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Results on Expansion Maps in Fuzzy Menger Space via Property-(E.A) and (E.A)-like Property

Satyanna K ^{*}
and Srinivas V [†]

Abstract

The main goal of this paper is to establish two results in fuzzy menger space by using property-(E.A), (E.A) like property and occasionally weakly compatible mappings. Furthermore these results are justified with proper examples. These are generalization of the theorem proved by Diwan and others.

Keywords: Property-(E.A); (E.A)-Like property; Occasionally weakly compatible mappings; fuzzy menger space.

2020 AMS subject classifications: 54H25 , 47H10. ¹

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1 Introduction

Menger space is one of the metric space generalizations. It is coined by Menger and Sklar [1959]. After words the effort of Karl Menger and Sklar [1960] the topology is generated with the neighborhood concept in menger space. It is rapidly growing and attracted many researchers. After introducing the concept of fuzziness by Zadeh [1978] we get various definitions on fuzzy metric space among one definition is popular which is proposed by George and Veeramani [1994] et al. By inserting the probabilistic notion into fuzzy metric space leads to generation of fuzzy menger space coined by Shrivastava and Dhagat [2013] et al. which is well suited for investigating physical quantities in different fields like partial differential equations, random differential, integral equations and integro-differential equations. Because of various applications, usefulness and simplicity it becomes very popular tool to solve existence problems in pure applied sciences. Some results in fuzzy menger space generated by Anil Goyal [2015] et al. using the Property (E.A). Recently Vijayabaskerreddy and Srinivasi [2021] using (E.A)- Like property generated some outcomes of fixed point results in metric space. Pathak and Rashmi [2014] et al. used the concepts of (P-1) compatible mappings to produce some of the fixed points in fuzzy menger space. Some of these outcomes can be witnessed like Ruchi Singh and Dhagat [2015], Y. Singh and Devi [2015]. S.D. Diwan and Raja [2016] and others applying the concepts of expansion mappings as well as CLR_S - property extracted fixed points in fuzzy menger space.

In this paper we established two results in fuzzy menger space by employing the notions (E.A) -like property, occasionally weakly compatible mappings and justified by suitable examples. These are generalization of the theorem proved by S.D. Diwan and Raja [2016] and others.

2 Preliminaries

Definition 2.1. A non empty set $\mathcal{X}$ and a mapping F_α from $\mathcal{X} \times \mathcal{X}$ into the collection of all fuzzy distribution functions $F_\alpha \in \mathfrak{R}$, $\forall \alpha \in [0, 1]$ make up a fuzzy probabilistic metric space ($\mathcal{FPM}$ -space) $(\mathcal{X}, F_\alpha)$ S.D. Diwan and Raja [2016] and the fuzzy distribution function is expressed by $F_\alpha(p, q)$ and $F_{\alpha(a,b)}(\mu)$ is the expansion of $F_{\alpha(p,q)}$ at $\mu \in \mathfrak{R}$.

The function $F_{\alpha(p,q)} \forall \alpha \in [0, 1]$ is to satisfy the following properties

- (a) $F_{\alpha(p,q)}(\mu) = 1 \iff p = q$,
- (b) $F_{\alpha(p,q)}(0) = 0$
- (c) $F_{\alpha(p,q)}(\mu) = F_{\alpha(q,p)}(\mu)$
- (d) If $F_{\alpha(p,q)}(u_1) = 1$ and $F_{\alpha(q,r)}(u_2) = 1 \implies F_{\alpha(p,r)}(u_1 + u_2) = 1$
 $\forall p, q, r \in \mathcal{X}$ and $u_1, u_2 > 0$.

Definition 2.2. A binary relation $t_\phi: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is stated as t - norm S.D. Diwan and Raja [2016] if

- (i) $t_\phi(p, 1) = p, t_\phi(0, 0) = 0$
 - (ii) $t_\phi(p, q) = t_\phi(q, p)$
 - (iii) $t_\phi(p, t_\phi(q, r)) = t_\phi(t_\phi(p, q), r)$
 - (iv) If $r \geq p, s \geq q \implies t_\phi(r, s) \geq t_\phi(p, q)$
- for all $p, q, r, s \in [0, 1]$.

Definition 2.3. A fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$ Ruchi Singh and Dhagat [2015] is formed by $(\mathcal{X}, F_\alpha)\mathcal{FPM}$ -space, t_ϕ where t_ϕ is t - norm and satisfy triangle inequality

$$F_{\alpha(p,r)}(u_1 + u_2) \geq t_\phi(F_{\alpha(p,q)}(u_1), F_{\alpha(q,r)}(u_2)) \forall p, q, r \in \mathcal{X} \text{ and } u_1, u_2 > 0 \text{ and } \alpha \in [0, 1].$$

2.1 Example

If $(\mathcal{X}, \rho)$ is metric space then $F_\alpha: \mathcal{X} \times \mathcal{X} \rightarrow L$ given by

$$(i) F_{\alpha(p,q)} = H_{\alpha(x-\rho(p,q))}, p, q \in \mathcal{X} \quad \forall \alpha \in [0, 1].$$

and if $t_\phi: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is given by

(ii) $t_\phi(r, s) = \min\{r, s\}$, then $(\mathcal{X}, F_\alpha, t_\phi)$ forms fuzzy menger space. Further it is complete whenever $(\mathcal{X}, \rho)$ is complete.

Definition 2.4. In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$ two mappings

$\gamma, \delta: \mathcal{X} \rightarrow \mathcal{X}$ are compatible S.D. Diwan and Raja [2016]

if $F_{\alpha(\gamma\delta a_n, \delta\gamma a_n)}(t_\phi) \rightarrow 1 \quad \forall t_\phi > 0$

whenever $(a_n) \in \mathcal{X}$ in order for $\gamma a_n, \delta a_n \rightarrow \mu$ for some $\mu \in \mathcal{X}$.

Definition 2.5. Let $(\mathcal{X}, F_\alpha, t_\phi)$ is fuzzy menger space.

Then the mappings $\gamma, \delta: \mathcal{X} \rightarrow \mathcal{X}$ satisfy

(a) CLR_δ -property S.D. Diwan and Raja [2016] if $\exists (c_n) \in \mathcal{X}$ in order for $\gamma c_n, \delta c_n \rightarrow \delta \eta$ for some $\eta \in \mathcal{X}$

(b) the (E.A) like property Vijayabaskerreddy and Srinivasi [2021] if there exist-
ing a sequence $(c_n) \in \mathcal{X}$ in order for $\gamma c_n, \delta c_n \rightarrow \eta$ where $\eta \in \gamma(\mathcal{X}) \cup \delta(\mathcal{X})$

(c) the property (E.A) Vijayabaskerreddy and Srinivasi [2021] if there exists a se-
quence $(c_n) \in \mathcal{X}$ in order for

$\gamma c_n, \delta c_n \rightarrow \eta$ for some $\eta \in \mathcal{X}$

From the definitions we can deduce that CLR_δ -property implies (E.A) like property and (E.A) like property implies the property (E.A). But the converse need not be true.

We are discussing through counter examples.

2.2 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
define the mapping $\gamma, \delta: \mathcal{X} \rightarrow \mathcal{X}$ as, where $\mathcal{X} = [0, 1]$

$$\gamma(a) = \sin^2(\pi x), \forall a \in [0, 1] \quad (1)$$

$$\delta(a) = \begin{cases} \cos^2(\pi x) & \text{if } a \in [0, 1] - \{\frac{1}{4}\} \\ \frac{1}{\sqrt{2}}, & \text{if } a = \frac{1}{4}. \end{cases} \quad (2)$$

From (1) and (2)

$$\gamma(\mathcal{X}) = [0, 1], \delta(\mathcal{X}) = [0, 1] - \{\frac{1}{2}\}.$$

choose $(c_m) = \frac{1}{4} - \frac{2}{m} \quad \forall m \geq 1$. Then

$$\lim_{m \rightarrow \infty} \gamma(c_m) = \lim_{m \rightarrow \infty} \gamma(\frac{1}{4} - \frac{2}{m}) = \lim_{m \rightarrow \infty} \sin^2(\pi(\frac{1}{4} - \frac{2}{m})) = \frac{1}{2}.$$

$$\lim_{m \rightarrow \infty} \delta(c_m) = \lim_{m \rightarrow \infty} \delta(\frac{1}{4} - \frac{2}{m}) = \lim_{m \rightarrow \infty} \cos^2(\pi(\frac{1}{4} - \frac{2}{m})) = \frac{1}{2}.$$

$$\lim_{m \rightarrow \infty} \gamma(c_m) = \lim_{m \rightarrow \infty} \delta(c_m) = \frac{1}{2}.$$

$$\frac{1}{2} \in \gamma(\mathcal{X}) \cup \delta(\mathcal{X}) \text{ but not } \frac{1}{2} \text{ in } \delta(\mathcal{X}).$$

Consequences the mappings γ, δ satisfy (E.A) like property but do not satisfy CLR_δ - property.

2.3 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$
define the mapping $\gamma, \delta: \mathcal{X} \rightarrow \mathcal{X}$ as, where $\mathcal{X} = [-1, 0]$

$$\gamma(a) = \begin{cases} \sin(\frac{\pi}{2}x) & \text{if } a \in [-1, 0] - \{\frac{-1}{2}\} \\ \frac{-1}{3}, & \text{if } a = \frac{-1}{2}. \end{cases} \quad (3)$$

$$\delta(a) = \begin{cases} -\cos(\frac{\pi}{2}x) & \text{if } a \in [-1, 0] - \{\frac{-1}{2}\} \\ \frac{-1}{2}, & \text{if } a = \frac{-1}{2}. \end{cases} \quad (4)$$

From (3) and (4)

$$\gamma(\mathcal{X}) = [-1, 0] - \{\frac{-1}{\sqrt{2}}\}, \delta(\mathcal{X}) = [-1, 0] - \{\frac{-1}{\sqrt{2}}\}$$

choose $(c_m) = \frac{-1}{2} + \frac{3}{m} \quad \forall m \geq 1$. Then

$$\lim_{m \rightarrow \infty} \gamma(c_m) = \lim_{m \rightarrow \infty} \gamma(\frac{-1}{2} + \frac{3}{m}) = \lim_{m \rightarrow \infty} \sin(\frac{\pi}{2}(\frac{-1}{2} + \frac{3}{m})) = -\frac{1}{\sqrt{2}}.$$

$$\lim_{m \rightarrow \infty} \delta(c_m) = \lim_{m \rightarrow \infty} \delta(\frac{-1}{2} + \frac{3}{m}) = \lim_{m \rightarrow \infty} \cos(\frac{\pi}{2}(\frac{-1}{2} + \frac{3}{m})) = -\frac{1}{\sqrt{2}}.$$

$$\lim_{m \rightarrow \infty} \gamma(c_m) = \lim_{m \rightarrow \infty} \delta(c_m) = -\frac{1}{\sqrt{2}} \in \mathcal{X}.$$

But $-\frac{1}{\sqrt{2}}$ not in $\delta(\mathcal{X}) \cup \gamma(\mathcal{X})$.

Consequences the mappings γ, δ satisfy (E.A) property but neither satisfy CLR_δ - property nor satisfy (E.A) like property

Note

(a) If $\delta(\mathcal{X})$ is closed then (E.A) property and CLR_δ - property are coincide.

(b) Further if $\delta(\mathcal{X})$ is closed as well as $\gamma(\mathcal{X})$ is closed then all the three properties coincide.

Definition 2.6. Two mappings $\gamma, \delta: \mathcal{X} \rightarrow \mathcal{X}$ are weakly compatible Y. Singh and Devi [2015] in fuzzy menger space if these are commuting at their coincidence points.

Definition 2.7. Two mappings $\gamma, \delta: X \rightarrow X$ are occasionally weakly compatible Y. Singh and Devi [2015] in fuzzy menger space if there is a coincidence point at which the mapping commutes.

The following example demonstrates that owc not necessarily weakly compatible.

2.4 Example

Define the mappings are $\gamma(a) = \sin^2(\pi x), \forall a \in [0, \frac{1}{2}]$.

$\delta(a) = \frac{1}{\sqrt{2}} \sin(\pi x), \forall a \in [0, \frac{1}{2}]$.

Then the mappings coincide at $a = \frac{1}{4}, 1, 0$ however at $a = \frac{1}{4}$

$\gamma(\frac{1}{4}) = \sin^2(\pi \frac{1}{4}) = \frac{1}{2}$.

$\delta(\frac{1}{4}) = \frac{1}{\sqrt{2}} \sin(\pi \frac{1}{4}) = \frac{1}{2}$,

$\gamma(\frac{1}{4}) = \delta(\frac{1}{4}) = \frac{1}{2}$.

$\gamma\delta(\frac{1}{4}) = \gamma(\frac{1}{2}) = \sin^2(\pi \frac{1}{2}) = 1$,

$\delta\gamma(\frac{1}{4}) = \delta(\frac{1}{2}) = \frac{1}{\sqrt{2}} \sin(\pi \frac{1}{2}) = \frac{1}{\sqrt{2}}$.

At $a = 1$,

$\gamma(1) = \sin^2(\pi 1) = 0$.

$\delta(1) = \frac{1}{\sqrt{2}} \sin(\pi 1) = 0$,

$\gamma(1) = \delta(1) = 0$.

$\gamma\delta(1) = \gamma(0) = 0$,

$\delta\gamma(1) = \delta(0) = \sin(\pi 0) = 0$.

Resulting $\gamma\delta(1) = \delta\gamma(1)$ but $\gamma\delta(\frac{1}{4}) \neq \delta\gamma(\frac{1}{4})$. Hence the result.

The following theorem proved by S.D. Diwan and Raja [2016] et al.

Theorem 2.1. “Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t)$ suppose that

(i) (A, S) satisfies CLR_S -property or (B, T) satisfies CLR_T -property,

(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$

(iii) (A, S) and (B, T) are weakly compatible

(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$

(v) there exists a constant $k > 1$ such that

$$F_{\alpha(Ax, By)}(kt_\phi) \leq F_{\alpha(Sx, Ty)}(t_\phi)$$

$\forall x, y \in \mathcal{X}$,for all $\alpha \in [0, 1]$ and $t_\phi > 0$.

Then A, B, S and T have a unique common fixed point in $\mathcal{X}$."

The above theorem generalized as following by using weaker conditions.

3 Main results

Theorem 3.1. Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that

- (i) (A, S) satisfied (E.A) property or (B, T) satisfies(E.A) property,
- (ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$
- (iii) (A, S) and (B, T) are occasionally weakly compatible
- (iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$
- (v) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax, By)}(kt_\phi) \leq F_{\alpha(Sx, Ty)}(t_\phi) \quad (5)$$

$\forall x, y \in \mathcal{X}$,for all $\alpha \in [0, 1]$ and $t_\phi > 0$.

Then A, B, S and T having unique common fixed point in $\mathcal{X}$.

Proof. Assume that the pair (A, S) satisfy (E.A)- Property so that $\exists(c_m) \in \mathcal{X}$ as

$$\lim_{m \rightarrow \infty} Ac_m = \lim_{m \rightarrow \infty} Sc_m = \mu, \mu \in \mathcal{X}. \quad (6)$$

By using $A(\mathcal{X}) \subseteq T(\mathcal{X})$ we can obtain sequence (d_m) in order for $Ac_m = Td_m$. Taking limit on both sides we get

$$\lim_{m \rightarrow \infty} Ac_m = \lim_{m \rightarrow \infty} Td_m \quad (7)$$

By (5)

$$F_{\alpha(Ac_m, Bd_m)}(kt_\phi) \leq F_{\alpha(Sc_m, Td_m)}(t_\phi) \quad (8)$$

as $m \rightarrow \infty$ using (6) and (7) in (8) we get

$$F_{\alpha(\mu, Bd_m)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1 \quad (9)$$

as $m \rightarrow \infty$ implies

$$\lim_{m \rightarrow \infty} Bd_m = \mu. \quad (10)$$

From (6),(7) and (10) we have

$$\lim_{m \rightarrow \infty} Ac_m = \lim_{m \rightarrow \infty} Sc_m = \lim_{m \rightarrow \infty} Td_m = \lim_{m \rightarrow \infty} Bd_m = \mu. \quad (11)$$

Results on Property-(E.A) & (E.A)- like properties

Assume $A(\mathcal{X})$ is closed sub set of $\mathcal{X}$.

Then $\mu = \lim_{m \rightarrow \infty} Ac_m \in A(\mathcal{X})$ and $A(\mathcal{X}) \subseteq T(\mathcal{X})$ implies $\mu \in T(\mathcal{X})$

Then there existing a point v in $\mathcal{X}$ as $\mu = Tv$. By (5)

$$F_{\alpha(Ac_m, Bv)}(kt_\phi) \leq F_{\alpha(Sc_m, Tv)}(t_\phi) \quad (12)$$

as $m \rightarrow \infty$ using (6) $\mu = Tv$ implies

$$F_{\alpha(\mu, Bv)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1 \quad (13)$$

This gives $\mu = Bv$ implies $\mu = Tv = Bv$.

$\mu = Bv \in B(\mathcal{X}) \subset S(\mathcal{X})$.

Then there existing w in $\mathcal{X}$ as $\mu = Tv = Bv = Sw$

Claim $Aw = Sw$. By (5)

$$F_{\alpha(Aw, Bv)}(kt_\phi) \leq F_{\alpha(Sw, Ty)}(t_\phi) = 1 \quad (14)$$

implies $Aw = Bv$ implies $Aw = Sw$.

Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \quad (15)$$

From (15) the pair (A, S) & (B, T) having coincidence points w and v respectively.

Further occasionally weakly compatibility of (A, S) & (B, T) resulting

$$A\mu = S\mu, B\mu = T\mu.. \quad (16)$$

By (5)

$$F_{\alpha(A\mu, B\mu)}(kt_\phi) \leq F_{\alpha(S\mu, T\mu)}(t_\phi). \quad (17)$$

Using (16) in (17) we get

$$F_{\alpha(A\mu, B\mu)}(kt_\phi) \leq F_{\alpha(A\mu, B\mu)}(t_\phi). \quad (18)$$

Then we have

$$A\mu = B\mu. \quad (19)$$

Claim $A\mu = \mu$. By (5) we have

$$F_{\alpha(A\mu, Bv)}(kt_\phi) \leq F_{\alpha(S\mu, Tv)}(t_\phi). \quad (20)$$

From (15) and (16)

$$F_{\alpha(A\mu, \mu)}(kt_\phi) \leq F_{\alpha(A\mu, \mu)}(t_\phi). \quad (21)$$

Then we have

$$A\mu = \mu. \quad (22)$$

From (16), (19) and (16) resulting

$$A\mu = S\mu = T\mu = B\mu = \mu. \quad (23)$$

Thus μ is combined fixed point for four self mappings.

In similar way when $S(\mathcal{X})$, $T(\mathcal{X})$ and $B(\mathcal{X})$ closed we can obtain fixed point.

Uniqueness

Let μ_1 be other similar point satisfy (23) implies

$$A\mu_1 = S\mu_1 = T\mu_1 = B\mu_1 = \mu_1. \quad (24)$$

By (5) we have

$$F_{\alpha(A\mu, B\mu_1)}(kt_\phi) \leq F_{\alpha(S\mu, T\mu_1)}(t_\phi) \quad (25)$$

From (23) and (24)

$$F_{\alpha(\mu, \mu_1)}(kt_\phi) \leq F_{\alpha(\mu, \mu_1)}(t_\phi) \quad (26)$$

Hence $\mu = \mu_1$

Consequently the four self mappings are having unique common fixed point in $\mathcal{X}$.

□

Now we validate our theorem by discussing suitable example.

3.1 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$

define the mapping A, B, S and $T: \mathcal{X} \rightarrow \mathcal{X}$ as, where $\mathcal{X} = (-3, 0]$

$$A(a) = B(a) = \begin{cases} -e^{1-|a|} & \text{if } a \in (-3, -1] \\ -1 - \frac{a}{2}, & \text{if } a \in (-1, 0] \end{cases} \quad (27)$$

$$S(a) = T(a) = \begin{cases} -e^{-(1+a)^2} & \text{if } a \in (-3, -1] \\ \frac{a}{3}, & \text{if } a \in (-1, 0]. \end{cases} \quad (28)$$

From (27) and (28)

$$A(\mathcal{X}) = [-1, -\frac{1}{e^2}] = B(\mathcal{X}) \& T(\mathcal{X}) = [-1, 0] = S(\mathcal{X})$$

This implies $A(\mathcal{X}) \subseteq T(\mathcal{X}) \& B(\mathcal{X}) \subseteq S(\mathcal{X})$.

Moreover the mappings A, S have coincidence the points at $a = -2, -1$.

At $a = -1$, $A(-1) = S(-1) = -1$ and $AS(-1) = SA(-1) = -1$.

But at $a = -2$, $A(-2) = S(-2) = -\frac{1}{e}$ and

$$AS(-2) = A(-\frac{1}{e}) = -1 + \frac{1}{2e}, SA(-2) = S(-\frac{1}{e}) = -\frac{1}{3e}.$$

Consequences both the pairs (A, S) & (B, T) satisfy OWC but are not weakly compatible. Define the sequence $(c_m) = -1 - \frac{3}{m} \quad \forall m \geq 1$. Then

Results on Property-(E.A) & (E.A)- like properties

$$\begin{aligned}\lim_{m \rightarrow \infty} A(c_m) &= \lim_{m \rightarrow \infty} A(-1 - \frac{3}{m}) = \lim_{m \rightarrow \infty} -e^{1-|-1-\frac{3}{m}|} = -1, \\ \lim_{m \rightarrow \infty} S(c_m) &= \lim_{m \rightarrow \infty} S(-1 - \frac{3}{m}) = \lim_{m \rightarrow \infty} -e^{-(1+(-1-\frac{3}{m}))^2} = -1, \\ \lim_{m \rightarrow \infty} A(c_m) &= \lim_{m \rightarrow \infty} S(c_m) = -1.\end{aligned}$$

This implies the mappings A, S satisfy Property-(E.A) and $S(\mathcal{X})$ is closed.

Further at point $a = -1$ the values are $A(-1) = B(-1) = S(-1) = T(-1) = -1$

Resulting the mappings A, B, S and T having unique common fixed point $a = -1$. Hence all the norms of the theorem satisfied.

Now we have one more generalization of the Theorem(2.1).

Theorem 3.2. *Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that*

- (i) *(A, S) satisfies (E.A) like property or (B, T) satisfies (E.A) like property,*
- (ii) *$A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$*
- (iii) *(A, S) & (B, T) are occasionally weakly compatible*
- (iv) *one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$*
- (v) *there exists a constant $k > 1$ such that*

$$F_{\alpha(Ax, By)}(kt_\phi) \leq F_{\alpha(Sx, Ty)}(t_\phi) \quad (29)$$

$\forall x, y \in \mathcal{X}$, for all $\alpha \in [0, 1]$ and $t_\phi > 0$.

Then A, B, S and T having unique common fixed point in $\mathcal{X}$.

Proof.

Case(i) Assuming that the pair (A, S) has (E.A) -Like property implies there existing a sequence $(c_m) \in \mathcal{X}$ as

$$\lim_{m \rightarrow \infty} Ac_m = \lim_{m \rightarrow \infty} Sc_m = \mu, \quad (30)$$

where $\mu \in A(\mathcal{X}) \cup S(\mathcal{X})$.

$\mu \in A(\mathcal{X}) \cup S(\mathcal{X}) \implies \mu \in A(\mathcal{X})$ or $\mu \in S(\mathcal{X})$.

Sub case (i) If $\mu \in A(\mathcal{X})$ and $A(\mathcal{X}) \subseteq T(\mathcal{X})$ implies $\mu \in T(\mathcal{X})$

Then there existing v in $\mathcal{X}$ as $\mu = Tv$. By (29)

$$F_{\alpha(Ac_m, Bv)}(kt_\phi) \leq F_{\alpha(Sc_m, Tv)}(t_\phi) \quad (31)$$

as $m \rightarrow \infty$ using (30) $\mu = Tv$ implies

$$F_{\alpha(\mu, Bv)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1 \quad (32)$$

This gives $\mu = Bv$ implies $\mu = Tv = Bv$.

$\mu = Bv \in B(\mathcal{X}) \subseteq S(\mathcal{X})$.

Then there exists w in X with $\mu = Tv = Bv = Sw$

Claim $Aw = Sw$. By (29)

$$F_{\alpha(Aw, Bv)}(kt_\phi) \leq F_{\alpha(Sw, Ty)}(t_\phi) = 1 \quad (33)$$

implies $Aw = Bv$ implies $Aw = Sw$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \quad (34)$$

From (34) the pairs (A,S) & (B,T) are having coincidence points w and v respectively.

Further occasionally weakly compatibility of (A,S) & (B,T) resulting

$$A\mu = S\mu, B\mu = T\mu.. \quad (35)$$

By (29)

$$F_{\alpha(A\mu, B\mu)}(kt_\phi) \leq F_{\alpha(S\mu, T\mu)}(t_\phi). \quad (36)$$

Using (34) in (35) we get

$$F_{\alpha(A\mu, B\mu)}(kt_\phi) \leq F_{\alpha(A\mu, B\mu)}(t_\phi). \quad (37)$$

By lemma we have

$$A\mu = B\mu. \quad (38)$$

Claim $A\mu = \mu$. By (29) we have

$$F_{\alpha(A\mu, Bv)}(kt_\phi) \leq F_{\alpha(S\mu, Tv)}(t_\phi). \quad (39)$$

From (34) and (35)

$$F_{\alpha(A\mu, \mu)}(kt_\phi) \leq F_{\alpha(A\mu, \mu)}(t_\phi). \quad (40)$$

By lemma we have

$$A\mu = \mu. \quad (41)$$

From(35), (38) and (41) resulting

$$A\mu = S\mu = T\mu = B\mu = \mu. \quad (42)$$

Thus μ is common fixed point for four self mappings.

Sub case (ii) If $\mu \in S(\mathcal{X})$ then there existing v in $\mathcal{X}$ as $\mu = Sw$. By (29)

$$F_{\alpha(Aw, Bc_m)}(kt_\phi) \leq F_{\alpha(Sw, Tc_m)}(t_\phi) \quad (43)$$

as $m \rightarrow \infty$ using (30) $\mu = Sw$ implies

$$F_{\alpha(Aw, \mu)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1. \quad (44)$$

Results on Property-(E.A) & (E.A)- like properties

This gives $\mu = Aw$ implies $\mu = Aw = Sw$.
 $\mu = Aw \in A(\mathcal{X}) \subseteq T(\mathcal{X})$.
 So that there existing v in $\mathcal{X}$ with $\mu = Tv = Aw = Sw$
 Claim $Bv = Tv$. By (29)

$$F_{\alpha(Aw, Bv)}(kt_\phi) \leq F_{\alpha(Sw, Tv)}(t_\phi) = 1 \quad (45)$$

implies $Aw = Bv$ implies $Bv = Tv$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \quad (46)$$

From here on words leads to sub case(i).

Consequently the four self mappings have unique common fixed point in $\mathcal{X}$.

Case(ii) Assuming that the pair (B,T) satisfy (E.A)- Like property so that there existing a sequence $(e_m) \in \mathcal{X}$ as

$$\lim_{m \rightarrow \infty} Be_m = \lim_{m \rightarrow \infty} Te_m = \mu, \quad (47)$$

$$\mu \in B(\mathcal{X}) \cup T(\mathcal{X}) \implies \mu \in B(\mathcal{X}) \text{ or } \mu \in T(\mathcal{X}).$$

Sub case (i) If $\mu \in B(\mathcal{X})$ and $B(\mathcal{X}) \subseteq S(\mathcal{X})$ implies $\mu \in S(\mathcal{X})$

Implies there existing v in $\mathcal{X}$ such that $\mu = Sw$. By (29)

$$F_{\alpha(Aw, Be_m)}(kt_\phi) \leq F_{\alpha(Sw, Te_m)}(t_\phi) \quad (48)$$

as $m \rightarrow \infty$ using (30) $\mu = Sw$ implies

$$F_{\alpha(Aw, \mu)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1 \quad (49)$$

This gives $\mu = Aw$ implies $\mu = Aw = Sw$.

$$\mu = Aw \in A(\mathcal{X}) \subseteq T(\mathcal{X}).$$

Then there existing w in $\mathcal{X}$ as $\mu = Tv = Aw = Sw$

Claim $Bv = Tv$. By (29)

$$F_{\alpha(Aw, Bv)}(kt_\phi) \leq F_{\alpha(Sw, Tv)}(t_\phi) = 1 \quad (50)$$

implies $Aw = Bv$ implies $Bv = Tv$. Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \quad (51)$$

Here on words leads to sub case(i)

Sub case (ii) If $\mu \in T(\mathcal{X})$ hence there existing w in $\mathcal{X}$ such that $\mu = Tv$. By (29)

$$F_{\alpha(Af_m, Bv)}(kt_\phi) \leq F_{\alpha(Sf_m, Tv)}(t_\phi) \quad (52)$$

as $m \rightarrow \infty$ using $\mu = Tv$ implies

$$F_{\alpha(\mu, Bv)}(kt_\phi) \leq F_{\alpha(\mu, \mu)}(t_\phi) = 1. \quad (53)$$

This gives $\mu = Bv$ implies $\mu = Bv = Tv$.

$\mu = Bv \in B(\mathcal{X}) \subset S(\mathcal{X})$.

Then there existing v in $\mathcal{X}$ as $\mu = Sw = Bv = Tv$

Claim $Aw = Sw$. By (29)

$$F_{\alpha(Aw, Bv)}(kt_\phi) \leq F_{\alpha(Sw, Tv)}(t_\phi) = 1 \quad (54)$$

implies $Aw = Bv$ implies $Aw = Sw$.

Therefore we have

$$\mu = Tv = Bv = Aw = Sw. \quad (55)$$

From here on words leads to the sub case (i).

□

Now our theory substantiated by following example.

3.2 Example

In fuzzy menger space $(\mathcal{X}, F_\alpha, t_\phi)$

define the mapping A, B, S and $T: \mathcal{X} \rightarrow \mathcal{X}$ as, where $\mathcal{X} = [-1, \frac{1}{2}]$

$$A(a) = B(a) = \begin{cases} -e^{-(1+a)} & \text{if } a \in [-1, 0] \\ -1, & \text{if } a \in (0, \frac{1}{2}] \end{cases} \quad (56)$$

$$S(a) = T(a) = \begin{cases} -e^{-(1+a)^3} & \text{if } a \in [-1, 0] \\ -(\frac{1}{2} - a), & \text{if } a \in (0, \frac{1}{2}]. \end{cases} \quad (57)$$

From (59) and (60)

$$B(\mathcal{X}) = A(\mathcal{X}) = [-1, -\frac{1}{e}], T(\mathcal{X}) = S(\mathcal{X}) = [-1, 0]$$

Hence $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$.

Moreover $a = 0, -1$ are the only coincidence points of the mappings A, S .

At the point $a = -1, A(-1) = S(-1) = -1$ and $AS(-1) = SA(-1) = -1$.

But at $a = 0, S(0) = A(0) = -\frac{1}{e}$ moreover

$$AS(0) = A(-\frac{1}{e}) = -e^{-(1-\frac{1}{e})},$$

$$SA(0) = S(-\frac{1}{e}) = -e^{-(1-\frac{1}{e})^3}.$$

Consequences these pairs (A, S) as well as (B, T) are OWC but are not weakly

compatible.

Define $(c_m) = -1 + \frac{1}{2m} \quad \forall \quad m \geq 1$. Then

$$\lim_{m \rightarrow \infty} A(c_m) = \lim_{m \rightarrow \infty} A(-1 + \frac{1}{2m}) = \lim_{m \rightarrow \infty} -e^{1-(-1+\frac{1}{2m})} = -1,$$

$$\lim_{m \rightarrow \infty} S(c_m) = \lim_{m \rightarrow \infty} S(-1 + \frac{1}{2m}) = \lim_{m \rightarrow \infty} -e^{-(1+(-1+\frac{3}{m}))^3} = -1.$$

$$\lim_{m \rightarrow \infty} A(c_m) = \lim_{m \rightarrow \infty} S(c_m) = -1.$$

This implies the mappings A, S satisfying (E.A) property and $A(\mathcal{X})$ is closed.

Further at $a = -1$ values are $A(-1) = S(-1) = B(-1) = T(-1) = -1$

Thus the four mappings A, B, S and T having unique common fixed point $a = -1$.

Hence all the conditions of the Theorem (3.2) satisfied.

4 Conclusion

We generated two results in fuzzy menger space by employing

(i) the pairs (A, S) or (B, T) are satisfying Property-(E.A) and both the pairs are OWC instead of the pairs (A, S) or (B, T) are satisfying CLR_S -property and both the pairs are weakly compatible in Theorem (3.1) .

(ii) The pairs (A, S) or (B, T) are satisfying (E.A)-Like property and both the pairs are OWC instead of the pairs (A, S) or (B, T) satisfies CLR_S -property and both the pairs are weakly compatible in Theorem (3.2).

These are generalizations of the theorem proved by Diwan et al. Moreover these two Theorems are supported with appropriate examples.

5 Remark

Further there is possibility to define the intimate mappings in fuzzy menger space and chance to obtain new results like as.

Definition 5.1. Let two mappings $\gamma, \delta: X \rightarrow X$ in fuzzy menger space. Then $\{A, S\}$ is said to be A-intimate if and only if

$$\beta F_{\alpha(ASx_n, Ax_n)}(t_\phi) \leq \beta F_{\alpha(SSx_n, Sx_n)}(t_\phi)$$

where $\beta = \text{limitsupremum or limitinfumum}$, (x_n) is a sequence such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \mu, \text{ for some } \mu \in X.$$

Theorem 5.1. Let A, B, S and T be four self-mappings on a fuzzy PM space $(\mathcal{X}, F_\alpha, t_\phi)$ suppose that

(i) (A, S) satisfied (E.A) property or (B, T) satisfies (E.A) property,

(ii) $A(\mathcal{X}) \subseteq T(\mathcal{X}), B(\mathcal{X}) \subseteq S(\mathcal{X})$

(iii) the pair (A, S) is S-intimate and (B, T) is T-intimate

(iv) one of the range of the mappings A, B, S or T is a closed subset of $\mathcal{X}$

(v) There exists a constant $k > 1$ such that

$$F_{\alpha(Ax,By)}(kt_{\phi}) \leq F_{\alpha(Sx,Ty)}(t_{\phi}) \quad (58)$$

$\forall x, y \in \mathcal{X}$, for all $\alpha \in [0, 1]$ and $t_{\phi} > 0$.

Then A, B, S and T having unique common fixed point in $\mathcal{X}$.

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Second Kind Chebyshev Wavelet Analysis of Abel's Integral Equations

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§

Abstract

This paper presents two approximations of the solution functions of Abel's integral equations belonging to classes $H^\alpha[0, 1)$, $H^\phi[0, 1)$ by $(\lambda^{k+1} - 1, M)^{th}$ partial sums of their second kind Chebyshev wavelet expansion in the interval $[0, 1)$, for $\lambda > 1$. These approximations are $E_{\lambda^{k+1}-1, M}^{(1)}(f)$, $E_{\lambda^{k+1}-1, M}^{(2)}(f)$. Chebyshev wavelets of the second kind were used to solve the Abel's integral equations. The Chebyshev wavelet of the second kind leads to a solution that is almost identical to their exact solution. This research paper's accomplishment in wavelet analysis is noteworthy.

Keywords: $H^\alpha[0, 1)$ class, $H^\phi[0, 1)$ class, Chebyshev wavelet of the second kind and Abel's integral equations.

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1 Introduction

In recent years, wavelets have become widely utilized across diverse fields such as biology, mathematics, and engineering. These are primarily used for

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signal analysis tasks such as time-frequency analysis and waveform segmentation, provide quick results for straightforward performance. Additionally, they utilize efficient organization of numerical data to compute solutions to integral and differential equations effectively. This capability presents a significant advantage in various scientific and engineering applications, where accurate and efficient computation is paramount. This research study investigates the properties associated with Hölder's classes $H^\alpha[0, 1)$ and $H^\phi[0, 1)$, presenting an innovative concept in the fields of mathematical sciences and physics. Numerous esteemed researchers have contributed to the understanding of functions within Hölder's class with order α ($0 < \alpha \leq 1$). Notable among them are Chui [1], Debnath [2], Kumar et al. ([4], [9]) Lal and Yadav ([7], [8]), Lepik [10], Meyer [11], Pandey et al. ([3], [12], [13], [14], [15], [16]), Ray and Sahu [17] and Sripathy et al. [18], whose extensive studies have paved the way for advancements in function approximation theory and related fields. Their work provides a solid foundation upon which this study builds, extending the understanding of Hölder's classes and their applications. This paper explores solving integral equations by employing the Chebyshev wavelet of the second kind within the interval $[0, 1)$, enhancing efficiency and broadening the toolkit for mathematical and computational scientists. For instance, the method for solving Abel's integral equations of the form:

$$y(t)\gamma(t) = f(t) + \int_0^t \frac{y(x)}{\sqrt{t-x}} dx \quad (1)$$

has been introduced using second kind Chebyshev wavelet.

This research paper aims to achieve several objectives. Firstly, it seeks to define the Chebyshev wavelet of the second kind within the interval $[0, 1)$, providing a clear understanding of its properties and characteristics. Secondly, it aims to define Hölder's classes $H^\alpha[0, 1)$ and $H^\phi[0, 1)$, portraying their significance in mathematical analysis. Thirdly, the paper aims to assess the approximation of function f within these classes, exploring the effectiveness of wavelet-based techniques in function approximation. Additionally, the procedure for solving Abel's integral equations utilizing the Chebyshev wavelet of the second kind have been discussed, offering insights into the practical application of wavelet methods in solving integral equations. Lastly, the paper intends to compare the exact solutions of Abel's integral equations with the approximate solutions obtained using various types of wavelets, including the Legendre wavelet and both kinds of Chebyshev wavelets, shedding light on the comparative efficacy of different wavelet approaches.

The subsequent sections of this paper are structured as follows: Section 2 outlines the definitions and properties of Hölder's class, the Chebyshev wavelet of the second kind, and their orthonormality. Section 3 examines the approximation of solution functions of Abel's integral equations using Chebyshev wavelets of

the second kind. Theorems are presented in Section 4, with their proofs in Section 5. Section 6 presents the algorithm for solving Abel's integral equations using the Chebyshev wavelet of the second kind within the interval $[0,1)$. Section 7 demonstrates the solutions of Abel's integral equations using this wavelet. Finally, the conclusions of this research paper are summarized in Section 8.

2 Definitions and Preliminaries

2.1 Function of Hölder's class

A function $f \in H^\alpha[0, 1)$, $0 < \alpha \leq 1$ if

$$f(x+t) - f(x) = O(|t|^\alpha), \quad \forall x+t, x, t \in [0, 1) \text{ (Titchmarsh[19])}.$$

2.2 Function of Hölder's class $H^\phi[0, 1)$

A function $f \in H^\phi[0, 1)$, $0 < \alpha \leq 1$ if

$$f(x+t) - f(x) = O(\phi(|t|)), \quad \forall x+t, x, t \in [0, 1),$$

where $\phi(t)$ is a positive monotonic increasing function of t such that $\phi(|t|) \rightarrow 0$ as $t \rightarrow 0$. If $\phi(t) = |t|^\alpha$ then $H^\phi[0, 1)$ class reduces to $H^\alpha[0, 1)$ class.

2.3 Chebyshev wavelet of the second kind

Chebyshev wavelet of the second kind is denoted by $\psi_{n,m}^{(\lambda)}$ and defined by

$$\psi_{n,m}^{(\lambda)}(t) = \begin{cases} \frac{2}{\sqrt{\pi}} \lambda^{\frac{k+1}{2}} U_m(2\lambda^{k+1}t - 2n - 1), & \text{if } t \in [\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}}); \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

where $n = 0, 1, 2, \dots, \lambda^{k+1} - 1$, $m = 0, 1, 2, \dots, M - 1$ and k, λ are positive integer. $U_m(t)$ are the second kind Chebyshev polynomial of degree m which are orthogonal with respect to weight function $\omega(t) = \sqrt{1-t^2}$ on $[-1,1]$ and satisfy $U_0(t) = 1$, $U_1(t) = 2t$ and $U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$, $m \in \{1, 2, 3, \dots\}$.

2.4 Orthonormality of the second kind Chebyshev wavelets

Proposition 2.1. $\{\psi_{n,m}^{(\lambda)}(t), n = 0, 1, 2, \dots, \lambda^{k+1} - 1, m = 0, 1, 2, \dots, M - 1\}$ forms an orthonormal set. i.e.

$$\langle \psi_{n,m}^{(\lambda)}, \psi_{n',m'}^{(\lambda)} \rangle_{\omega_{n,k}} = \begin{cases} 1, & \text{if } n = n' \text{ and } m = m'; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For $m = 0$ and $n = n'$,

$$\begin{aligned}
 \langle \psi_{n,0}^{(\lambda)}, \psi_{n,0}^{(\lambda)} \rangle_{\omega_{n,k}} &= \int_0^1 \psi_{n,0}^{(\lambda)}(t) \overline{\psi_{n,0}^{(\lambda)}(t)} \omega_{n,k}(t) dt \\
 &= \int_0^1 (\psi_{n,0}^{(\lambda)}(t))^2 \omega_{n,k}(t) dt \\
 &= \frac{4}{\pi} \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \lambda^{k+1} \omega_{n,k}(t) dt \\
 &= \frac{2}{\pi} \int_0^\pi \sin^2 \theta d\theta = 1, \quad 2\lambda^{k+1}t - 2n + 1 = \cos \theta.
 \end{aligned}$$

For $m \neq 0$,

1. For $m = m'$ and $n = n'$,

$$\begin{aligned}
 \langle \psi_{n,m}^{(\lambda)}, \psi_{n,m}^{(\lambda)} \rangle_{\omega_{n,k}} &= \int_0^1 \psi_{n,m}^{(\lambda)}(t) \overline{\psi_{n,m}^{(\lambda)}(t)} \omega_{n,k}(t) dt \\
 &= \int_0^1 (\psi_{n,m}^{(\lambda)}(t))^2 \omega_{n,k}(t) dt \\
 &= \frac{4}{\pi} \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \lambda^{k+1} U_m^2(2\lambda^{k+1}t - 2n + 1) \omega_{n,k}(t) dt \\
 &= \frac{2}{\pi} \int_0^\pi U_m^2(\cos \theta) \sin^2 \theta d\theta, \quad 2\lambda^{k+1}t - 2n + 1 = \cos \theta \\
 &= \frac{2}{\pi} \int_0^\pi \sin^2(m+1)\theta d\theta = 1.
 \end{aligned}$$

2. For $n \neq n'$,

$$\langle \psi_{n,m}^{(\lambda)}, \psi_{n',m}^{(\lambda)} \rangle_{\omega_{n,k}} = \int_0^1 \psi_{n,m}^{(\lambda)}(t) \overline{\psi_{n',m}^{(\lambda)}(t)} \omega_{n,k}(t) dt.$$

Since, $\psi_{n,m}^{(\lambda)}$ is defined in $[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}}) \subset [0, 1)$ and $\psi_{n',m}^{(\lambda)}$ is defined in $[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}}) \subset [0, 1)$, therefore if $n \neq n'$, then the intervals $[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}})$ and $[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}})$ are disjoint i.e.

$$\left[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}} \right) \cap \left[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}} \right) = \phi.$$

Therefore $\langle \psi_{n,m}^{(\lambda)}, \psi_{n',m}^{(\lambda)} \rangle_{\omega_{n,k}} = 0$ if $n \neq n' \forall m, m'$.

3. For $m \neq m'$,

$$\begin{aligned}\langle \psi_{n,m}^{(\lambda)}, \psi_{n,m'}^{(\lambda)} \rangle_{\omega_{n,k}} &= \int_0^1 \psi_{n,m}^{(\lambda)}(t) \overline{\psi_{n,m'}^{(\lambda)}(t)} \omega_{n,k}(t) dt \\ &= 0, \text{ by the orthogonality of Chebyshev polynomials.}\end{aligned}$$

□

3 Approximation of functions and Multiresolution analysis

This section contains the approximations of the solution functions of the Abel's integral equations and Multiresolution analysis by the second kind Chebyshev wavelet.

3.1 Approximation of functions

Since $\{\psi_{m,n}^{(\lambda)}\}$ forms an orthonormal basis for $L^2[0, 1)$, therefore the solution function $f \in L^2[0, 1)$ of the Abel's integral equations can be expressed into Chebyshev wavelet of second kind as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{m,n}^{(\lambda)}(t), \quad c_{n,m} = \langle f, \psi_{m,n}^{(\lambda)} \rangle_{\omega_{n,k}}. \quad (3)$$

The $(\lambda^{k+1} - 1, M)^{\text{th}}$ partial sum $(S_{\lambda^{k+1}-1, M} f)(t)$ of wavelet series (3) is given by

$$(S_{\lambda^{k+1}-1, M} f)(t) = \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{m,n}^{(\lambda)}(t) = C^T \psi^{(\lambda)}(t), \quad (4)$$

where $C = [c_{0,0}, c_{0,1}, \dots, c_{0,M-1}, c_{1,0}, \dots, c_{1,M-1}, \dots, c_{\lambda^{k+1}-1,0}, \dots, c_{\lambda^{k+1}-1,M-1}]^T$ and $\psi^{(\lambda)}(t) = [\psi_{0,0}^{(\lambda)}(t), \psi_{0,1}^{(\lambda)}(t), \dots, \psi_{0,M-1}^{(\lambda)}(t), \psi_{1,0}^{(\lambda)}(t), \dots, \psi_{1,M-1}^{(\lambda)}(t), \dots, \psi_{\lambda^{k+1}-1,0}^{(\lambda)}(t), \dots, \psi_{\lambda^{k+1}-1,M-1}^{(\lambda)}(t)]^T$.

The second kind Chebyshev wavelet approximation $E_{\lambda^{k+1}-1, M}(f)$ of f by $(\lambda^{k+1} - 1, M)^{\text{th}}$ partial sum $(S_{\lambda^{k+1}-1, M} f)$ of the wavelet series (3) is defined by

$$E_{\lambda^{k+1}-1, M}(f) = \min_{(S_{\lambda^{k+1}-1, M} f)} \|f - (S_{\lambda^{k+1}-1, M} f)\|_2 \quad (\text{Lal and Priya [5]}). \quad (5)$$

$E_{\lambda^{k+1}-1, M}(f)$ is said to be best approximation of the function f if $E_{\lambda^{k+1}-1, M}(f) \rightarrow 0$ as $k \rightarrow \infty, M \rightarrow \infty$ (Zygmund [20]).

3.2 Multiresolution analysis

The sequence of subspaces $\{V_n^{(\lambda)}\}_{n=0}^{\infty}$ of $L^2[0, 1)$ is defined by

$$V_n^{(\lambda)} = \bigoplus_{l=1}^{n-1} W_l^{(\lambda)} = W_1^{(\lambda)} \oplus W_2^{(\lambda)} \oplus \dots \oplus W_{n-1}^{(\lambda)};$$

where $W_n^{(\lambda)} = \text{clos}_{L^2[0,1)} \langle \psi_{n,m}^{(\lambda)} : m \in \mathbb{N} \cup \{0\} \rangle$, $n = 0, 1, 2, \dots, \lambda^{k+1} - 1$,

then $\{V_n^{(\lambda)}\}_{n=0}^{\infty}$ is a multiresolution analysis of $L^2[0, 1)$ (Chui [1]).

4 Theorems

The following theorems have been proved in this section:

4.1 Theorem 4.1

If a function $f \in H^\alpha[0, 1)$ and its second kind Chebyshev wavelet expansion be

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t)$$

having $(\lambda^{k+1} - 1, M)^{\text{th}}$ partial sums

$$(S_{\lambda^{k+1}-1, M} f)(t) = \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t),$$

then the second kind Chebyshev wavelet approximation of f is given by

$$E_{\lambda^{k+1}-1, M}^{(1)}(f) = \min \|f - \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t)\|_2 = O\left(\frac{1}{\lambda^{(k+1)\alpha} \sqrt{M}}\right), \quad M > 1.$$

4.2 Theorem 4.2

If a function $f \in H^\phi[0, 1)$ class such that $\phi(|t|) \rightarrow 0$ as $t \rightarrow 0$ then the second kind Chebyshev wavelet approximation of f by $(S_{\lambda^{k+1}-1, M} f)$ satisfies for $M > 1$

$$E_{\lambda^{k+1}-1, M}^{(2)}(f) = \min \|f - \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c'_{n,m} \psi_{n,m}^{(\lambda)}(t)\|_2 = O\left(\frac{1}{\lambda^{k+1} \sqrt{M}} \phi\left(\frac{1}{\lambda^{k+1}}\right)\right).$$

5 Proof of Theorems

5.1 Proof of Theorem 4.1

$$\begin{aligned}
 f(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t). \\
 c_{n,m} &= \langle f, \psi_{n,m}^{(\lambda)} \rangle_{\omega_{n,k}} \\
 &= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} f(t) \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \\
 &= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \left(f(t) - f\left(\frac{n}{\lambda^{k+1}}\right) \right) \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \\
 &\quad + f\left(\frac{n}{\lambda^{k+1}}\right) \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \\
 &= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \left(f(t) - f\left(\frac{n}{\lambda^{k+1}}\right) \right) \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \\
 |c_{n,m}| &\leq N \left(\frac{1}{\lambda^{k+1}} \right)^{\alpha} \left| \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \right|, \quad f \in H^{\alpha}[0, 1] \\
 &= N \left(\frac{1}{\lambda^{k+1}} \right)^{\alpha} \left| \int_0^{\pi} \frac{2}{\sqrt{\pi}} \lambda^{\frac{k+1}{2}} U_m(\cos \theta) \omega(\cos \theta) \frac{\sin \theta}{2\lambda^{k+1}} d\theta \right| \\
 &= \frac{N}{\pi \lambda^{(k+1)(\alpha+\frac{1}{2})}} \left| \int_0^{\pi} \sin(m+1)\theta \sin \theta d\theta \right|, \quad U_m(\cos \theta) = \frac{\sin(m+1)\theta}{\sin \theta} \\
 &= \frac{N}{2\pi \lambda^{(k+1)(\alpha+\frac{1}{2})}} \left| \int_0^{\pi} \cos m\theta - \cos(m+2)\theta d\theta \right| \\
 &\leq \frac{N}{2\pi \lambda^{(k+1)(\alpha+\frac{1}{2})}} \left(\left| \int_0^{\pi} \frac{1}{m} \frac{d}{d\theta} (\sin m\theta) d\theta \right| + \left| \int_0^{\pi} \frac{1}{m+2} \frac{d}{d\theta} (\sin(m+2)\theta) d\theta \right| \right) \\
 &\leq \frac{N}{2\pi \lambda^{(k+1)(\alpha+\frac{1}{2})}} \left(\frac{1}{m} \left(\max_{0 \leq \theta \leq \pi} (\sin m\theta) \right) + \frac{1}{m+2} \left(\max_{0 \leq \theta \leq \pi} (\sin(m+2)\theta) \right) \right) \\
 &= \frac{N}{\pi \lambda^{(k+1)(\alpha+\frac{1}{2})} (m+2)}. \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now,} \quad f(t) - (S_{\lambda^{k+1}-1, M} f)(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t) - \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t) \\
 &= \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=M}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t).
 \end{aligned}$$

$$\text{Then,} \quad (E_{\lambda^{k+1}-1, M}^{(1)}(f))^2 = \|f(t) - (S_{\lambda^{k+1}-1, M} f)(t)\|_2^2$$

$$\begin{aligned}
&= \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=M}^{\infty} |c_{n,m}|^2 \\
&\leq \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=M}^{\infty} \left(\frac{N}{\pi \lambda^{(k+1)(\alpha+\frac{1}{2})} (m+2)} \right)^2 \\
&= \frac{N^2}{\pi^2 \lambda^{2(k+1)\alpha}} \sum_{m=M}^{\infty} \frac{1}{(m+2)^2} \\
&\leq \frac{N^2}{\pi^2 \lambda^{2(k+1)\alpha}} \left(\frac{1}{(M+2)^2} + \int_M^{\infty} \frac{dm}{(m+2)^2} \right) \\
&\leq \frac{N^2}{\pi^2 \lambda^{2(k+1)\alpha}} \left(\frac{1}{M^2} + \frac{1}{M} \right) \leq \frac{2N^2}{\pi^2 \lambda^{2(k+1)\alpha} M}.
\end{aligned}$$

Therefore, $E_{\lambda^{k+1}-1,M}^{(1)}(f) \leq \frac{\sqrt{2}N}{\pi \lambda^{(k+1)\alpha} \sqrt{M}} = O\left(\frac{1}{\lambda^{(k+1)\alpha} \sqrt{M}}\right), M > 1.$

This completes the proof of the Theorem 4.1.

5.2 Proof of Theorem 4.2

Following the proof of Theorem 4.1 and for $f \in H^\phi[0, 1)$ class

$$\begin{aligned}
|c'_{n,m}| &\leq N \phi\left(\frac{1}{\lambda^{k+1}}\right) \left| \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \psi_{n,m}^{(\lambda)}(t) \omega_{n,k}(t) dt \right| \\
&= N \phi\left(\frac{1}{\lambda^{k+1}}\right) \left| \int_0^\pi \frac{2}{\sqrt{\pi}} \lambda^{\frac{k+1}{2}} U_m(\cos \theta) \omega(\cos \theta) \frac{\sin \theta}{2\lambda^{k+1}} d\theta \right| \\
&= \frac{N}{\pi \lambda^{\frac{k+1}{2}}} \phi\left(\frac{1}{\lambda^{k+1}}\right) \left| \int_0^\pi \sin(m+1)\theta \sin \theta d\theta \right| \\
&= \frac{N}{2\pi \lambda^{\frac{k+1}{2}}} \phi\left(\frac{1}{\lambda^{k+1}}\right) \left| \int_0^\pi \cos m\theta - \cos(m+2)\theta d\theta \right| \\
&= \frac{N}{\pi \lambda^{\frac{k+1}{2}} (m+2)} \phi\left(\frac{1}{\lambda^{k+1}}\right). \tag{7}
\end{aligned}$$

$$\begin{aligned}
(E_{\lambda^{k+1}-1,M}^{(2)}(f))^2 &= \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=M}^{\infty} |c'_{n,m}|^2 \\
&\leq \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=M}^{\infty} \left(\frac{N}{\pi \lambda^{\frac{k+1}{2}} (m+2)} \phi\left(\frac{1}{\lambda^{k+1}}\right) \right)^2
\end{aligned}$$

$$\begin{aligned}
 &= \frac{N^2}{\pi^2 \lambda^{k+1}} \phi^2 \left(\frac{1}{\lambda^{k+1}} \right) \sum_{m=M}^{\infty} \frac{1}{(m+2)^2} \\
 &\leq \frac{N^2}{\pi^2 \lambda^{k+1}} \phi^2 \left(\frac{1}{\lambda^{k+1}} \right) \left(\frac{1}{M^2} + \frac{1}{M} \right) \\
 &\leq \frac{2N^2}{\pi^2 \lambda^{k+1} M} \phi^2 \left(\frac{1}{\lambda^{k+1}} \right).
 \end{aligned}$$

$$\text{Therefore, } E_{\lambda^{k+1}-1, M}^{(2)}(f) = O \left(\frac{1}{\lambda^{k+1} \sqrt{M}} \phi \left(\frac{1}{\lambda^{k+1}} \right) \right), \quad M > 1.$$

This completes the proof of the Theorem 4.2.

6 Method for solving Abel's integral equations by second kind Chebyshev wavelet

Consider the Abel's integral equation (1). Let us consider the wavelet series expansion,

$$y(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t) \quad (8)$$

and $(\lambda^{k+1} - 1, M)^{th}$ partial sum of series (8) which is the approximate solution of (1) i.e.

$$y(t) = \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t) = C^T \psi^{(\lambda)}(t). \quad (9)$$

In Eqn. (9), C^T contains $\lambda^{k+1} M$ unknown coefficients $c_{n,m}$. For determining the values of $\lambda^{k+1} M$ unknown coefficients $c_{n,m}$, taking suitable collocation points near $t_i = \frac{i-1}{\lambda^{k+1} M}$, $i = 1, \dots, \lambda^{k+1} M$, the $\lambda^{k+1} M$ system of algebraic equations are obtained. By solving these $\lambda^{k+1} M$ system of algebraic equations, the values of $c_{n,m}$ have been obtained. Substituting these values in Eqn. (9), the second kind Chebyshev wavelet solution of the integral equation (1) is obtained.

7 Results and Discussion

In this section, the second kind Chebyshev wavelet method has been used to find the approximate solution of Abel's integral equations. The exact solutions of Abel's integral equations are compared with the Legendre wavelet, first kind, and second kind Chebyshev wavelet solutions (Sharma and Lal [6]). This is illustrated in the following examples:

7.1 Example 1

Consider the Abel's integral equation of the first kind

$$\int_0^t \frac{y(x)}{\sqrt{t-x}} dx = \frac{2\sqrt{t}(8t^2 + 10t + 15)}{15} \quad (10)$$

It is obtained by taking $\gamma(t) = 0$, $f(t) = -\frac{2\sqrt{t}(8t^2+10t+15)}{15}$ in Abel's integral equation (1). The exact solution of Eqn. (10) is $y(t) = t^2 + t + 1$.

The six basis function of the second kind Chebyshev wavelet for $k = 0$, $M = 3$, $\lambda = 2$ are as follows:

$$\left. \begin{array}{l} \text{If } 0 \leq t < \frac{1}{2}, \\ \psi_{0,0}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}} \\ \psi_{0,1}^{(2)}(t) = 4\sqrt{\frac{2}{\pi}}(4t-1) \\ \psi_{0,2}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1) \end{array} \right\} \left. \begin{array}{l} \text{If } \frac{1}{2} \leq t < 1, \\ \psi_{1,0}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}} \\ \psi_{1,1}^{(2)}(t) = 4\sqrt{\frac{2}{\pi}}(4t-3) \\ \psi_{1,2}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1) \end{array} \right\}$$

By using the algorithm of the second kind Chebyshev wavelet method described in Section 6, the first kind Abel's integral equation has been solved. The approximate solution $y(t)$ will be

$$\begin{aligned} y(t) &= c_{0,0}\psi_{0,0}^{(2)}(t) + c_{0,1}\psi_{0,1}^{(2)}(t) + c_{0,2}\psi_{0,2}^{(2)}(t), \quad t \in \left[0, \frac{1}{2}\right) \\ &= 2\sqrt{\frac{2}{\pi}}c_{0,0} + 4\sqrt{\frac{2}{\pi}}(4t-1)c_{0,1} + 2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1)c_{0,2}. \end{aligned} \quad (11)$$

$$\begin{aligned} y(t) &= c_{1,0}\psi_{1,0}^{(2)}(t) + c_{1,1}\psi_{1,1}^{(2)}(t) + c_{1,2}\psi_{1,2}^{(2)}(t), \quad t \in \left[\frac{1}{2}, 1\right) \\ &= 2\sqrt{\frac{2}{\pi}}c_{1,0} + 4\sqrt{\frac{2}{\pi}}(4t-3)c_{1,1} + 2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1)c_{1,2}. \end{aligned} \quad (12)$$

For values of unknowns $c_{0,0}$, $c_{0,1}$, $c_{0,2}$, $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$, we collocate equation Eqn. (10) at $t = 0.15, 0.3, 0.45, 0.6, 0.75$, and 0.9 , six system of algebraic equations are obtained. Solving these systems of algebraic equations, the values of the unknowns are obtained as follows:

$$\begin{aligned} c_{0,0} &= 0.832278919311074, \quad c_{0,1} = 0.117498200373328, \\ c_{0,2} &= 0.009791516697778, \quad c_{1,0} = 1.458935987968824, \\ c_{1,1} &= 0.195830333955546, \quad c_{1,2} = 0.009791516697777. \end{aligned} \quad (13)$$

Second Kind Chebyshev Wavelet Analysis of Abel's Integral Equations

Putting the values of $c_{0,0}$, $c_{0,1}$, $c_{0,2}$, $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$ from Eqn. (13) into Eqns. (11) and (12).

$$\begin{aligned}
 y(t) &= 0.832278919311074 \left(2\sqrt{\frac{2}{\pi}} \right) + 0.117498200373328 \left(4\sqrt{\frac{2}{\pi}}(4t-1) \right) \\
 &+ 0.009791516697778 \left(2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1) \right), \quad t \in \left[0, \frac{1}{2} \right), \\
 y(t) &= 1.458935987968824 \left(2\sqrt{\frac{2}{\pi}} \right) + 0.195830333955546 \left(4\sqrt{\frac{2}{\pi}}(4t-3) \right) \\
 &+ 0.009791516697777 \left(2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1) \right), \quad t \in \left[\frac{1}{2}, 1 \right).
 \end{aligned}$$

The approximate solution of the first kind Abel's integral equation (10) obtained by the second kind Chebyshev wavelet method (SKCWM) for different values of t in the interval $[0,1)$ has been obtained. The comparison of exact solutions (ES) of Abel's integral equation with the Legendre wavelet method (LWM), first kind Chebyshev wavelet method (FKCWM), and second kind Chebyshev wavelet method are given in Table 1.

Table (1)

t	ES	LWM	FKCWM	SKCWM
0.1	1.11	0.929532564	1.0719999995	1.10999999999999
0.2	1.24	1.115745774	1.1546666661	1.23999999999999
0.3	1.39	1.309633163	1.2479999994	1.38999999999999
0.4	1.56	1.511194733	1.3519999994	1.56000000000000
0.5	1.75	1.720430483	1.4666666667	1.75000000000000
0.6	1.96	1.937340412	1.5919999994	1.96000000000000
0.7	2.19	2.161924522	1.7279999994	2.18999999999999
0.8	2.44	2.394182812	1.8746666661	2.44000000000000
0.9	2.71	2.634115282	2.0319999994	2.70999999999999

Table(1): Comparison between exact and approximate solutions of Eqn. (10).

The graphs of the exact solution and approximate solution of the Abel's integral equation (10) are shown in Figure 1.

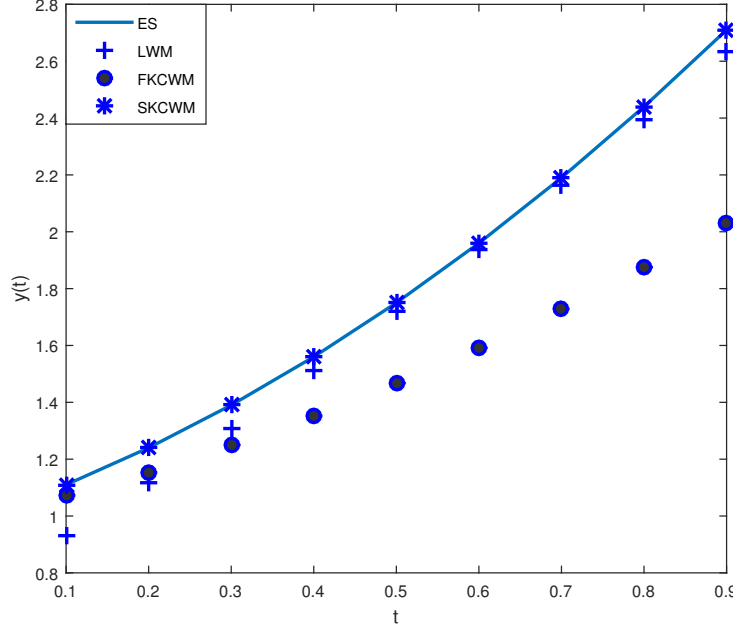


Fig.(1): The graphs of exact and approximate solutions of Eqn. (10).

7.2 Example 2

Consider the Abel's integral equation of the second kind

$$y(t) = 4t^{\frac{3}{2}} - \int_0^t \frac{y(x)}{\sqrt{t-x}} dx \quad (14)$$

It is obtained by taking $\gamma(t) = -1$, $f(t) = -4t^{\frac{3}{2}}$ in the Abel's integral equation (1). The exact solution of Eqn. (14) is $y(t) = 3t + \frac{3}{\pi}(1 - 2\sqrt{t} - \operatorname{erfc}(\sqrt{\pi t}))$. Following the procedure of example (1), the approximate solution $y(t)$ will be

$$\begin{aligned} y(t) &= 0.203127378663919 \left(2\sqrt{\frac{2}{\pi}} \right) + 0.1245710999898258 \left(4\sqrt{\frac{2}{\pi}}(4t-1) \right) \\ &+ 0.007508790407423 \left(2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1) \right), \quad t \in \left[0, \frac{1}{2} \right), \\ y(t) &= 0.788695399256314 \left(2\sqrt{\frac{2}{\pi}} \right) + 0.159580510841338 \left(4\sqrt{\frac{2}{\pi}}(4t-3) \right) \\ &+ 0.002685161812766 \left(2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1) \right), \quad t \in \left[\frac{1}{2}, 1 \right]. \end{aligned}$$

Second Kind Chebyshev Wavelet Analysis of Abel's Integral Equations

The exact solution and approximate solutions of the second kind Abel's integral equation (14) are shown in Table 2.

Table (2)

t	ES	LWM	FKCWM	SKCWM
0.1	0.0914468358	0.0296708741	0.0772954668400	0.09087255103875016
0.2	0.2313248892	0.1868894237	0.1890183222312	0.23456458416817604
0.3	0.3928020550	0.3529379159	0.3135382389570	0.39359395601475531
0.4	0.5682530617	0.5278163507	0.4467872378730	0.56796066657848785
0.5	0.7538699265	0.7115247282	0.5858830996710	0.76212315618636044
0.6	0.9473720681	0.9040630482	0.7291945119710	0.95487673780814230
0.7	1.1472485450	1.1054313110	0.8762823974830	1.15311498926330412
0.8	1.3524303100	1.3156295160	1.0263605224509	1.35683791055184616
0.9	1.5621248430	1.5346576640	1.1786727537049	1.56604550167376791

Table(2): Comparison between exact and approximate solutions of Eqn. (14).

The graphs of the exact solution and approximate solution of the second kind Abel's integral equation (14) are shown in Figure 2.

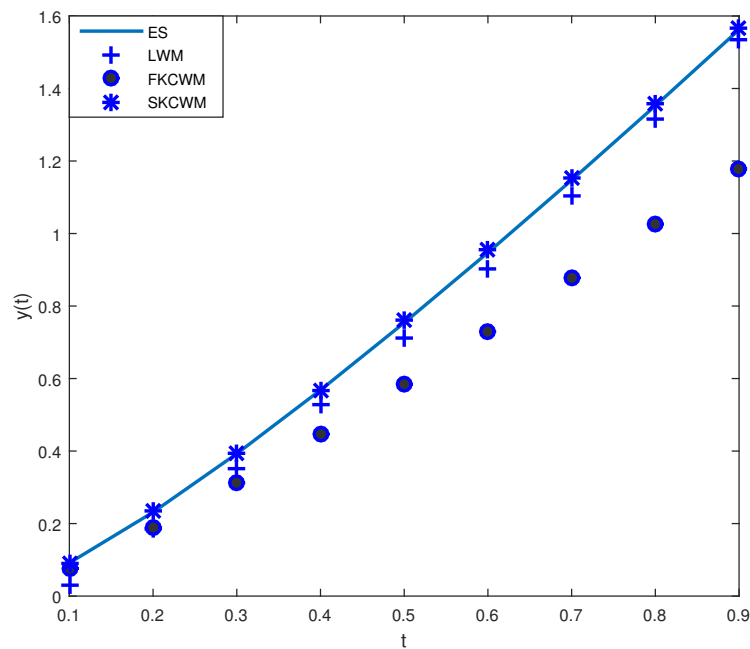


Fig.(2): The graphs of exact and approximate solutions of Eqn. (14).

7.3 Absolute Error

The absolute error in the approximate solution of the Abel's integral equations (10) and (14) are shown in Tables 3. This shows that absolute error is negligible by the second

kind Chebyshev wavelet method as compared to Legendre wavelet method, and first kind Chebyshev wavelet method.

Table (3)

	Absolute error in solution of Example 1			Absolute error in solution of Example 2		
t	LWM	FKCWM	SKCWM	LWM	FKCWM	SKCWM
0.1	0.18046	0.03800	0.222×10^{-15}	0.06177	0.01415	0.00057
0.2	0.12425	0.08533	0.222×10^{-15}	0.04443	0.04230	0.00323
0.3	0.08036	0.14200	0	0.03986	0.07926	0.00079
0.4	0.04880	0.20800	0	0.04043	0.12146	0.00029
0.6	0.02956	0.28333	0	0.04234	0.16798	0.00825
0.6	0.02265	0.36800	0	0.04330	0.21817	0.00750
0.7	0.02807	0.46200	0	0.04181	0.27096	0.00586
0.8	0.04581	0.56533	0	0.03680	0.32606	0.00441
0.9	0.07588	0.67800	0	0.02746	0.38345	0.00392

Table(3): Absolute error in approximate solutions of Eqns. (10) and (14).

8 Conclusions

1. The second kind Chebyshev wavelet approximations of Theorem 4.1 and 4.2 are given by $E_{\lambda^{k+1}-1,M}^{(1)}(f) = O\left(\frac{1}{\lambda^{(k+1)\alpha}\sqrt{M}}\right) \rightarrow 0$ as $k \rightarrow \infty$ and $M \rightarrow \infty$; and $E_{\lambda^{k+1}-1,M}^{(2)}(f) = O\left(\frac{1}{\lambda^{k+1}\sqrt{M}}\phi\left(\frac{1}{\lambda^{k+1}}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$ and $M \rightarrow \infty$. Therefore, estimators $E_{\lambda^{k+1}-1,M}^{(1)}(f)$ and $E_{\lambda^{k+1}-1,M}^{(2)}(f)$ of the solution function of the Abel's integral equations are best possible in wavelet analysis.

2. From the tables 1, 2, 3 and figures 1, 2, it is observed that the solutions of the Abel's integral equations by the second kind Chebyshev wavelet method coincide with their exact solution as compare to Legendre and first kind Chebyshev wavelet method. This shows the validity and applicability of the proposed algorithm.

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Second Kind Chebyshev Wavelet Analysis of Abel's Integral Equations

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Generalized soft multi connectedness and compactness

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Abstract

The significance of connectedness and compactness plays a vital role in studying the properties of topological spaces. Numerous authors explored several characterizations of topological spaces using these notions. Researchers had attempted to generalize the concepts of connectedness and compactness in various generalized topological spaces like ideal topological spaces, soft topological spaces, fuzzy topological spaces etc. In the year 2013, Babitha and John studied the concept of connectedness in soft multi topological spaces. Since then the study of these concepts has created a gap in the literature. The uniosity about the generalizations of connectedness and compactness in soft multi topological spaces explored this article.

Keywords: Soft multiset, soft multi topology, generalized semi closed soft multiset, generalized semi soft multi connectedness and generalized semi soft multi compactness.

2020 AMS subject classifications: 03E70, 54A05, 54B05, 54A10. ⁴

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1 Introduction

In mathematics, a multiset is regarded as the generalization of a set. In the classical set theory, set is a well-defined collection of distinct objects. If a repeated occurrence of any object is allowed in a set, then it creates another mathematical structure known as multiset. A multiset differs from a set in the sense that each element has a multiplicity that indicates how many times it occurs in that collection.

Multiset theory was introduced by Cerf et al.[1971]. Yager [1986] and Blizard [1989] added further contributions to it. An algebraic structure for multiset space was introduced by Ibrahim et al. [2011]. Multiset relations and multiset functions were introduced by Girish and John [2009].

In 2012, Girish and Sunil [2012] introduced multiset topologies induced by multiset relations. Later they introduced the concepts of basis, sub-basis, closure, interior, continuity and related properties in multiset topological spaces in [2012]. S. A. El. Sheikh et al. introduced the concept of separation axioms on multiset topological space [2015] and they studied the generalization of open multisets and its properties [2015]. In 2017, the notion of compactness in multiset topological space [2017] was introduced by Sougata Mahanta et al.

J. Mahanta et al.[2014] have introduced the semi compactness of multiset topological space. Many interesting findings concerning with the multisets were examined in [2015, 2018].

In 1999, Molodstov[1999] initiated the theory of soft sets as a new Mathematical tool to deal with uncertainties while modelling problems in various fields like Engineering and Medical Sciences. P. K. Maji et al.[2003] defined some basic properties of soft sets. In 2002, P. K. Maji et al. [2002] applied the theory of soft sets to solve a decision making problem. The notion of soft mapping was first initiated by P. Majumdar et al. [2010]. In 2011, N. Caugman et al. [2011] defined the soft topology on soft sets and investigated related properties. In 2012, I. Zorlutuna et al. [2012] acquainted the concept of soft point, soft continuity and soft compactness.

Soft set is a mapping from a parameter set to power set of universal. But among many practical situations some situation may occur, where the respective counts of objects in the universe of discourse are not single. To handle the situation mathematically, in 2013 Babitha & John [2013] were introduced the concept of soft multiset as a combination of soft sets and multisets, in which the universal set is a multiset. The topological and algebraic structures of soft multi set have large number of applications in soft computing, decision making, data analysis and information aggregation and information measure. Moreover the soft multi topology and its basic properties were given in [2013]. The same authors initiated the concept of connectedness on soft multi topological spaces [2013].

In 2015, Deniz Tokat et al. [2015] acquainted the idea of soft multi continuous functions and also soft multi semi continuity by using soft multi function. After that El. Sheikh et al. [2016] introduced the concepts of generalizations of open soft multi sets and mappings in soft multi topological space. In 2015, they have introduced the concept of g-

closed soft multisets in soft multi topological space. Ismail Osmanoglu et al. [2014] acquainted the concept of compact soft multi spaces. In 2015, Deniz Tokat et al. introduced the concept of soft multi continuous functions by using soft multi function. Muhammed Raafat et al. presented the concept of semi-compact soft multi spaces. Muhammed Riaz et al. [2020] gave the properties of soft multi topological spaces and also gave some real life applications. Seydakaya Pezukul et al. [2021] provided an application of soft multisets to a decision making problem concerning side effects of COVID-19 vaccines. V. Inthumathi et al. [2022] acquainted the concept of generalized semi closed soft multisets and established several of their properties. Further they introduced the concepts of generalized semi operators and generalized semi continuous functions in soft multi topological spaces [2023].

In this paper, we generalized the work of Muhammed Raafat et al. as generalized semi soft multi compactness and generalized semi soft multi connectedness by introducing generalized semi soft multi separated sets, soft multi adherent point, generalized semi soft multi adherent point, soft multi open cover and generalized semi soft multi open cover and also we explored new characterizations of soft multi topological spaces.

2 Preliminaries

Throughout this paper SMTS denotes the soft multi topological spaces.

Definition 2.1. [2013] Let U be an universal mset, E be a set of parameters and $A \subseteq E$. Then, an ordered pair (F, A) is called a soft mset where F is a mapping given by $F: A \rightarrow P^*(U)$; $P^*(U)$ is the power set of a mset U .

For all $e \in A$, $F(e)$ mset represents by count function $C_{F(e)}: U^* \rightarrow N$ where N represents the set of non-negative integers and U^* represents the support set of U .

Let $U = \{2/x, 3/y, 2/z\}$ be a mset. Then, the support set of U is $U^* = \{x, y, z\}$.

Definition 2.2. [2013] Let X be an universal mset and E be a set of parameters. Then, the collection of all soft multisets over X with parameters from E is called a soft multi class and is denoted as X_E .

Definition 2.3. [2013] Let (X, τ, E) be a SMTS over X and (G, E) be a soft multi set over X and $x \in X$. Then, (G, E) is said to be a soft multi neighbourhood of x if there exists a soft multi open set (F, E) such that $x \in (F, E) \subseteq (G, E)$. The set of all soft multi neighbourhood of α , denoted by $\tilde{N}(\alpha)$, is called the family of soft multi neighbourhoods of α ,

$$\text{i.e. } \tilde{N}(\alpha) = \{(G, E): (G, E) \in \tau, \alpha \in (G, E)\}.$$

Definition 2.4. [2022] Let f_E be a soft mset over X_E . f_E is called a soft multi point over X , if there exists $e \in E$ and $n/x \in X$, $1 \leq n \leq m$ such that

$$f(\epsilon) = \begin{cases} \{(n/x)\} & \text{if } \epsilon = e, \ 1 \leq n \leq m \\ \phi & \text{if } \epsilon \in E - \{e\} \end{cases}$$

We denote the soft multi point f_E by $[(n/x)_e]_E$. In this case, x is called support point of $[(n/x)_e]_E$, $\{x\}$ is called support set of $[(n/x)_e]_E$ and e is called the expressive parameter of $[(n/x)_e]_E$. The family of all soft multi points over X is denoted by $P(X, E)$ or P .

i.e. $P(X, E) = \{[(n/x_i)_{e_j}]_E : x_i \in X, e_j \in E, 1 \leq n \leq m\}$.

Definition 2.5. [2022] A soft mset S_E in a SMTS (X, τ, E) is said to be generalized semi closed soft multiset (briefly *gscs* mset) if $C_{scl(S)(e)}(x) \leq C_{U(e)}(x)$ whenever $C_{S(e)}(x) \leq C_{U(e)}(x)$ for all $x \in X^*$, $e \in E$ and $U_E \in OSM(X)_E$. The set of all *gscs* mset is denoted by $gscs(X_E)$.

Definition 2.6. [2023] A sub soft mset S_E in (X, τ, E) is called generalized semi open soft multiset (briefly *gsos* mset) if its complement is a *gscs* mset. The set of all *gsos* mset in (X, τ, E) is denoted by $gsos(X)_E$.

Definition 2.7. [2023] Let (X, τ, E) be a SMTS over X_E and S_E be a soft mset over X_E . Then the generalized semi soft multi closure of S_E , denoted by $gssm-cl(S_E)$ is the intersection of all *gscs* mset containing S_E .

Definition 2.8. [2023] Let (X, τ, E) be a SMTS over X_E and S_E be a soft mset over X_E . Then the generalized semi soft multi interior of S_E , denoted by $gssm-int(S_E)$ is the union of all *gsos* mset contained in S_E .

Definition 2.9. [2023] A sub soft mset N_E in SMTS (X, τ, E) is said to be generalized semi soft multi neighbourhood (briefly *gssm-nbd*) of a point $[(n/x)_e]_E$ in X_E if there exists an *gsos* mset U_E such that $[(n/x)_e]_E \leq C_{U(e)}(x) \leq C_{N(e)}(x)$.

If N_E is a *gsos* mset containing $[(n/x)_e]_E$, the N_E is called generalized semi soft multi open neighbourhood (briefly *gssm-open nbd*) of $[(n/x)_e]_E$.

Definition 2.10. [2020] Let $(\Omega_A, \tilde{\tau})$ be a SMTS and $\Omega_B \cong \Omega_A$ and $\alpha \in \Omega_A$. If every soft multi neighbourhood of α intersects Ω_B in some soft multi points other than α , then α is called a soft multi limit point of Ω_B .

Definition 2.11. [2014] Let (X_E, τ_1) and (X_E, τ_2) be SMTS. Then, the following hold:

- (a) If $\tau_2 \supseteq \tau_1$, then τ_2 is soft multi finer than τ_1 .
- (b) If $\tau_2 \supset \tau_1$, then τ_2 is soft multi strictly finer than τ_1 .
- (c) If either $\tau_2 \supseteq \tau_1$ or $\tau_2 \subseteq \tau_1$, then τ_1 is comparable with τ_2 .

Definition 2.12. [2023] Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ and $f: X_E \rightarrow Y_K$ be functions. Then the function f is called generalized semi soft multi continuous (briefly *gssm-cts*) if $f^{-1}(F_K)$ is a *gscs* mset over (X, τ, E) for every closed soft mset F_K over (Y, σ, K) .

Definition 2.13. [2014] A family of ψ of soft msets has the finite intersection property if the intersection of the members of each finite sub family of ψ is not the null soft mset.

Definition 2.14. [2013] Let (X_E, τ) be a SMTS over X . A soft multi separation of $\tilde{X}$ is a pair $((F, E), (G, E))$ of no-null soft multi open sets in X_E such that

$$\tilde{X} = (F, E) \tilde{\cup} (G, E), (F, E) \tilde{\cap} (G, E) = \phi.$$

Definition 2.15. [2013] A SMTS (X_E, τ) is said to be soft multi connected if there does not exist a soft multi separation of $\tilde{X}$. Otherwise, (X_E, τ) is said to be soft multi disconnected.

Definition 2.16. [2014] A family ψ of soft multi sets is a cover of a soft multiset (F, A) if $(F, A) \subseteq \bigcup \{(F_i, A) : (F_i, A) \in \psi, i \in I\}$. It is a soft multi open cover if each member of ψ is a soft multi open set. A subcover of ψ is a family of ψ which is also a cover.

Definition 2.17. [2014] Let (X_E, τ) be a SMTS and $(F, A) \in (X, E)$. Soft multi set (F, A) is called compact if each soft multi open cover of (F, A) has a finite subcover. Also SMTS (X_E, τ) is called compact if each soft multi open cover of $\tilde{X}$ has a finite subcover.

3 Generalized semi soft multi connectedness

In this section the notions of generalized semi soft multi separated sets, generalized semi soft multi connectedness are defined and some of their properties in SMTSs are studied.

Definition 3.1. Let (X, τ, E) be a SMTS. Two non-empty sub soft msets S_E and T_E are said to be generalized semi soft multi separated (briefly *gssm-separated*) if and only if $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$.

i.e. $C_{(S \cap gssm-cl(T)) \cup (gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$.

Example 3.1. Let $X = \{2/x_1, 2/x_2\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{\phi}, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where $F_1(e_1) = \{2/x_1\}$, $F_1(e_2) = \{\tilde{\phi}\}$, $F_2(e_1) = \{2/x_2\}$, $F_2(e_2) = \{\tilde{\phi}\}$, $F_3(e_1) = \{\tilde{X}\}$, $F_3(e_2) = \{\tilde{\phi}\}$. Let S_E , T_E and V_E be soft msets such that $S(e_1) = \{1/x_1\}$, $S(e_2) = \{\tilde{\phi}\}$, $T(e_1) = \{1/x_2\}$, $T(e_2) = \{\tilde{\phi}\}$ and $V(e_1) = \{1/x_1, 1/x_2\}$, $V(e_2) = \{\tilde{\phi}\}$. Then soft msets S_E and T_E are *gssm-separated* but the soft msets T_E and V_E are not *gssm-separated*.

Definition 3.2. Let (X, τ, E) be a SMTS and $C_{S(e)}(x) \leq C_{X(e)}(x)$. A soft multi point $[(n/x)_e]_E \in X_E$ is said to be soft multi adherent point of S_E if every open soft mset containing $[(n/x)_e]_E$, contains atleast one soft multi point of S_E .

Definition 3.3. Let (X, τ, E) be a SMTS and $C_{S(e)}(x) \leq C_{X(e)}(x)$. A soft multi point $[(n/x)_e]_E \in X_E$ is said to be generalized semi soft multi adherent point (briefly *gssm-adherent point*) of S_E if every *gsos* mset containing $[(n/x)_e]_E$, contains atleast one soft multi point of S_E .

Proposition 3.1. Two *gssm-separated* sets are always disjoint.

Proof. Let S_E and T_E be $gssm$ -separated sets. Then $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. Now $C_{(S \cap T)(e)}(x) \leq C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. This implies that $C_{(S \cap T)(e)}(x) = C_{\phi(e)}(x)$. Hence S_E and T_E are disjoint. $\square$

Proposition 3.2. Every soft multi separated sets are $gssm$ -separated but the converse is not true.

Proof. It follows from the definition. $\square$

Example 3.2. In Example 3.1, the soft msets S_E and T_E are $gssm$ -separated but not soft multi separated.

Theorem 3.1. Two soft msets are $gssm$ -separated if and only if they are disjoint soft msets and neither of them contains soft multi limit point of the other.

Proof. Let S_E and T_E be $gssm$ -separated sets if and only if $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. Now $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x) \Leftrightarrow C_{(S \cap (T \cup (n/x)_E))(e)}(x) = C_{\phi(e)}(x)$, where $[(n/x)_E]$ is the soft multi limit point of $T_E \Leftrightarrow C_{(S \cap T)(e)}(x) = C_{\phi(e)}(x)$ and $C_{(S \cap (n/x)_E)(e)}(x) = C_{\phi(e)}(x) \Leftrightarrow S_E$ and T_E are disjoint soft msets and S_E contains no soft multi limit point of T_E . Similarly $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$ if and only if S_E and T_E are disjoint soft msets and T_E contains no soft multi limit point of S_E . $\square$

Proposition 3.3. Sub soft msets of a $gssm$ -separated sets are also $gssm$ -separated.

Proof. Let S_E and T_E be $gssm$ -separated of a SMTS (X, τ, E) . Then $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. Let $C_{U(e)}(x) \leq C_{S(e)}(x)$ and $C_{V(e)}(x) \leq C_{T(e)}(x)$. Then $C_{(U \cap gssm-cl(V))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(U) \cap V)(e)}(x) = C_{\phi(e)}(x)$. Thus U_E and V_E are $gssm$ -separated. $\square$

Theorem 3.2. Two $gscs$ -sets of a SMTS (X, τ, E) are $gssm$ -separated if and only if they are disjoint.

Proof. Since $gssm$ -separated sets are disjoint, $gscs$ -msets are disjoint. Conversely, let S_E and T_E be two disjoint $gscs$ -msets. Then $C_{gssm-cl(S)(e)}(x) = C_{S(e)}(x)$, $C_{gssm-cl(T)(e)}(x) = C_{T(e)}(x)$ and $C_{(S \cap T)(e)}(x) = C_{\phi(e)}(x)$. Consequently, $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$ and $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. Hence S_E and T_E are $gssm$ -separated. $\square$

Theorem 3.3. Two sub $gsos$ msets of a SMTS (X, τ, E) are $gssm$ -separated if and only if they are disjoint.

Proof. Since $gssm$ -separated sets are disjoint, $gsos$ -msets are disjoint. Conversely, let S_E and T_E be two disjoint $gsos$ -msets. Suppose that $C_{(S \cap gssm-cl(T))(e)}(x) \neq C_{\phi(e)}(x)$. Let $[(n/x)_E] \leq C_{(S \cap gssm-cl(T))(e)}(x)$. Then $[(n/x)_E] \leq C_{S(e)}(x)$ and $[(n/x)_E]$ is a $gssm$ -adherent point of T_E , since S_E is a $gsos$ -mset containing $[(n/x)_E]$ and $[(n/x)_E]$

is a $gssm$ -adherent point of T_E , S_E contains at least one soft multi point of T_E . Thus $C_{(S \cap T)(e)}(x) \neq C_{\phi(e)}(x)$. This contradicts the fact that S_E and T_E are two disjoint $gsos$ -msets. Therefore $C_{(S \cap gssm-cl(T))(e)}(x) = C_{\phi(e)}(x)$. Similarly $C_{(gssm-cl(S) \cap T)(e)}(x) = C_{\phi(e)}(x)$. Hence S_E and T_E are $gssm$ -separated. $\square$

Definition 3.4. If $C_{X(e)}(x) = C_{S(e)}(x) \cup C_{T(e)}(x)$ such that S_E and T_E are non empty $gssm$ -separated sets, then S_E, T_E form $gssm$ -separation of X_E .

Definition 3.5. A SMTS (X, τ, E) is said to be generalized semi soft multi connected (briefly $gssm$ -connected) if X_E cannot be written as a union of two disjoint non empty $gsos$ -msets.

If X_E is not $gssm$ -connected then it is $gssm$ -disconnected.

Proposition 3.4. Every $gssm$ -connected space is soft multi connected but not conversely.

Proof. Let (X, τ, E) be a $gssm$ -connected space. Suppose that X_E is not soft multi connected. Then $C_{X(e)}(x) = C_{S(e)}(x) \cup C_{T(e)}(x)$, where S_E and T_E are disjoint non empty open soft msets in (X, τ, E) . Since every open soft mset is $gsos$ -mset, S_E and T_E are $gsos$ -mset. Therefore $C_{X(e)}(x) = C_{S(e)}(x) \cup C_{T(e)}(x)$, where S_E and T_E are disjoint non empty $gsos$ -msets in (X, τ, E) . This contradicts the fact that X_E is $gssm$ -connected and so that X_E is soft multi connected. $\square$

Proposition 3.5. A SMTS (X, τ, E) is a $gssm$ -disconnected space if there exists a non-empty proper sub soft mset of X_E which is both $gsos$ -mset and $gscs$ -mset.

Proof. Let S_E be a non-empty proper sub soft mset of X_E which is both $gsos$ -mset and $gscs$ -mset. Then clearly S_E^c is a non-empty proper sub soft mset of X_E which is both $gsos$ -mset and $gscs$ -mset. Thus $C_{(S \cap S^c)(e)}(x) = C_{\phi(e)}(x)$, and also $C_{X(e)}(x) = C_{S(e)}(x) \cup C_{S^c(e)}(x)$. Thus X_E is the union of two disjoint non empty $gsos$ -msets. Hence X_E is $gssm$ -disconnected. $\square$

Theorem 3.4. A SMTS (X, τ, E) is a $gssm$ -disconnected space if and only if X_E is the union of two disjoint non empty $gsos$ -msets.

Proof. Let X_E be $gssm$ -disconnected. Then there exists a non-empty proper sub soft mset of X_E which is both $gsos$ -mset and $gscs$ -mset. And therefore, S_E^c is a non-empty sub soft mset of X_E that is both $gsos$ -mset and $gscs$ -mset. Thus $C_{(S \cap S^c)(e)}(x) = C_{\phi(e)}(x)$, and also $C_{X(e)}(x) = C_{S(e)}(x) \cup C_{S^c(e)}(x)$. This shows that X_E is the union of two disjoint non empty $gsos$ -msets.

Conversely, let X_E is the union of two disjoint non-empty $gsos$ -msets S_E and T_E . Then $C_{T^c(e)}(x) = C_{S(e)}(x)$. Now T_E is a $gsos$ -mset, S_E is a $gscs$ -mset. Since T_E is non-empty S_E is a non-empty proper sub soft mset of X_E which is both $gsos$ -mset and $gscs$ -mset. Thus X_E is $gssm$ -disconnected. $\square$

Corollary 3.1. SMTS (X, τ, E) is a *gssm*-disconnected space if and only if X_E is the union of two disjoint non empty *gscs*-msets.

Proof. Obvious. $\square$

Theorem 3.5. For a SMTS (X, τ, E) , the following are equivalent

- (a) (X, τ, E) is *gssm*-connected
- (b) the only sub soft msets of (X, τ, E) which are both *gsos*-mset and *gscs*-mset are $\tilde{\phi}$ and $\tilde{X}$.

Proof. (a) $\Rightarrow$ (b) Let U_E be a *gsos*-mset and *gscs*-mset of X_E . Then U_E^c is both *gsos*-mset and *gscs*-mset of X_E . Since X_E is disjoint union of *gsos*-msets U_E and U_E^c , by assumption one of these must be empty. i.e. $C_{U(e)}(x) = C_{\phi(e)}(x)$ or $C_{U(e)}(x) = C_{X(e)}(x)$.
 (b) $\Rightarrow$ (a) Suppose that X_E *gssm*-disconnected. Then by Proposition 3.5, there exists a non-empty proper sub soft mset of X_E , which is both *gsos*-mset and *gscs*-mset, which is a contradiction. Thus X_E is *gssm*-disconnected. $\square$

Proposition 3.6. If (X, τ_1, E) is a *gssm*-connected topological space and τ_1 finer than τ_2 , then (X, τ_2, E) is also a *gssm*-connected space.

Proof. Suppose that (X, τ_2, E) is a *gssm*-disconnected space. Then there exists a non-empty proper sub soft mset S_E of X_E which is both *gsos*-mset and *gscs*-mset in (X, τ_2, E) .

So S_E and S_E^c are both *gsos*-mset in (X, τ_2, E) . Since $\tau_2 \subseteq \tau_1$, it follows that S_E and S_E^c are both *gsos*-mset in (X, τ_1, E) . This shows that S_E is a non-empty proper sub soft mset of X_E which is both *gsos*-mset and *gscs*-mset in (X, τ_1, E) . This is contradiction. Therefore (X, τ_2, E) is a *gssm*-connected space. $\square$

Proposition 3.7. Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ be functions. If $f: X_E \rightarrow Y_K$ is *gssm*-continuous surjection and (X, τ, E) is *gssm*-connected then (Y, σ, K) is soft multi connected.

Proof. Suppose that (Y, σ, K) is not soft multi connected. Let $C_{Y(k)}(y) = C_{S(k)}(y) \cup C_{T(k)}(y)$, where S_K and T_K are disjoint non-empty open soft msets in Y_K . Since f is *gssm*-continuous surjection, $X_E = f^{-1}(S_K) \cup f^{-1}(T_K)$, where $f^{-1}(S_K)$ and $f^{-1}(T_K)$ are disjoint non-empty *gsos*-msets in X_E . This contradicts the fact that X_E is *gssm*-connected. Hence Y_K is soft multi connected. $\square$

4 Generalized semi soft multi compactness

In this section, the notions of generalized semi soft multi compactness and locally generalized semi soft multi compactness are introduced. Some properties of generalized semi soft multi compact spaces are also discussed.

Definition 4.1. A collection $\{(S_E)_i: i \in \Lambda\}$ of *gsos*-msets in SMTS (X, τ, E) is called generalized semi soft multi open cover (briefly *gssm*-open cover) of a sub soft mset B_E of X_E if $C_{B(e)}(x) \leq C_{\cup\{(S_i)(e): i \in \Lambda\}}(x)$ holds.

Definition 4.2. A SMTS (X, τ, E) is generalized semi soft multi compact (briefly *gssm*-compact) if every *gssm*-open cover of X_E has a finite sub cover.

Example 4.1. In Example 3.1, the SMTS (X, τ, E) is *gssm*-compact.

Definition 4.3. A sub soft mset B_E of SMTS is said to be *gssm*-compact relative to X_E if for every collection $\{(S_E)_i: i \in \Lambda\}$ of *gsos*-msets of X_E such that $C_{B(e)}(x) \leq C_{\cup\{(S_i)(e): i \in \Lambda\}}(x)$, there exists a finite subset Λ_o of Λ such that $C_{B(e)}(x) \leq C_{\cup\{(S_i)(e): i \in \Lambda_o\}}(x)$.

Proposition 4.1. Every *gssm*-compact space is soft multi compact space.

Proof. Let (X, τ, E) be a *gssm*-compact space. Let $\{(S_E)_i: i \in \Lambda\}$ be a soft multi open cover of X_E . Since every open soft mset is a *gsos*-mset, $\{(S_E)_i: i \in \Lambda\}$ is a *gssm*-open cover of X_E . Then there exists a finite sub set Λ_o of Λ such that $C_{X(e)}(x) \leq C_{\cup\{(S_i)(e): i \in \Lambda_o\}}(x)$. Thus X_E is a soft multi compact space. $\square$

Proposition 4.2. Every *gscs*-mset of a *gssm*-compact space is *gssm*-compact relative to (X, τ, E) .

Proof. Let S_E be a *gscs*-mset of a *gssm*-compact space (X, τ, E) . Then S_E^c is a *gsos*-mset in (X, τ, E) . Let $C_{M(e)}(x) = C_{\cup\{(G_i)(e): i \in \Lambda\}}(x)$ be a soft multi open cover of S_E by *gsos*-msets. Therefore $M_E \cup S_E^c$ is a *gssm*-open cover of X_E . Since X_E is *gssm*-compact, there exists a finite subset Λ_o of Λ such that $C_{(M \cup S^c)(e)}(x) \leq C_{\cup\{(G_i)(e): i \in \Lambda_o\}}(x)$. Therefore $C_{S(e)}(x) \leq C_{\cup\{(G_i)(e): i \in \Lambda_o\}}(x)$. Hence S_E is *gssm*-compact relative to (X, τ, E) . $\square$

Proposition 4.3. Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ be functions. If $f: X_E \rightarrow Y_K$ is *gssm*-continuous and S_E is *gssm*-compact relative to X_E , then $f(S_E)$ is soft multi compact relative to (Y, σ, K) .

Proof. Let $\{(T_E)_i: i \in \Lambda\}$ be a soft multi open cover of $f(S_E)$. Then $C_{f(S)(e)}(x) \leq C_{\{(T_i)(e): i \in \Lambda\}}(x)$ and so $C_{S(e)}(x) \leq C_{\{f^{-1}(T_i)(e): i \in \Lambda\}}(x)$. Since f is *gssm*-continuous, $f^{-1}(T_E)_i$ is a *gsos*-mset for each $i \in \Lambda$. Thus $\{f^{-1}(T_E)_i: i \in \Lambda\}$ is a soft multi open cover of S_E by *gsos*-msets of X_E . Since S_E is *gssm*-compact relative to X_E , there exists a finite sub set Λ_o of Λ such that $C_{S(e)}(x) \leq C_{\{f^{-1}(T_i)(e): i \in \Lambda_o\}}(x)$ and hence $C_{f(S)(e)}(x) \leq C_{\{(T_i)(e): i \in \Lambda_o\}}(x)$. Thus $f(S_E)$ is soft multi compact relative to (Y, σ, K) . $\square$

Corollary 4.1. Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ be functions. If $f: X_E \rightarrow Y_K$ is *gssm*-continuous surjection and X_E is *gssm*-compact, then Y_K is soft multi compact.

Definition 4.4. Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ and $f: X_E \rightarrow Y_K$ be functions. Then f is said to be generalized semi soft multi irresolute (briefly *gssm*-irresolute) if the inverse image of every *gscs*-mset over (Y, σ, K) is *gscs*-mset over (X, τ, E) .

Proposition 4.4. Let (X, τ, E) and (Y, σ, K) be two SMTSs and let $u: X \rightarrow Y$, $p: E \rightarrow K$ be functions. If a soft multi function $f: X_E \rightarrow Y_K$ is *gssm*-irresolute and a sub soft mset B_E of X_E is *gssm*-compact relative to (X, τ, E) , then $f(B_E)$ is *gssm*-compact relative to (Y, σ, K) .

Proof. Let $\{(S_i)_e: i \in \Lambda\}$ be any collection of *gsos*-msets of (Y, σ, K) , such that $C_{f(T)(e)}(x) \leq C_{\{(S_i)_e: i \in \Lambda\}}(x)$. Then $C_{T(e)}(x) \leq C_{\{f^{-1}(S_i)_e: i \in \Lambda\}}(x)$ holds. By assumption, T_E is *gssm*-compact relative to X_E , there exists a finite sub collection Λ_o of Λ such that $C_{T(e)}(x) \leq C_{\{f^{-1}(S_i)_e: i \in \Lambda_o\}}(x)$. Therefore $C_{f(T)(e)}(x) \leq C_{\{(S_i)_e: i \in \Lambda_o\}}(x)$. Hence $f(T_E)$ is *gssm*-compact relative to Y_K . $\square$

Proposition 4.5. Let (X, τ, E) be a SMTS. If U_E and V_E are *gssm*-compact relative to X_E , then $U_E \cup V_E$ is *gssm*-compact relative to X_E .

Proof. Let $C_{F(e)}(x) = C_{\{(S_i)_e: i \in \Lambda\}}(x)$ be any soft multi open cover of $U_E \cup V_E$ by *gsos*-mset of X_E . Then F_E is a *gssm*-open cover of both U_E and V_E . By hypothesis, there exists a finite sub collection $\Lambda_{U_E}, \Lambda_{V_E}$ of Λ such that $C_{U(e)}(x) \leq C_{\{(S_i)_e: i \in \Lambda_{U_E}\}}(x)$ and $C_{V(e)}(x) \leq C_{\{(S_i)_e: i \in \Lambda_{V_E}\}}(x)$. Therefore $C_{U \cup V(e)}(x) \leq C_{\{(S_i)_e: i \in \Lambda_{U_E} \cup \Lambda_{V_E}\}}(x)$. This shows that $U_E \cup V_E$ is *gssm*-compact relative to X_E . $\square$

Proposition 4.6. Let (X, τ, E) be a SMTS. Let $C_{S(e)}(x) \leq C_{X(e)}(x)$ be *gssm*-compact relative to X_E and T_E be a *gscs*-mset of X_E . Then $S_E \cap T_E$ is *gssm*-compact relative to X_E .

Proof. Let $\{(U_E)_\alpha: \alpha \in \Lambda\}$ be any soft multi open cover of $S_E \cap T_E$ be *gsos*-msets of (X, τ, E) . Then $\{(U_E)_\alpha: \alpha \in \Lambda\} \cup (X_E - T_E)$ is soft multi open cover of *gsos*-msets of (X, τ, E) . Since S_E is *gssm*-compact relative to X_E , there exists a finite sub collection Λ_o of Λ such that $C_{S(e)}(x) \leq C_{\{(U_\alpha): \alpha \in \Lambda_o\} \cup (X - T)}(x)$. Then $\{(U_E)_\alpha: \alpha \in \Lambda_o\}$ is a finite *gssm*-open sub cover of $S_E \cap T_E$, which implies that $S_E \cap T_E$ is *gssm*-compact relative to X_E . $\square$

Theorem 4.1. A SMTS (X, τ, E) is *gssm*-compact if and only if every collection of *gscs*-msets of X_E with finite intersection property has a non-empty intersection.

Proof. Let (X, τ, E) be a *gssm*-compact and let $\{(F_E)_\alpha\}$ be a family of *gscs*-msets with the finite intersection property. Suppose that $C_{\cap_\alpha (F_\alpha)(e)}(x) = C_{\phi(e)}(x)$. Then $C_{\cap_\alpha (F_\alpha)^c(e)}(x) = C_{X(e)}(x)$. But X_E is *gssm*-compact, $C_{X(e)}(x) = C_{\cup_{i=1}^n (F_{\alpha_i})^c(e)}(x)$ and therefore $C_{\cap_{i=1}^n F_{\alpha_i}(e)}(x) = C_{\phi(e)}(x)$. This contradicts the finite intersection property of $\{(F_E)_\alpha\}$. Hence $C_{\cap_\alpha (F_\alpha)(e)}(x) \neq C_{\phi(e)}(x)$.

Conversely, suppose that every collection of *gscs*-msets with finite intersection property has a non-empty intersection. Let $\{(G_E)_\alpha\}$ be a *gssm*-open cover of X_E such that $C_{X(e)}(x) = C_{\cup_\alpha (G_\alpha)(e)}(x)$. Then $C_{\cap_\alpha (G_\alpha)^c(e)}(x) = C_{\phi(e)}(x)$, by assumption, this family does not have finite intersection property. That is $C_{\cap_{i=1}^n (G_{\alpha_i})^c(e)}(x) = C_{\phi(e)}(x)$ and so $C_{\cup_{i=1}^n G_{\alpha_i}(e)}(x) = C_{X(e)}(x)$. Hence (X, τ, E) is *gssm*-compact. $\square$

Corollary 4.2. A SMTS (X, τ, E) is *gssm*-compact if and only if every collection of *gscs*-msets of X_E with empty intersection has a finite sub family with empty intersection.

Definition 4.5. A SMTS (X, τ, E) is said to be locally *gssm*-compact if every soft multi point in X_E has at least one *gssm*-nbd whose closure is *gssm*-compact relative to X_E .

Proposition 4.7. Every *gssm*-compact space is locally *gssm*-compact but not conversely.

Proof. Let (X, τ, E) be a *gssm*-compact. Let $[(n/x)_e]_E \in X_E$. Then X_E is an *gssm*-nbd of $[(n/x)_e]_E$ such that $C_{cl(X)(e)}(x) = C_{X(e)}(x)$ is *gssm*-compact. Thus every soft multi point in X_E has at least one *gssm*-nbd, namely X_E whose closure is *gssm*-compact. Hence (X, τ, E) locally *gssm*-compact. $\square$

4 Discussion and Conclusions

The main focus of this work is the introduction of generalized semi soft multi connectedness and compactness by using generalized semi soft multi separated sets, soft multi adherent point, generalized semi soft multi adherent point and generalized semi soft multi open cover. Also, many related theorems and propositions are provided. As a continuation of this work, in future, more generalizations of soft multi connectedness and compactness will be investigated. Further we have an idea to extend the work in separation axioms and filters.

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Split Legendary Domination in Graphs

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Abstract

The concept of domination arises due to the necessity of various types of facility location problem in a network which includes the one to one correspondence between them. In a network, nodes with nearly equal capacity may interact with each other in better way. Motivating by this, the equivalent property of dominating set of a graph G and its line graph G_l are studied. This make us to introduce the legendary domination. In this paper, by posting the disconnected property on the legendary dominating set, the split legendary domination are introduced and defined. Further, the graph theoretical parameters are studied in terms of elements of G and its relationship with other domination parameters are presented.

Keywords: Split domination number, line domination number, legendary domination number, split legendary domination number.

2020 AMS subject classifications: 05C69, 05C76 ¹

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1 Introduction

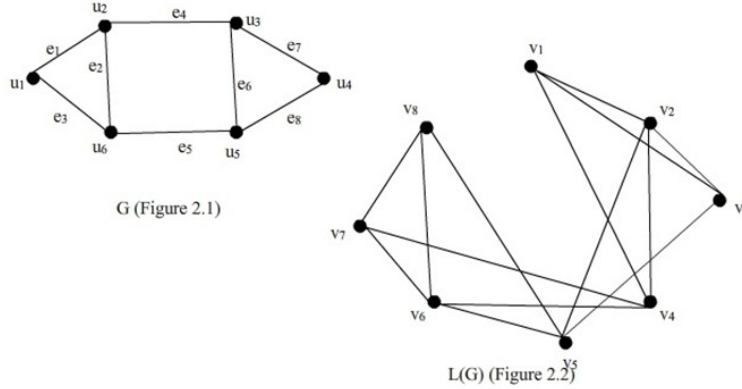
A graph $G = (V, E)$ we mean a finite, undirected, simple and connected graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [1]. In mathematical discipline of graph theory, the line graph of an undirected graph G is another G_l that represents the adjacencies between edges of G . A line graph can be viewed in several ways for example, the edges of G can be considered as elements of the graph and the adjacency in G_l determined by relations between the elements. The line graph of G , denoted G_l , is the graph with vertex set $E(G)$, where vertices x and y are adjacent in G_l if and only if edges x and y share a common vertex in G . This was introduced by Harary and Norman [2]. The basic motivation for the concept of domination arises due to the necessity of various types of facility location problem in a network which includes the one to one correspondence between them. A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a dominating set of G , if every vertex in $V \setminus D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . This concept was introduced by Ore in [14]. The study of domination in line graph is named as line domination, the study of domination in edges of a graph is called as edge domination. A set $D \subseteq V(G_l)$ is said to be line dominating set of G , if every vertex not in D is adjacent to a vertex in D . The line domination number of G is denoted by $\gamma_l(G)$ is the minimum cardinality of a line dominating set of G_l . It was introduced by S. R. Jayaram in the year 1987 [3]. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Motivating this, the equivalent property of the dominating set G and its line graph G_l is studied. This make us to defined the legendary domination parameters [7]. The legendary domination is relating vertex domination and edge domination of a graph G . A dominating set $D \subseteq V(G)$ of graph $G = (V, E)$ is said to be legendary dominating set of G , if D is a dominating set of G and G_l . The legendary domination number is the minimum cardinality taken over all legendary dominating sets of G and is denoted by $\gamma_{le}(G)$. A dominating set $D \subseteq V(G)$ is a split dominating set, if the induced subgraph $\langle V \setminus D \rangle$ is disconnected. This concept was introduced by V. R. Kulli and B. Janakiram in [6]. A line dominating set $D \subseteq V(L(G))$ is a split line dominating set, if the induced subgraph $\langle V(L(G)) \rangle$ is disconnected. The minimum cardinality of such a set is called split line dominating number of G and is denoted by $\gamma_{sl}(G)$. This was introduced by M.H.Muddebihal, V.A.Panfarosh, Anil R.Sedamkar [13]. In this paper, the split legendary domination are introduced by posting the disconnected property on the legendary dominating set. The bounds of this parameter was studied and its exact value was found for some standard graphs. The connectivity $k = K(G_l)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph.

2 Preliminaries

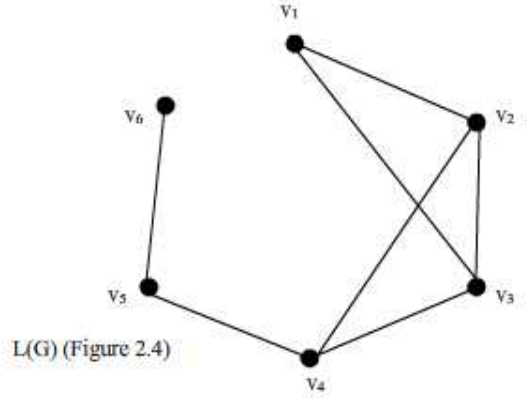
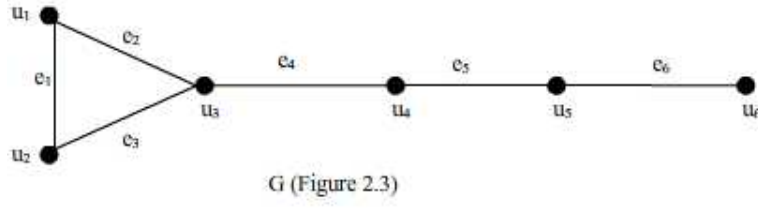
Definition 2.1. A split dominating set $X \subseteq V(G_l)$ is said to be split legendary dominating set (sle-d-set), if G has a split dominating set of cardinality X . The split legendary domination number is the minimum cardinality taken over all split legendary dominating set of G and is denoted by $\gamma_{sle}(G)$.

Remark 2.1. 1. Let $G, L(G) = G_l$ be respectively denote a (p, q) graph, and its line graph. Also, take $V(G) = \{u_1, u_2, \dots, u_p = u_n\}, E(G) = \{e_1, e_2, \dots, e_q = e_n\}$ $V(G_l) = V_l = \{v_1, v_2, \dots, v_{pl}\}, E(G_l) = \{f_1, f_2, \dots, f_{ql}\}$
2. All graphs consider in this paper are having at least one sled-set.

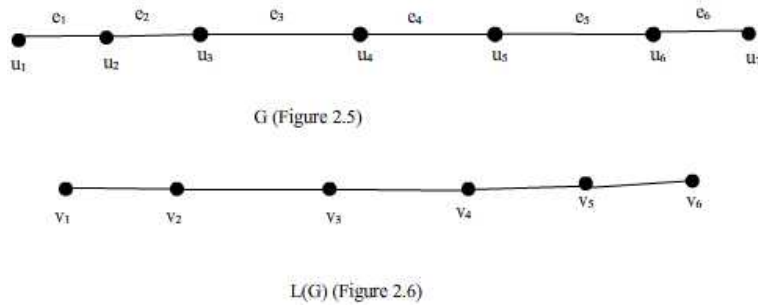
Example 2.1. For the graph G_l in Figure 2.2, the vertex set $D = \{v_1, v_3, v_4, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 4$. Also note that the graph G in Figure 2.1, the vertex set $D = \{u_1, u_3, u_7\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 3$. By the definition 2.1, the vertex set $X = \{v_1, v_3, v_4, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 4$. Here note that $\gamma_s(G) < \gamma_{sl}(G) \leq \gamma_{sle}(G)$.



Example 2.2. For the graph G_l in Figure 2.4, the vertex set $D = \{v_3, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 2$. Also note that the graph G in Figure 2.3, the vertex set $D = \{u_3, u_5\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 2$. By the definition 2.1, the vertex set $X = \{v_3, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$. Here note that $\gamma_s(G) = \gamma_{sl}(G) = \gamma_{sle}(G)$.



Example 2.3. For the graph G_l in Figure 2.6, the vertex set $D = \{v_2, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 2$. Also note that the graph G in Figure 2.5, the vertex set $D = \{u_2, u_5, u_7\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 3$. By the definition 2.1, the vertex set $X = \{v_1, v_2, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$. Here note that $\gamma_{sl}(G) < \gamma_s(G) \leq \gamma_{sle}(G)$.



Remark 2.2. (i) For the trivial graph, there is no split legendary dominating set.
(ii) For a connected graph, one can always find a split legendary dominating set. Therefore, for the existence of split legendary domination number; hereafter we assume that G is a connected graph with atleast two edge.

In view of the graphs given in Example 2.1, 2.2, 2.3, we have the following result.

Theorem 2.1. *For any graph G ,*

$$i \quad \gamma_s(G) \leq \gamma_{sle}(G).$$

$$ii \quad \gamma_{sl}(G) \leq \gamma_{sle}(G).$$

Proof. Since every split legendary dominating set of G is a split dominating set of G gives the result $\gamma_s(G) \leq \gamma_{sle}(G)$.

Since every split legendary dominating set of G is a split dominating set of G_l gives the result $\gamma_{sl}(G) \leq \gamma_{sle}(G)$. $\square$

Corollary 2.1. *For any graph G , $\gamma_{sle}(G) = \max(\gamma_s(G), \gamma_{sl}(G))$.*

Proof. By Theorem 2.1, we have the minimum cardinality of the split legendary domination number is atleast the minimum domination number of G or G_l .

Hence $\gamma_{sle}(G)$ is the maximum value of $\gamma_s(G)$ and $\gamma_{sl}(G)$. $\square$

3 Characterizations, Bounds and Exact values

The following result gives split legendary dominating sets of G which are minimal.

Theorem 3.1. *A split legendary dominating set $X \subseteq G_l$ is minimal for each vertex $x \in X$, one of the following three conditions holds:*

i There exists vertex $y \in V_l - X$, such that $N(y) \cap X = \{x\}$.

ii x is an isolated vertex in $\langle X \rangle$.

iii $\langle (V_l - x) \cup \{x\} \rangle$ is connected.

Proof. Let G be a graph and G_l be its line graph with vertex set V_l . Let $X \subseteq V_l$ be a split legendary split dominating set of G , since X is minimal and there exists a vertex $x \in X$ such that x does not hold any of the above conditions, then by condition (i) and (ii), $X_1 = X - \{x\}$ is a dominating set of G_l . Also by (iii) $\langle V_l - X \rangle$ is disconnected. This implies that X_1 is a split legendary dominating set of G , which is contradiction to the fact X is minimal. $\square$

The following theorem gives the necessary and sufficient conditions for a dominating set to be a split legendary dominating set of G .

Theorem 3.2. *A dominating set $X \subseteq G_l$ is a split legendary dominating set of G if and only if there exists two vertices v_1, v_2 in the induced subgraph $\langle V_l \setminus X \rangle$ such that $v_1 - v_2$ path of G_l contains atleast one vertex of X .*

Proof. Let G_l be a line graph of graph G and $X \subseteq V_l$ be a split legendary dominating set of G , since cardinality of X is a split dominating set of G .

It implies that the induced subgraph $\langle V_l \setminus X \rangle$ such that there is no path between v_1 and v_2 . In other words, G_l being connected, the original path between v_1 and v_2 in G_l does not exist due to the removal of X from V_l .

Thus, it follows that there exists two vertices v_1, v_2 in induced subgraph $\langle V_l \setminus X \rangle$ such that every $v_1 - v_2$ path of G_l contains at least one vertex in X .

Conversely, suppose the induced subgraph $\langle V_l \setminus X \rangle$ is connected. Then there exists a path between v_1 and v_2 in $\langle V_l \setminus X \rangle$, which is a contradiction to the fact that $v_1 - v_2$ path of G_l contains at least one vertex of X . Hence, $\langle V_l \setminus X \rangle$ is disconnected. therefore X is a split legendary dominating set of G . $\square$

Now we give a lower bound for split legendary dominating sets of G in terms of vertex connectivity $k(G_l)$

Theorem 3.3. For any graph G , $k(G_l) \leq \gamma_{sle}(G)$

Proof. Let G_l be a line graph of G . Let S be a set of minimum number of vertices of G such that whose removal makes the graph disconnected. By the definition of vertex connectivity of a graph G_l , we have

$$|S| = k(G_l) \dots \dots \dots (1)$$

Let $X \subseteq V_l$ be a split legendary dominating set with minimum cardinality, since cardinality of X is a split dominating set of G . Therefore,

$$|X| = \gamma_{sle}(G) \dots \dots \dots (2)$$

As X also has the property of disconnecting the graph, we have $|S| \leq |X|$. By equation (1) and (2) we have $k(G_l) \leq \gamma_{sle}(G)$. $\square$

The following result provides an upper bound for split legendary dominating sets of G in terms of maximum degree $\Delta(G_l)$.

Theorem 3.4. For any graph G , $\gamma_{sle}(G) \leq p_l \frac{\Delta(G_l)}{\Delta(G_l)+1}$.

Proof. Let G_l be a line graph of G and $X \subseteq V_l$ be the split legendary dominating set of G , since X is minimal by Theorem 3.1, it follows that for each vertex $x \in X$, there exists a vertex $y \in V_l - X$ such that x is adjacent to y . This implies that $V_l - X$ is a split legendary dominating set of G . Thus, $\gamma_l(G) \leq |V_l - X| p_l - \gamma_{sle}(G)$.

By the Theorem, "For any graph G , $\gamma_l(G) \geq \frac{p_l}{(\Delta(G_l)+1)}$ ".

Proof: Let X be a γ_l -set of G . Each vertex dominates at most itself and $\Delta(G_l)$ other vertices. Hence the result".

Hence, the upper bound follows from the fact $\gamma_l(G) \geq \frac{p_l}{(\Delta(G_l)+1)}$. $\square$

Split Legendary Domination in Graphs

The exact values of split legendary domination number $\gamma_{sle}(G)$ for some standard graphs are given below.

Theorem 3.5. For complete graph K_n , $\gamma_{sle}(G) = 2n - 4, n \geq 4$.

Proof. Let G be a complete graph K_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{\frac{n(n-1)}{2}}\}$. The vertex set $X = \{v_1, v_2, \dots, v_{n-2}, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n-3}\}$ is a sled – set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = 2n - 4 \dots \dots \dots (1)$$

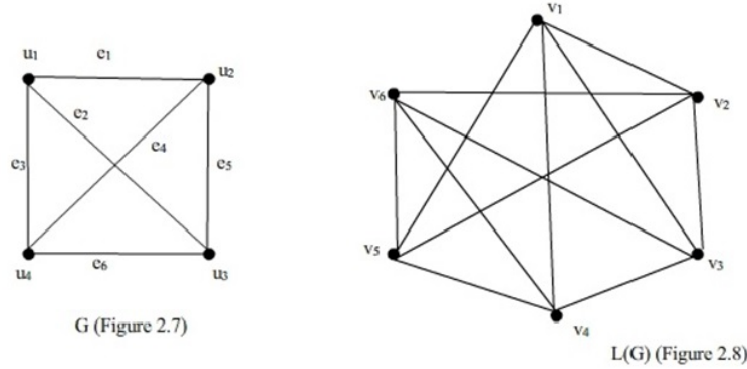
Let X be a γ_{sle} – set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a complete graph, X has atleast $2n - 4$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \geq 2n - 4 \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = 2n - 4, n \geq 4$. □

Example 3.1. For the graph G_l in Figure 2.8, the vertex set $X = \{v_1, v_2, v_4, v_5\} \subseteq V(G_l)$ is the minimum legendary split dominating set of G and hence $\gamma_{sle}(G) = 4$.



Theorem 3.6. For fan graph $F_{1,n}$, $\gamma_{sle}(G) = n, n \geq 3$.

Proof. Let G be a fan graph $F_{1,n}$ with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{2n-1}\}$. The vertex set $X = \{v_2, v_3, v_4, v_5, \dots, v_{n+1}\}$ is a sled – set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{les}(G) \leq |X| = n \dots \dots \dots (1)$$

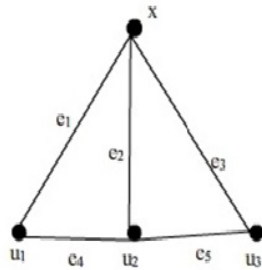
Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a fan graph, X has atleast n vertices.

Hence,

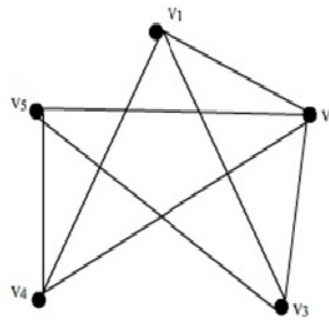
$$\gamma_{sled}(G) = |X| \leq n \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 2$. $\square$

Example 3.2. For the graph G_l in Figure 2.10, the vertex set $X = \{v_2, v_3, v_4\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.



G (Figure 2.9)



$L(G)$ (Figure 2.10)

Theorem 3.7. For friendship graph F_n , $\gamma_{sle}(G) = n + 1, n \geq 2$.

Proof. Let G be a friendship graph F_n with atleast five vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{3n}\}$. The vertex set $X = \{v_1, v_{3i}/i = 1, 2, \dots, n\}$ is a sled - set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1 \dots \dots \dots (1)$$

Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a friendship graph, X has atleast $n + 1$ vertices.

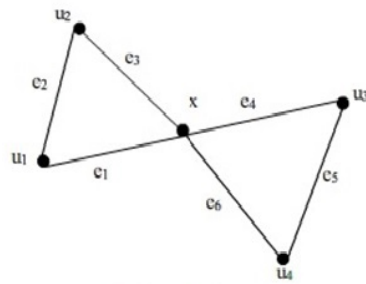
Hence,

$$\gamma_{sle}(G) = |X| \leq n + 1 \dots \dots \dots (2)$$

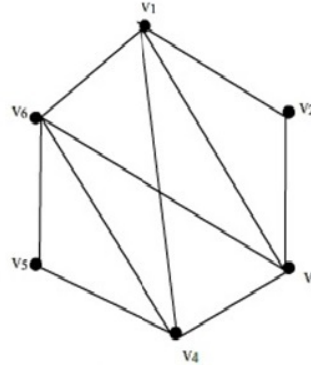
Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 2$. $\square$

Example 3.3. For the graph G_l in Figure 2.12, the vertex set $X = \{v_1, v_3, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.

Split Legendary Domination in Graphs



G (Figure 2.11)



L(G) (Figure 2.12)

Theorem 3.8. For ladder graph L_n ,

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 2 \\ n, & n \geq 3 \end{cases}$$

Proof. Let G be a ladder graph L_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{3(n-2)}\}$.

Case (i) $n = 2$

In this case, the vertex set $X = \{v_2, v_4\}$ is a minimum split Legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \geq 3$

In this case, the vertex set $X = \{v_1, v_{n+1}\} \cup \{v_4, v_7, v_{10}, \dots, v_{3n-5}\}$ is a minimum split Legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n \dots \dots \dots (1)$$

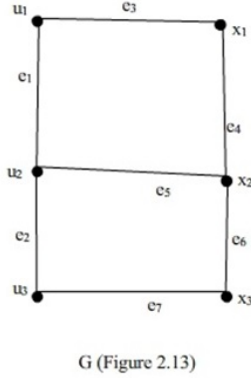
Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a ladder graph, X has atleast n vertices.

Hence,

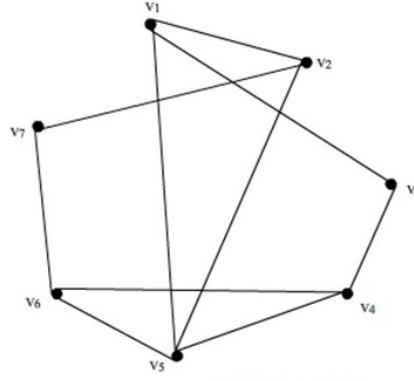
$$\gamma_{sle}(G) = |X| \leq n \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 3$. □

Example 3.4. For the graph G_l in Figure 2.14, the vertex set $X = \{v_1, v_4, v_7\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.



G (Figure 2.13)



L(G) (Figure 2.14)

Theorem 3.9. For wheel graph W_n ,

$$\gamma_{sle}(G) = \begin{cases} 4, & n = 3 \\ \lceil \frac{n+3}{2} \rceil, & n \geq 4 \end{cases}$$

Proof. Let G be a wheel graph W_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{2n}\}$.

Case (i) $n = 3$

In this case, the vertex set $X = \{v_2, v_3, v_4, v_5\}$ is a minimum split legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected and hence $\gamma_{sle}(G) = 4$

Case (ii) $n \equiv 0(mod 2)$

In this case, the vertex set $X = \{v_2, v_n, v_{n+1}, v_{n+2}\} \cup \{v_{2i}/i = 2, 3, \dots, n-4\}$ is a minimum split Legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+3}{2} = \lceil \frac{n+3}{2} \rceil \dots \dots \dots (1)$$

Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a wheel graph, X has atleast $\frac{n+3}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+3}{2} = \lceil \frac{n+3}{2} \rceil \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lceil \frac{n+3}{2} \rceil$.

Case (iii) $n \equiv 1(mod 2)$

In this case, the vertex set $X = \{v_2, v_n, v_{n+1}, v_{n+2}\} \cup \{v_{2i}/i = 2, 3, \dots, n-5\}$ is a minimum split legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

Split Legendary Domination in Graphs

$$\gamma_{sle}(G) \leq |X| = \frac{n+4}{2} = \lceil \frac{n+3}{2} \rceil \dots \dots \dots (3)$$

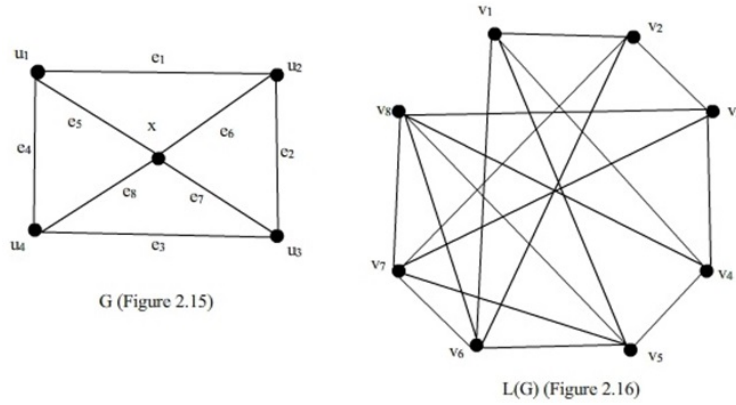
Let X be a γ_{sle} - set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a wheel graph, X has atleast $\frac{n+4}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+4}{2} = \lceil \frac{n+3}{2} \rceil \dots \dots \dots (4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lceil \frac{n+3}{2} \rceil$. □

Example 3.5. For the graph G_l in Figure 2.16, the vertex set $X = \{v_2, v_4, v_5, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 4$.



Theorem 3.10. For path P_n ,

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 4, 5, 6 \\ \lceil \frac{n}{3} \rceil, & n \geq 7 \end{cases}$$

Proof. Let G be a path P_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{n-1}\}$.

Case (i) $n = 4, 5, 6$

In this case, the vertex set $X = \{v_2, v_4\}$ is a minimum split Legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \equiv 0(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 1, 2, 3, \dots, n-7\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (1)$$

Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a path, X has atleast $\frac{n}{3}$ vertices.

Hence,

$$\gamma_{les}(G) = |X| \leq \frac{n}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$.

Case (iii) $n \equiv 1(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 2, 3, \dots, n-5\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (3)$$

Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a path, X has atleast $\frac{n+2}{3}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+2}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$.

Case (iv) $n \equiv 2(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 2, 3, \dots, n-6\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+1}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (5)$$

Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a path, X has atleast $\frac{n+1}{3}$ vertices.

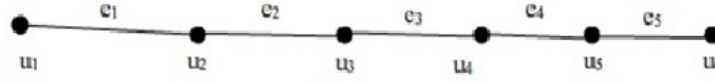
Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+1}{3} = \lceil \frac{n}{3} \rceil \dots \dots \dots (6)$$

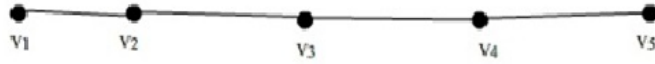
Then from equation (5) and (6) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$. □

Example 3.6. For the graph G_l in Figure 2.18, the vertex set $X = \{v_2, v_5\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$.

Split Legendary Domination in Graphs



G (Figure 2.17)



L(G) (Figure 2.18)

Theorem 3.11. For triangular snake graph T_n , $\gamma_{sle}(G) = n + 1, n \geq 2$.

Proof. Let G be a triangular snake graph T_n with atleast five vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{\frac{n(n-1)}{2}}\}$. The vertex set $X = \{v_2, v_{3i}/i = 1, 2, \dots, n\}$ is a lessd – set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1 \dots \dots \dots (1)$$

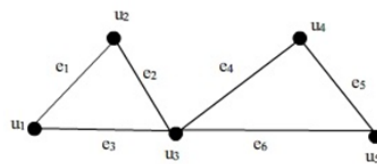
Let X be a γ_{sle} – set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a triangular snake graph, X has atleast $n + 1$ vertices.

Hence,

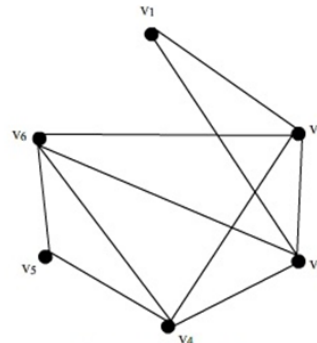
$$\gamma_{sle}(G) = |X| \leq n + 1 \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 2$. □

Example 3.7. For the graph G_l in Figure 2.20, the vertex set $X = \{v_2, v_3, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.



G (Figure 2.19)



L(G) (Figure 2.20)

Theorem 3.12. For book graph B_n ,

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 1 \\ n + 1, & n \geq 2 \end{cases}$$

Proof. Let G be a book graph B_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{3n}\}$.

Case (i) $n = 1$

In this case, the vertex set $X = \{v_1, v_3\}$ is a minimum split legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \geq 2$

In this case, the vertex set $X = \{v_2, v_4\} \cup \{v_{3i}/i = 2, 3, \dots, n\}$ is a minimum split legendary dominating set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1 \dots \dots \dots (1)$$

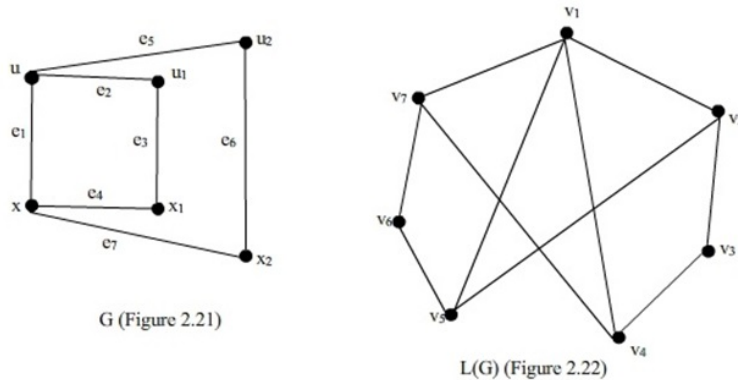
Let X be a γ_{sle} -set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a book graph, X has atleast $n + 1$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq n + 1 \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 3$. □

Example 3.8. For the graph G_l in Figure 2.22, the vertex set $X = \{v_2, v_4, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.



Theorem 3.13. For cycle graph C_n ,

$$\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor, \quad n \geq 4$$

Split Legendary Domination in Graphs

Proof. Let G be a cycle C_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_n\}$.

Case (i) $n \equiv 0(mod 2)$

In this case, the vertex set $X = \{v_2, v_n\} \cup \{v_4, v_6, v_8, \dots, v_{n-2}\}$ is a minimum split legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor \dots \dots \dots (1)$$

Let X be a γ_{sle} - set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a cycle graph, X has atleast $\frac{n}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n}{2} = \lfloor \frac{n}{2} \rfloor \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor$.

Case (ii) $n \equiv 1(mod 2)$

In this case, the vertex set $X = \{v_2, v_n\} \cup \{v_4, v_6, v_8, \dots, v_{n-3}\}$ is a minimum split legendary dominating set of G , since induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor \dots \dots \dots (3)$$

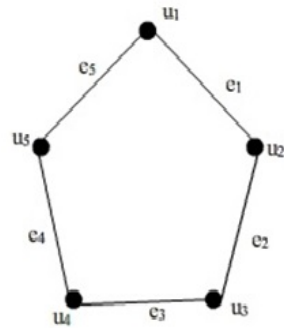
Let X be a γ_{sle} - set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a cycle graph, X has atleast $\frac{n-1}{2}$ vertices.

Hence,

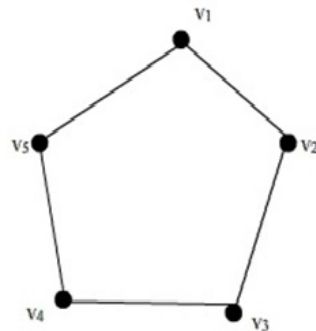
$$\gamma_{sle}(G) = |X| \leq \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor \dots \dots \dots (4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor$. □

Example 3.9. For the graph G_l in Figure 2.24, the vertex set $X = \{v_2, v_5\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$.



G (Figure 2.23)



L(G) (Figure 2.24)

Theorem 3.14. For crown graph C_n^+ , $\gamma_{sle}(G) = n, n \geq 3$.

Proof. Let G be a crown graph C_n^+ with atleast six vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, \dots, v_{2n}\}$. The vertex set $X = \{v_2, v_3, v_4, v_5, \dots, v_{n+1}\}$ is a sled – set of G , since the induced subgraph $\langle V_l \setminus X \rangle$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n \dots \dots \dots (1)$$

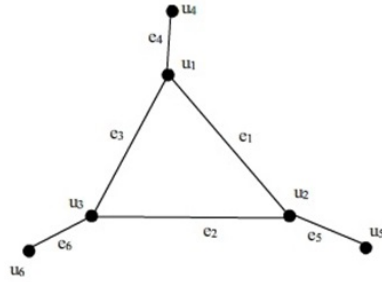
Let X be a γ_{sle} – set of G . Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l , where v_1, v_2 are the vertices in the disconnected subgraph $\langle V_l \setminus X \rangle$. As G being a crown graph, X has atleast n vertices.

Hence,

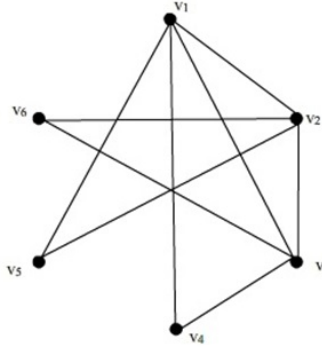
$$\gamma_{sle}(G) = |X| \leq n \dots \dots \dots (2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 3$. □

Example 3.10. For the graph G_l in Figure 2.26, the vertex set $X = \{v_2, v_3, v_4\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.



G (Figure 2.25)



L(G) (Figure 2.26)

4 Discussion

The applications of domination in graphs lies in various fields in solving real life problems. It includes social network theory, land surveying, radio stations, computer communication networks and school bus routing. The online social network has been developed significantly in the recent years as medium of communication, sharing the information and spreading the influence. Recent advances in technology have provided portable computers with wireless interfaces that allowed network communication among mobile users. The resulting computing environment, often referred to as wireless mobile computing, no longer requires

users to maintain a fixed and universally known position in the network and enables almost unrestricted mobility. A network with clustering is divided into several clusters. Within each cluster, one of the nodes is elected as a dominating nodes and with the rest being adjacent dominating nodes. The dominating node will collect data from its adjacent dominating nodes directly.

The present work, the split legendary dominating set is introduced and the corresponding split legendary domination number is defined. Further, the relationship with other domination parameter are obtained. The characterization of the split legendary dominating set is also given and the exact values of the parameter is found for complete graph, cycle graph, path, wheel graph, crown graph, ladder graph, book graph, triangular snake graph, fan graph and friendship graph. By using the split legendary domination one can optimize, the parameters involved in network theory.

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On an abstract nonlinear functional second order Volterra integrodifferential equation

P. M. Dhakane*

Abstract

In this paper, we prove the existence, uniqueness and boundedness of solutions of a nonlinear functional second order Volterra integrodifferential equation in a general Banach space. The theory of strongly continuous cosine family, modified version of Banach contraction mapping principle and Pachpatte's integral inequality are main techniques employed in our analysis.

Keywords: Volterra integro-differential equation, Cosine family, Modified version of Banach contraction mapping principle, Pachpatte's integral inequality.

2020 AMS subject classifications: 45D05, 34G20, 45N05, 45P05, 37C25.¹

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1 Introduction

In what follows, the symbol X stands for a Banach space with norm $\|\cdot\|$. Let $C = C([-r, 0], X)$, $0 < r < \infty$, be the Banach space of all continuous functions ψ from $[-r, 0]$ to X with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \leq \theta \leq 0\}.$$

Let $B = C([-r, T], X)$, $T > 0$ denotes the Banach space of all continuous functions $x : [-r, T] \rightarrow X$ with supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \leq t \leq T\}$. For any $x \in B$ and $t \in [0, T]$, x_t is the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Now, we study the abstract nonlinear functional second order Volterra integrodifferential equation of the form

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t, s)g(s, x_s)ds\right), \quad t \in [0, T] \quad (1)$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0, \quad x'(0) = \delta \quad (2)$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X , $f : [0, T] \times C \times X \rightarrow X$, $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$, $g : [0, T] \times C \rightarrow X$ are continuous functions, ϕ and δ are given elements of $C = C([-r, 0], X)$ and X respectively.

The equations of these types (1)-(2) and their special forms serve as an abstract formulation of many partial differential equations or partial integrodifferential equations arising in variety of physical phenomena, see, for example [2, 3, 6, 7, 11] and some of the references given therein. The problems of existence, uniqueness and other properties of solutions of various special forms of (1)-(2) have been studied by using different techniques by many authors, see, for example [1, 4, 5, 8, 9, 10, 14, 15, 17, 19, 20, 21, 22] and some of the references cited therein. Our endeavour is to generalize some results obtained by A Pazy [15] and C.C. Travis and G. F. Webb [20]. To treat second order abstract differential equations directly is worthwhile rather than to convert into first order systems, see, for example Fitzgibbon [11]. In [11], Fitzgibbon employed the second order abstract equation for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. Our work in this paper is motivated by the interesting results obtained by Fattorini H. O. in [10] and is influenced by the work of Travis C. C. and Webb G. F. [21]. The aim of the present paper is to investigate the existence, uniqueness and boundedness of solution to second order Volterra functional integrodifferential equations (1)-(2) by using the theory of cosine and sine families of operators, modified version of contraction mapping principle and Pachpatte's integral inequality.

On an abstract nonlinear functional second order Volterra integrodifferential equation

The paper is organized as follows: Section 2 presents the preliminaries and statements of our results. In section 3, we investigate Theorems 2.1 and 2.2. Finally, section 4 exhibits an example to illustrate the applications of our results.

2 Preliminaries and statement of results

Before proceeding to the statements of our main results, we set forth some preliminaries from [12, 18, 20] and hypotheses used in our further discussion.

Definition 2.1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if*

- (a) $C(0) = I$ (I is the identity operator);
- (b) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;
- (c) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

The associated strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \quad (3)$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : X \rightarrow X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$.

Definition 2.2. *Let $f \in L^1(0, T; X)$. The function $x \in B$ defined by*

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\delta \\ &\quad + \int_0^t S(t-s)f\left(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau\right)ds, \quad t \in [0, T] \end{aligned} \quad (4)$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0 \quad (5)$$

is called mild solution of the initial value problem (1)-(2).

we, now list the following hypotheses for our further discussion.

(H₁) There are constants $K \geq 1$ and $K_1 > 0$ such that

$$\|C(t)\| \leq K \quad \text{and} \quad \|S(t)\| \leq K_1,$$

for all $t \in [0, T]$.

(H₂) For every $t \in [0, T]$, $\psi \in C$ and $x \in X$, there exist a continuous function $p : [0, T] \rightarrow \mathbb{R}_+$ such that

$$\|f(t, \psi, x)\| \leq p(t) [\|\psi\|_C + \|x\|].$$

(H₃) There exist a continuous function $q : [0, T] \rightarrow \mathbb{R}_+$ such that

$$\|g(t, \psi)\| \leq q(t) \|\psi\|_C$$

for every $t \in [0, T]$ and $\psi \in C$.

(H₄) For every $t \in [0, T]$, $\psi_1, \psi_2 \in C$ and $x_1, x_2 \in X$, there exists a constant M such that

$$\|f(t, \psi_1, x_1) - f(t, \psi_2, x_2)\| \leq M [\|\psi_1 - \psi_2\|_C + \|x_1 - x_2\|]$$

(H₅) There exists a constant N such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq N \|\psi_1 - \psi_2\|_C,$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C$.

We use modified version of Banach contraction mapping principle to prove our results.

Lemma 2.1 ([16], p. 196). *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is positive integer. Then D has a unique fixed point in X .*

The following Pachpatte's inequality is the key instrument in our subsequent discussion.

Lemma 2.2 ([13], p. 758). *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s) \left[u(s) + \int_0^s q(\tau) u(\tau) d\tau \right] ds,$$

holds for all $t \in \mathbb{R}_+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0 \left[1 + \int_0^t p(s) \exp \left(\int_0^s (p(\tau) + q(\tau)) d\tau \right) ds \right],$$

holds for all $t \in \mathbb{R}_+$.

On an abstract nonlinear functional second order Volterra integrodifferential equation

With these preparations, now, we are in position to state our main results.

Theorem 2.1. *Suppose that the hypotheses (H_1) , (H_4) and (H_5) are satisfied. Then for each $\phi, \delta \in C \times X$, the initial value problem (1)-(2) has a unique mild solution $x \in B$ on $[-r, T]$. Moreover, the mapping $(\phi, \delta) \rightarrow x$ is Lipschitz continuous from $C \times X$ into B .*

Theorem 2.2. *Suppose that the hypotheses $(H_1) - (H_3)$ hold. Then, every solution of the initial value problem (1)-(2) is bounded on $[-r, T]$.*

3 Proofs of the Theorem 2.1 and 2.2

Proof of Theorem 2.1. Define a mapping $F : B \rightarrow B = C([-r, T], X)$ by

$$(Fx)(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ C(t)\phi(0) + S(t)\delta \\ \quad + \int_0^t S(t-s)f(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau)ds, & 0 \leq t \leq T. \end{cases} \quad (6)$$

In order to find mild solution of the initial value problem (1)-(2), it is sufficient to find the fixed point of an operator equation $x(t) = (Fx)(t)$. Since k is continuous real valued function on a compact set $[0, T] \times [0, T]$, there is a constant $L > 0$ such that

$$|k(t, s)| \leq L \quad \text{for } 0 \leq s \leq t \leq T \quad (7)$$

Let $x, y \in B = C([-r, T], X)$. Then by using equation (6), condition (7) and hypotheses (H_1) , (H_4) and (H_5) we have

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &\leq \int_0^t \|S(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau\right) \right. \\ &\quad \left. - f\left(s, y_s, \int_0^s k(s, \tau)g(\tau, y_\tau)d\tau\right) \right\| ds \\ &\leq K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \\ &\leq K_1 M \int_0^t \left[\|x - y\|_B + LN \int_0^s \|x - y\|_B d\tau \right] ds \\ &= K_1 t \left[M + LMN \frac{t}{2} \right] \|x - y\|_B. \end{aligned} \quad (8)$$

Case 1: Suppose $t \geq r$. Then, for every $\theta \in [-r, 0]$, we have $t + \theta \geq 0$. For such θ 's, from (8) we have

$$\begin{aligned} \|(Fx)_t(\theta) - (Fy)_t(\theta)\| &= \|(Fx)(t + \theta) - (Fy)(t + \theta)\| \\ &\leq K_1 M \int_0^{t+\theta} \{\|x - y\|_B + LN \int_0^s \|x - y\|_B d\tau\} ds \\ &\leq K_1 M \int_0^t \{\|x - y\|_B + LN \int_0^s \|x - y\|_B d\tau\} ds, \end{aligned}$$

which yields

$$\begin{aligned} \|(Fx)_t - (Fy)_t\|_C &\leq K_1 M \int_0^t \{\|x - y\|_B + LN \int_0^s \|x - y\|_B d\tau\} ds \\ &\leq t[K_1 M(1 + LN \frac{t}{2})] \|x - y\|_B \end{aligned} \quad (9)$$

Case 2: Suppose $0 \leq t < r$. Then for all $\theta \in [-r, -t]$, we have $t + \theta < 0$. For such θ 's, we observe, from (6), that

$$\begin{aligned} \|(Fx)_t(\theta) - (Fy)_t(\theta)\| &= \|(Fx)(t + \theta) - (Fy)(t + \theta)\| \\ &= 0, \end{aligned}$$

which implies

$$\|(Fx)_t - (Fy)_t\|_C = 0. \quad (10)$$

For $\theta \in [-t, 0]$, $t + \theta \geq 0$. Then, for such θ 's we obtain as in the case 1,

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN \frac{t}{2})] \|x - y\|_B \quad (11)$$

Thus, for every $\theta \in [-r, 0]$, ($0 \leq t \leq r$), from (10) and (11), we obtain

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN \frac{t}{2})] \|x - y\|_B \quad (12)$$

For every $t \in [0, T]$, from inequalities (9) and (12), we set

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN \frac{t}{2})] \|x - y\|_B \quad (13)$$

On an abstract nonlinear functional second order Volterra integrodifferential equation

By using the equation (6), the result (13), condition (7), and hypotheses (H_1) , (H_4) and (H_5) we get

$$\begin{aligned}
 \|(F^2x)(t) - (F^2y)(t)\| &= \|(F(Fx))(t) - (F(Fy))(t)\| \\
 &= \|(Fx_1)(t) - (Fy_1)(t)\| \\
 &\leq \int_0^t \|S(t-s)\| \left\| f\left(s, x_{1s}, \int_0^s k(s, \tau)g(\tau, x_{1\tau})d\tau\right) \right. \\
 &\quad \left. - f\left(s, y_{1s}, \int_0^s k(s, \tau)g(\tau, y_{1\tau})d\tau\right) \right\| ds \\
 &\leq K_1M \int_0^t \left[\|x_{1s} - y_{1s}\|_C + LN \int_0^s \|x_{1\tau} - y_{1\tau}\|_C d\tau \right] ds \\
 &\leq K_1M \int_0^t \left\{ K_1Ms \left[1 + LN \frac{s}{2}\right] \|x - y\|_B \right. \\
 &\quad \left. + LN \int_0^s K_1M\tau \left[1 + LN \frac{\tau}{2}\right] \|x - y\|_B d\tau \right\} ds \\
 &= \frac{t^2}{2} K_1^2 M^2 \|x - y\|_B \left[1 + 2LN \frac{t}{3} + L^2 N^2 \frac{t^2}{12}\right] \\
 &\leq \frac{t^2}{2} K_1^2 M^2 \|x - y\|_B \left[1 + 2LN \frac{t}{2} + L^2 N^2 \frac{t^2}{4}\right] \\
 &= \frac{t^2}{2} [K_1M(1 + LN \frac{t}{2})]^2 \|x - y\|_B \tag{14}
 \end{aligned}$$

By considering Case 1 and Case 2 as before and applying same arguments we prove that:

$$\|(F^2x)_t - (F^2y)_t\|_C \leq \frac{t^2}{2} [K_1M(1 + LN \frac{t}{2})]^2 \|x - y\|_B \tag{15}$$

for every $t \in [0, T]$.

Using equation (6), result (15), the condition (7), and hypotheses (H_1) , (H_4) and

(H_5) we obtain

$$\begin{aligned}
 \|(F^3x)(t) - (F^3y)(t)\| &= \|F(F^2x)(t) - F(F^2y)(t)\| \\
 &= \|(Fx_2)(t) - (Fy_2)(t)\| \\
 &\leq K_1M \int_0^t \left[\|x_{2s} - y_{2s}\|_C + LN \int_0^s \|x_{2\tau} - y_{2\tau}\|_C d\tau \right] ds \\
 &\leq K_1M \int_0^t \left\{ \frac{s^2}{2} \left[K_1M(1 + LN\frac{s}{2}) \right]^2 \|x - y\|_B \right. \\
 &\quad \left. + LN \int_0^s \frac{\tau^2}{2} \left[K_1M(1 + LN\frac{\tau}{2}) \right]^2 \|x - y\|_B d\tau \right\} ds \\
 &\leq K_1^3M^3 \|x - y\|_B \frac{t^3}{3!} \left[1 + LNt + L^2N^2\frac{t^2}{3} + L^3N^3\frac{t^3}{40} \right] \\
 &\leq K_1^3M^3 \|x - y\|_B \frac{t^3}{3!} \left[1 + 3LN\frac{t}{2} + 3L^2N^2\frac{t^2}{4} + L^3N^3\frac{t^3}{8} \right] \\
 &= \frac{t^3}{3!} \left[K_1M(1 + LN\frac{t}{2}) \right]^3 \|x - y\|_B \tag{16}
 \end{aligned}$$

By continuing the iteration process, it follows that

$$\begin{aligned}
 \|(F^n x)(t) - (F^n y)(t)\| &\leq \frac{t^n}{n!} [K_1M(1 + LN\frac{t}{2})]^n \|x - y\|_B \\
 &\leq \frac{T^n}{n!} [K_1M(1 + LN\frac{T}{2})]^n \|x - y\|_B
 \end{aligned}$$

which yields

$$\|F^n x - F^n y\|_B \leq \frac{1}{n!} [K_1MT(1 + LN\frac{T}{2})]^n \|x - y\|_B \tag{17}$$

For n large enough, $\frac{1}{n!} [K_1MT(1 + LN\frac{T}{2})]^n < 1$. This implies that F^n is a contraction in B for n sufficiently large and hence, by modified Banach contraction principle given in Lemma 2.1, it follows that F has a unique fixed point, say $x \in B$ which is the mild solution of initial value problem (1)–(2).

Next, we show that the mapping $(\phi, \delta) \rightarrow x$ is Lipschitz continuous from $C \times X$ into B . Assume that y is another mild solution of the initial value problem (1) on $[-r, T]$ with $y(t) = \chi(t)$, $-r \leq t \leq 0$ and $y'(0) = \sigma$. By making use of definition of mild solution given by (4)-(5) and hypotheses (H_1) , (H_4) and (H_5) ,

On an abstract nonlinear functional second order Volterra integrodifferential equation

we obtain

$$\begin{aligned}
 \|x(t) - y(t)\| &\leq \|C(t)\| \|\phi(0) - \chi(0)\| + \|S(t)\| \|\delta - \sigma\| \\
 &\quad + \int_0^t \|S(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau\right) \right. \\
 &\quad \left. - f\left(s, y_s, \int_0^s k(s, \tau)g(\tau, y_\tau)d\tau\right) \right\| ds \\
 &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^t K_1M \left[\|x_s - y_s\|_C \right. \\
 &\quad \left. + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \tag{18}
 \end{aligned}$$

Case1: Suppose $t \geq r$. Then, for every $\theta \in [-r, 0]$, we have $t + \theta \geq 0$. For such θ 's, from (18), we have

$$\begin{aligned}
 \|x(t + \theta) - y(t + \theta)\| &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^{t+\theta} K_1M \left[\|x_s - y_s\|_C \right. \\
 &\quad \left. + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \\
 &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^t K_1M \left[\|x_s - y_s\|_C \right. \\
 &\quad \left. + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds
 \end{aligned}$$

which yields

$$\begin{aligned}
 \|x_t - y_t\|_C &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| + \int_0^t K_1M \left[\|x_s - y_s\|_C \right. \\
 &\quad \left. + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \tag{19}
 \end{aligned}$$

Case2: Suppose $\theta \leq t < r$. Then, for all $\theta \in [-r, -t)$, we have $t + \theta < 0$. For such θ 's, we observe from (4)-(5) that

$$\begin{aligned}
 \|x(t + \theta) - y(t + \theta)\| &= \|x_t(\theta) - y_t(\theta)\| \\
 &= \|\phi(t + \theta) - \chi(t + \theta)\|
 \end{aligned}$$

which yields

$$\|x_t - y_t\|_C \leq \|\phi - \chi\|_C \tag{20}$$

For $\theta \in [-t, 0]$, $t + \theta \geq 0$. Then, for such θ 's, we get as in Case 1,

$$\begin{aligned} \|x_t - y_t\|_C &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \end{aligned} \quad (21)$$

Thus, for every $\theta \in [-r, 0]$, ($0 \leq t < r$), from (20) and (21) we obtain

$$\begin{aligned} \|x_t - y_t\|_C &\leq K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \end{aligned} \quad (22)$$

For every $t \in [0, T]$, from inequalities (19) and (22), we get

$$\begin{aligned} \|x_t - y_t\|_C &\leq K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \end{aligned} \quad (23)$$

Thanks to Pachpatte's inequality given in Lemma 2.2 and applying it to the inequality (23) with $u(t) = \|x_t - y_t\|_C$, we obtain

$$\begin{aligned} \|x_t - y_t\|_C &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\quad \left[1 + \int_0^t K_1 M \exp \left(\int_0^s (K_1 M + LN) d\tau \right) ds \right] \\ &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\quad [1 + K_1 M T \exp \{(K_1 M + LN)T\}] \end{aligned}$$

Thus, for $t \in [-r, T]$, we have

$$\begin{aligned} \|x(t) - y(t)\| &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\quad [1 + K_1 M T \exp \{(K_1 M + LN)T\}] \end{aligned}$$

and hence

$$\begin{aligned} \|x - y\|_B &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\quad [1 + K_1 M T \exp \{(K_1 M + LN)T\}] \end{aligned}$$

This proves Lipschitz continuity of the mapping $(\phi, \delta) \longrightarrow x$ and hence the proof of the Theorem is complete. $\square$

On an abstract nonlinear functional second order Volterra integrodifferential equation

Proof of Theorem 2.2. The solution of the initial value problem (1)-(2) is given by

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s)f\left(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau\right)ds$$

$$t \in [0, T] \quad (24)$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0 \quad (25)$$

If $t \in [0, T]$ then from (24) and using the hypotheses $(H_1) - (H_3)$, and condition (7) we have

$$\begin{aligned} \|x(t)\| &\leq \|C(t)\| \|\phi(0)\| + \|S(t)\| \|\delta\| \\ &\quad + \int_0^t \|S(t-s)\| \|f\left(s, x_s, \int_0^s k(s, \tau)g(\tau, x_\tau)d\tau\right)\| ds \\ &\leq K\|\phi(0)\| + K_1\|\delta\| + \int_0^t K_1p(s) \left[\|x_s\|_C + L \int_0^s q(\tau)\|x_\tau\|_C d\tau \right] ds \end{aligned}$$

Since $K \geq 1$, for $-r \leq t \leq T$, we get

$$\|x(t)\| \leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1p(s) \left[\|x_s\|_C + L \int_0^s q(\tau)\|x_\tau\|_C d\tau \right] ds \quad (26)$$

From (26) and considering case 1 and 2 as in the proof of the Theorem 2.1, we obtain

$$\begin{aligned} \|x_t\|_C &\leq K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1p(s)\|x_s\|_C ds \\ &\quad + \int_0^t K_1p(s) \int_0^s Lq(\tau)\|x_\tau\|_C d\tau ds \end{aligned} \quad (27)$$

Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (27) with $u(t) = \|x_t\|_C$, we get

$$\begin{aligned} \|x_t\|_C &\leq \left[K\|\phi\|_C + K_1\|\delta\| \right] \\ &\quad \left[1 + \int_0^t K_1p(s) \exp \left(\int_0^s (K_1p(\tau) + Lq(\tau))d\tau \right) ds \right] \\ &\leq \left[K\|\phi\|_C + K_1\|\delta\| \right] \left[1 + \{K_1P \exp(K_1P + LQ)T\}T \right] \end{aligned} \quad (28)$$

where

$$P = \max_{t \in [0, T]} p(t), \quad Q = \max_{t \in [0, T]} q(t).$$

It follows that solutions $x(t)$ of initial value problem (1)-(2) are bounded on closed interval $[-r, T]$ and proof of the Theorem is complete. $\square$

Remark 3.1. *It is to be noted that our result in above Theorem 2.2 also proves the stability of a solution $x(t)$ if $\|\phi\|_C, \|\delta\|$ are small enough.*

Remark 3.2. *It is important to note that cosine family $C(t)$ and sine family $S(t)$ are not bounded in $\mathbb{R}$. $C(t)$ and $S(t)$ are bounded only in finite interval and may have exponential growth on $\mathbb{R}$. Consequently, all solutions of initial value problem (1)-(2) are not bounded on $\mathbb{R}_+$.*

4 Application

In order to illustrate the applications of our results proved in previous section, we consider the following nonlinear partial integrodifferential equation of the type

$$z_{tt}(w, t) = z_{ww}(w, t) + \frac{z(w, t-r) \sin(z(w, t-r))}{1+t^2} + \int_0^t \frac{sz(w, s-r)}{1+t} ds, \\ t \in [0, T], \quad 0 \leq w \leq \pi \quad (29)$$

$$z(0, t) = z(\pi, t) = 0, \quad t \in [0, T], \quad (30)$$

$$z(w, t) = \phi(w, t), \quad 0 \leq w \leq \pi, -r \leq t \leq 0, \quad (31)$$

$$z_t(w, 0) = z_0(w), \quad 0 \leq w \leq \pi, \quad (32)$$

Let $X = L^2([0, \pi])$ be Banach space with norm $\|\cdot\|_{L^2}$. Define $z(w, t) = x(t)(w)$ and $z(w, t-r) = x_t(\cdot)w$. Now put

$$f(t, x_t, \int_0^t k(t, s)g(s, x_s)ds) = \frac{x_t \sin x_t}{1+t^2} + \int_0^t \frac{sx_s}{1+t} ds, \\ k(t, s) = \frac{1}{1+t} \quad \text{and} \quad g(s, x_s) = sx_s,$$

Then, we have

$$\|g(s, x_s) - g(s, y_s)\| = |s| \|x_s - y_s\|_C \\ = T \|x_s - y_s\|_C \quad (33)$$

$$|k(t, s)| = \left| \frac{1}{1+t} \right| = \frac{1}{1+t} \leq 1 \quad (34)$$

On an abstract nonlinear functional second order Volterra integrodifferential equation

and

$$\begin{aligned}
 & \left\| f(t, x_t, \int_0^t k(t, s)g(s, x_s)ds) - f(t, y_t, \int_0^t k(t, s)g(s, y_s)ds) \right\| \\
 & \leq \left\| \frac{x_t \sin x_t - y_t \sin y_t}{1 + t^2} \right\| + \int_0^t \frac{s \|x_s - y_s\|_C}{1 + t} ds \\
 & = \frac{1}{1 + t^2} \|x_t + y_t\|_C + \frac{t^2}{2(1 + t)} \|x - y\|_B \\
 & \leq \frac{N_1(t)}{1 + t^2} \|x_t - y_t\|_C + \frac{t^2}{2(1 + t)} N_2(t) \|x - y\| \\
 & \leq N_3 [\|x_t - y_t\|_C + \|x - y\|] \tag{35}
 \end{aligned}$$

where $N_1(t)$ and $N_2(t)$ are positive functions such that $\|x_t + y_t\|_C \leq N_1(t) \|x_t - y_t\|_C$ and $\|x - y\|_B \leq N_2(t) \|x - y\|$ and $N_3 = \max_{t \in [0, T]} \left(\frac{N_1(t)}{1 + t^2}, \frac{t^2}{2(1 + t)} N_2(t) \right)$.

Also we have

$$\|g(t, x_t)\| \leq t \|x_t\|_C \tag{36}$$

and

$$\begin{aligned}
 & \left\| f(t, x_t, \int_0^t k(t, s)g(s, x_s)ds) \right\| \leq \frac{\|x_t\|_C}{1 + t^2} + \int_0^t \frac{s \|x_s\|_C}{1 + t} ds \\
 & \leq \frac{\|x_t\|_C}{1 + t^2} + \int_0^t \frac{s \|x\|_B}{1 + t} ds \\
 & \leq \frac{\|x_t\|_C}{1 + t^2} + \frac{t^2 \|x\|_B}{2(1 + t)} \\
 & \leq \frac{\|x_t\|_C}{1 + t^2} + \frac{t^2}{2(1 + t)} N_4(t) \|x\| \\
 & \leq N_5(t) [\|x_t\|_C + \|x\|] \tag{37}
 \end{aligned}$$

where $N_4(t)$ is positive function such that $\|x\|_B \leq N_4(t) \|x\|$ and $N_5(t) = \max \left(\frac{1}{1 + t^2}, \frac{t^2 N_4(t)}{2(1 + t)} \right)$. Define the operator $A : X \rightarrow X$ by $Ay = y''$ with domain $D(A) = \{y \in X : y, y' \text{ are absolutely continuous, } y'' \in X \text{ and } y(0) = y(\pi) = 0\}$. Then

$$Ay = \sum_{n=1}^{\infty} -n^2 (y, y_n) y_n, \quad y \in D(A),$$

where $y_n(s) = (\sqrt{2/\pi}) \sin ns$, $n = 1, 2, 3, \dots$ is the orthogonal set of eigenvectors of A and it can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X and is given by

$$C(t)y = \sum_{n=1}^{\infty} \cos nt (y, y_n) y_n, \quad y \in X.$$

The associated sine family is given by

$$S(t)y = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(y, y_n)y_n, \quad y \in X.$$

Also $S(t)$ is compact for every $t > 0$ and $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0, T]$. For more details, we refer the readers to [10, 18].

With these choices of A, f, g and k it is to be observed that equations (1)-(2) is an abstract formulation of (29) - (32) and hence our Theorems 2.1 and 2.2 can be applied to guarantee the existence, uniqueness and boundedness of solutions of nonlinear partial integrodifferential equations (29) - (32).

5 Conclusions

We have developed modified version of Banach contraction mapping principle and proved the existence, uniqueness and boundedness of solutions of an abstract nonlinear functional second order Volterra integrodifferential equation (1)-(2). We have also presented an example to illustrate the applications of results proved in section 3.

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On an abstract nonlinear functional second order Volterra integrodifferential equation

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Heinz Quarter Mean Labeling of Graphs

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Abstract

Mean labeling is one of the best-known labeling methods for graphs. Despite the large number of papers published on the subject of graph labeling, there are some particular formulas to be used by researchers to mean-label graphs. In this paper, we introduced the concepts of Heinz Quarter Mean labeling graphs. Heinz Quarter Mean labeling for some graphs like Path, Cycle, Comb, Star graph and Complete Graph is proven as Heinz Quarter Mean Graphs.

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Keywords: Graph, Heinz – Quarter Mean Graph, Path, Cycle, Comb, Star Graph , Complete Graph.[‡]

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1. Introduction

Graph labeling started in 1960 by Ringel (1964) [1] and Rosa (1967)[2]. Ringel-Kotzig (1964, 1965) [3] conjecture that all trees are graceful. Rosa (1967) introduced β and α valuation and other labeling as a tool for the decomposition of complete graphs. Golomb (1972) [11] renamed the β - valuation as graceful labeling which is widely used. Graham and Sloane (1980)[4] have introduced the concept of harmonious labeling on additive bases problem starting from error-correcting codes. The two basic labeling methods which are widely studied are graceful and harmonious labeling. In the wide field of graph labeling, α -valuation, Elegant labeling and cordial labeling are used extensively.

A. Rosa published a original paper on labeling problems of graphs in 1967. Graph labeling is the way of providing labels, which are mentioned by integers to edges or vertices of a graph. Labeled graphs help as for collection of applications like Radar, X-ray, Crystallography, Communication networks, Circuit design, and models for Constraint programming over finite domain. In the duration of years different labeling of graphs such as graceful, prime, cordial, total cordial, k-graceful and odd graceful labeling etc.

Researchers have introduced and explored many labeling schemes. Graph labelling is widely applied for various problem in mathematics, computer science and communication networks. Yegnanaryanan and Vaidhyanathan [12] have discussed various applications of graph labeling . Though it seems that graph labelling deals with theoretical study, a lot of research works have been carried out in applied fields also.

A detailed literature review of graph labeling, is presented in J.A. Gallian [5]. The standard terminogy and notations are available in Harary [6]. S. Somasundaram and R. Ponraj [7,10] have introduced Mean labeling in 2004. In 2012, the concept of Harmonic mean labeling was developed by S.Somasundaram and S.S. Sandhya introduced [8]. In [9], Bhatia introduced the Heinz mean $f(u, v; y) = \frac{f(u)^y f(v)^{1-y} + f(u)^{1-y} f(v)^y}{2}$, for $0 \leq y \leq \frac{1}{2}$.

In graph theory, Path (P_n) is a single edge which is in between any of the two vertices. A closed path in which only the first and last vetices are equal is called as a Cycle (C_n). When the single pendant of each edge is connected to a path, then it is called as a Comb($P_n \odot K_1$). A Star ($K_{1,n}$) graph is defined as a complete bipartite graph. When each pair of distinct vertices of a graph is adjacent, then it is called as a Complete graph (G), denoted by K_n .

In this paper, we investigate the Heinz Quarter Mean Labeling of Path, Cycle, Comb, Star $K_{1,n}$ and Complete Graph K_n .

2. Main results

2.1 Heinz Mean Graphs

In this section, we first introduced Heinz Mean Graph using Heinz Mean [9]

Definition 2.1.1. A graph $G = (V, E)$ having number of vertices as p and number of edges as q can be called as **Heinz Mean graph**, if each vertex $x \in V$ can be labeled with distinct labels $f(x)$ from 1, 2, 3, ..., $q + 1$ and each edge $e = uv$ is labeled with $(e = uv) = \left\lfloor \frac{f(u)^y f(v)^{1-y} + f(u)^{1-y} f(v)^y}{2} \right\rfloor$ or $\left\lceil \frac{f(u)^y f(v)^{1-y} + f(u)^{1-y} f(v)^y}{2} \right\rceil$, with $0 \leq y \leq \frac{1}{2}$, which results in distinct edge labels. Here, f is called Heinz Quarter mean labeling of G .

Remark 2.1.2. When we assume $y = 0$, Heinz Mean is called Heinz-0-Mean and it can be viewed as Arithmetic mean. Heinz-0-Mean : $H(f(u), f(v)) = \frac{f(u) + f(v)}{2} = \text{A.M}$

When we assume $y = \frac{1}{2}$, Heinz Mean is called Heinz $\frac{1}{2}$ Mean and it can be viewed as Geometric mean. Heinz $\frac{1}{2}$ Mean : $H(f(u), f(v)) = \frac{2\sqrt{f(u)f(v)}}{2} = \text{G.M}$

2.2 Heinz Quarter Mean Graphs

In this section, using Heinz Mean Graph we assume $y = \frac{1}{4}$, newly formed Heinz Quarter Mean Graph.

Definition 2.2.1. A graph $G = (V, E)$ having number of vertices as p and number of edges as q can be called as **Heinz Quarter Mean graph**, if the vertices $x \in V$ can be labeled with distinct labels $f(x)$ from 1, 2, 3, ..., $q + 1$ and each edge $e = uv$ is labeled with $(e=uv) = \left\lfloor \frac{\sqrt[4]{f(u)f(v)}(\sqrt{f(u)} + \sqrt{f(v)})}{2} \right\rfloor$ or $\left\lceil \frac{\sqrt[4]{f(u)f(v)}(\sqrt{f(u)} + \sqrt{f(v)})}{2} \right\rceil$, then the resulting edge labels are distinct. Here, f is called Heinz Quarter mean labeling of G . This graph is said to Heinz Quarter Mean Graph.

Observation 2.2.2. If G is a Heinz Quarter mean graph, since one of the edge is getting label “1”, then, one of the vertices should get the label “1”.

Observation 2.2.3. If vertex u gets label 1, then any edges which are incident with u should get label 1 or 2 or 3.

The above statement we have provided seems to describe a labeling rule for the edges incident with a vertex u in a graph. Specifically it states that if vertex u is assigned the label 1 then any edges connected to u should be labeled with either 1, 2, or 3. Obviously the degree of this vertex $d(u)$ should be less than or equal to 3.

Theorem: 2.2.4. For a Heinz – Quarter mean graph G , the vertex are labeled from $1, 2, \dots, q + 1$, and the edges are labeled from $1, 2, \dots, q$.

Proof . Assign the labels from the set $1, 2, \dots, q + 1$ to the vertices of Heinz Quarter mean graph. Each vertex should be labeled with one of these numbers.

In a Heinz Quarter mean graph edges are labeled based on the difference between the labels of the vertices they connect. Specifically we assign the label from the set $(1, 2, \dots, q)$ to each edge based on the absolute difference between the labels of its two end points.

For each edge connecting vertices labeled i and j , where $i < j$, assign the label $[i - j]$ to that edge. We do this for all edges in the graph.

Theorem 2.2.5. Every Path P_n is Heinz – Quarter mean graph.

Proof. Consider a path graph G , with number of vertices as n and number of edges as $n-1$, with vertices $u_1, u_2, u_3, \dots, u_n$.

A function defined by $f: V(P_n) \rightarrow \{1, 2, 3, \dots, q + 1\}$ by $f(u_i) = i, 1 \leq i \leq n$.

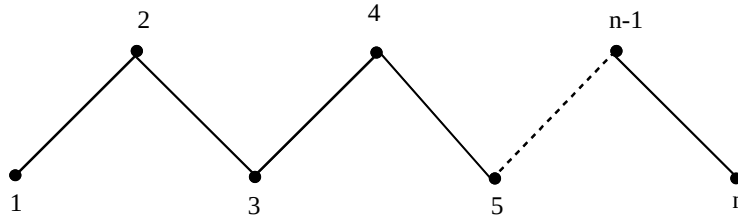


Figure 2.2.1 Path P_n

$$f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, \dots, f(u_{n-1}) = n - 1, f(u_n) = n$$

In view of the above defined labeling pattern f is the Heinz Quarter Mean labeling for the Path graph

Therefore, path P_n is Heinz – Quarter mean graph.

Theorem 2.2.6. For $n \geq 3$, all cycles C_n are Heinz-Quarter mean graphs.

Proof. Let G be a cycle graph of length n , with vertices $u_1, u_2, u_3, \dots, u_n$

$$\text{Consider } n = \begin{cases} 2m & \text{if } n = \text{even} \\ 2m + 1 & \text{if } n = \text{odd} \end{cases}$$

A function defined by $f: V(C_n) \rightarrow \{1, 2, 3, \dots, q + 1\}$

$$\text{Where, } f(u_i) = \begin{cases} 2i - 1 & 1 \leq j \leq m + 1 \text{ if } n \text{ is even} \\ 2i - 1 & 1 \leq j \leq m \text{ if } n \text{ is odd} \end{cases}$$

$$f(u_{m+j}) = \begin{cases} n - 2j + 2 & 2 \leq j \leq m \text{ if } n \text{ is even} \\ n - 2j + 3 & 1 \leq j \leq m + 1 \text{ if } n \text{ is odd} \end{cases}$$

Heinz Quarter Mean Labeling of Graphs

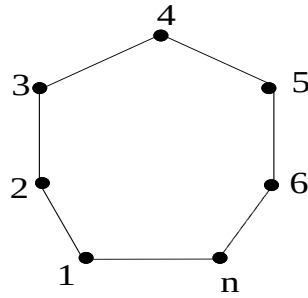


Figure 2.2.2 Cycle C_n

In view of the above defined labeling pattern, the edge labels are distinct. Hence f is the Heinz Quarter Mean labeling for the Cycle graph.

Hence Cycle C_n is a Heinz-Quarter mean graph.

Theorem 2.2.7. For all values of n , $\text{Comb } P_n \odot K_1$ is a Heinz - Quarter mean graph.

Proof. Let $P_n = u_1 u_2 u_3 \dots u_n$ be a path and v_i be the vertices corresponds to every vertex u_i , where $(1 \leq i \leq n)$. Let $P_n \odot K_1$ be a graph obtained by the number of vertices be $2n$ and the number of edges be $2n - 1$ in the graph G .

A function defined as $f: V(P_n \odot K_1) \rightarrow \{1, 2, 3, \dots, q + 1\}$

Where,

$$f(u_i) = 2i - 1, \quad 1 \leq i \leq n$$

$$\text{and } f(v_i) = 2i, \quad 1 \leq i \leq n.$$

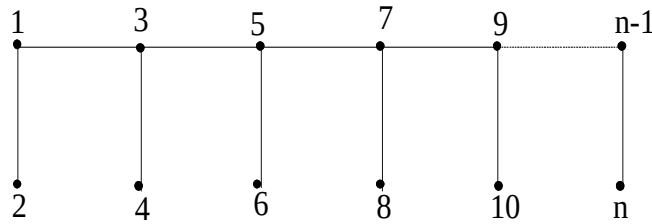


Figure 2.2.3 $\text{Comb } (P_n \odot K_1)$

Here, $f(u_i u_{i+1}) = 2i, \quad 1 \leq i \leq n - 1$

$f(u_i v_i) = 2i - 1, \quad 1 \leq i \leq n$, are the labels of edges.

In the above labeling pattern, we get the distinct edge labels. Hence $P_n \odot K_1$ is a Heinz-Quarter mean labeling. Therefore $P_n \odot K_1$ is a Heinz-Quarter mean graph.

Theorem: 2.2.8. Star $K_{1,n}$ is Heinz - Quarter mean graph if and only if $n \leq 5$.

Proof. Star $K_{1,1}$ and Star $K_{1,2}$ are same as P_2 and P_3 respectively. Clearly $K_{1,1}$ and $K_{1,2}$ are Heinz - Quarter Mean graphs. The central vertex of $K_{1,n}$ is considered as u and the remaining vertices are $v_1, v_2, v_3, \dots, v_n$ respectively.

Case (i). The vertex u is assigned with label 3 and the vertex v_i is assigned with i ($1 \leq i \leq 2$).

For $3 \leq i \leq n$, the vertex labels are $i + 1$. For the above, it is a Heinz - Quarter Mean Labeling, as shown below.

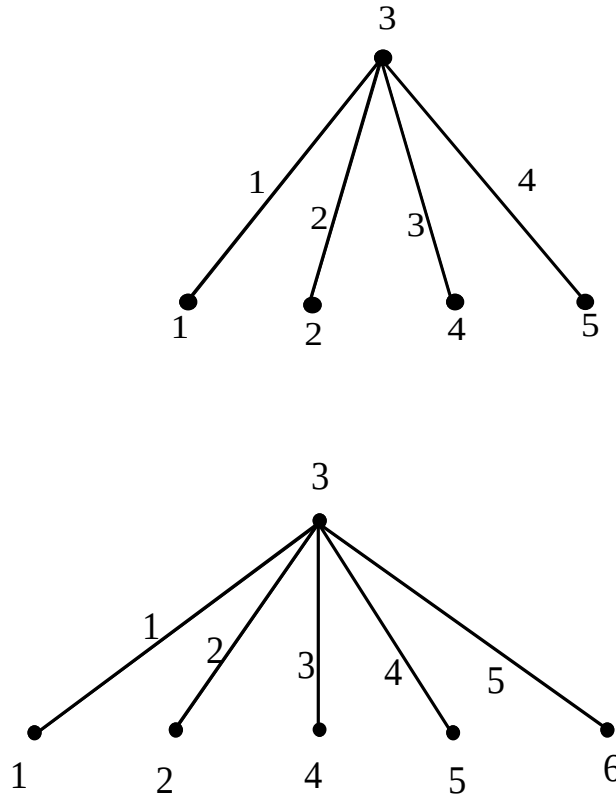


Figure 2.2.4 Star Graph $K_{1,4}$ and $K_{1,5}$

Case (ii) . Assume $n > 5$, if $K_{1,n}$ is Heinz - Quarter Mean labeling.

Here the following subcases are considered.

Subcase (ii) (a). Assuming that the central vertex u is labeled as 2.

Heinz Quarter Mean Labeling of Graphs

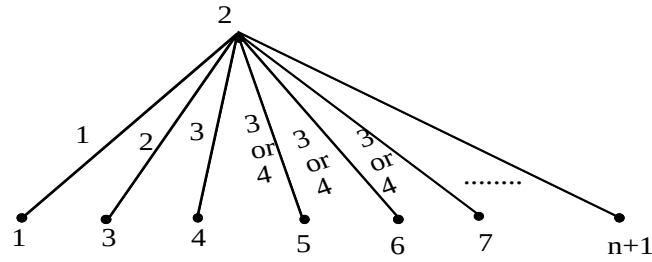


Figure 2.2.5 Star Graph $K_{1,n}$

The labeling of other vertices $v_1, v_3, v_4, v_5, v_6, \dots$ are taken as 1, 3, 4, 5, 6, ... respectively.

The edge labels of uv_4 is 3 and for uv_5, uv_6 and uv_7 are 3 or 4 (repeated for all 3 edges), which is not possible. Because f is a injective function.

Subcase (ii) (b). Assuming that the central vertex is labeled as 4.

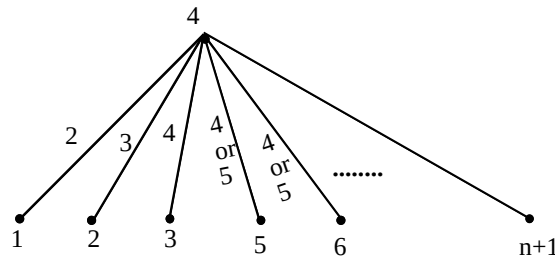


Figure 2.2.6 Star Graph $K_{1,n}$

The labeling of other vertices $v_1, v_2, v_3, v_5, v_6, v_7, \dots$ are taken as 1, 2, 3, 5, 6, 7, ... respectively. Here, no edges are given label 1.

The edge labels of uv_3, uv_5 and uv_6 are 4, 4 and 5, respectively, which is not possible. Because the function f is injective.

From the above, it is concluded that $K_{1,n}, n > 5$ is not a Heinz - Quarter Mean Graph.

Theorem 2.2.9. The complete graph K_n is Heinz-Quarter mean graph if and only if $n \leq 4$.

Proof . The following cases are studied

Case (i). $n = 2$ or 3 or 4

By theorem 2.2.5 and 2.2.6 K_2 and K_3 are Heinz – Quarter mean graph.

Also K_4 is Heinz-Quarter mean graph.

The pattern of labeling K_2, K_3 and K_4 are shown in Figure 7.

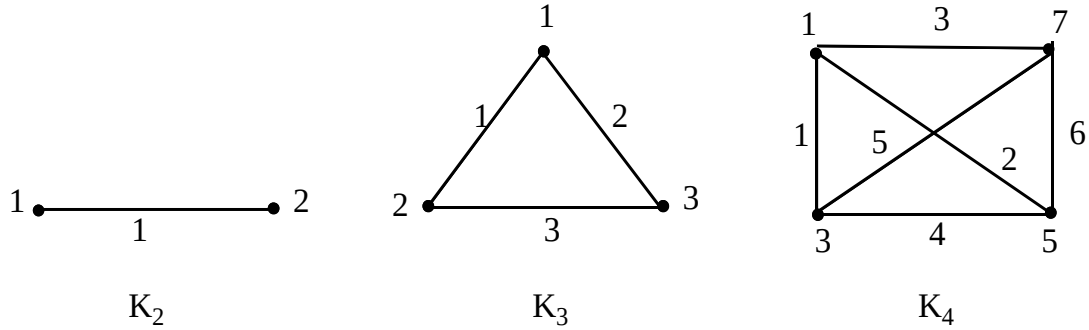


Figure 2.2.7 Complete Graph K_2 , K_3 and K_4

Case (ii). $n > 4$

In order to obtain an edge label 1, a vertex u having label 1 is essential . There are 4 more vertices namely u_1 to u_4 are incident with u . By considering observation 2.2.2, this is not allowed.

Therefore, the complete graph K_n is not Heinz – Quarter mean graph if $n > 4$.

The following example gives the labeling pattern of K_5

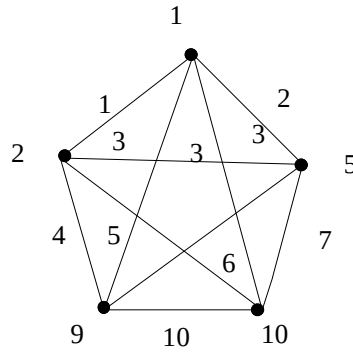


Figure 2.2.8 Complete Graph K_5

Hence K_n is Heinz mean graph if and only if $n \leq 4$

3. Conclusions

The Heinz Quarter mean labeling of various graphs, including Path, Cycle, Comb, Complete, and Star graphs, provides a unique and intriguing perspective on graph labeling schemes. The application of Heinz Quarter mean labeling involves assigning weights to the vertices of a graph based on the arithmetic mean of their distances from a designated set of vertices. This labeling scheme has been studied extensively for different graph classes.

In the case of Path graphs, Heinz Quarter mean labeling has been shown to yield interesting results, providing a systematic way to assign distinct labels to vertices. Similarly, for Cycle graphs, Comb graphs, Complete graphs, and Star graphs, the Heinz Quarter mean labeling presents a methodical approach to graph labeling, showcasing the versatility of this technique across various graph structures.

The exploration of Heinz Quarter mean labeling in different graph types contributes to the broader understanding of graph theory and labeling strategies. As researchers delve deeper into the properties and applications of this labeling scheme, it opens up avenues for further investigations into the mathematical characteristics and practical implications of Heinz Quarter mean labeled graphs.

Finally the study of Heinz Quarter mean labeling on Path, Cycle, Comb, Complete, and Star graphs enriches our understanding of graph labeling techniques, offering a systematic and structured approach for assigning labels to vertices in diverse graph structures. The insights gained from such investigations contribute to the ongoing development of graph theory and its applications in various fields.

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Some properties and extended Binet's formula for the class of bifurcating Fibonacci sequence

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Abstract

One of the generalizations of Fibonacci sequence is a k -Fibonacci sequence, which is further generalized in several other ways, some by conserving the initial conditions and others by conserving the related recurrence relation. In this paper, we generalize the sequence of k -Fibonacci numbers into the sequence of bifurcating Fibonacci numbers. The Binet-like formula for the terms of these numbers is obtained and further, we obtain several interesting properties related to the sequence.

Keywords: Fibonacci sequence, bifurcating Fibonacci sequence, generalization of Fibonacci sequence, Binet formula, identities related to the Fibonacci sequence.

2010 AMS subject classification: 11B39, 11B83, 11B99.[§]

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1. Introduction

The well-known sequence $\{F_n\}_{n \geq 0}$ of Fibonacci numbers is a sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... , where each term is the sum of two preceding terms. The corresponding recurrence relation for the terms of the sequence is $F_n = F_{n-1} + F_{n-2}$; $n \geq 2$ with the initial terms $F_0 = 0, F_1 = 1$. Many papers regarding various generalizations of this sequence have appeared in recent years [2, 3, 12]. There are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either by maintaining the recurrence relation but altering the first two terms of the sequence from 0, 1 to any arbitrary integers a, b or by preserving the first two terms of the sequence but altering the recurrence relation. The two techniques can even be combined. It is observed that the change in the recurrence relation leads to greater complexity in the properties of the resulting sequence. Edson, Yayenie [8] generalized this sequence into a new class of generalized sequence, which depends on two real parameters used in a recurrence relation. We define further generalizations of this sequence and call it the generalized Fibonacci sequence.

Definition: For any two positive numbers a and b , the class of bifurcating sequences $\{f_n^{L(a,b)}\} (= \{f_n^L\})$ is defined by the recurrence relation $f_n^L = a^{\chi(n)} b^{1-\chi(n)} f_{n-1}^L - f_{n-2}^L$, where $f_0^L = 0, f_1^L = 1$; a, b are any two positive integers and $\chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$.

This can be equivalently expressed as

$$f_n^{L(a,b)} = \begin{cases} af_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)}; & \text{when } n \text{ is odd} \\ bf_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)}; & \text{when } n \text{ is even} \end{cases} \quad (n \geq 2) \quad \text{with } f_0^L = 0, f_1^L = 1 \quad (1.1)$$

The first few terms of this sequence are shown below:

n	$f_n^{L(a,b)}$
0	0
1	1
2	b
3	$ab - 1$
4	$ab^2 - 2b$
5	$a^2b^2 - 3ab + 1$
6	$a^2b^3 - 4ab^2 + 3b$
7	$a^3b^3 - 5a^2b^2 + 6ab - 1$

2. Generating function for $\{f_n^{L(a, b)}\}$

Generating function is a powerful tool for solving linear homogeneous recurrence relations. In this section, we obtain the generating function for the bifurcating Fibonacci sequence and use it to obtain Binet-like formula for these numbers. We first prove a result which will be required to obtain the generating function.

We first prove an interesting recursive relation which connects three f_n^L 's into a linear relation.

Lemma 2.1: $f_{2n+5}^L - (ab - 2)f_{2n+3}^L + f_{2n+1}^L = 0$.

Proof: Using definition, we get

$$f_{2n+5}^L = af_{2n+4}^L - f_{2n+3}^L = a(bf_{2n+3}^L - f_{2n+2}^L) - f_{2n+3}^L = (ab - 1)f_{2n+3}^L - af_{2n+2}^L.$$

$$\begin{aligned} \text{Then, } f_{2n+5}^L - (ab - 2)f_{2n+3}^L + f_{2n+1}^L &= (ab - 1)f_{2n+3}^L - af_{2n+2}^L - (ab - 2)f_{2n+3}^L + f_{2n+1}^L \\ &= f_{2n+3}^L - (af_{2n+2}^L - f_{2n+1}^L) = 0. \end{aligned}$$

We next prove a relation which will be used to derive the generating of f_n^L .

Lemma 2.2: $\sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = \frac{x+x^3}{x^4-(ab-2)x^2+1}$.

Proof: Let $p(x) = \sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = f_1^L x + f_3^L x^3 + f_5^L x^5 + \dots$.

Then $x^4 p(x) = f_1^L x^5 + f_3^L x^7 + f_5^L x^9 + \dots$ and

$$(ab - 2)x^2 p(x) = (ab - 2)f_1^L x^3 + (ab - 2)f_3^L x^5 + (ab - 2)f_5^L x^7 + \dots$$

Using lemma 2.1, we get $(x^4 - (ab - 2)x^2 + 1)p(x) = x + x^3$.

$$\text{This gives } p(x) = \sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = \frac{x+x^3}{x^4-(ab-2)x^2+1}.$$

The next theorem provides the generating function for $\{f_n^L\}$.

Theorem 2.3: The generating function for $\{f_n^L\}$ is given by $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

Proof: Define $f(x) = \sum_{i=0}^{\infty} f_i^L x^i = f_0^L + f_1^L x + f_2^L x^2 + f_3^L x^3 + \dots$.

Then $-axf(x) = \sum_{i=0}^{\infty} f_i^L x^{i+1} = -af_0^L x - af_1^L x^2 - af_2^L x^3 - af_3^L x^4 + \dots$ and

$$x^2 f(x) = \sum_{i=0}^{\infty} f_i^L x^{i+2} = f_0^L x^2 + f_1^L x^3 + f_2^L x^4 + f_3^L x^5 + \dots$$

After the rearrangement of terms, we get

$$(1 - ax + x^2)f(x)$$

$$= f_0^L + (f_1^L - af_0^L)x + (f_2^L - af_1^L + f_0^L)x^2 + (f_3^L - af_2^L + f_1^L)x^3 + \dots$$

$$= x + (b-a)x^2 + (af_2^L - f_1^L - af_2^L + f_1^L)x^3 + (bf_3^L - f_2^L - af_3^L + f_2^L)x^4 + \dots$$

Then $(1-ax+x^2)f(x) = x + (b-a)x(\sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1})$. This gives

$$f(x) = \frac{x}{(1-ax+x^2)} + \frac{(b-a)x}{(1-ax+x^2)} \left\{ \frac{x+x^3}{x^4-(ab-2)x^2+1} \right\}$$

$$= \frac{x^5-(ab-2)x^3+x+bx^2+bx^4-ax^2-ax^4}{(1-ax+x^2)(x^4-(ab-2)x^2+1)} = \frac{(x^5-ax^4+x^3)+(bx^2-abx^3+bx^4)+(x-ax^2+x^3)}{(1-ax+x^2)(x^4-(ab-2)x^2+1)}$$

$$= \frac{(x+bx^2+x^3)(1-ax+x^2)}{(1-ax+x^2)(x^4-(ab-2)x^2+1)} = \frac{x(1+bx+x^2)}{(x^4-(ab-2)x^2+1)}.$$

This finally gives the required generating function as $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

3. Binet and additional formulae for $\{f_n^L\}$

For roots α and β of the equation $x^2 - abx + ab = 0$, we observe that $\alpha = \left(\frac{ab+\sqrt{a^2b^2-4ab}}{2}\right)$, $\beta = \left(\frac{ab-\sqrt{a^2b^2-4ab}}{2}\right)$. We note the following properties related with α, β :

$$1) \alpha + \beta = ab. \quad (3.1)$$

$$2) \alpha\beta = ab. \quad (3.2)$$

$$3) \alpha - \beta = \sqrt{a^2b^2 - 4ab}. \quad (3.3)$$

$$4) \frac{\alpha^2}{ab} = \alpha - 1. \quad (3.4)$$

This is since $\alpha^2 - ab\alpha + ab = 0$, we get $\alpha^2 = ab(\alpha - 1)$,

which gives the required result.

$$5) \frac{\beta^2}{ab} = \beta - 1. \quad (3.5)$$

$$6) (\alpha - 1)(\beta - 1) = 1. \quad (3.6)$$

$$\text{This is since } (\alpha - 1)(\beta - 1) = \left(\frac{\alpha^2}{ab}\right)\left(\frac{\beta^2}{ab}\right) = \frac{a^2b^2}{a^2b^2} = 1.$$

$$7) \beta(\alpha - 1) = \alpha. \quad (3.7)$$

$$\text{Here we note that } \beta(\alpha - 1) = \beta\left(\frac{\alpha^2}{ab}\right) = \frac{(\alpha\beta)\alpha}{ab} = \frac{(ab)\alpha}{ab} = \alpha.$$

$$8) \alpha(\beta - 1) = \beta. \quad (3.8)$$

We now derive an extended Binet's formula for the $\{f_n^L\}$.

Theorem 3.1: $F_n^L = \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$, where $\chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$.

Proof: By theorem 2.3, we have $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

This can be written as $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1} = \frac{(Ax+B)}{x^2-(\alpha-1)} + \frac{(Cx+D)}{x^2-(\beta-1)}$.

Then, $x + bx^2 + x^3 = (Ax + B)(x^2 - (\beta - 1)) + (Cx + D)(x^2 - (\alpha - 1))$.

Now comparing various powers of x on both sides, we get

$$1 = A + C; b = B + D; -1 = (A(\beta - 1) + C(\alpha - 1)) \text{ and } 0 = -B(\beta - 1) - D(\alpha - 1).$$

Solving them we get $A = \frac{\alpha}{\alpha - \beta}, B = \frac{b(\alpha - 1)}{(\alpha - \beta)}, C = \frac{-\beta}{\alpha - \beta}, D = \frac{-b(\beta - 1)}{\alpha - \beta}$.

Substituting these values in $f(x) = \frac{(Ax+B)}{x^2-(\alpha-1)} + \frac{(Cx+D)}{x^2-(\beta-1)}$, we get

$$f(x) = \frac{1}{(\alpha - \beta)} \left[\left\{ \frac{(\alpha - 1)b + \alpha x}{x^2 - (\alpha - 1)} \right\} - \left\{ \frac{(\beta - 1)b + \beta x}{x^2 - (\beta - 1)} \right\} \right].$$

Now by the Maclaurin's expansion of $\frac{P-Qx}{x^2-R}$, it is observed that

$$\frac{P-Qx}{x^2-R} = \sum_{n=0}^{\infty} QR^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} PR^{-n-1}x^{2n}.$$

Then $\frac{(\alpha-1)b - (\alpha x)}{x^2 - (\alpha-1)} = \left[\sum_{n=0}^{\infty} \frac{-\alpha}{(\alpha-1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{(\alpha-1)b}{(\alpha-1)^{n+1}} x^{2n} \right]$ and

$$\frac{-(\beta-1)b - \beta x}{x^2 - (\beta-1)} = \left[\sum_{n=0}^{\infty} \frac{\beta}{(\beta-1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{-b(\beta-1)}{(\beta-1)^{n+1}} x^{2n} \right].$$

$$\begin{aligned} \text{Then } f(x) &= \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{-\alpha}{(\alpha-1)^{n+1}} + \frac{\beta}{(\beta-1)^{n+1}} \right) x^{2n+1} \right\} \\ &\quad + \frac{b}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(\beta-1)}{(\beta-1)^{n+1}} - \frac{(\alpha-1)}{(\alpha-1)^{n+1}} \right) x^{2n} \right\} \\ &= \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{-\alpha(\beta-1)^{n+1} + \beta(\alpha-1)^{n+1}}{(\alpha-1)^{n+1}(\beta-1)^{n+1}} \right) x^{2n+1} \right\} \\ &\quad + \frac{b}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(\beta-1)(\alpha-1)^{n+1} - (\alpha-1)(\beta-1)^{n+1}}{(\beta-1)^{n+1}(\alpha-1)^{n+1}} \right) x^{2n} \right\} \end{aligned}$$

Now using (3.4) and (3.5) we get

$$\begin{aligned} f(x) &= \frac{1}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{\beta \left(\frac{\alpha^2}{ab} \right)^{n+1} - \alpha \left(\frac{\beta^2}{ab} \right)^{n+1}}{\left(\frac{\alpha^2}{ab} \right)^{n+1} \left(\frac{\beta^2}{ab} \right)^{n+1}} \right) x^{2n+1} \right\} \\ &\quad + \frac{b}{(\alpha - \beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{\left(\frac{\beta^2}{ab} \right) \left(\frac{\alpha^2}{ab} \right)^{n+1} - \left(\frac{\alpha^2}{ab} \right) \left(\frac{\beta^2}{ab} \right)^{n+1}}{\left(\frac{\beta^2}{ab} \right)^{n+1} \left(\frac{\alpha^2}{ab} \right)^{n+1}} \right) x^{2n} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^{n+1}(\beta\alpha^{2n+2}-\alpha\beta^{2n+2})}{\alpha^{2n+2}\beta^{2n+2}} \right) x^{2n+1} \right\} \\
 &\quad + \frac{b}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^n(\alpha^{2n+2}\beta^2-\alpha^2\beta^{2n+2})}{\alpha^{2n+2}\beta^{2n+2}} \right) x^{2n} \right\} \\
 &= \frac{1}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^{n+1}\alpha\beta(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha\beta)\alpha^{2n+1}\beta^{2n+1}} \right) x^{2n+1} \right\} \\
 &\quad + \frac{b}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^n(\alpha\beta)^2(\alpha^{2n}-\beta^{2n})}{(\alpha\beta)^2\alpha^{2n}\beta^{2n}} \right) x^{2n} \right\} \\
 &= \frac{1}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^{n+1}(\alpha\beta)(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha\beta)\alpha^{2n+1}\beta^{2n+1}} \right) x^{2n+1} \right\} \\
 &\quad + \frac{b}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^n(\alpha\beta)^2(\alpha^{2n}-\beta^{2n})}{(\alpha\beta)^2\alpha^{2n}\beta^{2n}} \right) x^{2n} \right\}
 \end{aligned}$$

Using (3.2), we get

$$\begin{aligned}
 f(x) &= \frac{1}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^{n+1}(\alpha^{2n+1}-\beta^{2n+1})}{(ab)^{2n+1}} \right) x^{2n+1} \right\} \\
 &\quad + \frac{b}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{(ab)^n(\alpha^{2n}-\beta^{2n})}{(ab)^{2n}} \right) x^{2n} \right\} \\
 &= \frac{1}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{1}{ab} \right)^n (\alpha^{2n+1} - \beta^{2n+1}) x^{2n+1} \right\} \\
 &\quad + \frac{b}{(\alpha-\beta)} \left\{ \sum_{n=0}^{\infty} \left(\frac{\alpha^{2n}-\beta^{2n}}{(ab)^n} \right) x^{2n} \right\}
 \end{aligned}$$

Considering $\chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$ we get $f(x) = \sum_{n=0}^{\infty} \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) x^n$.

This finally gives the required extended Binet Formula as $f_n^L = \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$.

We finally prove two the identities for f_{2k}^L and f_{2k+1}^L which expresses them in the series form.

Lemma 3.2:

$$\begin{aligned}
 \text{a) } f_{2k}^L &= \frac{b(ab)^{k-1}}{2^{2k-1}} \sum_{r=1}^{\infty} \binom{2k}{2r-1} \left(1 - \frac{4}{ab} \right)^{r-1} \\
 \text{b) } f_{2k+1}^L &= \frac{(ab)^k}{2^{2k}} \sum_{r=1}^{\infty} \binom{2k+1}{2r-1} \left(1 - \frac{4}{ab} \right)^{r-1}.
 \end{aligned}$$

Proof: By theorem 3.1, we have $f_n^L = \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$. If we consider $\mu = \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}(\alpha-\beta)}$, then this can be written as $f_n^L = \mu(\alpha^n - \beta^n)$.

$$\begin{aligned}
 \text{Then } f_n^L &= \mu \left(\left(\frac{ab + \sqrt{a^2b^2 - 4ab}}{2} \right)^n - \left(\frac{ab - \sqrt{a^2b^2 - 4ab}}{2} \right)^n \right) \\
 &= \frac{\mu}{2^n} \{ (ab + \sqrt{a^2b^2 - 4ab})^n - (ab - \sqrt{a^2b^2 - 4ab})^n \} \\
 &= \frac{\mu(ab)^n}{2^n} \left\{ \left(1 + \sqrt{1 - \frac{4}{ab}} \right)^n - \left(1 - \sqrt{1 - \frac{4}{ab}} \right)^n \right\}
 \end{aligned}$$

Some properties and extended Binet's formula for the class of bifurcating Fibonacci sequence

$$\begin{aligned}
&= \frac{\mu(ab)^{n \times 2}}{2^n} \left\{ \binom{n}{1} \sqrt{1 - \frac{4}{ab}} + \binom{n}{3} \left(\sqrt{1 - \frac{4}{ab}} \right)^3 + \binom{n}{5} \left(\sqrt{1 - \frac{4}{ab}} \right)^5 + \dots \right\} \\
&= \frac{b^{1-\chi(n)}(ab)^n}{(ab)^{\lfloor n/2 \rfloor} \sqrt{a^2 b^2 - 4ab} 2^{n-1}} \sqrt{1 - \frac{4}{ab}} \left\{ \binom{n}{1} + \binom{n}{3} \left(\sqrt{1 - \frac{4}{ab}} \right)^2 \right. \\
&\quad \left. + \binom{n}{5} \left(\sqrt{1 - \frac{4}{ab}} \right)^4 + \dots \right\} \\
&= \frac{b^{1-\chi(n)}(ab)^{n-1}}{(ab)^{\lfloor n/2 \rfloor} 2^{n-1}} \left\{ \binom{n}{1} + \binom{n}{3} \left(\sqrt{1 - \frac{4}{ab}} \right)^2 + \binom{n}{5} \left(\sqrt{1 - \frac{4}{ab}} \right)^4 + \dots \right\} \quad (3.9)
\end{aligned}$$

Now if n is even, say $n = 2k$, then $\chi(n) = 0$. In this case

$$\begin{aligned}
f_{2k}^L &= \frac{b(ab)^{2k-1}}{(ab)^k 2^{2k-1}} \left\{ \binom{2k}{1} + \binom{2k}{3} \left(\sqrt{1 - \frac{4}{ab}} \right)^2 + \binom{2k}{5} \left(\sqrt{1 - \frac{4}{ab}} \right)^4 + \dots \right\} \\
&= \frac{b(ab)^{k-1}}{2^{2k-1}} \left\{ \sum_{r=1}^{\infty} \binom{2k}{2r-1} \left(\sqrt{1 - \frac{4}{ab}} \right)^{2(r-1)} \right\} \\
&= \frac{b(ab)^{k-1}}{2^{2k-1}} \left\{ \sum_{r=1}^{\infty} \binom{2k}{2r-1} \left(1 - \frac{4}{ab} \right)^{(r-1)} \right\}.
\end{aligned}$$

Again, if n is odd, say $n = 2r + 1$, we have $\chi(n) = 1$.

Then by (3.9), we have

$$\begin{aligned}
f_{2k+1}^L &= \frac{(ab)^{(2k+1)-1}}{(ab)^k 2^{2k}} \left\{ \binom{2k+1}{1} + \binom{2k+1}{3} \left(\sqrt{1 - \frac{4}{ab}} \right)^2 \right. \\
&\quad \left. + \binom{2k+1}{5} \left(\sqrt{1 - \frac{4}{ab}} \right)^4 + \dots \right\} \\
&= \frac{(ab)^k}{2^{2k}} \left\{ \sum_{r=1}^{\infty} \binom{2k+1}{2r-1} \left(\sqrt{1 - \frac{4}{ab}} \right)^{2(r-1)} \right\} \\
&= \frac{(ab)^k}{2^{2k}} \left\{ \sum_{r=1}^{\infty} \binom{2k+1}{2r-1} \left(1 - \frac{4}{ab} \right)^{(r-1)} \right\}.
\end{aligned}$$

4. Conclusions

In this paper, we considered the sequence of ‘bifurcating Fibonacci numbers’ and obtained its Binet formula for $f_n^{L(a,b)}$ and also obtain some of its interesting properties.

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Nano fuzzy bitopological spaces

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Abstract

The objective of this paper is to introduce the concept of Nano fuzzy bitopological space with respect to the fuzzy subsets of any finite universe, which is defined in terms of Nano fuzzy lower and Nano fuzzy upper approximations of any fuzzy subset. After the introduction of Nano fuzzy topological space theory given by R. Navalkahe and P. Rajwade [16], efforts were made to enhance its precision through further expansion. So, in this paper there is also an attempt to define Nano fuzzy $(1,2)^*$ closure and Nano fuzzy $(1,2)^*$ interior with the help of Nano fuzzy $(1,2)^*$ closed sets and Nano fuzzy $(1,2)^*$ open sets respectively in Nano fuzzy bitopological space. We have expanded our work and defined Nano fuzzy $(1,2)^*$ generalized fuzzy closed sets in Nano fuzzy bitopological space and investigated some of its properties.

Keywords: Nano fuzzy bitopological spaces; Nano fuzzy $(1,2)^*$ open sets; Nano fuzzy $(1,2)^*$ closed sets; Nano fuzzy $(1,2)^*$ generalized closed sets.

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1. Introduction

In Mathematics, the significance of topological spaces can not be oversaturated as they represent a crucial and fundamental concept. Levine [14] introduced the concept of generalized closed sets in topological spaces. The theory of Nano topology was introduced and developed by L. Thivagar [12]. K. Bhuvanshwari et al. [9] introduced and investigated the concept of Nano generalized closed set in Nano topological space. The classical Nano topological space is based on an equivalence relation on a set. Kelly [6] introduced the concepts of bitopological spaces [4,5]. In addition to this theory, T. Fukutake [17,18] has defined and explored the theory of semi closed sets and generalized closed sets in bitopological spaces. K. Bhuvneshwari et al. [10] have introduced Nano bitopological spaces. In 2016, K. Bhuvaneshwari et al. [7,8] defined Nano generalized closed sets in Nano bitopological space. Fuzzy set is an important concept, which was introduced for the first time in 1965 by Zadeh [11], it was then used in many studies in various fields. Here, we are interested in fuzzy with topology. The fuzzy logic and fuzzifying topologies are two branches of fuzzy mathematics. The basic concepts and properties of fuzzy topologies were investigated by Chang in 1968 [3].

R. Navalakhe and P. Rajwade have introduced Nano fuzzy topological spaces [16] and Nano fuzzy generalized closed sets [15,16]. This was the source of motivation for fuzzifying Nano bitopological spaces and thus we tried to define Nano fuzzy bitopological spaces. In this paper, Nano fuzzy bitopological spaces are introduced and new class of Nano fuzzy closed sets in Nano fuzzy bitopological spaces which can be denoted by Nano fuzzy $(1,2)^*$ closed sets and Nano fuzzy $(1,2)^*$ generalized closed sets are introduced and their properties are studied.

2. Preliminaries

In this section, we briefly recall the definition and basic results on Nano topological space.

2.1. [10] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

(1) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$.

That is $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x .

(2) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$.

(3) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X) = U_R(X) - L_R(X)$.

2.2. [9] A subset A of a Nano topological space $(U, \tau_R(X))$ is called a Nano generalized closed set (briefly Ng-closed set) if $Ncl(A) \subseteq V$ Whenever $A \subseteq V$ and V is open in $(U, \tau_R(X))$.

2.3. [10] Let U be the universe, R be an equivalence relation on U and $\tau_{R_{1,2}}(X) = \cup\{\tau_{R_1}(X), \tau_{R_2}(X)\}$, where $X \subseteq U$. Then it satisfies the following axioms:

- (1) U and $\phi \in \tau_{R_{1,2}}(X)$.
- (2) The union of the elements of any sub-collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.
- (3). The intersection of the elements of any finite sub collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.

Then $\tau_{R_{1,2}}(X)$ is a topology on U called the Nano bitopology on U with respect to X . $(U, \tau_{R_{1,2}}(X))$ is called the Nano bitopological space. Elements of the Nano bitopology are known as Nano $(1,2)^*$ open sets in U . Elements of $\tau_{R_{1,2}}(X)^c$ are called Nano $(1,2)^*$ closed sets in $\tau_{R_{1,2}}(X)$.

2.4. [10] If $(U, \tau_{R_{1,2}}(X))$ is a Nano bitopological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

- (1) The Nano $(1,2)^*$ closure of A is defined as the intersection of all Nano $(1,2)^*$ closed sets containing A and it is denoted by $N\tau_{R_{1,2}}cl(A)$. It is the smallest Nano $(1,2)^*$ closed set containing A .
- (2) The Nano $(1,2)^*$ interior of A is defined as the union of all Nano $(1,2)^*$ open subsets of A contained in A and it is denoted by $N\tau_{R_{1,2}}Int(A)$. It is the largest Nano $(1,2)^*$ open subset of A .

2.5. [10] A subset A of a bitopological space $(U, \tau_{R_{1,2}}(X))$ is called Nano $(1,2)^*$ generalized closed set if $N\tau_{R_{1,2}}cl(A) \subseteq V$ where $A \subseteq V$ and V is Nano $(1,2)^*$ open.

2.6. [1] Let (X, R) be an approximation space and $X/R = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then for any $\lambda \leq F(X)$, $\underline{R}(\lambda)$ and $\overline{R}(\lambda)$, the lower and upper approximations of λ with respect to R are fuzzy sets in X/R . That is, $\underline{R}(\lambda), \overline{R}(\lambda): X/R \rightarrow [0,1]$, such that

- (1) $\underline{R}(\lambda)(\lambda_j) = \inf_{y \leq \lambda_j} \lambda(y)$ and
- (2) $\overline{R}(\lambda)(\lambda_j) = \sup_{y \leq \lambda_j} \lambda(y)$, for all $j = 1, 2, 3, \dots, n$.

2.7. [2] Let λ be a fuzzy set in a fuzzy topological space (X, τ) . then the fuzzy boundary of λ is defined as

- (1) $Bd(\lambda) = Cl(\lambda) \wedge Cl(\lambda^c)$
- (2) $Bd(\lambda) = Cl(\lambda) - Int(\lambda)$.

2.8. [2, 13] Let λ be a fuzzy subset of X . Then following are equivalent:

- (1) Upper approximation of λ denoted by $\bar{R}(\lambda)$ is the fuzzy closure operator.
- (2) Lower approximation of λ denoted by $\underline{R}(\lambda)$ is the fuzzy interior operator.

So, we can say that in a fuzzy topological space

$$Bd(\lambda) = Cl(\lambda) - Int(\lambda) = \bar{R}(\lambda) - \underline{R}(\lambda)$$

2.9. [2] Properties of Fuzzy Approximation Space

Let R be an arbitrary relation from X to Y . The lower and upper approximation operators of a fuzzy set $\underline{R}$ and $\bar{R}$ satisfies the following properties: for all $\alpha, \beta \in F(X)$,

- (FL1) $\underline{R}(\alpha) = (\bar{R}(\alpha^c))^c$
- (FU1) $\bar{R}(\alpha) = (\underline{R}(\alpha^c))^c$
- (FL2) $\underline{R}(\alpha \wedge \beta) = \underline{R}(\alpha) \wedge \underline{R}(\beta)$
- (FU2) $\bar{R}(\alpha \vee \beta) = \bar{R}(\alpha) \vee \bar{R}(\beta)$
- (FL3) $\alpha \leq \beta \Rightarrow \underline{R}(\alpha) \leq \underline{R}(\beta)$
- (FU3) $\alpha \leq \beta \Rightarrow \bar{R}(\alpha) \leq \bar{R}(\beta)$
- (FL4) $\underline{R}(\alpha \vee \beta) = \underline{R}(\alpha) \vee \underline{R}(\beta)$
- (FU4) $\bar{R}(\alpha \wedge \beta) = \bar{R}(\alpha) \wedge \bar{R}(\beta)$

2.10. [16] Let X be a non-empty finite set, R be an equivalence relation on X , $\lambda \leq X$ be a fuzzy subset and $\tau_R(\lambda) = \{X, \emptyset, \underline{R}(\lambda), \bar{R}(\lambda), Bd(\lambda)\}$. Then by property (2.9) $\tau_R(\lambda)$ satisfies the following axioms

- (1) $0_\lambda, 1_\lambda \in \tau_{(R)}(\lambda)$ where $0_\lambda: \lambda \rightarrow I$ denotes the null fuzzy sets and $1_\lambda: \lambda \rightarrow I$ denotes the whole fuzzy set.
- (2) Arbitrary union of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.
- (3) Finite intersection of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

That is, $\tau_{(R)}(\lambda)$ is a topology on X called the Nano fuzzy topology on X with respect to λ . We call $(X, \tau_{(R)}(\lambda))$ as the Nano fuzzy topological space (NFTS). The elements of the Nano fuzzy topological space that is $\tau_{(R)}(\lambda)$, are called Nano fuzzy open sets and elements of $[\tau_{(R)}(\lambda)]^c$ are called Nano fuzzy closed sets.

2.11. [16] Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space with respect to λ where $\lambda \leq X$ and if $\mu \leq X$ then the Nano fuzzy interior of μ is defined as union of all Nano fuzzy open subsets of μ and it is denoted by $NfInt(\mu)$. That is, it is the largest Nano

fuzzy open subset contained in μ . Similarly, the Nano fuzzy closure of μ is defined as the intersection of all Nano fuzzy closed sets containing μ . It is denoted by $NfCl(\mu)$ and it is the smallest Nano fuzzy closed set containing μ .

3. Nano Fuzzy Bitopological Space and Nano Fuzzy $(1, 2)^*$ Generalized Closed Sets

In this section we have introduced Nano fuzzy bitopological spaces and investigated some of its properties. We have also defined Nano fuzzy $(1,2)^*$ generalized closed sets in terms of Nano fuzzy $(1,2)^*$ closure and studied some of its properties. Throughout this paper $(X, \tau_{1,2R}(\lambda))$ is a Nano fuzzy bitopological space with respect to fuzzy subset λ and R is an equivalence relation on X , X/R denotes the family of equivalence classes of X by R .

Definition 3.1. Let X be a non-empty finite set, R be an equivalence relation on X , $\lambda_1, \lambda_2 \leq X$ be fuzzy subsets and $\tau_{1,2R}(\lambda) = \bigvee \{\tau_{1R}(\lambda_1), \tau_{2R}(\lambda_2)\}$. Then $\tau_{1,2R}(\lambda)$ satisfies the following axioms

- (1) $0_{\lambda_1}, 1_{\lambda_1} \in \tau_{1R}(\lambda_1)$ where $0_{\lambda_1}: \lambda_1 \rightarrow I$ denotes the null fuzzy sets and $1_{\lambda_1}: \lambda_1 \rightarrow I$ denotes the whole fuzzy set and $0_{\lambda_2}, 1_{\lambda_2} \in \tau_{2R}(\lambda_2)$ where $0_{\lambda_2}: \lambda_2 \rightarrow I$ denotes the null fuzzy sets and $1_{\lambda_2}: \lambda_2 \rightarrow I$ denotes the whole fuzzy set.
- (2) Arbitrary union of members of $\tau_{1,2R}(\lambda)$ is a member of $\tau_{1,2R}(\lambda)$.
- (3) Finite intersection of members of $\tau_{1,2R}(\lambda)$ is a member of $\tau_{1,2R}(\lambda)$.

That is, $\tau_{1,2R}(\lambda)$ is a topology on X called the Nano fuzzy topology on X with respect to λ_1 and λ_2 . We call $(X, \tau_{1,2R}(\lambda))$ as the Nano fuzzy bitopological space (NFBTS). The elements of the Nano fuzzy bitopological space are called Nano fuzzy $(1,2)^*$ open sets and elements of $[\tau_{1,2R}(\lambda)]^C$ are called Nano fuzzy $(1,2)^*$ closed sets.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ with

$X/R = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}$. Let λ_1 and λ_2 be a fuzzy subset of X

such that $\lambda_1 = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.7), (x_4, 0.4)\}$

and $\lambda_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.6), (x_4, 0.4)\}$. Then NFBTS is given by

$$\tau_{1,2R}(\lambda) = \{1_\lambda, 0_\lambda, \underline{R_{1,2}}(\lambda), \overline{R_{1,2}}(\lambda), Bd_{1,2}(\lambda)\}, \text{ where}$$

$$\underline{R_{1,2}}(\lambda) = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.4), (x_4, 0.4)\} \text{ and}$$

$$\overline{R_{1,2}}(\lambda) = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.7), (x_4, 0.7)\} \text{ with}$$

$$Bd_{1,2}(\lambda) = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.3), (x_4, 0.3)\}.$$

Definition 3.3. Let $(X, \tau_{1,2R}(\lambda))$ be a Nano fuzzy bitopological space with respect to λ_1 and λ_2 where $\lambda_1, \lambda_2 \leq X$ and if $\mu \leq X$ then the Nano fuzzy $(1,2)^*$ interior of μ is defined as union of all Nano fuzzy $(1,2)^*$ open subsets of μ and it is denoted by $Nf(1,2)^* Int(\mu)$. That is, it is the largest Nano fuzzy $(1,2)^*$ open subset contained in μ . Similarly, the Nano fuzzy $(1,2)^*$ closure of μ is defined as the intersection of all Nano fuzzy $(1,2)^*$ closed sets containing μ . It is denoted by $Nf(1,2)^* Cl(\mu)$ and it is the smallest Nano fuzzy $(1,2)^*$ closed set containing μ .

Definition 3.4. Let $(X, \tau_{1,2R}(\lambda))$ be a Nano fuzzy bitopological space. A subset μ of $(X, \tau_{1,2R}(\lambda))$ is called Nano fuzzy $(1,2)^*$ generalized closed set if $Nf(1,2)^* Cl(\mu) \leq \vartheta$ where $\mu \leq \vartheta$ and ϑ is Nano fuzzy $(1,2)^*$ open.

Example 3.5. Let $X = \{x_1, x_2, x_3, x_4\}$ with

$$X/R = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}. \text{ Let } \lambda_1 \text{ and } \lambda_2 \text{ be a fuzzy subset of } X \text{ such that } \lambda_1 = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.7), (x_4, 0.4)\}$$

and $\lambda_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.6), (x_4, 0.4)\}$. Then NFBTS is given by

$$\tau_{1,2R}(\lambda) = \{1_\lambda, 0_\lambda, \underline{R_{1,2}}(\lambda), \overline{R_{1,2}}(\lambda), Bd_{1,2}(\lambda)\}, \text{ where}$$

$$\underline{R_{1,2}}(\lambda) = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.4), (x_4, 0.4)\} \text{ and}$$

$$\overline{R_{1,2}}(\lambda) = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.7), (x_4, 0.7)\} \text{ with}$$

$$Bd_{1,2}(\lambda) = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.3), (x_4, 0.3)\}.$$

Then Nano fuzzy $(1,2)^*$ closed sets are

$$[\tau_{1,2R}(\lambda)]^c = \{1_\lambda, 0_\lambda, \underline{R_{1,2}}(\lambda)^c, \overline{R_{1,2}}(\lambda)^c, Bd_{1,2}(\lambda)^c\} \text{ where}$$

$$[\underline{R_{1,2}}(\lambda)]^c = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.6), (x_4, 0.6)\}. \quad \overline{R_{1,2}}(\lambda)^c =$$

$$\{(x_1, 0.8), (x_2, 0.2), (x_3, 0.3), (x_4, 0.3)\} \text{ and}$$

$$[Bd_{1,2}(\lambda)]^c = \{(x_1, 0.9), (x_2, 0.7), (x_3, 0.7), (x_4, 0.7)\}.$$

Let $\mu \leq X = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.2), (x_4, 0)\}$. Clearly, $Nf(1,2)^* Cl(\mu) \leq \vartheta$ where $\mu \leq \vartheta$ and ϑ is Nano fuzzy $(1,2)^*$ open. Hence, μ is Nano fuzzy $(1,2)^*$ generalized closed set.

Theorem 3.6. If μ and μ' are Nano fuzzy $(1,2)^*$ generalized closed, then $\mu \vee \mu'$ is Nano fuzzy $(1,2)^*$ generalized closed.

Proof: Let μ and μ' are Nano fuzzy $(1,2)^*$ generalized closed sets. Then $Nf(1,2)^* Cl(\mu) \leq \nu$ where $\mu \leq \nu$ and ν is Nano fuzzy $(1,2)^*$ open and $Nf(1,2)^* Cl(\mu') \leq \nu$ where $\mu' \leq \nu$ and ν is Nano fuzzy $(1,2)^*$ open. Since μ and μ' are subsets of ν , $\mu \vee \mu'$ is a subset of ν and ν is Nano fuzzy $(1,2)^*$ open. Then $Nf(1,2)^* Cl(\mu \vee \mu') = Nf(1,2)^* Cl(\mu) \vee Nf(1,2)^* Cl(\mu') \leq \nu$ which implies that $\mu \vee \mu'$ is Nano fuzzy $(1,2)^*$ generalized closed.

Theorem 3.7. If μ is Nano fuzzy $(1,2)^*$ generalized closed set and $\mu \leq \mu' \leq Nf(1,2)^* Cl(\mu)$, then μ' is Nano fuzzy $(1,2)^*$ generalized closed.

Proof: Let $\mu' \leq \nu$ where ν is Nano fuzzy $(1,2)^*$ generalized open in $\tau_{1,2R}(\lambda)$. Then $\mu \leq \mu'$ implies $\mu \leq \nu$. Since μ is Nano fuzzy $(1,2)^*$ generalized closed, $Nf(1,2)^* Cl(\mu) \leq \nu$. Also $\mu' \leq Nf(1,2)^* Cl(\mu)$ implies $Nf(1,2)^* Cl(\mu') \leq Nf(1,2)^* Cl(\mu) \leq \nu$ and so μ' is Nano fuzzy $(1,2)^*$ generalized closed.

Theorem 3.8. Every Nano fuzzy $(1,2)^*$ closed set is Nano fuzzy $(1,2)^*$ generalized closed set.

Proof: Let $\mu \leq \nu$ be a Nano fuzzy $(1,2)^*$ closed set and ν is Nano fuzzy $(1,2)^*$ open in Nano fuzzy bitopological space $\tau_{1,2R}(\lambda)$. Since μ is Nano fuzzy $(1,2)^*$ closed, $Nf(1,2)^* Cl(\mu) \leq \mu$. That is $Nf(1,2)^* Cl(\mu) \leq \mu \leq \nu$. Hence μ is Nano fuzzy $(1,2)^*$ generalized closed set.

Theorem 3.9. (for operation consider [1]) A Nano fuzzy $(1,2)^*$ generalized closed set μ is Nano fuzzy $(1,2)^*$ closed set if and only if $Nf_{\tau_{1,2R}} Cl(\mu) - \mu$ is Nano fuzzy $(1,2)^*$ closed.

Proof: Let μ be a Nano fuzzy $(1,2)^*$ closed set. Then $Nf_{\tau_{1,2R}} Cl(\mu) = \mu$ and so, $Nf_{\tau_{1,2R}} Cl(\mu) - \mu = \phi$ which is Nano fuzzy $(1,2)^*$ closed set.

Conversely, suppose $Nf_{\tau_{1,2R}} Cl(\mu) - \mu$ is Nano fuzzy $(1,2)^*$ closed. Then $Nf_{\tau_{1,2R}} Cl(\mu) - \mu = \phi$. Since μ be a Nano fuzzy $(1,2)^*$ closed set. This implies that $Nf_{\tau_{1,2R}} Cl(\mu) = \mu$ and so μ is Nano fuzzy $(1,2)^*$ closed.

Theorem 3.10. Suppose that $\beta \leq \alpha \leq X$, β is a Nano fuzzy $(1,2)^*$ generalized closed set relative to α and that α is a Nano fuzzy $(1,2)^*$ generalized closed set relative to X . Then β is Nano fuzzy $(1,2)^*$ generalized closed set relative to X .

Proof: Let $\beta \leq \nu$ and suppose that ν is Nano fuzzy $(1,2)^*$ open in X . Then $\beta \leq \alpha \wedge \nu$. Therefore, $Nf_{\tau_{1,2R}} Cl_{\alpha}(\beta) \leq \alpha \wedge \nu$. It follows that $\alpha \wedge Nf_{\tau_{1,2R}} Cl(\beta) \leq \alpha \wedge \nu$ and $\alpha \leq \nu \vee Nf_{\tau_{1,2R}} Cl(\beta)$. Since α is a Nano fuzzy $(1,2)^*$ generalized closed set relative to X , we have $Nf_{\tau_{1,2R}} Cl(\alpha) \leq \nu \vee Nf_{\tau_{1,2R}} Cl(\beta)$. Therefore, $Nf_{\tau_{1,2R}} Cl(\beta) \leq Nf_{\tau_{1,2R}} Cl(\alpha) \leq \nu \vee Nf_{\tau_{1,2R}} Cl(\beta)$. Then β is Nano fuzzy $(1,2)^*$ generalized closed set relative to X .

4 Conclusion

In this research work, we characterize Nano fuzzy bitopological spaces through the framework of topological spaces, specifically as an extension of Nano fuzzy topological spaces. To elaborate on the evolution of this theory, we have defined Nano fuzzy $(1,2)^*$ closed sets, Nano fuzzy $(1,2)^*$ open sets, Nano fuzzy $(1,2)^*$ generalized closed sets and Nano fuzzy $(1,2)^*$ interior and closure. We established the relationships among these defined Nano fuzzy sets. In the forthcoming work, Nano fuzzy $(1,2)^*$ continuous functions and Nano fuzzy $(1,2)^*$ open maps, Nano fuzzy $(1,2)^*$ closed maps and Nano fuzzy $(1,2)^*$ homeomorphisms. Expressing this differently, exploring these concepts may prove valuable in uncovering practical applications.

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Supra β -Paracompact Spaces

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Abstract

In this paper, authors present the concept of supra β -compact spaces and studied some basic properties of them. Further, supra β -locally finite collection and supra β -compact spaces are studied. Also, Some results regarding supra β -paracompact spaces are studied.

Keywords: Paracompact spaces, supra paracompact spaces, supra β -open sets, supra β -continuous functions.

2010 AMS subject classifications: 54A05, 54C08. ¹

1 Introduction

A space M is said to be paracompact if every open cover of M has a locally finite open refinement. Many researchers introduced and investigated the properties of α -paracompact [1], P_1 -paracompact [11] and S_1 -paracompact [13] spaces replacing the open cover by α -open, pre-open and semi-open cover in original definition.

Mashhour et al. [12] introduced the supra topological space and studied the properties of S -continuous maps, S^* -continuous maps in supra topology. Motivation for supra topology come from the need to obtain various examples that

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satisfy some properties on the finite sets, for instance, the only topology on a finite set that satisfies a T_1 -space is the discrete topology, where as there are many different types of supra topologies that satisfy a condition of T_1 -space.

Several authors, investigated the some topological notions on supra topological spaces such as compact spaces, [[2], [3]] and separation axioms [[4], [8]]. Some of the published paper showed that, any topological results became true on supra topology, but some fails to prove the results. Thus, supra topology provides a framework which is general enough to solve practical problems and to model phenomena.

Jassim T. H. [10] introduced the concept of supra compactness in supra topology and came out with many intersecting results. Paracompactness on the supra topology was introduced and studied by T.M.Al-shami [5] in 2020.

In this paper, authors are introduced and studied the concept of β -compact spaces, β -locally finite collection and β -paracompact spaces in supra topology. We discuss the main properties of these spaces.

2 Preliminaries

Throughout this paper (P, μ) and (Q, σ) (or simply P and Q) always mean supra topological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. [12] A sub collection μ of 2^P is called a supra topology on $P \neq \emptyset$ provided that it contains P and is closed under arbitrary union.

A pair (P, μ) is called a supra topological spaces.

Every member of μ is called a supra open set and its complement is called a supra closed sets.

Definition 2.2. [12] A map $s : (P, \mu) \rightarrow (Q, \sigma)$ is said to be an S^* -continuous if the inverse image of every supra open set in Q is supra open set in P .

Definition 2.3. [2] Let A_1 be a subset of a supra topological space (P, μ) . The family $\mu_{A_1} = \{A_1 \cap G_1 : G_1 \in \mu\}$ is called a supra relative topology on A_1 .

A pair (A_1, μ_{A_1}) is called a supra subspace of (P, μ) .

Definition 2.4. [2] A space (P, μ) is called a supra compact if every supra open cover of P has a finite subcover.

Definition 2.5. [7] Let $U_1 = \{U_i : i \in I\}$ and $V_1 = \{V_j : j \in J\}$ be two covers of P . The collection U_1 is said to be a refinement of V_1 provided that for each $U_i \in U_1$, there are some $V_j \in V_1$ such that $U_i \subseteq V_j$.

Definition 2.6. [7] The collection $U_1 = \{U_i : i \in I\}$ of subsets of a space P is said to be a locally finite provided that each point $p \in P$ has a neighborhood W such that $W \cap U_i \neq \phi$ holds for finitely many values of i .

Definition 2.7. [7] If the collection $\{U_i : i \in I\}$ is locally finite, then the collection $\{cl(U_i) : i \in I\}$ is locally finite.

Definition 2.8. [9] A function $s : (P, \mu) \rightarrow (Q, \sigma)$ is said to be

- (i) supra β -continuous if the inverse image of each open set in Q is supra β -open set in P .
- (ii) supra β -irresolute if the inverse image of every supra β -open set in Q is supra β -open in P .
- (iii) pre supra β -closed if the image of supra β -open set in P is supra β -open in Q .

3 supra β -compact spaces

Definition 3.1. A collection $\{A_i : i \in \Lambda\}$ of a supra β -open subsets of the space (P, μ) is called a supra β -open cover of the subsets B of P provided $B \subseteq \cup_{i \in \Lambda} A_i$.

Definition 3.2. A supra topological space (P, μ) is called supra β -compact provided that every supra β -open cover of P has a finite subcover.

Remark 3.1. Every finite supra topological space is supra β -compact.

Remark 3.2. A finite union of supra β -compact subsets of the space (P, μ) is supra β -compact.

Definition 3.3. A subset B of a supra topological space (P, μ) is said to be supra β -compact relative to (P, μ) if for every collection $\{A_i : i \in I\}$ of supra β -open subsets of P with $B \subseteq \cup\{A_i : i \in I\}$, there exists a finite subset I_0 of I such that $B \subseteq \cup\{A_i : i \in I_0\}$.

Theorem 3.1. Every supra β -compact space is supra compact.

Proof: Let $\{A_i : i \in \lambda\}$ be supra open cover of P . As every supra open set is supra β -open, so $\{A_i : i \in \lambda\}$ is a supra β -open cover of P . As P is supra β -compact, $\{A_i : i \in \lambda\}$ has a finite subcover. Thus, P is supra compact.

Definition 3.4. A subset A_1 of a supra space (P, μ) is said to be supra β -compact relative to P if every supra β -open cover of A_1 is reducible to a finite subcover.

Theorem 3.2. Every supra β -closed subset of a supra β -compact space is supra β -compact with respect to the space μ .

Proof: Let $S = \{G_i : i \in \lambda\}$ be a supra β -open cover of the supra β -closed subset F of the space P . Then $F^c = P \setminus F$ is supra β -open with $F \subseteq \bigcup \{G_i : i \in \lambda\}$. Let $S^* = \{G_i : i \in \lambda\} \cup \{F^c\}$ be a supra β -open cover of P , that is $P = \bigcup S^* = \bigcup \{G_i : i \in \lambda\} \cup \{F^c\}$. Then S^* is reducible to a finite subcover $G_1 \cup G_2 \cup \dots \cup G_n \cup F^c$. Hence $G \subseteq G_1 \cup G_2 \cup \dots \cup G_n$. Thus, every supra β -open cover of S contains a finite subcover. Thus, G is supra β -compact relative to μ .

Theorem 3.3. A space P is supra β -compact iff every collection of supra β -closed subsets has a finite intersection property has a non empty intersection.

Theorem 3.4. Supra β -irresolute image of a supra β -compact set is supra β -compact.

Proof: Let $S : (P, \mu) \rightarrow (Q, \sigma)$ be a supra β -irresolute map and A_1 be a supra β -compact subset of P . Let $\{G_i : i \in \lambda\}$ be supra β -open cover of Q . Then $A_1 \subseteq \bigcup_i S^{-1}(G_i)$. As S is supra β -irresolute, $S^{-1}(G_i)$ is supra β -open cover. Since A_1 is supra β -compact, $A_1 \subseteq \bigcup_{i=1}^n S^{-1}(G_i)$. Thus, $S(A_1) \subseteq \bigcup_{i=1}^n G_i$ and so $S(A_1)$ is supra β -compact.

4 Supra β -locally finite collection

Definition 4.1. A space (P, μ) is said to be Supra paracompact[2] if every supra open cover has a supra open locally finite refinement.

Definition 4.2. A collection $\Xi_1 = \{F_\alpha : \alpha \in \lambda\}$ of subsets of a space P is said to be supra β -locally finite provided that for each $p \in P$, there exists a supra β -open sets $U_1 \in P$ with U_1 intersects F_α at most finitely many values of α .

Example 4.1. Let us consider $P = \{p_1, p_2, r_1, r_2\}$ and $\mu = \{P, \phi, \{p_1, p_2, r_1\}, \{p_1, p_2, r_2\}, \{p_1, r_1, r_2\}, \{p_2, r_1, r_2\}, \{p_1, p_2\}, \{r_1, r_2\}, \{p_1, r_1\}, \{p_2, r_2\}, \{r_1, p_2\}\}$ be a supra topology on P . We can verify that, the subset $A = \{p_1, p_2\}$ is a supra β -locally finite in P .

Lemma 4.1. Let $\Xi = \{F_\alpha : \alpha \in \lambda\}$ be a collection of subsets of a space (P, μ) . Then the following properties holds:

- (i) if $\Xi = \{F_\alpha : \alpha \in \lambda\}$ is supra β -locally finite collection of subsets of P with $G_\alpha \subseteq F_\alpha$ holds for each $\alpha \in \lambda$, then $G_1 = \{G_\alpha : \alpha \in \lambda\}$ is also supra β -locally finite.
- (ii) Ξ is supra β -locally finite if and only if $\{\beta\text{-cl}(F_\alpha) : \alpha \in \lambda\}$ is also supra β -locally finite.
- (iii) if Ξ is supra β -locally finite then $\bigcup \beta\text{-cl}(F_\alpha) = \beta\text{-cl}(\bigcup F_\alpha)$.

Proof: (i) Follows from the definition of supra β -locally finite collection.

(ii) Assume that Ξ is supra β -locally finite. Then, for each $p \in P$, $\exists$ a β -open set U_1 containing the point p which meets finitely many of the supra sets of F_α , that is $F_{1_\alpha}, F_{2_\alpha}, F_{3_\alpha}, \dots, F_{n_\alpha}$. But $F_{\alpha_k} \subseteq \beta\text{-cl}(F_{\alpha_k})$ holds for each $k = 1, 2, \dots, n$. Then U_1 meets many supra β -open sets, say $\beta\text{-cl}(F_{\alpha_1}), \beta\text{-cl}(F_{\alpha_2}), \dots, \beta\text{-cl}(F_{\alpha_k})$. Thus U_1 meets finitely many of the supra sets $\beta\text{-cl}F_\alpha$. Thus, $\beta\text{-cl}(F_\alpha)$ is supra β -locally finite.

On the other hand, let $p \in P$. Then, there exists supra β -open set U_1 containing the point p which meets finitely many of the supra sets $\beta\text{-cl}(F_\alpha)$, that is $\beta\text{-cl}(F_{\alpha_1}), \beta\text{-cl}(F_{\alpha_2}), \dots, \beta\text{-cl}(F_{\alpha_n})$. Thus, $U_1 \cap \beta\text{-cl}(F_{\alpha_k}) \neq \phi$ holds for each $k = 1, 2, \dots, n$. Let $q \in U_1$ and $q \in \beta\text{-cl}(F_{\alpha_k})$. Then for every supra β -open set V_1 containing the point q such that $V_1 \cap F_{\alpha_k} \neq \phi$. Since U_1 is supra β -open set in P containing the point p , then $U_1 \cap F_{\alpha_k} \neq \phi$. Thus, Ξ is supra β -locally finite.

(iii) It is obvious.

5 Supra β -paracompact spaces

Definition 5.1. A space (P, μ) is said to be supra β -paracompact if every supra open cover has a supra β -locally finite supra β -open refinement.

Remark 5.1. Every supra compact space is supra β -paracompact.

Remark 5.2. The existence of supra topology does not forms a supra β -paracompact spaces follows from the following example.

Example 5.1. Let us consider a supra topology (P, μ) , where $\mu = \{\phi, G \subseteq Z : 2 \in G \text{ and } 3 \in G\}$ be a supra topology on set of all integers. Let U_1 be an infinite supra open cover of the space Z . Then any supra open refinement of U_1 contains an infinite supra open sets containing the points 2 and 3. Thus, we can not find a locally finite open refinement for a supra open cover U_1 . Thus, (Z, μ) is not supra β -paracompact.

Theorem 5.1. Union of any two supra β -paracompact space is supra β -paracompact.

Proof: Let $W_1 = \{w_i : i \in \lambda\}$ be a supra open cover of the set $A_1 \cup B_1$ where A_1 and B_1 are supra paracompact spaces. Then W_1 is a supra open cover of the sets A_1 and B_1 . Since W_1 has a supra β -locally finite open refinement say U_1 and V_1 of the sets A_1 and B_1 respectively. Again, from lemma 4.1, we have $U_1 \cup V_1$ is supra β -loaclly finite refinement of $A_1 \cup B_1$. Thus, $A_1 \cup B_1$ is supra β -paracompact.

Remark 5.3. [9] Union of supra open and supra β -open set is again supra β -open set.

Theorem 5.2. Every supra closed subset of a supra β -paracompact space is supra β -paracompact.

Proof: Let $U_1 = \{U_i : i \in \lambda\}$ be a supra open cover of the supra closed set F_1 . Then $U_1 \cup F_1^c$ is supra open cover of the space P . Since P is supra β -paracompact, then $U_1 \cup F_1^c$ has a supra open supra β -locally finite open refinement. Thus U_1 has a supra open, supra β -locally finite refinement of F_1 . Thus, F_1 is supra β -paracompact.

Theorem 5.3. Let $s : (P, \mu) \rightarrow (Q, \sigma)$ be surjective function. The S is supra pre β -closed if and only if for every $q \in Q$ and every supra β -open set $U_q \in (P, \mu)$ contains $s^{-1}(q)$, there exists a supra β -open set $V_1 \in Q$ with $s^{-1}(V_1) \subseteq U_1$.

Proof: Let us prove the necessity part first. Let $q \in Q$ and U_1 be supra β -open set in P with $s^{-1}(q) \subseteq U_1$. Put $V_1 = Q \setminus s(P \setminus U_1)$. Then from hypothesis, V_1 is supra β -open set in Q containing q with $q \in V_1$ and $s^{-1}(V_1) \subseteq U_1$.

Let us consider the sufficiency part. Let M be supra β -closed set in P with $q \in Q \setminus s(M)$. Then, $s^{-1}(q) \subset P \setminus M$. From hypothesis, there exists a supra β -open set V_{1q} such that $s^{-1}(V_{1q}) \subseteq Q \setminus s(M)$. Thus, $Q \setminus s(M) = \cup \{V_{1q} : q \in Q \setminus s(M)\}$ and so $Q \setminus s(M)$ is supra β -open set in Y . Hence $s(M)$ is supra β -closed in Q .

Theorem 5.4. Let $s : (P, \mu) \rightarrow (Q, \sigma)$ be supra continuous, supra open and supra pre β -closed surjective function with $s^{-1}(q)$ is supra β -compact for each $q \in Q$. Then Q is supra β -paracompact if P is supra β -paracompact.

Proof: Let $\Psi = \{U_\alpha : \alpha \in \lambda\}$ be an supra open cover of Q . Then $s^{-1}(\Psi) = \{s^{-1}(U_\alpha) : \alpha \in \lambda\}$ be an supra open cover of the supra β -paracompact space P and so it has a locally finite supra β -open refinement, that is $\nu = \{V_\gamma : \gamma \in B\}$. As s is supra pre β -open, the collection $s(\nu) = \{s(V_\gamma) : \gamma \in B\}$ is a supra β -open refinement of Ψ .

Further, we have to prove that the collection is supra β -locally finite in Q .

Let $q \in Q$. Then for each $p \in s^{-1}(q)$, there exists a supra β -open set U_p such that U_p intersects at most finitely many members of ν . Then the collection $\{U_p : p \in s^{-1}(q)\}$ is a supra β -open cover of $s^{-1}(q)$. Thus, there exists a finite supra subset M of $s^{-1}(q)$ with $s^{-1}(q) \subset \cup U_p$. By the definition of supra pre β -closed function, there exists a supra β -open set G_q containing q such that $s^{-1}(G_q) \subseteq \cup U_p$. Thus, $f^{-1}(G_q)$ intersects at most finitely many members of Ψ . Thus, G_q intersects at most finitely many members of $s(\Psi)$. Hence $s(\Psi)$ is supra β -locally finite in Q .

Theorem 5.5. Let $s : (P, \mu) \rightarrow (Q, \sigma)$ be supra β -irresolute, supra closed surjective function with $s^{-1}(q)$ is compact for each $q \in Q$. Then P is supra β -paracompact if Q is supra β -paracompact.

Proof: Let $\Psi = \{U_\alpha : \alpha \in \lambda\}$ be an supra open cover of P . As $s^{-1}(q)$ is supra compact, there exists a supra finite subset $M(q)$ such that $s^{-1}(q) \subseteq \cup_{\alpha \in M(q)} U_\alpha$. Since s is supra closed, there exists a supra open set V_q containing q such that

$s^{-1}(q) \subseteq \cup_{\alpha \in M(q)} U_{\alpha}$. Thus, $\{V_q : q \in Q\}$ is a supra open cover of the supra β -paracompact space Q and so it has a supra locally finite supra β -open refinement say $W = \{w_{\gamma} : \gamma \in B\}$. Since s is supra β -irresolute, $\{s^{-1}(w_{\gamma}) : \gamma \in B\}$ is a supra β -locally finite supra β -open cover of P . Then for each $\gamma \in B$, there exists a member $q(\gamma) \in Q$ such that $W_{\gamma} \subseteq V_{q(\gamma)}$. Thus, $s^{-1}(W_{\gamma}) \subseteq s^{-1}(V_{q(\gamma)}) \subseteq \cup_{\alpha \in M(q(\gamma))} U_{\alpha} = F_{q(\gamma)}$. Put $\Omega = \{s^{-1}(W_{\gamma}) \wedge F_{q(\gamma)} : \gamma \in B, q(\gamma) \in Q\}$, where $s^{-1}(W_{\gamma}) \wedge F_{q(\gamma)} = \{s^{-1}(W_{\gamma}) \cap U_{\alpha} : \gamma \in B, \alpha \in M(q(\gamma))\}$. By remark 5.1, each set of Ω is supra β -open subset of P . Thus, the family Ω is supra β -locally finite supra β -open refinement of Ψ . Thus, P is supra β -paracompact.

6 Discussion and Conclusion

Topology developed as a field of study of geometry and set theory, through analysis of such concepts as space, dimensions and transformation. The topological structures are modelled suitably in the field of Computer graphics, Pattern recognition, Artificial intelligence, Data mining, Rough set theory, Information systems, Quantum physics etc.

It is noted that the supra topological frame can be more convenient to solve some practical problems and to model some phenomena. Also, the possibility of applying semi-open sets to deal with some problems on digital topology.

In this way, this paper introduce some main concepts in supra topological spaces. We firstly introduce a supra β -compact spaces and some related properties and point out the relationship among them. Further, we introduce supra β -locally finite collection and and several results of these spaces in supra topological spaces. In the end, we presented the concept of supra β -paracompact spaces and several properties concerning to these spaces in supra topological spaces.

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Octagonal Prime Graceful Labeling

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Abstract

The main objectives of this research work is to explore and detect some new types of graphs that exhibit octagonal prime graceful labeling. The methodology entails developing a mathematical formulation for labeling a given graph's vertices and demonstrating that these formulations result in octagonal prime graceful labeling. Here we describe octagonal prime graceful labeling which is a new version of octagonal graceful labeling. In the present paper, we establish octagonal prime graceful labeling for fan graph and friendship graph. Octagonal graceful labeling was introduced by S. Mahendran, K. Kovusalya1 and P. Namasivayam, here we find the octagonal prime graceful labeling for the join of two graphs, shell graph, generalised butterfly graph, prism and polar grid graph. This is the first attempt of its sort, involving the investigation of octagonal prime graceful labeling for special graphs.

Keywords: Octagonal numbers, Octagonal graceful labeling, Octagonal graceful graph, Octagonal prime graceful labeling

2020 AMS subject classifications: 05C78 ¹

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1 Introduction

The finite, straightforward, as well as undirected graphs are examined in this paper. The graph definition with q edges and p vertices is $G = (V, E)$. Labeling is the process of giving numbers to a graph's vertices, edges, or both. If the mapping's domain is a set of edge (vertices/both), this labeling is referred to as a vertex labeling (edge/both). Graph labeling strategies include graceful labeling, incidence labeling, gracious labeling, radio labeling, antimagic labeling and prime labeling. One of these most often-used graph labeling approaches is graceful labeling [15]. The concept of β -valuation of a graph is proposed by Rosa[1]. According to Golomb[12], it was a graceful labeling.

Let's assume that G is a (p, q) graph. The graceful G labeling is considered to be a one-to-one function if the induced edge labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$ represented by $f^*(e) = |f(u) - f(v)|$ for all edges $e = uv$ of G is likewise one-to-one. A ' G ' graph with graceful labeling is referred to as a graceful graph. Several families of graceful graphs were built in [12]. A thorough examination of the many forms of graceful labeling can be found in [8].

Applications include creating an addressing system for communication networks, producing X-Ray crystallography, selecting the best circuit topologies and solving additive number theory problems and others, labelled graphs are becoming a more important family of mathematical models [7].

Numbers of the form $O_n = n(3n - 2)$ for all $n \geq 1$ are called octagonal numbers. Let G be a graph with p vertices and q edges. Let $f : V(G) \rightarrow \{0, 1, 2, \dots, O_m\}$ where O_m is the m^{th} octagonal number be an injective function. Define the function $f^* : E(G) \rightarrow \{1, 8, 21, \dots, O_m\}$ such that $f^*(uv) = |f(u) - f(v)|$ for all edges $uv \in E(G)$. If $f^*(E(G))$ is a sequence of distinct consecutive octagonal numbers $1, 2, \dots, O_q$, then the function f is said to be octagonal graceful labeling and the graph which admits such a labeling is called a octagonal graceful graph[7]. In this paper we have introduced octagonal prime graceful labeling and it is defined as follows. Let G be a graph with p vertices and q edges. Define a bijection $f : V(G) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_i) = i(3i - 2)$ for every i from 1 to p and define a 1 - 1 mapping $f_{opgl}^* : E(G) \rightarrow$ set of natural number N such that $f^*(uv) = |f(u) - f(v)|$ for all edges $(uv) \in E(G)$. The induced function f is said to be octagonal prime graceful labeling if the gcin of each vertex of degree atleast 2 is one.

This research examines the octagonal prime graceful labeling fan graph and friendship graph and some special graphs namely join of two graphs, shell graph, generalised butterfly graph, prism and polar grid graph.

2 Preliminaries

Definition 2.1. [9] A fan graph obtained by joining all vertices of F_n , $n \geq 2$ is a path P_n to a further vertex, called the centre. Thus F_n contains $n + 1$ vertices say C and $(2n - 1)$ edges, say cv_i , $1 \leq i \leq n$ and $v_i v_{i+1}$, $1 \leq i \leq n - 1$.

Definition 2.2. [5] If G_1 and G_2 are disjoint graphs then the join of G_1 and G_2 written as $G_1 + G_2$ is the graph consisting of the union $G_1 \cup G_2$, together with all edges of the type $v_1 v_2$ where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

Definition 2.3. [10] A shell graph is defined as a cycle C_n with $(n - 3)$ chords sharing a common end point called the apex. Shell graph are denoted as $C(n, n - 3)$. A shell S_n is also called fan f_{n-1} .

Definition 2.4. [14] A generalized butterfly graph, BF_n , obtained by inserting vertices to every wing with assumption that sum of inserting vertices to every wing are same then it has $2n + 1$ vertices and $4n - 2$ edges. Let the vertex set of BF_n be $V(BF_n) = \{v_i | i = 0, 1, 2, \dots, 2n\}$ and the edge set of BF_n be $E(BF_n) = \{(v_i, v_{i+1}) | i = 1, 2, \dots, n - 1, n + 1, \dots, 2n - 1\} \cup \{(v_0, v_i) | i = 1, 2, \dots, 2n\}$.

Definition 2.5. [3] The friendship graph $Fr_n^{(3)}$, is a planar undirected graph with $2n + 1$ vertices and $3n$ edges constructed by joining n copies of the cycle graph C_3 with a common vertex

Definition 2.6. [16] For $n \geq 3$, let $\{v_0, v_1, v_2, \dots, v_n\}$ be the vertices of W_n with hub vertex v_0 and $w'_n = \{u_0, u_1, u_2, \dots, u_n\}$ be a copy of W_n . Define the $Prism(W_n)$, called the prism of W_n by, $P(W_n) = K_2 \times W_n$ i.e., joining v_0 of W_n to the corresponding vertex u_0 of w'_n and each v_i of W_n to the corresponding vertex u_i of w'_n for all $i \in \{1, 2, \dots, n\}$. Thus, $E(P(W_n)) = E(W_n) \cup E(W'_n) \cup \{v_i u_i, i \in \{1, 2, \dots, n\}\} \cup \{v_0 u_0\}$.

Definition 2.7. [2] The polargrid graph $P_{m,n}$ is the graph consists of n copies of circles C_m which will be numbered from the inner most circle to the outer circle as $C_m^{(1)}, C_m^{(2)}, \dots, C_m^{(n-1)}, C_m^{(n)}$ and m copies of paths P_{n+1} intersected at the center vertex v_0 which will be numbered as $P_{n+1}^{(1)}, P_{n+1}^{(2)}, \dots, P_{n+1}^{(m-1)}, P_{n+1}^{(m)}$.

3 Main Results

Theorem 3.1. *The Fan graph F_n admits octagonal prime graceful labeling.*

Proof. Let $G = F_n$ and let $v_1, v_2, \dots, v_{n+1}$ are the vertices of G .
 Let p and q be the total number of vertices and edges of G respectively.
 The fan graph F_n has $n + 1$ vertices and $(2n - 1)$ edges.
 ie) $p = |V(F_n)| = n + 1$ and $q = |E(F_n)| = 2n - 1$.
 Define a function $f : V(G) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_i) = i(3i - 2)$,
 where $i = 1, 2, \dots, n + 1$ and the edge labelings are given as
 $f_{opgl}^*(v_i v_{i+1}) = 6i + 1$, where $i = 2, 3, \dots, n$
 $f_{opgl}^*(v_1 v_{i+1}) = 3i^2 + 4i$, where $i = 1, 2, \dots, n$
 Clearly f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(G)$ such that
 $f_{opgl}^*(uv) = |f(u) - f(v)|$.
 Also the $gcin$ of $(v_1) = \gcd$ of edges incident on $v_1 = 1$
 $gcin$ of $(v_{i+1}) = \gcd$ of edges incident on $v_{i+1} = 1$, where $i = 1, 2, \dots, n$
 In general, $gcin$ of $v_{n+1} = \gcd$ of edges incident on v_{n+1}
 Hence the $gcin$ of each vertex of degree atleast 2 is one.
 Therefore f is said to be an octagonal prime graceful labeling.
 Hence the graph F_n is an octagonal prime graceful graph.

Example 3.1. *Fan graph F_6 admits octagonal prime graceful labeling.*

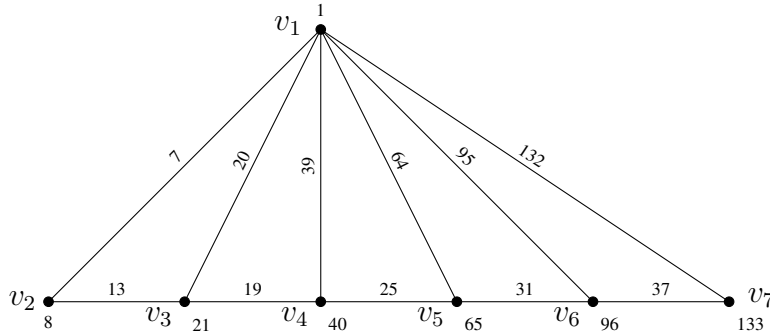


Figure 1: Octagonal prime graceful labeling of F_7

Theorem 3.2. *The graph $P_2 + mK_1$ is an octagonal prime graceful graph for $m \leq 6$.*

Proof. Consider a path P_2 with two vertices v_1, v_2 .
 Let $v_3, v_4, \dots, v_{m+2}$ be the m isolated vertices.
 Joining v_1, v_2 with v_i , $3 \leq i \leq m + 2$ we get $P_2 + mK_1$

Octagonal Prime Graceful Labeling

Let $G = P_2 + mK_1$

Let $V(G) = \{v_1, v_2, v_3, \dots, v_{m+2}\}$ and $E(G) = \{v_1v_2\}$

$\cup \{v_1v_i/3 \leq i \leq m+2\} \cup \{v_2v_i/3 \leq i \leq m+2\}$

Then $|V(G)| = 2 + m$ and $|E(G)| = 2m + 1$

Let p and q denote the total number of vertices and edges of graph G respectively.

Define a function $f : V(G) \rightarrow \{1, 8, \dots, p(3p-2)\}$ by $f(v_i) = i(3i-2)$, where $1 \leq i \leq m+2$ and the edge labels are as follows:

$f_{opgl}^*(v_1v_{i+1}) = 3i^2 + 4i$, where $i = 1, 2, \dots, m+1$ and

$f_{opgl}^*(v_2v_{i+2}) = f(v_1v_{i+2}) - f(v_1v_2)$, where $i = 1, 2, \dots, m$

Clearly, f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(G)$ such that $f_{opgl}^*(uv) = |f(u) - f(v)|$

Also $gcin$ of $v_1 = \gcd$ of edges incident on v_1

$= \gcd$ of edges $(\{v_1v_{i+2}\} \cup \{v_1v_2\}) = 1$, where $i = 1, 2, \dots, m$

$gcin$ of $v_2 = \gcd$ of edges incident on v_2

$= \gcd$ of edges $(\{v_2v_{i+2}\} \cup \{v_1v_2\}) = 1$, where $i = 1, 2, \dots, m$ and

$gcin$ of $v_{i+2} = \gcd$ of edges incident on $v_{i+2} = 1$, where $i = 1, 2, \dots, m$

In general, $gcin$ of $v_{m+2} = \gcd$ of edges incident on $v_{m+2} = 1$

Hence the $gcin$ of each vertex of degree greater than one is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the graph $P_2 + mK_1$ is an octagonal prime graceful graph.

Example 3.2. The octagonal prime graceful labeling of graph $P_2 + 6K_1$ is shown below.

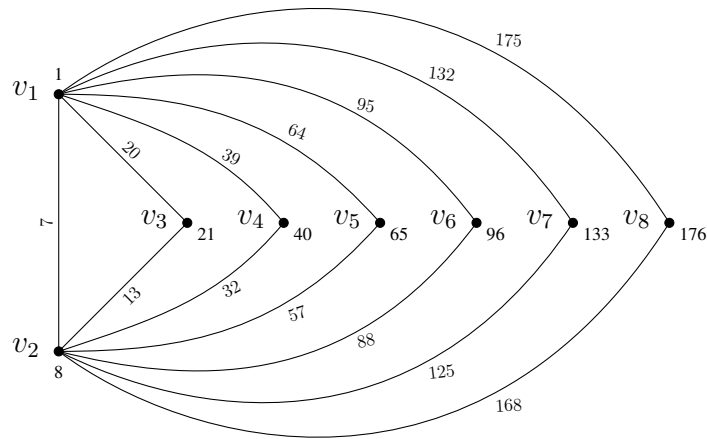


Figure 2: Octagonal prime graceful labeling of $P_2 + 6K_1$

3.1 Remark

The Graph $P_2 + mK_1$ is octagonal graceful but does not admits an octagonal prime graceful labeling for $m \geq 7$.

Proof. The Graph $P_2 + mK_1$ is octagonal graceful but not an octagonal prime graceful graph for $m \geq 7$ which is shown below by a figure $P_2 + 7K_1$ $\square$

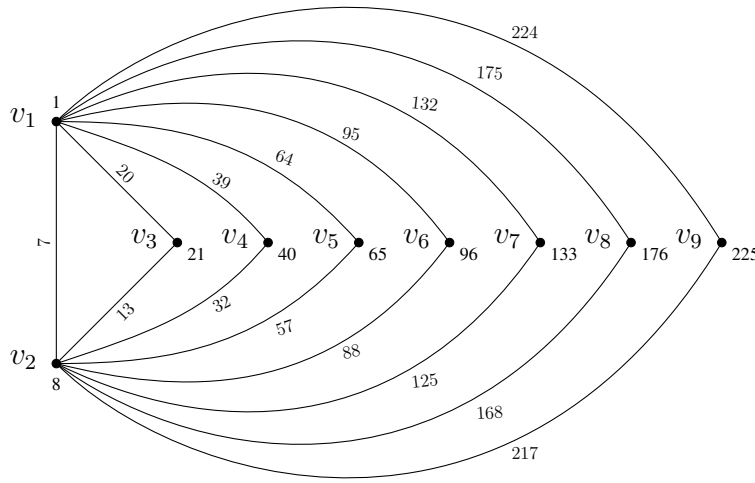


Figure 3: $P_2 + 7K_1$ which is not an Octagonal prime graceful labeling.
Here $gcin$ of vertex $v_9 = gcd\{224, 217\} \neq 1$

Theorem 3.3. The Shell graph S_n is an octagonal prime graceful graph.

Proof. Let S_n be a shell graph and let $v_1, v_2, \dots, v_n$ be the successive vertices of S_n .

Let $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$

Let p and q denote the total number of vertices and edges of graph S_n respectively. Define a function $f : V(S_n) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_i) = i(3i - 2)$, where $i = 1, 2, \dots, n$ and the edge labels are as follows:

$$f_{opgl}^*(v_i v_{i+1}) = 6i + 1, \text{ for } 1 \leq i \leq n - 1$$

$$f_{opgl}^*(v_1 v_n) = O_n - 1$$

Clearly f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(G)$ such that

$$f_{opgl}^*(uv) = |f(u) - f(v)|$$

Also $gcin$ of $v_1 = gcd$ of edges incident on v_1

Octagonal Prime Graceful Labeling

= gcd of edges $(v_1v_{i+1}) = 1$, where $1 \leq i \leq n-1$

$gcin$ of v_2 = gcd of edges incident on v_2

= gcd of edges $\{f_{opgl}^*(v_1v_2), f_{opgl}^*(v_2v_3)\} = 1$

$gcin$ of v_i = gcd of edges incident on v_i , where $3 \leq i \leq n-1$

$gcin$ of v_i = gcd of edges $\{v_iv_{i-1}, v_iv_1, v_iv_{i+1}\}$, where $3 \leq i \leq n-1$ and

In general, $gcin$ of v_n = gcd of edges incident on v_n

= gcd of edges $\{v_nv_1, v_nv_{n-1}\} = 1$

Hence the $gcin$ of each vertex of degree atleast two is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the graph S_n is an octagonal prime graceful graph

Example 3.3. The octagonal prime graceful labeling of shell graph S_8 is shown in the figure.

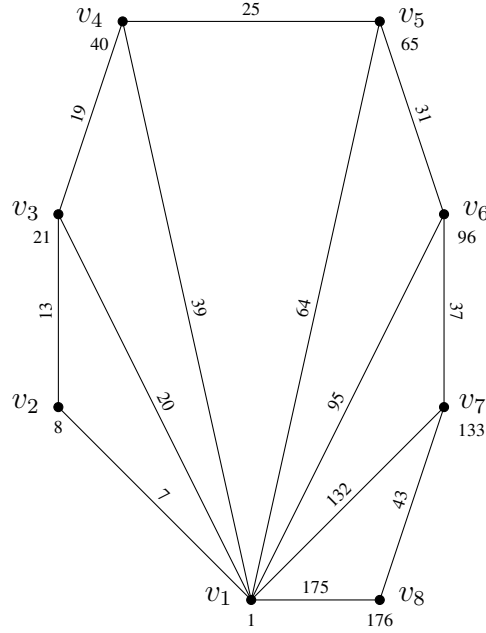


Figure 4: Octagonal prime graceful labeling of S_8

Theorem 3.4. For $n \geq 2$, a generalized butterfly graph BF_n admits octagonal prime graceful labeling.

Proof. A generalized butterfly graph, BF_n is obtained by inserting vertices to every wing with the assumption that sum of inserting vertices to every wing are same then it has $2n+1$ vertices and $4n-2$ edges.

Let the vertex set of BF_n be $V(BF_n) = \{v_i/i = 0, 1, 2, \dots, 2n\}$ and the edge set of BF_n be $E(BF_n) = \{(v_iv_{i+1})/i = 1, 2, \dots, n-1, n+1, \dots, 2n-1\} \cup \{(v_0v_i)/i = 1, 2, \dots, 2n\}$

Let BF_n has $\{v_0\}$ as an apex , $\{v_1, v_2, \dots, v_{n-1}, v_n\}$ as vertices on right wing and $\{v_{n+1}, v_{n+2}, \dots, v_{2n-1}, v_{2n}\}$ as vertices on left wing.

Let $f(v_0) = 1$

Define a function $f : V(BF_n) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_{i-1}) = i(3i - 2)$, where $i = 2, 3, \dots, 2n + 1$ and the edge labels are as follows:

$$f_{opgl}^*(v_0v_i) = 3i^2 + 4i , \text{ for } 1 \leq i \leq 2n + 1$$

$$f_{opgl}^*(v_{i-1}v_i) = 6i + 1 , \text{ for } 2, 3, \dots, n \text{ and}$$

$$f_{opgl}^*(v_{i-1}v_i) = 6i + 1 , \text{ for } i = n + 2, n + 3, \dots, 2n$$

Clearly, f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(G)$ such that $f_{opgl}^*(uv) = |f(u) - f(v)|$

Also the $gcin$ of each vertex of degree greater than one is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the generalized butterfly graph BF_n is an octagonal prime graceful graph for $n \geq 2$

Example 3.4. The Generalized butterfly graph BF_5 admits Octagonal prime graceful labeling.

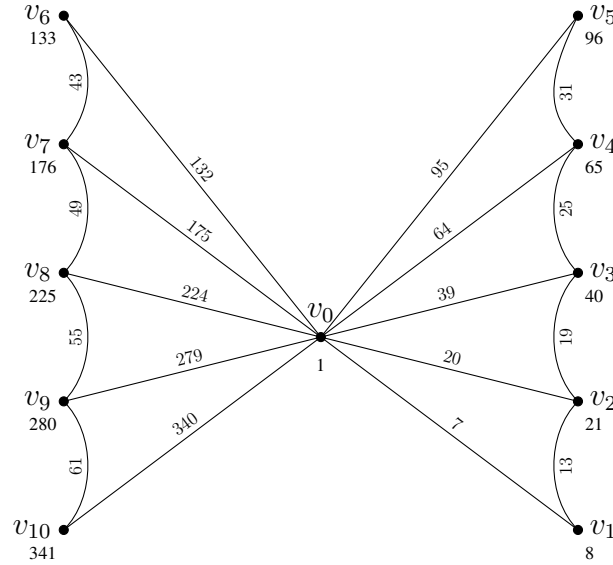


Figure 5: Octagonal prime graceful labeling of BF_5

Theorem 3.5. The Friendship graph Fr_n is an octagonal prime graceful graph.

Proof. The friendship graph Fr_n is a planar undirected graph constructed by joining n copies of cycle graph C_3 with common vertex.

Let $p = |V(Fr_n)| = 2n + 1$ and $q = |E(Fr_n)| = 3n$

Let the vertex set of Fr_n be $V(Fr_n) = \{v_i / i = 0, 1, 2, \dots, 2n\}$ with v_0 as the central vertex and edge set of Fr_n be $E(Fr_n) = \{v_0v_i, v_iv_{i+1}, i = 1, 2, \dots, 2n\}$

Octagonal Prime Graceful Labeling

Define a function $f : V(Fr_n) \rightarrow \{1, 8, \dots, p(3p-2)\}$ by $f(v_{i-1}) = i(3i-2)$, where $i = 2, 3, \dots, 2n+1$ and the edge labels are as follows:

$$f_{opgl}^*(v_0v_i) = 3i^2 + 4i, \text{ for } 1 \leq i \leq 2n$$

$$\text{and } f_{opgl}^*(v_{i-1}v_i) = 6i + 1, \text{ for } 2 \leq i \leq 2n$$

Clearly, f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(Fr_n)$ such that $f_{opgl}^*(uv) = |f(u) - f(v)|$

Also the $gcin$ of each vertex of degree atleast two is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the friendship graph Fr_n is an octagonal prime graceful graph.

Example 3.5. The Friendship graph Fr_7 is an Octagonal prime graceful graph.

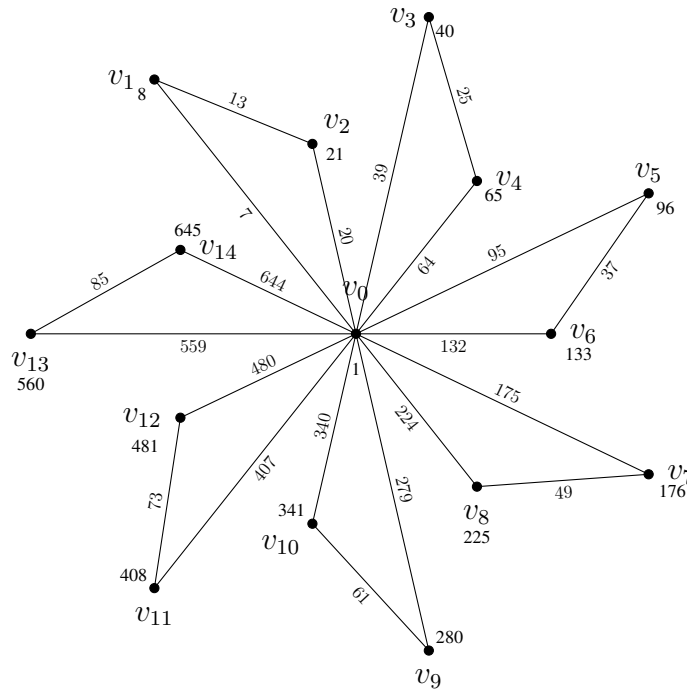


Figure 6: Octagonal prime graceful labeling of Fr_7

Theorem 3.6. The Prism of the wheel $P(W_n)$ is an Octagonal prime graceful graph for $n \geq 3$.

Proof. Let $P(W_n)$ be the prism of the wheel graph and let the vertices of $P(W_n)$ be $\{v_0, v'_0, v_i/1 \leq i \leq 2n\}$ and the corresponding edges of $P(W_n)$ be $E(P(W_n) = E(W_n) \cup E(W'_n) \cup \{v_iv_j/i = 1, 2, \dots, n, j = n+1, n+2, \dots, 2n\} \cup \{v_0v'_0\}$ where W'_n is the copy of W_n

Let $p = 2n + 2$ and $q = 5n + 1$ be the total number of vertices and edges of the prism of wheel $P(W_n)$ respectively.

Define a function $f : V(P(W_n)) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_{i-1}) = i(3i - 2)$, where $i = 2, 3, \dots, 2n + 1$ and let

$$f(v_0) = 1 \text{ and}$$

$$f(v'_0) = O_{2n+2}$$

Let the edge labels are $f_{opgl}^*(v'_0 v_i) = 3i^2 + 4i$, for $1 \leq i \leq n$

Also $f_{opgl}^*(v'_0 v_i) = f(v'_0) - O_{i+1}$, for $n + 1 \leq i \leq 2n$

$f_{opgl}^*(v_{i-1} v_i) = 6i + 1$, for $i = 2, 3, \dots, n, n + 2, n + 3, \dots, 2n$

$$f_{opgl}^*(v_0 v'_0) = O_{2n+2} - O_1$$

$$f_{opgl}^*(v_i v_{n+i}) = O_{n+1+i} - O_{i+1}, \text{ for } 1 \leq i \leq n$$

$$f_{opgl}^*(v_n v_1) = O_{n+1} - f(v_1)$$

$$f_{opgl}^*(v_{2n} v_{n+1}) = O_{2n+1} - f(v_{n+1})$$

Clearly, f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(P(W_n))$ such that $f_{opgl}^*(uv) = |f(u) - f(v)|$

Also the $gcin$ of each vertex of degree atleast two is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the prism of the wheel $P(W_n)$ is an octagonal prime graceful graph for $n \geq 3$.

Example 3.6. The prism graph $P(W_5)$ is an octagonal prime graceful graph .

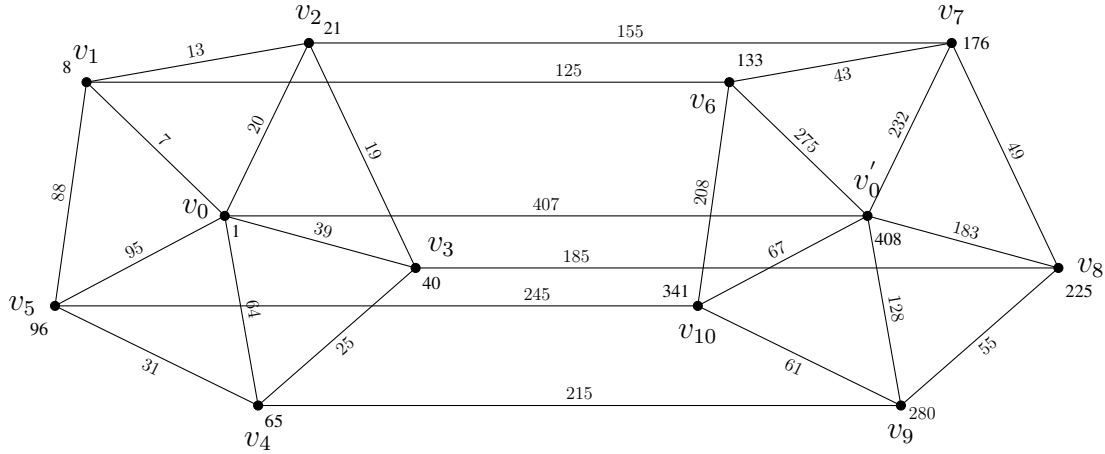


Figure 7: Octagonal prime graceful labeling of $P(W_5)$

Theorem 3.7. If m and n are positive integers such that $m \geq 3$ and $n \geq 2$, then the polar grid graph $P_{m,n}$ is an octagonal prime graceful graph.

Proof. Let $P_{m,n}$ be the polar grid graph and let the vertices and edges of $P_{m,n}$ be respectively

$$p = |V(P_{m,n})| = mn + 1 \text{ and } q = |E(P_{m,n})| = 2mn$$

Let v_0 be the central vertex and v_i where $1 \leq i \leq mn$ be the remaining vertices.

Octagonal Prime Graceful Labeling

Define a function $f : V(P_{m,n}) \rightarrow \{1, 8, \dots, p(3p - 2)\}$ by $f(v_{i-1}) = i(3i - 2)$, where $i = 2, 3, \dots, mn + 1$.

The edges of $P_{m,n}$ are labeled in such a way that

$$f_{opgl}^*(v_0v_i) = 3i^2 + 4i, \text{ for } 1 \leq i \leq m$$

$$f_{opgl}^*(v_{i-1}v_i) = 6i + 1, \text{ where } i = 2, 3, \dots, m, m + 2, \dots, 2m, 2m + 2, \dots, 3m, 3m + 2, \dots, mn$$

$$f_{opgl}^*(v_iv_j) = f(v_j) - f(v_i), \text{ for } 1 \leq i \leq m(n - 1) \text{ and } m + 1 \leq j \leq mn \text{ and}$$

$$f_{opgl}^*(v_iv_j) = f(v_i) - f(v_j), \text{ where } i = mk \text{ and } j = mk - (m - 1) \text{ for } 1 \leq k \leq n$$

Here all the edge labels are distinct.

Clearly, f_{opgl}^* is an injection and f induces the function f_{opgl}^* on $E(P_{m,n})$ such that $f_{opgl}^*(uv) = |f(u) - f(v)|$

Also the $gcin$ of each vertex of degree atleast two is 1

Therefore f is said to be an octagonal prime graceful labeling.

Hence the polar grid graph $P_{m,n}$ is an octagonal prime graceful graph for $m \geq 3$ and $n \geq 2$.

Example 3.7. An octagonal prime graceful labeling of polar grid $P_{m,n}$ where both m and n are even is given below

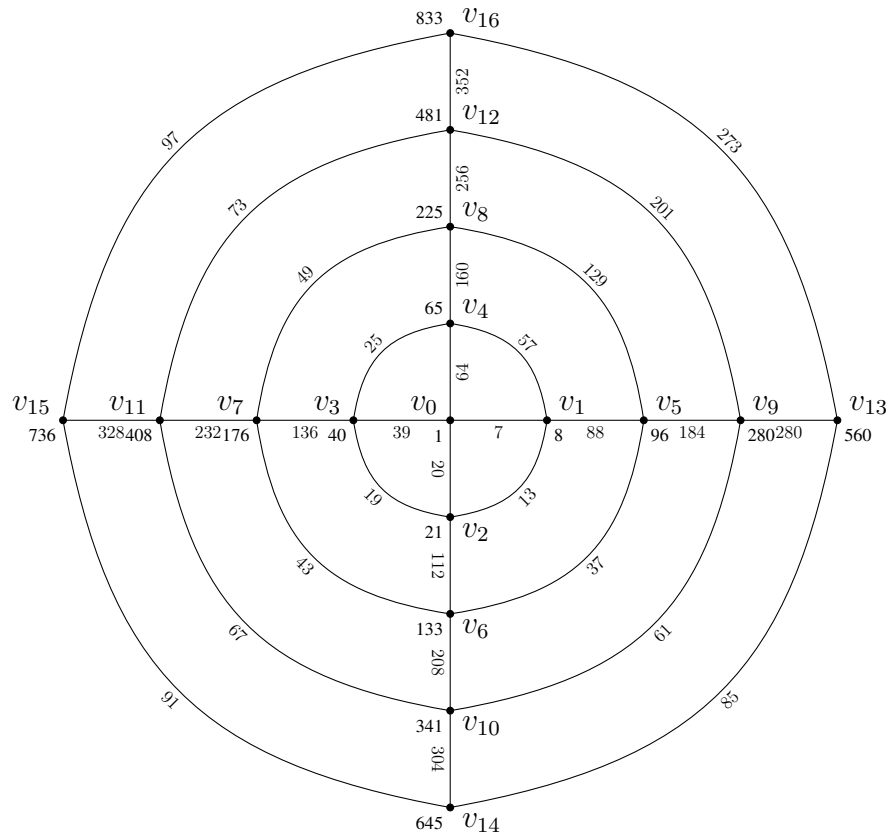


Figure 8: Octagonal prime graceful labeling of $P_{4,4}$

3.2 Remark.

If m and n are positive integers such that $m \leq 3$ and $n \leq 2$, then the polar grid graph $P_{m,n}$ is not an octagonal prime graceful graph.

Proof. We prove this by an example, that is the graph $P_{2,1}$ is not an octagonal prime graceful graph.

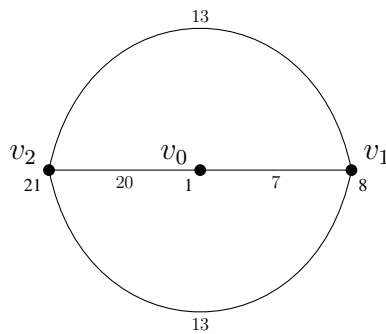


Figure 9: $P_{2,1}$

In the above figure the edge labels are not distinct
Hence $P_{2,1}$ is not an octagonal prime graceful graph

4 Conclusion

The authors of this study investigated the octagonal prime graceful labeling of fan F_n and friendship graph Fr_n . Further octagonal prime graceful labeling of some special graphs namely join of two graphs $P_2 + mK_1$, shell graph S_n , generalised butterfly graph BF_n , prism $P(W_n)$ and polar grid graph $P_{m,n}$ have been studied. We have also proved $P_2 + mK_1$ for $m \geq 7$ and polar grid graph $P_{m,n}$ for $m \leq 3$ & $n \leq 2$ are not octagonal prime graceful. Future development requires combination of the prime graceful labeling with some other polygonal numbers such as triangular, pentagonal, tetrahedral etc with reference from this study.

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Common fixed points for two pairs of selfmaps satisfying certain contraction condition in b -metric spaces

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Abstract

This study introduces generalized contraction for two pairs of selfmaps in complete b -metric spaces, and it then establishes the existence of common fixed points under the presumptions that these two pairs of maps are weakly compatible and satisfy the condition for generalized contraction. A sequence of selfmaps is added as an extension of the same. Additionally, we demonstrate the same using various hypotheses on two pairs of selfmaps that satisfy the b -(E.A)-property. Some of the conclusions in the literature are extended /generalized to two pairs of self maps by our theorems.

Keywords: common fixed points; b -metric space; weakly compatible; b -(E.A)-property. **2020 AMS subject classifications:** 47H10, 54H25. ¹

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1 Introduction

Czerwik (10) introduced the notion of b -metric space which is a generalization of metric space. Following that, numerous authors looked into fixed point theorems for single-valued and multi-valued mappings in b -metric spaces, we refer (3; 8; 9; 11; 15; 16; 17; 22; 23; 24).

The concept of property (E.A) was introduced by Aamari and Moutawakil (1). Several authors then used this idea to demonstrate the presence of common fixed points, we refer (2; 4; 5; 6; 19; 20; 21).

Definition 1.1. (10) Let X_b be a non-empty set and $s \geq 1$ be a given real number. A function $d_b : X_b \times X_b \rightarrow [0, \infty)$ is said to be a b -metric if the following conditions are satisfied: for any $x_b, y_b, z_b \in X_b$

- (i) $0 \leq d_b(x_b, y_b)$ and $d_b(x_b, y_b) = 0$ iff $x_b = y_b$,
- (ii) $d_b(x_b, y_b) = d_b(y_b, x_b)$,
- (iii) $d_b(x_b, z_b) \leq s[d_b(x_b, y_b) + d_b(y_b, z_b)]$.

The pair (X_b, d_b) is called a b -metric space with coefficient s .

Every metric space is a b -metric space with $s = 1$, but converse is need not be true.

Definition 1.2. (9) Let (X_b, d_b) be a b -metric space. Then a sequence $\{x_{b_n}\}$ in X_b is said to be

- (i) b -convergent if there exists $x_b \in X_b$ such that $d_b(x_{b_n}, x_b) \rightarrow 0$ as $n \rightarrow \infty$.
In this case, we write $\lim_{n \rightarrow \infty} x_{b_n} = x_b$.

- (ii) b -Cauchy if $d_b(x_{b_n}, x_{b_m}) \rightarrow 0$ as $n, m \rightarrow \infty$.

In general, a b -metric is not necessarily continuous (12).

Definition 1.3. (13) Let A and B be selfmaps of a metric space (X, d) . The pair (A, B) is said to be a compatible pair on X , if $\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$, for some $t \in X$.

Definition 1.4. (14) Let X be a nonempty set. Let $A : X \rightarrow X$ and $B : X \rightarrow X$ be two selfmaps. If $Ax = Bx$ implies that $ABx = BAx$ for x in X , then we say that the pair (A, B) is weakly compatible.

Common fixed points for two pairs of maps...

Definition 1.5. (19) Two selfmappings A_b and B_b of a b -metric space (X_b, d_b) are said to satisfy b -(E.A)-property if there exists a sequence $\{x_{b_n}\}$ in $X_b \ni \lim_{n \rightarrow \infty} A_b x_{b_n} = \lim_{n \rightarrow \infty} B_b x_{b_n} = z_b$ for some $z_b \in X_b$.

Lemma 1.1. (2) Let (X_b, d_b) be a b -metric space with coefficient $s \geq 1$. Suppose that $\{x_{b_n}\}$ and $\{y_{b_n}\}$ are b -convergent to x_b and y_b respectively, then we have

$$\frac{1}{s^2} d_b(x_b, y_b) \leq \liminf_{n \rightarrow \infty} d_b(x_{b_n}, y_{b_n}) \leq \limsup_{n \rightarrow \infty} d_b(x_{b_n}, y_{b_n}) \leq s^2 d_b(x_b, y_b)$$

In particular, if $x_b = y_b$, then we have $\lim_{n \rightarrow \infty} d_b(x_{b_n}, y_{b_n}) = 0$. Moreover for each $z_b \in X_b$ we have

$$\frac{1}{s} d_b(x_b, z_b) \leq \liminf_{n \rightarrow \infty} d_b(x_{b_n}, z_b) \leq \limsup_{n \rightarrow \infty} d_b(x_{b_n}, z_b) \leq s d_b(x_b, z_b).$$

Lemma 1.2. (7) Let (X_b, d_b) be a b -metric space with coefficient $s \geq 1$ and $T_b : X_b \rightarrow X_b$ be a self map. Suppose that $\{x_{b_n}\}$ is a sequence in X_b induced by $x_{b_{n+1}} = T_b x_{b_n}$ such that $d_b(x_{b_n}, x_{b_{n+1}}) \leq \lambda d_b(x_{b_{n-1}}, x_{b_n})$ for all $n \in N$, where $\lambda \in (0, 1)$ is a constant. Then x_{b_n} is a b -cauchy sequence in X_b .

Recently, Nagaraju, Raju and Thirupathi (18) proved a theorem in metric spaces as follows:

Theorem 1.1. (18) Let E, F, G and H be self-mappings of a metric space (X, d) satisfying the following conditions:

- (i) $E(X) \subseteq H(X)$ and $F(X) \subseteq G(X)$,
- (ii) (E, G) and (F, H) are weakly compatible and
- (iii) $[d(Ey, Fz)]^2 \leq \alpha \max\{[d(Gy, Ey)]^2, [d(Hz, Fz)]^2, [Gy, Hz]^2\}$
 $+ \beta \max\{d(Gy, Ey)d(Gy, Fz), d(Ey, Hz)d(Fz, Hz)\}$
 $+ \delta d(Gy, Fz)d(Hz, Ey)$
for all $y, z \in X$, where $\alpha, \beta, \delta \geq 0, \alpha + 2\beta < 1$ and $\alpha + \delta < 1$.
- (iv) Further, if the pair (E, G) satisfies (CLR-G)-property or the pair (F, H) satisfies (CLR-H)-property, then the self-maps E, F, G and H have a unique common fixed point.

We introduce generalized contraction for two pairs of selfmaps in b -metric spaces and prove the existence of common fixed points under the assumptions that these two pairs of maps are weakly compatible and satisfying a generalized

contraction condition in complete b -metric spaces. Our work is inspired by works of Nagaraju, Raju and Thirupathi (18). A series of selfmaps is added as an extension of the same. Additionally, we demonstrate the same using various hypotheses on two pairs of selfmaps that satisfy the b -(E.A)-property. Some of the conclusions in the literature are extended or generalized to two pairs of self maps by our theorems. We present examples to corroborate our findings and draw some conclusions from them.

2 Main Results

We introduce generalized contraction maps in b -metric spaces as follows.

Definition 2.1. Let (X_b, d_b) be a b -metric space with coefficient $s \geq 1$ and $A_b, B_b, S_b, T_b : X_b \rightarrow X_b$ be selfmaps. If there exist $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + s\lambda_2 + s^2\lambda_3 \leq 1$ such that

$$s^4[d_b(A_b x_b, B_b y_b)]^2 \leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}. \quad (1)$$

Then we call A_b, B_b, S_b and T_b are generalized contraction maps.

Example 2.1. Let $X_b = [0, 1]$ and let $d_b : X_b \times X_b \rightarrow [0, \infty)$ defined by

$$d_b(x_b, y_b) = \begin{cases} 0 & \text{if } x_b = y_b, \\ (x_b + y_b)^2 & \text{if } x_b \neq y_b. \end{cases}$$

Then clearly (X_b, d_b) is a complete b -metric space with $s = 2$.

We define $A_b, B_b, S_b, T_b : X_b \rightarrow X_b$ by $A_b(x_b) = \frac{1-x_b}{5}$, $B_b(x_b) = \frac{\log_{10}(1+x_b)}{5}$,

$S_b(x_b) = x_b^2$, $T_b(x_b) = x_b$ for all $x_b \in X_b$.

Take $\lambda_1 = \frac{1}{7}$, $\lambda_2 = \frac{1}{8}$, $\lambda_3 = \frac{1}{10}$.

Clearly, $\lambda_1 + s\lambda_2 + s^2\lambda_3 \leq 1$.

Then we have

$$s^4[d_b(A_b x_b, B_b y_b)]^2 = 16\left(\frac{1-e^{x_b}}{5} + \frac{\log_{10}(1+y_b)}{5}\right)^4 \\ \leq \frac{1}{7} \max\{(x_b^2 + y_b)^4, (x_b^2 + \frac{1-e^{x_b}}{5})^4, (y_b + \frac{\log_{10}(1+y_b)}{5})^4\} \\ + \frac{1}{8} \max\left\{\frac{(x_b^2 + \frac{1-e^{x_b}}{5})^2(x_b^2 + \frac{\log_{10}(1+y_b)}{5})^2}{2}, \frac{(y_b + \frac{\log_{10}(1+y_b)}{5})^2(y_b + \frac{1-e^{x_b}}{5})^2}{2}\right\} \\ + \frac{1}{10} \frac{(x_b^2 + \frac{\log_{10}(1+y_b)}{5})^2(y_b + \frac{1-e^{x_b}}{5})^2}{2} \\ \leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}.$$

Therefore A_b, B_b, S_b and T_b are generalized contraction maps.

Let A_b, B_b, S_b and T_b be mappings from a b -metric space (X_b, d_b) into itself and satisfying

$$A_b(X_b) \subseteq T_b(X_b) \text{ and } B_b(X_b) \subseteq S_b(X_b) \quad (2)$$

Now, by (2), for any $x_{b_0} \in X_b$, there exists $x_{b_1} \in X_b$ such that

$$y_{b_0} = A_b x_{b_0} = T_b x_{b_1}.$$

In the same way for this x_{b_1} , we can choose a point $x_{b_2} \in X_b$ such that

$$y_{b_1} = B_b x_{b_1} = S_b x_{b_2} \text{ and so on.}$$

In general, we can define a sequence $\{y_{b_n}\} \in X_b$ such that

$$y_{b_{2n}} = A_b x_{b_{2n}} = T_b x_{b_{2n+1}} \text{ and } y_{b_{2n+1}} = B_b x_{b_{2n+1}} = S_b x_{b_{2n+2}} \text{ for } n = 0, 1, 2, \dots \quad (3)$$

Proposition 2.1. *Let (X_b, d_b) be a b -metric space with coefficient $s \geq 1$. Suppose that A_b, B_b, S_b and T_b are generalized contraction maps. Then we have the following:*

(i) *If $A_b(X_b) \subseteq T_b(X_b)$ and the pair (B_b, T_b) is weakly compatible, and if x_b is a common fixed point of A_b and S_b then x_b is a common fixed point of A_b, B_b, S_b and T_b and it is unique.*

(ii) *If $B_b(X_b) \subseteq S_b(X_b)$ and the pair (A_b, S_b) is weakly compatible, and if x_b is a common fixed point of B_b and T_b then x_b is a common fixed point of A_b, B_b, S_b and T_b and it is unique.*

Proof. First, we assume that (i) holds. Let x_b be a common fixed point of A_b and S_b .

Then $A_b x_b = S_b x_b = x_b$.

Since $A_b(X_b) \subseteq T_b(X_b)$, there exists $y \in X_b$ such that $T_b y_b = x_b$.

Therefore $A_b x_b = S_b x_b = T_b y_b = x_b$. If $A_b x_b \neq B_b y_b$, then

$$\begin{aligned} s^4 [d_b(A_b x_b, B_b y_b)]^2 &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2} \\ &= \lambda_1 [d_b(A_b x_b, B_b y_b)]^2 \end{aligned}$$

which implies that $(s^4 - \lambda_1)[d_b(A_b x_b, B_b y_b)]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $d_b(A_b x_b, B_b y_b) = 0$ which implies that $A_b x_b = B_b y_b$.

Therefore $A_b x_b = B_b y_b = S_b x_b = T_b y_b = x_b$.

As (B_b, T_b) is weakly compatible and $T_b y_b = B_b y_b$, we have

$$B_b T_b y_b = T_b B_b y_b. \text{ i.e., } B_b x_b = T_b x_b.$$

Now, we prove that $B_b x_b = x_b$. If $B_b x_b \neq x_b$, then

$$\begin{aligned} s^4 [d_b(x_b, B_b x_b)]^2 &= s^4 [d_b(A_b x_b, B_b x_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b x_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b x_b, B_b x_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b x_b)}{2}, \frac{d_b(T_b x_b, B_b x_b)d_b(T_b x_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b x_b)d_b(T_b x_b, A_b x_b)}{2} \\ &= \lambda_1 [d_b(x_b, B_b x_b)]^2 + \lambda_3 \frac{[d_b(x_b, B_b x_b)]^2}{2} \end{aligned}$$

$$= (\lambda_1 + \frac{\lambda_3}{2})[d_b(x_b, B_b x_b)]^2$$

which implies that $[s^4 - (\lambda_1 + \frac{\lambda_3}{2})][d_b(x_b, B_b x_b)]^2 \leq 0$.

Since $[s^4 - (\lambda_1 + \frac{\lambda_3}{2})] \geq 0$, we have $d_b(x_b, B_b x_b) \leq 0$.

Hence, $B_b x_b = x_b$.

Therefore $A_b x_b = B_b x_b = S_b x_b = T_b x_b = x_b$.

Therefore, x_b is a common fixed point of A_b, B_b, S_b and T_b .

If x'_b is also a common fixed point of A_b, B_b, S_b and T_b with $x_b \neq x'_b$, then

$$\begin{aligned} s^4[d_b(x_b, x'_b)]^2 &= s^4[d_b(A_b x_b, B_b x'_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b x'_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b x'_b, B_b x'_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b x'_b)}{2}, \frac{d_b(T_b x'_b, B_b x'_b)d_b(T_b x'_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b x'_b)d_b(T_b x'_b, A_b x_b)}{2} \\ &= \lambda_1[d_b(x_b, x'_b)]^2 + \lambda_3 \frac{[d_b(x_b, x'_b)]^2}{2} \\ &= (\lambda_1 + \frac{\lambda_3}{2})[d_b(x_b, x'_b)]^2 \end{aligned}$$

which implies that $[s^4 - (\lambda_1 + \frac{\lambda_3}{2})][d_b(x_b, x'_b)]^2 \leq 0$.

Since $[s^4 - (\lambda_1 + \frac{\lambda_3}{2})] \geq 0$, we have $d_b(x_b, x'_b) \leq 0$.

Hence, $x'_b = x_b$.

Therefore x_b is the unique common fixed point of A_b, B_b, S_b and T_b .

The proof of (ii) is similar to (i) and hence is omitted. $\square$

Lemma 2.1. Let A_b, B_b, S_b and T_b be selfmaps of a b -metric space (X_b, d_b) and satisfy (2) and are generalized contraction maps. Then for any $x_{b_0} \in X_b$, the sequence $\{y_{b_n}\}$ defined by (3) is b -Cauchy in X_b .

Proof. Let $x_{b_0} \in X_b$ and let $\{y_{b_n}\}$ be a sequence defined by (3).

Assume that $y_{b_n} = y_{b_{n+1}}$ for some n .

Case (i): n even.

We write $n = 2m, m \in \mathbb{N}$.

Now, we consider

$$\begin{aligned} s^4[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 &= s^4[d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2 \\ &= s^4[d_b(y_{b_{2m+2}}, y_{b_{2m+1}})]^2 \\ &= s^4[d_b(A_b x_{b_{2m+2}}, B_b x_{b_{2m+1}})]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b x_{b_{2m+2}}, T_b x_{b_{2m+1}})]^2, [d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})]^2, \\ &\quad [d_b(T_b x_{b_{2m+1}}, B_b x_{b_{2m+1}})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+1}})}{2}, \right. \\ &\quad \left. \frac{d_b(T_b x_{b_{2m+1}}, B_b x_{b_{2m+1}})d_b(T_b x_{b_{2m+1}}, A_b x_{b_{2m+2}})}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+1}})d_b(T_b x_{b_{2m+1}}, A_b x_{b_{2m+2}})}{2} \\ &= \lambda_1 \max\{[d_b(y_{b_{2m+1}}, y_{b_{2m}})]^2, [d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2, [d_b(y_{b_{2m}}, y_{b_{2m+1}})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(y_{b_{2m+1}}, y_{b_{2m+2}})d_b(y_{b_{2m+1}}, y_{b_{2m+1}})}{2}, \frac{d_b(y_{b_{2m}}, y_{b_{2m+1}})d_b(y_{b_{2m}}, y_{b_{2m+2}})}{2}\right\} \end{aligned}$$

Common fixed points for two pairs of maps...

$$\begin{aligned}
 & + \lambda_3 \frac{d_b(y_{b_{2m+1}}, y_{b_{2m+1}})d_b(y_{b_{2m}}, y_{b_{2m+2}})}{2} \\
 & = \lambda_1 \max\{[d_b(y_{b_{n+1}}, y_{b_n})]^2, [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2, [d_b(y_{b_n}, y_{b_{n+1}})]^2\} \\
 & \quad + \lambda_2 \max\left\{\frac{d_b(y_{b_{n+1}}, y_{b_{n+2}})d_b(y_{b_{n+1}}, y_{b_{n+1}})}{2}, \frac{d_b(y_{b_n}, y_{b_{n+1}})d_b(y_{b_n}, y_{b_{n+2}})}{2}\right\} \\
 & \quad + \lambda_3 \frac{d_b(y_{b_{n+1}}, y_{b_{n+1}})d_b(y_{b_n}, y_{b_{n+2}})}{2} \\
 & = \lambda_1 [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2
 \end{aligned}$$

which implies that $(s^4 - \lambda_1)[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $d_b(y_{b_{n+1}}, y_{b_{n+2}}) \leq 0$

which implies that $y_{b_{n+2}} = y_{b_{n+1}} = y_{b_n}$.

In general, we have $y_{b_{n+k}} = y_{b_n}$ for $k = 0, 1, 2, \dots$.

Case (ii): n odd.

We write $n = 2m + 1$ for some $m \in \mathbb{N}$.

Now we consider

$$\begin{aligned}
 s^4[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 & = s^4[d_b(y_{b_{2m+2}}, y_{b_{2m+3}})]^2 \\
 & = s^4[d_b(A_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})]^2 \\
 & \leq \lambda_1 \max\{[d_b(S_b x_{b_{2m+2}}, T_b x_{b_{2m+3}})]^2, [d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})]^2, \\
 & \quad [d_b(T_b x_{b_{2m+3}}, B_b x_{b_{2m+3}})]^2\} \\
 & \quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})}{2}, \right. \\
 & \quad \left. \frac{d_b(T_b x_{b_{2m+3}}, B_b x_{b_{2m+3}})d_b(T_b x_{b_{2m+3}}, A_b x_{b_{2m+2}})}{2}\right\} \\
 & \quad + \lambda_3 \frac{d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})d_b(T_b x_{b_{2m+3}}, A_b x_{b_{2m+2}})}{2} \\
 & = \lambda_1 \max\{[d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2, [d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2, [d_b(y_{b_{2m+2}}, y_{b_{2m+3}})]^2\} \\
 & \quad + \lambda_2 \max\left\{\frac{d_b(y_{b_{2m+1}}, y_{b_{2m+2}})d_b(y_{b_{2m+1}}, y_{b_{2m+3}})}{2}, \right. \\
 & \quad \left. \frac{d_b(y_{b_{2m+2}}, y_{b_{2m+3}})d_b(y_{b_{2m+2}}, y_{b_{2m+2}})}{2}\right\} + \lambda_3 \frac{d_b(y_{b_{2m+1}}, y_{b_{2m+3}})d_b(y_{b_{2m+2}}, y_{b_{2m+2}})}{2} \\
 & = \lambda_1 \max\{[d_b(y_{b_n}, y_{b_{n+1}})]^2, [d_b(y_{b_n}, y_{b_{n+1}})]^2, [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2\} \\
 & \quad + \lambda_2 \max\left\{\frac{d_b(y_{b_n}, y_{b_{n+1}})d_b(y_{b_n}, y_{b_{n+2}})}{2}, \frac{d_b(y_{b_{n+1}}, y_{b_{n+2}})d_b(y_{b_{n+1}}, y_{b_{n+1}})}{2}\right\} \\
 & \quad + \lambda_3 \frac{d_b(y_{b_n}, y_{b_{n+2}})d_b(y_{b_{n+1}}, y_{b_{n+1}})}{2} \\
 & = \lambda_1 [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2
 \end{aligned}$$

which implies that $(s^4 - \lambda_1)[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $d_b(y_{b_{n+1}}, y_{b_{n+2}}) \leq 0$

which implies that $y_{b_{n+2}} = y_{b_{n+1}} = y_{b_n}$.

In general, we have $y_{b_{n+k}} = y_{b_n}$ for $k = 1, 2, 3, \dots$.

From Case (i) and Case (ii), we have $y_{b_{n+k}} = y_{b_n}$ for $k = 0, 1, 2, \dots$.

Therefore, $\{y_{b_{n+k}}\}$ is a constant sequence and hence $\{y_{b_n}\}$ is b -Cauchy.

Now we assume that $y_{b_n} \neq y_{b_{n+1}}$ for all $n \in \mathbb{N}$.

If n is odd then $n = 2m + 1$ for some $m \in \mathbb{N}$.

We now consider

$$\begin{aligned}
 s^4[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 &= s^4[d_b(y_{b_{2m+2}}, y_{b_{2m+3}})]^2 \\
 &= s^4[d_b(A_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})]^2 \\
 &\leq \lambda_1 \max\{[d_b(S_b x_{b_{2m+2}}, T_b x_{b_{2m+3}})]^2, [d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})]^2, \\
 &\quad [d_b(T_b x_{b_{2m+3}}, B_b x_{b_{2m+3}})]^2\} \\
 &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2m+2}}, A_b x_{b_{2m+2}})d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})}{2}, \right. \\
 &\quad \left. \frac{d_b(T_b x_{b_{2m+3}}, B_b x_{b_{2m+3}})d_b(T_b x_{b_{2m+3}}, A_b x_{b_{2m+2}})}{2}\right\} \\
 &\quad + \lambda_3 \frac{d_b(S_b x_{b_{2m+2}}, B_b x_{b_{2m+3}})d_b(T_b x_{b_{2m+3}}, A_b x_{b_{2m+2}})}{2} \\
 &= \lambda_1 \max\{[d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2, [d_b(y_{b_{2m+1}}, y_{b_{2m+2}})]^2, [d_b(y_{b_{2m+2}}, y_{b_{2m+3}})]^2\} \\
 &\quad + \lambda_2 \max\left\{\frac{d_b(y_{b_{2m+1}}, y_{b_{2m+2}})d_b(y_{b_{2m+1}}, y_{b_{2m+3}})}{2}, \frac{d_b(y_{b_{2m+2}}, y_{b_{2m+3}})d_b(y_{b_{2m+2}}, y_{b_{2m+2}})}{2}\right\} \\
 &\quad + \lambda_3 \frac{d_b(y_{b_{2m+1}}, y_{b_{2m+3}})d_b(y_{b_{2m+2}}, y_{b_{2m+2}})}{2} \\
 &= \lambda_1 \max\{[d_b(y_{b_n}, y_{b_{n+1}})]^2, [d_b(y_{b_n}, y_{b_{n+1}})]^2, [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2\} \\
 &\quad + \lambda_2 \max\left\{\frac{d_b(y_{b_n}, y_{b_{n+1}})d_b(y_{b_n}, y_{b_{n+2}})}{2}, \frac{d_b(y_{b_{n+1}}, y_{b_{n+2}})d_b(y_{b_{n+1}}, y_{b_{n+1}})}{2}\right\} \\
 &\quad + \lambda_3 \frac{d_b(y_{b_n}, y_{b_{n+2}})d_b(y_{b_{n+1}}, y_{b_{n+1}})}{2}
 \end{aligned} \tag{4}$$

If $[d_b(y_{b_n}, y_{b_{n+1}})]^2 < [d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2$ then from (4), we have

$$s^4[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq \lambda_1[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 + s\lambda_2[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2$$

which implies that $(s^4 - \lambda_1 - s\lambda_2)[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq 0$.

Since $(s^4 - \lambda_1 - s\lambda_2) \geq 0$, we have $[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq 0$

which implies that $y_{b_{n+1}} = y_{b_{n+2}}$,

which is a contradiction.

Therefore $[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2 \leq [d_b(y_{b_n}, y_{b_{n+1}})]^2$.

From the inequality (4), we have $s^4[[d_b(y_{b_{n+1}}, y_{b_{n+2}})]^2] \leq (\lambda_1 + s\lambda_2)[d_b(y_{b_n}, y_{b_{n+1}})]^2$

which implies that $d_b(y_{b_{n+1}}, y_{b_{n+2}}) \leq kd_b(y_{b_n}, y_{b_{n+1}})$, where $k = \frac{\sqrt{(\lambda_1 + s\lambda_2)}}{s^2} < 1$.

Similarly, we can prove that $d_b(y_{b_{n+1}}, y_{b_{n+2}}) \leq kd_b(y_{b_n}, y_{b_{n+1}})$ whenever n is even.

By Lemma 1.2, we have $\{y_{b_n}\}$ is a b -Cauchy sequence in X_b . $\square$

The following is the main result of this paper.

Theorem 2.1. *Let A_b, B_b, S_b and T_b be selfmaps on a complete b -metric space (X_b, d_b) and satisfy (2) and the maps are generalized contraction maps. If the pairs (A_b, S_b) and (B_b, T_b) are weakly compatible and one of the range sets $S_b(X_b), T_b(X_b), A_b(X_b)$ and $B_b(X_b)$ is closed, then for any $x_{b_0} \in X_b$, the sequence $\{y_{b_n}\}$ defined by (3) is Cauchy in X_b and $\lim_{n \rightarrow \infty} y_{b_n} = z_b$ (say), $z_b \in X_b$ and z_b is the unique common fixed point of A_b, B_b, S_b and T_b .*

Proof. By Lemma 2.1, the sequence $\{y_{b_n}\}$ is b -Cauchy in X_b .

Since X_b is b -complete, $\exists z_b \in X_b \ni \lim_{n \rightarrow \infty} y_{b_n} = z_b$.

Common fixed points for two pairs of maps...

Then

$$\begin{cases} \lim_{n \rightarrow \infty} y_{b_{2n}} = \lim_{n \rightarrow \infty} A_b x_{b_{2n}} = \lim_{n \rightarrow \infty} T_b x_{b_{2n+1}} = z_b \text{ and} \\ \lim_{n \rightarrow \infty} y_{b_{2n+1}} = \lim_{n \rightarrow \infty} B_b x_{b_{2n+1}} = \lim_{n \rightarrow \infty} S_b x_{b_{2n+2}} = z_b. \end{cases} \quad (5)$$

We consider the below cases.

Case (i). $S_b(X_b)$ is closed.

In this case $z_b \in S_b(X_b)$ and there exists $t_b \in X_b$ such that $z_b = S_b t_b$.

If $A_b t_b \neq z_b$, then

$$\begin{aligned} s^4 [d_b(A_b t_b, B_b x_{b_{2n+1}})]^2 &\leq \lambda_1 \max\{[d_b(S_b t_b, T_b x_{b_{2n+1}})]^2, [d_b(S_b t_b, A_b t_b)]^2, \\ &\quad [d_b(T_b x_{b_{2n+1}}, B_b x_{b_{2n+1}})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b t_b, A_b t_b) d_b(S_b t_b, B_b x_{b_{2n+1}})}{2}, \right. \\ &\quad \left. \frac{d_b(T_b x_{b_{2n+1}}, B_b x_{b_{2n+1}}) d_b(T_b x_{b_{2n+1}}, A_b t_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b t_b, B_b x_{b_{2n+1}}) d_b(T_b x_{b_{2n+1}}, A_b t_b)}{2} \end{aligned} \quad (6)$$

On letting limit superior as $n \rightarrow \infty$ in the inequality (6), using Lemma 1.1 and (5), we get

$$\frac{1}{s^2} (s^4 [d_b(A_b t_b, z_b)]^2) \leq \lambda_1 [d_b(A_b t_b, z_b)]^2$$

which implies that $(s^2 - \lambda_1) [d_b(A_b t_b, z_b)]^2 \leq 0$.

Since $(s^2 - \lambda_1) \geq 0$, we have $A_b t_b = z_b$.

Therefore, $A_b t_b = z_b = S_b t_b$.

Since (A_b, S_b) is weakly compatible and $A_b t_b = S_b t_b$, we have

$A_b S_b t_b = S_b A_b t_b$. i.e., $A_b z_b = S_b z_b$.

Suppose $A_b z_b \neq z_b$. We now consider

$$\begin{aligned} &s^4 [d_b(A_b z_b, B_b x_{b_{2n+1}})]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b z_b, T_b x_{b_{2n+1}})]^2, [d_b(S_b z_b, A_b z_b)]^2, [d_b(T_b x_{b_{2n+1}}, B_b x_{b_{2n+1}})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b z_b, A_b z_b) d_b(S_b z_b, B_b x_{b_{2n+1}})}{2}, \frac{d_b(T_b x_{b_{2n+1}}, B_b x_{b_{2n+1}}) d_b(T_b x_{b_{2n+1}}, A_b z_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b z_b, B_b x_{b_{2n+1}}) d_b(T_b x_{b_{2n+1}}, A_b z_b)}{2} \end{aligned} \quad (7)$$

On letting limit superior as $n \rightarrow \infty$ in the inequality (7), using Lemma 1.1 and (5), we get

$$\frac{1}{s^2} (s^4 [d_b(A_b z_b, z_b)]^2) \leq (\lambda_1 + \frac{s^2 \lambda_3}{2}) [d_b(A_b z_b, z_b)]^2$$

which implies that $(s^2 - \lambda_1 - \frac{s^2 \lambda_3}{2}) [d_b(A_b z_b, z_b)]^2 \leq 0$.

Since $(s^2 - \lambda_1 - \frac{s^2 \lambda_3}{2}) \geq 0$, we have $A_b z_b = z_b$.

Therefore $A_b z_b = S_b z_b = z_b$.

Hence, z_b is a common fixed point of A_b and S_b .

By Proposition 2.1, we get that z_b is a unique common fixed point of A_b, B_b, S_b and T_b .

Case (ii). $T_b(X_b)$ is closed.

In this case $z_b \in T_b(X_b)$ and there exists $u_b \in X_b \ni z_b = T_b u_b$.

If $B_b u_b \neq z_b$, then

$$\begin{aligned} & s^4 [d_b(A_b x_{b_{2n+2}}, B_b u_b)]^2 \\ & \leq \lambda_1 \max\{[d_b(S_b x_{b_{2n+2}}, T_b u_b)]^2, [d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}})]^2, [d_b(T_b u_b, B_b u_b)]^2\} \\ & \quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}})d_b(S_b x_{b_{2n+2}}, B_b u_b)}{2}, \frac{d_b(T_b u_b, B_b u_b)d_b(T_b u_b, A_b x_{b_{2n+2}})}{2}\right\} \\ & \quad + \lambda_3 \frac{d_b(S_b x_{b_{2n+2}}, B_b u_b)d_b(T_b u_b, A_b x_{b_{2n+2}})}{2} \end{aligned} \quad (8)$$

On letting limit superior as $n \rightarrow \infty$ in (8), using Lemma 1.1 and (5), we get

$$\frac{1}{s^2} (s^4 [d_b(B_b u_b, z_b)]^2) \leq \lambda_1 [d_b(B_b u_b, z_b)]^2 \text{ which implies that } (s^2 - \lambda_1) [d_b(B_b u_b, z_b)]^2 \leq 0.$$

Since $(s^2 - \lambda_1) \geq 0$, we have $B_b u_b = z_b$.

Therefore, $B_b u_b = z_b = T_b u_b$.

Since the pair (B_b, T_b) is weakly compatible and $B_b u_b = T_b u_b$, we have

$$B_b T_b u_b = T_b B_b u_b. \text{ i.e., } B_b z_b = T_b z_b.$$

We now prove that $B_b z_b = z_b$. Suppose that $B_b z_b \neq z_b$. We now consider

$$\begin{aligned} & s^4 [d_b(A_b x_{b_{2n+2}}, B_b z_b)]^2 \\ & \leq \lambda_1 \max\{[d_b(S_b x_{b_{2n+2}}, T_b z_b)]^2, [d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}})]^2, [d_b(T_b z_b, B_b z_b)]^2\} \\ & \quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}})d_b(S_b x_{b_{2n+2}}, B_b z_b)}{2}, \frac{d_b(T_b z_b, B_b z_b)d_b(T_b z_b, A_b x_{b_{2n+2}})}{2}\right\} \\ & \quad + \lambda_3 \frac{d_b(S_b x_{b_{2n+2}}, B_b z_b)d_b(T_b z_b, A_b x_{b_{2n+2}})}{2} \end{aligned} \quad (9)$$

On letting limit superior as $n \rightarrow \infty$ in (9), using Lemma 1.1 and (5), we get

$$\frac{1}{s^2} (s^4 [d_b(B_b z_b, z_b)]^2) \leq (\lambda_1 + \frac{s^2 \lambda_3}{2}) [d_b(B_b z_b, z_b)]^2 \text{ which implies that } (s^2 - \lambda_1 - \frac{s^2 \lambda_3}{2}) [d_b(B_b z_b, z_b)]^2 \leq 0.$$

Since $(s^2 - \lambda_1 - \frac{s^2 \lambda_3}{2}) \geq 0$, we have $B_b z_b = z_b$.

Therefore $B_b z_b = T_b z_b = z_b$.

Therefore, z_b is a common fixed point of B and T .

By Proposition 2.1, we get that z_b is the unique common fixed point of A_b, B_b, S_b and T_b .

Case (iii). $A_b(X_b)$ is closed.

From the inequality (2) and Case (ii), the conclusion follows.

Case (iv). $B_b(X_b)$ is closed.

From the inequality (2) and Case (i), the Proof follows. $\square$

Theorem 2.2. Let (X_b, d_b) be a b -metric space with coefficient $s \geq 1$. Assume that $A_b, B_b, S_b, T_b : X_b \rightarrow X_b$ are generalized contraction maps and satisfy (2). Suppose that one of the pairs (A_b, S_b) and (B_b, T_b) satisfies the b -(E.A)-property and that one of the subspace $A_b(X_b), B_b(X_b), S_b(X_b)$ and $T_b(X_b)$ is b -closed in X_b . Then the pairs (A_b, S_b) and (B_b, T_b) have a point of coincidence in X_b . Moreover, if the pairs (A_b, S_b) and (B_b, T_b) are weakly compatible, then A_b, B_b, S_b and T_b have a unique common fixed point in X_b .

Common fixed points for two pairs of maps...

Proof. We first assume that the pair (A_b, S_b) satisfies the b -(E.A)-property. So there exists a sequence $\{x_{b_n}\}$ in X_b such that

$$\lim_{n \rightarrow \infty} A_b x_{b_n} = \lim_{n \rightarrow \infty} S_b x_{b_n} = q_b \text{ for some } q_b \in X_b \quad (10)$$

Since $A_b(X_b) \subseteq T_b(X_b)$, there exists a sequence $\{y_{b_n}\}$ in X_b such that $A_b x_{b_n} = T_b y_{b_n}$, and hence

$$\lim_{n \rightarrow \infty} T_b y_{b_n} = q_b. \quad (11)$$

We now show that $\lim_{n \rightarrow \infty} B_b y_{b_n} = q_b$. Suppose that $\lim_{n \rightarrow \infty} B_b y_{b_n} \neq q_b$.

From (1), we have

$$\begin{aligned} s^4 [d_b(A_b x_{b_n}, B_b y_{b_n})]^2 &\leq \lambda_1 \max\{[d_b(S_b x_{b_n}, T_b y_{b_n})]^2, [d_b(S_b x_{b_n}, A_b x_{b_n})]^2, \\ &\quad [d_b(T_b y_{b_n}, B_b y_{b_n})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_n}, A_b x_{b_n}) d_b(S_b x_{b_n}, B_b y_{b_n})}{2}, \right. \\ &\quad \left. \frac{d_b(T_b y_{b_n}, B_b y_{b_n}) d_b(T_b y_{b_n}, A_b x_{b_n})}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_{b_n}, B_b y_{b_n}) d_b(T_b y_{b_n}, A_b x_{b_n})}{2} \end{aligned} \quad (12)$$

By taking limit superior as $n \rightarrow \infty$ in (12), and using (10) and (11), we obtain

$$\begin{aligned} \frac{1}{s^2} s^4 \liminf_{n \rightarrow \infty} [d_b(q_b, B_b y_{b_n})]^2 &\leq s^4 \limsup_{n \rightarrow \infty} [d_b(A_b x_{b_n}, B_b y_{b_n})]^2 \\ &\leq \limsup_{n \rightarrow \infty} (\lambda_1 \max\{[d_b(S_b x_{b_n}, T_b y_{b_n})]^2, [d_b(S_b x_{b_n}, A_b x_{b_n})]^2, \\ &\quad [d_b(T_b y_{b_n}, B_b y_{b_n})]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_n}, A_b x_{b_n}) d_b(S_b x_{b_n}, B_b y_{b_n})}{2}, \frac{d_b(T_b y_{b_n}, B_b y_{b_n}) d_b(T_b y_{b_n}, A_b x_{b_n})}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_{b_n}, B_b y_{b_n}) d_b(T_b y_{b_n}, A_b x_{b_n})}{2}) \\ &\leq s^2 \lambda_1 \limsup_{n \rightarrow \infty} [d_b(q_b, B_b y_{b_n})]^2. \end{aligned}$$

Since $(1 - \lambda_1) > 0$, we have

$$\lim_{n \rightarrow \infty} B_b y_{b_n} = q_b. \quad (13)$$

Case (i). Assume $T_b(X_b)$ is a b -closed subset of X_b .

In this case $q_b \in T_b(X_b)$, we can choose $r_b \in X_b \ni T_b r_b = q_b$.

Now, our claim is $B_b r_b = q_b$. Suppose $d_b(B_b r_b, q_b) > 0$. From (1), we have

$$\begin{aligned} s^4 [d_b(A_b x_{b_{2n+2}}, B_b r_b)]^2 &\leq \lambda_1 \max\{[d_b(S_b x_{b_{2n+2}}, T_b r_b)]^2, [d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}})]^2, [d_b(T_b r_b, B_b r_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_{b_{2n+2}}, A_b x_{b_{2n+2}}) d_b(S_b x_{b_{2n+2}}, B_b r_b)}{2}, \frac{d_b(T_b r_b, B_b r_b) d_b(T_b r_b, A_b x_{b_{2n+2}})}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_{b_{2n+2}}, B_b r_b) d_b(T_b r_b, A_b x_{b_{2n+2}})}{2} \end{aligned} \quad (14)$$

On letting limit superior as $n \rightarrow \infty$ in (14), using (10), (11), (12) and Lemma 1.1, we have $\frac{1}{s^2} s^4 d_b(q_b, B_b r_b) \leq \lambda_1 [d_b(q_b, B_b r_b)]^2$ which implies that

$$(s^2 - \lambda_1)[d_b(q_b, B_b r_b)]^2 \leq 0.$$

Since $(s^2 - \lambda_1) \geq 0$, we have $B_b r_b = q_b$.

Hence $B_b r_b = T_b r_b = q_b$, so that q_b is a coincidence point of B_b and T_b .

Since $B_b(X_b) \subseteq S_b(X_b)$, we have $q_b \in S_b(X_b)$, there exists $z_b \in X_b$ such that $S_b z_b = q_b = B_b r_b$.

Now we show that $A_b z_b = q_b$. Suppose $A_b z_b \neq q_b$. From the inequality (1), we have

$$\begin{aligned} s^4[d_b(A_b z_b, q_b)]^2 &= s^4[d_b(A_b z_b, B_b r_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b z_b, T_b r_b)]^2, [d_b(S_b z_b, A_b z_b)]^2, [d_b(T_b r_b, B_b r_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b z_b, A_b z_b)d_b(S_b z_b, B_b r_b)}{2}, \frac{d_b(T_b r_b, B_b r_b)d_b(T_b r_b, A_b z_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b z_b, B_b r_b)d_b(T_b r_b, A_b z_b)}{2} \end{aligned}$$

which implies that $(s^4 - \lambda_1)[d_b(q_b, A_b z_b)]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $A_b z_b = q_b$.

Therefore $A_b z_b = S_b z_b = q_b$ so that z_b is a coincidence point of A_b and S_b .

Since the pairs (A_b, S_b) and (B_b, T_b) are weakly compatible, we have $A_b q_b = S_b q_b$ and $B_b q_b = T_b q_b$.

Therefore q_b is also a coincidence point of the pairs (A_b, S_b) and (B_b, T_b) .

We now show that q_b is a common fixed point of A_b, B_b, S_b and T_b .

Suppose $A_b q_b \neq q_b$.

From the inequality (1), we have

$$\begin{aligned} s^4[d_b(A_b q_b, q_b)]^2 &= s^4[d_b(A_b q_b, B_b r_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b q_b, T_b r_b)]^2, [d_b(S_b q_b, A_b q_b)]^2, [d_b(T_b r_b, B_b r_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b q_b, A_b q_b)d_b(S_b q_b, B_b r_b)}{2}, \frac{d_b(T_b r_b, B_b r_b)d_b(T_b r_b, A_b q_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b q_b, B_b r_b)d_b(T_b r_b, A_b q_b)}{2} \end{aligned}$$

which implies that $[s^4 - (\lambda_1 + \frac{\lambda_3}{2})][d_b(q_b, A_b q_b)]^2 \leq 0$.

Since $(s^4 - (\lambda_1 + \frac{\lambda_3}{2})) \geq 0$, we have $A_b q_b = q_b$.

Therefore $A_b q_b = S_b q_b = q_b$ so that q_b is a common fixed point of A_b and S_b .

By Proposition 2.1, q_b is a unique common fixed point of A_b, B_b, S_b and T_b .

Case (ii). Suppose $A_b(X_b)$ is b -closed.

In this case, we have $q_b \in A_b(X_b)$ and $A_b(X_b) \subseteq T_b(X_b)$,

we choose $r_b \in X_b \ni q_b = T_b r_b$.

Rest of the proof follows as in Case (i).

Case (iii). Suppose $S_b(X_b)$ is b -closed.

We follow the argument similar as Case (i) and we get conclusion.

Case (iv). Suppose $B_b(X_b)$ is b -closed. As in Case (ii), we get the conclusion.

For the case of (B_b, T_b) satisfies the b -(E.A)-property, we follow the argument similar to the case (A_b, S_b) satisfies the b -(E.A)-property. $\square$

3 Corollaries and Examples

The following is an example in support of Theorem 2.1.

Example 3.1. Let $X_b = [0, \infty)$ and let $d_b : X_b \times X_b \rightarrow \mathbb{R}^+$ defined by

$$d_b(x_b, y_b) = \begin{cases} 0 & \text{if } x_b = y_b, \\ 4 & \text{if } x_b, y_b \in (0, 1), \\ \frac{9}{2} + \frac{1}{\frac{x_b+y_b}{12} + \frac{1}{5}} & \text{if } x_b, y_b \in [1, \infty), \\ \frac{12}{5} & \text{otherwise.} \end{cases}$$

Then clearly (X_b, d_b) is a complete b -metric space with coefficient $s = \frac{25}{24}$.

We define $A_b, B_b, S_b, T_b : X_b \rightarrow X_b$ by

$$A_b(x_b) = 1 \text{ if } x_b \in [0, \infty), B_b(x_b) = \begin{cases} x_b & \text{if } x_b \in [0, 1) \\ \frac{1}{x_b} & \text{if } x_b \in [1, \infty), \end{cases}$$

$$S_b(x_b) = \begin{cases} x_b & \text{if } x_b \in [0, 1) \\ \frac{1+x_b}{2} & \text{if } x_b \in [1, \infty), \end{cases} \text{ and } T_b(x_b) = \begin{cases} 2 & \text{if } x_b \in [0, 1) \\ 2x_b^2 - 1 & \text{if } x_b \in [1, \infty). \end{cases}$$

Clearly $A_b(X_b) \subseteq T_b(X_b)$, $B_b(X_b) \subseteq S_b(X_b)$ and $A_b(X_b)$ is closed.

Clearly the pairs (A_b, S_b) and (B_b, T_b) are weakly compatible.

We take $\lambda_1 = \frac{10}{51}$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = \frac{1}{2}$.

Then clearly $\lambda_1 + s\lambda_2 + s^2\lambda_3 \leq 1$.

With out loss generality, we assume that $x \geq y$.

Case (i). $x_b, y_b \in [0, 1)$.

$$d_b(A_b x_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, T_b y_b) = \frac{12}{5}, d_b(S_b x_b, A_b x_b) = \frac{12}{5},$$

$$d_b(T_b y_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, B_b y_b) = 4, d_b(T_b y_b, A_b x_b) = \frac{9}{2} + \frac{1}{x_b+y_b}.$$

We now consider

$$\begin{aligned} s^4[d_b(A_b x_b, B_b y_b)]^2 &= \left(\frac{25}{24}\right)^4 \left(\frac{12}{5}\right)^2 \\ &\leq \frac{10}{51} \left(\frac{12}{5}\right)^2 + \frac{1}{8} \left(\left(\frac{12}{5}\right) \left(\frac{9}{2} + \frac{1}{x_b+y_b}\right)\right) + \frac{1}{4} \left(\left(4\right) \left(\frac{9}{2} + \frac{1}{x_b+y_b}\right)\right) \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}. \end{aligned}$$

Case (ii). $x_b, y_b \in (1, \infty)$.

$$d_b(A_b x_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, T_b y_b) = \frac{9}{2} + \frac{1}{x_b+y_b}, d_b(S_b x_b, A_b x_b) = \frac{9}{2} + \frac{1}{x_b+y_b},$$

$$d_b(T_b y_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, B_b y_b) = \frac{12}{5}, d_b(T_b y_b, A_b x_b) = \frac{9}{2} + \frac{1}{x_b+y_b}.$$

We now consider

$$\begin{aligned} s^4[d_b(A_b x_b, B_b y_b)]^2 &= \left(\frac{25}{24}\right)^4 \left(\frac{12}{5}\right)^2 \\ &\leq \frac{10}{51} \left(\frac{9}{2} + \frac{1}{x_b+y_b}\right)^2 + \frac{1}{8} \left(\left(\frac{9}{2} + \frac{1}{x_b+y_b}\right) \left(\frac{12}{5}\right)\right) + \frac{1}{4} \left(\left(\frac{12}{5}\right) \left(\frac{9}{2} + \frac{1}{x_b+y_b}\right)\right) \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}. \end{aligned}$$

Case (iii). $x_b \in (1, \infty)$, $y_b \in (0, 1)$.

$$d_b(A_b x_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, T_b y_b) = \frac{9}{2} + \frac{1}{x_b+y_b}, d_b(S_b x_b, A_b x_b) = \frac{9}{2} + \frac{1}{x_b+y_b},$$

$$d_b(T_b y_b, B_b y_b) = \frac{12}{5}, d_b(S_b x_b, B_b y_b) = \frac{12}{5}, d_b(T_b y_b, A_b x_b) = \frac{9}{2} + \frac{1}{x_b + y_b},$$

We now consider

$$\begin{aligned} s^4[d_b(A_b x_b, B_b y_b)]^2 &= \left(\frac{25}{24}\right)^4 \left(\frac{12}{5}\right)^2 \\ &\leq \frac{10}{51} \left(\frac{9}{2} + \frac{1}{x_b + y_b}\right)^2 + \frac{1}{8} \left(\left(\frac{12}{5}\right) \left(\frac{9}{2} + \frac{1}{x_b + y_b}\right)\right) + \frac{1}{4} \left(\left(\frac{12}{5}\right) \left(\frac{9}{2} + \frac{1}{x_b + y_b}\right)\right) \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}. \end{aligned}$$

Therefore A_b, B_b, S_b and T_b satisfy all the hypotheses of Theorem 2.1 and 1 is the unique common fixed point in X_b .

The following is an example in support of Theorem 2.2.

Example 3.2. Let $X_b = [0, 1]$ and let $d_b : X_b \times X_b \rightarrow \mathbb{R}^+$ defined by

$$d_b(x_b, y_b) = \begin{cases} 0 & \text{if } x_b = y_b, \\ \frac{11}{15} & \text{if } x_b, y_b \in [0, \frac{2}{3}), \\ \frac{99}{100} + \frac{x_b + y_b}{200} & \text{if } x_b, y_b \in [\frac{2}{3}, 1], \\ \frac{12}{25} & \text{otherwise.} \end{cases}$$

Then clearly (X_b, d_b) is a complete b -metric space with coefficient $s = \frac{25}{24}$.

We define $A_b, B_b, S_b, T_b : X_b \rightarrow X_b$ by

$$\begin{aligned} A_b(x_b) &= \frac{2}{3} \text{ if } x_b \in [0, 1], \quad B_b(x_b) = \begin{cases} \frac{1}{2} & \text{if } x_b \in [0, \frac{2}{3}) \\ \frac{2}{3} & \text{if } x_b \in [\frac{2}{3}, 1], \end{cases} \\ S_b(x_b) &= \begin{cases} \frac{1}{2} & \text{if } x_b \in [0, \frac{2}{3}) \\ \frac{2 + 5x_b}{8} & \text{if } x_b \in [\frac{2}{3}, 1], \end{cases} \quad \text{and} \quad T_b(x_b) = \begin{cases} 1 & \text{if } x_b \in [0, \frac{2}{3}) \\ \frac{4 + x_b}{7} & \text{if } x_b \in [\frac{2}{3}, 1]. \end{cases} \end{aligned}$$

Clearly $A_b(X_b) \subseteq T_b(X_b)$ and $B_b(X_b) \subseteq S_b(X_b)$. $A_b(X_b) = \{\frac{2}{3}\}$ is b -closed.

We choose a sequence $\{x_{b_n}\}$ with $\{x_{b_n}\} = \frac{2}{3} + \frac{1}{n}, n \geq 4$ with

$$\lim_{n \rightarrow \infty} A_b x_{b_n} = \lim_{n \rightarrow \infty} S_b x_{b_n} = \frac{2}{3}, \text{ hence the pair } (A_b, S_b) \text{ satisfies the}$$

b -(E.A)-property.

Clearly the pairs (A_b, S_b) and (B_b, T_b) are weakly compatible.

We take $\lambda_1 = \frac{10}{51}, \lambda_2 = \frac{1}{4}, \lambda_3 = \frac{1}{2}$. Then clearly $\lambda_1 + s\lambda_2 + s^2\lambda_3 \leq 1$.

With out loss generality, we assume that $x \geq y$.

Case (i). $x_b, y_b \in (0, \frac{2}{3})$.

$$\begin{aligned} d_b(A_b x_b, B_b y_b) &= \frac{12}{25}, d_b(S_b x_b, T_b y_b) = \frac{12}{25}, d_b(S_b x_b, A_b x_b) = \frac{12}{25}, \\ d_b(T_b y_b, B_b y_b) &= \frac{12}{25}, d_b(S_b x_b, B_b y_b) = \frac{11}{15}, d_b(T_b y_b, A_b x_b) = \frac{99}{100} + \frac{x_b + y_b}{200}, \end{aligned}$$

We now consider

$$\begin{aligned} s^4[d_b(A_b x_b, B_b y_b)]^2 &= \left(\frac{25}{24}\right)^4 \left(\frac{12}{25}\right)^2 \\ &\leq \frac{10}{51} \left(\frac{12}{25}\right)^2 + \frac{1}{8} \left(\left(\frac{12}{25}\right) \left(\frac{99}{100} + \frac{x_b + y_b}{200}\right)\right) + \frac{1}{4} \left(\left(\frac{11}{15}\right) \left(\frac{99}{100} + \frac{x_b + y_b}{200}\right)\right) \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b x_b, A_b x_b)d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b)d_b(T_b y_b, A_b x_b)}{2}. \end{aligned}$$

Case (ii). $x_b, y_b \in (\frac{2}{3}, 1]$.

$d_b(A_b x_b, B_b y_b) = 0$. In this case the inequality (1) trivially holds.

Case (iii). $x_b \in (\frac{2}{3}, 1], y_b \in (0, \frac{2}{3})$.

$$d_b(A_b x_b, B_b y_b) = \frac{12}{25}, d_b(S_b x_b, T_b y_b) = \frac{99}{100} + \frac{x_b + y_b}{200}, d_b(S_b x_b, A_b x_b) = \frac{99 + x_b}{100},$$

$$d_b(T_b y_b, B_b y_b) = \frac{12}{25}, d_b(S_b x_b, B_b y_b) = \frac{12}{25}, d_b(T_b y_b, A_b x_b) = \frac{99}{100} + \frac{x_b + y_b}{200},$$

We now consider

$$\begin{aligned} s^4 [d_b(A_b x_b, B_b y_b)]^2 &= (\frac{25}{24})^4 (\frac{12}{25})^2 \\ &\leq \frac{10}{51} (\frac{99 + x_b}{100})^2 + \frac{1}{8} ((\frac{99 + x_b}{100})(\frac{12}{25})) + \frac{1}{4} (\frac{12}{25}) ((\frac{99}{100} + \frac{x_b + y_b}{200})) \\ &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_b x_b)]^2, [d_b(T_b y_b, B_b y_b)]^2\} \\ &\quad + \lambda_2 \max\{\frac{d_b(S_b x_b, A_b x_b) d_b(S_b x_b, B_b y_b)}{2}, \frac{d_b(T_b y_b, B_b y_b) d_b(T_b y_b, A_b x_b)}{2}\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, B_b y_b) d_b(T_b y_b, A_b x_b)}{2}. \end{aligned}$$

Therefore A_b, B_b, S_b and T_b satisfy all the hypotheses of Theorem 2.2 and $\frac{2}{3}$ is the unique common fixed point in X_b .

Corollary 3.1. Let $\{A_n\}_{n=1}^\infty, S_b$ and T_b be selfmaps on a complete b -metric space (X_b, d_b) satisfying $A_1 \subseteq S_b(X_b)$ and $A_1 \subseteq T_b(X_b)$. Assume that there exist positive reals $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 + s\lambda_2 + s^2\lambda_3 \leq 1$ such that

$$\begin{aligned} s^4 [d_b(A_1 x_b, A_j y_b)]^2 &\leq \lambda_1 \max\{[d_b(S_b x_b, T_b y_b)]^2, [d_b(S_b x_b, A_1 x_b)]^2, [d_b(T_b y_b, A_j y_b)]^2\} \\ &\quad + \lambda_2 \max\{\frac{d_b(S_b x_b, A_1 x_b) d_b(S_b x_b, A_j y_b)}{2}, \frac{d_b(T_b y_b, A_j y_b) d_b(T_b y_b, A_1 x_b)}{2}\} \\ &\quad + \lambda_3 \frac{d_b(S_b x_b, A_j y_b) d_b(T_b y_b, A_1 x_b)}{2}. \end{aligned} \tag{15}$$

for all $x_b, y_b \in X_b$ and $j = 1, 2, 3, \dots$. If the pairs (A_1, S_b) and (A_1, T_b) are weakly compatible and one of the range sets $A_1(X_b), S_b(X_b)$ and $T_b(X_b)$ is closed, then $\{A_n\}_{n=1}^\infty, S_b$ and T_b have a unique common fixed point in X_b .

Proof. Under the assumptions on A_1, S_b and T_b , the existence of common fixed point z_b of A_1, S_b and T_b follows by choosing $A_b = B_b = A_1$ in Theorem 2.1.

Therefore $A_1 z_b = S_b z_b = T_b z_b = z_b$.

Now, let $j \in \mathbb{N}$ with $j \neq 1$.

We now consider

$$\begin{aligned} s^4 [d_b(z_b, A_j z_b)]^2 &= s^4 [d_b(A_1 z_b, A_j z_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b z_b, T_b z_b)]^2, [d_b(S_b z_b, A_1 z_b)]^2, [d_b(T_b z_b, A_j z_b)]^2\} \\ &\quad + \lambda_2 \max\{\frac{d_b(S_b z_b, A_1 z_b) d_b(S_b z_b, A_j z_b)}{2}, \frac{d_b(T_b z_b, A_j z_b) d_b(T_b z_b, A_1 z_b)}{2}\} \\ &\quad + \lambda_3 \frac{d_b(S_b z_b, A_j z_b) d_b(T_b z_b, A_1 z_b)}{2}. \end{aligned} \tag{16}$$

From the inequality (16), we have

$$s^4 [d_b(z_b, A_j z_b)]^2 \leq \lambda_1 [d_b(z_b, A_j z_b)]^2$$

which implies that $(s^4 - \lambda_1)[d_b(z_b, A_j z_b)]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $A_j z_b = z_b$ for $j = 1, 2, 3, \dots$ and uniqueness of common fixed point follows from the inequality (15).

Therefore $\{A_n\}_{n=1}^\infty, S_b$ and T_b have a unique common fixed point in X_b . $\square$

Corolary 3.2. Let $\{A_n\}_{n=1}^\infty$, S_b and T_b be selfmaps on a b -metric space (X_b, d_b) satisfy the conditions $A_1 \subseteq S_b(X_b)$, $A_1 \subseteq T_b(X_b)$ and (15). If one of the pairs (A_1, S_b) and (A_1, T_b) satisfies the b -(E.A)-property and that one of the subspace $A_1(X)$, $S_b(X_b)$ or $T_b(X_b)$ is b -closed in X_b . Then the pairs (A_1, S_b) and (A_1, T_b) have a point of coincidence in X_b . Moreover, if the pairs (A_1, S_b) and (A_1, T_b) are weakly compatible, then $\{A_n\}_{n=1}^\infty$, S_b and T_b have a unique common fixed point in X_b .

Proof. Under the assumptions on A_1 , S_b and T_b , the existence of common fixed point z_b of A_1 , S_b and T_b follows by choosing $A_b = B_b = A_1$ in Theorem 2.2.

Therefore $A_1 z_b = S_b z_b = T_b z_b = z_b$.

Now, let $j \in \mathbb{N}$ with $j \neq 1$.

We now consider

$$\begin{aligned} s^4[d_b(z_b, A_j z_b)]^2 &= s^4[d_b(A_1 z_b, A_j z_b)]^2 \\ &\leq \lambda_1 \max\{[d_b(S_b z_b, T_b z_b)]^2, [d_b(S_b z_b, A_1 z_b)]^2, [d_b(T_b z_b, A_j z_b)]^2\} \\ &\quad + \lambda_2 \max\left\{\frac{d_b(S_b z_b, A_1 z_b)d_b(S_b z_b, A_j z_b)}{2}, \frac{d_b(T_b z_b, A_j z_b)d_b(T_b z_b, A_1 z_b)}{2}\right\} \\ &\quad + \lambda_3 \frac{d_b(S_b z_b, A_j z_b)d_b(T_b z_b, A_1 z_b)}{2}. \end{aligned} \tag{17}$$

From the inequality (17), we have

$s^4[d_b(z_b, A_j z_b)]^2 \leq \lambda_1[d_b(z_b, A_j z_b)]^2$ which implies that

$(s^4 - \lambda_1)[d_b(z_b, A_j z_b)]^2 \leq 0$.

Since $(s^4 - \lambda_1) \geq 0$, we have $A_j z_b = z_b$ for $j = 1, 2, 3, \dots$ and uniqueness of common fixed point follows from the inequality (15).

Therefore $\{A_n\}_{n=1}^\infty$, S_b and T_b have a unique common fixed point in X_b . $\square$

4 Conclusion

In this paper, we introduced generalized contraction for two pairs of selfmaps in complete b -metric spaces and proved the existence and of common fixed points. Our results extend/generalize the known results that are available in the literature. A sequence of selfmaps is added as an extension of the same. We provided examples in support of our results and some corollaries to our results are presented.

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Numerical solution of pollutant transport model with source term by RDTM

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Abstract

The focus of this paper is the study of mathematical model for the analysis of diffusion and transport processes of chemicals in river systems. In the present study, we have introduced a one-dimensional model that is described by time-dependent convection-diffusion differential equations. The focus of our study is the potential pollution of the river, both with and without a specific pollution source. The current investigation aims to analyze the impact of two different input source functions, namely constant and linear forms. In order to validate the model, we conducted a study on the diffusion and transport of chemicals, specifically focusing on NO_3 and PO_4 of the Tapi river. Reduced differential transform method (RDTM) is used to obtain solutions. For the accuracy of the solution, the convergence of solution function is examined in each case obtained from RDTM. The pollutant concentration at various distances and time levels has been shown numerically and graphically in each case. The pollution levels with and without sources are compared in 2D and 3D graphs. The methodology proposed in this study for river pollution prediction using a 1D pollution model can be applied to other rivers.

Keywords: Source, Pollutant transport equation; Reduced differential transform method; Convergence

2020 AMS subject classifications: 35A22, 35C10, 35G05, 35G10¹

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1 Introduction

Water, an essential element of the environment, plays a vital part in various ecological processes. However, the quality of both surface water and ground-water has been steadily declining over time. This degradation can be attributed to natural factors and human activities. Various natural factors influence water quality, including hydrological, atmospheric, climatic, topographical, and lithological factors. These factors play a significant role in shaping the overall quality of water in various aquatic systems. Hydrological factors, such as the flow rate and volume of water, can impact water quality by affecting the transport and dilution of pollutants. Atmospheric factors, such as temperature and precipitation, can influence water quality through the deposition of airborne pollutants and the alteration of water chemistry. Climatic factors, including seasonal variations and long-term climate patterns, can also impact water quality by influencing factors such as [Magesh et al., 2013, Kut et al., 2019].

In the modern era, there has been a conspicuous increase in the difficulties encountered by developing countries in maintaining water quality while attempting to improve water supply and sanitation. Many wealthy countries have faced substantial obstacles recently in their attempts to maintain or improve the quality of their water supplies. These difficulties are generally caused by problems like eutrophication and nutrient enrichment, which have negative impacts on the quality of the water. Additionally, these countries have had significant difficulties in meeting the needs of their expanding populations for water and wastewater services [Ortega et al., 2016, Abbasi and Abbasi, 2012].

The application of partial differential equations (PDEs) has been prevalent in formulating a diverse array of applied problems due to the inherent relationship between real-world phenomena and these equations. In numerous practical scenarios, such as the advection-diffusion equation, a significant partial differential equation (PDE) commonly observed in groundwater flows, the availability of ideal information is seldom encountered. The accuracy of our information regarding the soil's permeability, the source term's magnitude, and inflow or outflow conditions is questionable [Mirzaee et al., 2021].

Multiple methods approach the heat equation and advection–diffusion equation. The octic B-spline collocation, Exponential B-spline, Finite element and operator splitting methods improve the approximate solution of the heat and advection–diffusion equation by taking into consideration the inadequacies of previous numerical studies with high error accuracy. MATLAB simplifies tiresome computations [Jena and Gebremedhin, 2021, Mohammadi, 2013, Mojtabi and Deville, 2015, Bahar and Gürarslan, 2017].

The primary aim of the present study is to comprehensively integrate various components, including mathematical modelling, transformation methods, approx-

imations, and computer implementation, to investigate the time-dependent transport of substances in a water medium during the eutrophication process. The implementation of the proposed mathematical models and methods can be easily accomplished using the collected initial data. The model that has been developed offers an effective model for expeditious assessment of water quality in river systems. Furthermore, this concept can be extended to other disciplines within science and engineering. A model has been developed to analyze the distribution of polluting substances in rivers, considering multiple sources and types of pollutants. One-dimensional advection-diffusion models have been created. The presence of pollution sources in the studied river section is hypothesized. These sources may include point or volume sources such as pipe outlets used by industries to release industrial wastes into the river, underground sources, or other rivers that flow into the section.

For the present study, the water samples are collected from the Tapi river, Surat, India to develop the mathematical model for predicting pollutant level at different distance and time only.

In section 3, we describe the mathematical formulation of the problem. Section 4 presents a comprehensive overview of the fundamental concepts that underpin the RDTM. Section 5 contains the detailed process of converging the analytic series solution proposed by RDTM. Section 6 discusses the numerical outcomes and the method's convergence to evaluate its effectiveness. The utilization of 2D and 3D plots provides a visual depiction of the solutions that have been acquired. In section 7, a concise overview of the conclusion is presented.

2 Water Sample Collection

This investigation examines the Tapi river from the Pal-Umra bridge to the ONGC bridge in Surat. On November 12, 2022, seven 1 Litre capacity samples were taken from the beginning point of the study area in identically sized bottles. Using these water samples, NO_3 and PO_4 water pollutants concentration were analysed. This gathered information was transmitted to Pollucon Laboratories Pvt. Ltd., Nr. Navjivan Circle, Surat, Gujarat, for NO_3 (Nitrate) and PO_4 (Phosphate) pollutants concentration. Figure 1 show some pictures of research area.

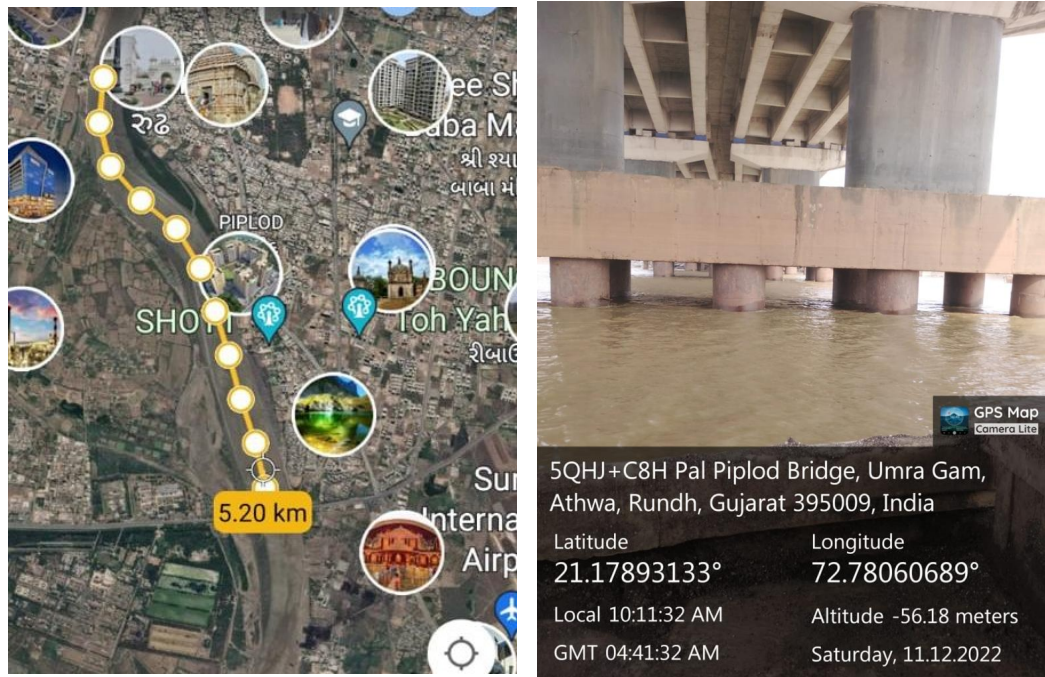


Figure 1: Research Area

$x^*(km)$	NO_3 concentration (mg/L)	PO_4 concentration (mg/L)
0	0.59	2.81
0.5	1.01	2.73
1	0.97	3.88
1.5	1.25	3.92
2.5	0.21	2.78
3.5	0.57	2.9
5	2.48	2.91

Table 1: Initial data of pollutant's concentration

Table 1 give numerical values of initial concentration of NO_3 and PO_4 pollutants in research area.

3 Mathematical Formulation of the problem

The entry of pollutants into the subsurface is primarily facilitated by two mechanisms. The first mechanism is known as advection, which occurs due to

the movement of groundwater. The second mechanism is dispersion, which is caused by a combination of mechanical mixing and molecular diffusion. The impact of molecular diffusions is often disregarded in the context of small seepage velocities [Yadav and Kumar, 2019]. The advection-diffusion equation in one dimension can be represented mathematically as a second order partial differential equation of parabolic nature. It can be expressed as:

$$A^* \frac{\partial c}{\partial t^*} = \frac{\partial}{\partial x^*} \left(A^* D_x^* \frac{\partial c}{\partial x^*} \right) - A^* v^* \frac{\partial c}{\partial x^*} - A^* k c + A^* f^* \quad (1)$$

where c [ML^{-3}] is the pollutant's solute concentration at any point x^* [L] and time t^* [T] while it is being transported through the medium along the flow field. D_x^* [L^2T^{-1}] is Longitudinal Diffusion coefficient. v^* [LT^{-1}] is the unsteady uniform seepage flow velocity along longitudinal direction. k [T^{-1}] is coefficient of non-conservation. A^* [L^2] is Flow area of cross section of river. f^* [$ML^{-3}T^{-1}$] is Source term. x^* is longitudinal distance from starting point of research area. t^* is Time dimension. The area of the river's cross-section is treated as a constant throughout the study and $k = 0$ [Kachiashvili et al., 2007]. Then equation (1) convert in the given form:

$$\frac{\partial c}{\partial t^*} = D_x^* \frac{\partial^2 c}{\partial x^{*2}} - v^* \frac{\partial c}{\partial x^*} + f^* \quad (2)$$

The presentation of the non-dimensional variables [Rubbab et al., 2016]:

$$x = \frac{x^*}{L}, t = \frac{D_x^* t^*}{L^2}, v = \frac{v^* L}{D_x^*}, S = \frac{L^2 f^*}{D_x^*} \quad (3)$$

Here L is the length of research portion of river. We derive dimensional less advection-diffusion equation as

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - v \frac{\partial c}{\partial x} + S(x, t) \quad (4)$$

To analyze this problem, we have used the concentration of pollutant substances NO_3 and PO_4 of river Tapi. In this paper, We have taken 5 km Research area of tapi river from ONGC bridge to Pal Umra bridge. So here, river's length(L) is 5 km. Using collected data of pollutants NO_3 and PO_4 concentration, at time $t = 0$ (any fixed time), the concentration is assumed as a polynomial form with good statistical indices.

In this article, the concentration level of contamination is discussed for both without and with various kind of input sources. Various cases have been examined to determine the concentration function's value and visual depiction. We're looking at two distinct sorts of known sources to see the pollution level in river. Here, S denotes the rate of the pollutant that enters the beginning border of the

portion of the river that is being examined per unit of time and per unit of volume. According to the value of $S(x, t)$, these two types of input sources include constant and linear source.

Initial function (5) for NO_3 pollutant concentration function

$$c(x, 0) = -1.4880e-05 x^5 + 8.1728e-04 x^4 - 0.0139 x^3 + 0.0728 x^2 + 0.0045 x + 0.6140 \quad (5)$$

Initial function (6) for PO_4 pollutant concentration function

$$c(x, 0) = -3.9488e-05 x^5 + 0.0023 x^4 - 0.0428 x^3 + 0.3006 x^2 - 0.5051 x + 2.791 \quad (6)$$

4 Reduced Differential Transform Method

Let $p(\Psi, t)$ be a two-variable function. Suppose $p(\Psi, t)$ is written as $p(\Psi, \tau) = u(\Psi)v(t)$. $p(\Psi, \tau)$ can be represented as the following using the features of the differential transform:

$$p(\Psi, t) = \sum_{i=0}^{\infty} U_{(i)} \Psi^i \sum_{j=0}^{\infty} V_{(j)} t^j = \sum_{k=0}^{\infty} P_k(\Psi) t^k \quad (7)$$

where $P_k(\Psi)$ is referred to as the t-dimensional spectrum function of $p(\Psi, t)$.

$$P_n(\Psi) = \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} p(\Psi, t) \right]_{t=0} \quad (8)$$

The original function is denoted by the lowercase $[p(\Psi, t)]$, whereas the altered function is denoted by the capital $[P(\Psi, t)]$. The way to define the differential inverse transform of $P_k(\Psi)$ is as follows:

$$p(\Psi, t) = \sum_{n=0}^{\infty} P_n(\Psi) t^n \quad (9)$$

From Equations (8) and (9), we get

$$p(\Psi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n}{\partial t^n} p(\Psi, t) \right]_{t=0} t^n \quad (10)$$

Let us consider the following nonlinear PDE, to understand the basic concept of

RDTM.

$$Tp(\Psi, t) + Kp(\Psi, t) + Op(\Psi, t) = f(\Psi, t) \quad (11)$$

with initial condition $p(\Psi, 0) = \Omega(\Psi)$, where $T = \frac{\partial}{\partial t}$, $Kp(\Psi, t)$ is a linear term that has partial derivatives, while $Op(\Psi, t)$ is a non-linear term, and $f(\Psi, t)$ is a source term [Keskin and Oturanc, 2009].

By applying the transform on equation (11), we get

$$(k+1)P_{k+1}(\Psi) = F_k(\Psi) - KP_k(\Psi) - OP_k(\Psi) \quad (12)$$

where $P_k(\Psi)$, $F_k(\Psi)$, $KP_k(\Psi)$ and $OP_k(\Psi)$ are transform of $p(\Psi, t)$, $f(\Psi, t)$, $Kp(\Psi, t)$ and $Op(\Psi, t)$ respectively. We are able to write this down based on the initial condition.

$$P_0(\omega) = \Omega(\Psi) \quad (13)$$

From equations (12) and (13), we get the values of $P_k(\Psi)$. After that, an approximation solution is produced by carrying out an inverse transformation on the set of values $\{P_k(\Psi)\}_{k=0}^n$. This transformation yields an approximation solution as

$$\tilde{p}_n(\Psi, t) = \sum_{k=0}^n P_k(\Psi) t^k \quad (14)$$

where n is the order of approximation answer. Consequently, the exact solution is given by [Al-Amr, 2014],

$$p(\Psi, t) = \lim_{n \rightarrow \infty} \tilde{p}_n(\Psi, t) \quad (15)$$

Original function	Transformation of function
$p(\Psi, t)$	$P_k(\Psi) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} p(\Psi, t) \right]_{t=0}$
$\gamma m(\Psi, t) \pm \delta n(\Psi, t)$	$\gamma M_k(\Psi) + \delta N_k(\Psi)$
$\Psi^m t^n$	$\Psi^m \delta(k - n)$
$\Psi^m t^n p(\Psi, t)$	$\Psi^m P_{k-n}(\Psi)$
$a(\Psi, t) = b(\Psi, t)c(\Psi, t)$	$A_k(\Psi) = \sum_{r=0}^k B_r(\Psi) C_{k-r}(\Psi)$
$\frac{\partial^r}{\partial t^r} p(\Psi, t)$	$\frac{(k+r)!}{k!} P_{k+r}(\Psi)$
$\frac{\partial}{\partial \Psi} p(\Psi, t)$	$\frac{\partial}{\partial \Psi} P_k(\Psi)$

Table 2: Transform Table[Maisuria and Tandel, 2023, Maisuria et al., 2023, Aljahdaly and Alharbi, 2022, Ganji et al., 2016]

Table 2 represents the fundamental operations of the reduced differential transform method.

5 Convergence of RDTM

Theorem 5.1. If $\sum_{m=0}^{\infty} P_m t^m$ is given series [Maisuria and Tandel, 2023, Moosavi Noori and Taghizadeh, 2021, Maisuria et al., 2023, Saeed and Mustafa, 2017],

$$[1] \exists 0 < \delta < 1 \ni \frac{\|P_{m+1}\|}{\|P_m\|} \leq \delta \Rightarrow \text{given series is convergent.}$$

$$[2] \exists \delta > 1 \ni \frac{\|P_{m+1}\|}{\|P_m\|} \geq \delta \Rightarrow \text{given series is divergent.}$$

Proof. Let $(C[E], \|\cdot\|)$ represent the Banach space that contains all continuous functions on E that satisfy the norm $\|\cdot\|$. Also, let's suppose that $\|P_0(\Psi)\| \leq K$, where K is an integer in the positive range. Let $\{\kappa_m\}_{m=0}^{\infty}$ be a partial sum

$$\kappa_m = P_0 + P_1 + P_2 + \dots + P_m$$

If we can prove that $\{\kappa_m\}_{m=0}^{\infty}$ is a Cauchy sequence in Banach Space, then we can conclude that the sequence of partial sum is convergent in Banach Space. As a result, at this point, we shall demonstrate that the series of partial sums follows the Cauchy sequence. We take

$$\|\kappa_{m+1} - \kappa_m\| = \|P_{m+1}\| \leq \delta \|P_m\| \leq \dots \leq \delta^{m+1} \|P_0\| \leq \delta^{m+1} K$$

Therefore, $\forall m, n \in \mathbb{N}, m \geq n$, We have

$$\begin{aligned}\|\kappa_m - \kappa_n\| &= \|(\kappa_m - \kappa_{m-1}) + (\kappa_{m-1} - \kappa_{m-2}) + \dots + (\kappa_{n+1} - \kappa_n)\| \\ &\leq \|\kappa_m - \kappa_{m-1}\| + \|\kappa_{m-1} - \kappa_{m-2}\| + \dots + \|\kappa_{n+1} - \kappa_n\| \\ &\leq \frac{1-\delta^{m-n}}{1-\delta} \delta^{n+1} \|P_0\|\end{aligned}$$

Now here $0 < \delta < 1$, We obtain

$$\lim_{m,n \rightarrow \infty} \|\kappa_m - \kappa_n\| = 0$$

Hence $\{\kappa_m\}_{m=0}^{\infty}$ is Cauchy sequence in Banach Space. Therefore, given series is convergent.

6 Result and discussion

6.1 NO₃ pollutant

(a) Without pollutant source

In this case, $S(x, t) = 0$ in equation (4). Solving equation (4) by RDTM with initial function (5), we get

$$c(x, t) = M_0 + M_1 t + M_2 t^2 + M_3 t^3 + \dots$$

where

$$\begin{aligned}M_0 &= -1.4880e-05 x^5 + 8.1728e-04 x^4 - 0.0139 x^3 + 0.0728 x^2 \\ &\quad + 0.0045 x + 0.6140\end{aligned}$$

$$M_1 = 3.7200e-05 x^4 - 0.0019 x^3 + 0.0307 x^2 - 0.1564 x + 0.1434$$

$$M_2 = -3.7200e-05 x^3 + 0.0017 x^2 - 0.0211 x + 0.0698$$

$$M_3 = 1.8600e-05 x^2 - 6.3184e-04 x + 0.0046$$

Here,

$$\frac{\|M_1\|}{\|M_0\|} = 0.2335 < 1, \frac{\|M_2\|}{\|M_1\|} = 0.4869 < 1, \frac{\|M_3\|}{\|M_2\|} = 0.0665 < 1$$

Therefore, given solution function $c(x, t)$ is converge to exact solution.

Numerical solution of pollutant transport model with source term by RDTM

$x \backslash t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.628653	0.64352	0.659796	0.677508	0.696684	0.717352	0.739538	0.763271	0.788577	0.815485
0.2	0.629702	0.643033	0.657731	0.673823	0.691336	0.710298	0.730734	0.752674	0.776142	0.801167
0.3	0.632103	0.643958	0.657139	0.671673	0.687585	0.704903	0.723654	0.743864	0.765559	0.788767
0.4	0.635777	0.646216	0.657939	0.670974	0.685347	0.701083	0.71821	0.736753	0.756739	0.778195
0.5	0.640646	0.649726	0.660051	0.671646	0.684539	0.698754	0.714318	0.731256	0.749595	0.769361
0.6	0.646633	0.654411	0.663395	0.673609	0.68508	0.697833	0.711894	0.727289	0.744042	0.76218
0.7	0.653666	0.660197	0.667895	0.676785	0.686892	0.698242	0.710859	0.724769	0.739996	0.756567
0.8	0.66167	0.66701	0.673478	0.681099	0.689899	0.699901	0.711132	0.723616	0.737377	0.752441
0.9	0.670577	0.674778	0.680069	0.686477	0.694024	0.702736	0.712637	0.723751	0.736104	0.749719
1	0.680316	0.683431	0.6876	0.692846	0.699196	0.706672	0.715298	0.7251	0.736101	0.748325

Table 3: NO₃ pollutant concentration at different x and t for $v = 0.5$

The values of the nitrate pollutant concentration at different values of x and t for this case are shown in table 3.

(b) Constant Input pollutant source

In this case, $S(x, t) = \frac{\alpha}{(x+L)^3}$, ($\alpha > 0$) in equation (4) then solving equation (4) by RDTM with initial function (5), we get

$$c(x, t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots$$

where

$$A_0 = -1.4880e - 05 x^5 + 8.1728e - 04 x^4 - 0.0139 x^3 + 0.0728 x^2 + 0.0045 x + 0.6140$$

$$A_1 = \frac{2}{(x+5)^3} - 0.1564 x + 0.0307 x^2 - 0.0019 x^3 + 3.7200e - 05 x^4 + 0.1434$$

$$A_2 = \frac{3x+39}{2(x+5)^5} - 0.0211 x + 0.0017 x^2 - 3.7200e - 05 x^3 + 0.0698$$

$$A_3 = \frac{x^2+30x+245}{(x+5)^7} - 6.3184e - 04 x + 1.8600e - 05 x^2 + 0.0046$$

Here,

$$\frac{\|A_1\|}{\|A_0\|} = 0.2596 < 1, \frac{\|A_2\|}{\|A_1\|} = 0.4772 < 1, \frac{\|A_3\|}{\|A_2\|} = 0.1023 < 1$$

Hence, it can be observed that the solution function $c(x, t)$ converges towards the exact solution.

Numerical solution of pollutant transport model with source term by RDTM

$x \backslash t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.630221	0.646786	0.664906	0.684627	0.705992	0.729045	0.753831	0.780393	0.808775	0.839021
0.2	0.631179	0.646105	0.662533	0.680502	0.700055	0.721234	0.74408	0.768636	0.794942	0.823041
0.3	0.633496	0.646853	0.661657	0.677948	0.695765	0.715148	0.736137	0.758771	0.783091	0.809135
0.4	0.637093	0.648946	0.662196	0.676878	0.693032	0.710695	0.729905	0.750699	0.773116	0.797194
0.5	0.64189	0.652305	0.664065	0.677208	0.691769	0.707784	0.72529	0.744323	0.764919	0.787115
0.6	0.647811	0.656849	0.667186	0.678855	0.691891	0.706329	0.722204	0.739551	0.758404	0.778798
0.7	0.654781	0.662505	0.671479	0.681739	0.693316	0.706245	0.72056	0.736292	0.753476	0.772146
0.8	0.662728	0.669196	0.67687	0.685782	0.695965	0.707451	0.720272	0.73446	0.750047	0.767066
0.9	0.671581	0.676851	0.683283	0.690909	0.699759	0.709866	0.721259	0.73397	0.74803	0.76347
1	0.68127	0.685399	0.690647	0.697045	0.704624	0.713412	0.723441	0.734741	0.74734	0.76127

Table 4: NO₃ pollutant concentration at different x and t for $\alpha = 2$ and $v = 0.5$

The values of nitrate pollutant concentration at various values of x and t with constant input pollutant source are presented in table 4.

(c) Linear Input pollutant source

In this case, $S(x, t) = \frac{\alpha t}{(x+L)^3}$, ($\alpha > 0$) in equation (4) then solving equation (4) by RDTM with initial function (5), we get

$$c(x, t) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \dots$$

where

$$P_0 = -1.4880e - 05 x^5 + 8.1728e - 04 x^4 - 0.0139 x^3 + 0.0728 x^2 + 0.0045 x + 0.6140$$

$$P_1 = 3.7200e - 05 x^4 - 0.0019 x^3 + 0.0307 x^2 - 0.1564 x + 0.1434$$

$$P_2 = \frac{1}{(x+5)^3} - 0.0211 x + 0.0017 x^2 - 3.7200e - 05 x^3 + 0.0698$$

$$P_3 = \frac{x+13}{2(x+5)^5} - 6.3184e - 04 x + 1.8600e - 05 x^2 + 0.0046$$

Here,

$$\frac{\|P_1\|}{\|P_0\|} = 0.2335 < 1, \frac{\|P_2\|}{\|P_1\|} = 0.5427 < 1, \frac{\|P_3\|}{\|P_2\|} = 0.0864 < 1$$

Therefore, the solution function $c(x, t)$ apparently exhibits convergence towards the exact solution.

Numerical solution of pollutant transport model with source term by RDTM

$x \backslash t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.628731	0.643837	0.660526	0.678836	0.698806	0.720476	0.743883	0.769068	0.796068	0.824922
0.2	0.629775	0.643331	0.658418	0.675072	0.693331	0.713233	0.734815	0.758114	0.783168	0.810015
0.3	0.632172	0.64424	0.657787	0.672849	0.689463	0.707665	0.727491	0.748977	0.772159	0.797074
0.4	0.635842	0.646481	0.65855	0.672084	0.687117	0.703685	0.721822	0.741565	0.762947	0.786004
0.5	0.640707	0.649977	0.660628	0.672694	0.686209	0.701207	0.717723	0.73579	0.755442	0.776713
0.6	0.646691	0.654649	0.663941	0.674599	0.686658	0.70015	0.715108	0.731565	0.749554	0.769109
0.7	0.653721	0.660422	0.668412	0.677722	0.688385	0.700432	0.713895	0.728807	0.7452	0.763106
0.8	0.661723	0.667223	0.673967	0.681986	0.691311	0.701973	0.714004	0.727434	0.742295	0.758617
0.9	0.670626	0.67498	0.680534	0.687318	0.695363	0.704699	0.715356	0.727365	0.740757	0.755556
1	0.680364	0.683623	0.688041	0.693645	0.700466	0.708533	0.717876	0.728524	0.740507	0.753854

Table 5: NO_3 pollutant concentration at different x and t for $\alpha = 2$ and $v = 0.5$

The values of nitrate pollutant concentration at different values of x and t with linear input pollutant source are shown in table 5.

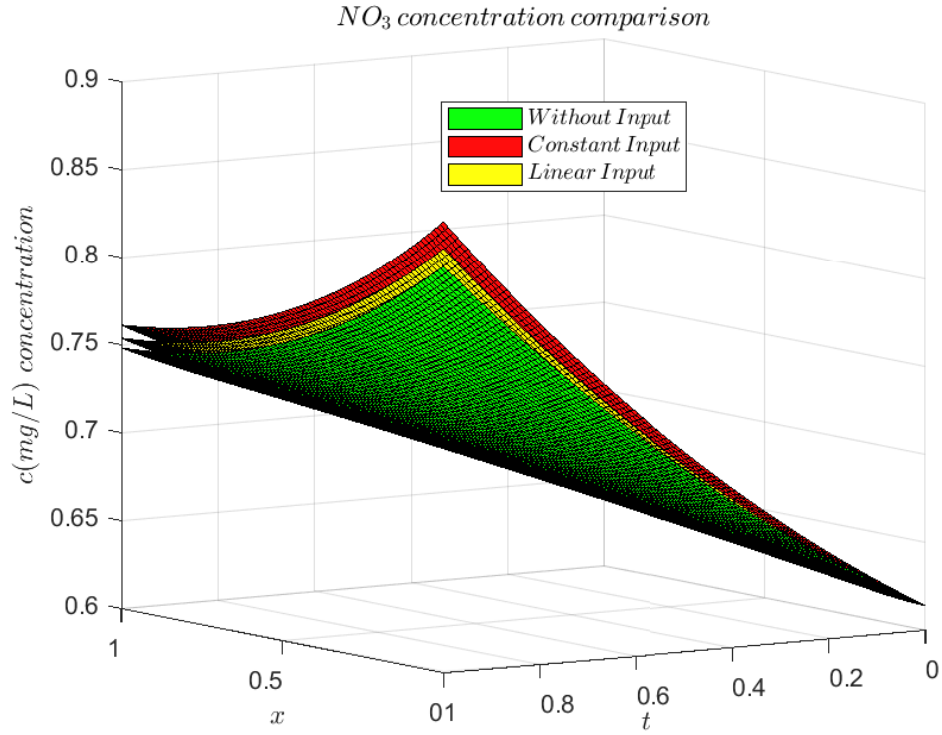


Figure 2: 3D Comparison of NO_3 pollutant concentration for $\alpha = 2, v = 0.5$

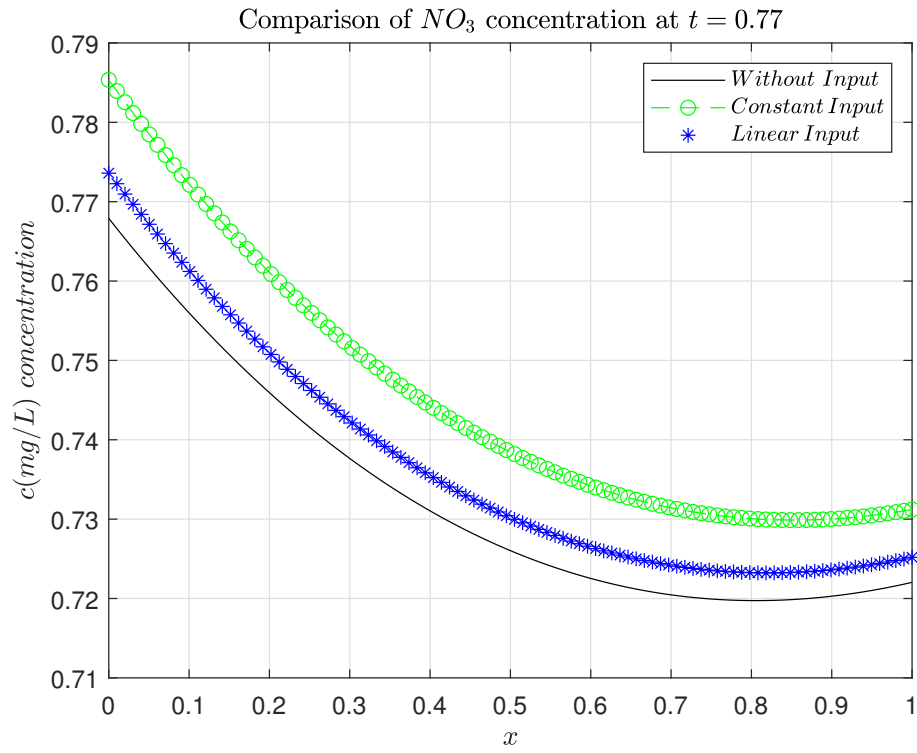


Figure 3: NO_3 pollutant concentration comparison for $t = 0.77, \alpha = 2, v = 0.5$

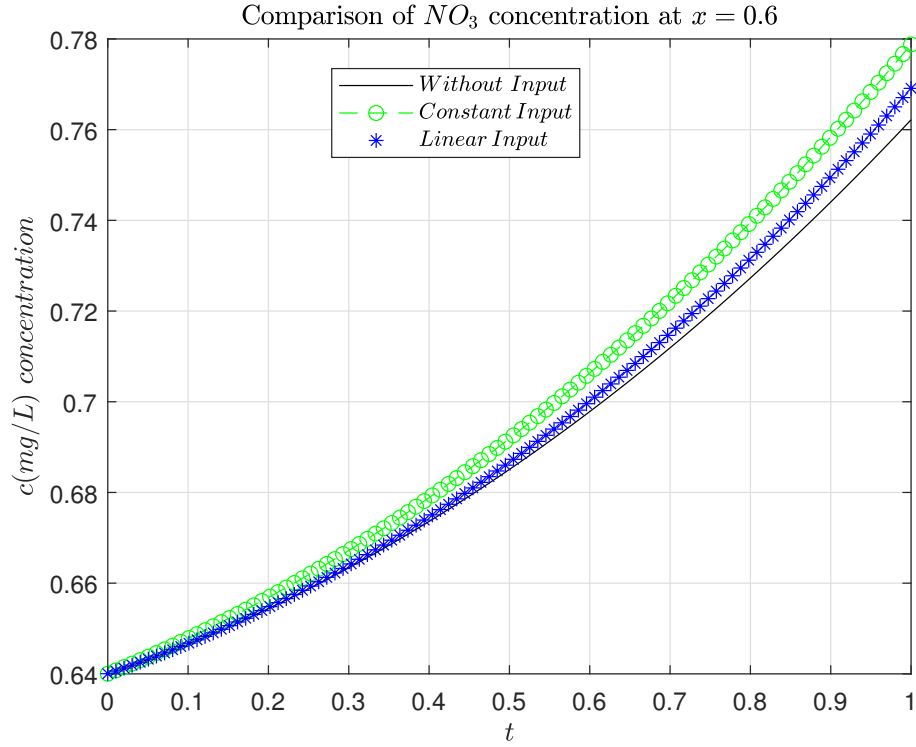


Figure 4: NO_3 pollutant concentration comparison for $x = 0.6, \alpha = 2, v = 0.5$

Figure 2 give the 3D graphical comparison of solution function obtained for all cases. Also figures 3 and 4 give the 2D graphical comparison of nitrate pollutant level obtained for all cases for fixed $t = 0.77$ and $x = 0.6$ respectively.

6.2 PO_4 pollutant

(a) Without pollutant source

In this case, $S(x, t) = 0$ in equation (4). Solving equation (4) by RDTM with initial function (6), we get

$$c(x, t) = N_0 + N_1t + N_2t^2 + N_3t^3 + \dots$$

Numerical solution of pollutant transport model with source term by RDTM

where

$$N_0 = -3.9488e - 05 x^5 + 0.0023 x^4 - 0.0428 x^3 + 0.3006 x^2 - 0.5051 x + 2.791$$

$$N_1 = 9.8720e - 05 x^4 - 0.0053 x^3 + 0.0912 x^2 - 0.5575 x + 0.8538$$

$$N_2 = -9.8720e - 05 x^3 + 0.0046 x^2 - 0.0615 x + 0.2306$$

$$N_3 = 4.9360e - 05 x^2 - 0.0017 x + 0.0133$$

Here,

$$\frac{\|N_1\|}{\|N_0\|} = 0.0063 < 1, \frac{\|N_2\|}{\|N_1\|} = 0.0025 < 1, \frac{\|N_3\|}{\|N_2\|} = 0.0013 < 1$$

Hence, the solution function $c(x, t)$ shows convergence toward the exact solution.

$x \backslash t$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	2.82561	2.912336	3.00371	3.099809	3.200714	3.306501	3.417251	3.533042	3.653952	3.78006
0.2	2.778455	2.859695	2.94546	3.035828	3.130876	3.230681	3.335323	3.444878	3.559424	3.679039
0.3	2.736988	2.812921	2.893257	2.978075	3.067451	3.161461	3.260183	3.363692	3.472066	3.58538
0.4	2.700962	2.771762	2.846848	2.926296	3.010181	3.098578	3.191564	3.289214	3.391604	3.498809
0.5	2.670135	2.735976	2.805986	2.880239	2.958811	3.041776	3.129208	3.221182	3.317773	3.419056
0.6	2.644271	2.705323	2.770428	2.83966	2.913094	2.990802	3.072859	3.159338	3.250312	3.345856
0.7	2.62314	2.679568	2.739936	2.804318	2.872785	2.94541	3.022267	3.103428	3.188965	3.278951
0.8	2.606515	2.658483	2.71428	2.773977	2.837647	2.90536	2.977189	3.053206	3.133481	3.218087
0.9	2.594176	2.641845	2.693232	2.748409	2.807447	2.870416	2.937386	3.008429	3.083615	3.163015
1	2.585908	2.629434	2.676571	2.727389	2.781957	2.840346	2.902624	2.968861	3.039128	3.113493

Table 6: PO_4 pollutant concentration at different x and t for $v = 0.5$

The values of phosphate pollutant concentration level at different values of x and t for this case are shown in table 6.

(b) Constant Input pollutant source

In this case, $S(x, t) = \frac{\alpha}{(x+L)^3}$, ($\alpha > 0$) in equation (4) then solving equation (4) by RDTM with initial function (6), we get

$$c(x, t) = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + \dots$$

Numerical solution of pollutant transport model with source term by RDTM

where

$$U_0 = -3.9488e - 05 x^5 + 0.0023 x^4 - 0.0428 x^3 + 0.3006 x^2 - 0.5051 x + 2.791$$

$$U_1 = \frac{2}{(x+5)^3} - 0.5575 x + 0.0912 x^2 - 0.0053 x^3 + 9.8720e - 05 x^4 + 0.8538$$

$$U_2 = \frac{3x+39}{2(x+5)^5} - 0.0615 x + 0.0046 x^2 - 9.8720e - 05 x^3 + 0.2306$$

$$U_3 = \frac{x^2+30x+245}{(x+5)^7} - 0.0017 x + 4.9360e - 05 x^2 + 0.0133$$

Here,

$$\frac{\|U_1\|}{\|U_0\|} = 0.0063 < 1, \frac{\|U_2\|}{\|U_1\|} = 0.0025 < 1, \frac{\|U_3\|}{\|U_2\|} = 0.0013 < 1$$

Hence, the solution function $c(x, t)$ converges to exact solution.

x\t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	2.827177	2.915601	3.00882	3.106928	3.210022	3.318195	3.431544	3.550164	3.674149	3.803596
0.2	2.779932	2.862768	2.950262	3.042507	3.139595	3.241618	3.348669	3.46084	3.578224	3.700913
0.3	2.738382	2.815815	2.897775	2.98435	3.075631	3.171706	3.272666	3.3786	3.489597	3.605748
0.4	2.702278	2.774493	2.851105	2.9322	3.017866	3.10819	3.203259	3.30316	3.407981	3.517809
0.5	2.671379	2.738555	2.81	2.885801	2.966041	3.050806	3.140181	3.234249	3.333098	3.43681
0.6	2.645449	2.707761	2.774219	2.844906	2.919904	2.999298	3.083169	3.1716	3.264674	3.362474
0.7	2.624255	2.681875	2.74352	2.809271	2.879209	2.953414	3.031968	3.114951	3.202445	3.294529
0.8	2.607573	2.660669	2.717672	2.77866	2.843713	2.91291	2.986329	3.06405	3.146151	3.232712
0.9	2.59518	2.643918	2.696446	2.752841	2.813182	2.877545	2.946008	3.018647	3.095541	3.176766
1	2.586862	2.631402	2.679619	2.731588	2.787385	2.847086	2.910767	2.978502	3.050367	3.126439

Table 7: PO₄ pollutant concentration at different x and t for $\alpha = 2$ and $v = 0.5$

Table 7 gives the values of phosphate pollutant concentration level at different values of x and t with constant input source.

(c) Linear Input pollutant source

In this case, $S(x, t) = \frac{\alpha t}{(x+L)^3}$, ($\alpha > 0$) in equation (4) then solving equation (4) by RDTM with initial function (6), we get

$$c(x, t) = Q_0 + Q_1 t + Q_2 t^2 + Q_3 t^3 + \dots$$

Numerical solution of pollutant transport model with source term by RDTM

where

$$Q_0 = -3.9488e - 05 x^5 + 0.0023 x^4 - 0.0428 x^3 + 0.3006 x^2 - 0.5051 x + 2.791$$

$$Q_1 = 9.8720e - 05 x^4 - 0.0053 x^3 + 0.0912 x^2 - 0.5575 x + 0.8538$$

$$Q_2 = \frac{1}{(x+5)^3} - 0.0615 x + 0.0046 x^2 - 9.8720e - 05 x^3 + 0.2306$$

$$Q_3 = \frac{x+13}{2(x+5)^5} - 0.0017 x + 4.9360e - 05 x^2 + 0.0133$$

Here,

$$\frac{\|Q_1\|}{\|Q_0\|} = 0.0063 < 1, \frac{\|Q_2\|}{\|Q_1\|} = 0.0025 < 1, \frac{\|Q_3\|}{\|Q_2\|} = 0.0013 < 1$$

Hence, the solution function $c(x, t)$ is convergent.

x\t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	2.825687	2.912653	3.004439	3.101137	3.202836	3.309625	3.421596	3.538839	3.661442	3.789497
0.2	2.778528	2.859994	2.946147	3.037077	3.13287	3.233617	3.339403	3.450319	3.56645	3.687887
0.3	2.737057	2.813202	2.893905	2.979252	3.069329	3.164223	3.264019	3.368805	3.478666	3.593688
0.4	2.701027	2.772028	2.847459	2.927405	3.011951	3.10118	3.195177	3.294026	3.397812	3.506619
0.5	2.670196	2.736227	2.806563	2.881287	2.960481	3.044229	3.132613	3.225716	3.32362	3.426408
0.6	2.64433	2.70556	2.770973	2.84065	2.914671	2.993119	3.076072	3.163614	3.255825	3.352785
0.7	2.623195	2.679793	2.740453	2.805254	2.874277	2.9476	3.025304	3.107467	3.194169	3.285489
0.8	2.606568	2.658697	2.71477	2.774865	2.83906	2.907433	2.980061	3.057024	3.138399	3.224263
0.9	2.594226	2.642047	2.693697	2.749251	2.808786	2.872378	2.940105	3.012043	3.088268	3.168856
1	2.585955	2.629626	2.677012	2.728188	2.783227	2.842207	2.905201	2.972285	3.043534	3.119023

Table 8: PO₄ pollutant concentration at different x and t for $\alpha = 2$ and $v = 0.5$

Table 8 gives the values of phosphate pollutant concentration level at different values of x and t with linear input pollutant source.

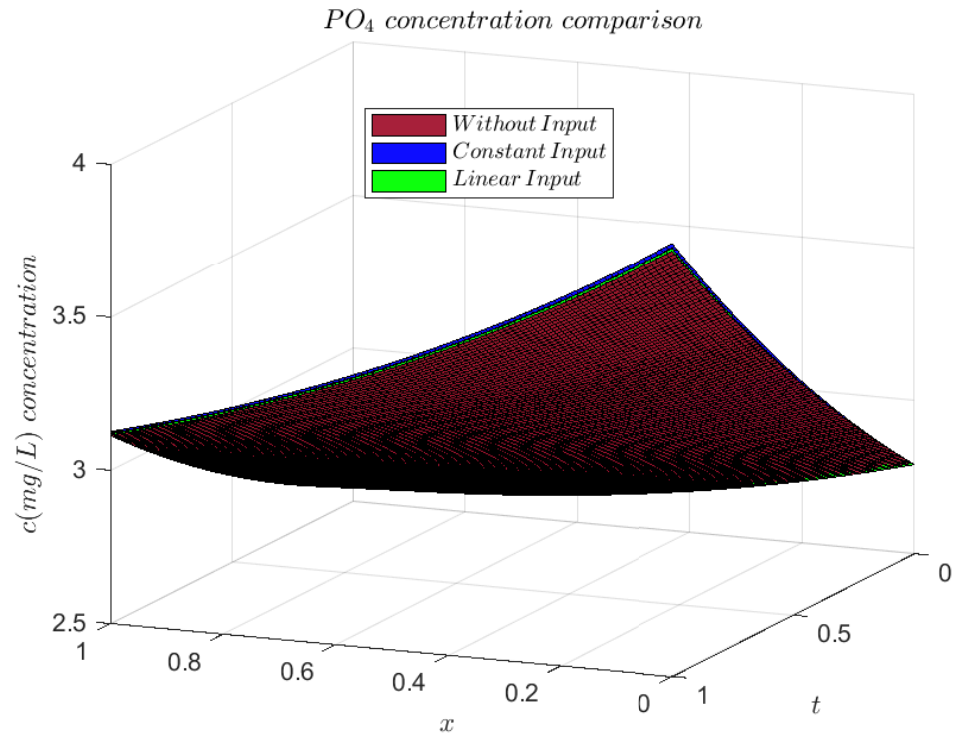


Figure 5: 3D Comparison of PO_4 pollutant concentration for $\alpha = 2, v = 0.5$

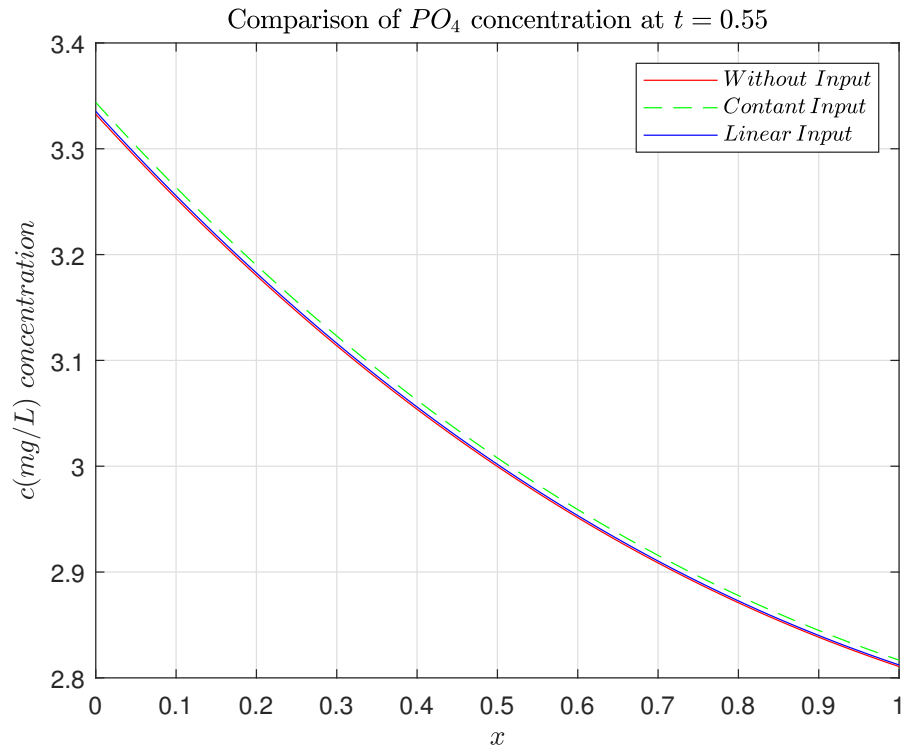


Figure 6: PO_4 pollutant concentration comparison for $t = 0.55, \alpha = 2, v = 0.5$

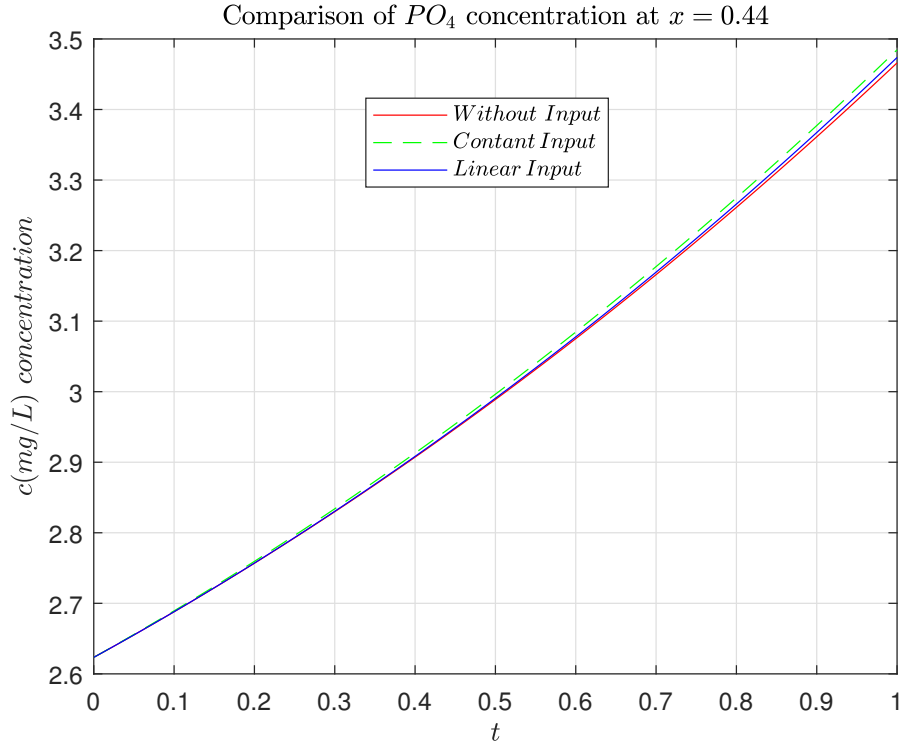


Figure 7: PO_4 pollutant concentration comparison for $x = 0.44, \alpha = 2, v = 0.5$

Figure 5 give the 3D graphical comparison of phosphate pollutant level obtained for all cases. Also figures 6 and 7 give the 2D graphical comparison of phosphate pollutant level obtained for all cases for fixed $t = 0.55$ and $x = 0.44$ respectively.

7 Conclusion

The analysis of the current problem has been conducted for two distinct cases: one in the absence of an input source and another with the presence of an input source. This research focuses on the identification and mitigation of two distinct types of input pollution sources: linear and constant. The objective is to propose effective solutions for addressing these specific forms of pollution in order to enhance the overall quality and reliability of the input data. By examining the characteristics and impacts of linear and constant input pollution, this study aims to contribute to the development of methods and approaches that can minimize the negative effects associated with these sources. Through our observations, it has been noted that the concentration value obtained from a constant input source

exceeds the concentration value acquired from a linear input source across various combinations of distance and time. In the context of concentration, it has been observed that the acquisition of a higher value is possible when the input source maintains linearity compared to situations where the input source is eliminated across various distances and time intervals. Based on our analysis, it can be inferred that the presence of an input source in the model leads to a higher level of pollutants compared to the model without an input source. In this study, the Reduced Differential Transform Method (RDTM) is employed to smoothly handle the problem of solving a one-dimensional advection-diffusion equation with a source term. The initial condition is derived from the collected data by fitting a polynomial function. The potential expansion of this approach encompasses the inclusion of advection-diffusion equations within the realms of both two and three-dimensional systems.

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On Nano Generalized α^{**} Continuous and Nano Generalized α^{**} Irresolute Mappings

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Abstract

The purpose of this study is to present nano generalized α^{**} -continuous functions and nano generalized α^{**} -irresolute mappings in Nano Topological Spaces. We introduce the terminologies for the sets, Nano Supremum and Nano Infimum. The fundamental properties of these functions with nano interior, nano closure, nano supremum, and nano infimum are investigated. The terms nano generalized α^{**} -open mapping, nano generalized α^{**} -closed mapping and nano generalized α^{**} -homeomorphism are also defined and their associations with other continuous functions are investigated.

Keywords: Nano Supremum, Nano Infimum, Nga^{**} -continuous, Nga^{**} -irresolute, Nga^{**} -homeomorphism.

2010 Mathematics Subject Classification: 54A05; 54B05*

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1. Introduction

Levine [6] first proposed the idea of generalized closed sets in topological spaces. The concept of rough sets and rough set approximations was introduced by Zdzislaw Pawlak[7], which incorporates the properties of sets. Lellis Thivagar M and Carmel Richard[5] introduced the concepts of nano topological spaces using rough set approximations and defined nano continuous functions, nano open mappings, nano closed mappings and nano homeomorphism. Bhuvaneswari K and Mythili Gnanapriya K[3] introduced the concepts of nano generalized continuous functions in nano topological spaces. The definition of nano generalized α^{**} closed sets in nano topological spaces was made by Kalarani M et al. [1].

This research aims to extend the study of nano generalized α^{**} closed and nano generalized α^{**} open sets using properties of sets and to define the new sets called Nano infimum and Nano supremum of the subsets in nano topological spaces. These sets are based on the set inclusion principles and are defined using the concepts of upper bound and lower bound. The Nano Hasse diagram for the nano generalized α^{**} closed sets and the nano generalized α^{**} open sets have been constructed by utilizing the set inclusion principles and examining the distinctive features of the new sets. The definition and study of a new class of continuous function called nano generalized α^{**} - continuous function in continuation aim to investigate the behaviour of the new sets under the function. Bhuvaneswari K and Ezhilarasi K[2] introduced nano semi generalized irresolute maps in Nano topological spaces. The concepts of nano generalized α^{**} - irresolute mappings are introduced and investigated their relationship with other nano continuous functions. Further, nano generalized α^{**} - open mappings, nano generalized α^{**} - closed mappings and nano generalized α^{**} - homeomorphism are presented and analyzed for their characteristics.

2. Preliminaries

Definition 2.1[5]. Given U is a nonempty finite set of objects and R is an equivalence relation on U called the indiscernibility relation. The U is divided into disjoint equivalence classes. The elements of the same equivalence class are said to be indiscernible with each other. The pair (U, R) is called an approximation space. Let $X \subseteq U$, then

- (i) The lower approximation of X with respect to R is $L_R(X) = \bigcup_{x \in X} \{R_X(x) / R_X(x) \subseteq X\}$, where $R_X(x)$ is the equivalence class determined by $L_R(X)$.
- (ii) The upper approximation of X with respect to R is $U_R(X) = \bigcup_{x \in X} \{R_X(x) / R_X(x) \cap X \neq \emptyset\}$.
- (iii) The boundary region of X is $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2[5]. Let U be the universe, R is an equivalence relation on U and $X \subseteq U$, $\tau_R(X) = \{\emptyset, U, L_R(X), U_R(X), B_R(X)\}$. Then $\tau_R(A)$ satisfies the axioms

(i) U and ϕ are in $\tau_R(X)$.

(ii) The arbitrary union of the elements of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The finite intersection of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

Here $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called a nano topological space. The elements of $\tau_R(X)$ are called the nano open sets and the complement of the nano open sets are called nano closed sets.

Definition 2.3[5]. If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

(i) The union of all nano open subsets contained in A is called the nano interior of A , denoted by $Nint(A)$. The $Nint(A)$ is the largest nano open subset of A .

(ii) The intersection of all nano closed sets containing A is called the nano closure of A denoted by $Ncl(A)$. The $Ncl(A)$ is the smallest nano closed set containing A .

Definition 2.4[1]. A subset A of a nano topological space $(U, \tau_R(X))$ is called a nano generalized α^{**} -closed set ($Ng\alpha^{**}$ -closed set) if $Ncl(A) \subseteq G$ whenever $A \subseteq G$ and G is $Ng\alpha^{**}$ -set. The complement of $Ng\alpha^{**}$ -closed sets are called $Ng\alpha^{**}$ -open sets.

Definition 2.5[4]. Let $(U, \tau_R(X))$ and $(V, \sigma_{R'}(Y))$ be two nano topological spaces. The function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is called nano continuous function on U , if the inverse image of every nano open set in V is nano open in U .

Definition 2.6[4]. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is nano open (nano closed) mapping if the image of every nano open (nano closed) set in $(U, \tau_R(X))$ is nano open (nano closed) set in $(V, \sigma_{R'}(Y))$.

Definition 2.7[3]. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is nano g -open (nano g -closed) mapping if the image of every nano g -open (nano g -closed) set in $(U, \tau_R(X))$ is nano g -open (Ng -closed) set in $(V, \sigma_{R'}(Y))$.

Definition 2.8[3]. A bijective function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is nano homeomorphism if f is both nano continuous and nano open.

Definition 2.9[3]. A bijective function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is nano g -homeomorphism if f is both nano g -continuous and nano g -open.

Definition 2.10[8]. A relation R on a set A is called a partially ordered set or poset if the relation R is reflexive, antisymmetric and transitive.

Definition 2.11[8]. A simplified form of the digraph of a partial ordering relation on a finite set that contains the information of the relation is called a Hasse diagram.

Definition 2.12[8]. Let A be subset of a partially ordered set $(P, \leq)$ and if u is an element of P such that $a \leq u$ for all elements $a \in A$, then u is called an upper bound of A .

Definition 2.13[8]. Let A be subset of a partially ordered set $(P, \leq)$ and if l is an element of P such that $l \leq a$ for all elements $a \in A$, then l is called the lower bound of A .

Definition 2.14[8]. The element x is called the least upper bound (LUB) or Supremum of the subset A of a poset $(P, \leq)$ if x is an upper bound that is less than every other upper bound of A .

Definition 2.15[8]. The element y is called the greatest lower bound (GLB) or Infimum of the subset A of a poset $(P, \leq)$ if y is a lower bound that is greater than every other lower bound of A .

Definition 2.16[4]. Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then the set $Nker(A) = \bigcap \{B/A \subseteq B, B \in \tau_R(X)\}$ is called the nano kernel of A .

3. Nano Infimum and Nano Supremum of subsets

This section defines the nano supremum and nano infimum of a subset in nano topological space and examines its behaviour with nano interior and nano closure. Additionally, the discussions cover the characteristics of nano supremum and nano infimum under nano generalized α^{**} -continuous functions.

Definition 3.1. Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The nano hasse diagram of the nano topological space is a digraph that contains the relation between its nano subsets.

Definition 3.2. Let $A \subseteq U$ be a subset of the nano topological space $(U, \tau_R(X))$. The nano generalized α^{**} - exterior of A or nano infimum of A is defined as the union of all Nga^{**} - closed sets of U contained in A and is denoted by $Nga^{**}-ext(A)$ or $Ninf(A)$.

$$Nga^{**}-ext(A) \text{ or } Ninf(A) = \bigcup \{Q/Q \subseteq A, Q \text{ is } Nga^{**}\text{-closed}\}.$$

Definition 3.3. Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The nano generalized α^{**} - kernel of A or nano supremum of A is defined as the intersection of all Nga^{**} - open sets of U which contains A and is denoted by $Nga^{**}-ker(A)$ or $Nsup(A)$.

$$Nga^{**}-ker(A) \text{ or } Nsup(A) = \bigcap \{P/A \subseteq P, P \text{ is } Nga^{**}\text{-open}\}.$$

Example 3.4. Consider $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. The nano topology is $\tau_R(X) = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the complement is $\tau_R^c(X) = \{\phi, U, \{c\}, \{a, c\}, \{b, c, d\}\}$. The nano hasse diagram of Nga^{**} - closed and Nga^{**} -open sets of the above nano topology are given below.

$Ng\alpha^{**}$ -closed sets = $\{\phi, U, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and
 $Ng\alpha^{**}$ -open sets = $\{\phi, U, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$

Nano Hasse Diagram of of $Ng\alpha^{**}$ - closed and $Ng\alpha^{**}$ -open sets

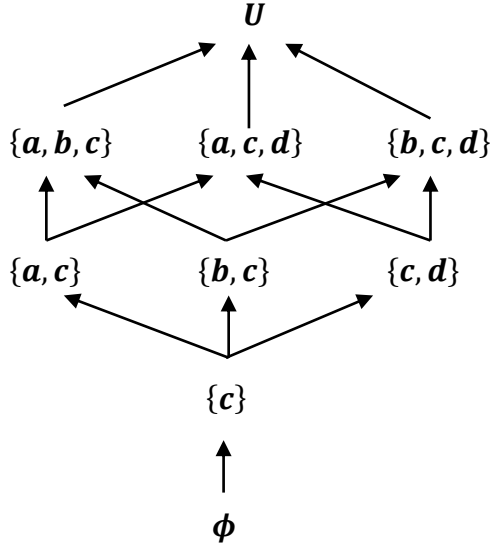


Figure.3.1

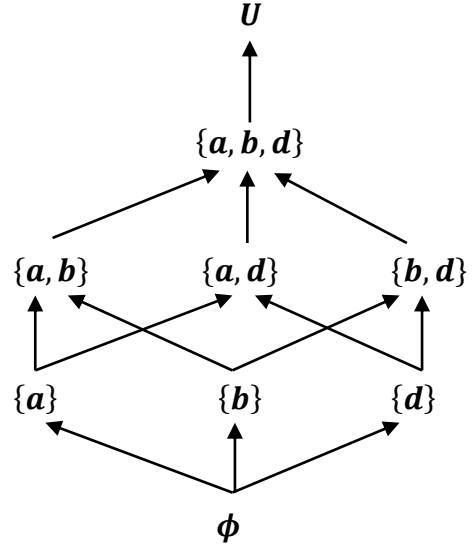


Figure. 3.2

In figure 3.1,

$$Ninf(\{\phi\}) = \phi$$

$$Ninf(\{a\}) = Ninf(\{b\}) = Ninf(\{d\}) = \phi, \quad Ninf(\{c\}) = \{c\}$$

$$Ninf(\{a, c\}) = \phi \cup \{c\} \cup \{a, c\} = \{a, c\}$$

$$Ninf(\{a, b, c\}) = \phi \cup \{c\} \cup \{a, c\} \cup \{b, c\} \cup \{a, b, c\} = \{a, b, c\}$$

And from figure 3.2,

$$Nsup(\{a\}) = \{a\} \cap \{a, b\} \cap \{a, d\} \cap \{a, b, d\} \cap U = \{a\}$$

$$Nsup(\{b\}) = \{b\} \cap \{a, b\} \cap \{b, d\} \cap \{a, b, d\} \cap U = \{b\}$$

$$Nsup(\{c\}) = U$$

$$Nsup(\{a, b\}) = \{a, b\} \cap \{a, b, d\} \cap U = \{a, b\}$$

Corollary 3.5. The nano infimum of a set A is the largest $Ng\alpha^{**}$ - closed subset of A or the greatest lower bound of A and the nano supremum of a set A is the smallest $Ng\alpha^{**}$ - open subset of A or the least upper bound of A.

Remark 3.6. The subsets which are not in the hasse diagram have the nano infimum (GLB) as ϕ and the nano supremum (LUB) as U.

Definition 3.7. If A and B are any two subsets of the nano topological space $(U, \tau_R(X))$. The nano infimum of A and B is defined as the greatest lower bound of A and B. The nano infimum is the largest Nga^{**} - closed subset of A and B.

$$Ninf(A, B) = GLB(A, B) = \text{Largest } Nga^{**}\text{- closed subset of A and B}$$

Definition 3.8. If A and B are any two subsets of the nano topological space $(U, \tau_R(X))$. The nano supremum of A and B is defined as the least upper bound of A and B. The nano supremum is the smallest Nga^{**} - open set that contains both A and B.

$$Nsup(A, B) = LUB(A, B) \\ = \text{Smallest } Nga^{**}\text{- open set that contains both A and B}$$

Example 3.9. Assume $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{b, d\}$. The nano topology is $\tau_R(X) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the complement is $\tau_R^c(X) = \{\phi, U, \{a\}, \{a, b\}, \{a, c, d\}\}$. The nano hasse diagram for the nano topology $(U, \tau_R(X))$ is given below.

Nga^{**} - closed sets = $\{\phi, U, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ and

Nga^{**} - open sets = $\{\phi, U, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$

The Nga^{**} -closed sets are highlighted in the nano hasse diagram.

Nano Hasse Diagram of Example.3.9

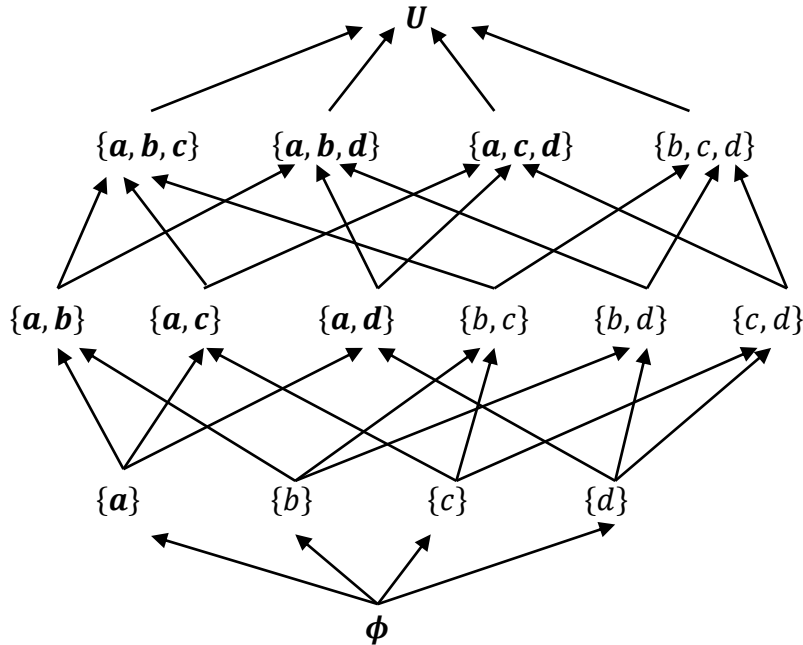


Figure.3.3

From figure.3.3,

$$Ninf(\{a, c\}, \{a, b, d\}) = GLB(\{a, c\}, \{a, b, d\}) = \phi$$

$$Ninf(\{a, c\}, \{b, c\}) = GLB(\{a, c\}, \{b, c\}) = \{c\}$$

$$Ninf(\{a, c\}, \{a, c, d\}) = GLB(\{a, c\}, \{a, c, d\}) = \{a, c\}$$

$$Ninf(\{a, c, d\}, \{b, c, d\}) = GLB(\{a, c, d\}, \{b, c, d\}) = \{c, d\}$$

$$Nsup(\{a\}, \{a, d\}) = LUB(\{a\}, \{a, d\}) = \{a, d\}$$

$$Nsup(\{b\}, \{a, b, c\}) = LUB(\{b\}, \{a, b, c\}) = \{a, b, c\}$$

$$Nsup(\{a\}, \{b, d\}) = LUB(\{a\}, \{b, d\}) = \{a, b, d\}$$

Theorem 3.10: Let P and Q be any two subsets of the nano topological space $(U, \tau_R(X))$. Then

(i). $Nsup(\phi) = \phi$, $Nsup(U) = U$.

(ii). $P \subseteq Nsup(P)$

(iii). If Q is a Nga^{**} - open subset of P , then $Q \subseteq Nsup(P)$

(iv). If $P \subseteq Q$, then $Nsup(P) \subseteq Nsup(Q)$

(v). $Nsup(Nsup(P)) = Nsup(P)$

Proof: (i). Assume P and Q are any two subsets of the nano topological space $(U, \tau_R(X))$.

(i) The sets ϕ and U are Nga^{**} - open sets. $Nsup(A)$ is the intersection of all Nga^{**} - open sets which contains A . So $Nsup(\phi) = \cap\{\phi\} = \phi$ and $Nsup(U) = \cap\{all\ Nga^{**}\ open\ sets\ which\ contains\ U\} = \cap\{U\} = U$

(ii) Let $P \subseteq U$, then $Nsup(P)$ is the smallest Nga^{**} - open set that contains P . Hence $P \subseteq Nsup(P)$.

(iii) Assume Q is a Nga^{**} - open subset and P is any subset with $Q \subseteq P$, then by (ii), $P \subseteq Nsup(P)$. But $Q \subseteq P \subseteq Nsup(P)$ which implies $Q \subseteq Nsup(P)$.

(iv) If $P \subseteq Q$. Let $x \in P$, then $x \in Nsup(P)$ by (ii) $P \subseteq Nsup(P)$ and $x \in P$ implies $x \in Q$. so $x \in Nsup(Q)$. Hence $Nsup(P) \subseteq Nsup(Q)$.

(v) Let $P \subseteq U$, then $Nsup(P) = \cap\{Nga^{**}\text{- open sets that contains } P\}$

$$Nsup(Nsup(P)) = Nsup\{\text{Intersection of } Nga^{**}\text{- open sets that contains } P\}$$

$$= \text{Intersection } \{Nga^{**}\text{- open sets that contains } P\} = Nsup(P)$$

$$Nsup(Nsup(P)) = Nsup(P)$$

Theorem 3.11: Let R and S be any two subsets of the nano topological space $(U, \tau_R(X))$. Then

(i). $Ninf(\phi) = \phi$, $Ninf(U) = U$.

(ii). $Ninf(R) \subseteq R$

(iii). If S is any subset of a Nga^{**} - closed set R , then $S \subseteq Ninf(R)$

(iv). If $R \subseteq S$, then $Ninf(R) \subseteq Ninf(S)$

(v). $Ninf(Ninf(R)) = Ninf(R)$

Proof: Similar to theorem.3.10

Theorem 3.12: Let P be a subset of a nano topological space $(U, \tau_R(X))$, then

(i). $Nint(P) \subseteq Nga^{**}int(P) \subseteq P \subseteq Nsup(P)$ for every subset P of U .

(ii). $Nint(P) = Nga^{**}int(P) = P = Nsup(P)$, if P is nano open in U .

Proof: (i) Let $x \in Nint(P)$, then x is in a nano open set in U . Every nano open set is a Nga^{**} -open set. $\Rightarrow x \in Nga^{**}int(P)$. $Nga^{**}int(P)$ is the largest Nga^{**} - open subset contained in P , $\Rightarrow Nga^{**}int(P) \subseteq P \Rightarrow Nint(P) \subseteq Nga^{**}int(P) \subseteq P$ ----(1)

If $x \in Nga^{**}int(P)$, then x is a Nga^{**} - open set of $U \Rightarrow x$ is in the intersection of Nga^{**} - open sets and intersection of any two Nga^{**} - open sets is again a Nga^{**} - open set, $\Rightarrow x \in Nsup(P) \Rightarrow Nga^{**}int(P) \subseteq Nsup(P)$ ----(2)

From (1) and (2), $Nint(A) \subseteq Nga^{**}int(A) \subseteq P \subseteq Nsup(P)$

(ii) Assume P is a nano open set and $x \in Nsup(P)$, then $x \in \cap\{B / P \subseteq B, B \text{ is } Nga^{**} - \text{open}\}$. Every nano open set is a Nga^{**} - open, $x \in \cap\{B / P \subseteq B, B \text{ is Nano open}\}$,

$\Rightarrow x \in Nint(P) \Rightarrow Nsup(P) \subseteq Nint(P)$. From (i), $Nint(A) \subseteq Nsup(P)$. Hence, $Nint(A) = Nsup(P)$ and $Nint(A) = Nga^{**}int(A) = P = Nsup(P)$ for every nano open set P of U .

Theorem 3.13: If P is a subset of a nano topological space $(U, \tau_R(X))$, then

(i). $Ninf(P) \subseteq P \subseteq Nga^{**}cl(P) \subseteq Ncl(P)$ for every subset P of U .

(ii). $Ninf(P) = P = Nga^{**}cl(P) = Ncl(P)$ if P is a nano closed set of U .

Proof: (i). Assume P is a subset of U , then $Ninf(P)$ is the largest Nga^{**} - closed subset of P , $Ninf(P)$ is a subset of $P \Rightarrow Ninf(P) \subseteq P$ ----(1)

$Nga^{**}cl(P)$ is the smallest Nga^{**} - closed set that contains P

$\Rightarrow P \subseteq Nga^{**}cl(P)$ ----(2)

and $Ncl(P)$ is the smallest nano closed set that contains $P \Rightarrow P \subseteq Ncl(P)$ ----(3)

From (1) and (2), $\Rightarrow Ninf(P) \subseteq P \subseteq Nga^{**}cl(P)$ ----(4)

Every nano closed set is a Nga^{**} - closed set and $Nga^{**}cl(P) \subseteq Ncl(P)$ ----(5)

From (4) and (5), $Ninf(P) \subseteq P \subseteq Nga^{**}cl(P) \subseteq Ncl(P)$

Hence the proof.

(ii). Assume P is a nano closed set. Every nano closed set is a Nga^{**} - closed set. Then $Ncl(P) = P = Nga^{**}cl(P)$. Let $x \in Ncl(P)$. Then $x \in Nga^{**}cl(P) \Rightarrow x \in \cup\{Nga^{**} \text{ closed sets of } P\} \Rightarrow x \in \cup\{Nga^{**} \text{ closed sets of } P \text{ containing } P\} \Rightarrow x \in Ninf(P)$

$\Rightarrow Ncl(P) \subseteq Ninf(P)$. But from (i), $Ninf(P) \subseteq Ncl(P) \Rightarrow Ncl(P) = Ninf(P)$
Hence, $Ninf(P) = P = Ng\alpha^{**}cl(P) = Ncl(P)$

4. Nano generalized α^{**} - continuous functions in Nano Topological Spaces

A new class functions called nano generalized α^{**} - continuous function is defined and its properties with nano interior, nano closure, nano infimum, nano supremum and other nano continuous functions are studied in this section.

Definition 4.1. Let $(U, \tau_R(X))$ and $(V, \sigma_{R'}(Y))$ be nano topological spaces, The mapping $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is said to be nano generalized α^{**} -continuous ($Ng\alpha^{**}$ -continuous) function on U if the inverse image of every nano open set in V is nano generalized α^{**} - open set in U.

Example 4.2: Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ is a nano topology with respect to X and the complement is $\tau_R^c(X) = \{\phi, U, \{c\}, \{a, c\}, \{b, c, d\}\}$. Let $V = \{x, y, z, w\}$ with $V/R' = \{\{x\}, \{w\}, \{y, z\}\}$ and $Y = \{x, z\}$. Then $\sigma_{R'}(Y) = \{\phi, V, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $\sigma_{R'}^c(Y) = \{\phi, V, \{w\}, \{x, w\}, \{y, z, w\}\}$. Define $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ as $f(a) = y, f(b) = x, f(c) = w, f(d) = z$. Then $f^{-1}(\{x\}) = \{b\}, f^{-1}(\{y, z\}) = \{a, d\}, f^{-1}(\{x, y, z\}) = \{b, a, d\}$. The inverse image of every nano open set in V is $Ng\alpha^{**}$ -open in U. Hence f is a $Ng\alpha^{**}$ -continuous function.

Theorem 4.3: A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is $Ng\alpha^{**}$ -continuous if and only if the inverse image of every nano closed set in V is $Ng\alpha^{**}$ -closed in U.

Proof: Assume f is a $Ng\alpha^{**}$ -continuous and $Q \subseteq V$ is a nano closed set, then $V - Q$ is nano open in V. Here f is $Ng\alpha^{**}$ -continuous, then $f^{-1}(V - Q)$ is $Ng\alpha^{**}$ -open in U. $f^{-1}(V - Q) = f^{-1}(V) - f^{-1}(Q) = U - f^{-1}(Q)$ is $Ng\alpha^{**}$ -open in U. So, $f^{-1}(Q)$ is $Ng\alpha^{**}$ -closed in U. The inverse image of every nano closed set in V is $Ng\alpha^{**}$ -closed in U.

Conversely, assume the inverse image of every nano closed set in V is $Ng\alpha^{**}$ -closed in U. Let P be a nano closed set in V. Then $V - P$ is nano open in V $\Rightarrow$ by assumption, $f^{-1}(V - P)$ is $Ng\alpha^{**}$ -open in U $\Rightarrow f^{-1}(V - P) = U - f^{-1}(P)$ is $Ng\alpha^{**}$ -open in U $\Rightarrow f^{-1}(P)$ is $Ng\alpha^{**}$ -closed in U. So if P is a nano closed set in V, then $f^{-1}(P)$ is $Ng\alpha^{**}$ -closed in U. Hence f is $Ng\alpha^{**}$ -continuous.

Theorem 4.4: A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is nano continuous if and only if f is $Ng\alpha^{**}$ -continuous.

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a nano continuous function and let B be a nano closed set in V . Then $V - B$ is nano open in $V \Rightarrow$ by assumption, $f^{-1}(V - B)$ is nano open in U . $f^{-1}(V - B) = f^{-1}(V) - f^{-1}(B) = U - f^{-1}(B)$ is nano open in $U \Rightarrow f^{-1}(B)$ is nano closed in U . We know every nano open set in V is a $Ng\alpha^{**}$ -open set in U . $f^{-1}(B)$ is $Ng\alpha^{**}$ -closed in U . Hence f is $Ng\alpha^{**}$ -continuous.

Conversely, f is $Ng\alpha^{**}$ -continuous, then every nano closed in V is $Ng\alpha^{**}$ -closed in U . It contains all the closed sets of U . So, the image of a nano closed set in $(V, \sigma_{R'}(Y))$ is a nano closed set in $(U, \tau_R(X))$. So f is a nano continuous function.

Example 4.5: Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $\tau_R^c(X) = \{\phi, U, \{c\}, \{a, c\}, \{b, c, d\}\}$.

Let $V = \{x, y, z, w\}$ with $V/R' = \{\{x, w\}, \{y\}, \{z\}\}$ and $Y = \{x, z\}$.

Then $\sigma_{R'}(Y) = \{\phi, V, \{z\}, \{x, w\}, \{x, z, w\}\}$ and $\sigma_{R'}^c(Y) = \{\phi, V, \{y\}, \{y, z\}, \{x, y, w\}\}$.

Define $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ as $f(a) = z, f(b) = x, f(c) = y, f(d) = w$. Then $f^{-1}(\{z\}) = \{a\}, f^{-1}(\{x, w\}) = \{b, d\}, f^{-1}(\{x, z, w\}) = \{a, b, d\}$. The inverse image of every nano open set in V is $Ng\alpha^{**}$ -open in U . Here f is $Ng\alpha^{**}$ -continuous function. Also $f^{-1}(\{y\}) = \{c\}, f^{-1}(\{y, z\}) = \{a, c\}, f^{-1}(\{x, y, w\}) = \{b, c, d\}$ which are all nano closed sets in U . Hence f is a nano continuous function.

Theorem 4.6: Let $(U, \tau_R(X))$ and $(V, \sigma_{R'}(Y))$ are two nano topological spaces and let $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ be a mapping. Then the following statements are equivalent.

- (i) f is $Ng\alpha^{**}$ -continuous
- (ii) $f(Ng\alpha^{**}cl(P)) \subseteq Ncl(f(P))$ for every subset P of U
- (iii) $Ng\alpha^{**}cl(f^{-1}(Q)) \subseteq f^{-1}(Ncl(Q))$ for every subset Q of V

Proof:

(i) $\Rightarrow$ (ii)

Assume f is $Ng\alpha^{**}$ -continuous and let $P \subseteq U$, then $Ncl(f(P))$ is nano closed in V . Now $f^{-1}(Ncl(f(P)))$ is $Ng\alpha^{**}$ -closed set containing P in U . Every nano closed set A is $Ng\alpha^{**}$ -closed set in U and which is the smallest set containing A . $f(P) \subseteq Ncl(f(P)) \Rightarrow f^{-1}(f(P)) \subseteq f^{-1}(Ncl(f(P))) \Rightarrow P \subseteq f^{-1}(Ncl(f(P)))$,

then $Ng\alpha^{**}cl(P) \subseteq Ng\alpha^{**}cl(f^{-1}(Ncl(f(P)))) = f^{-1}(Ncl(f(P)))$

$\Rightarrow f(Ng\alpha^{**}cl(P)) \subseteq Ncl(f(P))$ for every subset P of U .

(ii) $\Rightarrow$ (i)

Assume $f(Ng\alpha^{**}cl(P)) \subseteq Ncl(f(P))$. Let Q be a nano closed set in V , then $f^{-1}(Q)$ is nano closed in U , $f(Ng\alpha^{**}cl(Q)) \subseteq Ncl(f(f^{-1}(Q))) = Ncl(Q) = Q$
 $\Rightarrow f(Ng\alpha^{**}cl(Q)) \subseteq Q \Rightarrow Ng\alpha^{**}cl(Q) \subseteq f^{-1}(Q)$. But $f^{-1}(Q) \subseteq Ng\alpha^{**}cl(Q)$
 $\Rightarrow Ng\alpha^{**}cl(Q) = f^{-1}(Q)$. Hence $f^{-1}(Q)$ is $Ng\alpha^{**}$ - closed in U . Therefore, f is $Ng\alpha^{**}$ - continuous function.

(ii) $\Rightarrow$ (iii)

Assume $f(Ng\alpha^{**}cl(C)) \subseteq Ncl(f(C))$. Let Q be a nano closed set in V , then $f^{-1}(Q)$ is nano closed, $f(Ng\alpha^{**}cl(f^{-1}(Q))) \subseteq Ncl(f(f^{-1}(Q))) \subseteq Ncl(Q)$
 $\Rightarrow Ng\alpha^{**}cl(f^{-1}(Q)) \subseteq f^{-1}(Ncl(Q))$

(iii) $\Rightarrow$ (i)

Assume $Ng\alpha^{**}cl(f^{-1}(Q)) \subseteq f^{-1}(Ncl(Q))$ for any subset Q of V .

If Q is nano closed set in V , then $Ncl(Q) = Q$. From the assumption, $Ng\alpha^{**}cl(f^{-1}(Q)) \subseteq f^{-1}(Q)$. But for a nano closed set Q in V , $f^{-1}(Q) \subseteq Ng\alpha^{**}cl(f^{-1}(Q)) \Rightarrow Ng\alpha^{**}cl(f^{-1}(Q)) = f^{-1}(Q)$, a nano closed set $f^{-1}(Q)$ in U is a $Ng\alpha^{**}$ - closed in U . That is the inverse image of a nano closed set is a $Ng\alpha^{**}$ - closed set in U . So, f is $Ng\alpha^{**}$ - continuous.

Theorem 4.7: Let $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ be a function. Then (i) f is $Ng\alpha^{**}$ - continuous if and only if

(ii) $Nint(f(P)) \subseteq f(Ng\alpha^{**}int(P))$ for every subset P of U

(iii) $f^{-1}(Nint(Q)) \subseteq Ng\alpha^{**}int(f^{-1}(Q))$ for every subset Q of V

Proof:

(i) $\Rightarrow$ (ii)

Assume f is $Ng\alpha^{**}$ continuous and let $P \subseteq U$, then $Nint(f(P))$ is a nano open set in V . By the definition of f , $f^{-1}(Nint(f(P)))$ is $Ng\alpha^{**}$ - open in U . We know, $Nint(A) \subseteq A \Rightarrow f^{-1}(Nint(f(P))) \subseteq f^{-1}(f(P)) \subseteq P$, but $P \subseteq Ng\alpha^{**}int(P)$.

Hence $f^{-1}(Nint(f(P))) \subseteq Ng\alpha^{**}int(P) \Rightarrow Nint(f(P)) \subseteq f(Ng\alpha^{**}int(P))$.

(ii) $\Rightarrow$ (i)

Conversely, let $Nint(f(C)) \subseteq f(Ng\alpha^{**}int(C))$ for every subset C in U . Let D be a nano open set in V , then $Nint(D) = D \Rightarrow Nint(f(D)) = f(D)$. Using this in the assumption, $f(D) \subseteq f(Ng\alpha^{**}int(D)) \Rightarrow f^{-1}(f(D)) \subseteq Ng\alpha^{**}int(D)$
 $\Rightarrow D \subseteq Ng\alpha^{**}int(D)$

But for any open set D in V , $Ng\alpha^{**}int(D) \subseteq D$. Hence, $Ng\alpha^{**}int(D) = D$. Here D is $Ng\alpha^{**}$ - open in U , $\Rightarrow f$ is $Ng\alpha^{**}$ continuous.

(i) $\Rightarrow$ (iii)

Let f is a $Ng\alpha^{**}$ continuous function and let $Q \subseteq V$, then $Nint(Q)$ is nano open set in V and $Nint(Q) \subseteq Q$. Then $f^{-1}(Nint(Q))$ is $Ng\alpha^{**}$ open in U .

$Ng\alpha^{**}int(f^{-1}(Nint(Q))) = f^{-1}(Nint(Q))$, also $f^{-1}(Nint(Q)) \subseteq f^{-1}(Q)$

Hence, $Ng\alpha^{**}int(f^{-1}(Nint(Q))) \subseteq f^{-1}(Q) \Rightarrow f^{-1}(Nint(Q)) \subseteq Ng\alpha^{**}int(f^{-1}(Q))$.

(iii) $\Rightarrow$ (i)

Conversely, let us take F is a nano open set in V , then $Nint(F) = F$.

Using this in the assumption, $f^{-1}(Nint(F)) \subseteq Ng\alpha^{**}int(f^{-1}(F))$,

$$f^{-1}(F) \subseteq Ng\alpha^{**}int(f^{-1}(F)).$$

But for any open set F in V , $Ng\alpha^{**}int(f^{-1}(F)) \subseteq f^{-1}(F)$.

So, $Ng\alpha^{**}int(f^{-1}(F)) = f^{-1}(F)$. Hence, $f^{-1}(F)$ is $Ng\alpha^{**}$ - open in U , for any open set F in V , $f^{-1}(F)$ is $Ng\alpha^{**}$ - open in $U \Rightarrow f$ is $Ng\alpha^{**}$ continuous.

Theorem 4.8: If $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - continuous function, then

(i) $Nsup(f(P)) \subseteq f(Nsup(P))$

(ii) $Ninf(f(P)) \subseteq f(Ninf(P))$

Proof:

(i) Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - continuous function. Let $P \subseteq U$, then $f(P) \subseteq V$ and let $x \in Nsup(f(P))$, x is a $Ng\alpha^{**}$ -open set in V . For $P \subseteq U$, $Nsup(P)$ is the smallest $Ng\alpha^{**}$ -open set that contains P in $U \Rightarrow f(Nsup(P))$ is a $Ng\alpha^{**}$ -open set in $V \Rightarrow x \in f(Nsup(P)) \Rightarrow Nsup(f(P)) \subseteq f(Nsup(P))$.

(ii) Let $x \in Ninf(f(P))$, then x is a $Ng\alpha^{**}$ -closed set in V . For $Q \subseteq U$, $Ninf(f(Q))$ is the largest $Ng\alpha^{**}$ -closed subset of $f(Q)$ in V . Given $Q \subseteq U$, $f(Ninf(Q)) \subseteq V$ is also a $Ng\alpha^{**}$ -closed set. Given f is a $Ng\alpha^{**}$ - continuous, every nano closed set in V is a $Ng\alpha^{**}$ -closed set $\Rightarrow x \in f(Ninf(P)) \Rightarrow Ninf(f(P)) \subseteq f(Ninf(P))$.

Theorem 4.9: If $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - continuous function, then

(i) $Ng\alpha^{**}int(f(P)) \subseteq f(Nsup(P))$

(ii) $f(Ninf(P)) \subseteq Ng\alpha^{**}cl(f(P))$

Proof:

(i) Assume f is a $Ng\alpha^{**}$ - continuous function. Let $P \subseteq U$, and $x \in Ng\alpha^{**}int(f(P))$, then x is a $Ng\alpha^{**}$ -open set in V . $Nsup(P)$ is the smallest $Ng\alpha^{**}$ -open set that contains P in $U \Rightarrow f(Nsup(P))$ is either nano open or $Ng\alpha^{**}$ -open set in V . If it is nano open in V , then it is also $Ng\alpha^{**}$ -open set in $V \Rightarrow x \in f(Nsup(P)) \Rightarrow Ng\alpha^{**}int(f(P)) \subseteq f(Nsup(P))$

(ii) Let $x \in Ninf(P) \Rightarrow Ninf(P)$ is the largest $Ng\alpha^{**}$ -closed subset of P in U . So $f(Ninf(P)) \subseteq V$ is also a $Ng\alpha^{**}$ -closed set. For $P \subseteq U$, $f(P) \subseteq V$ and $Ng\alpha^{**}cl(f(P))$

is the largest $Ng\alpha^{**}$ -closed set in V . The $Ng\alpha^{**}$ - closed sets of $f(Ninf(P))$ is contained in $Ng\alpha^{**}cl(f(P))$. Hence $f(Ninf(P)) \subseteq Ng\alpha^{**}cl(f(P))$.

Corollary 4.10: If f is a $Ng\alpha^{**}$ - continuous function, then

(i) $Nint(f(P)) \subseteq f(Nsup(P))$

(ii) $f(Ninf(P)) \subseteq Ncl(f(P))$

Proof: Similar as theorem.4.9

Theorem 4.11: If $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - continuous function, then

$$f^{-1}(Ng\alpha^{**}int(P)) \subseteq Nsup(f^{-1}(P))$$

Proof: Let $P \subseteq V$ is a subset of V and $f^{-1}(P) \subseteq U$. $Nsup(f^{-1}(P))$ is the smallest $Ng\alpha^{**}$ -open set contains $f^{-1}(P)$ in U . That is $f^{-1}(P) \subseteq Nsup(f^{-1}(P))$ ----(1)

$$\text{But } Ng\alpha^{**}int(P) \subseteq P \Rightarrow f^{-1}(Ng\alpha^{**}int(P)) \subseteq f^{-1}(P) \text{ ----(2)}$$

From(1) and (2), $f^{-1}(Ng\alpha^{**}int(P)) \subseteq Nsup(f^{-1}(P))$.

5. Nano generalized α^{**} - open and Nano generalized α^{**} - closed mappings

In this section, nano generalized α^{**} - open and nano generalized α^{**} - closed mappings and its properties are described.

Definition 5.1. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is said to be nano $g\alpha^{**}$ - open (or nano $g\alpha^{**}$ - closed) function if the image of every nano open (or nano closed) set in $(U, \tau_R(X))$ is nano $g\alpha^{**}$ - open (or nano $g\alpha^{**}$ - closed) set in $(V, \sigma_{R'}(Y))$.

Definition 5.2. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is said to be nano $g\alpha^{**}$ - closed function if the image of every nano closed set in $(U, \tau_R(X))$ is nano $g\alpha^{**}$ - closed set in $(V, \sigma_{R'}(Y))$.

Example 5.3. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{b, d\}$. Then $\tau_R(X) = \{\phi, U, \{d\}, \{b, c\}, \{b, c, d\}\}$ is a nano topology with respect to X and its complement is $\tau_R^c(X) = \{\phi, U, \{a\}, \{a, d\}, \{a, b, c\}\}$. Let $V = \{x, y, z, w\}$ with $V/R' = \{\{z\}, \{y\}, \{x, w\}\}$ and $Y = \{y, w\}$. Then $\sigma_{R'}(Y) = \{\phi, V, \{y\}, \{x, w\}, \{x, y, w\}\}$ and $\sigma_{R'}^c(Y) = \{\phi, V, \{z\}, \{y, z\}, \{x, z, w\}\}$.

Define $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ as $f(a) = z, f(b) = x, f(c) = w, f(d) = y$. Then $f^{-1}(\{d\}) = \{y\}, f^{-1}(\{b, c\}) = \{x, w\}, f^{-1}(\{b, c, d\}) = \{x, y, w\}$. Thus, the image of every nano open set in U is $Ng\alpha^{**}$ - open in V . Hence f is a $Ng\alpha^{**}$ - open function.

Theorem 5.4. Let $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ be a function and if f is nano open, then f is $Ng\alpha^{**}$ - open function.

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a nano open function and let $A \subseteq U$ is a nano open set. By definition, $f(A)$ is nano open in V . Every nano open set in V is a nano $g\alpha^{**}$ -open in V . So $f(A)$ is nano $Ng\alpha^{**}$ - open in V . Hence f is $Ng\alpha^{**}$ - open function.

Theorem 5.5. Let $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ be a nano closed function, then f is $Ng\alpha^{**}$ - closed function.

Proof: Similar to Theorem. 5.4

Theorem 5.6. Every nano closed function is $Ng\alpha^{**}$ - closed function but not conversely.

Proof: Let $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a nano closed function and let A is a nano closed Set in U , then $f(A)$ is nano closed in V . Every nano closed set in V is a nano $g\alpha^{**}$ - closed. Hence $f(A)$ is nano $g\alpha^{**}$ - closed in V . Therefore f is $Ng\alpha^{**}$ - closed function.

Example 5.7. In example. 5.3, if f is defined as $f(a) = z, f(b) = x, f(c) = y, f(d) = w$. Then $f(\{a\}) = \{z\}, f(\{a, d\}) = \{x, w\}, f(\{a, b, c\}) = \{x, y, z\}$. Thus, the image of the nano closed sets are $Ng\alpha^{**}$ - closed but f is not nano closed.

Theorem 5.8. Every $Ng\alpha^{**}$ - closed function is nano g-closed function but not conversely.

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - closed function and let A is a nano closed set in U , then the image of A under f is $Ng\alpha^{**}$ - closed in V . Every $Ng\alpha^{**}$ -closed set is nano g-closed. Hence $f(A)$ is nano g-closed in V . The function f is Ng -closed function.

Conversely, a nano g-closed set need not be a $Ng\alpha^{**}$ - closed set. So, the converse need not be true.

Remark 5.9. The function f defined in example.5.7 is $Ng\alpha^{**}$ - closed but not nano g-closed.

Theorem 5.10. If $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - closed function, then f is a

- (i) nano $g\alpha$ - closed function.
- (ii) nano αg - closed function.
- (iii) nano g^* - closed function.

(iv) nano rg - closed function.

Proof: Similar to the above theorem.

6. Nano generalized α^{**} - irresolute mappings

This section discusses nano generalized α^{**} - irresolute mappings and their characterizations.

Definition 6.1. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is called nano $g\alpha^{**}$ - irresolute ($Ng\alpha^{**}$ - irresolute) function if the image of every $Ng\alpha^{**}$ - closed set in V is nano $g\alpha^{**}$ - closed set in U .

Example 6.2. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{b, d\}$. Then $\tau_R(X) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the closed sets are $\{\phi, U, \{a\}, \{a, b\}, \{a, c, d\}\}$. The $Ng\alpha^{**}$ - closed sets are $\phi, U, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$. Let $V = \{x, y, z, w\}$ with $V/R' = \{\{x\}, \{y, w\}, \{z\}\}$ and $Y = \{x, y\}$, then $\sigma_{R'}(Y) = \{\phi, V, \{x\}, \{y, w\}, \{x, y, w\}\}$ and $\sigma_{R'}^c(Y) = \{\phi, V, \{z\}, \{x, z\}, \{y, z, w\}\}$. Define $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ as $f(a) = z, f(b) = x, f(c) = y, f(d) = w$. Then inverse image of $Ng\alpha^{**}$ - closed sets in V is $f^{-1}(\{z\}) = \{a\}, f^{-1}(\{x, z\}) = \{a, b\}, f^{-1}(\{y, z\}) = \{a, c\}, f^{-1}(\{z, w\}) = \{a, d\}, f^{-1}(\{x, y, z\}) = \{a, b, c\}, f^{-1}(\{x, z, w\}) = \{a, b, d\}, f^{-1}(\{y, z, w\}) = \{a, c, d\}$. Hence f is a $Ng\alpha^{**}$ - irresolute function.

Theorem 6.3. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is $Ng\alpha^{**}$ - irresolute if and only if the inverse image of every $Ng\alpha^{**}$ - open set in V is $Ng\alpha^{**}$ - open set in U .

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ - irresolute map and let P be any $Ng\alpha^{**}$ - open set in V . Then $V - P$ is $Ng\alpha^{**}$ - closed in $V \Rightarrow$ by assumption, every $Ng\alpha^{**}$ - closed set in V is $Ng\alpha^{**}$ - closed in U . Then $f^{-1}(V - P)$ is $Ng\alpha^{**}$ - closed in $U \Rightarrow f^{-1}(V - P) = f^{-1}(V) - f^{-1}(P) = U - f^{-1}(P)$ is $Ng\alpha^{**}$ - closed in $U \Rightarrow f^{-1}(P)$ is $Ng\alpha^{**}$ - open in U . Thus the inverse image of every $Ng\alpha^{**}$ - open set in V is a $Ng\alpha^{**}$ - open set in U .

Conversely, assume the inverse image of any $Ng\alpha^{**}$ - open set $Q \subseteq V, f^{-1}(Q)$ is $Ng\alpha^{**}$ - open set in U . Let $Q \subseteq V$ is $Ng\alpha^{**}$ - closed, Then $V - Q$ is $Ng\alpha^{**}$ - open in $V \Rightarrow$ by assumption, $f^{-1}(V - Q) = U - f^{-1}(Q)$ is $Ng\alpha^{**}$ - open in $U \Rightarrow f^{-1}(Q)$ is $Ng\alpha^{**}$ - closed in U . Thus f is $Ng\alpha^{**}$ - irresolute.

Theorem 6.4. Every $Ng\alpha^{**}$ - irresolute map is $Ng\alpha^{**}$ - continuous and conversely.

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is an $Ng\alpha^{**}$ - irresolute map, then every $Ng\alpha^{**}$ - closed set in V is $Ng\alpha^{**}$ - closed in U . Let Q be any nano closed set in V . The

inverse image of $Q \subseteq V$, $f^{-1}(Q)$ is $Ng\alpha^{**}$ -closed in U , since every nano closed set is a $Ng\alpha^{**}$ -closed. So f is a nano continuous function.

Conversely, assume f is $Ng\alpha^{**}$ -continuous, then every nano closed set in V is $Ng\alpha^{**}$ -closed in U . But every nano closed set in V is $Ng\alpha^{**}$ -closed in V . So the inverse image of every $Ng\alpha^{**}$ -closed set in V is $Ng\alpha^{**}$ -closed in U .

7. Nano generalized α^{**} - Homeomorphism in Nano Topological

In this section, nano generalized α^{**} -homeomorphism in nano topological spaces is defined and its properties are discussed

Definition 7.1. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is called nano generalized $g\alpha^{**}$ -homeomorphism (or $Ng\alpha^{**}$ -homeomorphism) if

- (i) f is one to one and onto
- (ii) f is $Ng\alpha^{**}$ -continuous
- (iii) f is $Ng\alpha^{**}$ -open

Theorem 7.2. If $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a bijective mapping. Then f is a $Ng\alpha^{**}$ -homeomorphism if and only if f is $Ng\alpha^{**}$ -closed and $Ng\alpha^{**}$ -continuous.

Proof: Assume $f: (U, \tau_R(X)) \rightarrow (V, \sigma_{R'}(Y))$ is a $Ng\alpha^{**}$ -homeomorphism. Then by definition, f is bijective, $Ng\alpha^{**}$ -continuous and $Ng\alpha^{**}$ -open. Let A be a nano closed set in U , then $U - A$ is a nano open set in U . $f(U - A)$ is $Ng\alpha^{**}$ -open in V and $V - f(A)$ is $Ng\alpha^{**}$ -open in V . This implies $f(A)$ is $Ng\alpha^{**}$ -closed in V for the nano closed set A in U . So f is $Ng\alpha^{**}$ -closed function.

Conversely, assume f is $Ng\alpha^{**}$ -closed and $Ng\alpha^{**}$ -continuous function. Let B be a nano open set in U , then $U - B$ is a nano closed set in U . $f(U - B)$ is $Ng\alpha^{**}$ -closed in V and $V - f(B)$ is $Ng\alpha^{**}$ -closed. That is $f(B)$ is $Ng\alpha^{**}$ -open, hence f is $Ng\alpha^{**}$ -open mapping. Also, f is $Ng\alpha^{**}$ -continuous. So f is $Ng\alpha^{**}$ -homeomorphism. Hence the proof.

8. Discussion

This study uses $Ng\alpha^{**}$ -closed and $Ng\alpha^{**}$ -open sets to define two new classes of subsets called Nano Infimum and Nano Supremum of a nano topological space. These two sets are independent with $Ng\alpha^{**}$ -closure and $Ng\alpha^{**}$ -interior. The characteristics of nano infimum and nano supremum have been identified. Next, the new class of continuous function called the nano generalized α^{**} -continuous function is defined and explored together with other nano continuous functions. Additionally, nano generalized

α^{**} - open mappings, nano generalized α^{**} - closed mappings, nano generalized α^{**} - irresolute mappings, and nano generalized α^{**} homeomorphism are defined and their characteristics are examined.

In conclusion, the nano infimum of a subset is the inferior set which has the largest Nga^{**} - closed set contained in it and the nano supremum of a subset is the superior set in which it is the smallest Nga^{**} - open set that contains it. By drawing the nano hasse diagram for the Nga^{**} - closed and Nga^{**} -open sets, one can discover the nano infimum and nano supremum of a subset by finding the GLB and LUB of the set. Moreover, the ideas of Nga^{**} - closed and Nga^{**} -open sets are broadened to define Nga^{**} -continuous functions and the behaviours of nano infimum and nano supremum under this function are discovered.

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Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

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Abstract

In this paper, we introduce an innovative category of contractions termed cyclic Chatterjea type F –contraction and weak cyclic Kannan type F –contraction. Subsequently, we establish a theorem for determining the best proximity point in a uniformly convex Banach space, specifically focusing on weak cyclic Kannan type F –contractions. Furthermore, we extend our investigation to include cyclic Chatterjea type F –contractions in uniformly convex Banach spaces. To support our findings, we present illustrative examples.

Keywords: 47H10, 54H25, 47H09.

2020 AMS subject: weak cyclic Kannan type F –contraction, cyclic Chatterjea type F –contraction, cyclic Chatterjea type contraction, Kannan type cyclic contraction, F –contraction, best proximity point.¹

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1 Introduction

In 1922, Polish mathematician Banach proved an interesting result known as Banach contraction principle [4] states that a self-mapping T on a complete metric space X has a unique fixed point in X provided that there exists $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ is satisfied for all $x, y \in X$. many author extended and generalized Banach's result in many directions see [1, 5, 6, 8, 10, 11], Some other extensions of Banach contraction principle have been presented by considering the concept of best proximity point [20]. Let (X, d) be a metric space and A, B be two subsets of X , then $T : A \rightarrow B$ be a mapping. In this context the mapping T does not have a fixed point in case of $A \cap B = \phi$. It is reasonable to research a point x , which is closest to Tx , that is $d(x, Tx)$ is minimum. Since $d(x, Tx) \geq d(A, B)$ for all $x \in A$, then $d(x, Tx)$ is at least $d(A, B)$. Best proximity point theorems are related to find a point satisfying $d(x, Tx) = d(A, B)$ which is called a Best proximity point. There are many results about this subject in literature [2, 9, 12, 14, 15, 16, 19].

2 Preliminaries

In 2003, Kirk et al. [13] introduced the notion of cyclical contractive mappings, and generalized Banach fixed point result [4] to the class of cyclic mappings

Theorem 2.1. [13] *Let A and B be nonempty closed subsets of a complete metric space (X, d) and let $T : A \cup B \rightarrow A \cup B$ be such that $T(A) \subset B$ and $T(B) \subset A$. Assume that, for all $x \in A$ and $y \in B$*

$$d(Tx, Ty) \leq \alpha d(x, y),$$

where $\alpha \in (0, 1)$. Then, T has a unique fixed point $u \in A \cap B$.

The above mapping is called cyclic. In [7], Eldred and Veeramani are concerned with the case when $A \cap B = \phi$, and in this case they didn't seek for the existence of a fixed point of T but for the existence of a best proximity point. For instance, they [7] presented the following existence best proximity point result for cyclic contractions.

Theorem 2.2. [7] *Let A and B be non-empty closed and convex subsets of a complete metric space (X, d) and let $T : A \cup B \rightarrow A \cup B$ be cyclic. Assume that, for all $x \in A$ and $y \in B$*

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B).$$

Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

Where $\alpha \in (0, 1)$ and $d(A, B) = \inf\{d(x, y), x \in A, y \in B\}$. For $x_0 \in A$, define

$$x_{n+1} = Tx_n$$

for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \rightarrow x$ and $d(x, Tx) = d(A, B)$. Here, x is called a best proximity point of T .

In 2011 [17] M.A. Petric, introduced a new class of mappings known as weak cyclic Kannan contractions and proved some existence and convergence results for best proximity points in uniformly convex Banach spaces is proved as follows.

Theorem 2.3. [17] Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T : A \cup B \rightarrow A \cup B$ be a weak cyclic Kannan contraction map. Then

- (i) T has a unique best proximity point z in A .
- (ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
- (iii) z is the unique fixed point of T^2 .
- (iv) Tz is a best proximity point of T in B .

In [21], Dariusz Wardowski has proved fixed point theorem on F –contraction, which is perceived to be one of the most general non-linear contraction in complete metric spaces.

We first define weak cyclic Kannan type F –contraction and cyclic Chatterjea type F –contraction in this study. We also establish best proximity point result in the setting of uniformly convex Banach space using cyclic Chatterjea type F –contractions and prove the existence of best proximity point result in the setting of uniformly convex Banach space using weak cyclic Kannan type F –contraction.

Definition 2.1. [17] Let A and B be nonempty subsets of a metric space (X, d) . If a map $T : A \cup B \rightarrow A \cup B$ satisfies

- (i) $T(A) \subseteq B$ and $T(B) \subseteq A$;
- (ii) there exists a $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] + (1 - 2\alpha)d(A, B)$$

for any $x \in A$ and $y \in B$, then T is called a weak cyclic Kannan contraction on $A \cup B$.

Definition 2.2. [21] Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a mapping satisfying

- (F1) : F is strictly increasing i.e. for all $\alpha, \beta \in \mathbb{R}^+$ such that $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$,
- (F2) : For each sequence $\alpha_n, n \in \mathbb{N}$ of positive real numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$ if

and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$

(F3) : There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0+} \alpha^k F(\alpha) = 0$.

A mapping $T : X \rightarrow X$ is said to be an F -contraction if there exists $\tau > 0$, such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \quad (1)$$

Remark 2.1. [18] From (F1) and the equation (1) it is clear that every F -contraction T is a contractive mapping i.e.

$$d(Tx, Ty) < d(x, y), \forall x, y \in X, Tx \neq Ty.$$

Thus every F -contraction is a continuous mapping.

Theorem 2.4. [21] Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is convergent to x^* .

Definition 2.3. [3] Let (X, d) be a metric-like space. Let A and B be nonempty subsets of X . Take the cyclic mapping $T : A \cup B \rightarrow A \cup B$. T is said to be a cyclic Chatterjea type contraction if

$$d(Tx, Ty) \leq k(d(Tx, y) + d(Ty, x)) + (1 - 4k)d(A, B)$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1/4)$.

Definition 2.4. Uniformly convex space is a normed vector space such that, for every $0 < \epsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with

$$(\|x - y\|) \geq \epsilon \implies \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Intuitively, the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

Remark 2.2. Every closed subspace of a uniformly convex Banach space is uniformly convex.

Lemma 2.1. [7] Let A be a non-empty mapping closed and convex subset and B be a non-empty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:

(i) $\|z_n - y_n\| \rightarrow d(A, B)$

(ii) For every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$\|x_m - y_n\| \leq d(A, B) + \epsilon.$$

Then for every $\epsilon > 0$ there exists N_1 such that for all $m > n > N_1$, $\|x_m - z_n\| \leq \epsilon$.

Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

Lemma 2.2. [7] *Let A be a non-empty closed and convex subset and B be a non-empty closed subsets of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $\|x_n - y_n\| \rightarrow d(A, B),$

(ii) $\|z_n - y_n\| \rightarrow d(A, B).$

Then $\|x_n - z_n\|$ converges to 0.

3 Main Results

We first introduce a new concept of weak cyclic Kannan type F –contraction. We also prove the existence result for best proximity point of weak cyclic Kannan type F –contraction in the setting of uniformly convex Banach space.

Definition 3.1. *Let A and B be two non-empty subsets of a complete metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is called weak cyclic Kannan type F –contraction if the following conditions hold:*

(a) $T(A) \subseteq B$ and $T(B) \subseteq A,$

(b) $\forall x \in A, y \in B$ and if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq \alpha F(d(x, Tx) + d(y, Ty)) + (1 - 2\alpha)F(d(A, B)),$$

for some $\alpha \in (0, \frac{1}{2})$, provided $d(A, B) > 0$.

Note that (b) implies that T satisfies $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, for all $x \in A, y \in B$.

Example: Consider the complete metric space $X = \mathbb{R}$ with the usual metric d . If we consider the map $F(x) = -\frac{1}{\sqrt{x}}, x > 0$ and for any $k \in (\frac{1}{2}, 1)$. Then clearly F satisfies all the conditions from (F1) – (F3).

In this case, any $T : A \cup B \rightarrow A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ satisfying the condition

$$d(Tx, Ty) \leq \frac{[d(x, Tx) + d(y, Ty)]}{[2\alpha + (1 - 2\alpha)\sqrt{\frac{d(x, Tx) + d(y, Ty)}{d(A, B)}} + \tau\sqrt{d(x, Tx) + d(y, Ty)}]^2},$$

$\forall x \in A, \forall y \in B, d(Tx, Ty) > 0, \alpha \in (0, \frac{1}{2})$ and $\tau > 0$ provided $d(A, B) > 0$, is a weak cyclic Kannan type F –contraction.

Proposition 3.1. *Let A and B be non-empty closed subsets of a complete metric space (X, d) . Suppose that mapping $T : A \cup B \rightarrow A \cup B$ is weak cyclic Kannan type F –contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then we have*

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$. Let a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Now,

$$\begin{aligned}
F(d(x_2, x_1)) &= F(d(Tx_1, Tx_0)) \\
&\leq \alpha F[d(x_1, Tx_1) + d(x_0, Tx_0)] + (1 - 2\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(x_1, Tx_1) + d(x_0, Tx_0)] + (1 - 2\alpha)F(d(A, B)) \\
&= \alpha F[d(x_1, x_2) + d(x_0, x_1)] + (1 - 2\alpha)F(d(A, B)) \\
&< \alpha F[d(x_1, x_2) + d(x_0, x_1)] + (1 - 2\alpha)F(d(x_1, x_2)) \\
&< \alpha Fd(x_1, x_2) + \alpha F(d(x_0, x_1)) + (1 - 2\alpha)F(d(x_1, x_2)) \\
&< \alpha F(d(x_0, x_1)) + (1 - \alpha)F(d(x_1, x_2)) \\
&< \alpha F(d(x_0, x_1)) + (1 - \alpha)F(d(A, B)) \\
F(d(x_2, x_1)) - F(d(A, B)) &< \alpha\{F(d(x_0, x_1)) - F(d(A, B))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
F(d(x_3, x_2)) &= F(d(Tx_2, Tx_1)) \\
&\leq \alpha F[d(x_2, Tx_2) + d(x_1, Tx_1)] + (1 - 2\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(x_2, Tx_2) + d(x_1, Tx_1)] + (1 - 2\alpha)F(d(A, B)) \\
&= \alpha F[d(x_2, x_3) + d(x_1, x_2)] + (1 - 2\alpha)F(d(A, B)) \\
&< \alpha F[d(x_2, x_3) + d(x_1, x_2)] + (1 - 2\alpha)F(d(x_2, x_3)) \\
&< \alpha Fd(x_2, x_3) + \alpha F(d(x_1, x_2)) + (1 - 2\alpha)F(d(x_2, x_3)) \\
&< \alpha F(d(x_1, x_2)) + (1 - \alpha)F(d(x_2, x_3)) \\
&< \alpha F(d(x_1, x_2)) + (1 - \alpha)F(d(A, B)) \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha\{F(d(x_1, x_2)) - F(d(A, B))\} \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha^2\{F(d(x_0, x_1)) - F(d(A, B))\} \\
&\dots \\
&\dots
\end{aligned}$$

We say that $\alpha \in (0, 1)$,

$$F(d(x_{n+1}, x_n)) - F(d(A, B)) < \alpha^n\{F(d(x_0, x_1)) - F(d(A, B))\}.$$

For $n \rightarrow \infty$

$$F(d(x_{n+1}, x_n)) = F(d(A, B))$$

Since F is increasing.

$$\begin{aligned}
\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) &= d(A, B) \\
\lim_{n \rightarrow \infty} d(x_n, Tx_n) &= d(A, B)
\end{aligned}$$

Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

□

Theorem 3.1. *Let A and B be two non-empty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \rightarrow A \cup B$ is weak cyclic Kannan type F –contraction map and F is continuous map. Then there exists a best proximity point $x \in A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $d(A, B) \neq 0$. Then by proposition 3.1,

$$\begin{aligned} \|x_{2n} - x_{2n+1}\| &= \|x_{2n} - Tx_{2n}\| \rightarrow d(A, B) \\ \|x_{2n+2} - x_{2n+1}\| &= \|Tx_{2n+1} - Tx_{2n}\| \rightarrow d(A, B) \\ \|x_{2n+2} - x_{2n+1}\| &= \|T(Tx_{2n}) - Tx_{2n}\| \rightarrow d(A, B) \\ \|x_{2n+2} - x_{2n+1}\| &= \|Tx_{2n} - T(Tx_{2n})\| \rightarrow d(A, B) \end{aligned}$$

Then, by lemma 2.2

$$\|x_{2n} - x_{2n+2}\| \rightarrow 0$$

We now show that for every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$\|x_{2m} - Tx_{2n}\| \leq d(A, B) + \epsilon$$

Let $\epsilon > 0$, if possible, suppose for all $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$, such that,

$$\|x_{2m_k} - Tx_{2n_k}\| \geq d(A, B) + \epsilon$$

and

$$\|x_{2(m_k-1)} - Tx_{2n_k}\| \leq d(A, B) + \epsilon.$$

Now,

$$\begin{aligned} d(A, B) + \epsilon &\leq \|x_{2m_k} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + d(A, B) + \epsilon. \end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A, B) + \epsilon.$$

as F is continuous.

$$\begin{aligned}
\lim_{k \rightarrow \infty} F(\|x_{2m_k} - Tx_{2n_k}\|) &\leq \lim_{k \rightarrow \infty} F(\|Tx_{2m_k+1} - Tx_{2n_k+2}\|) \\
&\leq \lim_{k \rightarrow \infty} \{ \alpha F(\|x_{2m_k+1} - Tx_{2m_k+1}\| \\
&\quad + \|x_{2n_k+2} - Tx_{2n_k+2}\|) \} \\
&\quad + (1 - 2\alpha)F(d(A, B)) - \tau \\
&\leq \lim_{k \rightarrow \infty} \{ \alpha F(\|x_{2m_k+1} - Tx_{2m_k+1}\|) \\
&\quad + \alpha F(\|x_{2n_k+2} - Tx_{2n_k+2}\|) \} \\
&\quad + (1 - 2\alpha)F(d(A, B)) - \tau \\
&< \lim_{k \rightarrow \infty} \{ \alpha F(\|x_{2m_k+1} - Tx_{2m_k+1}\|) \\
&\quad + \alpha F(\|x_{2n_k+2} - Tx_{2n_k+2}\|) \} \\
&\quad + (1 - 2\alpha)F(d(A, B)) \\
&= \lim_{k \rightarrow \infty} \{ \alpha F(\|Tx_{2m_k} - Tx_{2m_k+1}\|) \\
&\quad + \alpha F(\|Tx_{2n_k+1} - Tx_{2n_k+2}\|) \} \\
&\quad + (1 - 2\alpha)F(d(A, B)) \\
&< \lim_{k \rightarrow \infty} \{ \alpha \{ \alpha F(\|x_{2m_k} - Tx_{2m_k}\| \\
&\quad + \|x_{2m_k+1} - Tx_{2m_k+1}\|) + \\
&\quad (1 - 2\alpha)F(d(A, B)) \} + \alpha \{ \alpha F(\|x_{2n_k+1} - Tx_{2n_k+1}\| \\
&\quad + \|x_{2n_k+2} - Tx_{2n_k+2}\|) \\
&\quad + (1 - 2\alpha)F(d(A, B)) \} \} + (1 - 2\alpha)F(d(A, B)) \\
&< \{ \alpha^2 F(d(A, B) + \epsilon) + \alpha^2 F(d(A, B) + \epsilon) \\
&\quad + \alpha(1 - 2\alpha)F(d(A, B)) \} \\
&\quad + \{ \alpha^2 F(d(A, B) + \epsilon) + \alpha^2 F(d(A, B) + \epsilon) \\
&\quad + \alpha(1 - 2\alpha)F(d(A, B)) \} + (1 - 2\alpha)F(d(A, B)) \\
&< 4\alpha^2 F(d(A, B) + \epsilon) + (\alpha - 2\alpha^2 + \alpha - 2\alpha^2 \\
&\quad + 1 - 2\alpha)F(d(A, B)) \\
&< 4\alpha^2 F(d(A, B) + \epsilon) + (1 - 4\alpha^2)F(d(A, B)) \\
&< 4\alpha^2 F(d(A, B) + \epsilon) + (1 - 4\alpha^2)F(d(A, B)) \\
&\quad F(d(A, B) + \epsilon) < 4\alpha^2 F(d(A, B) + \epsilon) + (1 - 4\alpha^2)F(d(A, B)) \\
&F(d(A, B) + \epsilon)(1 - 4\alpha^2) < (1 - 4\alpha^2)F(d(A, B)) \\
&F(d(A, B) + \epsilon) < F(d(A, B)).
\end{aligned}$$

Hence T is best proximity point. $\square$

Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

Now we introduce a new concept of cyclic Chatterjea type F –contraction. We also prove the existence result for best proximity point of cyclic Chatterjea type F –contraction in the setting of uniformly convex Banach space.

Definition 3.2. Let A and B be two non-empty subsets of a complete metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is called cyclic Chatterjea type F –contraction if the following conditions hold:

- (a) $T(A) \subseteq B$ and $T(B) \subseteq A$,
- (b) $\forall x \in A, y \in B$ and if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq \alpha F(d(Tx, y) + d(Ty, x)) + (1 - 4\alpha)F(d(A, B))$$

,
for some $\alpha \in (0, \frac{1}{4})$, provided $d(A, B) > 0$.

Note that (b) implies that T satisfies $\tau + F(d(Tx, Ty)) \leq F(d(Tx, y) + d(Ty, x))$, for all $x \in A, y \in B$.

Example: Consider the complete metric space $X = \mathbb{R}$ with the usual metric d . If we consider the map $F(x) = -\frac{1}{\sqrt{x}}$, $x > 0$ and for any $k \in (\frac{1}{2}, 1)$. Then clearly F satisfies all the conditions from $(F1) - (F3)$.

Let us define $X = \{0, 1\}$ and $d(0, 0) = d(1, 1) = 2$ and $d(0, 1) = d(1, 0) = 1$ and take $\alpha \in (0, \frac{1}{4})$,

Let $A = \{0\}$ and $B = \{1\}$. Note that $d(A, B) = 1$ and A, B are closed in (X, d)

Consider $T : A \cup B \rightarrow A \cup B$ defined by $T0 = 1$ and $T1 = 0$

Clearly, T is cyclic. Let $x \in A$ and $y \in B$, that is $x = 0$ and $y = 1$ satisfying the condition

$$d(Tx, Ty) \leq \frac{1}{[4\alpha + (1 - 4\alpha)\sqrt{\frac{d(Tx, y) + d(Ty, x)}{d(A, B)}} + \tau\sqrt{d(Tx, y) + d(Ty, x)}]^2} [d(Tx, y) + d(Ty, x)],$$

$\forall x \in A, \forall y \in B, d(Tx, Ty) > 0, \alpha \in (0, \frac{1}{4})$ and $\tau > 0$ provided $d(A, B) > 0$, is a cyclic Chatterjea type F –contraction.

Proposition 3.2. Let A and B be non-empty closed subsets of a complete metric space (X, d) . Suppose that mapping $T : A \cup B \rightarrow A \cup B$ is a cyclic Chatterjea type F –contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then we have

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$. Let a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Now,

$$\begin{aligned}
F(d(x_2, x_1)) &= F(d(Tx_1, Tx_0)) \\
&\leq \alpha F[d(Tx_1, x_0) + d(Tx_0, x_1)] + (1 - 4\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(Tx_1, x_0) + d(Tx_0, x_1)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[d(x_2, x_0) + d(x_1, x_1)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[\{d(x_2, x_1) + d(x_1, x_0)\} + \{d(x_1, x_2) + d(x_2, x_1)\}] \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[d(x_1, x_2) + d(x_1, x_0) + d(x_1, x_2) + d(x_1, x_2)] \\
&\quad + (1 - 4\alpha)F(d(x_1, x_2)) \\
&< \alpha F(d(x_1, x_0)) + 3\alpha F(d(x_1, x_2)) + (1 - 4\alpha)F(d(x_1, x_2)) \\
&< \alpha F[d(x_1, x_0)] + (1 - \alpha)F(d(x_1, x_2)) \\
&< \alpha F[d(x_0, x_1)] + (1 - \alpha)F(d(A, B)) \\
F(d(x_2, x_1)) - F(d(A, B)) &< \alpha\{F(d(x_0, x_1)) - F(d(A, B))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
F(d(x_3, x_2)) &= F(d(Tx_2, Tx_1)) \\
&\leq \alpha F[d(Tx_2, x_1) + d(Tx_1, x_2)] + (1 - 4\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(x_3, x_1) + d(x_2, x_2)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[\{d(x_3, x_2) + d(x_2, x_1)\} + \{d(x_2, x_3) + d(x_3, x_2)\}] \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[3d(x_3, x_2) + d(x_2, x_1)] + (1 - 4\alpha)F(d(x_3, x_2)) \\
&< \alpha F(d(x_2, x_1)) + (1 - \alpha)F(d(x_3, x_2)) \\
&< \alpha F(d(x_2, x_1)) + (1 - \alpha)F(d(A, B)) \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha\{F(d(x_2, x_1)) - F(d(A, B))\} \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha^2\{F(d(x_1, x_0)) - F(d(A, B))\} \\
&< \alpha^2\{F(d(x_0, x_1)) - F(d(A, B))\} \\
&\dots \\
&\dots
\end{aligned}$$

We say that $\alpha \in (0, 1)$,

$$F(d(x_{n+1}, x_n)) - F(d(A, B)) < \alpha^n\{F(d(x_0, x_1)) - F(d(A, B))\}.$$

For $n \rightarrow \infty$

$$F(d(x_{n+1}, x_n)) = F(d(A, B))$$

Best proximity point for weak cyclic Kannan type F –contraction map and cyclic Chatterjea type F –contraction map

Since F is increasing.

$$\begin{aligned}\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) &= d(A, B) \\ \lim_{n \rightarrow \infty} d(x_n, Tx_n) &= d(A, B).\end{aligned}$$

□

Theorem 3.2. *Let A and B be two non-empty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \rightarrow A \cup B$ is cyclic Chatterjea type F –contraction map and F is continuous map. Then there exists a best proximity point $x \in A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $d(A, B) \neq 0$. Then by proposition 3.2

$$\begin{aligned}\|x_{2n} - x_{2n+1}\| &= \|x_{2n} - Tx_{2n}\| \rightarrow d(A, B) \\ \|x_{2n+2} - x_{2n+1}\| &= \|Tx_{2n+1} - Tx_{2n}\| \rightarrow d(A, B)\end{aligned}$$

Then, by lemma 2.2

$$\|x_{2n} - x_{2n+2}\| \rightarrow 0$$

We now show that for every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$\|x_{2m} - Tx_{2n}\| \leq d(A, B) + \epsilon$$

Let $\epsilon > 0$, if possible, suppose for all $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$, such that

$$\|x_{2m_k} - Tx_{2n_k}\| \geq d(A, B) + \epsilon$$

and

$$\|x_{2(m_k-1)} - Tx_{2n_k}\| \leq d(A, B) + \epsilon$$

Now,

$$\begin{aligned}d(A, B) + \epsilon &\leq \|x_{2m_k} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + d(A, B) + \epsilon\end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A, B) + \epsilon$$

as F is continuous.

$$\begin{aligned}
\lim_{k \rightarrow \infty} F(||x_{2m_k} - Tx_{2n_k}||) &\leq \lim_{k \rightarrow \infty} F(||Tx_{2m_k+1} - Tx_{2n_k+2}||) \\
&\leq \lim_{k \rightarrow \infty} \{\alpha F(||Tx_{2m_k+1} - x_{2n_k+2}|| + ||Tx_{2n_k+2} - x_{2m_k+1}||)\} \\
&\quad + (1 - 4\alpha)F(d(A, B)) - \tau \\
&< \lim_{k \rightarrow \infty} \{\alpha F(||Tx_{2m_k+1} - Tx_{2n_k+1}|| + ||Tx_{2n_k+2} - Tx_{2m_k}||)\} \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&< \lim_{k \rightarrow \infty} [\alpha \{\alpha F(||Tx_{2m_k+1} - x_{2n_k+1}|| + ||Tx_{2n_k+1} - x_{2m_k+1}||) \\
&\quad + (1 - 4\alpha)F(d(A, B))\} + \alpha \{\alpha F(||Tx_{2n_k+2} - x_{2m_k}|| \\
&\quad + ||Tx_{2m_k} - x_{2n_k+2}||) + (1 - 4\alpha)F(d(A, B))\}] \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&\leq \{\alpha^2 F(d(A, B) + \epsilon) + \alpha^2 F(d(A, B) + \epsilon) + \alpha(1 - 4\alpha) \\
&\quad F(d(A, B))\} + \lim_{k \rightarrow \infty} \alpha^2 F(\{||Tx_{2n_k+2} - x_{2m_k+2}|| + ||x_{2m_k+2} \\
&\quad - x_{2m_k+1}|| \\
&\quad + ||x_{2m_k+1} - x_{2m_k}||\} + \{||Tx_{2m_k} - x_{2n_k}|| + ||x_{2n_k} \\
&\quad - x_{2n_k+1}|| + ||x_{2n_k+1} - x_{2n_k+2}||\}) + \alpha(1 - 4\alpha)F(d(A, B)) \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&\leq \{\alpha^2 F(d(A, B) + \epsilon) + \alpha^2 F(d(A, B) + \epsilon) + \alpha(1 - 4\alpha) \\
&\quad F(d(A, B))\} + \lim_{k \rightarrow \infty} \alpha^2 F(\{||Tx_{2n_k+2} - x_{2m_k+2}|| + ||Tx_{2m_k+1} \\
&\quad - x_{2m_k+1}|| + ||Tx_{2m_k} - x_{2m_k}||\} + \{||Tx_{2m_k} - x_{2n_k}|| + ||x_{2n_k} \\
&\quad - Tx_{2n_k}|| + ||x_{2n_k+1} - Tx_{2n_k+1}||\}) + \alpha(1 - 4\alpha) \\
&\quad F(d(A, B)) + (1 - 4\alpha)F(d(A, B)) \\
&\leq \{\alpha^2 F(d(A, B) + \epsilon) + \alpha^2 F(d(A, B) + \epsilon) + \alpha(1 - 4\alpha) \\
&\quad F(d(A, B)) + \alpha^2 F(\{(d(A, B) + \epsilon) + (d(A, B) + \epsilon) \\
&\quad + (d(A, B) + \epsilon)\} + \{(d(A, B) + \epsilon) + (d(A, B) + \epsilon) \\
&\quad + (d(A, B) + \epsilon)\}) + \alpha(1 - 4\alpha)F(d(A, B)) \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&\leq 8\alpha^2 F(d(A, B) + \epsilon) + \alpha(1 - 4\alpha + 1 - 4\alpha)F(d(A, B)) \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&\leq 8\alpha^2 F(d(A, B) + \epsilon) + (2\alpha - 8\alpha^2)F(d(A, B)) + \\
&\quad (1 - 4\alpha)F(d(A, B)) \\
&\leq 8\alpha^2 F(d(A, B) + \epsilon) + (2\alpha - 8\alpha^2 + 1 - 4\alpha)F(d(A, B)) \\
&\leq 8\alpha^2 F(d(A, B) + \epsilon) + (1 - 8\alpha^2 - 2\alpha)F(d(A, B)) \\
&< 8\alpha^2 F(d(A, B) + \epsilon) + (1 - 8\alpha^2)F(d(A, B)) \\
F(d(A, B) + \epsilon)(1 - 8\alpha^2) &< (1 - 8\alpha^2)F(d(A, B)) \\
F(d(A, B) + \epsilon) &< F(d(A, B)).
\end{aligned}$$

Best proximity point for weak cyclic Kannan type F -contraction map and cyclic Chatterjea type F -contraction map

Hence T is best proximity point. □

4 Conclusion

In this study, we have introduced the concepts of weak cyclic Kannan type F -contraction and cyclic Chatterjea type F -contraction. By employing these novel contraction mappings, we have successfully determined the best proximity point theorem within a uniformly convex Banach space. To further illustrate the effectiveness of our approach, we have presented a series of illustrative examples.

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The modified divisor graph of finite commutative ring

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§

Abstract

Let R be a finite commutative ring with unity. we have introduced a divisor graph of ring which is denoted by $\mathbb{D}[R]$. It is a undirected simple graph with vertex set $V = R - \{0, 1\}$ and two distinct vertices $u, v \in V$ are adjacent if and only if $\exists w \in R$ such that $v = uw$ or $u = vw$. Clearly 0 and unit elements are adjacent to all elements of ring. Thus, we adopt the idea of Anderson and Livingston and removed the zero and unit vertices from the vertex set. A new graph is called modified divisor graph noted by symbol $\mathbb{D}_0[R]$. In this study, we have focused on the structural properties of $\mathbb{D}_0[Z_n]$. Moreover, we have determined the diameter, girth, clique number, vertex connectivity and edge connectivity of the $\mathbb{D}_0[Z_n]$ for every natural number n . The purpose to study this graph is to find relationship between ring theoretic properties of R and graph theoretic properties of $\mathbb{D}_0[R]$.

Keywords: graph; eulerian graph; connected graph; clique number; etc...

2020 AMS subject classifications: 05C10, 05C12, 05C45, 05C15

1 Introduction

The study of zero divisor graphs was initiated by I. Beck [1998]. He mostly interested in coloring the graph. Later on D. F. Anderson and P. S. Livingston [1999] simplified definition of zero divisor graph by omitting 0 from the graph of I. Beck. D. F. Anderson and P. S. Livingston [1999] studied properties of zero divisor graph

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more clearly and find relations between ring-theoretic properties and graph theoretic ones. Anderson and Livingston have given surprising results on zero divisor graph $\Gamma[R]$. They have study structural properties and graph theoretic properties of graph $\Gamma[R]$. Many of researcher motivated by their work and attracted in this area of research. See, Redmond and P [2002], Akbari and Mohammadian [2004], Redmond [2003]. The study of vertex and edge connectivity of zero divisor graph is done in B.S. Reddy [2020]. Associating graph to an algebraic structure become popular research topic since the last 2 decades. The graph of various algebraic structure and their properties can be found in Mathil et al. [2022], B.S. Reddy [2020], Das [2016], S. Akabari [2014], R. A. Muneshwar [2021], M. S. Rahman [2005], Khandare et al. [2022].

The concept of divisor graph introduced in Khandare et al. [2022]. The divisor graph $\mathbb{D}[R]$ is graph with vertex set, $V = \{r \in R \mid r \neq 0, r \neq 1\}$ and two vertices $x, y \in V$ are adjacent if and only if $\exists z \in R$ such that $y = zx$ or $x = zy$. In Khandare et al. [2022], unit vertices are adjacent with all other element of ring. So we adopt the idea of Anderson and Livingston and omitted unit elements and non zero divisors of ring from vertex set. Clearly the new graph is a induced sub-graph of divisor graph $\mathbb{D}[R]$. we have denoted new graph as $\mathbb{D}_0[R]$. The section 2 includes the definition of divisor graph, modified divisor graph and some useful notation. In section 3, the graphical structure and graph theoretic properties like complete graph, connected, bipartite graph, planar graph is discussed. The section 4 contain diameter and girth of the graph $\mathbb{D}_0[Z_n]$. The section 5 includes condition for graph $\mathbb{D}_0[Z_n]$ to be an Eulerian and Hamiltonian. In section 6, a clique number is calculated and provides a short proof to Beck's conjecture. Also in the section 7, we derive formula to determine vertex and edge connectivity $\mathbb{D}_0[Z_n]$ for natural n .

2 Definition and Notation

Definition 2.1. divisor graph of ring R . Khandare et al. [2022]

Let R be finite commutative ring, then the graph $\mathbb{D}[R]$ is divisor graph of R , where $V = \{u \in R \mid u \neq 0, 1\}$ and for $u, v \in V, u \sim v$ if and only if $\exists w \in R$ such that $u = vw$ or $v = wu$.

Definition 2.2. modified divisor graph of R

Let R be finite commutative ring, then the graph $\mathbb{D}_0[R]$ is modified divisor graph of R , where $V = \{u \mid u \in Z^*(R)\}$ and for $u, v \in V, u \sim v$ if and only if $\exists w \in R$ such that $u = vw$ or $v = wu$.

Notation: Here, we discuss about the structure of the $\mathbb{D}_0[Z_n]$. Let $d_1, d_2, \dots, d_k$ be the proper divisors of n. For $1 \leq i \leq k$, consider the following sets,

$$A_{d_i} = \{x \in Z_n \mid \gcd(x, n) = d_i\}$$

Further, note that these sets $A_{d_1}, A_{d_2}, \dots, A_{d_k}$ forms a partition of the vertex set of the graph $\mathbb{D}_0[Z_n]$. Thus $V(\mathbb{D}_0[Z_n]) = A_{d_1} \cup A_{d_2} \cup \dots \cup A_{d_k}$. Reader also note that, we use another partition for $Z^*[R]$ in Das [2016] such that if $n = p_1 p_2 \dots p_m$, $Z^*[R] = \cup W_{d_i}$, $W_{d_i} = \{k(d_i) \mid k \in R^*, k \notin \langle d_j \rangle, d_i \neq d_j\}$, $W_{d_i} \cap W_{d_j} = \phi$ and if $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ then $Z^*[R] = \cup T_{d_i}$, where $T_{d_i} = \{k \cdot d_i \mid k \in R^*, k \notin \langle d_j \rangle, d_i^s \neq d_j, s \in N\}$ but $T_{d_i} \cap T_{d_j} \neq \phi$. It is clear that, $W_{d_i} = A_{d_i}$, $T_{d_i} \neq A_{d_i}$

, $A_{d_i} \subseteq T_{d_i}$ and $Z^*[R] = \cup T_{d_i}$ is not partition of set. Therefore we consider the partition, $Z^*[R] = \cup A_{d_i}$. It needs to be noted that non-trivial divisor of n that is neither 1 nor n .

3 Modified Divisor Graph of Z_n

In order to illustrate the difference between divisor graph and modified divisor graph, we will discuss the following figures.

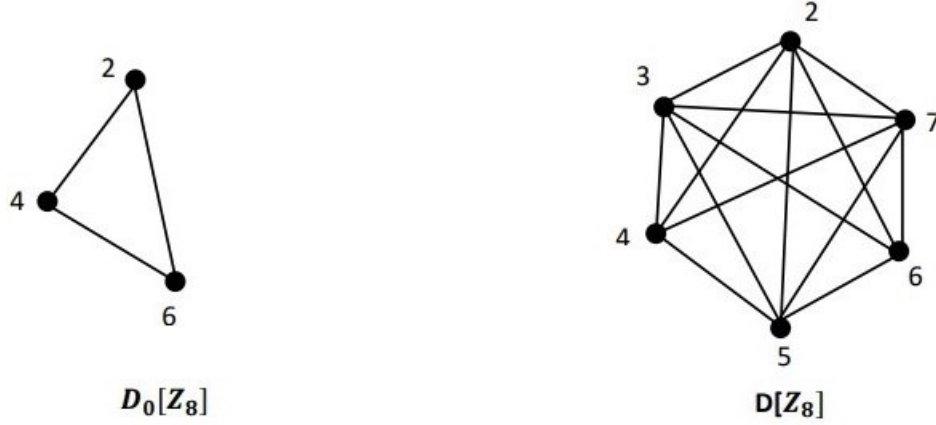


Figure 1: Modified Divisor Graph and Divisor Graph of Z_8

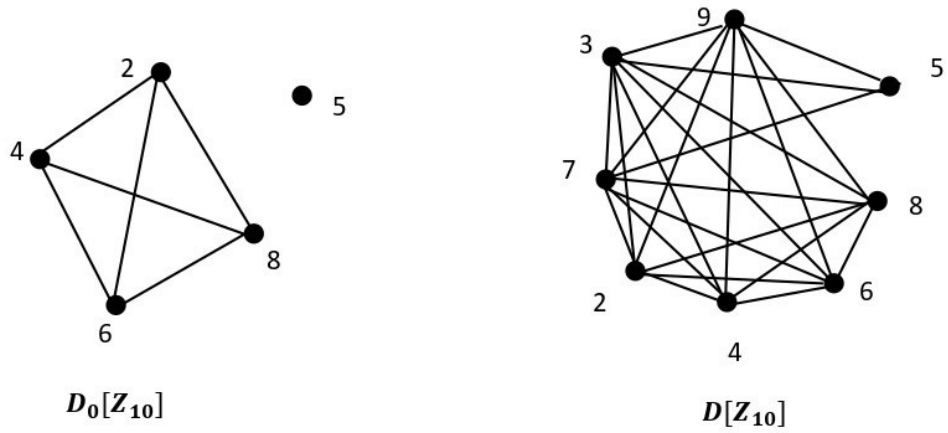


Figure 2: Modified Divisor Graph and Divisor Graph of Z_{10}

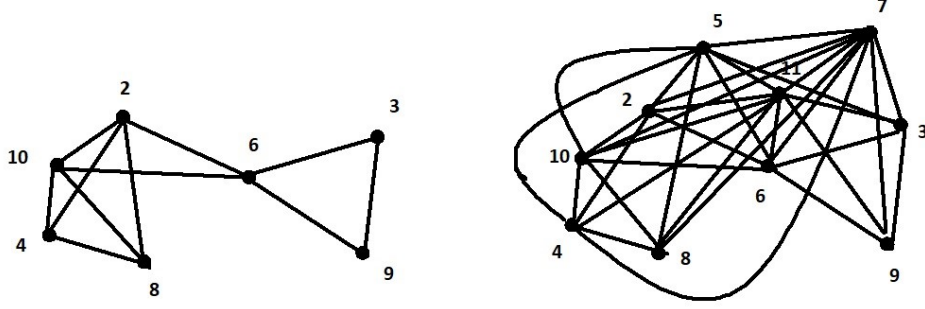


Figure 3: Modified Divisor Graph and Divisor Graph of Z_{12}

Remark 3.1. Mathil et al. [2022] If n be positive integer then $|A_{d_j}| = \phi(\frac{n}{d_j})$, for each proper divisor d_j of n .

Proposition 3.1. If $n = p_1 p_2 \cdots p_m$ then $|A_d| = (p_{(i+1)} - 1)(p_{(i+2)} - 1) \cdots (p_m - 1)$ for divisor $d = p_1 p_2 \cdots p_i, 3 \leq i \leq m$ of n .

Proof: Let $n = p_1 p_2 \cdots p_m$ and any divisor $d, d = p_1 p_2 \cdots p_i, 3 \leq i \leq m$ Given set $A_{d_j} = \{x \in Z_n \mid \gcd(x, n) = d_j\}$ and $|A_{d_j}| = \phi(\frac{n}{d_j})$. Therefore,

$$|A_d| = \phi(\frac{n}{d}) = \phi(p_i p_{(i+1)} \cdots p_m) = (p_{(i+1)} - 1)(p_{(i+2)} - 1) \cdots (p_m - 1)$$

Proposition 3.2. If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ and for divisor $d, d = p_1 p_2 \cdots p_i, 3 \leq i \leq m$ of n

$$|A_d| = \phi(p_3^{k_3} p_4^{k_4} \cdots p_m^{k_m}).$$

$$|T_d| = (p_1^{k_1-1} p_2^{k_2-1} \cdots p_m^{k_m-1})(p_1 - 1)(p_2 - 1) \cdots (p_{(i-1)} - 1)(p_{(i+1)} - 1)(p_m - 1).$$

Proof: Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ and d be non trivial divisor of n . We have, $A_d = \{x \in Z_n \mid \gcd(x, n) = d\}$ and $T_d = \{k \cdot d \mid k \in R^*, k \notin \langle d \rangle, d_i^s \neq d, s \in N\}$. To find $|A_d|, |T_d|$ consider simply $d = p_1^{k_1} p_2^{k_2}$. Thus,

$$|A_d| = \phi(\frac{n}{d}) = \phi(p_3^{k_3} p_4^{k_4} \cdots p_m^{k_m}).$$

Now suppose that $d = p_1$, then

$$\begin{aligned} T_{p_1} &= \{k \cdot p_1 \mid k \in R^*, k \notin \langle d_j \rangle, p_1^s \neq d_j, s \in N\} \\ \therefore |T_{p_1}| &= \left[\frac{n}{p_1} \right] - \left[\frac{n}{p_1 p_2} \right] - \left[\frac{n}{p_1 p_3} \right] \cdots - \left[\frac{n}{p_1 p_m} \right] + \left[\frac{n}{p_1 p_2 p_3} \right] + \left[\frac{n}{p_1 p_2 p_4} \right] + \cdots \\ &+ \left[\frac{n}{p_1 p_2 p_m} \right] + \left[\frac{n}{p_1 p_3 p_4} \right] + \left[\frac{n}{p_1 p_3 p_m} \right] \cdots + \cdots - \left[\frac{n}{p_1 p_2 p_3 p_4} \right] + \cdots \\ &+ (-1)^{(n-1)} \left[\frac{n}{p_1 p_2 p_3 \cdots p_m} \right] \end{aligned}$$

After simplifying, we will obtain,

$$|T_{p_1}| = (p_1^{k_1-1} p_2^{k_2-1} \cdots p_m^{k_m-1})(p_2 - 1)(p_3 - 1) \cdots ((p_m - 1).$$

Hence, if $d = p_1^{k_1} p_2^{k_2}$

$$|T_d| = (p_3^{k_3-1} p_4^{k_4-1} \cdots p_m^{k_m-1})(p_3 - 1)(p_4 - 1) \cdots (p_m - 1).$$

After the above discussion, we can now provide an important result of this article.

Theorem 3.1. Let R be a ring of integer modulo n and d_i, d_j be distinct non trivial divisors of n . if $u_i, u'_i \in A_{d_i}, v_j, v'_j \in A_{d_j}$ then following statements hold.

- (i) $u_i \sim u'_i$ and $v_j \sim v'_j$.
- (ii) $u_i \sim v_j$ if and only if $d_i \mid d_j$ or $d_j \mid d_i$,
- (iii) $u_i \approx v_j$ if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$

Proof: Let R be ring of residue modulo n and d_i, d_j be distinct non trivial divisors of n . we say $u \mid v$, when $\exists w \in R$ such that, $v = wu$.

(i) Let $u_i, u'_i \in A_{d_i}$ then $\gcd(u_i, n) = d_i$ and $\gcd(u'_i, n) = d_i$ i.e. $u_i = r_i \cdot d_i, u'_i = r'_i \cdot d_i$, for some $r_i, r'_i \in R$. Consider the following cases.

Case I: If r_i is multiple of r'_i then statement hold.

Case II: If either r_i or r'_i be unit element of ring then by theorem 3.6, Khandare et al. [2022], As every unit element is adjacent to other element of ring. Therefore $u_i \sim u'_i$. In the similar way, we obtain $v_j \sim v'_j$.

(ii) Let $u_i \in A_{d_i}, v_j \in A_{d_j}$. Suppose that $d_i \mid d_j$ i.e. $d_j = r \cdot d_i$ for some $r \in R$ and hence $v_j = r_j d_j = r_j \cdot r \cdot d_i$. This shows that $u_i \sim v_j$.

(iii) Let $u_i \in A_{d_i}, v_j \in A_{d_j}$. Suppose that $d_i \nmid d_j$ and $d_j \nmid d_i$. Assume that, $u_i \sim v_j$ then $\exists r \in R^*$ such that $v_j = r \cdot u_i = r \cdot r_i \cdot d_i$ i.e. $v_j \in A_{d_i}$. Which is contradiction to the set A_d . Hence $u_i \approx v_j$.

Corolary 3.1. The modified divisor graph $\mathbb{D}_0[R]$ is an induced subgraph of divisor graph $\mathbb{D}[R]$.

Proof: Theorem statement holds using definitions 2.1 and 2.2.

Corolary 3.2. If I be proper ideal of R then $\mathbb{D}_0[I]$ is proper subgraph of $\mathbb{D}_0[R]$.

Proof: The definition 2.2 provides the proof.

Theorem 3.2. A graph $\mathbb{D}_0[Z_n]$ is null if and only if $n = 1$ or $n = p$, for prime p .

Proof: Since $V(\mathbb{D}_0[Z_n]) = Z^*[R] = \phi$ when $n = 1$ and $n = p$. Hence the result is hold.

Theorem 3.3. If $n = p^k$ where $k > 1$ then modified divisor graph then $\mathbb{D}_0[Z_n]$ is complete. Moreover, $\mathbb{D}_0[Z_{p^k}] \cong K_{p^{k-1}-1}$.

Proof: Let $R = Z_n$ and $n = p^k, k > 1$ where p be prime. Then $|V| = n - \phi(n) - 1 = p^{k-1} - 1$. Here $p, p^2, p^3, \dots, p^k$ are proper divisors of n . Therefore $V = \cup A_{d_i} = A_p \cup A_{p^2} \cup A_{p^3} \cup \dots \cup A_{p^{k-1}}$. Also divisor $p \mid p^2 \mid p^3 \dots \mid p^{k-1}$ Hence by theorem 3.4, each vertex of the graph is adjacent to other vertex. Thus $\mathbb{D}_0[Z_{p^k}] \cong K_{p^{k-1}-1}$. Conversely assume $\mathbb{D}_0[Z_{p^k}]$ is a complete graph and $n \neq p^k$. So n may be expressed as a product of power of distinct primes. say $(n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_m^{k_m})$. Clearly $\exists p_1, p_2 \in V$ such that $p_1 \nmid p_2$. Thus we conclude, $\mathbb{D}_0[Z_{p^k}]$ is a complete graph if and only if $n = p^k$.

Theorem 3.4. *The modified divisor graph $\mathbb{D}_0[Z_n]$ is disconnected if and only if $n = pq$ for prime p and q . Moreover, it is a disjoint union of two connected components.*

Proof: Let R be a ring of residues modulo n and $n = p \cdot q$, where p and q are primes. As $p, q \in V$ and $p \nmid q$, Hence $\mathbb{D}_0[Z_n]$ is a disconnected graph. Conversely let $\mathbb{D}_0[Z_n]$ be disconnected. We claim that $n = p \cdot q$. Assume that, $n \neq pq$ i.e. n can be characterized in the following ways.

case I: If $n = p^k, k > 1$. Then from theorem 3.4, the result is true.

case II: If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$. With the given disjoint partition of vertex set $V = \cup A_{d_i}$, where $d_i \mid n$ and $A_{d_i} = \{x \in Z_n \mid \gcd(x, n) = d_i\}$. Let d_1, d_2 be any two proper divisor of n and $u_1 \in A_{d_1}, u_2 \in A_{d_2}$. If $\gcd(d_1, d_2) = 1$ then $\exists d_3 = d_1 d_2$ such that $d_1 \mid d_3$ and $d_2 \mid d_3$. Thus, using theorem 3.4, we obtain $u \in A_{d_3}$ such that $u_1 \sim u_3 \sim u_2$. If $\gcd(d_1, d_2) = d_4 > 1$ then again we can get the path $u_1 \sim u_4 \sim u_2$. Thus, a path is exist between any pair (u, v) of V . Hence $\mathbb{D}_0[Z_n]$ is disconnected graph if and only if $n = p \cdot q$.

Theorem 3.5. *If $n = pq$ then $\mathbb{D}_0[Z_n] \cong \bar{\Gamma}(Z_n)$.*

Proof: Let $R = Z_n, n = pq$. We have $V(\Gamma(R)) = V(\bar{\Gamma}(R)) = V(\mathbb{D}_0[R]) = Z^*(R)$. Since $u \sim v$ in $\Gamma(R)$ if and only if $uv = 0$. Let $u, v \in V$, such that, $uv = 0$. We claim that, $u \approx v$ in $\mathbb{D}_0[R]$. Clearly u is multiple of p and v multiple of q or vice versa. Using theorem 3.1, $u \approx v$ in $\mathbb{D}_0[R]$. Therefore $\mathbb{D}_0[R] \cong \bar{\Gamma}(R)$.

Theorem 3.6. *If R is ring of integers modulo n then $\mathbb{D}_0[R]$ is bipartite if and only if n is 1 or p or 9.*

Proof: Let R be a ring of integers modulo n . When n is 1 or p then there is nothing to prove as the graph is null. If $n = 9$ then the graph is isomorphic to K_2 , which is bipartite graph.

To prove converse part, suppose that graph $\mathbb{D}_0[R]$ is bipartite. i.e. graph without cycle. Clearly using theorem 3.1, we obtain the cycle of length 3 for $n \geq 6, n \neq 1, p, 4$. Thus $\mathbb{D}_0[R]$ is bipartite if and only if n is 1 or p or 9.

Theorem 3.7. *If $n \geq 14, n \neq p, pq, 15$ then $\mathbb{D}_0[R]$ is not planar and planar if $n \leq 15, n \neq 14, pq$.*

Proof: The number of non-zero zero divisor in Z_n are $n - \phi(n) - 1$. The Kuratowski's theorem, West [2001] says that graph is not planar if and only if it contains K_5 or $K_{3,3}$. Clearly for $n \leq 15, n \neq 14, n \neq pq$, $\mathbb{D}_0[Z_n]$ does not contains K_5 . When $n = 14$, an ideal of order 7 is exists in the ring. Hence by theorem 3.4, $\mathbb{D}_0[Z_7]$ contains K_5 . Thus $\mathbb{D}_0[Z_n]$ is planar if $n \leq 15$ unless $n = 14$.

Theorem 3.8. *The graph $\mathbb{D}_0[Z_n]$ is regular if and only if $n = p^k$*

Proof: The theorem 3.8 provides the proof.

4 Diameter and Girth :

In this section, diameter and girth of modified divisor graph $\mathbb{D}_0[Z_n]$ are investigated.

Theorem 4.1. *The diameter of the graph $\mathbb{D}_0[Z_n]$ is less than or equals to 4.*

Proof: Since $Diam(G) = Sup\{d(u, v) : u, v \text{ are vertices of graph.}$

Case I: If $n = 1$ or $n = pq$ or $n = p$ then the graph $\mathbb{D}_0[Z_n]$ is disconnected graph for or null. Hence $Diam(\mathbb{D}_0[Z_n]) = \infty$.

Case II: If $n = p^k, k > 1$, $\mathbb{D}_0[Z_n]$ is complete graph. Thus $Diam(\mathbb{D}_0[Z_n]) = 1$.

Case III: If $n \neq p^k$ for positive integer k . Then by theorem 3.4, if $n = p^2q$ then there is path $p^2 \sim p \sim pq \sim q$ of length 3. Also when $n = p^2q^2$ then there exists path $p^2 \sim p \sim pq \sim q \sim q^2$ of length 4. i.e. $Diam(\mathbb{D}_0[R]) = 4$. In general $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$. Clearly, $p_1 \mid p_1^{(k_1)} \nmid p_1^{(k_1-i)} p_2^{(k_2-j)} \nmid p_2^{k_2} \mid p_2$, for $i < k_1, j < k_2$. Thus, a path can be found between vertices $p_1^{(k_1)} \sim p_1 \sim p_1^{(k_1-i)} p_2^{k_2-j} \sim p_2 \sim p_2^{(k_2)}$ of length 4. Thus, diameter of the modified divisor graph $\mathbb{D}_0[Z_n]$ is less than or equals to 4.

Theorem 4.2. If R is ring of integer modulo n then

$$girth(\mathbb{D}_0[Z_n]) = \begin{cases} \infty & \text{if } n < 8, n = 9 \text{ or } n = p \\ 3 & \text{if } n \geq 8, n \neq 9 \text{ or } n \neq p \end{cases}$$

Proof: Let R be ring of integer modulo n . We prove this in following cases.

Case I: Let $n < 8$ or $n = 9$ or $n = p$ then $|V(\mathbb{D}_0[R])| < 3$ and hence girth is ∞ .

Case II: Let $n \geq 8$. Since $\mathbb{D}_0[Z_{p^k}]$ is complete graph of size $p^{k-1} - 1$. hence for $n = p^k$, we get smallest cycle of length 3 unless $n = 9$. Let $n \neq p^k, n \geq 8$ then there exists proper divisor $d \mid n$, such that $|A_{d_i}| \geq 3$. Therefore by theorem 3.4, we conclude that girth of $\mathbb{D}_0[Z_n]$ is 3.

5 Eulerian and Hamiltonian of $\mathbb{D}_0[Z_n]$:

In this section, the condition of graph to be Eulerian and Hamiltonian of $\mathbb{D}_0[Z_n]$ is discussed. Let we recall following propositions.

Proposition 5.1. A connected graph is Eulerian if and only if the degree of each vertex is even.

Proposition 5.2. The number of zero divisor in ring of integer modulo n are

$$n - \phi(n) = p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_m^{\alpha_m-1} [p_1 p_2 \cdots p_m - (p_1 - 1)(p_2 - 1) \cdots (p_m - 1)]$$

Theorem 5.1. The graph $\mathbb{D}_0[Z_n]$ is Eulerian if and only if n is an even number.

Proof: Let the following cases.

Case I: Let $n = p^k, k > 1$. Then using theorem 3.3, the graph $\mathbb{D}_0[Z_n]$ is complete of size $p^{k-1} - 1$. The degree of each vertex becomes $p^{k-1} - 2$. The number $p^{k-1} - 2$ is even only if p is an even prime. The proposition 5.1 provides the proof.

Case II: Let $n \neq p^k$ i.e. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ be prime factorization of n .

If n is odd. Then we get the prime vertex $p_2 \in V$ such that $d(p_2) = \left\lceil \frac{n}{p_2} \right\rceil - 2 = \text{odd} - 2 = \text{odd}$. Thus, the graph $\mathbb{D}_0[Z_n]$ is Eulerian if and only if n is even.

Theorem 5.2. *The graph $\mathbb{D}_0[Z_n]$ is Hamiltonian graph if and only if $n = p^k$, for positive integer k .*

Proof: Suppose that $n = p^k, k > 1$ then using Theorem 3.3, $d(u) = p^{k-1} - 2, u \in V$. Now,

$$\begin{aligned} p^{k-1} &\geq p^{k-1} \\ p^{k-1} - 1 &> p^{k-1} - 1 \\ p^{k-1} - 2 &\geq \frac{p^{k-1} - 1}{2} \\ \therefore d(u) &\geq \frac{p^{k-1} - 1}{2} = |V|, \forall u \in V \end{aligned}$$

Hence, using M. S. Rahman [2005], the graph $\mathbb{D}_0[Z_n]$ is Hamiltonian. In the other hand, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ is a prime factorisation of n . Assume that $\mathbb{D}_0[Z_n]$ is Hamiltonian graph. Let $p_1, p_2 \in V$ be any pair of non adjacent vertices. Without loss of generality, we suppose that, then $d(p_1) = p_1^{(k_1-1)} p_2^{k_2} - 2, d(p_2) = p_1^{k_1} p_2^{(k_2-1)} - 2$ and

$$\begin{aligned} d(p_1) + d(p_2) &= p_1^{(k_1-1)} p_2^{k_2} - 2 + p_1^{k_1} p_2^{(k_2-1)} - 2 \\ &= p_1^{(k_1-1)} p_2^{(k_2-1)} (p_2 + p_1) - 4 \\ &< p_1^{(k_1-1)} p_2^{(k_2-1)} (p_2 + p_1) - 1 \\ &< |V| \end{aligned}$$

Now using M. S. Rahman [2005], notice that, the assumption is wrong. Hence $\mathbb{D}_0[Z_n]$ is Hamiltonian graph if and only if $n = p^k$ for positive integer k .

6 Clique number

In this section, we provide formula to determine clique number of modified divisor graph $\mathbb{D}_0[Z_n]$.

Theorem 6.1. *The clique number of $\mathbb{D}_0[Z_n]$ is,*

$$cl(\mathbb{D}_0[Z_n]) = \begin{cases} \infty & \text{if } n < 4 \text{ or } n = p \\ p^{(k-1)} - 1 & \text{if } n = p^k, k > 1 \\ \sum_{i=1}^{k-1} \prod_{j=i+1}^m (p_j - 1) & \text{if } n = p_1 p_2 \cdots p_k \\ \sum_{i=1}^m \prod_{j=i}^m p_j^{k_j-1} (p_{j+1} - 1) & \text{if } n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_m^{k_m} \end{cases}$$

Proof: Clearly $\mathbb{D}_0[Z_n]$ is null graph when $n = 1$ or $n = p$. Therefore $cl(\mathbb{D}_0[Z_n]) = \infty$. Now, we consider following cases for the rest of values of n .

Case I: If $n = p^k, k > 1$ then graph $\mathbb{D}_0[Z_n]$ is complete graph of size $p^{(k-1)} - 1$. Therefore clique with maximum size is $p^{(k-1)} - 1$.

Case II: Let $n = p_1 p_2 p_3 \cdots p_m, m \geq 2$. With the given partition $V = \cup A_{d_i}$, $A_{d_i} = \{x \in Z_n \mid \gcd(x, n) = d_i\}$ and $|A_{d_i}| = \phi(\frac{n}{d_i}), 1 \leq i \leq k$. It is given that $A_{d_i} \leftrightarrow A_{d_j}, i \neq j$ when $d_i \mid d_j$ or $d_j \mid d_i$ and $A_{d_i} \nleftrightarrow A_{d_j}, i \neq j$ when

$d_i \nmid d_j$ and $d_j \nmid d_i$. As the clique number is size of largest complete subgraph of graph. note that, as $p_1 < p_i, \forall i$ then $| < p_1 > | > | < p_i > |$ Hence

$$\begin{aligned} cl(\mathbb{D}_0[Z_n]) &= |A_{p_1}| + |A_{p_1 p_2}| + \cdots + |A_{p_1 p_2 \cdots p_{(m-1)}}| \\ &= \phi\left(\frac{n}{p_1}\right) + \phi\left(\frac{n}{p_1 p_2}\right) + \cdots + \phi\left(\frac{n}{p_1 p_2 \cdots p_{(m-1)}}\right) \\ &= (p_2 - 1)(p_3 - 1) \cdots (p_m - 1) + (p_3 - 1) \cdots (p_m - 1) + \cdots + (p_m - 1) \\ &= \sum_{i=1}^{k-1} \prod_{k=i+1}^m (p_k - 1) \end{aligned}$$

Case III: Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_m^{k_m}, k_1 \geq 2$. Notice that $p_1^{k_1} \nmid p_1 p_2$ but $p_1^{k_1} \mid p_1^{k_1} p_2$, Also $p_1^{k_1} p_2^{k_2} \nmid p_1^{k_1} p_2 p_3$ but $p_1^{k_1} p_2^{k_2} \mid p_1^{k_1} p_2^{k_2} p_3$ and so on. Since set $T_{p_1} = \cup_{i=1}^{k_1} A_{p_1^i}$, so instead of using set A_{d_i} we must use T_{d_i} . Therefore

$$\begin{aligned} cl(\mathbb{D}_0[Z_n]) &= |T_{p_1}| + |T_{p_1^{k_1} p_2}| + |T_{p_1^{k_1} p_2 p_3}| + \cdots + |T_{p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_m}| \\ &= p_1^{k_1-1} p_2^{k_2-1} \cdots p_m^{k_m-1} (p_2 - 1)(p_3 - 1) \cdots (p_m - 1) + p_2^{k_2-1} \cdots p_m^{k_m-1} (p_3 - 1) \\ &\quad + \cdots (p_m - 1) + p_3^{k_3-1} \cdots p_m^{k_m-1} (p_4 - 1) \cdots (p_m - 1) + \cdots + p_m^{k_m-1} - 1 \\ &= \sum_{i=1}^m \prod_{j=i}^m p_j^{k_j-1} (p_{j+1} - 1) \end{aligned}$$

Example 6.1. For $n = 2 \times 3 \times 5$ then $cl(\mathbb{D}_0[Z_n]) = 12$

Let $n = 2 \times 3 \times 5$ then by theorem 6.1, $cl(\mathbb{D}_0[Z_n]) = \sum_{i=1}^{k-1} \prod_{k=i+1}^m (p_k - 1)$
 $\therefore cl(\mathbb{D}_0[Z_n]) = (p_2 - 1)(p_3 - 1) + (p_3 - 1) = 2 \times 4 + 4 = 12$

Example 6.2. For $n = 2^2 \times 3^2 \times 5^2$ then $cl(\mathbb{D}_0[Z_n]) = 304$

As $2^2 \nmid 2 \cdot 3$, $2^2 \cdot 3^2 \nmid 2^2 \cdot 3 \cdot 5^2$ Thus,

$$\begin{aligned} cl(\mathbb{D}_0[Z_n]) &= |T_2| + |T_{2^2 \cdot 3}| + |T_{2^2 \cdot 3^2 \cdot 5}| \\ &= 2 \cdot 3 \cdot 5(2 \times 4) + 3 \cdot 5(4) + 5 - 1 \\ &= 304 \end{aligned}$$

Theorem 6.2. If R is ring of integer modulo n then $cl(\mathbb{D}_0[Z_n]) = \chi(\mathbb{D}_0[Z_n]), \forall n$.

Proof: Let R be ring of integer modulo n and $cl(\mathbb{D}_0[Z_n]) = m$. Let $\{u_1, u_2, u_3, \cdots, u_m\}$ be maximal clique. Assume that there exists a vertex $u_i, i \neq m$ with i -color different from other color. Then the set $\{u_1, u_2, u_3, \cdots, u_m\} \cup \{u_i\}$ becomes maximal clique. Which is contradiction. Hence $cl(\mathbb{D}_0[Z_n]) = \chi(\mathbb{D}_0[Z_n]), \forall n$.

Theorem 6.3. If R is ring of integer modulo n then following results are equivalent.

- (i) $\mathbb{D}_0[Z_n]$ is perfect.
- (ii) $\mathbb{D}_0[Z_n]$ does not contain K_5 as a subgraph.
- (iii) $n \leq 15, n \neq 14$.

Proof: (i) $\Rightarrow$ (ii) Let $\mathbb{D}_0[Z_n]$ is perfect and assume that $\mathbb{D}_0[Z_n]$ have K_5 as a subgraph with vertices u_1, u_2, u_3, u_4, u_5 Then $u_1 \sim u_2 \sim u_3 \sim u_4 \sim u_5$ is an induced 5-cycle in $\mathbb{D}_0[Z_n]$, is a contradiction to perfectness. Hence $\mathbb{D}_0[Z_n]$ does not contain K_5 .

(ii) $\Rightarrow$ (iii) If $\mathbb{D}_0[Z_n]$ does not contain K_5 . Then by definition 2.1, It is easy to show that $n \leq 15, n \neq 14$.

(i) $\Rightarrow$ (iii) follows from $\mathbb{D}_0[Z_n]$ has an induced cycle if and only if $n \geq 15, n \neq 14$.

Corolary 6.1. *If $n \neq 15, n \geq 14$ then $cl(\mathbb{D}_0[Z_n])$ is weakly perfect graph.*

Proof: The proof is followed by theorem 6.5.

7 Vertex and Edge Connectivity in $\mathbb{D}_0[Z_n]$:

In this section, vertex and edge connectivity in $\mathbb{D}[Z_n]$ are calculated.

Theorem 7.1. *If R is ring of integer modulo n then a vertex connectivity of zero divisor graph $\kappa(\mathbb{D}[Z_n]) = p^{k-1} - 2$.*

Proof: Since the vertex set of $\mathbb{D}_0[R] = Z^*(R)$, set of non-zero zero divisor. If $n = p^k$ then $p, p^2, p^3, \dots, p^k$ are proper divisors of n . Therefore $V = \cup A_{d_i} = A_p \cup A_{p^2} \cup A_{p^3} \cup \dots \cup A_{p^{k-1}}$ and $|V| = p^{k-1} - 1$. As the graph corresponding to these vertices is complete of size $p^{k-1} - 1$. deletion of one or two vertices does not give a disconnected graph. If we delete $p^{k-1} - 2$ vertices then only one vertex remains and it results to disconnected graph. Thus minimum number of vertices required to obtain disconnected graph are $p^{k-1} - 2$. Hence $\kappa(\mathbb{D}[Z_n]) = p^{k-1} - 2$.

Theorem 7.2. *If R is ring of integer modulo n and $n = p_1 p_2 \dots p_m, p_1 < p_2 < \dots < p_m$ then vertex connectivity $\kappa(\mathbb{D}[Z_n]) = p_1 p_2 \dots p_{(m-1)} - \phi(p_1 p_2 \dots p_{(m-1)}) - 1$*

Proof: Let for the divisor $d_1, d_2, d_3, \dots, d_k$ of n , vertex set $V, V = \cup A_{d_i}$. Here notice that $|A_{p_m}| < |A_{p_i}|, \forall i = 1, 2, \dots, (m-1)$. As by theorem 3.4, vertices of set A_{p_m} are adjacent to each other vertex of A_{p_m} and all vertices of set A_{d_j} where $\gcd(d_j, p_m) = p_m$. So deletion of vertices of A_{d_j} leaves vertices of A_{p_m} isolated. Thus graph is disconnected when we remove minimal set $A_{d_j}, \gcd(d_j, p_m) = p_m, d_j \neq p_m$. Therefore vertex connectivity of the graph,

$$\begin{aligned} \kappa(\mathbb{D}[Z_n]) &= \sum_j |A_{d_j}|, \gcd(d_j, p_m) = p_m, d_j \neq p_m \\ &= |A_{p_1 p_m}| + |A_{p_2 p_m}| + \dots + |A_{p_{m-1} p_m}| + |A_{p_1 p_2 p_m}| + \dots + \dots \\ &+ |A_{p_1 p_2 \dots p_{m-2} p_m}| \\ &= p_1 p_2 \dots p_{(m-1)} - \phi(p_1 p_2 \dots p_{(m-1)}) - 1 \end{aligned}$$

Corolary 7.1. *If $n = p_1 p_2$, then vertex connectivity of $\mathbb{D}_0[Z_n]$ is zero.*

Proof: Since for $n = p_1 p_2, \mathbb{D}_0[Z_n]$ is disconnected graph. Hence vertex connectivity is 0.

$$\begin{aligned} \kappa(\mathbb{D}_0[Z_n]) &= \sum_j |A_{d_j}|, \gcd(d_j, p_2) = p_2, d_j \neq p_2 \\ &= p_1 - \phi(p_1) - 1 = 0 \end{aligned}$$

Corolary 7.2. *If $n = p_1 p_2 p_3$, then vertex connectivity of $\mathbb{D}_0[Z_n]$ is $p_1 + p_2 - 2$.*

Proof: Let $n = p_1 p_2 p_3$, then from theorem 7.2,

$$\begin{aligned} \kappa(\mathbb{D}_0[Z_n]) &= \sum_j |A_{d_j}|, \gcd(d_j, p_3) = p_3, d_j \neq p_3 \\ &= |A_{p_1 p_3}| + |A_{p_2 p_3}| = (p_2 - 1) + (p_1 - 1) = p_1 + p_2 - 2 \end{aligned}$$

Theorem 7.3. If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ then vertex connectivity $\kappa(\mathbb{D}_0[Z_n]) = \sum_j |T_{d_j}|, \gcd(d_j, p_m) = p_m, d_j \neq p_m$

Proof: If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ then vertex set V can be divide such that, $V = \cup T_{d_i}, d_i \mid n$. Here also we found that $p_1^{k_1} \nmid p_j^{k_j}, j = 2, 3, 4, \dots, m$ but $T_{p_1} \leftrightarrow T_{p_j}$, since $p_1^{k_1} \mid p_1^{k_1} p_j^{k_j}, \forall j = 2, 3, 4, \dots, m$ and $p_m^{k_m} \nmid p_m^{k_m-1} p_i, i = 1, 2, 3 \cdots (m-1)$. Therefore each vertex of T_{p_m} is adjacent to other vertices of A_{p_m} and some vertices are adjacent with T_{d_j} , where $\gcd(d_j, p_m) = p_m$. As the vertex connectivity is minimum number of vertices whose removal disconnected the graph are $\sum_i T_{d_j}, \gcd(d_j, p_m) = p_m, d_j \neq p_m$. Hence,

$$\begin{aligned} \kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|, \gcd(d_j, p_m) = p_m, d_j \neq p_m^s, s = 1, 2, 3, \dots, k_m \\ &= |T_{p_1 p_m}| + |T_{p_2 p_m}| + \cdots + |T_{p_{m-1} p_m}| + |T_{p_1 p_2 p_m}| + \cdots + \cdots \\ &+ |T_{p_1 p_2 \cdots p_{m-2} p_m}| \end{aligned}$$

since $T_{p_1^{k_1} p_m^{k_m}} \subset T_{p_1 p_m}, T_{p_1^{k_1} p_2^{k_2} p_m^{k_m}} \subset T_{p_1 p_2 p_m}$ and so on.

Corollary 7.3. If $n = p_1^{k_1} p_2^{k_2}, p_1 < p_2$ then $\kappa(\mathbb{D}_0[Z_n]) = p_1^{k_1-1} p_2^{k_2-1} - 1$.

Proof: Let $n = p_1^{k_1} p_2^{k_2}, p_1 < p_2$, from theorem 7.4,

$$\begin{aligned} \kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|, \gcd(p_2, d_j) = p_2, d_j \neq p_2^s, s = 1, 2, \dots, k_2 \\ &= |T_{p_1 p_2}| = p_1^{k_1-1} p_2^{k_2-1} - 1 \end{aligned}$$

Corollary 7.4. If $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}, p_1 < p_2 < p_3$, then $\kappa(\mathbb{D}_0[Z_n]) = p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1} (p_1 + p_2 - 1)$.

Proof: Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}, p_1 < p_2 < p_3$, from theorem 7.4,

$$\begin{aligned} \kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|, \gcd(p_3, d_j) = p_3, d_j \neq p_3^s, s = 1, 2, 3, \dots, k_3 \\ &= |T_{p_1 p_3}| + |T_{p_2 p_3}| = p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1} (p_1 + p_2 - 1) \end{aligned}$$

Theorem 7.4. If $s = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ and $t = p_1 p_2 \cdots p_m$ then vertex connectivity of the graph,

$$\kappa(\mathbb{D}_0[Z_s]) = p_1^{k_1-1} p_2^{k_2-1} \cdots p_m^{k_m-1} \kappa(\mathbb{D}_0[Z_t]) - 1, t \neq p_1 p_2$$

Proof: Theorem 6.2 and 6.3 follows the proof.

Theorem 7.5. If $n = p^k$ then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = p^{k-1} - 2$.

Proof: Let R be ring of integer modulo n and $n = p^k$. As the vertex set $V = Z^*[R]$, and $|V| = p^{k-1} - 1$. Since every vertex incident to $p^{k-1} - 2$ edges. So deletion of this $p^{k-1} - 2$ with respect to fixed vertex u , the graph becomes disconnected and vertex u becomes isolated. Hence $\kappa_e(\mathbb{D}_0[Z_{p^k}]) = p^{k-1} - 2$.

Theorem 7.6. If $n = p_1 p_2 \cdots p_m, p_1 < p_2 < \cdots < p_m$ then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = p_1 p_2 \cdots p_{(m-1)} - 1$.

Proof: We have the partition of the vertex set, $V = \cup A_{d_i}, d_i \mid n$. Clearly the prime p_m is largest, then the size of set A_{p_m} will be less. So degree of vertices of set A_{p_m} is less than the degree of vertices of other set $A_{p_i}, i = 1, 2, 3, \dots, (m-1)$. Therefore removal of vertex of set A_{p_m} gives disconnected graph. Since by theorem 3.4, each vertex of A_{p_m} is adjacent with each other and adjacent to vertices of set A_{d_j} , where $\gcd(d_j, p_m) = p_m$. If we remove all edges incident to the fixed vertex $u \in A_{p_m}$ then u becomes isolated. Thus edge connectivity,

$$\begin{aligned} \kappa_e(\mathbb{D}_0[Z_n]) &= d(u) \\ &= \left(\sum_j |A_{d_j}|, \gcd(d_j, p_m) = p_m \right) - 1 \\ &= |A_{p_m}| - 1 + |A_{p_1 p_m}| + |A_{p_2 p_m}| + \dots + |A_{p_{m-1} p_m}| + |A_{p_1 p_2 p_m}| \\ &\quad + \dots + \dots + |A_{p_1 p_2 \dots p_{m-2} p_m}| \\ &= p_1 p_2 \dots p_{(m-1)} - 1 \end{aligned}$$

Corollary 7.5. If $n = p_1 p_2$, then edge connectivity of $\mathbb{D}_0[Z_n]$ is zero.

Proof: The proof follows from theorem 7.6.

Corollary 7.6. If $n = p_1 p_2 p_3$, then edge connectivity of $\mathbb{D}_0[Z_n]$ is $\kappa_e(\mathbb{D}_0[Z_n]) = p_1 p_2 - 1$.

Proof: Let $n = p_1 p_2 p_3$, then from theorem 7.6, $\kappa_e(\mathbb{D}_0[Z_n]) = d(p_3) = |A_{p_3}| - 1 + |A_{p_1 p_3}| + |A_{p_2 p_3}| = p_1 p_2 - 1$

Theorem 7.7. If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}, p_1 < p_2 < \dots < p_m$ then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1} p_2^{k_2} \dots p_m^{k_{(m-1)}}) \times [p_m^{k_m} - 1] + p_1^{k_1} p_2^{k_2} \dots p_m^{k_{(m-1)}} - 2$.

Proof: Let $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}, p_1 < p_2 < \dots < p_m$. we should note that $p_m^{k_m} \nmid p_1^{k_1} p_2^{k_2} \dots p_m^{k_{m-1}}$ and $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$. Clearly the degree of vertex of $T_{p_m^{k_m}}$ is smaller than degree of all vertices of other set. Since $p_m^{k_m} \in V$, So removal of all edges incident to vertex $p_m^{k_m}$ makes graph disconnected. Therefore minimum number of edges whose removal disconnected the graph are ,

$$\begin{aligned} \kappa_e(\mathbb{D}_0[Z_n]) &= d(p_m^{k_m}) \\ &= \phi(p_1^{k_1} p_2^{k_2} \dots p_m^{k_{(m-1)}}) \times [p_m^{k_m} - 1] + p_1^{k_1} p_2^{k_2} \dots p_m^{k_{(m-1)}} - 2 \end{aligned}$$

Corollary 7.7. If $n = p_1^{k_1} p_2^{k_2}$, then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}) \times [p_2^{k_2} - 1] + p_1^{k_1} - 2$

Proof: Let $n = p_1^{k_1} p_2^{k_2}$, then from theorem 7.13,

$$\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}) \times [p_2^{k_2} - 1] + p_1^{k_1} - 2$$

Corollary 7.8. If $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$, then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1} p_2^{k_2}) \times [p_3^{k_3} - 1] + p_1^{k_1} p_2^{k_2} - 2$

Proof: Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$, then from theorem 7.13,

$$\begin{aligned} \kappa_e(\mathbb{D}_0[Z_n]) &= d(p_3^{k_3}) \\ &= \phi(p_1^{k_1} p_2^{k_2}) \times [p_3^{k_3} - 1] + p_1^{k_1} p_2^{k_2} - 2 \end{aligned}$$

Following table shows vertex connectivity, edge connectivity of $\mathbb{D}_0[Z_n]$ for some values n.

n	$\kappa(\mathbb{D}_0[Z_n])$	$\kappa_e(\mathbb{D}_0[Z_n])$
$n = 2^{15} = 32768$	16382	16382
$n = 3^{10} = 59049$	19681	19681
$n = 11 \times 13$	0	0
$n = 5 \times 7 \times 11$	10	33
$n = 2^4 \times 5^3$	199	206

Table 1: vertex and edge connectivity of $(\mathbb{D}_0[Z_n])$

Discussion and Conclusion

In this research article, The structure of modified divisor graph $\mathbb{D}_0[R]$ of ring R is introduced and discussed. Moreover, various interplay between Z_n and $\mathbb{D}_0[Z_n]$ were discussed. In this article, we characterized the value of n, for which the graph $\mathbb{D}_0[Z_n]$ is perfect, Eulerian, Hamiltonian, planar, regular, complete and connected. Apart from this, we determine diameter, girth, clique number, vertex connectivity and edge connectivity of the $\mathbb{D}_0[Z_n]$ for all natural values of n.

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Conflict of interest

There is no conflict of interest.

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Non-linear difference polynomials sharing a polynomial with finite weight

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Abstract

The uniqueness theory of meromorphic function mainly studies the conditions under which there exists only one function satisfying these conditions. The uniqueness theory of entire and meromorphic functions has grown up as an extensive sub-field of value distribution theory and the Nevanlinna's Five value and Four value theorems serves as the starting point of this uniqueness theory. In this paper, we consider a linear difference polynomial $\mathcal{L}_\eta(f) = f(z + \eta) + \eta_0 f(z)$, of the finite ordered non-constant meromorphic function f , with η and η_0 being finite non-zero complex constants, and with the help of Nevanlinna theory, we analyse the uniqueness results between two finite ordered non-constant meromorphic functions f and g , when their non-linear difference polynomials $f^n(z)\mathcal{L}_\eta(f)$ and $g^n(z)\mathcal{L}_\eta(g)$, with $n \geq 2$ being a positive integer shares a non-zero polynomial $p(z)$ with finite weights 0,1 and 2. Our results extend and improve some of the earlier results of Majumder (*Applied Mathematics E-Notes*, (17): 114-123, 2017).

Keywords: Meromorphic functions; difference polynomials; weighted sharing; uniqueness.

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1 Introduction

The Nevanlinna theory, is one of the several branches of complex analysis which has seen many fine works. It plays a key role in studying oscillation of complex differential equations. This theory mainly deals with studying the distribution of the zeros of the equation $f(z) = \beta$ in a disc $|z| \leq r$, where f is an entire or meromorphic function in the complex plane $\mathbb{C}$, $z \in \mathbb{C}$ and $\beta \in \mathbb{C} \cup \{\infty\}$. One can look into (Hayman [8], Yi and Yang [21], Yang [23]) for the standard definitions and notations of the theory.

Let $\mathcal{T} = \{f(z)/g(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a non-constant meromorphic function}\}$. For $f, g \in \mathcal{T}$, $\beta \in \mathbb{C} \cup \{\infty\}$ and $s \in \mathbb{Z}^+ \cup \{\infty\}$, the set $E(\beta, f) = \{z : f(z) - \beta = 0\}$ denotes all those β -points of f , where each β -point of f with multiplicity s is counted s times in the set and the set $\overline{E}(\beta, f) = \{z : f(z) - \beta = 0\}$, denotes all those β -points of f , where the multiplicities are ignored. If $f(z) - \beta$ and $g(z) - \beta$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value β CM (counting multiplicity) and we have $E(\beta, f) = E(\beta, g)$; Suppose, if $f(z) - \beta$ and $g(z) - \beta$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value β IM (ignoring multiplicity) and we will have $\overline{E}(\beta, f) = \overline{E}(\beta, g)$. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then f and g share ∞ CM.

In general, for a meromorphic function $f(z)$, the quantity $m(r, f)$ denotes the proximity function of $f(z)$, while $N(r, f)$ denotes the counting function of poles of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r, \beta; f)$ denotes the counting function of β points of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, \beta; f)$ denotes the reduced counting function when multiplicities are ignored). The notation $N(r, \beta; f | = 1)$ denotes the counting function of simple β -points of f and the notation $N(r, \beta; f | \geq 2)$ denotes the counting function of those β -points of f whose multiplicities are atleast 2 (respectively, $\overline{N}(r, \beta; f | = 1)$ and $\overline{N}(r, \beta; f | \geq 2)$ denotes the reduced counting functions).

Suppose f and g share 1 IM and z_0 is a zero of $f(z) - 1$ of order s and also a zero of $g(z) - 1$ of order t , then $\overline{N}_L(r, 1; f)$ counts those 1-points of $f(z)$ and $g(z)$ where $s > t$, $\overline{N}_E^{(1)}(r, 1; f)$ counts those 1-points of $f(z)$ and $g(z)$ where $s = t = 1$, $\overline{N}_E^{(2)}(r, 1; f)$ counts those 1-points of $f(z)$ and $g(z)$ where $s = t \geq 2$, $\overline{N}_{f>2}(r, 1; g)$ counts those 1-points of $f(z)$ and $g(z)$ where $s > t = 2$. It is to be noted that each point in these counting functions are counted only once. Similarly $\overline{N}_L(r, 1; g)$, $\overline{N}_E^{(2)}(r, 1; g)$, $\overline{N}_{g>2}(r, 1; f)$ are defined.

The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r, f)$. We have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non-

negative. A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity, which satisfies $S(r, f) = o(T(r, f))$ as $r \rightarrow +\infty$ possibly outside a set I with finite linear measure $\lim_{r \rightarrow \infty} \int_{(1,r] \cap I} \frac{dt}{t} < \infty$.

2 Definitions and Theorems

Let us recall the following standard definitions of Nevanlinna theory.

Definition 2.1. [8] The order $\rho(f)$ of a meromorphic function $f(z)$ is defined as,

$$\rho(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

Definition 2.2. [11] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with the weight k .

The definition implies that if f and g share a value a with the weight k , then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n . We write f, g share (a, k) to mean that, f, g share the value a with the weight k . Clearly, if f, g share (a, k) , then f, g share (a, p) for any integer p , such that, $0 \leq p < k$. Also we note that, f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2.3. [11] Let f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g .

Clearly, $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Around 2006, with the development of the difference analogue of Logarithmic Derivative Lemma [6], Halburd and Korhonen laid the foundation for the difference analogue of Nevanlinna theory and developed the Nevanlinna theory for the difference operator [5] wherein they extended the classical Nevanlinna theory based on certain estimates involving the derivative $f \mapsto f'$ of a meromorphic function to a theory for the exact difference $f \mapsto \Delta f = f(z+c) - f(z)$. Their results gave the difference analogue of the second main theorem of Nevanlinna theory, as well as the difference analogues of the Nevanlinna defect relation, Picards theorem and

Nevanlinna's five value theorem. In 2008, Chiang and Feng [4] investigated the growth of Nevanlinna characteristic of $f(z+c)$ for a fixed $c \in \mathbb{C}$ and obtained a precise asymptotic relation between $T(r, f(z+c))$ and $T(r, f)$, which is only true for finite ordered meromorphic functions. They also obtained the proximity function and pointwise estimates of $f(z+c)/f(z)$ which were discrete versions of the classical logarithmic derivative estimates of $f(z)$. Then they applied those results to give new growth estimates of meromorphic solutions to higher order linear difference equations. Since then many mathematicians showed keen interest on value distribution and uniqueness problems of meromorphic functions, their shifts, difference equations and difference operators. Some fine works can be seen in ([3], [7], [9], [10], [13], [14], [15], [16], [18], [19], [20], [24]). One such studies can be seen in Liu et al., [15], wherein they have studied the uniqueness of the difference monomials and obtained the following results.

Theorem A. [15] Let $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Let $f, g \in \mathcal{T}$. Suppose $\eta \in \mathbb{C} - \{0\}$ and n be any natural number, such that $n \geq 14$, with $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

Theorem B. [15] Let $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Let $f, g \in \mathcal{T}$. Suppose $\eta \in \mathbb{C} - \{0\}$, n be any natural number, such that $n \geq 26$, with $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 IM, then either $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$, where $t^{n+1} = 1$.

Later, using the concept of weighted sharing introduced by Lahiri [11] (given in Definition 2.2), in 2015 Liu et al., [16] improved Theorems A, B and obtained the following results.

Theorem C. [16] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Suppose $n, k \in \mathbb{N}$, such that $n \geq 14$, $k \geq 3$ and if $E_k(1, f^n(z)f(z+\eta)) = E_k(1, g^n(z)g(z+\eta))$, then either $f(z) \equiv t_1g(z)$ or $f(z)g(z) \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem D. [16] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 16$, and if $E_2(1, f^n(z)f(z+\eta)) = E_2(1, g^n(z)g(z+\eta))$, then either $f(z) \equiv t_1g(z)$ or $f(z)g(z) \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem E. [16] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 22$, and if $E_1(1, f^n(z)f(z+\eta)) = E_1(1, g^n(z)g(z+\eta))$, then either $f(z) \equiv t_1g(z)$ or $f(z)g(z) \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Later, in 2017 Majumder [17] studied Theorems C, D and E for polynomial sharing instead of sharing value 1 and obtained the following generalized results.

Theorem F. [17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/g(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 14$, $p(z) (\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$ and if $f^n(z)f(z+\eta) - p(z)$ and $g^n(z)g(z+\eta) - p(z)$, share $(0, 2)$, then one of the following two cases holds:

- (1) $f(z) \equiv tg(z)$ for some constant t such that $t^{n+1} = 1$,
- (2) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a non-zero constant η and t is a constant such that $t^{n+1} = \eta^2$.

Theorem G. [17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/g(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 16$, $p(z) (\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$ and if $f^n(z)f(z+\eta) - p(z)$ and $g^n(z)g(z+\eta) - p(z)$, share $(0, 1)$, then conclusion of Theorem F holds.

Theorem H. [17] Let $\eta \in \mathbb{C} - \{0\}$ and $\mathcal{T} = \{f(z)/g(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite order transcendental meromorphic function}\}$. Suppose $n \in \mathbb{N}$, such that $n \geq 26$, $p(z) (\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$ and if $f^n(z)f(z+\eta) - p(z)$ and $g^n(z)g(z+\eta) - p(z)$, share $(0, 0)$, then conclusion of Theorem F holds.

In the light of above mentioned results, it becomes sense to pose the following questions.

Question 1. Suppose $\mathcal{L}_\eta(f) = f(z+\eta) + \eta_0 f(z)$, is a linear difference polynomial of the finite ordered non-constant meromorphic functions f , with η and η_0 being finite non-zero complex constants, then what can we say about the relation between two finite ordered non-constant meromorphic functions f and g , if their non-linear difference polynomials $f^n(z)\mathcal{L}_\eta(f)$ and $g^n(z)\mathcal{L}_\eta(g)$ share a non-zero polynomial $p(z)$, with $n \geq 2$ being a positive integer?

Question 2. Is it possible to reduce the lower bound of n any further in Theorems F, G and H?

In this paper, we have attempted to answer the above posed questions successfully.

3 Main Results

Following are the main results of our paper.

Theorem 3.1. Let $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Let $\eta \in \mathbb{C} - \{0\}$ and $n \in \mathbb{N}$, such that $n > 11$. Suppose $f^n \mathcal{L}_\eta(f) - p(z)$ and $g^n \mathcal{L}_\eta(g) - p(z)$ share $(0, 2)$, where $p(z)$ be a non-zero polynomial such that $2\deg(p) < n - 2$ then one of the following two cases holds:

(1) $f \equiv tg$, for some constant t , such that $t^{n+1} = 1$,

(2) $fg \equiv d^2$, where $p(z)$ reduces to a non-zero constant d , such that

(i) when $\eta_0 = 0$, then f and g takes the form $f = e^U$ and $g = te^{-U}$, where U is a non-constant polynomial and t is a constant such that $t^{n+1} = d^2$.

(ii) when $\eta_0 \neq 0$, then f and g takes the form $f(z) = \eta_1 e^{az}$ and $g(z) = \eta_2 e^{-az}$, where a, η_1, η_2 are non-zero constants satisfying $(\eta_1 \eta_2)^{n+1} (e^{a\eta} + \eta_0)(e^{-a\eta} + \eta_0) = d^2$.

Theorem 3.2. Let $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Let $\eta \in \mathbb{C} - \{0\}$ and $n \in \mathbb{N}$, such that $n > 25/2$. Suppose $f^n \mathcal{L}_\eta(f) - p(z)$ and $g^n \mathcal{L}_\eta(g) - p(z)$ share $(0, 1)$, where $p(z)$ be a non-zero polynomial such that $2\deg(p) < n - 2$, then one of the conclusion of Theorem 3.1 holds.

Theorem 3.3. Let $\mathcal{T} = \{f(z)/f(z) : \mathbb{C} \rightarrow \mathbb{C} \text{ is a finite ordered transcendental meromorphic function}\}$. Let $\eta \in \mathbb{C} - \{0\}$ and $n \in \mathbb{N}$, such that $n > 14$. Suppose $f^n \mathcal{L}_\eta(f) - p(z)$ and $g^n \mathcal{L}_\eta(g) - p(z)$ share $(0, 0)$, where $p(z)$ be a non-zero polynomial such that $2\deg(p) < n - 2$, then one of the conclusion of Theorem 3.1 holds.

4 Remarks

If we take $\eta_0 = 0$ then,

1. Theorem 3.1, reduces to Theorem F with $n > 11$,
2. Theorem 3.2, reduces to Theorem G with $n > 25/2$,
3. Theorem 3.3, reduces to Theorem H with $n > 14$.

Thus, Theorems 3.1, 3.2, 3.3 are improvements of Theorems F, G, H respectively.

Clearly, for a specific choice of the constant $\eta_0 = -1$, we have $\mathcal{L}_\eta(f) = \Delta_\eta f$, and we can obtain similar uniqueness results for the relationship between $f^n \Delta_\eta f$ and $g^n \Delta_\eta g$, where $\Delta_\eta f = f(z + \eta) - f(z)$.

5 Lemmas

In this section we provide all the necessary lemmas required to prove our theorems.

Let us define,

$$H = \left(\frac{\mathfrak{F}''}{\mathfrak{F}'} - \frac{2\mathfrak{F}'}{\mathfrak{F} - 1} \right) - \left(\frac{\mathfrak{G}''}{\mathfrak{G}'} - \frac{2\mathfrak{G}'}{\mathfrak{G} - 1} \right). \quad (5.1)$$

Lemma 5.1. [4] Let $\eta \in \mathbb{C} - \{0\}$ be fixed and $\mathfrak{f}(z)$ be a non-constant finite ordered meromorphic function. Then for each $\epsilon > 0$, we have

$$m \left(r, \frac{\mathfrak{f}(z + \eta)}{\mathfrak{f}(z)} \right) + m \left(r, \frac{\mathfrak{f}(z)}{\mathfrak{f}(z + \eta)} \right) = O(r^{\rho-1+\epsilon}) = S(r, \mathfrak{f}).$$

Lemma 5.2. [22] Let $\mathfrak{f}(z)$ be a non-constant meromorphic function and for $i = 0, 1, 2, \dots, n$, let $\alpha_n(z)$ ($\neq 0$), $\alpha_{n-1}(z), \dots, \alpha_0(z)$ be meromorphic functions satisfying $T(r, \alpha_i(z)) = S(r, \mathfrak{f})$. Then

$$T(r, \alpha_n \mathfrak{f}^n + \alpha_{n-1} \mathfrak{f}^{n-1} + \dots + \alpha_1 \mathfrak{f} + \alpha_0) = nT(r, \mathfrak{f}) + s(r, \mathfrak{f}).$$

Lemma 5.3. [9] Let $\eta \in \mathbb{C}$ be fixed and $\mathfrak{f}(z)$ be a non-constant finite ordered meromorphic function. Then

$$\begin{aligned} N(r, 0; \mathfrak{f}(z + \eta)) &\leq N(r, 0; \mathfrak{f}(z)) + S(r, \mathfrak{f}), \\ N(r, \infty; \mathfrak{f}(z + \eta)) &\leq N(r, \infty; \mathfrak{f}) + S(r, \mathfrak{f}), \\ \overline{N}(r, 0; \mathfrak{f}(z + \eta)) &\leq \overline{N}(r, 0; \mathfrak{f}(z)) + S(r, \mathfrak{f}), \\ \overline{N}(r, \infty; \mathfrak{f}(z + \eta)) &\leq \overline{N}(r, \infty; \mathfrak{f}) + S(r, \mathfrak{f}). \end{aligned}$$

Lemma 5.4. [4] Let $\eta \in \mathbb{C}$ be fixed and $\mathfrak{f}$ be a finite ordered transcendental meromorphic function. Then

$$T(r, \mathfrak{f}(z + \eta)) = T(r, \mathfrak{f}) + S(r, \mathfrak{f}).$$

Lemma 5.5. [23] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two non-constant meromorphic functions. Then

$$N \left(r, \infty; \frac{\mathfrak{f}}{\mathfrak{g}} \right) - N \left(r, \infty; \frac{\mathfrak{g}}{\mathfrak{f}} \right) = N(r, \infty; \mathfrak{f}) + N(r, 0; \mathfrak{g}) - N(r, \infty; \mathfrak{g}) - N(r, 0; \mathfrak{f}).$$

Lemma 5.6. [1] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two non-constant meromorphic functions. Suppose $\mathfrak{f}, \mathfrak{g}$ share $(1, 1)$, then

$$2\overline{N}_L(r, 1; \mathfrak{f}) + 2\overline{N}_L(r, 1; \mathfrak{g}) + \overline{N}_E^{(2)}(r, 1; \mathfrak{f}) - \overline{N}_{\mathfrak{f} > 2}(r, 1; \mathfrak{g}) \leq N(r, 1; \mathfrak{g}) - \overline{N}(r, 1; \mathfrak{g}).$$

Lemma 5.7. [2] Let f and g be any two non-constant meromorphic functions. Suppose f, g share $(1, 1)$, then

$$\overline{N}_{f>2}(r, 1; g) \leq \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}N_0(r, 0; f') + S(r, f),$$

where $N_0(r, 0; f')$ represents counting function of those zeros of f' which are not the zeros of $f(f - 1)$.

Lemma 5.8. [2] Let f and g be any two non-constant meromorphic functions. Suppose f, g share $(1, 0)$, then

$$\begin{aligned} (i) \quad & \overline{N}_{f>1}(r, 1; g) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_0(r, 0; f') + S(r, f), \\ (ii) \quad & \overline{N}_{g>1}(r, 1; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_0(r, 0; g') + S(r, g). \end{aligned}$$

Lemma 5.9. [2] Let f and g be any two non-constant meromorphic functions. Suppose f, g share $(1, 0)$, then

$$\overline{N}_L(r, 1; f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 5.10. [2] Let f and g be any two non-constant meromorphic functions. Suppose f, g share $(1, 0)$, then

$$\begin{aligned} \overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2)}(r, 1; f) - \overline{N}_{f>1}(r, 1; g) - \overline{N}_{g>1}(r, 1; g) \\ \leq N(r, 1; g) - \overline{N}(r, 1; g). \end{aligned}$$

Lemma 5.11. [12] If $N(r, 0; f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then

$$N(r, 0; f^{(k)} | f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; |f| < k) + k\overline{N}(r, 0; |f| \geq k) + S(r, f).$$

Lemma 5.12. Let f be a finite ordered transcendental meromorphic function and let $\mathfrak{F} = f^n \mathcal{L}_\eta(f)$, where $n \in \mathbb{N}$. Then

$$(n - 1)T(r, f) \leq T(r, \mathfrak{F}) + S(r, f).$$

Proof. From Lemmas 5.1, 5.2 and the First fundamental theorem, we obtain

$$\begin{aligned} (n + 1)T(r, f) &= T(r, f^{n+1}) + S(r, f) \\ &\leq T\left(r, \frac{\mathfrak{F}f(z)}{\mathcal{L}_\eta(f)}\right) + S(r, f) \\ &\leq T(r, \mathfrak{F}) + T\left(r, \frac{\mathcal{L}_\eta(f)}{f(z)}\right) + S(r, f) \\ &\leq T(r, \mathfrak{F}) + N\left(r, \frac{\mathcal{L}_\eta(f)}{f(z)}\right) + S(r, f) \\ &\leq T(r, \mathfrak{F}) + 2T(r, f) + S(r, f). \end{aligned}$$

Therefore, we have $(n - 1)T(r, f) \leq T(r, \mathfrak{F}) + S(r, f)$. This completes the proof of Lemma 5.12. $\square$

Lemma 5.13. *Let $\eta \in \mathbb{C} - \{0\}$, $n \in \mathbb{N}$, such that $n > 2$. Let f, g be two finite ordered transcendental meromorphic functions. Let $p(z)$ be a non-zero polynomial such that $2\deg(p) < n - 2$. Then*

(1) *if $\deg(p(z)) \geq 1$, then $f^n \mathcal{L}_\eta(f) g^n \mathcal{L}_\eta(g) \not\equiv p^2(z)$,*

(2) *if $p(z)$ reduces to a non-zero constant d , then*

(i) *when $\eta_0 = 0$, f, g takes the form, $f = e^U$ and $g = te^{-U}$, where U is a non-constant polynomial and t is a constant, such that $t^{n+1} = d^2$, or*

(ii) *when $\eta_0 \neq 0$, f, g takes the form, $f = \eta_1 e^{az}$ and $g = \eta_2 e^{-az}$, where a, η_1, η_2 and d are non-zero constants satisfying $(\eta_1 \eta_2)^{n+1} (e^{a\eta} + \eta_0)(e^{-a\eta} + \eta_0) = d^2$.*

Proof. Suppose

$$f^n \mathcal{L}_\eta(f) g^n \mathcal{L}_\eta(g) \equiv p^2(z). \quad (5.2)$$

Let $h_1 = fg$, then by (5.2), we have

$$h_1^n(z) = \frac{p^2(z)}{\mathcal{L}_\eta(f) \mathcal{L}_\eta(g)}. \quad (5.3)$$

We now consider the following three cases.

Case 1. Suppose h_1 is a transcendental meromorphic function, then by Lemmas 5.1, 5.2 and 5.3, we get

$$\begin{aligned} nT(r, h_1) &= T(r, h_1^n) + S(r, h_1) \\ &= T\left(r, \frac{p^2(z)}{\mathcal{L}_\eta(f) \mathcal{L}_\eta(g)}\right) + S(r, h_1) \\ &\leq N(r, 0; \mathcal{L}_\eta(f)) + N(r, 0; \mathcal{L}_\eta(g)) + S(r, h_1) \\ &\leq 2[T(r, f) + T(r, g)] + S(r, h_1), \end{aligned}$$

from $h_1 = fg$, which implies $T(r, h_1) = T(r, f) + T(r, g)$, we get

$$n[T(r, f) + T(r, g)] \leq 2[T(r, f) + T(r, g)] + S(r, h_1),$$

which is a contradiction.

Case 2. Suppose h_1 is a rational function. Let

$$h_1 = \frac{h_2}{h_3}, \quad (5.4)$$

where h_2, h_3 are two non-zero relatively prime polynomials. By (5.4), we have

$$T(r, h_1) = \max \left\{ \deg(h_2), \deg(h_3) \right\} \log r + O(1). \quad (5.5)$$

Now by (5.3)-(5.5), we have

$$\begin{aligned} n \max \left\{ \deg(h_2), \deg(h_3) \right\} \log r &= T(r, h_1^n) + O(1) \\ &\leq 2 [T(r, \mathfrak{f}) + T(r, \mathfrak{g})] + 2T(r, p) + O(1) \\ &\leq 2 \max \left\{ \deg(h_2), \deg(h_3) \right\} \log r + 2T(r, p) + O(1). \end{aligned} \quad (5.6)$$

We see that $\max \left\{ \deg(h_2), \deg(h_3) \right\} \geq 1$.

Now by (5.6), we deduce that $(n - 2) \leq 2\deg(p(z))$, which contradicts our assumption that $2\deg(p(z)) < n - 2$. Hence h_1 must be a non-zero constant. Let

$$h_1 = t \in \mathbb{C} - \{0\}. \quad (5.7)$$

Now, when $\deg(p) \geq 1$, by (5.3) and (5.7), we arrive at a contradiction. In this case we have

$$\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) \neq p^2(z).$$

Case 3. suppose $p(z)$ is not a constant then, from (5.2), we have

$$\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z). \quad (5.8)$$

From (5.8) and the fact that $\mathfrak{f}$ and $\mathfrak{g}$ are meromorphic functions, one can immediately say that both $\mathfrak{f}$ and $\mathfrak{g}$ have either no or atmost finitely many zeros. So, we may write

$$\mathfrak{f}(z) = p_1(z)e^{Q_1(z)} \quad \text{and} \quad \mathfrak{g}(z) = p_2(z)e^{Q_2(z)}, \quad (5.9)$$

where p_1, p_2, Q_1, Q_2 are polynomials and Q_1, Q_2 are non-constants. Substituting (5.9) in (5.8), we obtain

$$\begin{aligned} (p_1 p_2)^n e^{n(Q_1(z) + Q_2(z))} &\left[p_1(z + \eta) p_2(z + \eta) e^{Q_1(z + \eta) + Q_2(z + \eta)} \right. \\ &+ \eta_0^2 p_1(z) p_2(z) e^{Q_1(z) + Q_2(z)} + \eta_0 p_1(z) p_2(z + \eta) e^{Q_1(z) + Q_2(z + \eta)} \\ &\left. + \eta_0 p_1(z + \eta) p_2(z) e^{Q_1(z + \eta) + Q_2(z)} \right] = p^2(z). \end{aligned} \quad (5.10)$$

Keeping in view of (5.9), we must have

$$n(Q_1(z) + Q_2(z)) + Q_1(z + \eta) + Q_2(z + \eta) = A_1, \quad (5.11)$$

$$n(Q_1(z) + Q_2(z)) + Q_1(z) + Q_2(z + \eta) = A_2, \quad (5.12)$$

$$n(Q_1(z) + Q_2(z)) + Q_1(z + \eta) + Q_2(z) = A_3, \quad (5.13)$$

$$(n + 1)(Q_1(z) + Q_2(z)) = A_4, \quad (5.14)$$

Non-linear difference polynomials sharing a polynomial with finite weight

where A_1, A_2, A_3, A_4 are constants. Let $Q_1(z) + Q_2(z) = W(z)$. Then (5.11) can be written as

$$nW(z) + W(z + \eta) = A_1, \quad (5.15)$$

for all $z \in \mathbb{C}$. Therefore, from (5.15), we must have $W = B$, where B is a constant, and therefore, we have

$$Q_2 = B - Q_1, \quad (5.16)$$

keeping in view of (5.16), we can write (5.9) as

$$f(z) = p_1(z)e^{Q_1(z)} \quad \text{and} \quad g(z) = p_2(z)e^B e^{-Q_1(z)}. \quad (5.17)$$

Now, (5.10) can be written as

$$\begin{aligned} (p_1 p_2)^n [p_1(z + \eta)p_2(z + \eta)e^{A_1} + \eta_0 p_1(z + \eta)p_2(z)e^{A_3} \\ + \eta_0 p_1(z)p_2(z + \eta)e^{A_2} + \eta_0^2 p_1(z)p_2(z)e^{A_4}] = p^2(z). \end{aligned} \quad (5.18)$$

If $p_1 p_2$ is not a constant, then the degree of the left side of (5.18) is atleast $n + 1$. But the condition $2deg(p) < n - 2$ implies that the degree of the right side of (5.18) is less than $n - 2$, which is a contradiction. Thus $p_1 p_2$ and p reduce to non-zero constants.

Since p_1, p_2 are both polynomials and their product is constant, each of them must be constant. Therefore, (5.17) can be written as

$$f(z) = e^U \quad \text{and} \quad g(z) = e^B e^{-U}, \quad (5.19)$$

where U is a non-constant polynomial. Using the above forms of f, g and keeping in mind that p is a constant, say d , (5.8) reduces to

$$e^{(n+1)B} (e^{[U(z+\eta)-U(z)]} + \eta_0) (e^{-[U(z+\eta)-U(z)]} + \eta_0) = d^2 \quad (5.20)$$

Subcase 3.1. If $\eta_0 = 0$, (5.20) reduces to $e^{(n+1)B} = d^2$. Set $e^B = t$. Then (5.19) can be written as $f(z) = e^U$, $g(z) = te^{-U}$, where $p(z)$ reduces to a non-zero constant d and t is a constant, such that $t^{n+1} = d^2$, and U is a non-constant polynomial.

Subcase 3.2. If $\eta_0 \neq 0$, then from (5.20), one can say that $e^{[U(z+\eta)-U(z)]} + \eta_0$ has no zeros. Then $\phi(z) = e^{[U(z+\eta)-U(z)]} \neq 0, -\infty, \infty$. By Picard's theorem, ϕ is constant and so $deg(U(z)) = 1$. Therefore, from (5.19), one may obtain $f(z) = \eta_1 e^{az}$ and $g(z) = \eta_2 e^{-az}$, where a, η_1, η_2 are non-zero constants. Using these in (5.8), we obtain $(\eta_1 \eta_2)^{n+1} (e^{a\eta} + \eta_0) (e^{-a\eta} + \eta_0) = d^2$.

This completes the proof of Lemma 5.13. $\square$

Lemma 5.14. *Let $\mathfrak{f}, \mathfrak{g}$ be two finite ordered transcendental meromorphic functions. Let $n \in \mathbb{N}$, such that $n > 9$. Let $\mathfrak{F} = \frac{\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G} = \frac{\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g})}{p(z)}$, where $p(z)$ is a non-zero polynomial. If $H \equiv 0$ (H as defined in (5.1)), then either*

(i) $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f})\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z)$, where $\mathfrak{f}^n(z)\mathcal{L}_\eta(\mathfrak{f}) - p(z)$ and $\mathfrak{g}^n(z)\mathcal{L}_\eta(\mathfrak{g}) - p(z)$ share 0 CM; or

(ii) $\mathfrak{f} \equiv t\mathfrak{g}$, for a constant t with $t^{n+1} = 1$.

Proof. Let H be as defined in (5.1). Since $H \equiv 0$, by integration, we get

$$\frac{1}{\mathfrak{F} - 1} = \frac{B\mathfrak{G} + A - B}{\mathfrak{G}}, \quad (5.21)$$

where $A \neq 0$, B are constants. From (5.21) it is clear that $\mathfrak{F}$ and $\mathfrak{G}$ share $(1, \infty)$. We now consider the following cases.

Case 4. Let $B \neq 0$ and $A \neq B$.

If $B = -1$, then from (5.21), we have

$$\mathfrak{F} = \frac{-A}{\mathfrak{G} - A - 1}.$$

Therefore, $\overline{N}(r, A + 1; \mathfrak{G}) = \overline{N}(r, \mathfrak{F}) \leq N\left(r, \frac{1}{p}\right) \leq T(r, p) = S(r, \mathfrak{g})$.

So in view of Lemma 5.12 and the Second fundamental theorem, we get

$$\begin{aligned} (n-1)T(r, \mathfrak{g}) &\leq T(r, \mathfrak{g}^n\mathcal{L}_\eta(\mathfrak{g})) + S(r, \mathfrak{g}) \\ &\leq T(r, \mathfrak{G}) + S(r, \mathfrak{g}) \\ &\leq \overline{N}(r, \infty; \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{G}) + \overline{N}(r, (A+1); \mathfrak{G}) + S(r, \mathfrak{g}) \\ &\leq 2T(r, \mathfrak{g}) + 3T(r, \mathfrak{g}) + S(r, \mathfrak{g}), \end{aligned}$$

which is a contradiction, since $n > 6$.

If $B \neq -1$, from (5.21), we obtain that

$$\mathfrak{F} - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2\left(\mathfrak{G} + \frac{A-B}{B}\right)}.$$

This implies that

$$\overline{N}\left(r, \frac{B-A}{B}; \mathfrak{G}\right) = S(r, \mathfrak{g}).$$

Using Lemma 5.12 and the same argument as used in the case when $B = -1$, we can get a contradiction.

Case 5. Let $B \neq 0$ and $A = B$.

If $B = -1$, then from (5.21), we have

$$\mathfrak{F}(z)\mathfrak{G}(z) \equiv 1,$$

i.e.,

$$\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) \equiv p^2(z),$$

where $\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) - p(z)$ and $\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}) - p(z)$ share 0 CM.

If $B \neq -1$, then from (5.21), we have

$$\frac{1}{\mathfrak{F}} = \frac{B\mathfrak{G}}{(1+B)\mathfrak{G}-1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+B}; \mathfrak{G}\right) = \overline{N}(r, 0; \mathfrak{F}) + S(r, \mathfrak{f}).$$

So, in view of Lemmas 5.3, 5.12 and the Second fundamental theorem, we get

$$\begin{aligned} (n-1)T(r, \mathfrak{g}) &\leq T(r, \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})) + S(r, \mathfrak{g}) \\ &\leq T(r, \mathfrak{G}) + S(r, \mathfrak{g}) \\ &\leq \overline{N}(r, \infty; \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{G}) + \overline{N}\left(r, \frac{1}{1+B}; \mathfrak{G}\right) + S(r, \mathfrak{g}) \\ &\leq \overline{N}(r, \infty; \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r, 0; \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r, 0; \mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f})) \\ &\quad + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\ &\leq 2T(r, \mathfrak{g}) + 3T(r, \mathfrak{g}) + 3T(r, \mathfrak{f}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}), \end{aligned}$$

for $r \in I$, we have

$$(n-9)T(r, \mathfrak{g}) \leq s(r, \mathfrak{g}),$$

which is a contradiction, since $n > 9$.

Case 6. Let $B = 0$. From (5.21), we obtain

$$\mathfrak{F} = \frac{\mathfrak{G} + A - 1}{A}. \quad (5.22)$$

If $A \neq 1$, then from (5.22), we obtain

$$\overline{N}(r, 1-A; \mathfrak{G}) = \overline{N}(r, 0; \mathfrak{F}).$$

We can similarly deduce a contradiction as in Case 5. Therefore $A = 1$ and from (5.22), we obtain

$$\mathfrak{F}(z) \equiv \mathfrak{G}(z),$$

i.e.,

$$\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f}) \equiv \mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g}). \quad (5.23)$$

Let $h = \mathfrak{f}/\mathfrak{g}$, and then substituting $\mathfrak{f} = \mathfrak{g}h$ in (5.23), we deduce

$$h^{n+1} = \frac{\mathfrak{f} \mathcal{L}_\eta(\mathfrak{g})}{\mathcal{L}_\eta(\mathfrak{f}) \mathfrak{g}}.$$

If h is not a constant, then we have

$$\begin{aligned} (n+1)T(r, h) &\leq T\left(r, \frac{\mathfrak{f}}{\mathcal{L}_\eta(\mathfrak{f})}\right) + T\left(r, \frac{\mathcal{L}_\eta(\mathfrak{g})}{\mathfrak{g}}\right) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\ &\leq N\left(r, \frac{\mathcal{L}_\eta(\mathfrak{f})}{\mathfrak{f}}\right) + N\left(r, \frac{\mathcal{L}_\eta(\mathfrak{g})}{\mathfrak{g}}\right) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\ &\leq 2[T(r, \mathfrak{f}) + T(r, \mathfrak{g})] + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \end{aligned}$$

Combining the above inequality with $T(r, h) = T\left(r, \frac{\mathfrak{f}}{\mathfrak{g}}\right) = T(r, \mathfrak{f}) + T(r, \mathfrak{g}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g})$, we obtain $(n-1)[T(r, \mathfrak{f}) + T(r, \mathfrak{g})] \leq S(r, \mathfrak{f}) + S(r, \mathfrak{g})$ which is impossible. Therefore h is a constant, then substituting $\mathfrak{f} = \mathfrak{g}h$ in to (5.23), we have $h^{n+1} \equiv 1$. Therefore $f \equiv t\mathfrak{g}$, where t is a constant with $t^{n+1} = 1$.

This completes the proof of Lemma 5.14. $\square$

6 Proof of Theorems

6.1 Proof of Theorem 2.1.

Proof. Let $\mathfrak{F} = \frac{\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G} = \frac{\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})}{p(z)}$. It follows that $\mathfrak{F}$ and $\mathfrak{G}$ share (1,2) except the zeros of $p(z)$.

Case 7. Let $H \neq 0$, from (5.1), we obtain

$$\begin{aligned} N(r, \infty; H) &\leq \overline{N}_*(r, 1; \mathfrak{F}, \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{F} \geq 2) + \overline{N}(r, 0; \mathfrak{G} \geq 2) \\ &\quad + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}'), \end{aligned} \quad (6.1)$$

where $\overline{N}_0(r, 0; \mathfrak{F}')$ is the reduced counting function of those zeros of $\mathfrak{F}'$ which are not the zeros of $\mathfrak{F}(\mathfrak{F} - 1)$. $\overline{N}_0(r, 0; \mathfrak{G}')$ is similarly defined.

Let z_0 be a simple zero of $\mathfrak{F} - 1$ such that $p(z_0) \neq 0$. Then z_0 is a simple zero of $\mathfrak{G} - 1$ and a zero of H . So

$$N(r, 1; \mathfrak{F} = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \quad (6.2)$$

From (6.1) and (6.2), we have

$$\begin{aligned}\overline{N}(r, 1; \mathfrak{F}) &\leq \overline{N}(r, 0; \mathfrak{F} \geq 2) + \overline{N}(r, 0; \mathfrak{G} \geq 2) + \overline{N}_*(r, 1; \mathfrak{F}, \mathfrak{G}) + \overline{N}_0(r, 0; \mathfrak{F}') \\ &\quad + \overline{N}(r, 1; \mathfrak{F} \geq 2) + \overline{N}_0(r, 0; \mathfrak{G}') + S(r, \mathfrak{f}) + S(r, \mathfrak{g}).\end{aligned}\quad (6.3)$$

Using Lemma 5.11, we get

$$\begin{aligned}\overline{N}_0(r, 0; \mathfrak{G}') + \overline{N}(r, 1; \mathfrak{F} \geq 2) + \overline{N}_*(r, 1; \mathfrak{F}, \mathfrak{G}) &\leq \overline{N}_0(r, 0; \mathfrak{G}') + \overline{N}(r, 1; \mathfrak{F} \geq 2) \\ &\quad + \overline{N}(r, 1; \mathfrak{F} \geq 3) \\ &\leq N(r, 0; \mathfrak{G}' | \mathfrak{G} \neq 0) \\ &\leq \overline{N}(r, 0; \mathfrak{G}) + S(r, \mathfrak{g}).\end{aligned}\quad (6.4)$$

Using (6.3), (6.4), Lemmas 5.1, 5.12, and the Second fundamental theorem, we get

$$\begin{aligned}(n-1)T(r, \mathfrak{f}) &\leq T(r, \mathfrak{F}) + S(r, \mathfrak{f}), \\ &\leq \overline{N}(r, \infty; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{F}) + \overline{N}(r, 1; \mathfrak{F}) - \overline{N}_0(r, 0; \mathfrak{F}') + s(r, \mathfrak{f}) \\ &\leq \overline{N}(r, \infty; \mathfrak{F}) + N_2(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\ &\leq 2\overline{N}(r, \mathfrak{f}) + 2\overline{N}(r, 0; \mathfrak{f}) + N(r, 0; \mathcal{L}_\eta(\mathfrak{f})) + N(r, 0; \mathcal{L}_\eta(\mathfrak{g})) \\ &\quad + 2\overline{N}(r, 0; \mathfrak{g}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\ &\leq 6T(r, \mathfrak{f}) + 4T(r, \mathfrak{g}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}).\end{aligned}\quad (6.5)$$

Similarly, we obtain

$$(n-1)T(r, \mathfrak{g}) \leq 6T(r, \mathfrak{g}) + 4T(r, \mathfrak{f}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}).\quad (6.6)$$

Combining (6.5) and (6.6), we have

$$(n-11)[T(r, \mathfrak{f}) + T(r, \mathfrak{g})] \leq S(r, \mathfrak{f}) + S(r, \mathfrak{g}),\quad (6.7)$$

which is a contradiction, since $n > 11$.

Case 8. Let $H \equiv 0$. Then the proof follows from Lemmas 5.13 and 5.14.

This completes the proof of Theorem 3.1. $\square$

6.2 Proof of Theorem 2.2.

Proof. Let $\mathfrak{F}(z) = \frac{\mathfrak{f}^n \mathcal{L}_\eta(\mathfrak{f})}{p(z)}$ and $\mathfrak{G}(z) = \frac{\mathfrak{g}^n \mathcal{L}_\eta(\mathfrak{g})}{p(z)}$. Then $\mathfrak{F}$ and $\mathfrak{G}$ share $(1, 1)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 9. Suppose $H \neq 0$.

Using Lemmas 5.12, 5.14, 5.6 and equations (6.1), (6.2), we get

$$\begin{aligned}
 \overline{N}(r, 1; \mathfrak{F}) &\leq N(r, 1; \mathfrak{F}| = 1) + \overline{N}_L(r, 1; \mathfrak{F}) + \overline{N}_L(r, 1; \mathfrak{G}) + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) \\
 &\leq \overline{N}(r, 0; \mathfrak{F}| \geq 2) + \overline{N}(r, 0; \mathfrak{G}| \geq 2) + \overline{N}_*(r, 1; \mathfrak{F}, \mathfrak{G}) + \overline{N}_L(r, 1; \mathfrak{F}) \\
 &\quad + \overline{N}_L(r, 1; \mathfrak{G}) + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}') \\
 &\quad + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 &\leq \overline{N}(r, 0; \mathfrak{F}| \geq 2) + \overline{N}(r, 0; \mathfrak{G}| \geq 2) + 2\overline{N}_L(r, 1; \mathfrak{F}) + 2\overline{N}_L(r, 1; \mathfrak{G}) \\
 &\quad + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}') + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 &\leq \overline{N}(r, 0; \mathfrak{F}| \geq 2) + \overline{N}(r, 0; \mathfrak{G}| \geq 2) + \overline{N}_{\mathfrak{F} > 2}(r, 1; \mathfrak{G}) + N(r, 1; \mathfrak{G}) \\
 &\quad - \overline{N}(r, 1; \mathfrak{G}) + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}') + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 &\leq \overline{N}(r, 0; \mathfrak{F}| \geq 2) + \frac{1}{2}\overline{N}(r, 0; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{G}| \geq 2) + N(r, 0; \mathfrak{G}' | \mathfrak{G} \neq 0) \\
 &\quad + \overline{N}_0(r, 0; \mathfrak{F}') + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 \overline{N}(r, 1; \mathfrak{F}) &\leq \overline{N}(r, 0; \mathfrak{F}| \geq 2) + \frac{1}{2}\overline{N}(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) + \overline{N}_0(r, 0; \mathfrak{F}') \\
 &\quad + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \tag{6.8}
 \end{aligned}$$

Hence by using Second fundamental theorem, (6.8), Lemmas 5.1 and 5.12, we get

$$\begin{aligned}
 (n-1)T(r, \mathfrak{f}) &\leq T(r, \mathfrak{F}) + S(r, \mathfrak{f}) \\
 &\leq \overline{N}(r, \infty; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{F}) + \overline{N}(r, 1; \mathfrak{F}) - N_0(r, 0; \mathfrak{F}') + S(r, \mathfrak{f}) \\
 &\leq \overline{N}(r, \infty; \mathfrak{F}) + \frac{1}{2}\overline{N}(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) \\
 &\quad + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 &\leq 2\overline{N}(r, \mathfrak{f}) + \frac{3}{2}T(r, \mathfrak{f}) + 2\overline{N}(r, 0; \mathfrak{f}) + N(r, 0; \mathcal{L}_\eta(\mathfrak{f})) \\
 &\quad + 2\overline{N}(r, 0; \mathfrak{g}) + N(r, 0; \mathcal{L}_\eta(\mathfrak{g})) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}) \\
 &\leq 2\overline{N}(r, \mathfrak{f}) + \frac{3}{2}T(r, \mathfrak{f}) + 2T(r, \mathfrak{f}) + 2T(r, \mathfrak{f}) + 2T(r, \mathfrak{g}) \\
 &\quad + 2T(r, \mathfrak{g}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \\
 (n-1)T(r, \mathfrak{f}) &\leq \frac{15}{2}T(r, \mathfrak{f}) + 4T(r, \mathfrak{g}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \tag{6.9}
 \end{aligned}$$

Similarly, we obtain

$$(n-1)T(r, \mathfrak{g}) \leq \frac{15}{2}T(r, \mathfrak{g}) + 4T(r, \mathfrak{f}) + S(r, \mathfrak{f}) + S(r, \mathfrak{g}). \tag{6.10}$$

Now, combining (6.9) and (6.10), we get

$$(n-1)[T(r, f) + T(r, g)] \leq \frac{23}{2}[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$$

$$\left(n - \frac{25}{2}\right)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

which is a contradiction, since $n > \frac{25}{2}$.

Case 10. Suppose $H \equiv 0$.

Then the proof follows from Lemmas 5.13 and 5.14.

This completes the proof of Theorem 3.2. $\square$

6.3 Proof of Theorem 2.3.

Proof. Let $\mathfrak{F}(z) = \frac{f^n \mathcal{L}_\eta(f)}{p(z)}$ and $\mathfrak{G}(z) = \frac{g^n \mathcal{L}_\eta(g)}{p(z)}$. Then $\mathfrak{F}$ and $\mathfrak{G}$ share $(1, 0)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 11. Suppose $H \not\equiv 0$.

In this case (6.2) changes to

$$N_E^{(1)}(r, 1; \mathfrak{F}) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, \mathfrak{F}) + S(r, \mathfrak{G}) \quad (6.11)$$

Using (6.2), (6.11) and Lemmas 5.8, 5.9, 5.10, 5.11, we get

$$\begin{aligned} \overline{N}(r, 1; \mathfrak{F}) &\leq N_E^{(1)}(r, 1; \mathfrak{F}) + \overline{N}_L(r, 1; \mathfrak{F}) + \overline{N}_L(r, 1; \mathfrak{G}) + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) \\ &\leq \overline{N}_*(r, 1; \mathfrak{F}, \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{F} | \geq 2) + \overline{N}(r, 0; \mathfrak{G} | \geq 2) + \overline{N}_L(r, 1; \mathfrak{F}) \\ &\quad + \overline{N}_L(r, 1; \mathfrak{G}) + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}') \\ &\quad + S(r, f) + S(r, g) \\ &\leq \overline{N}(r, 0; \mathfrak{F} | \geq 2) + \overline{N}(r, 0; \mathfrak{G} | \geq 2) + 2\overline{N}_L(r, 1; \mathfrak{F}) + 2\overline{N}_L(r, 1; \mathfrak{G}) \\ &\quad + \overline{N}_E^{(2)}(r, 1; \mathfrak{F}) + \overline{N}_0(r, 0; \mathfrak{F}') + \overline{N}_0(r, 0; \mathfrak{G}') + S(r, f) + S(r, g) \\ &\leq \overline{N}(r, 0; \mathfrak{F} | \geq 2) + \overline{N}(r, 0; \mathfrak{G} | \geq 2) + \overline{N}_{\mathfrak{F} > 1}(r, 1; \mathfrak{G}) + \overline{N}_{\mathfrak{G} > 1}(r, 1; \mathfrak{F}) \\ &\quad + \overline{N}_L(r, 1; \mathfrak{F}) + N(r, 1; \mathfrak{G}) - \overline{N}(r, 1; \mathfrak{G}) + \overline{N}_0(r, 0; \mathfrak{F}') \\ &\quad + \overline{N}_0(r, 0; \mathfrak{G}') + S(r, f) + S(r, g) \\ &\leq N_2(r, 0; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) + N(r, 0; \mathfrak{G}' | \mathfrak{G} | \neq 0) \\ &\quad + \overline{N}_0(r, 0; \mathfrak{F}') + S(r, f) + S(r, g) \\ \overline{N}(r, 1; \mathfrak{F}) &\leq N_2(r, 0; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{G}) \\ &\quad + \overline{N}_0(r, 0; \mathfrak{F}') + S(r, f) + S(r, g). \end{aligned} \quad (6.12)$$

Hence by using the Second fundamental theorem and Lemmas 5.1, 5.12, we obtain

$$\begin{aligned}
 (n-1)T(r, f) &\leq T(r, \mathfrak{F}) + S(r, f) \\
 &\leq \overline{N}(r, \infty; \mathfrak{F}) + \overline{N}(r, 0; \mathfrak{F}) + \overline{N}(r, 1; \mathfrak{F}) - N_0(r, 0; \mathfrak{F}') + S(r, f) \\
 &\leq \overline{N}(r, \infty; \mathfrak{F}) + 2N_2(r, 0; \mathfrak{F}) + N_2(r, 0; \mathfrak{G}) + \overline{N}(r, 0; \mathfrak{G}) + S(r, f) \\
 &\quad + S(r, \mathfrak{g}) \\
 &\leq 2\overline{N}(r, f) + 4\overline{N}(r, 0; f) + 2N(r, 0; \mathcal{L}_\eta(f)) + 2\overline{N}(r, 0; \mathfrak{g}) \\
 &\quad + \overline{N}(r, 0; \mathfrak{g}) + N(r, 0; \mathcal{L}_\eta(\mathfrak{g})) + \overline{N}(r, 0; \mathcal{L}_\eta(\mathfrak{g})) + S(r, f) + S(r, \mathfrak{g}) \\
 &\leq 2\overline{N}(r, f) + 4T(r, f) + 4T(r, f) + 2T(r, \mathfrak{g}) + 2T(r, \mathfrak{g}) \\
 &\quad + T(r, \mathfrak{g}) + 2T(r, \mathfrak{g}) + S(r, f) + S(r, \mathfrak{g}). \\
 (n-1)T(r, f) &\leq 6T(r, f) + 7T(r, \mathfrak{g}) + S(r, f) + S(r, \mathfrak{g}). \tag{6.13}
 \end{aligned}$$

Similarly, we obtain

$$(n-1)T(r, \mathfrak{g}) \leq 6T(r, \mathfrak{g}) + 7T(r, f) + S(r, f) + S(r, \mathfrak{g}). \tag{6.14}$$

Now, combining (6.13) and (6.14), we get

$$(n-14)[T(r, f) + T(r, \mathfrak{g})] \leq S(r, f) + S(r, \mathfrak{g}), \tag{6.15}$$

which is a contradiction, since $n > 14$.

Case 12. Suppose $H \equiv 0$.

Then the proof follows from Lemmas 5.13 and 5.14.

This completes the proof of Theorem 3.3. $\square$

7 Conclusions

Nevanlinna theory is an effective analytic technique with several applications in a variety of mathematical fields because of its insights into the behaviour of meromorphic functions, the distribution of values, and the characteristics of differential equation solutions. Mathematicians can develop mathematical theory in areas like communication networks, signal processing, the design of filters and controllers for systems, as well as certain practical applications, by using Nevanlinna's theory to better comprehend complex functions and their behaviour.

In this article, with the help of the Nevanlinna theory, we have studied the uniqueness results of two non-linear difference polynomials generated by two finite ordered non-constant meromorphic functions when they share a non-zero polynomial with finite weight and obtain some results which extend and improve some of the earlier results of Majumder [17].

Further, we would like to pose the following open questions.

Question 1. Can the linear difference polynomial $\mathcal{L}_\eta(f)$ in Theorems 2.1-2.3 be further generalized to a larger class of function $\Delta_\eta^n f$, where $\Delta_\eta^n f = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} f(z + (n-r)\eta)$?

Question 2. Is the condition for ‘ n ’, in Theorem 2.1, Theorem 2.2 and Theorem 2.3 sharp?

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Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

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Abstract

The main goal of this work is to present $G_{\mathcal{F}}$ -metric space, a new generalization of G -metric space. A comparison between the classes of G -metric spaces, GP -metric spaces, G_b -metric spaces, generalized G_b -metric spaces, and G^* -metric spaces and the class of $G_{\mathcal{F}}$ -metric spaces is also presented. We examine a few fundamental aspects of this newly defined abstract space. Proving the Banach contraction principle and the fixed point result for (ψ, ϕ) -contractive mapping in the context of $G_{\mathcal{F}}$ -metric spaces is the paper's secondary goal.

Keywords: Fixed point, G -metric space, G_b -metric space, $\mathcal{F}$ -metric space, Contractive mapping.

2020 AMS subject classifications: 47H10; 54H25.¹

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1 Introduction

Gahler (1) proposed the idea of 2-metric space, which is an extension of the famous concept of metric space (X, d) . Various writers have demonstrated that there is no relation between the two functions. For example, Ha *et al.* (2) demonstrate that the 2-metric does not necessarily need to be continuous. Dhage (3) introduced the concept of D -metric space, a new class of generalized metric space, in 1992. Most of the assertions about the basic topological structure of D -metric space were later proved inappropriate by Mustafa and Sims (4), Naidu *et al.* ((5; 6)). Therefore, Mustafa and Sims (7) created a more suitable concept, known as G -metric space. (7) Let $\mathcal{A}$ be a non-empty set and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function satisfying:

- (G1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (G2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (G3) $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (G4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G5) $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$, (rectangle inequality).

The pair $(\mathcal{A}, G)$ is a G -metric space, and the function G is referred to as a *generalized metric* or a G -metric on $\mathcal{A}$. Assume that the set of real numbers is $\mathcal{A}$, define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, \text{ for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

G is therefore a G -metric on $\mathcal{A}$.

(8; 9; 10; 11; 12; 13; 15; 14; 16) has more results and more information in G -metric spaces. As a generalisation of partial metric space (17) and G -metric space, Zand and Nezhad (18) presented GP -metric space in 2011. (18) Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function such that the following conditions hold:

- (G_p1) $\zeta = \eta = \vartheta$ if $G(\zeta, \eta, \vartheta) = G(\zeta, \zeta, \zeta) = G(\eta, \eta, \eta) = G(\vartheta, \vartheta, \vartheta)$;
- (G_p2) $G(\zeta, \zeta, \zeta) \leq G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$;
- (G_p3) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G_p4) $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta) - G(a, a, a)$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

Then, the function G is called a GP -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is a GP -metric space.

Later, in 2013, Parvaneh *et al.* (19) discovered that (G_p2) makes GP -metric spaces symmetric. Because those G -metric spaces are nonsymmetric, GP -metric spaces do not generalize them (see Example 1, (7)). Parvaneh *et al.* (19) redefined GP -metric space in light of this by modifying the inequality (G_p2) to read as follows:

$$(G_p2') \quad G(\zeta, \zeta, \zeta) \leq G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta), \text{ for all } \zeta, \eta, \vartheta \in \mathcal{A} \text{ with } \eta \neq \vartheta.$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

(18) Let $\mathcal{A} = [0, \infty)$ and define a map $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ by setting

$$G(\zeta, \eta, \vartheta) = \max\{\zeta, \eta, \vartheta\}, \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

Consequently, $(\mathcal{A}, G)$ is a GP -metric space but not a G -metric space since $G(1, 1, 1) = 1 \neq 0$, i.e., $(G1)$ does not hold.

Further details about GP -metric spaces are provided in papers (20; 21; 22; 23; 24; 25; 26; 27; 28). By merging the ideas of G -metric spaces and b -metric spaces(30), Aghajani *et al.* introduced the notion of G_b -metric spaces in (29) as follows: (29) Let $s \geq 1$ be a real number and let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function such that:

- (G_b1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (G_b2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (G_b3) $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (G_b4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G_b5) $G(\zeta, \eta, \vartheta) \leq s[G(\zeta, a, a) + G(a, \eta, \vartheta)]$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

Then, on $\mathcal{A}$, the function G is referred to as a G_b -metric or a *generalized b -metric*, and the pair $(\mathcal{A}, G)$ is a G_b -metric space or a *generalized b -metric space*. A G -metric space is a G_b -metric space with $s = 1$, but the opposite is not true in general. (29) Let $\mathcal{A} = \mathbb{R}$ represent the set of real numbers. Define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as follows:

$$G(\zeta, \eta, \vartheta) = \frac{1}{9}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)^2, \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

Hence, on $\mathcal{A}$, G is a G_b -metric but not a G -metric. Numerous researchers demonstrated different findings in G_b -metric spaces; refer to (31; 32; 33; 34; 35; 36; 37; 38; 39; 40; 41; 42). In (29), the authors proposed the idea of G_b -metric space. The term G_b -metric space was also used by Jain and Kaur in (43), although it referred to a different abstract space. Jain *et al.* (44) renamed this abstract space as ‘generalized G_b -metric space’, and its definition is as follows: (44) Let $\mathcal{A}$ be a non-empty set and $s \geq 1$ be a real number. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function satisfying:

- (gG_b1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (gG_b2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (gG_b3) $G(\zeta, \zeta, \eta) \leq s G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (gG_b4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (gG_b5) $G(\zeta, \eta, \vartheta) \leq s[G(\zeta, a, a) + G(a, \eta, \vartheta)]$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

The pair $(\mathcal{A}, G)$ is a *generalized G_b -metric space*, and the function G is referred to as a *generalized G_b -metric* on $\mathcal{A}$. The following example shows that while it is evident that every G_b -metric space is a generalized G_b -metric space, the converse is not true: (44) For every $\zeta, \eta, \vartheta \in \mathbb{R}$, define a mapping $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$

as follows:

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2.$$

In that case, $(\mathbb{R}, G)$ is not a G_b -metric space, but it is a generalized G_b -metric space with $s = 2$. To calculate $G(\zeta, \eta, \vartheta) = |1 - 3|^2 + |3 - 2|^2 + |2 - 1|^2 = 6$ and $G(\zeta, \eta, \eta) = 2|1 - 3|^2 = 8$, let $\zeta = 1$, $\eta = 3$, and $\vartheta = 2$. Cosequently, $G(\zeta, \eta, \eta) \not\leq G(\zeta, \eta, \vartheta)$, that is, (G_b3) , is not true.

After that, Jain *et al.* (44) introduced G^* -metric space to generalize GP -metric space and generalized G_b -metric space. (44) Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty]$ be a mapping, where $\mathcal{A}$ is a non-empty set. If there is an $\alpha > 0$ such that for every $\zeta, \eta, \vartheta \in \mathcal{A}$, the following conditions hold, then we say that G is a G^* -metric on $\mathcal{A}$:

- (Gg1) $G(\zeta, \eta, \vartheta) = 0$ implies $\zeta = \eta = \vartheta$;
- (Gg2) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in its variables);
- (Gg3) if $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$, then $G(\zeta, \eta, \vartheta) \leq \alpha \left(\limsup_{n \rightarrow \infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right)$,

$$\text{where } C_{\mathcal{A}}(G, \zeta) = \left\{ \{\zeta_n\} \subset \mathcal{A} \mid \lim_{n, m \rightarrow \infty} G(\zeta_n, \zeta_m, \zeta) = G(\zeta, \zeta, \zeta) < \infty \right\}.$$

The pair $(\mathcal{A}, G)$ in this instance is referred to as a G^* -metric space with constant α .

(44) Assume that $\mathcal{A} = \mathcal{B} \cup \{0\}$, where $\mathcal{B} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty]$ be a mapping defined so that G satisfies (Gg2) and

$$G(\zeta, \eta, \vartheta) = \begin{cases} \zeta + \eta + \vartheta, & \text{if atleast one of } \zeta, \eta, \vartheta \text{ is } 0; \text{ or} \\ & \text{if } \zeta = \frac{1}{n}, \eta = \frac{1}{n+m}, \vartheta = \frac{1}{n+l}, \text{ where } n, m, l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Consequently, $(\mathcal{A}, G)$ is a G^* -metric space with constant α . However, $G(0.5, 0.5, 0.5) = 5 \neq 0$, $(\mathcal{A}, G)$ is not a generalized G_b -metric space. Also, $(\mathcal{A}, G)$ is not a GP -metric space as for $\zeta = \frac{1}{10}$ and $\eta = \frac{1}{5}$, $G(\zeta, \zeta, \zeta) = 5 \not\leq \frac{2}{5} = G(\zeta, \zeta, \eta)$, that is, (G_p2') is not true.

Meanwhile, in 2018, Jleli and Samet (45) established an exciting generalization of metric space as follows.

Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

(F1) f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.

(F2) For every sequence $\{t_n\}$ in $(0, \infty)$, we have

$$\lim_{n \rightarrow \infty} t_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} f(t_n) = -\infty.$$

(45) Let $\mathcal{A}$ be a non-empty set and let $D : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that

(D1) $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, $D(\zeta, \eta) = 0$ if and only if $\zeta = \eta$.

(D2) $D(\zeta, \eta) = D(\eta, \zeta)$, for all $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$.

(D3) For every $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 2$, and for every $\{u_1, u_2, \dots, u_n\} \subset \mathcal{A}$ with $(u_1, u_n) = (\zeta, \eta)$, we have

$$D(\zeta, \eta) > 0 \text{ implies } f(D(\zeta, \eta)) \leq f\left(\sum_{i=1}^{n-1} D(u_i, u_{i+1})\right) + \alpha.$$

Then, the function D is said to be an $\mathcal{F}$ -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, D)$ is said to be an $\mathcal{F}$ -metric space.

We refer to (46; 48; 49; 47; 50) for more details on $\mathcal{F}$ -metric spaces. Now, motivated to the work done in (45), we define a new generalization of G -metric space as in the following section.

2 $G_{\mathcal{F}}$ -Metric Space

Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping. Let there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that

($G_{\mathcal{F}1}$) $G(\zeta, \eta, \vartheta) = 0$ if and only if $\zeta = \eta = \vartheta$.

($G_{\mathcal{F}2}$) $f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta, \zeta \neq \eta$.

($G_{\mathcal{F}3}$) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables).

($G_{\mathcal{F}4}$) For every $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 3$, and every $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$,

$$G(\zeta, \eta, \vartheta) > 0 \text{ implies } f(G(\zeta, \eta, \vartheta)) \leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha.$$

Then, the function G is called a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is said to be a $G_{\mathcal{F}}$ -metric space.

2.1 Examples

Every G -metric space is a $G_{\mathcal{F}}$ -metric space. Let $(\mathcal{A}, G)$ be a G -metric space. Then, G is a $G_{\mathcal{F}}$ metric on $\mathcal{A}$, as ($G_{\mathcal{F}1}$) and ($G_{\mathcal{F}3}$) can be obtained from ($G1$), ($G2$), ($G3$) and ($G4$). Also, with $\alpha = 0$ and $f(t) = \frac{-1}{t^2}$, ($G_{\mathcal{F}2}$) and ($G_{\mathcal{F}4}$) are satisfied using ($G3$) and ($G5$).

Now, we construct an example of a $G_{\mathcal{F}}$ -metric space which is a G_b -metric space as well, but not a G -metric space. Let $\mathcal{A} = \{a, b, c\}$ and define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as follows:

$G(a, a, a) = G(b, b, b) = G(c, c, c) = 0$, $G(a, a, b) = G(a, b, b) = 1$, $G(a, a, c) = G(a, c, c) = 1.2$, $G(b, b, c) = G(b, c, c) = 1.3$, $G(a, b, c) = 3.3$, and assume that ($G_{\mathcal{F}3}$) holds. Then G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$ with $f(t) = \ln(t)$, $t > 0$, and

$\alpha = \ln(1.5)$. Also, G is a G_b -metric on $\mathcal{A}$ with $s = 1.5$, but G is not a G -metric on $\mathcal{A}$ as $G(a, b, c) = 3.3 \not\leq 2.3 = G(a, b, b) + G(b, b, c)$.

See another example of a $G_{\mathcal{F}}$ -metric space which is a generalized G_b -metric space as well, but not a G_b -metric space. Let $\mathcal{A} = \{1, 2, 3, \dots, l-2\} \cup \{l - \frac{1}{n} \mid n \in \mathbb{N}\}$, where l be a fixed natural number such that $l \geq 5$ and $\mathcal{B} = \{1, 2, 3\}$. Consider a mapping $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2, & \text{if } (\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}, \\ |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, & \text{otherwise.} \end{cases}$$

Then, $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ are satisfied for mapping G .

Let $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$.

Let $P = \{i = 1, 2, 3, \dots, n-2 \mid a_i, a_{i+1} \in \mathcal{B}\}$ and $Q = \{1, 2, 3, \dots, n-2\} - P$.

Case 1: If $(\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}$ and $a_{n-1} \in \mathcal{B}$, we have

$$\begin{aligned} & G(\zeta, \eta, \vartheta) \\ &= |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2|\zeta - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}| + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}|^2 + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta|^2 + |\eta - \vartheta|^2 + 2|\vartheta - a_{n-1}|^2 \\ &\quad + 2|a_{n-1} - \zeta|^2 \\ &\leq 2 \left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta|^2 + 2|\eta - \vartheta|^2 + 2|\vartheta - a_{n-1}|^2 \\ &\quad + 2|a_{n-1} - \zeta|^2 \\ &\leq 10 \left(\left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + |a_{n-1} - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - a_{n-1}|^2 \right) \\ &= 10 \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right). \end{aligned}$$

Case 2: If $(\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}$ and $a_{n-1} \notin \mathcal{B}$, we have

$$\begin{aligned}
 G(\zeta, \eta, \vartheta) &= |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\
 &\leq 2|\zeta - \eta| + 2|\eta - \vartheta| + 2|\vartheta - \zeta| \\
 &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}| + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + 2|\eta - \vartheta| + 2|\vartheta - a_{n-1}| \\
 &\quad + 2|a_{n-1} - \zeta| \\
 &\leq 2 \left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + 2|\eta - \vartheta| + 2|\vartheta - a_{n-1}| \\
 &\quad + 2|a_{n-1} - \zeta| \\
 &\leq 2l \left(\left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + |a_{n-1} - \eta| + |\eta - \vartheta| + |\vartheta - a_{n-1}| \right) \\
 &= 2l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).
 \end{aligned}$$

Case 3: If $(\zeta, \eta, \vartheta) \notin \mathcal{B} \times \mathcal{B} \times \mathcal{B}$, then using Case 1 and Case 2, we have

$$\begin{aligned}
 G(\zeta, \eta, \vartheta) &= |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta| \\
 &\leq l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).
 \end{aligned}$$

Thus, combining all cases, we have

$$G(\zeta, \eta, \vartheta) > 0 \text{ implies } G(\zeta, \eta, \vartheta) \leq 2l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).$$

Therefore, by taking $f(t) = \ln(t)$, $t > 0$, and $\alpha = \ln(2l)$, $(G_{\mathcal{F}4})$ and $(G_{\mathcal{F}2})$ are obtained. Hence, G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$. Also, G is a generalized G_b -metric on $\mathcal{A}$ with $s = 2l$. But G is not a G_b -metric on $\mathcal{A}$ as $G(1, 1, 3) = 8 \not\leq 6 = G(1, 2, 3)$, i.e., (G_b3) does not hold.

In the following example, we see the existence of a $G_{\mathcal{F}}$ -metric and G^* -metric space, which is not a generalized G_b -metric space. Let $\mathcal{A} = \mathbb{R}$ and $\mathfrak{B} = [0, 1] \times [0, 1] \times [0, 1] - \{(\zeta, \zeta, \zeta) \mid \zeta \in [0, 1]\}$ and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} 0, & \text{if } \zeta = \eta = \vartheta, \\ \frac{1}{3}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|), & \text{if } (\zeta, \eta, \vartheta) \in \mathfrak{B}, \\ 2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|), & \text{otherwise.} \end{cases}$$

Then, G satisfies $(G_{\mathcal{F}1})$ and $(G_{\mathcal{F}3})$ directly from the definition of G .

Let $f(t) = \frac{-1}{\sqrt{t}}$, $t > 0$, and $\alpha = 1$. Then, for any $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$, we consider the following cases:

Case 1: If $(\zeta, \eta, \vartheta) \in \mathfrak{B}$, then

$$\begin{aligned} G(\zeta, \eta, \vartheta) &= \frac{1}{3}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|) \\ &\leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \end{aligned}$$

which implies that

$$\begin{aligned} f(G(\zeta, \eta, \vartheta)) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha. \end{aligned}$$

Case 2 : If $(\zeta, \eta, \vartheta) \notin \mathfrak{B}$, then

$$\begin{aligned} &f(G(\zeta, \eta, \vartheta)) - f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) - \alpha \\ &= \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} + \frac{1}{\sqrt{\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)}} - 1 \\ &\leq \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} + 1 - 1 \leq 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}4})$ holds.

Also, for $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\zeta \neq \eta, \eta \neq \vartheta$, we consider the following two cases:

Case 1: If $(\zeta, \zeta, \eta) \in \mathfrak{B}$, then

$$G(\zeta, \zeta, \eta) = \frac{1}{3}(2|\zeta - \eta|) \leq G(\zeta, \eta, \vartheta),$$

which implies that

$$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Case 2: If $(\zeta, \zeta, \eta) \notin \mathfrak{B}$, then

$$f(G(\zeta, \zeta, \eta)) = \frac{-1}{\sqrt{2(2|\zeta - \eta|)}} \leq \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

Thus, $(G_{\mathcal{F}}2)$ holds; hence, G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$. But G is not a generalized G_b -metric on $\mathcal{A}$ as for any $s \geq 1$, and for $n \in \mathbb{N}$,

$$\begin{aligned} 2^{4n} &= G(2n+1, 1, 1) \\ &\leq s(G(2n+1, n+1, n+1) + G(n+1, 1, 1)) \\ &= s(2^{2n} + 2^{2n}), \end{aligned}$$

which gives $2^{2n-1} \leq s$, therefore, by taking $n \rightarrow \infty$, we have a contradiction.

The next example assures the existence of a $G_{\mathcal{F}}$ -metric space which is not a G^* -metric space. Let $\mathcal{B} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $\mathcal{A} = \mathcal{B} \cup \mathbb{N} \cup \{0\}$. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} 0, & \text{if } \zeta = \eta = \vartheta; \\ |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, & \text{if } \zeta, \eta, \vartheta \in \mathcal{B}; \text{ or if atleast one of } \zeta, \eta, \vartheta \text{ is } 0; \\ 1, & \text{otherwise.} \end{cases}$$

First, we prove that G is not a G^* -metric. Suppose that G is a G^* -metric with constant $\beta > 0$.

Let $\zeta = 0$ and $\zeta_n = \frac{1}{n}$, then $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$. Therefore, for $\eta, \vartheta \in \mathbb{N}$, using $(Gg3)$, we get

$$G(\zeta, \eta, \vartheta) \leq \beta \left(\limsup_{n \rightarrow \infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right),$$

that is,

$$|\eta| + |\eta - \vartheta| + |\vartheta| \leq \beta(1 + 0) = \beta.$$

Taking limit $\eta, \vartheta \rightarrow \infty$, we have a contradiction. Thus, $(\mathcal{A}, G)$ is not a G^* -metric space.

We now prove that $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, where $f(t) = \frac{-1}{t}$ and $\alpha = 1$. $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ hold obviously. Now, for $(G_{\mathcal{F}}2)$, let $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\zeta \neq \eta, \eta \neq \vartheta$. Then consider the following two cases:

Case 1: If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$. Therefore,

$$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Case 2 : If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then $G(\zeta, \eta, \vartheta) \geq 1$. Thus,

$$\begin{aligned} &f(G(\zeta, \zeta, \eta)) - f(G(\zeta, \eta, \vartheta)) - \alpha \\ &= \frac{-1}{G(\zeta, \zeta, \eta)} + \frac{1}{G(\zeta, \eta, \vartheta)} - 1 \\ &\leq \frac{-1}{G(\zeta, \zeta, \eta)} + 1 - 1 < 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}2})$ holds.

Now, for $(G_{\mathcal{F}4})$ let $\zeta, \eta, \vartheta \in \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$. Consider the following two cases:

Case 1: If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then

$$G(\zeta, \eta, \vartheta) \leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta),$$

therefore,

$$\begin{aligned} f(G(\zeta, \eta, \vartheta)) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha. \end{aligned}$$

Case 2 : If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then $\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \geq$

1. Thus,

$$\begin{aligned} &f(G(\zeta, \eta, \vartheta)) - f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) - \alpha \\ &= \frac{-1}{G(\zeta, \eta, \vartheta)} + \frac{1}{\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)} - 1 \\ &\leq \frac{-1}{G(\zeta, \eta, \vartheta)} + 1 - 1 < 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}4})$ holds. Hence, $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$ -metric space.

Abstract spaces in Example 1, Example 1, Example 1, and Example 1 are not $G_{\mathcal{F}}$ -metric spaces. A relation among the classes of abstract spaces discussed so far is described in the following diagram (Figure 1):

Figure 1: A comparison among various classes of abstract space.

2.2 Some Basic Concepts

This section defines a few fundamental ideas and examines their characteristics within the context of $G_{\mathcal{F}}$ -metric space. Let A be a subset of $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$. Then A is said to be a $G_{\mathcal{F}}$ -open set if for every $\zeta \in A$, there exists $r > 0$ such that $B(\zeta, r) \subseteq A$, where

$$B(\zeta, r) = \{\eta \in \mathcal{A} \mid G(\zeta, \eta, \eta) < r\}.$$

Let $\tau_{G_{\mathcal{F}}}$ be the collection of all such open sets, then $\tau_{G_{\mathcal{F}}}$ is a topology on $\mathcal{A}$.

Let $\{\zeta_n\}$ be a sequence of points in $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$. Then sequence $\{\zeta_n\}$ is said to be $G_{\mathcal{F}}$ -convergent to $\zeta_0 \in \mathcal{A}$ if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(\zeta_n, \zeta_m, \zeta_0) < \epsilon$, for all $n, m \geq n_0$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, then

$$f(G(\zeta, \eta, \eta)) \leq f(2G(\zeta, \zeta, \eta)) + \alpha, \text{ for all } \zeta, \eta \in \mathcal{A} \text{ with } \zeta \neq \eta.$$

Proof. For $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$, using $(G_{\mathcal{F}}1)$, $(G_{\mathcal{F}}4)$ and $(G_{\mathcal{F}}3)$, we have,

$$f(G(\zeta, \eta, \eta)) \leq f(G(\eta, \zeta, \zeta) + G(\zeta, \zeta, \eta)) + \alpha = f(2G(\zeta, \zeta, \eta)) + \alpha.$$

□

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then, the following statements are equivalent.

- (I) $G(\zeta_n, \zeta_n, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.
- (II) $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.
- (III) $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. First, we prove (I) implies (II); for this, let $G(\zeta_n, \zeta_n, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$. If for infinitely many n , $\zeta_n \neq \zeta_0$, then for those n , using Proposition 2.2, we have

$$f(G(\zeta_n, \zeta_0, \zeta_0)) \leq f(2G(\zeta_n, \zeta_n, \zeta_0)) + \alpha.$$

Taking $n \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$. and for rest of n , $G(\zeta_n, \zeta_0, \zeta_0) = 0$, thus in overall $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$.

Now we prove (II) implies (III); for this, let $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.

If for infinitely many m and n , $G(\zeta_n, \zeta_m, \zeta_0) \neq 0$, then for those (m, n) , using $(G_{\mathcal{F}}4)$, we have

$$f(G(\zeta_n, \zeta_m, \zeta_0)) \leq f(G(\zeta_n, \zeta_0, \zeta_0) + G(\zeta_0, \zeta_0, \zeta_m)) + \alpha.$$

Taking $n, m \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$,

and for rest of n, m , $G(\zeta_n, \zeta_m, \zeta_0) = 0$, thus in overall $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$ as $n, m \rightarrow \infty$.

Also, (III) implies (I) obviously. □

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ which is $G_{\mathcal{F}}$ -convergent, converges to a unique element in $\mathcal{A}$.

Proof. Let, if possible that sequence $\{\zeta_n\}$ converges to ζ and η in $\mathcal{A}$ with $\zeta \neq \eta$, then, by $(G_{\mathcal{F}}4)$

$$f(G(\zeta, \eta, \eta)) \leq f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, \eta, \eta)) + \alpha,$$

taking $n \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction. $\square$

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ is said to be $G_{\mathcal{F}}$ -Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(\zeta_n, \zeta_m, \zeta_l) < \epsilon$, for all $n, m, l \geq n_0$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then, the following statements are equivalent.

(I) Sequence $\{\zeta_n\}$ in $\mathcal{A}$ is $G_{\mathcal{F}}$ -Cauchy sequence.

(II) $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

(III) $G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. (I) and (II) are equivalent using Definition 2.2.

Now (II) implies (III) obviously, so we prove (III) implies (II). If for infinitely many n, m and l , $G(\zeta_n, \zeta_m, \zeta_l) \neq 0$, then for those (n, m, l) , using $(G_{\mathcal{F}}4)$, we have

$$f(G(\zeta_n, \zeta_m, \zeta_l)) \leq f(G(\zeta_n, \zeta_m, \zeta_m) + G(\zeta_m, \zeta_m, \zeta_l)) + \alpha.$$

Taking $n, m, l \rightarrow \infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$,

and for rest of n, m, l , $G(\zeta_n, \zeta_m, \zeta_l) = 0$, thus in overall $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$. $\square$

Every $G_{\mathcal{F}}$ -convergent sequence is a $G_{\mathcal{F}}$ -Cauchy sequence.

In Example 2.1, sequence $\{l - \frac{1}{n}\}$ is $G_{\mathcal{F}}$ -Cauchy sequence but not a $G_{\mathcal{F}}$ -convergent sequence.

A $G_{\mathcal{F}}$ -metric space is said to be $G_{\mathcal{F}}$ -complete if every $G_{\mathcal{F}}$ -Cauchy sequence is a $G_{\mathcal{F}}$ -convergent sequence.

In Example 2.1, $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -complete metric space as every $G_{\mathcal{F}}$ -Cauchy sequence is a $G_{\mathcal{F}}$ -convergent sequence.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space and $\mathcal{B} \subseteq \mathcal{A}$. Then *closure* of $\mathcal{B}$ is denoted by $\overline{\mathcal{B}}$ and defined by

$$\overline{\mathcal{B}} = \{\zeta \in \mathcal{A} \mid \zeta \in \mathcal{B} \text{ or } B(\zeta, r) \cap \mathcal{B} \text{ is infinite set for every } r > 0\}.$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

The following propositions are needed in the main results of this paper. Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and f is a continuous function. If $\{\zeta_n\}$ is $G_{\mathcal{F}}$ -convergent to ζ with $\zeta \neq b$ or $\zeta \neq c$, where b and c are real constants, then

$$f(G(\zeta, b, c)) - \alpha \leq \liminf_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \leq f(G(\zeta, b, c)) + \alpha.$$

Proof. Without loss of generality, consider $\zeta \neq b$, so except first finitely many n , $\zeta_n \neq b$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned} f(G(\zeta, b, c)) - \alpha &\leq \liminf_{n \rightarrow \infty} f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, b, c)) \\ &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \\ &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \\ &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \zeta, \zeta) + G(\zeta, b, c)) + \alpha \\ &= \limsup_{n \rightarrow \infty} f(G(\zeta, b, c)) + \alpha \\ &= f(G(\zeta, b, c)) + \alpha. \end{aligned}$$

□

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, and f is a continuous function. If $\{\zeta_n\}$ and $\{\eta_n\}$ are $G_{\mathcal{F}}$ -convergent to ζ and η , respectively, and c is a real constant such that $c \neq \zeta$ or $c \neq \eta$, then

$$f(G(\zeta, \eta, c)) - 2\alpha \leq \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \leq f(G(\zeta, \eta, c)) + 2\alpha.$$

Proof. Without loss of generality, assume that $c \neq \zeta$; therefore, except first

finitely many n , $c \neq \zeta_n$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned}
 f(G(\zeta, \eta, c)) - 2\alpha &\leq \liminf_{n \rightarrow \infty} f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, \eta, c)) - \alpha \\
 &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta, c)) - \alpha \\
 &\leq \liminf_{n \rightarrow \infty} f(G(\eta, \eta_n, \eta_n) + G(\eta_n, \zeta_n, c)) \\
 &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \zeta, \zeta) + G(\zeta, \eta_n, c)) + \alpha \\
 &= \limsup_{n \rightarrow \infty} f(G(\zeta, \eta_n, c)) + \alpha \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\eta_n, \eta, \eta) + G(\eta, \zeta, c)) + 2\alpha \\
 &= \limsup_{n \rightarrow \infty} f(G(\eta, \zeta, c)) + 2\alpha. \\
 &= f(G(\zeta, \eta, c)) + 2\alpha.
 \end{aligned}$$

□

3 Fixed Point Results in $G_{\mathcal{F}}$ -Metric Space

This section deals with some fixed point results in the context of $G_{\mathcal{F}}$ -metric space. Our first result is the following theorem. Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that there exists a non-empty subset $\mathcal{B}$ of $\mathcal{A}$ with $T(\mathcal{B}) \subseteq \mathcal{B}$ and

$$G(T\zeta, T\eta, T\vartheta) \leq \lambda G(\zeta, \eta, \vartheta), \quad \text{for all } \zeta, \eta, \vartheta \in \overline{\mathcal{B}},$$

where $\lambda \in [0, 1)$. Then, T has a fixed point in $\mathcal{A}$. Moreover, if the fixed point belongs to $\mathcal{B}$, then T has a unique fixed point in $\mathcal{B}$.

Proof. Let $\zeta_0 \in \mathcal{B}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_n = T\zeta_{n-1}$, for all $n \in \mathbb{N}$. If $\zeta_n = \zeta_{n+1}$ for some $n \in \mathbb{N}$, then ζ_n is a fixed point of T . So let $\zeta_n \neq \zeta_{n+1}$, for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we have

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda G(\zeta_{n-1}, \zeta_n, \zeta_n).$$

Now, an easy induction gives that

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda^n G(\zeta_0, \zeta_1, \zeta_1).$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

Let $n, m \in \mathbb{N}$ with $m > n$; then, we have

$$\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1}) \leq \frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1).$$

If for infinitely many pairs (m, n) with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) \neq 0$, then for these m, n using $(G_{\mathcal{F}4})$, we have

$$\begin{aligned} f(G(\zeta_n, \zeta_m, \zeta_m)) &\leq f\left(\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1})\right) + \alpha \\ &\leq f\left(\frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1)\right) + \alpha, \end{aligned}$$

taking $n, m \rightarrow \infty$ and using $(\mathcal{F}2)$, we get $G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0$.

Also, for rest of m, n with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) = 0$. Thus in overall,

$$G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus, $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence in $\mathcal{A}$, but $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -complete metric space, therefore $\{\zeta_n\}$ is $G_{\mathcal{F}}$ -convergent to some $\zeta' \in \mathcal{A}$. Suppose, if possible $G(T\zeta', \zeta', \zeta') > 0$, then using $(G_{\mathcal{F}4})$, we have

$$\begin{aligned} f(G(T\zeta', \zeta', \zeta')) &\leq f(G(T\zeta', T\zeta_n, T\zeta_n) + G(T\zeta_n, \zeta', \zeta')) + \alpha \\ &\leq f(\lambda G(\zeta', \zeta_n, \zeta_n) + G(\zeta_{n+1}, \zeta', \zeta')) + \alpha. \end{aligned}$$

Taking $n \rightarrow \infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction.

Thus, $G(T\zeta', \zeta', \zeta') = 0$ and which implies that $T\zeta' = \zeta'$.

Now, if $\zeta' \in \mathcal{B}$ and $\eta \in \mathcal{B}$ be another fixed point of T , then

$$G(\zeta', \eta, \eta) = G(T\zeta', T\eta, T\eta) \leq \lambda G(\zeta', \eta, \eta)$$

which implies that $G(\zeta', \eta, \eta) = 0$ as $\lambda \in [0, 1)$. Therefore, $\zeta' = \eta$, that is, ζ' is a unique fixed point of T in $\mathcal{B}$. $\square$

Consider $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$ as in Example 2.1, which is a $G_{\mathcal{F}}$ -complete metric space. Define a mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ as $T\zeta = \frac{\zeta(\zeta+1)}{4}$, for all $\zeta \in \mathcal{A}$.

Now, for $\mathcal{B} = [0, 1]$, we see that $T(\mathcal{B}) \subseteq \mathcal{B}$, and for $\zeta, \eta, \vartheta \in \overline{\mathcal{B}}$, we have

$$\begin{aligned} &G(T\zeta, T\eta, T\vartheta) \\ &= \frac{1}{3} \left(\left| \frac{\zeta(\zeta+1)}{4} - \frac{\eta(\eta+1)}{4} \right| + \left| \frac{\eta(\eta+1)}{4} - \frac{\vartheta(\vartheta+1)}{4} \right| + \left| \frac{\vartheta(\vartheta+1)}{4} - \frac{\zeta(\zeta+1)}{4} \right| \right) \\ &\leq \frac{1}{3} \times \frac{3}{4} (|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|) \\ &= \frac{3}{4} G(\zeta, \eta, \vartheta). \end{aligned}$$

Thus, the hypothesis of Theorem 3 is satisfied. And we notice that T has two fixed points, 0 and 3. Also, 0 is the only fixed point of T in $\mathcal{B}$.

Consider $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$ as in Example 2.1, which is a $G_{\mathcal{F}}$ -complete metric space. Define a mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ as $T\zeta = \frac{\zeta}{2}$, for all $\zeta \in \mathcal{A}$. Then for $\mathcal{B} = [0, 1]$, we can easily see that hypothesis of Theorem 3 is satisfied. Also, we notice that T has a unique fixed point in $\mathcal{A}$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that

$$G(T\zeta, T\eta, T\vartheta) \leq \lambda G(\zeta, \eta, \vartheta), \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A},$$

where $\lambda \in [0, 1)$. Then, T has a unique fixed point in $\mathcal{A}$.

Proof. Take $\mathcal{B} = \mathcal{A}$ in Theorem 3. □

In the following result, we find the unique fixed point for (ψ, ϕ) -contractive mapping (see detail of (ψ, ϕ) -contractive mapping in (51; 52; 14; 53; 29; 32; 54)) in the setting of $G_{\mathcal{F}}$ -complete metric space.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that f is a continuous function. Let $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that

$$\psi(f(M(\zeta, \eta, \vartheta)) + 4\alpha) \leq \psi(f(G(T\zeta, T\eta, T\vartheta))) - \phi(M(\zeta, \eta, \vartheta)), \quad (1)$$

for all $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A} - \{(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A} \mid T\zeta = T\eta = T\vartheta\}$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous non-decreasing function and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi^{-1}(0) = \{0\}$ and

$$M(\zeta, \eta, \vartheta) = \max\{G(\zeta, \eta, \vartheta), G(\zeta, T\zeta, T\eta), G(\eta, T\eta, T\vartheta), G(\vartheta, T\vartheta, T\zeta)\}.$$

Then, T has a unique fixed point.

Proof. Let $\zeta_0 \in \mathcal{A}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_{n+1} = T\zeta_n$, $n = 0, 1, 2, \dots$.

If $\zeta_n = \zeta_{n+1}$, then ζ_n is a fixed point of T .

Now, assume that $\zeta_n \neq \zeta_{n+1}$, for all n . Let $\theta_n = f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$, $n = 1, 2, 3, \dots$.

Now,

$$\begin{aligned} \psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))) &\leq \psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha) \\ &\leq \psi(f(G(T\zeta_n, T\zeta_{n+1}, T\zeta_{n+2}))) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) \\ &\leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})), \end{aligned}$$

where

$$\begin{aligned} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) &= \max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), G(\zeta_n, T\zeta_n, T\zeta_{n+1}), \\ &\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}), G(\zeta_{n+2}, T\zeta_{n+2}, T\zeta_n)\} \\ &= \max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), \\ &\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}), G(\zeta_{n+2}, \zeta_{n+3}, \zeta_{n+1})\} \end{aligned}$$

and, therefore,

$$\begin{aligned} f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) &= \max\{f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})), f(G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}))\} \\ &= \max\{\theta_n, \theta_{n+1}\}. \end{aligned}$$

If $\theta_n < \theta_{n+1}$ for some n , then we have $\psi(\theta_{n+1}) \leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$, which gives that $M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, a contradiction, therefore $\theta_{n+1} \leq \theta_n$ for all n . Thus, $\{\theta_n\}$ is a non-increasing sequence. Suppose that $\{\theta_n\}$ is bounded below; then there exists a real θ such that $\lim_{n \rightarrow \infty} \theta_n = \theta$. Now,

$$\psi(\theta_n) \leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$$

taking limit supremum on both sides, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \psi(\theta_n) &\leq \limsup_{n \rightarrow \infty} \psi(\theta_{n+1}) - \liminf_{n \rightarrow \infty} \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) \\ \text{i.e.,} \quad \psi(\theta) &\leq \psi(\theta) - \phi\left(\liminf_{n \rightarrow \infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) \end{aligned}$$

which gives that $\liminf_{n \rightarrow \infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, therefore, by $(\mathcal{F}2)$

$\liminf_{n \rightarrow \infty} f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) = -\infty$, i.e., $\lim_{n \rightarrow \infty} \theta_n = -\infty$. Hence by $(\mathcal{F}2)$,

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0. \quad (2)$$

Since $\zeta_n \neq \zeta_{n+1}$ for every n , therefore, by $(G_{\mathcal{F}}2)$,

$$f(G(\zeta_n, \zeta_n, \zeta_{n+1})) \leq f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) + \alpha.$$

So, by using (2) and $(\mathcal{F}2)$, we have

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_n, \zeta_{n+1}) = 0. \quad (3)$$

Also, using Proposition 2.2, we have

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0. \quad (4)$$

Next, we prove that $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence. Suppose not, then there exists $\epsilon > 0$ such that we can find subsequences $\{\zeta_{m_k}\}$ and $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) \geq \epsilon \quad (5)$$

this means that

$$G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1}) < \epsilon. \quad (6)$$

Now further, consider only those k for which left-hand side of (6) is greater than 0, and clearly, such k exists infinitely many.

Now,

$$\begin{aligned} & \psi(f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) + 4\alpha) \\ &= \psi(f(G(T\zeta_{n_k}, T\zeta_{m_k-2}, T\zeta_{m_k-1}))) - \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) \\ &\leq \psi(f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k}))) - \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})), \end{aligned} \quad (7)$$

where

$$\begin{aligned} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) &= \max\{G(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}), G(\zeta_{n_k}, T\zeta_{n_k}, T\zeta_{m_k-2}), \\ & \quad G(\zeta_{m_k-2}, T\zeta_{m_k-2}, T\zeta_{m_k-1}), G(\zeta_{m_k-1}, T\zeta_{m_k-1}, T\zeta_{n_k})\} \\ &= \max\{G(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}), G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{m_k-1}), \\ & \quad G(\zeta_{m_k-2}, \zeta_{m_k-1}, \zeta_{m_k}), G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{n_k+1})\}. \end{aligned} \quad (8)$$

Also, using $(G_{\mathcal{F}4})$, (3), and (6), we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} f(G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{n_k+1})) \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{n_k}, \zeta_{n_k}) + G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1}) + G(\zeta_{m_k-1}, \zeta_{m_k-1}, \zeta_{m_k})) + \alpha \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1})) + \alpha \\ &\leq f(\epsilon) + \alpha. \end{aligned} \quad (9)$$

Now, using (5), $(G_{\mathcal{F}4})$, and $(G_{\mathcal{F}2})$, we have

$$\begin{aligned} f(\epsilon) &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k})) \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{n_k+1}) + G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})) + \alpha \\ &= \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})) + \alpha \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k-1})) + 2\alpha. \end{aligned} \quad (10)$$

Now, using (10), (7), and (9), we have

$$\begin{aligned}
 \psi(f(\epsilon) + 2\alpha) &\leq \psi \left(\limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k})) + 2\alpha + 2\alpha \right) \\
 &\leq \psi \left(\limsup_{k \rightarrow \infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) + 4\alpha \right) \\
 &\leq \psi \left(\limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k})) \right) - \liminf_{k \rightarrow \infty} \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) \\
 &\leq \psi(f(\epsilon) + \alpha) - \phi \left(\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) \right) \\
 &\leq \psi(f(\epsilon) + 2\alpha) - \phi \left(\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) \right)
 \end{aligned}$$

This gives

$$\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) = 0,$$

therefore, we have

$$\liminf_{k \rightarrow \infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) = -\infty,$$

which gives a contradiction in view of (8) and (10). Thus $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence in $(\mathcal{A}, G)$, but $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$ -complete. Therefore, there exists $b \in \mathcal{A}$ such that $\lim_{n \rightarrow \infty} \zeta_n = b$.

Next, we prove that b is a fixed point of T . Suppose that $Tb \neq b$, then

$$\begin{aligned}
 &\psi(f(M(b, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha) \\
 &\leq \psi(f(G(Tb, T\zeta_{n+1}, T\zeta_{n+2}))) - \phi(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &= \psi(f(G(Tb, \zeta_{n+2}, \zeta_{n+3}))) - \phi(M(b, \zeta_{n+1}, \zeta_{n+2})), \tag{11}
 \end{aligned}$$

where

$$\begin{aligned}
 M(b, \zeta_{n+1}, \zeta_{n+2}) &= \max\{G(b, \zeta_{n+1}, \zeta_{n+2}), G(b, Tb, T\zeta_{n+1}), \\
 &\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}), G(\zeta_{n+2}, T\zeta_{n+2}, Tb)\} \\
 &= \max\{G(b, \zeta_{n+1}, \zeta_{n+2}), G(b, Tb, \zeta_{n+2}), \\
 &\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}), G(\zeta_{n+2}, \zeta_{n+3}, Tb)\}.
 \end{aligned}$$

Thus, by Proposition 2.2 and Proposition 2.2, we have

$$\begin{aligned}
 f(G(Tb, b, b)) - 2\alpha &\leq \liminf_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq \limsup_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq f(G(Tb, b, b)) + 2\alpha. \tag{12}
 \end{aligned}$$

Now, using (11) and (12), we have

$$\begin{aligned}
 \psi(f(G(Tb, b, b)) + 2\alpha) &\leq \psi\left(\limsup_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha\right) \\
 &\leq \psi\left(\limsup_{n \rightarrow \infty} f(G(Tb, \zeta_{n+2}, \zeta_{n+3}))\right) - \liminf_{n \rightarrow \infty} \phi(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq \psi(f(G(Tb, b, b)) + 2\alpha) - \phi\left(\liminf_{n \rightarrow \infty} M(b, \zeta_{n+1}, \zeta_{n+2})\right).
 \end{aligned}$$

It gives that

$$\liminf_{n \rightarrow \infty} M(b, \zeta_{n+1}, \zeta_{n+2}) = 0,$$

therefore, we have

$$\liminf_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) = -\infty,$$

which gives a contradiction in (12); therefore, $Tb = b$. Next we prove that the fixed point of T is unique. For this, let c be another fixed point of T such that $c \neq b$. Then

$$\begin{aligned}
 M(b, b, c) &= \max\{G(b, b, c), G(b, Tb, Tb), G(b, Tb, Tc), G(c, Tc, Tb)\} \\
 &= \max\{G(b, b, c), G(b, b, b), G(b, b, c), G(c, c, b)\} \\
 &= \max\{G(b, b, c), G(c, c, b)\} \\
 &= M(c, c, b).
 \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
 \psi(f(G(b, b, c))) &\leq \psi(f(M(b, b, c)) + 4\alpha) \\
 &\leq \psi(f(G(Tb, Tb, Tc))) - \phi(M(b, b, c)) \\
 &= \psi(f(G(b, b, c))) - \phi(M(b, b, c)).
 \end{aligned} \tag{14}$$

It gives that $M(b, b, c) = 0$, and hence $G(b, b, c) = G(c, c, b) = 0$. Thus $b = c$. $\square$

4 Conclusion

With the aid of $\mathcal{F}$ -metric space, we have introduced a new generalization of G -metric space, which we call $G_{\mathcal{F}}$ -metric space. We have also shown a comparison between $G_{\mathcal{F}}$ -metric space and several abstract spaces found in literature. This newly defined abstract space is also studied in terms of some fundamental concepts. In the framework of $G_{\mathcal{F}}$ -metric space, we have demonstrated the Banach Contraction Principle and the fixed point result for (ψ, ϕ) -contractive mapping. In

this newly defined abstract space, fixed point results for different mappings existing in the literature and for some new mappings can be studied.

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Partial sharing of small functions with exact difference

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Abstract

We have proven certain uniqueness results related owing to the partial sharing of small functions of f with difference polynomials of the form $L(f) = b_k(z)f(z + kc) + \dots + b_0(z)f(z)$, where b_i are small functions of f . As a consequence of this, we have deduced the conditions under which, f is identically equal to its exact difference operator $\Delta_c^k f$.

Keywords: Meromorphic functions, difference polynomials, partial sharing.

2020 AMS subject classifications: 30D35 ¹

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1 Introduction

Two meromorphic functions f and g are said to share a value a CM (resp. IM) if whenever $f(z) = a$, then $g(z) = a$ with the same multiplicity (resp. ignoring multiplicity). The sets $E(a, f)$ and $\overline{E}(a, f)$ are also defined as the set of points z_0 where $f(z_0) = a$, z_0 being counted according to multiplicity and ignoring multiplicity respectively. The celebrated Nevanlinna's five value theorem which says that if for two meromorphic functions f and g , there are five distinct complex values a_i such that $\overline{E}(a_i, f) = \overline{E}(a_i, g)$, for $i = 1, \dots, 5$, then they are identical. This is also true for small functions of f and g as well. Let us denote the set of all small functions of f by $\mathcal{S}(f)$. Also, if the function g is replaced by the derivative of f , then the sharing conditions are changed from five to two shared values; that is, if f and f' share two finite values CM, then $f = f'$ Gundersen [1983]. Several further modifications to these theorems have been done thereafter, replacing the derivative f' by n -th order derivatives of f . However, in this paper, we are interested in the difference analogues of the uniqueness theorems. For the preliminary ideas and definitions of the classical Nevanlinna theory, we refer the reader to Hayman [1964], Lo [1993]. The reader can also obtain a comprehensive idea on the uniqueness theory of meromorphic functions from Yang and Yi [2003].

Let c be a non-zero complex number. Then the exact differences are defined as

$$\Delta_c f(z) = f(z+c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c (\Delta_c^{n-1} f(z)).$$

If $c = 1$, then we denote $\Delta_c^n f$ by $\Delta^n f$. Recent times have witnessed increasing interest in the difference analogues of the classical Nevanlinna theory owing to its wide applications in the theory of difference equations. The inception of the theory is credited to Halburd and Korhonen Halburd and Korhonen [2006a]; and Chiang and Feng Chiang and Feng [2008], who independently determined estimates of the proximity function of $\frac{f(z+c)}{f(z)}$ for a finite ordered meromorphic function f , respectively in the years 2006 and 2008. Thereafter, several existing results have been looked upon in this new perspective culminating in the disclosure of new theories in multitudes and methodically developing the theory of complex difference equations which runs parallel to the theory of complex differential equations Laine [1993]. We refer the reader to Chiang and Feng [2008, 2009], Halburd and Korhonen [2006a,b] for preliminaries in this field.

In their celebrated work on the difference analogue of the Nevanlinna theory, Halburd and Korhonen have deduced a uniqueness theorem by taking into account the counting function for the c -separated pairs Halburd and Korhonen [2006b]. Further studies on uniqueness theorems, and more specifically, periodicity conditions, for meromorphic functions have been done henceforth Heittokangas et al. [2009, 2011]. Since the difference $\Delta_c f$ can be thought of as the difference ana-

logue of the derivative of f , it is interesting to study the uniqueness results for values shared by f with its exact differences. We shall study a more generalised version of this by taking into account certain difference polynomials as stated later in the article. Significant contributions have been done in this field. Li et al. [2015] proved the following theorem for entire functions.

Theorem 1.1. *Li et al. [2015] Let $f(z)$ be a non-constant entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$. If $E(a_i, f) = E(a_i, \Delta_c f)$ where $a_i, i = 1, 2, 3$ are distinct complex values in the extended complex plane, then $f(z) \equiv \Delta_c f(z)$.*

However, Li and Gao gave a stronger version of the above theorem as follows.

Theorem 1.2. *Li and Gao [2011] Let $f(z)$ be a non-constant entire function of finite order, $c \in \mathbb{C}$ and n be a positive integer. Suppose that $E(a_i, f) = E(a_i, \Delta_c^n f)$ for two distinct complex numbers a_1 and a_2 and one of the following cases is satisfied:*

1. $a_1 a_2 = 0$;
2. $a_1 a_2 \neq 0$ and order of f $\rho(f) \notin \mathbb{N}$.

Then $f(z) \equiv \Delta_c^n f(z)$.

The sharing condition was improved further in the following theorem.

Theorem 1.3. *Li and Gao [2011] Let $f(z)$ be a non-constant entire function of finite order $\rho(f)$, $c \in \mathbb{C}$ and n be a positive integer. If $E(a, f) = E(a, \Delta_c^n f)$ for some $a \in \mathbb{C}$, and for a finite value $b \neq a$, $f(z) - b$ and $\Delta_c^n f(z) - b$ have $\max\{1, [\rho(f)]\}$ distinct common zeros of multiplicity ≥ 2 , then $f(z) \equiv \Delta_c^n f(z)$.*

The theorem has been extended to meromorphic conditions imposing further restrictions as in the following theorem by Gao et al.

Theorem 1.4. *Gao et al. [2019] Let $f(z)$ be a transcendental meromorphic function of hyper order less than 1 such that $E(a_i, f) = E(a_i, \Delta^n f)$ for three distinct periodic functions $a_i \in \mathcal{S}(f) \cup \{\infty\}$, $i = 1, 2, 3$ of period 1, the $\Delta^n f(z) \equiv f(z)$.*

Natural questions that can be posed are as follows:

1. Can we replace three functions by two functions?
2. Can we replace sharing of functions by partial sharing?
3. Can we replace the exact difference of f by any function of the form $b_k(z)f(z+k) + \dots + b_0(z)f(z)$, where $b_i(z) \in \mathcal{S}(f)$ for $i = 1, \dots, k$?

The first question has been answered by Gao et al. [2019]. They showed for any meromorphic solution f , of $\Delta^n f(z) - 1 = k(f(z) - 1)$, where $k \neq 0, 1$ is a constant, $E(1, f) = E(1, \Delta^n f)$ and $E(\infty, f) = E(\infty, \Delta^n f)$, but $\Delta^n f \not\equiv f$. In 2020, Chen and Xu have also discussed certain uniqueness results related to sharing of two values by entire functions with its differences Chen and Xu [2020]. We will however, examine further conditions that need to be imposed to ensure the equality, with the partial sharing condition defined as follows.

Definition 1.1. Let f and g be two meromorphic functions and $a \in \mathcal{S}(f) \cap \mathcal{S}(g)$. Then f share a with g partial if $E(a, f) \subseteq E(a, g)$.

In the year 2016, however, Charak et al. [2016] used a different definition for partial sharing where they considered $\overline{E}(a, f)$ instead of $E(a, f)$ and proved the following theorem.

Theorem 1.5. Charak et al. [2016] Let f be a meromorphic function of hyper order < 1 and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in \mathcal{S}(f)$ be distinct periodic functions with period c such that f share a_1, a_2 and a_3 partially with $f(z + c)$ and also $\sum_{i=1}^3 \delta(a_i, f) > 1$. Then $f(z) \equiv f(z + c)$.

Thereafter, in 2018, Chen [2018] deduced a result replacing the shift $f(z + c)$ by $\Delta_c f(z)$.

Theorem 1.6. Chen [2018] Let f be a non-constant meromorphic function of hyper order less than 1, and let c be a non-zero finite complex number such that $f(z + c) \not\equiv f(z)$. If $\Delta_c f(z) = f(z + c) - f(z)$, and $f(z)$ share the value 1 CM and satisfy

$$E(0, f) \subset E(0, \Delta_c f) \quad \text{and} \quad E(\infty, f) \supset E(\infty, \Delta_c f),$$

then $\Delta_c f \equiv f$.

In the next section, will deduce uniqueness results relating to partial sharing of small functions with a difference polynomial.

2 Main Results

Let us begin by defining the following function.

$$L(f) = b_k(z)f(z + kc) + \dots + b_0(z)f(z), \quad (1)$$

where $b_0, \dots, b_k \in \mathcal{S}(f)$, not all of which are equal to zero.

We have generalised Chen's result Chen [2018] in the following.

Theorem 2.1. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$. Also let f and $L(f)$ satisfy the following conditions:*

1. $E(a, f) \subseteq E(a, L(f))$, where $a \in \mathcal{S}(f)$ is a periodic function of period c ;
2. $E(0, f) \subseteq E(0, L(f))$ and $E(\infty, f) \supseteq E(\infty, L(f))$;

then $f \equiv L(f)$.

The following corollaries can be immediately deduced from the above theorem.

Corollary 2.1. *Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$. Also, let $E(a, f) \subseteq E(a, L(f))$ and $E(0, f) \subseteq E(0, L(f))$, where $a \in \mathcal{S}(f)$. Then $f \equiv L(f)$.*

Corollary 2.2. *Let f be a function as mentioned in theorem 2.1 and let for any positive integer n , $L(f) = \Delta_c^n f$. If f and $L(f)$ satisfies all the conditions of theorem 2.1, then $f \equiv \Delta_c^n f$.*

The first condition of theorem 2.1 is essential and non-replaceable as proved by the example below.

Example 2.1. *Consider $f(z) = e^z$, and $L(f) = (z + 1)f(z) - ef(z) = ze^{z+1}$. Then $E(0, f) \subseteq E(0, L(f))$ and $E(\infty, L(f)) = E(\infty, f)$. But, $f \not\equiv L(f)$.*

Henceforth, we have proved results using arbitrary functions $a_1, a_2, a_3 \in \mathcal{S}(f)$, periodic with period c .

Theorem 2.2. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ such that $E(a_i, f) \subseteq E(a_i, L(f))$ for small periodic functions a_i of f of period c , $i = 1, 2, 3$. Also, if*

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k + 1},$$

then $f \equiv L(f)$.

We can deduce the following corollaries from the above theorem.

Corollary 2.3. *Let f be a transcendental meromorphic function of hyper order less than 1. Let $a_i \in \mathcal{S}(f)$, $i = 1, 2, 3$ are periodic functions with period c .*

1. *Let $L(f) \equiv f(z + c)$. If $E(a_i, f) \subseteq E(a_i, L(f))$, $i = 1, 2, 3$, and also*

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{2},$$

then $f(z) \equiv f(z + c)$.

2. If $L(f) \equiv \Delta_c^k f(z)$, and all the conditions in the above theorem are satisfied, then $f(z) \equiv \Delta_c^k f(z)$.

Corolary 2.4. Let f be a transcendental entire function with hyper order less than 1 and a_i be defined as in the previous corollary. If $E(a_i, f) \subseteq E(a_i, L(f))$, $i = 1, 2, 3$, then $f \equiv L(f)$.

We have further tried to strengthen the results replacing $E(a, f)$ by $\overline{E}(a, f)$.

Theorem 2.3. If f is a transcendental meromorphic function of hyper order $\rho_2(f) < 1$, and for $a_i \in \mathcal{S}(f)$, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $i = 1, 2, \dots, k+4$, then, $f \equiv L(f)$.

Putting $k = 1$, we can immediately deduce the following corollary.

Corolary 2.5. If f be a function and we take $L(f) = \Delta_c f$ satisfying the conditions in theorem 2.3, such that f and $L(f)$ satisfy the condition $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $i = 1, 2, \dots, 5$, then $f \equiv \Delta_c f$.

Can we strengthen the sharing condition of the above theorem? We have deduced the following result in this context.

Theorem 2.4. Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathcal{S}(f)$, $i = 1, 2, 3, 4$. Then $f \equiv L(f)$.

Can we prove this for meromorphic functions as well? We see that in case of entire functions, the quantity $N(r, f)$ is "small". Can we do something by making the quantity somewhat "small" in the meromorphic case too? We have deduced the following theorem thus.

Theorem 2.5. Let f be a transcendental meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathcal{S}(f)$, $i = 1, 2, 3, 4$. Also, let

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2}.$$

Then $f \equiv L(f)$.

Theorem 2.4 can be deduced as a corollary of the above theorem. Here, we are inclined to make a conjecture that we can replace $N(r, f)$ by $\overline{N}(r, f)$. Under what conditions can we change the sharing condition from four functions to three functions?

Theorem 2.6. *Let k be a positive integer ≥ 3 and f be a non-constant meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathcal{S}(f)$, $i = 1, 2, 3$. Also let*

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2},$$

alongwith the condition $\delta(a_4, f) > \frac{2}{k}$, holds, where $a_4 \in \mathcal{S}(f) \setminus \{a_1, a_2, a_3\}$. Then $f \equiv L(f)$.

From the proof of the above theorem, we can easily show that we can replace $\delta(a_4, f)$ by $\Theta(a_4, f)$ which is defined by

$$\Theta(a_4, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)}.$$

What further conditions do we need to impose if f and $L(f)$ share only two small functions partially? Our next theorem points in that direction.

Theorem 2.7. *Let k be any positive integer ≥ 2 and f be a non-constant meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathcal{S}(f)$, for $i = 1, 2$. Also, let for some $a_3, a_4 \in \mathcal{S}(f) \setminus \{a_1, a_2\}$, be such that*

$$\limsup_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{f-a_3}\right) + \overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)} < \frac{k-1}{k}.$$

Further, if we assume that

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2},$$

then, $f \equiv L(f)$.

From all the theorems we so far stated, we can easily deduce the condition $\Delta_c^k f(z) \equiv f(z)$, under the required conditions, by replacing $L(f)$ by the difference operator.

3 Lemmas

We state a few lemmas in this section, that we will need in the sequel.

Lemma 3.1. Halburd et al. [2014] Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and $c \in \mathbb{C}$. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

outside an exceptional set of finite logarithmic measure.

Lemma 3.2. Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and $c \in \mathbb{C}$. Also, let $L(f)$ be as defined in equation (1). Then

$$m\left(r, \frac{L(f)}{f}\right) = S(r, f)$$

outside an exceptional set of finite logarithmic measure.

Proof. Follows by using the principle of mathematical induction on 3.1. $\square$

Lemma 3.3. Halburd et al. [2014] Let $T : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing continuous function and let $s \in (0, \infty)$. If the hyper order ρ_2 of T is strictly less than one, and $\delta \in (0, 1 - \rho_2)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 3.4. Let f be a meromorphic function of hyper order $\rho_2 < 1$ and a be any small periodic function of f of period c . Then

$$m\left(r, \frac{L(f)}{f-a}\right) = S(r, f),$$

outside an exceptional set with finite logarithmic measure.

Proof. Since we have

$$m\left(r, \frac{L(f)}{f-a}\right) = m\left(r, \frac{L(f) - L(a)}{f-a}\right) + O(1) = S(r, f-a),$$

so using the properties of $T(r, f)$ and lemma 3.3, the lemma is proved. $\square$

Lemma 3.5. Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$ and $c \in \mathbb{C}$ such that $L(f) \not\equiv 0$. Let $q \geq 2$ and $a_1, \dots, a_q$ be q meromorphic periodic functions in $\mathcal{S}(f)$ of period c . Then

$$\begin{aligned} m(r, f) + \sum_{i=1}^q m\left(r, \frac{1}{f-a_i}\right) &\leq 2T(r, f) - 2N(r, f) + N(r, L(f)) \quad (2) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + S(r, f), \end{aligned}$$

for all r outside an exceptional set of finite logarithmic measure.

Proof. Let $P(f) = (f - a_1)(f - a_2) \dots (f - a_q)$ which, on rewriting, gives

$$\frac{1}{P(f)} = \sum_{i=1}^q \frac{c_i}{f - a_i},$$

where $c_i \in \mathcal{S}(f)$ are periodic functions with period c . By previous lemma, we obtain

$$m\left(r, \frac{L(f)}{P(f)}\right) = m\left(r, L(f) \sum_{i=1}^q \frac{c_i}{f - a_i}\right) = S(r, f).$$

Thus,

$$m\left(r, \frac{1}{P(f)}\right) \leq m\left(r, \frac{1}{L(f)}\right) + S(r, f).$$

The preceding equation, first fundamental theorem and further properties of $T(r, f)$ gives

$$T(r, L(f)) \geq \sum_{i=1}^q m\left(r, \frac{1}{f - a_i}\right) + N\left(r, \frac{1}{L(f)}\right) + S(r, f).$$

Finally, we will have,

$$\begin{aligned} m(r, f) + \sum_{i=1}^q m\left(r, \frac{1}{f - a_i}\right) &\leq T(r, f) - N(r, f) + T(r, L(f)) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &\leq 2T(r, f) - 2N(r, f) + N(r, L(f)) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + S(r, f). \end{aligned}$$

□

Remark 3.1. The equation (2) can be rewritten as follows

$$\begin{aligned} (q - 1)T(r, f) &\leq N(r, f) + \sum_{i=1}^q N\left(r, \frac{1}{f - a_i}\right) - 2N(r, f) + N(r, L(f)) \\ &\quad - N\left(r, \frac{1}{L(f)}\right) + S(r, f). \end{aligned}$$

We also state the second fundamental theorem for small functions, due to Yamanoi.

Lemma 3.6. Yamanoi [2004] Let f be a non-constant meromorphic function and $a_i \in \mathcal{S}(f)$ be distinct for $i = 1, 2, \dots, n$. Then for any $\epsilon > 0$,

$$(k - 2 - \epsilon)T(r, f) \leq \sum_{i=1}^n \overline{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f).$$

4 Proof of Main Results

Proof of theorem 2.1

Proof. Since $E(a, f) \subseteq E(a, L(f))$, and $E(0, f) \subseteq E(0, L(f))$, so,

$$N\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{L(f)-a}\right) \quad \text{and} \quad N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{L(f)}\right). \quad (3)$$

Also, since $E(\infty, L(f)) \subseteq E(\infty, f)$, so $N(r, L(f)) \leq N(r, f)$. Now,

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &= m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) \\ &\leq m\left(r, \frac{L(f)}{f}\right) + m\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(f)}\right) + O(1) \\ &= T(r, L(f)) + S(r, f). \end{aligned} \quad (4)$$

Also,

$$T(r, L(f)) \leq \sum_{i=0}^k T(r, f(z+ic)) + S(r, f) = (k+1)T(r, f) + S(r, f). \quad (5)$$

Thus, we have, from equations (4) and (5), $S(r, L(f)) = S(r, f) = S(r)$, say.

Now, let us take

$$g = \frac{L(f)}{f}.$$

Then, by the given assumptions, we can say that g is an entire function of hyper order less than 1.

By lemma 3.2, we get

$$T(r, g) = S(r).$$

If possible, suppose that $L(f) \not\equiv f$. We show that

$$N\left(r, \frac{1}{f-a}\right) = S(r).$$

Since $E(a, f) \subseteq E(a, L(f))$, so,

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{g-1}\right) \leq S(r). \quad (6)$$

Next, we estimate $N_{(2)}\left(r, \frac{1}{f-a}\right)$, the counting function for the number of a -points of f with multiplicity greater than or equal to 2.

Also, we have

$$\frac{g'}{g} = \frac{[L(f)]'}{L(f)} - \frac{f'}{f}.$$

Let z_0 be an a -point of f of multiplicity p greater than or equal to 2. Then it is also so of $L(f)$. By the above equation, we can see that z_0 is a zero of $\frac{g'}{g}$ of multiplicity $\geq p-1$. Combining, if g is non-constant, we get

$$\begin{aligned} N_{(2)}\left(r, \frac{1}{f-a}\right) &\leq 2N\left(r, \frac{g}{g'}\right) \leq 2T\left(r, \frac{g}{g'}\right) \\ &= 2T\left(r, \frac{g'}{g}\right) + O(1) \\ &\leq 6T(r, g) + S(r, g) \\ &= S(r, g) = S(r). \end{aligned} \tag{7}$$

Thus, from equations (6) and (7), we get

$$N\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + N_{(2)}\left(r, \frac{1}{f-a}\right) = S(r).$$

Also, if g is constant and $L(f) \not\equiv f$, then $a(z)$ is not assumed by f and hence,

$$N\left(r, \frac{1}{f-a}\right) \leq S(r).$$

By remark 3.1, and equations (3), we get,

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f}\right) - 2N(r, f) \\ &\quad + N(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) + S(r) \\ &= N\left(r, \frac{1}{f-a}\right) + (N(r, L(f)) - N(r, f)) \\ &\quad + \left(N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{L(f)}\right)\right) + S(r) \\ &\leq S(r), \end{aligned}$$

which is a contradiction to the fact that f is transcendental. Thus, $L(f) \equiv f$. $\square$

Proof of theorem 2.2

Proof. Since

$$\limsup_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k+1},$$

so for $\epsilon > 0$, we have, for large r ,

$$N(r, f) \leq \frac{1-2\epsilon}{k+1} T(r, f).$$

If possible, let $f \not\equiv L(f)$. Since $E(a_i, f) \subseteq E(a_i, L(f))$, we have

$$N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) + N\left(r, \frac{1}{f-a_3}\right) \leq N\left(r, \frac{1}{L(f)-f}\right). \quad (8)$$

Now, by first main theorem, elementary Nevanlinna theory, and lemma 3.3,

$$\begin{aligned} N\left(r, \frac{1}{L(f)-f}\right) &\leq T(r, L(f)-f) + O(1) \\ &\leq m(r, L(f)-f) + (k+1)N(r+k|c|, f) + S(r, f) \\ &= m\left(r, f\left(\frac{L(f)}{f}-1\right)\right) + (k+1)N(r, f) + S(r, f) \\ &\leq T(r, f) + kN(r, f) + S(r, f), \end{aligned}$$

outside an exceptional set of finite logarithmic measure. Thus, by equation (8), we get,

$$\sum_{i=1}^3 N\left(r, \frac{1}{f-a_i}\right) \leq T(r, f) + kN(r, f) + S(r, f).$$

Finally, by second main theorem, it follows that

$$\begin{aligned} (2-\epsilon)T(r, f) &\leq N(r, f) + \sum_{i=1}^3 N\left(r, \frac{1}{f-a_i}\right) + S(r, f) \\ &\leq \frac{1-2\epsilon}{k+1} T(r, f) + T(r, f) + kN(r, f) + S(r, f) \\ &\leq (1-2\epsilon)T(r, f) + T(r, f) + S(r, f) \end{aligned}$$

$$\text{or,} \quad \epsilon T(r, f) \leq S(r, f),$$

which is a contradiction. Hence $f \equiv L(f)$. □

Proof of theorem 2.3

Proof. If $f \not\equiv L(f)$, then by the given assumption,

$$\begin{aligned} \sum_{i=1}^{k+4} \overline{N} \left(r, \frac{1}{f - a_i} \right) &\leq N \left(r, \frac{1}{L(f) - f} \right) \\ &\leq T(r, L(f) - f) + O(1) \\ &\leq (k+1)T(r, f) + O(1). \end{aligned}$$

Finally, by the second main theorem for small functions, we have, for $0 < \epsilon < 1$,

$$\begin{aligned} (k+2-\epsilon)T(r, f) &\leq \sum_{i=1}^{k+4} \overline{N} \left(r, \frac{1}{f - a_i} \right) + S(r, f) \\ &\leq (k+1)T(r, f) + S(r, f), \end{aligned}$$

which contradicts our assumption that f is a transcendental meromorphic function. Hence the result. $\square$

Proof of theorem 2.4

Proof. If $f \not\equiv L(f)$, then we have

$$\sum_{i=1}^4 \overline{N} \left(r, \frac{1}{f - a_i} \right) \leq N \left(r, \frac{1}{L(f) - f} \right). \quad (9)$$

Also, since f is an entire function of hyper order less than 1, we have

$$N \left(r, \frac{1}{L(f) - f} \right) \leq T(r, f) + S(r, f). \quad (10)$$

Finally, by lemma 3.6, and equations (9) and (10), we have, for $0 < \epsilon < 1$,

$$\begin{aligned} (2-\epsilon)T(r, f) &\leq \sum_{i=1}^4 \overline{N} \left(r, \frac{1}{f - a_i} \right) + S(r, f) \\ &\leq T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction. Hence the result. $\square$

Proof of theorem 2.5

Proof. For $\epsilon > 0$ and large r , we have

$$N(r, f) \leq \frac{1-\epsilon}{k^2} T(r, f). \quad (11)$$

Also, see that by the sharing assumptions and lemma 3.3, we have,

$$\begin{aligned} \sum_{i=1}^4 \overline{N} \left(r, \frac{1}{f - a_i} \right) &\leq N \left(r, \frac{1}{L(f) - f} \right) \\ &\leq T(r, f) + kN(r, f) + S(r, f). \end{aligned}$$

Using this theorem, we have, by lemma 3.6 and $0 \leq \epsilon < 1$,

$$\begin{aligned} (1 - \epsilon)T(r, f) &\leq kN(r, f) + S(r, f) \\ &\leq \frac{k - k\epsilon}{k^2} T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction. Hence the result. $\square$

Proof of theorem 2.6

Proof. We have, for large r and $\epsilon > 0$,

$$\frac{N(r, f)}{T(r, f)} \leq \frac{1 - \epsilon}{k^2}.$$

By the assumption, if $f \not\equiv L(f)$, then we have by previous computations,

$$\sum_{i=1}^3 \overline{N} \left(r, \frac{1}{f - a_i} \right) + \leq T(r, f) + kN(r, f) + S(r, f).$$

Thus, by the second main theorem, for the previously assumed $\epsilon > 0$, we have,

$$\begin{aligned} (2 - \epsilon)T(r, f) &\leq \sum_{i=1}^4 \overline{N} \left(r, \frac{1}{f - a_i} \right) + S(r, f) \\ &\leq kN(r, f) + T(r, f) + N \left(r, \frac{1}{f - a_4} \right) + S(r, f) \\ \text{or, } (1 - \epsilon)T(r, f) &\leq kN(r, f) + N \left(r, \frac{1}{f - a_4} \right) + S(r, f) \\ \text{or, } (1 - \epsilon) &\leq \frac{1 - \epsilon}{k} + 1 - \delta(a_4, f) \\ \text{or, } \delta(a_4, f) &\leq \frac{1}{k} + \frac{k - 1}{k} \epsilon. \end{aligned}$$

If we choose ϵ such that

$$\frac{1}{k} + \frac{k-1}{k}\epsilon < \frac{2}{k} \Rightarrow \epsilon < \frac{1}{k-1},$$

then we arrive at a contradiction to our assumption that $\delta(a_4, f) > \frac{2}{k}$. Thus, the result. $\square$

Proof of theorem 2.7

Proof. Let $f \neq L(f)$. Then by previous computations, we have,

$$\sum_{i=1}^2 \overline{N}\left(r, \frac{1}{f-a_i}\right) \leq T(r, f) + kN(r, f) + S(r, f).$$

Thus, by second main theorem, we have, for $\epsilon > 0$,

$$\begin{aligned} (2 - \epsilon)T(r, f) &\leq \sum_{i=1}^4 \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r, f) \\ &\leq kN(r, f) + T(r, f) + \overline{N}\left(r, \frac{1}{f-a_3}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f-a_4}\right) + S(r, f) \\ \text{or, } (1 - \epsilon) &\leq k \frac{N(r, f)}{T(r, f)} + \frac{\overline{N}\left(r, \frac{1}{f-a_3}\right) + \overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)} + S(r, f). \end{aligned}$$

Using limit supremum on both sides of the above equation, as $r \rightarrow \infty$ we get,

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{f-a_3}\right) + \overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)} &\geq 1 - \frac{1}{k} - \epsilon \\ &= \frac{k-1}{k} - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we arrive at a contradiction. Hence the result. $\square$

5 Results and Discussions

Our primary objective in this paper has been to deduce the uniqueness theorems related to the exact differences of meromorphic functions. In this process,

we have proven various results on the uniqueness of meromorphic functions with partially shared values with the difference polynomial given in (1), from which, we have shown, how many results done earlier by other authors can be improved substantially. We started with theorem 2.1 which has specific sharing conditions for a transcendental meromorphic function. Afterwards, from theorem 2.2 onwards, it can be seen that given the partial sharing condition, the “smallness” of the function $N(r, f)$ is somewhat necessary to ensure the equality of f with $L(f)$. It is needless saying that the results can be easily proven for entire functions without this additional condition on $N(r, f)$ since in that case, $N(r, f) = S(r, f)$. On changing the sharing conditions further, i.e., by replacing $E(a, f)$ by $\overline{E}(a, f)$, or by reducing the number of functions partially shared, we have been able to stick to the same equality. We have excluded the case when f shares one small function with $L(f)$, which we presume can be done using the similar tools that we have used in proving the previous theorems. It is to be noted that as we have kept reducing the number of functions shared, the counting function $N\left(r, \frac{1}{f-a}\right)$, or the reduced counting function $\overline{N}\left(r, \frac{1}{f-a}\right)$ becomes necessary to be “small” with respect to the characteristic function $T(r, f)$. It would be interesting to delve further into the topic and explore the possibilities of replacing the set $E(a, f)$ by $E_p(a, f)$ (resp. $E_{(p)}(a, f)$), which is the set of a -points of f having multiplicity p atmost (resp. at least). At this juncture, we are inclined to make a mention of the set $\tilde{E}_c(a, f)$, which is the number of a -points of f ignoring the c -separated a -pairs. It would also be worthy to study the case when $E(a, f)$ is replaced by $\tilde{E}_c(a, f)$. Further, in case of theorems 2.2, and thereafter, it can also be checked whether the limit supremum condition can be replaced by limit infimum. Finally, we would like to point that all discussions in this paper have been done for functions of “much higher” growth, which is measured by using the hyper order. But, the case of the functions of “much lower” growth may also be studied under the same light.

6 Open Problems

In this article, we have mainly concentrated on meromorphic functions of hyper order $\rho_2(f) < 1$. The case of $\rho_2(f) \geq 1$ is open. Also, the case of functions of “much lower” growth may also be studied under the same light. A possible measure of growth for such functions was developed by Chern in the year 2005 when he first came up with the idea of functions with logarithmic order of growth Chern [2006]. The topic itself is still largely unexplored. However, from the results deduced by Chern in Chern [2006], we conjecture that even stronger uniqueness

results can be obtained for such functions.

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Determining dependence among random variables across observations

Peter J. Veazie*

Abstract

It is common in applied research to analyze data from data generating processes with dependencies among random variables across observations. Such dependencies impact power calculations and standard errors. However, it is also common to mistake the structure of data for the structure of the data generating process and thereby to use inappropriate standard error estimators. The challenge is not merely to distinguish data from data generating processes but also to determine dependence. This paper discusses the problem and provides a four-step guide, with examples, for determining dependence of random variables across observations.

Keywords: clustered standard errors; data generating process; dependent random variables; hierarchical models; nested data.

2010 AMS subject classification: 62-02; 62P10; 62P20; 62P25.[†]

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1. Introduction

In applied research, particularly that which focuses on populations such as the social and health sciences, it is not uncommon to find studies that confuse the structure of data for the structure of the data generating process (DGP). In doing so, such work mistakes potential commonalities among the objects of measurement (e.g. students in the same school and patients in the same hospital) for dependence in the joint distributions of random variables across the observational process of the DGP. Such confusion can lead to the application of inappropriate standard error estimators. For example, see research on mortality rates among older hospitalized patients [1], readmission penalties among hospitals [2], and activities of daily living among nursing home residents [3], each of which used clustered standard errors that do not reflect the DGP. This problem is not merely due to the misreading of statistical texts, the texts themselves, intent on teaching the proper use of clustered standard errors, are sometimes misleading in how they frame the issue. For example, a recent manuscript in the *Journal of Human Resources* that instructs on cluster-robust inference frames the problem in terms of “where observations can be grouped into clusters” citing observational data that contain geographic location indicators such as village or state [4], which can lead to the mistaken interpretation that dependence is data specific.

To select proper standard error estimators, it is useful to determine the dependence of random variables among observations in a given study design based on the DGP. This paper provides guidance and examples for instructing students and applied researchers on how to understand a DGP to determine dependence, focusing examples on first-order linear dependence. The following is based on a simple conceptual understanding of probability spaces, a rigorous background is not required; indeed, readers without this background should be able to understand and apply the approach outlined here. A concise conceptual background on probability spaces can be found in books such as *What Makes Variables Random* [5], and a more comprehensive coverage can be found in books such as *Probability and Measure* [6] and *A Probability Path* [7]. The topic is also covered in discipline-specific works such as the econometric texts *Topics in Advanced Economics* [8] and *Stochastic Limit Theory* [9]. However, such texts do not provide a step-by-step guide to determining dependence, as is provided here.

2. Background

The question of dependence among random variables across observations is not one regarding the data but is instead one regarding the DGP. For example, suppose we have data from a sample of people in the United States that includes a variable indicating each person’s state of residence. If the outcome of interest is strongly influenced by the state in which a person lives, and this influence varies across states, then we may reason that data from persons within the same state are likely to be more similar than data from persons of different states, and consequently the random variables across observations are dependent. This is incorrect thinking. Instead, we should ask whether the distribution of possible results from one instance of generating an observation depends on the result of another instance of generating an observation and whether the

distributions of corresponding random variables vary across the observations. For example, if we randomly sample with replacement from all persons in the United States, then each observation is independent from all others, and all of the corresponding random variables across observations are independent, regardless of whether the data contains records on multiple individuals from the same states. Why is this the case? These observations are independent because the probability of obtaining an individual in one instance of generating an observation does not change due to whom we obtain on any other instance. Alternatively, suppose we generated the data by randomly sampling states, and then within each of those states we randomly sampled two individuals with replacement. In this case, we have the classic nested DGP. A process that, as is well documented, generates dependent observations within a state cluster, since if one such observation comes from a given state, then all other observations in the cluster must come from that state as well; consequently, their corresponding random variables may be dependent as well. I used the phrase “state cluster” rather than merely using “state” to differentiate the inherent clusters (or nesting) in the DGP from the states we obtain. If we sampled states with replacement and thereby have the same distribution of sampling probabilities across states for each observation, then two different instances of sampling states may yield the same state. Such a result would produce two different state-sampled clusters within which observations are dependent and across which observations are independent, even though both clusters have the same state.

3. Determining Dependence

We can determine whether random variables are dependent in four steps based on identifying the structure of our probability space (Ω, A, P) comprised of an outcome set Ω , a sufficiently granular sigma-algebra A , and probability measure P defined to represent a DGP. The outcome set Ω comprises the set of possible elements that a DGP might obtain on which random variables are defined, in many cases this is the set of objects or individuals that a DGP could produce on which measurements can be taken. For example, Ω might be identified by a sampling frame representing the set of students in a school and corresponding random variable are the measurements of student age and math aptitude; or, perhaps Ω is the set of cars manufactured at a factory in a given year and the corresponding random variables are gas mileage and color. The sigma-algebra A is a formally structured set of subsets of Ω to which the measure P assigns probabilities associated with the DGP. For example, perhaps the DGP entails an equal probability sample from the set of objects $\Omega = \{w_1, w_2, w_3\}$ and A is the set of all subsets of Ω , i.e. $A = \{\emptyset, \{w_1\}, \{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2, w_3\}\}$, and P assigns approximately 0.3333 to each singleton in A (e.g. $\{w_2\}$), approximately 0.6667 to each element with two objects (e.g. $\{w_1, w_2\}$), 1 to the element representing Ω (i.e. $\{w_1, w_2, w_3\}$), and 0 to the empty set, $\emptyset$. The details of the sigma-algebra are not relevant for this presentation, we will consider it sufficiently granular to allow us to assign P to any subset of outcomes required.

The four steps in determining dependence are (1) describe the DGP, (2) determine the structure of the potential outcomes (Ω), (3) determine the structure of the probability measure (P), and (4) determine the dependence of random variables.

3.1. Describe the data generating process

The first step is to describe the DGP. This can typically be done in a narrative or graphical form. Its mathematical translation is the role of the next two steps. It is, however, important to have a good understanding of the process that will be modeled in the following steps. Without it, it will be difficult to provide an accurately defined probability space that underlies analysis.

It is ambiguous to merely describe the DGP by stating “I will identify hospital patients and collect diagnosis and discharge information on them”. I need to know how I am going to achieve this: I need to know the process. I should say, for example, that I will use equal probability sampling of hospitals, with replacement, from the set of all hospitals in the United States. From each sampled hospital, I will use equal probability sampling of patients, with replacement, from the set of patients within each hospital during a specified time frame.

Consider the difference between a repeated measures design investigating an intervention targeting the reduction of systolic blood pressure in which (1) the times associated with measurements of systolic blood pressure are considered fixed, perhaps at baseline and every subsequent 3 months for a year, and (2) a design in which the times are randomly determined. In the first case we would need to include in our description of the design the fact that systolic blood pressure will be measured at baseline, and each of 3, 6, 9, and 12 months following an intervention under study. Whereas, in the second case we would need to state that 4 measurement times are randomly determined for each observation. This distinction is important to successfully engaging the remaining steps in determining whether observations are dependent.

3.2. Determining the structure of potential outcomes.

The second step in determining whether random variables across observations are dependent is to identify the structure of the outcome set associated with the probability space we are using to represent the DGP. For example, if we simply randomly select an individual from a population and measure qualities of interest, then the outcome set (Ω) is the population of individuals from which we will sample, i.e. $\Omega = \{w: w \text{ is a member of the population being sampled}\}$. In this case, denoting the set of individuals in the population being sampled as W , we have the outcome set specified as $\Omega = \{w: w \in W\}$. The random variables associated with the measurable space (Ω, A) , in which A denotes an appropriate sigma-algebra, are functions of the individuals (perhaps measurements on individuals) that compose the population W . If, instead, our DGP is to select a state and then select a resident of that state, then possible outcomes are pairs comprising a state (s) and person (w), i.e. $\Omega = \{(s, w): s \text{ is a member of a set of states and } w \text{ is a member of the set of residents who reside in the states}\}$. In this case, denoting the set of states as S and the set of residents across states as W , the outcome set is $\Omega = \{(s, w): (s, w) \in S \times W\}$. Corresponding random variables are a function of the pair; for example,

Determining dependence among random variables across observations

measurements on the state or individual in the pair. Notice, we can define random variables identifying the state in which the person lives on both of these outcome sets, and corresponding variables would exist in the resultant data sets. However, notwithstanding the existence of such variables in the data, the first case, with element (w) , cannot represent nested sampling, whereas the second case, with element (s, w) , can represent nested sampling. Further, suppose our DGP is to select a state (s) , then select a person (w) from that state, and then randomly assign a treatment (t) to the selected person. In this case, the outcome set comprises all potential triples (s, w, t) . For the set of possible treatments, denoted as T , the outcome set is $\Omega = \{(s, w, t): (s, w, t) \in S \times W \times T\}$. This is different from a DGP that selects an individual w from outcome set W and records the state within which that individual resides and her current treatment. Each DGP described above could result in data with variables that indicate the person, state of residents, and treatment; yet dependence of the random variables can be different.

How do we determine the structure of the outcome set? For the specific cases in this paper, in which we are structuring elements of a probability space used to represent a DGP, we can start by considering an arbitrary observation. Specifically, we consider how the process by which we generate the data incorporate degrees of freedom into the selection of potential outcomes. For example, if an observation entails sampling a state from the set of three states $S = \{s_1, s_2, s_3\}$ in which no element has a probability equal to 1, then the observation has a degree of freedom in determining the state that will be obtained. However, if it is determined a priori that the observation will be from state s_1 , then if we wish to frame it in terms of a probability space, we are sampling from the outcome set $S = \{s_1\}$, which being a singleton, has probability 1 of sampling s_1 , or more generally sampling from the set $S = \{s_1, \dots, s_j, \dots, s_J\}$ in which some element j has a probability of 1: $P(s_j) = 1$. In this case, the observation has no degree of freedom regarding state. We can exclude the set of states in the definition of the outcomes set; however, including the set can help track the fact that the state is pre-determined, which I will do in the examples below; but, this requires remembering that the probability of selecting this component of the outcome is 1.

The preceding paragraph showed how a potential component of the outcome set can be eliminated from the structure of the outcome set, specifically by identifying that there are no degrees of freedom allowed by this component in determining the outcome. Another consideration is to determine whether to expand the structure to include a component. Consider a DGP that entails sampling an individual from the set W and then randomly assigning a treatment from the set T . Can we use the probability space (W, A, P) to represent this process? In this case, because the outcome set represents only the population W and does not represent a set of possible treatments, treatment can only enter as a function of W , i.e. as a random variable. Consequently, true to the nature of a function, each individual has only one treatment assigned to it. Absent the DGP, does such a function exist? Clearly not, the value assigned by the function depends on the concrete outcome from the DGP, after the uncertainty has been resolved, which does not model what we seek to model: the uncertainty of treatment assignment. Consequently, T must be part of the outcome set, thereby allowing treatment to have a

probability assigned to it, and our probability space is $(W \times T, A, P)$. Of course, if T is a singleton or one of the elements of T has probability equal to 1 (i.e. we are determined to give whomever we select a specific treatment), then we can eliminate it from the structure as was done above regarding states S and treat it as a random variable associated with W . Another example is that of repeated measures to be recorded at predetermined times (e.g. a fixed pretreatment baseline, 6 months post treatment, and 12-month post treatment observations in a clinical research setting). Since the three measurement times are set prior to the DGP and determined by it, they can be excluded from the outcome set and treated as a random variable, or if included, they must be associated with probability equal to 1.

When considering dependence between two observations, we need to structure the elements of the outcome set for the pair of observations. If the first observation has outcome set Ω_1 and the second observation has outcome set Ω_2 , then the corresponding product space has the outcome set of $\Omega_1 \times \Omega_2$. If both have the same outcome set, Ω , then the outcome set for the pair of observations is $\Omega \times \Omega$, denoted also as Ω^2 . What is important in this step is to consider the arbitrary elements of each component in terms of whether they share elements. For example, suppose we are sampling states from the United States (denoted as the set S) and sampling residents from within states (denoted as the set W) the outcome set for each observation is $\Omega = \{(s, w): s \in S \text{ and } w \in W\}$. Consequently, we might indicate the outcome for the first observation as (s, w) and the second observation's outcome as (s', w') . The arbitrary element of the product set is then $((s, w), (s', w'))$. The question is whether the observations share elements. For example, if this is a nested DGP in which we sample a state s and then sample two residents from within the selected state, then two observations within a state cluster must necessarily have the same state (i.e., $s = s'$) and the arbitrary element of the product is $((s, w), (s, w'))$, whereas if we are considering two observations from different states, then the arbitrary element remains $((s, w), (s', w'))$ because the DGP does not restrict either the state or the resident to be the same across observations. It is important to identify shared elements in the outcome set across observations because this information will be used to calculate the quantities of interest in the following steps.

3.3. Determining the structure of the probability measure on the outcomes set.

In this step, we determine how the DGP imposes structure on the probability of obtaining elements from the outcome set. For example, consider an outcome set comprised of individuals (w), the probability is simply $P(w)$ for obtaining each individual w (note that the probability may be different for each w). For the state and resident outcome set it is $P(s, w) = P(w|s) \cdot P(s)$ for each pair. If, in this case, we were to randomly select a state, and then randomly select a person from the full population rather than from within the selected state (which seems an absurd thing to do, but suppose we did so anyway), then selection of state and person would be independent of each other, and we would structure the probability as the product of marginal probabilities: $P(s, w) = P(w) \cdot P(s)$. For the state, resident, and treatment outcome set, the probability of each possible outcome can be $P(s, w, t) = P(w|s, t) \cdot P(t|s) \cdot P(s)$. If, in this

Determining dependence among random variables across observations

case, the individual is independent of treatment within state, then $P(w|s, t) = P(w|s)$, and we further structure the probability as $P(s, w, t) = P(w|s) \cdot P(t|s) \cdot P(s)$. If the probability of treatment was the same across state, then treatment is independent of state, $P(t|s) = P(t)$, and we have $P(s, w, t) = P(w|s) \cdot P(t) \cdot P(s)$. On the other hand, if each person had their own probability of treatment assignment that is not dependent on the state, then it would be better to specify $P(s, w, t) = P(t|w) \cdot P(w|s) \cdot P(s)$. In each of these cases, and any other we wish to consider, the structure of the probability measure is usefully dictated by how the DGP produces the elements of the outcome set.

To understand dependence between any two observations, we need to structure the probability measure of the product space associated with the observations. We are concerned with structuring the probability of each element of the joint outcome set. For example, if we are considering whether observations within each state cluster in a DGP where state residents are nested within state, then based on the preceding step above, in which we determined the observations share state and the arbitrary pair is $((s, w), (s, w'))$, we are concerned with $P(s, w, s, w')$, which is $P(w, w'|s, s) \cdot P(s, s)$. However, given the redundancy in s , this is simply $P(w, w'|s) \cdot P(s)$. Now, if the DGP is one in which the residents are sampled with replacement independently within state, then $P(w, w'|s)$ is the product $P(w|s) \cdot P(w'|s)$, and overall probability is $P(s, w, s, w') = P(w|s) \cdot P(w'|s) \cdot P(s)$. We have therefore structured the joint probability of a pair of observations in terms of the DGP and their unique elements (in this example, in terms of s, w , and w').

3.4. Determine dependence of random variables between observations.

In this last step, we write random variables, as defined on the underlying probability space (Ω, A, P) , in terms of the elements of the outcome set Ω and not in terms of the derived probability measure of the random variable itself. In other words, we write a random variable Y as $Y(w)$ in which w is an arbitrary element of Ω . For example, if $\Omega = S \times W$, we write an associated random variable Y as $Y(s, w)$ for which $s \in S$ and $w \in W$, and we consider probabilities associated with (s, w) in a sufficient sigma-algebra (e.g. the power set of Ω).

For ease of presentation, I will consider only linear dependence (i.e. covariance); however, the same method can be used for higher order dependence, by substituting the appropriate moment, or calculation, for the covariance in the examples below. Consequently, because we are interested in the covariance, we require the expected value of the random variable in terms of the underlying probability space. For example, continuing the preceding, we insert the probability structure determined in step two above, i.e. $P(s, w) = P(w|s) \cdot P(s)$, into

$$E(Y_i) = \sum_s \sum_w Y(s, w) \cdot P(s, w)$$

and evaluate, say,

$$E(Y_i) = \sum_s \sum_w Y(s, w) \cdot P(w|s) \cdot P(s).$$

With the expected value in hand, we can evaluate the moment equation, or equations, that represent dependents.

Continuing our example, the covariance between random variables of two observations, Y_i and Y_j , would be based on the corresponding two observations (s, w) and (s', w') :

$$\text{Cov}(Y_i, Y_j) = \sum_s \sum_w \sum_{s'} \sum_{w'} (Y_i(s, w) - E(Y_i)) \cdot (Y_j(s', w') - E(Y_j)) \cdot P(s, w, s', w').$$

The required number of summations, the order of the summations, and the structure of the probability P is determined by the preceding steps applied to specific problem being addressed. Carefully reducing the covariance will provide the representation of dependence we set out to determine. Although here I have presented the steps in terms of simple random variables, Y , we can extend this analysis to functions of random variables. Details of these steps will become evident in the following examples.

4. Examples

4.1. Example 1. Standard nested sampling design

Suppose we have data on residents from a set of states in the United States of America. The data are from multiple observations from within any given state in the sample. Are random variables across observations within the same state dependent? As mentioned above, the answer depends on the nature of the DGP. To determine dependence, consider the four steps outlined above.

Describe the DGP. We randomly sample states with equal probability and replacement as the first stage of sampling, and we keep track of which state is obtained from each sample. Since we are sampling with replacement, it is possible that we obtain the same state more than once, in which case we need to denote these as separate primary sampling units. Next, we randomly sample residents with replacement from within each of the states obtained from the first stage of sampling and measure our quantities of interest.

Determine the structure of the outcome set. This DGP obtains states and residents, consequently the possible components of the outcome set are the set of states (S) from which we obtained the primary sampling units and the set of residents across the states (W). Should we include the set of states S as a component of the outcome set? To answer this, we consider whether there is any degree of freedom in determining the state that could be obtained for a given observation? Indeed, there is. For any given observation, the observation is not fixed a priori to any given state but could well have obtained any one of the states. Consequently, we must include S as a component of the outcome set.

Should we include the set of residents W as a component of the outcome set? Similar to the selection of states, the resident for any observations could well have been any one of the residents in the state that was obtained in the first phase of sampling. Consequently, as there is more than one resident in each state, the sampling of resident within state provides a degree of freedom in determining the outcome and the set of

Determining dependence among random variables across observations

residents must also be included in the outcome set. The outcome set is therefore $\Omega = S \times W$ with arbitrary element $\{(s, w)\}$.

For two observations from within the same primary sampling unit (i.e. the same state that was obtained for the observations) the outcome set is then Ω^2 with arbitrary element $\{((s, w), (s', w'))\}$ in which (s, w) is the state and resident associated with one observation and (s', w') is the state and resident associated with the other observation. However, since in this DGP we are investigating the dependence of random variables across observations with the same primary sampling unit (i.e. same state), then s is the same as s' . The pair of observations is therefore $\{((s, w), (s, w'))\}$ denoted with each observation having the same state s . This is important to identify because in the following steps we will need to account only for the unique individual components, which in this case is s , w , and w' .

Determine the structure of the probability measure. For an individual observation, we can structure the probability of an event $\{(s, w)\}$ as the probability of obtaining a state s multiplied by the conditional probability of obtaining a resident given a that state, i.e. $P(s, w) = P(s) \cdot P(w | s)$, or as the probability of obtaining an individual w multiplied by the probability of obtaining a state s given obtaining that individual, i.e. $P(s, w) = P(w) \cdot P(s | w)$. The former, however, more naturally aligns with how we conceptualize the DGP by which we obtain a state and then obtain a resident from within that state.

For a pair of observations from the same state as the primary sampling unit, the probability of the event $\{((s, w), (s, w'))\}$ can be written as $P(s, w, s, w') = P(s | s, w, w') \cdot P(s, w, w')$. However, because the probability of an event conditional on that event is 1, i.e. $P(s | s, w, w') = 1$, we can write the probability of event $\{((s, w), (s, w'))\}$ as $P(s, w, w')$ rather than the somewhat redundant $P(s, w, s, w')$. Moreover, the probability $P(s, w, w')$ can be expressed as the probability of obtaining state s multiplied by the probability of obtaining the pair of residents w and w' : $P(s, w, w') = P(s) \cdot P(w, w' | s)$. By the DGP, residents are selected at random with replacement, consequently the probability of obtaining residents is independent across observations, and therefore $P(w, w' | s) = P(w | s) \cdot P(w' | s)$. Overall, then, we structure the probability of the joint outcome event $\{((s, w), (s, w'))\}$ as $P(s, w, w') = P(s) \cdot P(w | s) \cdot P(w' | s)$.

Determine dependence. To determine dependence of random variables across observations, we need to express the random variables as a function of the outcome set. So, as mentioned above, let us consider an outcome set comprised of State (S) and Person (W), with arbitrary individual elements of $(s, w) \in S \times W$. In this case, we denote the outcome as $Y_i(s, w)$ and the expected value of the outcome as $\bar{Y}_i$. With the expected value of Y in hand, which is the same for each observation in a given state, we can determine the covariance between the Y 's of the observations within state. To calculate the covariance, we need to sum over the elements of w , w' , and s :

$$\text{Cov}(Y_1, Y_2) = \sum_s \sum_{w'} \sum_w (Y_1(s, w) - \bar{Y}_1) \cdot (Y_2(s, w') - \bar{Y}_2) \cdot P(w | s) \cdot P(w' | s) \cdot P(s).$$

Note, summing over w provides the expected value of Y_1 conditional on s :

P. Veazie

$$E(Y_1 | s) = \sum_w (Y_1(s, w) \cdot P(w | s)).$$

And, summing over w' provides the expected value of Y_2 conditional on s :

$$E(Y_2 | s) = \sum_{w'} (Y_2(s, w') \cdot P(w' | s)).$$

Consequently, we get

$$\text{Cov}(Y_1, Y_2) = \sum_s (E(Y_1 | s) - \bar{Y}_1) \cdot (E(Y_2 | s) - \bar{Y}_2) \cdot P(s).$$

Since the observations are independent within state, $(E(Y_1 | s) - \bar{Y}_1) = (E(Y_2 | s) - \bar{Y}_2)$ and the covariance between observations is the variance of the conditional expectation of Y across the states for a given observation. Remember, the correlation between Y_1 and Y_2 is the covariance divided by the product of the standard deviations for each of Y_1 and Y_2 . Because the standard deviations for each of Y_1 and Y_2 are identical, this product is simply the variance of either one. Consequently, the correlation is the between-state variance divided by the total variance, which is the intraclass correlation defined for a simple model with a state-level random intercept [10]. This is the expected result for the classic nested DGP.

4.2. Example 2. Fixed primary sampling units with unit-level random treatment assignment

For a more complicated example, suppose we have data on residents from each of the 50 states in the United States of America. Moreover, suppose each state has its own probability of adopting a policy that will affect everyone in the state. Are random variables associated with a pair of observation from within the same state dependent?

Describe the DGP. We randomly obtain a sample of residents, with replacement, from each of the 50 states in the United States of America. Each state has its own probability of adopting a policy, which we can generally consider as having a probability of being subject to a treatment or intervention. For any given state, all residents will either be subject to the policy or not subject to the policy; this is a state-level treatment.

Determine the structure of the outcome set. For this DGP, the outcome set for each observation has three possible components: states, individuals, and treatments (i.e., policies in this case). Should we include a set of states that underlie each observation as a component of the outcome set? Notice that since each observation comes from a fixed pre-determined state, the set of possible states that underlies an arbitrary observation i from state s is $S_i = \{s\}$ for which $P(s) = 1$. Consequently, we have the option of dropping S from the outcome set specification. Because, within each state, the DGP will randomly select a resident, each observation has the potential of obtaining one of many possible residents (assuming each state has more than one resident). Consequently, we must include the population of residents for each observation i , W_i , as

Determining dependence among random variables across observations

part of $W = \bigcup_i W_i$, as a component of the outcome set. Similarly, because each state has a probability of either adopting the policy or not adopting the policy (i.e. being subject to the treatment), then the DGP has a degree of freedom associated with treatment, $T = \{\text{adopts policy, does not adopt policy}\}$; therefore, T must be a component of the outcome set. Consequently, we could use the outcome set specified as $\Omega = W \times T$. However, to keep track of the fact that observations come from fixed states, we can use $\Omega = S \times W \times T$ as our outcome set and remember that $P(s) = 1$ for the state s underlying each observation.

Regarding joint observations, consider two from the same state: the first observation will select an element (s, w, t) , the second will select an element (s', w', t') . The DGP is such that they not only have the same state, but they will thereby have the same treatment (i.e. since t is a state-level assignment, it will be the same for all persons in the state). Therefore, across the pair of observations from the same selected state, $s = s'$ and $t = t'$, which I will denote simply as s and t . Our two elements will then be (s, w, t) and (s, w', t) in which both elements share state and treatment. Consequently, when we determine dependence, we will sum over s, w, w' , and t .

Determine the structure of the probability measure. Because in this example we are not sampling states but are using a fixed set of states, the probability of obtaining an individual depends on the state s from which we are sampling, but it does not depend on the treatment t . Consequently, $P(w | s, t) = P(w | s)$. Similarly, treatment depends on the state (each state can have their own probability of adopting the treatment/policy), but treatment does not depend on the individual, which is to say all residents of the state will either be subject to the treatment or not. So, we structure the probability of obtaining an element (s, w, t) as the probability of obtaining person w given state s , multiplied by the probability of being assigned treatment t given state s , multiplied by the probability of obtaining state s :

$$P(s, w, t) = P(w | s) \cdot P(t | s) \cdot P(s).$$

Also, for our DGP, conditional on state, the selection of individuals within state are independent, and consequently regarding the joint distribution for observations (s, w, t) and (s, w', t) the probability of the pair of individuals conditional on state is

$$P(w, w' | s) = P(w | s) \cdot P(w' | s).$$

Therefore, the overall probability of both observations can be structured as

$$P(w, w', s, t) = P(w | s) \cdot P(w' | s) \cdot P(t | s) \cdot P(s).$$

Determine dependence. In this example, let us represent an outcome Y as a linear function of policy or treatment T , such that for arbitrary observation i they are related as

$$Y_i = \alpha + \beta \cdot T_i + \varepsilon_i.$$

To determine dependence of random variables across observations, we need to express these random variables as a function of the outcome set. So, as mentioned

above, let us consider an outcome set Ω comprised of State (S), Person (W), and Treatment (T), with arbitrary individual elements of $(s, w, t) \in S \times W \times T$. In this case, we have

$$Y_i(s, w, t) = \alpha + \beta \cdot T_i(s, w, t) + \varepsilon_i(s, w, t).$$

Because the covariance requires knowing the expected value of the random variable itself, we need to first calculate the expected value by summing across the components of the outcome set:

$$E(Y_i) = \sum_s \sum_t \sum_w (\alpha + \beta \cdot T_i(s, w, t) + \varepsilon_i(s, w, t)) \cdot P(w | s, t) \cdot P(t | s) \cdot P(s).$$

In terms of the reduced probability structure, the expected value of Y is then

$$E(Y_i) = \sum_s \sum_t \sum_w (\alpha + \beta \cdot T_i(s, w, t) + \varepsilon_i(s, w, t)) \cdot P(w | s) \cdot P(t | s) \cdot P(s),$$

which, after distributing the summation over individuals, w, can be re-expressed as

$$E(Y_i) = \sum_s \sum_t \left(\alpha + \beta \cdot \sum_w T_i(s, w, t) \cdot P(w | s) + \sum_w \varepsilon_i(s, w, t) \cdot P(w | s) \right) \cdot P(t | s) \cdot P(s).$$

Because for this DGP $P(w | s) = P(w | s, t)$, each of the summations with respect to w provide expected values conditional on s and t, i.e.

$$E(T_i | s, t) = \sum_w T_i(s, w, t) \cdot P(w | s)$$

and

$$E(\varepsilon_i | s, t) = \sum_w \varepsilon_i(s, w, t) \cdot P(w | s).$$

Therefore, the expected value of Y_i is

$$E(Y_i) = \sum_s \sum_t (\alpha + \beta \cdot E(T_i | s, t) + E(\varepsilon_i | s, t)) \cdot P(t | s) \cdot P(s).$$

Distributing the summations over t yields

$$E(Y_i) = \sum_s \left(\alpha + \beta \cdot \sum_t E(T_i | s, t) \cdot P(t | s) + \sum_t E(\varepsilon_i | s, t) \cdot P(t | s) \right) \cdot P(s).$$

Notice that for a given state s, the expected value of T given $t = 1$ is just 1, and the expected value of T given $t = 0$ is just 0, consequently

$$\sum_t E(T_i | s, t) \cdot P(t | s) = P(t = 1 | s).$$

Therefore, the expected value of Y is

Determining dependence among random variables across observations

$$E(Y_i) = \sum_s (\alpha + \beta \cdot P(t=1|s) + E(\varepsilon_i | s)) \cdot P(s).$$

For simplicity of presentation, let us adopt the common assumption that the expectation of the error term is 0 within state, $E(\varepsilon_i | s) = 0$, for all s ; therefore, the expected value of Y is

$$E(Y_i) = \sum_s (\alpha + \beta \cdot P(t=1|s)) \cdot P(s).$$

For this DGP, remember that each observation comes from a given state with probability equal to 1: i.e. we determine to sample an individual from a particular state in each DGP instance, so the probability of s is 1 for the s from which the observation is sampled and 0 for all other states. Therefore, the summation across states reduces to the single summand associated with the state underlying the observation, and the expected value for all observations i from predetermined state s is

$$E(Y_i) = \alpha + \beta \cdot p_s,$$

in which p_s denotes $P(T=1 | s)$, representing the state-specific probability of treatment.

With the expected value of Y in hand, which is the same for each observation in a given state, we can now determine the covariance between the Y 's of the observations within state. To calculate the covariance, we need to sum over the elements of w , w' , t , and s :

$$\begin{aligned} Cov(Y_1, Y_2) = & \sum_s \sum_t \sum_{w'} \sum_w (\alpha + \beta \cdot T_1(s, w, t) + \varepsilon_1(s, w, t) - (\alpha + \beta \cdot p_s)) \\ & \cdot (\alpha + \beta \cdot T_2(s, w', t) + \varepsilon_2(s, w', t) - (\alpha + \beta \cdot p_s)) \cdot P(w' | s) \cdot P(w | s) \cdot P(t | s) \cdot P(s). \end{aligned}$$

Distributing the summations of the w 's and again assuming the errors have expectation 0 (we don't need to do this, but it simplifies the presentation here), we get, noting that the α 's cancel,

$$Cov(Y_1, Y_2) = \sum_s \sum_t (\beta \cdot E(T_1 | s, t) - \beta \cdot p_s) \cdot (\beta \cdot E(T_2 | s, t) - \beta \cdot p_s) \cdot P(t | s) \cdot P(s).$$

Expanding the summation over treatment $t \in \{0, 1\}$, and remembering that the expected value of T conditional on t is just the value of t , the covariance is

$$Cov(Y_1, Y_2) = \sum_s (\beta \cdot 1 - \beta \cdot p_s) \cdot (\beta \cdot 1 - \beta \cdot p_s) \cdot p_s + (0 - \beta \cdot p_s) \cdot (0 - \beta \cdot p_s) \cdot (1 - p_s) \cdot P(s)$$

Bringing β out from within the summation and expanding the terms under the summation yields

$$Cov(Y_1, Y_2) = \beta^2 \sum_s (1 - p_s) \cdot (1 - p_s) \cdot p_s + (0 - p_s) \cdot (0 - p_s) \cdot (1 - p_s) \cdot P(s),$$

and, again noting that, for this DGP, $P(s) = 1$ for the s underlying the observation and $P(s) = 0$ for all other states, the summation reduces to only the one summand associated with the s from which the observation was taken. Consequently, we have

$$\text{Cov}(Y_1, Y_2) = \beta^2 \cdot [(1 - p_s) \cdot (1 - p_s) \cdot p_s + (0 - p_s) \cdot (0 - p_s) \cdot (1 - p_s)],$$

which can be expressed simply as

$$\text{Cov}(Y_1, Y_2) = \beta^2 \cdot (1 - p_s) \cdot p_s.$$

The covariance of the Y 's between observations within state is equal to the squared treatment effect multiplied by the variance of the state-level indicator for treatment:

$$\text{Cov}(Y_1, Y_2) = \beta^2 \cdot \text{Var}(T | s).$$

Consequently, the random variables are dependent unless there is no treatment effect (i.e., $\beta = 0$), or if the treatment (i.e., in this case state policy) is determined (i.e., $\text{var}(T | s) = 0$).

4.3. Example 3. Fixed primary units with individual-level random treatment assignment

Suppose the DGP was one in which we had a fixed set of states, as in example 2, and independent samples of individuals from within each state, also as in example 2, but our state-level probability of treatment is applied to each individual. In other words, suppose that instead of everyone in the state either receives treatment or does not receive treatment, each person “flips the state-level coin” to determine their treatment status—i.e. each person has the same probability of treatment, but this will generate some people getting treatment and others not getting treatment according to that probability. Does this imply different results from those found for example 2?

Describe the DGP. We obtain a sample of individuals, with replacement, from each state in a fixed set of states. Treatment is assigned to each individual according to a state-specific probability: i.e., individuals in the same state have the same probability of treatment, but whether they obtain treatment is independent of whether other individuals in the state obtain treatment.

Determine the structure of the outcome set. As in example 2, there are three possible components of the outcome set: states, individuals, and treatments. Given that each observation comes from a fixed, predetermined, state, we can choose not to include the set of states (S) in the outcome set, or we can include it, remembering that the probability of the state for each observation is 1 for the state from which that observation comes. Assuming each state has more than one resident, and as individuals are being sampled such that no individual has a probability equal to 1 of being obtained, there is a degree of freedom in determining the individual component of the observation: the population of individuals (W) must be included in the outcome set. And, each observation may or may not be subject to treatment according to the treatment selection probability; consequently, treatment provides a degree of freedom in specifying the observation, and treatment (T) must be included in the outcome set. The outcome set is therefore the same as that of example 2: $\Omega = S \times W \times T$.

Determining dependence among random variables across observations

Unlike example 2 our observations from the same state will be (s, w, t) and (s, w', t') . The observations share a state but do not necessarily share treatments. Consequently, we allow treatment to be different across observations and t is not necessarily the same as t' . Therefore, when we calculate the covariance we sum over t' as well as w, w', s , and t used in the preceding example.

Determine the structure of the probability measure. For arbitrary observation (s, w, t) , the probability can be expressed as $P(s, w, t) = P(w | s, t) \cdot P(t | s) \cdot P(s)$. However, the selection probability of individual w from the state s does not depend on the treatment, consequently we can write $P(w | s, t)$ as $P(w | s)$. Also, because for each observation the state is fixed, we must remember that the probability of the state is 1 for the state underlying the specific observation.

The joint probability of observations (s, w, t) and (s, w', t') can be expressed as

$$P(w, w', s, t, t') = P(w, w' | t, t', s) \cdot P(t, t' | s) \cdot P(s).$$

However, again, because the selection of individuals is independent of treatment for a given state, $P(w, w' | t, t', s)$ can be written as $P(w, w' | s)$, and because individuals are selected with replacement, individual selection is independent given state and $P(w, w' | s)$ can be written as the product of probabilities $P(w | s) \cdot P(w' | s)$. Because the probability of treatment for one observation does not depend on the resulting treatment of another observation, $P(t, t' | s)$ can be rewritten as the product $P(t | s) \cdot P(t' | s)$. Consequently, the joint probability is

$$P(w, w', s, t, t') = P(w | s) \cdot P(w' | s) \cdot P(t | s) \cdot P(t' | s) \cdot P(s).$$

Determine dependence. The covariance between two observations within the same state is expressed as follows, noting that the expected value of Y is the same as in example 2:

$$\begin{aligned} \text{Cov}(Y_1, Y_2) = & \sum_s \sum_{t'} \sum_t \sum_{w'} \sum_w (\alpha + \beta \cdot T_1(s, w, t) + \varepsilon_1(s, w, t) - (\alpha + \beta \cdot p_s)) \\ & \cdot (\alpha + \beta \cdot T_2(s, w', t') + \varepsilon_2(s, w', t') - (\alpha + \beta \cdot p_s)) \cdot P(w' | s) \\ & \cdot P(w | s) \cdot P(t | s) \cdot P(t' | s) \cdot P(s). \end{aligned}$$

First, considering the summation over w , note that

$$\begin{aligned} \sum_w (\alpha + \beta \cdot T_1(s, w, t) + \varepsilon_1(s, w, t) - (\alpha + \beta \cdot p_s)) \cdot P(w | s) = & \beta \cdot E(T_1(s, w, t) | s, t) \\ & + E(\varepsilon_1(s, w, t) | s, t) - \beta \cdot p_s \end{aligned}$$

and for simplicity if we take the conditional expectation of the error to be 0, the summation is

$$\beta \cdot (E(T_1(s, w, t) | s, t) - p_s).$$

And similarly, the summation over w' is

$$\beta \cdot (E(T_1(s, w', t') | s, t') - p_s).$$

The covariance is the summation over state and treatments of these expected values multiplied by the corresponding probabilities. The covariance is therefore

$$\begin{aligned} Cov(Y_1, Y_2) = & \beta^2 \cdot \sum_s \sum_t \sum_{t'} [(E(T_1(s, w, t) | s, t) - p_s) \cdot (E(T_1(s, w', t') | s, t') - p_s) \\ & \cdot P(t | s) \cdot P(t' | s) \cdot P(s)] \end{aligned}$$

However, the summation over treatment t is equal to 0:

$$\begin{aligned} \sum_t (E(T_1(s, w, t) | s, t) - p_s) \cdot P(t | s) &= (1 - p_s) \cdot p_s + (0 - p_s) \cdot (1 - p_s) = (1 - p_s) \cdot (p_s - p_s) \\ &= 0 \end{aligned}$$

And, similarly, the summation over treatment t' is equal to 0:

$$\sum_{t'} (E(T_1(s, w, t') | s, t') - p_s) \cdot P(t' | s) = 0.$$

The covariance of Y_1 and Y_2 is therefore

$$Cov(Y_1, Y_2) = \beta^2 \cdot \sum_s (0) \cdot (0) \cdot P(s).$$

Which is clearly 0. However, to complete the derivation in light of the DGP, we should note that $P(s)$ is 1 for the state from which the observations come and $P(s)$ is 0 for states from which the observations do not come. Consequently,

$$Cov(Y_1, Y_2) = \beta^2 \cdot (0) \cdot (0) = 0.$$

From this, we conclude that the random variables of observations from within states in this DGP are independent.

In summary, for a fixed set of states (i.e., no sampling of states), if each state has a probability of treating everyone, as in example 2, then there is dependence of the random variables, if treatment has an effect and treatment is not determined. The standard errors should account for this state-level “clustering” effect. However, if each state has a probability of treatment that applies to each individual of the state (i.e. each individual obtains treatment with that probability), as in example 3, then the observations, and consequently the Y 's, are independent. In this case, the standard errors should not account for a state-level “clustering” effect. This statement is true regardless of values associated with statistics such as intraclass correlations, which are trumped by the independence in the structure of the DGP.

4.4. Example 4. Random primary units with individual-level random treatment assignment

In this example, we determine the covariance between Y_i and Y_j for two arbitrary observations (i, j) within an arbitrary nursing home; however, unlike the preceding

Determining dependence among random variables across observations

examples in which we used fixed primary units, in this case we will randomly sample the nursing homes.

Describe the DGP. We randomly sample nursing homes with replacement from all nursing homes in the United States and then randomly sample residents with replacement from within each nursing home that is obtained. Further, each nursing home has a probability of treatment for its residents and each resident is provided treatment according to the nursing home's probability of treatment, in other words, each resident "flips their nursing home treatment coin" to determine treatment status. Notice, this is similar to example 3, except that in this case we are sampling nursing homes rather than using predetermined nursing homes, as we did using predetermined states in example 3.

Structure of the outcome set. The potential components of this DGP may at first appear to be the set of nursing homes (H), the set of nursing home residents (R), and the set of treatments (T). If these were our potential components, which of these must be part of the outcome set? To help identify the components, we can first ask whether each component has any degrees of freedom available for an arbitrary observation: is the set of nursing homes with non-zero probabilities available for an arbitrary observation a singleton? If so, then H need not be part of the formal outcome set; although, as shown in the preceding examples, it can be, but one must be careful to assign the appropriate probability of 1. However, in this example the DGP is one of randomly selecting a nursing home from a large set of nursing homes; consequently, the set of nursing homes contains multiple possible nursing homes and is thereby not a singleton. H must be a component of the outcome set. Assuming nursing homes tend to have more than one resident, the random sampling of a resident from a selected nursing home means there is more than one resident with a nonzero probability of being selected; R must be a component of the outcome set. Similarly, each resident has a nonzero probability of getting the treatment and a nonzero probability of not getting the treatment: T must be a component of the outcomes set. The outcome set must therefore be $\Omega = H \times R \times T$ with arbitrary element of (h, r, t) .

Since our question is to determine the dependence of two observations from the same nursing home cluster, we need to consider the structure of the joint probability of $\{(h, r, t), (h', r', t')\}$. Here again, we need to make some reductions. First, we must consider whether these observations share any components. Since both observations will come from the same nursing home, i.e. $h = h'$, and we can just use one symbol in both observations. If the DGP assigned the same treatment to all residents of a nursing home, then it would be that $t = t'$; however, for this DGP, as in example 3, each resident "flips the coin" and therefore each resident can have a different treatment. Treatments must remain with different denotations. Our arbitrary pair of observations is therefore $\{(h, r, t), (h, r', t')\}$. Remember, the importance of this reduction is to identify the indices we will need to sum over when evaluating the covariance of random variables between observations. In this case, we must sum over h, r, t, r' , and t' .

Structure of the probability measure. How can we best represent $P(h, r, t)$? Mathematically we can decompose this probability many ways. However, there is a sense of hierarchy, or nesting, here. In such cases it can, though perhaps not always, be best to work from inside to outside. Rather than express the probability as $P((h, r, t)) = P(h | r, t) \cdot P(r | t) \cdot P(t)$, it seems more interpretable as $P((h, r, t)) = P(t | r, h) \cdot P(r | h) \cdot P(h)$ in which we think of selecting a nursing home, $P(h)$, then obtaining a resident given a nursing home, $P(r | h)$, and finally treatment assignment given the resident within a nursing home, $P(t | r, h)$.

Next, can we simplify this probability specification? Well, $P(h)$ is as simple as it gets. But, can we simplify $P(r | h)$? If the probability of obtaining a resident did not depend on which nursing home was selected, then we could reduce this to merely $P(r)$. However, since only residents of the selected nursing home have nonzero probability of selection for an observation, clearly the probability a resident is selected depends on whether the resident's nursing home is selected, and we cannot reduce $P(r | h)$. What about $P(t | r, h)$? In this case, the DGP is one in which each nursing home has its own probability of treatment, and this treatment probability does not depend on the resident once the nursing home is identified. Since $P(t | r, h)$ is independent of r but not of h , we can simplify it to $P(t | h)$. Consequently, for a given observation, we have $P((h, r, t)) = P(t | h) \cdot P(r | h) \cdot P(h)$.

The full expression of the probability for a pair of observations from an arbitrary nursing home, attending to the hierarchical structure of sampling, is

$$P(t, t', r, r', h) = P(t | t', r, r', h) \cdot P(t' | r, r', h) \cdot P(r | r', h) \cdot P(r' | h) \cdot P(h).$$

Can this be reduced? Well, as previously stated the probability of treatment, given nursing home, is independent of selected resident: consequently $P(t | t', r, r', h) = P(t | t', h)$ and $P(t' | r, r', h) = P(t' | h)$. And, the probability of one observation's treatment does not depend on the treatment provided to the other observation: consequently, $P(t | t', h) = P(t | h)$. Also, conditional on nursing home, the probability of obtaining one resident does not depend on the resident obtained in the other observation: $P(r | r', h) = P(r | h)$. Consequently, the reduced expression for the probability is

$$P(t, t', r, r', h) = P(t | h) \cdot P(t' | h) \cdot P(r | h) \cdot P(r' | h) \cdot P(h).$$

Determine dependence. With the probability space defined by these outcome and probability components, assuming the sigma algebra is sufficient for the task, we can evaluate the covariance between random variables. For this example, as above, we consider covariance between random variables Y_i and Y_j expressed as a function of treatment, for $k \in \{i, j\}$:

$$Y_k(h, r, t) = \alpha + \beta \cdot T_k(h, r, t) + \varepsilon_k(h, r, t).$$

First, to evaluate the covariance, we need the expected values:

$$E(Y_i) = \sum_h \sum_t \sum_r (\alpha + \beta \cdot T_i(h, r, t) + \varepsilon_i(h, r, t)) \cdot P(r | h) \cdot P(t | h) \cdot P(h).$$

Determining dependence among random variables across observations

Following example 3, this reduces to

$$E(Y_i) = \sum_h (\alpha + \beta \cdot P(t=1|h)) \cdot P(h).$$

However, unlike state s in example 3, the probability of the nursing home h is not 1 for the selected observation. Consequently, unlike example 3, this expected value does not reduce to a linear function of nursing home specific probability of treatment, instead we sum over all nursing homes and obtain a linear function of the marginal probability of treatment

$$E(Y_i) = \alpha + \beta \cdot P(t=1).$$

Which denoting the marginal probability of treatment as p can be expressed as

$$E(Y_i) = \alpha + \beta \cdot p.$$

In this example, the expected probability is the same across all observations regardless of nursing home.

The covariance of interest is then

$$\begin{aligned} Cov(Y_i, Y_j) = & \sum_h \sum_{t'} \sum_t \sum_{r'} \sum_r (\alpha + \beta \cdot T_i(h, r, t) + \varepsilon_i(h, r, t) - (\alpha + \beta \cdot p)) \\ & \cdot (\alpha + \beta \cdot T_j(h, r', t') + \varepsilon_j(h, r', t') - (\alpha + \beta \cdot p)) \cdot P(r|h) \cdot P(r'|h) \cdot P(t|h) \\ & \cdot P(t'|h) \cdot P(h). \end{aligned}$$

Notice there is a summation for each unique component across the two observations and the probability structure matches. Assuming the expected values of error terms are 0 across nursing homes, after summing across r , r' , t , and t' we are left with

$$Cov(Y_i, Y_j) = \beta^2 \cdot \sum_h (p_h - p)^2 \cdot P(h),$$

which is the squared treatment effect multiplied by the variance across nursing homes of the nursing home level treatment probability:

$$Cov(Y_i, Y_j) = \beta^2 \cdot \text{var}(p_h).$$

Consequently, the random variables Y_i and Y_j are dependent if there is a non-zero treatment effect and all nursing homes do not have the same probability of treatment.

5. Conclusion

The preceding three examples present distinct, though similar, DGPs. Examples 2 and 3 presented DGPs with fixed primary units (i.e., observations came from a priori fixed states) whereas example 4 presented a DGP with randomly sampled primary units (i.e. a random sample of nursing homes). However, in example 2 all individual within a primary unit received the same treatment according to a unit-specific treatment probability (as we would see in a cluster randomized trial [11]), whereas in examples 3 and 4, each individual received treatment according to a unit-specific treatment

probability (i.e. each individual metaphorically flipped a probability weighted coin to determine treatment status). Within-unit observations in example 2, in which all individuals in a unit receive treatment as determined by a unit-specific treatment probability, were dependent if there is a nonzero treatment effect and the probability of treatment is neither 0 nor 1. Within-unit observations in example 3, in which each individual flips a treatment coin with unit-specific probability of treatment, were independent regardless of treatment effect. And, within-unit observations in example 4, in which primary units are sampled and individuals flip a unit-specific treatment coin, were dependent if there is a nonzero treatment effect and treatment probabilities vary across primary units. These examples show that similar DGPs can produce different conclusions regarding the nature of dependence among random variables across within-unit observations.

Following the four-step strategy for identifying observational dependence presented here, it is easy to determine the dependence between observations in other DGPs. This is important to understand as it is a common mistake to use cluster standard errors when we have data that may appear to be “clustered” because we mistakenly reason data that share a group feature are more likely to be similar within group and should thereby use clustered standard errors. Consequent mistaken standard errors can lead to mistaken power calculations, confidence interval estimation, and statistical inferences in hypothesis testing. It is important to determine whether random variables across observations are dependent in virtue of analyzing the DGP—one way to accomplish this is to follow the four steps presented here.

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Status Connectivity indices of Middle graph

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Abstract

Topological index is sometimes also known as graph theoretic index, is a numerical invariant of a graph, the topological indices are classified on degree and distance based concepts. The status $\sigma(u)$ of a vertex $u \in V(G)$ is defined as the sum of its distances between each other vertex in $V(G)$ of the graph G . Ramane and Yalnaik defined the distance based topological indices such as first and second status connectivity indices. In this article bounds for the first and second status connectivity indices of middle graph of a graph are established and further status connectivity indices of middle graph of certain graphs are computed.

Keywords: Middle graph; Status connectivity indices; Sunlet graph; Tree.

2020 AMS subject classifications: 05C10, 05C38, 05C76, 05C05.

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1 Introduction

Let $G=\{V, E\}$ be a graph of order n and size m . The edge is formed by joining any two vertices. A vertex degree u denoted by $\deg(u)$ and is defined as the number of edges that are incident on it. The length of the shortest path connecting the any two vertices u and v in G is the distance between the vertices u and v and it is denoted by $d(u, v)$. The eccentricity of a vertex u is defined as $e(u) = \max\{d(u, v)|v \in V(G)\}$. The diameter of G is the maximum distance between any two vertices and it is denoted by $\text{diam}(G)$.

Topological index also known as molecular descriptors, play a vital role in the fields including chemical engineering, material science, chemistry and so on. Topological indices are classified on degree and distance based concepts. Oldest degree based index were introduced by Gutman and Trinajstić [1] in 1972 and the oldest distance based index was introduced by Wiener [14] in 1947 while working on the paraffin boiling point. The Wiener index of graph G is defined as

$$W(G) = \sum_{uv \in E(G)} d(u, v)$$

The status [4] of a vertex $u \in V(G)$ is defined as the sum of its distances between each other vertex in $V(G)$ and it is denoted by $\sigma(u)$.

$$i.e., \sigma(u) = \sum_{v \in V(G)} d(u, v)$$

Status connectivity indices were defined by Ramane and Yalnaik [11]

$$i.e., S_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]$$

and

$$S_2(G) = \sum_{uv \in E(G)} [\sigma(u)\sigma(v)]$$

$S_1(G)$ and $S_2(G)$ are the first and second status connectivity indices respectively.

For details on status connectivity indices see [7, 10, 11, 12].

Hamada and Yoshimura introduced the middle graph $M(G)$ concept [2]. $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and is defined as, two vertices are connected in $M(G)$ if and only if either they are adjacent edges in G or one vertex of G and the other is an edge incident with it and for related work on indices and

graph valued functions can be found in [6, 8, 9]. In this article status connectivity indices for middle graph of connected graph are established. Throughout this paper we consider graph which are simple and connected. For undefined terms related to graph theory, we refer the book [5].

Definition 1.1. [3] *The friendship graph F_n obtained by connecting n copies of triangles with a common vertex, which is a universal vertex for the graph.*

Definition 1.2. [13] *The cycle C_n is converted into a graph called the n -sunlet graph S_n by adding n - pendant edges to each vertices of C_n .*

The following remarks are immediately used in theorem

Remark 1: $S_1(G) \leq S_1(M(G))$

Remark 2: $S_2(G) \leq S_2(M(G))$

Status connectivity index of the graph increase, when new edges or new vertices are added to the graph. by the definition of middle graph, edges are consider as vertices. So, number of vertices and edges are more than the graph G hence, Status connectivity index of middle graph is more than the graph.

Remark 3: $S_1(M(K_{1,q})) \leq S_1(M(P_n))$, $n \geq 3$

Remark 4: $S_2(M(K_{1,q})) \leq S_2(M(P_n))$, $n \geq 3$

Status of the vertex is increase along with distance. If status of the vertex increase then, status connectivity index graph also increase. The middle graph of star graph has the $diam = 3$ but, in middle graph of path graph the diameter always $diam > 3$ so, status connectivity index of middle graph of star graph is always less than or equal to middle graph of path.

2 Bounds for status connectivity indices of middle graph

Theorem 2.1. *Let G be a connected graph of order $n \geq 3$ then,*

$$(n-1)[(14n-18)] + [(n^2-3n+2)](3n-4) \leq S_1(M(G)) \leq n(2n^3-3n^2+1) + 2nm(n-2)(n-1)$$

and

$$(n-1)[(24n^2-62n+40)] + \frac{1}{2}[n^2-3n+2](3n-4)^2 \leq S_2(M(G)) \leq n^2(n-1)^2[(n^2-1) + (nm-2m)]$$

equality for lower bound holds, when $G=K_{1,q}$ where $q=n-1$, $n > 1$ and equality for upper bound holds, when $G= K_n$.

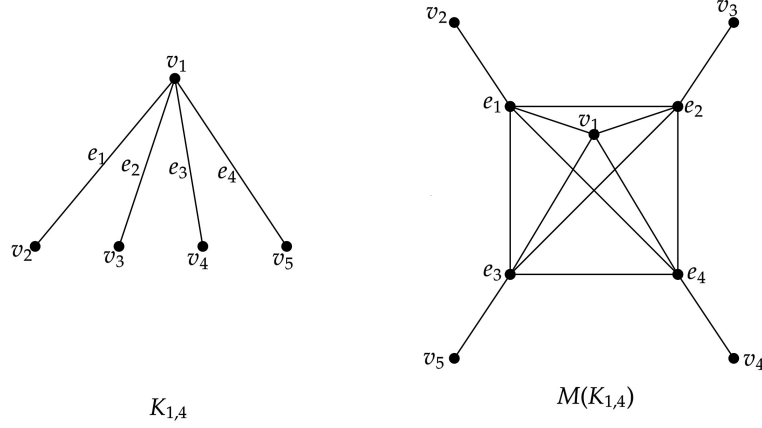


Figure 1: Star graph $K_{1,4}$ and their middle graph $M(K_{1,4})$.

Proof. For the equality part of the lower bound, let $G = K_{1,q}$ be the star graph with n vertices m edges. $M(K_{1,q})$ be the middle graph of star graph having $n+m$ vertices and $\frac{m^2+3m}{2}$ edges, it can be verified from the figure 1, then,

Status of the vertices having different degree in $M(K_{1,q})$ is as follows
Vertex having degree 1,

$$\begin{aligned}\sigma(u) &= \sum_{v \in V(M(K_{1,q}))} d(u, v) \\ &= 5n - 7\end{aligned}$$

Vertex having the degree n ,

$$\begin{aligned}\sigma(u) &= \sum_{v \in V(M(K_{1,q}))} d(u, v) \\ &= 3n - 4\end{aligned}$$

Vertex having the degree $n-1$,

$$\begin{aligned}\sigma(u) &= \sum_{v \in V(M(K_{1,q}))} d(u, v) \\ &= 3n - 3\end{aligned}$$

The edge set of $M(K_{1,q})$ is partitioned as,

$$E_1 = (uv | [deg(u) = n, deg(v) = 1]), |E_1| = n - 1.$$

$$E_2 = (uv | [deg(u) = n - 1, deg(v) = n]), |E_2| = n - 1.$$

Status Connectivity indices of Middle graph

$$E_3 = (uv | [deg(u) = n, deg(v) = n]), |E_3| = \frac{1}{2}[n^2 - 3n + 2].$$

by definition,

$$\begin{aligned} S_1(M(K_{1,q})) &= \sum_{uv \in E(M(K_{1,q}))} [\sigma(u) + \sigma(v)] \\ &= \sum_{uv \in E_1} (5n - 7 + 3n - 4) + \sum_{uv \in E_2} (3n - 3 + 3n - 4) \\ &\quad + \sum_{uv \in E_3} (3n - 4 + 3n - 4) \\ &= (n - 1)(8n - 11) + (n - 1)(6n - 7) \\ &\quad + \frac{1}{2}[n^2 - 3n + 2](6n - 8) \\ &= (n - 1)(14n - 18) + (n^2 - 3n + 2)(3n - 4) \end{aligned} \quad (1)$$

and

$$\begin{aligned} S_2(M(K_{1,q})) &= \sum_{uv \in E(M(K_{1,q}))} [\sigma(u)\sigma(v)] \\ &= \sum_{uv \in E_1(G)} (5n - 7)(3n - 4) + \sum_{uv \in E_2(G)} (3n - 3)(3n - 4) + \\ &\quad \sum_{uv \in E_3(G)} (3n - 4)(3n - 4) \\ &= (n - 1)(15n^2 - 41n + 28) + (n - 1)(9n^2 - 21n + 12) \\ &\quad + \frac{1}{2}[n^2 - 3n + 2](3n - 4)^2 \\ &= (n - 1)(24n^2 - 62n + 40) + \frac{1}{2}[(n^2 - 3n + 2) \\ &\quad (3n - 4)^2]. \end{aligned} \quad (2)$$

Equality For the upper bounds, let $G = K_n$ be the complete graph with n vertices and m edges. $M(K_n)$ be the middle graph of complete graph having $n+m$ vertices and nm edges. Status of vertex of $M(K_n)$ which having degree $n-1$ is $n^2 - 1$ and status of the vertex of $M(K_n)$ which having the degree $2(n-1)$ is $n^2 - n$. The edge set of $M(K_n)$ is partitioned as,

$$\begin{aligned} E_1 &= (uv | deg(u) = n - 1, deg(v) = 2n - 2), |E_1| = n(n - 1), \\ E_2 &= (uv | deg(u) = deg(v) = 2n - 2), |E_2| = nm - 2m. \end{aligned}$$

by definition,

$$\begin{aligned}
 S_1(M(K_n)) &= \sum_{uv \in E(M(K_n))} [\sigma(u) + \sigma(v)] \\
 &= \sum_{uv \in E_1} (n^2 - 1 + n^2 - n) + \sum_{uv \in E_2} (n^2 - n) + (n^2 - n) \\
 &= n(n-1)(2n^2 - n - 1) + m(n-2)(2n(n-1)) \\
 &= n(2n^3 - 3n^2 + 1) + 2nm(n-2)(n-1) \quad (3)
 \end{aligned}$$

$$\begin{aligned}
 S_2(M(K_n)) &= \sum_{uv \in E(M(K_n))} [\sigma(u)\sigma(v)] \\
 &= \sum_{uv \in E_1} (n^2 - 1)(n^2 - n) + \sum_{uv \in E_2} (n^2 - n)(n^2 - n) \\
 &= n(n-1)((n^2 - 1)(n^2 - n)) + (nm - 2m)n^2(n-1)^2 \\
 &= n^2(n-1)^2[(n^2 - 1) + (nm - 2m)]. \quad (4)
 \end{aligned}$$

For Lower bounds, G only has a minimum number of edges if it is a tree. $M(G)$ has least number of edges if $G \cong T$. $M(T)$ has minimum number of edges compared to every other graph $M(G)$.

Status connectivity index is decreases when number of vertices or edges are deleted from the graph.

by Remark (3) and (4),

$$\begin{aligned}
 S_1(M(G)) &\geq S_1(M(K_{1,q})) \\
 S_2(M(G)) &\geq S_2(M(K_{1,q}))
 \end{aligned}$$

from equation (1)

$$S_1(M(G)) \geq (n-1)[(14n-18)] + (n^2-3n+2)(3n-4) \quad (5)$$

from equation (2)

$$S_2(M(G)) \geq (n-1)[(24n^2-62n+40)] + \frac{1}{2}(n^2-3n+2)(3n-4)^2 \quad (6)$$

For upper bounds, G has maximum number of edges if and only if G is complete graph, $M(G)$ has maximum number of edges if and only if $G \cong K_n$. When additional edges or vertices are joined to the graph, the graph's status connectivity index extends and comparing to other middle graph of G , $M(K_n)$ has maximum number of edges.

Therefore,

Status Connectivity indices of Middle graph

$$\begin{aligned} S_1(M(G)) &\leq S_1(M(K_n)) \\ S_2(M(G)) &\leq S_2(M(K_n)) \end{aligned}$$

from equation (3)

$$S_1(M(G)) \leq n(2n^3 - 3n^2 + 1) + 2nm(n-2)(n-1) \quad (7)$$

from equation (4)

$$S_2(M(G)) \leq n^2(n-1)^2[(n^2-1) + (nm-2m)] \quad (8)$$

Hence, from the equation (5), (6), (7) and (8) the status connectivity indices of the middle graph will be,

$$(n-1)[(14n-18)] + [(n^2-3n+2)](3n-4) \leq S_1(M(G)) \leq n(2n^3 - 3n^2 + 1) + 2nm(n-2)(n-1)$$

and

$$(n-1)[(24n^2 - 62n + 40)] + \frac{1}{2}[n^2 - 3n + 2](3n-4)^2 \leq S_2(M(G)) \leq n^2(n-1)^2[(n^2-1) + (nm-2m)].$$

□

3 Status connectivity indices of middle graph of some class of graphs

Theorem 3.1. Let C_n be the cycle graph then,

$$\begin{aligned} S_1(M(C_n)) &= 2n(n^2 + 2n - 1) + m(n^2 + n) \\ \text{and} \\ S_2(M(C_n)) &= \frac{n(n^2+3n-2)(n^2+n)}{2} + \frac{m(n^2+n)}{4}. \end{aligned}$$

Proof. Let C_n be the cycle graph with n vertices and m edges. $M(C_n)$ be the middle graph of the C_n has $2n$ vertices, $2n+m$ edges. Status of vertex of $M(C_n)$ having the degree 2 is

$$\begin{aligned} \sigma(v) &= \sum_{u \in V(M(K_n))} d(u, v) \\ &= 1 + 2 + 3 + 4 + \cdots + (n-1) + (2n-1) \\ &= \frac{n^2 + 3n - 2}{2} \end{aligned}$$

Status of the vertex of $M(C_n)$ having the degree 4 is

$$\begin{aligned}\sigma(v) &= \sum_{u \in V(M(C_n))} d(u, v) \\ &= 1 + 2 + 3 + \cdots + n \\ &= \frac{n^2 + n}{2}\end{aligned}$$

The edge set of $M(C_n)$ is partitioned as,

$$E_1 = (uv | \deg(u) = 4, \deg(v) = 2), |E_1| = 2n$$

$$E_2 = (uv | \deg(u) = \deg(v) = 4), |E_2| = m.$$

by definition,

$$\begin{aligned}S_1(M(C_n)) &= \sum_{uv \in E(M(C_n))} [\sigma(u) + \sigma(v)] \\ &= \sum_{uv \in E_1} \left(\frac{n^2 + 3n - 2}{2} + \frac{n(n+1)}{2} \right) + \sum_{uv \in E_2} \left(\frac{n^2 + n}{2} \right) + \left(\frac{n^2 + n}{2} \right) \\ &= 2n(n^2 + 2n - 1) + m(n^2 + n).\end{aligned}$$

and

$$\begin{aligned}S_2(M(C_n)) &= \sum_{uv \in E(M(C_n))} [\sigma(u)\sigma(v)] \\ &= \sum_{uv \in E_1} \left(\frac{n^2 + 3n - 2}{2} \right) \left(\frac{n^2 + n}{2} \right) + \sum_{uv \in E_2} \left(\frac{n^2 + n}{2} \right) \left(\frac{n^2 + n}{2} \right) \\ &= \frac{n(n^2 + 3n - 2)(n^2 + n)}{2} + \frac{m(n^2 + n)}{4}.\end{aligned}$$

□

Theorem 3.2. Let $K_{p,q}$ be the complete bipartite graph then,

$$S_1(M(K_{p,q})) = 2m(n^2 + 3n - 5) + m\left(\frac{n}{2} - 1\right)(n^2 + 2n - 4)$$

and

$$S_2(M(K_{p,q})) = \frac{m}{2}(n^2 + 4n + 6)(n^2 + 2n - 4) + \frac{m}{4}\left(\frac{n}{2} - 1\right)(n^2 + 2n - 4)^2.$$

Status Connectivity indices of Middle graph

Proof. Let $K_{p,q}$, $p = q$ with $n=(p+q)$ vertices and $m=(pq)$ edges and $M(K_{p,q})$ be the middle graph of complete bipartite graph. Status of vertex of $M(K_{p,q})$ having degree $\frac{n}{2}$ is $\frac{n^2+4n-6}{2}$ and status of the vertex of $M(K_{p,q})$ having the degree n is $\frac{n^2+2n-4}{2}$.

The edge set of $M(K_{p,q})$ partitioned as,

$$E_1 = (uv | \deg(u) = \frac{n}{2}, \deg(v) = n), |E_1| = 2m$$

$$E_2 = (uv | \deg(u) = \deg(v) = n), |E_2| = m(\frac{n}{2} - 1).$$

by definition,

$$\begin{aligned} S_1(M(K_{p,q})) &= \sum_{uv \in E((M(K_{p,q})))} [\sigma(u) + \sigma(v)] \\ &= \sum_{uv \in E_1} \left(\frac{n^2 + 4n - 6}{2} + \frac{n^2 + 2n - 4}{2} \right) + \sum_{uv \in E_2} \left(\frac{n^2 + 2n - 4}{2} \right. \\ &\quad \left. + \frac{n^2 + 2n - 4}{2} \right) \\ &= 2m(n^2 + 3n - 5) + m(\frac{n}{2} - 1)(n^2 + 2n - 4) \end{aligned}$$

and

$$\begin{aligned} S_2(M(K_{p,q})) &= \sum_{uv \in E((M(K_{p,q})))} [\sigma(u)\sigma(v)] \\ &= \sum_{uv \in E_1(G)} \left(\frac{n^2 + 4n - 6}{2} \right) \left(\frac{n^2 + 2n - 4}{2} \right) + \sum_{uv \in E_2(G)} \left(\frac{n^2 + 2n - 4}{2} \right) \\ &\quad \left(\frac{n^2 + 2n - 4}{2} \right) \\ &= \frac{m}{2}(n^2 + 4n + 6)(n^2 + 2n - 4) + \frac{m}{4}(\frac{n}{2} - 1)(n^2 + 2n - 4)^2. \end{aligned}$$

□

Theorem 3.3. Let F_n be the friendship graph where n is number of triangles then,

$$S_1(M(F_n)) = 2n(16n^2 + 72n - 28)$$

and

$$S_2(M(F_n)) = 2n(64n^3 + 377n^2 - 300n + 57)$$

Proof. Let $M(F_n)$ be the middle graph of friendship graph, status of vertex of $M(F_n)$ which having degree 2 is $8 + 13(n - 1)$, status of the vertex of $M(F_n)$

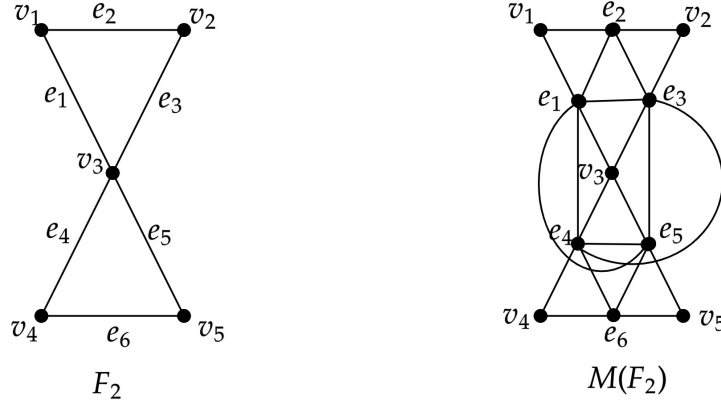


Figure 2: Friendship graph F_2 and their middle graph $M(F_2)$.

having degree $2n$ is $2n + 6n$, status of vertex of $M(F_n)$ having degree $2n+2$ is $6 + 8(n - 1)$ and status of vertex of $M(F_n)$ having degree 4 is $6 + 13(n - 1)$, it can be verified from the figure 2.

The edge set of $M(F_n)$ is partitioned as,

$$\begin{aligned} E_1 &= (uv | \deg(u) = 2, \deg(v) = 4), |E_1| = 2n \\ E_2 &= (uv | \deg(u) = 2, \deg(v) = 2n + 2), |E_2| = 2n \\ E_3 &= (uv | \deg(u) = 4, \deg(v) = 2n + 2), |E_3| = 2n \\ E_4 &= (uv | \deg(u) = \deg(v) = 2n + 2), |E_4| = n(2n - 1) \\ E_5 &= (uv | \deg(u) = 2n + 2, \deg(v) = 2n), |E_5| = 2n. \end{aligned}$$

by definition,

$$\begin{aligned} S_1(M(F_n)) &= \sum_{uv \in E(M(F_n))} \sigma(u) + \sigma(v) \\ &= \sum_{uv \in E_1} (8 + 13(n - 1) + 6 + 13n - 13) + \sum_{uv \in E_2} (8 + 13n - 13) + (6 + 8n - 8) \\ &\quad + \sum_{uv \in E_3} (6 + 13(n - 1) + 6 + 8(n - 1)) + \sum_{uv \in E_4} (6 + 8n - 8) + (6 + 8n - 8) \\ &\quad + \sum_{uv \in E_5} (6 + 8(n - 1) + 8n) \\ &= 2n(26n - 12) + 2n(21n - 7) + 2n(21n - 9) + n(2n - 1)(16n - 4) \\ &\quad + 2n(16n - 2) \\ &= 2n[8n^2 + 78n - 29] \end{aligned}$$

Status Connectivity indices of Middle graph

and

$$\begin{aligned}
 S_2(M(F_n)) &= \sum_{uv \in E(M(F_n))} \sigma(u)\sigma(v) \\
 &= \sum_{uv \in E_1} (8 + 13n - 13)(6 + 13n - 13) + \sum_{uv \in E_2} (8 + 33n - 13)(6 + 8n - 8) \\
 &\quad + \sum_{uv \in E_3} (6 + 13(n - 1))(6 + 8(n - 1)) + \sum_{uv \in E_4} (6 + 8n - 8)(6 + 8n - 8) \\
 &\quad + \sum_{uv \in E_5} (6 + 8(n - 1))(8n) \\
 &= 2n(13n - 5)(13n - 7) + 2n(13n - 5)(8n - 2) + 2n(13n - 7)(8n - 2) + \\
 &\quad n(2n - 1)(8n - 2)^2 + 2n(6 + 8(n - 1))(8n) \\
 &= 2n[32n^3 + 409n^2 - 310n + 58].
 \end{aligned}$$

□

Illustration 1: Let $M(F_2)$ be the middle graph of friendship graph F_2 .
 From the figure 2, $V(M(F_2)) = \{v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, e_5, e_6\}$.
 From the definition of status of the vertex,

$$\begin{aligned}
 \sigma(v_1) &= 21 = \sigma(v_2) = \sigma(v_5) = \sigma(v_5) \\
 \sigma(e_2) &= 19 = \sigma(e_6) \\
 \sigma(e_1) &= 14 = \sigma(e_3) = \sigma(e_4) = \sigma(e_5) \\
 \sigma(v_3) &= 16
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 S_1(M(F_2)) &= \sum_{uv \in E(M(F_2))} \sigma(u) + \sigma(v) \\
 &= [\sigma(v_1) + \sigma(e_2)] + [\sigma(v_2) + \sigma(e_2)] + [\sigma(e_1) + \sigma(e_2)] + [\sigma(e_3) + \sigma(e_2)] \\
 &\quad + [\sigma(e_1) + \sigma(e_3)] + [\sigma(e_1) + \sigma(e_4)] + [\sigma(e_1) + \sigma(e_5)] + [\sigma(e_3) + \sigma(e_5)] \\
 &\quad + [\sigma(e_3) + \sigma(e_4)] + [\sigma(e_1) + \sigma(v_3)] + [\sigma(e_3) + \sigma(v_3)] + [\sigma(v_3) + \sigma(e_4)] \\
 &\quad + [\sigma(v_3) + \sigma(e_5)] + [\sigma(e_4) + \sigma(e_5)] + [\sigma(e_4) + \sigma(e_6)] + [\sigma(e_5) + \sigma(e_6)] \\
 &\quad + [\sigma(v_4) + \sigma(e_4)] + [\sigma(e_5) + \sigma(v_5)] + [\sigma(v_4) + \sigma(e_6)] + [\sigma(v_5) + \sigma(e_6)] \\
 &\quad + [\sigma(v_1) + \sigma(e_1)] + [\sigma(v_2) + \sigma(e_3)].
 \end{aligned}$$

Substitute the values from the equation (9) we get,

$$S_1(M(F_2)) = 720 \quad (10)$$

From the Theorem 3.3

$$S_1(M(F_n)) = 2n(16n^2 + 72n - 28),$$

Put $n = 2$

$$\begin{aligned} S_1(M(F_2)) &= 2(2)(16(2^2) + 72(2) - 28) \\ S_1(M(F_2)) &= 720. \end{aligned} \quad (11)$$

The equation (10) and (11) are equal.

Hence the result.

Theorem 3.4. For S_n , $n \geq 3$ be the Sunlet graph having $2n$ vertices and $2n$ edges then,

$$S_1(M(S_n)) = \begin{cases} n[14n^2 + 55n - 15] & \text{if } n \text{ is even,} \\ n[14n^2 + 55n - 16] & \text{if } n \text{ is odd.} \end{cases}$$

$$S_2(M(S_n)) = \begin{cases} 7n^5 + 55n^4 + \frac{379}{4}n^3 - \frac{254}{4}n^2 + 10n & \text{if } n \text{ is even,} \\ 7n^5 + 55n^4 + \frac{375}{4}n^3 - 68n^2 + \frac{45}{4}n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $M(S_n)$ be the middle graph of sunlet graph has $4n$ vertices and $7n$ edges.

If n is even then, status of vertex of $M(S_n)$ which having degree 3 is $n^2 + \frac{9}{2}n - 1$, status of the vertex of $M(S_n)$ having the degree 1 is $n^2 + \frac{17}{2}n - 4$, status of vertex of $M(S_n)$ which having degree 4 is $n^2 + \frac{9}{2}n - 2$ and status of vertex of $M(S_n)$ which having degree 6 is $n^2 + \frac{5}{2}n$, it can be verified from the figure 3.

If n is odd then, the status of vertex which having degree 3 is $n^2 + \frac{9}{2}n - \frac{3}{2}$, status of the vertex of $M(S_n)$ which having the degree 1 is $n^2 + \frac{17}{2}n - \frac{9}{2}$, status of vertex which having degree 4 is $n^2 + \frac{9}{2}n - \frac{5}{2}$ and status of vertex of $M(S_n)$ which having degree 6 is $n^2 + \frac{5}{2}n + \frac{1}{2}$.

We partition edge set of $M(S_n)$ as,

$$E_1 = (uv | \deg(u) = 6, \deg(v) = 3), |E_1| = 2n$$

$$E_2 = (uv | \deg(u) = 4, \deg(v) = 3), |E_2| = n$$

$$E_3 = (uv | \deg(u) = 1, \deg(v) = 4), |E_3| = n$$

$$E_4 = (uv | \deg(u) = 4, \deg(v) = 6), |E_4| = 2n$$

$$E_5 = (uv | \deg(u) = \deg(v) = 6), |E_5| = n.$$

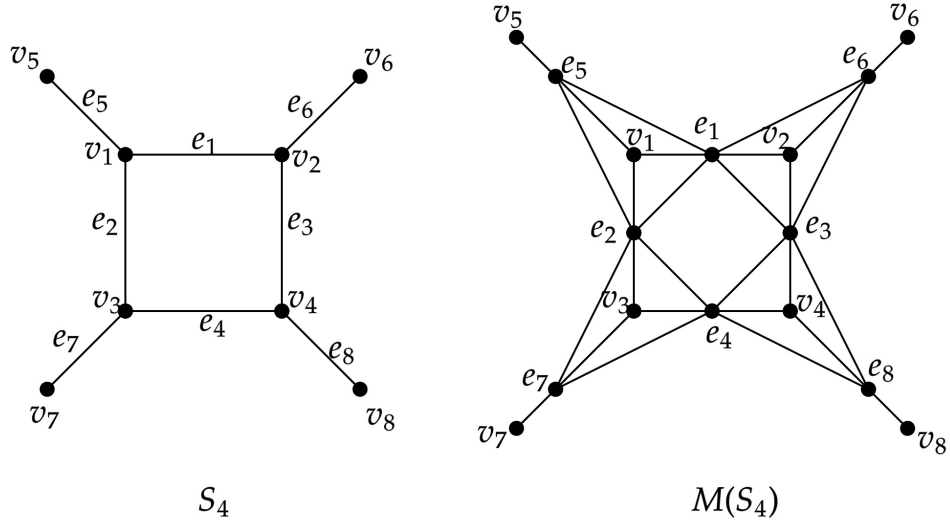


Figure 3: Sunlet graph S_4 and their middle graph $M(S_4)$.

by definition,

$$S_1(M(S_n))$$

$$\begin{aligned}
 &= \sum_{uv \in E(M(S_n))} (\sigma(u) + \sigma(v)) \\
 &= \sum_{uv \in E_1} \left(\frac{2n^2 + 9n - 2}{2} + \frac{2n^2 + 5n}{2} \right) + \sum_{uv \in E_2} \left(\frac{2n^2 + 9n - 2}{2} + \frac{2n^2 + 9n - 4}{2} \right) \\
 &\quad + \sum_{uv \in E_3} \left(\frac{2n^2 + 9n - 2}{2} + \frac{2n^2 + 17n - 8}{2} \right) + \sum_{uv \in E_4} \left(\frac{2n^2 + 9n - 4}{2} + \frac{2n^2 + 5n}{2} \right) \\
 &\quad + \sum_{uv \in E_5} \left(\frac{2n^2 + 5n}{2} + \frac{2n^2 + 5n}{2} \right) \\
 &= 2n(2n^2 + 13n - 6) + n(2n^2 + 9n - 3) + n(2n^2 + 7n - 1) + \\
 &\quad 2n(2n^2 + 7n - 2) + n(2n^2 + 5n) \\
 &= n[14n^2 + 55n - 15]
 \end{aligned}$$

$$\begin{aligned}
& S_1(M(S_n)) \\
&= \sum_{uv \in E(M(S_n))} \sigma(u) + \sigma(v) \\
&= \sum_{uv \in E_1} \frac{2n^2 + 9n - 3}{2} + \frac{n^2 + 5n + 1}{2} + \sum_{uv \in E_2} \frac{2n^2 + 9n - 5}{2} + \frac{2n^2 + 9n - 3}{2} + \\
&\quad \sum_{uv \in E_3} \frac{2n^2 + 9n - 5}{2} + \frac{2n^2 + 17n - 9}{2} + \sum_{uv \in E_4} \frac{2n^2 + 9n - 5}{2} + \frac{2n^2 + 5n + 1}{2} + \\
&\quad \sum_{uv \in E_5} \frac{2n^2 + 5n + 1}{2} + \frac{2n^2 + 5n + 1}{2} \\
&= 2n(2n^2 + 13n - 7) + n(2n^2 + 9n - 4) + n(2n^2 + 7n - 1) + 2n(2n^2 + 7n - 2) \\
&\quad + n(2n^2 + 5n + 1) \\
&= n[14n^2 + 55n - 16].
\end{aligned}$$

and

$$\begin{aligned}
& S_2(M(S_n)) \\
&= \sum_{uv \in E(M(S_n))} (\sigma(u) + \sigma(v)) \\
&= \sum_{uv \in E_1} \frac{(2n^2 + 9n - 2)(2n^2 + 5n)}{4} + \sum_{uv \in E_2} \frac{(2n^2 + 9n - 2)(2n^2 + 9n - 4)}{4} + \\
&\quad \sum_{uv \in E_3} \frac{(2n^2 + 9n - 2)(2n^2 + 17n - 8)}{4} + \sum_{uv \in E_4} \frac{(2n^2 + 9n - 4)(2n^2 + 5n)}{4} + \\
&\quad \sum_{uv \in E_5} \frac{(2n^2 + 5n)(2n^2 + 5n)}{4} \\
&= 2n \frac{(2n^2 + 9n - 2)(2n^2 + 5n)}{4} + n \frac{(2n^2 + 9n - 2)(2n^2 + 9n - 4)}{4} \\
&\quad + n \frac{(2n^2 + 9n - 2)(2n^2 + 17n - 8)}{4} + 2n \frac{(2n^2 + 9n - 4)(2n^2 + 5n)}{4} \\
&\quad + n \frac{(2n^2 + 5n)(2n^2 + 5n)}{4} \\
&= 7n^5 + 55n^4 + \frac{379}{4}n^3 - \frac{254}{4}n^2 + 10n.
\end{aligned}$$

$$\begin{aligned}
& S_2(M(S_n)) \\
&= \sum_{uv \in E(M(S_n))} (\sigma(u)\sigma(v)) \\
&= \sum_{uv \in E_1} \frac{(2n^2 + 9n - 3)(2n^2 + 5n + 1)}{4} + \sum_{uv \in E_2} \frac{(2n^2 + 9n - 5)(2n^2 + 9n - 3)}{4} + \\
&\quad \sum_{uv \in E_3} \frac{(2n^2 + 9n - 5)(2n^2 + 17n - 9)}{4} + \sum_{uv \in E_4} \frac{(2n^2 + 9n - 5)(2n^2 + 5n + 1)}{4} \\
&\quad + \sum_{uv \in E_5} \frac{(2n^2 + 5n + 1)(2n^2 + 5n + 1)}{4} \\
&= 2n\left(\frac{4n^2 + 14n - 2}{4}\right) + n\left(\frac{4n^2 + 18n - 8}{4}\right) + n\left(\frac{4n^2 + 26n - 14}{4}\right) + \\
&\quad 2n\left(\frac{4n^2 + 14n - 4}{4}\right) + n\left(\frac{4n^2 + 10n + 2}{4}\right) \\
&= 7n^5 + 55n^4 + \frac{375}{4}n^3 - 68n^2 + \frac{45}{4}n.
\end{aligned}$$

□

Illustration 2: Let $M(S_4)$ be the middle graph of Sunlet graph S_4 . From the figure 3, $V(M(S_4)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. From the definition of status of the vertex,

$$\begin{aligned}
\sigma(v_1) &= 33 = \sigma(v_2) = \sigma(v_3) = \sigma(v_4) \\
\sigma(v_5) &= 46 = \sigma(v_6) = \sigma(v_7) = \sigma(v_8) \\
\sigma(e_1) &= 26 = \sigma(e_2) = \sigma(e_3) = \sigma(e_4) \\
\sigma(e_5) &= 32 = \sigma(e_6) = \sigma(e_7) = \sigma(e_8)
\end{aligned} \tag{12}$$

$$\begin{aligned}
S_1(M(S_4)) &= \sum_{uv \in E(M(S_4))} \sigma(u) + \sigma(v) \\
&= [\sigma(v_5) + \sigma(e_5)] + [\sigma(e_5) + \sigma(e_1)] + [\sigma(e_5) + \sigma(v_1)] + [\sigma(e_5) + \sigma(e_2)] \\
&\quad + [\sigma(v_6) + \sigma(e_6)] + [\sigma(e_6) + \sigma(e_1)] + [\sigma(e_6) + \sigma(v_2)] + [\sigma(e_3) + \sigma(e_6)] \\
&\quad + [\sigma(v_7) + \sigma(e_7)] + [\sigma(e_7) + \sigma(v_3)] + [\sigma(e_7) + \sigma(e_2)] + [\sigma(e_7) + \sigma(e_4)] \\
&\quad + [\sigma(v_8) + \sigma(e_8)] + [\sigma(e_8) + \sigma(e_4)] + [\sigma(e_8) + \sigma(v_4)] + [\sigma(e_8) + \sigma(e_3)] \\
&\quad + [\sigma(v_1) + \sigma(e_1)] + [\sigma(e_1) + \sigma(v_2)] + [\sigma(v_2) + \sigma(e_3)] + [\sigma(e_3) + \sigma(v_4)] \\
&\quad + [\sigma(v_4) + \sigma(e_4)] + [\sigma(e_4) + \sigma(v_3)] + [\sigma(v_3) + \sigma(e_2)] + [\sigma(e_2) + \sigma(v_1)] \\
&\quad + [\sigma(e_1) + \sigma(e_2)] + [\sigma(e_1) + \sigma(e_3)] + [\sigma(e_3) + \sigma(e_4)] + [\sigma(e_4) + \sigma(e_2)].
\end{aligned}$$

Substitute the values from the equation (12) we get,

$$S_1(M(S_4)) = 1716 \quad (13)$$

From the Theorem 3.4

$$S_1(M(S_4)) = n[14n^2 + 55n - 15]$$

Put $n = 4$

$$\begin{aligned} S_1(M(S_4)) &= 4(14(4^2) + 55(4) - 15) \\ S_1(M(S_4)) &= 1716 \end{aligned} \quad (14)$$

The equation (13) and (14) are equal.
Hence the result.

Theorem 3.5. Let T be the tree with n vertices and $\text{diam}(T)=3$ then,

$$\begin{aligned} S_1(M(T)) &= 25k^2 + 28k + 10 + \frac{k^2-2k}{4}(8k+4) \\ &\text{and} \\ S_2(M(T)) &= 52k^3 + 82k^2 + 50k + 12 + (k^2 - 2k)[4k^2 + 4k + 1]. \end{aligned}$$

Proof. T be the tree obtained by joining equal number of pendent edges to both end vertices of path P_2 . $M(T)$ be the middle graph of tree containing $4n$ vertices and $7n$ edges, k be the pendent vertices of the tree. Status of vertex which having degree 1 is $6k + 3$, status of the vertex which having the degree $\frac{k}{2} + 2$ is $4k + 2$, status of vertex which having degree $\frac{k}{2} + 1$ is $4k + 3$ and status of vertex which having degree $k+2$ is $3k + 2$.

The edge set of $M(T)$ is partitioned as,

$$\begin{aligned} E_1 &= (uv | \deg(u) = 1, \deg(v) = \frac{k}{2} + 2), |E_1| = k \\ E_2 &= (uv | \deg(u) = \frac{k}{2} + 2, \deg(v) = \frac{k}{2} + 1), |E_2| = k \\ E_3 &= (uv | \deg(u) = \frac{k}{2} + 1, \deg(v) = k + 2), |E_3| = 2 \\ E_4 &= (uv | \deg(u) = \frac{k}{2} + 2, \deg(v) = k + 2), |E_4| = k \\ E_5 &= (uv | \deg(u) = \deg(v) = \frac{k}{2} + 2), |E_5| = \frac{k^2-2k}{4}. \end{aligned}$$

by definition,

$$\begin{aligned}
 S_1(M(T)) &= \sum_{uv \in E(M(T))} (\sigma(u) + \sigma(v)) \\
 &= \sum_{uv \in E_1} (6k + 3 + 4k + 2) + \sum_{uv \in E_2} (4k + 2 + 4k + 3) + \sum_{uv \in E_3} (4k + 3 \\
 &\quad + 3k + 2) + \sum_{uv \in E_4} (4k + 2 + 3k + 2) + \sum_{uv \in E_5} (4k + 2 + 4k + 2) \\
 &= k(10k + 5) + k(8k + 5) + 2(7k + 5) + k(7k + 4) + \frac{k^2 - 2k}{4} \\
 &\quad (8k + 4) \\
 &= 25k^2 + 28k + 10 + \frac{k^2 - 2k}{4}(8k + 4).
 \end{aligned}$$

and

$$\begin{aligned}
 S_2(M(T)) &= \sum_{uv \in E(M(T))} (\sigma(u)\sigma(v)) \\
 &= \sum_{uv \in E_1} 2(6k + 3)(2k + 1) + \sum_{uv \in E_2} 2(2k + 1)(4k + 3) + \sum_{uv \in E_3} [(4k + 3) \\
 &\quad (3k + 2)] + \sum_{uv \in E_4} 2(2k + 1)(3k + 2) + \sum_{uv \in E_5} (2(2k + 1))(2(2k + 1)) \\
 &= k(6k + 3)(4k + 2) + k(4k + 2)(4k + 3) + 2(4k + 3)(3k + 2) \\
 &\quad + k(4k + 2)(3k + 2) + \frac{k^2 - 2k}{4}[2(2k + 1)]^2 \\
 &= 52k^3 + 82k^2 + 50k + 12 + (k^2 - 2k)(4k^2 + 4k + 1).
 \end{aligned}$$

□

4 Conclusions

In the present research work bounds for first and second status connectivity indices of middle graph are obtained. Further, first and second status connectivity indices for middle graph of some class of graphs like cycle, complete bipartite, friendship graph, sunlet graph and tree of diameter 3 are established. For future research work, one can calculate the results on status connectivity indices of iterated middle graph of graph and other class of graphs including tree of diameter greater than 3.

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Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

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Abstract

In this paper, we studied and analysed some of the properties of soft μ - α -open sets and soft μ - β -open sets in soft generalized topological spaces. Also we investigate the inter-relationship among several weaker forms of soft μ -open sets in soft generalized topological spaces.

Keywords: soft $\alpha_{(\mu,\eta)}$ -continuous; soft $\beta_{(\mu,\eta)}$ -continuous; soft $\rho_{(\mu,\eta)}$ -continuous; soft $\delta_{(\mu,\eta)}$ -continuous.

2020 AMS subject classifications: 54A05, 54C05, 54G20. ¹

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1 Introduction

Topology is an important area of mathematics, with many applications in the domain of computer science and physical sciences. In 1999 D. Molodtsov [13] initiated the concept of soft set theory as a mathematical tool for modelling uncertainties. A soft set is a collection of approximate descriptions of an object. Molodtsov [14] applied the soft set theory in several areas of mathematics such as game theory, probability, Perron and Riemann Integration and theory of measurements. He has analysed that the soft set is used to eradicate problems occurring in the field of economics, social science, engineering, medical science, etc. Maji et al. [11] have further improved the theory of soft sets. N. Cagman et al. [1] modified the definition of soft sets which is similar to that of Molodtsov and also Cagman [5] formulated the maximum decision making method by means of soft matrix theory. Many researchers have worked on the topological structure of soft sets. Bashir Ahmad et al. [4] have defined the soft topological structures by using soft points. B.chen [6] was the first one who studied the weak forms of soft open sets and scrutinized soft semiopen sets in soft topological spaces and studied some of its properties. Arockiarani and Arokialancy [3] determined soft β -open sets and prolonged to study weak forms of soft open sets in soft topological space. Gunduz Aras et al. [2] investigated some soft continuous functions for a fixed number of parameters on the initial universe. Mahanta and Das [10] classified many forms of soft mappings such as irresolute, semicontinuous and semiopen soft mappings. Muhammad Shabir et al. [15] introduced soft topological spaces. In 2002 A. Csaszar [7] introduced the concept of generalized topology and also studied some of its properties.

Soft generalized topology is relatively new and promising domain which lead to the development of new mathematical models and innovative approaches that will give solution to complex problems in engineering and environment. Sunil Jacob John et al. [8] introduced the concept of soft generalized topological spaces in 2014. sunil Jacob John [9] also introduced some interesting properties of the soft mapping $\pi : S(U)_E \rightarrow S(U)_E$ which satisfy the condition $\pi F_B \subset \pi F_D$ whenever $F_B \subset F_D \subset F_{\bar{E}}$ in soft π -open sets in soft generalized topological spaces in 2015. These concepts promote us in elaborating our further study in soft μ - α -open sets and soft μ - β -open sets in soft generalized topological spaces.

2 Preliminaries

Definition 2.1. [9] A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \mid e \in E, f_A(e) \in \mathcal{P}(U)\}$, where $f_A : E \rightarrow \mathcal{P}(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$. Here f_A is called an approximate function of the soft set F_A .

Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

The value of $f_A(e)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection. The set of all soft sets over U with E as the parameter set will be denoted by $S(U)_E$ or simply $S(U)$.

Definition 2.2. [9] Let $F_A \in S(U)$. If $f_A(e) = \emptyset$, for all $e \in E$, then F_A is called an empty soft set, denoted by $F_\emptyset$. $f_A(e) = \emptyset$ means there is no element in U related to the parameter e in E . Therefore we do not display such elements in the soft sets as it is meaningless to consider such parameters.

Definition 2.3. [9] Let $F_A \in S(U)$. If $f_A(e) = U$ for all $e \in A$, then F_A is called an A -universal soft set, denoted by $F_{\tilde{A}}$. If $A = E$, then the A -universal soft set is called an universal soft set, denoted by $F_{\tilde{E}}$.

Definition 2.4. [8] Let $F_A \in S(U)$. Then, the soft complement of F_A , denoted by $(F_A)^c$, is defined by the approximate function $f_A^c(e) = (f_A(e))^c$, where $(f_A(e))^c$ is the complement of the set $f_A(e)$, that is, $(f_A(e))^c = U \setminus f_A(e)$, for all $e \in E$.

Definition 2.5. [8] Let $F_A \in S(U)$. A Soft Generalized Topology (SGT) on F_A , denoted by μ or μ_{F_A} is a collection of soft subsets of F_A having the following properties:

- i. $F_\emptyset \in \mu$
- ii. $\{F_{A_i} \subseteq F_A / i \in J \subseteq N\} \subseteq \mu \Rightarrow \bigcup_{i \in J} F_{A_i} \in \mu$

The pair (F_A, μ) is called a Soft Generalized Topological Space (SGTS). Observe that $F_A \in \mu$ must not hold.

Definition 2.6. [8] Let (F_A, μ) be a SGTS. Then every element of μ is called a soft μ -open set.

Definition 2.7. [8] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the soft μ -interior of F_B , denoted by $i_\mu(F_B)$ is defined as the soft union of all soft μ -open subsets of F_B . Note that $i_\mu(F_B)$ is the largest soft μ -open set that is contained in F_B .

Definition 2.8. [8] Let (F_A, μ) be a SGTS and $F_B \subseteq F_A$. Then the soft μ -closure of F_B , denoted by $c_\mu(F_B)$ is defined as the soft intersection of all soft μ -closed super sets of F_B . Note that $c_\mu(F_B)$ is the smallest soft μ -closed superset of F_B .

Definition 2.9. [9] Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ -pre-open set iff $F_G \subset i_\mu c_\mu F_G$ (i.e, the case when $\pi = i_\mu c_\mu$). The class of all soft μ -pre-open sets is denoted by $\rho_{(\mu)}$ or ρ_μ .

Definition 2.10. [9] Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ -semi-open set iff $F_G \subset c_\mu i_\mu F_G$ (i.e, the case when $\pi = c_\mu i_\mu$). The class of all soft μ -semi-open sets is denoted by $\delta_{(\mu)}$ or δ_μ .

Definition 2.11. [12] A soft mapping $g : A \rightarrow B$ is called soft α -continuous if the inverse image of each soft open subset of B is a soft α -open set in A.

Definition 2.12. [4] A soft mapping $g : A \rightarrow B$ is called soft β -continuous (resp., soft α -continuous, soft precontinuous, and soft semicontinuous) if the inverse image of each soft open set in B is soft β -open (resp., soft α -open, soft preopen, and soft semiopen) set in A.

Remark 2.1. [16] If $\{(G, A)_\alpha \mid \alpha \in I\}$ is a collection of soft sets, then

- (i) $\tilde{\cup} \tilde{int}(G, A)_\alpha \subseteq \tilde{int}(\tilde{\cup} (G, A)_\alpha)$
- (ii) $\tilde{\cup} \tilde{scl}(G, A)_\alpha \subseteq \tilde{scl}(\tilde{\cup} (G, A)_\alpha)$

Throughout this paper, we symbolize soft generalized topological space, soft generalized topological subspace and the set of all soft μ -semi-open sets, soft μ -pre-open sets, soft μ - α -open sets, soft μ - α -closed sets, soft μ - β -open sets, soft μ - β -closed sets, soft μ -interior and soft μ -closure in a soft generalized topological space $(F_{\tilde{E}}, \mu)$ by $SGTS$, $SGTSS$, $\mathcal{S}\mu\text{-}\mathcal{SO}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{PCO}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{CO}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{CC}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{PO}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{BC}(F_{\tilde{E}})$, $\mathcal{S}\mu\text{-}\mathcal{Int}(F_{\tilde{E}})$ and $\mathcal{S}\mu\text{-}\mathcal{Cl}(F_{\tilde{E}})$ respectively.

3 Soft μ - α -open sets

Definition 3.1. [9] Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ - α -open set iff $F_G \subset i_\mu c_\mu i_\mu F_G$ (i.e, the case when $\pi = i_\mu c_\mu i_\mu$). The class of all soft μ - α -open sets is denoted by $\alpha_{(\mu)}$ or α_μ .

Definition 3.2. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ - α -closed set iff its complement is a soft μ - α -open set.

Example 3.1. Let $\mathcal{K} = \{\kappa_1, \kappa_2, \kappa_3\}$, $\mathcal{E} = \{\hat{x}_1, \hat{x}_2\}$ and $\mu = \{F_\emptyset, F_{\mathcal{A}_1}, F_{\mathcal{A}_2}, F_{\mathcal{A}_3}, F_{\mathcal{A}_4}, F_{\mathcal{A}_5}, F_{\mathcal{A}_6}, F_{\tilde{\mathcal{E}}}\}$ where
 $F_{\mathcal{A}_1} = \{(\hat{x}_1, \{\kappa_1, \kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$
 $F_{\mathcal{A}_2} = \{(\hat{x}_1, \{\kappa_1, \kappa_2\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$
 $F_{\mathcal{A}_3} = \{(\hat{x}_1, \{\kappa_1\}), (\hat{x}_2, \{\kappa_1\})\}$
 $F_{\mathcal{A}_4} = \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$
 $F_{\mathcal{A}_5} = \{(\hat{x}_1, \{\kappa_2\}), (\hat{x}_2, \{\kappa_2\})\}$
 $F_{\mathcal{A}_6} = \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$

Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

Then $(F_{\tilde{E}}, \mu)$ is a $\mathcal{SGTS}$.

$$\mathcal{S}\mu\text{-}\alpha\mathcal{O}(F_{\tilde{E}}) = \{F_{\emptyset}, F_{\tilde{E}}, \{(\hat{x}_1, \{\kappa_1, \kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}, \{(\hat{x}_1, \{\kappa_1, \kappa_2\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}, \{(\hat{x}_1, \{\kappa_1\}), (\hat{x}_2, \{\kappa_1\})\}, \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}, \{(\hat{x}_1, \{\kappa_2\}), (\hat{x}_2, \{\kappa_2\})\}, \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}\}$$

Remark 3.1. $F_{\emptyset}$ and $F_{\tilde{E}}$ are always soft μ - α -open and soft μ - α -closed.

Theorem 3.1. Every soft μ -open set (soft μ -closed) is soft μ - α -open (soft μ - α -closed). But the converse need not be true.

Example 3.2. In the above example 3.3, $\{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\mathcal{K}\})\}, \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\mathcal{K}\})\}$ are soft μ - α -open sets but not soft μ -open.

Theorem 3.2. Arbitrary soft union of soft μ - α -open sets is a soft μ - α -open set.

Proof. Let $\{F_{A_i} / i \in I\}$ be a collection of soft μ - α -open sets in a $\mathcal{SGTS} (F_{\tilde{E}}, \mu)$.

Then by the definition of soft μ - α -open set,

$$F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}l(\mathcal{S}\mu\text{-}\mathcal{I}nt(F_{A_i}))), \forall i.$$

$$\text{Now, } \widetilde{\bigcup} F_{A_i} \subseteq \widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}l(\mathcal{S}\mu\text{-}\mathcal{I}nt(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (i), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{I}nt(\widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{C}l(\mathcal{S}\mu\text{-}\mathcal{I}nt(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (ii), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}l(\widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{I}nt(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (i), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}l(\mathcal{S}\mu\text{-}\mathcal{I}nt(\widetilde{\bigcup} F_{A_i}))), \forall i.$$

$$\text{Hence } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}l(\mathcal{S}\mu\text{-}\mathcal{I}nt(\widetilde{\bigcup} F_{A_i}))), \forall i. \quad \square$$

Remark 3.2. Finite soft intersection of soft μ - α -open sets need not be a soft μ - α -open set.

Example 3.3. In the above example 3.3, $F_{\mathcal{C}} = \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$ and $F_{\mathcal{D}} = \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$ are soft μ - α -open sets but $F_{\mathcal{C}} \widetilde{\cap} F_{\mathcal{D}} = \{(\hat{x}_1, \{\kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$ is not soft μ - α -open.

Theorem 3.3. Arbitrary soft intersection of soft μ - α -closed sets is a soft μ - α -closed set.

Proof. The proof is similar to that of theorem 3.7 by taking complements. $\square$

Remark 3.3. Finite soft union of soft μ - α -closed sets need not be a soft μ - α -closed set.

Example 3.4. In the above example 3.3, $F_{\mathcal{S}} = \{(\hat{x}_1, \{\kappa_1\}), (\hat{x}_2, \{\kappa_3\})\}$ and $F_{\mathcal{T}} = \{(\hat{x}_1, \{\kappa_2\}), (\hat{x}_2, \{\kappa_3\})\}$ are soft μ - α -closed sets but $F_{\mathcal{S}} \widetilde{\cup} F_{\mathcal{T}} = \{(\hat{x}_1, \{\kappa_1, \kappa_2\}), (\hat{x}_2, \{\kappa_3\})\}$ is not soft μ - α -closed.

Remark 3.4. Let $(F_{\tilde{E}}, \mu)$ be a $\mathcal{SGTS}$. Then every soft μ - α -open set is soft μ -pre-open, soft μ -semi-open and soft μ - β -open. But the converse need not be true.

Example 3.5. In the above example 3.3, $\{(\hat{x}_1, \{\kappa_1, \kappa_2\}), (\hat{x}_2, \{\kappa_3\})\}$ and $\{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_3\})\}$ are soft μ -pre-open sets but not soft μ - α -open sets. Similarly, $\{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_2\})\}$ and $\{(\hat{x}_1, \{\kappa_1\}), (\hat{x}_2, \{\kappa_1, \kappa_3\})\}$ are soft μ -semi-open sets but not soft μ - α -open sets. Also, $\{(\hat{x}_1, \{\mathcal{K}\}), (\hat{x}_2, \{\kappa_2\})\}$ and $\{(\hat{x}_1, \{\emptyset\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$ are soft μ - β -open sets but not soft μ - α -open sets.

Definition 3.3. Let $(F_{\tilde{E}}, \mu)$ be a $\mathcal{SGTS}$ and $F_Z \subseteq F_{\tilde{E}}$.

- (i) The soft μ - α -interior of F_Z is defined by
 $\mathcal{S}\alpha_{i_\mu}(F_Z) = \bigcup \{F_{\mathcal{W}} : F_{\mathcal{W}} \subseteq F_Z \text{ and } F_{\mathcal{W}} \tilde{\subseteq} \mathcal{S}\mu\text{-}\alpha\mathcal{O}(F_{\tilde{E}})\}$
- (ii) The soft μ - α -closure of F_Z is defined by
 $\mathcal{S}\alpha_{c_\mu}(F_Z) = \bigcap \{F_{\mathcal{W}} : F_Z \subseteq F_{\mathcal{W}} \text{ and } F_{\mathcal{W}} \tilde{\subseteq} \mathcal{S}\mu\text{-}\alpha\mathcal{C}(F_{\tilde{E}})\}$
 (i.e.) $\mathcal{S}\alpha_{i_\mu}(F_Z)$ is the largest soft μ - α -open set contained in F_Z and $\mathcal{S}\alpha_{c_\mu}(F_Z)$ is the smallest soft μ - α -closed set containing F_Z .

Theorem 3.4. Let $(F_{\tilde{E}}, \mu)$ be a $\mathcal{SGTS}$ and F_u, F_v be two soft sets over $F_{\tilde{E}}$. Then

- (i) $F_u \subseteq F_v \Rightarrow \mathcal{S}\alpha_{i_\mu}(F_u) \subseteq \mathcal{S}\alpha_{i_\mu}(F_v)$
- (ii) $F_u \subseteq F_v \Rightarrow \mathcal{S}\alpha_{c_\mu}(F_u) \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$
- (iii) $\mathcal{S}\alpha_{c_\mu}(F_u \cup F_v) = \mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v)$
- (iv) $\mathcal{S}\alpha_{i_\mu}(F_u \cap F_v) = \mathcal{S}\alpha_{i_\mu}(F_u) \cap \mathcal{S}\alpha_{i_\mu}(F_v)$
- (v) $\mathcal{S}\alpha_{c_\mu}(F_u \cap F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u) \cap \mathcal{S}\alpha_{c_\mu}(F_v)$
- (vi) $\mathcal{S}\alpha_{i_\mu}(F_u \cup F_v) \supseteq \mathcal{S}\alpha_{i_\mu}(F_u) \cup \mathcal{S}\alpha_{i_\mu}(F_v)$

Proof. (i) Since F_u is a soft μ -open set, $\mathcal{S}\alpha_{i_\mu}(F_u) \subseteq F_u \subseteq F_v$ which implies $\mathcal{S}\alpha_{i_\mu}(F_u) \subseteq F_v$ and $\mathcal{S}\alpha_{i_\mu}(F_v)$ is the largest soft μ - α -open set contained in F_v . Hence $\mathcal{S}\alpha_{i_\mu}(F_u) \subseteq \mathcal{S}\alpha_{i_\mu}(F_v)$.

(ii) since $F_u \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$ and $F_v \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$, $F_u \subseteq F_v \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$. Hence $F_u \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$. Also, $\mathcal{S}\alpha_{c_\mu}(F_u)$ is the smallest soft μ - α -closed containing F_u . Therefore, $\mathcal{S}\alpha_{c_\mu}(F_u) \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$.

(iii) We have $F_u \subseteq F_u \cup F_v$ and $F_v \subseteq F_u \cup F_v$. From (ii), $\mathcal{S}\alpha_{c_\mu}(F_u) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u \cup F_v)$ and $\mathcal{S}\alpha_{c_\mu}(F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u \cup F_v)$ which implies $\mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u \cup F_v)$. Since $F_u \subseteq \mathcal{S}\alpha_{c_\mu}(F_u)$ and $F_v \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$, $F_u \cup F_v \subseteq \mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v)$. Since $\mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v)$ is a soft μ - α -closed set containing $F_u \cup F_v$, $\mathcal{S}\alpha_{c_\mu}(F_u \cup F_v)$ is the smallest soft μ - α -closed set containing $F_u \cup F_v$. Therefore, $\mathcal{S}\alpha_{c_\mu}(F_u \cup F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v)$. Hence $\mathcal{S}\alpha_{c_\mu}(F_u \cup F_v) = \mathcal{S}\alpha_{c_\mu}(F_u) \cup \mathcal{S}\alpha_{c_\mu}(F_v)$.

Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

- (iv) We have $F_u \tilde{\cap} F_v \subseteq F_u$ and $F_u \tilde{\cap} F_v \subseteq F_v$. From (i), $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u)$ and $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_v)$ which implies $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{i_\mu}(F_v)$. Since $\mathcal{S}\alpha_{i_\mu}(F_u) \subseteq F_u$ and $\mathcal{S}\alpha_{i_\mu}(F_v) \subseteq F_v$, $\mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{i_\mu}(F_v) \subseteq F_u \tilde{\cap} F_v$. As $\mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{i_\mu}(F_v)$ is a soft μ - α -open set contained in $F_u \tilde{\cap} F_v$, $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v)$ is the largest soft μ - α -open set contained in $F_u \tilde{\cap} F_v$. Therefore, $\mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{i_\mu}(F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v)$. Hence $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cap} F_v) = \mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{i_\mu}(F_v)$.
- (v) We have $F_u \tilde{\cap} F_v \subseteq F_u$ and $F_u \tilde{\cap} F_v \subseteq F_v$ which implies $\mathcal{S}\alpha_{c_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u)$ and $\mathcal{S}\alpha_{c_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_v)$. Hence $\mathcal{S}\alpha_{c_\mu}(F_u \tilde{\cap} F_v) \subseteq \mathcal{S}\alpha_{c_\mu}(F_u) \tilde{\cap} \mathcal{S}\alpha_{c_\mu}(F_v)$.
- (vi) We have $F_u \subseteq F_u \tilde{\cup} F_v$ and $F_v \subseteq F_u \tilde{\cup} F_v$ which implies $\mathcal{S}\alpha_{i_\mu}(F_u) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cup} F_v)$ and $\mathcal{S}\alpha_{i_\mu}(F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cup} F_v)$. Hence $\mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cup} \mathcal{S}\alpha_{i_\mu}(F_v) \subseteq \mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cup} F_v)$.
(i.e.) $\mathcal{S}\alpha_{i_\mu}(F_u \tilde{\cup} F_v) \supseteq \mathcal{S}\alpha_{i_\mu}(F_u) \tilde{\cup} \mathcal{S}\alpha_{i_\mu}(F_v)$. □

Theorem 3.5. Let $(F_{\tilde{E}}, \mu)$ be a *SGTS* and F_N be a soft set over $F_{\tilde{E}}$. Then

- (i) F_N is a soft μ - α -closed set.
- (ii) $\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(F_N)^c)) \supseteq (F_N)^c$.
- (iii) $\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(F_N))) \subseteq F_N$.
- (iv) $(F_N)^c$ is a soft μ - α -open set.

Proof. (i) $\Rightarrow$ (ii):

Since F_N is a soft μ - α -closed set, $\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(F_N))) \subseteq F_N$ which implies $(F_N)^c \subseteq \mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(F_N)^c))$.

(i.e.) $\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(F_N)^c)) \supseteq (F_N)^c$.

(ii) $\Rightarrow$ (iii):

Suppose $\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(F_N)^c)) \supseteq (F_N)^c$. Then by taking the complement on both sides, we get $\mathcal{S}\mu\text{-Cl}(\mathcal{S}\mu\text{-Int}(\mathcal{S}\mu\text{-Cl}(F_N))) \subseteq F_N$.

(iii) $\Rightarrow$ (iv):

The result follows from the definition 3.2.

(iv) $\Rightarrow$ (i):

The result follows from the definition 3.1. □

Definition 3.4. Let $(F_{\tilde{E}}, \mu)$ and $(F_{\tilde{K}}, \eta)$ be any two *SGTS*'s. A soft function $\Psi_\chi : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is said to be soft $\alpha_{(\mu, \eta)}$ -continuous (briefly soft (μ, η) -alpha-continuous), if the inverse image of each soft η -open subset in $F_{\tilde{K}}$ is soft μ - α -open in $F_{\tilde{E}}$.

Theorem 3.6. Each soft (μ, η) -continuous is soft $\alpha_{(\mu, \eta)}$ -continuous. But the converse neednot be true.

Proof. Let $\Psi_\chi : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ be a soft (μ, η) -continuous mapping. Let $F_{\mathcal{D}}$ be any soft η -open set in $F_{\tilde{K}}$. since Ψ_χ is soft (μ, η) -continuous, its inverse image $\Psi_\chi^{-1}(F_{\mathcal{D}})$ is a soft μ -open set in $F_{\tilde{E}}$ and every soft μ -open set is soft μ - α -open, $\Psi_\chi^{-1}(F_{\mathcal{D}})$ is soft μ - α -open in $F_{\tilde{E}}$. Hence Ψ_χ is soft $\alpha_{(\mu, \eta)}$ -continuous. $\square$

Example 3.6. Let $\mathcal{L} = \{\ell_1, \ell_2, \ell_3\}$, $\mathcal{P} = \{\hat{p}_1, \hat{p}_2\}$ then $\mu = \{F_\emptyset, F_{\tilde{\mathcal{P}}}, F_{\mathcal{M}_1}, F_{\mathcal{M}_2}, F_{\mathcal{M}_3}, F_{\mathcal{M}_4}, F_{\mathcal{M}_5}, F_{\mathcal{M}_6}\}$ is a $\mathcal{SGTS}$ where
 $F_{\mathcal{M}_1} = \{(\hat{p}_1, \{\mathcal{L}\}), (\hat{p}_2, \{\ell_1, \ell_2\})\}$
 $F_{\mathcal{M}_2} = \{(\hat{p}_1, \{\ell_1, \ell_2\}), (\hat{p}_2, \{\ell_1, \ell_2\})\}$
 $F_{\mathcal{M}_3} = \{(\hat{p}_1, \{\ell_1, \ell_3\}), (\hat{p}_2, \{\ell_1, \ell_2\})\}$
 $F_{\mathcal{M}_4} = \{(\hat{p}_1, \{\ell_1\}), (\hat{p}_2, \{\ell_1\})\}$
 $F_{\mathcal{M}_5} = \{(\hat{p}_1, \{\ell_2\}), (\hat{p}_2, \{\ell_2\})\}$
 $F_{\mathcal{M}_6} = \{(\hat{p}_1, \{\ell_2, \ell_3\}), (\hat{p}_2, \{\ell_1, \ell_2\})\}$
 and

Let $\mathcal{D} = \{d_1, d_2, d_3\}$, $\mathcal{Q} = \{\hat{q}_1, \hat{q}_2\}$ then $\eta = \{F_\emptyset, F_{\tilde{\mathcal{Q}}}, F_{\mathcal{N}_1}, F_{\mathcal{N}_2}, F_{\mathcal{N}_3}, F_{\mathcal{N}_4}, F_{\mathcal{N}_5}\}$ is a $\mathcal{SGTS}$ where

$$\begin{aligned} F_{\mathcal{N}_1} &= \{(\hat{q}_1, \{d_2, d_3\}), (\hat{q}_2, \{d_1, d_3\})\} \\ F_{\mathcal{N}_2} &= \{(\hat{q}_1, \{d_2, d_3\}), (\hat{q}_2, \{d_1, d_2\})\} \\ F_{\mathcal{N}_3} &= \{(\hat{q}_1, \{d_2, d_3\}), (\hat{q}_2, \{\mathcal{D}\})\} \\ F_{\mathcal{N}_4} &= \{(\hat{q}_1, \{\mathcal{D}\}), (\hat{q}_2, \{d_1, d_3\})\} \\ F_{\mathcal{N}_5} &= \{(\hat{q}_1, \{d_3\}), (\hat{q}_2, \{d_3\})\} \end{aligned}$$

A mapping $\Psi : \mathcal{L} \rightarrow \mathcal{D}$ is defined by $\Psi(\ell_1) = d_2$, $\Psi(\ell_2) = d_3$, $\Psi(\ell_3) = d_1$ and $\chi : \mathcal{P} \rightarrow \mathcal{Q}$ by $\chi(p_1) = q_2$, $\chi(p_2) = q_1$. Then $\Psi_\chi : (F_{\tilde{\mathcal{P}}}, \mu) \rightarrow (F_{\tilde{\mathcal{Q}}}, \eta)$ is soft $\alpha_{(\mu, \eta)}$ -continuous. But it is not soft (μ, η) -continuous.

Theorem 3.7. A soft mapping $\Psi_\chi : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is soft $\alpha_{(\mu, \eta)}$ -continuous if the inverse image of each soft η -closed set in $F_{\tilde{K}}$ is soft μ - α -closed in $F_{\tilde{E}}$.

Proof. Let F_I be any soft η -closed set in $F_{\tilde{K}}$. Then $(F_I)^c$ is a soft η -open set in $F_{\tilde{K}}$. Since Ψ_χ is soft $\alpha_{(\mu, \eta)}$ -continuous, $\Psi_\chi^{-1}((F_I)^c)$ is a soft μ - α -open set in $F_{\tilde{E}}$. Hence $(\Psi_\chi^{-1}((F_I)^c))^c$ is a soft μ - α -closed set in $F_{\tilde{E}}$. $\square$

Theorem 3.8. Let $(F_{\tilde{E}}, \mu)$ and $(F_{\tilde{K}}, \eta)$ be two $\mathcal{SGTS}$'s. If $\Psi_\chi : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is a soft $\alpha_{(\mu, \eta)}$ -continuous mapping and $(F_{\mathcal{G}}, \xi)$ is a $\mathcal{SGTSS}$ of $F_{\tilde{E}}$, then the restricted map $\Psi_{\chi|F_{\mathcal{G}}} : (F_{\mathcal{G}}, \xi) \rightarrow (F_{\tilde{K}}, \eta)$ is soft $\alpha_{(\xi, \eta)}$ -continuous.

Proof. Let $(F_{\mathcal{B}_i})$ be any η -open set in F_B . Since Ψ_χ is soft $\alpha_{(\mu, \eta)}$ -continuous, $\Psi_\chi^{-1}(F_{\mathcal{B}_i})$ is a soft μ - α -open set in $F_{\tilde{E}}$. Then $\Psi_{\chi|F_{\mathcal{G}}}^{-1}(F_{\mathcal{B}_i}) = F_{\mathcal{G}} \hat{\cap} \Psi_\chi^{-1}(F_{\mathcal{B}_i})$,

Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

where $F_{\mathcal{G}} \bigcap \Psi_{\chi}^{-1}(F_{\mathcal{B}_i})$ is a soft ξ -open set in $F_{\mathcal{G}}$ which implies $\Psi_{\chi|F_{\mathcal{G}}}^{-1}(F_{\mathcal{B}_i})$ is a soft ξ -open set in $F_{\mathcal{G}}$. Hence $\Psi_{\chi|F_{\mathcal{G}}} : (F_{\mathcal{G}}, \xi) \rightarrow (F_{\tilde{K}}, \eta)$ is soft $\alpha_{(\xi, \eta)}$ -continuous. $\square$

Theorem 3.9. A soft mapping $\Psi_{\chi} : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is soft $\alpha_{(\mu, \eta)}$ -continuous iff for every soft μ -singleton set F_u in $F_{\tilde{E}}$ and every soft η -open set F_v in $F_{\tilde{K}}$ and $\Psi_{\chi}(F_u) \subseteq F_v, \exists$ a soft μ - α -open set $F_{\mathcal{H}}$ in $F_{\tilde{E}} \ni : F_u \subseteq F_{\mathcal{H}}$ and $\Psi_{\chi}(F_{\mathcal{H}}) \subseteq F_v$.

Proof. Let F_v be any soft η -open set in $F_{\tilde{K}}$. Since Ψ_{χ} is soft $\alpha_{(\mu, \eta)}$ -continuous, $\Psi_{\chi}^{-1}(F_v)$ is soft μ - α -open in $F_{\tilde{E}}$. By hypothesis, $\Psi_{\chi}(F_u) \subseteq F_v$ which implies $F_u \subseteq \Psi_{\chi}^{-1}(F_v)$. Take $F_{\mathcal{H}} = \Psi_{\chi}^{-1}(F_v)$ which implies $F_u \subseteq F_{\mathcal{H}}$ and $\Psi_{\chi}(F_{\mathcal{H}}) \subseteq F_v$. Conversely, Let F_v be any soft η -open set in $F_{\tilde{K}}$ and $\Psi_{\chi}(F_u) \subseteq F_v, \exists$ a soft μ - α -open set $F_{\mathcal{H}}$ in $F_{\tilde{E}}$ such that $F_u \subseteq F_{\mathcal{H}}$ and $\Psi_{\chi}(F_{\mathcal{H}}) \subseteq F_v$. Hence $F_u \subseteq F_{\mathcal{H}} \subseteq \Psi_{\chi}^{-1}(F_v) = \bigcup (F_{\mathcal{H}}) \in \mathcal{S}\mu\text{-}\alpha\mathcal{O}(F_{\tilde{E}})$. Hence Ψ_{χ} is soft $\alpha_{(\mu, \eta)}$ -continuous. $\square$

Definition 3.5. Let $(F_{\tilde{E}}, \mu)$ and $(F_{\tilde{K}}, \eta)$ be any two $\mathcal{SGTS}$'s. A soft function $\Psi_{\chi} : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is said to be soft $\rho_{(\mu, \eta)}$ -continuous (briefly soft (μ, η) -pre-continuous), if the inverse image of each soft η -open subset in $F_{\tilde{K}}$ is soft μ -pre-open in $F_{\tilde{E}}$.

Definition 3.6. Let $(F_{\tilde{E}}, \mu)$ and $(F_{\tilde{K}}, \eta)$ be any two $\mathcal{SGTS}$'s. A soft function $\Psi_{\chi} : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ is said to be soft $\delta_{(\mu, \eta)}$ -continuous (briefly soft (μ, η) -semi-continuous), if the inverse image of each soft η -open subset in $F_{\tilde{K}}$ is soft μ -semi-open in $F_{\tilde{E}}$.

Remark 3.5. Every soft $\alpha_{(\mu, \eta)}$ -continuous is soft $\rho_{(\mu, \eta)}$ -continuous and soft $\delta_{(\mu, \eta)}$ -continuous. But the converse neednot be true.

Example 3.7. Let $\mathcal{L} = \{\ell_1, \ell_2, \ell_3\}, \mathcal{P} = \{\hat{p}_1, \hat{p}_2\}$ then $\mu = \{F_{\emptyset}, F_{\mathcal{P}}, F_{\mathcal{M}_1}, F_{\mathcal{M}_2}, F_{\mathcal{M}_3}, F_{\mathcal{M}_4}, F_{\mathcal{M}_5}, F_{\mathcal{M}_6}\}$ is a $\mathcal{SGTS}$ where

$$\begin{aligned} F_{\mathcal{M}_1} &= \{(\hat{p}_1, \{\mathcal{L}\}), (\hat{p}_2, \{\ell_1, \ell_2\})\} \\ F_{\mathcal{M}_2} &= \{(\hat{p}_1, \{\ell_1, \ell_2\}), (\hat{p}_2, \{\ell_1, \ell_2\})\} \\ F_{\mathcal{M}_3} &= \{(\hat{p}_1, \{\ell_1, \ell_3\}), (\hat{p}_2, \{\ell_1, \ell_2\})\} \\ F_{\mathcal{M}_4} &= \{(\hat{p}_1, \{\ell_1\}), (\hat{p}_2, \{\ell_1\})\} \\ F_{\mathcal{M}_5} &= \{(\hat{p}_1, \{\ell_2\}), (\hat{p}_2, \{\ell_2\})\} \\ F_{\mathcal{M}_6} &= \{(\hat{p}_1, \{\ell_2, \ell_3\}), (\hat{p}_2, \{\ell_1, \ell_2\})\} \end{aligned}$$

and

Let $\mathcal{D} = \{d_1, d_2, d_3\}, \mathcal{Q} = \{\hat{q}_1, \hat{q}_2\}$ then $\eta = \{F_{\emptyset}, F_{\mathcal{Q}}, F_{\mathcal{N}_1}, F_{\mathcal{N}_2}, F_{\mathcal{N}_3}, F_{\mathcal{N}_4}\}$ is a $\mathcal{SGTS}$ where

$$\begin{aligned} F_{\mathcal{N}_1} &= \{(\hat{q}_1, \{d_2\}), (\hat{q}_2, \{d_1, d_3\})\} \\ F_{\mathcal{N}_2} &= \{(\hat{q}_1, \{d_1, d_2\}), (\hat{q}_2, \{d_2, d_3\})\} \\ F_{\mathcal{N}_3} &= \{(\hat{q}_1, \{d_1, d_2\}), (\hat{q}_2, \{\mathcal{D}\})\} \end{aligned}$$

$$F_{\mathcal{N}_4} = \{(\hat{q}_1, \{d_2, d_3\}), (\hat{q}_2, \{d_1, d_3\})\}$$

A mapping $\Psi : \mathcal{L} \rightarrow \mathcal{D}$ is defined by $\Psi(\ell_1) = d_2$, $\Psi(\ell_2) = d_3$, $\Psi(\ell_3) = d_1$ and $\chi : \mathcal{P} \rightarrow \mathcal{Q}$ by $\chi(p_1) = q_2$, $\chi(p_2) = q_1$. Then $\Psi_\chi : (F_{\tilde{\mathcal{P}}}, \mu) \rightarrow (F_{\tilde{\mathcal{Q}}}, \eta)$ is soft $\rho_{(\mu, \eta)}$ -continuous. But it is not soft $\alpha_{(\mu, \eta)}$ -continuous.

Example 3.8. Let $\mathcal{V} = \{\nu_1, \nu_2, \nu_3\}$, $\mathcal{E} = \{\hat{e}_1, \hat{e}_2\}$ then $\mu = \{F_\emptyset, F_{\tilde{\mathcal{E}}}, F_{\mathcal{A}_1}, F_{\mathcal{A}_2}, F_{\mathcal{A}_3}, F_{\mathcal{A}_4}, F_{\mathcal{A}_5}, F_{\mathcal{A}_6}\}$ is a $\mathcal{SGTS}$ where
 $F_{\mathcal{A}_1} = \{(\hat{e}_1, \{\mathcal{V}\}), (\hat{e}_2, \{\nu_1, \nu_2\})\}$
 $F_{\mathcal{A}_2} = \{(\hat{e}_1, \{\nu_1, \nu_2\}), (\hat{e}_2, \{\nu_1, \nu_2\})\}$
 $F_{\mathcal{A}_3} = \{(\hat{e}_1, \{\nu_1, \nu_3\}), (\hat{e}_2, \{\nu_1, \nu_2\})\}$
 $F_{\mathcal{A}_4} = \{(\hat{e}_1, \{\nu_1\}), (\hat{e}_2, \{\nu_1\})\}$
 $F_{\mathcal{A}_5} = \{(\hat{e}_1, \{\nu_2\}), (\hat{e}_2, \{\nu_2\})\}$
 $F_{\mathcal{A}_6} = \{(\hat{e}_1, \{\nu_2, \nu_3\}), (\hat{e}_2, \{\nu_1, \nu_2\})\}$
 and

Let $\mathcal{W} = \{w_1, w_2, w_3\}$, $\mathcal{Z} = \{\hat{z}_1, \hat{z}_2\}$ then $\eta = \{F_\emptyset, F_{\tilde{\mathcal{Z}}}, F_{\mathcal{B}_1}, F_{\mathcal{B}_2}, F_{\mathcal{B}_3}, F_{\mathcal{B}_4}\}$ is a $\mathcal{SGTS}$ where

$$\begin{aligned} F_{\mathcal{B}_1} &= \{(\hat{z}_1, \{w_2, w_3\}), (\hat{z}_2, \{\mathcal{W}\})\} \\ F_{\mathcal{B}_2} &= \{(\hat{z}_1, \{w_2, w_3\}), (\hat{z}_2, \{w_2, w_3\})\} \\ F_{\mathcal{B}_3} &= \{(\hat{z}_1, \{w_3\}), (\hat{z}_2, \{w_1, w_3\})\} \\ F_{\mathcal{B}_4} &= \{(\hat{z}_1, \{w_1, w_3\}), (\hat{z}_2, \{w_1, w_3\})\} \end{aligned}$$

A mapping $\Psi : \mathcal{V} \rightarrow \mathcal{W}$ is defined by $\Psi(\nu_1) = w_2$, $\Psi(\nu_2) = w_3$, $\Psi(\nu_3) = w_1$ and $\chi : \mathcal{E} \rightarrow \mathcal{Z}$ by $\chi(e_1) = z_2$, $\chi(e_2) = z_1$. Then $\Psi_\chi : (F_{\tilde{\mathcal{E}}}, \mu) \rightarrow (F_{\tilde{\mathcal{Z}}}, \eta)$ is soft $\delta_{(\mu, \eta)}$ -continuous. But it is not soft $\alpha_{(\mu, \eta)}$ -continuous.

Remark 3.6. The composition of two soft $\alpha_{(\mu, \eta)}$ -continuous function neednot be soft $\alpha_{(\mu, \eta)}$ -continuous.

Theorem 3.10. Let $\Phi_\zeta : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ be a soft $\alpha_{(\mu, \eta)}$ -continuous function and $\Psi_\chi : (F_{\tilde{K}}, \eta) \rightarrow (F_{\tilde{P}}, \xi)$ be a soft (η, ξ) -continuous map. Then $\Psi_\chi \circ \Phi_\zeta : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{P}}, \xi)$ is a soft $\alpha_{(\mu, \xi)}$ -continuous map.

Proof. Let $F_{\mathcal{C}_i}$ be any soft ξ -open subset of $F_{\tilde{P}}$. Since Ψ_χ is soft (η, ξ) -continuous, $\Psi_\chi^{-1}(F_{\mathcal{C}_i})$ is a soft η -open subset of $F_{\tilde{K}}$. Also, we have Φ_ζ is $\alpha_{(\mu, \eta)}$ -continuous, $\Phi_\zeta^{-1}(\Psi_\chi^{-1}(F_{\mathcal{C}_i}))$ is a soft μ - α -open set in $F_{\tilde{E}}$. Hence $\Psi_\chi \circ \Phi_\zeta$ is a soft $\alpha_{(\mu, \xi)}$ -continuous map. $\square$

4 Soft μ - β -open sets

Definition 4.1. [9] Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ - β -open set iff $F_G \subset c_\mu i_\mu c_\mu F_G$ (i.e, the case when $\pi = c_\mu i_\mu c_\mu$). The class of all soft μ - β -open sets is denoted by $\beta_{(\mu)}$ or β_μ .

Definition 4.2. Let $(F_{\tilde{E}}, \mu)$ be a SGTS. Then a soft set $F_G \subset F_{\tilde{E}}$ is said to be a soft μ - β -closed set iff its complement is a soft μ - β -open set.

Example 4.1. In the above example 3.3, $\mathcal{S}\mu\text{-}\beta\mathcal{O}(F_{\tilde{E}}) = \{F_\emptyset, F_{\tilde{E}}, \{(\hat{x}_1, \{\kappa_2\}), (\hat{x}_2, \{\kappa_3\})\}, \{(\hat{x}_1, \{\kappa_3\}), (\hat{x}_2, \{\kappa_1\})\}, \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_3\})\}, \{(\hat{x}_1, \{\emptyset\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}, \{(\hat{x}_1, \{\emptyset\}), (\hat{x}_2, \{\kappa_2, \kappa_3\})\}, \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\emptyset\})\}, \{(\hat{x}_1, \{\kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_3\})\}, \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_2, \kappa_3\})\}, \{(\hat{x}_1, \{\kappa_2\}), (\hat{x}_2, \{\kappa_2, \kappa_3\})\}\}$

Remark 4.1. Both $F_\emptyset$ and $F_{\tilde{E}}$ are soft μ - β -open and soft μ - β -closed.

Theorem 4.1. Every soft μ -open set (soft μ -closed) is soft μ - β -open (soft μ - β -closed). But the converse need not be true.

Example 4.2. In the above example 4.3, $\{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\emptyset\})\}, \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_2, \kappa_3\})\}$ are soft μ - β -open sets but not soft μ -open.

Theorem 4.2. Arbitrary soft union of soft μ - β -open sets is a soft μ - β -open set.

Proof. Let $\{F_{A_i} / i \in I\}$ be a collection of soft μ - β -open sets in a SGTS $(F_{\tilde{E}}, \mu)$.

Then by the definition of soft μ - β -open set,

$$F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_{A_i}))), \forall i.$$

$$\text{Now, } \widetilde{\bigcup} F_{A_i} \subseteq \widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (ii), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (i), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\widetilde{\bigcup} \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_{A_i}))), \forall i.$$

$$\text{By remark 2.13 (ii), } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\widetilde{\bigcup} F_{A_i}))), \forall i.$$

$$\text{Hence } \widetilde{\bigcup} F_{A_i} \subseteq \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}\mathcal{N}\mathcal{T}(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\widetilde{\bigcup} F_{A_i}))), \forall i. \quad \square$$

Remark 4.2. Finite soft intersection of soft μ - β -open sets need not be a soft μ - β -open set.

Example 4.3. In the above example 4.3, $F_{\mathcal{C}} = \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_3\})\}$ and $F_{\mathcal{D}} = \{(\hat{x}_1, \{\kappa_1, \kappa_3\}), (\hat{x}_2, \{\kappa_2, \kappa_3\})\}$ are soft μ - β -open sets but $F_{\mathcal{C}} \cap F_{\mathcal{D}} = \{(\hat{x}_1, \{\kappa_3\}), (\hat{x}_2, \{\kappa_3\})\}$ is not soft μ - β -open.

Theorem 4.3. Arbitrary soft intersection of soft μ - β -closed sets is a soft μ - β -closed set.

Proof. The proof is similar to that of theorem 4.7 by taking its complements. $\square$

Remark 4.3. Finite soft union of soft μ - β -closed sets need not be a soft μ - β -closed set.

Example 4.4. In example 4.3, $F_S = \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_2\})\}$ and $F_T = \{(\hat{x}_1, \{\kappa_3\}), (\hat{x}_2, \{\kappa_1\})\}$ are soft μ - β -closed sets but $F_S \tilde{\cup} F_T = \{(\hat{x}_1, \{\kappa_2, \kappa_3\}), (\hat{x}_2, \{\kappa_1, \kappa_2\})\}$ is not soft μ - β -closed.

Definition 4.3. Let $(F_{\tilde{E}}, \mu)$ be a $\mathcal{SGTS}$ and $F_Z \subseteq F_{\tilde{E}}$.

- (i) The soft μ - β -interior of F_Z is defined by
 $\mathcal{S}\beta_{i_\mu}(F_Z) = \tilde{\cup} \{F_W : F_W \subseteq F_Z \text{ and } F_W \tilde{\in} \mathcal{S}\mu\text{-}\beta\mathcal{O}(F_{\tilde{E}})\}$
- (ii) The soft μ - β -closure of F_Z is defined by
 $\mathcal{S}\beta_{c_\mu}(F_Z) = \tilde{\cap} \{F_W : F_Z \subseteq F_W \text{ and } F_W \tilde{\in} \mathcal{S}\mu\text{-}\beta\mathcal{C}(F_{\tilde{E}})\}$
 (i.e.) $\mathcal{S}\beta_{i_\mu}(F_Z)$ is the largest soft μ - β -open set contained in F_Z and $\mathcal{S}\beta_{c_\mu}(F_Z)$ is the smallest soft μ - β -closed set containing F_Z .

Theorem 4.4. Let $(F_{\tilde{E}}, \mu)$ be a $\mathcal{SGTS}$ and $F_Q, F_{\mathcal{R}}$ be two soft sets over $F_{\tilde{E}}$. Then

- (i) $F_Q \subseteq F_{\mathcal{R}} \Rightarrow \mathcal{S}\beta_{i_\mu}(F_Q) \subseteq \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$
- (ii) $F_Q \subseteq F_{\mathcal{R}} \Rightarrow \mathcal{S}\beta_{c_\mu}(F_Q) \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$
- (iii) $\mathcal{S}\beta_{c_\mu}(F_Q \tilde{\cup} F_{\mathcal{R}}) = \mathcal{S}\beta_{c_\mu}(F_Q) \tilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$
- (iv) $\mathcal{S}\beta_{i_\mu}(F_Q \tilde{\cap} F_{\mathcal{R}}) = \mathcal{S}\beta_{i_\mu}(F_Q) \tilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$
- (v) $\mathcal{S}\beta_{c_\mu}(F_Q \tilde{\cap} F_{\mathcal{R}}) \subseteq \mathcal{S}\beta_{c_\mu}(F_Q) \tilde{\cap} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$
- (vi) $\mathcal{S}\beta_{i_\mu}(F_Q \tilde{\cup} F_{\mathcal{R}}) \supseteq \mathcal{S}\beta_{i_\mu}(F_Q) \tilde{\cup} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$

Proof. (i) Since F_Q is a soft μ -open set, $\mathcal{S}\beta_{i_\mu}(F_Q) \subseteq F_Q \subseteq F_{\mathcal{R}}$ which implies $\mathcal{S}\beta_{i_\mu}(F_Q) \subseteq F_{\mathcal{R}}$ and $\mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$ is the largest soft μ - β -open set contained in $F_{\mathcal{R}}$. Hence $\mathcal{S}\beta_{i_\mu}(F_Q) \subseteq \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$.

(ii) since $F_Q \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$ and $F_{\mathcal{R}} \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$, $F_Q \subseteq F_{\mathcal{R}} \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$. Hence $F_Q \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$ and $\mathcal{S}\beta_{c_\mu}(F_Q)$ is the smallest soft μ - β -closed containing F_Q . Therefore, $\mathcal{S}\beta_{c_\mu}(F_Q) \subseteq \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$.

(iii) We have $F_Q \subseteq F_Q \tilde{\cup} F_{\mathcal{R}}$ and $F_{\mathcal{R}} \subseteq F_Q \tilde{\cup} F_{\mathcal{R}}$. From (ii), $\mathcal{S}\beta_{c_\mu}(F_Q) \subseteq \mathcal{S}\beta_{c_\mu}(F_Q \tilde{\cup} F_{\mathcal{R}})$ and $\mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}}) \subseteq \mathcal{S}\beta_{c_\mu}(F_Q \tilde{\cup} F_{\mathcal{R}})$ which

implies $\mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}})$. Since $F_Q \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q)$ and $F_{\mathcal{R}} \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$, $F_Q \widetilde{\cup} F_{\mathcal{R}} \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$. As $\mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$ is a soft μ - β -closed set containing $F_Q \widetilde{\cup} F_{\mathcal{R}}$, $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}})$ is the smallest soft μ - β -closed set containing $F_Q \widetilde{\cup} F_{\mathcal{R}}$. Therefore, $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$ which implies $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}}) = \mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$.

(iv) We have $F_Q \widetilde{\cap} F_{\mathcal{R}} \widetilde{\subseteq} F_Q$ and $F_Q \widetilde{\cap} F_{\mathcal{R}} \widetilde{\subseteq} F_{\mathcal{R}}$. From (i), $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q)$ and $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$ which implies $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$. Since $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\subseteq} F_Q$ and $\mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} F_{\mathcal{R}}$, $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} F_Q \widetilde{\cap} F_{\mathcal{R}}$. As $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$ is a soft μ - β -open set contained in $F_Q \widetilde{\cap} F_{\mathcal{R}}$, $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}})$ is the largest soft μ - β -open set contained in $F_Q \widetilde{\cap} F_{\mathcal{R}}$. Therefore, $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}})$. Hence $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) = \mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$.

(v) We have $F_Q \widetilde{\cap} F_{\mathcal{R}} \widetilde{\subseteq} F_Q$ and $F_Q \widetilde{\cap} F_{\mathcal{R}} \widetilde{\subseteq} F_{\mathcal{R}}$ which implies $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q)$ and $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$. Hence $\mathcal{S}\beta_{c_\mu}(F_Q \widetilde{\cap} F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{c_\mu}(F_Q) \widetilde{\cap} \mathcal{S}\beta_{c_\mu}(F_{\mathcal{R}})$.

(vi) We have $F_Q \widetilde{\subseteq} F_Q \widetilde{\cup} F_{\mathcal{R}}$ and $F_{\mathcal{R}} \widetilde{\subseteq} F_Q \widetilde{\cup} F_{\mathcal{R}}$ which implies $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}})$ and $\mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}})$. Hence $\mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}}) \widetilde{\subseteq} \mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}})$.

(i.e.) $\mathcal{S}\beta_{i_\mu}(F_Q \widetilde{\cup} F_{\mathcal{R}}) \widetilde{\supseteq} \mathcal{S}\beta_{i_\mu}(F_Q) \widetilde{\cup} \mathcal{S}\beta_{i_\mu}(F_{\mathcal{R}})$. □

Theorem 4.5. Let $(F_{\widetilde{E}}, \mu)$ be a $\mathcal{SGTS}$ and $F_{\mathcal{V}}$ be a soft set over $F_{\widetilde{E}}$. Then

- (i) $F_{\mathcal{V}}$ is a soft μ - β -closed set.
- (ii) $\mathcal{S}\mu\text{-}\mathcal{CL}(\mathcal{S}\mu\text{-}\mathcal{Int}(\mathcal{S}\mu\text{-}\mathcal{CL}(F_{\mathcal{V}})^c)) \widetilde{\supseteq} (F_{\mathcal{V}})^c$.
- (iii) $\mathcal{S}\mu\text{-}\mathcal{Int}(\mathcal{S}\mu\text{-}\mathcal{CL}(\mathcal{S}\mu\text{-}\mathcal{Int}(F_{\mathcal{V}}))) \widetilde{\subseteq} F_{\mathcal{V}}$.
- (iv) $(F_{\mathcal{V}})^c$ is a soft μ - β -open set.

Definition 4.4. Let $(F_{\widetilde{E}}, \mu)$ and $(F_{\widetilde{K}}, \eta)$ be any two $\mathcal{SGTS}$'s. A soft function $\Psi_{\chi} : (F_{\widetilde{E}}, \mu) \rightarrow (F_{\widetilde{K}}, \eta)$ is said to be soft $\beta_{(\mu, \eta)}$ -continuous (briefly, soft (μ, η) -beta-continuous), if the inverse image of each soft η -open subset in $F_{\widetilde{K}}$ is soft μ - β -open in $F_{\widetilde{E}}$.

Theorem 4.6. Each soft (μ, η) -continuous is soft $\beta_{(\mu, \eta)}$ -continuous.

Proof. Let $\Psi_{\chi} : (F_{\widetilde{E}}, \mu) \rightarrow (F_{\widetilde{K}}, \eta)$ be a soft (μ, η) -continuous mapping. Let $F_{\mathcal{W}}$ be any soft η -open set in $F_{\widetilde{K}}$. since Ψ_{χ} is soft (μ, η) -continuous, its inverse image $\Psi_{\chi}^{-1}(F_{\mathcal{W}})$ is a soft μ -open set in $F_{\widetilde{E}}$ and every soft μ -open set is soft μ - β -open, $\Psi_{\chi}^{-1}(F_{\mathcal{W}})$ is soft μ - β -open in $F_{\widetilde{E}}$. Hence Ψ_{χ} is soft $\beta_{(\mu, \eta)}$ -continuous. □

Example 4.5. Let $\mathcal{R} = \{r_1, r_2, r_3\}$, $\mathcal{E} = \{\hat{t}_1, \hat{t}_2\}$, then $\mu = \{F_\emptyset, F_{\tilde{\mathcal{E}}}, F_{\mathfrak{A}_1}, F_{\mathfrak{A}_2}, F_{\mathfrak{A}_3}, F_{\mathfrak{A}_4}, F_{\mathfrak{A}_5}, F_{\mathfrak{A}_6}\}$ is a $\mathcal{SGTS}$ where

$$F_{\mathfrak{A}_1} = \{(\hat{t}_1, \{r_1, r_2\}), (\hat{t}_2, \{r_1, r_2\})\}$$

$$F_{\mathfrak{A}_2} = \{(\hat{t}_1, \{r_2, r_3\}), (\hat{t}_2, \{r_1, r_2\})\}$$

$$F_{\mathfrak{A}_3} = \{(\hat{t}_1, \{r_2\}), (\hat{t}_2, \{r_2\})\}$$

$$F_{\mathfrak{A}_4} = \{(\hat{t}_1, \{r_1\}), (\hat{t}_2, \{r_1\})\}$$

$$F_{\mathfrak{A}_5} = \{(\hat{t}_1, \{\mathcal{R}\}), (\hat{t}_2, \{r_1, r_2\})\}$$

$$F_{\mathfrak{A}_6} = \{(\hat{t}_1, \{r_1, r_3\}), (\hat{t}_2, \{r_1, r_2\})\}$$

and

Let $\mathcal{H} = \{h_1, h_2, h_3\}$, $\mathcal{K} = \{\hat{i}_1, \hat{i}_2\}$, then $\eta = \{F_\emptyset, F_{\tilde{\mathcal{K}}}, F_{\mathfrak{B}_1}, F_{\mathfrak{B}_2}, F_{\mathfrak{B}_3}, F_{\mathfrak{B}_4}\}$ is a $\mathcal{SGTS}$ where

$$F_{\mathfrak{B}_1} = \{(\hat{i}_1, \{h_1, h_2\}), (\hat{i}_2, \{\mathcal{H}\})\}$$

$$F_{\mathfrak{B}_2} = \{(\hat{i}_1, \{h_1, h_3\}), (\hat{i}_2, \{h_3\})\}$$

$$F_{\mathfrak{B}_3} = \{(\hat{i}_1, \{h_1, h_2\}), (\hat{i}_2, \{h_1, h_2\})\}$$

$$F_{\mathfrak{B}_4} = \{(\hat{i}_1, \{h_1\}), (\hat{i}_2, \{h_3\})\}$$

A function $\Psi : \mathcal{R} \rightarrow \mathcal{H}$ is defined by $\Psi(r_1) = h_2$, $\Psi(r_2) = h_3$, $\Psi(r_3) = h_1$ and $\chi : \mathcal{E} \rightarrow \mathcal{K}$ by $\psi(t_1) = \hat{i}_2$, $\chi(t_2) = \hat{i}_1$. Then $\Psi_\chi : (F_{\tilde{\mathcal{E}}}, \mu) \rightarrow (F_{\tilde{\mathcal{K}}}, \eta)$ is soft $\beta_{(\mu, \eta)}$ -continuous. But it is not soft (μ, η) -continuous.

Remark 4.4. The composition of two soft $\beta_{(\mu, \eta)}$ -continuous function neednot be soft $\beta_{(\mu, \eta)}$ -continuous.

Theorem 4.7. Let $\Phi_\zeta : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{K}}, \eta)$ be a soft $\beta_{(\mu, \eta)}$ -continuous function and $\Psi_\chi : (F_{\tilde{K}}, \eta) \rightarrow (F_{\tilde{P}}, \xi)$ be a soft (η, ξ) -continuous map. Then $\Psi_\chi \circ \Phi_\zeta : (F_{\tilde{E}}, \mu) \rightarrow (F_{\tilde{P}}, \xi)$ is a soft $\beta_{(\mu, \xi)}$ -continuous map.

Proof. Let $F_{\mathcal{C}_i}$ be any soft ξ -open subset of $F_{\tilde{P}}$. Since Ψ_χ is soft (η, ξ) -continuous, $\Psi_\chi^{-1}(F_{\mathcal{C}_i})$ is a soft η -open subset of $F_{\tilde{K}}$. Also, we have Φ_ζ is $\beta_{(\mu, \eta)}$ -continuous, $\Phi_\zeta^{-1}(\Psi_\chi^{-1}(F_{\mathcal{C}_i}))$ is a soft μ - β -open set in $F_{\tilde{E}}$. Hence $\Psi_\chi \circ \Phi_\zeta$ is a soft $\beta_{(\mu, \xi)}$ -continuous map. $\square$

Theorem 4.8. Any soft μ - β -open set which is soft μ -semi-closed is soft μ -semi-open.

Proof. Let F_A be any soft μ - β -open set. Then $F_A \tilde{\subseteq} \mathcal{S}\mu\text{-}\beta\mathcal{O}(F_{\tilde{E}})$ which implies $F_A \tilde{\subseteq} \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_A)))$. Since F_A is a soft μ -semi-closed set, $F_A \tilde{\subseteq} \mathcal{S}\mu\text{-}\mathcal{S}\mathcal{C}(F_{\tilde{E}})$ which implies $\mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_A)) \tilde{\subseteq} F_A$. Now, $\mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_A)) \tilde{\subseteq} F_A \tilde{\subseteq} \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(\mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_A)))$. Since $\mathcal{S}\mu\text{-}\mathcal{I}nt(\mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_A)) = F_{\mathcal{H}}$ is a soft μ -open set, $F_{\mathcal{H}} \tilde{\subseteq} F_A \tilde{\subseteq} \mathcal{S}\mu\text{-}\mathcal{C}\mathcal{I}(F_{\mathcal{H}})$. Therefore F_A is a soft μ -semi-open set. $\square$

Corollary 4.1. Any soft set which is both soft μ - β -closed and soft μ -semi-open is soft μ -semi-closed.

Theorem 4.9. The soft intersection of a soft μ - α -open set and a soft μ - β -open set is a soft μ - β -open set.

Proof. Let $F_{\mathfrak{A}}$ be a soft μ - α -open set and $F_{\mathfrak{B}}$ be a soft μ - β -open set. Then

$$\begin{aligned} F_{\mathfrak{A}} \widetilde{\cap} F_{\mathfrak{B}} &\widetilde{\subseteq} \mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}}))) \widetilde{\cap} \mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{B}}))) \\ &\widetilde{\subseteq} \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}}))) \widetilde{\cap} \mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{B}}))] \\ &= \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} [\mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}})) \widetilde{\cap} \mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{B}}))] \\ &\widetilde{\subseteq} \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} [\mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}}) \widetilde{\cap} \mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{B}})]] \\ &= \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} [\mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}}) \widetilde{\cap} \mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{B}})]] \\ &\widetilde{\subseteq} \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} [\mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} (F_{\mathfrak{A}}) \widetilde{\cap} (F_{\mathfrak{B}})]] \\ &\widetilde{\subseteq} \mathfrak{S}\mu\text{-Cl} [\mathfrak{S}\mu\text{-Int} [\mathfrak{S}\mu\text{-Cl} [(F_{\mathfrak{A}}) \widetilde{\cap} (F_{\mathfrak{B}})]]]. \end{aligned}$$

Hence $F_{\mathfrak{A}} \widetilde{\cap} F_{\mathfrak{B}}$ is a soft μ - β -open set. □

Corollary 4.2. The soft union of a soft μ - α -closed set and a soft μ - β -closed set is a soft μ - β -closed set.

Theorem 4.10. Every soft μ - β -open set which is soft μ - α -closed is a soft μ -closed set.

Proof. Let $F_{\mathfrak{M}}$ be any soft μ - β -open set. Then $F_{\mathfrak{M}} \widetilde{\subseteq} \mathfrak{S}\mu\text{-}\beta\mathcal{O} (F_{\mathfrak{M}}^-)$ which implies $F_{\mathfrak{M}} \widetilde{\subseteq} \mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{M}})))$. Since $F_{\mathfrak{M}}$ is a soft μ - α -closed set, $\mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{M}}))) \subseteq F_{\mathfrak{M}}$ which implies $F_{\mathfrak{M}} = \mathfrak{S}\mu\text{-Cl} (\mathfrak{S}\mu\text{-Int} (\mathfrak{S}\mu\text{-Cl} (F_{\mathfrak{M}})))$. Hence $F_{\mathfrak{M}}$ is soft μ -closed. □

Corollary 4.3. Every soft μ - β -closed set which is soft μ - α -open is a soft μ -open set.

5 Conclusion

In the current work, we study some of the properties of soft μ - α -open sets, soft μ - α -closed sets, soft μ - β -open sets and soft μ - β -closed sets in soft generalized topological spaces using soft μ - α -interior, soft μ - α -closure, soft μ - β -interior and soft μ - β -closure respectively. We discuss the soft continuous functions related to soft μ -open sets and determine the necessary and sufficient condition for the composition mappings in soft μ -open sets. Also, we examine the inter-relationship between soft (μ, η) -continuous, soft $\rho_{(\mu, \eta)}$ -continuous, soft $\delta_{(\mu, \eta)}$ -continuous, soft $\alpha_{(\mu, \eta)}$ -continuous and soft $\beta_{(\mu, \eta)}$ -continuous mappings with suitable examples.

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Some Aspects of Soft μ - α -open sets and Soft μ - β -open sets in Soft Generalized Topological Spaces

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Value distribution of meromorphic functions whose differential polynomials share a small function

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Abstract

In this article, we study the uniqueness of differential polynomials $\mathcal{P}(f) = f_1^p P(f_1)$ and $P[f]$ generated by meromorphic functions f and g respectively sharing a small function. Our results generalises the result due to Harina P. Waghmare and Husna V. [7].

Keywords: Differential Polynomial, Small Function, Meromorphic Function etc.,
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1 Introduction

Throughout this paper, we use the standard notations of the Nevanlinna Theory of meromorphic function as explained in (see [8, 9, 10]). Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a -points with the same multiplicities, then we say that f and g share the value a counting multiplicities (CM) and if we do not consider the multiplicities, then f and g are said to share the value a ignoring multiplicities (IM). If $a = \infty$, then the zeros of $f - a$ means the poles of f . A meromorphic function $a = a(z)$ ($\not\equiv 0, \infty$) is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $r \rightarrow \infty$, $r \notin E$, where E is a set of positive real numbers with finite Lebesgue measure. If $a = a(z)$ is a small function, then we say that f and g share a IM (resp. CM) according to $f - a$ and $g - a$ share 0 IM (resp. CM).

In 2008, Zhang and Lü [14] obtained the following result.

Theorem 1.1. *be (see [14]) Let k, n be the positive integers, f be a non-constant meromorphic function, and a ($\not\equiv 0, \infty$) be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$, as $r \rightarrow \infty$. If f^n and $f^{(k)}$ share a IM and*

$$(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n$$

then $f^n \equiv f^{(k)}$.

In the same paper T. Zhang and W. Lü [14] asked the following question:

Question 1. What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function a ($\not\equiv 0, \infty$) satisfying $T(r, a) = o\{T(r, f)\}$ as $r \rightarrow \infty$?

In 2013, S. S. Bhoosnurmath and Kabbur [15] proved the following result.

Theorem 1.2. *(see [15]) Let f be a non-constant meromorphic function and a ($\not\equiv 0, \infty$) be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$, as $r \rightarrow \infty$. Let $P[f]$ be a non-constant differential polynomial in f . If f and $P[f]$ share a IM and*

$$(2Q + 6)\Theta(\infty, f) + (2 + 3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

or if f and $P[f]$ share a CM and

$$3\Theta(\infty, f) + (\underline{d}(P) + 1)\delta(0, f) > 4,$$

then $f \equiv P[f]$.

Value distribution of meromorphic functions whose differential polynomials share a small function

Banerjee and Majumder [16] considered the weighted sharing of f^n and $(f^m)^{(k)}$ and proved the following result.

Theorem 1.3. (see [16]) *Let f be a non-constant meromorphic function, $k, n, m \in \mathbb{N}$ and l be a non-negative integer. Suppose $a (\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$ as $r \rightarrow \infty$ such that f^n and $(f^m)^{(k)}$ share (a, l) . If $l \geq 2$ and*

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k+7-n,$$

or $l = 1$ and

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \left(k + \frac{9}{2}\right)\Theta(0, f) > 2k+8-n,$$

or $l = 0$ and

$$(2k+6)\Theta(\infty, f) + (2k+7)\Theta(0, f) > 4k+13-n,$$

then $f \equiv (f^m)^{(k)}$.

In 2015, Kuldeep S. Charak and Banarasi Lal [17] proved the following result.

Theorem 1.4. (see [17]) *Let f be a non-constant meromorphic function, n be a positive integer and $a (\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$ as $r \rightarrow \infty$. Let $P[f]$ be a non-constant differential polynomial in f . Suppose f^n and $P[f]$ share (a, l) such that any one of the following holds.*

(i) *when $l \geq 2$ and*

$$(Q+3)\Theta(\infty, f) + 2\Theta(0, f) + \bar{d}(P)\delta(0, f) > Q+5+2\bar{d}(P) - \underline{d}(P) - n,$$

(ii) *when $l = 1$ and*

$$\left(Q + \frac{7}{2}\right)\Theta(\infty, f) + \frac{5}{2}\Theta(0, f) + \bar{d}(P)\delta(0, f) > Q+6+2\bar{d}(P) - \underline{d}(P) - n,$$

(iii) *when $l = 0$ and*

$$(2Q+6)\Theta(\infty, f) + 4\Theta(0, f) + 2\bar{d}(P)\delta(0, f) > 2Q+10+4\bar{d}(P) - 2\underline{d}(P) - n.$$

Then $f^n \equiv P[f]$.

In 2017, H. P. Waghmare and Husna V. [7] proved the following.

Theorem 1.5. (see [7]) Let $k (\geq 1)$, $n (\geq 1)$, $p (\geq 1)$ and $m (\geq 0)$ be integers and f and $f_1 = f - \omega_p$ be two non-constant meromorphic functions and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \dots + a_nz^n + \dots + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a(z) (\neq 0, \infty)$ be a small function with respect to f . Suppose $\mathcal{P}(f) - a$ and $M[f] - a$ share $(0, l)$. If $l \geq 2$ and

$$(3 + 2\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + 2\mu_2 + 3 - p,$$

or $l = 1$

$$\left(\frac{7}{2} + 2\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + \mu_2 + 4 + \frac{(m+n) - 3p}{2},$$

or $l = 0$ and

$$(6+3\lambda)\Theta(\infty, f) + 2\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 3d_M\delta_{1+k}(0, f) > 3\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p$$

then $\mathcal{P}(f) \equiv M[f]$.

Question 2. What happens if differential monomial $M[f]$ is replaced by differential polynomial $P[f]$ in Theorem 1.5 ?

To answer the above question, we prove one Theorem which improves and generalizes Theorem 1.5.

2 Some Definitions

Definition 2.1. For any constant a , we define the quantities

$$\Theta(a, f) = 1 - \lim_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

$$\delta(a, f) = 1 - \lim_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)}.$$

Definition 2.2. (see [13]) We denote by $N_k\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f - a$ with multiplicity $\leq k$, and by $\overline{N}_k\left(r, \frac{1}{f-a}\right)$ the corresponding one

Value distribution of meromorphic functions whose differential polynomials share a small function

for which multiplicity is not counted. Let $N_{(k)}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f - a$ with multiplicity atleast k and $\overline{N}_{(k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f-a}\right) + \dots + \overline{N}_{(k)}\left(r, \frac{1}{f-a}\right).$$

Definition 2.3. (see [19]) For two positive integers n, p we define $\mu_p = \min\{n, p\}$ and $\mu_p^* = p + 1 - \mu_p$. Then it is clear that

$$N_p(r, 0; f^n) = \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 2.4. (see [20]) Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q . We denote by $\overline{N}_L(r, a; f)$ the counting function of those a -points of f and g where $p > q \geq 1$, by $N_E^{(1)}(r, a; f)$ the counting function of those a -points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2)}(r, a; f)$ the counting function of those a -points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{(1)}(r, a; g)$, $\overline{N}_E^{(2)}(r, a; g)$.

Definition 2.5. (see [21]) Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k . The definition implies that if f, g share a value a with weight k , then z_0 is an a -point of f with multiplicity $m (\leq k)$ if and only if it is an a -point of g with multiplicity $m (\leq k)$ and z_0 is an a -point of f with multiplicity $m (> k)$ if and only if it is an a -point of g with multiplicity $n (> k)$, where m not necessarily not equal to n . We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for all integers $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2.6. (see [1]) Suppose that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and $a(z)$ is a small function of $f(z)$. Then the differential monomial

is defined by $M[f] = \prod_{j=0}^k (f^{(j)})^{n_{ij}}$ where $n_{i_0}, \dots, n_{i_k}$ are non-negative integers.

Definition 2.7. (see [1]) Suppose that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and $M_1(f), \dots, M_m(f)$ be the differential monomials in $f(z)$ of

degree $n_1, \dots, n_m$ respectively. Then $P[f] = \sum_{j=0}^m M_j(f)$ is said to be differential polynomial in $f(z)$ and $n = \max\{n_1, \dots, n_m\}$ the degree of $P[f]$.

With the notion of weighted sharing of values Lahiri-Sarkar [11] improved the result of Zhang [12]. In [13] Zhang extended the result of Lahiri-Sarkar [11] and replaced the concept of value sharing by small function sharing.

3 Some Lemmas

Let F, G be two non-constant meromorphic functions. We define ψ as meromorphic function which is defined as follows.

$$\psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \quad (1)$$

Lemma 3.1. (see [1]) Let p, n be two positive integers. Then for $\epsilon > 0$

$$N_p(r, 0; f^n) \leq (n - n\delta_p(0; f) + \epsilon)T(r, f).$$

Lemma 3.2. (see [2]) Let $P[f]$ be a differential polynomial generated by non-constant meromorphic function f . Then

$$N(r, \infty; P) \leq \bar{d}(P)N(r, \infty; f) + (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; f).$$

Lemma 3.3. (see [3, 4]) Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then

$$m\left(r, \frac{P[f]}{f^{\bar{d}(P)}}\right) \leq (\bar{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 3.4. (see [5, 6]) Let $P[f]$ be a differential polynomial generated by non-constant meromorphic function f . Then

$$\begin{aligned} N\left(r, \infty; \frac{P[f]}{f^{\bar{d}(P)}}\right) &\leq (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; f) + (\bar{d}(P) - \underline{d}(P))N(r, 0; |f| \geq k+1) \\ &\quad + Q\bar{N}(r, 0; |f| \geq k+1) + \bar{d}(P)N(r, 0; |f| \leq k) + S(r, f). \end{aligned}$$

Lemma 3.5. (see [1]) For the differential polynomial $P[f]$,

$$\begin{aligned} N(r, 0; P[f]) &\leq (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; f) + \underline{d}(P)\bar{N}(r, 0; f) \\ &\quad + (\bar{d}(P) - \underline{d}(P))\left(m\left(r, \frac{1}{f}\right) + T(r, f)\right) + S(r, f). \end{aligned}$$

Value distribution of meromorphic functions whose differential polynomials share a small function

Lemma 3.6. (see [1]) Let j and p be two positive integers satisfying $j \geq p + 1$. Let $P[f]$ be a differential polynomial with $\Gamma_P > (k + 1)\underline{d}(P) - (p + 1)$. Then

$$\overline{N}_{(j+\Gamma_P-\underline{d}(P))}(r, 0; f^{\underline{d}(P)}) \leq \overline{N}_{(j)}(r, 0; P[f]).$$

Lemma 3.7. (see [1]) Let j and p be two positive integers satisfying $j \geq p + 1$. Let $P[f]$ homogeneous differential polynomial with $\Gamma_P > (k + 1)\underline{d}(P) - (p + 1)$. Then

$$N_p(r, 0; P[f]) \leq N_{p+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + (\Gamma_P - \underline{d}(P))\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 3.8. (see [19]) Let F and G share $(1, l)$. Then

$$\overline{N}_L(r, 1; F) \leq \frac{1}{l+1}\overline{N}(r, \infty; F) + \frac{1}{l+1}\overline{N}(r, 0; F) + S(r, F) \text{ if } l \geq 1,$$

and

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ if } l = 0.$$

Lemma 3.9. (see [19]) Let f be non-constant meromorphic function and $a(z)$ be a small function of f . Also, let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. If F and G shares $(1, \infty)$, then one of the following cases holds.

(i) $T(r) = N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) + S(r)$,

(ii) $F \equiv G$,

(iii) $FG \equiv 1$,

where $T(r) = \max\{T(r, F), T(r, G)\}$ and $S(r) = \{T(r)\}$, $r \in I$, I is set of infinite linear measure of $r \in (0, \infty)$.

Lemma 3.10. Let f be a non-constant meromorphic function and $a(z)$ be a small function in f . Let us define $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{P[f]}{a}$. Then $FG \not\equiv 1$.

Proof. On contrary, assume that $FG \equiv 1$ i.e., $P[f]f_1^p P(f_1) \equiv a^2$. Then

$$N(r, 0; f| \geq k + 1) = S(r, f)$$

Now applying Lemmas 3 and 4 and First fundamental Theorem of Nevanlinna,

we get

$$\begin{aligned}
 (n + m + \bar{d}(P))T(r, f) &= T\left(r, \frac{P[f]}{f^{\underline{d}(P)}}\right) + S(r, f) \\
 &\leq (\bar{d}(P) - \underline{d}(P))\left[T(r, f) - \{N(r, 0; f| \leq k) \right. \\
 &\quad \left. + N(r, 0; f| \geq k + 1)\}\right] \\
 &\quad + (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; f) + (\bar{d}(P) - \underline{d}(P))N(r, 0; f| \geq k + 1) \\
 &\quad + Q\bar{N}(r, 0; f| \geq k + 1) + \bar{d}(P)N(r, 0; f| \leq k) + S(r, f) \\
 &\leq (\bar{d}(P) - \underline{d}(P))T(r, f) + \underline{d}(P)N(r, 0; f| \leq k) \\
 &\quad + (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; f) + S(r, f) \\
 &\leq \bar{d}(P)T(r, f) + (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; P[f]f_1^p P(f_1)) + S(r, f) \\
 &\leq \bar{d}(P)T(r, f) + (\Gamma_P - \bar{d}(P))\bar{N}(r, \infty; a^2) + S(r, f) \\
 &\leq \bar{d}(P)T(r, f) + S(r, f).
 \end{aligned}$$

which is a contradiction. $\square$

4 Main Results

Theorem 4.1. Let $k (\geq 1)$, $n (\geq 1)$, $p (\geq 1)$ and $m (\geq 0)$ be integers and f and $f_1 = f - \omega_p$ be two non-constant meromorphic functions and $P[f]$ be a homogeneous differential polynomial of degree $\bar{d}(P)$ and weight Γ_P such that $\Gamma_P > (k + 1)\underline{d}(P) - 2$, where k is the highest derivative in $P[f]$. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \dots + a_n z^n + \dots + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m + n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a(z) (\not\equiv 0, \infty)$ be a small function with respect to f . Suppose $\mathcal{P}(f) = a$ and $P[f] = a$ share $(0, l)$.

(i) If $l \geq 2$ and

$$(\Gamma_P - \underline{d}(P) + 3)\Theta(\infty; f) + \mu_2 \delta_{\mu_2^*}(0; f) + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) > \Gamma_P + \mu_2 + 3 - n - m.$$

(ii) or $l = 1$ and

$$\begin{aligned}
 \left(\Gamma_P - \underline{d}(P) + \frac{7}{2}\right)\Theta(\infty; f) + \frac{1}{2}\Theta(0; f) + \mu_2 \delta_{\mu_2^*}(0; f) \\
 + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) > \Gamma_P + \mu_2 + 4 - n - m.
 \end{aligned}$$

(iii) or $l = 0$ and

$$\begin{aligned}
 (2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty; f) + 2\Theta(0; f) + \mu_2 \delta_{\mu_2^*}(0; f) + \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0; f) \\
 + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) \leq 2\Gamma_P + \mu_2 + 8 - n - m,
 \end{aligned}$$

then $\mathcal{P}(f) = P[f]$.

*Value distribution of meromorphic functions whose differential polynomials
share a small function*

Proof. Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{P[f]}{a}$. Then $F - 1 = \frac{f_1^p P(f_1) - a}{a}$ and $G - 1 = \frac{P[f] - a}{a}$. Since $\mathcal{P}(f)$ and $P[f]$ share (a, l) , it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$. Let ψ be defined as in equation (1). We consider the following cases.

Case 1. When $\psi \not\equiv 0$. Then from (1) we have $m(r, \psi) = S(r, f)$. By Nevanlinna Second Fundamental Theorem we have,

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F'}\right) \\ &\quad - N_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned} \quad (2)$$

where $N_0\left(r, \frac{1}{F'}\right)$ denotes the counting function of the zeros of F' which are not zeros of $F(F-1)$ and $N_0\left(r, \frac{1}{G'}\right)$ denotes the counting function of the zeros of G' which are not zeros of $G(G-1)$.

Subcase 1.1. When $l \geq 1$. Then from (1) we have

$$\begin{aligned} N_E^{1)}\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{1}{\psi}\right) + S(r, f) \\ &\leq T(r, \psi) + S(r, f) \\ &\leq N(r, \psi) + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) \\ &\quad + N_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

and so,

$$\begin{aligned}
 \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &= N_E^{(1)}\left(r, \frac{1}{F-1}\right) + N_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\
 &\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, f) \\
 &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
 &\quad + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) \\
 &\quad + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\
 &\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f).
 \end{aligned} \tag{3}$$

For $l \geq 2$

$$\begin{aligned}
 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) \\
 + \overline{N}\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{1}{G-1}\right) + S(r, f).
 \end{aligned}$$

Thus from (3) we obtain

$$\begin{aligned}
 \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
 &\quad + N\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) \\
 &\quad + N_0\left(r, \frac{1}{G'}\right) + S(r, f).
 \end{aligned} \tag{4}$$

From Lemma 3.7 and inequalities (2) and (4) we get

$$\begin{aligned}
 (n+m)T(r, f) &\leq (\Gamma_P - \underline{d}(P) + 3)\overline{N}(r, f) + \mu_2 N_{\mu_2^*}(r, \omega_p; f) + N_{2+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) \\
 &\quad + S(r, f) \\
 &\leq \{(\Gamma_P - \underline{d}(P) + 3) - (\Gamma_P - \underline{d}(P) + 3)\Theta(\infty; f) + \mu_2 - \mu_2 \delta_{\mu_2^*}(0; f) \\
 &\quad + \underline{d}(P) - \underline{d}(P) \delta_{2+\Gamma_P-\underline{d}(P)}(0; f) + \epsilon\}T(r, f) + S(r, f).
 \end{aligned}$$

i.e.,

$$(\Gamma_P - \underline{d}(P) + 3)\Theta(\infty; f) + \mu_2 \delta_{\mu_2^*}(0; f) + \underline{d}(P) \delta_{2+\Gamma_P-\underline{d}(P)}(0; f) \leq \Gamma_P + \mu_2 + 3 - n - m.$$

*Value distribution of meromorphic functions whose differential polynomials
share a small function*

which violates to the condition (i) of Theorem 4.1.

Subcase 1.2. Next, consider the case when $l = 1$. First note that

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2}N\left(r, \frac{1}{F'} \middle| F \neq 0\right) \leq \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right), \quad (5)$$

when $N\left(r, \frac{1}{F'} \middle| F \neq 0\right)$ denotes the zeros of F' that are not the zeros of F . From inequalities (1) and (5), we have

$$\begin{aligned} 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ \leq N\left(r, \frac{1}{G-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + S(r, f) \\ \leq N\left(r, \frac{1}{G-1}\right) + \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned} \quad (6)$$

Thus, from inequalities (2) and (6)

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + \frac{1}{2}\overline{N}(r, F) \\ &\quad + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + T(r, G) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) \\ &\quad + S(r, f). \end{aligned} \quad (7)$$

From Lemma 3.6, (2) and (7), we have

$$\begin{aligned} (n+m)T(r, f) &\leq \left(\Gamma_P - \underline{d}(P) + \frac{7}{2}\right)\overline{N}(r, f) + \frac{1}{2}\overline{N}\left(r, \frac{1}{f}\right) + \mu_2 N_{\mu_2^*}(r, 0; f) \\ &\quad + N_{2+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + S(r, f) \\ &\leq \left\{ \left(\Gamma_P - \underline{d}(P) + \frac{7}{2}\right) - (\Gamma_P - \underline{d}(P) + \frac{7}{2})\Theta(\infty; f) + \frac{1}{2} - \frac{1}{2}\Theta(0; f) \right. \\ &\quad \left. + \mu_2 - \mu_2\delta_{\mu_2^*}(0; f) + \underline{d}(P) - \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) + \epsilon \right\} T(r, f) \\ &\quad + S(r, f). \end{aligned}$$

i.e.,

$$\begin{aligned} \left(\Gamma_P - \underline{d}(P) + \frac{7}{2}\right)\Theta(\infty; f) + \frac{1}{2}\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0; f) \\ + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) \leq \Gamma_P + \mu_2 + 4 - n - m, \end{aligned} \quad (8)$$

which violates to the condition (ii) of Theorem 4.1.

Subcase 1.3. When $l = 0$. Then, we have

$$N_E^{(1)}\left(r, \frac{1}{F-1}\right) = N_E^{(1)}\left(r, \frac{1}{G-1}\right) + S(r, f),$$

$$\overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) = N_E^{(2)}\left(r, \frac{1}{G-1}\right) + S(r, f),$$

and also from (1), we get

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, f) \\ &\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ &\quad + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\ &\quad + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) \\ &\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned} \tag{9}$$

From Lemma 3.8, (3) and (6) we get

$$\begin{aligned} (n+m)T(r, f) &\leq (2(\Gamma_P - \underline{d}(P)) + 6)\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{f}\right) + \mu_2 N_{\mu_2^*}(r, 0; f) \\ &\quad + N_{2+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + N_{1+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + S(r, f) \\ &\leq \{(2(\Gamma_P - \underline{d}(P)) + 6) - (2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty; f) \\ &\quad + 2 - 2\Theta(0; f) + \mu_2 - \mu_2\delta\mu_2^*(0; f) \\ &\quad + \underline{d}(P) - \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0; f) + \underline{d}(P) \\ &\quad - \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) + \epsilon\}T(r, f) + S(r, f). \end{aligned}$$

i.e.,

$$\begin{aligned} (2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty; f) + 2\Theta(0; f) + \mu_2\delta\mu_2^*(0; f) + \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0; f) \\ + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) \leq 2\Gamma_P + \mu_2 + 8 - n - m. \end{aligned}$$

Value distribution of meromorphic functions whose differential polynomials share a small function

which violates condition (iii) of Theorem 4.1.

Case 2. If $H \equiv 0$. On integration of (1), we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B.$$

where $A (\neq 0)$ and B are complex constants. Clearly, F and G share $(1, \infty)$. Also, by construction of F and G , F and G share $(\infty, 0)$. So, by Lemma 3.6 and condition (2), we obtain

$$\begin{aligned} & N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_L(r, F) + \overline{N}_L(r, G) + S(r) \\ & \leq \mu_2 N_{\mu_2^*}(r, 0; f) + N_{2+\Gamma_P-\underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + (\Gamma_P - \underline{d}(P) + 3)\overline{N}(r, f) + S(r) \\ & \leq \{(\Gamma_P + \mu_2 + 3) - (\Gamma_P - \underline{d}(P) + 3)\Theta(\infty; f) - \mu_2 \delta \mu_2^*(0; f) \\ & \quad - \underline{d}(P) \delta_{2+\Gamma_P-\underline{d}(P)}(0; f) + \epsilon\} \\ & < T(r, F) + S(r), \end{aligned}$$

where $\epsilon > 0$ is any small quantity. Hence from the case $l \geq 2$ and Lemma 3.9 does not hold. Again in view of Lemma 3.10, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e., $\mathcal{P}(f) \equiv P[f]$. $\square$

5 Conclusion

In this research article, we have proved one result by using the concepts of differential polynomials. In this paper, we have replaced the differential monomial by differential polynomial and $\mathcal{P}(f) = f_1^p P(f_1)$ where f is a meromorphic function. Thus, the result have been improved and generalizes the previous result obtained in [7]. Finally, for further research we are posing a question What happens if $\mathcal{P}(f) = f_1^p P(f_1)$ is replaced with $\mathcal{P}(f) = [f_1^p P(f_1)]^{(k)}$ which shares “(0,1)” ?

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Value distribution of meromorphic functions whose differential polynomials share a small function

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Existence and Uniqueness of Solutions for System of Nonlinear Fractional Reaction-Diffusion Equations

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Abstract

In this paper, to develop monotone method for non-linear system of Riemann-Liouville (R-L) reaction-diffusion equations with initial and boundary conditions. The monotone method yields monotone sequences which converges uniformly and monotonically to minimal and maximal solutions. To investigate the existence and uniqueness of solutions by monotone method for R-L reaction-diffusion equations with initial and boundary conditions.

Keywords: Riemann-Liouville fractional derivative; eigenfunction; lower and upper solutions; monotone method.

2020 AMS subject classifications: 26A33, 26A48, 34A08. ¹

1 Introduction

Now a days, system of fractional reaction-diffusion equations is a leading topic in the area of fractional calculus. It is observed that the solution of non-linear reaction-diffusion equation is rarely possible. The system of fractional differential equations have many applications in the field of Physics, Chemistry, Biochemistry [Hilfer, 2000, A.A. Kilbas and Rogosin, 2014, A.A. Kilbas and Trujillo, 2006]. However the most of the natural phenomenon are nonlinear in nature.

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These situation are captured by nonlinear system of fractional reaction-diffusion equations. The leading researchers have studied the existence and uniqueness of solution of the fractional differential equations using fixed point theorem [Chhetri and Vatsala, 2017, 2018, D.B. Dhaigude and Nikam, 2012, Hilfer, 2000, Kilbas and Marzan, 2004, Lakshmikantham, 2008, Nanware and Dhaigude, 2012, Pao, 1992, Vatsala, 2015]. Also the monotone method combined with the lower and upper solution for system of R-L reaction-diffusion equations with initial and boundary conditions are discussed in [Denton and Vatsala, 2011, Z. Denton and Vatsala, 2011, V.Lakshmikantham and Leela, 2009, Vatsala, 2012b,a]. There are different kind of problems for which monotone method has been developed in [Lakshmikantham and Vatsala, 2012, Nanware and Kundgar, 2021, Nanware and Dhaigude, 2015, 2012, 2018, Jadhav and Dhaigude, 2014, 2017, 2020, Ramirez and Vatsala, 2012].

In this paper, we develop monotone method for the non-linear system of R-L reaction diffusion-equations using lower and upper solutions. We obtain integral representation form for the solution of linear system of R-L reaction-diffusion equations using eigen function expansion method and Green's theorem. The comparison results are obtained for system of R-L fractional reaction-diffusion equations. These sequences of solutions for linear fractional differential equations which converges uniformly and monotonically to minimal and maximal solutions of the non-linear system of fractional diffusion equations. We successfully apply monotone method to obtain existence and uniqueness of solutions of the problem.

The paper is arranged in the follows.

In section 2, definitions and basic results are discussed that plays vital role in the main results. The comparison results are obtained in section 3. These results are used to develop monotone method for the problem under study. The obtained results are given in section 4. At the end discussion and conclusion section is added.

2 Preliminaries

In this section, we introduce definitions and preliminary results which are used in main results.

Definition 2.1. ([Chhetri and Vatsala, 2018]) *The R-L fractional integral of $u(t)$ of order q is defined as*

$$D_t^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} u(\tau) d\tau, \quad 0 < q \leq 1.$$

*Existence and Uniqueness of Solutions for System of Nonlinear Fractional
Reaction-Diffusion Equations*

Definition 2.2. ([Chhetri and Vatsala, 2018]) The R-L (left-sided) fractional derivative of $u(t)$ is defined as

$$D_t^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\tau)^{q-1} u(\tau) d\tau, \quad t > 0, \quad 0 < q \leq 1.$$

Definition 2.3. ([Chhetri and Vatsala, 2018]) The Mittag-Leffler function with two parameters α and β , denoted by $E_{\alpha,\beta}$ is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$

Definition 2.4. ([Chhetri and Vatsala, 2018]) A function $h(t) \in C(J, \mathbb{R})$ is a C_p continuous function, if $t^{1-q}h(t) \in C(J_0, \mathbb{R})$ where $p = 1 - q, J = (0, T]$ and $J_0 = [0, T]$. The set of C_p continuous functions is denoted by $C_p(J, \mathbb{R})$. Further, given a function $h(t) \in C_p(J, \mathbb{R})$, we call the function $t^{1-q}h(t)$, the continuous extension of $h(t)$.

Consider the nonhomogeneous linear R-L fractional differential equation [A.A.Kilbas and Trujillo, 2006] with initial condition

$$D^q u = \lambda u + h(t), \quad \Gamma(q)u(t)t^{1-q} \big|_{t=0} = u^0 \quad (1)$$

where $\lambda \in \mathbb{R}$ and $h \in C[J_0, \mathbb{R}]$, has integral representation of the solution of equation (1) as

$$u(t) = u^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-\tau)^{q-1} E_{q,q}[\lambda(t-\tau)^q] h(\tau) d\tau. \quad (2)$$

Lemma 2.1. ([Denton and Vatsala, 2011, Z. Denton and Vatsala, 2011]) [Comparison Results] Let $\eta \in C_p[J_0, \mathbb{R}]$ be such that for some $t_1 \in (0, T]$, $\eta(t_1) = 0$, and $t^{1-q}\eta(t) \leq 0$ on $[0, t_1]$, then $D^q \eta(t_1) \geq 0$.

3 Comparison Results

In this section, we obtain a comparison results for the system of non-linear fractional reaction-diffusion equations with initial and boundary conditions. The comparison theorem is employed respect to the lower and upper solutions when the non-linear term is of the form $f_i(x, t, u_1, u_2)$, where $f_i(x, t, u_1, u_2)$ is quasi-monotone non-decreasing function and satisfies one sided Lipschitz condition.

Consider the initial and boundary value problem for system of non-linear R-L fractional reaction-diffusion equations.

$$\frac{\partial^q u_i}{\partial t^q} - k_i \frac{\partial^2 u_i}{\partial x^2} = f_i(x, t, u_1, u_2) \quad \text{on} \quad S_T. \quad (3)$$

$$u_i(0, t) = A_i(t), u_i(L, t) = B_i(t) \quad \text{in } I_T.$$

$$\Gamma(q)t^{1-q}u_i(x, t) \big|_{t=0} = f_i^0(x) \quad x \in I$$

where $i = 1, 2, f_i \in C[I \times J_0 \times \mathbb{R}^2, \mathbb{R}]$, $I = [0, L]$, $J = (0, T]$, $S_T = J \times I$, $k > 0$ and $I_T = (0, T) \times \partial I$.

Suppose that $f_i^0(0) = A_i(0) = f_i^0(L) = B_i(0) = 0$, $\Gamma(q)t^{1-q}u_i \big|_{t=0} = f_i^0(x)$. We assume the initial and boundary condition satisfy the compatibility conditions. Using the method of eigenfunction expansion, [Chhetri and Vatsala, 2018] the solution of (3), is of the form

$$u_i(x, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x), \quad (4)$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x) = \sin \frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n = \left[\left(\frac{n\pi}{L}\right)^2\right]$. We compute $b_n(t)$, where $b_n(t)$ is the solution of linear R-L fractional differential equations, as follows:

$$\begin{aligned} b_n(t) &= b_n^0 t^{q-1} E_{q,q}(-k_i \lambda_n t^q) + \int_0^t (t - \tau)^{q-1} E_{q,q}(-k_i \lambda_n \tau^q) q_n(\tau) \\ &\quad + k \frac{2n\pi}{L^2} [A_i(\tau) - (-1)^n B_i(\tau)] d\tau, \end{aligned}$$

where

$$b_n^0 = \frac{2}{L} \int_0^L f_i^0(y) \phi_n(y) dy, \quad q_n(t) = \frac{2}{L} \int_0^L f_i(x, t, u_1, u_2) \phi_n(y) dy.$$

Therefore,

$$\begin{aligned} b_n(t) &= \frac{2}{L} \int_0^L f_i^0(y) \phi_n(y) dy t^{q-1} E_{q,q}(-k_i \lambda_n t^q) \\ &\quad + \int_0^t (t - \tau)^{q-1} E_{q,q}(-k_i \lambda_n \tau^q) \frac{2}{L} \int_0^L f_i(x, t, u_1, u_2) \phi_n(y) dy d\tau \\ &\quad + k \frac{2n\pi}{L^2} \int_0^t (t - \tau)^{q-1} E_{q,q}(-k_i \lambda_n \tau^q) [A_i(\tau) - (-1)^n B_i(\tau)] d\tau. \end{aligned}$$

*Existence and Uniqueness of Solutions for System of Nonlinear Fractional
Reaction-Diffusion Equations*

So, using $b_n(t)$ in (4), the solution $u_i(x, t)$ has the form

$$\begin{aligned} u_i(x, t) = & \int_0^L t^{q-1} \left[\sum_{n=1}^{\infty} \frac{2}{L} E_{q,q}(-k_i \lambda_n t^q) \phi_n(x) \phi_n(y) \right] f_i^0(y) dy \\ & + \int_0^t \int_0^L \left[\sum_{n=1}^{\infty} \frac{2}{L} (t-\tau)^{q-1} E_{q,q}(-k_i \lambda_n (t-\tau)^q) \phi_n(x) \phi_n(y) \right] f_i dy d\tau \\ & + k_i \int_0^t \left[\frac{2n\pi}{L^2} (t-\tau)^{q-1} E_{q,q}(-k_i \lambda_n (t-\tau)^q) \phi_n(x) \right] A_i(\tau) d\tau \\ & - k_i \int_0^t \left[\frac{2n\pi}{L^2} (t-\tau)^{q-1} E_{q,q}(-k_i \lambda_n (t-\tau)^q) \phi_n(x) \right] B_i(\tau) d\tau. \end{aligned}$$

Thus we have,

$$\begin{aligned} u_i(x, t) = & \int_0^L t^{q-1} G(x, y, t) f_i^0(y) dy + \int_0^t \int_0^L G(x, y, t-\tau) f_i dy d\tau \\ & + k_i \int_0^t G_y(x, 0, t-\tau) A_i(\tau) d\tau - k_i \int_0^t G_y(x, L, t-\tau) B_i(\tau) d\tau, \end{aligned}$$

where

$$G(x, y, t) = \sum_{n=0}^{\infty} \frac{2}{L} E_{q,q}(-k_i \lambda_n t^q) \phi_n(x) \phi_n(y).$$

The following Lemmas are useful to obtain the convergence of the solutions.

Lemma 3.1. *Chhetri and Vatsala [2017] Let $E_{q,1}(-\lambda t^q)$ be the Mittag-Leffler function of order q , where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + c$ for $c > 0$.*

Lemma 3.2. *Chhetri and Vatsala [2017] Let $E_{q,q}(-\lambda t^q)$ be the Mittag-Leffler function of order q , where $0 < q \leq 1$. Then, $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + c$ for $c > 0$.*

Now, we show the convergence of the above solutions using Lemma 3.1 and Lemma 3.2 above. We can split the solution of (3) as $u_i^1(x, t)$, $u_i^2(x, t)$ and $u_i^3(x, t)$ respectively as follows:

- (a) $u_i^1(x, t)$ is the solution of (3), when $f_i = 0$, $A_i(t) = 0 = B_i(t)$,
- (b) $u_i^2(x, t)$ is the solution of (3), when $A_i(t) = 0 = B_i(t)$, $f_i^0 = 0$,
- (c) $u_i^3(x, t)$ is the solution of (3), when $f_i(x, t, u_1, u_2) = 0$, $f_i^0 = 0$.

Theorem 3.1. Chhetri and Vatsala [2017] $u_i^1(x, t)$, $u_i^2(x, t)$ and $u_i^3(x, t)$ converge when $|f_i^0(x)| < N_1$, $N_1 > 0$, $|f_i(x, t, u_1, u_2)| < N_2$, $N_2 > 0$; $|A_i(t)| < M_1$, $M_1 > 0$ and $|B_i(t)| < M_2$, $M_1, M_2 > 0$, respectively.

Now, we consider the initial and boundary value problem for system of non-linear R-L fractional reaction-diffusion equations of the type

$$\begin{aligned} \frac{\partial^q u_i}{\partial t^q} - k_i \frac{\partial^2 u_i}{\partial x^2} &= f_i(x, t, u_1, u_2), \quad (x, t) \in S_T, \\ \Gamma(q)t^{1-q}u_i(x, t) \big|_{t=0} &= f_i^0(x), \quad x \in I, \\ u_i(0, t) &= A_i(t), u_i(L, t) = B_i(t) \quad \text{on } I_T, \\ I &= [0, L], J = (0, T], S_T = J \times I, k > 0, \\ I_T &= (0, T) \times \partial I \quad i = 1, 2 \\ f_i &\in C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]. \end{aligned} \quad (5)$$

In this work, we can see the classical solution of (5) $u_i(x, t) \in C_p^{2,q}$ on S_T , and $u_i(x, t) \in C_p$ on $\overline{S_T}$. To develop monotone method for the problem, (5) we need to define the following important definitions and results

Definition 3.1. If the functions $v_i(x, t)$ & $w_i(x, t) \in C^{2,q}[S_T, \mathbb{R}]$ are called the natural lower and upper solutions of (5) if

$$\begin{aligned} \frac{\partial^q v_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i(x, t)}{\partial x^2} &\leq f_i(x, t, v_1, v_2), \quad \text{on } S_T \\ \Gamma(q)(t - t_0)^{1-q}v_i(x, t) \big|_{t=0} &\leq f_i^0(x), \quad x \in I \\ v_i(x, 0) &\leq A_i(t), v_i(L, t) \leq B_i(t) \quad \text{in } I_T \end{aligned} \quad (6)$$

and

$$\begin{aligned} \frac{\partial^q w_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i(x, t)}{\partial x^2} &\geq f_i(x, t, w_1, w_2), \quad \text{on } S_T \\ \Gamma(q)(t - t_0)^{1-q}w_i(x, t) \big|_{t=0} &\geq f_i^0(x), \quad x \in I \\ w_i(x, 0) &\geq A_i(t), w_i(L, t) \geq B_i(t) \quad \text{in } I_T. \end{aligned} \quad (7)$$

Definition 3.2. A function $f_i = f_i(x, t, u_1, u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ is said to be quasimonotone nondecreasing if $f_i(x, t, u_1, u_2) \leq f_i(x, t, v_1, v_2)$ if $u_i = v_i$ and $u_j \leq v_j$, $i \neq j$, $i = j = 1, 2$.

Definition 3.3. A function $f_i = f_i(x, t, u_1, u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ is said to be quasimonotone nonincreasing if $f_i(x, t, u_1, u_2) \geq f_i(x, t, v_1, v_2)$ if $u_i = v_i$ and $u_j \leq v_j$, $i \neq j$, $i = j = 1, 2$.

*Existence and Uniqueness of Solutions for System of Nonlinear Fractional
Reaction-Diffusion Equations*

Theorem 3.2. [Comparison result] Assume that

- (i) $v_i(x, t), w_i(x, t) \in C^{2,q}[S_T, \mathbb{R}]$ are natural lower and upper solutions of (5), respectively and $\Gamma(q)t^{1-q}v_i(x, t) \big|_{t=0} \leq \Gamma(q)t^{1-q}w_i(x, t) \big|_{t=0}$, $v_i(0, t) \leq w_i(0, t)$, $v_i(L, t) \leq w_i(L, t)$.
(ii) $f_i(x, t, u_1, u_2)$ satisfies the one sided Lipschitz condition

$$f_i(x, t, u_1, u_2) - f_i(x, t, u_1^*, u_2^*) \leq M_i(u_i - u_i^*),$$

whenever $u_i \leq u_i^*$ and $M_i > 0$, $i = 1, 2$.

Then $v_i(x, t) \leq w_i(x, t)$ on $J \times I$.

Proof. Initially, we prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m_i(x, t) = v_i(x, t) - w_i(x, t)$. We claim that $m_i(x, t) < 0$, $(x, t) \in I \times J$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in I$ such that $t^{1-q}m_i(x_1, t_1) < 0$ on $[0, t_1)$, $m_i(x_1, t_1) = 0$. It easy to check $\frac{\partial^q m_i(x_1, t_1)}{\partial t^q} = 0$ and $\frac{\partial^2 m_i(x_1, t_1)}{\partial x^2} \leq 0$.

Then, using Lemma 2.1 we get $\frac{\partial^q m_i(x, t)}{\partial t^q} \geq 0$

From the hypothesis, we also have

$$\begin{aligned} \frac{\partial^q m_i(x_1, t_1)}{\partial t_1^q} &= \frac{\partial^q v_i(x_1, t_1)}{\partial t_1^q} - \frac{\partial^q w_i(x_1, t_1)}{\partial t_1^q} \\ &< k_i \frac{\partial^2 v_i(x_1, t_1)}{\partial x^2} + f_i(x_1, t_1, v_1, v_2) \\ &\quad - k_i \frac{\partial^2 w_i(x_1, t_1)}{\partial x^2} - f_i(x_1, t_1, w_1, w_2) \\ &< f_i(x_1, t_1, v_1, v_2) - f_i(x_1, t_1, w_1, w_2) = 0 \end{aligned}$$

which is a contradiction. Therefore, $v_i(x, t) < w_i(x, t)$ on $\overline{S_T}$.

In order to prove the theorem for the non strict inequalities, let

$$\begin{aligned} w_i^*(x, t) &= w_i(x, t) + \epsilon t^{q-1} E_{q,q}[2Mt^q], \\ v_i^*(x, t) &= v_i(x, t) - \epsilon t^{q-1} E_{q,q}[2Mt^q]. \end{aligned}$$

From this it follows

$$\begin{aligned} w_i^*(0, t) &> v_i^*(0, t), \\ w_i^*(L, t) &> v_i^*(L, t), \end{aligned}$$

$\Gamma(q)t^{1-q}w_i^*|_{t=0} > \Gamma(q)t^{1-q}w_i|_{t=0} > \Gamma(q)t^{1-q}v_i|_{t=0} > \Gamma(q)t^{1-q}v_i^*|_{t=0}$. Then,

$$\begin{aligned} \frac{\partial^q w_i^*(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^*(x, t)}{\partial x^2} &= \frac{\partial^q w_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i(x, t)}{\partial x^2} + \frac{\partial^q}{\partial t^q} \epsilon t^{q-1} E_{q,q}[2Mt^q] \\ &\geq f_i(x, t, w_1, w_2) + \epsilon t^{q-1} 2M_i E_{q,q}[2Mt^q] \\ &= f_i(x, t, w_1, w_2) + M_i \epsilon t^{q-1} E_{q,q}[2Mt^q] \\ &= f_i(x, t, w_1^*, w_2^*) + \epsilon M_i t^{q-1} E_{q,q}[2Mt^q] \\ &> f_i(x, t, w_1^*, w_2^*) \quad \text{on } S_T. \end{aligned}$$

Similarly,

$$\frac{\partial^q v_i^*(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^*(x, t)}{\partial x^2} > f_i(x, t, v_1^*, v_2^*) \quad \text{on } S_T.$$

Since $v_i^* < w_i^*$ on $\overline{S_T}$ and as $\epsilon \rightarrow 0$ we have $v_i \leq w_i$ on $\overline{S_T}$. $\square$

Corolary 3.1. [Maximal principle] Let

$$\begin{aligned} \frac{\partial^q m_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 m_i(x, t)}{\partial x^2} &\leq 0 \quad \text{on } S_T, \\ m_i(0, t) &\leq 0, m_i(L, t) \leq 0 \quad \text{on } I_T, \\ \Gamma(q)t^{1-q}m_i(x, t)|_{t=0} &\leq 0 \quad \text{on } I. \end{aligned}$$

Then $m_i(x, t) \leq 0$ on $J \times I$.

Proof. Suppose $m_i(x, t)$ has positive maximum at (x_1, t_1) . Let $m_i(x_1, t_1) = K$. Let $\overline{m}_i(x, t) = m_i(x, t) - K$. Then $t^{1-q}\overline{m}_i(x, t) \leq 0$ on $(0, t_1]$ and $\overline{m}_i(x_1, t_1) = 0$. Using Lemma 2.1, we get $\frac{\partial^q m_i(x_1, t_1)}{\partial t_1^q} \geq 0$. Also $\frac{\partial^2 m_i(x_1, t_1)}{\partial x^2} \leq 0$. Combining these two, we get $\frac{\partial^q m_i(x_1, t_1)}{\partial t_1^q} - k_i \frac{\partial^2 m_i(x_1, t_1)}{\partial x^2} \geq 0$.

Also, we have

$$\begin{aligned} \frac{\partial^q \overline{m}_i(x, t)}{\partial t^q} - K \frac{\partial^2 \overline{m}_i(x, t)}{\partial x^2} &= \frac{\partial^q m_i(x, t)}{\partial t^q} - K \frac{\partial^2 m_i(x, t)}{\partial x^2} - K \frac{t^{q-1}}{\Gamma q} \\ &< \frac{\partial^q m_i(x, t)}{\partial t} - K \frac{\partial^2 m_i(x, t)}{\partial x^2} \\ &< 0. \end{aligned}$$

which gives a contradiction. Hence, $m_i(x, t) \leq 0$. $\square$

4 Main Results

This section is employed for monotone method for system of R-L fractional reaction-diffusion equation (5) using lower and upper solutions. Also obtain existence and uniqueness of solution of system(5).

Theorem 4.1. (i) Let $f_i = f_i(x, t, u_1, u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ be quasimonotone nondecreasing.

(ii) Let (v_i^0, w_i^0) be the lower and upper solutions of (5) such that $t^{1-q}v_i^0 \leq t^{1-q}w_i^0$ on $\overline{S_T}$.

(iii) Let $f_i(x, t, u_1, u_2)$ satisfies the one sided Lipschitz condition

$$f_i(x, t, u_1, u_2) - f_i(x, t, u_1^*, u_2^*) \geq -M_i(u_i - u_i^*),$$

whenever $u_i^* \leq u_i$ and $M_i > 0$. Then there exist monotone sequences $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ such that $t^{1-q}v_i^n(x, t) \rightarrow t^{1-q}\rho_i(x, t)$ and $t^{1-q}w_i^n(x, t) \rightarrow t^{1-q}\gamma_i(x, t)$ uniformly and monotonically on $\overline{S_T}$, where $\rho_i(x, t)$ and $\gamma_i(x, t)$ are minimal and maximal solutions of (5) respectively.

Proof. We construct the sequences $\{v_i^n(x, t)\}$ and $\{w_i^n(x, t)\}$ as follows:

$$\frac{\partial^q v_i^n(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^n(x, t)}{\partial x^2} = f_i(x, t, v_1^{n-1}(x, t), v_2^n(x, t)), \quad \text{on } S_T \quad (8)$$

$$\Gamma(q)(t)^{1-q}v_i^n(x, t) |_{t=0} = f_i^0(x), \quad x \in I$$

$$v_i^n(0, t) = A_i(t), v_i^n(L, t) = B_i(t) \quad \text{in } I_T,$$

and

$$\frac{\partial^q w_i^n(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^n(x, t)}{\partial x^2} = f_i(x, t, w_1^{n-1}(x, t), w_2^n(x, t)), \quad \text{on } S_T \quad (9)$$

$$\Gamma(q)(t)^{1-q}w_i^n(x, t) |_{t=0} = f_i^0(x), \quad x \in I$$

$$w_i^n(0, t) = A_i(t), w_i^n(L, t) = B_i(t) \quad \text{in } I_T,$$

Observe that $v_i^1(x, t)$ and $w_i^1(x, t)$ exist and unique solution of the system (5) and Corollary 3.1. By induction and the assumptions on $f_i(x, t, u_1, u_2)$, we show that the solution $v_i^n(x, t)$ and $w_i^n(x, t)$ exist and unique by Corollary 3.1, for any n . Let us prove first $v_i^0 \leq v_i^1$ and $w_i^1 \leq w_i^0$ on S_T .

Let $\rho_i(x, t) = v_i^0(x, t) - v_i^1(x, t)$. Then

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q v_i^0(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^0(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q v_i^1(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^1(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, v_1^0, v_2^0) - [f_i(x, t, v_1^0, v_2^0)] = 0 \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) |_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}v_i^0(x, t) \leq t^{1-q}v_i^1(x, t)$ on $\overline{S_T}$.

Suppose that $v_i^{k-1} \leq v_i^k$. Now we prove $v_i^k \leq v_i^{k+1}$.

Let $\rho_i(x, t) = v_i^k(x, t) - v_i^{k+1}(x, t)$. Then

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q v_i^k(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^k(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q v_i^{k+1}(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^{k+1}(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, v_1^k, v_2^k) - [f_i(x, t, v_1^{k+1}, v_2^{k+1})] \\ &\leq M_i(v_i^{k+1} - v_i^k) \\ &\leq -M_i \rho_i(x, t) \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) |_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}v_i^k(x, t) \leq t^{1-q}v_i^{k+1}(x, t)$ on $\overline{S_T}$.

Hence by mathematical induction, we have

$$t^{1-q}v_i^0 \leq t^{1-q}v_i^1 \leq \dots \leq t^{1-q}v_i^k \leq t^{1-q}v_i^{k+1} \leq \dots \leq t^{1-q}v_i^{n-1} \leq t^{1-q}v_i^n \quad (10)$$

We show that $w_i^1(x, t) \leq w_i^0(x, t)$ on $\overline{S_T}$.

Let $\rho_i(x, t) = w_i^1(x, t) - w_i^0(x, t)$. Then

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q w_i^1(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^1(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^0(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^0(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, w_1^0, w_2^0) - [f_i(x, t, w_1^0, w_2^0)] = 0 \end{aligned}$$

*Existence and Uniqueness of Solutions for System of Nonlinear Fractional
Reaction-Diffusion Equations*

Therefore, $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho(x, t) |_{t=0} = 0$ on I_T . By Corollary 3.1, we have $\rho_i(x, t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}w_i^0(x, t) \leq t^{1-q}w_i^1(x, t)$ on $\overline{S_T}$. Suppose that $w_i^k(x, t) \leq w_i^{k+1}(x, t)$. To show that $w_i^{k+1}(x, t) \leq w_i^k(x, t)$. Let $\rho_i(x, t) = w_i^{k+1}(x, t) - w_i^k(x, t)$. Then

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q w_i^{k+1}(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^{k+1}(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^k(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^k(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, w_1^k, w_2^k) - [f_i(x, t, w_1^{k+1}, w_2^k)] \\ &\leq M_i(w_i^k - w_i^{k+1}) \\ &\leq -M_i \rho_i(x, t) \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) |_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}w_i^{k+1}(x, t) \leq t^{1-q}w_i^k(x, t)$ on $\overline{S_T}$.

Hence by mathematical induction, we have

$$t^{1-q}w_i^n \leq t^{1-q}w_i^{n-1} \leq \dots \leq t^{1-q}w_i^{k+1} \leq t^{1-q}w_i^k \leq \dots \leq t^{1-q}w_i^1 \leq t^{1-q}w_i^0 \quad (11)$$

Then, we prove that $v_i^1(x, t) \leq w_i^1(x, t)$. Let $\rho_i(x, t) = v_i^1(x, t) - w_i^1(x, t)$. Then from hypothesis, we get

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q v_i^1(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^1(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^1(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^1(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, v_1^1, v_2^1) - [f_i(x, t, w_1^1, w_2^1)] \\ &\leq M_i(v_i^1 - w_i^1) \\ &\leq -M_i \rho_i(x, t) \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) |_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}v_i^1(x, t) \leq t^{1-q}w_i^1(x, t)$ on $\overline{S_T}$. Hence,

$$t^{1-q}v_i^0(x, t) \leq t^{1-q}v_i^1(x, t) \leq t^{1-q}w_i^1(x, t) \leq t^{1-q}w_i^0(x, t) \text{ on } \overline{S_T}.$$

By mathematical induction and equations (10), (11), we have

$$t^{1-q}v_i^0 \leq \dots \leq t^{1-q}v_i^n \leq t^{1-q}w_i^n \leq \dots \leq t^{1-q}w_i^0 \text{ on } \overline{S_T} \text{ for all } n.$$

Furthermore, if $t^{1-q}v_i^0 \leq t^{1-q}u_i \leq t^{1-q}w_i^0$ on $\overline{S_T}$, then for any $u_i(x, t)$ of (5), we establish the following inequality by the method of induction.

$$t^{1-q}v_i^0 \leq \dots \leq t^{1-q}v_i^n \leq t^{1-q}u_i \leq t^{1-q}w_i^n \leq \dots \leq t^{1-q}w_i^0 \quad (12)$$

on $\overline{S_T}$ for all n .

It is certainly true for $n = 0$, by hypothesis. Assume the inequality (10) to be true for $n = k$, that is

$$t^{1-q}v_i^0 \leq \dots \leq t^{1-q}v_i^k \leq t^{1-q}u_i \leq t^{1-q}w_i^k \leq \dots \leq t^{1-q}w_i^0 \quad (13)$$

on $\overline{S_T}$ for all n .

Let $\rho_i(x, t) = v_i^{k+1}(x, t) - u_i(x, t)$. Then from hypothesis, we get

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q v_i^{k+1}(x, t)}{\partial t^q} - k_i \frac{\partial^2 v_i^{k+1}(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q u_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 u_i(x, t)}{\partial x^2} \right] \\ &\geq f_i(x, t, v_1^{k+1}, v_2^k) - [f_i(x, t, u_1, u_2)] \\ &\geq -M_i(v_i^{k+1} - u_i) \\ &\geq -M_i \rho_i(x, t) \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) \big|_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \geq 0$ on $\overline{S_T}$. Therefore $t^{1-q}v_i^{k+1}(x, t) \leq t^{1-q}u_i(x, t)$ on $\overline{S_T}$. Similarly, we can show that $t^{1-q}u_i(x, t) \leq t^{1-q}w_i^{k+1}(x, t)$ on $\overline{S_T}$.

Let $\rho_i(x, t) = u_i(x, t) - w_i^{k+1}(x, t)$. Then from hypothesis, we get

$$\begin{aligned} \frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} &= \frac{\partial^q u_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 u_i(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^{k+1}(x, t)}{\partial t^q} - k_i \frac{\partial^2 w_i^{k+1}(x, t)}{\partial x^2} \right] \\ &\geq f_i(x, t, u_1, u_2) - [f_i(x, t, w_1^{k+1}, w_2^k)] \\ &\geq -M_i(u_i - w_i^{k+1}) \\ &\geq -M_i \rho_i(x, t) \end{aligned}$$

Thus we have $\rho_i(0, t) = 0, \rho_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x, t) \big|_{t=0} = 0$ on I_T . Therefore, by Corollary 3.1, we have $\rho_i(x, t) \geq 0$ on $\overline{S_T}$. Therefore $t^{1-q}u_i(x, t) \leq t^{1-q}w_i^{k+1}(x, t)$ on $\overline{S_T}$.

Hence we constructed the monotone sequences $\{v_i^n(x, t)\}, \{w_i^n(x, t)\}$ of lower and upper solutions of integral representation of linear problem and an appropriate computation process, we show that the sequences $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzelà theorem, we obtain subsequences of $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ which converge

*Existence and Uniqueness of Solutions for System of Nonlinear Fractional
Reaction-Diffusion Equations*

uniformly and monotonically on $\overline{S_T}$. Since the sequences $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ are monotone, the entire sequence $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ converges to $t^{1-q}\rho_i(x, t)$ and $t^{1-q}\gamma_i(x, t)$ respectively. From this it follows that

$$\begin{aligned} t^{1-q}v_i^0 &\leq t^{1-q}v_i^1 \leq \dots \leq t^{1-q}v_i^n \leq \dots \leq t^{1-q}\rho_i \leq t^{1-q}u_i \\ &\leq t^{1-q}\gamma_i \leq \dots \leq t^{1-q}w_i^n \leq \dots \leq t^{1-q}w_i^0 \quad \text{on } \overline{S_T}. \end{aligned}$$

Consequently, $\rho_i(x, t)$ and $\gamma_i(x, t)$ are minimal and maximal solutions of (5) since

$$t^{1-q}v_i^0 \leq t^{1-q}\rho_i \leq t^{1-q}u_i \leq t^{1-q}\gamma_i \leq t^{1-q}w_i^0 \quad \text{on } \overline{S_T}. \square$$

Theorem 4.2. *Let all the assumptions of Theorem 4.1 hold and $f_i(x, t, u_1, u_2)$ satisfy the one sided Lipschitz condition*

$$f_i(x, t, u_1, u_2) - f_i(x, t, u_1^*, u_2^*) \leq M_i(u_i - u_i^*), \quad M_i > 0.$$

Then the solution $u_i(x, t)$ of (5) is unique.

Proof. We have already proved (ρ_i, γ_i) are minimal and maximal solutions of (5) on $\overline{S_T}$. Hence it is enough to show that $\gamma_i(x, t) \leq \rho_i(x, t)$ on $\overline{S_T}$.

It is known from Theorem 4.1 that $\gamma_i(x, t) \leq \rho_i(x, t)$ on $\overline{S_T}$.

Let $p_i(x, t) = \gamma_i(x, t) - \rho_i(x, t)$. By the hypothesis, we get

$$\begin{aligned} \frac{\partial^q p_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 p_i(x, t)}{\partial x^2} &= \frac{\partial^q \gamma_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \gamma_i(x, t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q \rho_i(x, t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x, t)}{\partial x^2} \right] \\ &\leq f_i(x, t, \gamma_1(x, t), \gamma_2(x, t)) - [f_i(x, t, \rho_1(x, t), \rho_2(x, t))] \\ &\leq M_i | \gamma_i(x, t) - \rho_i(x, t) | \\ &\leq M_i | p_i(x, t) |, \end{aligned}$$

Thus we have $p_i(0, t) = 0, p_i(L, t) = 0$ on I and $\Gamma(q)t^{1-q}p_i(x, t) |_{t=0} = 0$ on Γ_T . Therefore, by Corollary 3.1, it follows that $p_i(x, t) \leq 0$. This proves that $\gamma_i(x, t) = \rho_i(x, t) = u_i(x, t)$ on $\overline{S_T}$ and proof is complete. $\square$

5 Discussion and Conclusions

In this work comparison results, we initially developed a monotone method for a system of R-L fractional reaction-diffusion equations with initial and boundary conditions (5) in which the function (f_1, f_2) is quasimonotone nondecreasing (nonincreasing). As an application of this method, the existence and uniqueness of the solution of the system of R-L fractional reaction-diffusion equations (5) are obtained when the function (f_1, f_2) is quasimonotone nondecreasing (nonincreasing).

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A Nonlinear Generalizations of Dynamic Inequalities on Time Scales

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Abstract

Considering the distinctive generalization of the Gronwall-Bellman inequality, in this paper we innovates new dynamical inequalities on time scale. These inequalities will be acted a profound tools for studying qualitative properties such as uniqueness, existence, stability and asymptotic behavior of solutions of dynamic equations on time scale.

Keywords: Dynamic and integral equations, Young's Inequality, Time Scale calculus.

2020 AMS subject classifications: 35A23, 37A50. ¹

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1 Introduction

Inequalities play pivoted role in the development of mathematics and sciences. Most of the inequalities have been acting as an extensive aid for the study of the qualitative properties of solutions of differential and integral equations. Consequently over the years many such new inequalities have been revealed, which are determined by uncontested applications. For example, see (1; 2; 3; 5; 10; 11; 13; 17) and the references therein.

In the analysis of some dynamic differential and dynamic integral equations, existing inequalities are inadequate. Thus, it is necessary to have some new inequalities in order to study these equations.

In this paper, we have established nonlinear dynamic inequalities on time scales and their applications. We generalized some crucial inequalities by making use of well-known Young's inequality. Here we have used certain analytical techniques to develop nonlinear dynamic inequalities on time scales. Our work is refinements and generality of existing results. Few applications of our are given to find importance of our work. It is an interesting and important techniques study differential and integral equations. Section 2 includes basic definitions, results and preliminaries which are useful for our main results. In Section 3, we are establishing some generalizations and variants of the inequalities established by A. A. El-Deeb (6). In Section 4, we present some applications of our main results. In particular, we discuss the boundedness behavior of solutions of a nonlinear dynamic equation on time scales.

2 Preliminaries

In this section, we are presenting some basic concepts on time scales which are required to set up our main result. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ represents the set of positive real numbers, $\mathbb{Z}$ represents the set of integers. $\mathbb{T}$, time scale has a topology that it inherits from standard topology on real numbers $\mathbb{R}$ and $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$.

Suppose $\phi : \mathbb{T} \rightarrow \mathbb{T}$ is defined as

$$\phi(t) := \inf \{x \in \mathbb{T} : x > t\}$$

and ϕ is called as forward jump operator and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined as

$$\rho(t) := \sup \{x \in \mathbb{T} : x < t\}.$$

Let $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ is the empty set. We say that a point $t \in \mathbb{T}$ is left dense if $\rho(t) = t$ and right dense if $\phi(t) = t$, where $\inf$

$\mathbb{T} < t < \sup \mathbb{T}$. A function $h : \mathbb{T} \rightarrow \mathbb{R}$ is said to be a right dense continuous (rd-continuous) function if h is continuous at right dense points and at left dense points in $\mathbb{T}$, left-sided limit exist and is finite. $C_{rd}(\mathbb{T})$ denotes the set of all such rd-continuous functions.

Let $\mu(t)$ and $\nu(t)$ be forward and backward graininess function on a time scale and are defined as $\mu(t) := \phi(t) - t$ and $\nu(t) = t - \rho(t)$ respectively.

Given a time scale $\mathbb{T}$ we introduce sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}_\kappa^\kappa$ as follows: If $\mathbb{T}$ has left scattered maximum t_1 , then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has right scattered minimum t_2 , then $\mathbb{T}^\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Finally $\mathbb{T}_\kappa^\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. We will give the definition of the generalized exponential function and its derivatives. We say that $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^\kappa$, $\mathfrak{R}$ be set of all regressive and rd-continuous functions. We define the set $\mathfrak{R}^+$ of all positive regressive elements of $\mathfrak{R}$ by $\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$. If $p \in \mathfrak{R}$, then we define the exponential function by

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h}, & h \neq 0 \\ z, & h = 0. \end{cases}$$

Lemma 2.1. [(12), Young's Inequality] If $x \geq 0, y \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

with equality holds if and only if $x = y$.

Lemma 2.2. Let $t_0 \in \mathbb{T}^\kappa$ and $k : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be continuous at (t, t) , where $t > t_0$ and $t \in \mathbb{T}^\kappa$. Let $k^\Delta(t, \cdot)$ be rd-continuous on $[t_0, \sigma(t)]$. If for any $\epsilon > 0$, there exist a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that

$$|[k(\sigma(t), \tau) - k(s, \tau)] - k^\Delta(t, \tau)[\sigma(t), s]| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$.

If k^Δ represent the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^t k(t, \tau) \Delta \tau,$$

yields

$$f^\Delta(t) = \int_{t_0}^t k^\Delta(t, \tau) \Delta \tau + k(\sigma(t), \tau).$$

Note 2.1. If $\mathbb{T} = \mathbb{R}$, then

$$\phi(t) = t, \mu(t) = 0, f^\Delta(t) = f'(t), \int_a^b f(t) \Delta t = \int_a^b f(t) dt \quad (1)$$

and if $\mathbb{T} = \mathbb{Z}$, then

$$\phi(t) = t + 1, \mu(t) = 1, f^\Delta(t) = \Delta f(t), \int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t) dt. \quad (2)$$

Lemma 2.3. Let $u, b \in C_{rd}$ and $a \in \mathfrak{R}^+$. Then

$$u^\Delta(t) \leq a(t)u(t) + b(t), t \geq t_0, t \in \mathbb{T}^k,$$

yields

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^t e_a(t, \sigma(\tau))b(\tau) \Delta \tau, t \geq t_0, t \in \mathbb{T}^k.$$

Lemma 2.4. If $p, q \in \mathfrak{R}$ and $a, b, c \in \mathbb{T}$, then

1. $e_p(t, t_0) = 1$ and $e_0(t, s) = 1$.
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$.

If $\mathbb{T} = \mathbb{R}$, then

$$e_a(t, t_0) = \exp \left(\int_{t_0}^t a(s) ds \right) \quad (3)$$

If $\mathbb{T} = \mathbb{Z}$, then

$$e_a(t, t_0) = \prod_{s=t_0}^{t-1} (1 + a(s)). \quad (4)$$

3 Main Results

In this section, we state and prove inequalities

Theorem 3.1. Suppose that $u(t), a(t), b(t), x(t), y(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ and $h(t, s), h^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. Let l, m, n be real numbers such that $l > m > 0, l > n > 0$. If $u^l(t)$ satisfies the inequality

$$u^l(t) \leq a(t) + b(t) \int_{t_0}^t [x(s)u^m(s) + y(s)u^n(s) + h(t, s)] \Delta s, \quad (5)$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq [a(t) + b(t) \int_{t_0}^t e_{\mu}(t, \phi(s)) \rho(s) \Delta s]^{\frac{1}{l}}, \quad (6)$$

for all $t \in \mathbb{T}_0$, where μ and ρ are

$$\mu(t) = b(t) \left(\frac{m}{l} x(t) + \frac{n}{l} y(t) \right) \quad (7)$$

and

$$\begin{aligned} \rho(t) = & x(t) \left[\frac{m}{l} a(t) + \frac{l-m}{l} \right] + y(t) \left[\frac{l-n}{l} + \frac{n}{l} a(t) \right] + \\ & h(\phi(t), t) + \int_{t_0}^t h^{\Delta}(t, s) \Delta s. \end{aligned} \quad (8)$$

Proof. Define a function $\tau(t)$ as

$$\tau(t) = \int_{t_0}^t [x(s)u^m(s) + y(s)u^n(s) + h(t, s)] \Delta s, \quad (9)$$

for all $t \in \mathbb{T}_0$. Then we have $\tau(t) \geq 0$ non decreasing function on $\mathbb{T}_0$ with $\tau(t_0) = 0$ and (5) can be written as

$$u^l(t) \leq a(t) + b(t)\tau(t), \quad (10)$$

for all $t \in \mathbb{T}_0$. From (10) and using lemma 2.1, we find

$$u^m(t) \leq \frac{l-m}{l} + \frac{m}{l} a(t) + \frac{m}{l} b(t)\tau(t), \quad (11)$$

for all $t \in \mathbb{T}_0$. Correspondingly, we get

$$u^n(t) \leq \frac{l-n}{l} + \frac{n}{l} a(t) + \frac{n}{l} b(t)\tau(t), \quad (12)$$

for all $t \in \mathbb{T}_0$. Now differentiating (9), we find

$$\tau^{\Delta}(t) = x(t)u^m(t) + y(t)u^n(t) + h(\phi(t), t) + \int_{t_0}^t h^{\Delta}(t, s) \Delta s, \quad (13)$$

for all $t \in \mathbb{T}_0$.

Now applying lemma 2.2 and using inequalities (11), (12), we obtain

$$\begin{aligned} \tau^\Delta(t) &\leq x(t) \left[\frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau(t) \right] + \\ &\quad y(t) \left[\frac{l-n}{l} + \frac{n}{l}a(t) + \frac{n}{l}b(t)\tau(t) \right] + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \\ \tau^\Delta(t) &\leq b(t) \left[\frac{m}{l}x(t) + \frac{n}{l}y(t) \right] \tau(t) + x(t) \left[\frac{m}{l}a(t) + \frac{l-m}{l} \right] + \\ &\quad y(t) \left[\frac{l-n}{l} + \frac{n}{l}a(t) \right] + h(\sigma(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \end{aligned} \quad (14)$$

$$\tau^\Delta(t) \leq \tau(t)\mu(t) + \nu(t). \quad (15)$$

for all $t \in \mathbb{T}_0$, where $\mu(t)$ and $\rho(t)$ are given by (3.3) and (3.4) respectively. Now a suitable application of lemma 2.3 to (15) with $\tau(t_0) = 0$, we have

$$\tau(t) \leq \int_{t_0}^t e_\mu(t, \phi(s)) \rho(s) \Delta s. \quad (16)$$

Finally, (6) follows from (10) and (16). □

Remark 3.1. If $\mathbb{T} = \mathbb{R}$ and $h(t, s) = 0$ in theorem 5 and using relation (1) and (3), then we obtain [(14), theorem 1 part(a₁)].

Remark 3.2. If $\mathbb{T} = \mathbb{Z}$ and $h(t, s) = 0$ in theorem 5 and using relation (2) and (4), then we obtain discrete analogue [(14), theorem 3 part(c₁)].

Remark 3.3. If $m = l$ and $n = 1$, then the theorem 5 reduces to the theorem 3.4 of (6).

Corolary 3.1. Let $u(t), x(t), y(t), b(t), h(t, s), h^\Delta(t, s), l, m, n$ be defined as same as in theorem 5 and $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ be a non decreasing function on $\mathbb{T}_0$. If $u^l(t)$ satisfies the inequality

$$u^l(t) \leq c^l(t) + b(t) \int_{t_0}^t [x(s)u^m(s) + y(s)u^n(s) + h(t, s)] \Delta s, \quad (17)$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq c(t) [1 + b(t) \int_{t_0}^t e_{\mu_1}(t, \phi(s)) \rho_1(s) \Delta s]^{\frac{1}{l}}, \quad (18)$$

for all $t \in \mathbb{T}_0$, where $\mu_1(t)$ and $\rho_1(t)$ are defined as

$$\mu_1(t) = b(t) \left[\frac{m c^{m-l}(t)x(t)}{l} + \frac{nc^{n-l}(t)y(t)}{l} \right] \quad (19)$$

and

$$\begin{aligned} \rho_1(t) = & c^{m-l}(t)x(t) \left[\frac{m}{l} + \frac{m-l}{l} \right] + c^{n-l}(t)y(t) \left[\frac{l-n}{l} + \frac{n}{l} \right] + \\ & h(\phi(t), t)c^{-l}(t) + \int_{t_0}^t h^\Delta(t, s)c^{-l}(s)\Delta s. \end{aligned} \quad (20)$$

Proof. Since $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is non decreasing function, from (17) we find,

$$\begin{aligned} \left[\frac{u(t)}{c(t)} \right]^l \leq & 1 + b(t) \int_{t_0}^t \left[c^{m-l}(s)x(s) \left(\frac{u(s)}{c(s)} \right)^m \right. \\ & \left. + y(s)c^{n-l}(s) \left(\frac{u(s)}{c(s)} \right)^n + c^{-l}(s)h(t, s) \right] \Delta s, \end{aligned} \quad (21)$$

for all $t \in \mathbb{T}_0$. By the theorem 5, we deduce that

$$u(t) \leq c(t) \left[1 + b(t) \int_{t_0}^t e_{\mu_1}(t, \phi(s)) \rho_1(s) \Delta s \right]^{\frac{1}{l}}, \quad (22)$$

where μ_1 and ρ_1 are given by (19) and (20). This completes the proof $\square$

Theorem 3.2. Let $u(t), x(t), y(t), a(t), b(t), h(t, s), h^\Delta(t, s), l, m, n$ be defined as same as in theorem 5 and $d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. If $u^l(t)$ satisfies the inequality

$$u^l(t) \leq a(t) + b(t) \int_{t_0}^t h(t, s) [x(s)u^m(s) + y(s)u^n(s) + d(s)] \Delta s, \quad (23)$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq \left[a(t) + b(t) \int_{t_0}^t e_{\mu_2}(t, \phi(s)) \rho_2(s) \Delta s \right]^{\frac{1}{l}}, \quad (24)$$

for all $t \in \mathbb{T}_0$ where μ_2 and ρ_2 are defined as

$$\begin{aligned} \mu_2(t) = & h(\phi(t), t) \left[b(t) \left(\frac{m}{l} x(t) + \frac{n}{l} y(t) \right) \right] + \\ & \int_{t_0}^t h^\Delta(t, s) b(s) \left(\frac{m}{l} x(s) + \frac{n}{l} y(s) \right) \Delta s \end{aligned} \quad (25)$$

and

$$\begin{aligned} \rho_2(t) = & h(\phi(t), t) \left[x(t) \left(\frac{l-m}{l} + \frac{m}{l} a(t) \right) + y(t) \left(\frac{l-n}{l} + \frac{n}{l} a(t) \right) + d(t) \right] + \\ & \int_{t_0}^t k^\Delta(t, s) \left[x(s) \left(\frac{l-m}{l} + \frac{m}{l} a(s) \right) + y(s) \left(\frac{l-n}{l} + \frac{n}{l} a(s) \right) + d(s) \right] \Delta s, \end{aligned} \quad (26)$$

for all $t \in \mathbb{T}_0$

Proof. Define a function $\tau_1(t)$ as

$$\tau_1(t) = \int_{t_0}^t h(t, s) [x(s)u^m(s) + y(s)u^n(s) + d(s)] \Delta s, \quad (27)$$

for all $t \in \mathbb{T}_0$. Then $\tau_1(t_0) = 0$ and $\tau_1(t) \geq 0$. Then as in the proof of theorem (5) from (23) we see that inequalities (11) and (12) holds for τ_1 . On differentiating (27) and applying lemma 2.2. we obtain

$$\begin{aligned} \tau_1^\Delta(t) = & h(\phi(t), t) [x(t)u^m(t) + y(t)u^n(t) + d(t)] + \\ & \int_{t_0}^t h^\Delta(t, s) [x(s)u^m(s) + y(s)u^n(s) + d(s)] \Delta s, \end{aligned} \quad (28)$$

for all $t \in \mathbb{T}_0$. From lemma (2.2) and using (11) and (12) in (28), we obtain,

$$\begin{aligned} \tau_1^\Delta(t) \leq & h(\phi(t), t) \left[x(t) \left(\frac{l-m}{l} + \frac{m}{l} a(t) + \frac{m}{l} b(t) \tau_1(t) \right) \right. \\ & \left. + y(t) \left(\frac{l-n}{l} + \frac{n}{l} a(t) + \frac{n}{l} b(t) \tau_1(t) \right) + d(t) \right] + \\ & \int_{t_0}^t h^\Delta(t, s) \left[x(s) \left(\frac{l-m}{l} + \frac{m}{l} a(s) + \frac{m}{l} b(s) \tau_1(s) \right) + \right. \\ & \left. y(s) \left(\frac{l-n}{l} + \frac{n}{l} a(s) + \frac{n}{l} b(s) \tau_1(s) \right) + d(s) \right] \Delta s. \end{aligned} \quad (29)$$

The inequality (29) gives

$$\begin{aligned}
 \tau_1^\Delta(t) \leq & \left[h(\phi(t), t) \left[b(t) \left(\frac{m}{l} x(t) + \frac{n}{l} y(t) \right) \right] \right. \\
 & + \int_{t_0}^t h^\Delta(t, s) b(s) \left(\frac{m}{l} x(s) + \frac{n}{l} y(s) \right) \Delta s \Big] \nu_1(t) \\
 & + h(\phi(t), t) \left[x(t) \left(\frac{l-m}{l} + \frac{m}{l} a(t) \right) \right. \\
 & + y(t) \left(\frac{l-n}{l} + \frac{n}{l} a(t) \right) + d(t) \Big] + \\
 & \int_{t_0}^t h^\Delta(t, s) \left[x(s) \left(\frac{l-m}{l} + \frac{m}{l} a(s) \right) + \right. \\
 & \left. y(s) \left(\frac{l-n}{l} + \frac{n}{l} a(s) \right) + d(s) \right] \Delta s.
 \end{aligned} \tag{30}$$

$$\tau_1^\Delta(t) \leq \mu_2(t) \tau_1(t) + \rho_2(t) \tag{31}$$

for all $t \in \mathbb{T}_0$ where $\mu_2(t)$ and $\nu_2(t)$ are given by (25) and (26) respectively. By (31), lemma 2.3 and with $\tau_1(t_0) = 0$, we obtain

$$\tau_1(t) \leq \int_{t_0}^t e_{\mu_2}(t, \phi(s)) \rho_2(s) \Delta s, \tag{32}$$

for all $t \in \mathbb{T}_0$.

Making use of the (32) in $u^l(t) \leq a(t) + b(t) \tau_1(t)$, we get the inequality (24). $\square$

Corollary 3.2. *Let $u(t), x(t), y(t), a(t), d(t), b(t), h(t, s), h^\Delta(t, s), l, m, n$ be defined as same as in theorem 6 and $c(t)$ be defined as same as in corollary 5. If*

$$u^l(t) \leq c^l(t) + b(t) \int_{t_0}^t h(t, s) [x(s) u^m(s) + y(s) u^n(s) + d(s)] \Delta s \tag{33}$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq c(t) \left[1 + b(t) \int_{t_0}^t e_{\mu_3}(t, \phi(s)) \rho_3(s) \Delta s \right]^{\frac{1}{l}}, \tag{34}$$

for all $t \in \mathbb{T}_0$, where μ_3 and ρ_3 are defined as

$$\begin{aligned}
 \mu_3(t) = & h(\phi(t), t) \left[b(t) \left(\frac{m c^{m-l} x(t)}{l} + \frac{n c^{n-l} y(t)}{l} \right) \right] + \\
 & \int_{t_0}^t h^\Delta(t, s) b(s) \left(\frac{m c^{m-l} b(s)}{l} + \frac{n c^{n-l} y(s)}{l} \right) \Delta s
 \end{aligned} \tag{35}$$

and

$$\begin{aligned} \nu_3(t) = & h(\phi(t), t) \left[b(t)c^{m-l}(t) \left(\frac{l-m}{l} + \frac{m}{l} \right) + \right. \\ & \left. y(t)c^{n-l}(t) \left(\frac{l-n}{l} + \frac{n}{l} \right) + d(t) \right] + \\ & \int_{t_0}^t h^\Delta(t, s) \left[x(s)c^{m-l}(s) \left(\frac{l-m}{l} + \frac{m}{l} \right) + \right. \\ & \left. y(s)c^{n-l}(s) \left(\frac{l-n}{l} + \frac{n}{l} \right) + d(s)c^{-l}(s) \right] \Delta s, \end{aligned}$$

for all $t \in \mathbb{T}_0$

Proof. Since $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is non decreasing function, from (17), we write

$$\begin{aligned} \left[\frac{u(t)}{c(t)} \right]^l \leq & 1 + b(t)h(t, s) \int_{t_0}^t \left[c^{m-l}(t)b(s) \left(\frac{u(t)}{c(t)} \right)^m + \right. \\ & \left. y(s)c^{n-l}(s) \left(\frac{u(t)}{c(t)} \right)^n + c^{-l}(s)d(s) \right] \Delta s, \end{aligned} \quad (36)$$

for all $t \in \mathbb{T}_0$. By theorem (6), we have

$$\frac{u(t)}{c(t)} \leq [1 + b(t) \int_{t_0}^t e_{\mu_3}(t, \phi(s)) \rho_3(s) \Delta s]^{\frac{1}{l}}, \quad (37)$$

where, $\mu_3(t)$ and $\nu_3(t)$ are given by (35) and (??) respectively for all $t \in \mathbb{T}_0$. This completes the proof $\square$

Theorem 3.3. Let $u(t), x(t), y(t), a(t), b(t), h(t, s), h^\Delta(t, s)$ be defined as same as in theorem 5, l, m be constant such that $l > m > 0$ and $d \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$, such that

$$0 \leq d(t, \alpha) - d(t, \beta) \leq \delta(t, \beta)(\alpha - \beta) \quad (38)$$

for all $t \in \mathbb{T}_0$ and $\alpha \geq 0, \beta \geq 0$, where $\delta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$. If

$$u^l(t) \leq a(t) + b(t) \int_{t_0}^t [d(s, u^m(s)) + h(t, s)] \Delta s, \quad (39)$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq [a(t) + b(t) \int_{t_0}^t e_{\mu_3}(t, \phi(s)) \rho_3(s) \Delta s]^{\frac{1}{l}}, \quad (40)$$

for all $t \in \mathbb{T}_0$, where $\mu_3(t)$ and $\rho_3(t)$ are defined as

$$\mu_3(t) = \delta\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) \frac{m}{l}b(t) \quad (41)$$

and

$$\rho_3(t) = d\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s. \quad (42)$$

Proof. Suppose that

$$\tau_2(t) = \int_{t_0}^t (d(s, u^m(s)) + h(t, s)) \Delta s, \quad (43)$$

for all $t \in \mathbb{T}_0$. Then from (39), we get

$$u(t) \leq [a(t) + b(t)\tau_2(t)]^{\frac{1}{l}},$$

$$u^m(t) \leq [a(t) + b(t)\tau_2(t)]^{\frac{m}{l}} (1)^{\frac{l-m}{l}}. \quad (44)$$

By lemma 2.1 and the inequality (44), we obtain

$$u^m(t) \leq \left[\frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau_2(t) \right] \quad (45)$$

From (43), (45), (39) and lemma 2.2, we deduce that

$$\begin{aligned} \tau_2^\Delta(t) &\leq \delta\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) \frac{m}{l}b(t)\tau_2(t) + \\ &\quad d\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \\ \tau_2^\Delta(t) &\leq \mu_3(t)\tau_2(t) + \rho_3(t), \end{aligned} \quad (46)$$

for all $t \in \mathbb{T}_0$ where $\mu_3(t)$ and $\rho_3(t)$ are given by (41) and (42) respectively. From (46) and lemma 2.3 with $\tau_2(t_0) = 0$, we obtain

$$\tau_2(t) = \int_{t_0}^t e_{\mu_3}(t, \phi(s)) \rho_3(s) \Delta s, \quad (47)$$

for all $t \in \mathbb{T}_0$. Using (47) in $u^l(t) \leq a(t) + b(t)\tau_2(t)$, we obtain (40). $\square$

Theorem 3.4. *Let $u(t), a(t), b(t), h(t, s), h^\Delta(t, s)$ be defined as same as in theorem 5, l, m be constants such that $l > m > 0$ and $d \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$ and $\lambda \in C(\mathbb{T}_0, \mathbb{R}_+)$ be a strictly increasing function with $\lambda(t_0) = 0$ such that*

$$0 \leq d(t, \alpha) - d(t, \beta) \leq \delta(t, \beta) \lambda^{-1}(\alpha - \beta), \quad (48)$$

for all $t \in \mathbb{T}_0$ and $\alpha, \beta \geq 0$ where $\delta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$ and

$$\lambda^{-1}(\alpha\beta) = \lambda^{-1}(\alpha)\lambda^{-1}(\beta), \quad (49)$$

for all $\alpha, \beta \in \mathbb{T}_0$ where λ^{-1} is the inverse of λ . If $u^l(t)$ satisfies inequality

$$u^l(t) \leq a(t) + b(t) \phi \int_{t_0}^t [d(s, u^m(s)) + h(t, s)] \Delta s, \quad (50)$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq a(t) + b(t) \int_{t_0}^t e_{\mu_5}(t, \phi(s)) \rho_5(s) \Delta s, \quad (51)$$

for all $t \in \mathbb{T}_0$, where $\mu_5(t)$ and $\rho_5(t)$ are defined as,

$$\mu_5(t) = \delta\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) \lambda^{-1}\left(\frac{nb(t)}{l}\right) \quad (52)$$

and

$$\rho_5(t) = d\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s. \quad (53)$$

Proof. Let

$$\tau_2(t) = \int_{t_0}^t [d(s, u^m(s)) + h(t, s)] \Delta s. \quad (54)$$

Then by lemma 2.1, we deduce that

$$u^l(t) \leq a(t) + b(t) \lambda(\tau_2(t)), \quad (55)$$

$$u^m(t) \leq \left[\frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l} \lambda(\tau_2(t)) \right], \quad (56)$$

$$\tau_2^\Delta(t) = d(t, u^m(t)) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \quad (57)$$

$$\begin{aligned} \tau_2^\Delta(t) \leq & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l} \lambda(\tau_2(t))\right) + \\ & h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \end{aligned}$$

$$\begin{aligned} \tau_2^\Delta(t) \leq & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l} \lambda(\tau_2(t))\right) - \\ & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l}\right) + d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l}\right) \\ & + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \end{aligned}$$

$$\begin{aligned} \tau_2^\Delta(t) \leq & \delta\left(t, \frac{l-m}{l} + \frac{m}{l} a(t)\right) \lambda^{-1}\left(\frac{m}{l} b(t) \lambda(\tau_2(t))\right) + \\ & d\left(t, \frac{l-m}{l} + \frac{m}{l} a(t)\right) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s \end{aligned}$$

$$\begin{aligned} \tau_2^\Delta(t) \leq & \delta\left(t, \frac{l-m}{l} + \frac{m}{l} a(t)\right) \lambda^{-1}\left(\frac{m}{l} a(t)\right) \lambda(\tau_2(t)) + \\ & d\left(t, \frac{l-m}{l} + \frac{m}{l} a(t)\right) + h(\phi(t), t) + \int_{t_0}^t h^\Delta(t, s) \Delta s, \end{aligned}$$

$$\tau_2^\Delta(t) \leq \mu_5(t) \tau_2(t) + \rho_5(t), \quad (58)$$

for all $t \in \mathbb{T}_0$

where μ_5 and τ_5 are given by (52) and (53) respectively.

From the inequality (49) and lemma (2.3), we get

$$\tau_2(t) \leq \int_{t_0}^t e_{\mu_5(t)}(t, \phi(s)) \rho_5(s) \Delta s, \quad (59)$$

for all $t \in \mathbb{T}_0$

The inequality (51) follows from (55) and (59). □

4 Applications

In this section, we present some applications of our main results. In particular, we discuss the boundedness behavior of solutions of a nonlinear dynamic equation on time scales.

Example 4.1. Consider the following dynamic integral equation:

$$u^l(t) = \Omega(t, \int_{t_0}^t \Gamma(s, u(s), h(t, s)) \Delta s, u(t_0) = a(t_0) = 0, \quad (60)$$

for all $t \in \mathbb{T}_0$, where $\Omega \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}, \mathbb{R})$ and $\Gamma \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ satisfying

$$|\Omega(t, j)| \leq a(t) + b(t)|j|, \quad (61)$$

and

$$|\Gamma(t, f, g)| \leq d(s, |f^m|) + g, \quad (62)$$

where $a, b, d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$

$$u(t) \leq [a(t) + b(t) \int_{t_0}^t e_{\mu_3}(t, \phi(s)) \rho_3(s) \Delta s]^{\frac{1}{l}}, \quad (63)$$

for all $t \in \mathbb{T}_0$, where l, m, μ_3, ρ_3 are given by the theorem 3.3. From (60) and by the assumption (61) and (62), we have

$$\begin{aligned} |u^l(t)| &= |\Omega(t, \int_{t_0}^t \Gamma(s, u(s), h(t, s)) \Delta s|, \\ u^l(t) &\leq \Omega(t, \int_{t_0}^t |\Gamma(s, u(s), h(t, s)) \Delta s|), \end{aligned}$$

$$|u^l(t)| \leq |a(t)| + |b(t)| \int_{t_0}^t [|d(s, u^m(s))| + h(t, s)] \Delta s, \quad (64)$$

for all $t \in \mathbb{T}_0$. By theorem 3.3 and (64) yields

$$u(t) \leq [a(t) + b(t) \int_{t_0}^t \mu_3(t, \phi(s)) \rho_3(s) \Delta s]^{\frac{1}{l}}. \quad (65)$$

Note that each function of the right hand side of (65) is known. Therefore if t is fix, then solution $u(t)$ is bounded.

Example 4.2. Consider the following dynamic equation on time scales:

$$(j^l)^\Delta(t) = \Theta(t, j(t), h(t, s)), t \in \mathbb{T}_0, \quad (66)$$

with initial condition $j(t_0) = C^{\frac{1}{l}}$, where $\Theta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^3, \mathbb{R})$.

Theorem 4.1. Assume that

$$|\Theta(t, j(t), h(t, s))| \leq h(t, s)[x(t)j^m(t) + y(t)j^n(t) + d(t)] \quad (67)$$

where $j, x, y, d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $h(t, s), h^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. If j is a solution of dynamic equation (66), then

$$j(t) \leq [C + \int_{t_0}^t e_{\mu_2}(t, \phi(s))\rho_2(s)\Delta s]^{\frac{1}{l}}, \quad (68)$$

for all $t \in \mathbb{T}_0$, where l, m, n, μ_2 and ρ_2 are given by theorem 3.2.

Proof. Clearly, the solution j of dynamic equation (66) with initial condition $j(t_0) = C^{\frac{1}{l}}$, satisfies the equivalent dynamic integral equation on time scales

$$j^l(t) = C + \int_{t_0}^t \Theta(s, j(s), h(t, s))\Delta s, \quad (69)$$

for all $t \in \mathbb{T}_0$ with initial condition $j(t_0) = C^{\frac{1}{l}}$. In fact from (69) and by using assumption (67), we have

$$\begin{aligned} |j^l(t)| &= |C + \int_{t_0}^t \Theta(s, j(s), h(t, s))\Delta s|, \\ |j^l(t)| &\leq |C| + \int_{t_0}^t |\Theta(s, j(s), h(t, s))\Delta s|, \end{aligned}$$

$$|j^l(t)| \leq |C| + \int_{t_0}^t h(t, s)[x(s)|j^m(s)| + d(s)|j^n(s)| + d(s)]\Delta s, \quad (70)$$

with initial condition $j(t_0) = C^{\frac{1}{l}}$. Then a suitable application of Theorem 3.2 with $a(t) = C$ and $b(t) = 1$ to (70), gives (68). $\square$

Example 4.3. Consider the following dynamic integral equation on time scales:

$$j^l(t) = \Phi\left(t, j(t), \int_{t_0}^t \Psi(s, j(s))\Delta s\right), t \in \mathbb{T}_0 \quad (71)$$

where $\Phi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ and $\Psi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R})$.

Theorem 4.2. Assume that

$$\left| \Phi(t) \left(t, j(t), \int_{t_0}^t \Psi(s, j(s)) \Delta s \right) \right| \leq a(t) + b(t) \left| \int_{t_0}^t \Psi(s, j(s)) \right|, \quad (72)$$

and

$$|\Psi(t, j(t))| \leq a(t)j^m(t) + y(t)j^n(t) + h(t, s), \quad (73)$$

where $j, x, y, a, b \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $h(t, s), h^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. If j is solution of dynamic equation (71), then

$$j(t) \leq \left[a(t) + b(t) \int_{t_0}^t e_\mu(t, \phi(s)) \rho(s) \Delta s \right]^{\frac{1}{l}}, \quad (74)$$

for all $t \in \mathbb{T}_0$, where l, m, n, μ and $\rho(t)$ are given by theorem 5.

Proof. In fact from (71) and by using assumption (72) and (73) we have

$$\begin{aligned} |j^l(t)| &= \left| \Phi \left(t, j(t), \int_{t_0}^t \Psi(s, j(s)) \Delta s \right) \right| \\ |j^l(t)| &\leq a(t) + b(t) \left| \int_{t_0}^t \psi(s, j(s)) \Delta s \right| \\ |j^l(t)| &\leq a(t) + b(t) \int_{t_0}^t |\psi(s, j(s))| \Delta s \\ |j^l(t)| &\leq a(t) + b(t) \int_{t_0}^t \left[x(s)|j^m(s)| + y(s)|j^n(s)| + h(t, s) \right] \Delta s. \end{aligned} \quad (75)$$

An application of theorem 5 to (75) we get (74). □

5 Conclusion

In this paper, we begin with review of nonlinear dynamic inequalities on time scales and their applications. We generalize nonlinear dynamic inequalities using Young's inequality and lemma 2.3. Our work includes several nonlinear dynamic inequalities on time scales. we have obtained new nonlinear dynamic inequalities. Our results are refinements and generality of some existing results in (6). Many existing results in the literature are reduced to the special case of our results when $m = n = 1$. Some applications to dynamic equations are provided. The findings of this study can be utilized in the analysis of dynamic equations. It is an interesting and new problems that upcoming researchers can develop similar inequalities for in their future research work, about differential and integral equations.

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ARMA and/or SETAR Estimation and Out-of-Sample Forecast of the Mean-Reversion Between Brent Crude Oil and Gasoline Prices on the Ghanaian Market

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Abstract

This study investigates linear and nonlinear mean reversion in bivariate price connections using VECM and SETAR models. It estimates relevant parameters and compares Self-Exciting Threshold Autoregressive (SETAR) prediction performance to linear models. Findings reveal that cointegration outperforms the linear Autoregressive (AR) models, with the 2-SETAR model showing significant asymmetric behavior and continuous price trends, indicating superior predictive capability.

Keywords: Mean-reversion; VECM; TVECM; SETAR; Time Series; ARMA; Ex-pump Ex-refinery

2020 AMS subject classifications: 62M10, 91B84 ¹

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1 Introduction

The observed skyrocketing movements of gasoline prices in positive response to significant fluctuation of the international crude oil market on the Ghanaian downstream market remain disheartening compared to slow smooth return to normal or perhaps decrements of same when the upstream declines Bacon [1991]. As a fast growing middle-income economy, Ghana joined the league of oil producers known as the Organization of Petroleum Exporting Countries (OPEC), when it discovered in commercial quantities the black gold in 2007. With the expectation of general public burnt on the benefit such discovery will inure cushioning on prices, rather it seems ill-disposed toward price mechanism in pricing the ex-pump gas oil for the ordinary citizens. In this study, the aim is to investigate the existence of long-run equilibrium or mean-reversion using bivariate analysis of paired prices and test for linear and nonlinear threshold-type mean-reversion of bivariate relationship, lastly, is to estimate the coefficient parameters of (non)linear VECM and threshold parameter value and to compared the forecast performance accuracies of the SETAR to linear models of the mean-reversion process.

Modeling of energy prices transmission mechanism is as-much a daunting issue essentially when the dynamics of crude oil and gasoline nexuses in-printed on the domestic retail markets are scrutinized daily, having been described as awkward and mischievous process. The puzzles over the facade of the implementation of the deregulation of energy market policy amid an automatic adjustment formula for pricing mechanism, and policy of hedging rewards. With such rolled-out programs being held in thrall by authorities notwithstanding the reduction in prices of international crude oil, tantamount to obvious public slanging and a call for the abrupt abrogation of such policy implementations.

In preceding two decades of the 21st century, large empirical research studies such as the works of Borenstein [1997], and Peltzman [2000] have identified that several of such distinctly separated vertical markets (i.e prices for a commodity at different stages of processing) to depict irregularities in price transmission response due to input differentials associated with consumption locally from their characteristic global value of extraction and production. Meyer and von Cramon-Taubadel [2004] and Tappata [2009] referred to this pricing phenomena as asymmetric price transmission (APT), which is often relaxedly referred to as "rocket and feathers" in crude oil and gasoline nexus.

The computational approach to unriddle the problem of this pricing phenomena was premiered by Engle [1987] traditional concept of cointegration that captures the existence of a stable long-run equilibrium relationship for any linear combinations of variables which were individually unit root. The concept has received extensive studies by works of Phillips and Ouliaris [1988], Stock and Watson [1988], and Johansen [1988]. The linear vector error correction model (VECM) technique

has proven to be an appropriate framework suitable for the modeling of symmetric mechanism in identifying the mean-reverting ratios that exist between two or more somewhat homogeneous variables. Unfortunately in the works of Huang et al. [2009], Natanelov et al. [2011] and Zhu et al. [2011] empirical investigations which incorporated the above linear technique to model price transmission from crude oil to gasoline unraveled the deficiency to explain for potential structural breaks or policy intervention acting as exogenous dummy variables inherent in the disequilibrium correction due to positive quick unusual peaking of the price and short-lived collapse readjustment just to soar in same direction.

The study of lack of symmetry in the phenomena of long-run equilibrium relationship was premiered by Tong and Lim [1980] and Balke [1997] for deviation of co-movements corrected within a band or percentage parameter. Their works permitted only threshold adjustment in the short-memory part if only the deviation from mean-reversion is above or below threshold specifications. Nonlinearities in cointegrated vector autoregressive models provides more supportive information about short-run price dynamics to external or internal shocks. Peltzman [2000] perceived such asymmetric response to imply a facetious potential gap in the price theory. Notwithstanding this perception, Bacon [1991] study of the U.K gasoline market and that of gal [2002] on Germany, France, UK, Italy, and Spain countries using an updated monthly data from 1985 to 2000 found evidence asymmetries in price adjustment between gasoline and crude oil. Likewise in the U.S market, Karenbrock et al. [1991], Shin [1992], Borenstein [1997], Balke [1998], Brown and Yucel [2000], bor [2002], and Bachmeier [2003] all found indications of quick upward response in gasoline when crude oil prices are rising than when it fall. This thesis follows closely to model the hypothesis of asymmetric relationship between before-tax and after-tax premium gasoline prices with one another and to the pass-through effect of international crude oil prices.

1.1 The mechanism of petroleum products price build-up.

The technique in determining prices of all ex-pump products sold in the country commence on the 'ex-refinery differential' which computation comprise of import parity of European prices determined by the Oil and LPG Marketing Companies (OMCs or LPGMCs) based on the cost price received from Bulk Oil Distribution Companies(BDCs) for each window. Afterwards, the build-up to 'ex-pump' constitutes the aggregated ex-depot of levies and margins, some special petroleum tax imputed by the authority. Particularly, the authority regulates the prices through negotiation and review of operational margins and charges of the marketers and dealers which incorporates transaction, distribution and administration cost that ensures reasonable profit margins to the computation of the 'ex-pump'. Shortfalls that emerges out of these negotiation process are covered by

the government funding through cross-subsidies of specific products to cushion consumers.

In June 2001, the policy makers introduced the deregulation policy and mechanization of the price build-up computation to the Automatic Petroleum Product Pricing Formula (APPPF). This policy was to equivocate speculation and believes of pass-on of extra undue cost to consumer and haven been remarked as not only a panacea that eradicate the leverage of cross-subsidy intervention for funding shortfalls, but more readily designed at full-cost recovery to alleviate the fiscal burden of products subsidization on the part of government except for imperative difficult times. The proponents of automatic adjustment mechanism also indicated that 'ex-pump' computations as true reflection of trends of global price indicators. Taxes and levies undulate ex-pump products through petroleum-related funds and road-fund which differ from year-in to year-out based on the fiscal policy of governments. The sum of taxes and levies imposed on products constitute an average of 41.91% from 2000 to 2009 and 27.22% from 2010 to 2015 of the indicative maximum price for ex-pump. Negative taxes and margins represent huge subsidies on products such as kerosine and premix which target rural area development and LPG for residents in cities to mitigate the severe suffering on consumers.

Government revenue received from the sector is earmarked for development from it taxes and levies are in the range of 10 to 14 percent. Subsidy intervention by the government inure infrastructural development which money is defrayed to the petroleum providers. For instant, government owed TOR debt is presently criticized over fairness of magnitude interventions, implications on socio-economic indicators and delays in developmental project. The question of 'whether or not these intervention in either cases were realistic', can be traced to the effects of such measures above, and its ability to keep prices low for a long time in the face of hiking international prices.

The strand of the price build-up computation twisted with the policy of price stabilization and recovery levy known as the Petroleum Debt Service Charge (PDSS) which was effective March 2002. Incorporation of such policy to recoup loses as a result of subsidies intervention and services the TOR debts. It was set at 95% of any decline in Ghana cedis value of oil import cost from the average of the levels prevailing world market prices. Thus, taxing petroleum consumers to pay directly for the cost of past consumed subsidies accrued. These two policy frameworks: *APPPF* and *PDSS* are assumed to absorb government burdensome expenditure of crude oil import and transactional cost subsidy and ensure full cost recovery for the BDCs and OMCs. Moreover, they seek to remove public sector dominance and promote liberalized market to ensure adequate supply.

2 Related Works

The influence of symmetric mean-reversion was investigated by overcoming the restriction of the linear model in cointegrated series Gregory [1996], Balke [1997]. These studies revealed the presence of nonlinear effects and possibly deterministic causes with inherent structural fractures that affects changes in policy interventions.

Failure to immediate adjustment after small shocks in equilibrium error is believed to follow a threshold autoregressive (TAR) process, a specified threshold bandwidth with unit roots lying within this range. As an extension of the Tsay's Tsay [1998] univariate approach, Hansen-Seo Hansen and Seo [2002] described a bivariate case of the Threshold of Vector Error Correction Model (from VECM to TVECM) with both unknown cointegrating vector and threshold value.³ However, if the adjustment behavior in the outer regimes is strong enough, the cointegration relationship is maintained in the longrun. A cointegration model with threshold adjustment was also proposed by Enders [2001]. Their model is restricted to a regime-specific coefficient of the first lag with threshold variable representing lagged equilibrium error in level series for self-exciting threshold autoregression (SETAR), or the first difference series for momentum threshold autoregression (MTAR) (see Enders [1998], Caner and Hansen [2001]).

The study into the regime-switching behavior of the bivariate cointegration or mean-reversion process permits extraction of a single paired price build-up mechanism of the Ghanaian gasoline markets application of threshold. The empirical research studies such as the works of Borenstein [1997], and Peltzman [2000] have identified that several of such distinctly separated vertical markets depict irregularities in price transmission response due to input differentials associated with consumption locally from their characteristic global value of extraction and production. Meyer and von Cramon-Taubadel [2004] and Tappata [2009] referred to this pricing phenomena as asymmetric price transmission (APT), which is often relaxedly referred to as "rocket and feathers" in crude oil and gasoline nexus.

Computationally approach to unriddle the problem of this pricing phenomena was premiered by Engle [1987] traditional concept of cointegration that captures the existence of a stable long-run equilibrium relationship for any linear combinations of variables which were individually unit root. The concept has received extensive studies by works of Phillips and Ouliaris [1988], Stock and Watson [1988], and Johansen [1988].

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³Seo [2006] discusses bootstrap tests of the null hypothesis of no cointegration in TVECM, while Gonzalo [2006] develop an asymptotic theory for testing the existence of a threshold effect in a VECM.

of co-movements corrected within a band or percentage parameter. Their works permitted only threshold adjustment in the short-memory part if only the deviation from mean-reversion is above or below threshold specifications.

In our empirical studies we have examined thoroughly the bivariate time series consisting of the monthly NIBOR rates of the maturities tomorrow next and 12 months. When modeling this bivariate time series, we find strong evidence for a two-regime TVECM being superior to a linear VECM, and in our out-of-sample forecasting the two-regime SETAR model gives much better prediction of the cointegration relation than a linear AR model. When testing a two-regime SETAR model for the cointegration relation against a three-regime model, the 2-regime model cannot be rejected at any reasonable significance level. In addition, we show how influential a few outliers may be by removing them from the time series and rerunning some of the statistical tests. Also, we have tested all the 66 possible pairs of Norwegian interest rates for cointegration, and we have tested the term spread of each pair for threshold effects, thus testing a linear model against a 2-regime model, as well as testing a 2-regime model against a three-regime model. We find a lot of cointegrated pairs, and we find evidence for a 2-regime model in approximately 50 % of the cases, and evidence for a three-regime model in some cases in this univariate time series analysis.

The term spread is modeled by a SETAR(3) model. When testing one regime against two regimes, the null hypothesis of one regime, i.e., a linear model, is strongly rejected, but when testing 2 regimes against 3 regimes, the null hypothesis of 2 regimes cannot be rejected at any reasonable significance level. Therefore, we choose a two-regime SETAR(3) model for the term spread. In the out-of-sample forecasting of the term spread, the SETAR(3) model with two regimes gives much better prediction of the term spread than the linear models.

In lieu of this, the paper set out to empirically model the asymmetric price build-up mechanism of the Ghanaian premium gasoline in nexuses with of Brent crude oil by investigating existence of long-run equilibrium or mean-reversion using bivariate analysis of paired prices; and to compared the forecast performance accuracies of the SETAR to linear models of the mean-reversion process to ARMA models.

The objective of this study is to empirically investigate asymmetric price build-up within the Ghanaian premium gasoline market. The question of whether the pricing mechanism of ex-refinery and ex-pump in nexuses with Brent crude oil mean-reverts in asymmetric manner. The study employs both Johansen and Juselius [1990] and Hansen and Seo [2002] techniques to test for the existence of long-run equilibrium and presence of threshold-type nonlinear effect in the disequilibrium correction term. The further estimation of coefficient parameters of the particular vector error correction model addresses the issue of percentage of adjustment speed and time elapsed to reoccur in mean-reversion. The techniques are lucid on

the insidious spontaneous adjustment responds or contemporaneity, and the extent of lag-delayed market prices respond or spontaneity to both external and internal shock.

The rest of the study in this paper is structured as follows: Section 2 discusses briefly the results of some empirical investigation that employed techniques in gasoline market. Section 3 outlines the statistical models and misspecification test methods employed to unravel the linear and nonlinear dynamics of the multivariate time series analysis. Section four contains the results and interpretation of the empirical analysis of investigation into both symmetric or asymmetric price build-up mechanism engulfed in the Ghanaian petroleum market.

3 Methodology

Following that each level pairs of price build-up transmission obeys the Engle-Granger [Engle, 1987] and Johansen [Johansen, 1988] traditional concept of cointegration: $w_t = X_t\beta \sim I(0)$ for each $x_{kt} \sim I(1)$ $k = 1, 2$ -dimensional or bivariate time series, and also exhibit presence nonlinear effects given the Hansen-Seo [Hansen and Seo, 2002] two step approach for 2R-TVECM(p) compact equation:

$$\Delta X_t = \alpha^{(1)}\beta' X_{t-1} \cdot I_{\omega_{t-d}(\beta) \leq \gamma} \quad (1)$$

$$+ \sum_{i=1}^{p-1} \Gamma^{(1)} \Delta X_{t-i} \cdot I_{\omega_{t-d}(\beta) \leq \gamma} + \epsilon_t^{(1)} \quad (2)$$

$$+ \alpha^{(2)}\beta' X_{t-1} \cdot I_{\omega_{t-d}(\beta) > \gamma} \quad (3)$$

$$+ \sum_{i=1}^{p-1} \Gamma^{(2)} \Delta X_{t-i} \cdot I_{\omega_{t-d}(\beta) > \gamma} + \epsilon_t^{(2)} \quad (4)$$

where α -coefficient of the loading matrix, β -cointegrating vectors of the coefficient matrix of the long-run equilibrium impact $\Pi = \alpha\beta'$, and $\Gamma^{(m)}$ -accumulating $p - 1$ lagged coefficient matrix of the short-run impact of the level representation for an $m = 2(or3)$ -regime switches defined for $I_{\{\cdot\}}$ is a dummy of the heavy-side indicator function:

$$I_{\omega_{t-d}(\beta)} = \begin{cases} 1, & \text{for } \omega_{t-d}(\beta) \leq \gamma^{(1)} \\ 0, & \text{for } \gamma^{(2)} < \omega_{t-d}(\beta) \end{cases} \quad (5)$$

and ϵ_t is a vector error such that $\epsilon_t \sim IN(0, \Sigma)$. This discrete linear models representation is similar to Tsay's Tsay [1998] generalization of Self-Exciting Autoregressive (m-SETAR(p)), m-SETAR(p) model due to discontinuity or structural breaks in policy implementation.

Seo [2006] suggests that the cointegration relation of a TVECM is could be modeled with univariate time series function such as SETAR model for either $m = 2(or3)$ -regimes, otherwise expressed as linear model. The sections below describes the expression for possible (non)linear univariate time series functions of the cointegrating relations, $w_t = X_t\beta$.

3.1 Test and estimation for linear or ARIMA models of the cointegrating process

From the Box and Jenkins [1994] procedure for fitting an autoregressive moving average denoted as ARMA(p, q) models of the random spread process is given as:

$$(1 - \sum_{j=1}^p \phi_j L^j) \omega_t = \sum_{j=1}^q \theta_j \epsilon_{t-j} + \epsilon_t \quad (6)$$

which follows that the residual series, $\epsilon_t \sim i.i.dN(0, \sigma^2)$ of uncorrelated white noise such that $\phi_j \neq \theta_j$ and $\phi_j, \theta_j \neq 0$ satisfying the stationary and linear stochastic difference property above. ARMA process has both causal and invertible properties. Thus, a restriction of the lag $q = 0$ gives an ARMA($p, 0$) or AR(p) model expressed in the form:

$$\omega_t - \sum_{j=1}^p \phi_j L^j \omega_t = \epsilon_t \quad (7)$$

such that $\phi_i \neq 0$, $\phi_i < 1$ and $\phi_0 < 1$ is causal process. The Markov process is AR(1) of the form $\omega_t = \phi_1 \omega_{t-1} + \epsilon_t$. Likewise, if $p = 0$ leads to a restricted Moving Average process, ARMA(0, q) or MA(q) model in the form:

$$\omega_t = \sum_{j=0}^q \theta_j L^j \epsilon_t \quad (8)$$

such that $\theta_i \neq 0$, $\theta_i < 1$, and $\theta_0 = 1$ is invertible process. The MA(∞) representation has unique solution

$$\omega_t = \frac{\sum_{j=0}^q \theta_j L^j}{1 - \sum_{j=1}^p \phi_j L^j} \epsilon_{t-j} \quad (9)$$

such that $\mu = 0$ and

$$(1 - \sum_{j=1}^p \phi_j L^j)^{-1} = 1 + \sum_{j=1}^p \phi_j L^j < \infty$$

if the characteristic root of the polynomial $1 - \sum_{j=1}^p \phi_j(z_j) \neq 0$ lie outside the unit circle, $|z| \leq 1$. The restrictions also apply.

However, for $\omega_t \sim I(1)$ non-stationary or unit root process may take several interesting deterministic components, D_t in form:

$$\omega_t = \sum_{j=0}^{\infty} \phi^j(\epsilon_{t-j} + D_{t-j}) \quad (10)$$

such that $\sum_{j=0}^{\infty} \phi^j \leq \infty$ with $E[\omega_t] = 0$ and $Var[\omega_t] = \frac{\delta_\epsilon^2}{1 - \phi}$. Thus, ϕ_ϵ does not decrease exponentially with ACF and PACF of ω_t persistently varying over time.

The deterministic components of integrated processes may be identified as $D_t=0$ for purely random walk process, $\Delta\omega_t = \epsilon_t$, or with constant drift $\Delta\omega_t = \delta + \epsilon_t$. D_t could also stem from either a linear trend-stationary $D_t = \delta + t$ or quadratic trend process respectively, $\delta + t^2$. An innovation dummy with an indicator function

$$|D_t| = \begin{cases} 1 & \text{for } D_t \geq 0 \\ 0 & \text{for } D_t < 0 \end{cases} \quad (11)$$

occurring due to some policy intervention causing break instability with an event phenomenon or perhaps seasonal component.

The effect of D_t 's in ω_t is that $\rho \geq 1$ with roots $z \geq 1$ trapped within the complex unit circle without unique stationary solution for certain unstable D_t cumulating behavior or trend embedded in ω_t . The unit root process expressed in form: $\phi_{p-1}^*(L)(1-L)^d\omega_t = \phi_{p-1}^*(L)\Delta^d\omega_t = \theta(L)\epsilon_t$ denoted as ARMA($p-d, q$) or ARIMA(p, d, q) of stationary covariance process with $\phi_{p-1}^*(L)$ roots avoiding the unit circle. The process of differencing, $\Delta^d\omega_t \sim I(d)$ with no deterministic term is said to be *integrated process* of order d stationary with $\epsilon_t \sim i.i.d, N(0, \sigma^2)$ expressed as:

$$\Delta^d\omega_t = \sum_{j=0}^{\infty} \psi_j\epsilon_{t-j} \quad (12)$$

such that $\psi_j(z) = \sum_{j=0}^{\infty} \psi_j z^j \leq \infty$ with $\psi_0 = 1$ convergent at $|z| \leq 1 + \delta$ for $\delta > 0$.⁴ Also, detrending or removing deterministic trend to achieve some filtered series of fluctuations around zero-horizontal line. By definition ω_t is trend-stationary if $\omega_t = a_j t + z_t$ where a_j are constants for $j = 0, 1, \dots$; while z_t is a stationary series such that $\omega_t = z_t$ is level-stationary if $a_j = 0$.

⁴Practically, after the second differencing, $I(d = 2)$, it is assumed that $\Delta^{d=1,2}\omega_t$ is stationary with exponentially decaying tail.

3.2 Estimation of the Parameters of Reduced form VECM

From the work of Johansen [1995] developed an estimate parameters of reduced form VECM using the Gaussian maximum likelihood (ML) procedure. The procedure is a maximum likelihood frame defined on the VECM given by

Removing the unrestricted deterministic variables, Δ_t and lagged differences and possibly additional non-modeled variables through a priori regression. For available T and p sample size and presample values respectively, we denote $\Delta X = [\Delta x_1, \dots, \Delta x_T]$, $X_{-1} = [x_0, \dots, x_{T-1}]$, $\varepsilon = [\epsilon_t, \dots, \epsilon_T]$, and $Y = [X_0, \dots, X_{T-1}]$ with $Y_{t-1} = [\Delta x'_{t-1}, \dots, \Delta x'_{t-p+1}]'$ and the parameters, $\Gamma = [\Gamma_1, \dots, \Gamma_{p-1}, \Psi_0]$; then compact matrix of VECM becomes

$$\Delta X = \alpha\beta' X_{-1} + \Gamma Y + \varepsilon \quad (13)$$

Assume estimates for matrix coefficient $\alpha\beta'$ of long-run part are known, we concentrating on the short-run dynamic adjustment effect using ordinary least square (OLS) estimator for each equation of the system $\hat{\Gamma} = (\Delta X - \alpha\beta' X_{-1})Y'(YY')^{-1}$. By substituting $\hat{\Gamma}$ in the above

$$\Delta X = \alpha\beta' X_{-1} + (\Delta X - \alpha\beta' X_{-1})Y'(YY')^{-1}Y + \varepsilon \quad (14)$$

$$\Delta X(I - Y'(YY')^{-1}Y) = \alpha\beta' X_{-1}(I - Y'(YY')^{-1}Y) + \hat{\varepsilon} \quad (15)$$

$$\Delta X M = \alpha\beta' X_{-1} M + \hat{\varepsilon}; \quad \text{for } M = I - Y'(YY')^{-1}Y \quad (16)$$

Now $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from Johansen [1995] reduced rank regression procedure based on the residual, $\hat{\varepsilon}$ above by defining a product of second moment matrix $S_{ij} = T^{-1} \Delta X_i M \Delta X_j'$ for $i, j = 0, 1$ such that:

$$S_{00} = T^{-1} \Delta X M \Delta X', \quad S_{01} = T^{-1} \Delta X M X_{-1}', \quad S_{11} = T^{-1} X_{-1} M X_{-1}' \quad (17)$$

for solving a generalized eigenproblem of the matrix $\lambda S_{11} V = S_{01}' S_{00}^{-1} S_{01} V$ with S_{11} and $S_{01}' S_{00}^{-1} S_{01}$ assumed to be symmetric and positive definite. First assume β is known, α can be estimated using OLS. Then substitute $\hat{\alpha}$ into the log-likelihood function which only is now a function of only β which is tantamount to reduce-rank regression to solve the k eigenvalue, λ_i for $0 \leq \lambda_i \leq 1$

$$|\lambda_i S_{11} - S_{01}' S_{00}^{-1} S_{01}| = 0 \quad (18)$$

such that $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_k \geq 0$ and the remaining $k_1 - k$ eigenvalue of the cofeature equal to zero. and it associated eigenvector, b_i of $V = [b_1, \dots, b_k]$ of normalized

$$\lambda_i S_{11} b_i = S_{01}' S_{00}^{-1} S_{01} b_i \quad (19)$$

such that $V'S_{11}V = I_K$. If estimator of β is obtained from the eigenvectors $\hat{\beta} = [b_1, \dots, b_r] \in b_k$ becomes the true value MLE $\hat{\beta}$ of β , then true value of $\hat{\alpha}$ will be given by $\hat{\alpha} = \alpha(\hat{\beta})$ below:

$$\hat{\alpha} = \Delta X M X'_{-1} \hat{\beta} (\hat{\beta}' X'_{-1} M X'_{-1} \hat{\beta})^{-1} \quad (20)$$

and finally the short-run parameter has a feasible estimator

$$\hat{I} = (\Delta X - \hat{\alpha} \hat{\beta}' X'_{-1}) X' (X X')^{-1} \quad (21)$$

The consistency and joint asymptotic normal are follow a more general assumptions than the Gaussianity condition.

3.2.1 Testing for cointegration and the estimation of VECM

We employ two kinds of tests to verify presence of long-run movement in series namely: the Enders [1998] two-step method; and the Johansen [1988] maximum likelihood ratio test for cointegration. Before we perform the estimate of the restricted parameters of the reduce form VECM using Johansen [1988] ML method in Appendix A1, the first assumption must be met first. Thus, the Johansen procedure jointly estimate for the $\Pi = \alpha\beta'$ (but not separating for only α or β); the rank of Π , the most significant r co-integrating vectors within β ; and hence α .⁸

The hypothesis test for cointegration rank is more defined under the null H_0 : $rank = k$ of full rank for $x_t \sim I(0)$ against an alternative H_a : $rank = r_0 < r \leq k$. This may be compared to the test statistic based on likelihood ratio (LR) test for finding r -cointegrating vectors

$$\Lambda_{LR}(H_a|H_0) = -T \ln[(1 - \lambda_{r+1}) \dots (1 - \lambda_k)] = -T \sum_{i=r+1}^k \ln(1 - \lambda_i) \quad (22)$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. The LR test discriminates empirically between the large eigenvalues, λ_i , for $i = 1, 2, \dots, r$ corresponding to stationary $\bar{\beta}' X_t \sim CI(1, 1)$ and smaller roots, λ_i , for $i = r + 1, r + 2, \dots, k$. LR is maximized by solving the generalized eigenvalue problem of the matrix

$$|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}| \quad (23)$$

⁸In practice, it obligatory that a stable system have $|\alpha_k| = [0, 1)$ for all k -dimension, since α explains the rate of the equilibrium error excluded for each variable movement for each period, t . The system may possibly diverge if α_k lies outside the boundary.

where $S_{ij} = T^{-1}Y_i'Y_j$, for $i, j = 0, 1$. The significance of the r largest canonical correlation for the reduced ranked of the Π matrix, which is established by the ML estimator of β is further confirmed by Johansen [1988]'s two different LR test for the cointegrating ranks.

3.3 Threshold Vector Error Correction Model (TVECM)

A m -regime TVECM for time series data, X_{it} of lag length p is generalized as

$$\Delta X_t = \alpha^{(\tau)}\beta'X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i^{(\tau)} \Delta X_{t-i} + \epsilon_t^{(\tau)} \quad ; \quad \gamma^{(\tau-1)} \leq \beta'X_{t-d} \leq \gamma^{(\tau)} \quad (24)$$

Thus, if the cointegrating vector representing the threshold variable at d lag-delay, $\beta'X_{t-d}$ is bounded and closed within some transition bandwidths of value, γ such that $\gamma^{(m-1)} \leq \beta'X_{t-d} \leq \gamma^{(m)}$ for $\tau = 1, \dots, m$ regimes of $m - 1$ threshold effects. The notation $\Gamma_i^{(m)}$ is m different matrices of short-run coefficient lagged differences of each regime, and $\Pi = \alpha^{(m)}\beta'$ is also m different matrices of long-run lagged level coefficients, and $\epsilon_t^{(m)}$ is m different innovation disturbance such that each $\epsilon_t \sim i.i.d.(0, \Sigma)$.

The threshold effect is active if only $0 < P(\beta'X_{t-d} \leq \gamma) < 1$, otherwise TVECM reduce to a regular VECM with $\Pi^{(1)} = \Pi^{(2)}$. Thus, allowing for 'threshold level stationary' equilibrium adjustments of linearly combine variables in different regimes.⁹ The estimation of multivariate TVECM developed by Tsay [1998] pose dimensionality problem in executing it, so we limit our analysis for $\tau = 1, m = 2$ regime of one threshold effect, TAR(1)

$$\begin{aligned} \Delta X_t &= \alpha^{(1)}\beta'X_{t-1}.I_{(\beta'x_{t-d} \leq \gamma)} + \alpha^{(2)}\beta'X_{t-1}.I_{(\beta'x_{t-d} > \gamma)} \\ &+ \sum_{i=1}^{p-1} \Gamma^{(1)} \Delta X_{t-i}.I_{(\beta'x_{t-d} \leq \gamma)} + \sum_{i=1}^{p-1} \Gamma^{(2)} \Delta X_{t-i}.I_{(\beta'x_{t-d} > \gamma)} + \epsilon_t \end{aligned}$$

where $I_{\{\cdot\}}$ is a dummy of the heavy-side indicator function given by

$$I_{\{\cdot\}} = \begin{cases} 1 & \text{for } \beta'X_{t-d} \leq \gamma \\ 0 & \text{for } \beta'X_{t-d} > \gamma \end{cases} \quad (25)$$

defines the different regimes encompassing both past changes in variables, ΔX_{it-p} and past deviation from equilibrating vector, $\beta'X_{t-1}$ likely to induce varying effects on current changes. This discrete adjustment explains that deviation from equilibrium relationship would not exist for system exceeding critical threshold

⁹It obvious that TVECM has more than 1 linear equilibrium correction relation.

at which point in time the benefits of adjustment are higher than the costs where economic agents move the system back to equilibrium. Thus, it captures the 'deepness' of asymmetric responses of the equilibrium error correction.

An alternative adjustment of the threshold specification is momentum TAR (M-TAR) model allows a variable to display differing amounts of AR decay depending on whether it is increasing or decreasing in $\beta' X_{t-d}$

$$I_{\{\cdot\}} = \begin{cases} 1 & \text{for } \Delta \hat{\epsilon}_{t-d} \leq \gamma \\ 0 & \text{for } \Delta \hat{\epsilon}_{t-d} > \gamma \end{cases} \quad (26)$$

the regime switch is initiated by the lagged changes in $\beta' X_{t-d}$. M-TAR describes the 'steepness' of the variation below or above the threshold value, γ for which the variable x_t follows two different stepwise stochastic processes.

In compact notation we obtain

$$\Delta X_t = \begin{cases} A_1' X_{t-1}(\beta) + \epsilon_t & \text{for } \omega_{t-1}(\beta) \leq \gamma \\ A_2' X_{t-1}(\beta) + \epsilon_t & \text{for } \omega_{t-1}(\beta) > \gamma \end{cases} \quad (27)$$

where $X_{t-1}(\beta) = (1, \omega_{t-1}(\beta), \Delta x_{t-i})'$ is a $K \times 1$ regressors, $A = [A_1 \ A_2]$ is $k \times 2 = (2p + 2) \times 2$ coefficient matrices which influences regime dynamics.

In an empirical estimation of the parameter $(\beta, A_1, A_2, \Sigma)$ TVECM, we may impose some form of constraints on the coefficients $A = [A_1 \ A_2]$ to be permanently fixed but allows only the constant and residual deviation to switch across regimes. The probability of existence of a threshold effect on the error correction is constrained in a balanced parameter, π_0 such that $\pi_0 < P(\omega_{t-1} \leq \gamma) < 1 - \pi_0$. The ML estimation of the TVECM which is given in appendix - A2 satisfies the assumption that the errors in each regime, $\epsilon_{it} \sim i.i.d$'s and the combined is vector martingale difference sequence that has a finite covariance matrix, $\Sigma = [\epsilon_t \epsilon_t']$. Thus, the normalization imposed on β and the threshold restriction band $\pi_0 \leq \frac{1}{n} \sum_{t=1}^n 1(x_t' \beta) \leq 1 - \pi_0$ is minimized with each regime possessing such minimum fraction, p_0 of the entire sample.

3.4 Estimation of the Parameters of Reduced form TVECM

From the technicalities of the Hansen and Seo [2002], and Hansen [2005], we rewrite the TVECM of the compact form as

$$\Delta x_t = M_1' X_{t-1}(\beta) \cdot d_{1t} + M_2' X_{t-1}(\beta) \cdot d_{2t} + \epsilon_t \quad (28)$$

where $d_{1t} = I_{\omega_{t-1}(\beta) \leq \gamma}$ and $d_{2t} = I_{\omega_{t-1}(\beta) > \gamma}$ and $\epsilon_t \sim i.i.d$ Gaussian applying a normalization on β . They proposed the quasi-MLE of the parameters $(\beta, M_1, M_2, \Sigma)$

denoted by $(\tilde{\beta}, \tilde{M}_1, \tilde{M}_2, \tilde{\Sigma})$ from the residual vector defined on the parameters of estimate $(\beta, M_1, M_2, \Sigma, \gamma)$ will be given as

$\tilde{\epsilon}_t(\beta, M_1, M_2, \Sigma, \gamma) = \Delta x_t - \tilde{M}' X_{t-1}(\tilde{\beta}) = \Delta x_{t-1} - seom_1' X_{t-1}(\beta) d_1(\beta, \gamma) - M_2' X_{t-1}(\beta) d_2(\beta, \gamma)$. Then the Gaussian likelihood of the residual is

$$InL(\beta, M_1, M_2, \Sigma, \gamma) = -\frac{n}{2} \log |\Sigma| - \frac{1}{2} \sum_{t=1}^n (\beta, M_1, M_2, \Sigma, \gamma)' \Sigma^{-1} \epsilon_t(\beta, M_1, M_2, \Sigma, \gamma)$$

Estimation of the ML on the above $InL(\beta, M_1, M_2, \Sigma, \gamma)$ focus partially first on the (M_1, M_2, Σ) parameters by holding fixed (β, γ) . This yields estimates of the coefficient matrices, $\hat{M}_1(\beta, \gamma)$ and $\hat{M}_2(\beta, \gamma)$, which are obtained by conditional sequential LS regression of Δx_t on $X_{t-1}(\beta)$ for each regime subsamples given below

$$\hat{M}_1(\beta, \gamma) = \left(\sum_{t=1}^n X_{t-1}(\beta) X_{t-1}(\beta)' d_1(\beta, \gamma) \right)' \left(\sum_{t=1}^n X_{t-1}(\beta) \Delta x_{t-1} d_1(\beta, \gamma) \right) \quad (29)$$

$$\text{for } \omega_{t-1} \leq \gamma \quad (30)$$

$$\hat{M}_2(\beta, \gamma) = \left(\sum_{t=1}^n X_{t-1}(\beta) X_{t-1}(\beta)' d_2(\beta, \gamma) \right)' \left(\sum_{t=1}^n X_{t-1}(\beta) \Delta x_{t-1} d_2(\beta, \gamma) \right) \quad (31)$$

$$\text{for } \omega_{t-1} > \gamma \quad (32)$$

Substituting into the residual vector and covariance matrix defined on (β, γ) given as

$$\hat{\epsilon}_t(\beta, \gamma) = \epsilon_t(\beta, \hat{M}_1'(\beta, \gamma), \hat{M}_2'(\beta, \gamma), \gamma) \text{ and} \quad (33)$$

$$\hat{\Sigma}(\beta, \gamma) = \frac{1}{n} \sum_{t=1}^n \hat{\epsilon}_t(\beta, \gamma) \hat{\epsilon}_t(\beta, \gamma)' \quad (34)$$

By so doing it produces a concentrated Gaussian likelihood function

$$L_n(\beta, \gamma) = (\beta, \hat{M}_1'(\beta, \gamma), \hat{M}_2'(\beta, \gamma), \Sigma(\beta, \gamma), \gamma) \quad (35)$$

$$= -\frac{n}{2} \log |\hat{\Sigma}(\beta, \gamma)| - \frac{np}{2} \quad (36)$$

The MLE of the $L_n(\beta, \gamma)$ is seen to act as a minimizer of $\log |\hat{\Sigma}(\beta, \gamma)|$ subject to the normalization imposed of β discussed in earlier in previous paragraph and the

constraint imposed within the probability of threshold effect of the trim parameter, π_0

$$\pi_0 \leq n^{-1} \sum_{t=1}^n 1(x'_t \beta \leq \gamma) \leq 1 - \pi_0 \quad (37)$$

The MLE solution for M_1 and M_2 are $\hat{M}_1 = \hat{M}_1(\beta, \gamma)$ and $\hat{M}_2 = \hat{M}_2(\beta, \gamma)$

3.5 The m-SETAR(p) of the cointegrating relationship

Following Enders & Granger [1998] generalization of the Dickey & Fuller [1979] of long-run process to assume the TAR(m, p) consisting of m -AR(p) parts with variation occurring within an 'embedding dimension' of $-\infty = \gamma_0 < \dots < \gamma_\tau = \infty$ for $\tau = 1, \dots, m$ -regimes as one AR(p) process switches to another according to the specific value of a threshold parameter, γ_τ

$$\omega_t = \left(\sum_{j=1}^p \phi_j \omega_{t-j} + \sum_{j=1}^p \sigma_{\tau j} \epsilon_t \right) I_\tau \{ \gamma_{j-1} \leq z_{t-d} < \gamma_j \} \quad (38)$$

where $I_\tau \{ \gamma_{j-1} \leq z_{t-d} < \gamma_j \}$ is the heavy-side indicator function that determines the observed or partially unobserved properties under different or discontinuous decay nonlinear AR(p) states as series crosses the threshold value to the next $m - 1$ -regime that quantify the magnitude(speed) of decay adjustment to new trend path. The term $\sum_{j=1}^p \sigma_{\tau j} \epsilon_t$ allows different error terms variances across regimes.

Tong [1983] shows that the self-exciting threshold autoregressive (SETAR(m)) is a special kind of TAR(m, p) with a dynamic threshold variable, $z_{t-1} \approx z_{t-p-d-1}$ where $d \in \tau$ is threshold time-delay well-defined in threshold range, $\gamma_\tau \in [\gamma_1, \gamma_m]$. TAR(2) or SETAR(1) is a simplified piece-wise linear approximation of AR($p - d - 1$) expressed in general form as

$$\omega_t = \begin{cases} \phi_{10} + \sum_{j=1}^p \phi_{1j} \omega_{t-j} + \sigma_1 \epsilon_t & \text{for } z_t \leq \gamma_1 \\ \phi_{20} + \sum_{j=1}^p \phi_{2j} \omega_{t-j} + \sigma_2 \epsilon_t & \text{for } \gamma_1 < z_t \leq \gamma_2 \end{cases} \quad (39)$$

where ϕ_i are the constant speed parameters.

Based on an ordinary least square estimation of TAR-model, we employ the Hansen [1997] and Hansen and Johansen [1999] to test a H_0 : linearity 1-regime model or TAR(1) against H_a : $m > 1$ -regime model, TAR(m) subject to the $\sup_{F\text{-type}}$ test statistics of the nonstandard F_{jk} distribution

$$F_{jk} = T \frac{SSR_j - SSR_k}{SSR_k} \quad ; \quad j < k \quad (40)$$

where SSR_j the minimum sum of squared residuals when fitting a j -regime SETAR against a k -regime SETAR model. A special estimation of the H_0 model is the SSR_1 . The threshold value for k -regime which minimizes the SSR_k is computed by performing a grid search conditional on the value of the previous threshold that minimizes SSR_j . Under the H_0 , vigorous re-sampling of the residuals is done using replicating bootstrap estimation of the threshold parameters and computation F_{jk} .

The residuals analysis of the models are

$$\hat{\epsilon}_t = \omega_t - \left\{ \hat{\phi}_{10} - \sum_{j=1}^p \hat{\phi}_{1j} \omega_{t-j} \right\} I_{\omega_{t-d} \leq \hat{\gamma}} - \left\{ \hat{\phi}_{20} - \sum_{j=1}^p \hat{\phi}_{2j} \omega_{t-j} \right\} I_{\omega_{t-d} > \hat{\gamma}} \quad (41)$$

The study employs the following performance criteria to compare the best Root mean square error

$$RMSE = \sqrt{\frac{\sum_{t=1}^T (\omega_t - f_t)^2}{T}} \quad (42)$$

Mean absolute error

$$MAE = \frac{1}{n} \sum_{t=1}^n |\omega_t - f_t| \quad (43)$$

Mean absolute percentage error

$$MAPE = 100 \times \frac{1}{n} \sum_{t=1}^n \left| \frac{\omega_t - f_t}{\omega_t} \right| \quad (44)$$

where f_t and x_t denote the forecast and actual values at t period respectively.

The RMSE and MAE are a measure of fit, which indicates how well model fits with the historical data. It should be noted that unlike the MAPE which is scale invariant, both RMSE and MAE criteria rely on scale of the dependent variable. This require that the criteria are implemented as relative measures in forecast comparison for similar series across varying models. A smaller value indicates stronger forecasting power for the respective model.

4 Analysis and Discussion of Results

4.1 Data Assimilation and Exploration

Since cointegrating relationship (ω_t) in a 2(3)-regime TVECM could be specified as 2(3)-regime TAR model, so we proceed to estimate the ω_t from our APT

models. Each computed vector of ω_t extraction comprises of 431 biweekly samples. An initial step to modeling these nonlinear price adjustment relationship is to understand graphically. Figure 1 displays pattern of the estimated mean-reversion at each stage. It indicates undoubtedly almost non-stationary strong upward trend for both initial and overall stages with respect to Brent crude oil in nexuses with

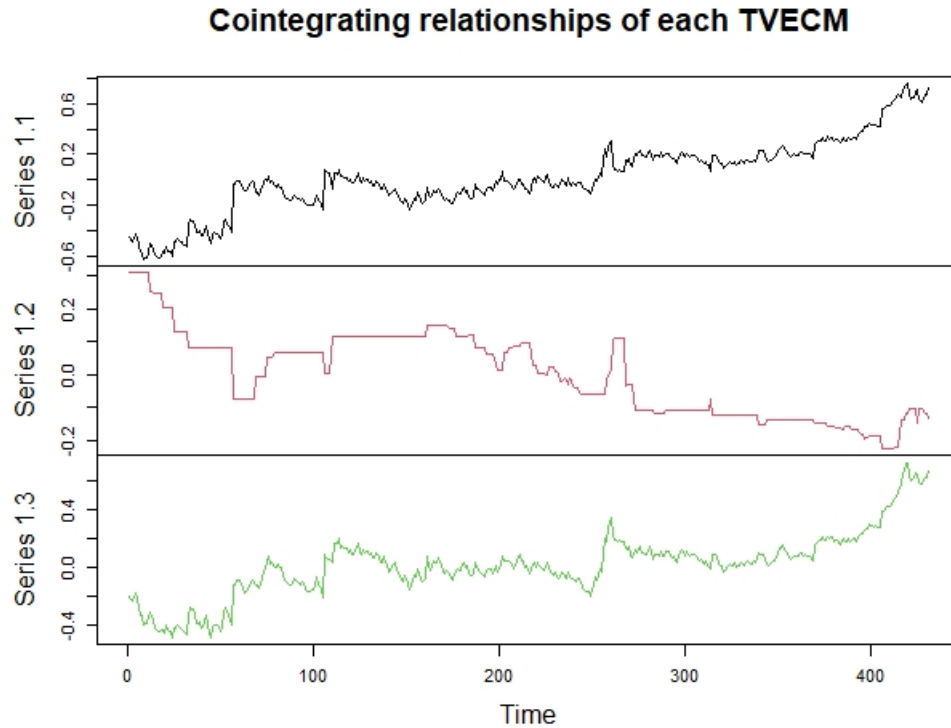


Figure 1: Estimated cointegrating relationship (ω_t) in a TVECM at each stage of price transmission 02/01/1999 - 27/06/2015.

both ex-pump and ex-refinery respectively, whereas the second stage show steady downward trend between paired localized co-movement in the sense of BDCs and OMCs. Figure 2 depicts the empirical densities for all three estimated cointegrating relationships in a TVECM.

4.2 Test for 2-regime against 3-regime TAR model

The study further employs the Hansen's testHansen [1999] procedure to confirm the linearity of the cointegrating relationships extracted from the TVECM with 1000 bootstrap replication and trimmed off at 10% sample observations. Table 1 and Figure 3-5 indicate the test results and plots respectively, for H_0 of linear

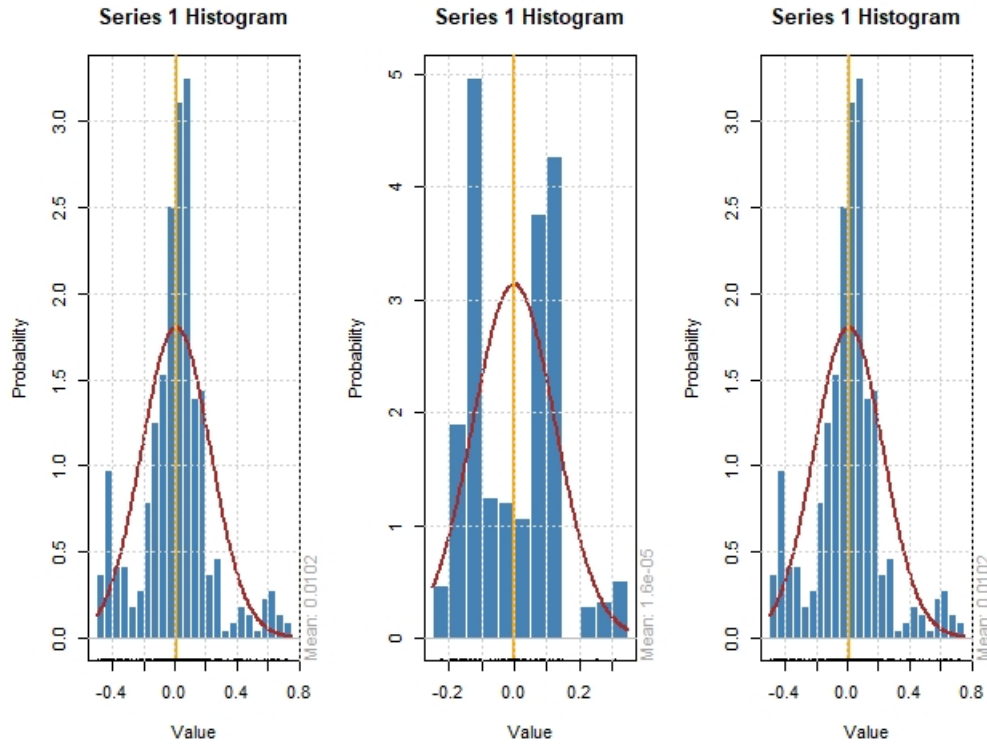


Figure 2: Plot of the empirical densities for estimated cointegrating relationship (ω_t) in a TVECM at each stage of price transmission 02/01/1999 - 27/06/2015.

cointegration relationship against an alternative of 2-regime threshold-type non-linear effect, with option for testing 2-regime versus 3-regime threshold-type AR model at each stage. We observe that H_{0_1vs2} of 1-regime tested against 2-regime is rejected at 1% and 5% significance levels for both initial and overall stages respectively. Whereas, on the contrary, the test for H_{0_2vs3} of 2-regime versus 3-regime cannot be rejected at any significance level for any of the modeling stage.

4.3 Modeling of SETAR model

As an initial model checking procedure, the study identifies appropriate SETAR models by employing the graphical diagnostic plots of the sample ACF and PACF of the residuals addressing degree of correlation between neighboring observations, and then test for normality which facilitates to represent $ARMA(p, d)$ modeling. A forecasting exercise of the term spread for each level paired PBU from their respective TVECMs following that of Di Narzo Di Narzo [2008] procedure of out-of-sample forecasting of the crack spread

Table 1: Hansen's Test for 1vs2, 1vs3 and/or 2vs3-regime TAR of the cointegrating relationship at each stage of APT

	SETAR Test					
	Test statistics		Critical Value			
	F-test	p_v	90%	95%	97.5%	99%
Stage 1:						
H_{1vs2}	40.86	0.02	24.82	31.12	37.93	46.85
H_{1vs3}	50.41	0.09	49.12	58.28	66.24	74.84
H_{2vs3}	8.71	0.85	20.91	25.92	30.92	40.26
Stage 2:						
H_{1vs2}	9.32	0.71	25.65	33.31	43.42	55.02
H_{1vs3}	31.27	0.48	62.10	74.61	87.77	103.41
H_{2vs3}	21.48	0.28	34.60	44.96	55.85	71.07
Overall:						
H_{1vs2}	27.42	0.05	22.97	27.12	30.59	40.14
H_{1vs3}	43.99	0.09	42.65	49.32	55.16	66.41
H_{2vs3}	15.57	0.38	22.24	26.72	30.13	34.24

Note:

$\sup LM$ is the supremum of the heteroskedastic-consistent Lagrange Multiplier test statistics, and p_{value} is the reasonable significant level for the reject of H_0 .

The term spread is modeled by a SETAR(3) model. When testing one regime against two regimes, the null hypothesis of one regime, i.e., a linear model, is strongly rejected, but when testing 2 regimes against 3 regimes, the null hypothesis of 2 regimes cannot be rejected at any reasonable significance level. Therefore, we choose a two-regime SETAR(3) model for the term spread. In the out-of-sample forecasting of the term spread, the SETAR(3) model with two regimes gives much better prediction of the term spread than the linear models.

4.4 Forecast Comparison for ARMA and SETAR of Mean Reversion in Price Transmission

Further estimation of coefficients of the linear and nonlinear mean-reversion in the univariate sense are made based on the result of Hansen and Johansen [1999] method which test parameter constancy for number of regimes in TAR model. The model considers only 2-regime threshold for price transmission at initial and the overall stages. A search order of linear model using the Box and Jenkins [1994]

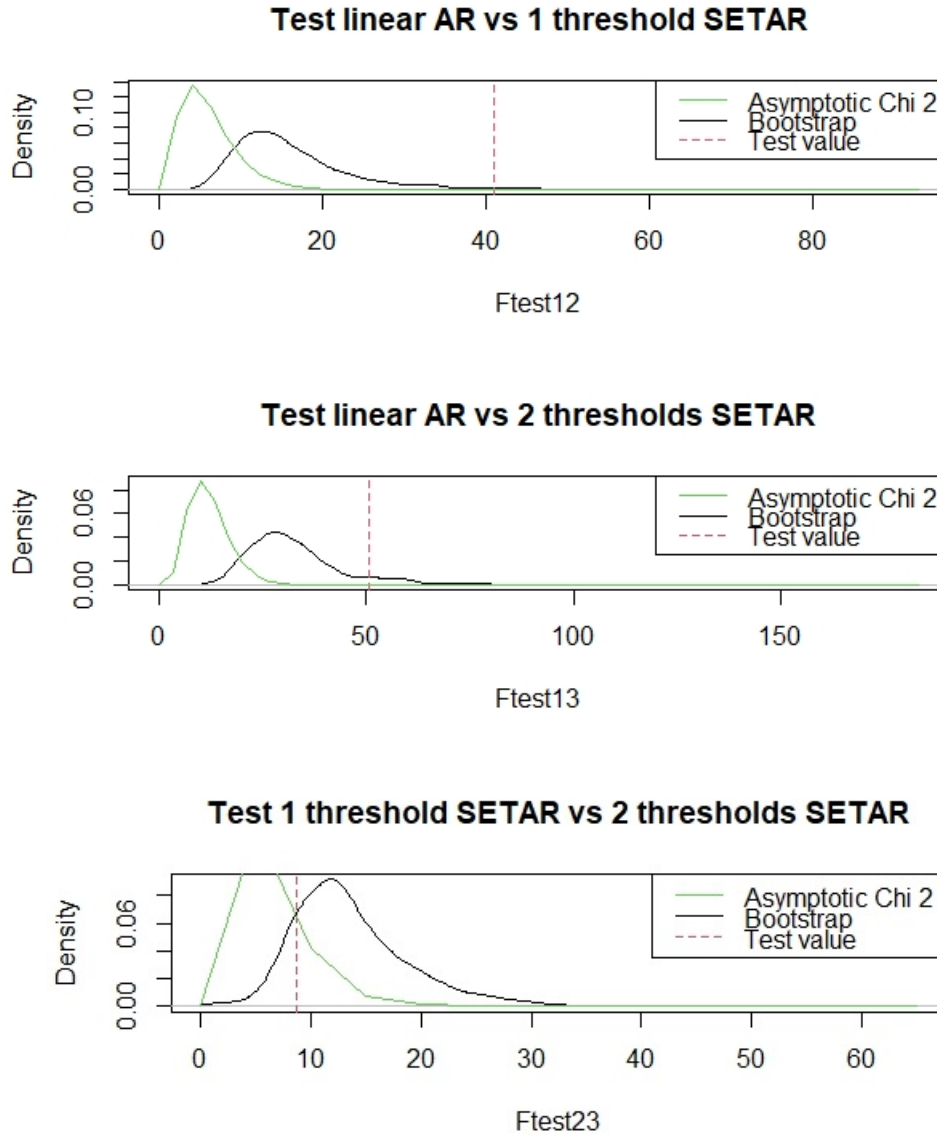


Figure 3: Biweekly Cointegrating relationship of the spread or mean-reversion at each stage of price transmission 02/01/1999 - 27/06/2015.

test of autocorrelation were conducted and concluded on the lag orders using ACF and PACF for both ARMA(p,q) and 2-SETAR(m,d) as in Larsen [2012]. Figure 6 - 8 represent the plot of volatile residuals, ACF and PACF for SETAR of ARMA model for cointegrating series of price transmission stage. The estimated ACFs and PACFs points to a SETAR of embedded dimensional orders, $m = 5$ and

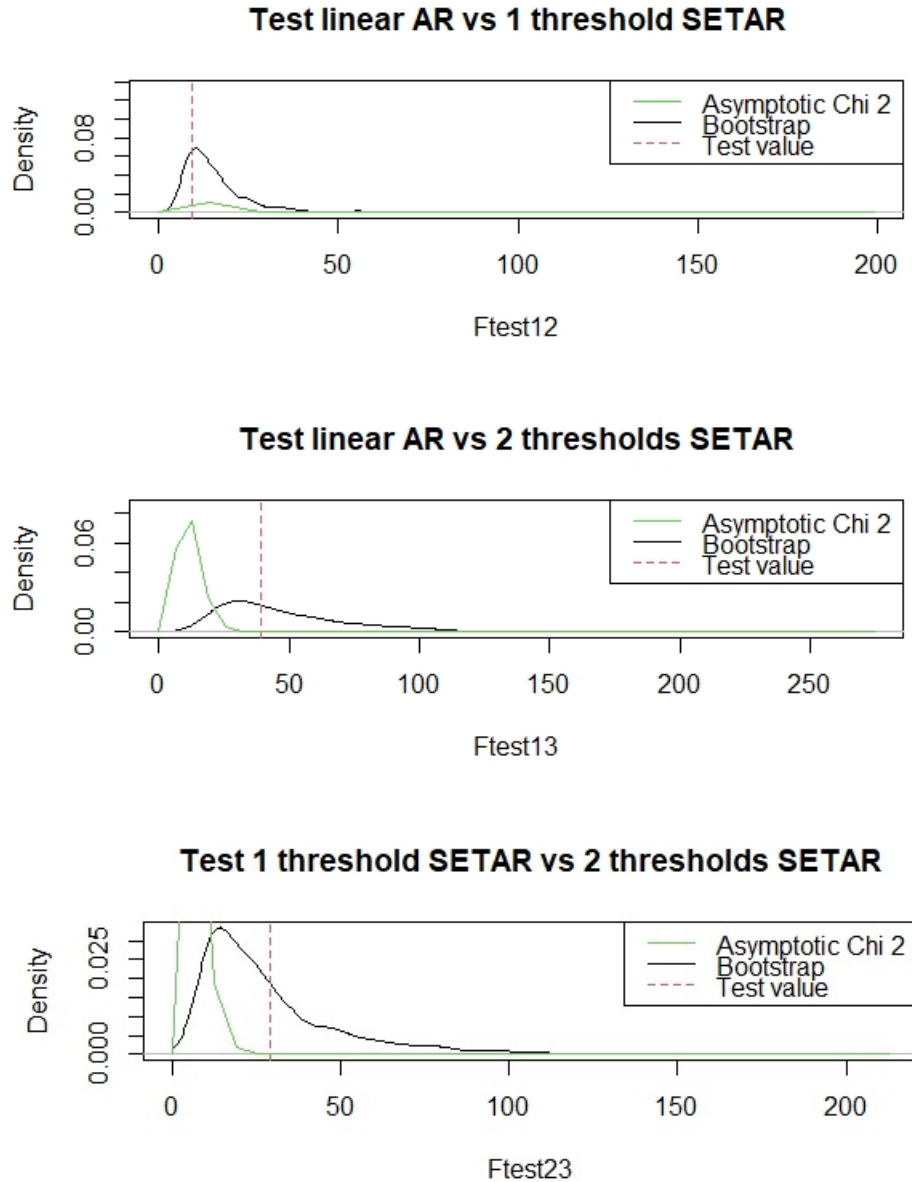


Figure 4: Biweekly Cointegrating relationship of the spread or mean-reversion at each stage of price transmission 02/01/1999 - 27/06/2015.

$m = 4$ for stage one and overall stage transmission respectively in Figure 8 & 10. Also from Figure 9, the estimated ACF and PACF confirmed a linear cointegrating relationship, as it points to ARMA of lag order $p = 4$ and $q = 2$ as the most suitable to estimate the second stage .

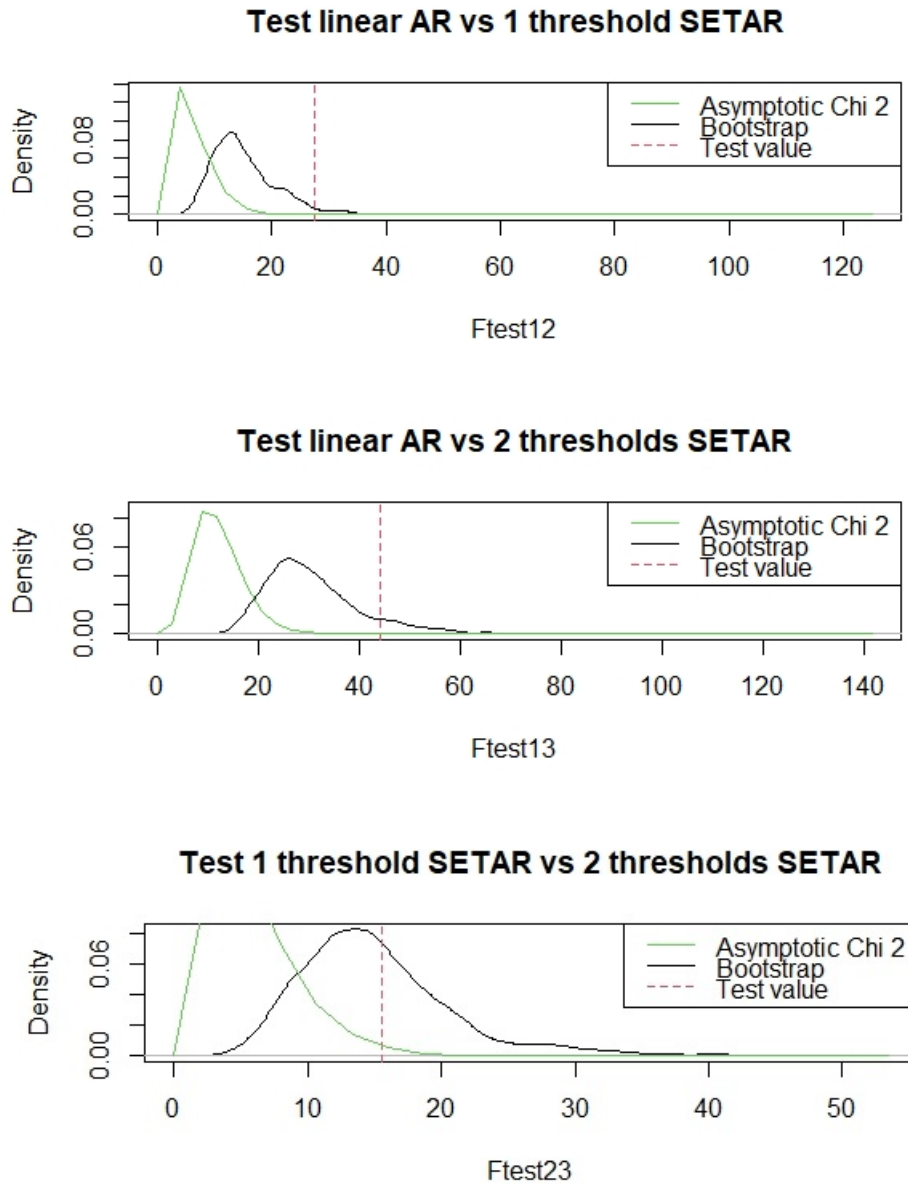


Figure 5: Biweekly Cointegrating relationship of the spread or mean-reversion at each stage of price transmission 02/01/1999 - 27/06/2015.

The results of long-run adjustment equation for each stage of price transmission displayed in Table 2 to Table 3 show the estimated coefficients and the vari-

ARMA and/or SETAR Estimation Models

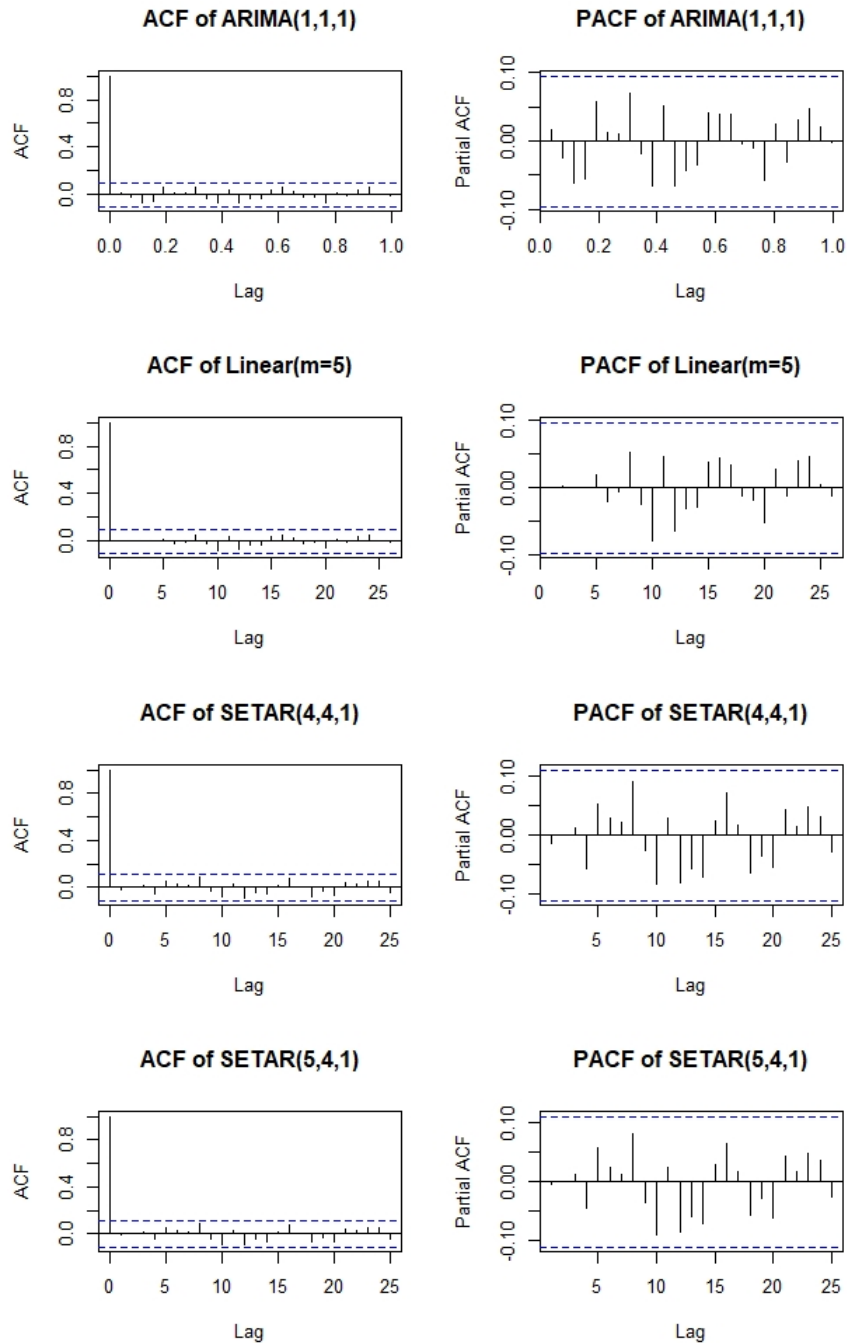


Figure 6: Plot of estimated ACF and PACF functions for the residuals of fitted AR(7), ARIMA(1,1,1), SETAR(5,5,1), STAR(11), NNET(4), AAR(4) in stage 1 PT mean-reversion

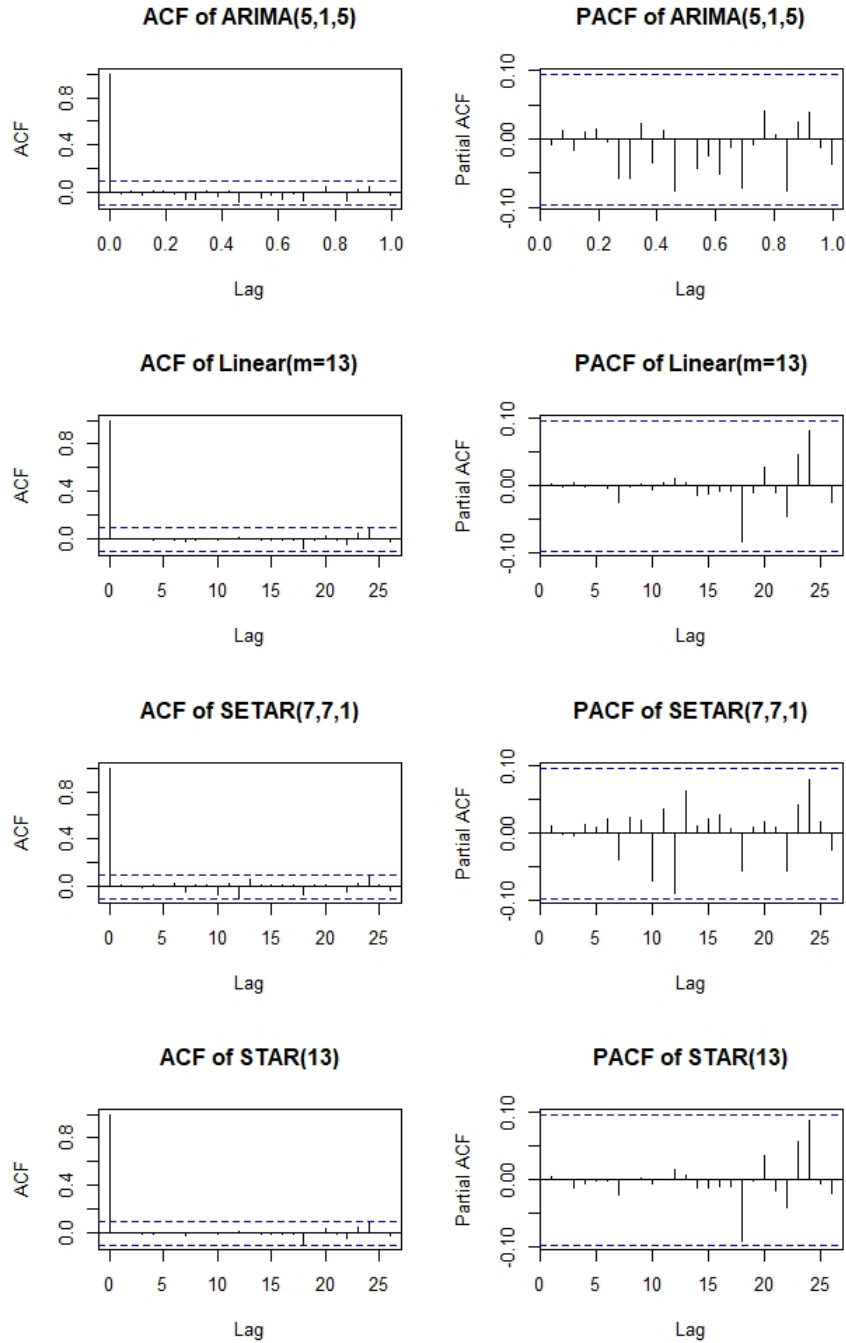


Figure 7: Plot of estimated ACF and PACF functions for the residuals of fitted AR(13), ARIMA(1,1,1), SETAR(7,7,1), STAR(13), NNET(13), AAR(13) in stage 1 PT mean-reversion

ARMA and/or SETAR Estimation Models

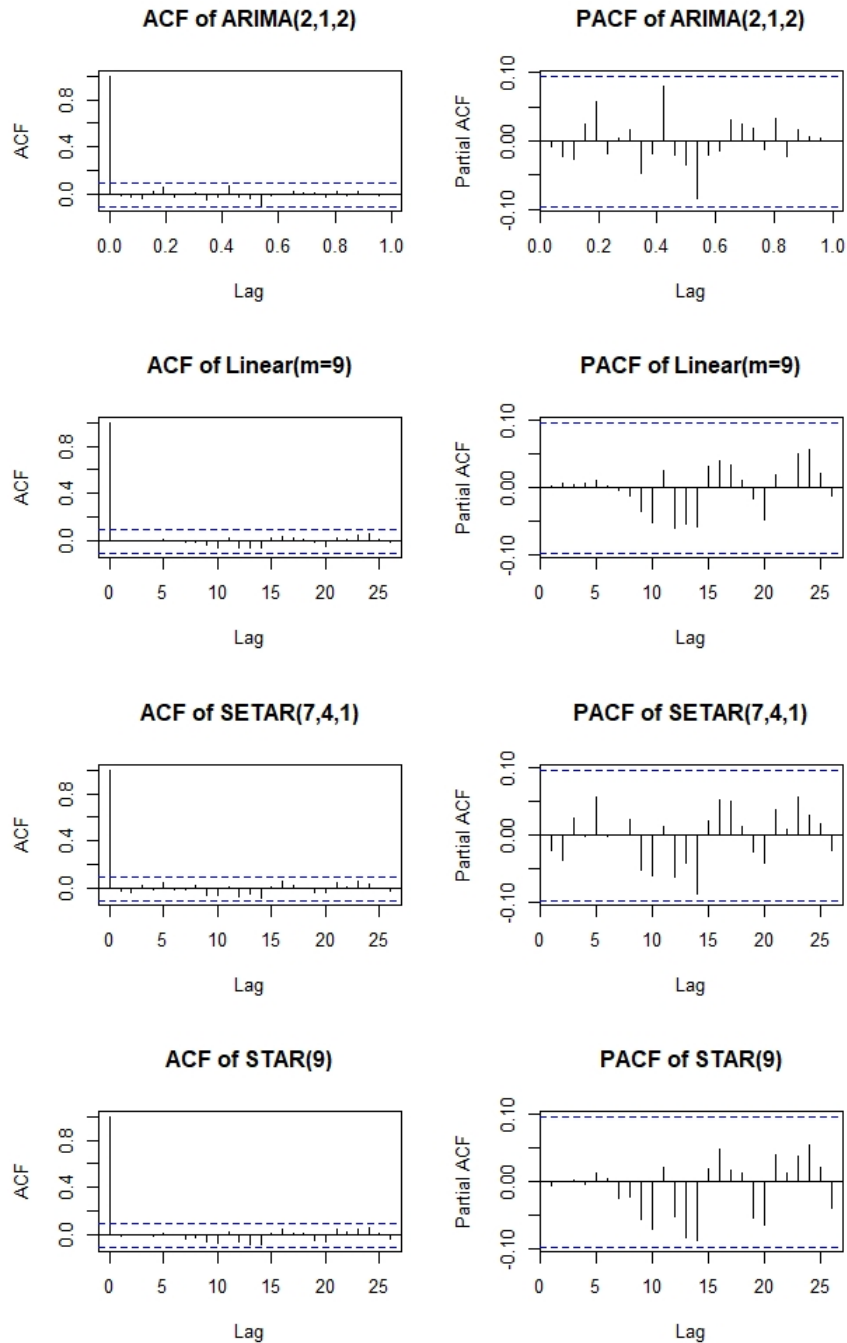


Figure 8: Plot of estimated ACF and PACF functions for the residuals of fitted AR(13), ARIMA(1,1,1), SETAR(7,4,1), STAR(9), NNET(13), AAR(13) in stage 1 PT mean-reversion

ance of fitted residual with it mean absolute prediction error and as displayed on Figures 11-14. The SETAR models are evaluated at the values of maximum threshold parameter attained using the Threshold Vector Error Correction Method. It is observe that the 2-regime SETAR model of mean-reversion for transmission from p_{co} to p_{exr} is best modeled at a maximum autoregressive order for both low and high regimes of embedding dimension $m = 5$ with the low regime order, $mL = 4$ and the high regime, $mH = 1$ to yield the minimum AIC and MAPE. The equation is significant for both constant and trend deterministic part in both regimes.

The linear cointegrating relationship of the transmission from p_{xrg} to p_{xpg} is modeled by ARMA of order $p = 4$ and $q = 2$ at the minimum AIC and MAPE, hence stand out to be best selection for the models for forecast.

Table 2: Estimated coefficients of 2-regime SETAR(m) models at each PT stage of the price build-up for Stage 1.

Equations		
Stage 1: 2-SETAR(5,4,1)	Regime 1 $\omega_t \leq -0.1619$ (16.83% Obs.)	Regime 2 $\omega_t > -0.1619$ (83.17% Obs.)
Intercept	- 4.2054e-02 (4.4317e-02)	- 1.8447e-02* (7.9242e-03)
Trend	4.2942e-04. (2.4712e-04)	9.2684e-05* (3.8082e-05)
ω_{t-1} :	8.3411e-01*** (1.2774e-01)	9.6617e-01*** (2.1341e-02)
ω_{t-2} :	- 1.9984e-02 (1.8212e-01)	
ω_{t-3} :	- 4.9994e-01** (1.7976e-01)	
ω_{t-4} :	6.0509e-01*** (1.2198e-01)	
Fitt. resid. variance: 0.001643 AIC: - 2673 MAPE: 87%		

Note: ***(or **,*,.) denotes the 0.1% (or 1%, 5%, 10%) level of significant for rejecting H_0 .

ARMA and/or SETAR Estimation Models

Table 3: Estimated coefficients of 2-regime SETAR(m) models at each PT stage of the price build-up for Stage 2.

Equations		
Stage 2: 2-SETAR(7,7,1)	Regime 1 (52.06%) $\omega_t \leq 0.01423$	Regime 2(47.94%) $\omega_t > 0.01423$
Intercept	1.2955e-02** (4.2977e-03)	5.1566e-05 (4.2908e-03)
Trend	- 3.9997e-05. (2.0805e-05)	- 8.7269e-06 (1.9142e-05)
ω_{t-1}	1.0279e+00*** (6.7593e-02)	9.7487e-01*** (2.4246e-02)
ω_{t-2}	1.5027e-02 (8.6038e-02)	
ω_{t-3}	7.3650e-02 (8.5459e-02)	
ω_{t-4}	- 1.8934e-01* (8.5242e-02)	
ω_{t-5}	2.2093e-01* (8.9211e-02)	
ω_{t-6}	- 9.5453e-02 (9.2777e-02)	
ω_{t-7}	- 6.3025e-02 (6.7460e-02)	
Fitt. resid. variance: 0.0002829 AIC: - 3406 MAPE: 28.45%		
Overall: 2-SETAR(7,4,1)	$\omega \leq -0.08925$ (23.97% Obs.)	$\omega > -0.08925$ (76.03% Obs.)
Intercept :	- 1.6853e-02 (1.4534e-02)	- 1.1776e-02. (6.2736e-03)
Trend :	1.4458e-04. (7.6403e-05)	4.9765e-05. (2.6596e-05)
ω_{t-1} :	9.1403e-01*** (8.9703e-02)	9.9883e-01*** (2.0604e-02)
ω_{t-2} :	8.6990e-03 (1.2746e-01)	
ω_{t-3} :	- 5.1775e-01*** (1.4559e-01)	
ω_{t-4} :	5.4928e-01*** (1.0792e-01)	
Fitt. resid. variance: 0.001438 AIC: - 2729 MAPE: 121%		

Note: ***(or **,*) denotes 0.1% (or 1%, 5%, 10%) level of sig. for rejecting H_0 .

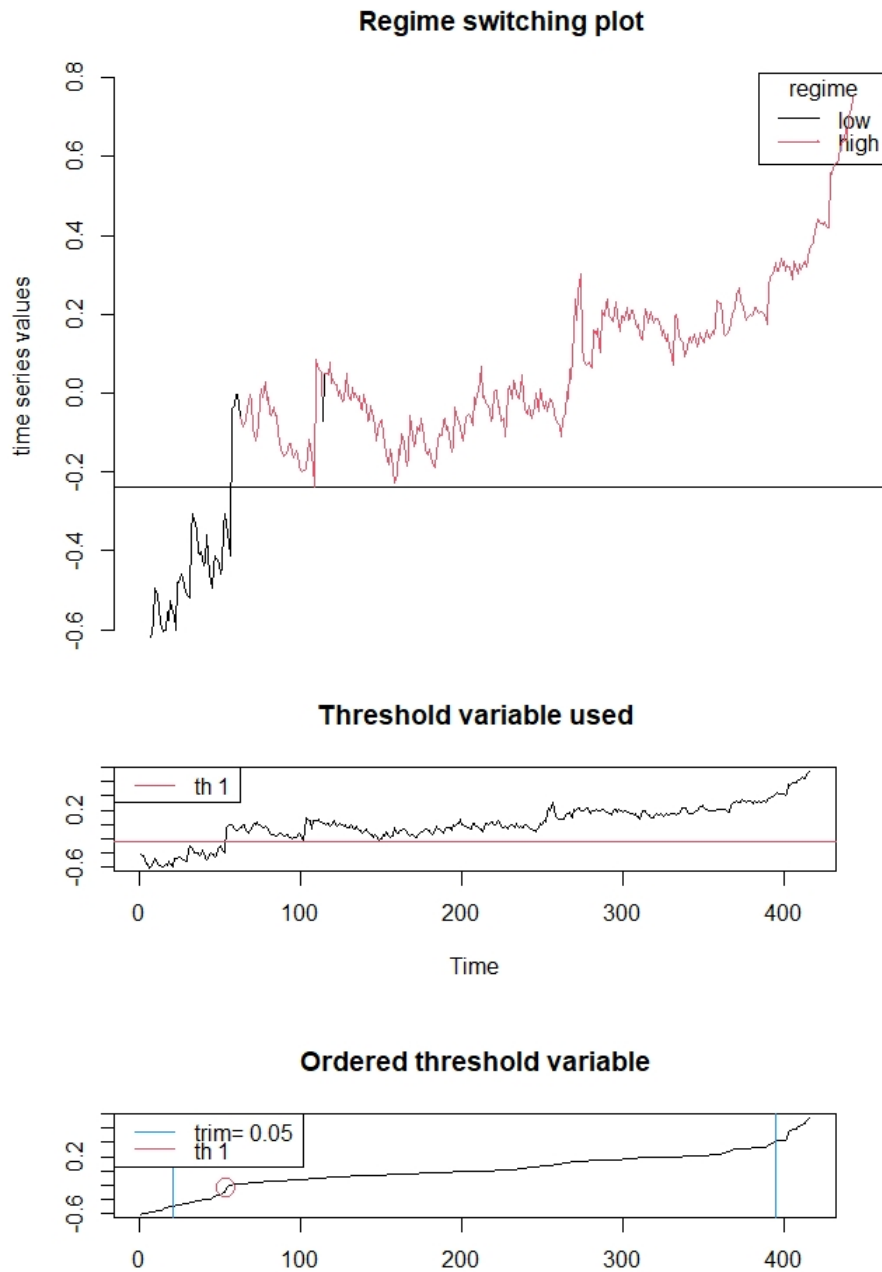


Figure 9: Plot of the regime switching level and threshold variable of the mean-reversion process in initial price transmission between crude oil and ex-refinery from 02/01/1999 - 27/06/2015.

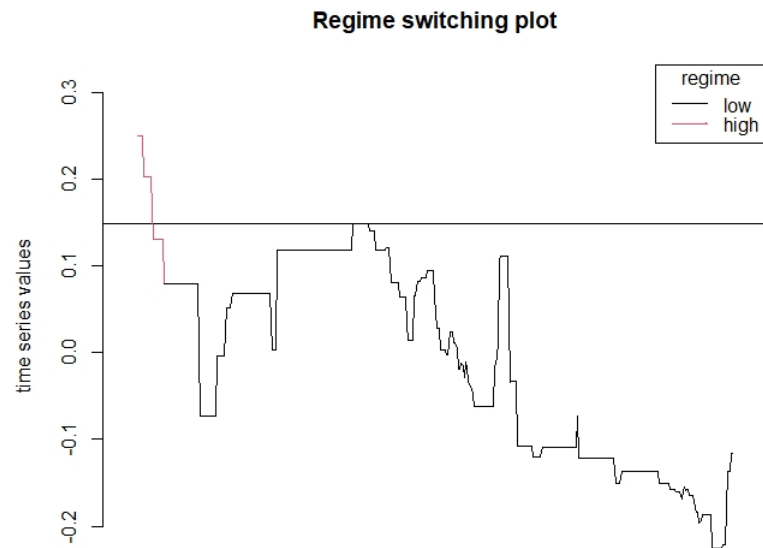


Figure 10: Plot of the regime switching level and threshold variable of the mean-reversion process in initial price transmission between ex-refinery and ex-pump from 02/01/1999 - 27/06/2015.

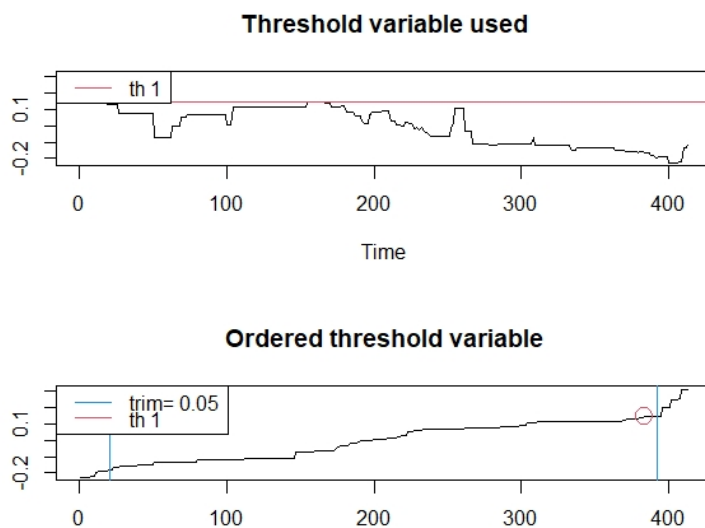


Figure 11: Ordered Threshold Plot of the regime switching level and threshold variable of the mean-reversion process in initial price transmission between ex-refinery and ex-pump from 02/01/1999 - 27/06/2015.

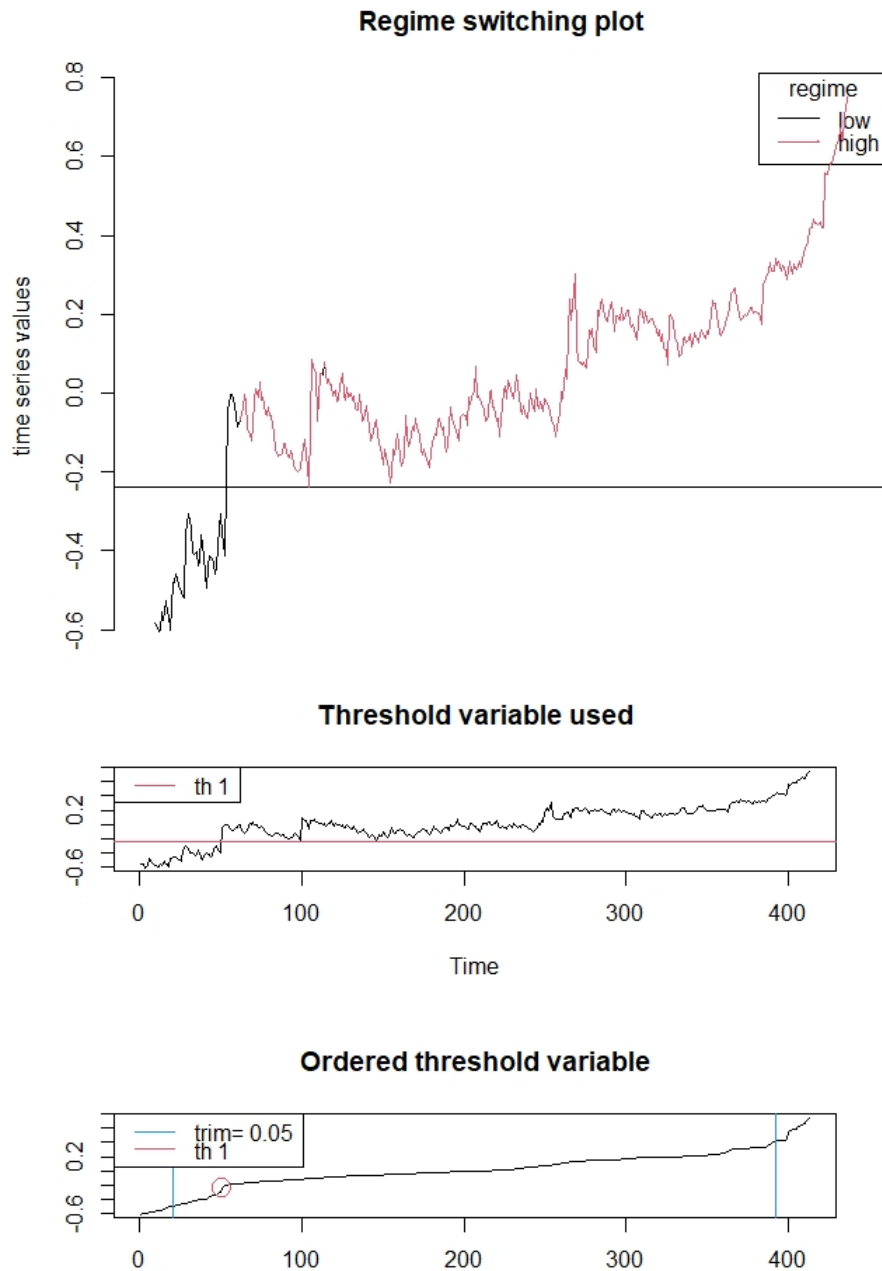


Figure 12: Plot of the regime switching level and threshold variable of the mean-reversion process in the overall PT between crude oil and ex-pump from 02/01/1999 - 27/06/2015.

ARMA and/or SETAR Estimation Models

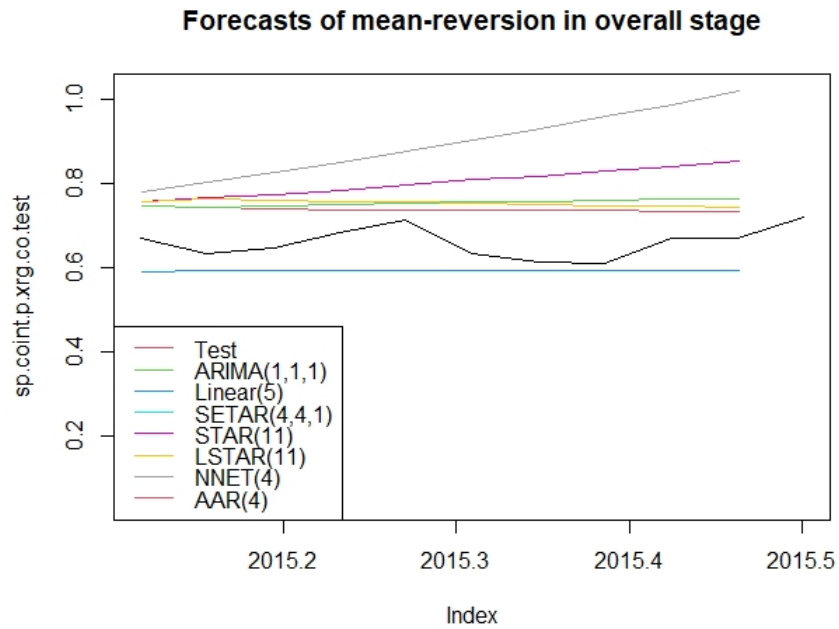


Figure 13: Plot of forecast comparison for nonlinear mean-reversion at stage 1

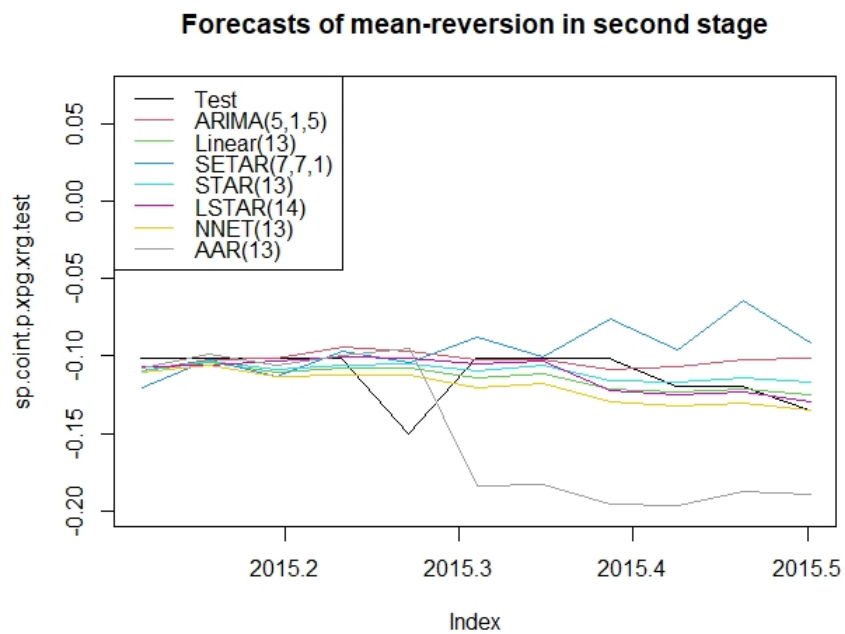


Figure 14: Plot of forecast comparison for nonlinear mean-reversion at stage 2

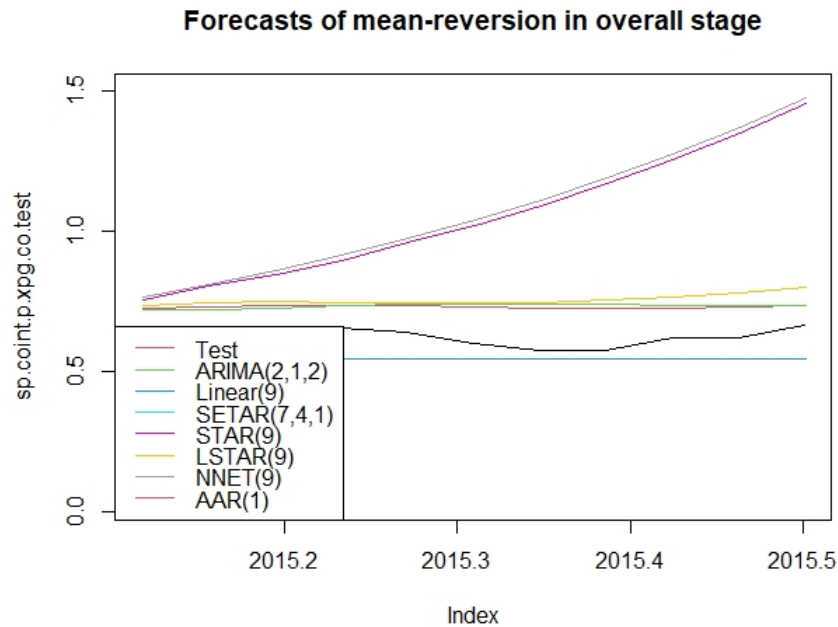


Figure 15: Plot of forecast comparison for nonlinear mean-reversion at overall stage

4.5 Evidence of Asymmetries price Transmission in Three Different Stages of Bivariate Time Series between Crude oil, ex-refinery and ex-pump Gasoline Prices in the Ghanaian petroleum market

The summary statistics of the mean return and variance coupled with graph of the indicates unstable or volatile residuals of the times series to random walk process. This observation enable the test for stationarity of the series using both the Dickey [1979] and Kwiatkowski et al. [1992] methods which indicate unit root processes $I(1)$ with significant trends and drifts deterministic forms in all three energy prices.

The choice of a suitable univariate technique took a naive approach to test linearity within the data. However, Hansen and Johansen [1999] tests for numbers of one regime AR versus two or three TAR model for univariate analysis pointed to threshold model of the initial and overall stage transmission and rather model the second stage with linear model. The major findings pertaining to the study, objectives and hypothesis are in all three stages of the price transmission of the price build-up formula are outlaid as follows:

The claim of presence of nonlinearities in terms of threshold effects is accepted evident from Hansen and Seo test under H_0 of linear cointegration. Additionally, the Hansen and Johansen test for 1-regime versus 2-regime and 2-regime versus 3-regime of the estimated univariate equations show that the initial price transmission is best characterized by two regimes.

More importantly, although the half-life of both p_{co} and Δp_{xrg} to return mean-reversion is approximately 2 months in quick unusual regime and it takes 7 months for the later to return in the slow usual regime for every unit of p_{co} . Also within the asymmetric price transmission, Δp_{xrg} does not respond spontaneously to itself in whether quick or slow direction although it responds positively to Δp_{co} . The change in downstream does not impact on the upstream.

The asymmetric behavior was still a dominant feature of our mean-reversion also in this stage with 2-SETAR(5,1,4) equation also exhibits significant constant and trending intervention properties of the price build-up mechanism. The threshold models outperformed the AR and ARMA process from the forecasts in Figure 8 although the MAPE is slightly high. That is, the mean reversion property depends significantly on what happens in the past 4 biweekly periods in the quick unusual directions, whereas in slow usual direction it depends on recurrence of the same in the just the immediate biweekly periods.

For a single equation of the mean-reversion, Hansen and Johansen test suggest we use a linear model of embedding dimension of $m \geq 2$ as seen Larsen [2012]. Both prices does not response spontaneously to dynamic external shocks from each other and within themselves.

This asymmetric behavior of the sample mean-reversion observation is best modeled with linear univariate process specifically the ARMA(4,0,2) since it forecasting performance slightly outweighs that of 2-regime SETAR or any order univariate method in Figure 10 according to the MAPE. We imply that domestic interaction through taxes, levies and margins added onto ex-pump price build-up is behave linearly upwards. Prices are not returning in the manner when even the upstream falls.

5 Conclusion

In the further investigation of whether this mean-reversion above for bivariate analysis is nonlinear, the Hansen and Seo test unexpectedly accepted the null hypothesis of linear cointegration over claim of asymmetric in paired mean-reversion in this stage. Additionally, the hansen and Johansen test for 1-regime versus 2-regime and 2-regime versus 3-regime of the univariate equation mean-reversion indicate we rather model by 2-regime or one threshold model. The 2-SETAR(5,1,4) equation, also showcases the robust and persistent intervention

elements of the price build-up process. The presence of asymmetric behavior continued to be a dominant factor in our mean-reversion strategy, and it was also evident in this stage. While the Mean Absolute Percentage Error (MAPE) is somewhat greater for the threshold models compared to the Autoregressive (AR) and Autoregressive Moving Average (ARMA) processes, the threshold models outperformed the AR and ARMA processes. In other words, the mean reversion property is significantly influenced by the events that have taken place in the previous four biweekly periods when there have been sudden and unexpected changes. However, when there are gradual and expected changes, it is influenced by the events in the biweekly periods immediately preceding it.

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Unicity Results Concerning of difference monomials of $\mathcal{L}$ –function and a meromorphic function

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Roopa M.[†]

Abstract

In this paper, we study the value distribution of $\mathcal{L}$ -function in the extend Selberg class and a non-constant transcendental meromorphic f function with finitely many zeros of finite order, sharing a polynomial with its difference monomial. Meromorphic functions and L -functions are central to Nevanlinna theory, a branch of complex analysis focusing on the distribution of zeros and poles of analytic functions. The Meromorphic functions are unique due to their combination of meromorphicity and analyticity. They are defined on the entire complex plane except for isolated poles in the complex analysis. On the other hand, L -functions, arising notably in number theory and automorphic forms, exhibit unique properties related to their zeros and poles. One classic example of a prime number distribution is the Riemann ζ -function. In this paper, we analyse the uniqueness results between a non-constant meromorphic function f having finitely many zeros and L -function, when their difference monomial share a non-zero polynomial p . Our results extends and improves the earlier results of Harina P. Waghamore and Manjunatha B E [6].

Keywords: Difference product; uniqueness; Meromorphic function; L –function.

2020 AMS subject classifications: 30D35¹

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1 Introduction

Nevanlinna theory can provide tools for proving uniqueness theorems for analytic functions. By analyzing the behavior of meromorphic functions and their singularities, one can often deduce properties that lead to uniqueness results. For instance, understanding the growth and distribution of values of a meromorphic function can help establish uniqueness within a certain class of functions.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We assume that the reader is familiar with the basic results and standard notations of Nevanlinna value distribution theory (see [7],[8],[23],[24],[25]). For some $a \in \mathbb{C} \cup \{\infty\}$, the zeros of $f - a$ and $g - a$ have same multiplicities, we say that f and g share the value a CM(counting multiplicities), otherwise if we do not consider the multiplicities, we say that f and g share the value a IM(ignoring multiplicities). If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then f and g share ∞ CM. While $N(r, f)$ denotes the counting function of poles of f whose multiplicities are taken into account(respectively $\bar{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). Also we have $N(r, a; f) = N\left(r, \frac{1}{f-a}\right)$, denotes the counting function of a points of $f(z)$ whose multiplicities are counted(respectively $\bar{N}(r, a; f)$ denotes for the reduced counting function when multiplicities are ignored). In general, the notation $N(r, a; f| = 1)$ denotes the counting function of simple a -points of f and the notation $N(r, a; f| \geq 2)$ denotes the counting function of those of a -points of f whose multiplicities are atleast 2(respectively, $\bar{N}(r, a; f| + 1)$ and $\bar{N}(r, a; f| \geq 2)$ denotes the reduced counting functions).

We denote by $T(r, f)$ the Nevanlinna characteristic function of f plays a key role in the value distribution theory. we have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non negative. $S(r, f)$ is small quantity defined by $S(r, f) = o\{T(r, f)\}$ as $(r \rightarrow \infty, r \notin E)$ where E denote any set of positive real numbers having a finite linear measure.

Let $\mathcal{L}$ be an L -function is defined as a meromorphic function in the complex plane $\mathbb{C}$ that is connected to one or more mathematical properties. A $\mathcal{L}$ -series is a Dirichlet series that typically converges on a half-plane and has the potential to analytically continued as an $\mathcal{L}$ -function. We first distinguish between the $\mathcal{L}$ -function, which is the function in the complex plane that is its analytic continuation, and the $\mathcal{L}$ -series, which is an infinite series representation (such as the Dirichlet series for the Riemann zeta function)(see.[20],[21]). Generally, the constructions begin with an $\mathcal{L}$ -series that is initially specified as a Dirichlet series, and subsequently expanded into an Euler product that is indexed by prime numbers. The $\mathcal{L}$ -function so defined analytically to the rest of the complex plane $\mathbb{C}$ atmost one pole. The Selberg class is an axiomatic definition of a class of $\mathcal{L}$ -functions. $\mathcal{L}$, consists essentially of those Dirichlet series that satisfy the Riemann hypothesis as well as

the Riemann zeta function, $F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

2 Definitions and Theorems

Let us recall the following standard definitions of Nevalinna theory.

Definition 2.1. [20, 21] *The Selberg class S consists of the functions $\mathcal{F}$ satisfying the following axioms:*

1. (Dirichlet series) $\mathcal{F}(s) = \sum_{n=1}^{\infty} a(n)/n^s$ where $s \in \mathbb{C}$, absolutely convergent for $\sigma > 1$.
2. (Analytic continuation) There exists a non-negative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.
3. (Functional equation) There exist an integer $k \geq 0$, positive real numbers Q, λ_j , complex numbers v_j with $\operatorname{Re} v_j \geq 0$ and ω with $|\omega| = 1$, such that $\wedge_{\mathcal{L}}(s)$ defined by

$$\wedge_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^K \Gamma(\lambda_j s + v_j) = \gamma(s) \mathcal{F}(s),$$

satisfies the functional equation $\wedge_{\mathcal{L}}(s) = \overline{\omega \wedge_{\mathcal{L}}(1 - \bar{s})}$. We would call the function $\gamma(s)$ the γ factor.

4. (Ramanujan conjecture) For every $\epsilon > 0$, $a(n) = O(n^{\epsilon})$.
5. (Euler product hypothesis) L can be written over prime as

$$\mathcal{L}(s) = \prod_p \exp \left(\sum_{k=1}^{\infty} b(p^k)/p^{ks} \right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p .

The degree $d_{\mathcal{L}}$ of an $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2 \sum_{j=1}^K \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

Definition 2.2. [9, 10, 11] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; h)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

Definition 2.3. [4] Let k be a positive integer for any constant a in the complex plane $\mathbb{C}$.

We denote

1. By $N_{(k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a -points of $f(z)$ with multiplicity $\leq k$.
2. By $N_{(k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a -points of $f(z)$ with multiplicity $\leq k$. Similarly, the reduced counting functions $\overline{N}_{(k)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(k)}\left(r, \frac{1}{f-a}\right)$ are defined.

Definition 2.4. [6] By a difference product, we mean a difference monomial and its shifts, that is an expression of the type $\prod_{j=1}^s f(z + c_j)^{\mu_j}$.

In 2011, K. Liu, X. L. Liu, and T. B. Cao [12] studied the uniqueness of the difference monomials and obtained the following results.

Theorem 2.1. [12] Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.

Theorem 2.2. [12] Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.

In 2015, Y.Liu, J. P. Wang, and F. H. Liu [13] improved Theorems A and B and obtained the following results.

Theorem 2.3. [13] Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n (\geq 14)$, $k (\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f \equiv t_1g$ or $fg \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem 2.4. [13] Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n (\geq 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f \equiv t_1g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

Theorem 2.5. [13] Let $c \in \mathbb{C} \setminus \{0\}$ and let f and h be two transcendental meromorphic functions with finite order; and $n (\geq 22)$ be a positive integer. If $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, then $f \equiv t_1g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

In 2017, Sujoy Majumder [14] replaced the sharing value 1 by a nonzero polynomial $p(z)$ in 2.3, 2.4, and 2.5 and obtained the following results.

Theorem 2.6. [14] Let f and g be two transcendental meromorphic functions of finite order; $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p (\neq 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. If $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$, then one of the following two cases holds:

- (1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,
- (2) $fg \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+1} = c^2$.

Theorem 2.7. [14] Let f and g be two transcendental meromorphic functions of finite order; $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p (\neq 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 1)$. Then the conclusion of Theorem 2.6 holds.

Theorem 2.8. [14] Let f and g be two transcendental meromorphic functions of finite order; $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p (\neq 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 0)$. Then conclusion of Theorem 2.6 holds.

In 2023, Harina P. Waghmare and Manjunatha B.E [6] replaced the shift by a difference monomial in 2.6, 2.7, and 2.8 and obtained the following results.

Theorem 2.9. [6] Let f and g be two transcendental meromorphic function of finite order and n be a positive integer such that $n \geq 2s + 2\sigma + 6$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z+c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^s g(z+c_j)^{\mu_j} - p(z)$ share $(0, 2)$ where p be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then one of the following two cases holds:

- (1) $ff \equiv tg$ for some constant t such that $t^{n+\sigma} = 1$,
- (2) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+\sigma} = c^2$.

Theorem 2.10. [6] Let f and g be two transcendental meromorphic function of finite order and n be a positive integer such that $n > \frac{5s+2\sigma+11}{2}$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^s g(z + c_j)^{\mu_j} - p(z)$ share $(0, 1)$ where p be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then the conclusion of Theorem 3.1 holds.

Theorem 2.11. [6] Let f and g be two transcendental meromorphic function of finite order and n be a positive integer such that $n \geq 5s + \sigma + 9$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^s g(z + c_j)^{\mu_j} - p(z)$ share $(0, 0)$ where p be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then the conclusion of Theorem 3.1 holds.

It is quite natural to ask the following question.

Question 1. When two finite order meromorphic functions f and g share a nonzero polynomial, what can be said about their uniqueness if we consider the difference monomial and its shift for an L -function instead of the difference polynomial? We have obtained the following results of Harina P. Waghamore and B. E. Manjunath [6] which improves and generalizes Theorems 2.9, 2.10 and 2.11.

3 Main Results

Following are the main results of our paper.

Theorem 3.1. Let f be non-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L -function of finite order and n be a positive integer such that $n \geq 2s + 2\sigma + 5$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} - p(z)$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j} - p(z)$ share $(0, 2)$ where p be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then one of the following two cases holds:

1. $f \equiv t\mathcal{L}$ for some constant t such that $t^{n+\sigma} = 1$,
2. $f(z)\mathcal{L}(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+\sigma} = c^2$.

Theorem 3.2. Let f be anon-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L function of finite order and n be a positive integer such that $n > \frac{3s+4\sigma+9}{2}$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} - p(z)$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j} - p(z)$ share $(0, 1)$ where p

be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then the conclusion of Theorem 3.1 holds.

Theorem 3.3. Let f be a non-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L function of finite order and n be a positive integer such that $n \geq 4s + \sigma + 7$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \dots, s\}$. Let $f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} - p(z)$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j} - p(z)$ share $(0, 0)$ where p be a nonzero polynomial such that $2\deg(p(z)) < n - \sigma$, then the conclusion of Theorem 3.1 holds.

Example 3.1. Let $f = \exp^{(z)}$ and $\mathcal{L} = \exp^{(-z)}$. By considering the values for $n > 9, \sigma = 1, s = 1$ and c_j, μ_j are complex constants. Then one can easily verify that $f^n \prod_{j=1}^s g(z + c_j)^{\mu_j}$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}$, where f and $\mathcal{L}$ satisfies the algebraic equations $R(f, \mathcal{L}) = 0$ with the conclusion of Theorem 3.1 holds.

Example 3.2. Let $f = \exp^{(z)}$ and $\mathcal{L} = t \exp^{(z)}$. By considering the values for $n > 9, \sigma = 1, s = 1$ and c_j, μ_j are complex constants. Then one can easily verify that $f^n \prod_{j=1}^s g(z + c_j)^{\mu_j}$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}$, where f and $\mathcal{L}$ satisfies the algebraic equations $R(f, \mathcal{L}) = 0$ with the conclusion of Theorem 3.1 holds.

4 Remarks

Theorem 3.1 - 3.3 is an improvement of Theorem 2.8 - 2.9 respectively.

Taking $s = 1$ we get $\sigma = \sum_{j=1}^s \mu_j = 1$, $f^n \prod_{j=1}^s g(z + c_j)^{\mu_j}$ and $\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}$ reduces to $f^n f(z + c)$ and $g^n g(z + c)$, then

- (1) In Theorem 3.1 we obtain $n \geq 8$ which is an improvement of Theorem 2.9,
- (2) In Theorem 3.2 we obtain $n > 8$ which is an improvement of Theorem 2.10,
- (3) In Theorem 3.3 we obtain $n \geq 13$ which is an improvement of Theorem 2.11.

5 Lemmas

In this section, we present few lemmas needed in the sequel. Let $\mathcal{F}$, and L be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $\mathcal{H}$ the function

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1} \right) - \left(\frac{L''}{L'} - \frac{2L'}{L - 1} \right). \quad (1)$$

Lemma 5.1. [5] Let f be a meromorphic function of finite order ρ , and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}) = S(r, f).$$

The following lemma has few modifications of the original version [Corollary 2.5 of [4]]

Lemma 5.2. [22] Let f be a non-constant meromorphic function and let $a_n(z)$ ($\neq 0$), $a_{n-1}(z), \dots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \dots, n$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 5.3. [8] Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r, f), \quad N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r, f),$$

$$\overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f(z)) + S(r, f), \quad \overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 5.4. [5] Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 5.5. [22] Let f and g be two non-constant meromorphic functions. Then

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r, \infty; f) + N(r, 0; g) - N(r, \infty; g) - N(r, 0; f).$$

Lemma 5.6. [6] Let f be a transcendental meromorphic function of finite order and let

$$F = f^n \prod_{j=1}^s (f(z+c_j))^{\mu_j}, \text{ where } n \text{ is positive integer. Then}$$

$$(n - \sigma)T(r, f) \leq T(r, F) + S(r, f).$$

Lemma 5.7. Let f be non-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L -function of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n > \sigma$. Let p be a nonzero polynomial such that $2\deg(p) < n - \sigma$. Then

$$(1) \text{ if } \deg(p) \geq 1, \text{ then } f^n \prod_{j=1}^s (f(z+c_j))^{\mu_j} \cdot \mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z+c_j))^{\mu_j} \not\equiv p^2(z)$$

Unicity Results concerning difference monomials of $\mathcal{L}$...

(2) if $p(z) = c \in \mathbb{C} \setminus \{0\}$, then the relation $f^n \prod_{j=1}^s (f(z + c_j))^{\mu_j} \mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j} \equiv p^2(z)$, always implies that $f \cdot \mathcal{L} = t$, where t is a constant such that $t^{n+\sigma} = c^2$.

Proof. Suppose

$$\left(f^n \prod_{j=1}^s (f(z + c_j))^{\mu_j} \right) \left(\mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j} \right) \equiv p^2(z). \quad (2)$$

Let $h_1 = f \mathcal{L}$ then by (2), we have

$$h_1^2(z) = \frac{p^2(z)}{\prod_{j=1}^s h_1(z + c_j)^{\mu_j}}. \quad (3)$$

We now consider the following two cases,

Case 1. Suppose h_1 is a transcendental meromorphic function. Now by Lemma 5.1, 5.2 and 5.3, we get

$$\begin{aligned} nT(r, h_1) &= T(r, h_1^n) + S(r, h_1), \\ &= T\left(r, \frac{p^2(z)}{\prod_{j=1}^s h_1(z + c_j)^{\mu_j}}\right) + S(r, h_1), \\ &\leq N\left(r, 0; \prod_{j=1}^s f(z + c_j)^{\mu_j}\right) + N\left(r, 0; \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}\right) + S(r, h_1), \\ &\leq \sigma [T(r, f) + T(r, \mathcal{L})] + S(r, h_1), \end{aligned}$$

from $h_1 = f \mathcal{L}$, which implies $T(r, h_1) = T(r, f) + T(r, \mathcal{L})$, we get

$$n [T(r, f) + T(r, \mathcal{L})] \leq \sigma [T(r, f) + T(r, \mathcal{L})] + S(r, h_1),$$

which is a contradiction.

Case 2. Suppose h_1 is a rational function. Let

$$h_1 = \frac{h_2}{h_3} \quad (4)$$

where h_2, h_3 are two non-zero relatively prime polynomials. By (3), we have

$$T(r, h_1) = \max\{\deg(h_2), \deg(h_3)\} \log r + O(1). \quad (5)$$

Now by (3) - (5), we have

$$\begin{aligned} \max\{\deg(h_2), \deg(h_3)\} \log r &= T(r, h_1^n) + O(1), \\ &\leq \sigma [T(r, f) + T(r, \mathcal{L})] + 2T(r, p) + O(1), \end{aligned}$$

$$\max\{\deg(h_2), \deg(h_3)\} \log r \leq \sigma \max\{\deg(h_2), \deg(h_3)\} \log r + 2T(r, p) + O(1). \quad (6)$$

We see that $\max\{\deg(h_2), \deg(h_3)\} \geq 1$. Now by (6), we deduce that $(n - \sigma) \leq 2\deg(p(z))$. Which contradicts our assumption that $2\deg(p(z)) < (n - \sigma)$. Hence h_1 must be a non-zero constant.

Let

$$h_1 = t \in \mathbb{C} \setminus \{0\}. \quad (7)$$

Now when $\deg(p) \geq 1$ by (3) and (7), we arrive at a contradiction.

$$\text{In this case we have } \left(f^n \prod_{j=1}^s (f(z + c_j))^{\mu_j} \right) \left(\mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j} \right) \not\equiv p^2(z).$$

Suppose $p(z) = c \in \mathbb{C} \setminus \{0\}$. So by (2), we see that $h_1^{n+\sigma} \equiv c^2$.

By (7), we get $t^{n+\sigma} \equiv c^2$. This completes the proof. $\square$

Lemma 5.8. Let f be non-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be finite complex constant such that $f(z + c_j) \neq f(z)$ for $i = 0, 1, 2, \dots, s$. and $\mathcal{L}(z + c_j) \neq$

$$\mathcal{L}(z) \text{ and let } n \text{ be an integer such that } n \geq (s+1) + \sigma. \text{ Let } \mathcal{F}(z) = \frac{f^n \prod_{j=1}^s (f(z+c_j))^{\mu_j}}{p(z)}$$

and $L(z) = \frac{\mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z+c_j))^{\mu_j}}{p(z)}$, where $p(z)$ is nonzero polynomial. If $\mathcal{H} \equiv 0$ then either

$$(1) \quad f^n \prod_{j=1}^s (f(z + c_j))^{\mu_j} \mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j} \equiv p^2(z),$$

where $f^n \prod_{j=1}^s (f(z + c_j))^{\mu_j} - p(z)$ and $\mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j} - p(z)$ share 0 CM,

$$(2) \quad f(z) = t\mathcal{L}(z) \text{ for a constant } t \text{ with } t^{n+\sigma} = 1.$$

Proof: From (1) we have,

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1} \right) - \left(\frac{L''}{L} - \frac{2L'}{L - 1} \right).$$

Since $\mathcal{H} \equiv 0$, by integration, we get

$$\frac{1}{\mathcal{F} - 1} = \frac{BL + A - B}{L - 1} \quad (8)$$

Unicity Results concerning difference monomials of $\mathcal{L}$...

where $A \neq 0$, B are constant. From (7) it is clear that $\mathcal{F}$ and L share $(1, \infty)$. We now consider the following cases.

Case 1. Let $B \neq 0$ and $A \neq B$, if $B = -1$ then from (8), we have $\mathcal{F} = \frac{-A}{L-A-1}$. Therefore $\overline{N}\left(r, \frac{1}{L-(A+1)}\right) = \overline{N}(r, \mathcal{F}) \leq N\left(r, \frac{1}{p}\right) \leq T(r, p) = S(r, \mathcal{L})$. So in view of Lemma 5.6 and the second fundamental theorem, we get

$$\begin{aligned} (n - \sigma)T(r, \mathcal{L}) &\leq T\left(r, \mathcal{L}^n \prod_{j=1}^s (\mathcal{L}(z + c_j))^{\mu_j}\right) + S(r, \mathcal{L}), \\ &\leq T(r, \mathcal{L}) + S(r, \mathbf{g}), \\ &\leq \overline{N}(r, \infty; \mathcal{L}) + \overline{N}(r, 0; \mathcal{L}) + \overline{N}(r, (A+1); \mathcal{L}) + S(r, \mathcal{L}), \\ &\leq (s+1)T(r, \mathcal{L}) + S(r, \mathcal{L}) \end{aligned}$$

which contradicts with $n > (s+1) + \sigma$.

If $B \neq 1$, from (8) we get that

$$\mathcal{F} - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2 \left(L + \frac{A-B}{B}\right)}.$$

So $\overline{N}\left(r, \frac{B-A}{B}; L\right) = \overline{N}(r, \infty; \mathcal{F}) \leq S(r, \mathcal{L})$. Using Lemma 5.6 and the same argument as used in the case when $B = -1$ we can get a contradiction.

Case 2. Let $B \neq 0$ and $A = B$. If $B = -1$, then from (8), we have

$$\mathbf{f}^n \prod_{j=1}^s \mathbf{f}(z + c_j)^{\mu_j} \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j} = p^2(z),$$

when $\left[\mathbf{f}^n \prod_{j=1}^s \mathbf{f}(z + c_j)^{\mu_j}\right] - p(z)$ and $\left[\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}\right] - p(z)$ share 0 CM.

If $B \neq 1$, from (8) we have

$$\frac{1}{\mathcal{F}} = \frac{BL}{(1+B)L-1}.$$

Therefore $\overline{N}(r, (1+B); L) = \overline{N}(r, \infty; \mathcal{F})$. So in view of Lemma 5.8 and the second fundamental theorem, we get

$$\begin{aligned} (n - \sigma)T(r, \mathcal{L}) &\leq T(r, L) + S(r, L), \\ &\leq \overline{N}(r, \infty; L) + \overline{N}(r, 0; L) + \overline{N}(r, (1+B); L) + S(r, L), \\ &\leq (s+1)T(r, \mathcal{L}) + S(r, \mathbf{f}) + S(r, \mathcal{L}), \end{aligned}$$

so for $r \in I$, we have

$$(n - \sigma - (s+1))T(r, \mathcal{L}) \leq S(r, \mathcal{L}).$$

Which is contradiction, since $n > (s + 1) + \sigma$.

Case 3. Let $B = 0$. From (8) we obtain

$$\mathcal{F} = \frac{L + A - 1}{A} \quad (9)$$

if $A \neq 1$, then from (9), we obtain $\overline{N}(r, (1 - A); L) = \overline{N}(r, 0; \mathcal{F})$.

We can similarly deduce a contradiction as in case 2. Therefore $A = 1$ and from (9), we obtain, $\mathcal{F}(z) \equiv L(z)$.

That is

$$f^n \prod_{j=1}^s f(z + c_j)^{\mu_j} \equiv \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}. \quad (10)$$

Let $h = \frac{f}{\mathcal{L}}$ and then substituting $f = h \mathcal{L}$ in (10), we deduce

$$h^n(z) = \frac{1}{\prod_{j=1}^s h(z + c_j)^{\mu_j}} \quad (11)$$

$$h^{n+1} = \frac{f}{\mathcal{L} \prod_{j=1}^s h(z + c_j)^{\mu_j}},$$

if h is not a constant, then we have

$$\begin{aligned} (n + 1)T(r, h) &\leq T\left(r, \frac{f}{\prod_{j=1}^s f(z + c_j)^{\mu_j}}\right) + T\left(r, \frac{\prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}}{\mathcal{L}}\right) \\ &\quad + S(r, f) + S(r, \mathcal{L}), \\ &\leq (1 + \sigma) [T(r, f) + T(r, \mathcal{L})] + S(r, f) + S(r, \mathcal{L}), \end{aligned}$$

we obtain

$$(n - \sigma) [T(r, f) + T(r, \mathcal{L})] \leq S(r, f) + S(r, \mathcal{L}),$$

which is impossible. Therefore h is constant, then substituting $f = h \mathcal{L}$ in (10), we have $h^{n+\sigma} \equiv 1$. Therefore $f = t \mathcal{L}$ where t is a constant with $t^{n+\sigma} = 1$.

Lemma 5.9. [1] If f, g be two non-constant meromorphic functions such that they share $(1, 1)$. Then

$$2\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2)}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 5.10. [3] Let f and g share $(1, 1)$. Then

$$\overline{N}_{f>2}(r, 1; g) \leq \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}N_0(r, 0; f') + S(r, g),$$

where $N_0(r, 0; f')$ is the counting function of the zeros of f' which are not the zero of $f(f - 1)$.

Lemma 5.11. [3] Let f, g share $(1, 0)$. Then

- (i) $\overline{N}_{f>1}(r, 1; g) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_0(r, 0; f') + S(r, f)$
- (ii) $\overline{N}_{g>1}(r, 1; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_0(r, 0; g') + S(r, g).$

Lemma 5.12. [25] Let f, g share $(1, 0)$. Then

$$\overline{N}_{\mathcal{L}}(r, 1; f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 5.13. [3] Let f and g be two non-constant meromorphic functions sharing $(1, 0)$. Then

$$\overline{N}_{\mathcal{L}}(r, 1; f) + 2\overline{N}_{\mathcal{L}}(r, 1; g) + \overline{N}_E^{(2)}(r, 1; f) - \overline{N}_{f>1}(r, 1; g) - \overline{N}_{g>1}(r, 1; f) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 5.14. [10, 19] If $N(r, 0; f^{(k)}|f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then

$$N(r, 0; f^{(k)}|f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; |f| < k) + k\overline{N}(r, 0; |f| \geq k) + S(r, f).$$

Lemma 5.15. [18, 20, 21] Let $\mathcal{L}$ be an L -function with degree q . Then $T(r, L) = \frac{q}{\pi}r \log r + O(r)$.

Lemma 5.16. [2, 15, 16, 17] Let $\mathcal{L}$ be an L -function. Then $N(r, \infty) = S(r, \mathcal{L}) = O(\log r)$.

Lemma 5.17. Let f be a non-constant meromorphic function and L be an L -function. If f and L share $(\infty, 0)$, then $\overline{N}(r, \infty; f) = S(r, \mathcal{L}) = O(\log r)$.

Proof:

Since f and $\mathcal{L}$ share $(\infty, 0)$, therefore by lemma 5.16, we have $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; \mathcal{L}) = S(r, \mathcal{L}) = O(\log r)$. This completes the proof.

6 Proof of main Theorems

6.1 Proof of Theorem 3.1

Let $\mathcal{F} = \frac{f^n \prod_{j=1}^s f(z+c_j)^{\mu_j}}{p(z)}$ and $L = \frac{\mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. It follows that $\mathcal{F}$ and L share $(1, 2)$ except the zeors of $p(z)$.

Case 1. Let $\mathcal{H} \not\equiv 0$ from (1), we obtain

$$N(r, \infty; \mathcal{H}) \leq \overline{N}_*(r, 1; \mathcal{F}, L) + \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L'). \quad (12)$$

$\overline{N}_0(r, 0; \mathcal{F}')$ is the reduced counting function of those zero's of $\mathcal{F}'$ which are not the zero's of $\mathcal{F}(\mathcal{F} - 1)$, and $\overline{N}_0(r, 0; \mathcal{F}')$ is similarly defined.

Let z_0 be a simple zero of $F - 1$ such that $p(z_0) \neq 0$. Then z_0 is a simple zero of $L - 1$ and a zero of H . So

$$N(r, 1; \mathcal{F} | = 1) \leq N(r, 0; L) \leq N(r, \infty; \mathcal{H}) + S(r, f) + S(r, \mathcal{L}). \quad (13)$$

From (12) and (13)

$$\overline{N}(r, 1; \mathcal{F}) \leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_*(r, 1; F, L) + \overline{N}(r, 1; \mathcal{F} | \geq 2) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, f) + S(r, L). \quad (14)$$

Now in the view of Lemma 5.14, we get

$$\begin{aligned} \overline{N}_0(r, 0; L') + \overline{N}(r, 1; \mathcal{F} | \geq 2) + \overline{N}_*(r, 1; \mathcal{F}, L) &\leq \overline{N}_0(r, 0; L') + \overline{N}(r, 1; \mathcal{F} | \geq 2) \\ &\quad + \overline{N}(r, 1; \mathcal{F} | \geq 3) \\ &\leq N(r, 0; L' | L \neq 0) \leq \overline{N}(r, 0; L) + S(r, L). \end{aligned} \quad (15)$$

Using equations (14), (15) and Lemma 5.6, we get from second fundamental theorem that

$$\begin{aligned} (n - \sigma)T(r, f) &\leq T(r, \mathcal{F}) + S(r, f), \\ &\leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, 1; \mathcal{F}) - N_0(r, 0; \mathcal{F}') + S(r, f), \\ &\leq \overline{N}(r, \infty; \mathcal{F}) + N_2(r, 0; \mathcal{F}) + N_2(r, 0; L) + S(r, f) + S(r, L), \\ &\leq \overline{N}(r, \infty; f) + \overline{N}\left(r, \infty; \prod_{j=1}^s f(z+c_j)^{\mu_j}\right) + 2\overline{N}(r, 0; f) \\ &\quad + N\left(r, 0; \prod_{j=1}^s f(z+c_j)^{\mu_j}\right) + N_2\left(r, 0; \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z+c_j)^{\mu_j}\right) \\ &\quad + S(r, f) + S(r, \mathcal{L}), \\ &\leq (2 + \sigma)T(r, f) + (2 + s)T(r, \mathcal{L}) + S(r, f) + S(r, \mathcal{L}). \end{aligned} \quad (16)$$

Similarly, we can obtain

$$(n - \sigma)T(r, \mathcal{L}) \leq (2 + \sigma)T(r, \mathcal{L}) + (2 + s)T(r, f) + S(r, f) + S(r, \mathcal{L}). \quad (17)$$

Combining (16) and (17) we have

$$(n - 2\sigma - s - 4) [T(r, f) + T(r, \mathcal{L})] \leq S(r, f) + S(r, \mathcal{L}). \quad (18)$$

Since $n > 2\sigma + s + 4$, equation (18) leads to contradiction.

Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

6.2 Proof of Theorem 3.2

Let $\mathcal{F}(z) = \frac{f^n(z) \prod_{j=1}^s f(z+c_j)^{\mu_j}}{p(z)}$ and $L(z) = \frac{\mathcal{L}^n(z) \prod_{j=1}^s \mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. Then $\mathcal{F}$ and L share (1,1) except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $\mathcal{H} \not\equiv 0$.

Using Lemmas 5.6, 5.8, 5.9 and equations (12) and (13), we get

$$\begin{aligned} \overline{N}(r, 1; \mathcal{F}) &\leq N(r, 1; \mathcal{F} | = 1) + \overline{N}_L(r, 1; \mathcal{F}) + \overline{N}_{\mathcal{L}}(r, 1; L) + \overline{N}_E^{(2)}(r, 1; F), \\ &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_*(r, 1; \mathcal{F}, L) + \overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) \\ &\quad + \overline{N}_{\mathcal{L}}(r, 1; L) + \overline{N}_E^{(2)}(r, 1; \mathcal{F}) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, f) + S(r, L), \\ &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + 2\overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) + 2\overline{N}_{\mathcal{L}}(r, 1; L) \\ &\quad + \overline{N}_E^{(2)}(r, 1; \mathcal{F}) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, f) + S(r, L), \\ &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_{\mathcal{F} > 2}(r, 1; L) + N(r, 1; L) - \overline{N}(r, 1; L) \\ &\quad + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, f) + S(r, L), \\ &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \frac{1}{2}\overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, 0; L | \geq 2) + N(r, 0; L' | L \neq 0) \\ &\quad + \overline{N}_0(r, 0; \mathcal{F}') + S(r, f) + S(r, L), \\ \overline{N}(r, 1; \mathcal{F}) &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \frac{1}{2}\overline{N}(r, 0; \mathcal{F}) + N_2(r, 0; L) + \overline{N}_0(r, 0; \mathcal{F}') \\ &\quad + S(r, f) + S(r, L). \end{aligned} \quad (19)$$

Hence by using second fundamental theorem, (19), Lemmas 5.1 and 5.6 we get

$$\begin{aligned}
 (n - \sigma)T(r, f) &\leq T(r, \mathcal{F}) + S(r, f), \\
 &\leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, 1; \mathcal{F}) - N_0(r, 0; \mathcal{F}') + S(r, f), \\
 &\leq \overline{N}(r, \infty; \mathcal{F}) + \frac{1}{2}\overline{N}(r, 0; \mathcal{F}) + N_2(r, 0; \mathcal{F}) + N_2(r, 0; L) + S(r, f) + S(r, L), \\
 &\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; \prod_{j=1}^s f(z + c_j)^{\mu_j}) + \frac{1}{2}\overline{N}(r, 0; f^n \prod_{j=1}^s f(z + c_j)^{\mu_j}) \\
 &\quad + 2\overline{N}(r, 0; f) + N(r, 0; \prod_{j=1}^s f(z + c_j)^{\mu_j}) + N_2\left(r, 0; \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}\right) \\
 &\quad + S(r, f) + S(r, \mathcal{L}), \\
 (n - \sigma)T(r, f) &\leq \left(\frac{5 + s + 2\sigma}{2}\right) T(r, f) + (2 + s)T(r, \mathcal{L}) + S(r, f) + S(r, \mathcal{L}).
 \end{aligned} \tag{20}$$

Similarly, we obtain

$$(n - \sigma)T(r, \mathcal{L}) \leq \left(\frac{5 + s + 2\sigma}{2}\right) T(r, \mathcal{L}) + (2 + s)T(r, f) + S(r, \mathcal{L}) + S(r, f). \tag{21}$$

Combining (20) and (21), we get

$$\left(n - \frac{3s + 4\sigma + 9}{2}\right) [T(r, f) + T(r, \mathcal{L})] \leq S(r, f) + S(r, \mathcal{L}). \tag{22}$$

Since $n > \frac{3s+4\sigma+9}{2}$, equation (22) leads to contradiction.

Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

6.3 Proof of Theorem 3.3

Let $\mathcal{F}(z) = \frac{f^n(z) \prod_{j=1}^s f(z+c_j)^{\mu_j}}{p(z)}$ and $L(z) = \frac{\mathcal{L}^n(z) \prod_{j=1}^s \mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. Then $\mathcal{F}$ and L share $(1, 0)$ except for the zeros of $p(z)$.

In this case equation (13) changes to

$$N_E^{(1)}(r, 1; \mathcal{F}) \leq N(r, 0; \mathcal{H}) \leq N(r, \infty; \mathcal{H}) + S(r, \mathcal{F}) + S(r, L). \tag{23}$$

Using Lemmas 5.11 - 5.14 and equations (13) and (23), we get

$$\begin{aligned}
 \overline{N}(r, 1; \mathcal{F}) &\leq N_E^{(1)}(r, 1; \mathcal{F}) + \overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) + \overline{N}_{\mathcal{L}}(r, 1; L) + \overline{N}_E^{(2)}(r, 1; \mathcal{F}) \\
 &\leq \overline{N}_*(r, 1; \mathcal{F}, L) + \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) \\
 &\quad + \overline{N}_{\mathcal{L}}(r, 1; L) + \overline{N}_E^{(2)}(r, 1; \mathcal{F}) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, \mathbf{f}) + S(r, L), \\
 &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + 2\overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) + 2\overline{N}_{\mathcal{L}}(r, 1; L) + \overline{N}_E^{(2)}(r, 1; \mathcal{F}) \\
 &\quad + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, \mathbf{f}) + S(r, L), \\
 &\leq \overline{N}(r, 0; \mathcal{F} | \geq 2) + \overline{N}(r, 0; L | \geq 2) + \overline{N}_{\mathcal{F} > 1}(r, 1; L) + \overline{N}_{\mathcal{L} > 1}(r, 1; \mathcal{F}) + \overline{N}_{\mathcal{L}}(r, 1; \mathcal{F}) \\
 &\quad + N(r, 1; L) - \overline{N}(r, 1; L) + \overline{N}_0(r, 0; \mathcal{F}') + \overline{N}_0(r, 0; L') + S(r, \mathbf{f}) + S(r, L), \\
 &\leq N_2(r, 0; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + N_2(r, 0; L) + N(r, 0; L' | L \neq 0) + \overline{N}_0(r, 0; L') + S(r, \mathbf{f}) \\
 &\quad + S(r, L), \\
 \overline{N}(r, 1; \mathcal{F}) &\leq N_2(r, 0; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + N_2(r, 0; L) + \overline{N}(r, 0; L) + \overline{N}_0(r, 0; \mathcal{F}') \\
 &\quad + S(r, \mathbf{f}) + S(r, L). \tag{24}
 \end{aligned}$$

Hence using (24), Lemmas 5.1 and 5.6, we obtain from second fundamental theorem that

$$\begin{aligned}
 (n - \sigma)T(r, \mathbf{f}) &\leq T(r, \mathcal{F}) + S(r, \mathbf{f}), \\
 &\leq \overline{N}(r, \infty; \mathcal{F}) + \overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, 1; \mathcal{F}) - N_0(r, 0; \mathcal{F}') + S(r, \mathbf{f}), \\
 &\leq \overline{N}(r, \infty; \mathcal{F}) + 2N_2(r, 0; \mathcal{F}) + N_2(r, 0; L) + \overline{N}(r, 0; L) + S(r, \mathbf{f}) + S(r, \mathcal{L}), \\
 &\leq \overline{N}(r, \infty; \mathbf{f}) + \overline{N}\left(r, \infty; \prod_{j=1}^s \mathbf{f}(z + c_j)^{\mu_j}\right) + 4\overline{N}(r, 0; \mathbf{f}) \\
 &\quad + 2N\left(r, 0; \prod_{j=1}^s \mathbf{f}(z + c_j)^{\mu_j}\right) + N_2\left(r, 0; \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}\right) \\
 &\quad + \overline{N}\left(r, 0; \mathcal{L}^n \prod_{j=1}^s \mathcal{L}(z + c_j)^{\mu_j}\right) + S(r, \mathbf{f}) + S(r, \mathcal{L}), \\
 (n - \sigma)T(r, \mathbf{f}) &\leq (4 + 2s)T(r, \mathbf{f}) + (3 + 2s)T(r, \mathcal{L}) + S(r, \mathbf{f}) + S(r, \mathcal{L}). \tag{25}
 \end{aligned}$$

Similarly, we obtain

$$(n - \sigma)T(r, \mathcal{L}) \leq (4 + 2s)T(r, \mathcal{L}) + (3 + 2s)T(r, \mathbf{f}) + S(r, \mathcal{L}) + S(r, \mathbf{f}). \tag{26}$$

Combining (25) and (26), we get

$$(n - \sigma - 4s - 7)[T(r, \mathbf{f}) + T(r, \mathcal{L})] \leq S(r, \mathbf{f}) + S(r, \mathcal{L}). \tag{27}$$

Since $n > 4s + \sigma + 7$, equation(27) leads to contradiction.

Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

7 Discussion and Conclusions

Nevanlinna theory has applications in various areas of mathematics, including number theory, differential equations, algebraic geometry, and mathematical physics. It provides powerful tools for understanding the intricate behavior of meromorphic functions and their connections to other mathematical structures. Also Applications of L-functions in various areas of mathematics often involve understanding their analytic properties, which can be studied using techniques from Nevanlinna theory.

In this article, with the help of the Nevanlinna theory and L -functions, we have studied the uniqueness results of two difference monomials generated by two finite order non-constant meromorphic function and L function when they share a non-zero polynomial and obtain some results which extend and improve some earlier results of Harina P. Waghamore and Manjunatha B.E.

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KURUVA SATYANNA, V SRINIVAS	RESULTS ON EXPANSION MAPS IN FUZZY Menger SPACE VIA PROPERTY- (E.A) AND (E.A)- LIKE PROPERTY	7-21
HARISH CHANDRA YADAV, SHYAM LAL	SECOND KIND CHEBYSHEV WAVELET ANALYSIS OF ABEL'S INTEGRAL EQUATIONS	22 - 37
A. GNANASOUNDARI, V. INTHUMATHI, M. MAHESWARI	GENERALIZED SOFT MULTI CONNECTEDNESS AND COMPACTNESS	38 - 50
P. KAVITHA	SPLIT LEGENDARY DOMINATION IN GRAPHS	51 - 68
PRAMOD M DHAKANE	ON AN ABSTRACT NONLINEAR FUNCTIONAL SECOND ORDER VOLTERRA INTEGRODIFFERENTIAL EQUATION	69 - 84
VITHYADEVI, ANNADURAI	HEINZ QUARTER MEAN LABELING OF GRAPHS	85 - 94
DAKSHA MANOJBHAI DIWAN, DEVBHADRA V SHAH, VANDANA R PATEL	SOME PROPERTIES AND EXTENDED BINET'S FORMULA FOR THE CLASS OF BIFURCATING FIBONACCI SEQUENCE	95 - 102
PURVA RAJWADE, RACHNA NAVALAKHE	NANO FUZZY BITOPOLOGICAL SPACES	103 - 111
RAJESHWARI K., T. D. RAYANAGOUDAR, SARIKA M. PATIL, P. S. MIRAJAKAR	SUPRA BETA-PARACOMPACT SPACES	112 - 119
V AKSHAYA	OCTAGONAL PRIME GRACEFUL LABELING	120 - 133
BHANU CHANDER, DASARI RATNA BABU, T V PRADEEP KUMAR	COMMON FIXED POINTS FOR TWO PAIRS OF SELFMAPS SATISFYING CERTAIN CONTRACTION CONDITION IN $\$B\$$ -METRIC SPACES	134 - 151
PRITI V. TANDEL, MANAN A. MAISURIA	NUMERICAL SOLUTION OF POLLUTANT TRANSPORT MODEL WITH SOURCE TERM BY RDTM	152 - 174
KALARANI M	ON NANO GENERALIZED $A^{(**)}$ CONTINUOUS AND NANO GENERALIZED $A^{(**)}$ IRRESOLUTE MAPPINGS	175 - 191
SATYAJ TIWARI, MONIKA DEWANGAN	BEST PROXIMITY POINT FOR WEAK CYCLIC KANNAN TYPE $\$F$ - $\$CONTRACTION$ MAP AND CYCLIC CHATTERJEA TYPE $\$F$ - $\$CONTRACTION$ MAP	192 - 206
PRAKASH DNYANBA KHANDARE, SURYAKANT M JOGDAND, RAJESH A MUNESHWAR	THE MODIFIED DIVISOR GRAPH OF COMMUTATIVE RING	207 - 220
PREETHAM NATARAJ RAJ, HARINA PANDIT WAGHAMORE	NON-LINEAR DIFFERENCE POLYNOMIALS SHARING A POLYNOMIAL WITH FINITE WEIGHT	221 - 241
KAPIL JAIN, JATINDERDEEP KAUR, SATVINDER SINGH BHATIA	FIXED POINT RESULTS FOR (Ψ, Φ) -CONTRACTIVE MAPPING IN GF-METRIC SPACE	242 - 267
AUDRIJA CHOUDHURY, RUPA PAL	PARTIAL SHARING OF SMALL FUNCTIONS WITH EXACT DIFFERENCE	268 - 285
PETER J. VEAZIE	DETERMINING DEPENDENCE AMONG RANDOM VARIABLES ACROSS OBSERVATIONS	286 - 306
ROOPA SUBHAS NAIKAR	STATUS CONNECTIVITY INDICES OF MIDDLE GRAPH	307 - 325
M SUPRIYA, R SELVI	SOME ASPECTS OF SOFT M-A-OPEN SETS AND SOFT M-B-OPEN SETS IN SOFT GENERALIZED TOPOLOGICAL SPACES	326 - 342
HUSNA V, NAGARJUN V	VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS WHOSE DIFFERENTIAL POLYNOMIALS SHARE A SMALL FUNCTION	343 - 358
PANDURANG D. KUNDGAR, JAGDISH A. NANWARE	EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR SYSTEM OF NONLINEAR FRACTIONAL REACTION DIFFUSION EQUATIONS	359 - 374
VIKAS SOPAN JADHAV, ASMA KAZI, SITARAM G. LATPATE, VANDANA S. GHODKE	A NONLINEAR GENERALIZATIONS OF DYNAMIC INEQUALITIES ON TIME SCALES	375 - 392
EMMANUEL DEGRAFT JOHNSON OWUSU ANSAH, JONATHAN AYIKU, SIMON KOJO APPIAH, EMMANUEL HARRIS	ARMA AND/OR SETAR ESTIMATION AND OUT-OF-SAMPLE FORECAST OF THE MEAN-REVERSION BETWEEN BRENT CRUDE OIL AND GASOLINE PRICES ON THE GHANAIAN MARKET	393 - 429
HARINA WAGHAMORE P	UNICITY RESULTS CONCERNING OF DIFFERENCE MONOMIALS OF L- FUNCTION AND A MEROMORPHIC FUNCTION	430 - 449