicated re-interpretation on isodual spaces over isodual fields [26].

3.3. Isoselfdual Dirac equation

In the transition from non-relativistic to relativistic isodual theory of antiparticles with spin 1/2, let us recall that the isodual theory of antimatter allows the *identical* reformulation of Dirac's equation (1) in the following form called *iso-self-dual Dirac's equation* because invariant under isoduality (Sect. 2.3.6, p. 118 of [26])

$$\begin{aligned} (i^d \times^d \eta^{d\mu\nu} \times^d \gamma_\mu^d \partial_\nu^d - m^d \times^d c^d) \times^d |\psi^d(x^d)\rangle = \\ = (i\eta^{\mu\nu} \bar{\gamma}_\mu \partial_\nu - mc) |\psi(x)\rangle = 0, \end{aligned} \quad (28)$$

where the 4×4 matrices $\bar{\gamma}_\mu$ are given by

$$\begin{aligned} \bar{\gamma}_k &= \begin{pmatrix} 0 & \sigma_k \\ \sigma_k^d & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} = \gamma_k, \\ \gamma_4 &= i \begin{pmatrix} +I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2}^d \end{pmatrix} \equiv i \begin{pmatrix} +I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix} = \gamma_4, \end{aligned} \quad (29)$$

and verify the isodual anti-commutation rules

$$\{\bar{\gamma}_\mu \bar{\gamma}_\nu\}^d = \bar{\gamma}_\mu \times^d \bar{\gamma}_\nu + \bar{\gamma}_\nu \times^d \bar{\gamma}_\mu = 2^d \times^d \eta_{\mu\nu}^d, \quad (30)$$

with total isodual spin symmetry [25]

$$S_{spin}^{total} = SU(2) \times SU^d(2), \quad (31)$$

and corresponding space-time symmetry [31]-[34]

$$S_{Dirac}^{total} = \mathcal{P}(3.1) \times \mathcal{P}^d(3.1), \quad (32)$$

both symmetries (31) and (32) being isoselfdual [26] [31]-[34].

In the author's view, a most important aspect of isodual mathematics is the axiomatic formulation of Dirac's particle-antiparticle annihilation into light via the notion of *isoselfduality* (invariance under the isoduality) with a number of important implications indicated later on.

Another important aspect is that, being a formulation for *point-like* antiparticles in vacuum, the isodual theory of antiparticles preserves special and general relativities in their entirety, since they are merely reformulated on isodual spaces over isodual fields. Intriguingly, the resulting *isodual special and general relativities* [26] are hidden in their conventional axioms, Eqs. (21), by therefore preventing a dismissal of their existence.

Finally, we should note that the isodual theory of antimatter predicts matter-antimatter *antigravity* at all levels, from Newtonian mechanics to general relativity [26]. Particularly instructive is to see the emergence of antigravity in the projection of the isodual curvature tensor in the Riemannian space of general relativity, by therefore illustrating that views on violation of general relativity by antigravity are essentially due to insufficient mathematics along the main line of Sect. 3.1.

3.4. Isodual rules for unstable antiparticles

The isoduality of *permanently stable* particles, i.e., the electron and the proton, is done via a simple use of the above isoselfdual Dirac equation. The isoduality of *unstable* particles is conceptually and technically difficult because instability suggests a composite structure that, in turn, suggests the use of physical, i.e., experimentally detectable (rather than hypothetical and undetectable) constituents as a condition for results to have a chance of passing the test of time.

According to the author's experience in the field dating back to the 1990's, the *numerically exact and time invariant* representation of *all* characteristics of unstable particles, including the mechanism for the spontaneous decay, the value of its mean life and the remaining characteristics could only be possible via the assumption that the physical constituents of unstable particles are ordinary particles are bonded together by nonlinear, nonlocal and nonpotential interactions due to deep mutual penetration and entanglement [38] of the wave packets according to the laws of *hadronic mechanics* (hm) [24] [25] [39] and produced free as a tunnel effect from hadronic to quantum mechanics in the spontaneous decays, generally those with the lowest modes.

As an example, the numerically exact and time invariant representation of all characteristics of the π^0 meson with physical constituents (which is needed for a consistent isodual conjugation) could only be reached by assuming that the π^0 is a *compressed positronium*, namely, the known quantum mechanical positronium [40] compressed down to mutual distances of the order of 10^{-13} cm under the laws of hadronic mechanics (see Table 5.1., p. 827 on of the 1978 Harvard University memoir [39] and Sect. 2.5.4 of the 2021 update [41])

$$\pi^0 = (e_{\downarrow}^-, e_{\uparrow}^{-d})_{hm} \equiv \pi^{0d}. \quad (33)$$

The presence of the positron in the π^0 structure then allowed the understanding of the mechanism of its spontaneous decay via electron-positron annihilation, the numeric value of its mean life, the particle produced in the spontaneous decays and the remaining characteristics.

Under the above conditions, the π^0 meson is isoselfdual, namely, a particle that exists identically in our world as well as in the antiparticle world. Note that representation (33) is impossible for quantum mechanics due to the lack of admission of contact non-Hamiltonian interactions between the wave packets of the electron and positron in deep EPR entanglement.

The exact and invariant representation of *all* characteristics of the $\mu^{\pm}$ lepton, including its anomalous magnetic moment [42], has been achieved via the structure [39] [41]

$$\mu^{\pm} = (e_{\downarrow}^-, e_{\uparrow}^{\pm}, e_{\downarrow}^{\pm})_{hm}, \quad (34)$$

in which case μ^+ can be interpreted as the antiparticle image of μ^- . The antiparticle interpretation of the remaining hadronic structure models of unstable leptons, mesons and baryons is left to the interested reader [41].

3.5. Intriguing predictions

The isodual theory of antimatter identifies the violation of the isoselfdual symmetry by annihilation (3) (because its l.h.s. is isoselfdual while the r.h.s. is not), but restores said symmetry via the prediction of the *antiphoton* (technically identified as the *isodual photon* and denoted γ^d) [43] consisting of photon with all characteristics opposite those of ordinary photons, with consequential reformulation of annihilation (3) into the isoselfdual form on both sides

$$e^- + (e^-)^d \rightarrow \gamma + \gamma^d. \quad (35)$$

An intriguing prediction of the isodual theory of antimatter is that, contrary to the rather popular belief for which nothing can escape from black holes, antiphotons are predicted to be *expelled* by black holes due to antigravity following internal particle-antiparticle creation and subsequent annihilation in the isoselfdual form (35).

The apparent lack of true antimatter character of laboratory antiprotons appears to be confirmed by the 1908 Tunguska explosion [52] in Siberia which (Fig. 1): had the power equivalent to hundreds of Hiroshima atomic bombs; ionized the entire Earth's atmosphere to the point that, the day after, people would read newspapers at midnight without artificial light; left no crater or debris in the ground. Since atomic bombs did not exist in 1908, the sole quantitative representation of the Tunguska explosions is that it was due to the entire annihilation into electromagnetic radiations of an antimatter asteroid of unknown astrophysical origin (as it is the case for cosmic antiprotons) when in contact with our matter atmosphere.

Therefore, the Tunguska explosion appear to confirm Dirac's particle-antiparticle annihilation into light and casts shadows on the antimatter character of laboratory antiprotons as well as of the nucleus of the anti-Hydrogen atom. Opposing interpretations of the Tunguska explosion are encouraged provided that they reach a numerical representation of the origin of the produced extreme energy.

Evidently, for the proton-antiproton annihilation (10) to be compatible with the lack of any debris in the Tunguska and other explosions (Fig. 1) it should have the isoselfdual form

$$p^+ + (p^+)^d \rightarrow \Sigma_K \gamma_k + \Sigma_k \gamma_k^d. \quad (36)$$

The number of similar large explosions, that also occurred without solid debris in the ground and caused the local termination of radio communications, is too long to be reported here (see, e.g., the 2017 explosion [53]). Among older similar events, we mention the 1871 Chicago fire [54] also without a crater on the ground but the electromagnetic radiations were such to instantly melt bricks. It is unfortunate for our own safety that, despite their magnitude and anomalous character, the study of these events is dismissed by mainstream academia because not predicted by special and general relativities.

4 Antigravity tests

As indicated the preceding paper [10], Theorem 2.1 casts shadows in the true antimatter character of laboratory antiprotions [14] and, consequently, on the full antimatter character of anti-Hydrogen atoms [22] with ensuing shadows on their conventional gravity [23].

In the author's view, these results are a consequence of this axiomatic representation of particle-antiparticle annihilation via isoselfduality and, therefore, are a consequence of its violation in the proton-antiproton annihilation (11) of the Bose-Einstein correlation [50] [51].

In view of the above problematic aspects we suggest that, prior to any final claim, the gravity between matter and antimatter should be tested via antiparticles

of clear and independently proved antimatter character, as it is the case for R. M. Santilli [44] [45] 1994 comparative test of the gravity of very low (thermal) energy electron and positron beams in horizontal flight in a supercooled vacuum tube in which the length of the tube can be selected for given eV in such a way to allow a *visible* deviation due to gravity when the beam hits a terminal scintillator, and the radius of the tube can be selected to render stray field fluctuations smaller than the expected terminal deviation due to gravity.

Test [44] [45] was presented at the 1996 international Workshop on Antimatter, Sepino, Italy, the 2011 San Marino Workshop on Antimatter, the 2016 SIPS Conference, Hainan Island, China, the 2023 SIPS Conference in Panama and other meeting. The feasibility, low cost and resolatory character of test [44] [45] have been confirmed by A. P. Mills [46] [47], V. de Haan [48] [49] and others.

5 Concluding remarks

In the first section, we recalled that Dirac's [1] characterizations of *negative energy* antielectrons (positrons) and antiprotons (called in this paper cosmic antiprotons [12] [13]) verify particle-antiparticle annihilation into light (3) but violate special relativity and have a number of causality problems.

In the second section, we recalled that, to achieve compatibility with special relativity, the physicists of the early 20th century [4] made the theoretical assumption (still valid nowadays) that antiparticles carry *positive energy* (called in this paper laboratory antiprotons [14]); we introduce, apparently for the first time, a theorem proving that positive energy antiparticles cannot verify Dirac's annihilation into light; and we present a number of arguments casting shadows on the true antimatter character of laboratory antiprotons [14], anti-Hydrogen atoms [22] and the recent claim of their conventional gravity [23].

Consequently, when treated with 20th century applied mathematics, the notion of antiparticles as not achieved compatibility with experimental evidence for both Dirac's and 20th century conception, by therefore warranting the use of a basically *new mathematics*.

Along the latter lines, we have reviewed in the third section decades of mathematical, theoretical and experimental studies on antiparticles based on *isodual mathematics* which is as an anti-Hermitean image of applied mathematics (called isodual and denoted with the letter d) characterized by the *negative multiplicative unit* $1^d = -1$ at all levels, beginning with basic numeric fields.

With the understanding that a technical knowledge of the paper requires the study of the original references due to decades of mathematical, theoretical and experimental studies on antimatter, we have shown that the ensuing isodual theory of antimatter provides:

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5.1) A causal representation of Dirac's negative energy antiparticles and their annihilation into light via the isoselfduality (invariance under isoduality).

5.2) A full admission of special and general relativities, only defined on isodual spaces over isodual numeric fields.

5.3) The prediction of the negative energy *antiphoton*[43]

5.4) Positive eigenvalues when projected on conventional spaces over conventional fields.

5.5) The representation of available experimental data on antiparticles.

The main scope of this paper is to indicate the existence of a number of intriguing open problems on antimatter whose study requires a collegial participatoin, among which we mention

5.A) The astrophysical origin of cosmic antiprotons as well as of large explosions in our atmosphere without debris on the ground.

5.B) The comparative test of the gravity of thermal electrons and positrons [44]-[49].

5.C) The possible *expulsion* of antiphotons by black holes following internal particle-antiparticle creation and annihilation into photon-antiphoton pairs.

All in all, a serious scientific process for our own safety (Fig. 1) requires the admission that our current knowledge on antiparticles, as well as antimatter at large, is quite limited, if not misleading, due to one century of restriction of the studies to be compatible with theories, such as special and general relativities, that were not conceived for antimatter.

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Antiprotons or pseudoprotons?

Emanuele Velardo *

Abstract

The author, an undergraduate physics student, reviews in this paper: 1) positive mass negative charge antiprotons according to 20th century particle physics; 2) positive mass negative charge pseudoprotons according to hadronic mechanics; 3) the notion of antiprotons according to the isodual theory of antimatter; 4) a theorem recently proved by R. M. Santilli essentially implying that, under the spinorial Poincaré symmetry, the 20th century conjugation from particles to antiparticles prohibits particle-antiparticle annihilation into light; and 5) consequential doubts on the recent gravity test for the anti-hydrogen atom. Therefore, the author examines the possibility that antiparticles are actually gravitationally rejected by matter and 20th century antiprotons are in reality pseudoprotons with a consequential number of intriguing open problems in antimatter, like the astrophysical origin of true antiprotons from cosmic rays, and others. We conclude by examining: the 1994 proposal for a resolatory test on the gravity of true antiparticles, such as the positrons; the possible existence of antimatter bodies in the universe which originate cosmic antiprotons; and related aspects.

Keywords: antiprotons; pseudoprotons; antimatter; hadronic mechanics¹

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1 Pseudoprotons according to hadronic mechanics

The positive mass, negative charge pseudoproton [1] is a *particle* predicted by hadronic mechanics [2] which proposes an alternative description of the structure of unstable particles compared to the standard model. More precisely, hadronic mechanics predicts that the multitude of known *unstable* particles are composed by generalized bound states of physical, i.e., detectable particles produced free in their spontaneous decays. Thanks to the first known exact representation of all characteristics of the neutron as a hadronic bound state of an electron and a proton in the core of stars [3], hadronic mechanics has confirmed the historical conception of stable matter in the universe as being solely composed by the permanently stable protons and electrons [4]. Additionally, hadronic mechanics has achieved industrial means for the resolution of the historical Coulomb barrier opposing controlled nuclear fusions [5].

1.1 Pseudoproton's structure

The characteristics of the pseudoproton resemble those of a proton in terms of mass, but the charge is negative since it is composed by a hadronic bound state of a proton and two singlet-coupled electrons (see Figure 1) or a proton and two unpaired electrons inside the proton (see Figure 2) which are two variants that differ in the overall spin. Pseudoprotons have been synthesized by using the Rutherford compression mechanism for the synthesis of the neutron from an electron and a proton [3] applied to the synthesis of an electron and, this time, a neutron [1]. Despite being unstable, the pseudoproton has a half-life on the order of seconds or fractions thereof, which is long enough for technological applications.

1.2 Experimental evidence of the existence of pseudoproton

The existence of pseudoprotons has been experimentally indicated in [6] where the pseudoprotons were detected by irradiation of a sample of silver by verifying the expected reaction

$$\begin{aligned} Ag(47, 109, 1/2) + \bar{p}(-1, 1, 1/2) &\rightarrow \\ &\rightarrow Pd(46, 110, 0) + 2\gamma(0, 0, 1), \end{aligned} \tag{1}$$

where $\bar{p}(-1, 1, 1/2)$ indicates pseudoprotons.

Antiprotons or pseudoprotons?

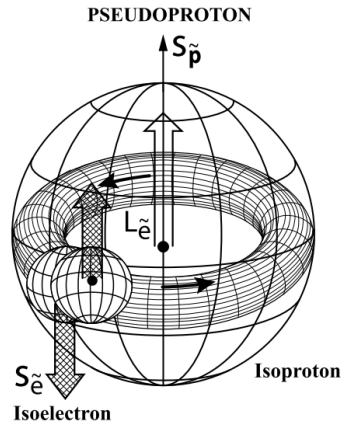


Figure 1: *Pseudoproton structure according to hadronic mechanics [1].*

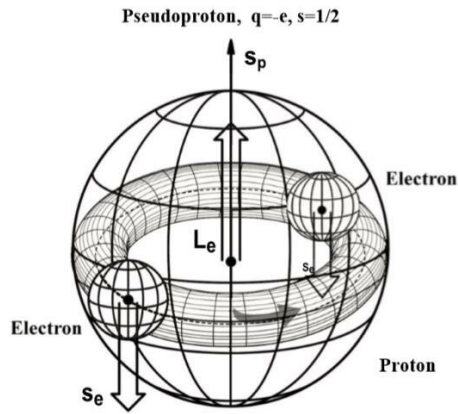


Figure 2: *Another possible structure of the pseudoproton that differs from that of Fig. 1 in spin [6].*

2 Basics elements of isodual mathematic

The most useful branch of mathematics for the representation of antimatter's behavior appears to be the isodual mathematic developed by R. M. Santilli [7]. Some of its basic concepts can be outlined as follows:

2.1 Isodual numbers

In mathematics, a "numeric field" is an algebraic structure in which a set of elements and two operations (an addition and a multiplication) are defined in such a way that, when applied to elements of the set, result in an element belonging to the same set.

The property that distinguishes the inverse of an element is that by applying the multiplication operation between an element and its inverse the unit is always obtained. The fundamental property of unity is that it is equivalent to its inverse.

Santilli conceived the isodual mathematics to study the numeric field in which the fundamental unit is *negative*,

$$1^d = -1, \quad (2)$$

in which case the multiplication, symbolized by $\times^d$, provides numeric values opposite those of a field of real numbers.

$$a \times^d b = a(1^d)^{-1}b. \quad (3)$$

It is then evident that in a theory where this type of multiplication is defined all numeric values are opposite those of an ordinary matter theory. More precisely, you can define for each given number its isodual which is anti-Hermitean

$$n^d = n1^d = -n^\dagger \quad (4)$$

2.2 Antimatter is represented as isodual matter on a isodual space over an isodual field

The isodual field turns out to be able to represent the properties of the anti-matter that can in fact be considered matter with the values of all the physical properties (for example mass) corresponding to the isodual values and placed in an isodual space over an isidual field.

For example consider the charge conjugation applied to a quantum wave function

$$C\psi(r) = -\psi(r)^\dagger, \quad (5)$$

it is evident that a wave function under the isodual transformation correctly represents the charge of the antiparticle

$$\psi^d(r^d) = -\psi(-r)^\dagger \quad (6)$$

with the significative difference that the isodual transformation, unlike charge conjugation, also applies to all other quantities (as in this case the variable r)

It is noteworthy that also the space and time that host antimatter are isodual with respect to space and time of matter (see also the excellent recent review by A. Muktibodh [8]).

3 The gravity of matter and antimatter

There is still no unanimous consensus on the behavior of antimatter in relation to matter gravity. The question is whether there is no gravitational difference between matter and antimatter, as expected from the positive mass of 20th century antiparticles, or antimatter produces an antigravity field in the field of matter, as expected from their negative mass, that rejects matter by therefore that matter and antimatter experience a *gravitational repulsion*, also called antigravity..

3.1 Behavior predicted by the isodual theory of antimatter

The isodual theory of antimatter [7] makes use of the isodual unit $1^d = -1$ to "conjugate" matter formulas and make them valid for the description of antimatter. By recalling Newton's gravitational *attraction* between two positive mass particles in the Euclidean space $\mathcal{E}$ with local coordinates r over the field of real numbers $\mathcal{R}$ with unit $1 > 0$

$$F = \frac{G \times m_1 \times m_2}{r^2} > 0 \quad (7)$$

the corresponding attraction between two negative mass particles in the isodual Euclidean space $\mathcal{E}^d$ with local coordinates $r^d = r1^d$ over the isodual field of real numbers $\mathcal{R}^d$ is given by

$$F^d = \frac{G^d \times^d m_1^d \times^d m_2^d}{r^{d2d}} < 0 \quad (8)$$

and denotes attraction because the negative value of F^d is measured in terms of the negative unit $1^d < 0$ of $\mathcal{R}^d$.

It then follows that the gravitational force between a positive mass particle and a negative mass antiparticle is *repulsive* in our Euclidean space $\mathcal{E}$ over $\mathcal{R}$

$$F = \frac{G \times m_1 \times m_2^d}{r^2} < 0. \quad (9)$$

as well as in the antimatter space $\mathcal{E}^d$ over $\mathcal{R}^d$

$$F^d = \frac{G^d \times^d m_1^d \times^d m_2}{r^{d2d}} > 0 \quad (10)$$

were, again, the negative value of F^d , this time, denotes repulsion because of the negative value of 1^d .

Consequently, the isodual theory of antimatter predicts that a matter particle and an antimatter particle should experience antigravity. Santilli [7] has proved the validity of this property at all levels of treatment, from Newtonian mechanics to the Riemannian geometry.

3.2 CERN's gravity experiment on antimatter

The mainstream interpretation of the 2023 experiment at CERN [9] is instead that antimatter would be gravitationally attracted by matter and there would be no gravitational difference in their behavior. In fact, the statistical data of the CERN's experiment indicates that anti-hydrogen atoms *fall* towards the center of the Earth.

3.3 Comparison between two lines of thought

A theorem recently proved by R. M. Santilli in Refs. [10] [11] states that, under the invariance by the spinorial Poincarè symmetry, the particle-antiparticle conjugation

$$PTC|p, j, \psi(t, r) \rangle = |p, -j, -\psi(t, r)^\dagger \rangle \quad (11)$$

is incompatible with the experimental evidence of electron-positron annihilation into light.

This suggests that antimatter particles actually have a negative mass as historically predicted by P. A. M. Dirac [12] who discovered himself that *negative energy antiparticles represented in our spacetime over the field of real numbers violate causality*.

Santilli [7] resolved Dirac's causality problem via the construction of the new isodual mathematics, which is inclusive of isodual spaces and isodual numbers, according to which *a negative energy antiparticle represented in the isodual spacetime over the field of isodual numbers all having the basic unit $1^d = -1$ are as causal as positive energy particles represented in our spacetime over ordinary numeric fields all having the basic unit 1* [7]. In particular, the isodual map coincides with charge conjugation at short distances and, therefore, the isodual theory of antimatter correctly describes all the experimental data known to date on antimatter [11].

4 Antiprotons or Pseudoprotons?

A different but valid interpretation of the CERN experiment that would resolve the apparent contradictions proposed in Ref. [11] is that, in reality, the nuclei of the anti hydrogen atoms are pseudoprotons since the latter have a positive mass and a negative charge and, therefore, can be easily mistaken for antiprotons.

4.1 Laboratory antiprotons could be pseudoprotons?

Antiprotons are produced in laboratory by sending high-energy protons against a matter target. It is possible that this process produces in reality pseudoprotons because the collision of high energy protons against the target does produce neutrons from the collision of protons with the target electron clouds with ensuing finite probability for the subsequent synthesis of pseudoprotons.

A strong argument in favor of the latter view is that laboratory antiprotons colliding with protons give rise to a rain of massive particles [13] instead of annihilating themselves into light, as expected in the event they were true antiprotons.

4.2 Cosmic antiprotons or cosmic pseudoprotons?

The presence of antiprotons among cosmic rays is the most reasonable explanation for the visions of light flashes witnessed by several astronauts during their travels in the upper Earth's atmosphere. These flashes could be caused by the interaction of antiparticles with the photoreceptor cells in the retina of the eye. Photoreceptor cells may be stimulated by photons emitted by particle antiparticle annihilation reactions. So it is extremely likely that there are true antiprotons among the cosmic rays.

However, the first detection of a negative-charged heavy cosmic ray was carried out by Chamberlain and Segre with the Bevatron in 1955 [14]. The crucial point on which we want to bring attention is the possibility that the particle detected, may not have been an antiproton, although it was logically considered such by the experimenters, but it's probably a pseudoproton as it shows the typical scattering of a particle of matter (see Figure 3) rather than annihilation at Dirac.

In fact, true antiprotons are expected to annihilate at contact with atmospheric nuclei, by therefore rendering virtually impossible they reaching us at sea level (the average free path appear to drop below 1 m already at 100 km elevation). The case for high energy pseudoprotons is different because they would have conventional scattering with atmospheric nuclei, rather than annihilation by therefore reaching us at sea level.

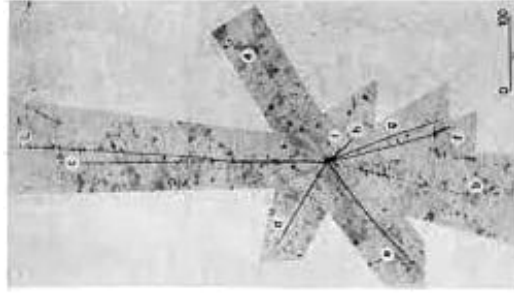


Figure 3: *Historical image of the first detection of an alleged antiproton. The visible scattering suggests that it was actually a matter particle, presumably a pseudoproton*

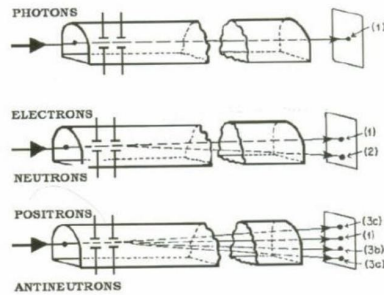


Figure 4: *Santilli's 1994 resolatory test on the gravity of antiparticles [15].*

5 A resolatory antimatter gravity test

A resolatory test on the gravity of antimatter was proposed by Santilli in 1964 [15] (see also the more recent study [16]) and submitted to CERN Directorate by the R. M. Santilli Foundation as well as by experts in antimatter. The proposed test essentially uses a beam of very low-energy positrons in a horizontal high-vacuum supercooled tube. The test was stated to be resolatory by A. P. Mills [17] [18], by V. O. de Haad [19] [20] and by others experts in antimatter because the positrons are unquestionable antiparticles and the point of incidence of positrons on a terminal scintillator can be visually detected and compared with the downward incidence due to gravity of a very low-energy electron beam and a thin beam of photons denoting lack of gravity deflection (see Figure 4).

6 Cosmic antimatter?

The current, widespread cosmological view is that the universe in which we live contains matter in an immensely greater quantity than antimatter and, consequently, the latter would have a marginal role in the processes taking place in the natural world. In fact, it is believed that the Big Bang produced matter and antimatter in equal quantities and that, in view of matter-antimatter annihilation, only a small part of antimatter "survived". Nevertheless, the author believes that the lack of antimatter at cosmological scale is far from having been resolved in a scientific way due to the *new light* emitted by antimatter, called isodual light [21] which is not expected to be visible by available telescopes [7].

6.1 Comments on particles decays

It is widely assumed that, during the Big Bang, positrons are converted into protons, in an attempt to explain the imbalance in favor of matter. However, such an assumption implies the prediction that low energy protons should decay into positrons in a finite period of time (the estimated half-life is of the order of 10^{33} years [22]). However, all attempts to detect proton decay have failed to date. This suggests that hadronic mechanics is a more accurate description of reality since it does not predict proton decay.

It may be useful to check whether the anti-hydrogen atoms produced at CERN do decay because in the event they really are antiparticles they should remain stable until they come into contact with the matter. On the other hand, in the event their nuclei are composed of pseudoprotons, hadronic mechanics foresees their rapid decay.

6.2 Antimatter asteroids

It is possible that there is no conversion between matter and antimatter and in this case it would be possible that matter and antimatter are still present in equal quantities in the universe since antimatter is predicted to emit the new isodual light [21]. Hence, there may be galaxies and celestial bodies in the universe composed entirely of antimatter since that would be the most logical origin of the true antiprotons detected in cosmic rays jointly with antimatter asteroids.

In the event the latter enter the gravitational field of the Sun, they are pushed back outwards according to the isodual theory of antimatter according to a trajectory that could intersect the orbit of our planet and impact our upper atmosphere under special conditions [23] [24].

Some past events are consistent with the entry of antimatter asteroids into the atmosphere. The famous and enigmatic event of Tunguska that devastated

2150 km² of taiga remained is not explained by the simple impact of a celestial body. For example there is no crater. Among the hypotheses about what may have caused the disaster is the encounter with an antimatter asteroid. But even if an isolated case can receive alternative explanations, there are other cases that can be explained by this origin, such as the Chicago fire of 1871, a highly destructive event that triggered a fire that spread very quickly, or the meteorite visible in the skies of Russia in 2013 that did not cause craters.

7 Concluding remarks

In this paper, we have reviewed:

- 1) positive mass negative charge *antiprotons* according to 20th century particle physics;
- 2) positive mass negative charge ordinary *particles* called pseudoprotons according to hadronic mechanics;
- 3) the notion of antiprotons according to the isodual theory of antimatter;
- 4) a theorem recently proved by R. M. Santilli essentially implying that, under the spinorial Poincaré symmetry, the 20th century conjugation from particles to antiparticles prohibits particle-antiparticle annihilation into light; and
- 5) consequential doubts on the recent gravity test for the anti-hydrogen atom.

The tentative conclusion of our comparative analysis is that 20th century antiprotons are, most likely, pseudoprotons, by therefore questioning the results of CERN gravity experiment [9] and leaving basically open the gravity of true antiparticles in favor of R. M. Santilli [15] 1964 antigravity test with clearly established antiparticles, such as the positrons (see also [16]- [20]).

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The incompleteness of the Schrödinger equation

Miroslav Pardy *

Abstract

The Schrödinger equation with the nonlinear term is derived in the framework of the Dirac heuristics. The particle behaves classically in case its mass is infinite. The nonlinear term is crucial and involves a new physical constant b which can be measured via the same methods used for the Casimir effect and accurate experimental procedures. New experiments, different from the Zeilinger ones, are proposed with Faraday simplicity for the determination of the new very small value of the constant b .

Keywords: Nonlinear Schrödinger equation, mass, Zeilinger constant, experiment. ¹

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1 Introduction

The non-Gödelian incompleteness of quantum mechanics was introduced by Einstein, Podolsky and Rosen (EPR) in his well known article [6], where they argued that the description of physical reality provided by quantum mechanics was incomplete.

The EPR thought experiment involves a pair of particles prepared in an so called entangled state. Einstein, Podolsky, and Rosen pointed out that, in this state, if the position of the first particle were measured, the result of measuring the position of the second particle could be predicted. If, instead, the momentum of the first particle were measured, then the result of measuring the momentum of the second particle could be predicted. They argued that no action taken on the first particle could instantaneously affect the other, since this would involve information being transmitted faster than light, which is forbidden by the theory of relativity. This contradicted the view associated with Niels Bohr and Werner Heisenberg, according to which a quantum particle does not have a definite value of a property like momentum until the measurement takes place.

While the EPR paradox is solved continuously many years without rigorous result [7], there is the crucial incompleteness of the Schrödinger equation. Namely the nonexistence of the classical limit of this equation for the infinite mass of a particle. We here investigate this problem using the nonlinear Schrödinger equation with the logarithmic correction and the adequate new constant.

2 The nonlinear wave mechanics

Many authors have suggested that the quantum mechanics based on linear Schrödinger equation is only an approximation of some more nonlinear theory with the nonlinear Schrödinger equation. The motivation for considering the nonlinear equations is to get some more nonstandard solution in order to get the better understanding of the synergism of wave and particle.

The program to create nonlinear wave mechanics was elaborated by de Broglie [8] and his group. Bialynicki-Birula and Mycielski [9] considered the generalized Schrödinger equation with the additional term $F(|\Psi|^2)\Psi$ where F is some arbitrary function which they later specified to $-b(\ln |\Psi|^2)$, $b > 0$. The nonlinear term was selected by assuming the factorization of the wave function for the composed system.

The most attractive feature of the logarithmic nonlinearity is the existence of the lower energy bound and validity of Planck's relation $E = \hbar\omega$. At the same time the Born interpretation of the wave function cannot be changed. In this theory the estimation of b was given by the relation $b < 4 \times 10^{-10} \text{ eV}$ following from the

agreement between theory and the observed $2S - 2P$ Lamb shift in hydrogen. This implies an upper bound to the electron soliton spatial width of $10 \mu\text{m}$.

Shimony [10] proposed an experiment which is based on idea that a phase shift occurs when an absorber is moved from one point to another along the path of one of the coherent split beams in a neutron interferometer. In case of the logarithmic nonlinearity Shull et al. [11] performed the experiment with a two-crystal interferometer. They searched for a phase shift when an attenuator was moved along the neutron propagation direction in one arm of the interferometer. A sheet of Cd, 0.086 mm thick, was used for the absorber. They obtained the upper bound on b of $3.4 \times 10^{-13} \text{eV}$ which is more than three orders of magnitude smaller than the bound estimated by Bialynicky-Birula and Mycielski [9].

The best upper limit on b has been reported by Gähler, Klein and Zeilinger [12] who has been searched for variations in the free space propagation of neutrons. $20 \text{ \AA}$ neutrons were diffracted from an abrupt highly absorbing knife edge at the object position. By comparing the experimental results with the solution to the ordinary Schrödinger equation they were able to get the limit $b < 3 \times 10^{-15} \text{eV}$, which corresponds to an electron soliton width of 3 mm. The similar results was obtained by the same group from diffraction a $100 \mu\text{m}$ boron wire.

To our knowledge the Mössbauer effect was not used to determine the constant b although this effect allows to measure energy losses smaller than 10^{-15}eV . Similarly the Josephson effect has been not applied for the determination of the constant b .

We see that the constant b is very small, nevertheless we cannot neglect it a priori, because we do not know its role in the future physics. The corresponding analog is the Planck constant which is also very small, however, it plays the fundamental role in physics.

The goal of this article is to give the new derivation of the logarithmic nonlinearity, to find the solution of the nonlinear Schrödinger equation of the one-dimensional case and to show that in the mass limit $m \rightarrow \infty$ we get exactly the delta-function behavior of the probability of finding the particle at point x . It means that there exists the classical motion of a particle with sufficient big mass. The nonlinearity of the Schrödinger equation also solves the collapse of the wave function and the Schrödinger cat paradox. We will start from the hydrodynamical formulation of quantum mechanics. The mathematical generalization of the Euler hydrodynamical equations leads automatically to the logarithmic term with $b > 0$. The article is the modified articles by author [3], [4], [5].

3 The derivation of the nonlinear Schrödinger equation

We respect here the so called Dirac heuristic principle [13] according to which it is useful to postulate some mathematical requirement in order to get the true information about nature. While the mathematical assumption is intuitive, the consequences have the physical interpretation, or, in other words they are physically meaningful. In derivation of the logarithmic nonlinearity we use just the Dirac method.

According to Madelung [14], Bohm and Vigier [15], Wilhelm [16], Rosen [17] and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n} e^{\frac{i}{\hbar} S}, \quad (1)$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \rightarrow 0$. The mass density is defined by relation $\varrho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (1) into the original Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi, \quad (2)$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V = \frac{\hbar^2}{2m} \frac{\Delta \sqrt{n}}{\sqrt{n}} \quad (3)$$

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0 \quad (4)$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \quad (5)$$

Equation (3) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{n}}{\sqrt{n}}, \quad (6)$$

which is called the quantum Bohm potential and equation (4) is the continuity equation.

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After application of operator ∇ on eq. (3), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{m} \nabla (V + V_q). \quad (7)$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\varrho^{-1} \nabla p$ where p is hydrodynamical pressure. We use here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the eq. (7) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla (V + V_q) - \nabla F, \quad (8)$$

where

$$\nabla F = \frac{1}{n} \nabla p. \quad (9)$$

The equation (8) can be obtained by the Madelung procedure from the following extended Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi + F\Psi \quad (10)$$

on the assumption that it is possible to determine F in term of the wave function. From the vector analysis follows that the necessary condition of the existence of F as the solution of the eq. (9) is $\text{rot grad } F = 0$, or,

$$\text{rot}(n^{-1} \nabla p) = 0, \quad (11)$$

which enables to take the linear solution in the form

$$p = -bn = -b|\Psi|^2, \quad (12)$$

where b is some arbitrary constant. We do not consider the more general solution of eq. (11). Then, from eq. (9) i.e. $\text{grad } F = \mathbf{a}$ we have:

$$F = \int a_i dx_i = -b \int \frac{1}{n} dn = -b \ln |\Psi|^2, \quad (13)$$

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which corresponds to the complete Euler equation (8) in the following form:

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$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi - b(\ln |\Psi|^2)\Psi. \quad (14)$$

Let us remark that the stochastic derivation of the nonlinear equation (14) was given by Lemos [18]. However, the author (Pardy) derivation is more attractive from the pedagogical view of point.

Let us approach the solving eq. (14).

4 The soliton-wave solution of the nonlinear Schrödinger equation

Let be $c, (\text{Im } c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x, t) = cG(x - vt)e^{ikx - i\omega t} \quad (15)$$

into the one-dimensional equation (14) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \quad (16)$$

and for function G we get the following nonlinear equation (the symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \quad (17)$$

where

$$A = \frac{2m}{\hbar} \omega - k^2 + \frac{2m}{\hbar^2} b \ln c^2 \quad (18)$$

$$B = \frac{4mb}{\hbar^2}. \quad (19)$$

After multiplication of eq. (17) by G' we get:

$$\frac{1}{2} [G'^2]' + \frac{A}{2} [G^2]' + B \left[\frac{G^2}{2} \ln G - \frac{G^2}{4} \right]' = 0, \quad (20)$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2} G^2 + \text{const.} \quad (21)$$

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If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B \ln G}} = d\xi, \quad (22)$$

where

$$a = \frac{B}{2} - A. \quad (23)$$

Eq. (22) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi+d)^2}, \quad (24)$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x, t) = c e^{\frac{a}{B}} e^{-\frac{B}{4}(x-vt+d)^2} e^{ikx-i\omega t}. \quad (25)$$

5 Normalization and the classical limit

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx = 1. \quad (26)$$

Using the Gauss integral

$$\int_0^{\infty} e^{-\lambda^2 x^2} dx = \frac{\sqrt{\pi}}{2\lambda}, \quad (27)$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \quad (28)$$

and the density probability $\Psi^* \Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \quad (29)$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \rightarrow \infty$ is just the Dirac δ -function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behavior of a particle cannot be obtained in the standard quantum mechanics because the plane wave

$$e^{ikx-i\omega t} \quad (30)$$

corresponds to the free particle with no possibility of localization for $m \rightarrow \infty$.

Let us still remark that coefficient c^2 is real and positive number because it is a result of the solution of eq. (28) which can be transformed into equation ($x = c^2$)

$$x^{1-r} = \text{const.} \quad (31)$$

6 The principle of superposition

The principle of superposition is in nonlinear theory broken. If φ_1 and φ_2 are two different solution of the nonlinear Schrödinger equation then the linear combination $\varphi = a\varphi_1 + b\varphi_2$ where a and b are the arbitrary constants is not the solution of the same equation because of its nonlinearity. In other words the original principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the resolution of the Schrödinger cat paradox [19].

7 The determination of the constant b by experiment

After insertion of the function

$$\Psi(x, t) = \exp - [i(E/\hbar)t]\phi(x) \quad (32)$$

into Eq. (14), we get, in the one-dimensional case,

$$\phi'' + k^2\phi = D\phi \ln |\phi|, \quad (33)$$

with

$$k^2 = \frac{2mE}{\hbar^2}, \quad D = -\frac{4mb}{\hbar^2}. \quad (34)$$

We suppose that the approximate solution of Eq. (33) is of the form

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$$\phi(x) = A \sin \kappa x, \quad (35)$$

where A and κ are to be determined. For $|\sin \kappa x \cdot \ln |A|| \gg |\sin \kappa x \cdot \ln |\sin \kappa x||$, we get

$$(k^2 - \kappa^2) = D \ln |A| \quad (36)$$

and, from the boundary conditions $\phi(0) = \phi(L) = 0$ and from the normalization condition of the wave function on the space interval $(0, L)$, we get

$$\phi(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x. \quad (37)$$

The corresponding energies of the stationary states then are

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 - b \lg \frac{2}{L}; \quad n = 1, 2, 3, \dots \quad (38)$$

The last formula indicates that it is not possible to determine the constant b spectroscopically, because relation $\hbar\omega = E_n - E_m$ does not contain the constant b .

On the other hand, the force between boundaries, due to the the existence of the particle in the box, is $F = -\partial E / \partial L$, i.e.,

$$F_n = \frac{\hbar^2 \pi^2}{mL^3} n^2 - \frac{b}{L} \quad (39)$$

for one particle in the box, and NF_n for the N noninteracting particles in the box. It can in principal be measured by the same methods that were used in the case of the Casimir effect [1], [2]. Of course, the difficulties will be greater than in the case of the Casimir effect.

The other possibility for the measurement of E_n , one which is here considered for the first time, is to consider the following experiment: Two rods with square cross section, are given, the near ends being at the distance L apart and forming a potential box of width L . Suppose particles are impinging on the rods in the locality of the boundary of the gap and are reflected. It is evident that the resonance absorption of particles by the gap occurs for a velocity that is determined by the equation

$$\frac{1}{2} m v_n^2 = E_n, \quad (40)$$

where v_n is the velocity perpendicular to the plane of the gap. If the source of particles is fixed to the rotating disk, then the velocity of the emitted particles can be continuously changed in order to get the resonance velocity, just as in case of

the Mössbauer experiment [20] , and in such a way that it gives the possibility for determination of the constant b .

For $\hbar = 1.05 \times 10^{-34}$ J, $L = 10^{-7}$ m, $m = 1,67 \times 10^{-27}$ kg, $b = 3.3 \times 10^{-15}$ eV $= 3.3 \times 1.6 \times 10^{-34}$ J, $n = 10$, , we get $E_{10} = 3,25 \times 10^{-25}$ J - 8.87×10^{-33} J.

In this calculation we have used the results of the interferometric search for the nonlinear term in the Schrödinger equation of Shull et al. [11] and Gähler et al. [12], who got the upper limit of b of $\leq 3.3 \times 10^{-15}$ eV.

8 Discussion

We have seen that the introduction of the logarithmic nonlinearity in the Schrödinger equation was logically supported by the fact that the nonlinear Schrödinger equation gives results which are physically meaningful. We have obtained the correct mass limit of the wave function.

The further strong point of the nonlinear Schrödinger equation (14) is the result (16) which is equivalent to the famous de Broglie relation

$$\lambda = \frac{h}{p} \quad (41)$$

because of $\lambda = 2\pi/k = 2\pi(\hbar/mv) = 2\pi(h/2\pi)(1/p)$ and it means that de Broglie relation is involved in this form of the nonlinear quantum mechanics.

The nonlinear equation (14) has also the normalized plane-wave solution

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} e^{ikx - i\omega t}. \quad (42)$$

After insertion of eq. (33) into eq. (14), we get the following dispersion relation:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + b \ln(2\pi), \quad (43)$$

from which the relations follows:

$$\hbar\omega = b \ln(2\pi); \quad k = 0 \quad (44)$$

and

$$k = \pm i \sqrt{\frac{2m}{\hbar^2} b \ln(2\pi)}; \quad \omega = 0. \quad (45)$$

It is no easy to give the physical interpretation of eqs. (44) and (45) and so we cannot say that the plane-solution of the nonlinear Schrödinger equation is physically meaningful. Only the soliton-wave solution of the nonlinear Schrödinger

equation can be taken as relevant. Only this solution is suitable for the physical verification. The possible new tests of the nonlinear quantum mechanics are discussed in the author article [4].

The generalization to the motion of particle in the electromagnetic field with potentials $\varphi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ can be performed by the standard transformation

$$\frac{\hbar}{i}\nabla \rightarrow \frac{\hbar}{i}\nabla - \left(\frac{e}{c}\right)\mathbf{A}(\mathbf{x}, t) \quad (46)$$

and adding the scalar potential energy $\varphi(x, t)$ in the Schrödinger equation for the free particles. According to Bialynicki-Birula et al. [9], the solution of the equation in this case can be taken in the form

$$\Psi(\mathbf{x}, t) = e^{\frac{i}{\hbar}S}G(\mathbf{x} - \mathbf{u}(t)), \quad (47)$$

where function G is necessary to determine. In the similar form the problem was yet solved [21].

Kamesberger and Zeilinger [22] have given the numerical solution of the original Schrödinger equation and this equation with the nonlinear term $-b(\ln |\Psi|^2)\Psi$ in order to visualize the spreading of the diffractive waves. When comparing the evolution patterns of the nonlinear case with the linear one, one notices that the maxims are more pronounced in the nonlinear solution. It can be understood as a mechanism compressing the wave maxims spatially. In the quantitative comparison of the both cases this enhancement of the maxims and minims can be seen very clearly.

Although we have given reasons for the introducing of the nonlinear Schrödinger equation it is obvious that only the crucial experiments can establish the physical and not only logical necessity of such equation. In case that the nonlinear Schrödinger equation will be confirmed by experiment, then it can be expected that it will influence other parts of theoretical physics.

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Small weak hyperfields in hadronic mechanics

Thomas Vougiouklis*

Abstract

It was in mid 90es when Professor R. M. Santilli realized, for the first time, that his innovating theories can be appropriate expressed by multi-valued systems. At that time the largest class of hyper-structures, the H_v -structures, based on the weak properties, were introduced and studied deeply. The extremely large number of weak hyper-structures defined on the same set can be reduced by demanding a lot of properties. Therefore, whenever the Santilli's Hadronic Mechanics need special elements satisfying a number of properties then the class of the H_v -structures is the appropriate one. The first H_v -structures in Hadronic Mechanics, introduced in 1996, are called e-hyper-structures. The main question in this theory proposed by Santilli is: What are the hyper-numbers? Hyper-numbers or H_v -numbers are called the elements of any H_v -field and they are important in representation theory, as well. In this presentation we give some constructions and properties on minimal H_v -fields.

Keywords: Lie-Santilli iso-theory, weak hyperstructures, H_v -fields.

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1. Introduction to H_V -structures

The hyper-structures were introduced by F.Marty in 1934 [13] when he first defined the *hypergroup* as a set equipped with an associative and reproductive hyper-operation. We deal with the largest class of hyper-structures called H_V -structures, introduced in 1990 [17], [29], [30]. These satisfy the *weak axioms* where the non-empty intersection replaces equality. Basic definitions [2], [4], [5], [6], [10], [11], [12], [21], [23], [24], [25], [26], [30], [31], [32], [34], [43], [51]:

A set H equipped with, at least, one *hyper-operation*: $\cdot : H \times H \rightarrow P(H) - \{\emptyset\}$, is called *hyper-structure*, we write $(H, \cdot)$. The $(\cdot)$ is *weak associative*, if $(xy)z \cap x(yz) \neq \emptyset$, $\forall x, y, z \in H$ and *weak commutative* if $xy \cap yx \neq \emptyset$, $\forall x, y \in H$. The $(H, \cdot)$ is an H_V -semigroup if it is weak associative, and it is an H_V -group if it is *reproductive* H_V -semigroup: $xH = Hx = H$, $\forall x \in H$.

Similarly, more complicated hyper-structures can be defined: $(R, +, \cdot)$ is H_V -ring if $(+)$ and $(\cdot)$ are weak associative, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset, \quad \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be H_V -ring, $(M, +)$ is a weak commutative H_V -group and there exists an external hyper-operation $\cdot : R \times M \rightarrow P(M)$: $(a, x) \rightarrow ax$ such that, $\forall a, b \in R$ and $\forall x, y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset,$$

then M is an H_V -module over R . In the case that we have an H_V -field F , defined later, instead of the H_V -ring R , then the H_V -vector space, is obtained.

In an H_V -semigroup $(H, \cdot)$, the *powers* are defined by

$$h^1 = \{h\}, \quad h^2 = h \cdot h \dots \quad h^n = h \circ h \circ \dots \circ h,$$

where $(\circ)$ is the *n-ary circle hope*: take the union of hyper-products n times, with all possible patterns of parentheses on them. An $(H, \cdot)$ is a *cyclic of period s* , if there is a *generator* h and the minimum natural s , such that

$$H = h^1 \cup h^2 \cup \dots \cup h^s.$$

Analogously, cyclicity for the infinite period is defined. If there are h and s , the minimum one, such that $H = h^s$, then we say that the $(H, \cdot)$, is a *single-power cyclic of period s* .

For more definitions and applications on H_V -structures one can see books and papers as [1], [2], [4], [5], [6], [11], [12], [22], [30], [32], [34], [38], [43], [49].

Let $(H, \cdot)$, $(H, *)$ be H_V -semigroups on H . $(\cdot)$ is called *smaller* than $(*)$, if there is an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x*y), \quad \forall x, y \in H,$$

then we say that $(H, *)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a classical structure then it the *basic structure*, and $(H, *)$ is called H_b -structure. *Minimal* is called an H_V -group if it contains no other H_V -group on the same set.

Theorem 1.1 (*The Little Theorem*). Greater hyper-operations than the ones which are weak associative or weak commutative, are also weak associative or weak commutative, respectively.

This Theorem leads to a partial order on H_v -structures and reveal the enormous number of defined on the same set. The problem of enumeration of hyper-structures, is complicate in H_v -structures because we have great numbers. The Little Theorem, transfer the problem into finding the minimal, up to isomorphisms, H_v -structures. For example, the number of H_v -groups with three elements, up to isomorphism, is 1.026.462, there are 7.926 abelian.

Interesting large classes of H_v -structures, are the following [25], [26], [30], [31], [41]:
Definition 1.2 An H_v -structure is called *very-thin* if all hyper-operations are operations except one, were all results are singletons except one, which is a set with more elements.

Definition 1.3 Let $(G, \cdot)$ groupoid, then $\forall P \subset G, P \neq \emptyset$, we define the P -hyper-operations:

$$\underline{P}: x \underline{P} y = (xP)y \cup x(Py), \forall x, y \in G,$$

$(G, \underline{P})$ is a P -hyper-structure. If $(G, \cdot)$ semigroup, then $x \underline{P} y = xPy$, $(G, \underline{P})$ is semi-hypergroup.

A generalization of P -hyper-operation, used in Lie Santilli's theory, is the following: Let $(G, \cdot)$ be abelian group and $P \subset G$, we define the P_e -hyper-operation $(\times_P)$ by

$$\begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

The hyper-structure $(G, \times_P)$ is an abelian H_v -group.

Hyper-products on any type of ordinary matrices can be defined [10], [21], [54], [55].

Definition 1.4 Let $A = (a_{ij}) \in M_{m \times n}$, $m \times n$ matrix and $s, t \in \mathbb{N}$, $1 \leq s \leq m$, $1 \leq t \leq n$. Denote $\underline{st}$ a map

$$\underline{st}: M_{m \times n} \rightarrow M_{s \times t}: A \rightarrow A \underline{st} = (\underline{a}_{ij}),$$

where $\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t, \kappa, \lambda \in \mathbb{N}, i+\kappa s \leq m, j+\lambda t \leq n\}$.

(a) Let $A = (a_{ij}) \in M_{m \times n}$, $B = (b_{ij}) \in M_{u \times v}$, $s = \min(m, u)$, $t = \min(n, v)$. The *helix-sum*, is

$$\oplus: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{s \times t}): (A, B) \rightarrow A \oplus B = A \underline{st} + B \underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s \times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

(b) Let $A = (a_{ij}) \in M_{m \times n}$, $B = (b_{ij}) \in M_{u \times v}$, $s = \min(n, u)$. The *helix-product*, is

$$\otimes: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{m \times v}): (A, B) \rightarrow A \otimes B = A \underline{ms} \cdot B \underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m \times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{it} \in \underline{a}_{it} \text{ and } b_{tj} \in \underline{b}_{tj}\}$.

The helix-sum is commutative, the helix-product is weak associative.

The *helix-Lie Algebra* is defined as well. We study the matrices $M_{m \times n}$ where $m < n$, since we have analogous cases if $m > n$. Several classes appropriate in helix-operations are defined. For example: Let $A = (a_{ij}) \in M_{m \times n}$ and $s, t \in \mathbb{N}$, $1 \leq s \leq m$, $1 \leq t \leq n$. We call A *S-helix matrix of type $s \times t$* , if $A \underline{st}$ has diagonal entries elements. An *S-helix* A , is an *S₀-helix matrix* if, moreover, $a_{11} \dots a_{mm} \neq 0$. The set of *S-helix* matrices $A = (a_{ij}) \in M_{m \times n}$, is closed under the helix-product. It has $\underline{st}$ left scalar unit I_c . The *S₀-helix* matrices X have inverses X^{-1} : $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.

2. Fundamental relations

The main tool to study the hyper-structures is the, so called, *fundamental relation*. In 1970 M. Koskas [30] introduced in classical hypergroups the relation β and its transitive closure β^* . This relation connects the hyperstructures with the corresponding classical structures and is defined in H_V -groups as well. T. Vougiouklis [11], [27], [29], [30], [31], [52] introduced, and gave new proofs, the analogous relations γ^* and ε^* , which are defined, in H_V -rings and H_V -vector spaces, respectively. He also named the relations β^* , γ^* and ε^* , *fundamental* since they play crucial role mainly in representation theory.

Definition 2.1 The *fundamental relations* β^* , γ^* , and ε^* are defined in H_V -groups, H_V -rings, and H_V -vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring, and vector spaces, respectively.

Let $(G, \cdot)$ be group and R a partition in G , then $(G/R, \cdot)$ is H_V -group, so $(G/R, \cdot)/\beta^*$ is a group, the *fundamental*. The classes of $(G/R, \cdot)/\beta^*$ are a union of the R -classes.

The main theorems, and a way to find the fundamental classes, are the following:

- (1) Let $(H, \cdot)$ be H_V -group and denote U the set of finite products in H . Define the relation β in H setting $x\beta y$ if $\{x, y\} \subset u$, where $u \in U$. Then, β^* is transitive closure of β .
- (2) Let $(R, +, \cdot)$ be H_V -ring, denote U the set of finite polynomials in R . Define the relation γ in R setting $x\gamma y$ if $\{x, y\} \subset u$, where $u \in U$. Then, γ^* is transitive closure of γ .
- (3) Let $(M, +)$ be H_V -vector space over the H_V -field $(F, +, \cdot)$, denote U the set of expressions in M . Define ε in M setting $x\varepsilon y$ if $\{x, y\} \subset u$, $u \in U$. Then, ε^* is transitive closure of ε .

We present a proof for the second of the above theorems, for the H_V -rings, and the rest are analogous [11], [29], [30], [33], [39], [50], [52]:

Theorem 2.2 Let $(R, +, \cdot)$ be an H_V -ring. Denote by U the set of all finite polynomials of elements of R . We define the relation γ in R as follows:

$$x\gamma y \text{ if } \{x, y\} \subset u, \text{ where } u \in U.$$

Then, the relation γ^* is the transitive closure of γ .

Proof. Let γ the transitive closure of γ , denote $\gamma(a)$ the class of a . First, we prove that the quotient R/γ is a ring.

In R/γ the sum $(\oplus)$ and the product $(\otimes)$ are defined in the usual manner:

$$\gamma(a) \oplus \gamma(b) = \{\gamma(c) : c \in \gamma(a) + \gamma(b)\}, \quad \gamma(a) \otimes \gamma(b) = \{\gamma(d) : d \in \gamma(a) \cdot \gamma(b)\}, \quad \forall a, b \in R.$$

Take $a' \in \gamma(a)$, $b' \in \gamma(b)$. Then, we have

$$a'\gamma a \text{ if } \exists x_1, \dots, x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, \dots, u_m \in U: \{x_i, x_{i+1}\} \subset u_i, i=1, \dots, m,$$

$$b'\gamma b \text{ if } \exists y_1, \dots, y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, \dots, v_n \in U: \{y_j, y_{j+1}\} \subset v_j, j=1, \dots, n.$$

From the above, we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, i=1, \dots, m-1, x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, j=1, \dots, n.$$

The sums $u_i + v_1 = t_i, i=1, \dots, m-1$ and $u_m + v_j = t_{m+j-1}, j=1, \dots, n$, are polynomials, so, $t_k \in U, \forall k \in \{1, \dots, m+n-1\}$.

Now, pick up elements $z_1, \dots, z_{m+n}$ such that

$$z_i \in x_i + y_1, i=1, \dots, n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1}, j=1, \dots, n,$$

therefore, we obtain $\{z_k, z_{k+1}\} \subset t_k, k=1, \dots, m+n-1$.

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is γ equivalent to every $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus, $\gamma(a) \oplus \gamma(b)$ is a singleton so we can write $\gamma(a) \oplus \gamma(b) = \gamma(c), \forall c \in \gamma(a) + \gamma(b)$.

In a similar way, we prove that $\gamma(a) \otimes \gamma(b) = \gamma(d), \forall d \in \gamma(a) \cdot \gamma(b)$.

The weak associativity and the weak distributivity on R give that the associativity and the distributivity are valid for the quotient R/γ^* . Therefore, R/γ^* is a ring.

Let σ equivalence with R/σ ring. Then, $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons, so $\forall a, b \in R$, we have $\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b)$, and $\sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b)$. Thus, $\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B), \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B), \forall a, b \in R, A \subset \sigma(a), B \subset \sigma(b)$. By induction, extend it on sums and products. So $\forall u \in U$, we have $\sigma(x) = \sigma(u), \forall x \in u$. So, $x \in \gamma(a)$ implies $x \in \sigma(a), \forall x \in R$. But σ is transitively closed, thus: $x \in \gamma(x)$ implies $x \in \sigma(a)$. That means that γ is the smallest equivalence in R such that R/γ is a ring, so, $\gamma = \gamma^*$. ■

An element is called *single* if its fundamental class is singleton, so if $\beta^*(s) = \{s\}$.

Denote S_H the set of singles. If $S_H \neq \emptyset$, then we can answer to the hard problem, which is to find the fundamental classes. There is the following [11], [30], [42], [56]:

Theorem 2.3 Let $(H, \cdot)$ be H_v -group and $s \in S_H \neq \emptyset$. Let $a \in H$, take a $v \in H$ such that $s \in av$, then $\beta^*(a) = \{h \in H: hv = s\}$, the core of H is $\omega_H = \{u \in H: us = s\} = \{u \in H: su = s\}$. Moreover, $sx = \beta^*(sx)$ and $xs = \beta^*(xs), \forall x \in H$.

The fundamental relations are used for general definitions.

Definition 2.4 An H_v -ring $(R, +, \cdot)$ is called *H_v -field* if R/γ^* is a field. In sequence the *H_v -vector space* is defined. The elements of an H_v -field are called *hyper-numbers* or *H_v -numbers*.

Definition 2.5 Let $(L, +)$ H_v -vector space on $(F, +, \cdot)$, $\phi: F \rightarrow F/\gamma^*, \omega_F = \{x \in F: \phi(x) = 0\}$, 0 zero of F/γ^* . Let ω_L core of $\phi': L \rightarrow L/\varepsilon^*, 0$ zero of L/ε^* . Take the *bracket hyper-operation*:

$$[,]: L \times L \rightarrow P(L): (x, y) \rightarrow [x, y],$$

then L is an *H_v -Lie algebra* over F if the following axioms are satisfied:

$$(L1) \quad [\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \forall x, x_1, x_2, y, y_1, y_2 \in L, \forall \lambda_1, \lambda_2 \in F$$

$$(L2) \quad [x, x] \cap \omega_L \neq \emptyset, \forall x \in L$$

$$(L3) \quad ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L.$$

Theorem 2.6 Let $(H, \cdot)$ be H_v -group, H/β^* fundamental group. If H/β^* is not commutative or not cyclic, then $(H, \cdot)$ is not weak commutative or cyclic, respectively.

Proof. Straightforward since if $(H, \cdot)$ is weak commutative or cyclic, then its fundamental group H/β^* is commutative or cyclic, respectively. ■

The *uniting elements* method, introduced by Corsini & Vougiouklis in 1989, [3], [30], is the following: Let G be structure and a no-valid property d , described by equations. Take the partition in G setting in the same class, all pairs of elements that cause the non-

validity of d . The quotient by this partition G/d is an H_V -structure. Then, quotient out G/d by β^* , is a stricter structure $(G/d)/\beta^*$ for which the property d is valid.

Theorem 2.7 Let $(R, +, \cdot)$ be ring, $F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$ system of equations on R consisting of subsystems $F_m = \{f_1, \dots, f_m\}$ and $F_n = \{f_{m+1}, \dots, f_{m+n}\}$. Let σ, σ_m the equivalence relations defined by uniting elements using F and F_m respectively, and σ_n the equivalence on F_n on the ring $R_m = (R/\sigma_m)/\gamma^*$. Then $(R/\sigma)/\gamma^* \cong (R_m/\sigma_n)/\gamma^*$.

Motivation for F. Marty to introduce the hypergroup was the quotient of a group by any not normal subgroup. Motivation to introduce the weak hyper-group, or H_V -group, is the quotient of a group by any partition of the group. Thus, we summarize:

Motivation for hyper-structures.

The quotient of a group by any invariant subgroup is a group.

The quotient of a group by any subgroup is a hypergroup.

The quotient of a group by any partition is an H_V -group.

3. Representation Theory. Small H_V -fields

The representations of H_V -groups, can be considered either by H_V -matrices [24], [29], [30], [36], [38], [46], [47], [49] or by generalized permutations [28].

Definition 3.1 H_V -matrix is called a matrix with entries elements of H_V -ring or H_V -field. The hyper-product of H_V -matrices $A = (a_{ij})$ and $B = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_V -matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) \mid c_{ij} \in \oplus \Sigma a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ is the n -ary circle hyper-sum.

The representation problem by H_V -matrices is the following:

Definition 3.2 Let $(H, \cdot)$ be H_V -group, $(R, +, \cdot)$ H_V -ring and $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then any

$$T: H \rightarrow M_R: h \mapsto T(h) \text{ with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H,$$

is called H_V -matrix representation. If the $T(h_1 h_2) \subset T(h_1)T(h_2)$, is valid, then T is called *inclusion* representation, if the $T(h_1 h_2) = T(h_1)T(h_2)$, is valid then T is called *good* representation. If T is one to one and good, then it is a *faithful* representation.

Theorem 3.3 A necessary condition in order to have an inclusion representation T of an H_V -group $(H, \cdot)$ by $n \times n$ H_V -matrices over the H_V -ring $(R, +, \cdot)$ is the following:

For all $\beta^*(x)$, $x \in H$ there are elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}.$$

Thus, every inclusion representation $T: H \rightarrow M_R: a \mapsto T(a) = (a_{ij})$ induces a homomorphic representation T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

The representation problem is simplified in cases such as if the H_V -rings have scalars 0 and 1, and they are used as singleton entries.

The representation problem by generalized permutations, is described as follows [28]:

Definitions 3.4 Let X be a set, then a map $f: X \rightarrow P(X) - \{\emptyset\}$, is a *generalized permutation* of X if it is reproductive: $\cup_{x \in X} f(x) = f(X) = X$. Denote by M_X the set of all generalized permutations on X . For an H_v -group $(X, \cdot)$ and $a \in X$, the generalized permutation f_a defined by $f_a(x) = ax$ is called *inner*. Arrow of f is any $(x, y) \in X^2$ with $y \in f(x)$. The $f_2 \in M_X$ contains $f_1 \in M_X$, if $f_1(x) \subset f_2(x)$, $\forall x \in X$, we write $f_1 \subset f_2$. If, moreover, $f_1 \neq f_2$, then f_1 is a *proper* sub-generalized permutation of f_2 . An $f \in M_X$ is called *minimal* if it has no proper sub-generalized permutation. Denote $\underline{M}_X$ the set of all minimal generalized permutations of M_X . The *converse* $\underline{f}$ of a generalized permutation f is obtained by reversing arrows: $\underline{f}(x) = \{z \in X: f(z) \ni x\}$. We call *associated* to $f \in M_X$ the generalized permutation $f \circ \underline{f}$.

Theorem 3.5 Let $f \in M_X$, then $f \in \underline{M}_X$ if, and only if, the following condition is valid: if $a \neq b$ and $f(a) \cap f(b) \neq \emptyset$, then $f(a) = f(b)$ and $f(a)$ is singleton. If $f \in \underline{M}_X$ then, $\underline{f} \in \underline{M}_X$. If $f \in \underline{M}_X$ then, $(f \circ \underline{f})(x) = \{y \in X: f(y) = f(x)\}$.

We summarize the representation theory of H_v -groups by H_v -matrices saying that the problem has two steps: First, find small H_v -fields and second, find appropriate H_v -matrices. We focus on small H_v -fields which can be applied in Hadronic Mechanics. These are the small non-degenerate H_v -fields, which in isothory, are:

(a) very-thin minimal, (b) weak commutative, (c) 0 and 1, scalars.

Definition 3.6 We call *raised very-thin* the H_v -fields which are obtained enlarging one result of a ring adding one element, such that the fundamental structure is a field.

Combining the uniting elements procedure with the raise theory we can obtain stricter structures or hyperstructures [3], [35], [36], [44], [46], [50], [51]. So, raising operations or hyper-operations we can obtain more strict structures as we can see in the following.

Theorem 3.7 In the ring of integers $(Z, +, \cdot)$, fix an $m > 1$. We raise in the product the result $0 \cdot m$ by $0 \otimes m = \{0, m\}$, the rest results are the same. Then $(Z, +, \otimes)$ becomes H_v -ring, with finite fundamental ring: $(Z, +, \otimes) / \gamma^* \cong (Z_m, +, \cdot)$.

If $m = p$, prime, then $(Z, +, \otimes)$ is raised very-thin H_v -field, with finite fundamental field. Raising $a \cdot b$ of fixed $a, b \in Z - \{0, 1\}$, setting $a \otimes b = \{a \cdot b, a \cdot b + m\}$, we have the same result, moreover, $(Z, +, \otimes)$ is raised very-thin H_v -field, where 0 and 1 are scalars.

Proof. Sums and products which contain more than one element are the ones which have $0 \otimes m$. Adding to $0 \otimes m$ the 1, several times, we have $\text{mod } m$ equivalence classes. On the other side, if add or multiply elements of the same class the results remain in one class, the class of the representatives. Thus, the γ^* -classes form a ring isomorphic to $(Z_m, +, \cdot)$. The rest of the proof is straightforward. Notice, that we can transfer the generalized raised case if we consider the expression $a \otimes b - a \cdot b = \{0, m\}$. ■

Theorem 3.8 In the ring $(Z_n, +, \cdot)$, with $n = ms$ we raise in the product the result $0 \cdot m$ by $0 \otimes m = \{0, m\}$ and the rest results are the same. Then, $(Z_n, +, \otimes) / \gamma^* \cong (Z_m, +, \cdot)$.

If $m = p$, prime, then $(Z_n, +, \otimes)$ is raised very-thin H_v -field.

Raising $a \cdot b$ of fixed $a, b \in Z_n - \{0, 1\}$, setting $a \otimes b = \{a \cdot b, a \cdot b + m\}$, we have the same result, $(Z_n, +, \otimes)$ is raised very-thin H_v -field, where 0 and 1, are scalars.

All H_v -fields obtained from a field raising any sum or product, are degenerate:

Theorem 3.9 In a field $(F, +, \cdot)$, we raise only the product of $a \cdot b$, by $a \otimes b = \{a \cdot b, c\}$, $c \neq a \cdot b$. Then we obtain the degenerate minimal very-thin H_v -field $(F, +, \otimes) / \gamma^* \cong \{0\}$.

Proof. Take any $x \in F - \{0\}$, then from $a \otimes b = \{ab, c\}$, we obtain $(a \otimes b) - ab = \{0, c - ab\}$ and then $(x(c - ab)^{-1}) \otimes ((a \otimes b) - ab) = \{0, x\}$ so, $0 \gamma x$, $x \in F - \{0\}$ or, every x is in the fundamental class of 0. Thus, $(F, +, \otimes) / \gamma^* \cong \{0\}$. ■

Theorem 3.10 In a field $(F, +, \cdot)$, we raise only the sum of $a + b$, by $a \oplus b = \{a + b, c\}$, $c \neq a + b$. Then we obtain the degenerate, minimal very-thin, H_v -field $(F, \oplus, \cdot) / \gamma^* \cong \{0\}$.

Proof. Take any $x \in F - \{0\}$, then from $a \oplus b = \{a + b, c\}$, we obtain $(a \oplus b) - (a + b) = \{0, c - (a + b)\}$ and then $[x(c - (a + b))^{-1}] \cdot [(a \oplus b) - (a + b)] = \{0, x\}$, so, $0 \gamma x$, $x \in F - \{0\}$ or, every x is in the fundamental class of 0. Thus, $(F, \oplus, \cdot) / \gamma^* \cong \{0\}$. ■

We present some ‘small’ H_v -fields.

Constructions 3.11 On rings $(\mathbb{Z}_4, +, \cdot)$ and $(\mathbb{Z}_6, +, \cdot)$ we define all multiplicative H_v -fields with non-degenerate fundamental field and, moreover they are,

(a) very-thin minimal, (b) weak commutative, (c) 0 and 1, scalars.

I. On $(\mathbb{Z}_4, +, \cdot)$ the isomorphic cases are: $2 \otimes 3 = \{0, 2\}$, $3 \otimes 2 = \{0, 2\}$.

Fundamental classes: $[0] = \{0, 2\}$, $[1] = \{1, 3\}$, and $(\mathbb{Z}_4, +, \otimes) / \gamma^* \cong (\mathbb{Z}_2, +, \cdot)$.

II. On $(\mathbb{Z}_6, +, \cdot)$, the isomorphic cases are:

(i) $2 \otimes 3 = \{0, 3\}$, $2 \otimes 4 = \{2, 5\}$, $3 \otimes 4 = \{0, 3\}$, $3 \otimes 5 = \{0, 3\}$, $4 \otimes 5 = \{2, 5\}$

Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$, and $(\mathbb{Z}_6, +, \otimes) / \gamma^* \cong (\mathbb{Z}_3, +, \cdot)$.

(ii) $2 \otimes 3 = \{0, 2\}$ or $2 \otimes 3 = \{0, 4\}$, $2 \otimes 4 = \{0, 2\}$ or $\{2, 4\}$, $2 \otimes 5 = \{0, 4\}$ or $2 \otimes 5 = \{2, 4\}$,

$3 \otimes 4 = \{0, 2\}$ or $3 \otimes 4 = \{0, 4\}$, $3 \otimes 5 = \{3, 5\}$, $4 \otimes 5 = \{0, 2\}$ or $\{2, 4\}$.

Fundamental classes: $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ and $(\mathbb{Z}_6, +, \otimes) / \gamma^* \cong (\mathbb{Z}_2, +, \cdot)$.

We can also define the analogous cases for the rings $(\mathbb{Z}_9, +, \cdot)$, and $(\mathbb{Z}_{10}, +, \cdot)$.

We present examples on the topic [44], [47], [49], [50], [51]:

Example 3.12 Take the 2×2 upper triangular H_v -matrices on the above H_v -field $(\mathbb{Z}_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct:

$$I = E_{11} + E_{22}, \quad a = E_{11} + E_{12} + E_{22}, \quad b = E_{11} + 2E_{12} + E_{22}, \quad c = E_{11} + 3E_{12} + E_{22},$$

$$d = E_{11} + 3E_{22}, \quad e = E_{11} + E_{12} + 3E_{22}, \quad f = E_{11} + 2E_{12} + 3E_{22}, \quad g = E_{11} + 3E_{12} + 3E_{22},$$

then, for $X = \{I, a, b, c, d, e, f, g\}$, we obtain the following multiplicative table:

$\otimes$	I	a	b	c	d	e	f	g
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	d	e	f
b	b	c	I	a	d, f	e, g	d, f	e, g
c	c	I	a	b	e	f	g	d
d	d	e	f	g	I	a	b	c
e	e	f	g	d	c	I	a	b
f	f	g	d	e	I, b	a, c	I, b	a, c
g	g	d	e	f	a	b	c	b

The $(X, \otimes)$ is weak commutative H_v -group with fundamental classes $\underline{I} = \{I, b\}$, $\underline{a} = \{a, c\}$, $\underline{d} = \{d, f\}$, $\underline{e} = \{e, g\}$, and fundamental group isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

Example 3.13 Consider the 2×2 upper triangular H_v -matrices on the H_v -field $(\mathbb{Z}_4, +, \otimes)$ where only $3 \otimes 2 = \{0, 2\}$ is hyper-product:

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$$a=E_{11}+E_{22}, \quad a_1=E_{11}+E_{12}+E_{22}, \quad a_2=E_{11}+2E_{12}+E_{22}, \quad a_3=E_{11}+3E_{12}+E_{22},$$

$$b=E_{11}+3E_{22}, \quad b_1=E_{11}+E_{12}+3E_{22}, \quad b_2=E_{11}+2E_{12}+3E_{22}, \quad b_3=E_{11}+3E_{12}+3E_{22},$$

$$c=3E_{11}+E_{22}, \quad c_1=3E_{11}+E_{12}+E_{22}, \quad c_2=3E_{11}+2E_{12}+E_{22}, \quad c_3=3E_{11}+3E_{12}+E_{22},$$

$$d=3E_{11}+3E_{22}, \quad d_1=3E_{11}+E_{12}+3E_{22}, \quad d_2=3E_{11}+2E_{12}+3E_{22}, \quad d_3=3E_{11}+3E_{12}+3E_{22},$$

then, for $X=\{a,a_1,a_2,a_3,b,b_1,b_2,b_3,c,c_1,c_2,c_3,d,d_1,d_2,d_3\}$, we obtain the multiplicative table.

$\otimes$	a	a₁	a₂	a₃	b	b₁	b₂	b₃	c	c₁	c₂	c₃	d	d₁	d₂	d₃
a	a	a ₁	a ₂	a ₃	b	b ₁	b ₂	b ₃	c	c ₁	c ₂	c ₃	d	d ₁	d ₂	d ₃
a₁	a ₁	a ₂	a ₃	a	b ₃	b	b ₁	b ₂	c ₁	c ₂	c ₃	c	d ₃	d	d ₁	d ₂
a₂	a ₂	a ₃	a	a ₁	b ₂	b ₃	b	b ₁	c ₂	c ₃	c	c ₁	d ₂	d ₃	d	d ₁
a₃	a ₃	a	a ₁	a ₂	b ₁	b ₂	b ₃	b	c ₃	c	c ₁	c ₂	d ₁	d ₂	d ₃	d
b	b	b ₁	b ₂	b ₃	a	a ₁	a ₂	a ₃	d	d ₁	d ₂	d ₃	c	c ₁	c ₂	c ₃
b₁	b ₁	b ₂	b ₃	b	a ₃	a	a ₁	a ₂	d ₁	d ₂	d ₃	d	c ₃	c	c ₁	c ₂
b₂	b ₂	b ₃	b	b ₁	a ₂	a ₃	a	a ₁	d ₂	d ₃	d	d ₁	c ₂	c ₃	c	c ₁
b₃	b ₃	b	b ₁	b ₂	a ₁	a ₂	a ₃	a	d ₃	d	d ₁	d ₂	c ₁	c ₂	c ₃	c
c	c	c ₃	c, c ₂	c ₁	d	d ₃	d, d ₂	d ₁	a	a ₃	a, a ₂	a ₁	b	b ₃	b, b ₂	b ₁
c₁	c ₁	c	c ₁ , c ₃	c ₂	d ₃	d ₂	d ₁ , d ₃	d	a ₁	a	a ₁ , a ₃	a ₂	b ₃	b ₂	b ₁ , b ₃	b
c₂	c ₂	c ₁	c, c ₂	c ₃	d ₂	d ₁	d, d ₂	d ₃	a ₂	a ₁	a, a ₂	a ₃	b ₂	b ₁	b, b ₂	b ₃
c₃	c ₃	c ₂	c ₁ , c ₃	c	d ₁	d	d ₁ , d ₃	d ₂	a ₃	a ₂	a ₁ , a ₃	a	b ₁	b	b ₁ , b ₃	b ₂
d	d	d ₃	d, d ₂	d ₁	c	c ₃	c, c ₂	c ₁	b	b ₃	b, b ₂	b ₁	a	a ₃	a, a ₂	a ₁
d₁	d ₁	d	d ₁ , d ₃	d ₂	c ₃	c ₂	c ₁ , c ₃	c	b ₁	b	b ₁ , b ₃	b ₂	a ₃	a ₂	a ₁ , a ₃	a
d₂	d ₂	d ₁	d, d ₂	d ₃	c ₂	c ₁	c, c ₂	c ₃	b ₂	b ₁	b, b ₂	b ₃	a ₂	a ₁	a, a ₂	a ₃
d₃	d ₃	d ₂	d ₁ , d ₃	d	c ₁	c	c ₁ , c ₃	c ₂	b ₃	b ₂	b ₁ , b ₃	b	a ₁	a	a ₁ , a ₃	a ₂

$(X, \otimes)$ is weak commutative H_v -group, fundamental classes: $\underline{a}=\{a, a_2\}$, $\underline{a}_1=\{a_1, a_3\}$, $\underline{b}=\{b, b_2\}$, $\underline{b}_1=\{b_1, b_3\}$, $\underline{c}=\{c, c_2\}$, $\underline{c}_1=\{c_1, c_3\}$, $\underline{d}=\{d, d_2\}$, $\underline{d}_1=\{d_1, d_3\}$, with table as the above example. In $(X, \otimes)$ there is unit a , and every element has unique double inverse. The c_2 is right inverse to c and c_2 , because $a \in cc_2$, $a \in c_2c_2$. The d_2 is right inverse to d and d_2 , because $a \in dd_2$, $a \in d_2d_2$. $(X, \otimes)$ is not cyclic, since, from Theorem 2.6, $(\underline{X}, \otimes)$ is not cyclic.

4. Santilli's iso-theory and e-hyper-structures

The H_v -structures have applications in mathematics and other sciences, as well. These range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, to mention but a few. The hyper-structure theory is related to fuzzy theory; consequently, hyper-structures can be widely applicable in linguistic, sociology, production and industry, too [1], [6], [7], [11], [37], [39], [52], [53], [56].

In 'The Santilli's theory 'invasion' in hyperstructures' [40], there is a description on how Santilli's theories effect in hyper-structures and how new theories in Mathematics appeared by Santilli's pioneer research. In 1996 Santilli & Vougiouklis [19], point out that in physics the most interesting hyper-structures are the e-hyperstructures.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit [11], [14], [15], [16], [17], [18], [19], [37], [45], [48].

Definition 4.1 According to Santilli's iso-theory on a field $F=(F,+, \cdot)$, a general isofield $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is defined to be a field with elements $\hat{a}=a \times \hat{1}$, called *isonumbers*, where $a \in F$, and $\hat{1}$ is positive-defined outside F , with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a} \hat{\times} \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F} \quad (i)$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \times \hat{1} = \hat{a}, \forall \hat{a} \in \hat{F} \quad (ii)$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyper-structure case we generalize only the new product $\hat{\times}$ from (i), replacing with a hyper-product including the old one. There is a general construction on this direction [9], [14], [15], [20], [33], [45], [46]:

Definition 4.2 *General enlargement*. On a field F and its isofield $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ replace in the results of the iso-product

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}$$

the $\hat{T}$ by a set $\hat{H}_{ab}=\{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}$, $\hat{x}_1, \hat{x}_2, \dots \in \hat{F}$, containing $\hat{T}$, $\forall \hat{a} \hat{\times} \hat{b}$ for which $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$, $\hat{x}_1, \hat{x}_2, \dots \in \hat{F} - \{\hat{0}, \hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore, the new iso-hyper-operation is

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F} \quad (iii)$$

$\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ becomes *very-thin isoH_v-field*. The elements of $\hat{F}$ are called *isoH_v-numbers* or *isonumbers*.

Important hyper-products are if only for few $(\hat{a}, \hat{b})$ the result enlarged, even more, the extra elements $\hat{x}_i$, are only few. Thus, if there exists only one pair $(\hat{a}, \hat{b})$ for which

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F}$$

and the rest are ordinary, then we have a *very-thin isoH_v-field*.

The assumption $\hat{H}_{ab}=\{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, and $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has a scalar absorbing $\hat{0}$, a scalar $\hat{1}$, and $\forall \hat{a} \in \hat{F}$, has one inverse.

The *isofields* corresponding to hyperstructures, introduced by Santilli & Vougiouklis in 1999, are called *e-hyperfields*. [9], [11], [14], [19], [33], [35], [40], [41], [42].

Definitions 4.3 A hyper-structure $(H, \cdot)$ is called *e-hyper-structure* if contains a unique scalar unit e , and every element x , has a, not unique, inverse x^{-1} : $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

A triad $(F, +, \cdot)$, with $(+)$ operation, $(\cdot)$ hyper-operation, is called *e-hyper-field* if: $(F, +)$ is abelian group; $(\cdot)$ weak associative; $(\cdot)$ weak distributive to $(+)$; 0 absorbing: $0 \cdot x = x \cdot 0 = 0$,

$\forall x \in F$; 1 scalar: $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$; and $\forall x \in F$ there is unique inverse x^{-1} : $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. The elements of F are called *e-hyper-numbers*. If $1 = x \cdot x^{-1} = x^{-1} \cdot x$, then $(F, +, \cdot)$ is *strong*.

Definition 4.4 *The Main e-Construction.* Given a group $(G, \cdot)$, e unit, define in G , a large number of hyper-operations $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, g_1, g_2, \dots \in G - \{e\}$$

$g_1, g_2, \dots$ are not the same for all pairs (x, y) . $(G, \otimes)$ is an H_v -group, it is an H_b -group which contains the $(G, \cdot)$. $(G, \otimes)$ is an *e-hypergroup*. Moreover, if for all x, y such that $xy = e$, we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong *e-hypergroup*.

The proof is immediate since we enlarge the results of the group by putting elements from G applying the Little Theorem. Moreover, the unit e is a unique scalar and $\forall x \in G$, there is a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Finally, if we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$ then the hyper-structure $(G, \otimes)$ is a strong *e-hypergroup*.

Remark that the main *e-construction* gives an extremely large number of *e-hyper-operations*.

Example 4.5 Consider the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$. Denote $\underline{i} = \{i, -i\}$, $\underline{j} = \{j, -j\}$, $\underline{k} = \{k, -k\}$, we may define a large number $(*)$ hyper-operations by enlarging few products. For example, we set $(-1) * \underline{k} = \underline{k}$, $k * i = \underline{j}$ and $i * j = \underline{k}$, then $(Q, *)$ is a strong *e-hyper-group*.

5. Applications in Lie-Santilli's admissibility

A new field in hyper-mathematics comes from Santilli's theory of admissibility. One can transfer Santilli's admissibility, either using ordinary matrices and a hyper-operation on them, or using hyper-matrices and ordinary operations. The Lie-Santilli admissibility in H_v -structures is defined as follows [1], [7], [8], [9], [11], [19], [20], [42], [43], [48]:

Definition 5.1 Let L be H_v -vector space on F , $\varphi: F \rightarrow F/\gamma^*$, canonical, $\omega_F = \{x \in F: \varphi(x) = 0\}$, 0 zero of F/γ^* , ω_L core of $\varphi': L \rightarrow L/\varepsilon^*$, 0 zero of L/ε^* . Take any $R, S \subseteq L$, then, a *Lie-Santilli admissible hyperalgebra* is obtained by taking the *Lie-bracket* which is hyper-operation:

$$[,]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = (xR)y - (yS)x = \{xry - ysx \mid r \in R, s \in S\}$$

Special cases: (a) Take only S , so, $[x, y]_S = xy - ySx$. (b) Take only R , so, $[x, y]_R = xRy - yx$.

On non square matrices we can define hyper-admissibility, as well:

Definition 5.2 Let $(L = M_{m \times n}, +)$ be H_v -vector space of $m \times n$ H_v -matrices on the H_v -field $(F, +, \cdot)$; $\varphi: F \rightarrow F/\gamma^*$, canonical map; $\omega_F = \{x \in F: \varphi(x) = 0\}$, 0 the zero of F/γ^* . Similarly, let ω_L core of $\varphi': L \rightarrow L/\varepsilon^*$ and 0 zero of L/ε^* . Take two subsets $R, S \subseteq L$, then, a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the hyper-operation Lie bracket:

$$[,]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x, y]_{RS} = xR^t y - yS^t x = \{xr^t y - ys^t x \mid r \in R \text{ and } s \in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) $R = \{e\}$ then $[x, y]_{RS} = xy - yS^t x = \{xy - ys^t x \mid s \in S\}$

- (b) $S=\{e\}$ then $[x,y]_{RS} = xR^t y - yx = \{xr^t y - yx \mid r \in R\}$
(c) $R=\{r_1, r_2\}$ and $S=\{s_1, s_2\}$ then

$$[x,y]_{RS} = xR^t y - yS^t x = \{xr_1^t y - ys_1^t x, xr_1^t y - ys_2^t x, xr_2^t y - ys_1^t x, xr_2^t y - ys_2^t x\}$$

Definition 5.3 *The P-hope.* Consider isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ with isonumbers $\hat{a} = a \times \hat{1}$, $a \in F$, and $\hat{1}$ is positive-defined element outside F , with operations $\hat{+}$, $\hat{\times}$, where $\hat{+}$ the sum with conventional unit 0, and $\hat{\times}$ the iso-product

$$\hat{a} \hat{\times} \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F}.$$

Take $\hat{P} = \{\hat{T}, \hat{p}_1, \dots, \hat{p}_s\}$, $\hat{p}_1, \dots, \hat{p}_s \in \hat{F} - \{\hat{0}, \hat{1}\}$, and define the isoP-H_v-field, $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times}_P)$, where the hyper-operation $\hat{\times}_P$ is defined by:

$$\hat{a} \hat{\times}_P \hat{b} := \begin{cases} \hat{a} \times \hat{P} \wedge \hat{b} = \{\hat{a} \times \hat{h} \wedge \hat{b} \mid \hat{h} \in \hat{P} \wedge\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \wedge \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases}$$

Special ‘small’ case: $\hat{P} = \{\hat{T}, \hat{p}\}$, so $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. Remark that we can take this small case on rings to obtain isoP-H_v-fields. We can see this in the following:

Example 5.4 For, $\hat{Z}_{10} = Z_{10}(\hat{a}, \hat{+}, \hat{\times})$, and if we take $\hat{P} = \{\hat{2}, \hat{7}\}$, then we have the table

$\hat{\times}$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$	$\hat{9}$
$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$
$\hat{1}$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$	$\hat{8}$	$\hat{9}$
$\hat{2}$	$\hat{0}$	$\hat{2}$	$\hat{8}$	$\hat{2}$	$\hat{6}$	$\hat{0}$	$\hat{4}$	$\hat{8}$	$\hat{2}$	$\hat{6}$
$\hat{3}$	$\hat{0}$	$\hat{3}$	$\hat{2}$	$\hat{3}, \hat{8}$	$\hat{4}$	$\hat{0}, \hat{5}$	$\hat{6}$	$\hat{2}, \hat{7}$	$\hat{8}$	$\hat{4}, \hat{9}$
$\hat{4}$	$\hat{0}$	$\hat{4}$	$\hat{6}$	$\hat{4}$	$\hat{2}$	$\hat{0}$	$\hat{8}$	$\hat{6}$	$\hat{4}$	$\hat{2}$
$\hat{5}$	$\hat{0}$	$\hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$	$\hat{0}$	$\hat{0}, \hat{5}$
$\hat{6}$	$\hat{0}$	$\hat{6}$	$\hat{4}$	$\hat{6}$	$\hat{8}$	$\hat{0}$	$\hat{2}$	$\hat{4}$	$\hat{6}$	$\hat{8}$
$\hat{7}$	$\hat{0}$	$\hat{7}$	$\hat{8}$	$\hat{2}, \hat{7}$	$\hat{6}$	$\hat{0}, \hat{5}$	$\hat{4}$	$\hat{3}, \hat{8}$	$\hat{2}$	$\hat{1}, \hat{6}$
$\hat{8}$	$\hat{0}$	$\hat{8}$	$\hat{2}$	$\hat{8}$	$\hat{4}$	$\hat{0}$	$\hat{6}$	$\hat{2}$	$\hat{8}$	$\hat{4}$
$\hat{9}$	$\hat{0}$	$\hat{9}$	$\hat{6}$	$\hat{4}, \hat{9}$	$\hat{2}$	$\hat{0}, \hat{5}$	$\hat{8}$	$\hat{1}, \hat{6}$	$\hat{4}$	$\hat{2}, \hat{7}$

The fundamental classes are $(0) = \{\hat{0}, \hat{5}\}$, $(1) = \{\hat{1}, \hat{6}\}$, $(2) = \{\hat{2}, \hat{7}\}$, $(3) = \{\hat{3}, \hat{8}\}$, $(4) = \{\hat{4}, \hat{9}\}$ and the multiplicative table of them is the following:

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$\times$	(0)	(1)	(2)	(3)	(4)
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{Z}_{10} = Z_{10}(\hat{\alpha}, \hat{\tau}, \hat{\times})$, is an H_V -field.

All biological structures, including cells, viruses and living organisms, are irreversible over time because they are born, grow and die. Santilli introduced in 1994 [16], [17], [18], the representation of biological structures via multivalued, Lie-admissible formulations on a 3-dimensional Euclidean space. They have genounits, *hyperunits*, who correspond the product of generic non-singular quantities, *classical hyper-products*, are equally multivalued, yet defined in our 3-dimensional Euclidean space. The introduced *classical hyper-fields*, are sets of multi-valued elements, products and units which verify the axioms of numeric fields. The hyperstructure representation in biological structures was initiated by T. Vouiouklis in 1999 [35], [37], with the lifting of the classical hyper-structures to Vouiouklis H_V -structures which are formulated via hyper-operations which are weak associative, weak commutative and so on. The advantages of lifting the classical hyper-structures to Vouiouklis H_V -structures are several [7], [8], [9], [16], [20], [40], [41], [45], [48]. The main advantage is a large increase of the representational capabilities which is necessary for a representation of biological structures such as the DNA.

In the transition from Lie theory to Lie-admissible theory, we must specify an element S on the right and an element R to the left. In hyper-structure realization we can use as S and R, sets instead of elements. But in this case, we have hyper-operations of constant length and the living organisms are not the case. Therefore, Santilli & Vouiouklis suggest the use of the special case of the main e-construction to face the problem. Therefore, as the Living Organism Construction the Main e-Construction is applied.

6. Conclusions

When applied sciences ask from mathematics models to express a new theory, one has to search on the existing mathematics if there is an appropriate theory. If there is not any, then the mathematicians try to create a new one to obey the axioms which are required. The hyper-structures offer models in multivalued problems as the ones appeared in real life. The largest class of hyperstructures, called H_V -structures, can offer to Lie-Santilli theory some of the models needed, because they define a huge number of H_V -structures defined on the same set. The appropriate H_V -fields, obtained by the Main e-Construction, can answer to the basic question of Santilli's isothory, which is: What are the hyper-numbers? The reason is that the Main e-Construction can enlarge every single result with the needed elements.

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Minimum Measurement Uncertainty in Quantum Systems Subject to High Energy Fluctuations

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Abstract

The stochastic generalization of the Madelung quantum hydrodynamics incorporates the fluctuations of the mass density into quantum equations induced by the gravitational background noise, a form of dark energy,. This model successfully addresses key aspects of quantum to classical transition through the definition of quantum potential length of interaction in addition to the De Broglie length, beyond which coherent quantum behavior and wavefunction evolution is perturbed or even deeply modified. The stochastic quantum hydrodynamic model emphasizes that an external environment is unnecessary, asserting that the stochastic behavior leading to wave-function collapse and macroscopic classical behavior with the Einstein realism can be an inherent property of physics in a spacetime with fluctuating metrics. The theory establishes a coherent link between the uncertainty principle and the constancy of light speed, aligning seamlessly with finite information transmission speed. Within quantum mechanics submitted to fluctuations, the stochastic quantum hydrodynamic model derives the indeterminacy relation between energy and time, offering insights into measurement processes impossible within a finite time interval in a truly quantum global system. The model offers also how the uncertainty relations modify themselves in an open quantum system submitted to fluctuations offering the possibility of describing how quantum mechanics can modify itself in nuclear and elementary particle physics up to the behavior of high energy black hole states. The self-consistency of the model lies in its ability to describe the dynamics of wavefunction collapse and the measure process within its mathematical structure. Additionally, the theory resolves the Einstein determinism and the pre-existing reality problem by showing that large-scale systems naturally self-decay into decoherent classical states stable in time without observer or measuring apparatus.

Keywords: quantum to classical transition, Einstein determinism, pre-measure reality, uncertainty relations in open quantum system, hidden variable, IsoRedShift Mechanics.

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1. Introduction

Any quantum theory that seeks to explain the development of a physical system across different scales, regardless of magnitude, must inherently deal with the shift from quantum mechanical characteristics to the classical behavior that emerges at larger scales. The essential differences between these two frameworks are highlighted by the uncertainty principle in quantum mechanics, which indicates the intrinsic impossibility of simultaneously measuring conjugate variables, and by the finite speed at which interactions and information propagate in local classical relativistic mechanics.

Several unresolved aspects persist within quantum theories. The Einstein–Podolsky–Rosen (EPR) paradox [1] and the question of the completeness of quantum mechanics are fundamental issues that have yet to be clearly integrated into a comprehensive theoretical framework.

A significant challenge to this view was posed by the Einstein–Podolsky–Rosen objection, which was subsequently criticized by Bohr [2]. Unable to find a compatible solution, Bohr advocated for the acceptance of the probabilistic foundations of quantum mechanics, regardless of the validity of the principle of causality.

In fact, Bohr's perspective significantly influenced subsequent scientific thought. The idea that quantum mechanics functions as a probabilistic process was revitalized by Nelson's research and has persisted over time. However, Nelson's hypotheses were ultimately limited [3] by the requirement of a specific stochastic derivative with time inversion symmetry, which restricted their general applicability. Additionally, Nelson's theory does not completely match the results of quantum mechanics, as shown by Von Neumann's proof [4-5] of the impossibility of replicating quantum mechanics using theories based on underlying classical probabilistic variables.

This approach was widely explored in local hidden variable theories attempting to replicate quantum mechanics, whose failure was demonstrated by Bell's theorem [6]. To address Bell's arguments, Bohm developed a non-local hidden variable theory that seeks to restore determinism in quantum mechanics by introducing the concept of a pilot wave [7]. The core idea is that, in addition to the particles, there is a “guidance” or influence from the pilot wave function that determines the particles' behavior. Although this pilot wave function is not directly observable, it affects the measurement probabilities of the particles.

The ambiguous definition and unclear logical foundations of the pilot wave have been significant obstacles to the broader acceptance of Bohm's theory among physicists. Nonetheless, Santilli has recently renewed interest in and the potential validity of the Bohm hidden variable hypothesis with his IsoRedShift Mechanics theory. This theory introduces a hidden variable to explain deviations in nuclear quantum mechanics [8] in the deuterium state. It is important to note that Santilli's hidden variable is integrated into quantum non-local theory rather than classical theory. Through this approach, Santilli has demonstrated that it is possible to recover Einstein's determinism [9-11].

More recently, the author [12], utilizing the Madelung hydrodynamic formulation of quantum mechanics—a specific case of Bohm's theory [13]—has demonstrated that by

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considering the fluctuating gravitational background due to relic metric tensor oscillations from the Big Bang and the dynamics of general relativity bodies, one can achieve a quantum-fluctuating dynamics without needing to introduce an external environment. From this standpoint, the derived stochastic quantum hydrodynamic model [12] (SQHM) highlights that conventional quantum theory, despite its mathematical precision, remains incomplete regarding its foundational postulates. Specifically, SQHM emphasizes that the measurement process cannot be fully explained within the deterministic "Hamiltonian" framework of quantum mechanics. Instead, it is fully described as a phenomenon within the framework of a quantum stochastic generalized approach.

The SQHM reveals that quantum mechanics represents the zero thermal noise limit of a broader stochastic theory capable of describing both wavefunction decay and the measurement process, thus achieving a comprehensive, self-consistent quantum theory. From this perspective, SQHM suggests that the term "probabilistic wave" in quantum mechanics is inaccurately introduced, stemming more from the inherent probabilistic nature of the measurement process than from its mathematical formulation. This approach acknowledges a curved spacetime characterized by stochastic fluctuations in both curvature and geodesic distance. Both SQHM and Bohm's theory restore the principle of determinism in quantum theory, leading to the wavefunction being understood as a "state wave" and explaining the probabilistic outcomes as resulting from an associated stochastic phenomenon (namely the GBN or the pilot wave, respectively).

The SQHM suggests that quantum mechanics serves as the deterministic boundary of a broader theory encompassing quantum stochasticity. The SQHM illustrates that in an ideal quantum deterministic universe, complete decoupling of its components—namely the system and the measuring apparatus—is unattainable. Rather, this scenario is only achievable within a self-fluctuating quantum system, where the interaction of the quantum potential, responsible for quantum entanglement, extends over a finite distance, thereby allowing for a macroscopic classical behavior on a large scale.

Therefore, in a hypothetical quantum universe devoid of fluctuating spacetime, measuring procedures are impracticable. Conversely, in a macroscopic classical realm formed through quantum decoherence triggered by fluctuating gravitational forces, measurement becomes viable.

While Bohm's theory attributes measurement uncertainty to an indeterminable pilot wave, the SQHM attributes it to fluctuating gravitational dark energy, which, due to its elusive nature, practically assumes the form of a genuine hidden variable. In this context, the author demonstrates that the hidden variable parameter of Santilli's IsoRedShift Mechanics arises from minor disturbances to quantum mechanics induced by fluctuating gravitational dark energy [14].

Moreover, it's important to note that the SQHM solves the enduring query concerning the existence of pre-measurement reality, thus bolstering Einsteinian realism. At this point, the SQHM introduces a nuanced advancement: the concept that the universe can undergo self-decay via macroscopic-scale decoherence, leading to the formation of stable macroscopic states. These states, remaining stable despite fluctuations, establish a continuous reality that predates measurement. Conversely, the Copenhagen interpretation

contends that the system only transitions into a stable state through the act of measurement, thereby establishing a continuous reality over time. Consequently, within this paradigm, it remains uncertain whether a continuous reality prevails prior to measurement.

2. The Stochastic Madelung Approach

The Madelung quantum hydrodynamic representation transforms the Schrodinger equation [15-17]

$$-i\hbar\partial_t\psi = \left(\frac{\hbar^2}{2m}\partial_i\partial_i - V_{(q)} \right)\psi \quad (1)$$

for the complex wave function $\psi = |\psi| e^{-\frac{iS}{\hbar}}$, into two equations of the real variables, $|\psi|^2$ and $p_i = \partial_i S_{(q,t)}$: the conservation equation for the mass density $|\psi|^2$

$$\partial_t |\psi|^2 + \partial_i (|\psi|^2 \dot{q}_i) = 0 \quad (2)$$

and the motion equation for the momentum,

$$\frac{\dot{p}}{m} = \ddot{q}_{j(t)} = -\frac{1}{m}\partial_j (V_{(q)} + V_{qu(n)}) \quad (3)$$

where $S_{(q,t)} = -\frac{\hbar}{2} \ln \frac{\psi}{\psi^*}$ and where

$$V_{qu} = -\frac{\hbar^2}{2m} \frac{1}{|\psi|} \partial_i \partial_i |\psi|. \quad (4)$$

These equations were derived by Madelung in a flat space. However, in the general relativity framework, the fluctuating energy content of gravitational background noise (GBN) causes fluctuations in the metric tensor, leading to local stochastic variations in mass density. These mass density fluctuations can be explained by recognizing that gravitons are metric fluctuations that cause space itself to oscillate. This oscillation contracts and elongates distances, similar to the effects observed with gravitational waves. As a result, when a mass element experiences a change in distance, its density decreases or increases accordingly. As demonstrated below, this causes the quantum potential to generate a stochastic force, thereby expanding the Madelung hydrodynamic analogy into a quantum-stochastic framework.. As shown in [12] the stochastic quantum hydrodynamic model (SQHM) is defined on the following assumptions:

1. The additional mass density generated by the GBN are described by the wave function ψ_{gbn} with density $|\psi_{gbn}|^2$;

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2. The associated energy density E of GBN is proportional to $|\psi_{gbn}|^2$;
3. The additional mass m_{gbn} is defined by the identity $E = m_{gbn}c^2 |\psi_{gbn}|^2$
4. The additional mass is assumed to not interact with the mass of the physical system (since the gravitational interaction is sufficiently weak to be disregarded).

Under this assumption, the wave function of the overall system ψ_{tot} reads

$$\psi_{tot} \cong \psi \psi_{gbn} \quad (5)$$

Moreover, given that the energy density of the GBN E is quite small, the mass density $m_{gbn} |\psi_{gbn}|^2$ is presumed to be significantly smaller than the body mass density $m |\psi|^2$ typically encountered in physical problems. Hence, considering the mass m_{gbn} much smaller than the mass of the system and assuming in Equation (3-4) $m_{tot} = m_{gbn} + m \cong m$, the overall quantum potential can be expressed as follows

$$\begin{aligned} V_{qu(n_{tot})} &= -\frac{\hbar^2}{2m_{tot}} |\psi|^{-1} |\psi_{gbn}|^{-1} \partial_i \partial_i |\psi| |\psi_{gbn}| = \\ &= -\frac{\hbar^2}{2m} \left(|\psi|^{-1} \partial_i \partial_i |\psi| + |\psi_{gbn}|^{-1} \partial_i \partial_i |\psi_{gbn}| \right) \quad (6) \\ &\quad + 2 |\psi|^{-1} |\psi_{gbn}|^{-1} \partial_i |\psi_{gbn}| \partial_i |\psi| \end{aligned}$$

By analyzing the variation in quantum potential energy generated by each Fourier component of mass fluctuation and applying the Maxwell-Boltzmann law, we can derive the spectrum of this noise and subsequently determine its correlation function. $G(\lambda)$.

To accomplish this, let's examine the fluctuating mass density with a wavelength λ

$$|\psi_{gbn}|^2_{(\lambda)} \propto \cos^2 \frac{2\pi}{\lambda} q \quad (7)$$

related to the wave function of the mass fluctuation

$$\psi_{gbn(\lambda)} \propto \pm \cos \frac{2\pi}{\lambda} q. \quad (8)$$

By inserting (7-8) in (6), the energy fluctuations of the quantum potential

$$\delta \bar{E}_{qu} = \int_V n_{tot(q,t)} \delta V_{qu(q,t)} dV, \quad (9)$$

following the procedure in reference [12], reads:

$$\delta \bar{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m} \left(\frac{2\pi}{\lambda} \right)^2 \quad (10)$$

that, in 3D space, read

$$\delta \bar{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m} \sum_i (k_i)^2 = \frac{\hbar^2}{2m} |k|^2 \quad (11)$$

The result shown in Equation (11) indicates that the energy arising from fluctuations in mass density increases inversely with the square of the wavelength λ . Moreover, since fluctuations in the quantum potential with extremely short wavelengths for $\lambda \rightarrow 0$ diverge they can lead to a finite contribution even when the noise amplitudes approach zero (i.e., $T \rightarrow 0$). This situation raises concerns regarding the achievement of the deterministic zero noise limit (2-4) that represents quantum mechanics.

Moreover, since fluctuations in the quantum potential with extremely short wavelengths diverge, they can result in a finite contribution even when the noise amplitudes approach zero (i.e., $T \rightarrow 0$). This situation raises concerns about reaching the deterministic zero noise limit that characterizes quantum mechanics.

This happens because the quantum potential's output, due to its second derivative, depends on the correlation distance of the noise. Therefore, to nullify the fluctuations in the quantum potential and achieve convergence to the deterministic limit of quantum mechanics for $T \rightarrow 0$, a condition on the correlation function $G(\lambda)$ of the quantum potential noise must be established for $\lambda \rightarrow 0$ [18].

Deriving the shape of the correlation function $G(\lambda)$ involves complex stochastic calculations [18], which can be obtained by considering the probability of uncorrelated fluctuations occurring at increasingly shorter distances.

A simpler and more direct method for calculating $G(\lambda)$ is through the noise spectrum, represented as $S(k)$, as outlined in [12].

$$S_{(k)} \propto \text{probability}_{(k=\frac{2\pi}{\lambda})} = \exp \left[-\frac{\delta \bar{E}_{qu(\lambda)}}{kT} \right] = \exp \left[-\left(\frac{k\lambda_c}{2} \right)^2 \right] \quad (12)$$

In Equation (12), k represents the Boltzmann constant, and T stands for the temperature (mean amplitude) of mass density fluctuations. It's important to highlight that Equation (12) displays a non-white characteristic, where the probability of wavelengths λ smaller than λ_c quickly diminishes toward zero. From (12), $G(\lambda)$ reads [12]:

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$$G_{(\lambda)} \propto \int_{-\infty}^{+\infty} \exp[ik\lambda] S_{(k)} dk \propto \int_{-\infty}^{+\infty} \exp[ik\lambda] \exp\left[-\left(k \frac{\lambda_c}{2}\right)^2\right] dk$$

$$\propto \frac{\pi^{1/2}}{\lambda_c} \exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right] \quad (13)$$

where

$$\lambda_c = \sqrt{2} \frac{\hbar}{(mkT)^{1/2}}. \quad (14)$$

At least for $\sqrt{2}$, λ_c represents the De Broglie length. The expression (12) reveals that uncorrelated mass density fluctuations are suppressed on increasingly shorter distances than λ_c . As a result, we reveal a novel function of the quantum potential within an open system: progressively damping fluctuations (due to their rapidly escalating energy) on a microscopic level. This clarifies the empirical observation that quantum mechanics governs the micro-scale.

This probability-mediated suppression enables the proposition of conventional quantum mechanics as the zero-noise 'deterministic' limit of the SQHM). Moreover, since this phenomenon pertains to systems with a physical length substantially smaller than the De Broglie length λ_c , directly extending quantum mechanical behavior to macroscopic large-scale scenarios is not feasible at $T > 0$. This is because, within the framework of SQHM, the De Broglie length λ_c disrupts scale invariance.

In the presence mass density fluctuations, $|\psi|^2$ becomes a stochastic function we name as $\tilde{n}$, where $\lim_{T \rightarrow 0} \tilde{n} \rightarrow |\psi|^2$. On this, we can conceptually separate $\tilde{n}$ into two parts: $\tilde{n} = \bar{n} + \delta n$, where δn is the fluctuating part and $\bar{n}$ is the regular part obeying to the condition $\lim_{T \rightarrow 0} \tilde{n} = \lim_{T \rightarrow 0} \bar{n} = |\psi|^2$.

Moreover, the features of the Madelung quantum potential in the presence of stochastic noise, can be determined by positing it as comprising a regular component $\overline{V_{qu(\tilde{n})}}$ (to be defined) along with a fluctuating component V_{st} , such as

$$V_{qu(\tilde{n})} = -\frac{\hbar^2}{2m} \tilde{n}^{-1/2} \partial_i \partial_i \tilde{n}^{1/2} = \overline{V_{qu(\tilde{n})}} + V_{st} \quad (15)$$

where the stochastic part of the quantum potential V_{st} results in force noise

$$-\partial_i V_{st} = m \varpi_{(q,t,T)} \quad (16)$$

leading to the stochastic motion equation

$$\ddot{q}_{j(t)} = -\frac{1}{m} \partial_j \left(V_{(q)} + \overline{V_{qu(\tilde{n})}} \right) + \varpi_{(q,t,T)}. \quad (17)$$

Moreover, the regular part $\overline{V_{qu(\tilde{n})}}$ for microscopic systems ($\frac{\mathcal{L}}{\lambda_c} \ll 1$), without loss of generality, can be reorganized as

$$\overline{V_{qu(\tilde{n})}} = -\frac{\hbar^2}{2m} \left(\tilde{n}^{-1/2} \partial_i \partial_i \tilde{n}^{1/2} \right) = -\frac{\hbar^2}{2m} \frac{1}{\rho^{1/2}} \partial_i \partial_i \rho^{1/2} + \overline{\Delta V} = V_{qu(\rho)} + \overline{\Delta V} \quad (18)$$

leading to the motion equation

$$\ddot{q}_{j(t)} = -\frac{1}{m} \partial_j \left(V_{(q)} + V_{qu(\rho)} + \overline{\Delta V} \right) + \varpi_{(q,t,T)} \quad (19)$$

where $\rho_{(q,t)}$ represents the probability mass density function (PMD) associated with the stochastic process (17) [19], obeying to the condition $\lim_{T \rightarrow 0} \rho_{(q,t)} = \lim_{T \rightarrow 0} \tilde{n} = \lim_{T \rightarrow 0} \bar{n} = |\psi|^2$.

For the sufficiently general case to be practically relevant, it can be assumed that the correlation function of $\varpi_{(q,t,T)}$ possesses zero correlation time, is isotropic in space, and is independent among different coordinates, taking the form

$$\lim_{T \rightarrow 0} \langle \varpi_{(q_\alpha,t)}, \varpi_{(q_\beta+\lambda,t+\tau)} \rangle \approx \langle \varpi_{(q_\alpha)}, \varpi_{(q_\beta)} \rangle_{(T)} G(\lambda) \delta(\tau) \delta_{\alpha\beta} \quad (20)$$

with

$$\lim_{T \rightarrow 0} \langle \varpi_{(q_\alpha)}, \varpi_{(q_\beta)} \rangle_{(T)} = 0 \quad . \quad (21)$$

Furthermore, given that for microscopic systems ($\frac{\mathcal{L}}{\lambda_c} \ll 1$) it follows that

$$\lim_{T \rightarrow 0} G(\lambda) \propto \frac{1}{\lambda_c} = \frac{1}{\hbar} \sqrt{\frac{mkT}{2}} \quad (22)$$

and that

$$\lim_{T \rightarrow 0} \langle \varpi_{(q_\alpha,t)}, \varpi_{(q_\beta+\lambda,t+\tau)} \rangle \approx \langle \varpi_{(q_\alpha)}, \varpi_{(q_\beta)} \rangle_{(T)} \frac{1}{\hbar} \sqrt{\frac{mkT}{2}} \delta(\tau) \delta_{\alpha\beta} . \quad (23)$$

Therefore, equation (17) takes the stochastic form of the Markov process [19]

$$\ddot{q}_{j(t)} = -\kappa \dot{q}_{j(t)} - \frac{1}{m} \frac{\partial \left(V_{(q)} + V_{qu(\rho)} \right)}{\partial q_j} + \kappa D^{1/2} \xi_{(t)}. \quad (24)$$

where

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$$D^{1/2} = \left(\frac{\mathcal{L}}{\lambda_c} \right) \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2} = \sqrt{\gamma_D} \frac{\mathcal{L}}{2} \sqrt{\frac{kT}{2\hbar}} \quad (25)$$

where γ_D is a non-zero pure number.

In this case, $\rho_{(q,t)}$ is the probability mass density determined by the probability transition function (PTF) $P(q,z|t,0)$ of the Markov process (24) [19] through the relation $\rho(q,t) = \int P(q,z|t,0) \rho_{(z,0)} d^{6N}z$ where $P(q,z|t,0)$ obeys to the Smoluchowski conservation equation [19]

$$P(q,q_0|t+\tau,t_0) = \int_{-\infty}^{\infty} P(q,z|\tau,t) P(z,q_0|t-t_0,t_0) d^{6N}z. \quad (26)$$

Finally, in summary, for the complex field

$$\psi = \rho^{1/2} e^{-\frac{iS}{\hbar}} \quad (27)$$

the SQHM system of equation reads:

$$\rho(q,t) = \int P(q,z|t,0) \rho_{(z,0)} d^r z \quad (28)$$

$$m\dot{q}_i = p_i = \partial_i S_{(q,t)}, \quad (29)$$

$$\ddot{q}_{j(t)} = -\kappa \dot{q}_{j(t)} - \frac{1}{m} \frac{\partial (V_{(q)} + V_{qu(\rho)})}{\partial q_j} + \kappa D^{1/2} \xi_{(t)} \quad (30)$$

$$V_{qu} = -\frac{\hbar^2}{2m} \frac{1}{\rho^{1/2}} \partial_i \partial_i \rho^{1/2}. \quad (31)$$

where

$$S_{(q,t)} = -\frac{\hbar}{2} \ln \frac{\psi}{\psi^*}. \quad (32)$$

3. Emerging classical behavior on large size systems

When attempting to nullify the quantum potential manually in the equations of motion for quantum hydrodynamics (2-4), the classical equation of motion emerges. However, despite the apparent validity of this assertion, such a manipulation lacks mathematical rigor as it alters the fundamental characteristics of the quantum hydrodynamic equations. Specifically, this process results in the removal of stationary configurations, such as eigenstates, as the counteracting force of the quantum potential against the Hamiltonian force—which establishes their stationary mass density distribution condition—is eliminated. Consequently, even a minor quantum potential is not disregarded in conventional quantum mechanics, as depicted by the zero-noise "deterministic" limit of SQHM (2-4).

Conversely, in the stochastic generalization, it is feasible to appropriately disregard the quantum potential in (3,24) when its force is significantly smaller than the force noise, such as $|\frac{1}{m}\partial_i V_{qu(\rho)}| \ll |\varpi_{(q,t,T)}|$ that by (25) leads to condition

$$|\frac{1}{m}\partial_i V_{qu(\rho)}| \ll \kappa \left(\frac{\mathcal{L}}{\lambda_c} \right) \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2} = \kappa \left(\frac{\mathcal{L}\sqrt{mkT}}{2\hbar} \right) \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2}, \quad (38)$$

and hence, in a coarse-grained description with elemental cell side Δq , to

$$\lim_{q \rightarrow \Delta q} |\partial_i V_{qu(\rho)}| \ll m\kappa \sqrt{\gamma_D} \frac{\mathcal{L}}{2} \sqrt{\frac{kT}{2\hbar}}, \quad (39)$$

where $\mathcal{L}$ is the physical length of the system.

It's important to note that, although the noise $\varpi_{(q,t,T)}$ has a zero mean, the mean of the fluctuations in the quantum potential, denoted as $\bar{V}_{st(n,S)} \cong \kappa S$, is not null. This non-null mean gives rise to the dissipative force $-\kappa \dot{q}_{(t)}$ in equation (24). As a result, the stochastic sequence of noise inputs disrupts the coherent evolution of the quantum superposition of states, leading them to decay to a stationary mass density distribution with $\dot{q}_{(t)} = 0$.

Furthermore, considering that the stochastic noise

$$\left(\frac{\mathcal{L}}{\lambda_c} \left(\gamma_D \frac{\hbar}{2m} \right)^{1/2} \right)^2 \ll \epsilon \quad (40)$$

grows with the size of the system, for macroscopic systems (i.e., $\frac{\mathcal{L}}{\lambda_c} \rightarrow \infty$), condition (38) is satisfied if

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$$\lim_{\frac{q}{\lambda_c} \rightarrow \infty \left(\frac{\mathcal{L}}{\lambda_c} = \infty \right)} \left| \frac{1}{m} \partial_i V_{qu(\eta(q))} \right| < \infty . \quad (41)$$

Furthermore, for achieving a comprehensive depiction without quantum correlations in any large-scale system of physical length $\mathcal{L}$, a more stringent criterion can be imposed, such as

$$\lim_{\frac{q}{\lambda_c} \rightarrow \infty} \left| \frac{1}{m} \partial_i V_{qu(\rho(q))} \right| = \lim_{\frac{q}{\lambda_c} \rightarrow \infty} \frac{1}{m} \sqrt{\partial_i V_{qu(\rho(q))} \partial_i V_{qu(\rho(q))}} = 0 . \quad (42)$$

Therefore, since for linear systems

$$\lim_{q \rightarrow \infty} V_{qu(q)} \propto \frac{1}{q}, \quad (43)$$

from the standpoint of SQHM promptly follows that they do not lead to the macroscopic classical phase.

Generally speaking, as the Hamiltonian potential strengthens, the wave function localization increases, and the quantum potential behavior becomes stronger.

This comes out clearly by considering the MDD

$$|\psi\rangle \propto \exp[-P_{(q)}^2], \quad (44)$$

where $P_{(q)}^k$ is a polynomial of order k , since a finite range of quantum potential interaction is achieved for $k < \frac{3}{2}$.

Linear systems, as well as the ballistic Gaussian coherent states, characterized by $k = 2$, have the quantum potential that is relevant even at infinite distance.

Conversely, for gas phases where particles interact via the Lennard-Jones potential, whose long-distance wave function reads [20]

$$\lim_{r \rightarrow \infty} |\psi\rangle \propto \frac{1}{r^2}, \quad (45)$$

the quantum potential reads

$$\lim_{r \rightarrow \infty} V_{qu(\rho)} \cong \lim_{q \rightarrow \infty} \frac{\hbar^2}{2m} \frac{1}{|\psi|} \partial_r \partial_r |\psi| = \frac{1}{r^2} = \frac{\hbar^2}{m} a |\psi|^2 \quad (46)$$

leading to the quantum force

$$\lim_{r \rightarrow \infty} -\partial_r V_{qu(\rho)} = \lim_{q \rightarrow \infty} \frac{\hbar^2}{2m} \partial_r \frac{1}{|\psi|} \partial_r \partial_r |\psi| = \frac{\hbar^2}{2m} \partial_r r \partial_r \partial_r \frac{1}{r} = -2 \frac{\hbar^2}{m} \frac{1}{r^3} = 0, \quad (47)$$

that satisfying conditions (38, 42), can lead to large-scale classical behavior [12] in a sufficiently rarefied phase.

It's worth mentioning that in equation (46), the output of the quantum potential corresponds to the hard sphere potential within the "pseudo-potential Hamiltonian model" of the Gross-Pitaevskij equation [21-22], where $\frac{a}{4\pi}$ represents the boson-boson s-wave scattering length.

By observing that, to satisfy (42), it is sufficient to have that

$$\int_0^{\infty} r^{-1} \left| \frac{1}{m} \partial_i V_{qu(\rho(q))} \right|_{(r,\theta,\varphi)} dr < \infty \quad \forall \theta, \varphi, \quad (48)$$

the finite value of the range of interaction of the quantum potential λ_{qu} such as [12]

$$\lambda_{qu} = \lambda_c \frac{\int_0^{\infty} r^{-1} \left| \partial_i V_{qu(\rho(q))} \right|_{(r,\theta,\varphi)} dr}{\left| \partial_i V_{qu(\rho(q))} \right|_{(r=\lambda_c,\theta,\varphi)}} = \lambda_c I_{qu} \quad . \quad I_{qu} > 1 \quad (49)$$

warrants the existence of classical macroscopic phase.

Relation (49) provides a measure of the physical length associated with quantum non-local interactions.

It's worth noting that the quantum non-local interactions extend themselves up to distances on the order of the largest length between λ_{qu} and λ_c . Below λ_c , an even weaker quantum potential arises due to noise damping, while above it but below λ_{qu} , the quantum potential becomes strong enough to surpass the fluctuations, establishing a quantum-stochastic behavior

The reach of quantum non-local effects can be expanded by increasing λ_c , achievable by reducing the temperature or mass of the bodies (refer to equation (14)), or by enhancing λ_{qu} through a more potent Hamiltonian potential. In the latter scenario, for example, augmenting the linear range of Hamiltonian interaction between particles.

4. Maximum Precision in Measure of Conjugated Mechanical Variable in a Quantum System Subject to fluctuations

During the measurement process, the interaction concludes when the measuring apparatus is relocated to a significant distance from the system. Within the SQHM framework, this relocation is crucial and must exceed specified distances λ_c and λ_{qu} . After this relocation, the measuring apparatus is responsible for interpreting and managing the "interaction output." This typically involves a classical, irreversible process characterized by a distinct temporal progression, culminating in the determination of the macroscopic measurement result. As a result, the phenomena of decoherence and quantum decoupling play a crucial role in the measurement process. They enable the establishment of a large-scale classical framework, ensuring genuine quantum isolation between the measuring apparatus and the system, both before and after the measurement event. This quantum-isolated state, at both the initial and final stages, is of paramount importance in determining the temporal duration of the measurement and in collecting statistical data through a series of independent repeated measurements.

It's important to emphasize that, within the framework of SQHM, simply moving the measured system to an infinite distance before and after the measurement, as commonly practiced, does not fully ensure the independence of the system and the measuring apparatus if either condition $\lambda_c = \infty$ or $\lambda_q = \infty$ is met. Therefore, the existence of a macroscopic classical reality remains essential for carrying out the measurement process.

Since the quantum potential suppresses fluctuations with wavelengths smaller than the De Broglie length λ_c , any system fully adheres to conventional quantum mechanics within a physical length $L_q < L_c$, where subparts lack individual identities. Based on this, the independent observer (experimental apparatus), seeking information about the system, must maintain a separation distance larger than L_q both before and after the process.

Therefore, due to the finite speed of propagation of interactions and information, the process cannot be executed in a time shorter than

$$\Delta\tau_{min} > \frac{L_q}{c} \propto \frac{\lambda_c}{c} \propto \frac{2\hbar}{(2mc^2kT)^{1/2}}. \quad (57)$$

Furthermore, considering the Gaussian noise (20) with the diffusion coefficient proportional to kT , we find that the mean value of energy fluctuation is $\delta E_{(T)} = \frac{kT}{2}$ for

the degree of freedom. As a result, a scalar structureless particle, with mass $m \gg \frac{kT}{c^2}$, exhibits an energy variance ΔE of

$$\begin{aligned}\Delta E &\approx (\langle (mc^2 + \delta E_{(T)})^2 - (mc^2)^2 \rangle)^{1/2} \cong (\langle (mc^2)^2 + 2mc^2 \delta E - (mc^2)^2 \rangle)^{1/2} \\ &\cong (2mc^2 \langle \delta E \rangle)^{1/2} \cong (mc^2 kT)^{1/2}\end{aligned}\quad (58)$$

from which it follows that

$$\Delta E \Delta t > \Delta E \Delta \tau_{min} \propto \frac{(mc^2 kT)^{1/2} \lambda_c}{c} \propto \sqrt{2} \hbar, \quad (59)$$

It's worth mentioning that the product $\Delta E \Delta \tau$ remains constant, as the increase in energy variance with the square root of T precisely counterbalances the corresponding decrease in the minimum acquisition time τ . This result holds true when establishing the uncertainty relations between the position and momentum of a particle with mass m .

When we obtain information about the spatial position of a particle with precision ΔL , we essentially exclude the space beyond this distance from the quantum non-local interaction of the particle, thereby

$$L_q < \Delta L. \quad (60)$$

the variance Δp of its relativistic momentum $(p^\mu p_\mu)^{1/2} = mc$ reads

$$\begin{aligned}\Delta p &\approx (\langle (mc + \frac{\delta E_{(T)}}{c})^2 - (mc)^2 \rangle)^{1/2} \cong (\langle (mc)^2 + 2m \delta E - (mc)^2 \rangle)^{1/2} \\ &\cong (2m \langle \delta E \rangle)^{1/2} \cong (mkT)^{1/2}\end{aligned}\quad (61)$$

leading to the indeterminacy relation

$$\Delta L \Delta p > L_q (mkT)^{1/2} \propto \lambda_c (mkT)^{1/2} \propto \sqrt{2} \hbar \quad (62)$$

Equating (62) to the uncertainty value, such as

$$\Delta L \Delta p > L_q (2mkT)^{1/2} = \frac{\hbar}{2} \quad (63)$$

or

$$\Delta E \Delta t > \Delta E \Delta \tau_{min} = \frac{(2mc^2 kT)^{1/2} L_q}{c} = \frac{\hbar}{2}, \quad (64)$$

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It follows that $L_q = \frac{\lambda_c}{2\sqrt{2}}$ represents the physical length below which quantum entanglement is fully effective establishing the zero-thermal noise limit of the SQHM, specifically the realization of quantum mechanics.

As far as it concerns the *theoretical minimum uncertainty* of quantum mechanics, obtainable from the *minimum indeterminacy* (59, 62) in the limit of quantum mechanics ($T = 0$ and $\lambda_c \rightarrow \infty$) in the non-relativistic limit (), we have that

$$\Delta\tau_{min} = \frac{\lambda_c}{2c\sqrt{2}} \rightarrow \infty \quad (65)$$

$$\Delta E \cong (mc^2 kT)^{1/2} = \sqrt{2} \frac{\hbar c}{\lambda_c} \rightarrow 0, \quad (66)$$

$$L_q = \frac{\lambda_c}{2\sqrt{2}} \rightarrow \infty \quad (67)$$

$$\Delta p \cong (mkT)^{1/2} \cong \frac{\sqrt{2}\hbar}{\lambda_c} \rightarrow 0 \quad (68)$$

and therefore that

$$\Delta E \Delta t > \Delta E \Delta\tau_{min} = \frac{\hbar}{2} \quad (69)$$

$$\Delta L \Delta p > L_q (mkT)^{1/2} = \frac{\hbar}{2} \quad (70)$$

That constitutes the *minimum uncertainty* in quantum mechanics, obtained as the zero noise-limit of the SQHM.

An important point to consider is that the SQHM broadens the uncertainty principles to encompass all paired characteristics of four-dimensional space-time. Unlike traditional quantum mechanics, where determining the uncertainty between energy and time is unattainable due to the lack of a defined time operator..

Furthermore, it is interesting to note that in the relativistic limit of quantum mechanics ($T = 0$ and $\lambda_c \rightarrow \infty$), by the finite speed of light, the minimum acquisition time of information in the quantum limit reads

$$\Delta\tau_{min} = \frac{L_q}{c} \rightarrow \infty. \quad (71)$$

The outcome of equation (71) demonstrates that conducting a measurement within a completely deterministic quantum mechanical global system is impractical since it would require an infinite duration.

Considering that non-locality is confined to domains with physical lengths on the order of $\frac{\lambda_c}{2\sqrt{2}}$, and given that information about a quantum system cannot travel faster than the speed of light (otherwise violating the uncertainty principle), local realism holds true within the coarse-grained macroscopic physics, where domains of order of λ_c^3 reduce to a point.

The paradox of "spooky action at a distance" is limited to microscopic distances (smaller than $\frac{\lambda_c}{2\sqrt{2}}$), where quantum mechanics is described in the low-velocity limit, assuming $c \rightarrow \infty$ and $\lambda_c \rightarrow \infty$. This results in the seemingly instantaneous transmission of interaction over a distance.

It is also noteworthy that in the presence of noise, the measured indeterminacy undergoes a relativistic correction, as $\Delta E \cong (mc^2 kT \left(1 + \frac{kT}{4mc^2}\right))^{1/2}$, leading at $T > 0$:

$$\Delta E \Delta t > \frac{\hbar}{2} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} \quad (72)$$

and

$$\Delta L \Delta p > \frac{\hbar}{2} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} \quad (73)$$

This can become significant for light nuclear and elementary particles (with $m \rightarrow 0$) and at high temperature T .

5. Cosmological and high energy physics consequences

Within the framework of the SQHM, utilizing the measure indeterminacy relations and the maximum attainable velocity of the speed of light

$$\square \quad x \leq c, \quad (74)$$

it follows that

$$\square \quad x \geq \Delta x = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x}, \quad (75)$$

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leading to $\frac{\hbar}{2m\Delta x} \leq c$ and to

$$\Delta x > \frac{\hbar}{2mc} = \frac{R_c}{2} . \quad (76)$$

where R_c is the Compton's length.

Equation (76) unveils that the highest density of mass within a body is found within an elemental volume with a side length equivalent to half of its Compton wavelength.

This discovery carries profound implications for the formation of black holes (BHs). In order for a BH to form, all of the mass must be compressed within its gravitational radius R_g , leading to the following relationship:

$$R_g = \frac{2Gm}{c^2} > \frac{\Delta x}{2} = r_{\min} = \frac{R_c}{4} , \quad (77)$$

which further leads to the condition:

$$\frac{R_c}{4R_g} = \frac{\hbar}{8mcR_g} = \frac{\hbar c}{8m^2 G} = \frac{m_p^2}{m^2} < 1 \quad (78)$$

showing that the BH mass

$$m > \sqrt{\frac{\hbar c}{8G}} = \sqrt{\pi} m_p . \quad (79)$$

where $m_p = \sqrt{\frac{\hbar c}{8\pi G}}$ is the reduced Planck's mass.

This holds true in a vacuum at zero background temperature, If we consider positive temperature as in (72-73) it follows that

$$\Delta x \geq \Delta x = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x} \left(1 + \frac{kT}{4mc^2} \right)^{1/2} , \quad (80)$$

leads to $\frac{\hbar}{2m\Delta x} \left(1 + \frac{kT}{4mc^2} \right)^{1/2} \leq c$ and, consequently, to

$$\Delta x > \frac{\hbar}{2mc} \left(1 + \frac{kT}{4mc^2} \right)^{1/2} = \frac{R_c}{2} \left(1 + \frac{kT}{4mc^2} \right)^{1/2} . \quad (81)$$

which leads to the condition:

$$\begin{aligned} \frac{R_c}{4R_g} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} &= \frac{\hbar}{8mcR_g} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} \\ &= \frac{\hbar c}{8m^2 G} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} = \frac{m_p^2}{m^2} \left(1 + \frac{kT}{4mc^2}\right)^{1/2} < 1 \end{aligned} \quad (82)$$

with

$$m > \sqrt{\frac{\hbar c}{G}} \left(1 + \frac{kT}{16mc^2}\right) = \sqrt{\pi} m_p \left(1 + \frac{kT}{16mc^2}\right) \quad (83)$$

Form (83) we can see that in presence of thermal fluctuations, the black hole of Planck mass becomes unstable and more mass is necessary to stabilize the BH.

If we consider black holes of mass hugely larger than m_p , considered that they are constituted by N sub-units (particles of mass m_i), we have that the energy fluctuations results $\delta E_{(T)} = 3N \frac{kT}{2}$ and relation (83) reads

$$m > \sqrt{\pi} m_p \left(1 + \frac{NkT}{16mc^2}\right) \quad (84)$$

Applying this result to the pre-big-bang BH (PBBH), we can derive an order of magnitude of the temperature necessary at the big bang to make thermally unstable the PBBH letting to expelling matter outside.

Assuming the universe energy $m_{PBBH} c^2 \approx 10^{53}$, the PBBH made of N particles becomes unstable if

$$\left(1 + \frac{NkT}{16mc^2}\right) > \sqrt{\pi} \frac{m_p}{m} \quad (85)$$

and therefore, for a temperature such as

$$\frac{NkT}{16m_{PBBH} c^2} = \frac{NkT}{16 \times 10^{53}} > \frac{m_{PBBH} - m_p}{m_p} \approx \frac{m_{PBBH}}{m_p} \approx 4,6 \times 10^{44} \quad (86)$$

leading to the value

$$T > \frac{10^{120}}{N} \text{ } ^\circ\text{K}. \quad (87)$$

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If we assume that the elementary particles in the Primordial Black Hole were black holes of Planck mass, it follows that

$$N = \frac{m_{PBBH}}{m_p} \approx 10^{45} \quad (88)$$

we obtain,

$$\frac{kT}{16m_{PBBH}c^2} = \frac{kT}{16 \times 10^{53}} \gg 1 \quad (89)$$

$$T_{big-bang} > 10^{77} \text{ }^\circ\text{K} \quad (90)$$

while, if we assume that the particles weigh into the PBBH on the order of the elemental baryonic values, we have that:

$$T_{big-bang} > 10^{63} \text{ }^\circ\text{K} \quad (91)$$

this is larger than the Planck temperature necessary to vaporize a black hole of small Planck mass.

Moreover, in the case of high energy fluctuations, the Marcovian term $\delta E_{(T)} = 3N \frac{kT}{2}$ has to be substituted by the more general expression $\delta E_{(T)} = \frac{3NkT}{2} + \sum_{i=1}^{\infty} \alpha_i \left(\frac{3NkT}{2} \right)^{i+1}$ that for particles with internal structure reads

$$\begin{aligned} \delta E_{(T)} &= f_{s_0} \frac{3NkT}{2} + \sum_{i=1}^{\infty} f_{s_i} \alpha_i \left(\frac{3NkT}{2} \right)^{i+1} \\ &= \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^{i+1} \cong \sum_{i=0}^{h-1} \beta_i \left(\frac{3NkT}{2} \right)^{i+1} \end{aligned} \quad (92)$$

where f_{s_i} are structural factors. Therefore (72) becomes:

$$\Delta E \Delta t > \frac{\hbar}{2} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} \quad (93)$$

and

$$\Delta L \Delta p > \frac{\hbar}{2} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} \quad (94)$$

In this case it follows that

$$\Delta x \geq \Delta x = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2}, \quad (95)$$

leads to $\frac{\hbar}{2m\Delta x} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} \leq c$ and, consequently, to

$$\Delta x > \frac{\hbar}{2mc} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} = \frac{R_c}{2} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2}. \quad (96)$$

which leads to the condition:

$$\begin{aligned} \frac{R_c}{4R_g} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} &= \frac{\hbar}{8mcR_g} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} \\ &= \frac{\hbar c}{8m^2 G} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} = \frac{m_p^2}{m^2} \left(1 + \frac{3NkT}{4mc^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)^{1/2} < 1 \end{aligned} \quad (97)$$

with

$$\begin{aligned} m_{BH} &> \sqrt{\frac{\hbar c}{m_{BH}^2 G} \left(1 + \frac{3NkT}{16m_{BH}^2 c^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right)} \\ &= m_p \left(1 + \frac{3NkT}{16m_{BH}^2 c^2} \sum_{i=0}^{\infty} \beta_i \left(\frac{3NkT}{2} \right)^i \right) \end{aligned} \quad (98)$$

leading to

$$\frac{3NkT}{16m_{PBH}^2 c^2} \sum_{i=0}^{h-1} \beta_i \left(\frac{3NkT}{2} \right)^i > 1 \quad (99)$$

that, for BH of elemental baryonic particles, leads to

$$T_{big-bang} > 10^{\frac{63}{h}} \text{ } ^\circ\text{K} \quad (100)$$

where h is the polynomial degree of temperature series expansion (92). For quadratic dependence by temperature it follows that

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$$T_{big-bang} > \approx 3,3 \times 10^{31} \text{ }^\circ\text{K}, \quad (101)$$

while for cubic dependence we have

$$T_{big-bang} > \approx 10^{21} \text{ }^\circ\text{K} \quad (102)$$

The validity of the results (76) and subsequent ones are substantiated by the quantum-gravitational effects produced by the quantum mass distribution within spacetime [23-24]. This demonstration elucidates that when mass density is condensed into a sphere with a diameter equal to half the Compton length, it engenders a quantum potential force that is able to counter the compressive gravitational force within a black hole. Moreover, in presence of thermal fluctuations, for $T > 0$, the repulsive thermal force adds itself to the quantum potential one rendering the Planck's BH unstable and leading to its evaporation.

Regarding high-energy physics, the pure quantum realm is limited to roughly one-third of the De Broglie length. When the system experiences significant thermal energy fluctuations and the De Broglie length approaches its physical length, quantum mechanics begins to be influenced by environmental fluctuations.

From the standpoint of the SQHM, classical behavior begins to emerge at increasingly smaller dimensions, leading to deviations from quantum mechanical behavior. At higher energies and fluctuation amplitudes, it becomes necessary to account for perturbations to the quantum theory. By employing a semiempirical approach at the first order of approximation, this effect can be addressed by introducing a measurable parameter. In this regard, Santilli's IsoRedShift theory, which incorporates hidden variables, seems to provide a solid and dependable model for describing high-energy dynamics in the presence of significant fluctuations.

6. Discussion and Conclusions

The stochastic extension of Madelung quantum hydrodynamics effectively integrates mass density fluctuations into quantum equations, driven by gravitational background noise, which is a form of dark energy. This model adeptly tackles significant elements of the quantum-to-classical transition by introducing the concept of quantum potential interaction length alongside the De Broglie wavelength. Beyond this length, coherent quantum behavior and the evolution of the wavefunction are disrupted or profoundly changed.

The stochastic quantum hydrodynamic framework asserts that an external environment is not required, indicating that the stochastic behavior causing wavefunction collapse and macroscopic classical phenomena, consistent with Einsteinian realism, is an intrinsic aspect of physics in a fluctuating spacetime. The theory provides a coherent connection between the uncertainty principle and the constancy of light speed, corresponding with the finite speed of information transmission. In quantum mechanics subjected to fluctuations, the model derives the uncertainty relationship between energy and time,

shedding light on measurement processes that cannot be concluded within a finite time interval in a truly quantum global system.

Moreover, the model elucidates how uncertainty relations evolve in an open quantum system under fluctuations, presenting a framework to describe how quantum mechanics adapts in the contexts of nuclear and elementary particle physics and the behavior of high-energy black hole states. The model's self-consistency is demonstrated through its ability to depict the dynamics of wavefunction collapse and the measurement process within its mathematical structure. Additionally, the theory addresses the issue of Einsteinian determinism and pre-existing reality by showing that large-scale systems naturally transition into stable, decoherent classical states over time, without the necessity for an observer or measurement apparatus. The SQHM presented here is limited to the non-relativistic Madelung theory. The generalization to the relativistic representation of Klein-Gordon equation as well Dirac's one represent the possible future developments.

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