ISSN 1592-7415 **eISSN 2282-8214**

Volume n. 50 – 2023

RATIO MATHEMATICA

Journal of Mathematics, Statistics, and Applications

Special Issue:

RECENT INNOVATIONS IN APPLICATIONS OF MATHEMATICS

Publisher

A.P.A.V.
Accademia Piceno – Aprutina dei Velati in Teramo, Italy

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Journal of Mathematics, Statistics, and Applications

ISSN 1592-7415 - eISSN 2282-8214

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Publisher: **APAV** - Accademia Piceno-Aprutina dei Velati in Teramo, Italy
Tax Code 92036140678, **Vat Id. Number** 02184450688
Registered office: Contrada Salice, 28 - 87023 - Diamante (CS)
Operational headquarters: c / o Hotel Cristina - via Pietrarossa, 24 - Diamante (CS)
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Ratio Mathematica Volume 50, 2023

On NS $^{s}(\mathscr{I})$ Semi Open Sets

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Abstract

In this paper we present Nano soft(NS) $^{s}(\mathscr{I})$ semi open(SO) sets and NS $^{s}(\mathscr{I})$ semi closed(SC) sets in nano soft ideal topological space. Further we investigate NS $^{s}(\mathscr{I})$ semi interior and NS $^{s}(\mathscr{I})$ semi closure with theorems and examples. Also their properties and characterization are discussed.

Keywords: NS $^{s}(\mathscr{I})$ SO; NS $^{s}(\mathscr{I})$ SC; NS $^{s}(\mathscr{I})$ semi- interior; NS $^{s}(\mathscr{I})$ semi- closure.
2020 AMS subject classifications: 54A10, 54C50, 54G20. [1]

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[1]Received on May 12, 2023. Accepted on December 12, 2023. Published on December, 31, 2023. doi: 10.23755/rm.v50i0.ID. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Molodstov[9] created the soft set (SS) theory in 1999 to address the issue of modelling uncertainty mathematically. The soft topological spaces (STS) were first described by M. Shabir and M. Naz[12]. Many other authors contributed to this area. Soft Semi Open sets and Soft Semi Closed sets were developed by Binchen [2] and their characteristics in soft topological spaces were examined. As for the nano topological spaces, it seems that their study was initiated by Lellis Thivagar et al.[6] in 2013. Since then, various mathematicians proved many important results on nano topologies. In 1990, Jankovic and Hamlett[3] developed the ideal topological space. The notion of soft ideal was initiated by Kandil et al.[4]. S. P. R. Priyalatha et al.[10] invented the nano soft ideal topology. In this paper we investigate NS $^s(\mathscr{I})$ Semi Open sets, NS $^s(\mathscr{I})$ Semi Closed sets, NS $^s(\mathscr{I})$ Semi Interior sets and NS $^s(\mathscr{I})$ Semi Closure sets in nano soft ideal topological space based on the concept of nano topology with soft ideal sets. Further, the characterization and properties are discussed with theorems and examples.

2 Preliminaries

Definition 2.1. *Let $\tilde{U}$ be a non empty finite set of objects called the universe, R be an equivalence relation on $\tilde{U}$ named as the indiscerniblity relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\tilde{U}, R)$ is said to be approximation space. Let $X \subseteq \tilde{U}$.*

(i) The Lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \left\{ \bigcup\limits_{x \in \tilde{\mathcal{U}}} \{R(x) : R(x) \subseteq X\} \right\}$, where $R(x)$ denotes the equivalence class determined by x.

(ii) The Upper approximation of X with respect to R is the set of all objects, which can be for possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \left\{ \bigcup\limits_{x \in \tilde{\mathcal{U}}} \{R(x) : R(x) \cap X \neq \emptyset\} \right\}$.

(iii) The Boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X) = U_R(X) - L_R(X)$".

Definition 2.2. *Let $\tilde{U}$ be the universe, R be an equivalence relation on $\tilde{U}$ and $\tau_R(x) = \left\{ \tilde{U}, \tilde{\emptyset}, L_R(X), U_R(X), B_R(X) \right\}$ where $X \subseteq \tilde{U}$ and $\tau_R(X)$ satisfies the* *following axioms.*

(i) $\tilde{U}$ *and* $\tilde{\emptyset} \in \tau_R(X)$.

(ii) The union of the elements of any subcollection $\tau_R(X)$ *is in* $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub collection $\tau_R(X)$ *is in* $\tau_R(X)$.

That is, $\tau_R(X)$ *forms a topology on* $\tilde{U}$ *and it is called as the nano topology on* $\tilde{U}$ *with respect to* X. *The elements of* $\tau_R(X)$ *are called as nano open sets".*

Definition 2.3. *If* $(\tilde{U}, \tilde{\tau}_R(X))$ *is a nano topological space with respect to* X *where* $X \subseteq \tilde{U}$ *and if* $A \subseteq \tilde{U}$*, then the nano interior of* A *is defined as the union of all nano open subsets of* A *and it is denoted by* $NInt(A)$

Definition 2.4. *If* $(\tilde{U}, \tilde{\tau}_R(X))$ *is a nano topological space with respect to* X *where* $X \subseteq \tilde{U}$ *and if* $A \subseteq \tilde{U}$*, then the nano closure of* A *is defined as the intersection of all nano closed sets containing* A *and it is denoted by* $NCl\,(A)$.

Definition 2.5. *If* $(\tilde{U}, \tilde{\tau}_R(X))$ *is a nano topological space and* $A \subseteq \tilde{U}$. *Then A is said to be Nano SO if* $A \subseteq NCl(NInt(A))$.

Definition 2.6. *A SS* $\mathscr{F}_{\mathscr{A}}$ *on the universe* $\tilde{U}$ *is defined by the set of ordered pairs* $\mathscr{F}_{\mathscr{A}} = \Big\{(e, F(e)) : e \in E, F(e) \in P(\tilde{U})\Big\}$*, where* $F : E \to P(\tilde{U})$ *such that* $F(e) = \emptyset$*, if* $e \notin \mathscr{A}$ *and* $\mathscr{A} \subseteq E$

Definition 2.7. *Let* $\tilde{\tau}$ *be the collection of SSs over* $\tilde{U}$*, then* $\tilde{\tau}$ *is said to be soft topology on* $\tilde{U}$ *if*

(i) $\tilde{U}, \tilde{\emptyset} \in \tilde{\tau}$

(ii) Union of any number of SSs in $\tilde{\tau}$ *belongs to* $\tilde{\tau}$.

(iii) Intersection of any two SSs in $\tilde{\tau}$ *belongs to* $\tilde{\tau}$.

The triplet $(\tilde{U}, \tilde{\tau}, E)$ *is called soft topological space over* $\tilde{U}$. *The members in* $\tilde{\tau}$ *are said to be soft open sets in* $\tilde{U}$."

Definition 2.8. *A soft subset* $\mathscr{F}_{\mathscr{A}}$ *of a soft toplogical space* $\tilde{U}$ *is said to be soft closed if* $\tilde{U} - \mathscr{F}_{\mathscr{A}}$ *is soft open.*

Definition 2.9. *Let $(\tilde{U}, \tilde{\tau}, E)$ be a soft topological space over $\tilde{U}$ and $\mathscr{F}_{\mathscr{A}}$ be a SS over $\tilde{U}$. Then the soft closure of $\mathscr{F}_{\mathscr{A}}$, denoted by $\bar{\mathscr{F}}_{\mathscr{A}}$ is the intersection of all soft closed supersets of $\mathscr{F}_{\mathscr{A}}$."*

Definition 2.10. *Let $(\tilde{U}, \tilde{\tau}, E)$ be a soft topological space over $\tilde{U}$, $\mathscr{F}_{\mathscr{A}}$ be a SS over $\tilde{U}$ and $u \in \tilde{U}$. Then u is said soft interior of $\mathscr{F}_{\mathscr{A}}$ if there exists a soft open set $\mathscr{G}_{\mathscr{A}}$ such that $u \in \mathscr{G}_{\mathscr{A}} \subset \mathscr{F}_{\mathscr{A}}$.*

Definition 2.11. *A SS $\mathscr{F}_{\mathscr{A}}$ in a soft topological space $(\tilde{U}, \tilde{\tau}, E)$ is said to be soft SO if there exists a soft open set $\mathscr{G}_{\mathscr{A}}$ such that $\mathscr{G}_{\mathscr{A}} \subset \mathscr{F}_{\mathscr{A}} \subset Cl(\mathscr{G}_{\mathscr{A}})$.*

Definition 2.12. *An ideal $\mathscr{I}$ on a topological space $(\tilde{U}, \tilde{\tau})$ is a non empty collection of subsets of $\tilde{U}$ which satisfies*

(i) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$ and

(ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$.

Definition 2.13. *Let $\mathscr{I}$ be the non empty collection of SSs over $\tilde{U}$, with the same set of parameters E. Then $\mathscr{I} \subseteq SS(\tilde{U})_E$ is called a soft ideal on $\tilde{U}$ with the same set E if,*

(i) $\mathscr{F}_E \in \mathscr{I}$ and $\mathscr{G}_E \in \mathscr{I}$ implies $\mathscr{F}_E \cup \mathscr{G}_E \in \mathscr{I}$.

(ii) $\mathscr{F}_E \in \mathscr{I}$ and $\mathscr{G}_E \subseteq \mathscr{F}_E$ implies $\mathscr{G}_E \in \mathscr{I}$."

Definition 2.14. *Let $\tilde{U}$ be a non empty finite set of objects called the universe, $\mathscr{F}_{\mathscr{A}} \subseteq \mathscr{G}_{\mathscr{A}}$ is an SS over $\tilde{U}$ and $\mathscr{I}$ is an ideal on $\mathscr{G}_{\mathscr{A}}$. Then $(\tilde{U}, \mathscr{F}_{\mathscr{A}}, \mathscr{I})$ is an triplet ordered pair of soft ideal approximation space and $\tilde{\tau}_R(\mathscr{I}) = \left\{\tilde{U}, \emptyset, L_R(\mathscr{I}), U_R(\mathscr{I}), B_R(\mathscr{I})\right\}$ where $\mathscr{I} \subseteq \mathscr{G}_{\mathscr{A}}$ and $\tilde{\tau}_R(\mathscr{I})$ satisfies the following axiom*

(i) $\tilde{U}, \emptyset \in \tilde{\tau}_R(\mathscr{I})$

(ii) The union of the elements of any subcollection of soft ideal $\tilde{\tau}_R(\mathscr{I})$is in $\tilde{\tau}_R(\mathscr{I})$.

(iii) The intersection of the elements of any finite subcollection of soft ideal $\tilde{\tau}_R(\mathscr{I})$is in $\tilde{\tau}_R(\mathscr{I})$.

That is, $\tilde{\tau}_R(\mathscr{I})$ forms a soft ideal topology on $\tilde{U}$ having atmost five elements of soft ideal and four ordered pair $(\tilde{U}, \tilde{\tau}_R, E, \mathscr{I})$ is called a NS ideal topological space over $\tilde{U}$ with respect to $\mathscr{I}$, then the members of $\tilde{\tau}_R$ are said to be NS ideal open sets in $\tilde{U}$.

Definition 2.15. *Let $(\tilde{U}, \tilde{\tau}, E, \mathscr{I})$ be a NS ideal topological space over $\tilde{U}$. Then NS ideal closure of SS $\mathscr{F}_E$ over $\tilde{U}$ is denoted by $\mathscr{N}_{SI}Cl(\mathscr{F}_E)$. Thus $\mathscr{N}_{SI}Cl(\mathscr{F}_E)$ is the smallest NS ideal closed set which containing $\mathscr{F}_E$ and is defined as the intersection of all NS ideal closed supersets of $\mathscr{F}_E$.*

Definition 2.16. *Let $\mathscr{F}_{\mathscr{B}}$ be a soft subset of a soft topological space $(\mathscr{F}_A, \tilde{\tau})$. $\mathscr{F}_B$ is said to be a soft SO set if,$\mathscr{F}_B \subseteq cl(int(\mathscr{F}_B))$*

Definition 2.17. *Let $\mathscr{F}_{\mathscr{B}}$ be a soft subset of a soft topological space $(\mathscr{F}_A, \tilde{\tau})$. $\mathscr{F}_B$ is said to be a soft SC set if its relative complement is soft SO.*

Definition 2.18. *Let $(\mathscr{F}_A, \tilde{\tau})$ be a soft topological space and let $\mathscr{F}_B$ be a SS in $\mathscr{F}_A$. The soft semi- interior of $\mathscr{F}_B$ is the SS $\cup\{\mathscr{F}_C : \mathscr{F}_C$ is soft SO and $\mathscr{F}_C \subseteq \mathscr{F}_B\}$ and is denoted by S- int$(\mathscr{F}_B)$.*

Definition 2.19. *Let $(\mathscr{F}_A, \tilde{\tau})$ be a soft topological space and let $\mathscr{F}_B$ be a SS in $\mathscr{F}_A$. The soft semi- closure of $\mathscr{F}_B$ is the SS $\cap\{\mathscr{F}_C : \mathscr{F}_C$ is soft SC and $\mathscr{F}_B \subseteq \mathscr{F}_C\}$ and is denoted by S- cl$(\mathscr{F}_B)$.*

3 NS $^s(\mathscr{I})$ SO Sets

In this section we present NS $^s(\mathscr{I})$ SO sets, NS $^s(\mathscr{I})$ SC sets, NS $^s(\mathscr{I})$ semi interior and NS $^s(\mathscr{I})$ semi closure with theorems and examples. Throughout this paper we represent nano soft ideal semi open sets by NS $^s(\mathscr{I})$ SO sets, nano soft ideal semi closed sets by NS $^s(\mathscr{I})$ SC sets, nano soft ideal topological space by NSITS.

Definition 3.1. *"Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NS ideal topological space over $\tilde{Y}$ and soft ideal $\mathscr{F}_A$ is said to be NS $^s(\mathscr{I})$ SO set if $\mathscr{F}_A \subseteq \mathscr{N}_{SI}Cl(\mathscr{N}_{SI}Int(\mathscr{F}_A)$.*

Example 3.1. *Let $\tilde{Y} = \{y_1, y_2\}$ and $E = \{e_1, e_2\}$ Let $A = \{e_1\}$. Then $\mathscr{F}_A = \{(e_1, \tilde{Y})\}$ and $^s(\mathscr{I}) \subseteq \mathscr{F}_E$ where $^s(\mathscr{I}) = \{\emptyset, (e_1, \{y_1\})\}$ and $R = \{\mathscr{F}(y_1) \times \mathscr{F}(y_1)\}$. Here $\iota^s_R(\mathscr{I}) = \{\tilde{Y}, \emptyset, (e_1, \{y_1, y_2\})\}$ and $(\iota^s_R(\mathscr{I}))^c = \{\tilde{Y}, \emptyset, (e_2, \{y_1, y_2\})\}$. Then the family of all NS$^s(\mathscr{I})$ SO sets are $\emptyset, \tilde{Y}, \{e_1, \{y_1, y_2\}\}, \{(e_1, \{y_1, y_2\}), (e_2, \{y_1\})\}, \{(e_1, \{y_1, y_2\}), (e_2, \{y_2\})\}$ and the family of all NS$^s(\mathscr{I})$ SC sets are $\emptyset, \tilde{Y}, \{e_2, \{y_1\}\}, \{(e_2, \{y_1, y_2\})\}, \{e_2, \{y_2\}\}$.*

Theorem 3.1. *Every NS $^s(\mathscr{I})$ open set in NS ideal space, $\mathscr{F}_A$ is a NS $^s(\mathscr{I})$ SO set.*

Proof. This proof is deduced from definition 3.1. □

Remark 3.1. *The above theorem does not hold true in its opposite implication. Therefore, not all NS $^s(\mathscr{I})$ SO sets are NS $^s(\mathscr{I})$ open sets.*

Example 3.2. *Consider the NS $^{s}(\mathscr{I})$ SO set in the example 3.1 are* $\{(e_1, \{y_1, y_2\}), (e_2, \{y_1\})\}, \{(e_1, \{y_1, y_2\}), (e_2, \{y_2\})\}$ *but these are not NS $^{s}(\mathscr{I})$ open sets.*

Remark 3.2. *$\mathscr{F}_{\emptyset}, \mathscr{F}_A$ are always NS $^{s}(\mathscr{I})$ S- closed and NS$^{s}(\mathscr{I})$ S- open sets.*

Theorem 3.2. *A soft ideal $\mathscr{F}_B$ in a nano soft ideal topological space is a NS $^{s}(\mathscr{I})$ S- open set iff there exists a NS $^{s}(\mathscr{I})$ open set $\mathscr{F}_C \ni \mathscr{F}_C \subset \mathscr{F}_B \subset N_{SI}cl(\mathscr{F}_C)$.*
Proof. *Suppose that $\mathscr{F}_B \subseteq N_{SI}cl(N_{SI}int(\mathscr{F}_B))$.Then for $\mathscr{F}_C = N_{SI}int(\mathscr{F}_B)$, we have $\mathscr{F}_C \subseteq \mathscr{F}_B \subseteq N_{SI}cl(\mathscr{F}_C)$. Therefore the condition holds. Conversely suppose that $\mathscr{F}_C \subset \mathscr{F}_B \subset N_{SI}cl(\mathscr{F}_C))$ for some NS $^{s}(\mathscr{I})$ open set $\mathscr{F}_C$. Since $\mathscr{F}_C \subseteq N_{SI}int(\mathscr{F}_B)$ and so $N_{SI}cl(\mathscr{F}_C) \subseteq N_{SI}cl(N_{SI}int(\mathscr{F}_B))$. Hence $\mathscr{F}_B \subseteq N_{SI}cl(\mathscr{F}_C) \subseteq N_{SI}cl(N_{SI}int(\mathscr{F}_B))$. Hence $\mathscr{F}_B$ is NS $^{s}(\mathscr{I})$ SO set.* □

Theorem 3.3. *Let $(\tilde{Y}, \tilde{\iota}_R(^{s}(\mathscr{I})))$ be a NS ideal topological space over $\tilde{Y}$ and $\{(\mathscr{F}_{B_\alpha}) : \alpha \in \Delta\}$ be a set of NS $^{s}(\mathscr{I})$ S- open sets in $\tilde{Y}$. Then $\cup_{\alpha\in\Delta}\mathscr{F}_{B_\alpha}$ is also a NS $^{s}(\mathscr{I})$ S- open set.*
Proof. *Let $\{(\mathscr{F}_{B_\alpha}) : \alpha \in \Delta\}$ be a collection of NS $^{s}(\mathscr{I})$ SO sets in $\tilde{Y}$. Then for each $\alpha \in \Delta$, we have a NS $^{s}(\mathscr{I})$ open set $\mathscr{F}_{C_\alpha} \subseteq \mathscr{F}_{B_\alpha}$ such that $\mathscr{F}_{C_\alpha} \subseteq \mathscr{F}_{B_\alpha} \subseteq N_{SI}cl(\mathscr{F}_{C_\alpha})$. Then $\cup_{\alpha\in\Delta}\mathscr{F}_{C_\alpha} \subseteq \mathscr{F}_{B_\alpha} \subseteq \cup_{\alpha\in\Delta}N_{SI}cl(\mathscr{F}_{C_\alpha}) \subseteq N_{SI}cl(\cup_{\alpha\in\Delta}\mathscr{F}_{C_\alpha})$.*
Hence $\cup_{\alpha\in\Delta}\mathscr{F}_{B_\alpha}$ is a NS $^{s}(\mathscr{I})$ SO set. □

Definition 3.2. *A soft ideal $\mathscr{F}_B$ in a NS ideal space $(\tilde{Y}, \tilde{\iota}_R(^{s}(\mathscr{I})))$ is called a NS $^{s}(\mathscr{I})$ SC set, if its complement is a NS $^{s}(\mathscr{I})$ SO set.*

Theorem 3.4. *For each NSI closed set in a NS ideal TS $(\tilde{Y}, \tilde{\iota}_R(^{s}(\mathscr{I})))$ is NS $^{s}(\mathscr{I})$ SC set.*
Proof. *Assume that $\mathscr{F}_B$ in a NSITS $(\tilde{Y}, \tilde{\iota}_R(^{s}(\mathscr{I})))$ is a NS $^{s}(\mathscr{I})$ closed set. By the definition of NS $^{s}(\mathscr{I})$ closed set, its complement is NS $^{s}(\mathscr{I})$ open set. Then $(\mathscr{F}_B)^c$ is a NS $^{s}(\mathscr{I})$ SO set. Hence $\mathscr{F}_B$ is a NS $^{s}(\mathscr{I})$ SC set.* □

Remark 3.3. *The converse of the above theorem is not true. In other words, not every NS $^{s}(\mathscr{I})$ SC set is an NS $^{s}(\mathscr{I})$ closed set.*

Example 3.3. *From the example 3.2, consider the NS $^{s}(\mathscr{I})$ SC sets $\{(e_1, y_1)\}$, $\{(e_2, y_2)\}$ which are not NS $^{s}(\mathscr{I})$ closed sets.*

Theorem 3.5. *Let $\mathscr{F}_C$ be a NS $^{s}(\mathscr{I})$ SC set in a NS ideal topological space $(\tilde{Y}, \tilde{\iota}_R(^{s}(\mathscr{I})))$ iff $N_{SI}int\mathscr{F}_E \subseteq \mathscr{F}_C \subseteq \mathscr{F}_E$ for some NS $^{s}(\mathscr{I})$ closed set $\mathscr{F}_E$.*
Proof : *Since $\mathscr{F}_C$ is NS $^{s}(\mathscr{I})$ SC*

$\iff$ *$(\mathscr{F}_C)^c$ is NS $^{s}(\mathscr{I})$ SO*

$\iff$ *there exists $\mathscr{F}_D \ni \mathscr{F}_D \subseteq (\mathscr{F}_C)^c \subseteq N_{SI}cl(\mathscr{F}_D)$*
by theorem 3.2

$\Longleftrightarrow$ *there is a NS $^s(\mathscr{I})$ open set* $\mathscr{F}_D \ni (N_{SI}cl(\mathscr{F}_D))^c \subseteq \mathscr{F}_C \subseteq (\mathscr{F}_D)^c$.
$\Longleftrightarrow$ *a NS $^s(\mathscr{I})$ open set* $\mathscr{F}_D \ni N_{SI}int(\mathscr{F}_D^c) \subseteq \mathscr{F}_C \subseteq (\mathscr{F}_D)^c$.
$\Longleftrightarrow$ *there exists* $\mathscr{F}_D \ni N_{SI}int(\mathscr{F}_E) \subseteq \mathscr{F}_C \subseteq \mathscr{F}_E$
where $\mathscr{F}_E = \mathscr{F}_D^c$. □

Theorem 3.6. *A soft ideal $\mathscr{F}_B$ in a NSITS $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ is NS $^s(\mathscr{I})$ SC iff* $N_{SI}int(N_{SI}cl(\mathscr{F}_B)) \subseteq \mathscr{F}_B$.
Proof : *$\mathscr{F}_B$ is NS $^s(\mathscr{I})$ SC*
$\Longleftrightarrow$ $(\mathscr{F}_B)^c \subseteq N_{SI}cl(N_{SI}int(\mathscr{F}_B)^c)$
$\Longleftrightarrow$ $(\mathscr{F}_B)^c \subseteq N_{SI}cl(N_{SI}cl(\mathscr{F}_B)^c)$, *by definition*
$\Longleftrightarrow$ $(\mathscr{F}_B)^c \subseteq (N_{SI}int(N_{SI}cl(\mathscr{F}_B)))^c$
$\Longleftrightarrow$ $(\mathscr{F}_B)^c \subseteq N_{SI}int(N_{SI}cl(\mathscr{F}_B)) \subseteq \mathscr{F}_B$
This completes the proof. □

Theorem 3.7. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and $\{(\mathscr{F}_{B_\alpha}) : \alpha \in \Delta\}$ be a collection of NS $^s(\mathscr{I})$ SC sets in $\tilde{Y}$. Then $\cap_{\alpha\in\Delta}\mathscr{F}_{B_\alpha}$ is also NS $^s(\mathscr{I})$ SC set.*
Proof : *Let $\{(\mathscr{F}_{B_\alpha}) : \alpha \in \Delta\}$ be a collection of NS $^s(\mathscr{I})$ SC sets in $\tilde{Y}$. Then for each $\alpha \in \Delta$, we have a NS $^s(\mathscr{I})$ closed set $\mathscr{F}_{C_\alpha}$ such that $N_{SI}int\mathscr{F}_{C_\alpha} \subseteq \mathscr{F}_{B_\alpha} \subseteq \mathscr{F}_{C_\alpha}$. Then $N_{SI}int(\cap_{\alpha\in\Delta}\mathscr{F}_{C_\alpha}) \subseteq \cap_{\alpha\in\Delta}N_{SI}int(\mathscr{F}_{C_\alpha}) \subseteq \cap_{\alpha\in\Delta}\mathscr{F}_{B_\alpha} \subseteq \cap_{\alpha\in\Delta}\mathscr{F}_{C_\alpha}$. Because $\cap_{\alpha\in\Delta}\mathscr{F}_{C_\alpha} = \mathscr{F}_C$ is NS $^s(\mathscr{I})$ closed set then $\cap_{\alpha\in\Delta}\mathscr{F}_{B_\alpha}$ is NS $^s(\mathscr{I})$ SC set.* □

Definition 3.3. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and let $\mathscr{F}_B$ be a soft ideal in $\tilde{Y}$.*

(i) The NS $^s(\mathscr{I})$ semi- interior of $\mathscr{F}_B$ is the $\cup\{\mathscr{F}_C : \mathscr{F}_C$ NS$^s(\mathscr{I})$ SO and $\mathscr{F}_C \subseteq \mathscr{F}_B\}$ and is represented by NS$^s(\mathscr{I})$ S- int($\mathscr{F}_B$).

(ii) The NS $^s(\mathscr{I})$ semi- closure of $\mathscr{F}_B$ is the $\cap\{\mathscr{F}_C : \mathscr{F}_C$ NS$^s(\mathscr{I})$ SC and $\mathscr{F}_B \subseteq \mathscr{F}_C\}$ and is represented by NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$).

Clearly NSI $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$) is the smallest NS$^s(\mathscr{I})$ SC set containing $\mathscr{F}_B$ and NS $^s(\mathscr{I})$ S- int($\mathscr{F}_B$) is the largest NS$^s(\mathscr{I})$ SO set contained in $\mathscr{F}_B$.
By theorem 3.8 and 3.15 we have NSI S-int($\mathscr{F}_B$) is NS$^s(\mathscr{I})$ SO and NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) is NS $^s(\mathscr{I})$ SC sets.

Example 3.4. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and the soft ideal $\mathscr{F}_B = \{(e_1, y_1), (e_2, y_1, y_2)\}$ as in example 3.2 we get NS $^s(\mathscr{I})$ S-int($\mathscr{F}_B$) $= \mathscr{F}_B$.*

Example 3.5. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and the soft ideal $\mathscr{F}_B = \{(e_1, y_1), (e_2, y_2)\}$ as in example 3.2 we get NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) $= \mathscr{F}_B$.*

Theorem 3.8. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and $\mathscr{F}_B$ be a soft ideal in $\tilde{Y}$. We have $N_{SI}int(\mathscr{F}_B) \subseteq$NS $^s(\mathscr{I})$ S-int($\mathscr{F}_B$) $\subseteq \mathscr{F}_B \subseteq$NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) $\subseteq N_{SI}cl(\mathscr{F}_B)$.*
Proof: *This proof follows from the theorem 3.1, theorem 3.4 and definition 3.3.*

Theorem 3.9. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and $\mathscr{F}_B$ be a soft ideal in $\mathscr{F}_A$. Then the following hold.*

(i) (NS $^s(\mathscr{I})S - cl(\mathscr{F}_B))^c$ =NS $^s(\mathscr{I})S - int(\mathscr{F}_B^c)$

(ii) (NS $^s(\mathscr{I})S - int(\mathscr{F}_B))^c$ =NS $^s(\mathscr{I})S - cl(\mathscr{F}_B^c)$

Proof :

(i) (NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B))^c = (\cap\{\mathscr{F}_C : \mathscr{F}_C$ *is NS* $^s(\mathscr{I})$ *semi closed and* $\mathscr{F}_C \subseteq \mathscr{F}_B\})^c$.
$= \cup\{\mathscr{F}_C^c : \mathscr{F}_C$ *is NS*$^s(\mathscr{I})$ *SC and* $\mathscr{F}_C \subseteq \mathscr{F}_B\}$.

$= \cup\{\mathscr{F}_C^c : \mathscr{F}_C^c$ *is NS* $^s(\mathscr{I})$ *SO and* $\mathscr{F}_C^c \subseteq \mathscr{F}_B^c\}$.

=NS $^s(\mathscr{I})$ *int* $(\mathscr{F}_B)^c$.

(ii) (NS $^s(\mathscr{I})$ S-int$(\mathscr{F}_B))^c = (\cup\{\mathscr{F}_C : \mathscr{F}_C$ *is NS* $^s(\mathscr{I})$ *semi open and* $\mathscr{F}_C \subseteq \mathscr{F}_B\})^c$.

$= \cap\{\mathscr{F}_C^c : \mathscr{F}_C$ *is NS* $^s(\mathscr{I})$ *SO and* $\mathscr{F}_C \subseteq \mathscr{F}_B\}$.

$= \cap\{\mathscr{F}_C^c : \mathscr{F}_C^c$ *is NS* $^s(\mathscr{I})$ *SC and* $\mathscr{F}_C^c \subseteq \mathscr{F}_B^c\}$.

=NS $^s(\mathscr{I})$ *cl* $(\mathscr{F}_B)^c$.

Theorem 3.10. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and let $\mathscr{F}_B$ and $\mathscr{F}_C$ are soft ideals in $\tilde{Y}$. Then the next holds.*

(i) NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_\emptyset) = \mathscr{F}_\emptyset$ *and* NS $^s(\mathscr{I})$ S- cl$(\mathscr{F}_A) = \mathscr{F}_A$.

(ii) $\mathscr{F}_B$ *is NS* $^s(\mathscr{I})$ *SC set iff* $\mathscr{F}_B =$ NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B)$.

(iii) NS $^s(\mathscr{I})$ S-cl(NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B)$) =NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B)$.

(iv) $\mathscr{F}_B \subset \mathscr{F}_C$ *implies* NS $^s(\mathscr{I})$ S-cl $(\mathscr{F}_B) \subseteq$ NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_C)$.

(v) NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B \cap \mathscr{F}_C) \subseteq$NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B)\cap$NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_C)$.

(vi) NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B \cup \mathscr{F}_C) \subseteq$NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_B)\cup$NS $^s(\mathscr{I})$ S-cl$(\mathscr{F}_C)$.

Proof:

(i) *It is trivial.*

(ii) *If $\mathscr{F}_B$ is NS $^s(\mathscr{I})$ S- closed set, then $\mathscr{F}_B$ is itself a NS $^s(\mathscr{I})$ S-closed in $\mathscr{F}_A$ containing $\mathscr{F}_B$. So NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) is the smallest NS $^s(\mathscr{I})$ SC set containing $\mathscr{F}_B$ and $\mathscr{F}_B$ =NS $^s(\mathscr{I})$ S-cl(F_B). Conversely, suppose that $\mathscr{F}_B$ =NS$^s(\mathscr{I})$ S-cl($\mathscr{F}_B$). Since NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) is a NS $^s(\mathscr{I})$ SC set, so $\mathscr{F}_B$ is NS $^s(\mathscr{I})$ SC set.*

(iii) *Since NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) is a NS$^s(\mathscr{I})$ SC set therefore by part (ii) we have NS $^s(\mathscr{I})$ S-cl(NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$)) =NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$).*

(iv) *Suppose that $\mathscr{F}_B \subseteq \mathscr{F}_C$. Then every NS $^s(\mathscr{I})$ SC super set of $\mathscr{F}_C$ will also contain $\mathscr{F}_B$. This means every NS $^s(\mathscr{I})$ SC super set of $\mathscr{F}_C$ is also a NS $^s(\mathscr{I})$ SC super set of $\mathscr{F}_B$. Hence the NS $^s(\mathscr{I})$ intersection of NS $^s(\mathscr{I})$ semi- closd super sets of $\mathscr{F}_B$ is contained in the NS$^s(\mathscr{I})$ intersection of NS $^s(\mathscr{I})$ SC super sets of $\mathscr{F}_C$. Thus NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) $\subseteq$NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_C$).*

(v) *Since $\mathscr{F}_B \cap \mathscr{F}_C \subseteq \mathscr{F}_B$ and $\mathscr{F}_B \cap \mathscr{F}_C \subseteq \mathscr{F}_C$ and so by part(iv) NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B \cap \mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S- cl ($\mathscr{F}_B$) and NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B \cap \mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$).Thus NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B \cap \mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$)$\cap$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$).*

(vi) *Since $\mathscr{F}_B \subseteq \mathscr{F}_B \cup \mathscr{F}_C$ and $\mathscr{F}_C \subseteq \mathscr{F}_B \cup \mathscr{F}_C$, so by part(iv) $\mathscr{F}_B \subseteq \mathscr{F}_C$ implies NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$) $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$). Then NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$) $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B \cup \mathscr{F}_C$) and NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B \cup \mathscr{F}_C$), which implies NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S- int($\mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S- cl ($\mathscr{F}_B \cup \mathscr{F}_C$). Now, NS$^s(\mathscr{I})$ S- cl($\mathscr{F}_B$), NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$) belongs to NS $^s(\mathscr{I})$ SC set in $\mathscr{F}_A$ which implies that NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$) belong to NS $^s(\mathscr{I})$ SC set in $\mathscr{F}_A$. Then $\mathscr{F}_B$ $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$) and $\mathscr{F}_C$ $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$) imply $\mathscr{F}_B \cup \mathscr{F}_C \subseteq$ NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_C$). That is, NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_C$) is a NS $^s(\mathscr{I})$ SC set containing $\mathscr{F}_B \subseteq \mathscr{F}_C$. Hence NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B \cup \mathscr{F}_C$) $\subseteq$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S- cl ($\mathscr{F}_C$). So, NS $^s(\mathscr{I})$ S-cl($\mathscr{F}_B \cup \mathscr{F}_C$) =NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_B$)$\cup$NS $^s(\mathscr{I})$ S- cl($\mathscr{F}_C$).*

Theorem 3.11. *Let $(\tilde{Y}, \tilde{\iota}_R(^s(\mathscr{I})))$ be a NSITS and let $\mathscr{F}_B$ and $\mathscr{F}_C$ are soft ideals in $\tilde{Y}$. Then the following holds.*

(i) *NS $^s(\mathscr{I})$ S-int($\mathscr{F}_\emptyset$) $= \mathscr{F}_\emptyset$ and NS $^s(\mathscr{I})$ S- int($\mathscr{F}_A$) $= \mathscr{F}_A$.*

(ii) *$\mathscr{F}_B$ is NS $^s(\mathscr{I})$ S- open set iff $\mathscr{F}_B =$ NS $^s(\mathscr{I})$ S-int($\mathscr{F}_B$).*

(iii) *NS $^s(\mathscr{I})$ S-int(NS $^s(\mathscr{I})$ S-int($\mathscr{F}_B$)) =NS $^s(\mathscr{I})$ S-int($\mathscr{F}_B$).*

(iv) *$\mathscr{F}_B \subseteq \mathscr{F}_C$ implies NS $^s(\mathscr{I})$ S-int ($\mathscr{F}_B$) $\subseteq$ NS $^s(\mathscr{I})$ S-int($\mathscr{F}_C$).*

(v) *NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)\cap$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B \cap \mathscr{F}_C)$.

(vi) *NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B \cup \mathscr{F}_C)$=*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$.

Proof:

(i) *Clear.*

(ii) *First part of the proof is obvious from the definition 3.1. Conversely, suppose that* $\mathscr{F}_B$ *=NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$. *Since NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$ *is a NS* $^{s}(\mathscr{I})$ *SO set, so* $\mathscr{F}_B$ *is NS* $^{s}(\mathscr{I})$ *S- open set in* $\mathscr{F}_A$.

(iii) *Since NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$ *is a NS* $^{s}(\mathscr{I})$ *S- open set therefore by part (ii) we have NS* $^{s}(\mathscr{I})$ *S-int(NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$) *=NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$.

(iv) *Suppose that* $\mathscr{F}_B \subseteq \mathscr{F}_C$. *Since NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B) \subseteq \mathscr{F}_B \subseteq \mathscr{F}_C$. *NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$ *is a NS* $^{s}(\mathscr{I})$ *SO subset of* $\mathscr{F}_C$, *so by the definition of NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$, *NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$.

(v) *Since*$\mathscr{F}_B \subseteq \mathscr{F}_B \cap \mathscr{F}_C$ *and* $\mathscr{F}_C \subseteq \mathscr{F}_B \cap \mathscr{F}_C$ *and so by part(iv) NS* $^{s}(\mathscr{I})$ *S-int* $(\mathscr{F}_B)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B \cap \mathscr{F}_C)$ *and NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C) \subseteq$ *NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B \cap \mathscr{F}_C)$. *So that NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)\cap$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C) \subseteq$ *NS* $^{s}(\mathscr{I})$ *S- int*$(F_B \cap \mathscr{F}_C)$, *since NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B \cap \mathscr{F}_C)$ *is a NS* $^{s}(\mathscr{I})$ *SO set.*

(vi) *Since* $\mathscr{F}_B \subseteq \mathscr{F}_B \cup \mathscr{F}_C$ *and* $\mathscr{F}_C \subseteq \mathscr{F}_B \cup \mathscr{F}_C$, *so by part(iv)* $\mathscr{F}_B \subseteq \mathscr{F}_C$ *implies NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$. *Then NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B \cup \mathscr{F}_C)$ *and NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B \cup \mathscr{F}_C)$, *which implies NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int* $(\mathscr{F}_B \cup \mathscr{F}_C)$. *Now, NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)$, *NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$ *belongs to NS* $^{s}(\mathscr{I})$ *SO set in* $\mathscr{F}_A$ *which implies that NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$ *belong to NS* $^{s}(\mathscr{I})$ *semi- open set in* $\mathscr{F}_A$. *Then* $\mathscr{F}_B$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)$ *and* $\mathscr{F}_C$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$ *imply* $\mathscr{F}_B \cup \mathscr{F}_C \subseteq$ *NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$. *That is, NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_C)$ *is a NS* $^{s}(\mathscr{I})$ *SO set containing* $\mathscr{F}_B \cup \mathscr{F}_C$. *Hence NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_B \cup \mathscr{F}_C)$ $\subseteq$*NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)\cup$*NS*$^{s}(\mathscr{I})$ *S- int* $(\mathscr{F}_C)$. *So, NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B \cup \mathscr{F}_C)$ *=NS* $^{s}(\mathscr{I})$ *S- int*$(\mathscr{F}_B)\cup$*NS* $^{s}(\mathscr{I})$ *S-int*$(\mathscr{F}_C)$.

4 Conclusion

This paper investigate the concept of NS $^{s}(\mathscr{I})$ SO sets in nano soft ideal topological space. Some of the theorems and examples are discussed. Moreover, we defined NS $^{s}(\mathscr{I})$ SC sets, NS $^{s}(\mathscr{I})$ semi interior, NS $^{s}(\mathscr{I})$ semi closure sets and go through their properties.

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Ratio Mathematica Volume 50, 2023

Neutrosophic Bipolar Vague Binary Topological Space

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Abstract

In this paper, we make an interesting connection between the mathematical approaches to vagueness and the bipolar binary set. The concept of bipolar binary sets over two universal sets. To eliminate ambiguity in a data set, bipolarity is primarily introduced in imprecise binary sets. This article's main objective is to familiarize the reader with the novel concept of a neutrosophic bipolar vague binary set. We also discuss operations like 'union', 'intersection' and 'complement'. The concepts of neutrosophic bipolar vague binary topology, zero, unit, open and closed set, neutrosophic bipolar vague binary '**𝔟**-open' and '**𝔟**-closed' sets are introduced and their properties are also discussed. Furthermore, the neutrosophic bipolar vague binary 'b-interior' and 'b-closure' set are defined. Theorems related to these concepts are stated and proved.

Keywords: Bipolar binary set, Neutrosophic bipolar vague binary 𝔟 -open and 𝔟 -closed, interior, closure

2010 AMS subject classification: 05C72, 54A05, 54A40.§

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§ Received on May 12, 2023. Accepted on December 15, 2022. Published on December 31, 2023. doi:10.23755/rm.v50i0.1544. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

The fuzzy set was introduced by Zadeh [24] in 1965 where each element had a degree of membership. The bifuzzy set on a universe was proposed by K. Atanasov [15] in 1983, as a generalization of fuzzy set, which discussed both the degree of membership and the degree of non-membership of each element. The concept of Neutrosophic set was proposed by F. Smarandache [15] as a generalization of the intuitionistic fuzzy set, which straight away proved to be very effective in the domains of decision-making, artificial intelligence, and other areas. The 2020 generalized neutrosophic b-open sets were introduced by Das and Pramanik. 2012 saw the introduction of neutrosophic topological space by Salama and Alblowi [16]. Topological spaces and generalized neutrosophic sets were studied by Salama and Alblowi [17]. Then Deli et al. [8] proposed a bipolar neutrosophic set as an extension of neutrosophic sets. A combination of neutrosophic set and vague set is what Shawkat Alkhazaleh [2] refers to as neutrosophic vague set. The neutrosophic vague theory can be used to practice dealing with vague, biased, and inaccurate information. Neutrosophic bipolar vague sets were first introduced by Satham Hussain [18]. In topological spaces, neutrosophic vague binary sets were presented by P.B.Remya.and A.Francina Shalinl. [14]. According to their definition, Arokiarani et al. [3] developed some relationships between the two neutrosophic semi-open functions. neutrosophic pre-open sets and pre-closed sets were first described by Rao and Srinivasa [13] in neutrosophic topological spaces. Iswaraya and Bageerathi were the first to introduce the ideas of neutrosophic semi-closed sets and neutrosophic semi-open sets [12]. In neutrosophic topological spaces, generalized neutrosophic closed sets were studied by Dhavaseelan and Jafari [9]. Imranet et al. [10] later presented the neutrosophic semi-open sets in neutrosophic topological space. Imranet et al. [11] used neutrosophic topological spaces to define neutrosophic generalized alpha generalized continuity. Pushpalatha and Nandhini [20] established the concept of neutrosophic generalized closed sets in neutrosophic topological spaces. Ebenanjar et al. [21] went on to introduce neutrosophic b-open sets in neutrosophic topological spaces. C. Maheswari, M. Sathyabama and S. Chandrasekar. [23] were the first to introduce the concept of neutrosophic simplistic b-closed sets in neutrosophic topological spaces. Afterward, Das and Pramanik [6] proposed neutrosophic soft open sets in neutrosophic soft topological spaces. Das and Tripathy [7] recently pioneered the idea of pairwise neutrosophic-b-open sets in neutrosophic bitopological spaces. M., Shabir, M., Naz [19] who was the first to identify a bipolar soft set and its operations, such as union, intersection, and complement. Then, Karaaslan and Karata's [22] new approximation redefined bipolar soft sets, giving researchers a chance to examine their topological structures. On two initial universal sets, Binary soft set theory was proposed by Ackgoz and Tas [1]. Then, S.S., Benchalli., [4] related basic properties have appropriate parameters that are defined over two initial universal sets. The concept of neutrosophic bipolar vague binary topological space is discussed in the current work along with its investigation. The basic operation in this paper uses a bipolar binary set. Additionally, this study examines neutrosophic bipolar vague binary b-open, b-closed, b-interior, and b-closure operators in addition to the connections between a number of other operators.

2. Preliminaries

Definition 2.1 [18]

Within the context of discussion S, the word Neutrosophic Bipolar Vague Set (abbreviated as *NBVS*) is expressed as A_{NBVS}. $A_{NBVS} = \{< x, \hat{T}^P{}_{ANBVS}(x), \hat{F}^P{}_{ANBVS}(x), \hat{I}^P{}_{ANBVS}(x), \hat{T}^N{}_{ANBVS}(x), \hat{F}^N{}_{ANBVS}(x), \hat{I}^N{}_{ANBVS}(x), >; x \in S\}$. Their extension pertains to 'truth membership', 'indeterminacy membership', and 'falsity membership, $\{\hat{T}^P{}_{ANBVS}(x) = [(T^-)^P(x), (T^+)^P(x)]$; $\hat{I}^P{}_{ANBVS}(x) = [(I^-)^P(x), (I^+)^P(x)]$; $\hat{F}^P{}_{ANBVS}(x) = [(F^-)^P(x), (F^+)^P(x)]\}$, where $(T^+)^P(x) = 1 - (F^-)^P(x)$, $(F^+)^P(x) = 1 - (T^-)^P(x)$ and given that $0 \leq (T^-)^P(x) + (I^-)^P(x) + (F^-)^P(x) \leq 2$.

Also $\{\hat{T}^N{}_{ANBVS}(x) = [(T^-)^N(x), (T^+)^N(x)]$; $\hat{I}^N{}_{ANBVS}(x) = [(I^-)^N(x), (I^+)^N(x)]$; $\hat{F}^N{}_{ANBVS}(x) = [(F^-)^N(x), (F^+)^N(x)]\}$ where $(T^+)^N(x) = 1 - (F^-)^N(x)$, $(F^+)^N(x) = 1 - (T^-)^N(x)$ and given that $0 \geq (T^-)^N(x) + (I^-)^N(x) + (F^-)^N(x) \geq 2$.

Example 2.1

If $S = \{x_1, x_2, x_3\}$ is a set of universes. We define the NBVS A_{NBV} as follows. $A_{NBV} =$

$$\{\frac{x_1}{[0.4,0.7]^P, [0.6,0.6]^P, [0.5,0.8]^P, [-0.4,-0.7]^N, [-0.5,-0.5]^N, [-0.6,-0.8]^N};$$
$$\frac{x_2}{[0.5,0.7]^P, [0.5,0.7]^P, [0.5,0.7]^P, [-0.5,-0.5]^N, [-0.6,-0.6]^N, [-0.7,-0.7]^N};$$
$$\frac{x_3}{[0.4,0.8]^P, [0.7,0.5]^P, [0.4,0.8]^P, [-0.5,-0.7]^N, [-0.6,-0.7]^N, [-0.5,-0.7]^N}\}$$

Definition 2.2 [14]

A Neutrosophic vague binary set M_{NVB} (NVBS in short) over a standard universe $S_1 = \{x_j, 1 \leq j \leq n\}$; $S_2 = \{y_k, 1 \leq k \leq p\}$ is an object of the form

$$M_{NVB} = \left\{ \begin{array}{c} \langle \frac{\hat{T}_{M_{NVB}}(x_j), \hat{I}_{M_{NVB}}(x_j), \hat{F}_{M_{NVB}}(x_j)}{x_j}; \ \forall x_j \in S_1 \rangle \\ \langle \frac{\hat{T}_{M_{NVB}}(y_k), \hat{I}_{M_{NVB}}(y_k), \hat{F}_{M_{NVB}}(y_k)}{y_k}; \forall y_k \in S_2 \rangle \end{array} \right\}$$

is defined as $\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$; $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$; $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$; $x_j \in S_1$ and $\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$; $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$; $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$; $y_k \in S_2$ where

1. $T^+(x_j) = 1 - F^-(x_j)$; $F^+(x_j) = 1 - T^-(x_j)$; $x_j \in S_1$ and $T^+(y_k) = 1 - F^-(y_k)$; $F^+(y_k) = 1 - T^-(y_k)$; $y_k \in S_2$

2. $-0 \leq T^{-}(x_j) + I^{-}(x_j) + F^{-}(x_j) \leq 2^{+};$
   $0 \leq T^{-}(y_k) + I^{-}(y_k) + F^{-}(y_k) \leq 2^{+}$ and
   $-0 \leq T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) \leq 2^{+};$
   $-0 \leq T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \leq 2^{+}$
3. $T^{-}(x_j), I^{-}(x_j), F^{-}(x_j): V(S_1) \rightarrow [0,1]$ and $T^{-}(y_k), I^{-}(y_k), F^{-}(y_k): V(S_2) \rightarrow [0,1]. T^{+}(x_j), I^{+}(x_j), F^{+}(x_j): V(S_1) \rightarrow [0,1]$ and $T^{+}(y_k), I^{+}(y_k), F^{+}(y_k): V(S_2) \rightarrow [0,1]$.
   Here $V(S_1), V(S_2)$ represent power set of vague sets on S_1, S_2 respectively.

Example 2.2
Let $S_1 = \{x_1, x_2, x_3\}$, $S_2 = \{y_1, y_2\}$ be the common universe under consideration. A_{NVBS} is given below:

$$M_{NVB} = \left\{ \begin{array}{c} \langle \frac{[0.2,0.3][0.6,0.7][0.7,0.8]}{x_1}; \frac{[0.3,0.7][0.5,0.6][0.3,0.7]}{x_2}; \frac{[0.1,0.9][0.4,0.8][0.1,0.9]}{x_3} \rangle \\ \langle \frac{[0.6,0.8][0.5,0.7][0.2,0.4]}{y_1}; \frac{[0.2,0.7][0.6,0.9][0.3,0.8]}{y_2} \rangle \end{array} \right\}$$

3. Operations on Neutrosophic bipolar vague binary set

Definition 3.1
The bipolar binary set $\widehat{A_B}$ is defined on a conventional universe $S_1 = \{x_j, 1 \leq j \leq n\}$; $S_2 = \{y_k, 1 \leq k \leq p\}$ is an object of the form

$$\widehat{A_B} = \{\langle \frac{\left(T_A(x_j), I_A(x_j), F_A(x_j)\right)^P \left(T_A(x_j), I_A(x_j), F_A(x_j)\right)^N}{x_j} \forall x_j \in S_1 \rangle$$
$$\langle \frac{(T_A(y_k), I_A(y_k), F_A(y_k))^P (T_A(y_k), I_A(y_k), F_A(y_k))^N}{y_k} \forall y_k \in S_2 \rangle\}$$

$\left(T_A(x_j), I_A(x_j), F_A(x_j)\right)^P : S_1 \rightarrow [0,1]$ and $\left(T_A(x_j), I_A(x_j), F_A(x_j)\right)^N : S_1 \rightarrow [-1,0]$ provide the truth, indeterminacy and false membership values of the components x_j in S_1 and $(T_A(y_k), I_A(y_k), F_A(y_k))^P: S_2 \rightarrow [0,1]$ and $(T_A(y_k), I_A(y_k), F_A(y_k))^N: S_2 \rightarrow [-1,0]$ provides the truth, indeterminacy and false membership values of the component y_k in S_2.

Definition 3.2
A Neutrosophic Bipolar Vague Binary Set M_{NBVB} (NBVBS for brevity) over a standard universe $\{S_1 = \{x_j, 1 \leq j \leq n\}; S_2 = \{y_k, 1 \leq k \leq p\}\}$ is conceptual entity that may be represented as

$M_{NBVB} =$

$$= \langle (\frac{\hat{T}_{M_{NBVB}}(x_j), \hat{I}_{M_{NBVB}}(x_j), \hat{F}_{M_{NBVB}}(x_j)}{x_j})^P; \forall x_j \in S_1\rangle \langle (\frac{\hat{T}_{M_{NBVB}}(y_k), \hat{I}_{M_{NBVB}}(y_k), \hat{F}_{M_{NBVB}}(y_k)}{y_k})^P \forall y_k \in S_2\rangle$$

$$U_2$$

$$\langle \left(\frac{\hat{T}_{M_{NBVB}}(x_j), \hat{I}_{M_{NBVB}}(x_j), \hat{F}_{M_{NBVB}}(x_j)}{x_j}\right)^N; \forall x_j \in S_1\rangle \langle \left(\frac{\hat{T}_{M_{NBVB}}(y_k), \hat{I}_{M_{NBVB}}(y_k), \hat{F}_{M_{NBVB}}(y_k)}{y_k}\right)^N \forall y_k \in S_2\rangle$$

is defined as,

$\hat{T}_{M_{NBVB}}(x_j)^P = [T^-(x_j), T^+(x_j)]^P$; $\hat{T}_{M_{NBVB}}(x_j)^N = [T^-(x_j), T^+(x_j)]^N$

$\hat{I}_{M_{NBVB}}(x_j)^P = [I^-(x_j), I^+(x_j)]^P$; $\hat{I}_{M_{NBVB}}(x_j)^N = [I^-(x_j), I^+(x_j)]^N$

$\hat{F}_{M_{NBVB}}(x_j)^P = [F^-(x_j), F^+(x_j)]^P$; $\hat{F}_{M_{NBVB}}(x_j)^N [F^-(x_j), F^+(x_j)]^N$;

$\hat{T}_{M_{NBVB}}(y_k)^P = [T^-(y_k), T^+(y_k)]^P$; $\hat{T}_{M_{NBVB}}(y_k)^N = [T^-(y_k), T^+(y_k)]^N$

$\hat{I}_{M_{NBVB}}(y_k)^P = [I^-(y_k), I^+(y_k)]^P$; $\hat{I}_{M_{NBVB}}(y_k)^N = [I^-(y_k), I^+(y_k)]^N$

$\hat{F}_{M_{NBVB}}(y_k)^P = [F^-(y_k), F^+(y_k)]^P$; $\hat{F}_{M_{NBVB}}(y_k)^N = [F^-(y_k), F^+(y_k)]^N$

$\forall x_j \in S_1$ and $\forall y_k \in S_2$. where, 1. $(T^+)^P(x_j) = 1 - (F^-)^P(x_j)$; $(F^+)^P(x_j) = 1 - (T^-)^P(x_j)$; $(T^+)^N(x_j) = 1 - (F^-)^N(x_j)$; $(F^+)^N(x_j) = 1 - (T^-)^N(x_j)$ *and* $(T^+)^P(y_k) = 1 - (F^-)^P(y_k)$; $(F^+)^P(y_k) = 1 - (T^-)^P(y_k)$ $(T^+)^N(y_k) = 1 - (F^-)^N(y_k)$; $(F^+)^N(y_k) = 1 - (T^-)^N(y_k)$ $\forall x_j \in S_1$ *and* $\forall y_k \in S_2$.

1. $-0 \leq (T^-)^P(x_j) + (I^-)^P(x_j) + (F^-)^P(x_j) \leq 2^+$
   $0 \geq (T^-)^N(x_j) + (I^-)^N(x_j) + (F^-)^N(x_j) \geq 2^-$;
2. $(T^-)^P(x_j), (I^-)^P(x_j), (F^-)^P(x_j), (T^+)^P(x_j), (I^+)^P(x_j), (F^+)^P(x_j): V(S_1) \to [0,1]$
   $(T^-)^N(x_j), (I^-)^N(x_j), (F^-)^N(x_j), (T^+)^N(x_j), (I^+)^N(x_j), (F^+)^N(x_j): V(S_1) \to [-1,0]$
   $(T^-)^P(y_k), (I^-)^P(y_k), (F^-)^P(y_k), (T^+)^P(y_k), (I^+)^P(y_k), (F^+)^P(y_k): V(S_2) \to [0,1]$
   $(T^-)^N(y_k), (I^-)^N(y_k), (F^-)^N(y_k), (T^+)^N(y_k), (I^+)^N(y_k), (F^+)^N(y_k): V(S_2) \to [-1,0]$

Here $V(S_1), V(S_2)$ denotes power set of bipolar set on S_1 and S_2 respectively.

Example 3.1

Let $S_1 = \{x_1, x_2, x_3\}$ and $S_2 = \{y_1, y_2\}$ be the standard universe under consideration. A NBVBS is given below:

$$M_{NBVBS} = \left\{ \begin{array}{l} \dfrac{[0.3,0.6]^P, [0.5,0.5]^P, [0.4,0.7]^P, [-0.3,-0.5]^N, [-0.4,-0.4]^N, [-0.5,-0.7]^N}{x_1}; \\ \dfrac{[0.4,0.6]^P, [0.4,0.6]^P, [0.4,0.6]^P, [-0.4,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.6]^N}{x_2}; \\ \dfrac{[0.3,0.7]^P, [0.6,0.4]^P, [0.3,0.7]^P, [-0.4,-0.6]^N, [-0.5,-0.6]^N, [-0.4,-0.6]^N}{x_3}; \\ \dfrac{[0.6,0.8]^P, [0.5,0.7]^P, [0.2,0.4]^P, [-0.3,-0.6]^N, [-0.5,-0.7]^N, [-0.4,-0.7]^N}{y_1}; \\ \dfrac{[0.2,0.7]^P, [0.6,0.9]^P, [0.3,0.8]^P, [-0.5,-0.7]^N, [-0.3,-0.7]^N, [-0.3,-0.5]^N}{y_1} \end{array} \right\}$$

Definition 3.3

Let M_{NBVBS} and $\mathrm{P_{NBVBS}}$ be two NBVBS on a common universe S_1 and S_2 the $\mathrm{M_{NBVBS}}$ is included by P_{NBVBS} denoted by $M_{NBVBS} \subseteq P_{NBVBS}$ if the below condition is true:

If $\forall x_j \in S_1$ *and* $1 \le j \le n$

(1) $\widehat{T}_{M_{NBVB}}(x_j)^P \le \widehat{T}_{P_{NBVB}}(x_j)^P$ *and* $\widehat{T}_{M_{NBVB}}(x_j)^N \ge \widehat{T}_{P_{NBVB}}(x_j)^N$

(2) $\widehat{I}_{M_{NBVB}}(x_j)^P \ge \widehat{I}_{P_{NBVB}}(x_j)^P$ *and* $\widehat{I}_{M_{NBVB}}(x_j)^N \le \widehat{I}_{P_{NBVB}}(x_j)^N$

(3) $\widehat{F}_{M_{NBVB}}(x_j)^P \ge \widehat{F}_{P_{NBVB}}(x_j)^P$ *and* $\widehat{F}_{M_{NBVB}}(x_j)^N \le \widehat{F}_{P_{NBVB}}(x_j)^N$

and $\forall y_k \in S_2$ *and* $1 \le k \le p$

(1) $\widehat{T}_{M_{NBVB}}(y_k)^P \le \widehat{T}_{P_{NBVB}}(y_k)^P$ *and* $\widehat{T}_{M_{NBVB}}(y_k)^N \ge \widehat{T}_{P_{NBVB}}(y_k)^N$

(2) $\widehat{I}_{M_{NBVB}}(y_k)^P \ge \widehat{I}_{P_{NBVB}}(y_k)^P$ *and* $\widehat{I}_{M_{NBVB}}(y_k)^N \le \widehat{I}_{P_{NBVB}}(y_k)^N$

(3) $\widehat{F}_{M_{NBVB}}(y_k)^P \ge \widehat{F}_{P_{NBVB}}(y_k)^P$ *and* $\widehat{F}_{M_{NBVB}}(y_k)^N \le \widehat{F}_{P_{NBVB}}(y_k)^N$

Example 3.2

Let $S_1 = \{x_1, x_2\}$ and $S_2 = \{y_1\}$ be the standard universe. A NBVBS is given below:

$$M_{NBVBS} = \left\{ \begin{array}{l} \dfrac{[0.1,0.2]^P,[0.6,0.7]^P,[0.8,0.9]^P,[-0.1,-0.2]^N,[-0.6,-0.7]^N,[-0.8,-0.9]^N}{x_1}; \\ \dfrac{[0.2,0.6]^P,[0.5,0.6]^P,[0.4,0.8]^P,[-0.2,-0.6]^N,[-0.5,-0.6]^N,[-0.4,-0.8]^N}{x_2}; \\ \dfrac{[0.1,0.3]^P,[0.6,0.7]^P,[0.7,0.9]^P,[-0.1,-0.3]^N,[-0.6,-0.7]^N,[-0.7,-0.9]^N\text{"}}{y_1} \end{array} \right\}$$

$$P_{NBVBS} = \left\{ \begin{array}{l} \dfrac{[0.2,0.3]^P, [0.5,0.6]^P, [0.7,0.8]^P, [-0.2,-0.3]^N, [-0.5,-0.6]^N, [-0.7,-0.8]^N}{x_1}; \\ \dfrac{[0.3,0.7]^P, [0.4,0.5]^P, [0.3,0.7]^P, [-0.3,-0.7]^N, [-0.4,-0.5]^N, [-0.3,-0.7]^N}{x_2}; \\ \dfrac{[0.2,0.4]^P, [0.5,0.6]^P, [0.6,0.8]^P, [-0.2,-0.4]^N, [-0.5,-0.6]^N, [-0.6,-0.8]^N}{y_1}\text{"} \end{array} \right\}$$

Clearly $M_{NBVBS} \subseteq P_{NBVBS}$.

Definition 3.4

Let M_{NBVBS} and P_{NBVBS} are two NBVBS

(i) Union of two NBVBS, M_{NBVBS} and P_{NBVBS} is a NBVBS given as,

$M_{NBVBS} \cup P_{NBVBS} = S_{NBVBS}$

$$= \begin{Bmatrix} \langle (\frac{\hat{T}_{S_{NBVBS}}(x_j), \hat{I}_{S_{NBVBS}}(x_j), \hat{F}_{S_{NBVBS}}(x_j)}{x_j})^P ; \forall x_j \in S_1 \rangle \\ \langle (\frac{\hat{T}_{S_{NBVBS}}(y_k), \hat{I}_{S_{NBVBS}}(y_k), \hat{F}_{S_{NBVBS}}(y_k)}{y_k})^P \forall y_k \in S_2 \rangle \\ \langle \left(\frac{\hat{T}_{S_{NBVBS}}(x_j), \hat{I}_{S_{NBVBS}}(x_j), \hat{F}_{S_{NBVBS}}(x_j)}{x_j}\right)^N ; \forall x_j \in S_1 \rangle \\ \langle (\frac{\hat{T}_{S_{NBVBS}}(y_k), \hat{I}_{S_{NBVBS}}(y_k), \hat{F}_{S_{NBVBS}}(y_k)}{y_k})^N \forall y_k \in S_2 \rangle \end{Bmatrix}$$

The relationship between the 'truth-membership, indeterminacy-membership, and false-membership functions' of M_{NBVBS} and P_{NBVBS} is provided as

$$\hat{T}_{S_{NBVBS}}(x_j)^P = [max\left(T_{M_{NBVBS}}{}^{-}(x_j), T_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(T_{M_{NBVBS}}{}^{+}(x_j), T_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{I}_{S_{NBVBS}}(x_j)^P = [min\left(I_{M_{NBVBS}}{}^{-}(x_j), I_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(I_{M_{NBVBS}}{}^{+}(x_j), I_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{F}_{S_{NBVBS}}(x_j)^P = [min\left(F_{M_{NBVBS}}{}^{-}(x_j), F_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(F_{M_{NBVBS}}{}^{+}(x_j), F_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{T}_{S_{NBVBS}}(x_j)^N = [min\left(T_{M_{NBVBS}}{}^{-}(x_j), T_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(T_{M_{NBVBS}}{}^{+}(x_j), T_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{I}_{S_{NBVBS}}(x_j)^N = [max\left(I_{M_{NBVBS}}{}^{-}(x_j), I_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(I_{M_{NBVBS}}{}^{+}(x_j), I_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{F}_{S_{NBVBS}}(x_j)^N = [max\left(F_{M_{NBVBS}}{}^{-}(x_j), F_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(F_{M_{NBVBS}}{}^{+}(x_j), F_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

Similarly, for $\forall y_k \in S_2$.

Example 3.3

From above example 3.2,

$$S_{NBVBS} = \left\{ \begin{array}{l} \frac{[0.2,0.3],[0.5,0.6][0.7,0.8][-0.1,-0.2][-0.6,-0.7][-0.8,-0.9]}{x_1}; \\ \langle \frac{[0.3,0.7][0.4,0.5][0.3,0.7][-0.2,-0.6][-0.5,-0.6][-0.4,-0.8]}{x_2}; \rangle \\ \frac{[0.2,0.4][0.5,0.6][0.6,0.8][-0.2,-0.4][-0.5,-0.6][-0.7,-0.9]}{y_1} \end{array} \right\}$$

Definition 3.5

Assume M_{NBVBS} and P_{NBVBS} as two NBVBS

(i) Intersection of two NBVBS, M_{NBVBS} and P_{NBVBS} is a NBVBS given as,

$M_{NBVBS} \cap P_{NBVBS} = S_{NBVBS}$

$$= \left\{ \begin{array}{l} \langle (\frac{\hat{T}_{S_{NBVBS}}(x_j), \hat{I}_{S_{NBVBS}}(x_j), \hat{F}_{S_{NBVBS}}(x_j)}{x_j})^P; \forall x_j \in S_1 \rangle \\ \langle (\frac{\hat{T}_{S_{NBVBS}}(y_k), \hat{I}_{S_{NBVBS}}(y_k), \hat{F}_{S_{NBVBS}}(y_k)}{y_k})^P \forall y_k \in S_2 \rangle \\ \langle \left(\frac{\hat{T}_{S_{NBVBS}}(x_j), \hat{I}_{S_{NBVBS}}(x_j), \hat{F}_{S_{NBVBS}}(x_j)}{x_j}\right)^N; \forall x_j \in S_1 \rangle \\ \langle (\frac{\hat{T}_{S_{NBVBS}}(y_k), \hat{I}_{S_{NBVBS}}(y_k), \hat{F}_{S_{NBVBS}}(y_k)}{y_k})^N \forall y_k \in S_2 \rangle \end{array} \right\}$$

It is given that the 'truth-membership, indeterminacy-membership and false-membership function are similar to that of M_{NBVBS} and P_{NBVBS}.

$$\hat{T}_{S_{NBVBS}}(x_j)^P = [min\left(T_{M_{NBVBS}}{}^{-}(x_j), T_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(T_{M_{NBVBS}}{}^{+}(x_j), T_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{I}_{S_{NBVBS}}(x_j)^P = [max\left(I_{M_{NBVBS}}{}^{-}(x_j), I_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(I_{M_{NBVBS}}{}^{+}(x_j), I_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{F}_{S_{NBVBS}}(x_j)^P = [max\left(F_{M_{NBVBS}}{}^{-}(x_j), F_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(F_{M_{NBVBS}}{}^{+}(x_j), F_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{T}_{S_{NBVBS}}(x_j)^N = [max\left(T_{M_{NBVBS}}{}^{-}(x_j), T_{P_{NBVBS}}{}^{-}(x_j)\right), max\left(T_{M_{NBVBS}}{}^{+}(x_j), T_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{I}_{S_{NBVBS}}(x_j)^N = [min\left(I_{M_{NBVBS}}{}^{-}(x_j), I_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(I_{M_{NBVBS}}{}^{+}(x_j), I_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

$$\hat{F}_{S_{NBVBS}}(x_j)^N = [min\left(F_{M_{NBVBS}}{}^{-}(x_j), F_{P_{NBVBS}}{}^{-}(x_j)\right), min\left(F_{M_{NBVBS}}{}^{+}(x_j), F_{P_{NBVBS}}{}^{+}(x_j)\right)]$$

Similarly, for $\forall y_k \in S_2$.

Example 3.4
From above example 3.2

$$S_{NBVBS} = \left\{ \begin{array}{c} \dfrac{[0.1,0.2],[0.6,0.7][0.7,0.8][-0.2,-0.3][-0.6,-0.7][-0.8,-0.9]}{\mathrm{x}_1}; \\ \langle \dfrac{[0.2,0.6][0.5,0.6][0.4,0.8][-0.2,-0.6][-0.4,-0.6][-0.3,-0.7]}{\mathrm{x}_2}; \rangle \\ \dfrac{[0.1,0.3][0.6,0.7][0.7,0.9][-0.1,-0.3][-0.5,-0.6][-0.6,-0.8]}{\mathrm{y}_1} \end{array} \right\}$$

Definition 3.6

Let ${M_{NBVBS}}^c$ is defined as,

$${M_{NBVBS}}^c = \left\{ \begin{array}{l} \langle \dfrac{\left(\widehat{\mathrm{T}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}}, \hat{\mathrm{I}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}} \widehat{\mathrm{F}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}}\right)^{\mathrm{P}}}{\mathrm{x_j}}; \dfrac{\left(\widehat{\mathrm{T}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}}, \hat{\mathrm{I}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}} \widehat{\mathrm{F}}_{\mathrm{M_{NBVBS}}}(\mathrm{x_j})^{\mathrm{c}}\right)^{\mathrm{N}}}{\mathrm{x_j}} \forall \mathrm{x_j} \in \mathrm{U_1} \rangle \\ \langle \dfrac{\left(\widehat{\mathrm{T}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}}, \hat{\mathrm{I}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}} \widehat{\mathrm{F}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}}\right)^{\mathrm{P}}}{\mathrm{y_k}}; \dfrac{\left(\widehat{\mathrm{T}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}}, \hat{\mathrm{I}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}} \widehat{\mathrm{F}}_{\mathrm{M_{NBVBS}}}(\mathrm{y_k})^{\mathrm{c}}\right)^{\mathrm{N}}}{\mathrm{y_k}} \forall \mathrm{y_k} \in \mathrm{U_2} \rangle \end{array} \right.$$

Where

$$\hat{T}_{M_{NBVBS}}(x_j)^{c^P} = \left[1 - T^+(x_j), 1 - T^-(x_j)\right]^P;$$
$$\hat{T}_{M_{NBVBS}}(x_j)^{c^N} = \left[-1 - T^+(x_j), -1 - T^-(x_j)\right]^N$$
$$\hat{I}_{M_{NBVBS}}(x_j)^{c^P} = \left[1 - I^+(x_j),\ 1 - I^-(x_j)\right]^P;$$
$$\hat{I}_{M_{NBVBS}}(x_j)^{c^N} = \left[-1 - I^+(x_j), -1 - I^-(x_j)\right]^N$$
$$\hat{F}_{M_{NBVBS}}(x_j)^{c^P} = \left[1 - F^+(x_j),\ 1 - F^-(x_j)\right]^P;$$
$$\hat{F}_{M_{NBVBS}}(x_j)^{c^N} = \left[-1 - F^+(x_j), -1 - F^-(x_j)\right]^N$$
$$\hat{T}_{M_{NBVBS}}(y_k)^{c^P} = [1 - T^+(y_k),\ 1 - T^-(y_k)]^P;$$
$$\hat{T}_{M_{NBVBS}}(y_k)^{c^N} = [-1 - T^+(y_k), -1 - T^-(y_k)]^N$$
$$\hat{I}_{M_{NBVBS}}(y_k)^{c^P} = [1 - I^+(y_k),\ 1 - I^-(y_k)]^P;$$
$$\hat{I}_{M_{NBVBS}}(y_k)^{c^N} = [-1 - I^+(y_k), -1 - I^-(y_k)]^N$$
$$\hat{F}_{M_{NBVBS}}(y_k)^{c^P} = [1 - F^+(y_k),\ 1 - F^-(y_k)]^P;$$
$$\hat{F}_{M_{NBVBS}}(y_k)^{c^N} = [-1 - F^+(y_k), -1 - F^-(y_k)]^N$$

$\forall x_j \in S_1$ *and* $\forall y_k \in S_2$

Example 3.5

Let M_{NBVB} is defined as in above example. Its complement is given by,

$$M_{NBVBS}{}^{c} = \left\{\begin{array}{l} \frac{\langle([0.4,0.7][0.5,0.5][0.3,0.6])^{P}([-0.5,-0.7],[-0.6,-0.6],[-0.3,-0.5])^{N}\rangle}{x_1}; \\ \frac{\langle([0.4,0.6][0.4,0.6][0.6,0.4])^{P}([-0.6,-0.6],[-0.5,-0.5],[-0.4,-0.4])^{N}\rangle}{x_2}; \\ \frac{\langle([0.3,0.7][0.6,0.4][0.3,0.7])^{P}([-0.4,-0.6],[-0.4,-0.5],[-0.4,-0.6])^{N}\rangle}{x_3}; \\ \frac{\langle([0.2,0.4][0.3,0.5][0.6,0.8])^{P}([-0.4,-0.7],[-0.3,-0.5],[-0.3,-0.6])^{N}\rangle}{y_1}; \\ \frac{\langle([0.3,0.8][0.1,0.4][0.2,0.7])^{P}([-0.3,-0.75],[-0.3,-0.3],[-0.5,-0.7])^{N}\rangle}{y_2} \end{array}\right\}$$

Proposition 3.1

Let A_{NBVBS}, B_{NBVBS}, and C_{NBVBS} *be NBVBS* then

(a) $A_{NBVBS} \subseteq B_{NBVBS}$ *and* $C_{NBVBS} \subseteq D_{NBVBS} \Rightarrow A_{NBVBS} \cup C_{NBVBS} \subseteq B_{NBVBS} \cup D_{NBVBS}$ and

(b) $A_{NBVBS} \cap C_{NBVBS} \subseteq B_{NBVBS} \cap D_{NBVBS}$

(c) $A_{NBVBS} \subseteq B_{NBVBS}$ *and* $A_{NBVBS} \subseteq C_{NBVBS} \Rightarrow A_{NBVBS} \subseteq B_{NBVBS} \cap C_{NBVBS}$

(d) $A_{NBVBS} \subseteq C_{NBVBS}$ *and* $B_{NBVBS} \subseteq C_{NBVBS} \Rightarrow A_{NBVBS} \cup B_{NBVBS} \subseteq C_{NBVBS}$

(e) $A_{NBVBS} \subseteq B_{NBVBS}$ *and* $B_{NBVBS} \subseteq C_{NBVBS} \Rightarrow A_{NBVBS} \subseteq C_{NBVBS}$

Proof:

Proof is clear.

Theorem 3.2

A bipolar neutrosophic vague binary set is the bipolar fuzzy set.

Proof:

Let *S* is a "bipolar neutrosophic vague binary set.

Then by setting the positive component;$\hat{I}_{M_{NBVB}}(x_j)^{P}, \widehat{F}_{M_{NBVB}}(x_j)^{P}; \forall x_j \in S_1$ and $\hat{I}_{M_{NBVB}}(y_k)^{P}; \hat{F}_{M_{NBVB}}(y_k)^{P} \forall y_k \in S_2$ equal to zero as well as the negative component $\hat{I}_{M_{NBVB}}(x_j)^{N}; \hat{F}_{M_{NBVB}}(x_j)^{N}; \forall x_j \in U_1$ and $\hat{I}_{M_{NBVB}}(y_k)^{N}; \hat{F}_{M_{NBVB}}(y_k)^{N}; \forall y_k \in U_2$ equal to zero reduces the bipolar neutrosophic vague binary set to bipolar fuzzy set.

4. Neutrosophic Bipolar Vague Binary Set Topology

The present paper deals with Neutrosophic bipolar vague binary topology and provides definitions and features for key terminology such as unit, zero, interior, course, ' ƀ -open, and ' ƀ c-closed sets.

Definition 4.1
A family of 'neutrosophic bipolar vague binary sets' in S_1, S_2 that adhere to the following criteria constitutes a Neutrosophic bipolar vague binary topology in a same universe with S_1, S_2
(i) $\emptyset_{NBVB}, \psi_{NBVB} \in \tau_{\Delta}^{NBVB}$
(ii) For any M_{NBVB}, $P_{NBVB} \in \tau_{\Delta}^{NBVB}$, $M_{NBVB} \cap P_{NBVB} \in \tau_{\Delta}^{NBVB}$
i.e., the finite intersection of a collection of sets belonging to the NBVB topology, denoted as τ_{Δ}^{NBVB}, is also an element of τ_{Δ}^{NBVB}.
(iii) Let $\{M_{NBVB}^{i}; i \in I\} \subseteq \tau_{\Delta}^{NBVB}$ then $\bigcup_{i\in I} \tau_{\Delta}^{NBVB} \subseteq \tau_{\Delta}^{NBVB}$
i.e., the arbitrarily union of a set of NBVB sets in τ_{Δ}^{NBVB} is also belongs to τ_{Δ}^{NBVB}.

Example 4.1
Consider the sets $S_1 = \{x_1, x_2\}, S_2 = \{y_1\}$ The topology under consideration is a "neutrosophic bipolar vague binary topology.
$\tau_{\Delta}^{NBVB} =$

$$\left\{\begin{array}{c}
\emptyset_{\mathrm{NBVB}} = \left\langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{x_1}\right\rangle \left\langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{x_2}\right\rangle \left\langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{y_1}\right\rangle \\
\mathrm{M}_{\mathrm{NBVB}} = \left\langle\frac{[0.2,0.4][0.6,0.8][0.6,0.8][-0.2,-0.4][-0.6,-0.8][-0.6,-0.8]}{x_1}\right\rangle \left\langle\frac{[0.3,0.6][0.7,0.8][0.4,0.7][-0.3,-0.6][-0.7,-0.8][-0.4,-0.7]}{x_2}\right\rangle \\
\left\langle\frac{[0.6,0.8][0.7,0.9][0.2,0.4][-0.6,-0.8][-0.7,-0.9][-0.2,-0.4]}{y_1}\right\rangle \\
\mathrm{P}_{\mathrm{NBVB}} = \left\langle\frac{[0.6,0.7][0.1,0.9][0.3,0.4][-0.6,-0.7][-0.1,-0.9][-0.3,-0.4]}{x_1}\right\rangle \left\langle\frac{[0.7,0.8][0.3,0.7][0.2,0.3][-0.7,-0.8][-0.3,-0.7][-0.2,-0.3]}{x_2}\right\rangle \\
\left\langle\frac{[0.6,0.7][0.2,0.5][0.3,0.4][-0.6,-0.7][-0.2,-0.5][-0.3,-0.4]}{y_1}\right\rangle \\
\mathrm{K}_{\mathrm{NBVB}} = \mathrm{M}_{\mathrm{NBVB}} \cap \mathrm{P}_{\mathrm{NBVB}} = \left\langle\frac{[0.2,0.4][0.6,0.9][0.6,0.8][-0.2,-0.4][-0.6,-0.8][-0.6,-0.8]}{x_1}\right\rangle \\
\left\langle\frac{[0.3,0.6][0.7,0.8][0.4,0.7][-0.7,-0.8][-0.3,-0.7][-0.2,-0.3]}{x_2}\right\rangle \left\langle\frac{[0.6,0.7][0.7,0.9][0.3,0.4][-0.6,-0.8][-0.2,-0.5][-0.2,-0.4]}{y_1}\right\rangle \\
\mathrm{H}_{\mathrm{NBVB}} = \mathrm{M}_{\mathrm{NBVB}} \cup \mathrm{P}_{\mathrm{NBVB}} = \left\langle\frac{[0.6,0.7][0.1,0.8][0.3,0.4][-0.2,-0.4][-0.6,-0.9][-0.6,-0.8]}{x_1}\right\rangle \\
\left\langle\frac{[0.7,0.8][0.3,0.7][0.2,0.3][-0.3,-0.6][-0.7,-0.8][-0.4,-0.7]}{x_2}\right\rangle \left\langle\frac{[0.6,0.8][0.2,0.5][0.2,0.4][-0.6,-0.7][-0.7,-0.9][-0.3,-0.4]}{y_1}\right\rangle \\
\psi_{\mathrm{NBVB}} = \left\langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{x_1}\right\rangle \left\langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{x_2}\right\rangle \left\langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{y_1}\right\rangle
\end{array}\right\}$$

Definition 4.2
Neutrosophic bipolar vague binary open set is defined as an element of neutrosophic bipolar vague binary topology.

Example 4.2
$\emptyset_{\mathrm{NBVB}}, \psi_{\mathrm{NBVB}}$, M_{NBVB}, P_{NBVB}, K_{NBVB}, H_{NBVB} are as a NBVBoS.

Definition 4.3
A NBVBCS is the complement of an NBVBoS.

Example 4.3

$$\emptyset_{\mathrm{NBVB}}{}^{\mathrm{C}}=\begin{Bmatrix}\langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{x_1}\rangle\\ \langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{x_2}\rangle\\ \langle\frac{[1,1][0,0],[0,0][-1,-1],[0,0],[0,0]}{y_1}\rangle\end{Bmatrix}=\psi_{\mathrm{NBVB}}$$

$$M_{NBVB}{}^{C}=\begin{Bmatrix}\langle\frac{[0.6,0.8][0.2,0.4][0.2,0.4][-0.6,-0.8][-0.2,-0.4][-0.2,-0.4]}{x_1}\rangle\\ \langle\frac{[0.4,0.7][0.2,0.3][0.3,0.6][-0.4,-0.7][-0.2,-0.3][-0.3,-0.6]}{x_2}\rangle\\ \langle\frac{[0.2,0.4][0.1,0.3][0.6,0.8][-0.2,-0.4][-0.1,-0.3][-0.6,-0.8]}{y_1}\rangle\end{Bmatrix}$$

$$P_{NBVB}{}^{C}=\begin{Bmatrix}\langle\frac{[0.3,0.4][0.1,0.9][0.6,0.7][-0.3,-0.3][-0.1,-0.9][-0.6,-0.7]}{x_1}\rangle\\ \langle\frac{[0.2,0.3][0.3,0.7][0.7,0.8][-0.2,-0.3][-0.3,-0.7][-0.7,-0.8]}{x_2}\rangle\\ \langle\frac{[0.3,0.4][0.5,0.8][0.6,0.7][-0.3,-0.4][-0.5,-0.8][-0.6,-0.7]}{y_1}\rangle\end{Bmatrix}$$

$$\psi_{NBVB}{}^{C}=\begin{Bmatrix}\langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{x_1}\rangle\\ \langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{x_2}\rangle\\ \langle\frac{[0,0][1,1][1,1],[0,0][-1,-1][-1,-1]}{y_1}\rangle\end{Bmatrix}=\emptyset_{NBVB}$$

Definition 4.4
The triple $(S_1, S_2, \tau_{\Delta}^{NBVB})$ is known as a NBVB topological space where τ_{Δ}^{NBVB} is a NBVB topology.

Example 4.4
If $S_1=\{x_1, x_2\}$ and $S_2=\{y_1\}$ $\tau_{\Delta}^{NBVB}=\{\emptyset_{NBVB}, \psi_{NBVB}, M_{NBVB}, P_{NBVB}, K_{NBVB}, H_{NBVB}\}$ then the triplet $(S_1, S_2, \tau_{\Delta}^{NBVB})$ is clearly a NBVBTS.

Definition 4.5
Let $S_1=\{x_j, \leq j \leq n\}, S_2=\{y_k, 1 \leq k \leq p\}$ be the two universes under study.
Over this common universe, a zero NBVBS is represented as ψ_{NBVB} and is given by

$$\psi_{NBVB}=\begin{Bmatrix}\langle\frac{\langle\{[1,1][0,0][0,0]\}^{P}\rangle\langle\{[-1,-1][0,0][0,0]\}^{N}\rangle}{x_j}\forall x_j\in S_1\rangle\\ \langle\frac{\langle\{[1,1][0,0][0,0]\}^{P}\rangle\langle\{[-1,-1][0,0][0,0]\}^{N}\rangle}{y_k}\forall y_k\in S_2\rangle\end{Bmatrix}$$

Definition 4.6

consider $S_1 = \{x_j, 1 \leq j \leq n\}, S_2 = \{y_k, 1 \leq k \leq p\}$ as the two universes. Over this common universe, a Unit NBVBS defined as $\emptyset_{\mathrm{NBVB}}$ is given by

$$\emptyset_{NBVB} = \begin{Bmatrix} \langle \frac{\langle\{[0,0][1,1][1,1]\}^P\rangle\langle\{[0,0][-1,-1][-1,-1]\}^N\rangle}{x_j} \forall x_j \in S_1\rangle \\ \langle \frac{\langle\{[0,0][1,1][1,1]\}^P\rangle\langle\{[0,0][-1,-1][-1,-1]\}^N\rangle}{y_k} \forall y_k \in S_2\rangle \end{Bmatrix}$$

Definition 4.7

Let $(S_1, S_2, \tau_\Delta^{NBVB})$ be a NBVBTS and also let *MNBVB*=

$$\begin{Bmatrix} \langle(\frac{\hat{T}_{M_{NBVB}}(x_j),\hat{I}_{M_{NBVB}}(x_j),\hat{F}_{M_{NBVB}}(x_j)}{x_j})^P; \forall x_j \in S_1\rangle \langle(\frac{\hat{T}_{M_{NBVB}}(y_k),\hat{I}_{M_{NBVB}}(y_k),\hat{F}_{M_{NBVB}}(y_k)}{y_k})^P \forall y_k \in S_2\rangle \\ \langle(\frac{\hat{T}_{M_{NBVB}}(x_j),\hat{I}_{M_{NBVB}}(x_j),\hat{F}_{M_{NBVB}}(x_j)}{x_j})^N; \forall x_j \in S_1\rangle \langle(\frac{\hat{T}_{M_{NBVB}}(y_k),\hat{I}_{M_{NBVB}}(y_k),\hat{F}_{M_{NBVB}}(y_k)}{y_k})^N \forall y_k \in S_2\rangle \end{Bmatrix}$$

A NBVBS defined over a common universe $\mathrm{S_1, S_2}$ based on definition 3.2. The neutrosophic bipolar vague binary interior (designated as M^0_{NBVB}) and closure (designated as a $\overline{M_{NBVB}}$) are then defined as follows:

$M^0_{NBVB} = \cup \{M^i_{NBVB}; i \in I \backslash M^i_{NBVB}\ is\ a\ NBVBOS\ over\ S_1, S_2\ with\ M^i_{NBVB} \subseteq M_{NBVB}; \forall i\}$

$\overline{M_{NBVB}} = \cap \{M^i_{NBVB}; i \in I \backslash M^i_{NBVB}\ is\ a\ NBVBCS\ over\ S_1, S_2\ with\ M_{NBVB} \subseteq M^i_{NBVB}; \forall i\}$

Example 4.5 In example 4.1,

$$H^0_{NBVB} = \left(\begin{Bmatrix} \langle\frac{[0.6,0.7][0.1,0.8][0.3,0.4][-0.2,-0.4][-0.6,-0.9][-0.6,-0.8]}{x_1}\rangle \\ \langle\frac{[0.7,0.8][0.3,0.7][0.2,0.3][-0.3,-0.6][-0.7,-0.8][-0.4,-0.7]}{x_2}\rangle \\ \langle\frac{[0.6,0.8][0.2,0.5][0.2,0.4][-0.6,-0.7][-0.7,-0.9][-0.3,-0.4]}{y_1}\rangle \end{Bmatrix}\right) = H_{NBVB}$$

From example 4.1,

$$\overline{M^C_{NBVB}} = \left(\begin{Bmatrix} \langle\frac{[0.6,0.8][0.2,0.4][0.2,0.4][-0.6,-0.8][-0.2,-0.4][-0.2,-0.4]}{x_1}\rangle \\ \langle\frac{[0.4,0.7][0.2,0.3][0.3,0.6][-0.4,-0.7][-0.2,-0.3][-0.3,-0.6]}{x_2}\rangle \\ \langle\frac{[0.2,0.4][0.1,0.3][0.6,0.8][-0.2,-0.4][-0.1,-0.3][-0.6,-0.8]}{y_1}\rangle \end{Bmatrix}\right) = M^C_{NBVB}$$

Note that NBVBC (M_{NBVB}) is a NBVBCS and NBVBint (M_{NBVB}) is a NBVBOS in S_1 and S_2. Further,

1. M_{NBVB} *is a NBVBCS* $\Leftrightarrow \overline{M_{NBVB}} = M_{NBVB}$
2. M_{NBVB} *is a NBVBOS* $\Leftrightarrow M^0_{NBVB} = M_{NBVB}$

Definition 4.8

Let τ_Δ^{NBVB} be NBVBT and M_{NBVB} is defined as

(i) 'Neutrosophic $\alpha -$ open set$[N_\alpha O]$ iff $M_{NBVB} \subseteq M^0_{NBVB}\overline{M_{NBVB}}M^0_{NBVB}(M_{NBVB})$

(ii) 'Neutrosophic semi open set[NSO] iff $M_{NBVB} \subseteq \overline{M_{NBVB}}M^0_{NBVB}(M_{NBVB})$

(iii) ‘Neutrosophic pre open set[NPO] iff $M_{NBVB} \subseteq M^0_{NBVB}\overline{M_{NBVB}}(M_{NBVB})$

Definition 4.9

In an NBVBT τ_Δ^{NBVB}, an NBVB M_{NBVB} is said to be,

(i) Neutrosophic Bipolar Vague Binary ‘𝔟 -Open (NBVBbO) set is defined if and only if $M_{NBVB} \subseteq M^0_{NBVB}\overline{(M_{NBVB})} \cup \overline{M_{NBVB}}(M^0_{NBVB})$

(ii) Neutrosophic bipolar vague binary’𝔟 -closed (NBVBbC) set is defined if and only if $M_{NBVB} \supseteq M^0_{NBVB}\overline{(M_{NBVB})} \cap \overline{M_{NBVB}}(M^0_{NBVB})$

It is obvious that (NBVBPO ∪ NBVBSO) ⊆NBVBbO. Inequalities cannot be substituted for the inclusion.

For an NBVBS M_{NBVB} in a NBVBT τ_Δ^{NBVB}

(i) M_{NBVB} is an NBVBbO set iff $\overline{M_{NBVB}}$ is an NBVBbC set.

(ii) M_{NBVB} is an NBVBbC set iff $\overline{M_{NBVB}}$ is an NBVBbO set.

Proof: evident from the definition.

Definition 4.10

Let $(S_1, S_2, \tau_\Delta^{NBVB})$ be a NBVBTS and M_{NBVB} be NBVBS over S_1, S_2.

(i) Briefly $[M^{bo}_{NBVB}]$,Neutro 𝔟 – interior of M_{NBVB} is the set of all union of “neutrosophic bipolar vague binary ‘𝔟 – open sets that occur in M_{NBVB}. (ie), $M^{bo}_{NBVB} = \cup \{M^i_{NBVB}; i \in I \backslash M^i_{NBVB}$ is a NBVbOS over S_1, S_2 with $M^i_{NBVB} \subseteq M_{NBVB}; \forall i\}$

(ii) Briefly $[\overline{M^b_{NBVB}}]$, Neutro 𝔟 -closure of M_{NBVB} is the set of all intersection of neutrosophic bipolar vague ‘𝔟 -closed set of S_1, S_2 that occur in M_{NBVB}. (ie), $\overline{M^b_{NBVB}} = \cap \{M^i_{NBVB}; i \in I \backslash M^i_{NBVB}$ is a $NBVBbCS$ S_1, S_2 with $M_{NBVB} \subseteq M^i_{NBVB}; \forall i\}$

It is evident that the complement of the set $\overline{M^b_{NBVB}}$ is the smallest ‘𝔟 -closed neutrosophic bipolar vague binary set over S_1 and S_2 that includes M_{NBVB}”. Additionally, $M^{bo}_{NBVB}$is the largest ‘𝔟 -open neutrosophic bipolar vague binary set over S_1 and S_2 that is contained inside M_{NBVB}.

Theorem 4.1

Let M_{NBVB} neutrosophic bipolar vague binary set in a NBVBT τ_Δ^{NBVB} then

(i) $(M^{bo}_{NBVB})^c = \overline{M^c_{NBVB}}^{\,b}$

(ii) $(\overline{M^b_{NBVB}})^c = {M^c_{NBVB}}^{bo}$

Proof:

Let M_{NBVB} be a NBVBS in NBVBT.

Now ${M^b_{NBVB}}^o = [\cup \{'D; D$ is a NBVBbO set in S_1, S_2 and $D \subseteq M_{NBVB}'\}]^c$

$= \cap \{'D^C; D^C$ is a NBVBbC set in C $M^C_{NBVB} \subseteq D^{C'}\}$

Replacing D^C *by ‘M’*, we get,

$(M^{bo}_{NBVB})^c = \cap \{'M; M$ *is a* NBVBbC set in S_1, S_2 and $M \supseteq M^C_{NBVB}{}' \}$

$(M^{bo}_{NBVB})^c = \overline{M^C_{NBVB}}^{\,b}$.

This shows how (i) equivalent (ii) can be shown.

Theorem 4.2

In the topological space τ^{NBVB} ,"every neutrosophic bipolar vague binary set of neutrosophic bipolar vague binary pre-opens is equivalent to a set of neutrosophic bipolar vague binary ƀ -opens".

Proof:

Assuming $'M_{NBVB}'$ as a 'neutrosophic bipolar vague binary pre-open set' in an NBVBTS τ^{NBVB}.then

$$M_{NBVB} \subseteq (M^{O}_{NBVB})(\overline{M_{NBVB}})$$

$$M_{NBVB} \subseteq (M^{O}_{NBVB})(\overline{M_{NBVB}}) \cup (M^{O}_{NBVB})$$

$$\subseteq (M^{O}_{NBVB})\overline{M_{NBVB}}(M_{NBVB}) \cup \overline{M_{NBVB}}(M_{NBVB})$$

$$\subseteq (M^{O}_{NBVB})[\overline{M_{NBVB}}(M_{NBVB})] \cup \overline{M_{NBVB}}[M^{O}_{NBVB}(M_{NBVB})]$$

This M_{NBVB} is a NBVBbO set.

(iii) Let M_{NBVB} be a NBVBO set in a NBVBT τ^{NBVB}.then

$$M_{NBVB} \subseteq \overline{M_{NBVB}}M^{O}_{NBVB}(M_{NBVB})$$

$$M_{NBVB} \subseteq \overline{M_{NBVB}}M^{O}_{NBVB}(M_{NBVB}) \cup M^{O}_{NBVB}(M_{NBVB})$$

$$M_{NBVB} \subseteq \overline{M_{NBVB}}M^{O}_{NBVB}(M_{NBVB}) \cup M^{O}_{NBVB}\overline{M_{NBVB}}(M_{NBVB})$$

M_{NBVB} *is therefore an NBVBbO set.*

Theorem 4.3

Assuming M_{NBVB} is an NBVBO set in a NBVBT τ^{NBVB}.then

(i) $\overline{M^{b}_{NBVB}} = \overline{M_{NBVB}} \cap \overline{M^{p}_{NBVB}}$

(ii) $M^{O}_{NBVB} = M^{O}_{NBVB} \cup {M^{p}_{NBVB}}^{o}$

Proof:

(i) Given that $\overline{M^{b}_{NBVB}}$ is an open NBVB ƀ set

There is *a* $\overline{M^{b}_{NBVB}} \supseteq M^{O}_{NBVB}M_{NBVB}[\overline{M^{b}_{NBVB}}] \cap \overline{M_{NBVB}}M^{O}_{NBVB}(\overline{M^{b}_{NBVB}}) \supseteq M^{O}_{NBVB}[\overline{M_{NBVB}}] \cap \overline{M_{NBVB}}(M^{O}_{NBVB})$

And also $\overline{M^{b}_{NBVB}} \supseteq M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}} \cap \overline{M_{NBVB}}M^{O}_{NBVB} = \overline{M^{s}_{NBVB}} \cap \overline{M^{p}_{NBVB}}$

The inclusion in reverse is evident.

Hence, $\overline{M^{b}_{NBVB}} = \overline{M^{s}_{NBVB}} \cap \overline{M^{p}_{NBVB}}$

In the same way, (ii) can be shown

Theorem 4.4

If M_{NBVB} is an NBVB set contained in an NBVTS τ^{NBVB}.then

(i) $\overline{M^{s}_{NBVB}} = M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}}$ *and*

${M^{s}_{NBVB}}^{o} = M_{NBVB} \cap \overline{M_{NBVB}}M^{O}_{NBVB}$

(ii) $\overline{M^{p}_{NBVB}} = M_{NBVB} \cup \overline{M_{NBVB}}M^{O}_{NBVB}$

${M^{p}_{NBVB}}^{o} = M_{NBVB} \cap M^{O}_{NBVB}\overline{M_{NBVB}}$

Proof:

$\overline{M^{s}_{NBVB}} \supseteq M^{O}_{NBVB}\overline{M_{NBVB}}\,\overline{M^{s}_{NBVB}} \supseteq M^{O}_{NBVB}\overline{M_{NBVB}}$

$$M_{NBVB} \cup \overline{M^{s}_{NBVB}} = \overline{M^{s}_{NBVB}} \supseteq M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}}$$

So $M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}} \subseteq \overline{M^{s}_{NBVB}}$-----------*(1)*

Also $M_{NBVB} \subseteq \overline{M^{s}_{NBVB}}$,

$$M^{O}_{NBVB}\overline{M_{NBVB}} \subseteq M^{O}_{NBVB}\overline{M_{NBVB}}\,\overline{M^{s}_{NBVB}} \subseteq \overline{M^{s}_{NBVB}}$$

$M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}} \subseteq \overline{M^{s}_{NBVB}} \cup M_{NBVB} \subseteq \overline{M^{s}_{NBVB}}$-----------*(2)*

From (1) and (2) we get,

$$\overline{M^{s}_{NBVB}} = M_{NBVB} \cup M^{O}_{NBVB}\overline{M_{NBVB}}$$
$${M^{s}_{NBVB}}^{o} = M_{NBVB} \cap \overline{M_{NBVB}}\,M^{O}_{NBVB}$$

The hypothesis can be illustrated by using the complement of $\overline{M^{s}_{NBVB}} = M_{NBVB} \cup M^{O}_{NBVB}$

This shows that (i)

The evidence for (ii) is similar.

Theorem 4.5

If M_{NBVB} is an NBVB set in a NBVTS τ^{NBVB} then

(i) $\overline{M^{s}_{NBVB}}{M^{s}_{NBVB}}^{o} = {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\,M^{O}_{NBVB}$

(ii) ${M^{s}_{NBVB}}^{o}\overline{M^{s}_{NBVB}} = \overline{M^{s}_{NBVB}} \cap \overline{M_{NBVB}}\,M^{O}_{NBVB}\overline{M_{NBVB}}$

Proof:

We have $\overline{M^{s}_{NBVB}}{M^{s}_{NBVB}}^{o} = {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\left({M^{s}_{NBVB}}^{o}\right)$

$$= {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}(\overline{M_{NBVB}})[M_{NBVB} \cap \overline{M_{NBVB}}\,M^{O}_{NBVB}]$$
$$\subseteq {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}[\overline{M_{NBVB}} \cap \overline{M_{NBVB}}\,(M^{O}_{NBVB})]$$
$$= {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}[\overline{M_{NBVB}}\,(M^{O}_{NBVB})]$$

In order to establish the converse inclusion, it is observed that,

$\overline{M^{s}_{NBVB}}\left({M^{s}_{NBVB}}^{o}\right) = {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\,({M^{s}_{NBVB}}^{o}) \supseteq {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\,(M^{O}_{NBVB})$

We have $\overline{M^{s}_{NBVB}}\left({M^{s}_{NBVB}}^{o}\right) = {M^{s}_{NBVB}}^{o} \cup \overline{M_{NBVB}}\,M^{O}_{NBVB}$

This provides evidence for (i)

The proof for (ii) can be considered equivalent.

Theorem4.6

If M_{NBVB} is an NBVB set in a NBVTS τ^{NBVB} then

(i) $\overline{M^{p}_{NBVB}}{M^{p}_{NBVB}}^{o} = {M^{p}_{NBVB}}^{o} \cup \overline{M_{NBVB}}\,M^{O}_{NBVB}$

(ii) ${M^{p}_{NBVB}}^{o}\overline{M^{p}_{NBVB}} = \overline{M^{p}_{NBVB}} \cap M^{O}_{NBVB}\overline{M_{NBVB}}$

Proof:

We have $\overline{M^{p}_{NBVB}}{M^{p}_{NBVB}}^{o} = {M^{p}_{NBVB}}^{o} \cup \overline{M_{NBVB}}\,M^{O}_{NBVB}\left({M^{p}_{NBVB}}^{o}\right)$

$= {M^{p}_{NBVB}}^{o} \cup \overline{(M_{NBVB}}\,M^{O}_{NBVB}[M_{NBVB} \cap M^{O}_{NBVB}\overline{M_{NBVB}}]) = {M^{p}_{NBVB}}^{o} \cup \overline{M_{NBVB}}\,[M^{O}_{NBVB}]$

In order to establish the converse inclusion, it is observed that,

$\overline{M^{s}_{NBVB}}\left({M^{s}_{NBVB}}^{o}\right) = {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\,({M^{s}_{NBVB}}^{o}) \supseteq {M^{s}_{NBVB}}^{o} \cup M^{O}_{NBVB}\overline{M_{NBVB}}\,(M^{O}_{NBVB})$

Therefore, we have $\overline{M_{NBVB}^{s}} {M_{NBVB}^{s}}^{o} = {M_{NBVB}^{s}}^{o} \cup M_{NBVB}^{O} \overline{M_{NBVB}} M_{NBVB}^{O}$
This provides evidence for (i)
The proof for (ii) can be considered equivalent.

5. Conclusions

In this paper, we have proposed the idea of neutrosophic bipolar vague binary topological spaces with suitable examples. Also, we have discussed neutrosophic bipolar vague binary 𝔟 -open and 𝔟 -closed set and neutrosophic bipolar vague binary interior and closure with respect to neutrosophic theory and we have given their properties also. In future, we will investigate the relationship between NBVBPOS and NBVBS separation theorem.

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Ratio Mathematica Volume 50, 2023

Detour Self-Decomposition of Corona Product of Graphs

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Abstract

Decomposition of a graph G is the collection of edge-disjoint subgraphs of G. The longest distance between any two vertices of G is its detour distance. A subset S of $V(G)$ is a detour set if every vertex of G lie on some $u-v$ detour path, where $u, v \in S$. If a graph G can be decomposed into subgraphs $G_1, G_2, ..., G_n$ with same detour number as G then the decomposition $\Pi = (G_1, G_2, ..., G_n)$ is called detour self-decomposition. The cardinality of maximum such possibility of detour self-decomposition in G is the detour self-decomposition number of G and is denoted by $\pi_{sdn}(G)$. The bounds of detour self-decomposition number of corona product of graphs based on few properties have been discussed here.

Keywords: Detour number; Detour self-decomposition; Detour self-decomposition number; Corona product;
2020 AMS subject classifications: 05C12, 05C99. [1]

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[1]Received on September 15, 2023. Accepted on December 15, 2023. Published on December 31, 2023. DOI: 10.23755/rm.v50i0.1545. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

The graphs $G = (V, E)$ that we have used in this work are all finite, simple, connected and undirected. We refer to [4] for important graph theory terms.

G. Chartrand, P. Zhang, and G.L. Johns [1] introduced the notion of the detour number. In the graph G for any two vertices x and y, the notation $D(x, y)$ refers to the detour distance which is the longest $x - y$ path of length in G. The $x - y$ detour means a $x - y$ path of $D(x, y)$ length. Vertices lying at any $x - y$ detour of G are represented by $I_D[x, y]$ and for any of the subset S of $V(G)$, $I_D[S]$ implies $\cup_{x,y \in S} I_D[x, y]$. If $I_D[S] = V$, S is considered as a detour set and also the detour set with the least number of vertices in G is the minimum detour set and also the cardinality of this set is a detour number. It has applications in game theory, telephone switching centres, facility location, distributed computing, information retrieval and communication networks.

Graph decompositions is one of the interesting areas of research in Graph Theory. The edge disjoint subgraphs collection $G_1, G_2, ..., G_n$ of G represents the decomposition of G if $G's$ each edge is in exactly one $G_i, 1 \leq i \leq n$ [6].

The idea behind H-decomposition was introduced by L.Posa, P. Erdos, and A. W. Goodman [3] and various problems related to H-decomposition has been studied in recent years. Recent researchers found interest in graph decompositions based on some graph properties. The concept ”steiner decomposition number of graphs” [7] introduced by M. Mahiba & E.E.R. Merly, in which the graph and its subgraphs had the same steiner number motivated us to introduce the concept of detour self-decomposition of graphs. In [5], we introduced the concept of ”detour self-decomposition of graphs”, some general properties and bounds of detourself-decomposition of some graphs has been studied.

Many research articles have been published based on graph operations. Researchers are particularly drawn to the corona product among the several graph operations now in use because of its complicated yet distinctive structure. The corona $G \odot H$ of the given graphs G and H is well-defined as the graph acquired with taking one copy of G and $|V(G)|$ number of copies of H. Here, the i^{th} vertex of G has been adjacent to every vertex at the i^{th} copy of H. Due to the structural properties of the corona product of graphs, it is interesting to study its detour self-decomposition.

In this paper, keeping the first graph as any general graph and giving certain restrictions to the second graph, the detour number and the bounds of the detour self-decomposition number of the corona product of such graphs have been studied.

2 Preliminaries

The subsequent work makes use of the following definitions and theorems.

Definition 2.1. *[5] The decomposition $\Pi = (G_1, G_2, ..., G_n)$ of G is stated to be detour self-decomposition if $dn(G) = dn(G_i), 1 \leq i \leq n$ and detour self-decomposition number of G is the greatest cardinality of such decomposition and is represented as $\pi_{sdn}(G)$.*

Theorem 2.1. *[2] Every detour-set for the non-trivial connected graph G consists every end vertices of that graph.*

Theorem 2.2. *[2] A tree T having k end-vertices has detour number k.*

Theorem 2.3. *[9] Any G of an order $p \geq 2, 2 \leq dn(G) \leq p$*

3 Main Results

Lemma 3.1. *Consider v as the cut vertex for G. If $G - \{v\}$ has m components then $dn(G) \geq m$ i.e., the detour set at the least consists one of the vertex from every component of $G - \{v\}$.*

Proof. Let v represent a cut vertex of the graph, G.

Let $G - \{v\}$ has m components $H_1, H_2, ..., H_m$.

To prove: $dn(G) \geq m$ i.e., to prove the minimum number of detour set of G contains atleast m vertices.

Suppose $dn(G) = m_1$,where $m_1 < m$.

Since $m_1 < m$, some components of $G - \{v\}$ does not have any vertices in S, let H_i be one of the components of $G - \{v\}$ that does not have any vertices at the S. Meanwhile $I_D[S] = V(G)$ means there is a vertex pair $v_j, v_k \in S$ such that the vertices of H_i lies in some $v_j - v_k$ detour path in G. Also since v is a cut vertex and $\{v_j, v_k\} \notin V(H_i)$ the $v_j - v_k$ detour paths contains a detour path $v_j - v$ followed by a $v - v$ detour path covering all the vertices of H_i again followed by another $v - v_k$ detour path. But this does not form a path since the cut vertex v is repeated twice in this sequence which is a contradiction to the definition of path. Hence for G the detour set contains at least one vertex from every $G - \{v\}$ components. $\square$

Lemma 3.2. *Consider G and H as any two graphs and detour number of H is 2. Then $dn(G \odot H) = |V(G)|$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$ and the vertex set of i^{th} copy of H be $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}, 1 \leq i \leq |V(G)|$. Meanwhile $dn(H) = 2$ so rename the vertices of i^th copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ is the detour set of i^th copy of $H, 1 \leq i \leq |V(G)|$.

In $G \odot H$, every vertex of G is a cutvertex and one of the components of $G \odot H - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the minimum detour set of $G \odot H$ contains atleast one vertex from the i^th copy of H .

Also for any i^{th} and j^{th} copy of H, there exists two vertices u_{i_1}, u_{j_1} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_1} - u_{j_1}$ detour paths.

In this way the set $S = \{u_{1_1}, u_{2_1}, ..., u_{|V(G)|_1}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1.

Hence $dn(G \odot H) = |S| = |V(G)|$. $\square$

Theorem 3.1. *Let G and H be any of the two graphs and detour number of H is 2. If $|V(H)| > |V(G)|$ then $G \odot H$ is detour self-decomposable and $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$*

Proof. Let G and H be the graphs and $dn(H) = 2$.

By lemma 3.2, $dn(G \odot H) = |V(G)|$.

Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|V(H)|}}\}$.

Since $dn(H) = 2$, rename the vertices of i^{th} copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ as the detour set for the graph H.

Assume $|V(H)| > |V(G)|$

Then $|V(H)| = a|V(G)| + b$,where $a, b \in \mathbb{Z}^+, 0 \leq b < |V(G)|$

In $G \odot H, deg(v_i) \geq |V(H)| + 1 = a|V(G)| + b + 1$.

Consider the case when $b = 0$.

Take $G_{j|V(G)|+k}$ as the star graph with center v_k and pendant vertices $u_{k_{j|V(G)|+2}}, u_{k_{j|V(G)|+3}}, ..., u_{k_{(j+1)|V(G)|}}, u_{k_{(j+1)|V(G)|+1}}$, for all $0 \leq j \leq a - 2$ and $1 \leq k \leq |V(G)|$.

Removal of edges of $G_i, 1 \leq i \leq (a-1)|V(G)|$ from $G \odot H$ results into a connected graph G' since i^{th} copy of H is connected to G through edges $v_i u_{i_1}, v_i u_{i_{(a-1)|V(G)|+2}}, ..., v_i u_{i_{a|V(G)|+b}}$.

Again there exists $|V(G)|$ stars with $|V(G)|$ pendant vertices in G' but removal of these subgraphs from G' results into a disconnected graph and the components may have different detour number which may or may not be further decomposed into subgraphs with same detour number.

In G', each $v_i, 1 \leq i \leq |V(G)|$ is a cutvertex of G' and one of the components of $G' - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the minimum detour-set of G' consists atleast a vertex from the i^{th} copy of the graph H.

Also for any i^{th} and j^{th} copy of H in G', there exist two vertices u_{i_2}, u_{j_2} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G' lie in some $u_{i_2} - u_{j_2}$ detour paths.

In this way the set $S = \{u_{1_2}, u_{2_2}, ..., u_{|V(G)|_2}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1 Hence $dn(G') = |S| = |V(G)|$.

Thus G' itself is a subgraph of $G \odot H$ with same detour number. Depending upon G and H it can further be decomposed into subgraphs with same detour number.

Hence $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$.

□

Theorem 3.2. *Let G and H be any of the two graphs and detour number of H is 2. If $|V(H)| > |V(G)|$ and $|V(G) \nmid |V(H)|$ then $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$*

Proof. Let G and H be the two graphs and $dn(H) = 2$.

By lemma 3.2, $dn(G \odot H) = |V(G)|$.

Let $V(G) = \{v_1, v_2, ..., v_|V(G)|\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}$.

Since $dn(H) = 2$, rename the vertices of i^{th} copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ as the detour set for the graph H.

Assume $|V(H)| > |V(G)|$ and $|V(G)| \nmid |V(H)|$

Then $|V(H)| = a|V(G)| + b$,where $a, b \in \mathbb{Z}+, 0 < b < |V(G)|$

In $G \odot H, deg(v_i) \geq |V(H)| + 1 = a|V(G)| + b + 1$.

Take $G_{j|V(G)|+k}$ as the star graph with center v_k and pendant vertices $u_{k_{j|V(G)|+2}}, u_{k_{j|V(G)|+3}}, ..., u_{k_{(j+1)|V(G)|}}, u_{k_{(j+1)|V(G)|+1}}$, for all $0 \leq j \leq a - 1$ and $1 \leq k \leq |V(G)|$.

Removal of edges of $G_i, 1 \leq i \leq a|V(G)|$ from $G \odot H$ results into a connected graph G' since $H's$ i^{th} copy is connected to G atleast through an edge $v_i u_{i_1}$.

In G', each $v_i, 1 \leq i \leq |V(G)|$ is the cut vertex. Also one of the components of $G' - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 for G' the minimum detour set consists atleast a vertex from each copy of H.

Also for any i^{th} and j^{th} copy of H in G', there exist two vertices u_{i_2}, u_{j_2} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_2} - u_{j_2}$ detour paths.

In this way the set $S = \{u_{1_2}, u_{2_2}, ..., u_{|V(G)|_2}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1. Hence $dn(G') = |S| = |V(G)|$.

Thus G' itself is the $G \odot H$ subgraph with same detour number. Depending upon G and H it can further be decomposed into subgraphs with same detour number. Hence $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$. □

Lemma 3.3. *Let G be any graph and H be any tree with exactly one center then $dn(G \odot H) = V(G)$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}$. Let the center of i^{th} copy of H be u_{i_1}.

In $G \odot H$, each vertex of G is a cutvertex and one of the components of $G \odot H - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the $G \odot H$ minimum detour set consists at least a vertex from the i^{th} copy of H .

Again, considering the subgraph $H + \{v_i\}$ in $G \odot H$, the set $\{u_{i_1}, v_i\}$ is the minimum detour set for this subgraph.

Also for any i^{th} and j^{th} copy of H, there exists two vertices u_{i_1}, u_{j_1} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_1} - u_{j_1}$ detour paths. In this way the set $S = \{u_{1_1}, u_{2_1}, ..., u_{|V(G)|_1}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1. Hence $dn(G \odot H) = |S| = |V(G)|$. □

Theorem 3.3. *Let G be any of the graph along with the H be any tree with exactly one center. If $|V(H)| > |V(G)|$ then $G \odot H$ is detour self-decomposable and $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$*

Proof. From lemma 3.3 $dn(G \odot H) = |V(G)|$.

The proof is same as the theorem 3.1. □

Theorem 3.4. *Let G be the any of the graph and the H be any tree with exactly one center. If $|V(H)| > |V(G)|$ and $|V(G) \nmid |V(H)|$ then $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$*

Proof. From lemma 3.3 $dn(G \odot H) = |V(G)|$.

The proof is same as the theorem 3.2. □

4 Conclusions

In this paper, a decomposition parameter on the basis of the detour number is studied. The detour number and few lower bounds for the detour self-decomposition number on corona product of some graphs that satisfy such decomposition is discussed. Although the first graph in the corona product is general, the lower bound for general graphs remains unsolved. Further research can be obtained by analyzing the graphs satisfying this decomposition with other graph parameters and graph operations.

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Ratio Mathematica Volume 50, 2023

On Pseudo Regular Spherical Fuzzy Graphs

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Abstract

In this paper, pseudo regular and totally pseudo regular spherical fuzzy graphs are defined using pseudo degree and total pseudo degree of a vertex. A necessary and sufficient condition for pseudo regular spherical fuzzy graph is given. Some properties of pseudo regular and totally pseudo regular spherical fuzzy graphs are derived. Theorems related to these concepts are stated and proved.

Keywords: pseudo degree; pseudo regular; totally pseudo regular; spherical fuzzy graphs

2010 AMS subject classification: 05C22, 05C90.[‡]

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[‡] Received on May 12, 2023. Accepted on December 15, 2023. Published on December 31, 2023. doi: 10.23755/rm.v50i0.1546. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

Zadeh [16] introduced the concept of fuzzy set (FS) as a generalization of the crisp set, and it has numerous applications in networking, decision making, telecommunication, artificial intelligence, management sciences, computer science, social science, and the chemical industry. The fuzzy set theory was proposed in order to deal with problems involving a lack of information and impreciseness. It can deal with uncertainty and vague problems whose membership functions lie in the closed interval [0, 1]. Atanassov [2] proposed the intuitionistic fuzzy sets (IFSs) as an extension of Zadeh's [16] fuzzy set theory. Cuong [4, 5] initiated the concept of the picture fuzzy set (PFS) as a direct extension of intuitonistic fuzzy sets, which may be adequate in cases when human opinions are of types: yes, abstain, no, and refusal. Mahmood et al. [10] introduced the concept of spherical fuzzy set which gives an additional strength to the concept of picture fuzzy set by enlarging the space for the grades for all the four parameters. Kifayat et al. [7] studied the geometrical comparison of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, picture fuzzy sets with spherical fuzzy sets. Cen Zuo, et al. [15] introduced the some new concepts of picture fuzzy graph. Akram et.al [1] introduced the notion of spherical fuzzy graphs and Abhishek Guleria [6] also introduced generalized version spherical fuzzy graphs using T-spherical fuzzy sets. Mordeson and Peng [11] introduced some operations on fuzzy graphs. Mohamed Harif and Nazeera Begam [8] defined regular spherical fuzzy graphs and studied some properties of regular spherical fuzzy graphs. Muhammad Shoaib et al. [9] introduced complex spherical fuzzy graphs. In this paper, pseudo regular and totally pseudo regular spherical fuzzy graphs are defined using pseudo degree and total pseudo degree of a vertex. A necessary and sufficient condition for pseudo regular spherical fuzzy graph is given. Some comparative study between pseudo regular and totally pseudo regular spherical fuzzy graphs are done.

2. Preliminaries

Definition 2.1[10] A spherical fuzzy set S in U (universe of discourse) is given by $S = \{< \alpha, \mu_S(\alpha), \eta_S(\alpha), \nu_S(\alpha) >: \alpha \in U\}$ where $\mu_S: U \to [0,1]$, $\eta_S: U \to [0,1]$ and $\nu_S: U \to [0,1]$ denote degree of membership, degree of neutral membership and degree of non-membership respectively, and for each $\alpha \in U$ satisfying the condition $0 \le \mu_S^2(\alpha)+\eta_S^2(\alpha)+ \nu_S^2(\alpha) \le 1$. The degree of refusal for any spherical fuzzy set S and $\alpha \in U$ is given by $r_S(\alpha) = \sqrt{1 - (\mu_S^2(\alpha) + \eta_S^2(\alpha) + \nu_S^2(\alpha))}$.

Definition 2.2 [1] A spherical fuzzy graph (SFG) $\mathcal{G} = (N, L)$ where

- $N = \{v_1, v_2, \dots, v_n\}$ such that $\sigma_1: N \to [0,1]$, $\sigma_2: N \to [0,1]$ and $\sigma_3: N \to [0,1]$ denote the degree of membership, degree of neutral membership and degree of

non-membership of each element $v_i \in N$ respectively, and
$0 \leq \sigma_1^2(v_i) + \sigma_2^2(v_i) + \sigma_3^2(v_i) \leq 1$, for every $v_i \in N$,$(i = 1,2,3 \dots, n)$.

- $L \subseteq N \times N$ where $\mu_1 : L \to [0,1]$, $\mu_2 : L \to [0,1]$ and $\mu_3 : L \to [0,1]$ are such that

$$\mu_1(u_i, u_j) \leq min\{\sigma_1(u_i), \sigma_1(u_j)\},$$
$$\mu_2(u_i, u_j) \leq min\{\sigma_2(u_i), \sigma_2(u_j)\},$$
$$\mu_3(u_i, u_j) \leq max\{\sigma_3(u_i), \sigma_3(u_j)\}$$

$0 \leq \mu_1^2(u_i, u_j) + \mu_2^2(u_i, u_j) + \mu_3^2(u_i, u_j) \leq 1$, for every $(u_i, u_j) \in L$,
$(i = 1,2,3 \dots, n)$.

Definition 2.3 [1] Let $\mathcal{G} = (N, L)$ be a SFG. The degree of a vertex u of a SFG is

$$d_{\mathcal{G}}(u) = \left(\sum_{u \neq v} \mu_1(u, v), \sum_{u \neq v} \mu_2(u, v), \sum_{u \neq v} \mu_3(u, v)\right).$$

Definition 2.4[1] The minimum degree of a SFG $\mathcal{G} = (N, L)$ is defined by $\delta(\mathcal{G}) = (\delta_1(\mathcal{G}), \delta_2(\mathcal{G}), \delta_3(\mathcal{G}))$, where

$$\delta_1(\mathcal{G}) = min\{d_1(u)/\, u \in N\},$$
$$\delta_2(\mathcal{G}) = min\{d_2(u)/\, u \in N\}$$
$$\delta_3(\mathcal{G}) = min\,\{d_3(u)/\, u \in N\}.$$

Definition 2.5 [1] The maximum degree of a SFG $\mathcal{G} = (N, L)$ is defined by $\Delta(\mathcal{G}) = (\Delta_1(\mathcal{G}), \Delta_2(\mathcal{G}), \Delta_3(\mathcal{G}))$ where

$$\Delta_1(\mathcal{G}) = max\{d_1(u)/\, u \in N\}\},$$
$$\Delta_2(\mathcal{G}) = max\{d_2(u)/\, u \in N\},$$
$$\Delta_3(\mathcal{G}) = max\{d_3(u)/\, u \in N\}.$$

Definition 2.6 [8] Let $\mathcal{G} = (N, L)$ be a SFG. If $d_{\mathcal{G}}(u) = (c_1, c_2, c_3)$, $\forall\, u \in N$. (i,e) each vertex has same degree (c_1, c_2, c_3), then $\mathcal{G}$ is said to be regular spherical fuzzy graph (RSFG) of degree (c_1, c_2, c_3) or $(c_1, c_2, c_3) -$ regular spherical fuzzy graph.

Definition 2.7 [8] Let $\mathcal{G} = (N, L)$ be a SFG, then the order of $\mathcal{G}$ is denoted by $O(\mathcal{G})$ and defined by

$$O(\mathcal{G}) = \left(\sum_{v \in N} \sigma_1(v), \sum_{v \in N} \sigma_2(v), \sum_{v \in N} \sigma_3(v)\right).$$

Definition 2.8 [8] Let $\mathcal{G} = (N, L)$ be a SFG, then the size of $\mathcal{G}$ is denoted by $S(\mathcal{G})$ and defined by

$$S(\mathcal{G}) = \left(\sum_{uv \in L} \mu_1(uv), \sum_{uv \in L} \mu_2(uv), \sum_{uv \in L} \mu_3(uv)\right).$$

Definition 2.9 [3] Let $G^* = (V, E)$ be a crisp graph. Then the 2-degree of v is defined as the sum of the degrees of the vertices adjacent to v and it is denoted by $t(v)$.

Definition 2.10 [14] Let $G^* = (V, E)$ be a crisp graph. The average degree of v is defined as $\frac{t(v)}{d(v)}$ where $t(v)$ is the $2-$degree of v and $d(v)$ is the degree of v and it is denoted by $d_a(v)$.

Definition 2.11 [14] A graph $G^* = (V, E)$ is called pseudo regular if every vertex of $\mathcal{G}$ has equal average degree.

3. Pseudo regular spherical fuzzy graphs

Definition 3.1 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$. The $2-$degree of a vertex v in $\mathcal{G}$ is defined as the sum of degrees of the vertices adjacent to v and is denoted by $t_{\mathcal{G}}(v) = (t_1(v), t_2(v), t_3(v))$ where $t_i(v) = \sum d_i(u) = \sum_{u \neq w} \mu_i(u, w)$; $i = 1, 2, 3$ and $d_i(u)$ is the degree of the vertex u which is adjacent with the vertex v.

Definition 3.2 Let $\mathcal{G} = (N, L)$ be a SFG on $G^* = (V, E)$. A pseudo (average) degree of a vertex v in spherical fuzzy graph $\mathcal{G}$ is denoted by $d_{\mathcal{G}}^a(v)$ and is defined by $d_{\mathcal{G}}^a(v) = (d_1^a(v), d_2^a(v), d_3^a(v))$ where $d_i^a(v) = \frac{t_i(v)}{d_{G^*}(v)}$, $i = 1, 2, 3$ and $d_{G^*}(v)$ is the number of edges incident at v.

Definition 3.3 Let $\mathcal{G} = (N, L)$ be a SFG on $G^* = (V, E)$. If $d_{\mathcal{G}}^a(v) = (k_1, k_2, k_3)$ for all $v \in N$ then $\mathcal{G}$ is called (k_1, k_2, k_3)-pseudo regular spherical fuzzy graph or $(k_1, k_2, k_3)-$PRSFG.

Example 3.4 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.1.

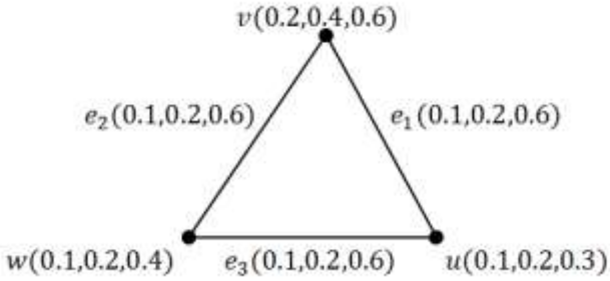

Figure 3.1: Pseudo Regular Spherical Fuzzy Graph $\mathcal{G}$

Here, $d(u) = (0.2, 0.4, 1.2)$, $d(v) = (0.2, 0.4, 1.2)$, $d(w) = (0.2, 0.4, 1.0)$ and $d_{G^*}(u) = 2$, for all $u \in N$.

Now u is adjacent to v and w.

Then $d_{\mathcal{G}}^a(u) = (0.2, 0.4, 1.2)$.

Similarly, $d_{\mathcal{G}}^a(v) = d_{\mathcal{G}}^a(w) = (0.2, 0.4, 1.2)$.

Thus $\mathcal{G}$ is $(0.2, 0.4, 1.2)-$PRSFG.

Definition 3.5 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$. The total pseudo degree of a vertex v in $\mathcal{G}$ is denoted by $td_{\mathcal{G}}^{a}(v)$ and is defined by $td_{\mathcal{G}}^{a}(v) = \left(td_1^a(v), td_2^a(v), td_3^a(v)\right)$ where $td_i^a(v) = d_i^a(v) + \sigma_i(v)$ for all $v \in V$ and $i = 1, 2, 3$.

Definition 3.6 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$. If all the vertices of G have the same total pseudo degree (k_1, k_2, k_3), then $\mathcal{G}$ is said to be a (k_1, k_2, k_3)-totally Pseudo regular spherical fuzzy graph $\left((k_1, k_2, k_3) - TPRSFG\right)$.

Example 3.7 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.2.

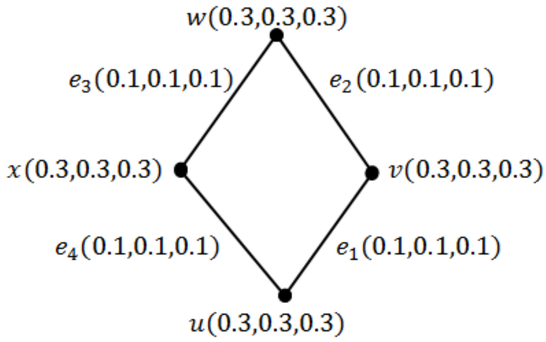

Figure 3.2: Totally Pseudo Regular Spherical Fuzzy Graph $\mathcal{G}$

Here $d_{\mathcal{G}}^{a}(u) = d_{\mathcal{G}}^{a}(v) = d_{\mathcal{G}}^{a}(w) = d_{\mathcal{G}}^{a}(x) = (0.2,0.2,0.2)$ and $td_{\mathcal{G}}^{a}(u) = (0.5,0.5,0.5)$ for all $u \in V$. Hence $\mathcal{G}$ is $(0.5,0.5,0.5)$ −TPRSFG.

Remark 3.8 A Pseudo regular spherical fuzzy graph need not be a totally Pseudo regular spherical fuzzy graph. It can be verified by the following example.

Example 3.9 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.3.

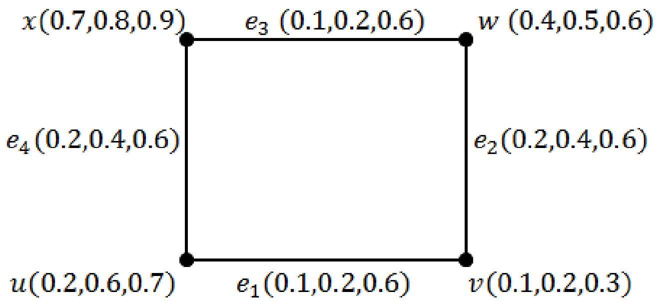

Figure 3.3: Illustration of Remark 3.8

Here $d_{\mathcal{G}}^{a}(u) = d_{\mathcal{G}}^{a}(v) = d_{\mathcal{G}}^{a}(w) = d_{\mathcal{G}}^{a}(x) = (0.3,0.6,1.2)$;

$td_{\mathcal{G}}^{a}(u) = (0.5,1.2,1.9)$ and $td_{\mathcal{G}}^{a}(v) = (0.4,1.8,1.5)$

Therefore $\mathcal{G}$ is PRSFG.

But $td_{\mathcal{G}}^{a}(u) \neq td_{\mathcal{G}}^{a}(v)$

Hence $\mathcal{G}$ is not TPRSFG.

Remark 3.10 A totally pseudo regular spherical fuzzy graph need not be a pseudo regular spherical fuzzy graph.

Example 3.11 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.4.

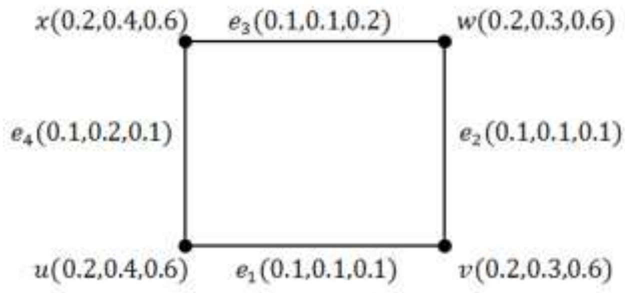

Figure 3.4: Illustration of Remark 3.10

Here $td_{\mathcal{G}}^{a}(u) = td_{\mathcal{G}}^{a}(v) = td_{\mathcal{G}}^{a}(w) = td_{\mathcal{G}}^{a}(x) = (0.4, 0.6, 0.8)$;

$d_{\mathcal{G}}^{a}(u) = (0.2, 0.2, 0.2)$ and $d_{\mathcal{G}}^{a}(v) = (0.2, 0.3, 0.2)$

Therefore $\mathcal{G}$ is TPRSFG.

But $d_{\mathcal{G}}^{a}(u) \neq d_{\mathcal{G}}^{a}(v)$

Hence $\mathcal{G}$ is not PRSFG.

Note: From the above examples, it is clear that in general there does not exist any relationship between regular spherical fuzzy graphs and totally regular spherical fuzzy graphs. However, a necessary and sufficient condition under which these two types of spherical fuzzy graphs are equivalent is provided in the following theorem.

Theorem 3.12 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$. Then N is a constant function if the following are equivalent.

1. $\mathcal{G}$ is pseudo regular spherical fuzzy graph.
2. $\mathcal{G}$ is a totally pseudo regular spherical fuzzy graph.

Proof:

Assume that N is a constant function.

Let $\sigma_1(u) = c_1, \sigma_2(u) = c_2, \sigma_3(u) = c_3$ for all $u \in N$.

Suppose $\mathcal{G}$ is a pseudo regular spherical fuzzy graph.

Then $d_{\mathcal{G}}^{a}(u) = (k_1, k_2, k_3)$ for all $u \in N$.

Now,

$$\begin{aligned} t\, d_{\mathcal{G}}^{a}(u) &= d_{\mathcal{G}}^{a}(u) + \sigma(u) \\ &= (k_1, k_2, k_3) + (c_1, c_2, c_3) \\ &= (k_1 + c_1,\ k_2 + c_2,\ k_3 + c_3) \end{aligned}$$

Hence, $\mathcal{G}$ is a totally pseudo regular spherical fuzzy graph.

Thus (i) implies (ii) is proved.
Suppose $\mathcal{G}$ is a totally pseudo regular spherical fuzzy graph.
Then,
$td_{\mathcal{G}}^{a}(u) = (k_1, k_2, k_3)$ for all $u \in N$.

$$d_{\mathcal{G}}^{a}(u) + \sigma_i(u) = (k_1, k_2, k_3) \text{ for all } u \in N$$

$$d_{\mathcal{G}}^{a}(u) + (c_1, c_2, c_3) = (k_1, k_2, k_3) \text{ for all } u \in N.$$

$$d_{\mathcal{G}}^{a}(u) = (k_1, k_2, k_3) - (c_1, c_2, c_3) \text{ for all } u \in N$$

Hence $\mathcal{G}$ is a Pseudo regular spherical fuzzy graph.
Thus (ii) implies (i) proved.
Conversely Suppose (i) and (ii) are equivalent.
Let $\mathcal{G}$ be a pseudo regular spherical fuzzy graph and a totally pseudo regular spherical fuzzy graph.
Then, $d_i^a(u) = k_i$ and
$td_i^a(u) = k_j$, for all $u \in N$ and $i, j = 1,2,3$.

$$d_i^a(u) + \sigma_i(u) = k_j \text{ for all } u \in n \text{ and } i, j = 1,2,3.$$

$$k_i + \sigma_i(u) = k_j \text{ and } i, j = 1,2,3.$$

$$\sigma_i(u) = k_j - k_i \text{ for all } u \in V \text{ and } i, j = 1,2,3.$$

Hence N is a constant function.
Example 3.13 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.5.

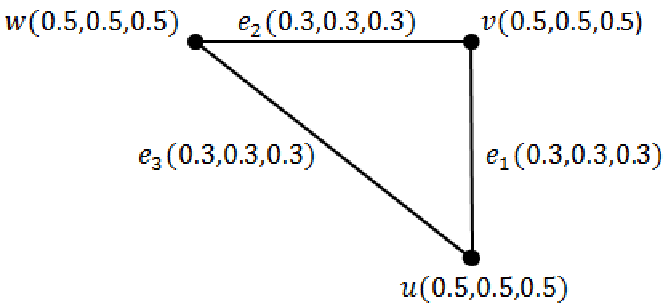

Figure 3.5: Spherical Fuzzy Graph $\mathcal{G}$

$d_{\mathcal{G}}^{a}(u) = (0.6,0.6,0.6)$ and $td_{\mathcal{G}}^{a}(u) = (1.1,1.1,1.1)$
The graph is $(0.6,0.6,0.6)$ −PRSFG and $(1.1,1.1,1.1)$ −TPRSFG.

Theorem 3.14 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$. If $\mathcal{G}$ is both pseudo regular and totally pseudo regular spherical fuzzy graph, then N is a constant function.
Proof:

Assume that $\mathcal{G}$ is both pseudo regular and totally pseudo regular spherical fuzzy graph. Then $d_i^a(u) = c_i$ and $td_i^a(u) = k_i$, for all $u \in V$.

Now,

$$td_i^a(u) = k_i$$
$$d_i^a(u) + \sigma_i(u) = k_i$$
$$c_i + \sigma_i(u) = k_i$$
$$\sigma_i(u) = k_i + c_i$$
$$= \text{constant}$$

Hence N is a constant function.

Remark 3.15 The converse of the theorem 3.14 need not be true.

Example 3.16 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.6.

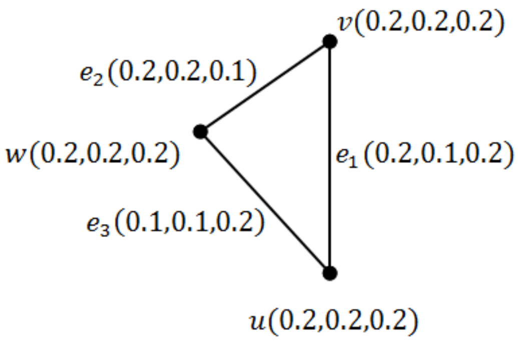

Figure 3.6: Illustration of Remark 3.15

Here N is a constant function.

But $\mathcal{G}$ is neither PRSFG nor a totally PRSFG.

Theorem 3.17 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^*(V, E)$ a cycle of length n. If L is a constant function, then $\mathcal{G}$ is a pseudo regular spherical fuzzy graph.

Proof:

If L is a constant function then $\mu_1(uv) = c_1$, $\mu_2(uv) = c_2$, $\mu_3(uv) = c_3$ for all $uv \in L$. Then $d_1^a(u) = 2c_1$, $d_2^a(u) = 2c_2$, $d_3^a(u) = 2c_3$ for all $u \in N$.

Hence $\mathcal{G}$ is (c_1, c_2, c_3)-Pseudo regular spherical fuzzy graph.

Remark 3.18 Converse of the above theorem 3.17 need not be true**.**

Example 3.19 Consider a SFG $\mathcal{G}$ given in figure 3.3. Here $\mathcal{G}$ is a pseudo regular spherical fuzzy graph but L is not a constant function.

Theorem 3.20 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$, an even cycle of length n. If the alternate edges have same membership values, then $\mathcal{G}$ is a pseudo regular spherical fuzzy graph.

Proof: If the alternate edges have the same membership values, then

$L(e_j) = \begin{cases} (m_1, m_2, m_3), & if\ j\ is\ odd \\ (n_1, n_2, n_3), & if\ j\ is\ even \end{cases}$ for all $i = 1,2,3$.

If $(m_1, m_2, m_3) = (n_1, n_2, n_3)$, then L is a constant function.

So by the theorem 3.17, $\mathcal{G}$ is a pseudo regular spherical fuzzy graph.

If $(m_1, m_2, m_3) \neq (n_1, n_2, n_3)$, then $d_{\mathcal{G}}(v) = (m_1, m_2, m_3) + (n_1, n_2, n_3)$ for all $v \in V$.

So $t_{\mathcal{G}}(v) = (2m_1 + 2n_1, 2m_2 + 2n_2, 2m_3 + 2n_3)$ and $d_{G^*}(v) = 2$.

Hence $\mathcal{G}$ is a pseudo regular spherical fuzzy graph.

Remark 3.20 The above theorem 3.20 does not hold for a totally pseudo regular spherical fuzzy graph.

Example 3.21 Consider a SFG $\mathcal{G}$ on $G^* = (V, E)$ given in figure 3.7.

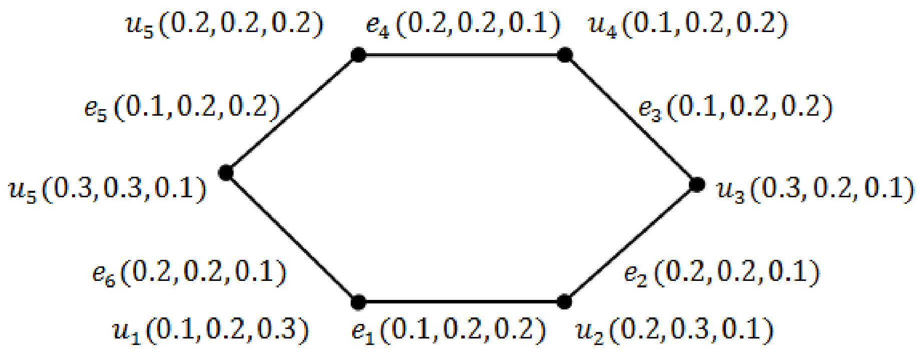

Figure 3.7: Illustration of Remark 3.20

Here alternate edges have the same membership values.

$d_{\mathcal{G}}^a(u_1) = d_{\mathcal{G}}^a(u_2) = d_{\mathcal{G}}^a(u_3) = d_{\mathcal{G}}^a(u_4) = d_{\mathcal{G}}^a(u_5) = (0.3,0.4,0.3)$ and

$$td_{\mathcal{G}}^a(u_1) = (0.4,0.6,0.6) \text{ and } td_{\mathcal{G}}^a(u_2) = (0.5,0.7,0.4)$$

$$td_{\mathcal{G}}^a(u_1) \neq td_{\mathcal{G}}^a(u_2)$$

Therefore, $\mathcal{G}$ is not TPRSFG.

Theorem 3.22 If $\mathcal{G} = (N, L)$ is a regular spherical fuzzy graph on $G^* = (V, E)$, an r −regular graph, then $d_i^a(v) = d_i^{\mathcal{G}}(v)$ for all $v \in N$.

Proof:

Let $\mathcal{G} = (N, L)$ is a (k_1, k_2, k_3) regular spherical fuzzy graph on $G^*(V, E)$ an (r_1, r_2, r_3) regular spherical fuzzy graph.

Then $d_1(v) = k_1, d_2(v) = k_2, d_3(v) = k_3$ for all $v \in \mathcal{G}$ and $d_{G^*}(v) = r$ for all $v \in \mathcal{G}$.

So, $t_G(v) = (\sum d_1(u), \sum d_2(u), \sum d_3(u))$ where each u is adjacent with vertex v.

Which implies

$$t_{\mathcal{G}}(v) = \left(\sum d_1(u), \sum d_2(u), \sum d_3(u)\right)$$
$$= (rk_1, rk_2, rk_3,)$$

Also $d_i^a(v) = \frac{t_{\mathcal{G}}(v)}{d_{G^*}(v)}$

Which implies $d_i^a(v) = \frac{t_{\mathcal{G}}(v)}{r}$.

Which implies $d_i^a(v) = (\frac{rk_1}{r}, \frac{rk_2}{r}, \frac{rk_3}{r})$

$= (k_1, k_2, k_3)$

Which implies $d_i^a(v) = d_i(v)$ for all $i = 1, 2, 3$.

Theorem 3.23 Let $\mathcal{G} = (N, L)$ be a spherical fuzzy graph on $G^* = (V, E)$, an r −regular graph. Then $\mathcal{G}$ is pseudo regular spherical fuzzy graph, if $\mathcal{G}$ is a regular spherical fuzzy graph.

Proof: Let $G = (N, L)$ be a (k_1, k_2, k_3) regular spherical fuzzy graph on $G^*(V, E)$ an (r_1, r_2, r_3) regular spherical graph.

Then $d_i^a(v) = d_i(v)$ for all $v \in N$.

$d_i^a(v) = k_i$ for all $v \in N$.

Which implies all the vertices have same pseudo degree (k_1, k_2, k_3).

Hence $\mathcal{G}$ is (k_1, k_2, k_3) pseudo regular spherical fuzzy graph.

4 Conclusions

Graph theory has become a most powerful conceptual framework for modelling and for solutions of combinatorial problems that arise in various areas, including mathematics, computer science, and engineering. Many real life problems can be represented by graphs. The spherical fuzzy model is a more flexible model because it broadly resolves the ambiguity of actual phenomena. To solve many real-world problems and loosen restrictive constraints, the need for SFGs will arise. In this paper, we have defined pseudo regular and totally pseudo regular spherical fuzzy graphs using pseudo degree and total pseudo degree of a vertex. A necessary and sufficient condition for pseudo regular spherical fuzzy graph is given. Some results between Pseudo regular and totally Pseudo regular spherical fuzzy graphs are derived. In the future, we plan to extend this study to (i) operations on pseudo regular spherical fuzzy graphs; (ii) complex spherical fuzzy graphs.

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Ratio Mathematica Volume 50, 2023

Parameter uniform convergence of a finite element method for a singularly perturbed linear reaction diffusion system with discontinuous source terms

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Abstract

A linear system of 'n' second order ordinary differential equations of reaction-diffusion type with discontinuous source terms is considered. On a piecewise uniform Shishkin mesh, a numerical system is built that employs the finite element method. The numerical approximations obtained by this approach are proven to be effectively almost second order convergent.
Keywords: singular perturbation problems, system of differential equations, reaction - diffusion equations, overlapping boundary and interior layers, finite element method, Shishkin mesh, parameter - uniform convergence, discontinuous source terms.
2020 AMS subject classifications: 65L10, 65L11, 65L50, 65L60, 65L70. [1]

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[1]Received on May 12, 2023. Accepted on December 14, 2023. Published on December 31, 2023. doi:10.23755/rm.v50i0.1547. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Differential equations with small parameters reflect the rapid progress of science and technology and many practical problems, including mathematical boundary layer theory and approximation of solutions are described in [Doolan et al., 1980], [Roos et al., 1996] and [Miller et al., 1996]. Sometimes, solving mathematical problems accurately can be very challenging, if even impossible; especially in these situations, approximations of the solutions are needed. Its derivative often does not converge uniformly at $x = 0$ or $x = 1$ as $\varepsilon \to 0$, whereas the generated asymptotic solution converges uniformly to the solution of reduced difficulty in the prescribed problem throughout the interval $[0, 1]$. Several finite difference methods have been suggested as problems of this type have been taken up for discussion in [Paramasivam et al., 2013, 2010]. In this paper, we consider the type of problem below, but we assume discontinuity of the source term at an interior point of the domain. For second-order singular perturbation problems of reaction diffusion type with discontinuous source term, many authors have studied the finite-difference and finite-element methods; References are included [Paramasivam et al., 2014, Linβ and Madden, 2009]. Motivated by the works of [Miller et al., 1996], in the present paper we discussed a approximate solutions generated by the numerical approach must be globally established at every point throughout the domain of the exact solution to represent a boundary layer with that method. A basic interpolation technique, such as piecewise linear interpolation, takes the numerical solution from a finite-element approach limited to mesh points for the entire domain. Since our method should be extended to complex situations in higher dimensions, we consider only finite-element subspaces via piecewise polynomial basis functions. For small values of parameter ε, the strategy proposed in this work gives better findings and is more suitable.
In the interval $\Omega = \{x : 0 < x < 1\}$, a singularly perturbed linear system of 'n' second order ordinary differential equations of reaction - diffusion type with discontinuous source terms is considered. Assume that the point $d \in \Omega$ occurs as a single discontinuity in the source terms. The jump at d in any function $\vec{\phi}$ is defined by $[\vec{\phi}](d) = \vec{\phi}(d+) - \vec{\phi}(d-)$.
The self-adjoint two-point boundary value problem that corresponds is

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \text{ on } \Omega^- \cup \Omega^+,\ \vec{u} \text{ given on } \Gamma \text{ and } \vec{f}(d+) \neq \vec{f}(d-) \tag{1}$$

where $\Gamma = \{0, 1\},\ \Omega^- = \{x : 0 < x < d\},\ \Omega^+ = \{x : d < x < 1\}$.
Here $\vec{u}$ is a column $n-$vector, E and $A(x)$ are $n \times n$ matrices, $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1,\ \cdots,\ \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \ldots, n$. The parameters are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

Cases in which any of the parameters are coincident are not considered here for the sake of convenience. The number of layer functions and, as a result, the number of transformation parameters in the Shishkin mesh specified in Section 4 is reduced in these situations. The problem can also be written in the operator form

$$\vec{L}\vec{u} \;=\; \vec{f} \text{ on } \; \Omega^- \cup \Omega^+, \; \vec{u} \text{ given on } \Gamma \;, \text{ and } \vec{f}(d+) \neq \vec{f}(d-),$$

where the operator $\vec{L}$ is defined by

$$\vec{L} \;=\; -ED^2 + A, \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}(x)| \;\text{ for }\; 1 \leq i \leq n \;\text{ and }\; a_{ij}(x) \leq 0 \;\text{ for }\; i \neq j \tag{2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x \in [0,1] \\ 1 \leq i \leq n}} \Big(\sum_{j=1}^{n} a_{ij}(x)\Big). \tag{3}$$

It is assumed that $a_{ij},\, f_i \;\in\; C^{(2)}(\overline{\Omega})$, for $i, j \;=\; 1, \ldots, n.$ Then (1) has a solution $\vec{u} \;\in\; C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$.
It is also assumed that

$$\sqrt{\varepsilon}_n \leq \frac{\sqrt{\alpha}}{6}. \tag{4}$$

C is a generalised positive constant that is independent of x as well as all singular perturbation and discretization parameters used in this article. The empirical results are discussed for the continuous problem are presented on the following section. In Section 3, piecewise-uniform Shishkin meshes can be used to solve the boundary and interior layers. The discrete problem is described in Section 4, and the corresponding maximum principle and stability result are defined. Interpolation error bounds is defined in Section 5. The parameter-uniform error estimation is defined in Section 6. The numerical diagrams in Section 8 are included. Discussion and conclusions is described in Section 9.

2 Analysis of the finite element method

Let V be a given Hilbert space with norm $|| \cdot ||_V$ and scalar product $(\cdot, \cdot)$. V is usually a subspace of the Sobolev space $H^1(\Omega^- \cup \Omega^+) = H^1(\Omega^-) \cup H^1(\Omega^+)$.

Consider the weak formulation, find $\vec{u} \in H_0^1(\Omega^- \cup \Omega^+)^n$ in particular $u_i \in H_0^1(\Omega^- \cup \Omega^+)$ for $i = 1, \ldots, n$ such that

$$\beta_i(u_i, v_i) = f_i(v_i) \quad \forall\, v_i \in H_0^1(\Omega^- \cup \Omega^+) \tag{5}$$

$$\beta_i(u_i, v_i) = -\varepsilon_i(u_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}u_j), v_i\right)$$

and

$$f_i(v_i) = (f_i, v_i)$$

where $(u_i, v_i) = \int_0^1 u_i v_i \, dx$. $\beta_i(u_i, v_i)$ is a bilinear form on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i)$, a given continuous linear functional on $H_0^1(\Omega^- \cup \Omega^+)^n$ and $f_i(v_i(d+)) \neq f_i(v_i(d-))$.

Lemma 2.1. *Suppose that the bilinear form $\beta_i(\cdot,\cdot)$, $i = 1, \ldots, n$, is continuous on $H_0^1(\Omega^- \cup \Omega^+)^n$ is coercive, that*

$$|\beta_i(u_i, v_i)| \leq \gamma ||u_i||\, ||v_i|| \tag{6}$$

$$\beta_i(v_i, v_i) \geq \alpha ||v_i||^2 \tag{7}$$

where α and γ are constants that are indepentent of u_i and v_i. Then for any continuous linear functional $f_i(\cdot)$, the problem (5) *has a unique solution.*

A natural norm on $H_0^1(\Omega^- \cup \Omega^+)^n$ associated with the bilinear form $\beta_i(\cdot,\cdot)$ is the energy norm

$$||v_i||_{\varepsilon_i} = (\varepsilon_i ||v_i||_1^2 + \alpha ||v_i||_0^2)$$

where $||v_i||_1 = (v_i', v_i')^{1/2}$, $||v_i||_0 = (v_i, v_i)^{1/2}$ on $H_0^1(\Omega^- \cup \Omega^+)^n$.

Lemma 2.2. *A bilinear functional $\beta_i(u_i, v_i)$, $i = 1, \ldots, n$, satisfies the coercive property with respect to*

$$||v_i||_{\varepsilon_i}^2 \leq \beta_i(v_i, v_i)$$

Proof. For $i = 1, \ldots, n$

$$\begin{aligned}
\beta_i(v_i, v_i) &= -\varepsilon_i(v_i', v_i') + \left(\sum_{j=1}^{n}(a_{ij}v_j), v_i\right) \\
&= \varepsilon_i ||v_i||_1^2 + \int_0^1 \left(\sum_{j=1}^{n}(a_{ij}v_j) \cdot v_i\right) dx \\
&\geq \varepsilon_i ||v_i||_1^2 + \alpha ||v_i||_0^2
\end{aligned}$$

□

3 The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $\Omega^- \cup \Omega^+$ as follows. Let $\Omega^N = \Omega^{-N} \cup \Omega^{+N}$ where $\Omega^{-N} = \{x_k\}_{k=1}^{\frac{N}{2}-1}, \Omega^{+N} = \{x_k\}_{k=\frac{N}{2}+1}^{N-1}$ $\overline{\Omega}^N = \{x_k\}_{k=0}^{N}$ and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on $[0,1]$ that was generated by dividing $[0,d]$ into $2n+1$ mesh-intervals as follows:

$$[0,\sigma_1] \cup \cdots \cup (\sigma_{n-1},\sigma_n] \cup (\sigma_n, d-\sigma_n] \cup (d-\sigma_n, d-\sigma_{n-1}] \cup \cdots \cup (d-\sigma_1, d].$$

The points separating the uniform meshes are determined by the n parameters $\sigma_r,$ which are defined by $\sigma_0 = 0,$ $\sigma_{n+1} = \frac{1}{2},$

$$\sigma_n = \min\left\{\frac{d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}}\ln N\right\} \tag{8}$$

and, for $r = n-1, \ldots 1,$

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}\ln N\right\}. \tag{9}$$

Clearly

$$0 < \sigma_1 < \ldots < \sigma_n \le \frac{d}{4}, \qquad \frac{3d}{4} \le 1-\sigma_n < \ldots < 1-\sigma_1 < d.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(\sigma_n, d-\sigma_n]$, and a uniform mesh of $\frac{N}{8n}$ mesh-points is placed on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(d-\sigma_{r+1}, d-\sigma_r],$ $r = 0,1,\ldots,n-1,$ respectively.
The remaining was generated by dividing $[d,1]$ into $2n+1$ mesh-intervals as follows:

$$[d, d+\tau_1] \cup \cdots \cup (d+\tau_{n-1}, d+\tau_n] \cup (d+\tau_n, 1-\tau_n] \cup (1-\tau_n, 1-\tau_{n-1}] \cup \cdots \cup (1-\tau_1, 1].$$

The points separating the uniform meshes are determined by the n parameters $\tau_r,$ which are defined by $\tau_0 = \frac{1}{2},$ $\tau_{n+1} = 1,$

$$\tau_n = \min\left\{\frac{1-d}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}}\ln N\right\} \tag{10}$$

and, for $r = n-1, \ldots 1,$

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}\ln N\right\}. \tag{11}$$

Clearly

$$d \;<\; d+\tau_1 \;<\; \ldots \;<\; d+\tau_n \;\le\; \frac{1-d}{4}, \qquad \frac{3(1-d)}{4} \le 1-\tau_n < \;\ldots\; < 1-\tau_1 < 1.$$

Then a uniform mesh of $\frac{N}{4}$ mesh-points is placed on the sub-interval $(d+\tau_n, 1-\tau_n]$, and a uniform mesh of $\frac{N}{8n}$ mesh-points is placed on each of the sub-intervals $(d+\tau_r, d+\tau_{r+1}]$ and $(1-\tau_{r+1}, 1-\tau_r],\ \ r = 0, 1, \ldots, n-1,$ respectively.
In practice, it is convenient to take

$$N = 8n\delta, \;\; \delta \ge 3, \tag{12}$$

where n denotes the number of distinct singular perturbation parameters involved in the experiment (1). This produces a class of 2^{n+1} piecewise uniform Shishkin meshes $\overline{\Omega}^N$.
When all of the parameters σ_r and $\tau_r,\ r = 1, \ldots, n,$ are set to the left, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh with the transformation parameters σ_r, τ_r and a scale N^{-1} from 0 to 1.
The following inequalities hold for the mesh Ω^N , $s = 1, \ldots, n-1$

$$
\begin{array}{lll}
h_k \le 2/N & \text{for} & 1 \le k \le N \\
h_k \ge 1/N & \text{for} & \dfrac{N}{8} \le k \le \dfrac{3N}{8} \text{ and } \dfrac{5N}{8} \le k \le \dfrac{7N}{8} \\
h_k \le 1/N & \text{for} & 1 \le k \le \dfrac{N}{8} \text{ and } \dfrac{3N}{8} \le k \le \dfrac{N}{2} \\
h_k \le 1/N & \text{for} & \dfrac{N}{2} \le k \le \dfrac{5N}{8} \text{ and } \dfrac{7N}{8} \le k \le N \\
h_k \ge \dfrac{N}{8s} & \text{for} & \dfrac{N}{8s} \le k \le \dfrac{N}{8(s+1)} \text{ and } (d - \dfrac{N}{8(s+1)}) \le k \le (d - \dfrac{N}{8s}) \\
h_k \ge \dfrac{N}{8s} & \text{for} & d + \dfrac{N}{8s} \le k \le d + \dfrac{N}{8(s+1)} \text{ and } (1 - \dfrac{N}{8(s+1)}) \le k \le (1 - \dfrac{N}{8s}) \\
h_k \le \dfrac{N}{8s} & \text{for} & 1 \le k \le \dfrac{N}{8(s)} \text{ and } (d - \dfrac{N}{8(s)}) \le k \le \dfrac{N}{2} \\
h_k \le \dfrac{N}{8s} & \text{for} & \dfrac{N}{2} \le k \le d + \dfrac{N}{8(s)} \text{ and } (1 - \dfrac{N}{8(s)}) \le k \le N
\end{array}
\tag{13}
$$

4 The discrete problem

In this segment, a numerical method for (5) is constructed using a finite element method with a suitable Shishkin mesh. Let for $i = 1, \ldots, n$ and $k = 1, \ldots, N-1 \backslash \{\frac{N}{2}\},\ V_{i,k} \subset H_0^1(\Omega^- \cup \Omega^+)^n$ be the space of piecewise linear functionals on $\Omega^- \cup \Omega^+$, that vanish $x = 0, d$, and 1.

The finite element approach is now established for the discrete two-point boundary value problem, $U_{i,k} \in V_{i,k}$

$$\beta_i(U_{i,k}, v_{i,k}) = f(v_{i,k}), \quad \forall\, v_{i,k} \in V_{i,k}, \quad v_{i,\frac{N}{2}} = 0. \tag{14}$$

By $Lax - Migram\; Lemma$ implies that

1. The discrete problem has a unique solution,

2. The discrete problem is stable.

From (3) on A implies that for arbitrary $x \in (\Omega^- \cup \Omega^+)$

$$\xi^T A\, \xi \geq \alpha\, \xi^T \xi \qquad \forall\, \xi \text{ on } \; V^*_{i,k}$$

$V^*_{i,k}$ is dual space for $V_{i,k}$.
Let $\{\phi_{i,k} : k = 1, \cdots, N-1 \backslash \{\frac{N}{2}\}\}$ be a basis for $V_{i,k}$, where $N = N(i,k)$ is the dimension of $V_{i,k}$. Then

$$U_{i,k} = \sum_{k=1}^{\frac{N}{2}-1} C_{i,k}\phi_{i,k} + \sum_{k=\frac{N}{2}+1}^{N-1} C_{i,k}\phi_{i,k}, \qquad \phi_{i,\frac{N}{2}} = 0$$

where the unknowns $C_{i,k}$ satisfiy the linear system

$$AU = B$$

with $A = \beta_i(\phi_{i,k_1}, \phi_{i,k_2})$, $U = C_{i,k}$, $B = f_i(\phi_{i,k})$.
The corresponding difference scheme is

$$\begin{pmatrix} \beta_1(\phi_{1,1},\phi_{1,1}) & \beta_1(\phi_{1,1},\phi_{1,2}) & \cdots & \beta_1(\phi_{1,1},\phi_{n,N-1}) \\ \beta_1(\phi_{1,2},\phi_{1,1}) & \beta_1(\phi_{1,2},\phi_{1,2}) & \cdots & \beta_1(\phi_{1,2},\phi_{n,N-1}) \\ \vdots & \vdots & & \vdots \\ \beta_n(\phi_{n,N-1},\phi_{n,1}) & \beta_n(\phi_{n,N-1},\phi_{n,2}) & \cdots & \beta_n(\phi_{n,N-1},\phi_{n,N-1}) \end{pmatrix} \begin{pmatrix} C_{1,1} \\ C_{1,2} \\ \vdots \\ C_{n,N-1} \end{pmatrix} = \begin{pmatrix} (f_1,\phi_{1,1}) \\ (f_1,\phi_{1,2}) \\ \vdots \\ (f_n,\phi_{n,N-1}) \end{pmatrix}$$

For $k = 1, \ldots, N-1$

$$\phi_{1,k} = \phi_{2,k} = \cdots = \phi_{n,k}$$

$$C_{1,k} = C_{2,k} = \cdots = C_{n,k}$$

The nonzero contribution from a particular element is

$$A_{i,k} = \begin{pmatrix} \int\limits_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k-1}dx & \int\limits_{x_{k-1}}^{x_k} \phi_{i,k-1}.\phi_{i,k}dx \\ \int\limits_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k}dx & \int\limits_{x_k}^{x_{k+1}} \phi_{i,k}.\phi_{i,k+1}dx \end{pmatrix}$$

Similarly, the local load vector is

$$B_{i,k} = \begin{pmatrix} \int\limits_{x_k}^{x_{k+1}} f_i.\phi_{i,k}dx \\ \int\limits_{x_k}^{x_{k+1}} f_i.\phi_{i,k+1}dx \end{pmatrix}$$

For $k = \frac{N}{2}$, the local load vector $(\int\limits_{x_{k-1}}^{x_k} f_i(\frac{N}{2} - 1) + \int\limits_{x_k}^{x_{k+1}} f_i(\frac{N}{2} + 1))/2$

5 Interpolation error bounds

Lemma 5.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of (1) on the fitted mesh Ω^N. Then*

$$\max_{i=1,\dots,n} \sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k} - u_{i,k}||_{\Omega^N} \leq C(N^{-2}lnN)^2,$$

where C is a constant independent of the parameters ε_i.

Proof. The estimate is obtained separately on each subinterval $\Omega_k = (x_{k-1}, x_k) \in \Omega^- \cup \Omega^+$,
$k = 1, \dots, N-1 \backslash \{\frac{N}{2}\}$. Note that for any function $g_{i,k}$ on Ω_k

$$g^*_{i,k} = g_{i,k-1}\phi_{i,k-1} + g_{i,k}\phi_{i,k},$$

and so it is obvious that, on Ω_k,

$$|g^*_{i,k}(x)| \leq \max_{\Omega_k} |g_{i,k}(x)|, \tag{15}$$

and it's easy to see that by using sufficient Taylor expansions

$$|g^*_{i,k}(x) - g_{i,k}(x)| \leq Ch_k^2 \max_{\Omega_k} |g^{''}_{i,k}(x)|. \tag{16}$$

For $i = 1, \dots, n$ from (16) and using Lemma 3 in [Paramasivam et al., 2014], on $\Omega_k \in \Omega^- \cup \Omega^+$,

$$|u^*_{i,k}(x) - u_{i,k}(x)| \leq Ch_k^2 \max_{\Omega_k} |u^{''}_{i,k}(x)|$$

$$\leq C\frac{h_k^2}{\varepsilon_i}. \tag{17}$$

Also, (17) using Lemma 6 and Lemma 7 are in [Paramasivam et al., 2014] on $\Omega_k \in \Omega^-$, for $k = 1, \ldots, \frac{N}{2} - 1$

$$|u^*_{i,k}(x) - u_{i,k}(x)| = |v^*_{i,k}(x) + w^*_{i,k}(x) - v_{i,k}(x) - w_{i,k}(x)|$$

$$\leq |v^*_{i,k}(x) - v_{i,k}(x)| + |w^{*L}_{i,k}(x) - w^L_{i,k}(x)| + |w^{*R}_{i,k}(x) - w^R_{i,k}(x)|$$

$$\leq Ch_k^2 \max_{\Omega_k} |v''_{i,k}(x)| + Ch_k^2 \max_{\Omega_k} |w^{L''}_{i,k}(x)| + Ch_k^2 \max_{\Omega_k} |w^{R''}_{i,k}(x)|$$

$$\leq C(1 + \sum_{q=i}^{n} B_q(x)) + C\sum_{q=i}^{n} \frac{B^L_q(x)}{\varepsilon_q} + C\sum_{q=i}^{n} \frac{B^R_q(x)}{\varepsilon_q} \tag{18}$$

The discussion now centres on whether $2\sqrt{\varepsilon_n}\, lnN/\sqrt{\alpha} \geq d/4$ or $2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha} \leq d/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (17). In the second case $\sigma_n = 2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha}$. Suppose that k satisfies $N/8 \leq k \leq 3N/8$. Then $h_k = 2(d - 2\sigma_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{d - 2\sigma_n}{\varepsilon_n},$$

$\sigma_n \leq 1 - x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\sigma_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{19}$$

Using (19) and (13) in (18) gives the required result.
On the other hand, if k satisfies $1 \leq k \leq N/8$ and $3N/8 \leq k \leq N/2$ and $r = n-1, \ldots, 1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\, lnN/\sqrt{\alpha} \geq r\sigma_{r+1}/r + 1$ or $2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha} \leq r\sigma_{r+1}/r + 1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (17).
In the second case $\sigma_r = 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$ and $s = 1, \ldots, n-2$.

1. Suppose that k satisfies $N/8(s+1) \leq k \leq N/8(s)$ and $d - (N/8(s)) \leq k \leq d - (N/8(s+1))$. Then $h_k = 8n(\sigma_{r+1} - \sigma_r)/N$ or $8n(\sigma_r - \sigma_{r+1})/N$ and $\sigma_r \leq 1 - x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\sigma_{r+1} - \sigma_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\sigma_r - \sigma_{r+1}}{\varepsilon_r}, \tag{20}$$

   Using (20) and (13) in (18) gives the required result.

2. If k satisfies $1 \leq k \leq N/8(s+1)$ and $d-(N/8(s+1)) \leq k \leq N/2$ Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_{r+1}-\sigma_r)/N$ and therefore
$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\sigma_r-\sigma_{r+1})}{\varepsilon_r}, \tag{21}$$
Using (21) and (13) in (18) gives the required result.

Also, (17) using Lemma 6 and Lemma 7 are in [Paramasivam et al., 2014] on $\Omega_k \in \Omega^+$, for $k = \frac{N}{2}+1, \ldots, N-1$
$$|u^*_{i,k}(x) - u_{i,k}(x)| = |v^*_{i,k}(x) + w^*_{i,k}(x) - v_{i,k}(x) - w_{i,k}(x)|$$
$$\leq |v^*_{i,k}(x) - v_{i,k}(x)| + |w^{*L}_{i,k}(x) - w^L_{i,k}(x)| + |w^{*R}_{i,k}(x) - w^R_{i,k}(x)|$$
$$\leq Ch_k^2 \max_{\Omega_k} |v''_{i,k}(x)| + Ch_k^2 \max_{\Omega_k} |w^{L''}_{i,k}(x)| + Ch_k^2 \max_{\Omega_k} |w^{R''}_{i,k}(x)|$$
$$\leq C(1+\sum_{q=i}^{n} B_q(x)) + C\sum_{q=i}^{n}\frac{B^L_q(x)}{\varepsilon_q} + C\sum_{q=i}^{n}\frac{B^R_q(x)}{\varepsilon_q} \tag{22}$$
The discussion now centres on whether $2\sqrt{\varepsilon_n}\, lnN/\sqrt{\alpha} \geq (1-d)/4$ or $2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha} \leq (1-d)/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (17). In the second case $\tau_n = 2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha}$. Suppose that k satisfies $5N/8 \leq k \leq 7N/8$. Then $h_k = 2(1-2\tau_n)/N$ and therefore
$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{1-2\tau_n}{\varepsilon_n},$$
$\tau_n \leq 1 - x_k$, and so
$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\tau_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{23}$$
Using (23) and (13) in (22) gives the required result.
On the other hand, if k satisfies $N/2 < k \leq 5N/8$ and $7N/8 \leq k < N$ and $r = n-1, \ldots, 1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\, lnN/\sqrt{\alpha} \geq r\tau_{r+1}/r+1$ or $2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha} \leq r\tau_{r+1}/r+1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (17).
In the second case $\tau_r = 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$ and $s = 1, \ldots, n-2$.

1. Suppose that k satisfies $5N/8(s+1) \leq k \leq 5N/8(s)$ and $1-(N/8(s)) \leq k \leq 1-(N/8(s+1))$. Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_r-\tau_{r+1})/N$ and $\tau_r \leq 1-x_k$ therefore
$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\tau_{r+1}-\tau_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\tau_r-\tau_{r+1}}{\varepsilon_r}, \tag{24}$$
Using (24) and (13) in (22) gives the required result.

2. If k satisfies $N/2 < k \leq N/8(s+1)$ and $1-(N/8(s+1)) \leq k < N$ Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_{r+1}-\tau_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\tau_{r+1}-\tau_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\tau_r-\tau_{r+1})}{\varepsilon_r}, \tag{25}$$

Using (25) and (13) in (22) gives the required result.

For $k=\frac{N}{2}$, the source terms is assumed by

$$(\int\limits_{x_{k-1}}^{x_k} f_i(\frac{N}{2}-1) + \int\limits_{x_k}^{x_{k+1}} f_i(\frac{N}{2}+1))/2$$

$h_k = (h_{k-1}+h_{k+1})/2,\ h_{k-1} = (x_{k-2}-x_{k-1})$ and $h_{k+1} = (x_{k+1}-x_{k+2})$, $h_{k-1} = 8n(\sigma_{n-1}-\sigma_n)/N,\ \ h_{k+1} = 8n(\tau_1-\tau_2)/N$

$$\frac{h_k}{\varepsilon_i} = \frac{h_{k+1}+h_{k-1}}{2\varepsilon_i} = \frac{4nN^{-1}((\sigma_{n-1}-\sigma_n)+(\tau_1-\tau_2))}{\varepsilon_i} \tag{26}$$

Using (26) and (13) in (22) gives the required result. □

Lemma 5.2. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\ldots,n}\sup_{0<\varepsilon_i\leq 1} ||u^*_{i,k}-u_{i,k}||_{\varepsilon_i} \leq C(N^{-1}lnN)^2,$$

where C is a constant independent of ε_i.

Proof. For $i=1,\ldots,n$ from the definition of the energy norm

$$||u^*_{i,k}-u_{i,k}||_{\varepsilon_i} = \varepsilon_i((u^*_{i,k}-u_{i,k})', (u^*_{i,k}-u_{i,k})') + \alpha(u^*_{i,k}-u_{i,k}, u^*_{i,k}-u_{i,k}). \tag{27}$$

Each term on the right is now treated separately. It is easy to see that the second term satisfies

$$(u^*_{i,k}-u_{i,k}, u^*_{i,k}-u_{i,k}) \leq ||u^*_{i,k}-u_{i,k}||^2. \tag{28}$$

Using integration by parts and noting that $(u^*_{i,k}-u_{i,k})(x_k)=0$, for each k, the first term can be bounded as follows

$$\varepsilon_i((u^*_{i,k}-u_{i,k})', (u^*_{i,k}-u_{i,k})') = \varepsilon_i \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \int\limits_{x_{i-1}}^{x_i} ({u^*_{i,k}}'(s)-{u_{i,k}}'(s))^2 ds$$

$$= -\varepsilon_i \sum_{k=1,k\neq\frac{N}{2}}^{N-1} \int\limits_{x_{i-1}}^{x_i} ({u^*_{i,k}}''(s)-{u_{i,k}}''(s))(u^*_{i,k}(s)-u_{i,k}(s))ds$$

$$= \varepsilon_i \sum_{k=1, k\neq \frac{N}{2}}^{N-1} \int_{x_{i-1}}^{x_i} u_{i,k}''(s)(u_{i,k}^*(s) - u_{i,k}(s))ds$$

$$= (\varepsilon_i u_{i,k}'', u_{i,k}^* - u_{i,k}),$$

where the fact that $u_{i,k}^*{}'' = 0$ on each Ω_k has been used.
The estimate for the second derivative of the components of $u_{i,k}$ are contains in [Paramasivam et al., 2014], using lemma 6 and lemma 7 in[Paramasivam et al., 2014] then gives

$$|(\varepsilon_i u_{i,k}^{''}, u_{i,k}^* - u_{i,k})| \leq ||u_{i,k}^* - u_{i,k}|| \int_0^d \varepsilon_i |u_{i,k}^{''}| ds + \int_d^1 \varepsilon_i |u_{i,k}^{''}| ds$$

$$|(\varepsilon_i u_{i,k}^{''}, u_{i,k}^* - u_{i,k})| \leq ||u_{i,k}^* - u_{i,k}||(\int_0^d + \int_d^1)(\varepsilon_i |v_{i,k}^{''}| + \varepsilon_i |w_{i,k}^{L''}| + \varepsilon_i |w_{i,k}^{R''}|)ds$$

$$\leq C||u_{i,k}^* - u_{i,k}||(\int_0^d + \int_d^1)(1 + \sum_{q=i}^n B_q(s)) + C\sum_{q=i}^n \frac{B_q^L(s)}{\varepsilon_q} + C\sum_{q=i}^n \frac{B_q^R(s)}{\varepsilon_q} ds$$

$$\leq C||u_{i,k}^* - u_{i,k}||,$$

and so

$$\varepsilon_i((u_{i,k}^* - u_{i,k})^{'}, (u_{i,k}^* - u_{i,k})^{'}) \leq C||u_{i,k}^* - u_{i,k}||. \tag{29}$$

Combining (27)-(29) leads to

$$||u_{i,k}^* - u_{i,k}||_{\varepsilon_i} \leq C||u_{i,k}^* - u_{i,k}||(1 + \alpha ||u_{i,k}^* - u_{i,k}||),$$

and the proof is completed using the estimate of $||u_{i,k}^* - u_{i,k}||$ from Lemma 5.1. □

Lemma 5.3. *Let $u_{i,k}^*$ be the $V_{i,k}$ -interpolant of the solution $u_{i,k}$ of* (1) *on the fitted mesh Ω^N. Then*

$$\max_{i=1,\ldots,n} \sup_{0<\varepsilon_i\leq 1} ||u_{i,k}^* - u_{i,k}||_{\varepsilon_i, \overline{\Omega}^N} \leq C(N^{-1}lnN)^2.$$

Proof. Since $u_{i,k}^*(x_k) - u_{i,k}(x_k) = 0$, it follows from the definitions of the norms that

$$||u_{i,k}^* - u_{i,k}||_{\varepsilon_i, \vec{\Omega}^N}^2|| = \varepsilon_i((u_{i,k}^* - u_{i,k})^{'}, (u_{i,k}^* - u_{i,k})^{'}) \leq ||u_{i,k}^* - u_{i,k}||_{\varepsilon_i}^2.$$

Using the estimate in Lemma 5.2 completes the proof. □

6 Interpolation error estimate

Lemma 6.1. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14)*. Suppose that $V_{i,k}$. Then*

$$\max_{i=1,\dots,n} |\beta_i(U_{i,k} - u_{i,k}, v_i)| \leq C(N^{-1}lnN)^2 ||v_{i,k}||_{l^2(\overline{\Omega}^N)},$$

where the constant C is independent of ε_i.

Proof. Since v_i is in $V_{i,k}$, it can be written in the form

$$v_i = \sum_{k=1, k\neq \frac{N}{2}}^{N-1} v_{i,k}\phi_{i,k},$$

and so

$$\beta_i(U_{i,k} - u_{i,k}, v_i) = \sum_{k=1, k\neq \frac{N}{2}}^{N-1} v_{i,k}\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k}) \tag{30}$$

Then, for each $k, 1 \leq k \leq N-1\backslash\{\frac{N}{2}\}$, using (1), (14) and the fact that $(1, \phi_{i,k})_{\Omega^N} = (1, \phi_{i,k}) = \frac{h_k+h_{k+1}}{2}$,

$$\begin{aligned}\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k}) &= \sum_{j=1}^{n}(a_{ij}U_{i,k}, \phi_{i,k}) - \sum_{j=1}^{n}(a_{ij}u_{i,k}, \phi_{i,k}) \\ &= \sum_{j=1}^{n}(a_{ij}u_{j,k}(x_k), \phi_{i,k}) - \sum_{j=1}^{n}(a_{ij}u_{j,k}, \phi_{i,k}) \\ &= \sum_{j=1}^{n}(a_{ij}(u_{j,k}(x_k) - u_{j,k}), \phi_{i,k})\end{aligned}$$

Since

$$|u_{j,k}(x_k) - u_{j,k}| = |\int_{x}^{x_k} u'_{j,k}(s)ds| \leq I_k,$$

where

$$I_k = \int_{x_{k-1}}^{x_{k+1}} |u'_{j,k}(s)|ds,$$

it follows from (13) that

$$|\beta_i(U_{i,k} - u_{i,k}, \phi_{i,k})| \leq C\frac{(h_k + h_{k+1})}{2}(I_k + N^{-1}). \tag{31}$$

Assume for the moment that

$$I_k \leq C(N^{-1}lnN)^2. \tag{32}$$

Then (30)-(32) and the Cauchy-Schwarz inequality give

$$|\beta_i(U_{i,k} - u_{i,k}, v_i)| \leq C(N^{-1}lnN)^2 \sum_{k=1,k\neq \frac{N}{2}}^{N-1} \frac{(h_k + h_{k+1})}{2}^{1/2} |v_{i,k}| \frac{(h_k + h_{k+1})}{2}^{1/2}$$

$$\leq C(N^{-1}lnN)^2 ||v_{i,k}||_{l^2(\overline{\Omega}^N)},$$

as required.
It remains therefore to verify (32). From the estimate are contain Lemma 3 in [Paramasivam et al., 2014], for the first derivative of the solution, it is clear that

$$I_k \leq C \int_{x_{k-1}}^{x_{k+1}} \varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}||_\Omega)dx$$

It follows that

$$I_k \leq C(h_k + h_{k+1})^2/\varepsilon_i, \tag{33}$$

and that

$$I_k \leq C\frac{(h_k + h_{k+1})^2}{\varepsilon_i} + e^{-\sqrt{\alpha}(1-x_{k+1})/\sqrt{\varepsilon_n}} \tag{34}$$

For $i = 1, \dots, n,\ k = 1, \dots, \frac{N}{2} - 1$, then the discussion now centres on whether $2\sqrt{\varepsilon_n}\, lnN/\sqrt{\alpha} \geq d/4$ or $2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha} \leq d/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (34). In the second case $\sigma_n = 2\sqrt{\varepsilon_n} lnN/\sqrt{\alpha}$. Suppose that k satisfies $N/8 \leq k \leq 3N/8$. Then $h_k = 2(d - 2\sigma_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{d - 2\sigma_n}{\varepsilon_n},$$

$\sigma_n \leq 1 - x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\sigma_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{35}$$

Using (35) and (13) in (34) gives the required result.
On the other hand, if k satisfies $1 \leq k \leq N/8$ and $3N/8 \leq k \leq N/2$ and $r = n-1, \dots, 1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\, lnN/\sqrt{\alpha} \geq r\sigma_{r+1}/r + 1$ or $2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha} \leq r\sigma_{r+1}/r + 1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (34).
In the second case $\sigma_r = 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$ and $s = 1, \dots, n-2$.

1. Suppose that k satisfies $N/8(s+1) \leq k \leq N/8(s)$ and $d-(N/8(s)) \leq k \leq d-(N/8(s+1))$. Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_r-\sigma_{r+1})/N$ and $\sigma_r \leq 1-x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\sigma_{r+1}-\sigma_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\sigma_r-\sigma_{r+1}}{\varepsilon_r}, \tag{36}$$

Using (36) and (13) in (34) gives the required result.

2. If k satisfies $1 \leq k \leq N/8(s+1)$ and $d-(N/8(s+1)) \leq k \leq N/2$ Then $h_k = 8n(\sigma_{r+1}-\sigma_r)/N$ or $8n(\sigma_{r+1}-\sigma_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\sigma_{r+1}-\sigma_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\sigma_r-\sigma_{r+1})}{\varepsilon_r}, \tag{37}$$

Using (37) and (13) in (34) gives the required result.

3. Finally, suppose that $k = \{N/8(s), d-(N/8(s)), N/8n, d-(N/8n)\}$. Then

$$I_k \leq (\int_{k-1}^{k} + \int_{k}^{k+1})|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$

$$\leq C(N^{-1}lnN)^2$$

For $i = 1,\ldots,n,\ k = \frac{N}{2}+1,\ldots,N-1$, then the discussion now centres on whether $2\sqrt{\varepsilon_n}\,lnN/\sqrt{\alpha} \geq (1-d)/4$ or $2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha} \leq (1-d)/4$ should be used. In the first case $1/\varepsilon_n \leq C\,(lnN)^2$ and the result follows at once from (13) and (34). In the second case $\tau_n = 2\sqrt{\varepsilon_n}lnN/\sqrt{\alpha}$. Suppose that k satisfies $5N/8 \leq k \leq 7N/8$. Then $h_k = 2(1-2\tau_n)/N$ and therefore

$$\frac{h_k}{\varepsilon_n} = 2N^{-1}\frac{1-2\tau_n}{\varepsilon_n},$$

$\tau_n \leq 1-x_k$, and so

$$e^{-\sqrt{\alpha}(1-x_k)/\sqrt{\varepsilon_n}} \leq e^{-\sqrt{\alpha}\tau_n/\sqrt{\varepsilon_n}} = e^{-2\,lnN} = N^{-2}. \tag{38}$$

Using (38) and (13) in (34) gives the required result.
On the other hand, if k satisfies $N/2 < k \leq 5N/8$ and $7N/8 \leq k < N$ and $r = n-1,\ldots,1$, then the discussion now centres on whether $2\sqrt{\varepsilon_r}\,lnN/\sqrt{\alpha} \geq r\tau_{r+1}/r+1$ or $2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha} \leq r\tau_{r+1}/r+1$should be used.
In the first case $1/\varepsilon_r \leq C\,(lnN)^2$ and the result follows at once from (13) and (34).
In the second case $\tau_r = 2\sqrt{\varepsilon_r}lnN/\sqrt{\alpha}$ and $s = 1,\ldots,n-2$.

1. Suppose that k satisfies $5N/8(s+1) \leq k \leq 5N/8(s)$ and $1-(N/8(s)) \leq k \leq 1-(N/8(s+1))$. Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_r - \tau_{r+1})/N$ and $\tau_r \leq 1 - x_k$ therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{\tau_{r+1}-\tau_r}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{\tau_r-\tau_{r+1}}{\varepsilon_r}, \tag{39}$$

   Using (39) and (13) in (34) gives the required result.

2. If k satisfies $N/2 < k \leq N/8(s+1)$ and $1-(N/8(s+1)) \leq k < N$ Then $h_k = 8n(\tau_{r+1}-\tau_r)/N$ or $8n(\tau_{r+1}-\tau_r)/N$ and therefore

$$\frac{h_k}{\varepsilon_r} = 8nN^{-1}\frac{(\tau_{r+1}-\tau_r)}{\varepsilon_r} \text{ or } 8nN^{-1}\frac{(\tau_r-\tau_{r+1})}{\varepsilon_r}, \tag{40}$$

   Using (40) and (13) in (34) gives the required result.

3. Finally, suppose that $k = \{d + N/8(s), 1-(N/8(s)), d + N/8n, 1-(N/8n)\}$. Then

$$I_k \leq (\int\limits_{k-1}^{k} + \int\limits_{k}^{k+1})|u'_{i,k}|dx < I_{k-1} + I_{k+1}$$

$$\leq C(N^{-1}lnN)^2$$

   For $k = \frac{N}{2}$, the source terms is assumed by

$$(\int\limits_{x_{k-1}}^{x_k} f_i(\frac{N}{2}-1) + \int\limits_{x_k}^{x_{k+1}} f_i(\frac{N}{2}+1))/2$$

   $h_k = (h_{k-1}+h_{k+1})/2,\ h_{k-1} = (x_{k-2}-x_{k-1})$ and $h_{k+1} = (x_{k+1}-x_{k+2})$, $h_{k-1} = 8n(\sigma_{n-1}-\sigma_n)/N,\ \ h_{k+1} = 8n(\tau_1-\tau_2)/N$

$$\frac{h_k}{\varepsilon_i} = \frac{h_{k+1}+h_{k-1}}{2\varepsilon_i} = \frac{4nN^{-1}((\sigma_{n-1}-\sigma_n)+(\tau_1-\tau_2))}{\varepsilon_i} \tag{41}$$

   Using (41) and (13) in (34) gives the required result.

□

7 Discretization error

Lemma 7.1. *Let $u^*_{i,k}$ be the $V_{i,k}$-interpolant of the solution $u_{i,k}$ of* (1) *and $U_{i,k}$ the solution of* (14)*. Then*

$$\max_{i=1,\ldots,n} ||U_{i,k} - u^*_{i,k}|_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. From the coercivity of $\beta_i(.,.)$ in Lemma 2.1 and since $U_{i,k} - u^*_{i.k} \in V_{i,k}$,

$$||U_{i,k} - u^*_{i,k}||^2_{\varepsilon_i,\overline{\Omega}^N} \leq C\beta_i(U_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})$$

$$\leq C[\beta_i(U_{i,k} - u_{i,k}, U_{i,k} - u^*_{i,k}) + \beta_i(u_{i,k} - u^*_{i,k}, U_{i,k} - u^*_{i,k})].$$

Using Lemma 6.1, with $v_i = U_{i,k} - u^*_{i,k}$, then gives

$$||U_{i,j} - u^*_{i,k}||^2_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2||U_{i,k} - u^*_{i,k}||_{\varepsilon_i,\overline{\Omega}^N},$$

Cancelling the common factor gives

$$||U_{i,k} - u^*_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

as required. □

Theorem 7.1. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14)*. Then*

$$\max_{i=1,\ldots,n} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. Since

$$||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq ||U_{i,k} - u^*_{i.k}||_{\varepsilon_i,\overline{\Omega}^N} + ||u^*_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N},$$

the result follows by combining Lemma 5.1 and Lemma 7.1. □

Theorem 7.2. *Let $u_{i,k}$ be the solution of* (1) *and $U_{i,k}$ the solution of* (14)*. Then the following parameter uniform error estimate holds*

$$\max_{i=1,\ldots,n} \sup_{0<\varepsilon_i\leq 1} ||U_{i,k} - u_{i,k}||_{\varepsilon_i,\overline{\Omega}^N} \leq C(N^{-1}lnN)^2,$$

where the constant C is independent of the parameters ε_i.

Proof. Since $\sigma_r \leq 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$, $r = n, \dots, 1$, consider k satisfies, $1 \leq k \leq N/8s$ and $(3N/8s) \leq k \leq \frac{N}{2}$, $s = 1, \dots, n-1$ on a neighbourhood of the boundary layers.
Using the Cauchy Schwarz inequality and Theorem 7.1,

$$|(U_{i,k} - u_{i,k})(x_k)| = |\int\limits_{\Omega_k} (U_{i,k} - u_{i,k})(s) ds|$$

$$\leq (\frac{1}{\varepsilon_r} \int\limits_{\Omega_k} 1^2 ds)^{1/2} (\varepsilon_r \int\limits_{\Omega_k} |(U_{i,k} - u_{i,k})'(s)|^2 ds)^{1/2}$$

$$\leq \frac{\sigma_r}{\varepsilon_r} ||U_{i,k} - u_{i,k}||_{\varepsilon_r, \overline{\Omega}^N}$$

$$\leq C(N^{-1} lnN)^2. \tag{42}$$

On the other hand, Suppose that k satisfies $N/8 \leq k \leq 3N/8$, outside the boundary layers, $h_k \geq 1/N$ and so

$$|(U_{i,k} - u_{i,k})(x_k)|^2 \leq N h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$

$$\leq N \sum_{k=N/8}^{\frac{3N}{8}} h_k |(U_{i,k} - u_{i,k})(x_k)|^2$$

$$\leq N ||U_{i,k} - u_{i,k}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,k} - u_{i,k})(x_k)| \leq ||U_{i,k} - u_{i,k}||_{l^2(\Omega^N)})$$

$$\leq C(N^{-1} lnN)^2. \tag{43}$$

Combining (42) and (43) as required results
Since $\tau_r \leq 2\sqrt{\varepsilon_r} lnN/\sqrt{\alpha}$, $r = n, \dots, 1$, consider k satisfies, $\frac{N}{2} < k \leq 5N/8s$ and $1 - (7N/8s) \leq k \leq N-1$, $s = 1, \dots, n-1$ on a neighbourhood of the boundary layers.
Using the Cauchy Schwarz inequality and Theorem 7.1,

$$|(U_{i,k} - u_{i,k})(x_k)| = |\int\limits_{\Omega_k} (U_{i,k} - u_{i,k})(s) ds|$$

$$\leq (\frac{1}{\varepsilon_r} \int\limits_{\Omega_k} 1^2 ds)^{1/2} (\varepsilon_r \int\limits_{\Omega_k} |(U_{i,k} - u_{i,k})'(s)|^2 ds)^{1/2}$$

$$\leq \frac{\sigma_r}{\varepsilon_r}||U_{i,k}-u_{i,k}||_{\varepsilon_r,\overline{\Omega}^N}$$
$$\leq C(N^{-1}(lnN)^2. \quad (44)$$

On the other hand, Suppose that k satisfies $5N/8 \leq k \leq 7N/8$, outside the boundary layers, $h_k \geq 1/N$ and so

$$|(U_{i,k}-u_{i,k})(x_k)|^2 \leq Nh_k|(U_{i,k}-u_{i,k})(x_k)|^2$$
$$\leq N\sum_{k=N/4}^{\frac{3N}{4}} h_k|(U_{i,k}-u_{i,k})(x_k)|^2 \leq N||U_{i,k}-u_{i,k}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,k}-u_{i,k})(x_k)| \leq ||U_{i,k}-u_{i,k}||_{l^2(\Omega^N)})$$
$$\leq C(N^{-1}lnN)^2. \quad (45)$$

For $k=\frac{N}{2}$, $h_{\frac{N}{2}}=(h_{\frac{N}{2}-1}+h_{\frac{N}{2}+1})/2$, $h_{\frac{N}{2}-1}=(x_{\frac{N}{2}-2}-x_{\frac{N}{2}-1})$ and $h_{\frac{N}{2}+1}=(x_{\frac{N}{2}+1}-x_{\frac{N}{2}+2})$,

$$|(U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2 \leq Nh_{\frac{N}{2}}|(U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2$$
$$\leq N\frac{h_{\frac{N}{2}-1}+h_{\frac{N}{2}+1}}{2}|(U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}})(x_{\frac{N}{2}})|^2 \leq N||U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}}||^2_{l^2(\Omega^N)}.$$

Using Theorem 7.1 then leads to

$$|(U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}})(x_{\frac{N}{2}})| \leq ||U_{i,\frac{N}{2}}-u_{i,\frac{N}{2}}||_{l^2(\Omega^N)})$$
$$\leq C(N^{-1}lnN)^2. \quad (46)$$

Combining (44) and (46) completes the proof. □

8 Numerical Illustrations

Example 8.1. *Consider the BVP*

$$-E\vec{u}''(x)+A(x)\vec{u}(x)=\vec{f}(x),\ \textit{for}\ x\in(0,1),\ \vec{u}(0)=\vec{0},\ \vec{u}(1)=\vec{0}$$

where $E=\text{diag}(\varepsilon_1,\ \varepsilon_2)$, $A=\begin{pmatrix}5 & -1\\ -1 & 5(x+1)\end{pmatrix}$, $\vec{f_1}=(1+x^2,\,2)^T$ $\vec{f_2}=(4,\,x^2)^T$. For various values of $\varepsilon_1,\ \varepsilon_2$ $N=8k,\ k=2^r,\ r=3,\cdots,8,\ d=0.3,$ and $\alpha=1.9$,

Using the general methods from [Miller et al., 1996] , the ε uniform order of convergence and the ε uniform error constant are computed by applying fitted mesh method to the Example 8.1 shown in the Figure 1. In the following Table 8 outlines the conclusions.

η	Number of mesh points N				
	64	128	256	512	1024
2^{0}	0.7544E-03	0.1717E-03	0.6677E-04	0.2797E-04	0.1303E-04
2^{-2}	0.1786E-02	0.2975E-03	0.1115E-03	0.4510E-04	0.2050E-04
2^{-4}	0.3974E-02	0.7429E-03	0.1842E-03	0.7169E-04	0.3064-04
2^{-6}	0.8120E-02	0.1769E-02	0.3029E-03	0.1139E-03	0.4607E-04
2^{-8}	0.1492E-01	0.3948E-02	0.7378E-03	0.1837E-03	0.7132E-04
2^{-10}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
2^{-12}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
2^{-14}	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
D^N	0.2426E-01	0.8082E-02	0.1761E-02	0.3010E-02	0.1129E-03
p^N	0.1319E+01	0.1389E+01	0.1453E+01	0.1473E+01	
C_p^N	0.9233E+00	0.9053E+00	0.7898E+00	0.5031E+00	0.5032E+00
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^* = 1.319$					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.9233$					

Table 1: Values of $D_{\varepsilon}^N,\ D^N,\ p^N,\ p^*$ and $C_{p^*}^N$ for $\varepsilon_1 = \dfrac{\eta}{64},\ \varepsilon_2 = \dfrac{\eta}{16}$.

9 Conclusions

The research work presented in this article is based on the fundamental concept developed by [Miller et al., 1996]. They considered convection diffusion problems in one dimension. In this paper, second order parameter uniform convergence has been established for system of n second order differential equations of reaction diffusion type with discontinuous source terms. The proposed method can be extended to higher dimensional problems.

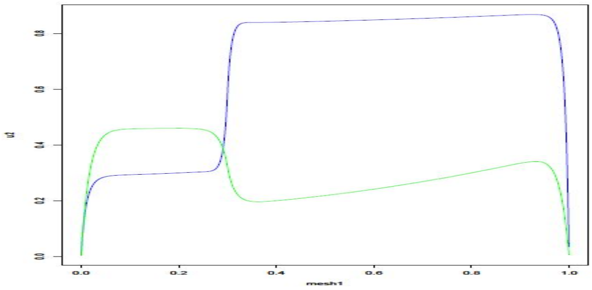

Figure 1: Graphical representation of solution for $\varepsilon = 2^{-4}$ and $N = 512$ of Example 8.1.

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Ratio Mathematica Volume 50, 2023

Isolate g-eccentric domination in fuzzy graph

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Abstract

In a fuzzy graph $G(\rho, \mu)$, a dominating set $D \subseteq P(G)$ is said to be g-eccentric if at least one g-eccentric vertex a of every vertex b in $P - D$ exists in D. If the induced fuzzy sub graph $< D >$ has at least one isolated vertex, then a g-eccentric dominating set D of G is said to be an isolate g-eccentric dominating set. The isolate g-eccentric domination number is defined as the smallest cardinality over all isolate g-eccentric dominating sets of G. This article introduces the isolate g-eccentric point set, isolate g-eccentric dominating set, and their numbers in fuzzy graphs. In some standard fuzzy graphs, bounds are found for an isolate g-eccentric domination number.

Keywords: Fuzzy graph(FG); g-Distance; Isolate g-eccentric number; Isolate domination number; g-eccentric domination number; Isolate g-eccentric domination number.
2020 AMS subject classifications: 05C72. [1]

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[1]Received on May 19, 2023. Accepted on December 16, 2023. Published on December 31, 2023. doi: 10.23755/rm.v50i0.1548. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Rosenfeld [1975] created the idea of fuzzy graphs(FG). The g-eccentric node and related concepts were introduced by Linda and Sunitha [2012]. Janakiraman et al. [2010] initiated the eccentric domination in graph. Arriola [2015] introduced the concept of isolate domination in graph. Isolate domination in graphs discussed by Hamid and Balamurugan [2016]. The isolate eccentric domination in graphs introduced by Bhanumathi and Niroja [2018]. Ismayil and S.Muthupandiyan [2020] initiated the g-eccentric domination in FG. Ismayil and Muthupandiyan [2020] initiated the complmentary nil g-eccentric domination in FG. Muthupandiyan and Ismayil [2021] initiated the idea of perfect g-eccentric domination in FG.

The isolate g-eccentric point set, isolate g-eccentric dominating set and associated its numbers in FG are introduced in this article. Additionally, the bounds for various well-known FG's isolate g-eccentric domination number are determined. There are a few theorems that are presented and demonstrated based on isolate g-eccentric domination in FG. Graph and FG theoretic terminologies can refer Harary [1992] and Rosenfeld [1975] respectively. Here after, the notation G means connected simple FG.

A FG $G(\rho,\mu)$ on $G^*(P,Q)$ is characterized with two functions ρ on P and μ on $Q \subseteq P \times P$, where $\rho : P \rightarrow [0,1]$ (P is a vertex set on G^*, ρ is a vertex set on G) and $\mu : Q \rightarrow [0,1]$ (Q is an edge set on G^* , μ is an edge set on G) such that $\mu(a,b) \leq \rho(a) \wedge \rho(b), \forall a,b \in P$. We anticipate that μ will be reflexive and symmetric functions, and ρ will be a non-empty finite fuzzy set. We indicate the crisp graph $G^*(\rho^*,\mu^*)$ of the FG $G(\rho,\mu)$ where $\rho^* = \{a \in P : \rho(a) > 0\}$ and $\mu^* = \{(a,b) \in Q : \mu(a,b) > 0\}$. The order and size of a FG $G(\rho,\mu)$ are defined by $p = \sum_{a \in P} \rho(a)$ and $q = \sum_{(ab) \in Q} \mu(a,b)$ respectively.

A path P of length n is a sequence of different vertices $a_0, a_1, ..., a_n$ such that $\mu(a_{i-1}, a_i) > 0$, $i = 1, 2, .., n$ and strength of the path P is $s(P) = min\{\mu(a_{i-1}, a_i), i = 1, 2 \cdots n\}$. An edge (a,b) is said to be strong (or strong arc) if its weight is greater than or equal to maximum strength of all paths between a and b. Symbolically $\mu(a,b) \geq CONN_{G-(a,b)}(a,b)$.

The strong neighbourhood of a node a is defined as $N_s(a) = \{b \in P : (a,b)$ is a strong arc $\}$. The minimum strong degree of a FG $G(\rho,\mu)$ is $\delta_s(G) = min\{d_s(b) : b \in \rho^*\}$ and maximum strong degree of G is $\Delta_s(G) = max\{d_s(b) : b \in \rho^*\}$.

If there is no shorter strong path from a to b, the strong path π from a to b is referred to as a geodesic in a FG. The length of a geodesic from a to b is the geodesic distance (g-distance) from a to b and is indicated by $d_g(a,b)$. The expression $e_g(a) = max\{d_g(a,b), b \in P\}$ describes the geodesic eccentricity (g-eccentricity) of a node $a \in P$ in a connected FG. The g-radius and g-diameter of the vertices of G are both defined in terms of g-eccentricity, with the former being denoted by $r_g(G) = min\{e_g(a), a \in P\}$ and the latter by $d_g(G) = max\{e_g(a) : a \in P\}$. Let $a, b \in P(G)$ be any two vertices in a FG $G(\rho,\mu)$, a vertex a at a g-distance $e_g(b)$ from b is a g-eccentric point of b. The g-eccentric set of a vertex a is defined and denoted by $E_g(b) = \{a : d_g(a,b) = e_g(b)\}$. The set $S \subseteq P$ in a FG is called g-eccentric point set if $\forall\ a \in P - S$, $\exists$ at least one g-eccentric point $b \in S$ of a.

A set $D \subseteq P(G)$ in a FG $G(\rho,\mu)$ is called a dominating set (D-set) for $\forall$ $b \in P - D$, $\exists$ at least a vertex $a \in D$, such that (a,b) is a strong arc. A D-set $D \subseteq P(G)$ of a graph G is a g-eccentric dominating set (g-ED set), if for every $b \in P - D$, there exists at least a g-eccentric vertex a of b in D. A g-ED set D is a minimal g-ED set if any subset $D' \subset D$ is not a g-ED set. The g-ED number is denoted and defined by $\gamma_{ged}(G) = min\{|D_i|/D_i,$ is a minimal g-ED set$\}$. The upper g-ED number is denoted and defined by $\Gamma_{ged}(G) = max\{|D_i|/D_i,$ is a minimal g-ED set$\}$. A vertex a is called isolate if $\mu(a,b) = 0$, for all $b \in P(G)$.

2 Isolate g-eccentric point set in a fuzzy graph

In a FG $G(\rho,\mu)$ an isolate g-eccentric point set and its numbers are introduced with suitable examples and certain points are observed in this part. We stated and demonstrated the theorems based on the isolated g-eccentric point set.

Definition 2.1. *A g-eccentric point set $S \subseteq P(G)$ of a FG $G(\rho,\mu)$ is an isolate g-eccentric point set (Ig-EP set) if the induced fuzzy sub graph $< S >$ contains at least an isolate vertex. An Ig-EP set S is a minimal Ig-EP set if any subset $S' \subset S$ is not an Ig-EP set. The Ig-EP number is denoted and defined by $e_{og}(G)$= min $\{|S_i|/S_i$ is a minimal Ig-EP set$\}$. The upper Ig-EP number is denoted and defined by $E_{og}(G) = max\{|S_i|/S_i$ is a minimal Ig-EP set$\}$.*

Example 2.1.

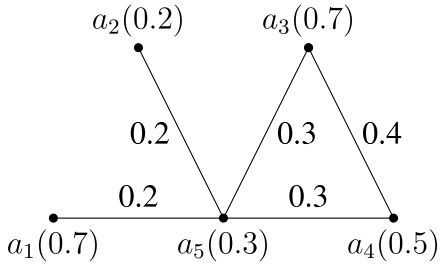

Figure 1: Ig-EP - set in FG

The FG given in figure 1, the Ig-EP sets are $S_1 = \{a_1, a_2\}$, $S_2 = \{a_2, a_3, \}$, $S_3 = \{a_2, a_4\}$, $S_4 = \{a_1, a_3\}$ and $S_5 = \{a_1, a_4\}$. The Ig-EP number is $e_{og}(G) = 0.7$ and the upper Ig-EP number is $E_{og}(G) = 1.4$.

Observation 2.1.

(i) If S is a minimal Ig-EP set, then $S' \subset S$ is not an Ig-EP set.

(ii) In a fuzzy tree, every Ig-EP set has at least one pendent vertex.

(iii) For any FG $G(\rho, \mu)$, $e_{og}(G) \leq E_{og}(G)$.

(iv) The complement of an Ig-EP set is need not be an Ig-EP set.

(v) If S is an Ig-EP set, then $S' \supset S$ is need not be an Ig-EP set.

Example 2.2. *In a FG given in example 2.1, the set $S_1 = \{a_1, a_3\}$ is an Ig-EP set, the complement of $S_1 = \{a_2, a_4, a_5\}$ is a g-eccentric point set (g-EP- set) not an IgEP-set.*

Example 2.3. *Add the vertex a_5 in the set S_1, S_2, S_3, S_4 and S_5 in example 2.1, all the sets are not an IgEP-set.*

Result 2.1.

(i) $e_{og}(K_\rho) = \rho_0$, where $\rho_0 = min\{\rho(a), a \in P(K_\rho)\}$.

(ii) $e_{og}(P_\rho) = \sum \rho(a)$, where a is a pendent vertex .

Theorem 2.1. *Let $G(\rho, \mu)$ be a FG and let S be an Ig-EP set. Then S is a minimal Ig-EP $\Leftrightarrow \forall\ a \in S$, satisfies one of the following condition:*

1. *a has no Ig-EP in S.*

2. *There exists some $b \in P - S$ such that $E_g(b) \cap S = \{a\}$.*

Proof. Assume that S is a minimal Ig-EP set of a FG $G(\rho, \mu)$. Then $\forall\ a \in S, S - \{a\}$ is not an Ig-EP set. Then $\exists$ some node $b \in P - S \cup \{a\}$ which is not an isolate g-eccentric vertex in $S - \{a\}$ or $\exists\ b \in P - S \cup \{a\}$, i.e b has no Ig-EP in $S - \{a\}$.
Case(i) If $b = a$, then a has no Ig-EP in S.
Case (ii) If $b \neq a$, then assume that b has no Ig-EP in $S - \{a\}$ but b has an Ig-EP in S. Then a is the only IgEP of b in S. That is $E_g(b) \cap S = \{a\}$.
Conversly, assume that S is an Ig-EP- set and $\forall\ a \in S$ one of the condition holds. Suppose that S is not a minimal Ig-EP set, $\exists$ a vertex $a \in S$ such that $S - \{a\}$ is an Ig-EP set. Therefore, a is strong neighbor to at least one node b in$S - \{a\}$ and a has an Ig-EP in $S - \{a\}$. Hence, condition (i) does not hold.
Also, if $S - \{a\}$ is an Ig-EP set, then $\forall\ b$ in $P - S$ is strong neighbor to at least one vert ex in $S - \{a\}$ and b has an Ig-EP in $S - \{a\}$. Therefore, condition (ii) does not hold. this is a contradiction to our assumption that $\forall\ a \in D$ satisfies one of the condition. Hence, S is minimal Ig-EP set. $\square$

3 Isolate g-eccentric domination in a fuzzy graph

This section studies an Ig-ED set, a novel domination parameter as well as an Ig-ED number for some FG. A few observations and theorems are discussed.

Definition 3.1. *A g-ED set $D \subseteq P(G)$ of a FG $G(\rho, \mu)$ is called an Ig-ED set (Ig-ED-set)if the induced fuzzy sub graph $< D >$ contains at least an isolate node. An Ig-ED set D is a minimal Ig-ED set if a subset $D' \subset D$ is not a Ig-ED set. The Ig-ED number is denoted and defined by $\gamma_{oged}(G)$= min$\{|D_i|/D_i$ is a minimal Ig-ED set$\}$. The upper Ig-ED number is denoted and defined by $\Gamma_{oged}(G) = max\{|D_i|/D_i$ is a minimal Ig-ED set$\}$.*

Example 3.1.

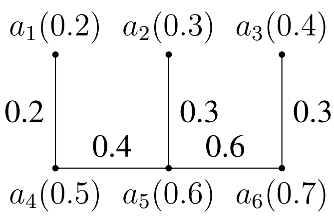

Figure 2: Ig-ED - set in FG

The FG in figure 2, the Ig-ED-sets are $D_1 = \{a_1, a_2, a_3\}$ and $D_2 = \{a_1, a_3, a_5\}$. Hence, $\gamma_{oged}(G) = 0.9$ and $\Gamma_{oged}(G) = 1.2$.

Example 3.2. *The FG given in figure 1, the D-set* $D_1 = \{a_5\}$*, ID- set* $D_2 = \{a_1, a_2, a_4\}$*, gED-set* $D_3 = \{a_2, a_3, a_5\}$ *and Ig-ED set* $D_4 = \{a_1, a_2, a_3\}$*. Then,* $\gamma(G) = 0.3, \gamma_{od}(G) = 1.4, \gamma_{ged}(G) = 1.2$ *and* $\gamma_{oged}(G) = 1.6$.

Observation 2: For any FG, $G(\rho, \mu)$

(i) $\gamma(G) \leq \gamma_o(G) \leq \gamma_{oged}(G)$

(ii) $\gamma_{ged}(G) \leq \gamma_{oged}(G)$

(iii) $\rho_0 \leq \gamma_{oged}(G) \leq p$.

(iv) Every subset $D' \subset D$ is not an Ig-ED set.

(v) Every super set $D" \supset D$ is need not be an Ig-ED set.

(vi) The complement of an Ig-ED set is need not be an Ig-ED set.

Example 3.3. *Let us consider the FG figure 1 and in example 3.2, the set* $D_4 = \{a_1, a_2, a_3\}$ *is an Ig-ED set. Add one more vertex* a_5 *in* D_4 *then the set* D_4 *is not an Ig-ED set.*

Example 3.4.

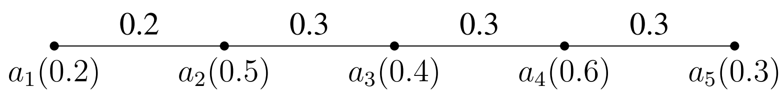

Figure 3: Path FG

The FG given in figure 3, $D = \{a_1, a_3, a_5\}$ is a Ig-ED-set. The Complement $P - D = \{a_2, a_4\}$ of D is ID-set but not a Ig-ED- set.

Result 3.1.

(i) If $G = \bar{K}_{\rho_1} + K_{\rho_2} + K_{\rho_3} + \bar{K}_{\rho_4}$ where $\mid \rho_1^* \mid= 2$, $\mid \rho_2^* \mid= 1$, $\mid \rho_3^* \mid= 1$ and $\mid \rho_4^* \mid= 2$, then $\gamma(G) =\mid \rho_2 \mid + \mid \rho_3 \mid$, $\gamma_{od}(G) = min\{\mid \rho_1 \mid + \mid \rho_3 \mid, \mid \rho_2 \mid + \mid \rho_4 \mid\}$ and $\gamma_{ged}(G) = \gamma_{oged}(G) =\mid \rho_1 \mid + \mid \rho_4 \mid$.

(ii) If $G = K_{\rho_1} + K_{\rho_2} + K_{\rho_3} + K_{\rho_4}$ where $\mid \rho_1^* \mid=\mid \rho_4^* \mid= n, n > 2 \mid \rho_2^* \mid= 1$, $\mid \rho_3^* \mid= 1$, then $\gamma(G) = \gamma_{od}(G) = \gamma_{ged}(G) = \gamma_{oged}(G) \leq 2$.

(iii) If $G = \bar{K_{\rho_1}} + K_{\rho_2} + K_{\rho_3} + \bar{K_{\rho_4}}$ where $\mid \rho_1^* \mid= q$, $\mid \rho_2^* \mid= 1$, $\mid \rho_3^* \mid= 1$ and $\mid \rho_4^* \mid= p$,and $q \leq p$, then $\gamma(G) =\mid \rho_2^* \mid + \mid \rho_3^* \mid$, $\gamma_{od}(G) =\mid \rho_1^* \mid +min\{\mid \rho_2^* \mid, \mid \rho 3^* \mid\}$, $\gamma_{ged}(G) \leq 4$ and $\gamma_{oged}(G) \leq min\{\mid \rho_1^* \mid, \mid \rho_4^* \mid\} + 2$.

(iv) $\gamma_{oged}(k_\rho) = \rho_0$.

(v) $\gamma_{oged}(G) = p$ if and only if $G(\sigma, \mu) = \bar{K_\sigma}$.

(vi) $\gamma_{oged}(K_{\rho_1,\rho_2}) = p, \mid \rho_1^* \mid= 1, \mid \rho_2 \mid= \emptyset$.

4 Bounds on isolate g-eccentric domination in fuzzy graph

This section discusses the bounds for an Ig-ED number of a FG under certain conditions.

Theorem 4.1. *Let P_ρ be a path FG, $\mid \rho^* \mid= n$ then*
$$\gamma_{oged}(P_\rho) \leq \begin{cases} \frac{p}{3} + 1, \text{ n=3k+1} \\ \frac{p}{3} + 2, \text{ n =3k+2} . \end{cases}$$

Proof. Let P_ρ be a path FG, $\mid \rho^* \mid= n$.
Case(i) $n = 3k$.
Let $D = \{a_1, a_4, a_7, ..., a_{3k-2}, a_{3k}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated vertex. Hence D is an Ig-ED set. Therefore, $\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 1$.
Case(ii) $n = 3k + 1$.
Let $D = \{a_1, a_4, a_7, ..., a_{3k-2}, a_{3k+1}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated vertex. Hence $\langle D \rangle$ is an Ig-ED set. Therefore,$\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 1$.
Case(iii) $n = 3k + 2$.
Let $D = \{a_1, a_4, a_5, a_8, ..., a_{3k-1}, a_{3k+1}a_{3k+2}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated node. Hence D is an Ig-ED set. Therefore, $\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 2$.
Hence, From cases (i), (ii) and (iii) we conclude that
$$\gamma_{oged}(P_\rho) \leq \begin{cases} \frac{p}{3} + 1, \text{ n=3k, 3k+1} \\ \frac{p}{3} + 2, \text{ n=3k+2} . \end{cases} .$$
□

Theorem 4.2. *Let C_ρ be a cycle FG, $\mid \rho^* \mid= n$ then*
$$\gamma_{oged}(C_\rho) \leq \begin{cases} \frac{p}{3}, \text{ n is even} \\ \frac{p}{3} + 1, \text{ n is odd} . \end{cases}$$

Proof. Proof of (i): If $n = 4$, any two adjacent nodes of C_ρ is a g-ED set but not an Ig-ED set.
Let $n = 2k$ and $k > 2$.
Let the cycle $C_\rho = \{a_1 a_2 a_3 ..., a_2 k a_1\}$. Each vertex of C_ρ has exactly one g-eccentric vertex (unique g-eccentric node FG).
Case(i) k is odd.
Let $D = \{a_1, a_3, ..., a_k, a_{k+2}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D \rangle$ contains an isolated node. Hence $\langle D \rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (ii) k is even.
Let $D = \{a_1, a_3, ..., a_{k-1}, a_{k+2}, ..., a_{2k}\}$ be the minimum g-ED set and $\langle D \rangle$ contains an isolated node. Hence $\langle D \rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Proof of (ii)
Let $n = 2k + 1$, is odd, each vertex of C_ρ has exactly two g-eccentric node. If $b_i \in P(G)$ has b_{i+k}, b_{i+k+1} is a g-eccentric vertex.
Case (i): $n = 3k, n-$ odd implies m is odd.
Let $D = \{a_1, a_4, x_7, ..., a_k, a_{k+3}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D \rangle$ contains an isolated node. Hence $\langle D \rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (ii): $n = 3k + 1$, $n-$ odd implies k is even.
Let $D = \{a_1, a_4, ..., a_{k+1}, a_{k+3}, a_{k+6}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D \rangle$ contains an isolated node. Hence $\langle D \rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (iii): $n = 3k + 2$, $3k-$ odd implies k is odd.
Let $D = \{a_1, a_4, ..., a_{k-1}, a_k, a_{k+3}, ..., a_{2k+1}\}$ be the minimum g-ED set and $\langle D \rangle$ contains an isolated node. Hence $\langle D \rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3} + 1$.
From the cases (i), (ii) and (iii) we obtain $\gamma_{oged}(C_\rho) \leq \begin{cases} \frac{p}{3}, \text{ n is even} \\ \frac{p}{3} + 1, \text{ n is odd .} \end{cases}$ □

Theorem 4.3. *Let G be a FG of order p with n nodes. Then $\gamma_{oged}(G \circ K_{\rho_1}) \leq n$, where $\mid \rho_1^* \mid = 1$.*

Proof. Let $V(G) = \{a_1, a_2, a_3, ...a_n\}$. Let b'_t be the pendent vertex adjacent to b_t in $G \circ K_\rho$. Then $\{b'_1, b'_2, ...b'_n\}$ is an Ig-ED set for $G \circ K_{\rho_1}$. Hence, $\gamma_{oged}(G \circ K_\rho) \leq n$. □

5 Discussion and conclusion

The geodesic distance and geodesic eccentricity(g-eccentricity) in FG are already defined and discussed in Ismayil and Muthupandiyan [2020] using strong arc. In this article, Ig-EP set, Ig-ED set, and their corresponding numbers in FG are introduced. For some standard FG, these numbers ($\gamma_{oged}(G)$ & $\gamma_{og}(G)$) are obtained and bounds for $\gamma_{oged}(G)$ are also derived. Monophonic Ig-ED set and monophonic connected Ig-ED set and their number in FG may be extended using strong arcs.

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Ratio Mathematica Volume 50, 2023

sc-graphs

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Abstract

Any graph operation of a graph drive only one graph structure. But eccentric graph operation of almost all the graphs drives more than one graph structure, which are interesting and marvelous one. The eccentric graph of G is denoted by $ecc(G)$. Only few families of prime cordial eccentric graphs are investigated. A graph G is separation cordial graph, if it admits separation cordial labeling. In this paper, we investigate some new families of separation cordial eccentric graphs such as $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m,n))$.
Keywords: Eccentric graph; separation cordial graph; crown graph; double star graph.
2020 AMS subject classifications: 05C78 [1]

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[1]Received on May 25, 2023. Accepted on December 20, 2023. Published on December 31, 2023. DOI: 10.23755/rm.v50i0.1549. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Labeling of graph is one of the interesting and active area in the field of graph theory. Rosa [1967] introduced the concept of graceful. Graham and Sloane [1980] introduced the concept of harmonious. Graceful labeling and harmonious labeling are the origin of the graph labeling methods in graph theory. Cahit [1987] introduced another important type of labeling, which contains aspects of both graceful and harmonious, is called cordial. Separation cordial graph was introduced by Elachini V. Lal [2015a] and presented the separation cordial labeling of some standard graph. Elachini V. Lal [2015b] investigated some star and bistar related separation cordial graphs.
Eccentricity related study of graphs are interesting and marvelous, one such eccentricity related study of graphs is eccentric graphs. The special type of graph operation, which is called eccentric graph of graph was defined by Akiyama et al. [1985] and they also contributed few interesting characterizations of such graphs. Any graph operation of a graph drive only one graph structure. But eccentric graph operation of almost all the graphs drives more than one graph structure, which are interesting and marvelous. Such a interesting graph structures of eccentric graphs of some particular classes of graphs are found and characterized by Kaspar et al. [2018]. Some few families of prime cordial eccentric graphs are investigated by Poulomi Ghosh and Anita Pal [2017].
Graphs discussed here are simple, finite and undirected graphs. For graph theoretical terminologies we refer Harary [1972]. For all standard terminologies and notations in Number theory we refer Tom M Apostol [1998] and we refer Gallian [2019] for the literature on graph labeling. In this paper, we obtained some new families of separation cordial eccentric graphs such as $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m,n))$.
We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.*

Definition 1.2. *A separation cordial labeling (or sc-labeling) of a graph G with vertex set V is a bijection f from V to 1, 2,$\cdots$, $|V(G)|$ such that if each edge uv assigned the label 1 if $f(u) + f(v)$ is an odd number and label 0 if $f(u) + f(v)$ is an even number then the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph G is separation cordial graph (or sc-graph), if it admits separation cordial labeling.*

Definition 1.3. *Let G be a graph with vertex set V and edge set E. The eccentric graph of G is denoted by $ecc(G)$ = (V, E'), where $E' = \{(u,v) : d_G(u,v) = ecc(u)$ or $d_G(u,v) = ecc(v)$ for $u,v \in V\}$.*

Definition 1.4. *Double star $S(m,n)$ graph where $m,n \geqslant 1$ is the graph consisting of the union of two starts $K_{1,m}$ and $K_{1,n}$ together with a line joining their centers.*

Definition 1.5. *Shee and Ho [1996] Let $G_1, G_2, \cdots, G_n$ be $n(\geqslant 2)$ copies of a graph G. It is denote by $G(n)$ the graph obtained by adding an edge to G_i and G_{i+l}, $i = 1, 2, \cdots, n-1$, and call $G(n)$ the path-union of n copies of the graph G.*

2 sc-graphs

We will present some new families of separation cordial eccentric graphs.

Theorem 2.1. *$ecc(C_n \odot K_1)$ is sc-graph for $n \geqslant 3$.*

Proof. $C_n \odot K_1$ has $2n$ vertices and $2n$ edges. $v_1, v_2, v_3, \cdots, v_n$ are vertices of C_n; $w_1, w_2, w_3, \cdots, w_n$ are pendent vertices and $v_1w_1, v_2w_2, v_3w_3, \cdots, v_nw_n$ are pendent edges of $C_n \odot K_1$
Case 1 : when n is even.
$ecc(C_n \odot K_1)$ is $\frac{n}{2}P_4$ and the vertices of $ecc(C_n \odot K_1)$ are $v_i, w_{\frac{n}{2}+i}, w_i, v_{\frac{n}{2}+i}$, for $i = 1, 2, \cdots, \frac{n}{2}$.
Then, $|V(ecc(C_n \odot K_1))| = 2n$ and $|E(ecc(C_n \odot K_1))| = \frac{3n}{2}$. Define $f : V(ecc(C_n \odot K_1)) \to \{1, 2, \cdots, 2n\}$ as follows.
Sub case 1(a) : $n = 4$.
$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$. Now $e_f(0) = e_f(1) = 3$, then $|e_f(0)–e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for $n = 4$.
Sub case 1(b) : $n \equiv 0(mod4)$ and $n > 4$.
$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$, $f(v_{i+2}) = f(v_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(w_{i+2}) = f(w_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(v_{i+2}) = f(v_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-2$, $f(w_{i+2}) = f(w_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-2$.
Now $e_f(0) = e_f(1) = \dfrac{3n}{4}$, then $|e_f(0)–e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for $n \equiv 0(mod4)$ and $n > 4$.
Sub case 1(c) : $n \equiv 2(mod4)$ and $n > 4$.
$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$, $f(v_{i+2}) = f(v_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(w_{i+2}) = f(w_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(v_{i+2}) = f(v_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-3$, $f(w_{i+2}) = f(w_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-3$, $f(v_n) = 2n$ and $f(w_n) = 2n - 3$.

Now $e_f(0) = \frac{3n+2}{4}$ and $e_f(1) = \frac{3n-2}{4}$, then $|e_f(0)-e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for $n \equiv 2(mod4)$ and $n > 4$. Therefore, $ecc(C_n \odot K_1)$ is sc-graph for n is even.

Case 2 : when n is odd.

$ecc(C_n \odot K_1)$ is a subdivision of outer edges of $C_n(1, \frac{n-1}{2})$, where $w_1, w_2, w_3, \cdots, w_n$ are vertices of $C_n(1, \frac{n-1}{2})$ and $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \cdots, v_n, v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}}$ are the subdivided vertices of the edges $w_1w_2, w_2w_3, w_3w_4, \cdots, w_nw_1$ of $C_n(1, \frac{n-1}{2})$. Then, $|V(ecc(C_n \odot K_1))| = 2n$ and $|E(ecc(C_n \odot K_1))| = 3n$. Define $f : V(ecc(C_n \odot K_1)) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(w_i) = i$, for $i = 1, 2, \cdots, n$, $f(v_{\frac{n+1}{2}+i}) = n + i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(v_i) = \frac{3n-1}{2} + i$, for $i = 1, 2, \cdots, \frac{n+1}{2}$.

Now $e_f(0) = \frac{3n-1}{2}$ and $e_f(1) = \frac{3n+1}{2}$, when $n \equiv 1(mod4)$ and $e_f(0) = \frac{3n+1}{2}$ and $e_f(1) = \frac{3n-1}{2}$, when $n \equiv 3(mod4)$. Then $|e_f(0)-e_f(1)| \leqslant 1$. Therefore, $ecc(C_n \odot K_1)$ is sc-graph for n is odd.

Hence, $ecc(C_n \odot K_1)$ is sc-graph for $n \geqslant 3$. □

Example 2.1. *$C_5 \odot K_1$, $ecc(C_5 \odot K_1)$ and sc-labeling of $ecc(C_5 \odot K_1)$ are shown in below figure.*

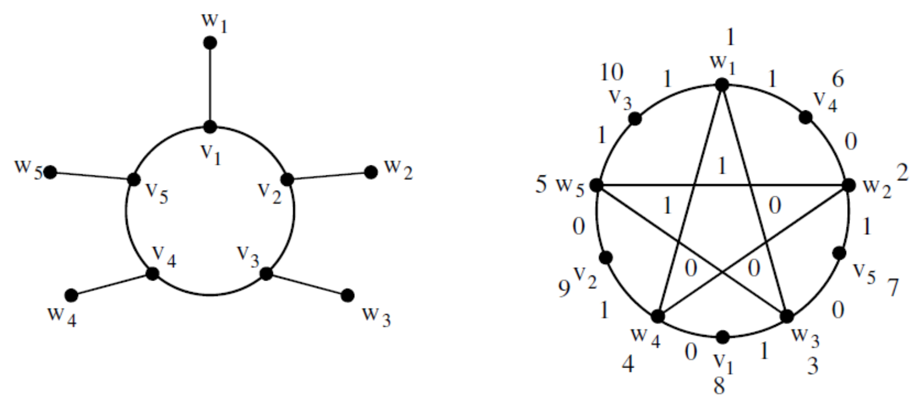

Figure 2.1 : $C_5 \odot K_1$, $ecc(C_5 \odot K_1)$ and sc-labeling of $ecc(C_5 \odot K_1)$

Theorem 2.2. *$ecc(P(C_n(2)))$ is sc-graph for $n \geqslant 3$.*

Proof. $P(C_n(2))$ has $2n$ vertices and $2n+1$ edges. The vertices of $P(C_n(2))$ are $v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_n$.

Case 1 : when n is even.

$ecc(P(C_n(2)))$ is $S(n-1, n-1)$ and the vertices of $ecc(P(C_n(2)))$ are two centers $w_{\frac{n+2}{2}}, v_{\frac{n+2}{2}}$, and two set of pendent vertices $v_1, v_2, v_3, \cdots, v_n$ and $w_1, w_2, w_3, \cdots, w_n$.

$|V(ecc(P(C_n(2))))| = 2n$ and $|E(ecc(P(C_n(2))))| = 2n-1$ and define $f : V(ecc(P(C_n(2)))) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(v_{\frac{n+2}{2}}) = 1, f(v_1) = 2$ and $f(w_{\frac{n+2}{2}}) = 3$, $f(v_i) = i+2$, for $i = 2, 3, \cdots, \frac{n}{2}$, $f(v_i) = i+1$, for $i = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n$, $f(w_i) = i+n+1$, for $i = 1, 2, \cdots, \frac{n}{2}$, $f(w_i) = i+n$, for $i = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n$. Now $e_f(0) = n-1$ and $e_f(1) = n$, then $|e_f(0) - e_f(1)| \leqslant 1$. Thus, $ecc(P(C_n(2)))$ is sc-graph for n is even.

Case 2 : when n is odd.

$ecc(P(C_n(2)))$ is $\overline{K_{n-2}} + \overline{K_2} + \overline{K_2} + \overline{K_{n-2}}$ and the vertices of $ecc(P(C_n(2)))$ are $w_1, w_2, \cdots, w_{\frac{n-1}{2}}, w_{\frac{n+5}{2}}, \cdots, w_n$, $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}$, $w_{\frac{n+1}{2}}, w_{\frac{n+3}{2}}$, $v_1, v_2, \cdots, v_{\frac{n-1}{2}}$, $v_{\frac{n+5}{2}}, \cdots, v_n$. $|V(ecc(P(C_n(2))))| = 2n$ and $|E(ecc(P(C_n(2))))| = 4n-4$. Define $f : V(ecc(P(C_n(2)))) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(v_{\frac{n+1}{2}}) = 1$ and $f(v_{\frac{n+3}{2}}) = 2$, $f(w_{\frac{n+1}{2}}) = 3$ and $f(w_{\frac{n+3}{2}}) = 4$, $f(v_i) = n+2+i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(v_i) = n+i$, for $i = \frac{n+5}{2}, \frac{n+7}{2}, \cdots, n$, $f(w_i) = 4+i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(w_i) = 2+i$, for $i = \frac{n+5}{2}, \frac{n+7}{2}, \cdots, n$.
Now $e_f(0) = e_f(1) = 2n-2$, then $|e_f(0) - e_f(1)| \leqslant 1$. Thus, $ecc(P(C_n(2)))$ is sc-graph for n is odd.
Hence, $ecc(P(C_n(2)))$ is sc-graph for $n \geqslant 3$. □

Example 2.2. *$P(C_5(2))$, $ecc(P(C_5(2)))$ and sc-labeling of $ecc(P(C_5(2)))$ are shown in below figure.*

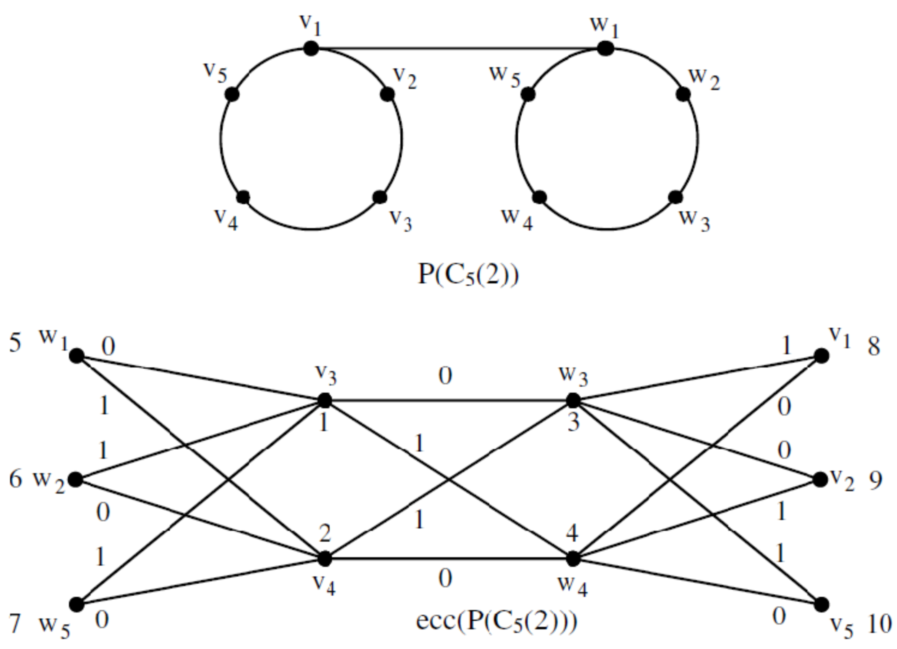

Figure 2.2: $P(C_5(2))$, $ecc(P(C_5(2)))$ and sc-labeling of $ecc(P(C_5(2)))$

Theorem 2.3. *$ecc(S(m,n))$ is sc-graph for $m, n \geqslant 2$.*

Proof. $S(m,n)$ is a double star for $m, n \geqslant 2$.
$S(m,n)$ has $m+n+2$ vertices and $m+n+1$ edges. The vertices of S(m,n) are $v, v_1, v_2, \cdots, v_m$, $w, w_1, w_2, \cdots, w_n$. $ecc(S(m,n))$ is $K_1 + \overline{K_n} + \overline{K_m} + K_1$, the vertices of $ecc(S(m,n))$ are $v, w_1, w_2, \cdots, w_n$, $w, v_1, v_2, , \cdots, v_n$. In $ecc(S(m,n))$, v is adjacent to $w_1, w_2, \cdots, w_n$; w is adjacent to $v_1, v_2, \cdots, v_m$, and, also $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_m$ are mutually adjacent.

$|V(ecc(S(m,n)))| = m+n+2$ and $|E(ecc(S(m,n)))| = nm+m+n$. Define $f : V(ecc(S(m,n))) \to 1, 2, \cdots, m+n+2$ as follows.

$f(v) = 1$ and $f(w) = 2$, $f(w_i) = i+2$, for $i = 1, 2, \cdots, m$, $f(v_i) = i+n+2$, for $i = 1, 2, \cdots, n$. Now $e_f(0) = e_f(1) = \frac{mn+m+n}{2}$, when m and n are even. $e_f(0) = \frac{mn+m+n-2}{2}$ and $e_f(1) = \frac{mn+m+n+2}{2}$, when either m is even and n is odd or m is odd and n is even. $e_f(0) = \frac{mn+m+n+2}{2}$ and $e_f(1) = \frac{mn+m+n-2}{2}$, when m and n are odd. Thus, $|e_f(0) - e_f(1)| \leqslant 1$.
Hence, $ecc(S(m,n))$ is sc-graph for $m, n \geqslant 2$. □

Example 2.3. *$S(4,5)$, $ecc(S(4,5))$ and sc-labeling of $ecc(S(4,5))$ are shown in below figure.*

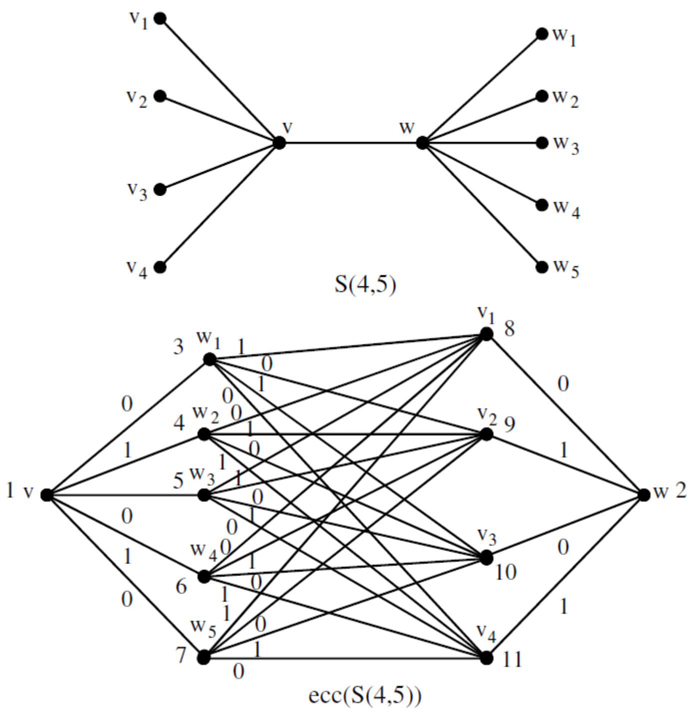

Figure 2.3 : $S(4,5)$, $ecc(S(4,5))$ and sc-labeling of $ecc(S(4,5))$

3 Conclusions

Interesting and marvelous graph structures of some more new families of eccentric graph are derived and verified all are separation cordial eccentric graphs. We discussed only the eccentric graphs of graphs, which are obtained by some more graph operations of graphs. The construction of these eccentric graphs are difficult but interesting one. We presented $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m,n))$ are separation cordial eccentric graphs.

Acknowledgements

The author would like to express his sincere thanks to the editor and anonymous reviewers for their time and valuable suggestions.

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Ratio Mathematica, 50, 2023

Published by Accademia Piceno - Aprutina dei Velati in Teramo (A.P.A.V.)