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A new form of continuity in fuzzy soft topological spaces

Sandhya G Venkatachala Rao*

Anil P Narappanavar[†]

P. G. Patil[‡]

Abstract

The current work introduces a new class of fuzzy soft b continuous functions such as slightly b continuous, semi b continuous, pre b continuous functions and their relation with the existing fuzzy soft continuous functions in fuzzy soft topological spaces. Further optimal definitions of totally b continuous functions have also been brought out in the paper. A new space such as fuzzy soft b compact space is also initiated.

Keywords: fuzzy soft (fs) open set, fs semi-open set, fs pre-open set, fs b -open set, fs continuous functions.

2020 AMS subject classifications: 54A05, 54A40.¹

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1 Introduction

Topology is significant and a significant zone of arithmetic, and it can give numerous connections between logical regions and numerical models. Both mathematicians and computer scientists have concentrated on fuzzy set theory, and numerous utilizations of these have emerged throughout the long term. The soft set hypothesis has been applied to various fields with incredible achievement and rich potential for application in every engineering and sciences gambit. The idea of fuzzy soft sets is presented as a comprehensive numerical tool for managing vulnerability.

In the past years, issues in the field of Engineering, physics, social sciences, and medical sciences etc., in recent times involving uncertainties, cannot be dealt with crisp data. Zadeh [30] in 1965 introduced a general mathematical device recognized as a "fuzzy set" to address uncertainties. The topological structure of fuzzy sets was introduced by Chang [4]. To overcome the existing difficulties in fuzzy set theory, soft sets were introduced by Molodtsov [9] in 1999. This theory of soft sets can be successfully applied in many directions such as Game theory, Riemann integration, Smoothness of functions, Probability theory etc. Maji et al. [8] introduced the merger of fuzzy set and soft set known as a fuzzy soft set. The notion of the topological structure of fuzzy soft (fs) set was introduced by Tanay and Kandemir [26] in 2011 and studied further by Varol and Aygun [29], Roy and Samanta [16] and Pradeep[14]. Continuity is the core of any topological space. The authors Patil et. al. [11] and Missier and Rodrigo[12] many others contributed significantly to the continuous functions in topology. Kharal and Ahmad [7] studied mappings of fuzzy soft classes. The concept of a fuzzy soft semi-open set was introduced by A.Kandil et al. [6].

Fuzzy soft pre-open and regular open sets were introduced by Sabir Hussain in 2016 [19]. Sabir Hussain[20] has also proposed fuzzy soft semi-open and semi-permanent functions in fuzzy soft topological spaces. The concept of fs b-open sets was introduced by Anil P. N. [2] in 2016. Anil P. N et al. [3] has also introduced fs strongly b-continuous and perfectly b-continuous functions. Fuzzy soft pre continuous functions were introduced and studied by Ponselvakumari and Selvi [13]. Sabir Hussain [21] has also introduced fs locally connected spaces and the concept of fs semi pre-open set. The idea of generalized fs b open set and fs gb continuous functions were initiated by Sandhya and Anil P. N. [23]. Fuzzy soft connectedness through fs b open set was formed by Rodyna [15]. Further Abbas et al. [1] and Ruth and Selvam [18] contributed to the concept of fuzzy soft connectedness in 2018. Ibedou and Abbas [5] defined a fuzzy soft net consisting of fuzzy soft points and their convergence. This powerful tool called net

is applied to study some important properties of fuzzy soft topological spaces by Rui Gao and Jianrong Wu [17] in 2018. Sabir Hussain [22] introduced compactness and locally compactness in fuzzy soft topological spaces. Smitha and Sindhu [24] introduced gb-closed and gb-open sets in intuitionistic fuzzy soft topological spaces in 2019. Tingshui Ping [27] investigated a few mappings on fuzzy soft topological spaces. Alkouri [28] introduced a new mathematical tool called complex generalised fuzzy soft set, a combination of generalised fuzzy soft set and complex fuzzy set. Parimala and Karthika [10] reviewed fuzzy soft topological spaces and neutrosophic soft topological spaces in 2020. Smitha and Sindhu [25] studied gb-continuous functions in intuitionistic fuzzy soft topological spaces in 2021. Zhi Kong and Lifu Wang [31] applied a fuzzy soft set in decision-making problems based on grey theory in 2021.

In this work, a new class of fs b continuous functions known as fs slightly b continuous, fs semi b continuous functions, fs pre b continuous, and fs totally b continuous mappings in fs topological spaces are introduced, and some of their properties are studied. Further, the concept of fs b compact spaces is initiated.

2 Preliminaries

Definition 2.1. [8] Let U be the initial universe and K be the set of parameters. I^U be the set of all fuzzy sets on U . Let $A \subseteq K$ and a mapping $f : A \rightarrow I^U$. A pair (f, A) is called fuzzy soft (fs) set over U . It is also denoted by f_A . That is for each $a \in A$, $f(a) = f_a : U \rightarrow I$ is a fuzzy set on U .

Example 2.1. [8] Let the set of shirts be U , and the set of parameters be K . A fs set describes the attractiveness of shirts with respect to the given parameters. The set of all fuzzy sets of U is I^U . $X = \{x_1, x_2, x_3\}$ and $K = \{e_1, e_2, e_3, e_4, e_5\}$. Let $E = \{e_1, e_2, e_3\}$ be the subset of K . A fs set is denoted by (F, E) or f_E . Where $e_1 = \text{colourful}$, $e_2 = \text{bright}$, $e_3 = \text{reasonable price}$, $e_4 = \text{good quality}$, $e_5 = \text{modern}$. $(F, E) = \{\{0.5/x_1, 0.9/x_2, 1/x_3\}, \{0.3/x_1, 0.6/x_2, 0/x_3\}, \{0.2/x_1, 0.9/x_2, 1/x_3\}\}$ describes three shirts with respect to parameters e_1, e_2 , and e_3 . The shirt x_1 w.r.t $e_1 = \text{colourful}$ has a graded value 0.5 out of 1. Similarly, x_2 with respect to e_1 has a graded value of 0.9, and x_3 has 1 out of 1... so on.

Definition 2.2. [26] Let τ be a collection of all fs sets over a universe U and K be a fixed parameter set. A triplet (U, τ, K) is called fuzzy soft topological space [fst_s] if the following hypotheses are satisfied:

- i. $\tilde{O}_K, \tilde{I}_K \in \tau$

- ii. Arbitrary union of members of τ is a member of τ .
- iii. Finite intersection of members of τ is a member of τ .

Each member of τ is called *fs open set*. If $f_K \in \tau$ then $1 - f_K$ is known as *fs closed set*.

Definition 2.3. If (U, τ, K) is *fsts*, then a *fs set* f_K in U is called a

- i. *fs semi-open* [6] if $f_K \leq fsclfsint(f_K)$, *fs semi-closed* if $fsintfscl(f_K) \leq f_K$.
- ii. *fs pre-open*[19] if $f_K \leq fsintfscl(f_K)$, *fs pre-closed* if $fsclfsint(f_K) \leq f_K$.
- iii. *fs b-open* [2] if $f_K \leq fsclfsint(f_K) \vee fsintfscl(f_K)$. The complement of the *fs b open set* is *fs b closed*. A *fs set* that is both *b open* and *b closed* is called *fs b clopen*. And *fs b open set* is referred to as *fsbo*.
- iv. *fs semi pre-open* [20] if $f_K \leq fsclfsintfscl(f_K)$, *fs semi pre-closed* if $fsclfsintfscl(f_K) \leq f_K$.
- v. *fs generalised b open*[23] if $fsbint(f_K) \geq g_K$ whenever $(f_K) \geq g_K$ and g_K is *fs closed set* in U .

Example 2.2. Consider a *fsts* (U, τ, K) and $K = \{e_1, e_2\}$ where $U = \{a, b\}$, $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K), (F_3, K)\}$ $(F_1, K) = \{\{0.6/a, 0.8/b\}, \{0.7/a, 1/b\}\}$ $(F_2, K) = \{\{0.4/a, 0.4/b\}, \{0.3/a, 0.1/b\}\}$ $(F_3, K) = \{\{0.3/a, 0.3/b\}, \{0.2/a, 0.1/b\}\}$. In (U, τ, K) , $(G_1, K) = \{\{0.5/a, 0.6/b\}, \{0.4/a, 0.3/b\}\}$, is *fs semi open*. $(G_2, K) = \{\{0.4/a, 0.3/b\}, \{0.3/a, 0.1/b\}\}$, is *fs pre open*, *fs b open* and also *fs gb open*. $(G_3, K) = \{\{0.6/a, 0.4/b\}, \{0.3/a, 0.2/b\}\}$, is *fs semi pre open*.

Definition 2.4. [8] If f_K is a *fs set*, then

- i. the intersection of all *fs closed supersets* of f_K is *fs closure* of f_K .
- ii. the union of all *fs open subsets* of f_K is called *fs interior* of f_K .

Definition 2.5. [2] If f_K is a *fs set*, then

- i. the intersection of all *fs b closed supersets* of f_K is *fs b closure (fsbcl)* of f_K .
- ii. the union of all *fs b open subsets* of f_K is called *fs b interior (fsbint)* of f_K .

Definition 2.6. Let (U, τ, K) and (V, σ, K) be fsts and f be a function from U to V . Then f is said to be a

- i. f is continuous [7] if the inverse of every f s open set in V is f s open in U .
- ii. f is semi-continuous (resp. f s pre continuous) [13] if the inverse of every f s open set in V is f s semi-open (resp f s pre-open) in U .
- iii. f is semi pre continuous [21] if the inverse of every f s open set in V is f s semi pre-open in U .
- iv. f is b -continuous [23] if the inverse of every f s open set in V is f s b open in U .
- v. f is b -irresolute [23] if the inverse of every f s b open set in V is f s b open in U .
- vi. f is contra b continuous [3] if the inverse of every f s open set in V is f s b closed in U .
- vii. f is strongly continuous [3] if the inverse of every f s set in V is f s clopen in U .
- viii. f is perfectly continuous [3] if the inverse of every f s open set in V is f s clopen in U .
- ix. f is strongly b -continuous [3] if the inverse of each f s b open set in V is f s open set in U .
- x. f is perfectly b -continuous [3] if for each f s b -open set in V its inverse is f s clopen in U .
- xi. f is gb continuous [23] if for each f s open set in V its inverse is f s gb open set in U .

Definition 2.7. Any fsts (U, τ, K) is called

- i. f s discrete space [23] if every f s set is f s open in τ .
- ii. f s locally indiscrete space [21] if every f s open set is closed in τ .
- iii. f s $bT_{1/2}$ space [3] if every f s b open set is f s open.
- iv. f s b connected [15] if there are no f s b separations of $\widetilde{1}_K$, otherwise (U, τ, K) is said to be f s b disconnected space.

Definition 2.8. [15] Let (U, τ, K) be a fsts. An f s b separation on $\widetilde{1}_K$ is a pair of non-null proper f s b open sets f_K and g_K where $f_K \cap g_K = \widetilde{0}_K$, $\widetilde{1}_K = f_K \cup g_K$.

3 Fuzzy soft slightly b-continuous functions

Consider two fsts (U, τ, K) , (V, σ, K) and f is a function from U to V and K is the set of parameters throughout this section.

Definition 3.1. A function f is said to be *fs slightly continuous (fssc)* if the inverse of each *fs clopen* set in V is *fs open* in U .

Definition 3.2. A function f is said to be *fs slightly b continuous (fssbc)* if the inverse of each *fs clopen* set in V is *fs b open (fsbo)* in U .

Example 3.1. Suppose f is an identity map and $U = \{a, b\}$, $V = \{c, d\}$, $K = \{e_1, e_2\}$, $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ $(F_1, K) = \{\{1/a, 0.9/b\}, \{0.8/a, 0.8/b\}\}$
 $(F_2, K) = \{\{0/a, 0.1/b\}, \{0.2/a, 0.2/b\}\}$
 $(G_1, K) = \{\{0.7/c, 0.6/d\}, \{0.5/c, 0.6/d\}\}$
 $(G_2, K) = \{\{0.3/c, 0.4/d\}, \{0.5/c, 0.4/d\}\}$. The inverse images of (G_1, K) and (G_2, K) are *fs b open* sets. Therefore f is *fs slightly b continuous*.

Theorem 3.1. Every *fs slightly continuous* function is *fs slightly b continuous*.

Proof: Let f be *fs slightly continuous*. Let (G, K) be *fs clopen* set in V , then $f^{-1}(G, K)$ is *fs open* and hence *fs b-open* in U . Hence f is *fs slightly b-continuous*.

Converse need not be confirmed, as seen from the below example.

In example 3.1, $f^{-1}(G_1, K)$ and $f^{-1}(G_2, K)$ are *fs b-open* sets but not *fs open* in U . Therefore, f is *slightly b continuous* but not *fs slightly continuous*.

Theorem 3.2. Every *fs contra b continuous* function is *fs slightly b-continuous*.

Proof: If f is *fs contra b continuous* map and (G, K) is *fs clopen* set in V , then $f^{-1}(G, K)$ is *fs b open* in U . Hence the theorem.

The reverse implication is not valid.

Example 3.2. Let f be an *fs identity* map. Let $U = \{a, b\}$, $V = \{c, d\}$ and $K = \{e_1, e_2\}$ and $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$, where $(F_1, K) = \{\{1/a, 0.9/b\}, \{0.8/a, 0.8/b\}\}$
 $(F_2, K) = \{\{0/a, 0.1/b\}, \{0.2/a, 0.2/b\}\}$
 $(G_1, K) = \{\{0.7/c, 0.6/d\}, \{0.5/c, 0.6/d\}\}$
 $(G_2, K) = \{\{0.3/c, 0.4/d\}, \{0.5/c, 0.4/d\}\}$
 $(G_3, K) = \{\{0.2/c, 0.3/d\}, \{0.4/c, 0.3/d\}\}$. It is verified that $f^{-1}(G_1, K)$ and $f^{-1}(G_2, K)$ are *fs b open* sets but $f^{-1}(G_3, K)$ is not *fs b closed* in U . Thus f is *fs slightly b-continuous* but not *fs contra b continuous*.

Theorem 3.3. Every fs b continuous function is fs slightly b continuous.

Proof: Let (G, K) be fs clopen set in V and f be fs b continuous. Then $f^{-1}(G, K)$ is fs b clopen in U . Hence f is fs slightly b continuous. The reverse implication need not be true in general.

Example 3.3. Consider fs identity map. Let $U = \{a, b\}$, $V = \{c, d\}$, $K = \{e_1, e_2\}$, $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$, where $(F_1, K) = \{\{0.5/a, 0.4/b\}, \{0.3/a, 0.4/b\}\}$
 $(F_2, K) = \{\{0.3/a, 0.3/b\}, \{0.2/a, 0.3/b\}\}$
 $(G_1, K) = \{\{0.4/c, 0.5/d\}, \{0.4/c, 0.6/d\}\}$
 $(G_2, K) = \{\{0.6/c, 0.5/d\}, \{0.6/c, 0.4/d\}\}$
 $(G_3, K) = \{\{0.4/c, 0.5/d\}, \{0.3/c, 0.3/d\}\}$. Since the inverse of fs clopen sets (G_1, K) and (G_2, K) are fs b open sets in U , but f is fs slightly b continuous and $f^{-1}(G_3, K)$ is not fs b open in U . Hence, f is not fs b continuous.

Theorem 3.4. Composition of fs slightly b continuous functions need not be fs slightly b continuous.

Example 3.4. Let $f : (U, \tau, K) \rightarrow (V, \tau', K)$ and $g : (V, \tau', K) \rightarrow (w, \sigma, K)$ be fs identity mappings. So, $g \circ f : (U, \tau, K) \rightarrow (w, \sigma, K)$ is also fs identity map. Let $U = \{a, b\}$, $V = \{c, d\}$, $w = \{g, h\}$, $K = \{e_1, e_2\}$, $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$, $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ and $\sigma = \{\tilde{O}, \tilde{1}, (H_1, K), (H_2, K)\}$ be fuzzy soft topological spaces. Here, $(F_1, K) = \{\{0.4/a, 0.3/b\}, \{0.4/a, 0.3/b\}\}$
 $(F_2, K) = \{\{0.5/a, 0.4/b\}, \{0.4/a, 0.4/b\}\}$
 $(G_1, K) = \{\{0.5/c, 0.4/d\}, \{0.5/c, 0.5/d\}\}$
 $(G_2, K) = \{\{0.5/c, 0.6/d\}, \{0.5/c, 0.5/d\}\}$
 $(H_1, K) = \{\{0.7/g, 0.8/h\}, \{0.6/g, 0.5/h\}\}$
 $(H_2, K) = \{\{0.3/g, 0.2/h\}, \{0.4/g, 0.5/h\}\}$. Then $f^{-1}(G_1, K)$, $f^{-1}(G_2, K)$ in U and $g^{-1}(H_1, K)$, $g^{-1}(H_2, K)$ in V are fs b open sets but $(g \circ f)^{-1}(H_2, K)$ is not fs b open in U .

Theorem 3.5. Let $f : (U, \tau, K) \rightarrow (V, \tau', K)$ and $g : (V, \tau', K) \rightarrow (w, \sigma, K)$ be two fs mappings, then

- i. If f is fs b -irresolute and g is $fssbc$, then $g \circ f$ is $fssbc$.
- ii. If f is fs b -irresolute and g is fs b -continuous, then $g \circ f$ is $fssbc$.
- iii. If f is fs b -irresolute and g is $fssc$, then $g \circ f$ is fs b continuous.

- iv. If f is fs b -continuous and g is $fssc$, then $g \circ f$ is $fssbc$.
- v. If f is strongly b continuous and g is $fssbc$, then $g \circ f$ is $fssc$.
- vi. If f is $fssbc$ and g is fs perfectly b continuous, then $g \circ f$ is fs b -irresolute.
- vii. If f is $fssbc$ and g is fs contra continuous, then $g \circ f$ is $fssbc$.
- viii. If f is fs b irresolute and g is fs contra b continuous, then $g \circ f$ is $fssbc$.

Proof:

- i. Let (H, K) be fs clopen set in W , since g is $fssbc$, $g^{-1}(H, K)$ is fs b open in V and f is fs b irresolute $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b open in U . Thus $g \circ f$ is $fssbc$.
- ii. Let (H, K) be fs clopen set in W , since g is fs b continuous, $g^{-1}(H, K)$ is fs b open in V and f is fs b irresolute, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b open in U . Therefore $g \circ f$ is $fssbc$.
- iii. Let (H, K) be fs clopen set in W , since g is $fssc$, $g^{-1}(H, K)$ is fs open set in V and also fs b open in V since f is fs b irresolute, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b -open. Therefore, $g \circ f$ is $fssbc$.
- iv. Let (H, K) be fs clopen set in W , since g is fs slightly continuous, $g^{-1}(H, K)$ is fs b open set in V , since f is fs b continuous, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b -open in U and hence $g \circ f$ is $fssbc$.
- v. Let (H, K) be fs clopen set in W . Since g is fs slightly continuous, $g^{-1}(H, K)$ is fs b open in V and f is fs strongly b continuous. $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ Is fs open in U . Consequently $g \circ f$ is $fssc$.
- vi. Let (H, K) be fs b open set in W , since g is fs perfectly b continuous, $g^{-1}(H, K)$ is fs open. fs closed in V since f is $fssbc$, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b open in U . Accordingly $g \circ f$ is fs b irresolute.

- vii. Let (H, K) be fs clopen set in W , since g is fs contra continuous, $g^{-1}(H, K)$ is fs open and fs closed in V since, f is $fssbc$, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b open in U . Hence $g \circ f$ is $fssbc$.
- viii. Let (H, K) be fs b clopen set in W , since g is fs contra b continuous, $g^{-1}(H, K)$ is fs b open and fs b closed in V . Since f is fs b irresolute, $(g \circ f)^{-1}(H, K) = f^{-1}(g^{-1}(H, K))$ is fs b open, and fs b closed in U . So $g \circ f$ is $fssbc$.

Theorem 3.6. If f is $fssbc$ and U is fs $bT_{1/2}$ topological space, then f is $fssc$.

Proof: Let (H, K) be fs clopen set in W . Since f is $fssbc$, $f^{-1}(H, K)$ is fs b open in the space U and U is fs $bT_{1/2}$ space, so $f^{-1}(H, K)$ is fs open in U . Hence f is $fssc$.

Theorem 3.7. If f is $fssbc$ and U is fs b connected space, then V is not fs discrete space.

Proof: Let us assume V as fs discrete space. Let (H, K) be a proper non-empty fs open subset of V . Since, f is fs slightly b continuous, so $f^{-1}(H, K)$ is proper non-empty fs b clopen subset of U , which contradicts that U is fs b connected. Therefore V is not fs discrete space.

Theorem 3.8. If f is $fssbc$ and V is fs locally indiscrete space, then f is fs b continuous.

Proof: Let (H, K) be fs open set in V and V is locally indiscrete space with (H, K) is fs closed in V . And function is fs slightly b continuous, $f^{-1}(H, K)$ is fs b open in U . Hence f is fs b continuous.

Remark 3.1. From the above observations of stronger and weaker forms of fs slightly b continuous functions in $fsts$ we have the following implications.

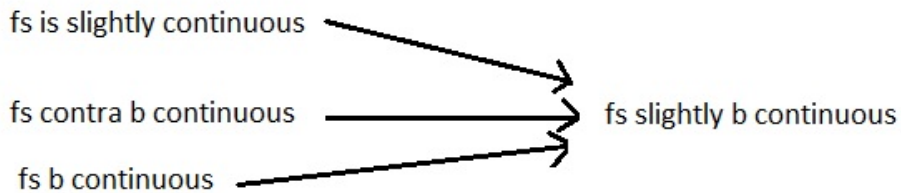


Figure 1:

4 Fuzzy soft semi b continuous functions

Throughout this section, (U, τ, K) and (V, σ, K) be any two fsts where K is the set of parameters and f be a mapping from U to V

Definition 4.1. A function f is *fs semi b continuous* if the inverse of every *fs b open (fsbo)* set is *fs semi-open*. The family of all *fs semi b continuous functions* is denoted by *fssmbc*.

Theorem 4.1. If f is a member of *fssmbc* then it is *fs semi-continuous*.

Proof: If $f \in \text{fssmbc}$ and (G, K) is *fs open set* in V , since every *fs open set* is *fsbo*, $f^{-1}(G, K)$ is *fs semi-open* in U . Hence f is *fs semi-continuous*.

The converse of the this theorem need not be true in general.

Example 4.1. Consider the *fs identity map* f from U to V .

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$, $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fsts. Let $U = \{a, b\}$, $V = \{c, d\}$, $K = \{e_1, e_2\}$, where

$$(F_1, K) = \{\{0.5/a, 0.3/b\}, \{0.2/a, 0.4/b\}\}$$

$$(F_2, K) = \{\{0.3/a, 0.1/b\}, \{0.2/a, 0.3/b\}\},$$

$$(G_1, K) = \{\{0.5/c, 0.7/d\}, \{0.3/c, 0.5/d\}\},$$

$$(G_2, K) = \{\{0.4/c, 0.5/d\}, \{0.2/c, 0.3/d\}\}.$$

Consider $(H, K) = \{\{0.3/c, 0.4/d\}, \{0.3/c, 0.1/d\}\}$ a *fsbo set* in V . Since $f^{-1}(H, K)$ it is not *fs semi-open* in U , f does not belong to *fssmbc*. But it is *fs semi-continuous*.

Theorem 4.2. If $f \in \text{fssmbc}$ then f is *fs b continuous*.

Proof: If f is in *fssmbc*, the inverse of every *fsbo set* is *fs semi-open*. Consider *fs open set* (G, K) in V , $f^{-1}(G, K)$ is *fs semi-open* and hence *fsbo* in U , f is *fs b continuous*.

But the converse is not as seen from the above example 4.1, $f^{-1}(G_1, K)$ and $f^{-1}(G_2, K)$ are *fs b -open sets* in U . Therefore, f is *fs b continuous*. But $f^{-1}(H, K)$ is not *fs semi-open* in U , Hence $f \notin \text{fssmbc}$.

Theorem 4.3. If $f \in \text{fssmbc}$ then f is *fs gb continuous*.

Proof: For an *fs semi b -continuous function*, the inverse image of a *fsbo set* is *fs semi-open*. Each *fs semi-open set* is *fs gb open*. Hence f is *fs gb continuous*. With the counter example, we can prove that converse is not valid.

Example 4.2. Let f be *fs identity mapping* and $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$, $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fsts.

Let $U = \{a, b\}$, $V = \{c, d\}$, $K = \{e_1, e_2\}$, where

$$(F_1, K) = \{\{0.3/a, 0.4/b\}, \{0.5/a, 0.6/b\}\},$$

$$(F_2, K) = \{\{0.4/a, 0.4/b\}, \{0.6/a, 0.6/b\}\},$$

$$(G_1, K) = \{\{0.5/c, 0.4/d\}, \{0.1/c, 0.8/d\}\},$$

$$(G_2, K) = \{\{0.4/c, 0.3/d\}, \{0.1/c, 0.6/d\}\}.$$

Consider f b open set, $(H, K) = \{\{0.4/c, 0.2/d\}, \{0.5/c, 0.4/d\}\}$ in V . Then $f^{-1}(G_1, K)$ and $f^{-1}(G_2, K)$ are f s gb -open sets in U but $f^{-1}(H, K)$ is not f s semi available in U . Thus f is f s gb continuous but $f \notin fssmbc$.

Theorem 4.4. If f is a member of $fssmbc$ then it is f s semi pre continuous.

Proof: Every f s semi-open set is f s semi pre-open proof is evident.

Theorem 4.5. If $\theta \in fssmbc$ and U is f s $bT_{1/2}$ space, then θ is f s continuous.

Proof: Since θ is f s semi b continuous function, for any f s open set (G, K) in V , $\theta^{-1}(G, K)$ is f s semi-open in U . And every f s semi-open set is f s bo and hence f s open in f s $bT_{1/2}$ space, θ is f s continuous.

The converse is not true.

Example 4.3. Let $\theta : (U, \tau, K) \rightarrow (V, \sigma, K)$ be a f s mapping defined by $\theta(a) = d$ and $\theta(b) = c$. Let $U = \{a, b\}$, $V = \{c, d\}$ and $K = \{e_1, e_2\}$.

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\sigma = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be f s t s .

Where $(F_1, K) = \{\{0.6/a, 0.5/b\}, \{0.4/a, 0.5/b\}\}$,

$$(F_2, K) = \{\{0.5/a, 0.4/b\}, \{0.3/a, 0.4/b\}\},$$

$$(G_1, K) = \{\{0.5/c, 0.6/d\}, \{0.5/c, 0.4/d\}\},$$

$(G_2, K) = \{\{0.4/c, 0.5/d\}, \{0.4/c, 0.3/d\}\}$. Consider f s b open set $(H, K) = \{\{0.6/c, 0.5/d\}, \{0.7/c, 0.7/d\}\}$ in V and $\theta^{-1}(G_1, K)$, $\theta^{-1}(G_2, K)$ are f s open sets but $\theta^{-1}(H, K)$ is not f s semi-open in U . Thus θ is f s continuous but $\theta \notin fssmbc$.

Theorem 4.6. If $\alpha : (U, \tau, K) \rightarrow (V, \tau', K)$ is f s semi b continuous and $\beta : (V, \tau', K) \rightarrow (W, \sigma, K)$ is f s b continuous, then $\beta \circ \alpha : (U, \tau, K) \rightarrow (W, \sigma, K)$ is f s gb continuous.

Proof: Let (H, K) be f s open set in W , since β is f s b continuous, $\beta^{-1}(H, K)$ is f s bo in V and α is f s semi b continuous, so $(\beta \circ \alpha)^{-1}(H, K) = \alpha^{-1}(\beta^{-1}(H, K))$ is f s semi-open and hence f s gb open in U . Thus $(\beta \circ \alpha)$ is f s gb continuous.

Theorem 4.7. If $\alpha : (U, \tau, K) \rightarrow (V, \tau', K)$ is f s semi b continuous and $\beta : (V, \tau', K) \rightarrow (W, \sigma, K)$ is f s semi-continuous, then $\beta \circ \alpha : (U, \tau, K) \rightarrow (W, \sigma, K)$ is f s semi pre continuous.

Proof: Let (H, K) be f s open set in W , since β is f s semi-continuous, $\beta^{-1}(H, K)$ is f s semi-open and also f s bo in V . Since α is f s semi b continuous $(\beta \circ \alpha)^{-1}(H, K) = \alpha^{-1}(\beta^{-1}(H, K))$, is f s semi-open in U . Every f s semi-open set is f s semi pre-open. Hence $(\beta \circ \alpha)$ is f s semi pre continuous.

Remark 4.1. The relations of stronger and weaker forms of f s semi-continuous functions in f s t s is represented as :

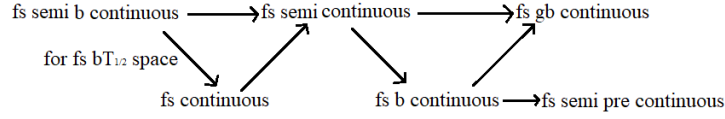


Figure 2:

5 Fuzzy soft pre b continuous functions

In this section $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ is defined as *fs* mapping, and parameter set be K where U and V are fst's.

Definition 5.1. A function η is said to be *fs pre b continuous* if the inverse of each *fsbo* in V is *fs pre-open* in U . The family of *fs pre b continuous* functions is denoted by *fspbc*.

Theorem 5.1. Every *fs pre b continuous* function is *fs pre continuous*.

Proof: Let η be *fs pre b continuous* mapping, (G, K) be *fs open set* in V . Since every *fs open set* is *fsbo*, $\eta^{-1}(G, K)$ is *fs pre-open* in U . Hence η is *fs pre continuous*. But converse need not be true in general.

Example 5.1. Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be a function defined by $\eta(x_1) = y_2$ and $\eta(x_2) = y_1$ where $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$, $K = \{e_1, e_2\}$.

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fst's.

$(F_1, K) = \{\{0.5/x_1, 0.6/x_2\}, \{0.3/x_1, 0.4/x_2\}\}$

$(F_2, K) = \{\{0.4/x_1, 0.3/x_2\}, \{0.2/x_1, 0.4/x_2\}\}$

$(G_1, K) = \{\{0.3/y_1, 0.3/y_2\}, \{0.2/y_1, 0.2/y_2\}\}$

$(G_2, K) = \{\{0.5/y_1, 0.3/y_2\}, \{0.3/y_1, 0.3/y_2\}\}$.

Consider $(H, K) = \{\{0.4/y_1, 0.1/y_2\}, \{0.3/y_1, 0.2/y_2\}\}$ which is a *fsbo* in V . $\eta^{-1}(G_1, K)$ and $\eta^{-1}(G_2, K)$ are *fs pre-open sets* in U , But $\eta^{-1}(H, K)$ is not *fs pre-open* in U . Therefore, η is *fs pre-continuous* but not *fs pre b continuous*.

Theorem 5.2. If $\eta \in \text{fspbc}$ then η is *fs b continuous*.

Proof: Let η be *fs pre b-continuous* function. So the inverse of every *fsbo set* is *fs pre-open* and each *fs pre-open set* is *fsbo*. Hence η is *fs b continuous*.

Converse of the above theorem need not be accurate.

Example 5.2. Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be *fs identity map*, where $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $K = \{e_1, e_2\}$.

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fuzzy soft topological spaces. $(F_1, K) = \{\{0.3/x_1, 0.2/x_2\}, \{0.2/x_1, 0.3/x_2\}\}$

$(F_2, K) = \{\{0.3/x_1, 0.3/x_2\}, \{0.8/x_1, 0.5/x_2\}\}$

$(G_1, K) = \{\{0.5/y_1, 0.6/y_2\}, \{0.2/y_1, 0.3/y_2\}\}$

$(G_2, K) = \{\{0.4/y_1, 0.3/y_2\}, \{0.2/y_1, 0.3/y_2\}\}$.

Consider $(H, K) = \{\{0.5/y_1, 0.6/y_2\}, \{0.3/y_1, 0.3/y_2\}\}$ which is a fsbo set in V . $\eta^{-1}(H, K)$ is not fs pre-open in U . Hence $\eta \notin \text{fspbc}$. But it is fs b continuous.

Theorem 5.3. If $\eta \in \text{fspbc}$ then η is fsgb continuous.

Proof: Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be fs pre b- continuous function and (G, K) be fs open set in V since every fs open set is fsbo and η is fs pre-b-continuous, $\eta^{-1}(G, K)$ is fs pre-open, and hence fs gb open in U .

Converse is not be true.

Example 5.3. Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be fs identity mapping, where $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $K = \{e_1\}$.

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fuzzy soft topological spaces. $(F_1, K) = \{\{0.3/x_1, 0.2/x_2\}\}$ $(F_2, K) = \{\{0.3/x_1, 0.3/x_2\}\}$ $(G_1, K) = \{\{0.5/y_1, 0.6/y_2\}\}$ $(G_2, K) = \{\{0.4/y_1, 0.3/y_2\}\}$.

Consider fs b open set $(H, K) = \{\{0.8/y_1, 0.7/y_2\}\}$ in V . Since, $\eta^{-1}(G_1, K)$ and $\eta^{-1}(G_2, K)$ are fs gb-open but $\eta^{-1}(H, K)$ is not fs pre-open in U , η is fs gb-continuous but not fs pre b- continuous.

Theorem 5.4. If $\eta \in \text{fspbc}$ then it is fs semi pre continuous.

Proof: Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be fs pre b- continuous function. Let (G, K) be fs open set in V . Hence fs b open and $\eta^{-1}(G, K)$ is fs pre-open in U . Every fs pre-open set is fs semi pre-open, η is fs semi pre continuous.

Converse need not be accurate as seen from the example 6.1, $\eta^{-1}(G_1, K)$ and $\eta^{-1}(G_2, K)$ are fs semi pre-open sets in U . Therefore η is fs semi pre continuous. But $\eta^{-1}(H, K)$ is not fs pre-open in U . Hence η is not fs pre b- continuous.

Theorem 5.5. If η is fs pre b continuous and (U, τ, K) is fs $bT_{1/2}$ space, then η is fs continuous.

Proof: Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be fs pre b- continuous function. So the inverse of each fsbo set is fs pre-open and hence fsbo in U . But U is fs $bT_{1/2}$ space each fs b open set is fs open, η is fs continuous.

Example 5.4. Let $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ be defined by $\eta(x_1) = y_2$ and $\eta(x_2) = y_1$, where $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$, $K = \{e_1, e_2\}$

$\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$ and $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be fuzzy soft topological spaces.

$(F_1, K) = \{\{0.6/x_1, 0.5/x_2\}, \{0.4/x_1, 0.5/x_2\}\}$

$(F_2, K) = \{\{0.5/x_1, 0.4/x_2\}, \{0.3/x_1, 0.4/x_2\}\}$

$(G_1, K) = \{\{0.5/y_1, 0.6/y_2\}, \{0.5/y_1, 0.4/y_2\}\}$

$(G_2, K) = \{\{0.4/y_1, 0.5/y_2\}, \{0.4/y_1, 0.3/y_2\}\}$.

Consider fs b open set $(H, K) = \{\{0.6/y_1, 0.5/y_2\}, \{0.6/y_1, 0.7/y_2\}\}$ in V .

Since $\eta^{-1}(G_1, K)$ and $\eta^{-1}(G_2, K)$ are fs open sets and $\eta^{-1}(H, K)$ is not fs pre-open in U , η is fs continuous but not fs pre b continuous.

Theorem 5.6. If $\eta : (U, \tau, K) \rightarrow (V, \tau', K)$ is *fs pre b continuous* and $\mu : (V, \tau', K) \rightarrow (W, \sigma, K)$ is *fs b continuous*, then $\mu \circ \eta : (U, \tau, K) \rightarrow (W, \sigma, K)$ is *fs gb continuous*.

Proof: Let (H, K) be *fs open* set in W , since g is *fs b continuous*, $\mu^{-1}(H, K)$ is *fsbo* in V and η is *fs pre b continuous* $(\mu \circ \eta)^{-1}(H, K) = \eta^{-1}(\mu^{-1}(H, K))$ is *fs pre-open* in U . Every *fs pre-open* set is *fs gb open*, $(\mu \circ \eta)$ is *fs gb continuous*.

Remark 5.1. From the above observations we have the following implication:

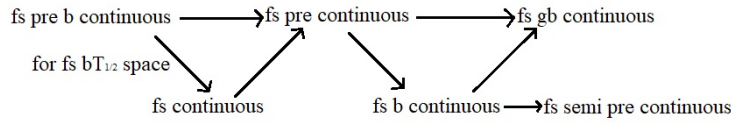


Figure 3:

6 Fuzzy soft totally *b* continuous functions

Definition 6.1. A function $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ is *fs totally continuous* if the inverse of every *fs open* set is *fs clopen*.

Definition 6.2. A function $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ is *fs totally b continuous* if inverse image of every *fs open* set in V is *fs b clopen* in U .

Theorem 6.1. Every *fs totally continuous* function is *fs totally b continuous*.

Proof: Since every *fs open(closed)* set is *fs b open (b closed)*, it was evident that every *fs totally continuous* function is *fs totally b-continuous*.

But converse need not be true.

Example 6.1. Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be defined by $\psi(x_1) = y_2$ and $\psi(x_2) = y_1$. Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$, $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$, be fuzzy soft topological spaces. Let $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $K = \{e_1, e_2\}$
 $(F_1, K) = \{\{1/x_1, 0.9/x_2\}, \{0.8/x_1, 0.8/x_2\}\}$
 $(F_2, K) = \{\{0/x_1, 0.1/x_2\}, \{0.2/x_1, 0.2/x_2\}\}$
 $(G_1, K) = \{\{0.7/x_1, 0.6/x_2\}, \{0.5/x_1, 0.6/x_2\}\}$
 $(G_2, K) = \{\{0.3/x_1, 0.4/x_2\}, \{0.5/x_1, 0.4/x_2\}\}$. Then $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are neither *fs open*, nor *fs closed* sets in U . But they are *fs b clopen* sets in U . Therefore ψ , *fs totally b continuous* but not *fs totally continuous*.

Theorem 6.2. Every *fs perfectly b-continuous* function is *fs totally b continuous*.

Proof: Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be *fs perfectly b continuous* function. Consider an *fs open* set f_K in V . Since f_K is *fs b open* and ψ is *fs perfectly b*

continuous. $\psi^{-1}(f_K)$ is *fs* open and *fs* closed in U . Every *fs* open (closed) set is *fsbo* (closed), thus ψ is *fs* totally *b* continuous.

But the converse is not valid.

In example 6.1, $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are *fs* *b* open, and *b* closed sets in U . Therefore ψ is *fs* totally *b*-continuous. Consider a *fsbo* set $(H, K) = \{\{0.2/y_1, 0.3/y_2\}, \{0.3/y_1, 0.4/y_2\}\}$. But $\psi^{-1}(H, K)$ it is neither *fs* open nor *fs* closed in U . Therefore ψ is not perfectly *b* continuous.

Remark 6.1. The concepts of *fs* strongly *b*-continuous function and *fs* totally *b* continuous functions are independent of each other.

In Example 6.1, $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are *fs* *b* open, and *b* closed sets in U . Therefore ψ is *fs* totally *b*-continuous. Consider a *fsbo* set $(H, K) = \{\{0.2/y_1, 0.3/y_2\}, \{0.3/y_1, 0.4/y_2\}\}$ in V , $\psi^{-1}(H, K)$ is not *fs* open in U . Therefore ψ is not strongly *b*-continuous.

Theorem 6.3. Every *fs* totally *b*-continuous function is *fs* *b* continuous.

Proof: Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be *fs* totally *b* continuous function. Then inverse of each *fs* open set is *fsbo*, and *fs* *b* closed in U . So ψ is *fs* *b* continuous. Following example shows that, *fs* *b* continuous function need not be *fs* totally *b* continuous.

Example 6.2. Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be a *fs* identity map. $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $K = \{e_1, e_2\}$,

Let $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K), (F_3, K), (F_4, K), (F_5, K), (F_6, K), (F_7, K)\}$,
 $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$ be *fst*s.

$$\begin{aligned} (F_1, K) &= \left\{ \left\{ \frac{1/2}{x_1}, \frac{1/3}{x_2} \right\}, \left\{ \frac{1/4}{x_1}, \frac{2/3}{x_2} \right\} \right\} & (F_2, K) &= \left\{ \left\{ \frac{1/3}{x_1}, \frac{1/4}{x_2} \right\}, \left\{ \frac{0}{x_1}, \frac{1/6}{x_2} \right\} \right\} \\ (F_3, K) &= \left\{ \left\{ \frac{1/2}{x_1}, \frac{1}{x_2} \right\}, \left\{ \frac{2/3}{x_1}, \frac{1/6}{x_2} \right\} \right\} & (F_4, K) &= \left\{ \left\{ \frac{1/5}{x_1}, \frac{1/3}{x_2} \right\}, \left\{ \frac{1/4}{x_1}, \frac{1/6}{x_2} \right\} \right\} \\ (F_5, K) &= \left\{ \left\{ \frac{1/5}{x_1}, \frac{1/4}{x_2} \right\}, \left\{ \frac{0}{x_1}, \frac{1/6}{x_2} \right\} \right\} & (F_6, K) &= \left\{ \left\{ \frac{1/2}{x_1}, \frac{1}{x_2} \right\}, \left\{ \frac{2/3}{x_1}, \frac{2/3}{x_2} \right\} \right\} \\ (F_7, K) &= \left\{ \left\{ \frac{1/3}{x_1}, \frac{1/3}{x_2} \right\}, \left\{ \frac{1/4}{x_1}, \frac{1/6}{x_2} \right\} \right\} \\ (G_1, K) &= \left\{ \left\{ \frac{1/2}{y_1}, \frac{1/4}{y_2} \right\}, \left\{ \frac{1/5}{y_1}, \frac{0}{y_2} \right\} \right\} & (G_2, K) &= \left\{ \left\{ \frac{1/4}{y_1}, \frac{1/5}{y_2} \right\}, \left\{ \frac{1/6}{y_1}, \frac{0}{y_2} \right\} \right\} \end{aligned}$$

Then $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are *fsbo* sets, but they are not *fs* *b* closed sets in U . Therefore ψ is *fs* *b*-continuous but not *fs* totally *b* continuous.

Theorem 6.4. Every *fs* totally *b*-continuous function is *fs* *gb* continuous.

Proof: Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be *fs* totally *b*-continuous function. Then

inverse of fs open set is fs b -open, and fs b -closed in U . Since the inverse of every $fsbo$ set is fs gb -open, ψ is fs gb -continuous. Example 6.2, give the converse is not true, $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are fs gb open sets but not fs b closed sets in U . Therefore ψ , fs gb continuous but not fs totally b -continuous.

Theorem 6.5. Every fs totally b continuous function is fs semi pre continuous.

Proof: Let $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ be fs totally b - continuous function. Let (G, K) be fs open set in V , then $\psi^{-1}(G, K)$ is fs b open and fs b closed in U and every fs b open set is fs semi pre-open, ψ is fs semi pre continuous. In example 6.2, $\psi^{-1}(G_1, K)$ and $\psi^{-1}(G_2, K)$ are fs semi pre-open sets but not fs b open, and fs b closed sets in U . Therefore ψ , fs semi pre continuous but not fs totally b continuous. Hence converse of this is not true in general.

Theorem 6.6. If $f : (U, \tau, K) \rightarrow (V, \tau', K)$ is fs totally b continuous and $\lambda : (V, \tau', K) \rightarrow (W, \sigma, K)$ is fs b continuous, then $\lambda \circ f : (U, \tau, K) \rightarrow (W, \sigma, K)$ is fs gb continuous.

Proof: Let (H, K) be fs open set in W , since λ is fs b continuous, $\lambda^{-1}(H, K)$ is $fsbo$ in V and f is fs totally b continuous, $(\lambda \circ f)^{-1}(H, K) = f^{-1}(\lambda^{-1}(H, K))$ is fs b open and fs b closed in U . Every $fsbo$ is fs gb open, $(\lambda \circ f)$ is fs gb continuous.

Theorem 6.7. If $f : (U, \tau, K) \rightarrow (V, \tau', K)$ is fs totally b continuous and $\lambda : (V, \tau', K) \rightarrow (W, \sigma, K)$ is fs b continuous, then $\lambda \circ f : (U, \tau, K) \rightarrow (W, \sigma, K)$ is fs semi pre continuous.

Proof: Let (H, K) be fs open set in W , since λ is fs b continuous, $\lambda^{-1}(H, K)$ is $fsbo$ in V and f is fs totally b continuous, $(\lambda \circ f)^{-1}(H, K) = f^{-1}(\lambda^{-1}(H, K))$ is fs b open and fs b closed in U . Since every fs b open set is fs semi pre-open, $(\lambda \circ f)$ is fs semi pre continuous.

Remark 6.2. The concept of fs pre b continuous and totally b continuous functions in $fsts$ are independent of each other.

Example 6.3. Suppose $\psi : (U, \tau, K) \rightarrow (V, \tau', K)$ is defined by $\lambda(x_1) = y_2$ and $\lambda(x_2) = y_1$ and $\tau = \{\tilde{O}, \tilde{1}, (F_1, K), (F_2, K)\}$, $\tau' = \{\tilde{O}, \tilde{1}, (G_1, K), (G_2, K)\}$, be any two $fsts$. Let $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $K = \{e_1, e_2\}$,

$$(F_1, K) = \left\{ \left\{ \frac{1}{x_1}, \frac{0.9}{x_2} \right\}, \left\{ \frac{0.8}{x_1}, \frac{0.8}{x_2} \right\} \right\} \quad (F_2, K) = \left\{ \left\{ \frac{0}{x_1}, \frac{0.1}{x_2} \right\}, \left\{ \frac{0.2}{x_1}, \frac{0.2}{x_2} \right\} \right\}$$

$$(G_1, K) = \left\{ \left\{ \frac{0.7}{y_1}, \frac{0.6}{y_2} \right\}, \left\{ \frac{0.5}{y_1}, \frac{0.6}{y_2} \right\} \right\} \quad (G_2, K) = \left\{ \left\{ \frac{0.3}{y_1}, \frac{0.4}{y_2} \right\}, \left\{ \frac{0.5}{y_1}, \frac{0.4}{y_2} \right\} \right\}$$

Here, $\lambda^{-1}(G_1, K)$ and $\lambda^{-1}(G_2, K)$ are fs b open, and b closed sets in U . Thus λ is fs totally b continuous. Consider an fs b open set

$(H, K) = \{ \{0.2/y_1, 0.3/y_2\}, \{0.3/y_1, 0.4/y_2\} \}$. Since $\lambda^{-1}(H, K)$ it is not fs pre-open in U , λ it is not fs pre b continuous.

Remark 6.3. From the above observations, we have the following :

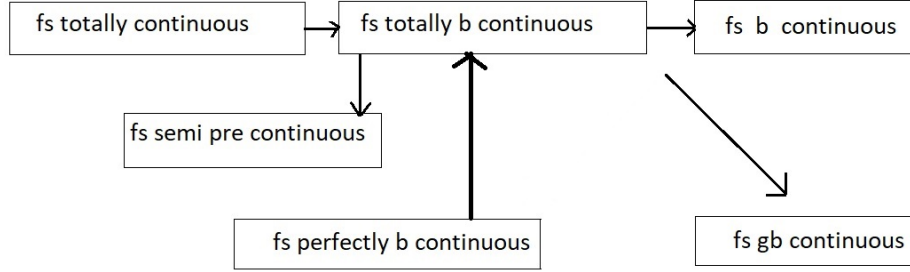


Figure 4:

7 Fuzzy soft b compact spaces

Definition 7.1. Let (U, τ, K) be fsts and $f_K \in fss(U, \tau, K)$ the set of all fs sets. An fs set f_K is called b compact if each fs b open cover of f_K has a finite subcover. Also (U, τ, K) is called fs b compact if each fs open cover of $\widetilde{1}_K$ has a finite subcover.

Remark 7.1. A fsts is fs b compact if U is finite.

Example 7.1. Let (U, τ, K) and (V, σ, K) be two fsts and $\tau \subset \sigma$. Then fsts (U, τ, K) is fs b compact if (V, σ, K) is fs b compact.

Definition 7.2. A fsts (U, τ, K) is called a

- i. strongly compact if and only if every fs pre-open cover of U has a finite subcover.
- ii. semi-compact if and only if every fs semi-open cover of U has a finite subcover.
- iii. semi pre compact if and only if every fs semi pre-open cover of U has a finite subcover.
- iv. S -closed if and only if every fs semi-open cover of U has a finite subcollection whose closures cover U .

Remark 7.2. Each fs semi-open and fs pre-open sets implies fs b open sets. Every fs b compact space means each of fs strongly compact and fs semi-compact spaces. Also, since fs b -open set implies fs semi pre-open set, it is clear that fs semi pre compact space means fs b compact space. The finite intersection property for fs b compact spaces is provided as follows.

Definition 7.3. A family ψ of fs b open sets has the finite intersection property if the intersection of members of each finite subfamily of ψ is not the null fs set.

Theorem 7.1. A fs b compact space U is fs b compact if and only if each family of fs b closed sets with the finite intersection property has a non-null intersection.

Proof: Let ψ be an arbitrary family of fs b closed sets with the finite intersection property. We assume that $\bigcap_{i \in I} \{(f_i, K) : (f_i, K) \in \psi\}$ is non-null, that is

$\bigcap_{i \in I} (f_i, K) = \tilde{O}_K$. Then $(\bigcap_{i \in I} (f_i, K))^c = \bigcup_{i \in I} (f_i, K)^c = \tilde{1}_K$. Since each (f_i, K) is b closed, the family $\{(f_i, K)^c : i \in I\}$ is fs b open cover of U . But U is fs b compact, therefore $\bigcup_{i \in I} (f_i, K) = \tilde{1}_K$. Thus we have $K = (\bigcap_{i \in I} (f_i, K))^c \bigcap_{i \in I} (f_i, K) =$

$\tilde{O}_K$ a contradiction to assumption.

Suppose U is such that each family of fs b closed sets with the finite intersection property has a non-null intersection. Let $\psi = \{(f_i, K) : i \in I\}$ be a family of fs b open sets. Let ψ has a finite subfamily that also covers U . Assume that $\bigcup_{i \in J} (f_i, K) = \tilde{1}_K$ for any finite $J < I$.

Then $\bigcap_{i \in J} (f_i, K)^c = (\bigcup_{i \in J} (f_i, K))^c \neq \tilde{O}_K$, since J is finite. Thus $\{(f_i, K)^c : i \in I\}$ has finite intersection property. By assumption $\bigcap_{i \in I} (f_i, K)^c \neq \tilde{O}_K$, and we have

$\bigcup_{i \in I} (f_i, K) \neq \tilde{1}_K$. This is a contradiction. Thus U is fs b compact.

Theorem 7.2. Let g_K be fs closed set in fs b compact space (U, τ, K) . Then g_K is also fs b compact.

Proof: Let $\psi = \{(h_i, K) : i \in I\}$ be fs b open cover of g_K . Then $\tilde{1}_K \subseteq \{(\bigcup_{i \in I} (h_i, K)) \cup (g, K)^c\}$. Therefore there exists a finite sub covering

$(h_1, K), (h_2, K), (h_3, K), \dots, (g, K)^c$. Hence we get $\tilde{1}_K \subseteq (h_1, K) \cup (h_2, K) \cup (h_3, K) \cup \dots \cup (h_n, K) \cup (g, K)^c$. Therefore $(g, K) \subseteq (h_1, K) \cup (h_2, K) \cup (h_3, K) \cup \dots \cup (h_n, K) \cup (g, K)^c$, which implies $(g, K) \subseteq (h_1, K) \cup (h_2, K) \cup (h_3, K) \cup \dots \cup (h_n, K)$ since $(g, K) \cap (g, K)^c = \tilde{O}_K$. Hence (g, K) has a finite subcover thus g_K is fs b compact.

8 Conclusion

The present work zeroed in on introducing slightly b , semi b , pre b and totally b continuous mappings in fuzzy soft topological spaces. The correlation with the existing fs continuous functions are studied, established and compared. It is proved that every fs is slightly continuous, fs contra b continuous, and fs b continuous function is fs slightly b continuous. In contrast, the composition of fs

slightly b continuous function need not be fs slightly b continuous. In fs $bT_{1/2}$ space fs slightly b continuous function becomes fs slightly continuous. Counter examples have been shown to illustrate and evidence that the reverse implications do not imply either. It is deduced that every fs pre b continuous and fs semi b continuous function is fs b continuous, fs gb continuous, and fs semi pre continuous function. It is also enumerated that the converse is not valid with evidence using a counter example. In addition, it is implicated that fs totally b continuous function is also fs b continuous, fs gb continuous, and fs semi pre continuous with clear inferences that the reverse implication is not true. Further, it is also concluded that fs pre b continuous and fs totally b continuous, fs strongly b continuous and fs totally b continuous functions are independent of each other. A new form of topological space such as fs b compact space is also introduced. The current work forms the basis for further work to be computed, emphasising fuzzy soft topology as the fulcrum of computations.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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Relation-theoretic contraction principle in metric spaces using multiplicative contraction

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Abstract

Alam and Imdad have presented a novel application of the Banach contraction principle on a complete metric spaces with a binary relation. We have extended the concept of binary relation with the multiplicative contraction in a complete metric spaces. We have also included corollary to demonstrate our results.

Keywords: fixed point, metric spaces, binary relation, multiplicative contraction

2020 AMS subject classifications: 47H10, 54H25.¹

1 Introduction

In many scientific domains, particularly in fixed point theory, the concept of a metric space is extremely useful. This notion has been generalised in numerous directions in recent years, and many notions of a metric-type space have been introduced (b-metric, dislocated space, generalised metric space, quasi-metric space, symmetric space, etc.). The Banach contraction principle's [3] contraction condition has been generalised to numerous forms in the last fifty years. Furthermore, the metric space in the Banach contraction principle has been generalised to a variety of generalised metric spaces. Many authors researched other sorts of fixed point theorems in metric spaces later on, as seen by the and references therein.

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2 Preliminaries

We give the required background material needed to prove our results in this part to make our exposition self-contained. In what follows, N , N_0 , Q and R denote the sets of positive integers, non-negative integers, rational numbers and real numbers, respectively.

Aftab and Alam [1] proved new relation-theoretic fixed point theorems on metric spaces in 2015, and then inferred comparable findings in metric spaces.

Metric spaces are sets in which there is a defined a notion of 'distance between pair of points'. The concept of metric spaces was formulated in 1906 by M.Frechet [7], though the definition presently in use given by the German mathematician, Felix Hausdorff.

Definition 2.1. Let M be a non empty arbitrary set and d be a real function from $M \times M$ into R^+ such that for all $u, v, w \in M$ we have

1. $d(u, v) \geq 0$,
2. $d(u, v) = 0 \iff u = v$,
3. $d(u, v) = d(v, u)$ and
4. $d(u, w) \leq d(u, v) + d(v, w)$,

Here (M, d) is called a metric in R and (R, d) is a metric space.

Example 2.1. 1. $d(u, v) = |u - v|$ is a metric space in R .

2. If $d(u, v)$ defined by

$$d(u, v) = \begin{cases} 1 & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

Definition 2.2. [10] Let M be a nonempty set. A subset R of M^2 is called a binary relation on M . Notice that for each pair $u, v \in M$, one of the following conditions holds:

1. $(u, v) \in R$; which amounts to saying that "u is R-related to v" or "u relates to v under R". Sometimes, we write uRv instead of $(u, v) \in R$;
2. $(u, v) \notin R$; which means that "u is not R-related to v" or "u does not relate to v under R".

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Definition 2.3. [10] Let R be a binary relation defined on a nonempty set M and $u, v \in M$. We say that u and v are R -comparative if either $(u, v) \in R$ or $(v, u) \in R$. We denote it by $[u, v] \in R$.

Definition 2.4. [6, 10, 11, 12, 15] A binary relation R defined on a nonempty set M is called

1. reflexive if $(u, u) \in R$ for all $u \in M$,
2. irreflexive if $(u, u) \notin R$ for all $u \in M$,
3. symmetric if $(u, v) \in R$ implies $(v, u) \in R$,
4. antisymmetric if $(u, v) \in R$ implies $(v, u) \notin R$,
5. transitive if $(u, w) \in R$ and $(w, v) \in R$ implies $(u, v) \in R$,
6. complete, connected or dichotomous if $[l, n] \in R$ for all $l, n \in M$,
7. weakly complete, weakly connected or trichotomous if $[u, v] \in R$ or $u = v$ for all $u, v \in M$.
8. strict order or sharp order if R is irreflexive and transitive,
9. near-order if R is antisymmetric and transitive,
10. pseudo-order if R is reflexive and antisymmetric,
11. quasi-order or preorder if R is reflexive and transitive,
12. partial order if R is reflexive, antisymmetric and transitive,
13. simple order if R is weakly complete strict order,
14. weak order if R is complete preorder,
15. total order, linear order or chain if R is complete partial order,
16. tolerance if R is reflexive and symmetric,
17. equivalence if R is reflexive, symmetric and transitive.

Definition 2.5. [4] Let M be a nonempty set and R a binary relation on M . A sequence $\{u_n\} \subset M$ is called R -preserving if

$$(u_n, u_{n+1}) \in R \quad \forall n \in N_0$$

The notion of d - self closeness of a partial order $\preceq$ defined by Turinici [16] is extended to an arbitrary binary relation in the following lines.

Now, we state and prove our main result, which is as follows:

Theorem 2.1. *Let (M, d) be a complete metric space, R a binary relation on M and T a self-mapping on M . Suppose that the following conditions hold: a) $M(f; R)$ is nonempty, b) R is f -closed, c) either f is continuous or R is p -self-closed, d) there exists $\lambda \in [0, 1)$ $d(f(u), f(v)) \leq d(u, v)^\lambda$ for all $u, v \in M$ with $(u, v) \in R$. Then f has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$, then f has a unique fixed point.*

Proof. Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, \dots$. From the multiplicative contraction property [13] of f for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition (b), we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^nu_0, f^{n+1}u_0), \dots \in R$$

so that

$$(u_n, u_{n+1}) \in R \quad n \in N_0. \quad (1)$$

Thus the sequence $\{u_n\}$ is R -preserving. Applying the contractivity condition (d) to equation (1), we deduce, for all $n \in N_0$, that

$$d(u_{n+1}, u_n) \leq d(u_n, u_{n-1})^\lambda \leq d(u_{n-1}, u_{n-2})^{\lambda^2} \leq \dots \leq d(u_1, u_0)^{\lambda^n}.$$

which by induction yields that

$$d(u_{n+1}, u_{n+2}) \leq d(u_0, fu_0)^{\lambda^{n+1}} \quad n \in N_0. \quad (2)$$

Using equation (2) and triangular inequality, for all $n \in N_0, p \in N, p \geq 2$, we have

$$\begin{aligned} d(u_{n+1}, u_{n+p}) &\leq d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_{n+3}) + \dots + d(u_{n+p-1}, u_{n+p}) \\ &\leq d(u_1, u_0)^{\lambda^{n+1} + \dots + \lambda^p} \\ &\leq d(u_1, u_0)^{\frac{\lambda^p}{1-\lambda}} \end{aligned}$$

This implies that $d(u_n, u_p) \rightarrow 0$ as $(n, p \rightarrow \infty)$. Hence the sequence $(x_n) = (f^n u_0)$ is multiplicative Cauchy. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Moreover,

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$$d(fz, z) \leq d(fu_n, fz) \cdot d(fu_n, z) \leq d(u_n, z)^\lambda \cdot d(fu_n, z) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

which implies $d(fz, z) = 0$. Therefore this says that z is a fixed point of f ; that is $fz = z$.

Now, if there is another point y such that $fy = y$, then

$$d(z, y) = d(fz, fy) \leq d(z, y)^\lambda.$$

Therefore $d(z, y) = 0$ and $y = z$. This implies that z is the unique fixed point of f .

Alternatively, let us assume that R is d -self-closed. As u_n is an R -preserving sequence and

$$u_n \rightarrow^d u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d), $[u_{n_k}, u] \in R$ and $u_n \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq d(u_{n_k}, u)^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \rightarrow^d f(u)$. Again, owing to the uniqueness of limit, we get $f(u) = u$ so that u is a fixed point of f . To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (3)$$

By assumption (d), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{foreach } i(0 \leq i \leq k-1). \quad (4)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{foreach } i(0 \leq i \leq k-1) \quad \text{and} \quad \text{for each } n \in N_0. \quad (5)$$

Making use of equations (3),(4),(5),, triangular inequality and assumption (d), we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{k-1}^{i=0} d(f^{n-1} z_i, f^{n-1} z_{i+1})^\lambda \\ &\leq \sum_{k-1}^{i=0} d(f^{n-2} z_i, f^{n-2} z_{i+1})^{\lambda^2} \end{aligned}$$

$$\leq \dots \leq \sum_{k=1}^{i=0} d(z_i, z_{i+1})^{\lambda^n} \\ \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $u = v$. Hence f has a unique fixed point. $\square$

Corolary 2.1. Let (M, d) be a complete metric space. For ϵ with $\epsilon > 1$ and $u_0 \in M$, consider the multiplicative closed ball, $\overline{B}_\epsilon(u_0)$. Suppose the mapping $f : M \rightarrow M$ satisfies the contraction condition

$$d(f(u), f(v)) \leq d(u, v)^\lambda \quad \text{for all } u, v \in \overline{B}_\epsilon(u_0)$$

where $\lambda \in [0, 1)$ is a constant R is a relation such that $d(fu_0, u_0) \leq \epsilon^{1-\lambda}$. Then f has a unique fixed point in $\overline{B}_\epsilon(u_0)$.

Corolary 2.2. Let (M, d) be a complete metric space. If a mapping $f : M \rightarrow M$ satisfies for some positive integer n ,

$$d(f^n u, f^n v) \leq d(u, v)^\lambda \quad \text{for all } u, v \in M,$$

where $\lambda \in [0, 1)$ is a constant, then f has a unique fixed point in M .

Theorem 2.2. Let (M, d) be a complete metric space, R a binary relation on M and T a self-mapping on M . Suppose that the following conditions hold: a) $M(f; R)$ is nonempty,

b) R is f -closed,

c) either f is continuous or R is p -self-closed,

d) there exists $\lambda \in [0, \frac{1}{2})$ $d(fu, fv) \leq (d(f(u, v).d(fv, u)))^\lambda$ for all $u, v \in M$ with $(u, v) \in R$

Then f has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$,

then f has a unique fixed point.

Proof. Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, \dots$ From the multiplicative contraction property of f for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition 2, we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^nu_0, f^{n+1}u_0), \dots \in R$$

we have

$$d(u_{n+1}, u_n) = d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n).d(fu_{n-1}, u_{n-1}))^\lambda \\ = (d(u_{n+1}, u_n).d(u_n, u_{n-1}))^\lambda$$

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Thus we have

$$d(u_{n+1}, u_n) \leq (d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2}) \dots d(u_{m+1}, u_m) \\ &\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+\dots+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}} \end{aligned}$$

This implies $d(u_n, u_m) \rightarrow 1$ as $(n, m \rightarrow \infty)$. Hence (u_n) is a Cauchy sequence. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Since

$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz).d(fu_n, z) \\ &\leq (d(fu_n, u_n).d(fz, z))^\lambda.d(u_{n+1}, z), \end{aligned}$$

we have

$$d(u_{n+1}, u_n) \leq (d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2}) \dots d(u_{m+1}, u_m) \\ &\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+\dots+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}} \end{aligned}$$

This implies $d(u_n, u_m) \rightarrow 1$ as $(n, m \rightarrow \infty)$. Hence (u_n) is a Cauchy sequence. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Since

$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz).d(fu_n, z) \\ &\leq (d(fu_n, u_n).d(fz, z))^\lambda.d(u_{n+1}, z), \end{aligned}$$

we have

$$d(fz, z) \leq (d(fu_n, u_n)^\lambda.d(u_{n+1}, z))^{\frac{1}{1-\lambda}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence $d(fz, z) = 0$. This implies $fz = z$. Finally, it is easy to prove that the fixed point of f is unique.

Alternatively, let us assume that R is d -self-closed. As $\{u_n\}$ is an R -preserving sequence and

$$u_n \xrightarrow{d} u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d), $[u_{n_k}, u] \in R$ and $u_{n_k} \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \rightarrow^d f(u)$. Again, owing to the uniqueness of limit, we obtain $f(u) = u$, so that u is a fixed point of f .

To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (6)$$

By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{for each } i(0 \leq i \leq k-1). \quad (7)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{for each } i(0 \leq i \leq k-1) \quad \text{and for each } n \in N_0. \quad (8)$$

Making use of equations (6),(7),(8),, triangular inequality and assumption (d) we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^\lambda \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2} \\ &\leq \dots \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

so that $u = v$. Hence f has a unique fixed point. $\square$

Theorem 2.3. Let (M, d) be a complete metric space, R a binary relation on M and T a self-mapping on M . Suppose that the following conditions hold: a) $M(f; R)$ is nonempty,

b) R is f -closed,

c) either f is continuous or R is p -self-closed,

d) there exists $\lambda \in [0, \frac{1}{2})$ $d(fu, fv) \leq (d(f(u, u).d(fv, v)))^\lambda$ for all $u, v \in M$ with $(u, v) \in R$

Then f has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$,

then f has a unique fixed point.

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Proof. Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, \dots$. From the multiplicative contraction property of f for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition 2, we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^n u_0, f^{n+1}u_0), \dots \in R$$

we have

$$\begin{aligned} d(u_{n+1}, u_n) &= d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n).d(fu_{n-1}, u_{n-1}))^\lambda \\ &= (d(u_{n+1}, u_n).d(u_n, u_{n-1}))^\lambda \end{aligned}$$

Thus we have

$$d(u_{n+1}, u_n) \leq d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2}) \dots d(u_{m+1}, u_m) \\ &\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+\dots+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}} \end{aligned}$$

This implies $d(u_n, u_m) \rightarrow 1$ as $(n, m \rightarrow \infty)$. Hence (u_n) is a Cauchy sequence. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Since

$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz).d(fu_n, z) \\ &\leq (d(fu_n, u_n).d(fz, z))^\lambda.d(u_{n+1}, z), \end{aligned}$$

we have

$$d(fz, z) \leq (d(fu_n, u_n)^\lambda.d(u_{n+1}, z))^{\frac{1}{1-\lambda}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence $d(fz, z) = 0$. This implies $fz = z$. Finally, it is easy to prove that the fixed point of f is unique.

Alternatively, let us assume that R is d -self-closed. As $\{u_n\}$ is an R -preserving sequence and

$$u_n \rightarrow^d u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d) , $[u_{n_k}, u] \in R$ and $u_{n_k} \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \xrightarrow{d} f(u)$. Again, owing to the uniqueness of limit, we obtain $f(u) = u$, so that u is a fixed point of f .

To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (9)$$

By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{foreach} \quad i(0 \leq i \leq k-1). \quad (10)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{foreach} \quad i(0 \leq i \leq k-1) \quad \text{and} \quad \text{for each } n \in N_0. \quad (11)$$

Making use of equations (9),(10),(11), triangular inequality and assumption (d), we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^\lambda \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2} \\ &\leq \dots \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

so that $u = v$. Hence f has a unique fixed point. $\square$

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Some topological properties of revised fuzzy cone metric spaces

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Abstract

In this paper, we introduced Revised fuzzy cone Metric space with its topological properties. Likewise A necessary and sufficient condition for a Revised fuzzy cone metric space to be precompact is given. We additionally show that each distinct Revised fuzzy cone metric space is second countable and that a subspace of a separable Revised fuzzy cone metric space is separable.

Keywords: Revised fuzzy metric space; Revised fuzzy cone metric space; separable; second countable.

2020 AMS subject classifications: 54A40, 54E35, 54E15, 54H25.¹

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1 Introduction

After Zadeh [1965] introduced the idea of fuzzy sets, several authors have introduced and studied many notions of metric indistinctness [Huang and Zhang, 2007, Kramosil and Michalek, 1975, Ahmad and Mesiarová-Zemánková, 2007, Navara, 2007, Muraliraj and Thangathamizh, 2021a,b,d, Oner and Tanay, 2015, Grigorenko et al., 2020] and metric cone indistinctness. By modifying the idea of metric indistinctness introduced by George and Veeramani [1994], Zadeh [1965] studied the notion of fuzzy cone metric areas. Especially, they evidenced that every fuzzy cone topological space generates a Hausdorff first-countable topology. Here we tend to study additional topological properties of these areas whose fuzzy metric version are usually found in George and Veeramani [1994], Ghareeb and Al-Omeri [2018], Grabiec [1988], Gregori et al. [2011].

Sostak [2018] additionally represented the idea of “George–Veeramani Fuzzy Metrics Revised”. Presently Olga Grigorenko, Juan Jose Minana, Alexander Sostak, Muraliraj and Thangathamizh [2021c] have introduced “On t -conorm primarily based Fuzzy (Pseudo) metrics”. Recently Muraliraj and Thangathamizh [2021a,c,d] proved various fixed point theorems in revised fuzzy metric spaces. Muraliraj and Thangathamizh [2021b] introduce the concept of Revised fuzzy modular metric space. Moreover, we tend to prove that a Revised fuzzy cone topological space is precompact if and providing each sequence in it’s a Cauchy subsequence. Further, we tend to show that $X_1 \times X_2$ may be a complete Revised fuzzy cone topological space if and providing X_1 and X_2 are complete Revised fuzzy cone metric areas. Finally it’s tried that each divisible Revised fuzzy cone topological space is second calculable and a mathematical space of a separable Revised fuzzy cone topological space is separable.

2 Preliminaries

Definition 2.1 (Gregori et al. [2011]). *Let E be a real Banach space, θ the zero of E and P a subset of E . Then P is called a cone if and only if*

- (i) P is closed, nonempty, and $P \neq \{\theta\}$,
- (ii) if $ab \in R, ab \geq 0$ and $xy \in P$, then $ax + by \in P$,
- (iii) if both $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone P , a partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if only if $y - x \in P$. The notation $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(P)$. Throughout this paper, we assume that all the cones have nonempty interiors.

There are two kinds of cones: normal and non-normal ones. A cone P is called normal if there exists a constant $K \geq 1$ such that for all $t, s \in E$, $\theta \lesssim t \lesssim s$ implies $\|t\| \leq K \|s\|$, and the least positive number K having this property is called normal constant of P Gregori et al. [2011]. It is clear that $K \geq 1$.

Definition 2.2 (Sostak [2018]). A binary operation $\oplus : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t -conorm if it satisfies the following conditions:

- (i) $\oplus$ is associative and commutative,
- (ii) $\oplus$ is continuous,
- (iii) $a \oplus 0 = a$ for all $a \in [0, 1]$,
- (iv) $a \oplus b \leq c \oplus d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

2.1 Examples Sostak and Öner [2020]

- (i) Lukasiewicz t -conorm: $a \oplus b = \max\{a, b\}$;
- (ii) Product t -conorm: $a \oplus b = a + b - ab$;
- (iii) Minimum t -conorm: $a \oplus b = \min(a + b, 1)$.

Definition 2.3 (Öner and Tanay [2015]). A 3-tuple $(U, M_0, *)$ is called a fuzzy cone metric space (FCM space) if C is a cone of E , U is an arbitrary set, $(*)$ is a continuous t -norm, and M_0 is a fuzzy set on $U^2 \times \text{int}(P)$ satisfying the following conditions:

- (i) $M_0(\lambda_1, \lambda_2, t) > 0$ and $M_0(\lambda_1, \lambda_2, t) = 1 \Leftrightarrow \lambda_1 = \lambda_2$,
- (ii) $M_0(\lambda_1, \lambda_2, t) = M_0(\lambda_2, \lambda_1, t)$
- (iii) $M_0(\lambda_1, \lambda_2, t) * M_0(\lambda_2, \lambda_3, s) \leq M_0(\lambda_1, \lambda_3, t + s)$,
- (iv) $M_0(\lambda_1, \lambda_2,) : \text{int}(P) \rightarrow [0, 1]$ is continuous $\forall \lambda_1, \lambda_2, \lambda_3 \in U$ and $t, s \in \text{int}(p)$.

Definition 2.4 (Sostak [2018]). A Revised fuzzy metric space is an ordered triple $(X, \mu, \oplus)$ such that X is a non empty set, $\oplus$ is a continuous t -conorm and μ is a Revised fuzzy set on $\mu : X^2 \times R^+ \rightarrow [0, 1]$ satisfies the following conditions:

- (RFM1) $\mu(x, y, t) < 1$;
- (RFM2) $\mu(x, y, t) = 0$ if and only if $x = y$;
- (RFM3) $\mu(x, y, t) = \mu(y, x, t)$;
- (RFM4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$;
- (RFM5) $\mu(x, y, -) : (0, \infty) \rightarrow [0, 1]$ is continuous $\forall x, y, z \in X$ and $t, s \in R^+$.
Then μ is called a Revised fuzzy metric on X .

3 Main Results

Definition 3.1. A Revised fuzzy cone metric space is an 3-triple $(X, \mu, \oplus)$ such that P is a cone of E , X is a non empty set, $\oplus$ is a continuous t -conorm and μ is a Revised fuzzy set on $X^2 \times \text{int}(P)$ satisfies the following conditions, $\forall x, y, z \in X$ and $s, t \in \text{int}(P)$ (that is $s \gg \theta, t \gg \theta$),

(RFCM 1) $\mu(x, y, t) < 1$,

(RFCM 2) $\mu(x, y, t) = 0$ if and only if $x = y$,

(RFCM 3) $\mu(x, y, t) = \mu(y, x, t)$,

(RFCM 4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$,

(RFCM 5) $\mu(x, y, -) : \text{int}(P) \rightarrow [0, 1]$ is continuous.

Then μ is called a Revised fuzzy cone metric on X .

If $(X, \mu, \oplus)$ is a Revised fuzzy cone metric space, we will say that μ is a Revised fuzzy cone metric on X .

Every revised fuzzy cone metric space $(X, \mu, \oplus)$ induces a Hausdorff first-countable topology τ_{fc} on X which has as a base the family of sets of the form $\{B(x, r, t) : x \in X; 0 < r < 1, t \gg \theta\}$,

where $\{B(x, r, t) : y \in X; \mu(x, y, t) < r\}$ for every r with $0 < r < 1$ and $t \gg \theta$.

A Revised fuzzy cone metric space $(X, \mu, \oplus)$ is called complete if every Cauchy sequence in it is convergent, where a sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $\varepsilon \in (0, 1)$ and any $t \gg \theta$ there exists a natural number n_0 such that $\mu(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$, and a sequence $\{x_n\}$ is said to converge to x if for any $t \gg \theta$ and any $r \in (0, 1)$ there exists a natural number n_0 such that $\mu(x_n, x, t) < r$ for all $n \geq n_0$. A sequence $\{x_n\}$ converges to x if and only if $\mu(x_n, x, t) \rightarrow 0$ for each $t \gg \theta$.

Definition 3.2. Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. For $t \gg \theta$, the closed ball $B[x, r, t]$ with center x and radius $r \in (0, 1)$ is defined by

$$B[x, r, t] = \{y \in X; \mu(x, y, t) < r\}.$$

Lemma 3.3. Every closed ball in a Revised fuzzy cone metric space $(X, \mu, \oplus)$ is a closed set.

Proof. Let $y \in \overline{B[x, r, t]}$. Since X is first countable, there exists a sequence $\{y_n\}$ in $B[x, r, t]$ converging to y . Therefore $\mu(y_n, y, t)$ converges to 0 for all $t \gg \theta$. For a given $\epsilon \gg 0$, we have, $\mu(x, y, t + \epsilon) \leq \mu(x, y_n, t) \oplus \mu(y_n, y, \epsilon)$

Hence, $\mu(x, y, t + \epsilon) \leq \mu(x, y_n, t) \oplus \mu(y_n, y, \epsilon) \leq 0 \oplus 0 = 0$.

(If $\mu(x, y_n, t)$ is bounded, then the sequence y_n has a subsequence, which we again denote by y_n , for which $\mu(x, y_n, t)$ exists.) In particular for $n \in N$, take $\epsilon = \frac{t}{n}$. Then, $\mu(x, y, t + \frac{t}{n}) < r$. Hence, $\mu(xyt) \leq \mu(x, y, t + \frac{t}{n}) < r$. Thus $y \in B[x, r, t]$. Therefore $B[x, r, t]$ is a closed set. $\square$

Definition 3.4. A Revised fuzzy cone metric space $(X, \mu, \oplus)$ is called precompact if for each r , with $0 < r < 1$, and each $t \gg \theta$, there is a finite subset A of X , such that $X = \bigcup_{a \in A} B(a, r, t)$. In this case, we say that μ is a precompact Revised fuzzy cone metric on X .

Lemma 3.5. A Revised fuzzy cone metric space is precompact if and only if every sequence has a Cauchy subsequence.

Proof. Suppose that $(X, \mu, \oplus)$ is a precompact Revised fuzzy cone metric space. Let x_n be a sequence in X . For each $m \in N$ there is a finite subset A_m of X such that $X = \bigcup_{a \in A_m} B(a, \frac{a}{m}, \frac{t_0}{m\|t_0\|})$ where $t_0, t \gg \theta$ is a constant. Hence, for $m = 1$, there exists an $a_1 \in A_1$ and a subsequence $x_{1(n)}$ of x_n such that $x_{1(n)} \in B(a_1, 1, \frac{t_0}{m\|t_0\|})$ for every $n \in N$. Similarly, there exist an $a_2 \in A_2$ and a subsequence $\{x_{2(n)}\}$ of $x_{1(n)}$ such that $x_{2(n)} \in B(a_2, \frac{1}{2}, \frac{t_0}{m\|t_0\|})$ for every $n \in N$. By continuing this process, we get that for $m \in N$, $m > 1$, there is an $a_m \in A_m$ and a subsequence $\{x_{m(n)}\}$ of $x_{m-1(n)}$ such that $x_{m(n)} \in B(a_m, \frac{1}{m}, \frac{t_0}{m\|t_0\|})$ for every $n \in N$. Now, consider the subsequence $x_{n(n)}$ of x_n . Given r with $0 < r < 1$ and $t \gg \theta$ there is an $n_0 \in N$ such that $\frac{1}{n_0} \oplus \frac{1}{n_0} < r$ and $\frac{2t_0}{n_0\|t_0\|} \ll t$. Then, for every $km \geq n_0$, we have

$$\begin{aligned} \mu(x_{k(n)}, x_{m(n)}, t) &\leq \mu\left(x_{k(n)}, x_{m(n)}, \frac{t_0}{n_0\|t_0\|}\right) \\ &\leq \mu\left(x_{k(n)}, a_{n_0}, \frac{t_0}{n_0\|t_0\|}\right) \oplus \mu\left(a_{n_0}, x_{m(n)}, \frac{t_0}{n_0\|t_0\|}\right) \\ &\leq \frac{1}{n_0} \oplus \frac{1}{n_0} < r \end{aligned}$$

Hence $(x_{n(n)})$ is a Cauchy sequence in $(X, \mu, \oplus)$.

Conversely, suppose that $(X, \mu, \oplus)$ is a nonprecompact Revised fuzzy cone metric space. Then there exist an r with $0 < r < 1$ and $t \gg \theta$ such that for each finite subset A of X , we have $X \neq \bigcup_{a \in A} B(a, r, t)$ fix $x_1 \in X$. There is $x_2 \in X - B(x_1, r, t)$. Moreover, there is an $x_3 \in X - \bigcup_{k=1}^2 B(x_k, r, t)$. By continuing this process, we construct a sequence x_n of distinct points in X such that $x_{n+1} \notin \bigcup_{k=1}^n B(x_k, r, t)$ for every $n \in N$. Therefore x_n has no Cauchy subsequence. This completes the proof. $\square$

Lemma 3.6. *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. If a Cauchy sequence clusters around a point $x \in X$, then the sequence converges to x .*

Proof. Let x_n be a Cauchy sequence in $(X, \mu, \oplus)$ having a cluster point $x \in X$. Then, there is a subsequence $\{x_{k(n)}\}$ of x_n that converges to x with respect to τf_c . Thus, given r with $0 < r < 1$ and $t \gg \theta$, there is an $n_0 \in N$ such that for each $n \geq n_0$, $\mu(x, x_{k(n)}, \frac{t}{2}) < s$ where $s > 0$ satisfies $s \oplus s < r$. On the other hand, there is $n_0 \geq k(n_0)$ such that for each $nm \geq n_1$, we have $\mu(x_n, x_m, \frac{t}{2}) < s$. Therefore, for each $n \geq n_1$, we have

$$\mu(x, x_n, t) \leq \mu(x, x_{k(n)}, \frac{t}{2}) \oplus \mu(x_{k(n)}, x, \frac{t}{2}) \leq s \oplus s < r.$$

We conclude that the Cauchy sequence x_n converges to x . $\square$

Proposition 3.7. *Let $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ be Revised fuzzy cone metric spaces. For $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, let $\mu((x_1, x_2), (y_1, y_2), t) = \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t)$. Then μ is a Revised fuzzy cone metric on $X_1 \times X_2$.*

Proof. **RFCM 1:** Since $\mu_1(x_1, y_1, t) < 1$ and $\mu_2(x_2, y_2, t) < 1$, this implies that $\mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t) < 1$. Therefore, $\mu((x_1, x_2), (y_1, y_2), t) < 1$.

RFCM 2: Suppose that for all $t \gg \theta$, $(x_1, y_1, t) = (x_2, y_2, t)$. This implies that $x_1 = y_1$ and $x_2 = y_2$ for all $t \gg \theta$. Hence, $\mu_1(x_1, y_1, t) = 0$ and $\mu_2(x_2, y_2, t) = 0$.

It follows that, $\mu((x_1, x_2), (y_1, y_2), t) = 0$.

Conversely, suppose that $\mu((x_1, x_2), (y_1, y_2), t) = 0$.

This implies that $\mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t) = 0$.

Since, $0 < \mu_1(x_1, y_1, t) < 1$ and $0 < \mu_2(x_2, y_2, t) < 1$. It follows that, $\mu_1(x_1, y_1, t) = 0$ and $\mu_2(x_2, y_2, t) = 0$. Thus $x_1 = y_1$ and $x_2 = y_2$. Therefore $(x_1, x_2) = (y_1, y_2)$.

RFCM 3: To prove that $\mu((x_1, x_2), (y_1, y_2), t) = \mu((y_1, y_2), (x_1, x_2), t)$ we observe that $\mu_1(x_1, y_1, t) = \mu_1(y_1, x_1, t)$ and $\mu_2(x_2, y_2, t) = \mu_2(y_2, x_2, t)$. It follows that for all $(x_1, x_2)(y_1, y_2) \in X_1 \times X_2$ and $t \gg \theta$,

$$\mu((x_1, x_2), (y_1, y_2), t) = \mu((y_1, y_2), (x_1, x_2), t)$$

RFCM 4: Since $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ are Revised fuzzy cone metric spaces, we have that, $\mu_1(x_1, z_1, t + s) \leq \mu_1(x_1, y_1, t) \oplus \mu_1(y_1, z_1, s)$ and

$\mu_2(x_2, z_2, t + s) \leq \mu_2(x_2, y_2, t) \oplus \mu_2(y_2, z_2, s)$, for all

$(x_1, x_2)(y_1, y_2)(z_1, z_2) \in X_1 \times X_2$ and $t, s \gg \theta$. Therefore,

$$\begin{aligned} & \mu((x_1, x_2), (z_1, z_2), t + s) \\ &= \mu_1(x_1, z_1, t + s) \oplus \mu_2(x_2, z_2, t + s) \\ &\leq \mu_1(x_1, y_1, t) \oplus \mu_1(y_1, z_1, s) \oplus \mu_2(x_2, y_2, t) \oplus \mu_2(y_2, z_2, s) \\ &\leq \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t) \oplus \mu_1(y_1, z_1, s) \oplus \mu_2(y_2, z_2, s) \\ &\leq \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t) \oplus \mu_1(y_1, z_1, s) \oplus \mu_2(y_2, z_2, s) \\ &\leq \mu((x_1, x_2), (y_1, y_2), t) \oplus \mu((y_1, y_2), (z_1, z_2), t) \end{aligned}$$

RFCM 5: Note that $\mu_1(x_1, y_1, t)$ and $\mu_2(x_2, y_2, t)$ are continuous with respect to t and $\oplus$ is continuous too. It follows that,

$\mu((x_1, x_2), (y_1, y_2), t) = \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t)$ is also continuous. $\square$

Proposition 3.8. *Let $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ be Revised fuzzy cone metric spaces. We define, $\mu((x_1, x_2), (y_1, y_2), t) = \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t)$. Then μ is a complete Revised fuzzy cone metric on $X_1 \times X_2$ if and only if $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ are complete.*

Corollary 3.9. *Every separable Revised fuzzy cone metric space is second countable.*

Proof. Let $(X, \mu, \oplus)$ be the given separable Revised fuzzy cone metric space. Let $A = \{a_n : n \in N\}$ be a countable dense subset of X . Consider

$$B = \left\{ \left(a_j, \frac{1}{k}, \frac{t_1}{k \|t_1\|} \right) : j, k \in N \right\}$$

where $t_1 \gg \theta$ is constant. Then B is countable. We claim that B is a base for the family of all open sets in X . Let G be an open set in X . Let $x \in G$ then there exists r with $0 < r < 1$ and $t \gg \theta$ such that $B(x, rt) \subset G$.

Since $r \in (0, 1)$, we can find an $s \in (0, 1)$ such that $s \oplus s < r$. Choose $m \in N$ such that $\frac{1}{m} < s$ and $\frac{t_1}{m \|t_1\|} \ll \frac{t}{2}$. Since A is dense in X , there exists an $a_j \in A$ such that $a_j \in B\left(x, \frac{1}{m}, \frac{t_1}{m \|t_1\|}\right)$. Now if $y \in B\left(a_j, \frac{1}{m}, \frac{t_1}{m \|t_1\|}\right)$, then

$$\begin{aligned} \mu(x, y, t) &\leq \mu\left(x, a_j, \frac{t}{2}\right) \oplus \mu\left(y, a_j, \frac{t}{2}\right) \\ &\leq \mu\left(x, a_j, \frac{t_1}{k \|t_1\|}\right) \oplus \mu\left(x, a_j, \frac{t_1}{k \|t_1\|}\right) \\ &\leq \frac{1}{m} \oplus \frac{1}{m} \\ &\leq s \oplus s < r < r. \end{aligned}$$

Thus $y \in B(x, r, t)$ and hence B is a basis. $\square$

Proposition 3.10. *A subspace of a separable Revised fuzzy cone metric space is separable.*

Proof. Let X be a separable Revised fuzzy cone metric space and Y a subspace of X . Let $A = \{x_n : n \in N\}$ be a countable dense subset of X . For arbitrary but fixed $nk \in N$, if there are points $x \in X$ such that $\mu\left(x_n, x, \frac{t_1}{k \|t_1\|}\right) < \frac{1}{k}$ where $t_1 \gg \theta$ is constant, choose one of them and denote it by x_{n_k} .

Let $B = \{x_{n_k} : n, k \in N\}$ then B is countable. Now we claim that $Y \subset \bar{B}$. Let $x \in Y$. Given r with $0 < r < 1$ and $t \gg \theta$ we can find $k \in N$ such that $\frac{1}{k} \oplus \frac{1}{k} < r$ and $\frac{t_1}{k\|t_1\|} \ll \frac{t}{2}$.

Since A is dense in X , there exists an $m \in N$ such that $\mu\left(x_m, y, \frac{t_1}{k\|t_1\|}\right) < \frac{1}{k}$. But by definition of B , there exists an x_{m_k} such that $\mu\left(x_{m_k}, x_m, \frac{t_1}{k\|t_1\|}\right) < \frac{1}{k}$. Now

$$\begin{aligned} \mu\left(x_{m_k}, y, \frac{t_1}{k\|t_1\|}\right) &\leq \mu\left(x_{m_k}, x_m, \frac{t}{2}\right) \oplus \mu\left(x_m, y, \frac{t}{2}\right) \\ &\leq \mu\left(x_{m_k}, x_m, \frac{t_1}{k\|t_1\|}\right) \oplus \mu\left(x_m, y, \frac{t_1}{k\|t_1\|}\right) \\ &\leq \frac{1}{k} \oplus \frac{1}{k} < r. \end{aligned}$$

Thus $Y \subset \bar{B}$ and hence Y is separable. $\square$

Corollary 3.11. *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. Then $(X, \tau f_c)$ is Hausdorff.*

Corollary 3.12. *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. Define $\tau f_c = A \subset X : x \in A$ if and only if there exist $r \in (0, 1)$, and $t \gg \theta$ such that $B(x, r, tt) \subset A$, then τf_c is a topology on X .*

Corollary 3.13. *In a Revised fuzzy cone metric space, every compact set is closed and RFC-bounded.*

4 Conclusion

In this paper we proved a necessary and sufficient condition for a revised fuzzy cone metric space to be precompact. We also show that every separable revised fuzzy cone metric space is second countable and that a subspace of a separable revised fuzzy cone metric space is separable.

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Moduli of continuity of functions in Hölder's class $H_{\omega_k}^{\alpha,2}[0, 1)$ by first kind Chebyshev wavelets and its applications in the solution of Lane-Emden differential equations

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Abstract

In this paper, two new moduli of continuity $W((f - S_{2^{k-1},0}f), \frac{1}{2^k})$, $W((f - S_{2^{k-1},M}f), \frac{1}{2^k})$ and two estimators $E_{2^{k-1},0}(f)$ and $E_{2^{k-1},M}(f)$ of a functions f in Hölder's class $H_{\omega_k}^{\alpha,2}[0, 1)$ by First kind Chebyshev wavelets have been determined. These moduli of continuity and estimators are new and best possible in wavelet analysis. Applying this technique, Lane -Emden differential equations have been solved by the first of kind Chebyshev wavelet method. These solutions obtained by first kind Chebyshev wavelet method approximately coincided with their exact solutions. This is a significant achievement of this research paper in wavelet analysis.

Keywords: Chebyshev wavelet, Modulus of continuity, Wavelet approximation, Hölder's class, Orthonormal basis, Operational matrix of integration .

2020 AMS subject classifications: 42C40, 65T60, 65L10 , 65L60. ¹

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1 Introduction

Recently, researchers are making attention on wavelets. Wavelets have connections between several branches of mathematical sciences and play an important role in signal processing, engineering and technology. The approximation of functions of a certain class by trigonometric series is a common places of analysis. Approximation of functions belonging to some class by wavelet method has been discussed by many researchers like DeVore[1], Morlet[4] , Meyer[3] and Debnath[2]. Wavelets are new tools to solve differential equations and to estimate the moduli of continuity & the approximation of functions. Wavelets help in the most accurate representation of functions $f \in H_{\omega_k}^{\alpha,2}[0, 1)$ class. Several wavelets are known like that Haar wavelet , Legendre wavelet , Chebyshev wavelet. Haar wavelet is one of simplest in wavelet analysis . Due to its simplicity and better applications, it is used in solution of integral as well as differential equations . Haar wavelet contains a non-smooth character. This is a difficiency of Haar wavelet to estimate the moduli of continuity and the approximation of the smooth function by it. This weak point is almost removed by Chebyshev wavelets and more accurate the moduli of continuity and approximations of functions are obtained. Sripathy[14] discussed the chebyshev wavelet based approximation for solving linear and non-linear differential equations.

Moduli of continuity of functions have been studies by Alexander Babenko[10]. In best of our knowledge, there is no work associated to the modulus of continuity and approximation of a function f in Hölder's class by first kind Chebyshev wavelet method. To make an advanced study in this direction, in this paper, the moduli of continuity and approximation of functions have been determined in Hölder's class $H_{\omega_k}^{\alpha,2}[0, 1)$.

Several linear, as well as non-linear differential equations are solvable by Galerkin, Collocation, and other known methods. These equations can be solved by Chebyshev wavelet technique in very efficient and suitable manners. This motivates us to consider first kind Chebyshev wavelet method for the solution of differential equations. Also, Babolian and Fattahzadeh[5] suggested a method to solve the differential equations by using Chebyshev wavelet operational matrix of integration.

In this paper, the Lane-Emden differential equations has been solved by Chebyshev wavelet technique. The main characteristic of this techique is that it reduces the problem to a system of algebraic equations. The approach is based on converting the given differential equations into integral equations through integration by approximating various signals involved in the equation through truncated orthogonal Chebyshev wavelet series and using the operational matrix P of integration , to eliminate the integral operations.

This paper is organized as follows: Section(1) is introductory in which the

importance of moduli of continuity and Chebyshev wavelet of first kind related literature are studied. In section(2), Chebyshev wavelet of first kind ,approximation of function and moduli of continuity of functions in class $H_{\omega_k}^{\alpha,2}[0, 1)$ are defined. In section(3), theorem concerning the moduli of continuity of $f - S_{2^{k-1},M}(f)$ has been established and also its detail proof is discussed in section(4). In section(5) corollaries are deduced from theorem of section (3) . In section(6) first kind Chebyshev wavelet operational matrix of integration has been constructed and the product operational matrix has been obtained in section(7). In section(8), Lane-Emden differential equations of index 0, 1 & 2 are solved by Chebyshev wavelet method. Finally, the main conclusions are summarized in section (9).

2 Definitions and Preliminaries

2.1 Chebyshev wavelets of first kind

Wavelets constitute a family of functions constructed from dialation and translation of a single function $\Psi \in L^2(\mathbb{R})$ called mother wavelet .We write

$$\Psi_{b,a}(t) = |a|^{\frac{-1}{2}} \Psi \left(\frac{t-b}{a} \right), \quad a \neq 0. \quad (\text{Daubechies [6]})$$

If we restrict the values of dialation and translation parameter to $a = a_0^{-k}, b = (2n-1)b_0 a_0^{-k}, a_0 > 1, b_0 > 0$ respectively, the following family of discrete wavelets are constructed:

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \Psi(a_0^k t - (2n-1)b_0).$$

Now, taking $a_0 = 2, \Psi(t) = \tilde{T}_m(t)$ & $b_0 = 1$ the Chebyshev wavelet $\Psi(k, n, m, t)$ of first kind , generally denoted by $\Psi_{n,m}^{(c_1)}(t)$ over the interval $[0,1)$, is obtained as (Babolian [5])

$$\Psi_{n,m}^{(c_1)}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^k-1} \leq t < \frac{n}{2^k-1} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$\text{where} \quad \tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t) & , \text{otherwise.} \end{cases}$$

where $n = 1, 2, \dots, 2^{k-1}, m = 0, 1, 2, \dots, M$ and k is the positive integer. In above definition, T_m are the first kind Chebyshev polynomials of degree m on the interval $[-1,1]$ which are defined by

$$T_m(t) = \cos(m\theta), \quad \theta = \arccos(t) \quad (2)$$

and also satisfy the following recursive formula:

$$T_0(t) = 1 \quad T_1(t) = t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, 3, \dots$$

The set of $\{T_m(t) : m = 0, 1, 2, 3, \dots\}$ in the Hilbert space $L^2[-1, 1]$ is an orthogonal set with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$. Orthogonality of Chebyshev polynomial of first kind on the interval $[-1, 1]$ implies that

$$\langle T_m(t), T_n(t) \rangle = \int_{-1}^1 \frac{T_m(t) \overline{T_n(t)}}{\sqrt{1-t^2}} dt = \begin{cases} \pi, & m = n = 0 \\ 0, & n \neq m. \\ \frac{\pi}{2}, & n = m \neq 0. \end{cases}$$

In dealing with Chebyshev wavelets, the weight function $\omega(t)$ for orthogonal Chebyshev polynomials has to be dilated and translated to construct orthonormal wavelets. So the first kind Chebyshev wavelets are an orthonormal set with weight function (S. Dhawan[7])

$$\omega_k(t) = \begin{cases} \omega_{1,k}(t), & 0 \leq t < \frac{1}{2^{k-1}}, \\ \omega_{2,k}(t), & \frac{1}{2^{k-1}} \leq t < \frac{2}{2^{k-1}}, \\ \vdots & \\ \omega_{2^{k-1},k}(t), & \frac{2^{k-1}-1}{2^{k-1}} \leq t < 1, \end{cases} \quad (3)$$

where $\omega_{n,k}(t) = \omega(2^k t - 2n + 1)$. Furthermore, the set of wavelets $\psi_{n,m}(t)$ makes an orthonormal basis in Hilbert space $L^2_{\omega_k}[0, 1]$, i.e.

$$\langle \Psi_{n,m}^{(c_1)}, \Psi_{n',m'}^{(c_1)} \rangle_{\omega_k} = \int_0^1 \Psi_{n,m}^{(c_1)}(t) \Psi_{n',m'}^{(c_1)}(t) \omega_k(t) dt = \delta_{n,n'} \delta_{m,m'}$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,n'} = \begin{cases} 1, & n=n' \\ 0, & \text{otherwise} \end{cases}$$

2.2 First kind Chebyshev wavelet expansion and approximation of function

The function $f \in L^2_{\omega_k}[0, 1]$ is expressed in the Chebyshev wavelet series as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t), \quad (4)$$

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where $c_{n,m} = \langle f, \Psi_{n,m}^{(c_1)} \rangle_{\omega_k}$. The $(2^{k-1}, M+1)^{th}$ partial sums of above series (4) is given by

$$S_{2^{k-1},M}(f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t) = C^T \Psi^{(c_1)}(t) \quad (5)$$

in which C and $\Psi^{(c_1)}(t)$ are $2^{k-1}(M+1)$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, \dots, c_{1,M}, c_{2,0}, c_{2,1}, \dots, c_{2,M}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M}] \quad \text{and}$$

$$\Psi^{(c_1)} = [\Psi_{1,0}^{(c_1)}, \Psi_{1,1}^{(c_1)}, \dots, \Psi_{1,M}^{(c_1)}, \Psi_{2,0}^{(c_1)}, \Psi_{2,1}^{(c_1)}, \dots, \Psi_{2,M}^{(c_1)}, \dots, \Psi_{2^{k-1},0}^{(c_1)}, \dots, \Psi_{2^{k-1},M}^{(c_1)}]^T$$

The Chebyshev wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L_{\omega_k}^2[0, 1)$ by $(2^{k-1}, (M+1))^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its Chebyshev wavelet series is given by

$$E_{2^{k-1},M}(f) = \min_{S_{2^{k-1},M}(f)} \|f - S_{2^{k-1},M}(f)\|_2$$

$$\text{where, } \|f\|_2 = \left(\int_0^1 |f(t)|^2 \omega_k(t) dt \right)^{\frac{1}{2}}$$

If $E_{2^{k-1},M}(f) \rightarrow 0$ as $k, M \rightarrow \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M+1)$ (Zygmund[8]).

2.3 Modulus of continuity

The Modulus of continuity of a function $f \in L_{\omega_k}^2[0, 1)$ is defined as

$$W(f, \delta) = \sup_{0 < h \leq \delta} \|f(\cdot + h) - f(\cdot)\|_2$$

$$= \sup_{0 < h \leq \delta} \left(\int_0^1 |f(t+h) - f(t)|^2 \omega_k(t) dt \right)^{\frac{1}{2}}.$$

It is remarkable to note that $W(f, \delta)$ is a non-decreasing function of δ and $W(f, \delta) \rightarrow 0$ as $\delta \rightarrow 0^+$ (Chui[9]).

2.4 Function of Hölder's class

A function f is said to be in Hölder's class $H_{\omega_k}^\alpha[0, 1)$ of order α , $0 < \alpha \leq 1$, (G.Das[11]) if f satisfies

$$(f(x+t) - f(x))\omega_k(x) = O(|t|^\alpha), \quad \forall x, t, x+t \in [0, 1).$$

$$\text{If } \left(\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \right)^{\frac{1}{2}} = O(|t|^\alpha), \quad \forall x, t, x+t \in [0, 1).$$

then f is said to be a function of Hölder's class $H_{\omega_k}^{\alpha,2}[0, 1)$ of order α , $0 < \alpha \leq 1$.

The class $H_{\omega_k}^{\alpha}[0, 1)$ is a proper subclass of $H_{\omega_k}^{\alpha,2}[0, 1)$.

Let $f \in H_{\omega_k}^{\alpha}[0, 1)$, $\alpha = 1$. Then

$$\begin{aligned} \left(f(x+t) - f(x) \right) \omega_k(x) &= O(|t|) \\ \left(\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \right) &= \int_0^1 \frac{O(|t|)^2}{\omega_k(x)} dx \\ &= O(|t|^2) \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \sqrt{1 - (2^k x - 2n + 1)^2} dx \\ &= O(|t|^2) \sum_{n=1}^{2^{k-1}} \int_0^\pi \sin^2 \theta \frac{d\theta}{2^k} \\ &= O(|t|^2) \frac{\pi}{4} \\ \left(\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \right)^{\frac{1}{2}} &= (O(|t|)^2)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2} = O(|t|) \end{aligned}$$

$$\text{i.e. } f \in H_{\omega_k}^{\alpha,2}[0, 1), \quad \alpha = 1.$$

$$\text{Let, } f(t) = \begin{cases} t^2, & 0 \leq t < \frac{1}{2} \\ 1, & \frac{1}{2} \leq t < 1 \end{cases}$$

$$f\left(\frac{1}{2} + \frac{\epsilon}{2}\right) - f\left(\frac{1}{2} - \frac{\epsilon}{2}\right) = 1 - \left(\frac{1}{2} - \frac{\epsilon}{2}\right)^2 = \frac{3}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{4}.$$

$$\left(\frac{f\left(\frac{1}{2} + \frac{\epsilon}{2}\right) - f\left(\frac{1}{2} - \frac{\epsilon}{2}\right)}{\epsilon} \right) \omega_k(x) = \left(\frac{3}{4\epsilon} + \frac{1}{2} - \frac{\epsilon}{4} \right) \omega_k\left(\frac{1}{2} - \frac{\epsilon}{2}\right) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0^+$$

Therefore, $f \notin H_{\omega_k}^{\alpha}[0, 1)$, $\alpha = 1$.

$$\begin{aligned} \int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx &= \int_0^{\frac{1}{2}} |f(x+t) - f(x)|^2 \omega_k(x) dx \\ &\quad + \int_{\frac{1}{2}}^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \\ &= \int_0^{\frac{1}{2}} |(x+t)^2 - x^2|^2 \omega_k(x) dx + 0 \\ &\leq \int_0^{\frac{1}{2}} |2xt + t^2|^2 \frac{2}{\sqrt{3}} dx, \quad 1 \leq \omega(x) \leq \frac{2}{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \int_0^{\frac{1}{2}} (4x^2 t^2 + t^4 + 4xt^3) dx \end{aligned}$$

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$$\begin{aligned}
\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx &= \frac{2}{\sqrt{3}} \left[\frac{4}{3} t^2 \times \frac{1}{8} + t^4 \times \frac{1}{2} + 2t^3 \times \frac{1}{4} \right] \\
&= \frac{2}{\sqrt{3}} \left[\frac{1}{6} \times t^2 + \frac{1}{2} \times t^2 + \frac{1}{2} \times t^2 \right] \\
&= \frac{7}{3\sqrt{3}} t^2 \\
\left(\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \right)^{\frac{1}{2}} &\leq \sqrt{\frac{7}{3\sqrt{3}}} |t| \\
\left(\int_0^1 |f(x+t) - f(x)|^2 \omega_k(x) dx \right)^{\frac{1}{2}} &= O(|t|) \\
f &\in H_{\omega_k}^{\alpha,2}[0,1), \alpha = 1
\end{aligned}$$

Thus, $H_{\omega_k}^{\alpha}[0,1) \subsetneq H_{\omega_k}^{\alpha,2}[0,1), \alpha = 1$.

3 Theorems

In this paper, following theorems have been proved.

Theorem 3.1. *If $f \in H_{\omega_k}^{\alpha,2}[0,1)$ and its first kind Chebyshev wavelet expansion be*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t)$$

having $(2^{k-1}, M+1)^{th}$ partial sums

$$(S_{2^{k-1},M}(f))(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t)$$

Then the Moduli of continuity of $f - (S_{2^{k-1},M}(f))$ satisfies

$$\begin{aligned}
\text{i. } W((f - S_{2^{k-1},0}f), \frac{1}{2^k}) &= \sup_{0 < h \leq \frac{1}{2^k}} \|(f - S_{2^{k-1},0}f)(\cdot + h) - (f - S_{2^{k-1},0}f)(\cdot)\| \\
&= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \\
\text{ii. } W((f - S_{2^{k-1},M}f), \frac{1}{2^k}) &= \sup_{0 < h \leq \frac{1}{2^k}} \|(f - S_{2^{k-1},M}f)(\cdot + h) - (f - S_{2^{k-1},M}f)(\cdot)\| \\
&= O\left(\frac{1}{2^{k(\alpha+1/2)} \sqrt{M+1}}\right), \quad M \geq 1.
\end{aligned}$$

4 Proof

Proof of the theorem (3.1)

(i) Error between $f(x)$ and its Chebyshev wavelet expansion in interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ is given by

$$\begin{aligned}
 e_n(t) &= c_{n,0} \Psi_{1,0}^{(c_1)}(t) - f \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)}(t) \\
 &= \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) \Psi_{1,0}^{(c_1)}(x) \omega_k(x) dx \right) \Psi_{1,0}^{(c_1)}(t) - f \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)}(t) \\
 &= \frac{2^{\frac{k}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) \omega_k(x) dx \right) \frac{2^{\frac{k}{2}}}{\sqrt{\pi}} - f \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)}(t) \\
 &= \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) \omega_k(x) dx \right) - \frac{2^k}{\pi} f \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)}(t) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \omega_k(x) dx \\
 &= \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (f(x) - f(t)) \omega_k(x) dx \right). \\
 |e_n(t)| &\leq \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |(f(x) - f(t))| |\omega_k(x)| dx \right) \\
 &= \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |(f(x) - f(t))| |\sqrt{\omega_k(x)}| |\sqrt{\omega_k(x)}| dx \right) \\
 &\leq \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (|f(x) - f(t)|)^2 |\omega_k(x)| dx \right)^{\frac{1}{2}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |\omega_k(x)| dx \right)^{\frac{1}{2}} \\
 \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |\omega_k(x)| dx &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |\omega(2^k x - 2n + 1)| dx = \frac{\pi}{2^k} \\
 |e_n(t)| &\leq \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (|f(x) - f(t)|)^2 |\omega_k(x)| dx \right)^{\frac{1}{2}} \left(\frac{\pi}{2^k} \right)^{\frac{1}{2}} \\
 &= \frac{2^k}{\pi} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (|f(x) - f(t)|)^2 |\omega(2^k x - 2n + 1)| dx \right)^{\frac{1}{2}} \left(\frac{\pi}{2^k} \right)^{\frac{1}{2}} \\
 &= \frac{2^k}{\pi} \left(\int_{-1}^1 \left(\left| f\left(\frac{v + 2n - 1}{2^k}\right) - f(t) \right| \right)^2 |\omega(v)| \frac{dv}{2^k} \right)^{\frac{1}{2}} \left(\frac{\pi}{2^k} \right)^{\frac{1}{2}}
 \end{aligned}$$

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$$\begin{aligned}
|e_n(t)| &\leq \frac{1}{\sqrt{\pi}} \left(\int_{-1}^1 \left(\left| f\left(\frac{v+2n-1}{2^k}\right) - f(t) \right| \right)^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\pi}} \left(\int_0^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(v)| dv \right. \\
&\quad \left. + \int_{-1}^0 \left| f\left(\frac{v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\pi}} \left(\int_0^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(v)| dv \right. \\
&\quad \left. + \int_0^1 \left| f\left(\frac{-v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(-v)| dv \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\pi}} \left(\int_0^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{\sqrt{\pi}} \left(\int_0^1 \left| f\left(\frac{-v+2n-1}{2^k}\right) - f(t) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}}, \\
&\quad \frac{v+2n-1}{2^k} \in \left[\frac{n-\frac{1}{2}}{2^{k-1}}, \frac{n}{2^{k-1}} \right) \quad \text{and} \quad \frac{-v+2n-1}{2^k} \in \left(\frac{n-1}{2^{k-1}}, \frac{n-\frac{1}{2}}{2^{k-1}} \right] \\
|e_n(t)| &\leq \frac{1}{\sqrt{\pi}} O\left(\frac{1}{2^{(k-1)}}\right)^\alpha + \frac{1}{\sqrt{\pi}} O\left(\frac{1}{2^{(k-1)}}\right)^\alpha \\
&= \frac{2}{\sqrt{\pi}} O\left(\frac{1}{2^{(k-1)}}\right)^\alpha = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \tag{6}
\end{aligned}$$

$$\begin{aligned}
\|e_n\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |e_n(t)|^2 |\omega_k(t)| dt \\
&= O\left(\frac{1}{2^{(k-1)}}\right)^{2\alpha} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |\omega_k(t)| dt \\
&= O\left(\frac{1}{2^{2(k-1)\alpha}}\right) \frac{\pi}{2^k} \\
&= \frac{1}{2^k} O\left(\frac{1}{2^{2(k-1)\alpha}}\right) \\
&\leq \frac{1}{2^{k-1}} O\left(\frac{1}{2^{2(k-1)\alpha}}\right) \\
\|e_n\|_2^2 &= O\left(\frac{1}{2^{(k-1)(2\alpha+1)}}\right) \tag{7}
\end{aligned}$$

$$\begin{aligned}
 \|f - (S_{2^{k-1},0}f)\|_2^2 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} e_n(t) \right)^2 \omega_k(t) dt \\
 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} (e_n(t))^2 \omega_k(t) \right) dt \\
 &\quad + \sum_{1 \leq n \neq n' \leq 2^{k-1}} \int_0^1 e_n(t) e_{n'}(t) \omega_k(t) dt,
 \end{aligned} \tag{8}$$

Due to disjointness of supports of $e_n(t)$ & $e_{n'}(t)$, equation(8) becomes,

$$\begin{aligned}
 &= \int_0^1 \left(\sum_{n=1}^{2^{k-1}} (e_n(t))^2 \omega_k(t) \right) dt \\
 &= \sum_{n=1}^{2^{k-1}} \int_0^1 (e_n(t))^2 \omega_k(t) dt \\
 &= \sum_{n=1}^{2^{k-1}} \|e_n(t)\|_2^2 \\
 &= \sum_{n=1}^{2^{k-1}} O\left(\frac{1}{2^{(k-1)(2\alpha+1)}}\right) \\
 &= O\left(\frac{2^{k-1}}{2^{(k-1)(2\alpha+1)}}\right) = O\left(\frac{1}{2^{2(k-1)\alpha}}\right) \\
 \|f - (S_{2^{k-1},0}f)\|_2 &= O\left(\frac{1}{2^{(k-1)\alpha}}\right)
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 W\left((f - S_{2^{k-1},0}f), \frac{1}{2^k}\right) &= \sup_{0 < h \leq \frac{1}{2^k}} \|(f - S_{2^{k-1},0}f)(\cdot + h) - (f - S_{2^{k-1},0}f)(\cdot)\|_2 \\
 &\leq \|(f - S_{2^{k-1},0}f)\|_2 + \|(f - S_{2^{k-1},0}f)\|_2 \\
 &= 2\|(f - S_{2^{k-1},0}f)\|_2 \\
 &= 2.O\left(\frac{1}{2^{(k-1)\alpha}}\right) \\
 &= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \\
 W\left((f - S_{2^{k-1},0}f), \frac{1}{2^k}\right) &= O\left(\frac{1}{2^{(k-1)\alpha}}\right).
 \end{aligned} \tag{10}$$

(ii) Consider

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{m,n}^{(c_1)}(t) \\
 c_{n,m} &= \langle f, \Psi_{m,n}^{(c_1)}(t) \rangle_{\omega_k} \\
 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) \Psi_{m,n}^{(c_1)}(t) \omega_k(t) dt \\
 &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) T_m(2^k t - 2n + 1) \omega_k(t) dt \right) \\
 &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{2n-1}{2^k}\right) + f\left(\frac{2n-1}{2^k}\right) \right) g(t) \omega_k(t) dt \\
 &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{2n-1}{2^k}\right) \right) g(t) \omega_k(t) dt \right) \\
 &\quad + \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} f\left(\frac{2n-1}{2^k}\right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t) \omega_k(t) dt
 \end{aligned} \tag{11}$$

where $g(t) = T_m(2^k t - 2n + 1)$

$$\begin{aligned}
 \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t) \omega_k(t) dt &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_m(2^k t - 2n + 1) \omega(2^k t - 2n + 1) dt \\
 &= \int_{\pi}^0 T_m(\cos \theta) \omega(\cos \theta) \frac{(-\sin \theta)}{2^k} d\theta \\
 &= \int_0^{\pi} \frac{\cos m\theta}{\sin \theta} \frac{(\sin \theta)}{2^k} d\theta = \frac{1}{2^k} \left[\frac{\sin m\theta}{m} \right]_0^{\pi} = 0
 \end{aligned} \tag{12}$$

Therefore equation (11) reduces to,

$$\begin{aligned}
 c_{n,m} &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{2n-1}{2^k}\right) \right) g(t) \omega_k(t) dt \right) \\
 &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{2n-1}{2^k}\right) \right) g(t) \sqrt{\omega_k(t) \omega_k(t)} dt \right) \\
 |c_{n,m}| &\leq \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left| f(t) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega_k(t)| dt \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |T_m(2^k t - 2n + 1)|^2 |\omega_k(t)| dt \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left| f(t) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(2^k t - 2n + 1)| dt \right)^{\frac{1}{2}} \\
&\times \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |T_m(2^k t - 2n + 1)| |\omega(2^k t - 2n + 1)| dt \right) \\
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| \frac{dv}{2^k} \right)^{\frac{1}{2}} \\
&\times \left(\int_0^\pi |T_m(\cos \theta)| |\omega(\cos \theta)| \frac{\sin \theta}{2^k} d\theta \right) \\
&= \frac{1}{2^k} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\left(\int_0^\pi |\cos m\theta| \left| \frac{1}{\sin \theta} \right| \sin \theta d\theta \right) \\
&= \frac{1}{2^k} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \cos\left(\frac{m\pi}{2} - mt\right) \right| dt \right) \\
&= \frac{\sqrt{\frac{2}{\pi}}}{2^k} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \cos \frac{m\pi}{2} \cos mt + \sin \frac{m\pi}{2} \sin mt \right| dt \\
&\leq \frac{\sqrt{\frac{2}{\pi}}}{2^{k-1}} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\left(\int_0^{\pi/2} (|\cos(mt)| + |\sin(mt)|) dt \right) \\
&\leq \frac{1}{2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^1 \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}} \\
&\times \left[\frac{|\sin(mt)|}{m} + \frac{|\cos(mt)|}{m} \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{m \cdot 2^k} \sqrt{\frac{2}{\pi}} \left(\left(\frac{1}{2^k} \right)^\alpha + \left(\frac{1}{2^k} \right)^\alpha \right) = \frac{1}{m \cdot 2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2^{k\alpha}} \right)
\end{aligned}$$

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$$|c_{n,m}| \leq \frac{1}{m \cdot 2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2^{k\alpha}} \right). \quad (13)$$

$$\text{Let } I(t) = f(x) - S_{2^{k-1},M}(f)(t)$$

$$\begin{aligned} I(t) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t) \\ &= \left(\sum_{n=1}^{2^{k-1}} + \sum_{\substack{n=1 \\ 2^{k-1}+1}}^{\infty} \right) \left(\sum_{m=0}^M + \sum_{\substack{m=M+1}}^{\infty} \right) c_{n,m} \Psi_{n,m}^{(c_1)}(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t) \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t), \quad \text{by def}^n \text{ of } \Psi_{n,m}^{(c_1)} \\ (I(t))^2 &= \left(\sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t) \right)^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m}^2 \left(\Psi_{n,m}^{(c_1)}(t) \right)^2 \\ &\quad + \sum_{n=1}^{2^{k-1}} \sum_{M+1 \leq m \neq m' \leq \infty} c_{n,m} c_{n,m'} \Psi_{n,m}^{(c_1)}(t) \Psi_{n,m'}^{(c_1)}(t) \\ &\quad + \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} c_{n',m} \Psi_{n,m}^{(c_1)}(t) \Psi_{n',m}^{(c_1)}(t) \\ &\quad + \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{M+1 \leq m \neq m' \leq \infty} c_{n,m} c_{n',m'} \Psi_{n,m}^{(c_1)}(t) \Psi_{n',m'}^{(c_1)}(t) \\ \|I\|_2^2 &= \int_0^1 |f(t) - S_{2^{k-1},M}(f)(t)|^2 \omega_n(t) dt \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} |c_{n,m}|^2 \int_0^1 |\Psi_{n,m}^{(c_1)}(t)|^2 \omega_n(t) dt \\ &\quad + \sum_{n=1}^{2^{k-1}} \sum_{M+1 \leq m \neq m' \leq \infty} c_{n,m} c_{n,m'} \int_0^1 \left(\Psi_{n,m}^{(c_1)}(t) \Psi_{n,m'}^{(c_1)}(t) \right) \omega_n(t) dt \\ &\quad + \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{m=M+1}^{\infty} c_{n,m} c_{n',m} \int_0^1 \left(\Psi_{n,m}^{(c_1)}(t) \Psi_{n',m}^{(c_1)}(t) \right) \omega_n(t) dt \\ &\quad + \sum_{1 \leq n \neq n' \leq 2^{k-1}} \sum_{M+1 \leq m \neq m' \leq \infty} c_{n,m} c_{n',m'} \int_0^1 \Psi_{n,m}^{(c_1)}(t) \Psi_{n',m'}^{(c_1)}(t) \omega_n(t) dt \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} |c_{n,m}|^2, \quad \text{by orthonormality of } \{\Psi_{n,m}^{(c_1)}\}. \end{aligned}$$

$$\begin{aligned}
 \|f - S_{2^{k-1},M}(f)\|_2^2 &\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left[\frac{1}{m \cdot 2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2^{k\alpha}} \right) \right]^2 \\
 &= \frac{2}{\pi} \cdot 2^{k-1} \cdot \frac{1}{2^{2(k-1)}} \cdot \frac{1}{2^{2k\alpha}} \sum_{m=M+1}^{\infty} \frac{1}{m^2} \\
 &= \frac{2^2}{\pi \cdot 2^k \cdot 2^{2k\alpha}} \sum_{m=M+1}^{\infty} \frac{1}{m^2} \\
 &\leq \frac{2^2}{\pi \cdot 2^k \cdot 2^{2k\alpha}} \left[\frac{1}{(M+1)^2} + \int_{M+1}^{\infty} \frac{1}{m^2} dm \right]
 \end{aligned}$$

By Cauchy integral test,

$$\begin{aligned}
 &\leq \frac{2^2}{\pi \cdot 2^k \cdot 2^{2k\alpha}} \left[\frac{1}{(M+1)} + \left(\frac{-1}{m} \right)_{M+1}^{\infty} \right] \\
 &= \frac{2^2}{\pi \cdot 2^k \cdot 2^{2k\alpha}} \left[\frac{1}{(M+1)} + \frac{1}{M+1} \right] \\
 \|f - S_{2^{k-1},M}(f)\|_2 &\leq 2 \sqrt{\frac{2}{\pi}} \frac{1}{2^{k(\alpha+1/2)} \sqrt{M+1}} \\
 &= O \left(\frac{1}{2^{k(\alpha+1/2)} \sqrt{M+1}} \right)
 \end{aligned}$$

$$W \left((f - S_{2^{k-1},M}f), \frac{1}{2^k} \right) = \sup_{0 < h \leq \frac{1}{2^k}} \| (f - S_{2^{k-1},M}f)(\cdot + h) - (f - S_{2^{k-1},M}f)(\cdot) \|_2$$

$$\leq \| (f - S_{2^{k-1},0}f) \|_2 + \| (f - S_{2^{k-1},M}f) \|_2$$

$$= 2 \| (f - S_{2^{k-1},M}f) \|_2$$

$$= 2 \cdot O \left(\frac{1}{2^{k(\alpha+1/2)} \sqrt{M+1}} \right)$$

$$= O \left(\frac{1}{2^{k(\alpha+1/2)} \sqrt{M+1}} \right).$$

Thus, this theorem is completely established.

5 Corollary

Corollary 5.1. *If $f \in H_{\omega_k}^{\alpha,2}[0, 1)$ and its first kind Chebyshev wavelet expansion be*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(c_1)}(t)$$

having $(2^{k-1}, M+1)^{th}$ partial sums

$$(S_{2^{k-1},M}(f))(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t)$$

Then the first kind Chebyshev wavelet approximation $E_{2^{k-1},M}(f)$ of f is given by

$$\begin{aligned} (i) \ E_{2^{k-1},0}(f) &= \min \|f - (S_{2^{k-1},0}f)\|_2 = \min \|f - \sum_{n=1}^{2^{k-1}} c_{n,0} \Psi_{n,0}^{(c_1)}(t)\|_2 \\ &= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \\ (ii) \ E_{2^{k-1},M}(f) &= \min \|f - (S_{2^{k-1},M}f)\|_2 = \min \|f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^M c_{n,m} \Psi_{n,m}^{(c_1)}(t)\|_2 \\ &= O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right) \end{aligned}$$

The proof of corollary (5.1) can be developed parallel to the proof of theorem (3.1) independently.

Remark

If $f \in H_{\omega_k}^{\alpha,2}[0, 1)$, then the moduli of continuity $W\left((f - S_{2^{k-1},M}f), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$ and approximation $E_{2^{k-1},M}(f) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$ tends to 0 as $k \rightarrow \infty$, $M \rightarrow \infty$. Hence $W\left((f - S_{2^{k-1},M}f), \frac{1}{2^k}\right)$ and $E_{2^{k-1},M}(f)$ are best possible modulus of continuity and approximation of functions respectively in wavelet analysis. It is also observed that

$$W\left((f - S_{2^{k-1},M}f), \frac{1}{2^k}\right) \leq E_{2^{k-1},M}(f).$$

Hence modulus of continuity is sharper than the approximation of function in $H_{\omega_k}^{\alpha,2}[0, 1)$ by first kind Chebyshev wavelet method.

6 First kind Chebyshev wavelet operational matrix of integration

In this section, the operational matrix P of integration of order $2^{k-1}(M+1) \times 2^{k-1}(M+1)$ has been derived. This plays a great role in dealing with the problem of Lane-Emden differential equations (Babolian [5]) .

First the matrix $P_{8 \times 8}$ has been obtained for $M = 3$ and $k = 2$. There are eight basis wavelet functions given below:

$$\begin{aligned}\Psi_{1,0}(t) &= \begin{cases} \frac{2}{\sqrt{\pi}}, & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{1,1}(t) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(4t-1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{1,2}(t) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(32t^2-16t+1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{1,3}(t) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(256t^3-192t^2+36t-1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{2,0}(t) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{2}{\sqrt{\pi}}, & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{2,1}(t) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(4t-3), & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{2,2}(t) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(32t^2-48t+17), & \frac{1}{2} \leq t < 1, \end{cases} \\ \Psi_{2,3}(t) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(256t^3-576t^2+420t-99), & \frac{1}{2} \leq t < 1. \end{cases}\end{aligned}$$

Let,

$$\Psi(t) = [\Psi_{10}, \Psi_{11}, \Psi_{12}, \Psi_{13}, \Psi_{20}, \Psi_{21}, \Psi_{22}, \Psi_{23}]^T \quad (14)$$

By integrating the basis functions from 0 to t and expressing them in term of Chebyshev wavelet series ,

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$$\begin{aligned}
\int_0^t \Psi_{1,0}(x) &= \begin{cases} \frac{2}{\sqrt{\pi}}t, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{\sqrt{\pi}}, & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[\frac{1}{4}, \frac{1}{4\sqrt{2}}, 0, 0, \frac{1}{2}, 0, 0, 0 \right] \Psi(t) \\
\int_0^t \Psi_{1,1}(x) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(4t-1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[-\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0, 0, 0, 0, 0 \right] \Psi(t) \\
\int_0^t \Psi_{1,2}(x) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(32t^2-16t+1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[-\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, \frac{1}{24}, -\frac{1}{3\sqrt{2}}, 0, 0, 0 \right] \Psi(t) \\
\int_0^t \Psi_{1,3}(x) &= \begin{cases} 2\sqrt{\frac{2}{\pi}}(256t^3-192t^2+36t-1), & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[\frac{1}{16\sqrt{2}}, 0, -\frac{1}{16}, 0, 0, 0, 0, 0 \right] \Psi(t) \\
\int_0^t \Psi_{2,0}(x) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{2}{\sqrt{\pi}}, & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[0, 0, 0, 0, \frac{1}{4}, \frac{1}{4\sqrt{2}}, 0, 0 \right] \Psi(t) \\
\int_0^t \Psi_{2,1}(x) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(4t-3), & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[0, 0, 0, 0, -\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0 \right] \Psi(t) \\
\int_0^t \Psi_{2,2}(x) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(32t^2-48t+17), & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[0, 0, 0, 0, -\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, \frac{1}{24} \right] \Psi(t) \\
\int_0^t \Psi_{2,3}(x) &= \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(256t^3-576t^2+420t-99), & \frac{1}{2} \leq t < 1, \end{cases} \\
&= \left[0, 0, 0, 0, \frac{1}{16\sqrt{2}}, 0, -\frac{1}{16}, 0 \right] \Psi(t)
\end{aligned}$$

$$\text{Thus, } \int_0^t \Psi(x) dx = P_{8 \times 8} \Psi(t) \quad (\text{M. A. Fariborzi [13]}) \quad (15)$$

$$\text{where } P_{8 \times 8} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{8} & 0 & \frac{1}{24} & -\frac{1}{3\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{16\sqrt{2}} & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{6\sqrt{2}} & -\frac{1}{8} & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & 0 & -\frac{1}{16} & 0 \end{pmatrix} \quad (16)$$

7 The Product Operation matrix (POM)

In this section , the product operation matrix has been obtained. This has important role in solving differential equations (Babolian [5]).
By Equation (14),

$$\Psi \Psi^T = \begin{pmatrix} \Psi_{1,0}\Psi_{1,0} & \Psi_{1,0}\Psi_{1,1} & \Psi_{1,0}\Psi_{1,2} & \Psi_{1,0}\Psi_{1,3} & \Psi_{1,0}\Psi_{2,0} & \Psi_{1,0}\Psi_{2,1} & \Psi_{1,0}\Psi_{2,2} & \Psi_{1,0}\Psi_{2,3} \\ \Psi_{1,1}\Psi_{1,0} & \Psi_{1,1}\Psi_{1,1} & \Psi_{1,1}\Psi_{1,2} & \Psi_{1,1}\Psi_{1,3} & \Psi_{1,1}\Psi_{2,0} & \Psi_{1,1}\Psi_{2,1} & \Psi_{1,1}\Psi_{2,2} & \Psi_{1,1}\Psi_{2,3} \\ \Psi_{1,2}\Psi_{1,0} & \Psi_{1,2}\Psi_{1,1} & \Psi_{1,2}\Psi_{1,2} & \Psi_{1,2}\Psi_{1,3} & \Psi_{1,2}\Psi_{2,0} & \Psi_{1,2}\Psi_{2,1} & \Psi_{1,2}\Psi_{2,2} & \Psi_{1,2}\Psi_{2,3} \\ \Psi_{1,3}\Psi_{1,0} & \Psi_{1,3}\Psi_{1,1} & \Psi_{1,3}\Psi_{1,2} & \Psi_{1,3}\Psi_{1,3} & \Psi_{1,3}\Psi_{2,0} & \Psi_{1,3}\Psi_{2,1} & \Psi_{1,3}\Psi_{2,2} & \Psi_{1,3}\Psi_{2,3} \\ \Psi_{2,0}\Psi_{1,0} & \Psi_{2,0}\Psi_{1,1} & \Psi_{2,0}\Psi_{1,2} & \Psi_{2,0}\Psi_{1,3} & \Psi_{2,0}\Psi_{2,0} & \Psi_{2,0}\Psi_{2,1} & \Psi_{2,0}\Psi_{2,2} & \Psi_{2,0}\Psi_{2,3} \\ \Psi_{2,1}\Psi_{1,0} & \Psi_{2,1}\Psi_{1,1} & \Psi_{2,1}\Psi_{1,2} & \Psi_{2,1}\Psi_{1,3} & \Psi_{2,1}\Psi_{2,0} & \Psi_{2,1}\Psi_{2,1} & \Psi_{2,1}\Psi_{2,2} & \Psi_{2,1}\Psi_{2,3} \\ \Psi_{2,2}\Psi_{1,0} & \Psi_{2,2}\Psi_{1,1} & \Psi_{2,2}\Psi_{1,2} & \Psi_{2,2}\Psi_{1,3} & \Psi_{2,2}\Psi_{2,0} & \Psi_{2,2}\Psi_{2,1} & \Psi_{2,2}\Psi_{2,2} & \Psi_{2,2}\Psi_{2,3} \\ \Psi_{2,3}\Psi_{1,0} & \Psi_{2,3}\Psi_{1,1} & \Psi_{2,3}\Psi_{1,2} & \Psi_{2,3}\Psi_{1,3} & \Psi_{2,3}\Psi_{2,0} & \Psi_{2,3}\Psi_{2,1} & \Psi_{2,3}\Psi_{2,2} & \Psi_{2,3}\Psi_{2,3} \end{pmatrix}$$

Substituting the values of each entry of above symmetric matrix , and after simplification expressing them in term of Chebyshev wavelet series ,

$$\Psi \Psi^T = \begin{pmatrix} A & O_{4 \times 4} \\ O_{4 \times 4} & B \end{pmatrix}$$

where $O_{4 \times 4}$ is a 4×4 zero matrix and

$$A = \begin{pmatrix} \frac{2}{\sqrt{\pi}}\Psi_{1,0} & \frac{2}{\sqrt{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,2} & \frac{2}{\sqrt{\pi}}\Psi_{1,3} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} + \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} + \sqrt{\frac{2}{\pi}}\Psi_{1,3} & \sqrt{\frac{2}{\pi}}\Psi_{1,2} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} + \sqrt{\frac{2}{\pi}}\Psi_{1,3} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,3} & \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{2}{\sqrt{\pi}}\Psi_{2,0} & \frac{2}{\sqrt{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{2,2} & \frac{2}{\sqrt{\pi}}\Psi_{2,3} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} + \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} + \sqrt{\frac{2}{\pi}}\Psi_{2,3} & \sqrt{\frac{2}{\pi}}\Psi_{2,2} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} + \sqrt{\frac{2}{\pi}}\Psi_{2,3} & \frac{2}{\sqrt{\pi}}\Psi_{2,0} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,3} & \sqrt{\frac{2}{\pi}}\Psi_{2,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{2,0} \end{pmatrix}$$

8 Solution of the Lane -Emden differential equation by first kind Chebyshev wavelet

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Lane-Emden type as follows:

$$\frac{d^2y}{dt^2} + \frac{2}{t} \frac{dy}{dt} + L(y) = 0 \quad (\text{A-M Wazwaz}[12])$$

where $L(y)$ is some given function of y .

A difficult element in the analysis of this type of equations is the singularity behaviour that occurs at $t = 0$. Most algorithms currently in use for handling the Lane-Emden type problems are based on either series solution or perturbation techniques. In recent years, a lot of attention has been devoted to the study of wavelet theory. The first kind Chebyshev wavelet method accurately computes the solution of differential equations and it is of great interest to solve this type of problem.

The most popular form of $L(y)$ attracted by the scientific community is

$$L(y) = y^n$$

where n is constant parameter.

The standard Lane-Emden equation of index n is of the form

$$\frac{d^2y}{dt^2} + \frac{2}{t} \frac{dy}{dt} + y^n = 0, \quad \text{with} \quad y(0) = 1, \quad y'(0) = 0 \quad (17)$$

It is a basic equation in the theory of stellar structure. The equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules .

It was physically shown that interesting values of n lie in the interval $[0, 5]$. In addition, exact solutions exist only for $n = 0, 1, 5$. Notice that equation (17) is linear for $n = 0, 1$ and non-linear otherwise.

Example (1). For $n = 0$, the Lane-Emden differential equation (17) reduces to

$$t y'' + 2 y' + t = 0 \quad \text{with} \quad y(0) = 1, \quad y'(0) = 0 \quad (18)$$

The exact solution of the equation (18) is $y(t) = 1 - \frac{t^2}{6}$.

The equation (18) has been solved using first kind Chebyshev wavelet. First it is assumed that $y''(t)$ can be expanded in terms of Chebyshev wavelet as

$$y''(t) = C^T \Psi(t) \quad (19)$$

where $\Psi(t)$ is given by equation (14). By integration equation (19) twice with respect to t from 0 to t and using conditions of equation (18) & (15) following equations are obtained ,

$$y'(t) = C^T P \Psi(t) \quad (20)$$

$$\text{and, } y(t) = C^T P^2 \Psi(t) + d^T \Psi(t) \quad (21)$$

$$\text{where, } d^T = [\frac{\sqrt{\pi}}{2}, 0, 0, 0, \frac{\sqrt{\pi}}{2}, 0, 0, 0] \quad (22)$$

Using equation (14), t is expressed as

$$t = e^T \Psi(x) \quad (23)$$

$$\text{where, } e^T = [\frac{\sqrt{\pi}}{8}, \frac{\sqrt{\frac{\pi}{2}}}{8}, 0, 0, \frac{3\sqrt{\pi}}{8}, \frac{\sqrt{\frac{\pi}{2}}}{8}, 0, 0] \quad (24)$$

Substituting values from Equation (19) to (24) in Equation (18),

$$e^T \Psi(t) \Psi^T(t) C + 2C^T P \Psi(t) + e^T \Psi(t) = 0 \quad (25)$$

The following property of the product of two first kind Chebyshev wavelet vectors will also be used:

$$e^T \Psi(t) \Psi^T(t) \simeq \Psi^T(t) \tilde{e} \quad (26)$$

where $\tilde{e}$ is 8×8 matrix. Let us establish equation (26) & find $\tilde{e}$,

$$e^T \Psi \Psi^T = \frac{1}{4} \left[\Psi_{1,0} + \frac{1}{\sqrt{2}} \Psi_{1,1}, \frac{1}{\sqrt{2}} \Psi_{1,0} + \Psi_{1,1} + \frac{1}{2} \Psi_{1,2}, \Psi_{1,2} + \frac{1}{2} \Psi_{1,1} + \frac{1}{2} \Psi_{1,3}, \Psi_{1,3} + \frac{\Psi_{1,2}}{2}, 3\Psi_{2,0} + \frac{\Psi_{2,1}}{\sqrt{2}}, 3\Psi_{2,1} + \frac{1}{\sqrt{2}} \Psi_{2,0} + \frac{\Psi_{2,2}}{2}, \frac{\Psi_{2,1}}{2} + 3\Psi_{2,2} + \frac{1}{2} \Psi_{2,3}, 3\Psi_{2,3} + \frac{1}{2} \Psi_{2,2} \right]$$

$$e^T \Psi \Psi^T = \begin{pmatrix} \Psi_{1,0} \\ \Psi_{1,1} \\ \Psi_{1,2} \\ \Psi_{1,3} \\ \Psi_{2,0} \\ \Psi_{2,1} \\ \Psi_{2,2} \\ \Psi_{2,3} \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 \end{pmatrix}$$

where

$$\tilde{e} = \frac{1}{4} \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 \end{pmatrix}_{8 \times 8} \quad (27)$$

Using Equation (26), Equation (25) becomes

$$\begin{aligned} (\Psi^T(t)\tilde{e})C + \Psi^T(t)(2P^T C) + \Psi^T(t)e &= 0 \\ \Psi^T(t) [\tilde{e} C + (2 P^T C) + e] &= 0 \\ \tilde{e} C + (2 P^T C) + e &= 0 \end{aligned} \quad (28)$$

Equation (28) is a set of algebraic equations which is solved for C in following form

$$C = (\tilde{e} + 2 P^T)^{-1} (-e)$$

$$C = \begin{pmatrix} -0.295408975150919 \\ 0.000000000000000 \\ -0.000000000000000 \\ -0.000000000000000 \\ -0.295408975150919 \\ 0.000000000000000 \\ -0.000000000000000 \\ 0.000000000000000 \end{pmatrix} \quad (29)$$

By substituting the Chebyshev wavelet coefficients C from equation (29) into equation (21), the explicit form of the approximate solution of equation (18) is,

$$y(t) = [0.872379629742559, -0.013055355597036, -0.003263838899259, 0, 0.798527385954829, -0.039166066791109, -0.003263838899259, 0] \Psi(t)$$

Using the simplified value of $\Psi(t)$ from equation (14), $y(t)$ becomes

$$y(t) = \begin{cases} 1 - 0.1666666666666660t^2, & 0 \leq t < \frac{1}{2}, \\ 1 - 6.357719098970234 \cdot 10^{-15} t - 0.1666666666666661 t^2, & \frac{1}{2} \leq t < 1. \end{cases}$$

The exact solution and the solution obtained by Chebyshev wavelet method of differential equation (18) at different values of t are given in Table (1):

Variable (t)	Exact solution	Chebyshev solution	Absolute error
0	1.0000000000000000	1.0000000000000000	0
0.1	0.9983333333333333	0.9983333333333333	0
0.2	0.9933333333333333	0.9933333333333334	0.0000000000000001
0.3	0.9850000000000000	0.9850000000000001	0.0000000000000001
0.4	0.9733333333333333	0.9733333333333334	0.0000000000000001
0.5	0.9583333333333333	0.9583333333333333	0
0.6	0.9400000000000000	0.9399999999999999	0.0000000000000001
0.7	0.9183333333333333	0.9183333333333333	0
0.8	0.8933333333333333	0.8933333333333333	0
0.9	0.8650000000000000	0.8650000000000000	0

Table(1): Comparison table for the exact and Chebyshev wavelet solution of Lane-Emden equation of index 0.

This table shown that the solution of equation (18) obtained by first kind Chebyshev wavelet method nearly coincides with its exact solution. The graphs of exact and first kind Chebyshev wavelet solutions of Lane -Emden differential equation for $n = 0$ are drawn in figure (1).

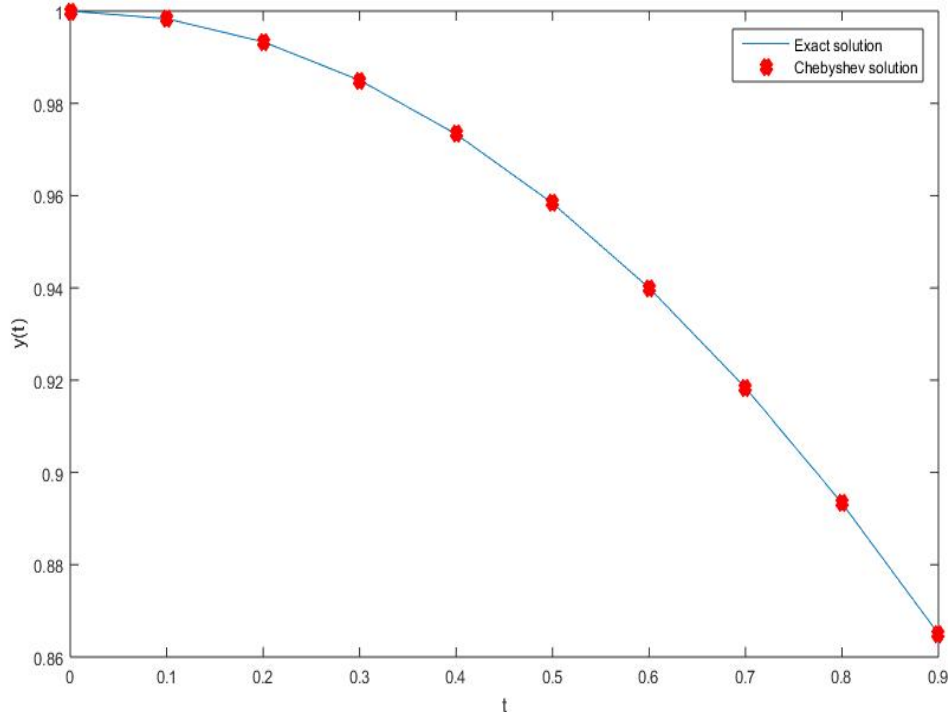


Figure 1: The graphs of first kind Chebyshev wavelet and exact solutions

Example (2). For $n = 1$, the Lane-Emden differential equation (17) reduces to

$$t y'' + 2 y' + t y = 0 \quad \text{with} \quad y(0) = 1, y'(0) = 0 \quad (30)$$

The exact solution of the equation (30) is $y(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1 & t = 0. \end{cases}$

Substituting values from Equation (19) – (24) in equation (30) ,

$$e^T \Psi(t) \Psi^T(t) C + 2 C^T P \Psi(t) + e^T \Psi(t) (C^T P^2 + d^T) \Psi(t) = 0$$

$$C = [\tilde{e} + 2 P^T + \tilde{e} (P^2)^T]^{-1} (-\tilde{e} d)$$

$$C = [-0.2871902685, 0.0077317973, 0.0019076755, -0.0000134466, \\ -0.2449764764, 0.0218383818, 0.0015733213, -0.0000401720]^T \quad (31)$$

By substituting the Chebyshev wavelet coefficients C from equation (31) into equation (21), the explicit form of the approximate solution of equation (30) is,

$$y(t) = [0.872505218190184, -0.012913303990151, -0.003192905781766, \\ 0.000020169906045, 0.801594934932526, -0.036835535048721, \\ -0.002723022009203, 0.000056975401025] \Psi(t)$$

$$\text{or, } y(t) = \begin{cases} 0.9999960361853 + 2.5435469133605 \cdot 10^{-4} t - 0.1692243050960 t^2 + \\ 0.0082397473925 t^3, & 0 \leq t < \frac{1}{2} \\ 0.9979746687982 + 0.0116372814448 t - 0.1914197434042 t^2 + \\ 0.0232754139255 t^3 & \frac{1}{2} \leq t < 1. \end{cases}$$

Table (2): Comparisons between the exact solution and numerical solutions for various values of t

Variable (t)	Exact solution	Chebyshev solution	Absolute error
0	1	0.999996036185	$0.0396381467104 \times 10^{-4}$
0.1	0.99833416646	0.998337468350	$0.0330188261299 \times 10^{-4}$
0.2	0.99334665397	0.993343852898	$0.0280107641192 \times 10^{-4}$
0.3	0.98506735553	0.985064628313	$0.0272722411598 \times 10^{-4}$
0.4	0.97354585577	0.973549233079	$0.0337730799104 \times 10^{-4}$
0.5	0.95885107720	0.958847800410	$0.0327679807199 \times 10^{-4}$
0.6	0.94107078899	0.941073419447	$0.0263045585602 \times 10^{-4}$
0.7	0.92031098176	0.920308558518	$0.0242325006294 \times 10^{-4}$
0.8	0.89669511362	0.896692870105	$0.0224351906097 \times 10^{-4}$
0.9	0.87036323291	0.870366006692	$0.0277377353607 \times 10^{-4}$

This table shown that the solution of equation (30) obtained by first kind Chebyshev wavelet method nearly coincides with its exact solution. The graphs of exact and first kind Chebyshev wavelet solutions of Lane -Emden differential equation for $n = 1$ are drawn in figure (2).

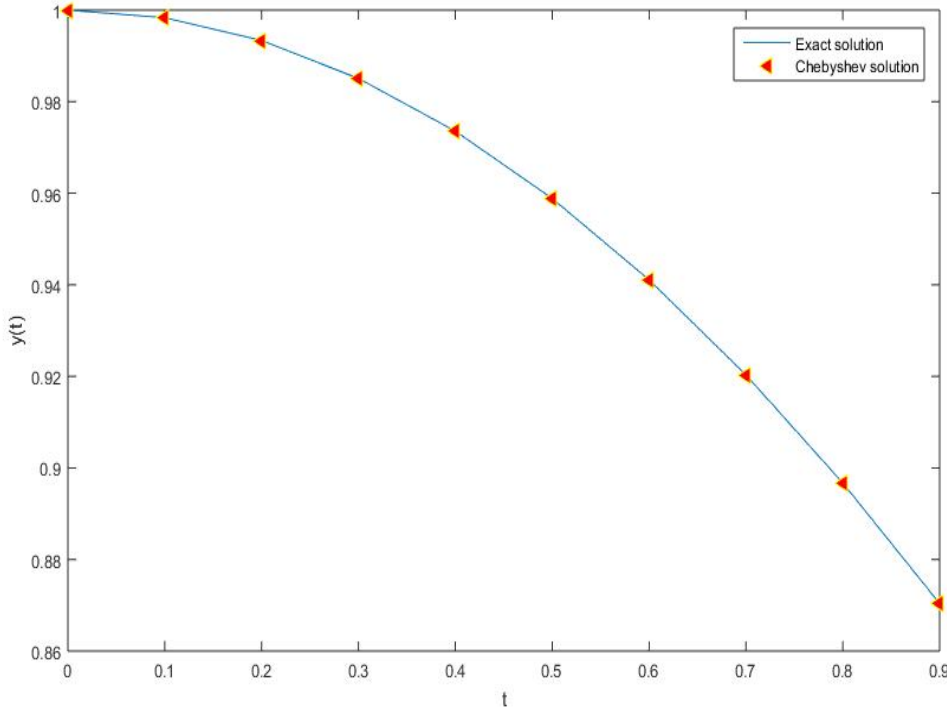


Figure 2: The graphs of first kind Chebyshev wavelet and exact solutions

Example (3). For $n = 2$, The Lane-Emden differential equation is

$$t y'' + 2 y' + t y^2 = 0 \quad \text{with} \quad y(0) = 1, y'(0) = 0 \quad (32)$$

The Taylor series expansion of function $L(y)$ about a point $y = 1$ is given as,

$$L(y) = \sum_{m=0}^{\infty} \frac{(y-1)^m}{m!} \frac{d^m L(y)}{dy^m} \Big|_{y=1}$$

$$y^2 = 1 + 2(y-1) + (y-1)^2$$

From equation (21) & (22),

$$= d^T \Psi(t) + 2 C^T P^2 \Psi(t) + C^T P^4 \Psi(t)$$

Substituting the Taylor series expansion of y^2 and values from equation (19) to (24) in equation (32) ,

$$e^T \Psi(t) \Psi^T(t) C + 2 C^T P \Psi(t) + e^T \Psi(t) \Psi^T(t) [d + 2 (P^T)^2 C + (P^T)^4 C] = 0$$

Moduli of continuity of functions in Hölder's class....

$$C = [\tilde{e} + 2P^T + 2\tilde{e}(P^T)^2 + \tilde{e}(P^T)^4]^{-1} (-\tilde{e}d) \quad (33)$$

$$C = [-0.279001698770205, 0.015429503002548, 0.003798307476540, \\ -0.000031441187989, -0.195293820350412, 0.043103314900479, \\ 0.003011315345934, -0.000095144707040]^T \quad (34)$$

By substituting the Chebyshev wavelet coefficients C from equation (34) into equation (21), the explicit form of the approximate solution of equation (32) is,

$$y(t) = [0.872630597117721, -0.012771501223632, -0.003122128096052, \\ 0.000040262875496, 0.804642516403461, -0.034524067315646, \\ -0.002189080129024, 0.000112495988561] \Psi(t) \\ \text{or, } y(t) = \begin{cases} 0.9999921077841 + 5.0666971784118 \cdot 10^{-4} t - 0.1717663081650 t^2 \\ + 0.016448064886675 t^3, & 0 \leq t < \frac{1}{2} \\ 0.996061399263954 + 0.022704430920508 t - 0.215186679101439 t^2 \\ + 0.045956511961640 t^3 & \frac{1}{2} \leq t < 1. \end{cases}$$

9 Conclusion

(i) In theorem (3.1), the moduli of continuity has been estimated as following:

$$W\left((f - S_{2^{k-1},0}f), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty, \\ W\left((f - S_{2^{k-1},M}f), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty, M \rightarrow \infty.$$

(ii) In corollary (5.1),

$$E_{2^{k-1},0}(f) = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty, \\ E_{2^{k-1},M}(f) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty, M \rightarrow \infty.$$

Thus $W((f - S_{2^{k-1},0}f), \frac{1}{2^k})$, $W((f - S_{2^{k-1},M}f), \frac{1}{2^k})$, $E_{2^{k-1},0}(f)$ & $E_{2^{k-1},M}(f)$ are best possible estimators in wavelet analysis.

(iii) From theorem (3.1) and corollary (5.1) , it is observed that

$$W \left((f - S_{2^{k-1},0}f), \frac{1}{2^k} \right) \leq 2 E_{2^{k-1},0}(f)$$
$$W \left((f - S_{2^{k-1},M}f), \frac{1}{2^k} \right) \leq 2 E_{2^{k-1},M}(f)$$

Hence moduli of continuity $W \left((f - S_{2^{k-1},0}f), \frac{1}{2^k} \right)$, $W \left((f - S_{2^{k-1},M}f), \frac{1}{2^k} \right)$ are better and sharper than approximations $E_{2^{k-1},0}(f)$, $E_{2^{k-1},M}(f)$ respectively.

(iv) Solution of Lane-Emden differential equations by first kind Chebyshev wavelet is approximately same as exact solution of differential equations. This is the significant achievement of this research paper . Calculations performed in section(8) demonstrate that the accuracy of Chebyshev wavelet method is quite high. In this method , there is no complex integral or methodology. Application of this proposed method is very simple & gives the explicite form of approximate solutions to the Lane - Emden differential equations. These are the main advantages of the method. This method is also very convenient for solving the boundary value problems. Hence, this proposed method is very reliable, simple, fast & computationally efficient method.

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On Real Roots of Complement Degree Polynomial of Graphs

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Abstract

Many studies have been carried out on the roots of graph polynomials such as the matching polynomials, the characteristic polynomial, the chromatic polynomial, and many others. In this paper, we study the real roots of the complement degree polynomials of some graphs. Moreover, we investigate the location of the roots of the complement degree polynomials of some graphs.

Keywords: complement degree polynomial, cd -roots.

2020 AMS subject classifications: 05C31, 30C15 ¹

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1 Introduction

In mathematics, a graph polynomial is a graph invariant whose values are polynomials. In algebraic graph theory, invariants of this kind are explored [Shi et al., 2016]. These are some crucial graph polynomials: characteristic polynomial, chromatic polynomial, dichromatic polynomial, flow polynomial, Ihara zeta function, Martin polynomial, matching polynomial, reliability polynomial, Tutte polynomial. There is a lot of research on the roots of graph polynomials, including the characteristic polynomial, the chromatic polynomial, the matching polynomial, and many others. The location and nature of the roots have been important research areas for several graph polynomials. Recently, the present authors [Safeera and Kumar, a] introduced the complement degree polynomial of a graph.

Definition 1.1. Let $G = (V, E)$ be a finite simple graph of order n and let $CD(G, i)$ be the set of vertices of degree i in complement graph $\bar{G}$ and let $Cd_i(G) = |CD(G, i)|$. Then complement degree polynomial of G is the polynomial:

$$CD[G, x] = \sum_{i=\delta(\bar{G})}^{\Delta(\bar{G})} Cd_i(G)x^i, \quad (1)$$

where $\delta(\bar{G})$ and $\Delta(\bar{G})$ respectively denote the minimum degree and maximum degree of the complement graph $\bar{G}$ [Safeera and Kumar, a].

The authors also derived the complement degree polynomial of some well-known graphs and some graph operations [Safeera and Kumar, a,b]. In this paper, we study the real roots of the complement degree polynomial of some graphs obtained in [Safeera and Kumar, a]. In particular, we investigate the location of the roots of the polynomials so obtained.

2 Main Results

The roots of several graph polynomials have drawn a lot of interest, both for what they represent and what their nature and location indicated. In this section, we investigate the roots of the complementary degree polynomial investigated in [Safeera and Kumar, a].

Definition 2.1. The roots of polynomial defined in equation (1) are called cd-roots of G . The number of real cd-roots of a graph G where the multiplicities counted, is denoted by $cd(G)$.

Theorem 2.1. *Zero is a cd -root of a complement degree polynomial of a graph G with n vertices if and only if $\Delta(G) \leq n - 2$.*

Proof. Let G be a graph of order n and zero is a cd -root of the polynomial $CD[G, x]$. If G has a vertex, say v which is adjacent to all other vertices, then v is an isolated vertex in $\overline{G}$. This implies that $CD[G, x]$ has a constant term. This is a contradiction because zero is a cd -root of $CD[G, x]$. Therefore, G has no vertices adjacent to all other vertices.

Conversely, assume that $\Delta(G) \leq n - 2$. Then $\delta(\overline{G}) \geq 1$. Equivalently, $Cd_0(G) = 0$. This tells us that the constant term of $CD[G, x]$ is zero, and hence the result follows. $\square$

Corollary 2.1. *If $\overline{G}$ has no isolated vertices, then zero is a root of $CD[G, x]$ with multiplicity $\delta(\overline{G})$.*

Theorem 2.2. *If G is the non complete graph of order n , then zero is the only cd -root of $CD[G, x]$ if and only if G is a regular graph.*

Proof. First, assume that zero is the only cd -root of a graph G with n vertices. Then it follows that the complement degree polynomial of G is $CD[G, x] = nx^r$. This implies that the degree of every vertex in $\overline{G}$ is the same. Equivalently, G is regular.

Conversely, assume that G is a r -regular graph. Then we have $CD[G, x] = nx^{n-r-1}$. It follows that zero is only cd -root of $CD[G, x]$. $\square$

Corollary 2.2. *If G is the r -regular graph with n vertices, then $cd(G) = n - r - 1$.*

Theorem 2.3. *Let G be a graph with n vertices. Then*

- (1) $CD[G, x]$ is a strictly increasing function in $[0, \infty)$.
- (2) Let G be a graph and H be any spanning subgraph of G . Then the degree of $CD[G, x]$ is less than or equal to the degree of $CD[H, x]$.
- (3) Let G be a graph and H be any induced subgraph of G . Then the degree of $CD[G, x]$ is greater than equal to the degree of $CD[H, x]$.
- (4) Let G be a graph of order n with t isolated vertices in G and r isolated vertices in $\overline{G}$. Then $Cd_0(G) = r$ and $Cd_{n-1}(G) = t$.

Proof Proof of the above result follows from the definition of complement degree polynomial of a graph.

Theorem 2.4. *For a cosplitting graph $CS(G)$ of r -regular graph G with n vertices, $cd(CS(G)) = n$.*

Proof. Observe that $CD[CS(G), x] = nx^{n-1}(1+x^r)$ [Safeera and Kumar, a]. It is clear that $x = 0$ is a cd -root of G with multiplicity $n - 1$. Note that the polynomial $1 + x^r$ has no real roots if r is even and one real root if r is odd. Thus we have,

$$cd(CS(G)) = \begin{cases} n - 1 & \text{if } r \text{ is even,} \\ n & \text{if } r \text{ is odd} \end{cases}$$

. This completes the proof. $\square$

Theorem 2.5. For a path graph P_n ,

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n - 2, & n \geq 3. \end{cases}$$

Proof. For a path graph P_n , we have [Safeera and Kumar, a]:

$$CD[P_n, x] = \begin{cases} 2x^{n-2}, & n = 2 \\ (n - 2)x^{n-3} + 2x^{n-2}, & n \geq 3 \end{cases}.$$

Here we consider two cases:

If $n = 2$, then $CD[P_2, x] = 2$, which has no zeros. If $n > 2$, then we have $CD[P_n, x] = x^{n-3}(2x + n - 2)$. Obviously, $x = 0$ is the cd -root of $CD[P_n, x]$ with multiplicity $n - 3$ and $x = -(n - 2)/2$ is the another cd -root of $CD[P_n, x]$. Thus

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n - 2, & n \geq 3. \end{cases}$$

This completes the proof. $\square$

Theorem 2.6. Let G be a graph with order n and $\mathbf{G} = G \cup G \cup \dots \cup G$ (m times). Then $cd(\mathbf{G}) = n(m - 1) + cd(G)$.

Proof. Let G be a graph of order n and $\mathbf{G} = G \cup G \cup \dots \cup G$ (m times). Then, $CD[\mathbf{G}, x] = mx^{(m-1)n}CD[G, x]$. Observe that $x = 0$ is a zero of $CD[\mathbf{G}, x]$ of multiplicity $n(m - 1)$. Consequently, $cd(\mathbf{G}) = n(m - 1) + cd(G)$. This completes the proof. $\square$

Theorem 2.7. For a ladder graph L_n , $cd(L_n) = 2n - 3$ for $n \geq 2$.

Proof. Obviously, cd -roots of $CD[L_n, x]$ are $x = 0$ with multiplicity $2n - 4$ and $x = -n - 2/2$ with multiplicity one. Hence the result follows. $\square$

Theorem 2.8. For a cocktail party graph CP_n , $cd(CP_n) = 3$ for $n \geq 2$.

Proof. In [Safeera and Kumar, a], the authors proved that $CD[CP_n, x] = 2x^2(2+(n-2)x)$. It follows that $CD[CP_n, x]$ has *cd-roots* $x = 0$ with multiplicity 2 and $x = -2/(n-2)$ with multiplicity one. Thus $cd(CP_n) = 3$. Hence the proof follows. $\square$

Theorem 2.9. *If $S(G)$ is a splitting graph of a graph G with n vertices, then $cd(S(G)) \geq 1$.*

Proof. Observe that $CD[S(G), x]$ do not have a constant term(see [Safeera and Kumar, a]). Hence the result $\square$.

Theorem 2.10. *For a bull graph Bl , $cd(Bl) = 1$.*

Proof. Note that $CD[Bl, x] = x(2x^2 + x + 2)$ [Safeera and Kumar, a]. The roots of this polynomial are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$. Obviously, $x = 0$ is the only real root of $CD[Bl, x]$. Hence the result follows. $\square$

Theorem 2.11. *For a sunlet graph Sl_n , $cd(Sl_n) = 2n - 4$ for $n \geq 3$.*

Proof. Note that $CD[Sl_n, x] = nx^{2n-4}(1+x^2)$ [Safeera and Kumar, a]. Then the *cd-roots* are $x = 0$ with multiplicity $2n - 4$ and $x = \pm i$. Thus $cd(Sl_n) = 2n - 4$. $\square$

Theorem 2.12. *For a tadpole graph $T_{m,n}$, $cd(T_{m,n}) = m + n - 2$ for $m \geq 3$ and $n \geq 1$.*

Proof. Note that $CD[T_{m,n}, x] = x^{m+n-4}(x^2 + (m+n-2)x + 1)$ [Safeera and Kumar, a]. Since the discriminant of the polynomial $x^2 + (m+n-2)x + 1$ is always greater than or equal to zero, it follows that $cd(T_{m,n}) = m + n - 2$. This completes the proof. $\square$

Theorem 2.13. *For a bistar graph $B_{n,n}$ ($n \geq 1$),*

$$cd(B_{n,n}) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Note that $CD[B_{n,n}, x] = 2x^n(nx^n + 1)$. If n is even, then $nx^n + 1$ has only complex roots. If n is odd, then $nx^n + 1$ has only one real root and $n - 1$ complex roots. Hence ,

$$cd(B_{n,n}) = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

This completes the proof. $\square$

Theorem 2.14. For a web graph Wb_n , $cd(Wb_n) = 3n - 4$ for $n \geq 3$.

Proof. Note that $CD[Wb_n, x] = nx^{3n-5}(x^3 + x + 1)$. Obviously, the cd -roots of $CD[Wb_n, x]$ are 0 with multiplicity $3n - 5$, $-0.68233, 0.34116 \pm 1.16154i$. Hence $cd(Wb_n) = 3n - 4$. $\square$

Theorem 2.15. For a armed crown graph $C_n \odot P_m$,

$$cd(C_n \odot P_m) = \begin{cases} n(m+1) - 4, & \text{if } m=1,2 \\ n(m+1) - 2, & \text{if } m \geq 3. \end{cases}$$

Proof. Note that $CD[C_n \odot P_m, x] = x^2 + (m-1)x + 1$. For $m = 1, 2$, the zeros of $x^2 + (m-1)x + 1$ are complex numbers. If $m > 2$, the zeros of $x^2 + (m-1)x + 1$ are real numbers. Thus

$$cd(C_n \odot P_m) = \begin{cases} n(m+1) - 4 & \text{if } m=1,2 \\ n(m+1) - 2 & \text{if } m \geq 3. \end{cases}$$

This completes the proof. $\square$

Theorem 2.16. For a sungraph S_n , $n \geq 3$,

$$cd(S_n) = \begin{cases} n - 2, & \text{if } n \text{ is odd} \\ n - 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Note that $CD[S_n, x] = nx^{n-2}(x^{n-1} + 1)$. Since $x^{n-1} + 1$ has real roots if and only if n is even. This tells us that the real cd -roots of $CD[S_n, x]$ are 0 and -1 if n is even. If n is odd $x = 0$ is the only real root of $CD[S_n, x]$. Therefore, we have

$$cd(S_n) = \begin{cases} n - 2, & \text{if } n \text{ is odd} \\ n - 1, & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. $\square$

Theorem 2.17. For a bipartite cocktaill party graph b_n ($n \geq 2$), we have $cd(B_n) = n$.

Proof. The result follows from the fact that $CD[B_n, x] = 2nx^n$. $\square$

3 Location of the cd -roots of the some graphs

In this section, we investigate the location of the roots of some complement degree polynomials. Here need the following result [Prasolov, 2009].

Theorem 3.1. *Let $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$, where $a_i \in \mathbb{C}$. Then, inside the circle $|z| = 1 + \max |a_i|$, there are exactly n roots of f , multiplicities counted.*

Theorem 3.2. *All the cd -roots of the gear graph G_n lie inside the circle with center $(0, 0)$ and radius $n + 1$.*

Proof. Observe that $CD[G_n, x] = x^n + nx^{2n-3} + nx^{2n-2}$. In this case $\max |a_i| = n$, where a_i 's are the coefficients of $CD[G_n, x]$ for $i = 1, 2, \dots, 2n-2$. Then by theorem 3.1, the result follows. $\square$

Theorem 3.3. *All the cd -roots of the wheel graph W_n lie inside the circle with center $(0, 0)$ and radius n .*

Proof. It follows from the fact that $CD[W_n, x] = (n-1)x^{n-4} + 1$. $\square$

Theorem 3.4. *All the cd -roots of the bull graph B_l lie on the unit circle centered at the origin.*

Proof. Note that the cd -roots of B_l are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$. These three roots lie on the unit circle centered at the origin. $\square$

Theorem 3.5. *All the cd -roots of the sunlet graph Sl_n lies in the disk $|z| \leq 1$.*

Proof. The cd -roots of the sunlet graph $CD[Sl_n, x]$ are $x = 0$ and $x = \pm i$. Hence the result follows. $\square$

Theorem 3.6. *All the cd -roots of the sun graph S_n lies in the disk $|z| \leq 1$.*

Proof. Note that $CD[S_n, x] = nx^{2n-3} + nx^{n-2} = nx^{n-2}(x^{n-1} + 1)$. Obviously, roots of $x^{n-1} + 1$ lie on the unit circle. Hence the result. $\square$

4 Conclusions

In this paper, we introduced cd -roots of the complement degree polynomial of some graphs. Moreover, we investigated the location of the cd - roots of some complementary degree polynomials.

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Equitable eccentric domination in graphs

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Abstract

In this paper, we define equitable eccentric domination in graphs. An eccentric dominating set $S \subseteq V(G)$ of a graph $G(V, E)$ is called an equitable eccentric dominating set if for every $v \in V - S$ there exist at least one vertex $u \in S$ such that $|d(v) - d(u)| \leq 1$ where $vu \in E(G)$. We find equitable eccentric domination number $\gamma_{eqed}(G)$ for most popular known graphs. Theorems related to $\gamma_{eqed}(G)$ have been stated and proved.

Keywords: eccentricity, equitable domination number, equitable eccentric domination number.

2020 AMS subject classifications: 05C69. ¹

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1 Introduction

A graph is a representation of a pair of sets (V, E) , where V is the set of vertices and E is the set of edges which are connecting the pair of vertices. Graph theory has its application in many fields such as computation, social and natural science etc. Any problems of mathematics, science and engineering can be represented in the form of a graph. The concept of graph theory was first introduced by Leonard Euler in the year 1736. He created the first graph as a solution to solve the problem of seven bridges of Konigsberge built across the pregel river of prussia. Graph theory has experienced tremendous growth, the main reason for this phenomena is applicability of graph theory in different disciplines. Graph theory becomes interesting because graphs can be used to model situations that occur in real world problems. These problems can be studied with the aid of graphs.

The concept of domination in graphs was studied by Ore and Berge. Ore[11] introduced domination in graphs in his famous book 'Theory of graphs' in 1962. Cockyane and Hedetniemi[3] also contributed several results pertaining to domination. They unfolded different aspects, by swaying all available results bringing to light new ideas and emphasizing its applicable potential in a variety of scientific ideas in their paper 'Towards a theory of domination in graphs'. T.W.Haynes, S.Hedetniemi and P.Slater[6] have briefly discussed on various domination parameters in the book Fundamentals of domination in graphs.

T.N. Janakiraman et al[9] introduced the concept of eccentric domination in graphs in 2010. Kuppusamy Markandan Dharmalingam[4] introduced equitable graph of a graph. E. Sampathkumar et al[1] introduced degree equitable sets in a graph. V Swaminathan and K.M. Dharmalingam[12] introduced degree equitable domination in graphs. Basavanagoud et al[2] introduced equitable dominating graph.

The concept of eccentricity by T.N. Janakiraman et al has inspired researchers which has led to many invariants of eccentric dominations in graphs. Some of the extended eccentric dominations are accurate eccentric domination[7] and equal eccentric domination[8]. The concept of geodesic distance is very important. The existing eccentric domination only highlighted the idea based on an eccentric vertex and its domination. The proposed equitable eccentric domination was mainly necessary because it highlights the properties of a vertex in a graph, it considers the connectivity between the vertices where the difference between their vertex degrees is less than or equal to one. Equitable domination when incorporated with eccentric domination yeilds equitable eccentric domination which concentrates on the vertex degree, geodesic distance, eccentricity, eccentric vertex and domination. In this paper, we introduce equitable eccentric domination in graphs. We

find equitable eccentric dominating set, equitable eccentric domination number $\gamma_{eqed}(G)$, upper equitable eccentric dominating set and upper equitable eccentric domination number $\Gamma_{eqed}(G)$ of different standard graphs. For undefined graph terminologies refer the book 'Graph theory' by Frank Harary[5].

2 Preliminaries

Definition 2.1 ([11]). Let G be a graph with the vertex set V . A subset D of V is a dominating set for G when every vertex not in D is the endpoint of some edge from a vertex in D .

Definition 2.2 ([10]). Let $\gamma(G)$ (called the domination number) and $\Gamma(G)$ (called the upper domination number) be the minimum cardinality and the maximum cardinality of a minimal dominating set of G , respectively.

Definition 2.3 ([6]). The degree $\deg(v)$ of v is the number of edges incident with v .

Definition 2.4 ([9]). The eccentricity $e(v)$ of v is the distance to a vertex farthest from v . Thus, $e(v) = \max\{d(u, v) : u \in V\}$. For a vertex v , each vertex at a distance $e(v)$ from v is an eccentric vertex. Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) / d(u, v) = e(v)\}$.

Definition 2.5 ([9]). The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $\text{diam}(G)$ is the maximum eccentricity.

Definition 2.6 ([9]). v is a central vertex if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices. v is a peripheral vertex if $e(v) = \text{diam}(G)$. The periphery $P(G)$ is the set of all peripheral vertices.

Definition 2.7 ([9]). A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D . If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set. An eccentric dominating set D is a minimal eccentric dominating set if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition 2.8 ([9]). The eccentric domination number $\gamma_{ed}(G)$ of a graph G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in $\mathcal{D}$, where $\mathcal{D}$ is the set of all minimal eccentric dominating sets of G .

Definition 2.9 ([4]). A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ^e and is called the equitable domination number of G .

3 Equitable eccentric domination in graphs

In this section we introduce equitable eccentric domination, theorems related to equitable eccentric domination number of family of graphs are stated and proved.

Definition 3.1. An eccentric dominating set $S \subseteq V(G)$ is called an equitable eccentric dominating set (EQED-set) if for every $v \in V - S$ there exist at least one vertex $u \in S$ such that $vu \in E(G)$ and $|d(v) - d(u)| \leq 1$.

Definition 3.2. An equitable eccentric dominating set S is called a minimal equitable eccentric dominating set if no proper subset of S is equitable eccentric dominating set.

Definition 3.3. The equitable eccentric domination number $\gamma_{eqed}(G)$ of a graph G is the minimum cardinality among the minimal equitable eccentric dominating sets of G .

Definition 3.4. The upper equitable eccentric domination number $\Gamma_{eqed}(G)$ of a graph G is the maximum cardinality among the minimal equitable eccentric dominating sets of G .

Example 3.1.

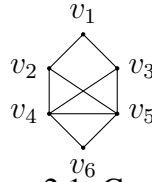


Figure 2.1: Graph G

Consider the graph G consists of 6 vertices given in figure 2.1. Here the dominating set is $S = \{v_1, v_4\}$ but not eccentric dominating set since $E(v_3) = \{v_2, v_6\}$ not in S . The eccentric dominating set is $S = \{v_1, v_6\}$ but not equitable eccentric dominating set since $|d(v_4) - d(v_6)| = 2$. The equitable eccentric dominating set is $S = \{v_1, v_2, v_6\}$.

Remark 3.1. For any path P_n where $n \geq 3$,

1. Every minimum EQED-set contains the pendant vertices.
2. If D_1, D_2, D_3 are minimum EQED-sets of paths P_{n-1}, P_n, P_{n+1} consecutively where $n = 3k$ and $k > 1$. Then $|D_1| = |D_2| = |D_3|$. Therefore for $k = 2$, $\gamma_{eqed}(P_5) = \gamma_{eqed}(P_6) = \gamma_{eqed}(P_7) = 3$.

Theorem 3.1. For complete graph K_n , $\gamma_{eqed}(K_n) = 1, \forall n \geq 2$.

Proof. In a complete graph K_n all the vertices are eccentric vertices to each other. If $v \in V(K_n)$ then the eccentric vertex $E(v) = V(K_n) - \{v\}$ and every singleton set forms a dominating set. For every vertex $v \in D \exists$ a vertex $u \in V(K_n) - D \ni |deg(u) - deg(v)| \leq 1$ where $uv \in E(K_n)$. Therefore every single vertex of K_n is an EQED-set. Hence $\gamma_{eqed}(K_n) = 1$.

Theorem 3.2. For path graph P_n where $n > 1$,

$$\gamma_{eqed}(P_n) = \begin{cases} 1, & \text{for } n = 2 \\ \lfloor \frac{n+1}{3} \rfloor + 1, & \forall n \geq 3 \end{cases}$$

Proof. Case(i): For a path P_2 , $V(P_2) = \{v_1, v_2\}$. Both the vertices are eccentric vertices to each other. Therefore $D = \{v_1\}$ or $\{v_2\}$ and $|deg(v_2) - deg(v_1)| = 0$, where $v_1v_2 \in E(P_2)$. Hence $\gamma_{eqed}(P_2) = 1$.

Case(ii): For a path P_n where $n \geq 3$. The pendant or end vertices of the path form the eccentric vertices ie, if $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$, $E(v_1) = \{v_n\}$ and $E(v_n) = \{v_1\}$. $E(v_i) = \{v_1\}$ or $\{v_n\}$ for any $v_i \in V(P_n)$ where n is even. If n is odd then $E(v_i) = \{v_1\}$ or $\{v_n\}$. For P_n where ' n ' is odd, the central vertex v_i has two eccentric vertices ie, $E(v_i) = \{v_1, v_n\}$. Degree of end vertices is 1 and degree of all the intermediate vertices is 2. The EQED-set contains both the pendant vertices. Both v_1 and v_n being pendant vertices dominate the vertices adjacent to them and the minimum dominating set among the intermediate vertices along with two pendant vertices forms an EQED-set. Since $|deg(u) - deg(v)| \leq 1$ where $uv \in E(P_n)$ for all $u \in D$ and $v \in V(P_n) - D$ and $\exists$ an eccentric vertex $u \in D$ for every $v \in V(P_n) - D$. For P_n where $n = 3k$ and $k > 2$, number of vertices of $P_{3k-1}, P_{3k}, P_{3k+1}$ are same. Every minimum equitable eccentric domination set of D contains $\lfloor \frac{n+1}{3} \rfloor + 1$ number of vertices.

Theorem 3.3. For star graph S_n ,

$$\gamma_{eqed}(S_n) = \begin{cases} 2, & \text{if } n = 3 \\ 0, & \text{if } n \neq 3 \end{cases}$$

Proof. Case(i): If $n = 3$, then the star graph S_3 is isometric to P_3 . From the theorem-3.2 $\gamma_{eqed}(P_3) = \gamma_{eqed}(S_3) = 2$.

Case(ii): If $n \neq 3$ then S_n is of the form $S_4, S_5, S_6, \dots$. For any graph S_n where $n \neq 3$, there can be many dominating sets and eccentric dominating sets but we cannot find a EQED-set because of the central vertex v_i of the star graph has degree ≥ 3 . The degree of every pendant vertex u of a star graph is 1, $deg(u) = 1, u \in V(S_n) - \{v_i\}$. The degree of central vertex v_i of a star graph is given by $deg(v_i) = n - 1$. Since, central vertex $v_i \in V(S_n)$ then either $v_i \in D$ or $v_i \in V(S_n) - D$. Therefore $|deg(v_i) - deg(u)| > 1$ always which doesnot satisfy the condition to be a EQED-set. Hence $\gamma_{eqed}(S_n) = 0$ where $n \neq 3$.

Theorem 3.4. For cycle graph C_n where $n \geq 3$,

$$\gamma_{eqed}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even } \forall n \geq 4 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \text{ is odd \& } n = 3k \forall k = 1, 3, 5, 7, \dots \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise} \end{cases}$$

Proof. Case(i): If ' n ' is even and $n \geq 4$. Let the cycles C_n be of the form $C_4, C_6, C_8, C_{10}, \dots, C_{2n}$. In an even cycle if $u \in V(C_n)$ the eccentric vertex of u , $E(u) = \{v\}$ is always placed at a distance of $\frac{n}{2}$ edges from it and every vertex has a unique eccentric vertex to form the first eccentric dominating set. The set D must contain $\frac{n}{2}$ vertices in such a way that for every $v \in D$ then $E(v) \notin D$ or for some $u \in V(C_n) - D$, $E(u) \notin V(C_n) - D$. Then if the vertex u and $E(u) \in D$ then we cannot construct a eccentric dominating set. Further if we reduce the cardinality of D to less than $\frac{n}{2}$ we will have u and $E(u)$ in $V - D$. Therefore D must contain $\frac{n}{2}$ vertices with all the unique eccentric vertices in $V - D$. Then for any $u \in V(C_n) - D \exists$ a vertex $v \in D$ such that $|deg(u) - deg(v)| \leq 1$ where $uv \in E(C_n)$ for every vertex $v_i \in C_n$, $deg(v_i) = 2$. Therefore $|deg(u) - deg(v)| = |2 - 2| = 0$. Therefore $\gamma_{eqed}(C_n) = \frac{n}{2}$.

Case(ii): Now we have the odd cycles of the form $C_3, C_9, C_{15}, C_{21}, \dots, C_{3k}$. Every vertex $u \in V(C_n)$ has two eccentric vertices v_i, v_j such that $E(u) = \{v_i, v_j\}$. The eccentric vertices v_i, v_j will always be adjacent i.e., $v_i, v_j \in E(C_n)$. v_i, v_j are placed at a distance of $\frac{n-1}{2}$ edges from u . Since every vertex u can dominate its adjacent vertices v, w . $\frac{n}{3}$ set of vertices form a dominating set of a cycle. The dominating set $D = \{v_i, v_j, v_n\}$ forms the EQED-set such that no eccentric vertices of $v_i \in D$ are in D . Then $\forall v \in V(C_n) - D \exists$ a vertex $u \in D \ni |deg(u) - deg(v)| = |2 - 2| = 0$. Therefore $\gamma_{eqed}(C_n) = \lceil \frac{n}{3} \rceil$.

Case(iii): If $n = 3k+1$ where k is even. The cycles are of the form $C_7, C_{13}, C_{19}, \dots, C_{3k+1}$ and if $n = 3k+1$ where k is odd, the cycles are of the form $C_5, C_{11}, C_{17}, \dots, C_{3k+2}$. Totally we have $C_5, C_7, C_{11}, C_{13}, C_{17}, C_{19}, \dots, C_{3k+1}, C_{3k+2}$. Similar to case(ii) every vertex $v_i \in V(C_n)$ has two eccentric vertices v_l, v_m , $E(v_i) = \{v_l, v_m\}$ such that v_l and v_m are adjacent i.e., $v_l, v_m \in E(C_n)$. Eccentric vertex v_l and v_m of v_i are placed at a distance of $\frac{n-1}{2}$ from v_i . If $n = 3k$ we get 3, 9, 15, 21, $\dots$ which are the multiples of 3 we get a whole number which forms the cardinality of a EQED-set as proved in case(ii). But when $n = 3k + 1$ or $n = 3k + 2$ then $n = 5, 7, 11, 13, 17, 19, \dots, 3k + 1, 3k + 2$ which are not multiples of 3 we get a fraction value and also we are left out with a vertex which is to be dominated. Therefore the cardinality of the EQED-set of a cycles of the form C_{3k+1}, C_{3k+2} increases by 1. Hence $\gamma_{eqed}(C_n) = \lceil \frac{n}{3} \rceil + 1$.

Theorem 3.5. Every EQED-set in a wheel graph $W_n, n \geq 6$ contains the central vertex.

Proof. Let v_1 be the central vertex of the wheel graph $W_n, n \geq 6$ then $\deg(v_1) = n - 1 = \Delta(W_n)$. The degree of any non-central vertex $u \in V(W_n)$ is $\deg(u) = 3 = \delta(W_n)$.

Suppose the central vertex $v_1 \in V(W_n) - D, u \in D$ and D is an minimal eccentric dominating set we need to check for the condition of equitable domination then for $v_1 \in V(W_n) - D$ and $u \in D$, we have $uv_1 \in E(W_n)$

$$|\deg(v_1) - \deg(u)| = |\Delta(G) - \delta(G)|$$

$$|\deg(v_1) - \deg(u)| = |(n - 1) - 3|$$

$$|\deg(v_1) - \deg(u)| = |n - 4| \text{ where } n \geq 6$$

$$|\deg(v_1) - \deg(u)| > 1. \text{ which is a contradiction.}$$

Therefore the central vertex v_1 must belong to D , if the set D is a equitable eccentric dominating set of W_n .

Theorem 3.6. *Let W_n be a wheel graph where $n \geq 5$ then EQED-set contains more than one vertex.*

Proof. In any wheel graph W_n where $n \geq 5$. If the set $D \subseteq V(W_n)$ contains the central vertex v_1 then D forms a dominating set as $\deg(v_1) = n - 1 = \Delta(G)$. But the eccentric vertices of a central vertex v_1 is given by $E(v_1) = V - \{v_1\}$ and the eccentric vertex of any non-central vertex u is given by $E(u) = V - N[u]$. Therefore there is no eccentric dominating or equitable eccentric dominating set of cardinality 1 for W_n where $n \geq 5$.

Theorem 3.7. *For wheel graph W_n , where $n \geq 4$ we have*

$$\gamma_{eqed}(W_n) = \begin{cases} 1, & \text{if } n = 4 \\ 4, & \text{if } n = 6 \\ \lfloor \frac{n}{2} \rfloor, & \text{if } n \text{ is odd and } n \geq 5 \\ \lfloor \frac{n+1}{3} \rfloor + 1, & \forall n \geq 8 \text{ and } n \text{ is even} \end{cases}$$

Proof. Case(i): If $n = 4$, W_4 is isometric to K_4 , then by theorem-3.1 $\gamma_{eqed}(W_4) = \gamma_{eqed}(K_4) = 1$.

Case(ii): If $n = 6$, in a wheel graph W_6 , there are no eccentric dominating sets of cardinality 1 or 2. Therefore we do not get an EQED-set of cardinality 1 or 2. There are sets of cardinality 3 which are eccentric dominating sets. But they do not form an EQED-set as the central vertex should not be present in $V - D$. Since the degree of central vertex v_i is $\deg(v_i) = n - 1 = 5$ and degree of any other non-central vertex is $\deg(v_j) = 3$. Therefore $|\deg(v_i) - \deg(v_j)| = 2 > 1$ and in other cases if $v_i \notin V - D$ then we find a combination of vertices of 3 cardinality which are eccentric dominating set but they dont form an INED-set since for some vertex $v \in V - D$ there is no vertex $u \in D$ such that $u, v \notin E(W_6)$. But we find a EQED-set with cardinality 4 as we have the central vertex in D . Then $|\deg(v_i) - \deg(v_j)| \leq 1, (v_i, v_j) \in E(W_6)$ where $v_i \in D$ and $v_j \in V(W_6) - D$.

Therefore $\gamma_{eqed}(W_6) = 4$.

Case(iii): If n is odd and $n \geq 5$ we have the wheel graph of order $W_5, W_7, W_9, W_{11}, \dots$

If $v \in V(W_n)$ then $|E(v)| = n - 4$. There will always be $n - 4$ vertices which form the eccentric vertex $E(v)$ for every vertex v . And for any wheel graph where ' n ' is odd. The set $D \subseteq V(W_n)$ forms an eccentric dominating sets only when $|D| = \lfloor \frac{n}{2} \rfloor$. Then for every $v \in V(W_n) - D$ there exists a vertex $u \in D$ such that $|deg(u) - deg(v)| \leq 1$ and $(u, v) \in E(W_n)$. Therefore $\gamma_{eqed}(W_n) = \lfloor \frac{n}{2} \rfloor$.

Case(iv): The wheel graph W_n where n is even and $n \geq 8$ has $n - 4$ eccentric vertices. We have wheel graphs $W_8, W_{10}, W_{12}, \dots$. For every vertex $v \in V(W_n)$, $|E(v)| = n - 4$. From theorem-3.6,3.5, $\gamma_{eqed}(W_n) \neq 1$ and the central vertex $v_i \in D$ then D contains other vertices of W_n where cardinality of D is of the form $\lfloor \frac{n+1}{3} \rfloor + 1$. For every $v \in V - D$ there exists a vertex $u \in D$ such that $E(v)$ lies in D and $|deg(u) - deg(v)| \leq 1$ such that there exists an edge between u and v . Therefore $\gamma_{eqed}(W_n) = \lfloor \frac{n+1}{3} \rfloor + 1$.

Theorem 3.8. *An EQED-set D is a minimal EQED-set if one of the following conditions holds,*

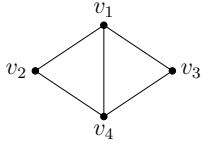
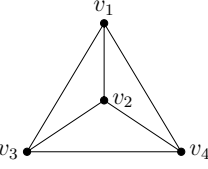
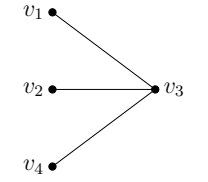
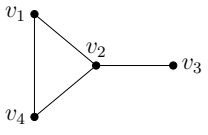
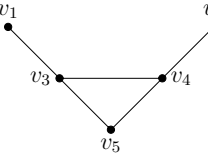
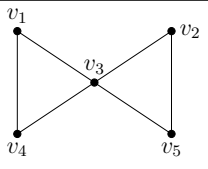
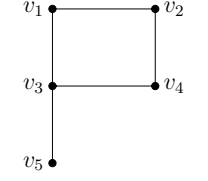
1. *For every vertex u in $V - D$ there does not exists v in D such that $E(u) = \{v\}$ ie, u has no eccentric vertex in D .*
2. *There exists some $u \in V - D$ such that $N(u) \cap D = \{v\}$, $E(u) \cap D = \{v\}$ and $|d(u) - d(v)| \leq 1$ where $uv \in E(G)$.*

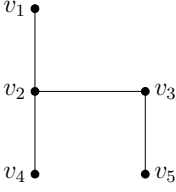
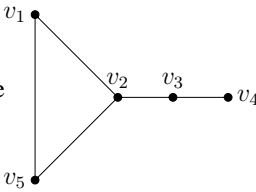
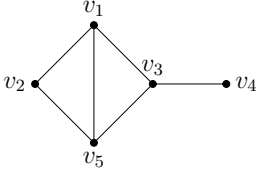
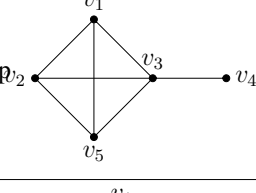
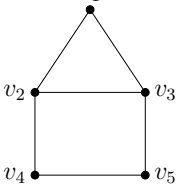
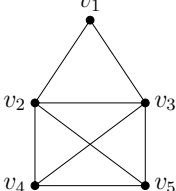
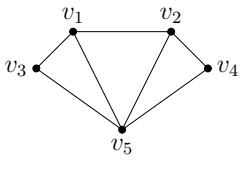
Proof. Suppose D is a minimal EQED-set of G . Then for every vertex v in D , $D - \{v\}$ is not an EQED-set. Thus there exists some vertex u in $V - D \cup \{v\}$ which is not dominated by any vertex in $D - \{v\}$ or there exists $u \in V - D \cup \{v\}$ such that u does not have an eccentric vertex in $D - \{v\}$ ie, $E(u) \neq D - \{v\}$ or $|d(u) - d(v)| \not\leq 1$ or $uv \notin E(G)$. $\therefore$ The concept of equitable condition does not hold. Case(i): If $v = u$ then u does not have an eccentric vertex in D ie, $E(u) \neq D$. Case(ii): If $v \neq u$, (a) If $u \in V - D$ and u is not dominated by $D - \{v\}$, but dominated by D then u is adjacent to only v in D ie, $N(u) \cap D = \{v\}$. (b) If $u \in V - D$ and u does not have an eccentric vertex in $D - \{v\}$ but u has an eccentric vertex in D . Thus v is the only eccentric vertex of u in D ie, $E(u) \cap D = \{v\}$. (c) If $u \in V - D$ and $|d(u) - d(x)| \not\leq 1$ or $ux \notin E(G)$ where $x \in D - \{v\}$ but $|d(u) - d(v)| \leq 1$ and $uv \in E(G)$. Conversely, Suppose D is an EQED-set and for each $v \in D$, one of the two conditions holds. Now we show that D is a minimal EQED-set. Suppose D is not a minimal EQED-set ie, there exists a vertex $v \in D$ such that $D - \{v\}$ is an EQED-set. Hence v is adjacent to at least one vertex x in $D - \{v\}$, v has an eccentric vertex in $D - \{v\}$ ie, $E(v) \in D - \{v\}$ and $|d(u) - d(x)| \leq 1$ where $ux \in E(G)$. $\therefore$ Equitable condition holds and EQED-set exists. Also if $D - \{v\}$ is an EQED-set, then every vertex u in $V - D$ is adjacent to at least one vertex x in $D - \{v\}$, u has an eccentric vertex in $D - \{v\}$ ie, $E(u) \in D - \{v\}$ and $|d(u) - d(x)| \leq 1$ and $ux \in E(G)$. Therefore condition-(2)

Equitable eccentric domination

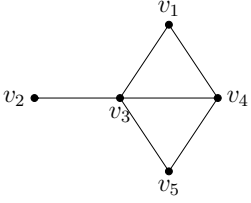
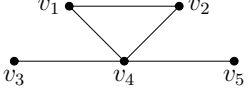
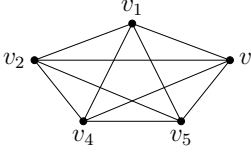
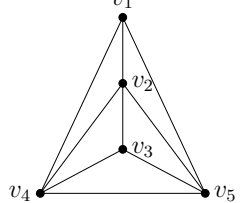
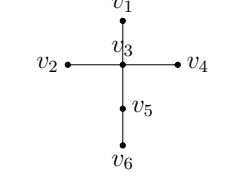
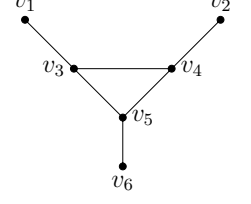
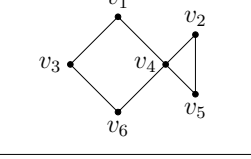
does not hold. Hence neither condition-(1) nor (2) holds, which is a contradiction to our assumption. Hence for each $v \in D$ one of the 2 conditions holds.

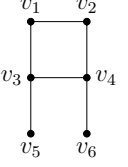
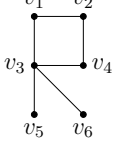
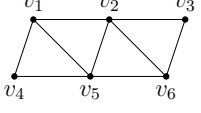
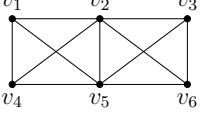
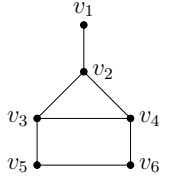
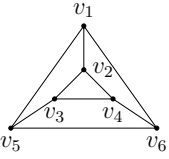
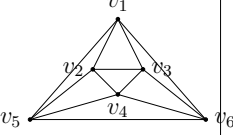
The equitable eccentric dominating set, $\gamma_{eqed}(G)$, upper equitable eccentric dominating set and $\Gamma_{eqed}(G)$ of standard graphs are tabulated.

Graph	Figure	D - Minimum EQED set. $ D = \gamma_{eqed}(G)$	$\gamma_{eqed}(G)$	S - Upper EQED set. $ S = \Gamma_{eqed}(G)$	$\Gamma_{eqed}(G)$
Diamond graph		$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_2, v_3\},$ $\{v_2, v_4\},$ $\{v_3, v_4\}.$	2	$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_2, v_3\},$ $\{v_2, v_4\},$ $\{v_3, v_4\}.$	2
Tetrahedral graph		$\{v_1\},$ $\{v_2\},$ $\{v_3\},$ $\{v_4\}.$	1	$\{v_1\},$ $\{v_2\},$ $\{v_3\},$ $\{v_4\}.$	1
Claw graph		Does not exist	0	Does not exist	0
Paw graph		$\{v_1, v_3\},$ $\{v_2, v_3\},$ $\{v_3, v_4\}.$	2	$\{v_1, v_3\},$ $\{v_2, v_3\},$ $\{v_3, v_4\}.$	2
Bull graph		$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_4\},$ $\{v_1, v_2, v_5\}.$	3	$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_4\},$ $\{v_1, v_2, v_5\}.$	3
Butterfly graph		$\{v_1, v_2, v_3\},$ $\{v_1, v_3, v_5\},$ $\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\}.$	3	$\{v_1, v_2, v_3\},$ $\{v_1, v_3, v_5\},$ $\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\}.$	3
Banner graph		$\{v_1, v_2, v_5\},$ $\{v_1, v_3, v_5\},$ $\{v_2, v_3, v_5\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_5\}.$	3	$\{v_1, v_2, v_5\},$ $\{v_1, v_3, v_5\},$ $\{v_2, v_3, v_5\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_5\}.$	3

Graph	Figure	D - Minimum EQED set. $ D = \gamma_{eqed}(G)$	$\gamma_{eqed}(G)$	S - Upper EQED set. $ S = \Gamma_{eqed}(G)$	$\Gamma_{eqed}(G)$
Fork graph		$\{v_1, v_2, v_3, v_4\},$ $\{v_1, v_2, v_4, v_5\},$ $\{v_1, v_3, v_4, v_5\}.$	4	$\{v_1, v_2, v_3, v_4\},$ $\{v_1, v_2, v_4, v_5\},$ $\{v_1, v_3, v_4, v_5\}.$	4
(3,2)-Tadpole graph		$\{v_1, v_4\},$ $\{v_4, v_5\}.$	2	$\{v_1, v_2, v_3, v_5\}.$	4
Kite graph		$\{v_1, v_2, v_4\},$ $\{v_1, v_3, v_4\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_5\}.$	3	$\{v_1, v_2, v_4\},$ $\{v_1, v_3, v_4\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_5\}.$	3
(4,1)-Lollipop graph		$\{v_1, v_4\},$ $\{v_2, v_4\},$ $\{v_3, v_4\},$ $\{v_4, v_5\}.$	2	$\{v_1, v_4\},$ $\{v_2, v_4\},$ $\{v_3, v_4\},$ $\{v_4, v_5\}.$	2
House graph		$\{v_2, v_4\},$ $\{v_3, v_5\}.$	2	$\{v_1, v_2, v_3\},$ $\{v_1, v_4, v_5\}.$	3
House X graph		$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_1, v_4\},$ $\{v_1, v_5\}.$	2	$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_1, v_4\},$ $\{v_1, v_5\}.$	2
Gem graph		$\{v_1, v_2\}.$	2	$\{v_1, v_3, v_4\},$ $\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\}.$	3

Equitable eccentric domination

Graph	Figure	D - Minimum EQED set. $ D = \gamma_{eqed}(G)$	$\gamma_{eqed}(G)$	S - Upper EQED set. $ S = \Gamma_{eqed}(G)$	$\Gamma_{eqed}(G)$
Dart graph		$\{v_2, v_4\}.$	2	$\{v_1, v_2, v_3, v_5\}.$	4
Cricket graph		$\{v_1, v_3, v_4, v_5\},$ $\{v_2, v_3, v_4, v_5\}.$	4	$\{v_1, v_3, v_4, v_5\},$ $\{v_2, v_3, v_4, v_5\}.$	4
Pentatope graph		$\{v_1\},$ $\{v_2\},$ $\{v_3\},$ $\{v_4\},$ $\{v_5\}.$	1	$\{v_1\},$ $\{v_2\},$ $\{v_3\},$ $\{v_4\},$ $\{v_5\}.$	1
Johnson solid skeleton-12 graph		$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_1, v_4\},$ $\{v_1, v_5\},$ $\{v_2, v_3\},$ $\{v_3, v_4\},$ $\{v_3, v_5\}.$	2	$\{v_1, v_2\},$ $\{v_1, v_3\},$ $\{v_1, v_4\},$ $\{v_1, v_5\},$ $\{v_2, v_3\},$ $\{v_3, v_4\},$ $\{v_3, v_5\}.$	2
Cross graph		$\{v_1, v_2, v_3, v_4, v_5\},$ $\{v_1, v_2, v_3, v_4, v_6\}.$	5	$\{v_1, v_2, v_3, v_4, v_5\},$ $\{v_1, v_2, v_3, v_4, v_6\}.$	5
Net graph		$\{v_1, v_2, v_3, v_6\},$ $\{v_1, v_2, v_4, v_6\},$ $\{v_1, v_2, v_5, v_6\}.$	4	$\{v_1, v_2, v_3, v_6\},$ $\{v_1, v_2, v_4, v_6\},$ $\{v_1, v_2, v_5, v_6\}.$	4
Fish graph		$\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\}.$	3	$\{v_1, v_2, v_4, v_5, v_6\}.$	5

Graph	Figure	D - Minimum EQED set. $ D = \gamma_{eqed}(G)$	$\gamma_{eqed}(G)$	S - Upper EQED set. $ S = \Gamma_{eqed}(G)$	$\Gamma_{eqed}(G)$
A graph		$\{v_1, v_2, v_5, v_6\},$ $\{v_1, v_3, v_5, v_6\},$ $\{v_1, v_4, v_5, v_6\},$ $\{v_2, v_3, v_5, v_6\},$ $\{v_2, v_4, v_5, v_6\},$ $\{v_3, v_4, v_5, v_6\}.$	4	$\{v_1, v_2, v_5, v_6\},$ $\{v_1, v_3, v_5, v_6\},$ $\{v_1, v_4, v_5, v_6\},$ $\{v_2, v_3, v_5, v_6\},$ $\{v_2, v_4, v_5, v_6\},$ $\{v_3, v_4, v_5, v_6\}.$	4
R graph		$\{v_2, v_3, v_5, v_6\}.$	4	$\{v_1, v_3, v_4, v_5, v_6\}.$	5
4-polynomial graph		$\{v_1, v_2, v_3\},$ $\{v_1, v_3, v_4\},$ $\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\},$ $\{v_3, v_4, v_6\},$ $\{v_4, v_5, v_6\}.$	3	$\{v_1, v_2, v_3\},$ $\{v_1, v_3, v_4\},$ $\{v_2, v_3, v_4\},$ $\{v_3, v_4, v_5\},$ $\{v_3, v_4, v_6\},$ $\{v_4, v_5, v_6\}.$	3
(2,3)-King graph		$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_6\},$ $\{v_1, v_3, v_5\},$ $\{v_1, v_5, v_6\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_6\},$ $\{v_3, v_4, v_5\},$ $\{v_4, v_5, v_6\}.$	3	$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_6\},$ $\{v_1, v_3, v_5\},$ $\{v_1, v_5, v_6\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_6\},$ $\{v_3, v_4, v_5\},$ $\{v_4, v_5, v_6\}.$	3
Antenna graph		$\{v_1, v_2, v_5\},$ $\{v_1, v_2, v_6\},$ $\{v_1, v_3, v_5\},$ $\{v_1, v_3, v_6\},$ $\{v_1, v_4, v_5\},$ $\{v_1, v_4, v_6\}.$	3	$\{v_1, v_2, v_3, v_4\}.$	4
3-prism graph		$\{v_1, v_2\},$ $\{v_3, v_5\},$ $\{v_4, v_6\}.$	2	$\{v_1, v_5, v_6\},$ $\{v_2, v_3, v_4\}.$	3
Octahedral graph		$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_5\},$ $\{v_1, v_3, v_6\},$ $\{v_1, v_5, v_6\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_6\},$ $\{v_4, v_5, v_6\}.$	3	$\{v_1, v_2, v_3\},$ $\{v_1, v_2, v_5\},$ $\{v_1, v_3, v_6\},$ $\{v_1, v_5, v_6\},$ $\{v_2, v_3, v_4\},$ $\{v_2, v_4, v_5\},$ $\{v_3, v_4, v_6\},$ $\{v_4, v_5, v_6\}.$	3

4 Conclusions

Inspired by eccentric dominating set and equitable dominating set we introduce the equitable eccentric dominating set. We find minimum equitable eccentric dominating set, minimum equitable eccentric domination number $\gamma_{eqed}(G)$, upper equitable eccentric dominating set and upper equitable eccentric domination number $\Gamma_{eqed}(G)$ of different standard graphs. We have discussed the properties and proved theorems related to equitable eccentric dominating set of family of graphs.

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Square sum and square difference labelings of semitotal-block graph for some class of graphs

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Abstract

A graph G is said to be square sum and square difference labeling, if there exists a bijection f from $V(G)$ to $\{1, 2, 3, \dots, (p-1)\}$ which induces the injective function f^* from $E(G)$ to N , defined by $f^*(uv) = f(u)^2 + f(v)^2$ and $f^*(uv) = f(u)^2 - f(v)^2$ respectively, for each $uv \in E(G)$ and the resulting edges are distinctly labeled. G is said to be square sum and square difference graph, if it admits a square sum and square difference labeling respectively. The present work investigates, square sum and square difference labeling of semitotal-block graph for some class of graphs which are proved using number theory concept.

Keywords: Square Sum Labeling; Square Difference Labeling; Semitotal-Block Graph; Comb Graph; Crown Graph; and Friendship Graph.

2020 AMS subject classifications: 05C05, 05C38, 05C76, 05C78.¹

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1 Introduction

Graph labeling is one of the fascinating and active area in the field of graph theory and it is the numbering of integers to the vertices or edges, or both under particular conditions and is mainly divided into two parts namely, vertex labeling function and edge labeling function. The numbers considered here are positive integers which was introduced by A. Rosa in 1967 by identifying four types of labelings namely, α , β , γ , and δ . β -labeling was later renamed as graceful and the name has been popular since. Graph labeling has a wide range of applications such as X-ray crystallography, coding theory, radar, astronomy, circuit design, network theory, communication networks and database management.

Nowadays, research in labeling of graph is increasingly expanding. In the previous 60 years, more than 200 kinds of labeling have been studied and nearly 2500 articles have been published. One such labeling is square sum labeling defined by V. Ajitha, S. Arumugam, and K. A. Germina in 2009. They investigated square sum labeling of trees, unicyclic graphs, mC_n , $m \geq 1$, cycle with a chord, the graph obtained by joining two copies of cycle C_n by a path P_k , and the graph defined by path union of k copies of C_n and also investigated that complete graph, when $n \geq 6$ is not square sum. R. Sebastian et al. [Sebastian and Germina, 2015] discussed on this concept by introducing square sum labeling of class of planar graphs. G.V. Ghodasara and M.J. patel [Ghodasara and Patel, 2017] introduced some bistar related square sum graphs. Afterwords, J. B. Babujee and S. Babitha [Babujee and Babitha, 2012] worked on some graphs which admits square sum labeling. One of the interesting labeling called square difference labeling which was first introduced by J. Shiama [Shiama, 2012] in 2012 and investigated the results on cycles, complete graphs, cycle cactus, ladder, lattice grids, wheels and quadrilateral snakes which admits square difference labeling. Also proved that the path graph is an odd square difference graph. In this article, square sum and square difference labelings of semitotal-block graphs for some class of graphs are obtained.

All the graphs mentioned in this article are finite, simple, undirected, and connected. For graph theoretic terms and system of symbols refer [F.Harary, 1969] and [Bondy and Murthy, 1976]. For latest and regular updates of different forms of labeling skills refer [Gallian, 2020] and for the terms related to number theory refer [Burton, 2006]

2 Preliminaries

Definition 2.1. [Ghudasara and Patel, 2018] A (p, q) connected graph G is said to be **square sum labeling (SSL)** if there exists a bijection f from $V(G)$ to $\{0, 1, 2, \dots, (p-1)\}$, which induces the injection f^* and values from $E(G)$ to N given by $f^*(uv) = |f(u)^2 + f(v)^2|$ for all $e = uv$ are distinct.

Definition 2.2. [Shiama, 2012] A (p, q) connected graph G is said to be **square difference labeling (SDL)** if there exists a bijection f from $V(G)$ to $\{0, 1, 2, \dots, (p-1)\}$, which induces the injection f^* and values from $E(G)$ to N given by $f^*(uv) = |f(u)^2 - f(v)^2|$ for all $e = uv$ are distinct.

Definition 2.3. [Kulli, 1976] **Semitotal-block graph** $T_b(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and blocks of G and in which two points are adjacent iff the corresponding vertices of G are adjacent or the corresponding blocks are incident.

Definition 2.4. [Shiama, 2012] The **comb graph** $(P_n \odot K_1)$ is a graph aquired by joining a pendant edge to each vertex of a path graph P_n with $2n$ vertices and $(2n-1)$ edges.

Definition 2.5. [Govindan and Dhivya, 2019] The **crown graph** $(C_n \odot K_1)$ is a graph aquired by joining a pendant edge to each vertex of a cycle C_n . The **friendship graph** F_n [Govindan and Dhivya, 2019] can be aquired by joining n copies of the cycle graph C_3 with a common vertex.

The following theorem is used for an immediate results.

Theorem A: [Kulli, 1976] If G is connected graph with p vertices and q edges and if b_r is the number of blocks to which vertex v_i belongs in G , then the semitotal-block graph $T_b(G)$ has $\sum b_i + 1$ vertices and $q + \sum b_i$ edges.

3 Main Results

Theorem 3.1. For $n \geq 4$, $T_b(F_n)$ admits SSL.

Proof. By Theorem A, the vertex set of $T_b(F_n)$ be $V(T_b(F_n)) = 3n + 1$ and the edge set be $E(T_b(F_n)) = 6n$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of F_n and the total number of blocks are equal to n copies of C_3 in F_n i.e., $|B(F_n)| = n$. There are four cases to be consider

Case 1. $n \equiv 0 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ thus, the vertices and ten classes of edges, which are distinctly labeled as follows,

$$f(b_i) = 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 1$$

$$f(u_i) = 2i, \text{ for } 0 \leq i \leq \frac{3n-2}{2}$$

$$f(u_{\frac{3n-2}{2}+1}) = 3n, \quad f(u_{\frac{3n}{2}+i}) = (3n-1) - 2(i-1) \text{ for } 1 \leq i \leq \frac{n}{2} - 1.$$

E_1 : Edges connecting consecutive distinct even numbers and edge labels

$f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \frac{3n-4}{4}\}$ are in increasing sequence of even order.

$$E_2: f^*(u_{\frac{3n}{2}}u_{\frac{3n}{2}+2}) = (18n^2 - 6n + 1).$$

E_3 : Edges connecting consecutive distinct odd numbers and edge labels

$f^*\{u_{\frac{3n}{2}+2}u_{\frac{3n}{2}+2+j}; 1 \leq i \leq \frac{n-4}{4} \text{ where } i \text{ is odd and } 2 \leq j \leq \frac{n}{4} \text{ where } j \text{ is even}\}$ are in decreasing sequence of even order.

E_4 : Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of even numbers of the form $(4K + 4m + 2)$, where $K, m \in N$ and $K = i^2, m = i$.

E_5 : The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_6 : The edge labels $f^*\{uu_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n-2}{2}\}$ are in decreasing order of the

form $(9n^2 - 12nm + 6n + 4K - 4m + 2)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_7 : The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3n}{4} \text{ and } 1 \leq j \leq \frac{3n+4}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_8 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n}{4} \text{ and } 1 \leq j \leq \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_9 : Edges joining distinct odd numbers and the edge labels,

$f^*\{u_{\frac{3n}{2}+i}b_{\frac{n}{2}+(j-1)}; 2 \leq i \leq \frac{n}{2} \text{ } i \text{ is even and } 1 \leq j \leq \frac{n}{4}\}$, are in descending order.

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E_{10} : Edges joining distinct odd numbers and the edge labels

$f^*\{u_{\frac{3n}{2}+i} b_{\frac{3n+4}{2}+(j-1)} ; 1 \leq i \leq \frac{n-2}{2} \text{ } i \text{ is odd and } 1 \leq j \leq \frac{n}{4}\}$, are in descending order.

Case 2. $n \equiv 1 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ so, the vertices are labeled as,

$$f(b_i) = 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 1$$

$$f(u_i) = 2i, \text{ for } 0 \leq i \leq \frac{3n-1}{2}$$

$$f(u_{\frac{3n-1}{2}+i}) = 3n - 2(i-1) \text{ for } 1 \leq i \leq \frac{n-1}{2}.$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are nine classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels,

$f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of even order.

E_2 : Edges connecting consecutive distinct odd numbers and edge labels

$f^*\{u_{\frac{3n-1}{2}+i} u_{\frac{3n-1}{2}+j} ; 1 \leq i \leq \frac{n-3}{2} \text{ where } i \text{ is odd and } 2 \leq j \leq \frac{n-1}{2} \text{ where } j \text{ is even}\}$ are in decreasing sequence of even order.

E_3 : Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K + 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_4 : The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n-1}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_5 : The edge labels $f^*\{u u_{\frac{3n-1}{2}+i} ; 1 \leq i \leq \frac{n-1}{2}\}$ are in decreasing order of

the form $(9n^2 - 12nm + 12n + 4K - 8m + 5)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_6 : The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3(n-1)}{4} \text{ and } 1 \leq j \leq \frac{3n+1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i, \text{ and } l = j$.

E_7 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n+1}{4} \text{ and } 1 \leq j \leq \frac{3n+1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i, \text{ and } l = j$.

E_8 : Edges joining distinct odd numbers and the edge labels,

$$f^*\{u_{\frac{3(n-1)}{2}+i} b_{\frac{3n+5}{4}+(j-1)}; 2 \leq i \leq \frac{n-1}{2} \text{ } i \text{ is even and } 1 \leq j \leq \frac{n-1}{4}\},$$

edge labels are in descending order of distinct even numbers.

E_9 : Edges joining distinct odd numbers and the edge labels

$$f^*\{u_{\frac{3(n-1)}{2}+i} b_{\frac{3n+5}{4}+(j-1)}; 3 \leq i \leq \frac{n+1}{2} \text{ } i \text{ is odd and } 1 \leq j \leq \frac{n-1}{4}\},$$

edge labels are in descending order of distinct even numbers.

Case 3. $n \equiv 2 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 1, \quad f(u_i) = 2i \text{ for } 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n - 1) - 2(i - 1) \text{ for } 1 \leq i \leq \frac{n}{2} - 1. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are nine classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels,

$$f^*\{u_{2i} u_{2i+1}; 0 \leq i \leq \frac{3n-2}{4}\} \text{ are in increasing sequence of even order.}$$

E_2 : Edges connecting consecutive distinct odd numbers and edge labels,

$$f^*\{u_{\frac{3n-2}{2}+i} u_{\frac{3n-2}{2}+j}; 1 \leq i \leq \frac{n-2}{2} \text{ where } i \text{ is even and } 3 \leq j \leq \frac{n}{2} \text{ where } j \text{ is odd}\} \text{ are in decreasing sequence of even numbers.}$$

E_3 : Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_4 : The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_5 : The edge labels $f^*\{uu_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n-2}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 6n + 4K - 4m + 2)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_6 : The edge labels $f^*\{u_{2i} b_j; 0 \leq i \leq \frac{3n-2}{4} \text{ and } 1 \leq j \leq \frac{3n+2}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and

Square sum and square difference labelings of semitotal-block graph for some class of graphs

$K = i^2, L = j^2, m = i, \text{ and } l = j.$

E_7 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n+2}{4} \text{ and } 1 \leq j \leq \frac{3n+2}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i, \text{ and } l = j.$

E_8 : Edges joining distinct odd numbers and the edge labels

$f^*\{u_{\frac{3n-2}{2}+i}b_{\frac{3(n+2)}{4}+(j-1)}; 2 \leq i \leq \frac{n-2}{2}, i \text{ is even and } 1 \leq j \leq \frac{n-2}{4}\}$

are in descending sequence of distinct even numbers.

E_9 : Edges joining distinct odd numbers and the edge labels

$f^*\{u_{\frac{3n-2}{2}+i}b_{\frac{3(n+2)}{4}+(j-1)}; 3 \leq i \leq \frac{n}{2}, i \text{ is odd and } 1 \leq j \leq \frac{n-2}{4}\}$, are

in descending order of distinct even numbers.

Case 4. $n \equiv 3 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 1 \\ f(u_i) &= 2i, \text{ for } 0 \leq i \leq \frac{3(n-1)}{2} \\ f(u_{\frac{3(n-1)}{2}+i}) &= 3n - 2(i-1) \text{ for } 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are ten classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels

$f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \frac{3n-5}{4}\}$ are in increasing sequence of even order.

E_2 : $f^*(u_{\frac{3n-1}{2}}u_{\frac{3n+1}{2}}) = (18n^2 - 6n + 1).$

E_3 : Edges connecting consecutive distinct odd numbers and edge labels

$f^*\{u_{\frac{3n+1}{2}+i}u_{\frac{3n+1}{2}+j}; 1 \leq i \leq \frac{n-5}{2} \text{ where } i \text{ is odd and } 2 \leq j \leq \frac{n-3}{2} \text{ where } j \text{ is even}\}$ are in decreasing sequence of even order.

E_4 : Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K + 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i.$

E_5 : The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n-1}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex

vertex 1 are distinct.

E_6 : The edge labels $f^*\{u \frac{3n-1}{2} + i; 1 \leq i \leq \frac{n-1}{2}\}$ are in decreasing order of

the form $(9n^2 - 12nm + 6n + 4K - 8m + 5)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_7 : The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3n-1}{4} \text{ and } 1 \leq j \leq \frac{3(n+1)}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.

E_8 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n-1}{4} \text{ and } 1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.

E_9 : Edges joining distinct odd numbers and the edge labels

$f^*\{u \frac{3(n-1)}{2} + i \quad b \frac{3n-1}{4} + j; 2 \leq i \leq \frac{n+1}{2}, i \text{ is even and } 1 \leq j \leq \frac{n+1}{4}\}$

are in descending order of distinct even numbers.

E_{10} : Edges joining distinct odd numbers and the edge labels

$f^*\{u \frac{3(n-1)}{2} + i \quad b \frac{3n-1}{4} + j; 1 \leq i \leq \frac{n-1}{2}, i \text{ is odd and } 1 \leq j \leq \frac{n+1}{4}\}$ are

in descending order of distinct even numbers. Clearly, all the ten classes of edges are distinctly labeled. Hence, for $n \geq 4$, $T_b(F_n)$ admits SSL. $\square$.

Square sum and square difference labelings of semitotal-block graph for some class of graphs

Example 3.1: The square sum labeling of semitotal-block graph of F_4 , F_5 , F_6 and F_7 are shown in below figure.

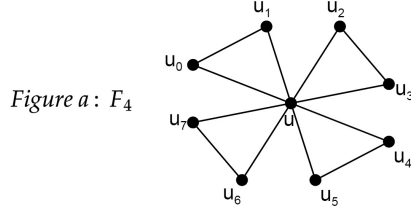


Figure A: $T_b(F_4)$
Case 1: $n \equiv 0(mod 4)$

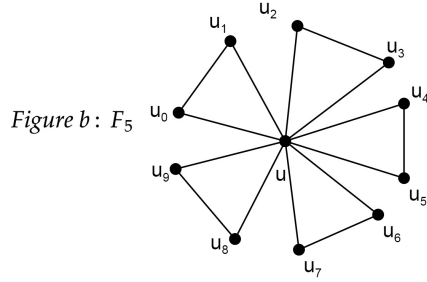
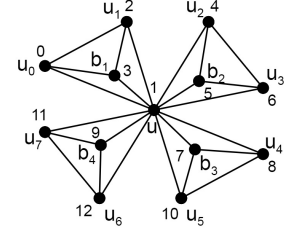


Figure B: $T_b(F_5)$
Case 2: $n \equiv 1(mod 4)$

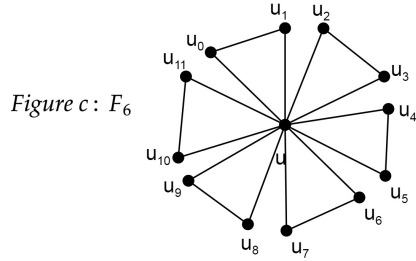
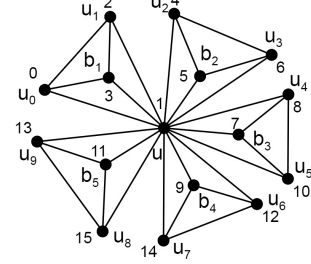


Figure C: $T_b(F_6)$
Case 3: $n \equiv 2(mod 4)$

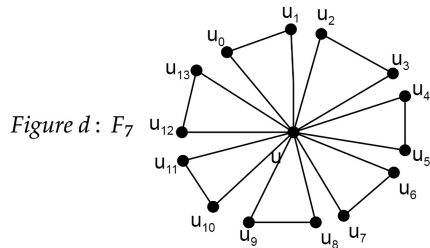
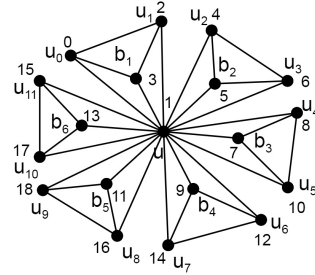
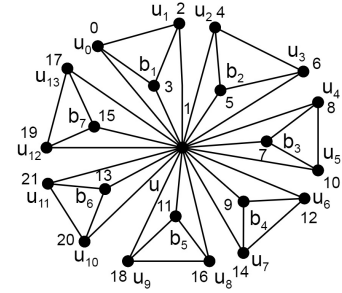


Figure D: $T_b(F_7)$
Case 4: $n \equiv 3(mod 4)$



Theorem 3.2. For $n \geq 3$, $T_b(P_n \odot K_1)$ admits SSL.

Proof. By Theorem A, the vertex set of $T_b(P_n \odot K_1)$ be $V(T_b(P_n \odot K_1)) = 2n + r$ and the edge set be $E(T_b(P_n \odot K_1)) = 6n - 3$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of $(P_n \odot K_1)$ and the total number of blocks (r) are equal to $|E(P_n \odot K_1)|$ i.e., $|B(P_n \odot K_1)| = 2n - 1$.

Define a function $f : V(T_b(P_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, (2n + r - 1)\}$ thus, the vertices are assigned as,

$$\begin{aligned}
 \text{if } n = 3 \quad f(u_i) &= 2i \text{ for } 1 \leq i \leq n \\
 f(b_i) &= 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(b_{n+1}) = 0, \quad f(b_r) = 1 \\
 f(v_i) &= (2n + 1) + i \text{ for } 1 \leq i \leq n \\
 \text{if } n > 3 \quad f(u_i) &= 2i \text{ for } 1 \leq i \leq n \\
 f(b_i) &= 2i + 1 \text{ for } 1 \leq i \leq n, \quad f(b_{n+1}) = 0, \quad f(b_{n+2}) = 1 \\
 f(v_i) &= (2n + 1) + i \text{ for } 1 \leq i \leq n \\
 f(b_{(n+3)+(i-1)}) &= (3n + 1) + i \text{ for } 1 \leq i \leq (n - 3).
 \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are seven classes of edges, which are distinctly labeled as follows,

E_1 : Edges joining naturally distinct consecutive even numbers and the edge labels $f^*\{u_i u_{i+1}; 1 \leq i \leq (n - 1)\}$ are in strictly increasing sequence.

E_2 : The labeling of edges $f^*\{u_i v_i; 1 \leq i \leq n\}$ are in ascending order of the form $(4^2 + 4n + 4nm + 5K + 2m + 1)$, where $K, m \in N$ and $K = i^2, m = i$, which are distinctly labeled.

E_3 : Edges connecting consecutive distinct even and odd numbers and the edge labels $f^*\{u_i b_i; 1 \leq i \leq n\}$ are in increasing sequence of the form $(8K + 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_4 : The edge labels $f^*\{v_i b_i; 1 \leq i \leq n\}$ are in ascending order of the form $(4n^2 + 4n + 4nm + 5K + 6m + 2)$, where $K, m \in N$ and $K = i^2, m = i$, which are distinctly labeled.

E_5 : The edge labels $f^*\{u_i b_{r-1}; 1 \leq i \leq (n - 1)\}$ are in ascending order of the form $4K$, where $K \in N$ and $K = i^2$ which are distinctly labeled.

E_6 : The edge labels $f^*\{u_i b_r; 2 \leq i \leq n\}$ are in ascending order of the form $(4K + 1)$, where $K \in N$ and $K = i^2$, which are distinctly labeled.

E_7 : If $n > 3$ the edge labels $f^*\{u_i b_{(n+3)+(i-1)}; 3 \leq i \leq (n - 1) \text{ and } 1 \leq j \leq (n - 3)\}$ and $f^*\{u_i b_{(n+3)+(i-1)}; 4 \leq i \leq n \text{ and } 1 \leq j \leq (n - 3)\}$ are in ascending orders of the form $(9n^2 + 6nl + 6n + 4K + L + 2l + 1)$, where $K, L, m, l \in N$ and $K = i^2, m = i, L = j^2, l = j$, which are distinctly labeled. Clearly, the assignment pattern of all the classes of edge are distinct. So, f is SSL and $T_b(P_n \odot K_1)$ admits SSG. $\square$

Square sum and square difference labelings of semitotal-block graph for some class of graphs

Example 3.2: The square sum labeling of semitotal-block graph of $P_4 \odot K_1$ is shown in below figure.

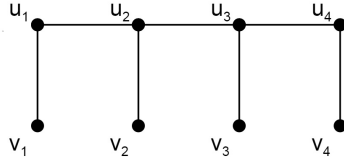


Figure : $P_4 \odot K_1$

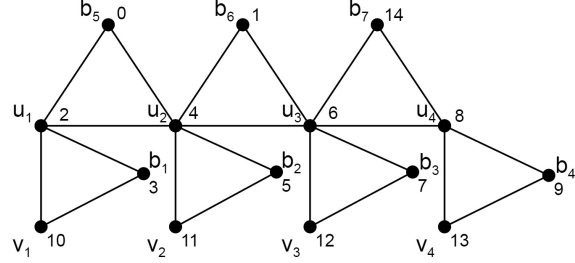


Figure : $T_b(P_4 \odot K_1)$

Theorem 3.3. For $n \geq 3$, $T_b(C_n \odot K_1)$ admits SSL except $n \equiv 0 \pmod{8}$.

Proof. The vertex set of $T_b(C_n \odot K_1)$ be $V(T_b(C_n \odot K_1)) = 3n + 1$ and the edge set be $E(T_b(C_n \odot K_1)) = 5n$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of $(C_n \odot K_1)$ and the total number of blocks in $(C_n \odot K_1)$ are equal to $(n + 1)$ i.e., $|B(C_n \odot K_1)| = n+1$.

Define a function $f : V(T_b(C_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned} f(b_r) &= 3n, \quad f(u_n) = 2n, \quad f(v_n) = n, \quad f(b_{r-1}) = 0 \\ f(u_i) &= f(b_r) - i \quad \text{for } 1 \leq i \leq n \\ f(v_i) &= f(u_n) - i \quad \text{for } 1 \leq i \leq n \\ f(b_i) &= f(v_n) - i, \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are five classes of edges and all edge labels are in the form of decreasing sequence, which are distinctly labeled as follows,

E_1 : The edge assignments $f^*\{b_r u_i; 1 \leq i \leq n\}$ are in the form $(18n^2 + K - 6nm)$, where $K, m \in N$ and $K = i^2, m = i$.

E_2 : The edge assignments $f^*\{u_i v_i; 1 \leq i \leq n\}$ are in the form $(2K + 13n^2 - 10nm)$, where $K, m \in N$ and $K = i^2, m = i$.

E_3 : The edge assignments $f^*\{v_i b_i; 1 \leq i \leq n\}$ are in the form $(2K + 9n^2 - 8nm)$, where $K, m \in N$ and $K = i^2, m = i$.

E_4 : The edge assignments $f^*\{u_i u_{j+1}; 1 \leq i \leq (n-1) \text{ and } 1 \leq j \leq (n-1)\}$ are in the form $(K + 9n^2 - 6nm) + (L + 9n^2 - 6nl)$, where $K, L, m, n \in N$ and $K = i^2, L = j^2; m = i, l = j$.

E_5 : $f^*(u_n u_1) = (13n^2 - 6n + 1)$. Clearly, the labeling pattern of all the classes

of edges are not being same. So, the $T_b(C_n \odot K_1)$, is SSL. $\square$

Example 3.3: The square sum labeling of semitotal-block graph of $C_5 \odot K_1$ is shown in below figure.

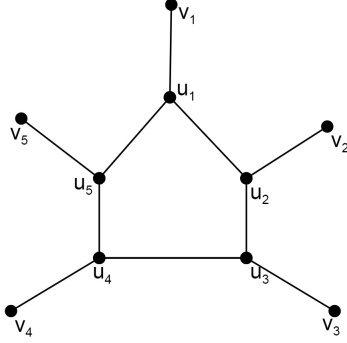


Figure : $C_5 \odot K_1$

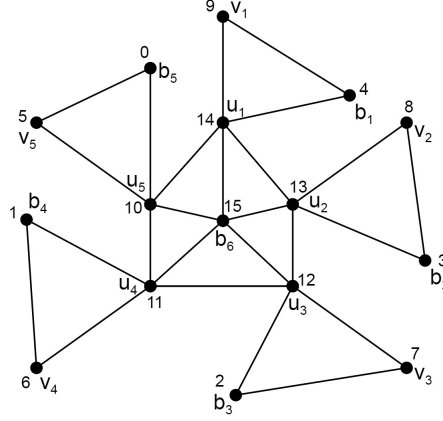


Figure : $T_b(C_5 \odot K_1)$

Theorem 3.4. For $n \geq 3$, $T_b(F_n)$ admits SDL except $n \equiv 1 \pmod{4}$.

Proof. By Theorem A, the vertex set of $T_b(F_n)$ be $V(T_b(F_n)) = 3n + 1$ and the edge set be $E(T_b(F_n)) = 6n$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of F_n and the total number of blocks are equal to n copies of C_3 in F_n i.e., $|B(F_n)| = n$.

We consider three cases

Case 1. $n \equiv 0 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned} f(b_i) &= 2i - 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 0 \\ f(u_i) &= 2i \text{ for } 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n - 1) - 2(i - 1) \text{ for } 1 \leq i \leq \frac{n}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are nine classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels

$f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of even order.

E_2 : Edges connecting consecutive distinct odd numbers and edge labels,

$f^*\{u_{\frac{3n}{2}+i}u_{\frac{3n}{2}+i+1}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of even order.

Square sum and square difference labelings of semitotal-block graph for some class of graphs

E_3 : The edge labels $f^*\{u_0b_i; 1 \leq i \leq n\}$ are in increasing order of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining odd numbers and apex vertex 0 are distinct.

E_4 : The edge labels $f^*\{u_0u_i; 1 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_5 : The edge labels $f^*\{u_0u_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n}{2}\}$ are in increasing order of the form $(9n^2 + 4K - 12nm + 6n - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_6 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n}{4} \text{ and } 1 \leq j \leq \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_7 : The edge labels $f^*\{u_{2i}b_j; 1 \leq i \leq \frac{3n}{4} \text{ and } 1 \leq j \leq \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_8 : Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n}{2}+i}b_{\frac{3n+4}{4}+(j-1)}; 1 \leq i \leq \frac{n}{2} - 1, i \text{ is odd and } 1 \leq j \leq \frac{n}{2} - 1\}$ are in descending order of distinct even numbers.

E_9 : Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n}{2}+i}b_{\frac{3n+4}{4}+(j-1)}; 1 \leq i \leq \frac{n}{2} - 1, i \text{ is even and } 1 \leq j \leq \frac{n}{2} - 1\}$ are in descending order of distinct even numbers.

Case 2. $n \equiv 2 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i - 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 0 \\ f(u_i) &= 2i, \text{ for } 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n - 1) - 2(i - 1) \text{ for } 1 \leq i \leq \frac{n}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are ten classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels

$f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of even order.

$$E_2: f^*(u_{\frac{3n}{2}} u_{\frac{3n+2}{2}}) = (6n - 1)$$

E_3 : Edges connecting consecutive distinct odd numbers and edge labels,

$$f^*\{u_{\frac{3n}{2}+i} u_{\frac{3n+2}{2}+j}; 1 \leq i \leq \frac{n-2}{4}, i \text{ is odd and } 2 \leq j \leq \frac{n+2}{4} j \text{ is even}\}$$

are in decreasing sequence of even order.

E_4 : The edge labels $f^*\{u_0 b_i; 1 \leq i \leq n\}$ are in increasing order of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edge joining odd numbers and apex vertex 0 are distinct.

E_5 : The edge labels $f^*\{u_0 u_i; 1 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_6 : The edge labels $f^*\{u_0 u_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n}{2}\}$ are in decreasing order of the form

$(9n^2 - 12nm + 6n + 4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_7 : The edge labels $f^*\{u_{2i-1} b_j; 1 \leq i \leq \frac{3n+2}{4} \text{ and } 1 \leq j \leq \frac{3n+2}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_8 : The edge labels $f^*\{u_{2i} b_j; 1 \leq i \leq \frac{3n-2}{4} \text{ and } 1 \leq j \leq \frac{3n-2}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i$, and $l = j$.

E_9 : Edges joining distinct odd numbers and the edge labels

$$f^*\{u_{\frac{3n-2}{2}+i} b_{\frac{3(n+2)}{4}+(j-1)}; 3 \leq i \leq \frac{n}{2}, i \text{ is odd and } 1 \leq j \leq \frac{n-2}{4}\}$$

are in descending order of distinct even order.

E_{10} : Edges joining distinct odd numbers and the edge labels

$$f^*\{u_{\frac{3n-2}{2}+i} b_{\frac{3n+2}{4}+(j-1)}; 2 \leq i \leq \frac{n+2}{2}, i \text{ is even and } 1 \leq j \leq \frac{n+2}{4}\}$$

are in descending order of distinct even numbers.

Case 3. $n \equiv 3 \pmod{4}$

Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i - 1 \text{ for } 1 \leq i \leq n, \quad f(u) = 0 \\ f(u_i) &= 2i, \text{ for } 0 \leq i \leq \frac{3n-1}{2} \end{aligned}$$

Square sum and square difference labelings of semitotal-block graph for some class of graphs

$$f(u_{\frac{3n-1}{2}+i}) = 3n-2(i-1) \text{ for } 1 \leq i \leq \frac{n+1}{2}.$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N . There are nine classes of edges, which are distinctly labeled as follows,

E_1 : Edges connecting consecutive distinct even numbers and edge labels

$f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n-1}{4}\}$ are in increasing sequence of even order.

E_2 : Edges connecting consecutive distinct odd numbers and edge labels

$f^*\{u_{\frac{3n-1}{2}+i}u_{\frac{3n-1}{2}+j}; 1 \leq i \leq \frac{n-1}{2} \text{ where } i \text{ is odd and } 2 \leq j \leq \frac{n+1}{2} \text{ where } j \text{ is even}\}$ are in decreasing sequence of even order.

E_3 : The edge labels $f^*\{u_0b_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K-4m+1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining odd numbers and apex vertex 0 are distinct.

E_4 : The edge labels $f^*\{u_0u_i; 1 \leq i \leq \frac{3n-1}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_5 : The edge labels $f^*\{u_0u_{\frac{3n-1}{2}+i}; 1 \leq i \leq \frac{n+1}{2}\}$ are in increasing order of

the form $(9n^2 - 12nm + 12n + 4K - 8m + 4)$, where $K, m \in N$ and $K = i^2, m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_6 : The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n-1}{4} \text{ and } 1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K+4L-4l+1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i, \text{ and } l = j$.

E_7 : The edge labels $f^*\{u_{2i}b_j; 1 \leq i \leq \frac{3n-1}{4} \text{ and } 1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K+4L-4l+1)$, where $K, L, m, l \in N$ and $K = i^2, L = j^2, m = i, \text{ and } l = j$.

E_8 : Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n-1}{2}+i}b_{\frac{3n-1}{4}+j};$

$1 \leq i \leq \frac{n-1}{2}, i \text{ is odd and } 1 \leq j \leq \frac{n+1}{4}\}$ are in descending order of distinct even order.

E_9 : Edges joining distinct odd numbers and the edge labels

$f^*\{u_{\frac{3n-1}{2}+i}b_{\frac{3n-1}{4}+j}; 2 \leq i \leq \frac{n+1}{4} \text{ where } i \text{ is even and } 1 \leq j \leq \frac{n+1}{4}\}$ are in

descending order of distinct even numbers. Clearly, the assignment pattern of all the classes of edge are distinct. So, $T_b(F_n)$ admits SDL. $\square$

Example 3.4: The square difference labeling of semitotal-block graph of F_4 , F_6 and F_7 are shown in the below figure.

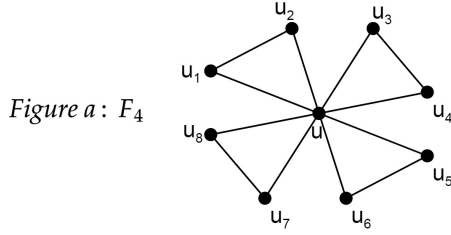


Figure A: $T_b(F_4)$
Case 1: $n \equiv 0(mod 4)$

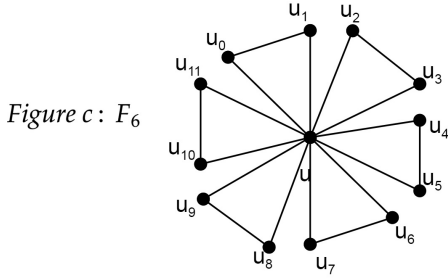
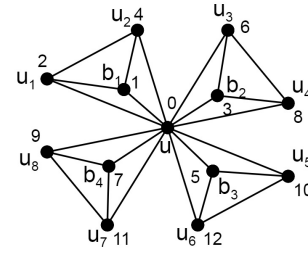


Figure C: $T_b(F_6)$
Case 3: $n \equiv 2(mod 4)$

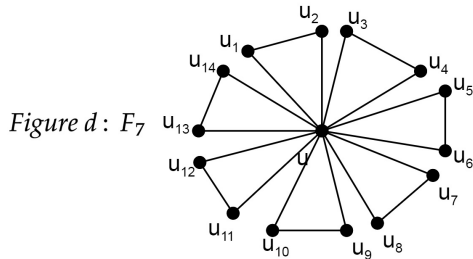
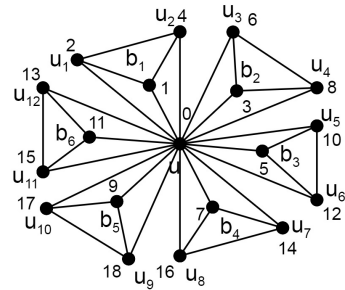
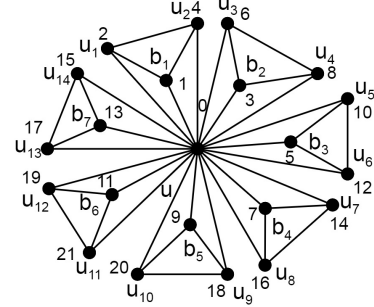


Figure D: $T_b(F_7)$
Case 4: $n \equiv 3(mod 4)$



Theorem 3.5. For $n \geq 3$, $T_b(P_n \odot K_1)$ admits SDL.

Proof. By Theorem A, the vertex set of $T_b(P_n \odot K_1)$ be $V(T_b(P_n \odot K_1)) = 2n + r$ and the edge set be $E(T_b(P_n \odot K_1)) = 6n - 3$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of $(P_n \odot K_1)$ and the total number of blocks (r) are equal to $|E(P_n \odot K_1)|$ i.e., $|B(P_n \odot K_1)| = 2n - 1$.

Define a function $f : V(T_b(P_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, (2n + r - 1)\}$ thus, the

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vertices are assigned as,

$$\begin{aligned} f(u_i) &= 2i - 1 \text{ for } 1 \leq i \leq n \\ f(b_i) &= f(u_n) + 2 + 2(i - 1) \text{ for } 1 \leq i \leq (n - 1) \\ f(v_j) &= 2j \text{ for } 0 \leq j \leq (n - 1), \quad f(b_n) = (2n + r - 1) \\ f(b_{(n+1)+(i-1)}) &= f(v_n) + 2 + 2(i - 1) \text{ for } 1 \leq i \leq (n - 1). \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are eight classes of edges which are distinctly labeled as follows,

E_1 : Edges joining (naturally distinct)consecutive odd numbers and the labels $f^*\{u_i u_{i+1}; 1 \leq i \leq (n - 1)\}$ are in strictly increasing sequence of even numbers.

E_2 : The edge labels $f^*\{u_i v_j; 1 \leq i \leq n \text{ and } 0 \leq j \leq (n - 1)\}$ are in strictly increasing sequence, having end vertices with the labels in which one of them is even and the other is odd, which receive odd edge labels.

E_3 : Edges joining distinct odd numbers and the labels $f^*\{u_i b_i; 1 \leq i \leq (n - 1)\}$ are in ascending order of the form $(f(u_n) + 2 + 2(i + 1))^2 - (4K - 4i + 1)$, which are distinct odd edge labels.

E_4 : $f^*(u_n b_{r-(n-1)}) = 3(4n^2 - 4n + 1)$.

E_5 : Edges joining distinct odd and even numbers and the labels $f^*\{v_j b_i; 0 \leq j \leq (n - 2) \text{ and } 1 \leq i \leq (n - 1)\}$ are in ascending order of the form $(f(u_n) + 2 + 2(i + 1))^2 - (4K)$, where $K = i^2$ which contains distinct odd edge labels.

E_6 : $f^*(v_n b_{r-(n-1)}) = (12n^2 - 8)$.

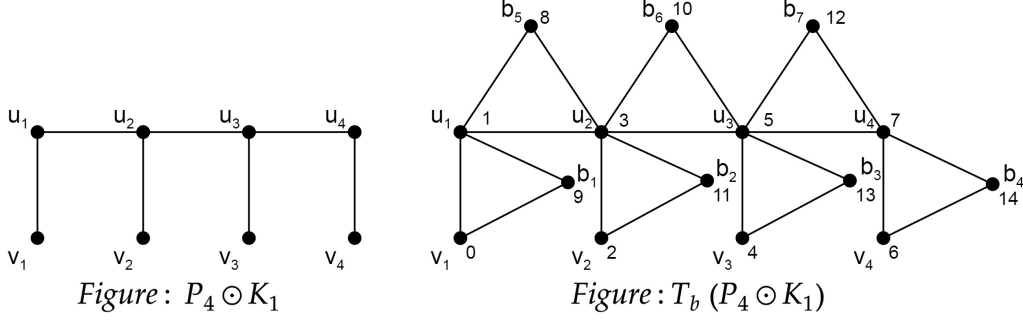
E_7 : Edges joining distinct odd and even numbers and the labels

$f^*\{u_i b_{(n+1)+(i-1)}; 1 \leq i \leq (n - 1)\}$ are in ascending order of the form $(f(v_n) + 2) + 2(i + 1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$ and $m = i$ which contains distinct odd edge labels.

E_8 : Edges joining distinct odd and even numbers and the labels

$f^*\{u_{i+1} b_{(n+1)+(i-1)}; 1 \leq i \leq (n - 1)\}$ are in ascending order of the form $(f(v_n) + 2) + 2(i + 1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$ and $m = i$ which contains distinct odd edge labels. Clearly, all the classes of edges are not being same. Hence, $T_b(P_n \odot K_1)$ admits SDL. $\square$

Example 3.5: The square difference labeling of semitotal-block graph of $P_4 \odot K_1$ is shown in the below figure.



Theorem 3.6. For $n \geq 4$, $(T_b(C_n \odot K_1))$ is SDL.

Proof. By Theorem A, the vertex set of $T_b(C_n \odot K_1)$ be $V(T_b(C_n \odot K_1)) = 3n + 1$ and the edge set be $E(T_b(C_n \odot K_1)) = 5n$. Let $B = b_1, b_2, b_3, \dots, b_r$ be the blocks of $(C_n \odot K_1)$ and the total number of blocks are equal to $(n + 1)$ i.e., $|B(C_n \odot K_1)| = n + 1$.

We consider four cases

Case 1. $n \equiv 0 \pmod{4}$

Define a function $f : V(T_b(C_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned} f(b_i) &= 4i \text{ for } 1 \leq i \leq \frac{3n}{4}, \quad f(b_0) = 0 \\ f(u_i) &= 2i - 1, \text{ for } 1 \leq i \leq n, \quad f(v_i) = 2(2i - 1) \text{ for } 1 \leq i \leq \frac{3n}{4} \\ f(b_{\frac{3n}{4}+i}) &= f(b_{\frac{3n}{4}}) - 3 - 4(i - 1) \text{ for } 1 \leq i \leq \frac{n}{4} \\ f(v_{\frac{3n}{4}+i}) &= f(v_{\frac{3n}{4}}) + 1 - 4(i - 1) \text{ for } 1 \leq i \leq \frac{n}{4}. \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are nine classes of edges which are distinctly labeled as follows,

E_1 : Edges joining consecutive (naturally distinct) odd numbers and common end vertex 0. The labeling of edges $f^*\{b_0 u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_2 : Edges joining consecutive (naturally distinct) odd numbers and the labeling

Square sum and square difference labelings of semitotal-block graph for some class of graphs

of edges $f^*\{u_i u_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.

E_3 : $f^*(u_n u_1) = 4n(n-1)$.

E_4 : Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_i v_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $3(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.

E_5 : Edges joining (naturally distinct) odd numbers, the edge labels

$f^*\{u_{\frac{3n}{4}+i} v_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are of the form $(f(v_{\frac{3n}{4}}) + 1 - 4(m-1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_6 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_i b_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $(12K + 4m - 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_7 : Edges joining (naturally distinct) odd numbers and the edge labels

$f^*\{u_{\frac{3n}{4}+i} b_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of even numbers of the form $(f(b_{\frac{3n}{4}}) - 3 - 4(m-1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_8 : Edges joining consecutive (naturally distinct) odd and even numbers and the edge labels $f^*\{v_i b_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and $m = i$.

E_9 : Edges joining (naturally distinct) odd numbers and the edge labels

$f^*\{v_{\frac{3n}{4}+i} b_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of the form $(f(v_{\frac{3n}{4}}) + 1 - 4(m-1))^2 - (f(b_{\frac{3n}{4}}) - 3 - 4(m-1))^2$, where $K, m \in N$ and $K = i^2, m = i$.

Case 2. $n \equiv 1 \pmod{4}$

Define a function $f : V(T_b(C_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, 3n\}$ so, the vertices are labeled as,

$$f(b_i) = 4i \text{ for } 1 \leq i \leq \frac{3(n-1)}{4}, \quad f(b_0) = 0$$

$$f(u_i) = 2i - 1 \text{ for } 1 \leq i \leq n, \quad f(v_i) = 2(2i - 1) \text{ for } 1 \leq i \leq \frac{3n+1}{4}$$

$$f\left(\frac{b_{3(n-1)}}{4} + i\right) = f\left(\frac{b_{3(n-1)}}{4}\right) + 3 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n+3}{4}$$

$$f\left(\frac{v_{3n+1}}{4} + i\right) = f\left(\frac{v_{3n}}{4}\right) - 1 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n-1}{4}.$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are ten classes of edges which are distinctly labeled as follows,

E_1 : Edges joining consecutive (naturally distinct) odd numbers and common end vertex. The labeling of edges $f^*\{b_0u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_2 : Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_iu_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.

E_3 : $f^*(u_nu_1) = 4n(n-1)$.

E_4 : Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_iv_i; 1 \leq i \leq \frac{3n+1}{4}\}$ are in increasing sequence of the form $3(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.

E_5 : Edges joining (naturally distinct) odd numbers and the edge labels,

$f^*\left\{u_{\frac{3n+1}{4}+i}v_{\frac{3n+1}{4}+i}; 1 \leq i \leq \frac{n}{4}\right\}$ are of the form $(f(v_{\frac{3n+1}{4}}) - 1 - 4(m-1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_6 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_ib_i; 1 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of the form $(12K + 4m - 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_7 : Edges joining (naturally distinct) odd numbers and the edge labels,

$f^*\left\{u_{\frac{3(n-1)}{4}+i}b_{\frac{3(n-1)}{4}+i}; 1 \leq i \leq \frac{n+3}{4}\right\}$ are in decreasing sequence of the form $(f(b_{\frac{3(n-1)}{4}}) + 3 - 4(m-1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_8 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{v_ib_i; 1 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and $m = i$.

E_9 : $f^*\left(\frac{v_{3n+1}}{4}b_{\frac{3(n+1)}{4}}\right) = (6n - 1)$.

E_{10} : Edges joining distinct odd numbers and the the edge labels

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$f^*\{v_{\frac{3n+1}{4}+i} b_{\frac{3(n+1)}{4}+1} ; 1 \leq i \leq \frac{n-1}{4}\}$ are in decreasing order of the form $| (f(v_{\frac{3n+1}{4}}) - 1 - 4(m-1))^2 - (f(b_{\frac{3n+1}{4}} - 4 - 4(m-1))^2 |$, where $m \in N$ and $m = \frac{4}{i}$.

Case 3. $n \equiv 2 \pmod{4}$

Define a function $f : V(T_b(C_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 4i \text{ for } 1 \leq i \leq \frac{3n-2}{4}, \quad f(b_0) = 0 \\ f(u_i) &= 2i - 1, \text{ for } 1 \leq i \leq n \\ f(v_i) &= 2(2i - 1) \text{ for } 1 \leq i \leq \frac{3n+2}{4} \\ f(b_{\frac{3n-2}{4}+i}) &= f(b_{\frac{3n-2}{4}}) + 1 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n+2}{4} \\ f(v_{\frac{3n+2}{4}+i}) &= f(v_{\frac{3n+2}{4}}) - 3 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n-2}{4}. \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are nine classes of edges which are distinctly labeled as follows,

E_1 : Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{b_0 u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_2 : Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_i u_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.

$E_3 : f^*(u_n u_1) = 4n(n-1)$.

E_4 : Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_i v_i; 1 \leq i \leq \frac{3n+2}{4}\}$ are in increasing sequence of the form $3(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.

E_5 : Edges joining (naturally distinct) odd numbers and the edge labels,

$f^*\{u_{\frac{3n+2}{4}+i} v_{\frac{3n+2}{4}+i} ; 1 \leq i \leq \frac{n-2}{4}\}$ are of the form $| (f(v_{\frac{3n+2}{4}}) - 3 - 4(m-1))^2 - (4K - 4m + 1) |$, where $K, m \in N$ and $K = i^2, m = i$.

E_6 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_i b_i; 1 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of the form $(12K + 4m - 1)$,

where $K, m \in N$ and $K = i^2, m = i$.

E_7 : Edges joining (naturally distinct) odd numbers and the edge labels,

$f^*\{u_{\frac{3n+2}{4}+i} b_{\frac{3n+2}{4}+i}; 1 \leq i \leq \frac{n+2}{4}\}$ are in decreasing sequence of the form $| (f(b_{\frac{3n+2}{4}}) + 3 - 4(m-1))^2 - (4K - 4m + 1) |$, where $K, m \in N$ and $K = i^2, m = i$.

E_8 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{v_i b_i; 1 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and $m = i$.

E_9 : Edges joining naturally distinct odd numbers except first end vertex and the edge labels $f^*\{v_{\frac{3n-2}{4}+i} b_{\frac{3n-2}{4}+i}; 1 \leq i \leq \frac{n+2}{4}\}$ are in decreasing sequence of the form $| (f(v_{\frac{3n-2}{4}}) + 4 - 3(m-1))^2 - (f(b_{\frac{3(n-1)}{4}}) + 1 - 4(m-1))^2 |$, where $m \in N$ and $m = i$.

Case 4. $n \equiv 3 \pmod{4}$

Define a function $f : V(T_b(C_n \odot K_1)) \rightarrow \{0, 1, 2, \dots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 4i \text{ for } 1 \leq i \leq \frac{3n-1}{4}, \quad f(b_0) = 0 \\ f(u_i) &= 2i - 1, \text{ for } 1 \leq i \leq n \\ f(v_i) &= 2(2i - 1) \text{ for } 1 \leq i \leq \frac{3n-1}{4} \\ f(b_{\frac{3n-1}{4}+i}) &= f(b_{\frac{3n-1}{4}}) - 1 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n+1}{4} \\ f(v_{\frac{3n-1}{4}+i}) &= f(v_{\frac{3n-1}{4}}) + 3 - 4(i-1) \text{ for } 1 \leq i \leq \frac{n+1}{4}. \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N . There are nine classes of edges which are distinctly labeled as follows,

E_1 : Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{b_0 u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_2 : Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_i u_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.

$E_3 : f^*(u_n u_1) = 4n(n-1)$.

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E_4 : Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_i v_i; 1 \leq i \leq \frac{3n-1}{4}\}$ are in increasing sequence of the form $3(4K-4m+1)$, where $K, m \in N$ and $K = i^2, m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.

E_5 : Edges joining naturally distinct odd numbers, the edge labels such as $f^*\{u_{\frac{3n-1}{4}+i} v_{\frac{3n-1}{4}+i}; 1 \leq i \leq \frac{n+1}{4}\}$ are of the form $(f(v_{\frac{3n-1}{4}}) + 1 - 4(m-1))^2 - (4K-4m+1)$, where $K, m \in N$ and $K = i^2, m = i$.

E_6 : Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_i b_i; 1 \leq i \leq \frac{3n-1}{4}\}$ are in increasing sequence of the form $(12K+4m-1)$, where $K, m \in N$ and $K = i^2, m = i$.

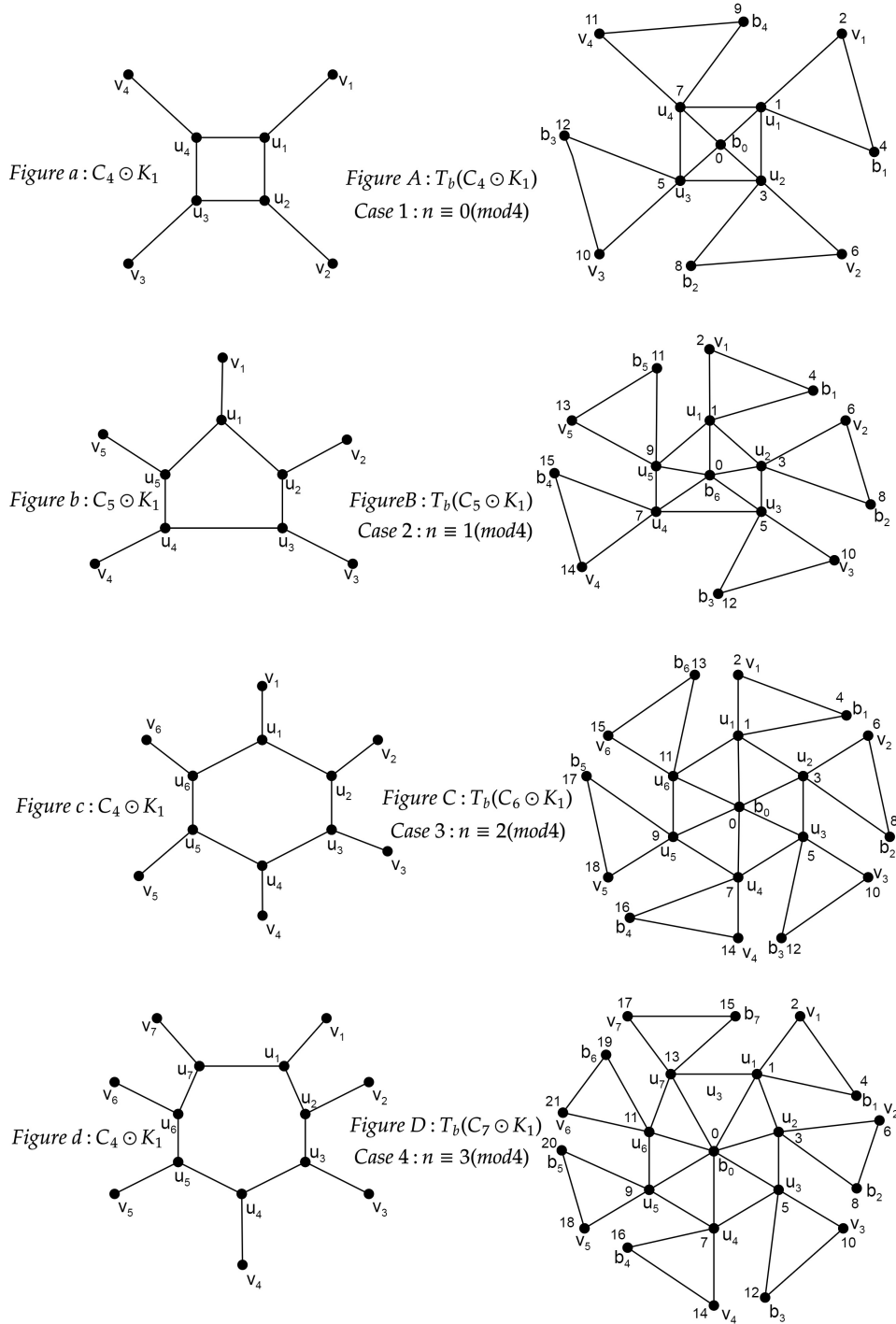
E_7 : Edges joining (naturally distinct) odd numbers, the edge labels $f^*\{u_{\frac{3n-1}{4}+i} b_{\frac{3n-1}{4}+i}; 1 \leq i \leq \frac{n+1}{4}\}$ are in decreasing sequence of the form $| (f(b_{\frac{3n-1}{4}}) - 1 - 4(m-1))^2 - (4K-4m+1) |$, where $K, m \in N$ and $K = i^2, m = i$.

E_8 : Edges joining consecutive distinct even numbers and the edge labels $f^*\{v_i b_i; 1 \leq i \leq \frac{3n-1}{4}\}$ are in increasing sequence of the form $(16m-4)$, where $m \in N$ and $m = i$.

E_9 : Edges joining (naturally distinct) odd numbers and the edge labels $f^*\{v_{\frac{3n-1}{4}+i} b_{\frac{3n-1}{4}+i}; 1 \leq i \leq \frac{n+1}{4}\}$ are in decreasing sequence of the form $| (f(v_{\frac{3n-1}{4}}) + 3 - 4(m-1))^2 - (f(b_{\frac{3(n-1)}{4}}) - 1 - 4(m-1))^2 |$, where $m \in N$ and $m = i$.

Clearly, the labeling of all the classes of edges are not being same and the function f is SDL and the graph $T_b(C_n \odot K_1)$ admits SDG. $\square$

Example 3.6: The square difference labeling of semitotal-block graph of $C_4 \odot K_1$, $C_5 \odot K_1$, $C_6 \odot K_1$, and $C_7 \odot K_1$ are shown in the below figure



4 Conclusion

In this article, we have studied on vertex labeling functions called square sum and square difference labeling and obtained the results for semitotal-block graph for some class of graphs which admits both SSL and SDL. This work contributes several new results to the theory of graph labeling.

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Common fixed point theorem for weakly compatible mappings in S_m metric space

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Abstract

In the present paper, at first, we study the structure of the newly S_m - metric space, which is a combination of S-metric space and multiplicative metric space. We have proved a common fixed point theorem for four self-maps in S_m metric space with a new contraction condition by applying the concepts of weakly compatible mappings, semi-compatible mappings, and reciprocally continuous mappings. Further, we also provide some examples to support our results.

Keywords: Multiplicative metric space, S-metric space, S_m -metric space, weakly compatible mappings, reciprocally continuous mappings, and semi-compatible mappings.

2020 AMS subject classifications: 54H25 ¹

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1 Introduction

The notion of multiplicative metric space (MMS) was first developed by Bashirove [1]. Following that, several theorems came to light in this area of MMS [2] ,[3] and [4]. On the other side, Sedghi.S et al.[5] presented a new structure to S-metric space which modified D-metric and G-metric spaces, and then several fixed point theorems [6] and [7] were obtained. Pant et al. [8] generalized the notion of reciprocally continuous mapping which is weaker than continuous and compatible mappings. Recently, Mukesh Kumar Jain [9] introduced a more general form of semi-compatible mappings and proved many fixed point theorems in metric space.

In this article, we use a new generalized metric space referred to as S_m -metric space, which is a combination of both MMS and S -metric space. Using this concept, we establish a common fixed point theorem by applying weakly compatible mappings(WCM), reciprocally continuous mappings, and semi-compatible mappings. Furthermore, some examples are also discussed to support our conclusions.

2 Preliminaries:

Now we give some definitions and examples which are used in this theorem.

Definition 2.1. [1] “Let χ be a non-empty set and $\delta : \chi^2 \rightarrow \mathbb{R}^+$ be a multiplicative metric space (MMS) satisfying the properties :

- (i) $\delta(\psi, \phi) \geq 1$ and $\delta(\psi, \phi) = 1 \iff \psi = \phi$
- (ii) $\delta(\psi, \phi) = \delta(\phi, \psi)$
- (iii) $\delta(\psi, \phi) \leq \delta(\psi, \sigma)\delta(\sigma, \phi), \forall \psi, \phi, \sigma \in \chi.$ ”

Definition 2.2. [5] “ Let χ be a non-empty set defined $S : \chi^3 \rightarrow [0, \infty)$ satisfying:

- (i) $S(\psi, \phi, \sigma) \geq 0$
- (ii) $S(\psi, \phi, \sigma) = 0 \iff \psi = \phi = \sigma$
- (iii) $S(\psi, \phi, \sigma) \leq S(\psi, \psi, \rho) + S(\phi, \phi, \rho) + S(\sigma, \sigma, \rho), \forall \psi, \phi, \sigma, \rho \in \chi.$

A mapping S together with $\chi, (\chi, S)$ is called a S -metric space.”

Definition 2.3. [10] “ Let χ be a non-empty set .A function $S_m : \chi^3 \rightarrow \mathbb{R}^+$ satisfying the conditions :

S_m metric space

- (i) $S_m(\psi, \phi, \sigma) \geq 1$
- (ii) $S_m(\psi, \phi, \sigma) = 1 \iff \psi = \phi = \sigma$
- (iii) $S_m(\psi, \phi, \sigma) \leq S_m(\psi, \psi, \rho)S_m(\phi, \phi, \rho)S_m(\sigma, \sigma, \rho), \forall \psi, \phi, \sigma, \rho \in \chi.$

The pair (χ, S_m) is called as S_m -metric space”.

Definition 2.4. [10] “ Let (χ, S_m) be a S_m -metric space, a sequence $\{\psi_\theta\} \in \chi$ is said to be

- (i) *cauchy sequence* $\iff S_m(\psi_\theta, \psi_\theta, \psi_l) \rightarrow 1$, for all $\theta, l \rightarrow \infty$;
- (ii) *convergent* $\iff \exists \psi \in \chi$ such that $S_m(\psi_\theta, \psi_\theta, \psi) \rightarrow 1$ as $\theta \rightarrow \infty$;
- (iii) *is complete if every cauchy sequence is convergent.”*

Definition 2.5. [11] ” Two self-maps M and K of a S_m metric space are said to be

- (i) **Compatible:** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ for some } \omega \in \chi.$$

- (ii) **Weakly-compatible mappings:** *if they commute at their coincidence points,*

$$\text{i.e. } \omega \in \chi, S_m(M\omega, M\omega, K\omega) = 1, \implies S_m(MK\omega, MK\omega, KM\omega) = 1.”$$

Definition 2.6. [9] “Two self maps M and K of S_m -metric space are said to be **Semi-compatible:** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1$$

whenever there exists a sequence $\{\psi_\theta\} \in X$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ for all } \omega \in \chi.”$$

Now we present an example in which semi-compatible is weaker than compatible.

Example 2.6.1

Consider $\chi = [0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \frac{\cos^2(\pi\psi)+1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ \sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} \frac{2\sin(\pi\psi)-1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ 1 - \sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{\pi}{2} - \frac{1}{\theta}\}$ for $\theta \geq 0$.

Then

$$\lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} M(\frac{1}{2} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} \frac{\cos^2\pi(\frac{1}{2} - \frac{1}{\theta}) + 1}{2} = \lim_{\theta \rightarrow \infty} \frac{\sin^2(\frac{\pi}{\theta}) + 1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\frac{1}{2} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} \frac{2\sin\pi(\frac{1}{2} - \frac{1}{\theta}) - 1}{2} = \lim_{\theta \rightarrow \infty} \frac{2\cos(\frac{\pi}{\theta}) - 1}{2} = \frac{1}{2}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} M\psi_\theta = \lim_{\theta \rightarrow \infty} K\psi_\theta = \frac{1}{2} = \omega \text{ (say).}$$

Now

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M(\frac{2\cos\frac{\pi}{\theta} - 1}{2}) = \lim_{\theta \rightarrow \infty} \frac{\cos^2\pi(\frac{2\cos\frac{\pi}{\theta}-1}{2}) + 1}{2} = \frac{\cos^2\frac{\pi}{2} + 1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\frac{\sin^2\frac{\pi}{\theta} + 1}{2}) = \lim_{\theta \rightarrow \infty} [1 - \sin\pi(\frac{\sin^2\frac{\pi}{\theta} + 1}{2})] = 0.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) \neq 0.$$

This implies these two self-maps M and K are not compatible.

But $K(\omega) = K(\frac{1}{2}) = \frac{1}{2}$.

$$\text{Therefore } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = \lim_{\theta \rightarrow \infty} S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 1.$$

Hence these two self maps M and K are semi-compatible but not compatible.

S_m metric space

Definition 2.7. [8] “Two self-maps M, K of S_m -metric space are said to be **reciprocally continuous** if

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \text{ and } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ such that

$$\lim_{\theta \rightarrow \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ some } \omega \in \chi.”$$

Now we present an example in which satisfies reciprocally continuous is weaker but not compatible.

Example 2.7.1 Consider $\chi = (0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \psi^2 + 2 & \text{if } 0 < \psi \leq 1; \\ 4 - \psi & \text{if } 1 < \psi \leq 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} 1 - 2\psi & \text{if } 0 < \psi \leq 1; \\ \psi - 2 & \text{if } 1 < \psi \leq 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{3 - \frac{1}{\theta}\}$, for $\theta \geq 0$.

Now

$$\lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} [4 - (3 - \frac{1}{\theta})] = 1 \text{ and } \lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} [(3 + \frac{1}{\theta}) - 2] = 1$$

$$\therefore \lim_{\theta \rightarrow \infty} M\psi_\theta = \lim_{\theta \rightarrow \infty} K\psi_\theta = 1 = \omega_1 \neq \phi.$$

Also

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M[(3 - \frac{1}{\theta}) - 2] = \lim_{\theta \rightarrow \infty} M(1 - \frac{1}{\theta}) = 3$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(4 - (3 - \frac{1}{\theta})) = \lim_{\theta \rightarrow \infty} K(1 + \frac{1}{\theta}) = -1.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(3, 3, -1) \neq 1.$$

This gives the self maps M and K are not compatible in S_m - metric space.

Moreover, $M(\omega_1) = 3$ and $K(\omega_1) = -1$.

Which gives

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(3, 3, 3) = 1,$$

and

$$\lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(-1, -1, -1) = 1.$$

This implies the self-maps M and K are reciprocally continuous but not compatible in S_m metric space.

Now we proceed to the main theorem.

3 Main Theorem

Theorem 3.1. *Let M, H, K , and J be self-mapping of a complete S_m -metric space satisfying the following*

$$(3.1.1) \quad M(\chi) \subseteq J(\chi) \text{ and } H(\chi) \subseteq K(\chi)$$

$$(3.1.2)$$

$$S_m(M\psi, M\psi, H\phi) \leq \left\{ \begin{aligned} &\max[S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, J\phi), \\ &S_m(M\psi, M\psi, J\phi)S_m(K\psi, K\psi, H\phi), \\ &S_m(M\psi, M\psi, J\phi)S_m(H\phi, H\phi, J\phi), \\ &S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, K\psi)] \end{aligned} \right\}^\lambda$$

where $\lambda \in (0, \frac{1}{2})$

$$(3.1.3) \quad \text{the pair } M \text{ and } K \text{ are reciprocally continuous and semi-compatible,}$$

$$(3.1.4) \quad \text{the pair } H \text{ and } J \text{ are weakly compatible.}$$

Then the self-maps M, H, K , and J have a unique common fixed point in χ .

Proof:

Let there is a point $\psi_0 \in \chi$, and the sequence $\{\psi_\theta\}$ be defined as $M\psi_0 = J\psi_1 = \phi_0$. For this point ψ_1 then there exists $\psi_2 \in \chi$ such that $H\psi_1 = K\psi_2 = \phi_1$. In general, by induction choose $\psi_{\theta+1}$, construct a sequence $\{\phi_\theta\} \in \chi$ such that

$$\phi_{2\theta} = M\psi_{2\theta} = J\psi_{2\theta+1} \text{ and } \phi_{2\theta+1} = H\psi_{2\theta+1} = K\psi_{2\theta+2}, \text{ for } \theta \geq 0.$$

S_m metric space

On putting $\psi = \psi_{2\theta}$ and $\phi = \phi_{2\theta+1}$ in (3.1.2) we get.

$$\begin{aligned} S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) &= S_m(M\psi_{2\theta}, M\psi_{2\theta}, H\psi_{2\theta+1}) \\ &\leq \max \left\{ S_m(M\psi_{2\theta}, M\psi_{2\theta}, \theta\psi_{2\theta}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}), \right. \\ &\quad S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, \theta\psi_{2\theta}), \\ &\quad S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}), \\ &\quad \left. S_m(M\psi_{2\theta}, M\psi_{2\theta}, K\psi_{2\theta}) S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, K\psi_{2\theta}) \right\}^\lambda \end{aligned}$$

$$\begin{aligned} S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) &\leq \max \left\{ S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}), \right. \\ &\quad S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}), \\ &\quad S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}), \\ &\quad \left. S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1}) S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}) \right\}^\lambda \end{aligned}$$

this implies that

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta+1})^\lambda.$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq \{S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}) S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1})\}^\lambda.$$

$$S_m^{1-\lambda}(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^\lambda(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^{\frac{\lambda}{1-\lambda}}(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^p(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}). \text{ where } p = \frac{\lambda}{1-\lambda}.$$

Now this gives

$$S_m(\phi_\theta, \phi_\theta, \phi_{\theta+1}) \leq S_m^p(\phi_{\theta-1}, \phi_{\theta-1}, \phi_\theta) \leq S_m^{p^2}(\phi_{\theta-2}, \phi_{\theta-2}, \phi_{\theta-1}) \leq \cdots S_m^{p^n}(\phi_0, \phi_0, \phi_n).$$

By using triangular inequality

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \leq S_m^{p^\theta}(\phi_0, \phi_0, \phi_l) \leq S_m^{p^{\theta+1}}(\phi_0, \phi_0, \phi_n) \leq \cdots S_m^{p^{n-1}}(\phi_0, \phi_0, \phi_n)$$

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \leq S_m^{\frac{p^\theta}{1-p}}(\phi_0, \phi_0, \phi_l) \text{ for all } \theta \geq 1.$$

Hence $\{\phi_\theta\}$ is a cauchy sequence in S_m -metric space.

Since the self-maps, M and K are weakly reciprocally continuous.

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \text{ or } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, \theta\omega) = 1. \quad (1)$$

Also, the pair (M, K) is semi compatible, we have

$$\lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1. \quad (2)$$

From (1) and (2) we get

$$S_m(M\omega, M\omega, K\omega) = 1. \quad (3)$$

Since $M(\chi) \subseteq J(\chi)$ which gives then there exists $\nu \in \chi$ such that $J\nu = M\psi_\theta$, since $M\psi_\theta \rightarrow \omega$ as $\theta \rightarrow \infty$. Which implies

$$S_m(J\nu, J\nu, \omega) = 1. \quad (4)$$

Now, we have to prove $S_m(J\nu, H\nu, \omega) = 1$.

Substitute $\psi = \psi_\theta$ and $\phi = \nu$ in (3.1.2) we have

$$S_m(M\psi_\theta, M\psi_\theta, H\nu) \leq \left\{ \begin{aligned} &\max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, J\nu), \\ &S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(K\psi_1, K\psi_1, H\nu), \\ &S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(H\nu, H\nu, J\nu), \\ &S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, K\psi_\theta)] \end{aligned} \right\}^\lambda$$

$$\begin{aligned} S_m(\omega, \omega, H\nu) &\leq \\ &\left\{ \begin{aligned} &\max[S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\nu), \\ &S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega)] \end{aligned} \right\}^\lambda \end{aligned}$$

$$\begin{aligned} S_m(\omega, \omega, H\nu) &\leq \{S_m(\omega, \omega, H\nu)\}^\lambda \\ S_m^{(1-\lambda)}(\omega, \omega, H\nu) &\leq 1 \implies S_m(H\nu, H\nu, \omega) = 1. \\ \therefore S_m(J\nu, H\nu, \omega) &= 1. \end{aligned}$$

S_m metric space

Since the pair (H, J) is WCM and ν is a coincidence point then $HJ\nu = JH\nu$

$$S_m(H\omega, H\omega, J\omega) = 1. \quad (5)$$

Substitute $\psi = \psi_\theta$ and $\phi = \omega$ in (3.1.2)
we have

$$\begin{aligned} S_m(M\psi_\theta, M\psi_\theta, H\omega) \leq & \\ & \left\{ \max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, J\omega), \right. \\ & S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(K\psi_\theta, K\psi_\theta, H\omega), \\ & S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(H\omega, H\omega, J\omega), \\ & \left. S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, K\psi_\theta)] \right\}^\lambda \end{aligned}$$

also

$$\begin{aligned} S_m(H\omega, \omega, \omega) \leq & \\ & \left\{ \max[S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\omega), \right. \\ & \left. S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega)] \right\}^\lambda \end{aligned}$$

and this gives

$$S_m(H\omega, \omega, \omega) \leq S_m(H\omega, \omega, \omega)^\lambda$$

$$S_m^{(1-\lambda)}(H\omega, \omega, \omega) \leq 1 \implies H\omega = \omega$$

$$\therefore S_m(H\omega, J\omega, \omega) = 1. \quad (6)$$

Replace $\psi = \omega$ and $\phi = \nu$ in (3.1.2) then we have

$$\begin{aligned} S_m(M\omega, M\omega, H\nu) \leq & \left\{ \max[S_m(M\omega, M\omega, K\omega)S_m(J\nu, H\nu, H\nu), \right. \\ & S_m(M\omega, M\omega, J\nu)S_m(K\omega, K\omega, H\nu), \\ & S_m(M\omega, M\omega, J\nu)S_m(J\nu, J\nu, H\nu), \\ & \left. S_m(M\omega, M\omega, K\omega)S_m(H\nu, H\nu, K\omega)] \right\}^\lambda \end{aligned}$$

$$S_m(M\omega, M\omega, \omega) \leq \left\{ \begin{aligned} &\max[S_m(M\omega, M\omega, M\omega)S_m(\omega, \omega, \omega), \\ &S_m(M\omega, M\omega, \omega)S_m(M\omega, M\omega, \omega), \\ &S_m(M\omega, M\omega, \omega)S_m(\omega, \omega, \omega), \\ &S_m(M\omega, M\omega, M\omega)S_m(\omega, \omega, M\omega)] \end{aligned} \right\}^\lambda$$

$$S_m(M\omega, M\omega, \omega) \leq \{S_m(M\omega, M\omega, \omega)\}^\lambda$$

$$S_m^{(1-\lambda)}(M\omega, M\omega, \omega) \leq 1 \implies M\omega = \omega$$

$$\therefore S_m(M\omega, J\omega, \omega) = 1. \quad (7)$$

From (6) and (7) we get

$$M\omega = J\omega = H\omega = K\omega = \omega. \quad (8)$$

Therefore “ ω ” is a common fixed point of M , H , K , and J .

Uniqueness

Let ρ be one more fixed point, we assume that $\rho \neq \omega$ then we have

$$M\rho = K\rho = H\rho = J\rho = \rho.$$

In the condition (3.1.2) put $\psi = \omega$ and $\phi = \rho$ we get

$$S_m(M\omega, M\omega, H\rho) \leq \left\{ \begin{aligned} &\max[S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, J\rho), \\ &S_m(M\omega, M\omega, J\rho)S_m(K\omega, K\omega, H\rho), \\ &S_m(M\omega, M\omega, J\rho)S_m(H\rho, H\rho, J\rho), \\ &S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, K\omega)] \end{aligned} \right\}^\lambda$$

$$S_m(\omega, \omega, \rho) \leq$$

$$\left\{ \begin{aligned} &\max[S_m(\omega, \omega, \omega)S_m(\rho, \rho, \rho), S_m(\omega, \omega, \rho)S_m(\omega, \omega, \rho), \\ &S_m(\omega, \omega, \rho)S_m(\rho, \rho, \rho), S_m(\omega, \omega, K\omega)S_m(\rho, \rho, \omega)] \end{aligned} \right\}^\lambda$$

$$S_m(\omega, \omega, \rho) \leq \left\{ S_m(\omega, \omega, \rho) \right\}^\lambda$$

this implies that $S_m(\omega, \omega, \rho) = 1 \implies \omega = \rho$.

This shows that “ ω ” is the unique common fixed point of M, H, J and K .

S_m metric space

Now, the following example substantiates our theorem.

Example 3.2

Suppose $\chi = (0, 1)$, S_m - metric space by $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$,

when $\psi, \phi, \sigma \in \chi$. Define $M, K, H, J: \chi \times \chi \times \chi \rightarrow \chi$ as follows

$$M(\psi) = \begin{cases} \frac{2-\psi}{5} & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$K(\psi) = \begin{cases} 1 - 2\psi & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{1+\psi}{2} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$H(\psi) = \begin{cases} 3\psi^2 - 3\psi + 1 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{2+\psi}{7} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$J(\psi) = \begin{cases} 1 - 6\psi^2 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ 1 - \psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

Then $M(\chi) = (\frac{1}{3}, 1] \subseteq J(\chi) = (0, 1]$ and $H(\chi) = (\frac{1}{3}, 1] \subseteq K(\chi) = (\frac{1}{3}, 1]$.

Therefore the condition (3.1.1) holds.

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{1}{3} - \frac{1}{\theta}\}$ as $\theta \geq 0$.

$$\text{Then } \lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} M(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} \frac{2 - (\frac{1}{3} - \frac{1}{\theta})}{5} = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} K(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} [1 - 2(\frac{1}{3} - \frac{1}{\theta})] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} M(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Further

$$\lim_{\theta \rightarrow \infty} H(\psi_\theta) = \lim_{\theta \rightarrow \infty} H(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} [3(\frac{1}{3} - \frac{1}{\theta})^2 - 3(\frac{1}{3} - \frac{1}{\theta}) + 1] = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} J(\psi_\theta) = \lim_{\theta \rightarrow \infty} J(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta \rightarrow \infty} [1 - 6(\frac{1}{3} - \frac{1}{\theta})^2] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} H(\psi_\theta) = \lim_{\theta \rightarrow \infty} J(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Moreover

$$\lim_{\theta \rightarrow \infty} MK(\psi_\theta) = \lim_{\theta \rightarrow \infty} M[1 - (\frac{2}{3} - \frac{2}{\theta})] = \lim_{\theta \rightarrow \infty} M(\frac{1}{3} + \frac{2}{\theta}) = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} KM(\psi_\theta) = \lim_{\theta \rightarrow \infty} K(\frac{1}{3} + \frac{1}{5\theta}) = \lim_{\theta \rightarrow \infty} \frac{1 + 2(\frac{1}{3} + \frac{1}{5\theta})}{2} = \frac{2}{3}.$$

$$\therefore \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \neq 1$$

which implies that the pair (M, K) is not compatible.

Furthermore

$$\lim_{\theta \rightarrow \infty} HJ(\psi_\theta) = \lim_{\theta \rightarrow \infty} H(\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2}) = \lim_{\theta \rightarrow \infty} (\frac{2 + (\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2})}{7}) = \frac{1}{3}$$

and

$$\lim_{\theta \rightarrow \infty} JH(\psi_\theta) = \lim_{\theta \rightarrow \infty} J(\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2}) = \lim_{\theta \rightarrow \infty} [1 - (\frac{1}{3} + \frac{4}{\theta} - \frac{1}{\theta^2})] = \frac{2}{3}$$

$$\text{Therefore } \lim_{\theta \rightarrow \infty} S_m(HJ\psi_\theta, HJ\psi_\theta, JH\psi_\theta) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \neq 1.$$

Which shows that the pair (H, J) is not compatible .

Also $M(\frac{1}{3}) = \frac{1}{3}$, $K(\frac{1}{3}) = \frac{1}{3}$.

$$\text{This implies } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$$

$$\text{and } \lim_{\theta \rightarrow \infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1.$$

This shows that the pair (M, K) is reciprocally continuous in S_m metric space.

$$\text{Also } \lim_{\theta \rightarrow \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega_1) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1.$$

This shows that the pair (M, K) is semi-compatible in S_m metric space.

Hence the inequality (3.1.3) holds.

Further

$$S_m(H(\frac{1}{3}), J(\frac{1}{3}), \frac{1}{3}) = 1 \text{ and } S_m(HJ(\frac{1}{3}), JH(\frac{1}{3}), \frac{1}{3}) = 1.$$

S_m metric space

This implies that $S_m(HJ(\frac{1}{3}), HJ(\frac{1}{3}), JH(\frac{1}{3})) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$.
Which indicates that the pair (H, J) is weakly compatible.

Now, we prove the condition (3.1.2) in various cases

CASE-I

Let $\psi, \phi \in [0, \frac{1}{2}]$, while we have $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.

Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{5}$ then $M(\frac{1}{4}) = \frac{7}{20}$, $K(\frac{1}{4}) = \frac{1}{2}$, $H(\frac{1}{5}) = \frac{13}{25}$ and $J(\frac{1}{5}) = \frac{19}{25}$
substitute the above values in (3.1.2)

$$S_m(\frac{7}{20}, \frac{7}{20}, \frac{13}{25}) \leq \left\{ \max[S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2})] \right\}^\lambda$$

$$we have e^{0.34} \leq \left\{ \max[e^{0.3}e^{0.48}, e^{0.82}e^{0.34}, e^{0.3}e^{0.04}, e^{0.82}e^{0.48}] \right\}^\lambda$$

$$e^{0.34} \leq \{ \max[e^{0.78}, e^{1.16}, e^{0.0.34}, e^{1.3}] \}^\lambda \implies e^{0.34} \leq e^{1.16\lambda}$$

which gives $\lambda = 0.2$ where $\lambda \in (0, \frac{1}{3})$.

CASE-II

Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.

Take $\psi = \frac{1}{2}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{2}) = \frac{1}{2}$, $K(\frac{1}{2}) = \frac{3}{4}$, $H(\frac{1}{2}) = \frac{5}{14}$ and $J(\frac{1}{2}) = \frac{1}{2}$
substitute the above values in (3.1.2)

$$S_m(\frac{1}{2}, \frac{1}{2}, \frac{5}{14}) \leq \left\{ \max[S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{3}{4}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{3}{4})] \right\}^\lambda$$

which implies that

$$e^{0.285} \leq \left\{ \max[e^{0.5}e^{0.285}, e^{0.0}e^{0.786}, e^{0.0}e^{0.28}, e^{0.5}e^{0.786}] \right\}^\lambda$$

$$e^{0.285} \leq \{ \max[e^{0.785}, e^{0.786}, e^{0.28}, e^{1.286}] \}^\lambda \implies e^{0.285} \leq e^{1.286\lambda}$$

which gives $\lambda = 0.22$ where $\lambda \in (0, \frac{1}{2})$.

CASE-III

Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$

Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{4}) = \frac{7}{20}$, $K(\frac{1}{4}) = \frac{1}{2}$, $H(\frac{1}{5}) = \frac{5}{14}$ and $J(\frac{1}{5}) = \frac{1}{2}$
substitute the above values in (3.1.2)

$$S_m(\frac{7}{20}, 0, \frac{7}{20}, \frac{5}{14}) \leq \left\{ \max[S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2})] \right\}^\lambda$$

which implies that

$$e^{0.014} \leq \left\{ \max[e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}] \right\}^\lambda$$

$$e^{0.014} \leq \{ \max[e^{0.58}, e^{0.58}, e^{0.58}, e^{0.58}] \}^\lambda \implies e^{0.014} \leq e^{0.5.8\lambda}$$

this gives that $\lambda = 0.14$ where $\lambda \in (0, \frac{1}{2})$.

Hence the inequality (3.3.2) holds.

It can be seen that " $\frac{1}{2}$ " is a unique common fixed point for four self mappings M, K H, and J.

4 Conclusions

In this article, we established a common fixed point theorem in S_m -metric space by using weakly-compatible mappings, semi-compatible mappings, and reciprocally continuous mappings for four self-maps. Furthermore, our results are also justified with suitable examples.

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On Neutrosophic filter of BL-algebras

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Abstract

In this paper, we introduce the notion of neutrosophic filter in BL-algebras and investigate several key characteristics with illustrations. Additionally, we obtain few conditions, order properties of neutrosophic filter of BL-algebras. Moreover, we prove that intersection of two neutrosophic filters of a BL-algebra is also a neutrosophic filter.

Keywords: BL-algebra; filter; neutrosophic set; neutrosophic filter.

2020 AMS subject classification: 03G25, 03E72, 03F55, 06F35[‡].

1. Introduction

In 1965, L. A. Zadeh [2, 3, 14] was the first to introduce the notion of fuzzy sets to describe vagueness mathematically. He rectified those problems by designating every feasible individual in the universe a number resembling its grade of membership in the fuzzy set. In lattice implication algebras, Xu and Qin[12] first proffered the concept of filters. An important aspect of researching various logical algebras is filter theory. In the argument of the completeness of various logical algebra, filters are crucial. Several academicians have investigated the filter theory of different logical algebras. In 1998, the ideology of neutrosophy was introduced by Smarandache [1,9]. This became a new idea of philosophy to characterize the neutralites. The main idea of neutrosophy is

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that behind every concept there also exists an indeterminant degree in addition to truth and falsity. Neutrosophy, a discipline of philosophy that has just recently been recognised as a science, examines the genesis, character, and range of neutralities as well as how they interact with various ideational spectra. Hajek[6] introduced the concept of BL-algebras (Basic Logic), is a type of logical algebra.

In this paper, the concept of neutrosophic filters of BL-algebras is discussed. In section 2, few fundamental definitions and results are explained. In section 3, we introduce the notion of neutrosophic filters of BL-algebras along with some of its related features.

2. Preliminaries

In this section, we recall few fundamental definitions and their characteristics that are useful for developing the primary findings.

Definition 2.1[6,11] A BL-algebra is an algebra $(\mathcal{B}, \vee, \wedge, \circ, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ such that the following are satisfied for all $j_1, k_1, l_1 \in \mathcal{B}$,

- (i) $(\mathcal{B}, \vee, \wedge, 0, 1)$ is a bounded lattice,
- (ii) $(\mathcal{B}, \circ, 1)$ is a commutative monoid,
- (iii) $' \circ '$ and $' \rightarrow '$ form an adjoint pair, that is, $l_1 \leq j_1 \rightarrow k_1$ if and only if $j_1 \circ l_1 \leq k_1$ for all $j_1, k_1, l_1 \in \mathcal{B}$,
- (iv) $j_1 \wedge k_1 = j_1 \circ (j_1 \rightarrow k_1)$,
- (v) $(j_1 \rightarrow k_1) \vee (k_1 \rightarrow j_1) = 1$.

Proposition 2.2[7,13] The following axioms are satisfied in a BL- algebra $\mathcal{B}$ for all $j_1, k_1, l_1 \in \mathcal{B}$,

- (i) $k_1 \rightarrow (j_1 \rightarrow l_1) = j_1 \rightarrow (k_1 \rightarrow l_1) = (j_1 \circ k_1) \rightarrow l_1$,
- (ii) $1 \rightarrow j_1 = j_1$,
- (iii) $j_1 \leq k_1$ if and only if $j_1 \rightarrow k_1 = 1$,
- (iv) $j_1 \vee k_1 = ((j_1 \rightarrow k_1) \rightarrow k_1) \wedge ((k_1 \rightarrow j_1) \rightarrow j_1)$,
- (v) $j_1 \leq k_1$ implies $k_1 \rightarrow l_1 \leq j_1 \rightarrow l_1$,
- (vi) $j_1 \leq k_1$ implies $l_1 \rightarrow j_1 \leq l_1 \rightarrow k_1$,
- (vii) $j_1 \rightarrow k_1 \leq (l_1 \rightarrow j_1) \rightarrow (l_1 \rightarrow k_1)$,
- (viii) $j_1 \rightarrow k_1 \leq (k_1 \rightarrow l_1) \rightarrow (k_1 \rightarrow l_1)$,
- (ix) $j_1 \leq (j_1 \rightarrow k_1) \rightarrow k_1$,
- (x) $j_1 \circ (j_1 \rightarrow k_1) = j_1 \wedge k_1$,
- (xi) $j_1 \circ k_1 \leq j_1 \wedge k_1$
- (xii) $j_1 \rightarrow k_1 \leq (j_1 \circ l_1) \rightarrow (k_1 \circ l_1)$,
- (xiii) $j_1 \circ (k_1 \rightarrow l_1) \leq k_1 \rightarrow (j_1 \circ l_1)$,
- (xiv) $(j_1 \rightarrow k_1) \circ (k_1 \rightarrow l_1) \leq j_1 \rightarrow l_1$,
- (xv) $(j_1 \circ j_1^*) = 0$.

Note. In the above sequence, $\mathcal{B}$ is used to intend the BL- algebras and the operations

' $\vee$ ', ' $\wedge$ ', ' $\circ$ ' have preference on the way to the operations ' $\rightarrow$ '.

Note. In a BL- algebra $\mathcal{B}$, ' $*$ ' is a complement defined as $j_1^* = j_1 \rightarrow 0$ for all $j_1 \in \mathcal{B}$.

Definition 2.3[15] A non-empty subset F of a BL- algebra $\mathcal{B}$ is a filter of $\mathcal{B}$ if the following axioms hold for all $j_1, k_1 \in \mathcal{B}$,

- (i) If $j_1, k_1 \in F$, then $j_1 \circ k_1 \in F$,
- (ii) If $j_1 \in F$ and $j_1 \leq k_1$, then $k_1 \in F$.

Proposition 2.4[15] A nonempty subset F of a BL- algebra $\mathcal{B}$ is a filter of $\mathcal{B}$ if and only if the following are satisfied for all $j_1, k_1 \in \mathcal{B}$,

- (i) $1 \in F$,
- (ii) $j_1, j_1 \rightarrow k_1 \in F$ implies $k_1 \in F$.

A filter F of a BL-algebra $\mathcal{B}$ is proper if $F \neq \mathcal{B}$.

Definition 2.5[8,9] Let X be a set. A neutrosophic subset R of X is a triple (T_R, I_R, F_R) where $T_R: X \rightarrow [0,1]$ is truth membership function, $I_R: X \rightarrow [0,1]$ is indeterminacy function and $F_R: X \rightarrow [0,1]$ is false membership function and $0 \leq T_R(j_1) + I_R(j_1) + F_R(j_1) \leq 3$ for all $j_1 \in X$. Hence, for each $j_1 \in X$, $T_R(j_1)$, $I_R(j_1)$ and $F_R(j_1)$ are all standard real numbers in $[0,1]$.

Note. The values of $T_R(j_1)$, $I_R(j_1)$ and $F_R(j_1)$ have no limitations and we have the obvious condition $0 \leq T_R(j_1) + I_R(j_1) + F_R(j_1) \leq 3$.

Definition 2.6[4, 9] Let R and S be two neutrosophic sets on X . Define $R \leq S$ if and only if

$$T_R(j_1) \leq T_S(j_1), I_R(j_1) \geq I_S(j_1), F_R(j_1) \geq F_S(j_1) \text{ for all } j_1 \in X.$$

Definition 2.7[5,9] Let R and S be two neutrosophic sets on X . Define $R \wedge S = (T_R \wedge T_S, I_R \vee I_S, F_R \vee F_S)$; $R \vee S = (T_R \vee T_S, I_R \wedge I_S, F_R \wedge F_S)$

Where ' $\wedge$ ' is the minimum and ' $\vee$ ' is the maximum between real numbers.

Definition 2.8[10] Let R be a neutrosophic set in X and $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and (α, β, γ) – level set of R denoted by $R^{(\alpha, \beta, \gamma)}$ is defined as

$$R^{(\alpha, \beta, \gamma)} = \{j_1 \in X / T_R(j_1) \geq \alpha, I_R(j_1) \leq \beta \text{ and } F_R(j_1) \leq \gamma\}.$$

3. Properties of neutrosophic filter

In this section, we introduce the definition of neutrosophic filter of BL-algebra and obtain some relevant properties with illustrations.

Definition 3.1 A neutrosophic set R of algebra $\mathcal{B}$ is called a neutrosophic filter if it satisfies the subsequent conditions:

- (i) $T_R(j_1) \leq T_R(1)$, $I_R(j_1) \geq I_R(1)$ and $F_R(j_1) \geq F_R(1)$,
- (ii) $\min\{T_R(j_1 \rightarrow k_1), T_R(j_1)\} \leq T_R(k_1)$, $\min\{I_R(j_1 \rightarrow k_1), I_R(j_1)\} \geq I_R(k_1)$
and $\min\{F_R(j_1 \rightarrow k_1), F_R(j_1)\} \geq F_R(k_1)$ for all $j_1, k_1 \in \mathcal{B}$.

Example 3.2 Let $\mathcal{B} = \{0, u, v, 1\}$. The binary operations ' $\circ$ ' and ' $\rightarrow$ ' are given by the subsequent tables (3.1) and (3.2).

$\circ$	0	u	v	1
0	0	0	0	0
u	0	0	u	v
v	0	u	v	v
1	0	u	v	1

Table 3.1: ' $\circ$ ' Operation

$\rightarrow$	0	u	v	1
0	1	1	1	1
u	u	1	1	1
v	0	u	1	1
1	0	u	v	1

Table 3.2: ' $\rightarrow$ ' Operation

Then, $(\mathcal{B}, \vee, \wedge, \circ, \rightarrow, 0, 1)$ is a BL-algebra.

Define a neutrosophic set R of $\mathcal{B}$ as follows:

$$R = \{(1, [0.9, 0.2, 0.1]), (u, [0.5, 0.3, 0]), (v, [0.5, 0.3, 0]), (0, [0.9, 0.2, 0.1])\}.$$

It is evident that R is a neutrosophic filter of $\mathcal{B}$ and assure the conditions (i) and (ii) of the definition 3.1.

Example 3.3 Let $\mathcal{B} = \{0, a, b, 1\}$. The binary operations are given by the tables (3.3) and (3.4). Let S of $\mathcal{B}$ be a neutrosophic set as follows:

$$S = \{(1, [0.5, 0.3, 0.2]), (a, [0.9, 0.2, 0.1]), (b, [0.9, 0.2, 0.1]), (0, [0.9, 0.2, 0.1])\}$$

Here, S is not a neutrosophic filter of $\mathcal{B}$.

$$\text{Since } T_S(1) = 0.5 \not\geq 0.9 = \min\{T_S(a \circ 1), T_S(a)\},$$

$$T_S(1) = 0.5 \not\geq 0.9 = \min\{T_S(b \circ 1), T_S(b)\}.$$

$\circ$	0	a	b	1
0	0	0	0	b
a	0	0	a	b
b	0	a	b	b
1	0	a	b	1

Table 3.3: ' $\circ$ ' Operation

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Table 3.4: ' $\rightarrow$ ' Operation

Proposition 3.4 Let R be a neutrosophic filter in a BL- algebra $\mathcal{B}$. If $j_1 \leq k_1$ then

$$T_R(j_1) \leq T_R(k_1), I_R(j_1) \geq I_R(k_1), F_R(j_1) \geq F_R(k_1) \text{ for all } j_1, k_1 \in \mathcal{B}.$$

Proof: Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$. If $j_1 \leq k_1$, then $j_1 \rightarrow k_1 = 1$ for all $j_1, k_1 \in \mathcal{B}$ [From(iii) of the proposition 2.2]

Then, from (i) and(ii) of the definition 3.1,

$$\begin{aligned} T_R(j_1) &= \min\{T_R(1), T_R(j_1)\} \\ &= \min\{T_R(j_1 \rightarrow k_1), T_R(j_1)\} \\ &\leq T_R(k_1), \\ I_R(j_1) &= \min\{I_R(1), I_R(j_1)\} \\ &= \min\{I_R(j_1 \rightarrow k_1), I_R(j_1)\} \\ &\geq I_R(k_1), \\ F_R(j_1) &= \min\{F_R(1), F_R(j_1)\} \\ &= \min\{F_R(j_1 \rightarrow k_1), F_R(j_1)\} \\ &\geq F_R(k_1) \text{ for all } j_1, k_1 \in \mathcal{B}. \blacksquare \end{aligned}$$

Proposition 3.5 Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$. If $j_1 \leq k_1$ then $T_R(j_1)$ is order preserving and $I_R(j_1), F_R(j_1)$ are order reversing.

Proof: Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$.

To prove: If $j_1 \leq k_1$ then $T_R(j_1) \leq T_R(k_1)$, $I_R(j_1) \geq I_R(k_1)$, $F_R(j_1) \geq F_R(k_1)$ for all $j_1, k_1 \in \mathcal{B}$.

Then, from the proposition 3.4, the proof is straight forward. Thus $T_R(j_1)$ is order preserving and $I_R(j_1), F_R(j_1)$ are order reversing. $\blacksquare$

Proposition 3.6 Let R be a neutrosophic set of a BL-algebra $\mathcal{B}$. R is a neutrosophic filter of $\mathcal{B}$ if and only if $j_1 \rightarrow (k_1 \rightarrow l_1) = 1$ implies $T_R(l_1) \geq \min\{T_R(j_1), T_R(k_1)\}$, $I_R(l_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and $F_R(l_1) \leq \min\{F_R(j_1), F_R(k_1)\}$ for all $j_1, k_1, l_1 \in \mathcal{B}$.

Proof: Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$.

Then, from (ii) of the definition 3.1, we have

$$T_R(l_1) \geq \min\{T_R(l_1 \rightarrow k_1), T_R(k_1)\} \text{ for all } j_1, k_1, l_1 \in \mathcal{B}.$$

$$\text{Now, } T_R(l_1 \rightarrow k_1) \geq \min\{T_R(j_1 \rightarrow (k_1 \rightarrow l_1)), T_R(j_1)\}.$$

$$\begin{aligned} \text{If } j_1 \rightarrow (k_1 \rightarrow l_1) = 1, \text{ then we have } T_R(l_1 \rightarrow k_1) &\geq \min\{T_R(1), T_R(j_1)\} \\ &= T_R(j_1). \end{aligned}$$

$$\text{So, } T_R(l_1) \geq \min\{T_R(j_1), T_R(k_1)\}.$$

$$\text{Similarly, } I_R(l_1) \leq \min\{I_R(j_1), I_R(k_1)\} \text{ and}$$

$$F_R(l_1) \leq \min\{F_R(j_1), F_R(k_1)\}.$$

Conversely, Let $j_1 \rightarrow (j_1 \rightarrow 1) = 1$ for all $j_1 \in \mathcal{B}$.

$$\text{Then, } T_R(1) \geq \min\{T_R(j_1), T_R(j_1)\} = T_R(j_1).$$

On the other hand, from $(j_1 \rightarrow k_1) \rightarrow (j_1 \rightarrow k_1) = 1$ implies

$$\begin{aligned} T_R(k_1) &\geq \min\{T_R(j_1 \rightarrow k_1), T_R(j_1)\}, I_R(k_1) \leq \min\{I_R(j_1 \rightarrow k_1), I_R(j_1)\} \text{ and} \\ F_R(k_1) &\leq \min\{F_R(j_1 \rightarrow k_1), F_R(j_1)\}. \end{aligned}$$

Then, from the definition 3.1, R is a neutrosophic filter of $\mathcal{B}$. ■

Corollary 3.7 Let R be a neutrosophic set of BL-algebra $\mathcal{B}$. R is a neutrosophic filter of $\mathcal{B}$ if and only if $j_1 \circ k_1 \leq l_1$ or $k_1 \circ l_1 \leq l_1$ implies $T_R(l_1) \geq \min\{T_R(j_1), T_R(k_1)\}$, $I_R(l_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and $F_R(l_1) \leq \min\{F_R(j_1), F_R(k_1)\}$ for all $j_1, k_1, l_1 \in \mathcal{B}$.

Proof: From (i) of the proposition 2.2 and the proposition 3.6 the proof is obvious. ■

Proposition 3.8 Let R be a neutrosophic set of a BL-algebra $\mathcal{B}$. R is a neutrosophic filter of $\mathcal{B}$ if and only if

- (i) If $j_1 \leq k_1$ then $T_R(j_1) \leq T_R(k_1)$, $I_R(j_1) \geq I_R(k_1)$ and $F_R(j_1) \geq F_R(k_1)$,
- (ii) $T_R(j_1 \circ k_1) \geq \min\{T_R(j_1), T_R(k_1)\}$, $I_R(j_1 \circ k_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and $F_R(j_1 \circ k_1) \leq \min\{F_R(j_1), F_R(k_1)\}$ for all $j_1, k_1 \in \mathcal{B}$.

Proof: Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$.

Then, from the proposition 3.4, we have $T_R(j_1) \leq T_R(k_1)$, $I_R(j_1) \geq I_R(k_1)$ and $F_R(j_1) \geq F_R(k_1)$ when $j_1 \leq k_1$.

Since $j_1 \circ k_1 \leq j_1 \circ k_1$ and from the corollary 3.7, we have $T_R(j_1 \circ k_1) \geq \min\{T_R(j_1), T_R(k_1)\}$, $I_R(j_1 \circ k_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and $F_R(j_1 \circ k_1) \leq \min\{F_R(j_1), F_R(k_1)\}$.

Conversely, Let R be a neutrosophic set and satisfies (i) and (ii).

If $j_1 \circ k_1 \leq l_1$ then from (i) and (ii) we get, $T_R(l_1) \geq \min\{T_R(j_1), T_R(k_1)\}$, $I_R(l_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and $F_R(l_1) \leq \min\{F_R(j_1), F_R(k_1)\}$ for all $j_1, k_1, l_1 \in \mathcal{B}$.

Then, from the corollary 3.7, we have R is a neutrosophic filter. ■

Proposition 3.9 Let R be a neutrosophic set of a BL-algebra $\mathcal{B}$. If R is a neutrosophic filter of $\mathcal{B}$, then it satisfies the following for all $j_1, k_1, l_1 \in \mathcal{B}$.

- (i) If $T_R(j_1 \rightarrow k_1) = T_R(1)$, then $T_R(j_1) \leq T_R(k_1)$, $I_R(j_1 \rightarrow k_1) = I_R(1)$, then $I_R(j_1) \geq I_R(k_1)$, $F_R(j_1 \rightarrow k_1) = F_R(1)$, then $F_R(j_1) \geq F_R(k_1)$.
- (ii) $T_R(j_1 \rightarrow k_1) \leq T_R(j_1 \circ l_1 \rightarrow k_1 \circ l_1)$, $I_R(j_1 \rightarrow k_1) \geq I_R(j_1 \circ l_1 \rightarrow k_1 \circ l_1)$ and $F_R(j_1 \rightarrow k_1) \geq F_R(j_1 \circ l_1 \rightarrow k_1 \circ l_1)$.
- (iii) $T_R(j_1 \rightarrow k_1) \leq T_R((k_1 \rightarrow l_1) \rightarrow (j_1 \rightarrow l_1))$, $I_R(j_1 \rightarrow k_1) \geq I_R((k_1 \rightarrow l_1) \rightarrow (j_1 \rightarrow l_1))$ and $F_R(j_1 \rightarrow k_1) \geq F_R((k_1 \rightarrow l_1) \rightarrow (j_1 \rightarrow l_1))$.
- (iv) $T_R(j_1 \rightarrow k_1) \leq T_R((l_1 \rightarrow j_1) \rightarrow (l_1 \rightarrow k_1))$, $I_R(j_1 \rightarrow k_1) \geq I_R((l_1 \rightarrow j_1) \rightarrow (l_1 \rightarrow k_1))$ and $F_R(j_1 \rightarrow k_1) \geq F_R((l_1 \rightarrow j_1) \rightarrow (l_1 \rightarrow k_1))$.

Proof: (i) Let R be a neutrosophic filter of a BL-algebra $\mathcal{B}$.

Then from the definition 3.1, and since $T_R(j_1 \rightarrow k_1) = T_R(1)$,

$$\begin{aligned} \text{we have } T_R(k_1) &\geq \min\{T_R(j_1), T_R(j_1 \rightarrow k_1)\} \\ &= \min\{T_R(j_1), T_R(1)\} = T_R(j_1). \end{aligned}$$

Thus, $T_R(j_1) \leq T_R(k_1)$. Similarly we get, $I_R(j_1) \geq I_R(k_1)$, $F_R(j_1) \geq F_R(k_1)$.

From the proposition 2.2 and (i) of the proposition 3.8, we can prove (ii), (iii) and (iv) easily. ■

Proposition 3.10 Let R be a neutrosophic set of a BL-algebra $\mathcal{B}$. R is a neutrosophic filter of $\mathcal{B}$ if and only if

- (i) $T_R(j_1) \leq T_R(1)$, $I_R(j_1) \geq I_R(1)$ and $F_R(j_1) \geq F_R(1)$,
- (ii) $T_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \geq \min\{T_R(j_1), T_R(k_1)\}$,
 $I_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \leq \min\{I_R(j_1), I_R(k_1)\}$
and $F_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \leq \min\{F_R(j_1), F_R(k_1)\}$ for all $j_1, k_1, l_1 \in \mathcal{B}$.

Proof: Let R be a neutrosophic filter of BL-algebra $\mathcal{B}$.

From the definition 3.1 (i) is straight forward.

$$\text{Since, } T_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \geq \min\{T_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1), T_R(k_1)\} \quad (3.1)$$

Now, we have, $(j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow (k_1 \rightarrow l_1) = j_1 \vee (k_1 \rightarrow l_1) \geq j_1$.

$$T_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow (k_1 \rightarrow l_1)) \geq T_R(j_1) \quad (3.2)$$

[From the proposition 3.5]

Using (3.2) in (3.1), we have $T_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1)$

$$\geq \min\{T_R(j_1), T_R(k_1)\}$$

Similarly, we get $I_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \leq \min\{I_R(j_1), I_R(k_1)\}$ and

$$F_R((j_1 \rightarrow (k_1 \rightarrow l_1)) \rightarrow l_1) \leq \min\{F_R(j_1), F_R(k_1)\}.$$

Conversely, assume (i) and (ii) hold.

$$\text{Since } T_R(k_1) = T_R(1 \rightarrow k_1)$$

$$= T_R((j_1 \rightarrow k_1) \rightarrow (j_1 \rightarrow k_1) \rightarrow k_1)$$

$$\geq \min\{T_R(j_1 \rightarrow k_1), T_R(k_1)\}.$$

Similarly, $I_R(k_1) \leq \min\{I_R(j_1 \rightarrow k_1), I_R(k_1)\}$ and

$$F_R(k_1) \leq \min\{F_R(j_1 \rightarrow k_1), F_R(k_1)\}.$$

From (i), R is a neutrosophic filter of $\mathcal{B}$. ■

Proposition 3.11 The intersection of two neutrosophic filters of $\mathcal{B}$ is also a neutrosophic filter of $\mathcal{B}$.

Proof: Let R and S be two neutrosophic filters of $\mathcal{B}$,

To Prove: $R \cap S$ is a neutrosophic filter of $\mathcal{B}$.

We have $T_R(k_1) \geq \min\{T_R(l_1), T_R(j_1)\}$ and $T_S(k_1) \geq \min\{T_S(l_1), T_S(j_1)\}$ for all $j_1, k_1, l_1 \in \mathcal{B}$.

$$\text{Since } T_{R \cap S}(k_1) = \min\{T_R(k_1), T_S(k_1)\}$$

$$\geq \min\{\min\{T_R(l_1), T_R(j_1)\}, \min\{T_S(l_1), T_S(j_1)\}\}$$

$$= \min\{\min\{T_R(l_1), T_S(l_1)\}, \min\{T_R(j_1), T_S(j_1)\}\}$$

$$= \min\{\min\{T_{R \cap S}(l_1), T_{R \cap S}(j_1)\}\}.$$

Similarly, $I_{R \cap S}(k_1) = \min\{\min\{I_R(l_1), I_R(j_1)\}\}$ and

$$F_{R \cap S}(k_1) = \min\{\min\{F_R(l_1), F_R(j_1)\}\}.$$

Hence, $T_{R \cap S}(k_1) \geq \min\{T_{R \cap S}(l_1), T_{R \cap S}(j_1)\}$,

$$I_{R \cap S}(k_1) \leq \min\{I_{R \cap S}(l_1), I_{R \cap S}(j_1)\}$$
$$\text{and } F_{R \cap S}(k_1) \leq \min\{F_{R \cap S}(l_1), F_{R \cap S}(j_1)\}.$$

Thus $R \cap S$ is a neutrosophic filter of $\mathcal{B}$. ■

Corollary 3.12 Let P_i be a family of neutrosophic filters of $\mathcal{B}$, where $i \in I$, I is a index set, then $\cap_{i \in I} P_i$ is a neutrosophic filter of $\mathcal{B}$. ■

4. Discussion and Conclusions

In this paper, we have introduced the notion of a neutrosophic filter in BL-algebras with illustrations. Moreover, several features of the neutrosophic filters are conferred. Also, we have derived a few equivalent conditions for a neutrosophic set of a BL-algebra to be a neutrosophic filter. In BL-algebras, fuzzy, vague, and many additional filters have already been defined. The main focus of this paper is to establish the neutrosophical nature of BL-algebras. Further, research on BL-algebras structure and the creation of the associated many-valued logical system will benefit from the aforementioned study. In the future, these neutrosophic filters can be extended to include fantastic filters, implicative filters, normal filters, and so on.

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Superior domination polynomial of cycles

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Abstract

Superior domination polynomial $SD(G, x) = \sum_{t=\gamma_{sd}(G)}^n |sd(G, t)|x^t$ is a polynomial in which the power of the variable denotes the cardinality of a superior dominating set and the total number of sets of same cardinality forms the coefficient of the variable. In this paper we find the $SD(G, S_n)$ of stars and $SD(G, C_n)$ of cycles and properties of the coefficients are discussed. The $SD(G, x)$ different standard graphs are obtained and the roots of the polynomial are tabulated.

Keywords: Superior distance, superior domination, neighbourhood vertex, superior domination polynomial.

2020 AMS subject classifications: 05C12, 05C69, 05C31.¹

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1 Introduction

The graph $G = (V, E)$ is a finite, undirected, simple, ordered pair where $V(G)$ is a set of vertices and $E(G)$ is the set of edges. In 2009 Saeid Alikhani and Yee-hock Peng[1] conceptualized the concept of domination polynomial. Domination is a vast arena in graph theory, Ore[8] coined the term domination in graphs. A vast literature about domination can be found in domination in graphs[3].

There are different types of distances in graph theory one being superior distance, Kathiresan and Marimuthu[7] were the pioneers of superior distance in graphs. The same authors[6] put forth the concept of superior domination in 2008. A Mohamed Ismayil and Tejaskumar R[4] introduced eccentric domination polynomial which was the hybrid idea of combining eccentric domination[5] and domination polynomial.

In this paper, a distance based domination polynomial called superior domination polynomial is introduced by coalescence of superior domination and domination polynomial. Standard formulas to find the coefficients or the superior dominating sets of stars S_n and cycles C_n for any value of n . Theorems related to properties of these coefficients are stated and proved. Superior domination polynomial $SD(G, x)$ of different standard graphs are calculated, their roots are tabulated. For all the undefined terminologies and basic concepts of graphs refer the book Graph theory by Frank Harary[2].

2 Preliminaries

Definition 2.1. [1]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. Then the domination polynomial $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G .

Definition 2.2. [7]. Let $D_{uv} = N[u] \cup N[v]$. A D_{uv} -walk is defined as a $u - v$ walk in G that contains every vertex of D_{uv} . The superior distance $d_D(u, v)$ from u to v is the length of a shortest $D_{u,v}$ walk.

Definition 2.3. [7]. The superior neighbour of a vertex u is given by $d_D(u) = \min\{d_D(u, v) : v \in V(G) - \{u\}\}$. A vertex $v (\neq u)$ is called a superior neighbour of u if $d_D(u, v) = d_D(u)$.

Definition 2.4. [6]. A vertex u is said to be a superior dominate a vertex v if v is a superior neighbour of u .

Definition 2.5. [6]. A set S of vertices of G is called a superior dominating set of G if every vertex $V(G) - S$ is superior dominated by some vertex in S . A superior dominating set G of minimum cardinality is a minimum superior dominating set and its cardinality is called superior domination number of G and denoted by $\gamma_{sd}(G)$.

Theorem 2.1. [6]. For a cycle C_n the superior domination number is given by

$$\gamma_{sd}(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0(\text{mod } 3) \\ \frac{n+2}{3}, & \text{if } n \equiv 1(\text{mod } 3) \\ \frac{n+1}{3}, & \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

3 Superior Domination Polynomial of Graphs

In this section, we defined superior domination polynomial, properties and results related to superior domination polynomial are observed, stated and proved.

Definition 3.1. Superior domination polynomial is given by $SD(G, x) = \sum_{t=\gamma_{sd}(G)}^n |sd(G, t)| x^t$ where $|sd(G, t)|$ is the number of distinct superior dominating set with cardinality t and $\gamma_{sd}(G)$ is the superior domination number.

Example 3.1. .

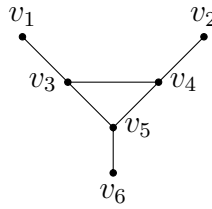


Figure 1: Net graph

Vertex	Minimum superior distance d_D	Superior neighbour
v_1	3	v_2, v_6
v_2	3	v_1, v_6
v_3	4	v_1
v_4	4	v_2
v_5	4	v_6
v_6	3	v_1, v_2

From figure-1 we get $\{v_3, v_4, v_5\}$ is a superior dominating set with cardinality 3, $\{v_1, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_3, v_4, v_5, v_6\}$ are superior dominating sets of cardinality 4, $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_3, v_4, v_5, v_6\}$, $\{v_2, v_3, v_4, v_5, v_6\}$ are superior dominating sets of cardinality 5 and $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is superior dominating set with cardinality 6. Therefore superior domination polynomial is given by $SD(G, x) = x^6 + 3x^5 + 3x^4 + x^3$.

Theorem 3.1. For a complete graph K_n the superior domination polynomial is given by $SD(K_n, x) = (1 + x)^n - 1$.

Proof. The degree of every vertex $v \in K_n$ is $n - 1$. For any two vertices u and v the number of vertices on their D_{uv} -walk is given by $|V(K_n)|$. Since $|N[u]| = n$ and $|N[v]| = n$ both the vertices have common neighbours and both u and v are incident to each other. Therefore a $D_{u,v}$ -walk between u and v contains all vertices of K_n and all the vertices of K_n forms the superior neighbour of any $v \in V(K_n)$ other than itself. By the definition of superior distance, the distance between any two vertices is $n - 1$. Now by the definition of superior domination, for every vertex of $V(K_n) - S$ is superior dominated by some vertex in S which is a superior dominating set and every vertex of $V(K_n) - S$ has a superior neighbour in S . Therefore $SD(K_n, x) = (1 + x)^n - 1$.

Theorem 3.2. If two graphs are isomorphic then $SD(G_1, x) = SD(G_2, x)$.

Proof. Let G_1 and G_2 be any two isomorphic graphs. Then there exist a one-one and onto function between the vertex sets such that $f : V(G_1) \rightarrow V(G_2)$ such that V_m and V_n are superior neighbours in G_1 if and only if $f(V_m)$ and $f(V_n)$ are superior neighbour of some vertex in G_2 . Therefore $|SD(G_1, n)| = |SD(G_2, n)| \forall n$. Therefore $SD(G_1, x) = SD(G_2, x)$.

Example 3.2. In the figure 2 and 3 both the tetrahedral graph and complete graph K_4 are isomorphic to each other.

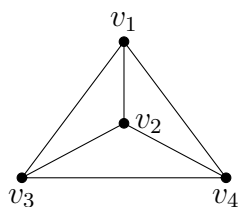


Fig:2-Tetrahedral graph T_g

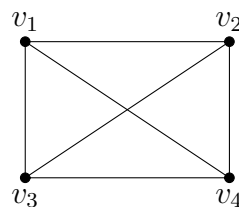


Fig:3-Complete graph K_4

$$SD(T_g, x) = x^4 + 4x^3 + 6x^2 + 4x.$$

$$SD(K_4, x) = x^4 + 4x^3 + 6x^2 + 4x.$$

Hence $T_g \cong K_4$ implies $SD(T_g, x) = SD(K_4, x)$.

Definition 3.2. Superior domination polynomial of a star graph S_n is given by $SD(S_n, x) = \sum_{t=\gamma_{sd}(S_n)}^n |sd(S_n, t)| x^t$ where $|sd(S_n, t)|$ is the number of distinct superior dominating sets with cardinality t and $\gamma_{sd}(S_n)$ is the superior domination number of a star graph.

Theorem 3.3. For a star graph S_n of order n where $n \geq 3$, the following are true.

1. $|sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|, t \in \mathbb{Z}^+, t \leq n$.
2. $SD(S_n, x) = xSD(S_{n-1}, x) + SD(S_{n-1}, x)$.
3. $SD(S_n, x) = x(x+1)^{n-1}$.

Proof.

1. Let $V(S_n) = \{v_1, v_2, \dots, v_n\}$. All the pendant vertices form the superior neighbours of central vertex v_1 since $\deg(v_1) = \Delta(S_n) = n-1$. Here we have ${}^{(n-1)}C_{t-1}$ superior dominating sets of cardinality t . Therefore $|sd(S_n, t)| = {}^{(n-1)}C_{t-1}$, $|sd(S_{n-1}, t-1)| = {}^{(n-2)}C_{t-2}$ and $|sd(S_{n-1}, t)| = {}^{(n-2)}C_{t-1}$.

$$\text{But } {}^{(n-1)}C_{t-1} = {}^{(n-2)}C_{t-2} + {}^{(n-2)}C_{t-1}.$$

$$\text{Therefore } |sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|.$$

2. By theorem-3.3-(1) we have

$$|sd(S_n, t)| = |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|.$$

$$\text{When } t = 1, |sd(S_n, 1)| = |sd(S_{n-1}, 0)| + |sd(S_{n-1}, 1)|.$$

$$\implies x|sd(S_n, 1)| = x|sd(S_{n-1}, 0)| + x|sd(S_{n-1}, 1)|.$$

$$\text{When } t = 2, |sd(S_n, 2)| = |sd(S_{n-1}, 1)| + |sd(S_{n-1}, 2)|.$$

$$\implies x^2|sd(S_n, 2)| = x^2|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)|.$$

$$\text{When } t = 3, |sd(S_n, 3)| = |sd(S_{n-1}, 2)| + |sd(S_{n-1}, 3)|.$$

$$\implies x^3|sd(S_n, 3)| = x^3|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)|.$$

$$\text{When } t = 4, |sd(S_n, 4)| = |sd(S_{n-1}, 3)| + |sd(S_{n-1}, 4)|.$$

$$\implies x^4|sd(S_n, 4)| = x^4|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)|.$$

$\vdots$

$$\text{When } t = n-1, |sd(S_n, n-1)| = |sd(S_{n-1}, n-2)| + |sd(S_{n-1}, n-1)|.$$

$$\implies x^{n-1}|sd(S_n, n-1)| = x^{n-1}|sd(S_{n-1}, n-2)| + x^{n-1}|sd(S_{n-1}, n-1)|.$$

$$\text{When } t = n, |sd(S_n, n)| = |sd(S_{n-1}, n-1)| + |sd(S_{n-1}, n)|.$$

$$\implies x^n|sd(S_n, n)| = x^n|sd(S_{n-1}, n-1)| + x^n|sd(S_{n-1}, n)|.$$

$$\begin{aligned} \text{Hence } x|sd(S_n, 1)| + x^2|sd(S_n, 2)| + x^3|sd(S_n, 3)| + x^4|sd(S_n, 4)| + \dots + x^{n-1}|sd(S_n, n-1)| + x^n|sd(S_n, n)| \\ = x|sd(S_{n-1}, 0)| + x|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 1)| + x^2|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 2)| + x^3|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 3)| + x^4|sd(S_{n-1}, 4)| + \dots + x^{n-1}|sd(S_{n-1}, n-2)| + x^{n-1}|sd(S_{n-1}, n-1)| \end{aligned}$$

$$1)| + x^n |sd(S_{n-1}, n-1)| + x^n |sd(S_{n-1}, n)|.$$

$$= x |sd(S_{n-1}, 0)| + x^2 |sd(S_{n-1}, 1)| + x^3 |sd(S_{n-1}, 2)| + x^4 |sd(S_{n-1}, 3)| + \dots + x^{n-1} |sd(S_{n-1}, n-2)| + x^n |sd(S_{n-1}, n-1)| + x |sd(S_{n-1}, 1)| + x^2 |sd(S_{n-1}, 2)| + x^3 |sd(S_{n-1}, 3)| + x^4 |sd(S_{n-1}, 4)| + \dots + x^{n-1} |sd(S_{n-1}, n-1)| + x^n |sd(S_{n-1}, n)|.$$

$$= x [x |sd(S_{n-1}, 1)| + x^2 |sd(S_{n-1}, 2)| + x^3 |sd(S_{n-1}, 3)| + x^4 |sd(S_{n-1}, 4)| + \dots + x^{n-1} |sd(S_{n-1}, n-1)|] + x |sd(S_{n-1}, 1)| + x^2 |sd(S_{n-1}, 2)| + x^3 |sd(S_{n-1}, 3)| + x^4 |sd(S_{n-1}, 4)| + \dots + x^{n-1} |sd(S_{n-1}, n-1)| + x^n |sd(S_{n-1}, n)|.$$

$$\text{Since } |sd(S_{n-1}, 0)| = |sd(S_{n-1}, n)| = 0.$$

$$= x \sum_{t=1}^{n-1} |sd(S_{n-1}, t)| x^t + \sum_{t=1}^{n-1} |sd(S_{n-1}, t)| x^t.$$

$$SD(S_n, x) = x SD(S_{n-1}, x) + SD(S_{n-1}, x).$$

3. We prove this by mathematical induction.

When $n = 3$,

$$\begin{aligned} SD(S_n, x) &= x(x+1)^{n-1} \\ &= x(x+1)^{3-1} \\ &= x(x+1)^2 \end{aligned}$$

The result is true for $n = 3$.

When $n = 4$,

$$SD(S_n, x) = x(x+1)^3$$

The result is true for $n = 4$.

Assume the result is true for all natural numbers less than n .

$$\begin{aligned} SD(S_{n-1}, x) &= x(x+1)^{(n-1)-1} \\ &= x(x+1)^{n-2} \end{aligned}$$

Now we prove the result for n .

$$\begin{aligned} SD(S_n, x) &= x SD(S_{n-1}, x) + SD(S_{n-1}, x) \text{ using theorem 3.3-(2)} \\ &= x[x(x+1)^{n-2}] + x(x+1)^{n-2} \\ &= x(x+1)^{n-2}[x+1] \\ &= x(x+1)^{n-2+1} \\ &= x(x+1)^{n-1} \end{aligned}$$

$\therefore$ The result is true for all n .

Table: $|sd(S_n, t)|$ is the number of superior dominating sets of S_n with cardinality t where $1 \leq t \leq 15$.

Superior domination polynomial

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0														
2	2	1													
3	1	2	1												
4	1	3	3	1											
5	1	4	6	4	1										
6	1	5	10	10	5	1									
7	1	6	15	20	15	6	1								
8	1	7	21	35	35	21	7	1							
9	1	8	28	56	70	56	28	8	1						
10	1	9	36	84	126	126	84	36	9	1					
11	1	10	45	120	210	252	210	120	45	10	1				
12	1	11	55	165	330	462	462	330	165	55	11	1			
13	1	12	66	220	495	792	924	792	495	220	66	12	1		
14	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
15	1	14	91	364	1001	2002	3003	3423	3003	2002	1001	364	91	14	1

Theorem 3.4. *The following properties for the co-efficients of $SD(S_n, x)$ holds.*

1. $|sd(S_n, 1)| = 1$ for all $n > 2$.
2. $|sd(S_n, n)| = 1$ for all $n \geq 2$.
3. $|sd(S_n, n - 1)| = n - 1$ for all $n > 2$.
4. $|sd(S_n, n - 2)| = \frac{(n - 1)(n - 2)}{2}$ for all $n \geq 3$.
5. $|sd(S_n, n - 3)| = \frac{(n - 1)(n - 2)(n - 3)}{6}$ for all $n \geq 4$.
6. $|sd(S_n, n - 4)| = \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24}$ for all $n \geq 5$.
7. $|sd(S_n, t)| = |sd(S_n, n - t + 1)|$ for all $n \geq 3$.
8. If $SD_n = \sum_{t=1}^n |sd(S_n, t)|$ for all $n \geq 3$ then $SD_n = 2(SD_{n-1})$ with initial condition $SD_3 = 4$.
9. $SD_n = \text{Total number of superior dominating sets in } S_n = 2^{n-1}$ for all $n \geq 3$.

Proof.

1. Let $V(S_n) = \{v_1, v_2, \dots, v_n\}$. In a star graph S_n all the vertices form a superior neighbour of central vertex v_1 except itself. Therefore the only set with single cardinality $D = \{v_1\}$ forms the superior dominating set of every star graph S_n where $n > 2$. Therefore $|sd(S_n, 1)| = 1$ for all $n > 2$.

2. The whole set of vertices $V(S_n)$ forms the superior dominating set $|sd(S_n, n)| = 1$ for all $n \geq 2$.
3. By mathematical induction on n . The result is true for $n = 3$, since $|sd(S_3, 3 - 1)| = |sd(S_3, 2)| = 2$. Assume the result is true for all natural numbers less than n . Now we prove it for n ie, $|sd(S_{n-1}, n - 2)| = n - 2$. By theorem-3.3-(1) and 3.4-(2)

$$\begin{aligned} |sd(S_n, n - 1)| &= |sd(S_{n-1}, n - 2)| + |sd(S_{n-1}, n - 1)| \\ &= (n - 2) + 1 \\ &= n - 1 \end{aligned}$$

$\therefore$ The result is true for all n .

4. By mathematical induction on n . For $n = 3$, $|sd(S_3, 1)| = 1$. For $n = 4$, $|sd(S_4, 2)| = 3$. Assume the result is true for all natural numbers less than n , ie, for $n = n - 1$, $|sd(S_{n-1}, n - 3)| = \frac{(n-2)(n-3)}{2}$. Now we prove it for n . By theorem-3.3-(1) and 3.4-(3),

$$\begin{aligned} |sd(S_n, n - 2)| &= |sd(S_{n-1}, n - 3)| + |sd(S_{n-1}, n - 2)| \\ &= \frac{(n - 2)(n - 3)}{2} + (n - 2) \\ &= \frac{(n - 2)(n - 3) + 2(n - 2)}{2} \\ &= \frac{(n - 2)(n - 3 + 2)}{2} \\ &= \frac{(n - 2)(n - 1)}{2} \\ &= \frac{(n - 1)(n - 2)}{2} \end{aligned}$$

$\therefore$ The result is true for all n .

5. By mathematical induction on n . For $n = 4$, $|sd(S_4, 1)| = 1$. For $n = 5$, $|sd(S_5, 2)| = 4$. Assume the result is true for all natural numbers less than n . For $n = n - 1$, $|sd(S_{n-1}, n - 4)| = \frac{(n-2)(n-3)(n-4)}{6}$

Superior domination polynomial

Now we prove it for n . By theorem-3.3-(1) and 3.4-(4),

$$\begin{aligned}
 |sd(S_n, n-3)| &= |sd(S_{n-1}, n-4)| + |sd(S_{n-1}, n-3)| \\
 &= \frac{(n-2)(n-3)(n-4)}{6} + \frac{(n-2)(n-3)}{2} \\
 &= \frac{(n-2)(n-3)(n-4+3)}{6} \\
 &= \frac{(n-1)(n-2)(n-3)}{6}
 \end{aligned}$$

$\therefore$ The result is true for all n .

6. By mathematical induction on n .

The result is true for $n = 5$ since $|sd(S_5, 1)| = 1$

For $n = 6$, $|sd(S_6, 2)| = 5$.

Assume the result is true for all natural numbers less than n .

For $n = n-1$, $|sd(S_{n-1}, n-5)| = \frac{(n-2)(n-3)(n-4)(n-5)}{24}$

Now we prove it for n . By theorem-3.3-(1) and 3.4-(5),

$$\begin{aligned}
 |sd(S_n, n-4)| &= |sd(S_{n-1}, n-5)| + |sd(S_{n-1}, n-4)| \\
 &= \frac{(n-2)(n-3)(n-4)(n-5)}{24} + \frac{(n-2)(n-3)(n-4)}{6} \\
 &= \frac{(n-2)(n-3)(n-4)(n-5+4)}{24} \\
 &= \frac{(n-1)(n-2)(n-3)(n-4)}{24}
 \end{aligned}$$

$\therefore$ The result is true for all n .

7. By mathematical induction on n .

The result is true for $n = 3$ since $|sd(S_3, 1)| = |sd(S_3, 3-1+1)| = |sd(S_3, 3)| = 1$.

For $n = 4, t = 2$, $|sd(S_4, 2)| = |sd(S_4, 4-2+1)| = |sd(S_4, 3)| = 3$.

Assume the result is true for all natural numbers less than n .

For $n = n-1$, $|sd(S_{n-1}, t-1)| = |sd(S_{n-1}, (n-t+1))|$

Now we prove it for n . By theorem-3.3 we have,

$$\begin{aligned}
 |sd(S_n, t)| &= |sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)| \\
 &= |sd(S_{n-1}, (n-1-(t-1)+1))| + |sd(S_{n-1}, (n-1-(t)+1))| \\
 &= |sd(S_{n-1}, (n-1-t+1+1))| + |sd(S_{n-1}, (n-1-t+1))| \\
 &= |sd(S_{n-1}, (n-t+1))| + |sd(S_{n-1}, (n-t))| \\
 &= |sd(S_n, (n-t+1))|
 \end{aligned}$$

$\therefore$ The result is true for all n .

8. $SD_n = \sum_{i=1}^n |sd(S_n, t)|$
By theorem-3.3 we have

$$\begin{aligned} SD_n &= \sum_{i=1}^n [|sd(S_{n-1}, t-1)| + |sd(S_{n-1}, t)|] \\ &= \sum_{i=1}^{n-1} |sd(S_{n-1}, t)| + \sum_{i=1}^{n-1} |sd(S_{n-1}, t)| \\ &= SD_{n-1} + SD_{n-1} \\ SD_n &= 2[SD_{n-1}] \end{aligned}$$

9. By mathematical induction on n .

When $n = 3$,

$$SD_3 = 2^{3-1} = 2^2 = 4.$$

$$SD_4 = 2^{4-1} = 2^3 = 8.$$

Assume the result is true for all natural numbers less than n . Now we prove it for n . Therefore $SD_{n-1} = 2^{n-1-1} = 2^{n-2}$

Now

$$\begin{aligned} SD_n &= 2[SD_{n-1}] \text{ from theorem-3.4-(8)} \\ &= 2[2^{n-2}] \\ &= 2^{n-2+1} \\ &= 2^{n-1} \end{aligned}$$

$\therefore$ The result is true for all n .

Hence the theorem.

4 Superior domination polynomial of cycle

Let $sd(C_n, m)$ be the superior dominating set of cycle C_n with cardinality m . K.M. Kathiresan and G. Marimuthu[6] proved theorem-2.1, for our convenience we reframe theorem-2.1 as $\gamma_{sd}(C_n) = \lceil \frac{n}{3} \rceil \forall n$. Hereafter we denote the vertex set $V(G) = \{v_1, v_2, \dots, v_n\} = [n]$.

Lemma 4.1. For a cycle C_n , $sd(C_n, m) = \emptyset$ if $m > n$ or $m < \lceil \frac{n}{3} \rceil$

Proof. Let C_n be a cycle, a superior dominating set D has the minimum cardinality among the minimum superior dominating set with cardinality $\lceil \frac{n}{3} \rceil$ by theorem-2.1. Therefore there is no proper subset of D which forms a superior dominating set. Hence

$$sd(C_n, m) = \emptyset \text{ where } m < \lceil \frac{n}{3} \rceil = |D| \quad (1)$$

There can not exists a superior dominating set greater than the order of the graph. Therefore

$$sd(C_n, m) = \emptyset \text{ if } m > n \quad (2)$$

From equation-(1) and (2) we obtain the result.

Observation 4.1. *If a cycle C_n contains a maximal simple path of length $3k - 1$, $3k$ or $3k + 1$ then every dominating set of C_n must contain at least k , $k + 1$ or $k + 1$ vertices respectively.*

Lemma 4.2. *Let L be a subset of the vertex set, $L \subseteq [n]$. If L is in $sd(C_{n-4}, m-1)$ or $sd(C_{n-5}, m-1) \ni L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$ then $L \in sd(C_{n-3}, m-1)$.*

Proof. Let $L \in sd(C_{n-4}, m-1)$ and $L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$ then by lemma-4.3 we consider $\{1, n-4\}$, $\{2, n-4\}$ and $\{1, n-5\}$ as a subset of L . Then $L \in sd(C_{n-3}, m-1)$ suppose $L \in sd(C_{n-5}, m-1)$ and $L \cup \{v\} \in sd(C_n, m)$ for $v \in [n]$. Then by lemma-4.3 $\{1, n-5\}$ must be a subset of L . Hence $L \in sd(C_{n-3}, m-1)$.

Lemma 4.3. .

1. If $sd(C_{n-1}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ then $sd(C_{n-2}, m-1) = \emptyset$.
2. If $sd(C_{n-1}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_{n-2}, m-1) \neq \emptyset$.
3. If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ then $sd(C_n, m) = \emptyset$.

Proof.

1. Since $sd(C_{n-1}, m-1) = sd(C_{n-3}, m-1) = \emptyset$, by lemma-4.1, $m-1 > n-1$ or $m-1 < \lceil \frac{n-3}{3} \rceil$. In both cases we have $sd(C_{n-2}, m-1) = \emptyset$.
2. Suppose that $sd(C_{n-2}, m-1) = \emptyset$ by lemma-4.1, $m-1 > n-2$ or $m-1 < \lceil \frac{n-2}{3} \rceil$. If $m-1 > n-2$ then $m-1 > n-3$. Hence $sd(C_{n-3}, m-1) = \emptyset$, a contradiction. Hence $m-1 < \lceil \frac{n-2}{3} \rceil$. So $m-1 < \lceil \frac{n-1}{3} \rceil$, $sd(C_{n-1}, m-1) = \emptyset$, also a contradiction.
3. Suppose $sd(C_n, m) \neq \emptyset$. Let $L \in sd(C_n, m)$ such that at least one vertex labelled as v_n or v_{n-1} is in L . If $v_n \in L$, then by observation-4.1 at least one vertex labelled as v_{n-1} , v_{n-2} or v_{n-3} is in L . If $v_{n-1} \in L$ or $v_{n-2} \in L$, then $L - \{v_n\} \in sd(C_{n-1}, m-1)$, a contradiction. If $v_{n-3} \in L$, then $L - \{v_n\} \in sd(C_{n-2}, m-1)$ a contradiction. Now suppose that $v_{n-1} \in L$. Then by observation-4.1 at least one vertex labelled v_{n-2} , v_{n-3} or v_{n-4} is in L . If $v_{n-2} \in L$ or $v_{n-3} \in L$, then $L - \{v_{n-1}\} \in sd(C_{n-2}, m-1)$, a contradiction. If $v_{n-4} \in L$ then $L - \{v_{n-1}\} \in sd(C_{n-3}, m-1)$, a contradiction. Therefore $sd(C_n, m) = \emptyset$.

Lemma 4.4. Suppose that $sd(C_n, m) \neq \emptyset$ then we have

1. $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ if and only if $n = 3k$ and $m = k$ for some $k \in N$.
2. $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ if and only if $m = n$.
3. $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ if and only if $n = 3k + 2$ and $m = \lceil \frac{3k+2}{3} \rceil$ for some $k \in N$.
4. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) = \emptyset$ if and only if $m = n - 1$.
5. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ if and only if $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n - 2$.

Proof.

1. Since $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$. By lemma-4.1 $m-1 > n-1$ or $m-1 < \lceil \frac{n-2}{3} \rceil$. If $m-1 > n-1$ then $m > n$ by lemma-4.1 $sd(C_n, m) = \emptyset$ a contradiction. Therefore $m < \lceil \frac{n-2}{3} \rceil + 1$ since $sd(C_n, m) \neq \emptyset$ together we have $\lceil \frac{n}{3} \rceil \leq m \leq \lceil \frac{n-2}{3} \rceil + 1$. Hence $n = 3k$ and $m = k$ for $k \in N$.
Conversely suppose if $n = 3k$, $m = k$ for $k \in N$ then by lemma-4.1 $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$.
2. Since $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ by lemma-4.1 $m-1 > n-2$ or $m-1 < \lceil \frac{n-3}{3} \rceil$. If $m-1 < \lceil \frac{n-3}{3} \rceil$ then $m-1 < \lceil \frac{n-1}{3} \rceil$. Hence $sd(C_{n-1}, m-1) = \emptyset$ a contradiction. So we have $m > n-1$ also since $sd(C_{n-1}, m-1) \neq \emptyset$ we have $m-1 \leq n-1$. Hence $m = n$.
Conversely suppose if $m = n$ then by lemma-4.1 we have $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$.
3. Since $sd(C_{n-1}, m-1) = \emptyset$ by lemma-4.1 $m-1 > n-1$ or $m-1 < \lceil \frac{n-1}{3} \rceil$. If $m-1 > n-1$ then $m-1 > n-2$ and by lemma-4.1 $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ a contradiction so we must have $m < \lceil \frac{n-1}{3} \rceil + 1$. But also we have $m-1 \geq \lceil \frac{n-2}{3} \rceil$ because $sd(C_{n-2}, m-1) \neq \emptyset$. Hence we have $\lceil \frac{n-2}{3} \rceil + 1 \leq m < \lceil \frac{n-1}{3} \rceil + 1$. Therefore $n = 3k + 2$ and $m = k + 1 = \lceil \frac{3k+2}{3} \rceil$ for some $k \in N$.
Conversely suppose if $n = 3k + 2$, $m = \lceil \frac{3k+2}{3} \rceil$ for some $k \in N$ then by lemma-4.1 $sd(C_{n-1}, m-1) = sd(C_{3k+1}, k) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$.

4. Since $sd(C_{n-3}, m-1) = \emptyset$ by lemma-4.1 we have $m-1 > n-3$ or $m-1 < \lceil \frac{n-3}{3} \rceil$ since $sd(C_{n-2}, m-1) \neq \emptyset$ by lemma-4.1 we have $\lceil \frac{n-2}{3} \rceil + 1 \leq m \leq n-1$. Therefore $m-1 < \lceil \frac{n-3}{3} \rceil$ is not possible. Hence we must have $m-1 > n-3$. Then $m = n-1$ or n but $m \neq n$ as $sd(C_{n-2}, m-1) \neq \emptyset$. Therefore $m = n-1$.

Conversely suppose if $m = n-1$ then by lemma-4.1 $sd(C_{n-1}, m-1) \neq \emptyset$ $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) = \emptyset$.

5. Since $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then by applying lemma-4.1 we have $\lceil \frac{n-1}{3} \rceil \leq m-1 \leq n-1$, $\lceil \frac{n-2}{3} \rceil \leq m-1 \leq n-2$ and $\lceil \frac{n-3}{3} \rceil \leq m-1 \leq n-3$ so $\lceil \frac{n-1}{3} \rceil \leq m-1 \leq n-3$ and hence $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n-2$.

Conversely suppose if $\lceil \frac{n-1}{3} \rceil + 1 \leq m \leq n-2$ then by lemma-4.1 we have the result.

Theorem 4.1. For every $n \geq 4$ and $m \geq \lceil \frac{n}{3} \rceil$,

1. If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = sd(C_n, \frac{n}{3}) = \{\{1, 4, \dots, n-2\}, \{2, 5, \dots, n-1\}, \{3, 6, \dots, n\}\}$.

2. If $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = sd(C_n, n) = \{[n]\}$.

3. If $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then

$$sd(C_n, m) = \{\{1, 4, \dots, n-4, n-1\}, \{2, 5, \dots, n-3, n\}, \{3, 6, \dots, n-2, n\}\} \cup$$

$$\left\{ S \cup \begin{cases} \{n-2\}, & \text{if } 1 \in S \\ \{n-1\}, & \text{if } 1 \notin S, 2 \in S \\ \{n\}, & \text{otherwise} \end{cases} \mid S \in sd(C_{n-3}, m-1) \right\}$$

where $S \subseteq V(C_n)$.

4. If $sd(C_{n-3}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{[n] - \{P\} \mid P \in [n]\}$

5. If $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then

$$sd(C_n, m) = \{\{n\} \cup S \mid S \in sd(C_{n-1}, m-1) \cup \left\{ S_1 \begin{cases} \{n\}, & \text{if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-1}, m-1) \\ \{n-1\}, & \text{if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \cap sd(C_{n-2}, m-1) \end{cases} \right\} \cup$$

$$\{S_2 \begin{cases} \{n-2\}, \text{ if } 1 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } S_2 \in sd(C_{n-3}, m-1) \cap sd(C_{n-2}, m-1) \\ \{n-1\}, \text{ if } n-3 \in S_2 \text{ or } n-4 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1) \end{cases} \}$$

where S, S_1 and S_2 are subsets of $V(C_n)$.

Proof.

1. $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ by lemma-4.4-(1) $n = 3k$, $m = k$ for some $k \in N$. Hence $sd(C_n, m) = sd(C_n, \frac{n}{3}) = \{\{1, 4, 7, \dots, n-2\}, \{2, 5, 8, \dots, n-1\}, \{3, 6, 9, \dots, n\}\}$.
2. $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ by lemma-4.4-(2) $m = n$. Therefore $sd(C_n, m) = (C_n, n) = \{[n]\}$.
3. $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ by lemma-4.4-(3) $n = 3k+2$, $m = k+1$ for some $k \in N$ we denote the families
 $\{\{1, 4, \dots, 3k-2, 3k+1\}, \{2, 5, \dots, 3k-1, 3k+2\}, \{3, 6, \dots, 3k, 3k+3\}\}$
and

$$\{S \cup \begin{cases} \{3k\}, \text{ if } 1 \in S \\ \{3k+1\}, \text{ if } 1 \notin S, 2 \in S \mid S \in sd(C_{3k-1}, k) \\ \{3k+2\}, \text{ otherwise} \end{cases}$$

by L_1 and L_2 respectively. We shall prove that $sd(C_{3k+2}, k+1) = L_1 \cup L_2$. Since $sd(C_k, 3k) = \{\{1, 4, 7, \dots, 3k-2\}, \{2, 5, 8, \dots, 3k-1\}, \{3, 6, 9, \dots, 3k\}\}$ then $L_1 \subseteq sd(C_{3k+2}, k+1)$. Also it is obvious that $L_2 \subseteq sd(C_{3k+2}, k+1)$. Hence $L_1 \cup L_2 \subseteq sd(C_{3k+2}, k+1)$.

Now let $L \in sd(C_{3k+2}, k+1)$ then by observation-4.1 at least one of the vertices labelled $3k+2, 3k+1$ or $3k$ is in L . Suppose that $3k+2 \in L$ then by observation-4.1 at least one of vertices say $1, 2, 3, 3k+1, 3k$ or $3k-1$ are in L . If $3k+1$ and at least one of $\{1, 2, 3\}$ and also $3k$ and at least one of $\{1, 2\}$ are in L . Then $L - \{3k+2\} \in sd(C_{3k+1}, k)$ a contradiction. If $\{3, 3k\}$ or $2, 3k-1$ is a subset of L , then $L = S \cup \{3k+2\}$ for some $S \in sd(C_{3k}, k)$ therefore $L \in L_1$ if $\{1, 3k-1\}$ is a subset of L then $L - \{3k+2\} \in sd(C_{3k+1}, k)$ a contradiction. If $\{3, 3k-1\}$ is a subset of L and $\{3k, 3k+1\}$ is not a subset of L then $L - \{3k+2\} \in sd(C_{3k-1}, k)$ hence $L \in L_2$. If $3k+1$ or $3k$ is in L we have the result.

4. By lemma-4.4-(4) $m = n-1$ therefore $sd(C_n, m) = sd(C_n, n-1) = \{[n] - \{x\} \mid x \in [n]\}$.
5. $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$. First suppose that $S \in sd(C_{n-1}, m-1)$ then $S \cup \{n\} \in sd(C_n, m)$. So

$L_1 = \{\{n\} \cup S \mid S \in sd(C_{n-1}, m-1)\} \subseteq sd(C_n, m)$. Now suppose that $sd(C_{n-2}, m-1) \neq \emptyset$. Let $S_1 \in sd(C_{n-2}, m-1)$ we denote $\{S_1 \cup \begin{cases} \{n\}, & \text{if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-1}, m-1) \\ \{n-1\}, & \text{if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \cap sd(C_{n-2}, m-1) \end{cases}$ simply by L_2 by observation-4.1 at least one vertices labeled $n-3, n-2$ or 1 is in S_1 if $n-2$ or $n-3$ is in S_1 . Then $S_1 \cup \{n\} \in sd(C_n, m)$ otherwise $S_1 \cup \{n-1\} \in sd(C_n, m)$. Hence $L_2 \subseteq sd(C_n, m)$ there we shall consider $sd(C_{n-3}, m-1) \neq \emptyset$. Let $S_2 \in sd(C_{n-3}, m-1)$ we denote $\{S_2 \cup \begin{cases} \{n-2\}, & \text{if } 1 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } S_2 \in sd(C_{n-3}, m-1) \cap sd(C_{n-2}, m-1) \\ \{n-1\}, & \text{if } n-3 \in S_2 \text{ or } n-4 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1) \end{cases}$ Simply by L_3 . If $n-3$ or $n-4$ is in S then $S \cup \{n-1\} \in sd(C_n, m)$, otherwise $S_2 \cup \{n-2\} \in sd(C_n, m)$. Hence $L_3 \subseteq L$. Therefore we proved that $L_1 \cup L_2 \cup L_3 \subseteq sd(C_n, m)$.
Now suppose that $L \in sd(C_n, m)$ so by observation-4.1 L contains at least one of the vertices say $n, n-1$ or $n-2$. If $n \in L$ so by observation-4.1 at least one of the vertices labelled $n-1, n-2$ or $n-3$ and $1, 2$, or 3 in L . If $n-2 \in L$ or $n-3 \in L$ then $L = S \cup \{n\}$ for some $S \in sd(C_{n-2}, m-1)$. Hence $L \in L_2$ otherwise $L = S \cup \{n-1\}$ for some $S \in sd(C_{n-2}, m-1)$. Hence $L \in L_2$. If $n-1$ or $n-2$ is in L , we have the result.

Theorem 4.2. *If $sd(C_n, m)$ is the family of superior dominating sets of C_n with cardinality m then $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$*

Proof. We consider the five cases in theorem-4.1 we write theorem-4.1 in the following form.

1. If $sd(C_{n-1}, m-1) = sd(C_{n-2}, m-1) = \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n-2\} \cup S_1, \{n-1\} \cup S_2, \{n\} \cup S_3 \mid 1 \in S_1, 2 \in S_2, 3 \in S_3, S_1, S_2, S_3 \in sd(C_{n-3}, m-1)\}$.
2. If $sd(C_{n-2}, m-1) = sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \cup S \mid S \in sd(C_{n-1}, m-1)\}$.
3. If $sd(C_{n-1}, m-1) = \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \cup S_1, \{n-1\} \cup S_2 \text{ or } S_1, S_2 \in sd(C_{n-2}, m-1), 1 \in S_2\} \cup (S \cup \begin{cases} \{n-2\}, & \text{if } 1 \in S \\ \{n-1\}, & \text{if } 1 \in S, 2 \in S \\ \{n\}, & \text{otherwise} \end{cases}) \text{ where } S \in sd(C_{n-3}, m-1)$
4. If $sd(C_{n-3}, m-1) = \emptyset$ and $sd(C_{n-2}, m-1) \neq \emptyset$, $sd(C_{n-1}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \cup S_1, \{n-1\} \cup S_2 \text{ or } S_1 \in sd(C_{n-1}, m-1), S_2 \in sd(C_{n-2}, m-1)\}$.

5. If $sd(C_{n-1}, m-1) \neq \emptyset$, $sd(C_{n-2}, m-1) \neq \emptyset$ and $sd(C_{n-3}, m-1) \neq \emptyset$ then $sd(C_n, m) = \{\{n\} \cup S \mid S \in sd(C_{n-1}, m-1)\} \cup \{S_1 \cup \begin{cases} \{n\}, & \text{if } n-2 \text{ or } n-3 \in S_1 \text{ for } S_1 \in sd(C_{n-2}, m-1) \text{ or } sd(C_{n-2}, m-1) \\ \{n-1\}, & \text{if } n-2 \notin S_1, n-3 \notin S_1 \text{ or } S_1 \in sd(C_{n-1}, m-1) \cap sd(C_{n-2}, m-1) \end{cases}\} \cup \{S_2 \cup \begin{cases} \{n-2\}, & \text{if } 1 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } S_2 \in sd(C_{n-3}, m-1) \cap sd(C_{n-2}, m-1) \\ \{n-2\}, & \text{if } n-3 \in S_2 \text{ or } n-4 \in S_2 \text{ for } S_2 \in sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1) \end{cases}\}$ where $S_1 \in sd(C_{n-2}, m-1)$ or $sd(C_{n-1}, m-1)$ and $S_2 \in \{sd(C_{n-3}, m-1) \text{ or } sd(C_{n-2}, m-1)\} \cap sd(C_{n-1}, m-1)$. Hence we have $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$.

Definition 4.1. Let $sd(C_n, m)$ be the family of superior dominating sets of a cycle C_n with cardinality n . Then the superior domination polynomial $SD(C_n, x)$ of C_n is defined as $SD(C_n, x) = \sum_{m=\lceil \frac{n}{3} \rceil}^n |sd(C_n, m)| x^m$ where $sd(C_n, m)$ is the number of distinct superior dominating sets of same cardinality.

Theorem 4.3. For every $n \geq 4$ $SD(C_n, x) = x[SD(C_{n-1}, x) + SD(C_{n-2}, x) + SD(C_{n-3}, x)]$ with initial values $SD(C_1, x) = x$, $SD(C_2, x) = x^2 + 2x$, $SD(C_3, x) = x^3 + 3x^2 + x$.

Table: The co-efficients of $SD(C_n, x)$ for $1 \leq n \leq 16$

$\begin{matrix} t \\ n \end{matrix}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0															
2	0	0														
3	3	3	1													
4	0	6	4	1												
5	0	5	10	5	1											
6	0	3	14	15	6	1										
7	0	0	14	28	21	7	1									
8	0	0	8	38	48	28	8	1								
9	0	0	3	36	81	75	36	9	1							
10	0	0	0	25	102	150	110	45	10	1						
11	0	0	0	11	99	231	253	154	55	11	1					
12	0	0	0	3	72	282	456	399	208	66	12	1				
13	0	0	0	0	39	273	663	819	598	273	78	13	1			
14	0	0	0	0	14	210	786	1372	1372	861	350	91	14	1		
15	0	0	0	0	3	125	765	1905	2590	2178	1200	440	105	15	1	
16	0	0	0	0	0	56	608	2214	4096	4560	3312	1628	544	120	16	1

Theorem 4.4. The following properties hold for co-efficients of $SD(C_n, x)$.

1. $|sd(C_{3n}, n)| = 3, \forall n \in N$.
2. $|sd(C_n, m)| = |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)|$,
 $\forall n \geq 4, m \geq \lceil \frac{n}{3} \rceil$,

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3. $|sd(C_{3n+2}, n+1)| = 3n+2, \forall n \in N.$
4. $|sd(C_{3n+1}, n+1)| = \frac{n(3n+7)+2}{2}, \forall n \in N.$
5. $|sd(C_n, m)| = 1, \forall n \geq 3.$
6. $|sd(C_n, n-1)| = n, \forall n \geq 3.$
7. $|sd(C_n, n-2)| = \frac{(n-1)n}{2}, \forall n \geq 3.$
8. $|sd(C_n, n-3)| = \frac{(n-4)(n)(n+1)}{6}, \forall n \geq 4.$
9. $\sum_{n=m}^{3m} |sd(C_n, m)| = 3 \sum_{n=m-1}^{3m-3} |sd(C_n, m-1)|, \forall m \geq 4.$
10. $1 = |sd(C_n, n)| < |sd(C_{n+1}, n)| < |sd(C_{n+2}, n)| < \dots < |sd(C_{2n-1}, n)| < |sd(C_{2n}, n)| > |sd(C_{2n+1}, n)| > \dots > |sd(C_{3n-1}, n)| > |sd(C_{3n}, n)| = 3, \forall n \geq 3.$
11. If $A_n = \sum_{m=\lceil \frac{n}{3} \rceil}^n |sd(C_n, m)|$ then for every $n \geq 4, A_n = A_{n-1} + A_{n-2} + A_{n-3}$ with initial values $A_1 = 1, A_2 = 3$ and $A_3 = 7.$

Proof.

1. Since $|sd(C_n, 3n)| = \{\{1, 4, 7, \dots, 3n-2\}, \{2, 5, 8, \dots, 3n-1\}, \{3, 6, 9, \dots, 3n\}\},$
so $|sd(C_{3n}, n)| = 3.$

2. It follows from theorem-4.2.

3. By induction on n , the result is true for $n = 1$, because
 $|sd(C_2, 5)| = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}.$ Suppose result is true
for all $n - 1$ then we prove for n . By (1),(2) and induction we have

$$\begin{aligned} |sd(C_{3n+2}, n+1)| &= |sd(C_{3n+1}, n)| + |sd(C_{3n}, n)| + |sd(C_{3n-1}, n)| \\ &= 3n+2 \end{aligned}$$

4. By mathematical induction $|sd(C_2, 4)| = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$
So $|sd(C_4, 2)| = 6$, the result is true for $n = 1$. Now suppose that the result
is true for all natural numbers less than n and we prove it for n .
By (1),(2),(3) and induction we have

$$\begin{aligned} |sd(C_{3n+1}, n+1)| &= |sd(C_{3n}, n)| + |sd(C_{3n-1}, n)| + |sd(C_{3n-2}, n)| \\ &= 3 + 3(n-1) + 2 + (n-1) \frac{3(n-1)+7}{2} \\ &= \frac{n(3n+7)+2}{2} \end{aligned}$$

5. It is obvious that for a graph with n vertices $|sd(G, n)| = 1$.
6. It is obvious that for a graph G with n vertices $|sd(G, n - 1)| = n$.
7. By induction on n . The result is true for $n = 3$. Since $|sd(C_3, 1)| = 3$. Assume it is true for all $n - 1$. We prove for n . By parts (1), (2), (4), (5) and induction

$$\begin{aligned}
 |sd(C_n, n - 2)| &= |sd(C_{n-1}, n - 3)| + |sd(C_{n-2}, n - 3)| + |sd(C_{n-3}, n - 3)| \\
 &= \frac{(n-2)(n-1)}{2} + n - 2 + 1 \\
 &= \frac{(n-1)n}{2}
 \end{aligned}$$

8. By induction on n . The result is true for $n = 4$. Since $|sd(C_4, 1)| = 0$. Assume the result is true for all $n - 1$. We prove for n , by parts (2), (6), (7) and induction we have

$$\begin{aligned}
 |sd(C_n, n - 3)| &= |sd(C_{n-1}, n - 4)| + |sd(C_{n-2}, n - 4)| + |sd(C_{n-3}, n - 4)| \\
 &= \frac{(n-5)(n-1)n}{6} + \frac{(n-2)(n-3)}{2} + n - 3 \\
 &= \frac{(n-4)n(n+1)}{6}
 \end{aligned}$$

9. Proof by induction on m . Suppose $m = 3$ then $\sum_{n=3}^9 sd(C_n, 3) = 54 = 3 \sum_{n=2}^6 |sd(C_n, 2)|$. Now suppose the result is true for every $m < t$ and we prove for $m = t$

$$\begin{aligned}
 \sum_{n=t}^{3t} |sd(C_n, t)| &= \sum_{n=t}^{3t} |sd(C_{n-1}, t-1)| + \sum_{n=t}^{3t} |sd(C_{n-2}, t-1)| + \sum_{n=t}^{3t} |sd(C_{n-3}, t-1)| \\
 &= 3 \sum_{n=t-1}^{3(t-1)} |sd(C_{n-1}, t-2)| + 3 \sum_{n=t-1}^{3(t-1)} |sd(C_{n-2}, t-2)| + 3 \sum_{n=t-1}^{3(t-1)} |sd(C_{n-3}, t-2)| \\
 &= 3 \sum_{n=t-1}^{3t-3} |sd(C_n, t-1)|
 \end{aligned}$$

10. We plan for every m , $|sd(C_n, m)| < |sd(C_{n+1}, m)|$ for $m \leq n \leq 2m - 1$ and $|sd(C_n, m)| > |sd(C_{n+1}, m)|$ for $2m \leq n \leq 3m - 1$. We prove first inequality by induction on m . The result hold for $m = 3$. Suppose that result is true for all $m \leq t$. Now we prove it for $m = t + 1$. That is $|sd(C_n, t + 1)| < |sd(C_{n+1}, t + 1)|$ for $t + 1 \leq n \leq 2t + 1$.

$$\begin{aligned}
 |sd(C_n, t + 1)| &= |sd(C_{n-1}, t)| + |sd(C_{n-2}, t)| + |sd(C_{n-3}, t)| \\
 &< |sd(C_n, t)| + |sd(C_{n-1}, t)| + |sd(C_{n-2}, t)| \\
 &= |sd(C_{n+1}, t + 1)|
 \end{aligned}$$

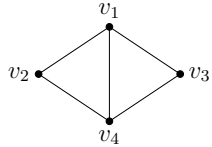
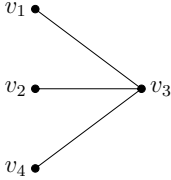
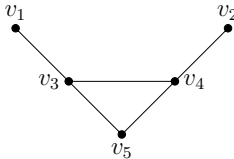
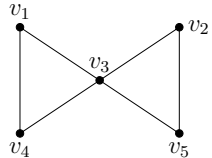
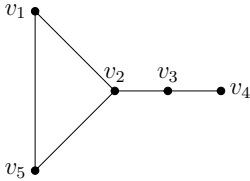
The other inequality follows in same way.

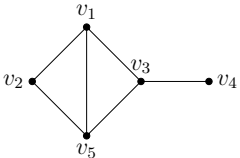
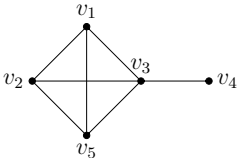
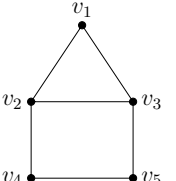
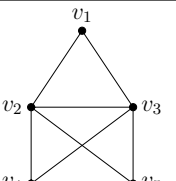
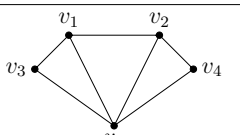
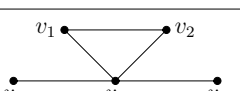
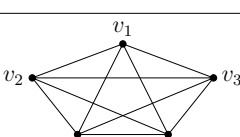
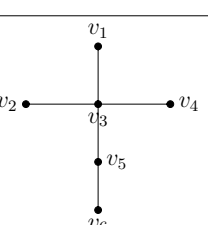
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11. By theorem-4.2, we have

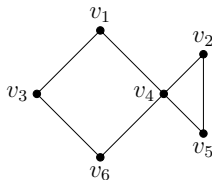
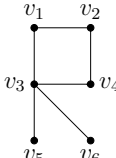
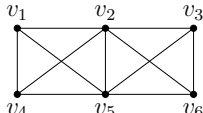
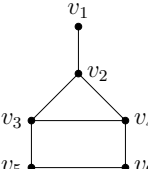
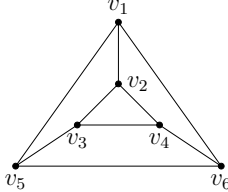
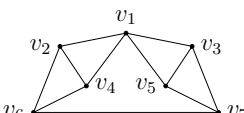
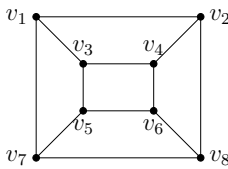
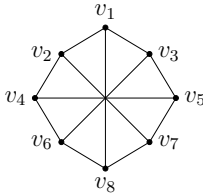
$$\begin{aligned}
 A_n &= \sum_{m=\lceil \frac{n}{3} \rceil}^n |sd(C_n, m)| \\
 &= \sum_{m=\lceil \frac{n}{3} \rceil}^n |sd(C_{n-1}, m-1)| + |sd(C_{n-2}, m-1)| + |sd(C_{n-3}, m-1)| \\
 &= \sum_{m=\lceil \frac{n}{3} \rceil-1}^{n-1} |sd(C_{n-1}, m)| + \sum_{m=\lceil \frac{n}{3} \rceil-1}^{n-2} |sd(C_{n-2}, m)| + \sum_{m=\lceil \frac{n}{3} \rceil-1}^{n-3} |sd(C_{n-3}, m-1)| \\
 &= A_{n-1} + A_{n-2} + A_{n-3}
 \end{aligned}$$

Table: $SD(G, x)$ of different standard graphs and their roots are tabulated in the table below

Graph	Figure	Superior domination polynomial $SD(G, x)$	Roots
Diamond graph		$x^4 + 4x^3 + 6x^2 + 2x$	$x_1 = 0,$ $x_2 = -0.4563,$ $x_3 = -1.7718 + 1.1151i,$ $x_4 = -1.7718 - 1.1151i.$
Claw graph		$x^4 + 3x^3 + 3x^2 + x$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1.$
Bull graph		$x^5 + 3x^4 + 3x^3 + x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1.$
Butterfly graph		$x^5 + 4x^4 + 6x^3 + 4x^2 + x$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1,$ $x_5 = -1.$
(3,2)-Tadpole graph		$x^5 + 4x^4 + 5x^3 + 2x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -2,$ $x_4 = -1.$

Graph	Figure	Superior domination polynomial $SD(G, x)$	Roots
Kite graph		$x^5 + 5x^4 + 9x^3 + 5x^2 + x$	$x_1 = 0,$ $x_2 = -0.378 + 0.1877i,$ $x_3 = -0.378 - 0.1877i,$ $x_4 = -2.122 + 1.0538i,$ $x_5 = -2.122 - 1.0538i.$
(4,1)-Lollipop graph		$x^5 + 4x^4 + 6x^3 + 4x^2 + x$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1,$ $x_5 = -1.$
House graph		$x^5 + 5x^4 + 8x^3 + 4x^2 + x$	$x_1 = 0,$ $x_2 = -0.3076 + 0.3182i,$ $x_3 = -0.3076 - 0.3182i,$ $x_4 = -2.1924 + 0.5479i,$ $x_5 = -2.1924 - 0.5479i.$
House X graph		$x^5 + 5x^4 + 8x^3 + 5x^2 + x$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -0.382,$ $x_5 = -2.618.$
Gem graph		$x^5 + 4x^4 + 4x^3$	$x_1 = 0,$ $x_2 = -2.$
Cricket graph		$x^5 + 4x^4 + 6x^3 + 4x^2 + x$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1, x_5 = -1.$
Pentatope graph		$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x$	$x_1 = 0,$ $x_2 = -0.691 + 0.9511i,$ $x_3 = -0.691 - 0.9511i,$ $x_4 = -1.809 + 0.5878i,$ $x_5 = -1.809 - 0.5878i.$
Cross graph		$x^6 + 5x^5 + 9x^4 + 7x^3 + 2x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -2,$ $x_4 = -1,$ $x_5 = -1.$

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Graph	Figure	Superior domination polynomial $SD(G, x)$	Roots
Fish graph		$x^6 + 5x^5 + 10x^4 + 9x^3 + 3x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i,$ $x_5 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i.$
R graph		$x^6 + 5x^5 + 10x^4 + 9x^3 + 3x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i,$ $x_5 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i.$
(2,3)-King graph		$x^6 + 4x^5 + 6x^4 + 4x^3 + x^2$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1,$ $x_5 = -1.$
Antenna graph		$x^6 + 4x^5 + 5x^4 + 2x^3$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1.$
3-prism graph		$x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x$	$x_1 = 0,$ $x_2 = -2,$ $x_3 = -0.5 + 0.866i,$ $x_4 = -0.5 - 0.866i,$ $x_5 = -1.5 + 0.866i,$ $x_6 = -1.5 - 0.866i.$
Moser Spindle graph		$x^7 + 4x^6 + 6x^5 + 4x^4 + x^3$	$x_1 = 0,$ $x_2 = -1,$ $x_3 = -1,$ $x_4 = -1,$ $x_5 = -1.$
Cubical graph		$x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 48x^3 + 16x^2$	$x_1 = 0,$ $x_2 = -0.6714 + 0.5756i,$ $x_3 = -0.6714 - 0.5756i,$ $x_4 = -0.8352 + 1.4854i,$ $x_5 = -0.8352 - 1.4854i,$ $x_6 = -2.4934 + 0.9097i,$ $x_7 = -2.4934 - 0.9097i.$
Wagner graph		$x^8 + 8x^7 + 24x^6 + 32x^5 + 16x^4$	$x_1 = 0,$ $x_2 = -2,$ $x_3 = -2,$ $x_4 = -2,$ $x_5 = -2.$

5 Conclusions

In this paper we introduced superior domination polynomial, this is a distance based domination polynomial. Emphasis was given to the family of stars and cycles. Formulas to find the coefficients of the superior domination polynomials of cycles and stars were stated and proved. These formulas helps us to calculate the number of superior dominating sets of a specific desired cardinality for any given value of n . The superior domination polynomial of different standard graphs and their roots are calculated.

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Some fixed point results in fuzzy metric space using intimate mappings

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Srinivas Veladi[†]

Abstract

The aim of this research paper is to prove the existence and uniqueness of common fixed point theorems for four self-mappings in fuzzy metric space using the notion of Intimate mappings. We also provide appropriate illustrations to justify the key points mentioned in the main results.

Keywords: Fuzzy metric space, Intimate mappings, E.A property, Common E.A property.

2010 AMS subject classification: 54H25, 47H10.[‡]

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1. Introduction

L.A Zadeh [1] initiated the new concept, as extension an of classical set namely Fuzzy set. Lateron the notion of fuzzy metric space was introduced by Kramosil and Mechalek in [2]. Further this was altered by George and veeramani [4] in order to obtain Harsdorff topology for the class of fuzzy metric spaces. Thereafter many fixed point theorems came into light under various conditions like ([5],[6],[9],[10],[11],[13],[16],[19]) in fuzzy metric space.

Under other conditions, Sahu and others [12] developed the notion of generalized compatible mappings of type $(\mathcal{A})$ called Intimate mappings. These were further extended by Chugh and Madhu Aggarwal [13] which resulted in the formation of some results in Hausdorff uniform spaces. Further some more results can be witnessed like [14] using intimate mappings in complex valued metric space. Apart from this Praveenkumar and others [15] proved some theorems in multiplicative metric space (MMS) using the notion of intimate mappings and subsequently many results came into existence on this platform like ([17],[18]).

The concept of non-compatible mappings extended as the E. A property was introduced in metric space by Aamri and Matouwakil [20]. Consequently, the concept of improved E.A property resulted in the formation of common property E.A was introduced by Yicheng liu et al. [21].

The important note of this article is to extend the notion of intimate mappings in fuzzy metric space using recent concepts like the different forms of E.A properties. In this process we prove three unique common fixed point theorems using these concepts. Cocequently these results stand as generalizations of some of the existing results like [16] [19]. Furthermore, some illustrations are provided to support our findings.

2. Definitions and Preliminaries

Definition 2.1 (B.Schweizer and A.Sklar [7]): A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is said to be continuous triangular norm (i.e continuous t – norm) if the following assertions hold: (CT-i) $*$ is continuous; (CT-ii) $a * b \leq c * d$ where $a \leq b, c \leq d$ and $a, b, c, d \in [0,1]$; (CT-iii) $a * 1 = a$ for $a \in [0,1]$; (CT-iv) $*$ is associative and commutative.

Definition 2.2 (Kramosil and Mechalek [2]): A triplet $(\mathbb{X}, M_{KM}, *)$ is fuzzy metric space (i.e., FMS) if $\mathbb{X}$ is a arbitrary set, $*$ is continuous t – norm and M_{KM} is fuzzy set on $\mathbb{X}^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in \mathbb{X}$ such that $t, s \in (0, \infty)$:

(KMFM-i) $M_{KM}(x, y, 0) = 0$

(KMFM-ii) $M_{KM}(x, y, t) = 1 \quad \forall t > 0 \Leftrightarrow x = y$

(KMFM-iii) $M_{KM}(y, x, t) = M_{KM}(x, y, t)$

(KMFM-iv) $M_{KM}(x, z, t + s) \geq M_{KM}(x, y, t) * M_{KM}(y, z, s)$

(KMFM-v) $M_{KM}(x, y, .): [0,1] \rightarrow [0,1]$ left continuous.

Example 2.3 (George & Veeramani [4]): Consider $(\mathbb{X}, d_u)$ is a metric space and define

$$M_{KM}(x, y, t) = \frac{t}{t + d_u(x, y)} \text{ then } (\mathbb{X}, M_{KM}, *) \text{ is FMS where } \forall x, y \in \mathbb{X}, t > 0$$

and $*$ is continuous t – norm with $a * b = \min \{a, b\}$.

In the entire paper, $(\mathbb{X}, M_{KM}, *)$ is to be assumed FMS with the condition (KFFM-6) : $\lim_{t \rightarrow \infty} M_{KM}(x, y, t) = 1$ for all $x, y, \in \mathbb{X}$.

Definition 2.4 (Grabiec [3]): Let $\langle x_n \rangle$ be sequence in FMS $(\mathbb{X}, M_{KM}, *)$, $\langle x_n \rangle$ then converges to a point $\ell \in \mathbb{X}$ if $\lim_{n \rightarrow \infty} M_{KM}(x_n, \ell, t) = 1, \forall t > 0$.

Definition 2.5 (Garbaic [3]): Let $\langle x_n \rangle$ be a sequence in FMS $(\mathbb{X}, M_{KM}, *)$, this sequence $\langle x_n \rangle$ in $\mathbb{X}$ is said to be Cauchy sequence in FMS if $\lim_{n \rightarrow \infty} M_{KM}(x_{n+p}, x_n, t) = 1, \forall t > 0$ and $p > 0$.

Definition 2.6 (Garbiec [3]): If every Cauchy sequence is convergent in $(\mathbb{X}, M_{KM}, *)$ then we say that it is complete.

Lemma 2.7 (S.N. Mishra et al [5]): Let $(\mathbb{X}, M_{KM}, *)$ be a FMS if there exists $k \in (0, 1)$ such that $M_{KM}(x, y, kt) \geq M_{KM}(x, y, t)$ then $x = y$.

Definition 2.8 ([5],[10]): Let $\mathfrak{S}$ and $\mathfrak{T}$ be two self mappings of a FMS $(\mathbb{X}, M_{KM}, *)$. Then $\mathfrak{S}$ and $\mathfrak{T}$ are

- (1) compatible if $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{T}x_n, \mathfrak{T}\mathfrak{S}x_n, t) = 1$ whenever a sequence $\langle x_n \rangle$ in $\mathbb{X}$ provided $\lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{T}x_n = t$ for some $t \in \mathbb{X}$
- (2) compatible of type $(\mathcal{A})$ if

$\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{T}x_n, \mathfrak{T}\mathfrak{T}x_n, t) = 1$ $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{T}\mathfrak{S}x_n, \mathfrak{S}\mathfrak{S}x_n, t) = 1$ whenever $\langle x_n \rangle$ in $\mathbb{X}$ such that $\lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{T}x_n = t$ for some $t \in \mathbb{X}$.

Now we discuss some definitions related to intimate mappings in FMS.

Definition 2.9: Let $\mathfrak{U}$ and $\mathfrak{S}$ be two mappings of a FMS $(\mathbb{X}, M_{KM}, *)$ into itself. Then $\mathfrak{U}$ and $\mathfrak{S}$ are said to be

- (1). $\mathcal{A}$ -Intimate mappings if $\alpha M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) \geq \alpha M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{S}x_n, t)$ where $\alpha = \lim_{n \rightarrow \infty} \text{Sup}$ or $\lim_{n \rightarrow \infty} \text{Inf}$ and $\langle x_n \rangle$ is a sequence in $\mathbb{X} \ni \lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = t$ for some $t \in \mathbb{X}$.
- (2). $\mathcal{S}$ -Intimate mapping if $\alpha M_{KM}(\mathfrak{S}\mathfrak{U}x_n, \mathfrak{S}x_n, t) \geq \alpha M_{KM}(\mathfrak{U}\mathfrak{U}x_n, \mathfrak{U}x_n, t)$ where $\alpha = \lim_{n \rightarrow \infty} \text{Sup}$ or $\lim_{n \rightarrow \infty} \text{Inf}$ and a sequence $\langle x_n \rangle$ in $\mathbb{X} \ni \lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = t$ for some $t \in \mathbb{X}$.

Proposition 2.10: Let $\mathfrak{U}$ and $\mathfrak{S}$ be two self mappings of a FMS $(\mathbb{X}, M_{KM}, *)$. Suppose $\mathfrak{U}$ and $\mathfrak{S}$ are compatible mappings of type $(\mathcal{A})$ then the pair of mappings $\mathfrak{U}$ and $\mathfrak{S}$ are $\mathcal{A}$ – intimate mappings and $\mathcal{S}$ -intimate mappings.

Proof: Since $\mathfrak{U}$ and $\mathfrak{S}$ are compatible of type $(\mathcal{A})$, we have $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{S}\mathfrak{S}x_n, t) = 1$ and

$\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{U}x_n, \mathfrak{U}\mathfrak{U}x_n, t) = 1$ whenever $\langle x_n \rangle$ in $\mathbb{X} \ni \lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = t$ for some $t \in \mathbb{X}$.

Now $M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, (2 - \beta)t) = M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, (1 + k_1)t)$

$$\geq M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{S}\mathfrak{S}x_n, k_1 t) * M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{U}x_n, t).$$

By taking $k_1 = 1 - \beta$ and $0 < k_1 < 1$ and letting $n \rightarrow \infty$ and $\beta \rightarrow 1$ we obtain

$$\begin{aligned} M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) &\geq M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{S}\mathfrak{S}x_n, k_1 t) * M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{U}x_n, t) \\ &= M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{A}x_n, t). \end{aligned}$$

By applying limit supremum on both sides,

$\alpha M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) \geq \alpha M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{U}x_n, t)$ this implies $\mathfrak{U}$ and $\mathfrak{S}$ are $\mathcal{A}$ -intimate mappings whenever $\{x_n\}$ is a sequence in $\mathbb{X}$ such that $\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = t$ for some $t \in \mathbb{X}$. Likewise, we can prove that the pair of these mappings is $\mathcal{S}$ -intimate.

Proposition 2.11: Let $\mathfrak{U}$ and $\mathfrak{S}$ be two self mappings on FMS. $\mathfrak{U}$ and $\mathfrak{S}$ are $\mathcal{A}$ -intimate mappings and $\mathfrak{U}t_1 = \mathfrak{S}t_1 = p, p \in \mathbb{X}$ then $M_{KM}(\mathfrak{U}p, p, t) \geq M_{KM}(\mathfrak{S}p, p, t)$.

Proof: Suppose that $\{x_n\} \in \mathbb{X}$ is a sequence such that $\mathfrak{U}x_n = \mathfrak{S}x_n \rightarrow \mathfrak{U}t_1 = \mathfrak{S}t_1 = p$ for some $p, t \in \mathbb{X}$.

Since the pair of mappings $\mathfrak{U}$ and $\mathfrak{S}$ are $\mathcal{A}$ – intimate, then we obtain

$$\begin{aligned} M_{KM}(\mathfrak{U}p, p, t) &= \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{S}x_n, t) \\ &= M_{KM}(\mathfrak{S}p, p, t). \end{aligned}$$

Thus $M_{KM}(\mathfrak{U}p, p, t) \geq M_{KM}(\mathfrak{S}p, p, t)$.

Remark 2.12: A pair of mappings $\mathfrak{U}$ and $\mathfrak{S}$ is $\mathcal{A}$ -intimate or $\mathcal{S}$ -intimate but not compatible mapping of type $(\mathcal{A})$.

The following example revels the relation between intimate mappings and compatible mappings of type $(\mathcal{A})$.

Example 2.13: Suppose $\mathbb{X} = [0,1]$. Define two self-mappings $\mathfrak{U}$ and $\mathfrak{S}$ as follows

$$\mathfrak{U}(x) = \frac{5}{x+5} \quad \mathfrak{S}(x) = \frac{1}{x+1} \text{ for every } x \text{ in } [0,1].$$

Consider a sequence $\langle x_n \rangle = \frac{1}{n} \quad n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = 1$.

Consequently, $\lim_{n \rightarrow \infty} M(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) = \frac{6t}{6t+1}$ and $\lim_{n \rightarrow \infty} M(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{S}x_n, t) = \frac{2t}{2t+1}$.

Hence $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{U}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{S}x_n, \mathfrak{S}x_n, t)$, for all $t > 0$.

Thus, the pair $(\mathfrak{U}, \mathfrak{S})$ is $\mathcal{A}$ -intimate.

On the other hand, the $(\mathfrak{U}, \mathfrak{S})$ are not compatible of type $(\mathcal{A})$, since

$$\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{S}x_n, \mathfrak{S}\mathfrak{S}x_n, t) = \frac{3t}{3t+1} \neq 1 \text{ and } \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{U}x_n, \mathfrak{U}\mathfrak{U}x_n, t) = \frac{3t}{3t+1} \neq 1.$$

Definition 2.14[20]: Define $\mathfrak{U}$ and $\mathfrak{S}$ as two self maps of FMS $(\mathbb{X}, M_{KM}, *)$ then we say that $\mathfrak{U}$ and $\mathfrak{S}$ satisfy the property E.A if there exists a sequence $\langle x_n \rangle \in \mathbb{X}$ such that $\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = t$ for some $t \in \mathbb{X}$.

Definition 2.15[21]: Suppose $\mathfrak{U}, \mathfrak{P}, \mathfrak{B}$ and $\mathfrak{T}$ are four self maps on FMS $(\mathbb{X}, M_{KM}, *)$ then we say that $(\mathfrak{U}, \mathfrak{P})$ and $(\mathfrak{B}, \mathfrak{T})$ satisfy common property E.A whenever two sequences $\langle x_n \rangle$ and $\langle \gamma_n \rangle$ in $\mathbb{X}$ satisfying

$$\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{B}\gamma_n = \lim_{n \rightarrow \infty} \mathfrak{T}\gamma_n = t \text{ for some } t \in \mathbb{X}.$$

3. Main results

3.1 Theorem: Let $(\mathbb{X}, M_{KM}, *)$ be a complete fuzzy metric space. Suppose $\mathfrak{P}, \mathfrak{Q}, \mathfrak{S}$ and $\mathfrak{U}$ are self maps on $\mathbb{X}$ satisfying the conditions

(C – 1) $\mathfrak{P}(\mathbb{X}) \subseteq \mathfrak{S}(\mathbb{X})$ and $\mathfrak{Q}(\mathbb{X}) \subseteq \mathfrak{U}(\mathbb{X})$

(C – 2) $M_{KM}(\mathfrak{P}x, \mathfrak{Q}\gamma, kt) \geq M_{KM}(\mathfrak{U}x, \mathfrak{S}\gamma, t) * M_{KM}(\mathfrak{P}x, \mathfrak{U}x, t) * M_{KM}(\mathfrak{Q}\gamma, \mathfrak{S}\gamma, t) * M_{KM}(\mathfrak{P}x, \mathfrak{S}\gamma, t)$

where $k \in (0, 1)$ and for all $x, \gamma \in \mathbb{X}$

(C – 3) $\mathfrak{U}(\mathbb{X})$ is complete

(C – 4) the pair of mappings $\mathfrak{U}$ and $\mathfrak{P}$ is $\mathcal{A}$ – intimate and the other pair of mappings also $\mathfrak{S}$ and $\mathfrak{Q}$ is $\mathcal{S}$ – intimate.

Then $\mathfrak{P}, \mathfrak{Q}, \mathfrak{S}$ and $\mathfrak{U}$ have a unique common fixed point in $\mathbb{X}$.

Proof:

Let x_0 be any arbitrary point of $\mathbb{X}$.

Since from the condition $\mathfrak{P}(\mathbb{X}) \subseteq \mathfrak{S}(\mathbb{X})$ of (C – 1), there exists a point $x_1 \in \mathbb{X}$ such that $\mathfrak{P}x_0 = \mathfrak{S}x_1 = \gamma_0$.

Now for this x_1 and applying the (C – 1) [i.e. $\mathfrak{Q}(\mathbb{X}) \subseteq \mathfrak{U}(\mathbb{X})$] $\exists x_2 \in \mathbb{X}$ such that $\mathfrak{Q}x_1 = \mathfrak{U}x_2 = \gamma_1$.

Inductively, we establish two real sequences $\langle x_n \rangle$ and $\langle \gamma_n \rangle$ in $\mathbb{X} \ni \gamma_{2n} = \mathfrak{P}x_{2n} = \mathfrak{S}x_{2n+1}$ and $\gamma_{2n+1} = \mathfrak{Q}x_{2n+1} = \mathfrak{U}x_{2n+2}$ for $n \geq 0$.

By taking $x = x_{2n}, \gamma = x_{2n+1}$ in the inequality (C – 2),

$$M_{KM}(\mathfrak{P}x_{2n}, \mathfrak{Q}x_{2n+1}, kt) \geq M_{KM}(\mathfrak{U}x_{2n}, \mathfrak{S}x_{2n+1}, t) * M_{KM}(\mathfrak{P}x_{2n}, \mathfrak{U}x_{2n}, t) * M_{KM}(\mathfrak{Q}x_{2n+1}, \mathfrak{S}x_{2n+1}, t) * M_{KM}(\mathfrak{P}x_{2n}, \mathfrak{S}x_{2n+1}, t)$$

which implies that an $n \rightarrow \infty$

$$M_{KM}(\gamma_{2n}, \gamma_{2n+1}, kt) \geq M_{KM}(\gamma_{2n-1}, \gamma_{2n}, t) * M_{KM}(\gamma_{2n}, \gamma_{2n-1}, t) * M_{KM}(\gamma_{2n+1}, \gamma_{2n}, t) * M_{KM}(\gamma_{2n}, \gamma_{2n}, t).$$

This yield

$$M_{KM}(\gamma_{2n}, \gamma_{2n+1}, kt) \geq M_{KM}(\gamma_{2n-1}, \gamma_{2n}, t) * M_{KM}(\gamma_{2n+1}, \gamma_{2n}, t) * M_{KM}(\gamma_{2n}, \gamma_{2n-1}, t) * 1.$$

Again, by the condition KMFM-3, we get

$$M_{KM}(\gamma_{2n}, \gamma_{2n+1}, kt) \geq M_{KM}(\gamma_{2n-1}, \gamma_{2n}, t) * M_{KM}(\gamma_{2n}, \gamma_{2n+1}, t)$$

which implies (since $a * b = \min\{a, b\}$.)

$$M_{KM}(\gamma_{2n}, \gamma_{2n+1}, kt) \geq M_{KM}(\gamma_{2n-1}, \gamma_{2n}, t).$$

In general

$$M_{KM}(\gamma_{n+1}, \gamma_{n+2}, kt) \geq M_{KM}(\gamma_n, \gamma_{n+1}, t) \dots (\sigma - 1)$$

for all $n = 1, 2, 3, \dots$, and $t > 0$.

From $(\sigma - 1)$,

$$[M_{KM}(\gamma_n, \gamma_{n+1}, t)] \geq M_{KM}\left(\gamma_{n-1}, \gamma_n, \frac{t}{k}\right) \geq M_{KM}\left(\gamma_{n-2}, \gamma_{n-1}, \frac{t}{k^2}\right) \geq \dots$$

$$\dots \geq M_{KM}\left(\gamma_0, \gamma_1, \frac{t}{k^n}\right) \rightarrow 1 \text{ as } n \rightarrow \infty \dots (\sigma - 2)$$

For any $t > 0$ and $\lambda_{MK} \in (0, 1)$ we consider $\forall n > n_0 \in \mathbb{N}$ such that $M_{KM}(\gamma_n, \gamma_{n+1}, t) > (1 - \lambda_{MK}) \dots (\sigma - 3)$.

For $m, n \in \mathbb{N}$. Suppose $m \geq n$, then we have that

$$[M_{MK}(\gamma_n, \gamma_m, t)]$$

$$\geq \min \{M_{MK}\left(\gamma_n, \gamma_{n+1}, \frac{t}{m-n}\right) * M_{MK}\left(\gamma_{n+1}, \gamma_{n+2}, \frac{t}{m-n}\right) * \dots$$

$$M_{MK}\left(\gamma_{m-1}, \gamma_m, \frac{t}{m-n}\right) \geq (1 - \lambda_{MK}) * (1 - \lambda_{MK}) * \dots (1 - \lambda_{MK}).. (m - n) \text{ times.}$$

This implies $M_{MK}(\gamma_{m-1}, \gamma_m, t) \geq (1 - \lambda_{MK})$

Therefore $\langle \gamma_n \rangle$ is cauchy sequence in FMS.

Since $(X, M_{KM}, *)$ is complete FMS, so sequence $\{\gamma_n\}$ converges to $p^* \in X$.

Further fuzzy cauchy sequence $\{\gamma_n\}$ has convergent subsequence $\{\gamma_{2n+1}\}$ and $\{\gamma_{2n}\}$.

From the above argument,

$$\gamma_{2n+1} = Qx_{2n+1} = \mathcal{U}x_{2n+2} \rightarrow p^* \text{ and}$$

$$\gamma_{2n} = \mathcal{P}x_{2n} = \mathcal{S}x_{2n+1} \rightarrow p^* \text{ as } n \rightarrow \infty \dots (\sigma - 4)$$

Now suppose that the range set $\mathcal{U}(X)$ is complete then $\exists$ a point $u \in X \ni \mathcal{U}u = p^*.. (\sigma - 5)$.

Now we claim that $\mathcal{P}u = p^*$ from the inequality, put $x = u$ and $\gamma = x_{2n+1}$ we have

$$M_{KM}(\mathcal{P}u, Qx_{2n+1}, kt) \geq M_{KM}(\mathcal{U}u, \mathcal{S}x_{2n+1}, t) * M_{KM}(\mathcal{P}u, \mathcal{U}u, t)$$

$$* M_{KM}(Qx_{2n+1}, \mathcal{S}x_{2n+1}, t) * M_{KM}(\mathcal{P}u, \mathcal{S}x_{2n+1}, t).$$

Taking limit as $n \rightarrow \infty$

$$M_{KM}(\mathcal{P}u, p^*, kt) \geq M_{KM}(p^*, p^*, t) * M_{KM}(\mathcal{P}u, p^*, t)$$

$$* M_{KM}(p^*, p^*, t) * M_{KM}(\mathcal{P}u, p^*, t).$$

This gives $\mathcal{P}u = p^*$. That is $\mathcal{P}u = \mathcal{U}u = p^* \dots (\sigma - 6)$

Let us prove that $Qv = p^*$.

Using the equation $((\sigma - 6)$ with contained inequality $\mathcal{P}(X) \subseteq \mathcal{S}(X)$,

$p^* = \mathcal{P}u \in \mathcal{P}(X) \subseteq \mathcal{S}(X)$ then $\exists$ a point $v \in X \ni$

$$\mathcal{S}v = \mathcal{P}u = p^* \dots (\sigma - 7).$$

Put $x = u$ and $\gamma = v$ in $(\mathcal{C} - 2)$ then we obtain

$$M_{KM}(\mathcal{P}u, Qv, kt)$$

$$\geq M_{KM}(\mathcal{U}u, \mathcal{S}v, t) * M_{KM}(\mathcal{P}u, \mathcal{U}u, t) * M_{KM}(Qv, \mathcal{S}v, t)$$

$$* M_{KM}(\mathcal{P}u, \mathcal{S}v, t).$$

By using $(\sigma - 7)$ we get

$$M_{KM}(p^*, Qv, kt) \geq M_{KM}(p^*, \mathcal{S}v, t) * M_{KM}(p^*, p^*, t)$$

$$* M_{KM}(Qv, p^*, t) * M_{KM}(p^*, p^*, t)$$

this gives

$$M_{KM}(p^*, Qv, kt) \geq M_{KM}(Qv, p^*, kt).$$

Consequently $M_{KM}(p^*, Qv, kt) \geq M_{KM}(p^*, Qv, kt)$

this implies $\mathfrak{Q}v=p^*$.

This shows that $\mathfrak{Q}v=\mathfrak{S}v=p^*..... (\sigma - 8)$

Since $\mathfrak{P}u=\mathfrak{U}u=p^*$ and

$(\mathfrak{U}, \mathfrak{P})$ is $\mathcal{A}$ -intimate we have $M_{KM}(\mathfrak{U}p^*, p^*, t) \geq M_{KM}(\mathfrak{P}p^*, p^*, t).... (\sigma - 9)$.

Suppose that $\mathfrak{P}p^* \neq p^*$.

Put $x = p^*, \gamma = v$ in $(\mathcal{C} - 2)$ then we get,

$$M_{KM}(\mathfrak{P}p^*, \mathfrak{Q}v, kt) \geq M_{KM}(\mathfrak{U}p^*, \mathfrak{S}v, t) * M_{KM}(\mathfrak{P}p^*, \mathfrak{U}p^*, t) * \\ M_{KM}(\mathfrak{Q}v, \mathfrak{S}v, t) * M_{KM}(\mathfrak{P}p^*, \mathfrak{S}v, t).$$

Using $(\sigma - 8)$ we get,

$$M_{KM}(\mathfrak{P}p^*, p^*, kt) \\ \geq M_{KM}(\mathfrak{U}p^*, p^*, t) * M_{KM}(\mathfrak{P}p^*, \mathfrak{U}p^*, t) * M_{KM}(p^*, p^*, t) \\ * M_{KM}(\mathfrak{P}p^*, p^*, t).$$

By applying (KMFM-iv) we get

$$M_{KM}(\mathfrak{P}p^*, p^*, kt) \\ \geq M_{KM}(\mathfrak{P}p^*, p^*, t) * M_{KM}(\mathfrak{P}p^*, p^*, t/2) * M_{KM}(p^*, \mathfrak{U}p^*, t/2) \\ * M_{KM}(p^*, p^*, t) * M_{KM}(\mathfrak{P}p^*, p^*, t).$$

By using $(\sigma - 9)$ we get

$$M_{KM}(\mathfrak{P}p^*, p^*, kt) \geq M_{KM}(\mathfrak{P}p^*, p^*, t/2).$$

This gives $\mathfrak{P}p^*=p^*(\sigma - 10)$.

From $(\sigma - 9)$ and $(\sigma - 10)$ we write $M_{KM}(\mathfrak{U}p^*, p^*, t) \geq 1$

this gives $\mathfrak{U}p^*=p^*.....(\sigma - 11)$

using $(\sigma - 10)$ and $(\sigma - 11)$ we get

$$\mathfrak{U}p^*=\mathfrak{P}p^*=p^*..... (\sigma - 12)$$

Also, $\mathfrak{Q}v=\mathfrak{S}v=p^*$ and using the pair $(\mathfrak{S}, \mathfrak{Q})$ as $\mathcal{S}$ -intimate then we have

$$M_{KM}(\mathfrak{S}p^*, p^*, t) \geq M_{KM}(\mathfrak{Q}p^*, p^*, kt).... (\sigma - 13)$$

Suppose that $\mathfrak{Q}p^* \neq p^*$.

Put $x = u$ and $\gamma = p^*$ in the inequality

$$M_{KM}(\mathfrak{P}u, \mathfrak{Q}p^*, kt) \\ \geq M_{KM}(\mathfrak{U}u, \mathfrak{S}p^*, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}p^*, \mathfrak{S}p^*, t) \\ * M_{KM}(\mathfrak{P}u, \mathfrak{S}p^*, t)$$

using $(\sigma - 6)$ and (KMFM-iv) we get,

$$M_{KM}(p^*, \mathfrak{Q}p^*, kt) \geq M_{KM}(p^*, \mathfrak{S}p^*, t) * M_{KM}(p^*, p^*, t) * \\ M_{KM}\left(\mathfrak{P}p^*, p^*, \frac{t}{2}\right) * M_{KM}\left(p^*, \mathfrak{S}p^*, \frac{t}{2}\right) * M_{KM}(p^*, \mathfrak{S}p^*, t)$$

on using $(\sigma - 13)$ we get

$$M_{KM}(p^*, \mathfrak{Q}p^*, kt) \geq M_{KM}(p^*, \mathfrak{Q}p^*, t) * M_{KM}\left(\mathfrak{Q}p^*, p^*, \frac{t}{2}\right) * \\ M_{KM}(\mathfrak{Q}p^*, p^*, t/2) * M_{KM}(p^*, \mathfrak{Q}p^*, t).$$

This implies $M_{KM}(p^*, \mathfrak{Q}p^*, kt) \geq M_{KM}(p^*, \mathfrak{Q}p^*, t/2)$.

This gives $\mathcal{Q}p^* = p^* \dots (\sigma - 14)$.

From $(\sigma - 13)$ and $(\sigma - 14)$ we get

$$M_{KM}(\mathcal{S}p^*, p^*, t) \geq 1$$

$$\mathcal{S}p^* = p^* \dots (\sigma - 15).$$

Using $(\sigma - 14)$ and $(\sigma - 15)$ we get

$$\mathcal{Q}p^* = \mathcal{S}p^* = p^* \dots (\sigma - 16).$$

Using $(\sigma - 12)$ and $(\sigma - 16)$ we conclude that $\mathcal{U}p^* = \mathcal{P}p^* = \mathcal{Q}p^* = \mathcal{S}p^* = p^*$. Hence the result.

We can prove the uniqueness of the fixed point easily.

Example 3.1.1: Suppose $(\mathbb{X}, M_{KM}, *)$ is a standard FMS with $a * a \geq a \forall a \in [0, 1]$, where $\mathcal{U}, \mathcal{S}, \mathcal{P}$ and $\mathcal{Q}: \mathbb{X} \rightarrow \mathbb{X}$ as

$$\mathcal{P}(x) = \mathcal{Q}(x) = \begin{cases} x + 0.125 & \text{if } 0 \leq x < 0.125 \\ 0.25 & \text{if } 0.125 \leq x \leq 1 \end{cases}$$

$$\mathcal{U}(x) = \mathcal{S}(x) = \begin{cases} 2x & \text{if } 0 \leq x < 0.125 \\ 0.25 & \text{if } 0.125 \leq x \leq 1 \end{cases}$$

$\mathcal{P}(\mathbb{X}) = \mathcal{Q}(\mathbb{X}) = [0.125, 0.25]$ and $\mathcal{U}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) = [0, 0.25]$ these sets satisfy the condition $(\mathcal{C} - 1)$.

Now assume $\langle x_n \rangle = \left\{0.125 + \frac{1}{n}\right\}$ then $\lim_{n \rightarrow \infty} \mathcal{U}x_n = \lim_{n \rightarrow \infty} \mathcal{P}x_n = 0.25$.

Also we have, $\lim_{n \rightarrow \infty} \mathcal{U}\mathcal{P}x_n = \lim_{n \rightarrow \infty} \mathcal{U}\mathcal{P}\left(0.125 + \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \mathcal{U}(0.25) = 0.125$.

$$\lim_{n \rightarrow \infty} M_{KM}(\mathcal{U}\mathcal{P}x_n, \mathcal{U}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathcal{P}\mathcal{P}x_n, \mathcal{P}x_n, t), \text{ for } t > 0.$$

Thus, the pair $(\mathcal{U}, \mathcal{P})$ is $\mathcal{A}$ -intimate.

$$\text{Further } \lim_{n \rightarrow \infty} M_{KM}(\mathcal{S}\mathcal{Q}x_n, \mathcal{S}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathcal{Q}\mathcal{Q}x_n, \mathcal{Q}x_n, t).$$

Thus, the pair $(\mathcal{S}, \mathcal{Q})$ is $\mathcal{S}$ -intimate.

Moreover, it satisfies the contraction condition of the theorem. Clearly 0.25 is the unique common fixed point for these four mappings.

Theorem.3.2: Let $(\mathbb{X}, M_{KM}, *)$ be a fuzzy metric space. Suppose $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{U}$ are self maps on $\mathbb{X}$ satisfies the conditions $(\mathcal{C} - 1)$, $(\mathcal{C} - 2)$, $(\mathcal{C} - 3)$ and $(\mathcal{C} - 4)$ with $(\mathcal{C} - 5): (\mathcal{P}, \mathcal{U})$ or $(\mathcal{Q}, \mathcal{S})$ satisfy E.A property then $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{U}$ have a unique common fixed point in $\mathbb{X}$.

Proof: Suppose the pair $(\mathcal{Q}, \mathcal{S})$ satisfies E.A property then $\exists$ sequence $\langle x_n \rangle$ in $\mathbb{X}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{Q}x_n = \lim_{n \rightarrow \infty} \mathcal{S}x_n = p^* \text{ for some } p^* \in \mathbb{X}.$$

Since $\mathcal{Q}(\mathbb{X}) \subseteq \mathcal{U}(\mathbb{X})$ then $\exists \langle x_n \rangle$ in $\mathbb{X}$ such that $\mathcal{Q}x_n = \mathcal{U}y_n$.

$$\text{Hence } \lim_{n \rightarrow \infty} \mathcal{U}y_n = p^* \dots (\varphi - 1).$$

$$\text{Now we show that } \lim_{n \rightarrow \infty} \mathcal{P}y_n = p^*.$$

Put $x = y_n$ and $y = x_n$ we obtain,

$$\begin{aligned} M_{KM}(\mathfrak{P}\gamma_n, \mathfrak{Q}x_n, kt) \\ \geq M_{KM}(\mathfrak{U}\gamma_n, \mathfrak{S}x_n, t) * M_{KM}(\mathfrak{P}\gamma_n, \mathfrak{U}\gamma_n, t) * M_{KM}(\mathfrak{Q}x_n, \mathfrak{S}x_n, t) \\ * M_{KM}(\mathfrak{P}\gamma_n, \mathfrak{S}x_n, t). \end{aligned}$$

Letting $n \rightarrow \infty$ and using $p\gamma_n \rightarrow p *$ we get

$$\lim_{n \rightarrow \infty} \mathfrak{Q}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{U}\gamma_n = \lim_{n \rightarrow \infty} \mathfrak{P}\gamma_n = p *.$$

Suppose that $\mathfrak{U}(\mathbb{X})$ is closed subspace of $\mathbb{X}$, $\exists u \in \mathbb{X}$ such that

$$p * = \mathfrak{U}u \dots (\varphi - 2).$$

We show that $\mathfrak{U}u = \mathfrak{P}u$.

Put $x = u$ and $\gamma = x_n$ in $(\mathcal{C} - 2)$ then we get

$$\begin{aligned} M_{KM}(\mathfrak{P}u, \mathfrak{Q}x_n, kt) \\ \geq M_{KM}(\mathfrak{U}u, \mathfrak{S}x_n, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}x_n, \mathfrak{S}x_n, t) \\ * M_{KM}(\mathfrak{P}u, \mathfrak{S}x_n, t). \end{aligned}$$

This implies $\mathfrak{P}u = p * \dots (\varphi - 3)$.

From $(\varphi - 2)$ and $(\varphi - 2)$ we get

$$\mathfrak{U}u = \mathfrak{P}u = p * \dots (\varphi - 4).$$

And since $(\mathfrak{U}, \mathfrak{P})$ is $\mathcal{A}$ - intimate then we get $\mathfrak{U}p * = \mathfrak{P}p * = p * \dots (\varphi - 5)$.

Since $\mathfrak{P}(\mathbb{X}) \subseteq \mathfrak{S}(\mathbb{X})$ then there exists a point $v \in \mathbb{X}$ such that

$$\mathfrak{P}u = \mathfrak{S}v = p * \dots (\varphi - 6).$$

Now put $x = u$ and $\gamma = v$ in $(\mathcal{C} - 2)$ then this gives

$$\begin{aligned} M_{KM}(\mathfrak{P}u, \mathfrak{Q}v, kt) \geq M_{KM}(\mathfrak{U}u, \mathfrak{S}v, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}v, \mathfrak{S}v, t) * \\ M_{KM}(\mathfrak{P}u, \mathfrak{S}v, t) \text{ implies} \end{aligned}$$

$$\begin{aligned} M_{KM}(p *, \mathfrak{Q}v, kt) \\ \geq M_{KM}(p *, p *, t) * M_{KM}(p *, p *, t) * M_{KM}(\mathfrak{Q}v, p *, t) \\ * M_{KM}(p *, p *, t). \end{aligned}$$

This implies $\mathfrak{Q}v = p *$ therefore $\mathfrak{S}v = \mathfrak{Q}v = p * \dots (\varphi - 7)$,

and since $(\mathfrak{S}, \mathfrak{Q})$ is $\mathcal{S}$ - intimate then we get

$$\mathfrak{S}p * = \mathfrak{Q}p * = p * \dots (\varphi - 8).$$

Using $(\varphi - 7)$ $(\varphi - 8)$ and we conclude that $\mathfrak{U}p * = \mathfrak{P}p * = \mathfrak{Q}p * = \mathfrak{S}p * = p *$.

We can prove the uniqueness of the common fixed point easily.

Example 3.2.1: Suppose $(\mathbb{X}, M_{KM}, *)$ is a standard FMS with $a * a \geq a \ \forall a \in [1, 11)$, where $\mathfrak{U}, \mathfrak{S}, \mathfrak{P}$ and $\mathfrak{Q}: \mathbb{X} \rightarrow \mathbb{X}$ as

$$\mathfrak{P}(x) = \mathfrak{Q}(x) = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 11) \\ 1 + x & \text{if } 1 < x \leq 3 \end{cases}$$

$$\mathfrak{S}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 6 & \text{if } 1 < x \leq 3 \\ x - 2 & \text{if } 3 < x < 11 \end{cases} \quad \mathfrak{U}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 4 & \text{if } 1 < x \leq 3 \\ \frac{3x-1}{8} & \text{if } 3 < x < 11 \end{cases}$$

$\mathfrak{P}(\mathbb{X}) = \mathfrak{Q}(\mathbb{X}) = \{1\} \cup (2,4]$ and $\mathfrak{S}(\mathbb{X}) = \{1\} \cup \{6\} \cup (1,9)$ $\mathfrak{U}(\mathbb{X}) = \{1\} \cup \{4\} \cup (1,4) = [1,4]$ these sets satisfy the conditions $(\mathcal{C} - 2)$ and $(\mathcal{C} - 3)$.

Now assume $\langle x_n \rangle = \left\{3 + \frac{1}{n}\right\}$ then $\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{P}x_n = 1$ and this implies $(\mathfrak{P}, \mathfrak{U})$ satisfies E.A property and also we have, $\lim_{n \rightarrow \infty} \mathfrak{U}\mathfrak{P}x_n = \lim_{n \rightarrow \infty} \mathfrak{P}\mathfrak{P}x_n = 1$. This gives $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{P}x_n, \mathfrak{U}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{P}\mathfrak{P}x_n, \mathfrak{P}x_n, t) > 0$.

Thus, the pair $(\mathfrak{U}, \mathfrak{P})$ is $\mathcal{A}$ -intimate.

Since $\lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{Q}x_n = 1$ and $\lim_{n \rightarrow \infty} \mathfrak{S}\mathfrak{Q}x_n = \lim_{n \rightarrow \infty} \mathfrak{Q}\mathfrak{Q}x_n = 1$ this gives $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{Q}x_n, \mathfrak{S}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{Q}\mathfrak{Q}x_n, \mathfrak{Q}x_n, t)$.

Thus, the pair $(\mathfrak{S}, \mathfrak{Q})$ is $\mathcal{S}$ -intimate. Moreover, it satisfies the contraction condition of the theorem. Clearly 1 is the unique common fixed point for these four mappings.

Finally, we discuss another theorem.

3.3 Theorem: Let $(\mathbb{X}, M_{KM}, *)$ be a FMS. Suppose $\mathfrak{P}, \mathfrak{Q}, \mathfrak{S}$ and $\mathfrak{U}$ are self maps on $\mathbb{X}$ satisfying the conditions $(\mathcal{C} - 2)$ and $(\mathcal{C} - 4)$ in addition to $(\mathcal{C} - 6)$ $\mathfrak{U}(\mathbb{X})$ and $\mathfrak{S}(\mathbb{X})$ are closed subsets of $\mathbb{X}$ $(\mathcal{C} - 7)$ the pairs $(\mathfrak{P}, \mathfrak{U})$ and $(\mathfrak{Q}, \mathfrak{S})$ share the common property E. A. Then $\mathfrak{P}, \mathfrak{Q}, \mathfrak{S}$ and $\mathfrak{U}$ have a unique common fixed point in $\mathbb{X}$.

Proof: In view of the condition $(\mathcal{C} - 7)$ there exists two sequences $\langle x_n \rangle$ and $\langle \gamma_n \rangle$ in $\mathbb{X}$ such that $\lim_{n \rightarrow \infty} \mathfrak{P}x_n = \lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{Q}\gamma_n = \lim_{n \rightarrow \infty} \mathfrak{S}\gamma_n = p^*$ for some $p^* \in \mathbb{X}$.

From the $(\mathcal{C} - 6)$ we have $\mathfrak{U}(\mathbb{X})$ is closed subset of $\mathbb{X}$, consequently $\lim_{n \rightarrow \infty} \mathfrak{P}x_n = p^* \in \mathfrak{U}(\mathbb{X})$. This means there exists a point $u \in \mathbb{X}$ such that $\mathfrak{U}u = p^*$.

Now we assert that $\mathfrak{P}u = \mathfrak{U}u$.

Put $x = u$ and $\gamma = \gamma_n$, we get

$$\begin{aligned} M_{KM}(\mathfrak{P}u, \mathfrak{Q}\gamma_n, kt) &\geq M_{KM}(\mathfrak{U}u, \mathfrak{S}\gamma_n, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}\gamma_n, \mathfrak{S}\gamma_n, t) \\ &\quad * M_{KM}(\mathfrak{P}u, \mathfrak{S}\gamma_n, t) \end{aligned}$$

Which on making $n \rightarrow \infty$, with $\mathfrak{U}u = p^*$ reduces to $\mathfrak{P}u = p^*$. This implies

$\mathfrak{P}u = \mathfrak{U}u = p^*$ which signifies that u is coincident point of the pair $(\mathfrak{P}, \mathfrak{U})$.

On the other hand, $\mathfrak{S}(\mathbb{X})$ is closed subset of $\mathbb{X}$ therefore $\lim_{n \rightarrow \infty} \mathfrak{S}\gamma_n = p^* \in \mathfrak{S}(\mathbb{X})$ and

hence we can find a point $w \in \mathbb{X}$ $\mathfrak{S}w = p^*$.

Now we show that $\mathfrak{S}w = \mathfrak{Q}w$. On using condition $(\mathcal{C} - 2)$ with

$x = u$ and $\gamma = w$ then we get

$$\begin{aligned} M_{KM}(\mathfrak{P}u, \mathfrak{Q}w, kt) &\geq M_{KM}(\mathfrak{U}u, \mathfrak{S}w, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}w, \mathfrak{S}w, t) * \\ &\quad M_{KM}(\mathfrak{P}u, \mathfrak{S}w, t). \end{aligned}$$

This implies $\mathfrak{Q}w = p^*$. This gives $\mathfrak{S}w = \mathfrak{Q}w = p^*$.

Since the pair $(\mathfrak{Q}, \mathfrak{S})$ is $\mathcal{S}$ -intimate this gives

$$M_{KM}(\mathfrak{S}p^*, p^*, t) \geq M_{KM}(\mathfrak{Q}p^*, p^*, t).$$

Suppose that $\mathfrak{S}p * \neq p *$.

Put $x = u$ and $\gamma = p *$ in contraction condition (C – 2)

$$\begin{aligned} M_{KM}(\mathfrak{P}u, \mathfrak{Q}p *, kt) \\ \geq M_{KM}(\mathfrak{U}u, \mathfrak{S}p *, t) * M_{KM}(\mathfrak{P}u, \mathfrak{U}u, t) * M_{KM}(\mathfrak{Q}p *, \mathfrak{S}p *, t) \\ * M_{KM}(\mathfrak{P}u, \mathfrak{S}p *, t) \end{aligned}$$

implies $\mathfrak{Q}p * = p *$.

Using $M_{KM}(\mathfrak{S}p *, p *, t) \geq M_{KM}(p *, p *, t)$

we get $\mathfrak{S}p * = p *$.

Therefore $\mathfrak{Q}p * = \mathfrak{S}p * = p * \dots (\psi - 1)$.

Since $\mathfrak{P}u = \mathfrak{U}u = p *$ and using $(\mathfrak{P}, \mathfrak{U})$ is $\mathcal{A}$ –intimate then we get $\mathfrak{U}p * = p *$.

By putting $x = \gamma = p *$ we get

$$\begin{aligned} M_{KM}(\mathfrak{P}p *, \mathfrak{Q}p *, kt) \\ \geq M_{KM}(\mathfrak{U}p *, \mathfrak{S}p *, t) * M_{KM}(\mathfrak{P}p *, \mathfrak{U}p *, t) * M_{KM}(\mathfrak{Q}p *, \mathfrak{S}p *, t) \\ * M_{KM}(\mathfrak{P}p *, \mathfrak{S}p *, t). \end{aligned}$$

This implies $\mathfrak{P}p * = p *$ and this gives $\mathfrak{U}p * = \mathfrak{P}p * = p * \dots (\psi - 2)$.

From $(\psi - 1)$ and $(\psi - 2)$ we conclude that $\mathfrak{U}p * = \mathfrak{P}p * = \mathfrak{Q}p * = \mathfrak{S}p * = p *$.

We can prove the uniqueness of the fixed point easily.

Example 3.3.1: Suppose $(\mathbb{X}, M_{KM}, *)$ is a standard FMS with $a * a \geq a \quad \forall a \in [1, 20]$, where $\mathfrak{U}, \mathfrak{S}, \mathfrak{P}$ and $\mathfrak{Q}: \mathbb{X} \rightarrow \mathbb{X}$ as

$$\begin{aligned} \mathfrak{P}(x) = \mathfrak{Q}(x) = \begin{cases} 1 & \text{if } x = 1, \quad 2 \leq x < 20 \\ x & \text{if } 1 \leq x < 2 \end{cases} \\ \mathfrak{S}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 12 & \text{if } 1 < x < 2 \\ \frac{x+1}{3} & \text{if } 2 \leq x \leq 20 \end{cases} \quad \mathfrak{U}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 7 & \text{if } 1 < x < 2 \\ \frac{2x+5}{9} & \text{if } 2 \leq x \leq 20 \end{cases} \end{aligned}$$

$\mathfrak{P}(\mathbb{X}) = \mathfrak{Q}(\mathbb{X}) = \{1\} \cup (1, 2), \mathfrak{S}(\mathbb{X}) = \{1\} \cup \{12\} \cup [1, 5]$ and

$\mathfrak{U}(\mathbb{X}) = \{1\} \cup \{7\} \cup [1, 9]$ these sets satisfy the conditions (C – 1) and (C – 3).

Now assume $\langle x_n \rangle = \left\{2 + \frac{1}{n}\right\}$ and $\langle \gamma_n \rangle = \{1\}$ then

$$\lim_{n \rightarrow \infty} \mathfrak{U}x_n = \lim_{n \rightarrow \infty} \mathfrak{P}x_n = \lim_{n \rightarrow \infty} \mathfrak{S}\gamma_n = \lim_{n \rightarrow \infty} \mathfrak{Q}\gamma_n = 1.$$

This implies the pairs $(\mathfrak{P}, \mathfrak{U})$ and $(\mathfrak{S}, \mathfrak{Q})$ share the common E. A property and also we

have, $\lim_{n \rightarrow \infty} \mathfrak{U}\mathfrak{P}x_n = \lim_{n \rightarrow \infty} \mathfrak{P}\mathfrak{P}x_n = 1$ this gives

$$\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{U}\mathfrak{P}x_n, \mathfrak{U}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{P}\mathfrak{P}x_n, \mathfrak{P}x_n, t), \text{ for } t > 0.$$

Thus, the pair $(\mathfrak{U}, \mathfrak{P})$ is $\mathcal{A}$ -intimate.

Since $\lim_{n \rightarrow \infty} \mathfrak{S}x_n = \lim_{n \rightarrow \infty} \mathfrak{Q}x_n = 1$ and $\lim_{n \rightarrow \infty} \mathfrak{S}\mathfrak{Q}x_n = \lim_{n \rightarrow \infty} \mathfrak{Q}\mathfrak{Q}x_n = 1$

this gives $\lim_{n \rightarrow \infty} M_{KM}(\mathfrak{S}\mathfrak{Q}x_n, \mathfrak{S}x_n, t) \geq \lim_{n \rightarrow \infty} M_{KM}(\mathfrak{Q}\mathfrak{Q}x_n, \mathfrak{Q}x_n, t).$

Thus, the pair $(\mathfrak{S}, \mathfrak{Q})$ is $\mathcal{S}$ -intimate. Moreover, it satisfies the contraction condition of the theorem. Clearly 1 is the unique common fixed point for these four mappings.

4 Conclusion

This paper aimed to prove three common fixed point theorems to generalize the class of compatible mappings by using the class of non compatible mappings like different forms of E.A properties along with intimate mappings in fuzzy metric space. In Theorem 3.1, one of the range of mappings is assumed to be complete. Further, in Theorem 3.2, one of the pairs is assumed to satisfy E.A property along with one of the range of mappings is complete without being complete fuzzy metric space. Finally in Theorem 3.3, improved version of EA property namely common EA property is assumed along with completeness of fuzzy metric space. Moreover, all these results are justified with suitable examples.

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Common Fixed Point Theorems for $(\phi, \mathfrak{F})$ -Integral Type Contractive Mapping on C^* -Algebra Valued b -Metric Space

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Abstract

The object of this paper, we establish the concept of integral type of common fixed point theorem for new type of generalized C^* -valued contractive mapping. The main theorem is an existence and uniqueness of common fixed-point theorem for self-mappings with $(\phi, \mathfrak{F})$ -contractive conditions on complete C^* -algebra valued b -metric space. Moreover, some illustrated examples are also provided.

Keywords: C^* -algebra valued, common fixed point, b -metric spaces.

2020 AMS subject classifications: 47H10, 54H25, 54M20.¹

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1 Introduction

In 2002, Branciari [2002] introduced the concept of integral type contraction on fixed point solution. Many writers researched at the presence of fixed points for a variety of integral type contractive mappings, see Liu et al. [2018]. Especially, Liu et al. [2014] several more fixed point theorems for integral type contractive mappings in complete metric spaces. After that Ma et al. [2014] and Ma and Jiang [2015] presented the notion of C^* -algebra-valued metric space, C^* -algebra-valued $\mathfrak{b}$ -metric space and investigated certain fixed point results for self-mapping under certain contractive conditions. Alsulami et al. [2016] investigated that fixed point theorem in the classical Banach fixed point theorem can be used to produce C^* -algebra-valued $\mathfrak{b}$ -metric space in fixed point results Kamran et al. [2016]. We symbolize $\mathcal{A}$ as an unital C^* -algebra, and $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. Especially, an element $a \in \mathcal{A}$ is a positive factor, if $a = a^*$. A natural partial order on $\mathcal{A}_h$ given by $a \leq b$ if $f\theta \leq (b - a)$, where q signifies the zero element in $\mathcal{A}$. Then, let $\mathcal{A}^+$ and $\mathcal{A}'$ symbolize the set $\{a \in \mathcal{A} : \theta \leq a\}$ and the set $\{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$, respectively and $|a| = (a^*a)^{\frac{1}{2}}$.

2 Preliminaries

Definition 2.1 (Ma and Jiang [2015]). *Let χ be a non-empty set and $\omega \in \mathcal{A}$ such that $\omega \geq I$. Suppose that the mapping $D_b : X \times X \rightarrow \mathcal{A}$ is held, the following constraints exist.*

- (i) $\theta \leq D_b(\zeta, \eta)$ and $D_b(\zeta, \eta) = \theta$ iff $\zeta = \eta$;
- (ii) $D_b(\zeta, \eta) = D_b(\eta, \zeta)$;
- (iii) $D_b(\zeta, \eta) \leq \omega(D_b(\zeta, \vartheta) + D_b(\vartheta, \eta))$ for all $\zeta, \eta, \vartheta \in \chi$.

Then, D_b is called C^* -algebra-valued $\mathfrak{b}$ -metric on X and $(\chi, \mathcal{A}, D_b)$ is called C^* -algebra-valued $\mathfrak{b}$ -metric space.

Definition 2.2 (Ma and Jiang [2015]). *Let $(\chi, \mathcal{A}, D_b)$ be C^* -algebra valued $\mathfrak{b}$ -metric space. Assume that $\{\zeta_n\}$ is a sequence in χ and $\zeta \in \chi$. If for each $\epsilon > \theta$, there exists $\mathcal{N}$ such that $\forall n > \mathcal{N}$, $\|d(\zeta_n, \zeta)\| \leq \epsilon$ then $\{\zeta_n\}$ is alleged to be convergent with regard to $\mathcal{A}$, and $\{\zeta_n\}$ converges to ζ , i.e., we take $\lim_{n \rightarrow \infty} \zeta_n = \zeta$. If for each $\epsilon > \theta$, there exists $\mathcal{N}$ such that $\forall l, n, m > \mathcal{N}$, $\|d(\zeta_n, \zeta_m)\| \leq \epsilon$, then $\{\zeta_n\}$ is referred to as a Cauchy sequence in χ . $(\chi, \mathcal{A}, D_b)$ is referred to as a complete C^* -algebra-valued $\mathfrak{b}$ -metric space if every Cauchy sequence is convergent in χ .*

Common fixed point theorems for $(\phi, \mathfrak{F})$ integral type contractive mapping on C^* -algebra valued $\mathfrak{b}$ -metrix space

Definition 2.3 (Mustafa et al. [2021]). *Let the non-decreasing function $\mathfrak{F} : \mathcal{A}^+ \rightarrow \mathcal{A}^+$ be positive linear mapping satisfying following constraints:*

- (i) $\mathfrak{F}$ is continuous;
- (ii) $\mathfrak{F}(\mathfrak{a}) = \theta$ iff $\mathfrak{a} = \theta$;
- (iii) $\lim_{n \rightarrow \infty} \mathfrak{F}^n(\mathfrak{a}) = \theta$.

Definition 2.4 (Mustafa et al. [2021]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^* -algebra. $\mathcal{A}$ mapping $\mathfrak{F} : \mathcal{A} \rightarrow \mathcal{B}$ is said to be C^* -homomorphism if :*

- (i) $\mathfrak{F}(\mathfrak{a}\zeta + \mathfrak{b}\eta) = \mathfrak{a}\mathfrak{F}(\zeta) + \mathfrak{b}\mathfrak{F}(\eta)$ for all $\mathfrak{a}, \mathfrak{b} \in C$ and $\zeta, \eta \in \mathcal{A}$;
- (ii) $\mathfrak{F}(\zeta\eta) = \mathfrak{F}(\zeta)\mathfrak{F}(\eta)$ for all $\zeta, \eta \in \mathcal{A}$;
- (iii) $\mathfrak{F}(\zeta^*) = \mathfrak{F}(\zeta)^*$ for all $\zeta \in \mathcal{A}$;
- (iv) $\mathfrak{F}$ maps the unit in $\mathcal{A}$ to the unit in $\mathcal{B}$.

Lemma 2.1. *Let $(\chi, \mathcal{A}, D_b)$ be a C^* -algebra valued $\mathfrak{b}$ -metric space such that $D_b(\zeta, \eta) \in \mathcal{A}$, for all $\zeta, \eta \in \chi$ where $\zeta \neq \eta$. Let $\phi : \mathcal{A}^+ \rightarrow \mathcal{A}^+$ be a function with the following properties:*

- (i) $\phi(\mathfrak{a}) = \theta$ if and only if $\mathfrak{a} = \theta$;
- (ii) $\phi(\mathfrak{a}) < \mathfrak{a}$, for all $\mathfrak{a} \in \mathcal{A}$;
- (iii) either $\phi(\mathfrak{a}) \leq D_b(\zeta, \eta)$ or $D_b(\zeta, \eta) \leq \phi(\mathfrak{a})$, where $\mathfrak{a} \in \mathcal{A}$ and $\zeta, \eta \in \chi$.

Corollary 2.1. *Every C^* -homomorphism is bounded.*

Lemma 2.2. *Every $*$ -homomorphism is positive.*

Definition 2.5 (Branciari [2002]). *The function $\xi : \chi \rightarrow \chi$ is called sub-additive integrable function iff $\forall \mathfrak{a}, \mathfrak{b} \in \chi$, $\int_0^{\mathfrak{a}+\mathfrak{b}} \xi d_t \leq \int_0^{\mathfrak{a}} \xi d_t + \int_0^{\mathfrak{b}} \xi d_t$.*

3 Main Results

Definition 3.1. Let $(\chi, \mathcal{A}, D_b)$ is a complete C^* -algebra valued $\mathfrak{b}$ -metric space. Let $\mathfrak{L}, \mathfrak{M}, : \chi \rightarrow \chi$ be a integral C^* -valued contractive mapping and

$$\begin{aligned} \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}\zeta, \mathfrak{M}\eta)} \xi d_t \right) &\leq \mathfrak{F} \left(\int_0^{\mathfrak{I}(\zeta, \eta)} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta, \eta)} \xi d_t \right) \quad (1.1) \\ \mathfrak{I}(\zeta, \eta) &\leq \left(\alpha \int_0^{D_b(\zeta, \eta)} \xi d_t + \gamma \int_0^{[D_b(\zeta, \mathfrak{L}\zeta) + D_b(\eta, \mathfrak{M}\eta)]} \xi d_t \right. \\ &\quad \left. + \delta \int_0^{[D_b(\zeta, \mathfrak{M}\eta) + D_b(\eta, \mathfrak{L}\zeta)]} \xi d_t \right) \end{aligned}$$

For all $\zeta, \eta \in \chi$, where $\omega \in \mathcal{A}'_+$, $\alpha + \gamma + \delta \geq 0$ with $\omega\alpha + \gamma(\omega + 1) + \delta(\omega(\omega + 1)) < 1$. $\mathfrak{F} \in \Psi$ and $\phi \in \Phi$ and $\xi : \chi \rightarrow \chi$ is the Lebesgue-integral function.

Theorem 3.1. Let $(\chi, \mathcal{A}, D_b)$ is a complete C^* -algebra valued $\mathfrak{b}$ -metric space. 3.1 are $*$ -homomorphisms and with the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$ and $\xi : \chi \rightarrow \chi$ is a Lebesgue-integral mapping which is summable, non-negative and such that for all $\varepsilon > 0$, $\int_0^\varepsilon \xi d_t > 0$. Then $\mathfrak{L}$ and $\mathfrak{M}$ have a unique common fixed point in χ .

Proof. Let $\zeta_0 \in \chi$ and define $\zeta_n = \mathfrak{L}\zeta_{n-1}$, $\zeta_{n+1} = \mathfrak{M}\zeta_n$ we have

$$\begin{aligned} \mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t \right) &= \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}\zeta_{n-1}, \mathfrak{M}\zeta_n)} \xi d_t \right) \\ &\leq \mathfrak{F} \left(\int_0^{\mathfrak{I}(\zeta_{n-1}, \zeta_n)} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) \\ &= \mathfrak{F} \left(\alpha \int_0^{D_b(\zeta, \eta)} \xi d_t + \gamma \int_0^{[D_b(\zeta, \mathfrak{L}\zeta) + D_b(\eta, \mathfrak{M}\eta)]} \xi d_t \right. \\ &\quad \left. + \delta \int_0^{[D_b(\zeta, \mathfrak{M}\eta) + D_b(\eta, \mathfrak{L}\zeta)]} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) \\ &= \left(\mathfrak{F}(\alpha) \mathfrak{F} \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) + \mathfrak{F}(\gamma) \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \mathfrak{L}\zeta_{n-1}) + D_b(\zeta_n, \mathfrak{M}\zeta_n)]} \xi d_t \right) \right. \\ &\quad \left. + \mathfrak{F}(\delta) \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \mathfrak{M}\zeta_n) + D_b(\zeta_n, \mathfrak{L}\zeta_{n-1})]} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t \right) \right\| &= \left\| \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}\zeta_{n-1}, \mathfrak{M}\zeta_n)} \xi d_t \right) \right\| \\ &\leq \left(\begin{aligned} &\left\| \mathfrak{F}(\alpha) \right\| \left\| \mathfrak{F} \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) \right\| \\ &+ \left\| \mathfrak{F}(\gamma) \right\| \left\| \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \mathfrak{L}\zeta_{n-1}) + D_b(\zeta_n, \mathfrak{M}\zeta_n)]} \xi d_t \right) \right\| \\ &+ \left\| \mathfrak{F}(\delta) \right\| \left\| \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \mathfrak{M}\zeta_n) + D_b(\zeta_n, \mathfrak{L}\zeta_{n-1})]} \xi d_t \right) \right\| \\ &- \left\| \phi \left(\int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right) \right\| \end{aligned} \right) \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

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Give that ϕ (2.1) and $\mathfrak{F}$ (2.4) are strongly monotone functions. We have

$$\begin{aligned}
 \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t &= \int_0^{D_b(\mathfrak{L}\zeta_{n-1}, \mathfrak{M}\zeta_n)} \xi d_t \\
 &\leq \left(\alpha \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t + \gamma \int_0^{[D_b(\zeta_{n-1}, \mathfrak{L}\zeta_{n-1}) + D_b(\zeta_n, \mathfrak{M}\zeta_n)]} \xi d_t \right. \\
 &\quad \left. + \delta \int_0^{[D_b(\zeta_{n-1}, \mathfrak{M}\zeta_n) + D_b(\zeta_n, \mathfrak{L}\zeta_{n-1})]} \xi d_t \right) \\
 &= \left(\alpha \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t + \gamma \int_0^{[D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1})]} \xi d_t \right) \\
 &\quad + \delta \int_0^{[D_b(\zeta_{n-1}, \zeta_{n+1}) + D_b(\zeta_n, \zeta_n)]} \xi d_t \\
 &\leq (\alpha + \gamma) \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t + \gamma \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t + \delta \int_0^{D_b(\zeta_{n-1}, \zeta_{n+1})} \xi d_t \\
 &\leq (\alpha + \gamma) \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t + \gamma \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t + \omega \delta \int_0^{(D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1}))} \xi d_t \\
 &\leq (\alpha + \gamma + \omega \delta) \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t + (\gamma + \omega \delta) \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t &\leq \frac{\alpha + \gamma + \omega \delta}{\gamma + \omega \delta} \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \\
 \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t &\leq h \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t
 \end{aligned}$$

where, $h = \frac{\alpha + \gamma + \omega \delta}{\gamma + \omega \delta} < 1$. Thus, we have

$$\left\| \int_0^{D_b(\zeta_{n-1}, \zeta_n)} \xi d_t \right\| \left\| \int_0^{D_b(\zeta_n, \zeta_{n+1})} \xi d_t \right\| \leq h \left\| \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right\| \rightarrow 0,$$

as $n, m \rightarrow +\infty$. If $n > m$

$$\int_0^{D_b(\zeta_n, \zeta_m)} \xi d_t \leq \left(\omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t + \omega^2 \int_0^{D_b(\zeta_{n-1}, \zeta_{n-2})} \xi d_t \right. \\
 \left. + \dots + \omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right).$$

Applying the constraint of theorem then,

$$\mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_m)} \xi d_t \right) \leq \left(\mathfrak{F} \left(\omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) + \mathfrak{F} \left(\omega^2 \int_0^{D_b(\zeta_{n-1}, \zeta_{n-2})} \xi d_t \right) \right. \\
 \left. + \dots + \mathfrak{F} \left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right) \right)$$

$$\begin{aligned}
&\leq \left(\mathfrak{F}(\omega) \mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) + \mathfrak{F}(\omega^2) \mathfrak{F} \left(\int_0^{D_b(\zeta_{n-1}, \zeta_{n-2})} \xi d_t \right) \right. \\
&\quad \left. + \dots + \mathfrak{F}(\tau^{n-m}) \mathfrak{F} \left(\int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right) \right) \\
&\leq \left(\begin{aligned} &\mathfrak{F} \left(\omega \int_0^{\mathfrak{I}(\zeta_n, \zeta_{n-1})} \xi d_t \right) - \phi \left(\omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) \\ &+ \mathfrak{F} \left(\omega^2 \int_0^{\mathfrak{I}(\zeta_{n-1}, \zeta_{n-2})} \xi d_t \right) - \phi \left(\omega^2 \int_0^{D_b(\zeta_{n-1}, \zeta_{n-2})} \xi d_t \right) \\ &+ \dots + \mathfrak{F} \left(\omega^{n-m} \int_0^{\mathfrak{I}(\zeta_{m-1}, \zeta_m)} \xi d_t \right) - \phi \left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right) \end{aligned} \right) \\
&= \left(\begin{aligned} &\mathfrak{F} \left(\begin{aligned} &\alpha \omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \\ &+ \gamma \omega \int_0^{[D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1})]} \xi d_t \\ &+ \delta \omega \int_0^{[D_b(\zeta_{n-1}, \zeta_{n+1}) + D_b(\zeta_n, \zeta_n)]} \xi d_t \end{aligned} \right) \\ &- \phi \left(\omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) + \dots \\ &+ \mathfrak{F} \left(\begin{aligned} &\alpha \omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \\ &+ \gamma \omega^{n-m} \int_0^{[D_b(\zeta_m, \zeta_{m-1}) + D_b(\zeta_{m-1}, \zeta_{m-2})]} \xi d_t \\ &+ \delta \omega^{n-m} \int_0^{[D_b(\zeta_m, \zeta_{m-2}) + D_b(\zeta_{m-1}, \zeta_{m-1})]} \xi d_t \end{aligned} \right) \\ &- \phi \left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right) \end{aligned} \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_m)} \xi d_t \right) &= \mathfrak{F}(\alpha) \mathfrak{F}(\omega) \mathfrak{F} \left(\int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) \\
&+ \mathfrak{F}(\gamma) \mathfrak{F}(\omega) \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1})]} \xi d_t \right) \\
&+ \mathfrak{F}(\delta) \mathfrak{F}(\omega) \mathfrak{F} \left(\int_0^{[D_b(\zeta_{n-1}, \zeta_{n+1}) + D_b(\zeta_n, \zeta_n)]} \xi d_t \right) \\
&- \phi \left(\omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \right) + \dots \\
&+ \mathfrak{F}(\alpha) \mathfrak{F}(\omega^{n-m}) \mathfrak{F} \left(\int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right) \\
&+ \mathfrak{F}(\gamma) \mathfrak{F}(\omega^{n-m}) \mathfrak{F} \left(\int_0^{[D_b(\zeta_m, \zeta_{m-1}) + D_b(\zeta_{m-1}, \zeta_{m-2})]} \xi d_t \right) \\
&+ \mathfrak{F}(\delta) \mathfrak{F}(\omega^{n-m}) \mathfrak{F} \left(\int_0^{[D_b(\zeta_m, \zeta_{m-2}) + D_b(\zeta_{m-1}, \zeta_{m-1})]} \xi d_t \right) \\
&- \phi \left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \right).
\end{aligned}$$

Since the property of ϕ (2.1) and $\mathfrak{F}$ (2.4) is strongly monotone, we have

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$$\begin{aligned}
 \int_0^{D_b(\zeta_n, \zeta_m)} \xi d_t &\leq \alpha \omega \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t + \gamma \omega \int_0^{[D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1})]} \xi d_t \\
 &+ \delta \omega \int_0^{[D_b(\zeta_{n-1}, \zeta_{n+1}) + D_b(\zeta_n, \zeta_n)]} \xi d_t + \dots \\
 &+ \alpha \omega^{n-m} \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \\
 &+ \gamma \omega^{n-m} \int_0^{[D_b(\zeta_m, \zeta_{m-1}) + D_b(\zeta_{m-1}, \zeta_{m-2})]} \xi d_t \\
 &+ \delta \omega^{n-m} \int_0^{[D_b(\zeta_m, \zeta_{m-2}) + D_b(\zeta_{m-1}, \zeta_{m-1})]} \xi d_t
 \end{aligned}$$

so we get

$$\int_0^{D_b(\zeta_n, \zeta_m)} \xi d_t \leq \left(\begin{aligned} &+ \|\alpha\| \|\omega\| \int_0^{D_b(\zeta_n, \zeta_{n-1})} \xi d_t \| \\ &+ \|\gamma\| \|\omega\| \int_0^{[D_b(\zeta_{n-1}, \zeta_n) + D_b(\zeta_n, \zeta_{n+1})]} \xi d_t \| \\ &+ \|\delta\| \|\omega\| \int_0^{[D_b(\zeta_{n-1}, \zeta_{n+1}) + D_b(\zeta_n, \zeta_n)]} \xi d_t \| + \dots \\ &+ \|\alpha\| \|\omega^{n-m}\| \int_0^{D_b(\zeta_{m-1}, \zeta_m)} \xi d_t \| \\ &+ \|\gamma\| \|\omega^{n-m}\| \int_0^{[D_b(\zeta_m, \zeta_{m-1}) + D_b(\zeta_{m-1}, \zeta_{m-2})]} \xi d_t \| \\ &+ \|\delta\| \|\omega^{n-m}\| \int_0^{[D_b(\zeta_m, \zeta_{m-2}) + D_b(\zeta_{m-1}, \zeta_{m-1})]} \xi d_t \| \end{aligned} \right) \rightarrow 0,$$

as $n, m \rightarrow +\infty$.

Then $\{\zeta_n\}$ is Cauchy sequence. Since $(\chi, \mathcal{A}, D_b)$ is a complete C^* -algebra valued b -metric space there exists $u \in \chi$ such that $\zeta_n \rightarrow u$ as $n \rightarrow \infty$. Now since

$$\begin{aligned}
 \int_0^{D_b(u, \mathfrak{M}u)} \xi d_t &\leq \omega \left[\int_0^{D_b(u, \zeta_{n+1})} \xi d_t + \int_0^{D_b(\zeta_{n+1}, \mathfrak{M}u)} \xi d_t \right] \\
 &= \omega \left[\int_0^{D_b(\zeta_{n+1}, \mathfrak{M}u)} \xi d_t + \int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right] \\
 &= \omega \left[\int_0^{D_b(\zeta_n, \mathfrak{M}u)} \xi d_t + \int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right]
 \end{aligned}$$

$$\mathfrak{F} \left(\int_0^{D_b(u, \mathfrak{M}u)} \xi d_t \right) = \omega \left[\mathfrak{F} \left(\int_0^{D_b(\zeta_n, \mathfrak{M}u)} \xi d_t \right) + \mathfrak{F} \left(\int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right) \right]$$

$$\leq \omega \left[\mathfrak{F} \left(\int_0^{\mathfrak{I}(\zeta_n, u)} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta_n, u)} \xi d_t \right) \right] + \omega \left[\mathfrak{F} \left(\int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right) \right]$$

$$\| \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}\zeta_n, \mathfrak{M}u)} \xi d_t \right) \| \leq \left(\begin{array}{c} \| \omega \| \| \mathfrak{F} \left(\int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right) \| \\ + \| \omega \| \| \mathfrak{F} \alpha \| \| \int_0^{D_b(\zeta_n, u)} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F} \gamma \| \| \int_0^{[D_b(\zeta_n, \zeta_{n+1}) + D_b(u, \mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F} \delta \| \| \int_0^{[D_b(\zeta_n, \mathfrak{M}u) + D_b(u, \mathfrak{L}\zeta_n)]} \xi d_t \| \\ - \| \omega \| \| \phi \left(\int_0^{D_b(\zeta_n, u)} \xi d_t \right) \| \end{array} \right)$$

Using the property of ϕ (2.1), we get

$$\| \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}\zeta_n, \mathfrak{M}u)} \xi d_t \right) \| \leq \left(\begin{array}{c} \| \omega \| \| \mathfrak{F} \left(\int_0^{D_b(u, \zeta_{n+1})} \xi d_t \right) \| \\ + \| \omega \| \| \mathfrak{F} \alpha \| \| \int_0^{D_b(\zeta_n, u)} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F} \gamma \| \| \int_0^{[D_b(\zeta_n, \zeta_{n+1}) + D_b(u, \mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F} \delta \| \| \int_0^{[D_b(\zeta_n, \mathfrak{M}u) + D_b(u, \mathfrak{L}\zeta_n)]} \xi d_t \| \end{array} \right).$$

Where $\mathfrak{F}$ (2.4) is strongly monotone, then

$$\begin{aligned} \| \int_0^{(D_b(\mathfrak{L}\zeta_n, \mathfrak{M}u))} \xi d_t \| &\leq \left(\begin{array}{c} \| \omega \| \| \int_0^{(D_b(u, \zeta_{n+1}))} \xi d_t \| \\ + \| \omega \| \| \alpha \| \| \int_0^{D_b(\zeta_n, u)} \xi d_t \| \\ + \| \omega \| \| \gamma \| \| \int_0^{[D_b(\zeta_n, \zeta_{n+1}) + D_b(u, \mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \delta \| \| \int_0^{[D_b(\zeta_n, \mathfrak{M}u) + D_b(u, \mathfrak{L}\zeta_n)]} \xi d_t \| \end{array} \right) \\ &= \| \omega \| \| \int_0^{(D_b(u, \zeta_{n+1}))} \xi d_t \| + \| \omega \| \left[\begin{array}{c} \| \alpha \| \| \int_0^{D_b(\zeta_n, u)} \xi d_t \| \\ + \| \gamma \| \| \int_0^{[D_b(\zeta_n, \zeta_{n+1}) + D_b(u, \mathfrak{M}u)]} \xi d_t \| \\ + \| \delta \| \| \int_0^{[D_b(\zeta_n, \mathfrak{M}u) + D_b(u, \mathfrak{L}\zeta_n)]} \xi d_t \| \end{array} \right] \end{aligned}$$

as $\zeta_n \rightarrow u$ and $\zeta_{n+1} \rightarrow u$ as $n \rightarrow \infty$, we get

$$\| 1 - \omega\gamma - \omega\delta \| \| \int_0^{D_b(u, \mathfrak{M}u)} \xi d_t \| \leq \left[\begin{array}{c} \| \omega \| \| \alpha \| \| \int_0^{D_b(\zeta_n, u)} \xi d_t \| \\ + \| \omega \| \| 1 + \delta \| \| \int_0^{D_b(u, \zeta_{n+1})} \xi d_t \| \end{array} \right] \rightarrow 0$$

as $n \rightarrow \infty$.

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Hence $\| \int_0^{D_b(\mathfrak{M}u, u)} \xi d_t \| = 0$ since $\| 1 - \omega\gamma - \omega\delta \| > 0$. As a result, $\mathfrak{M}u = u$ that is u is a fixed point of $\mathfrak{M}$. Similarly we are able to demonstrate that $\mathfrak{L}u = u$. Hence $\mathfrak{L}u = \mathfrak{M}u = u$. This demonstrates that u is common fixed point of $\mathfrak{L}$ and $\mathfrak{M}$.

Let v be a different fixed point common to $\mathfrak{L}$ and $\mathfrak{M}$. (i.e) $\mathfrak{L}v = \mathfrak{M}v = v$ such that $u \neq v$ we have $\int_0^{D_b(u, v)} \xi d_t = \int_0^{D_b(\mathfrak{L}u, \mathfrak{M}v)} \xi d_t$ then

$$\begin{aligned} \mathfrak{F} \left(\int_0^{D_b(u, v)} \xi d_t \right) &= \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}u, \mathfrak{M}v)} \xi d_t \right) \\ &\leq \mathfrak{F} \left(\int_0^{\mathfrak{J}(u, v)} \xi d_t \right) - \phi \left(\int_0^{D_b(u, v)} \xi d_t \right) \\ \left\| \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}u, \mathfrak{M}v)} \xi d_t \right) \right\| &\leq \left(\begin{array}{l} \|\mathfrak{F}\alpha\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \\ + \|\mathfrak{F}\gamma\| \left\| \int_0^{[D_b(u, \mathfrak{L}u) + D_b(v, \mathfrak{M}v)]} \xi d_t \right\| \\ + \|\mathfrak{F}\delta\| \left\| \int_0^{[D_b(u, \mathfrak{M}v) + D_b(v, \mathfrak{L}u)]} \xi d_t \right\| \\ - \|\phi\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \end{array} \right) \end{aligned}$$

Using the property of ϕ (2.1), we get

$$\left\| \mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}u, \mathfrak{M}v)} \xi d_t \right) \right\| \leq \left(\begin{array}{l} \|\mathfrak{F}\alpha\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \\ + \|\mathfrak{F}\gamma\| \left\| \int_0^{[D_b(u, \mathfrak{L}u) + D_b(v, \mathfrak{M}v)]} \xi d_t \right\| \\ + \|\mathfrak{F}\delta\| \left\| \int_0^{[D_b(u, \mathfrak{M}v) + D_b(v, \mathfrak{L}u)]} \xi d_t \right\| \end{array} \right)$$

where $\mathfrak{F}$ (2.4) is strongly monotone, then

$$\begin{aligned} \left\| \int_0^{(D_b(\mathfrak{L}u, \mathfrak{M}v))} \xi d_t \right\| &\leq \left(\begin{array}{l} \|\alpha\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \\ + \|\gamma\| \left\| \int_0^{[D_b(u, \mathfrak{L}u) + D_b(v, \mathfrak{M}v)]} \xi d_t \right\| \\ + \|\delta\| \left\| \int_0^{[D_b(u, \mathfrak{M}v) + D_b(v, \mathfrak{L}u)]} \xi d_t \right\| \end{array} \right) \\ &\leq \|\alpha + 2\delta\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \\ &\leq \|\omega\alpha + (\omega + 1)\gamma + \omega(\omega + 1)\delta\| \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \\ &< \left\| \int_0^{D_b(u, v)} \xi d_t \right\| \end{aligned}$$

Which is a contradiction. Hence $\left\| \int_0^{D_b(u, v)} \xi d_t \right\| = 0$ and $u = v$. Thus u is a unique common fixed point of $\mathfrak{L}$ and $\mathfrak{M}$.

Corollary 3.1. Let $(\chi, \mathcal{A}, D_b)$ is a complete C^* -algebra valued $\mathfrak{b}$ -metric space. Let $\mathfrak{L} : \chi \rightarrow \chi$ be a contractive mapping and

$$\mathfrak{F} \left(\int_0^{D_b(\mathfrak{L}^n \zeta, \mathfrak{L}^n \eta)} \xi d_t \right) \leq \mathfrak{F} \left(\int_0^{\mathfrak{J}(\zeta, \eta)} \xi d_t \right) - \phi \left(\int_0^{D_b(\zeta, \eta)} \xi d_t \right)$$

$$\mathfrak{I}(\zeta, \eta) \leq \left(\begin{array}{l} \alpha \int_0^{D_b(\zeta, \eta)} \xi d_t + \beta \int_0^{\frac{[1+D_b(\zeta, \mathfrak{L}^n \zeta)]D_b(\eta, \mathfrak{L}^n \eta)}{1+D_b(\zeta, \eta)}} \xi d_t \\ + \gamma \int_0^{[D_b(\zeta, \mathfrak{L}^n \zeta)+D_b(\eta, \mathfrak{L}^n \eta)]} \xi d_t \\ + \delta \int_0^{[D_b(\zeta, \mathfrak{L}^n \eta)+D_b(\eta, \mathfrak{L}^n \zeta)]} \xi d_t \end{array} \right)$$

for all $\zeta, \eta \in \chi$, where $\omega \in \mathcal{A}'_+$, $\alpha + \beta + \gamma + \delta \geq 0$ with $\omega\alpha + \beta + \gamma(\omega + 1) + \delta(\omega(\omega + 1)) < 1$. $\mathfrak{F}$ and ϕ are $*$ -homomorphisms and with the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$ and $\xi : \chi \rightarrow \chi$ is a Lebesgue-integral mapping which is summable, non-negative and such that for all $\varepsilon > 0$, $\int_0^\varepsilon \xi d_t > 0$. Then $\mathfrak{L}$ have a unique fixed point in χ .

Example 3.1. Let $\chi = [0, 1]$ and $\mathcal{A} = \mathcal{R}^2$ with a norm $\|\zeta\| = |\zeta|$ be a real C^* -algebra. We define $\mathfrak{p} = \{(\zeta, \eta) \in \mathcal{R}^2 : \zeta \geq 0, \eta \geq 0\}$. The partial order $\leq$ with respect to the C^* -algebra $\mathcal{R}^2$. $\zeta_1 \leq \zeta_2$ and $\eta_1 \leq \eta_2$ for all $(\zeta_1, \eta_1), (\zeta_2, \eta_2) \in \mathcal{R}^2$. Let $D_b : \chi \times \chi \rightarrow \mathcal{R}^2$ suppose that $D_b(\zeta, \eta) = 2(|\zeta - \eta|, |\zeta - \eta|)$ for $\zeta, \eta \in \chi$.

Then, $(\chi, \mathcal{A}, D_b)$ is a C^* -algebra valued $\mathfrak{b}$ -metric space where $\omega = 1$ in theorem 3.1.

Let $\mathfrak{F}, \phi : \mathfrak{p} \rightarrow \mathfrak{p}$ be the mappings defined as follows: For $\mathcal{T} = (\zeta, \eta) \in \mathfrak{p}$

$$\mathfrak{F}(\mathcal{T}) = \begin{cases} (\zeta, \eta), & \text{if } \zeta \leq 1 \text{ and } \eta \leq 1, \\ (\zeta^2, \eta), & \text{if } \zeta > 1 \text{ and } \eta \leq 1, \\ (\zeta, \eta^2), & \text{if } \zeta \leq 1 \text{ and } \eta > 1, \\ (\zeta^2, \eta^2), & \text{if } \zeta > 1 \text{ and } \eta > 1. \end{cases}$$

and for $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) \in \mathfrak{p}$ with $\mathcal{V} = \min\{\mathcal{S}_1, \mathcal{S}_2\}$,

$$\phi(\mathcal{S}) = \begin{cases} \left(\frac{\mathcal{V}^2}{2}, \frac{\mathcal{V}^2}{2}\right), & \text{if } \mathcal{V} \leq 1 \\ \left(\frac{1}{2}, \frac{1}{2}\right), & \text{if } \mathcal{V} > 2 \end{cases}$$

Then, $\mathfrak{F}$ and ϕ have the properties mentioned in (2.4) and (2.1). Let $\mathfrak{L}, \mathfrak{M} : \chi \rightarrow \chi$ be defined as follows:

$$\mathfrak{L}(\zeta) = \begin{cases} \frac{1}{32}, & \text{if } 0 \leq \zeta \leq \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < \zeta \leq 1 \end{cases} ; M(\zeta) = \frac{1}{32}, \text{ for } \zeta \in \chi$$

Then, $\mathfrak{L}$ and $\mathfrak{M}$ have the required properties mentioned in theorem 3.1.

Let $\alpha = \frac{1}{16}$, $\beta = 0$, $\gamma = \frac{1}{64}$ and $\delta = \frac{1}{64}$. It can be verified that:

$$\mathfrak{F}(D_b(\mathfrak{L}\zeta, \mathfrak{M}\eta)) \leq \mathfrak{F}(N(\zeta, \eta)) - \phi(D_b(\zeta, \eta)), \forall \zeta, \eta \in \chi \text{ with } \eta \leq \zeta$$

. Hence, Theorem 3.1 is satisfied. Then demonstrate that 0 is a unique common fixed point of $\mathfrak{L}$ and M .

4 Conclusions

In Theorem 3.1 we have formulated a contractive conditions to modify and extend the concept of common fixed point theorem for C^* -algebra valued b -metric space via $(\phi, \mathfrak{F})$ -integral type contractive mapping. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the contractive condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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Fixed point theorems in uniformly convex Banach spaces

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Abstract

In this article, we establish a concept of fixed point result in Uniformly convex Banach space. Our main finding uses the Ishikawa iteration technique in uniformly convex Banach space to demonstrate strong convergence. Additionally, we use our primary result to demonstrate some corollaries.

Keywords: Fixed point; Mann iteration; Ishikawa iteration.

2020 AMS subject classifications: 55M20, 46B80, 52A21.¹

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1 Introduction

Mann [Mann [1953]] defined mean value methods in an iterative scheme in 1953, and Ishikawa [Ishikawa [1974]] established fixed points using a new iteration method technique in 1954. Takahashi [Takahashi [1970]] introduced the idea of convexity in metric spaces and non-expansive mappings in 1970. Then Machado [Machado [1973]] went on to discuss about a classification of convex subsets of normed spaces. After that, Luis Bernal-Gonzalez [Bernal-Gonzalez [1996]] discussed convex domain in uniformly Banach spaces. Berinde [Berinde [2004]] investigates iterative scheme to finding fixed points using quasi contractive mappings in uniformly convex Banach spaces(stands for CBS), extended to uniformly convex Banach spaces(CBS).

Throughout this paper, we use strong convergence of ishikawa iterations to prove such fixed point results in uniformly CBS using different type of contractions.

2 Preliminaries

Definition 2.1. Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \rightarrow X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space, which is denoted by (X, d, W) .

Definition 2.2. Let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

Definition 2.3. Let $f : X \rightarrow X$. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

Definition 2.4. Let E be a uniformly Banach space and $T : E \rightarrow E$ a map for which there is a real constant $k_1 \in (0, 1/5)$ such that each pair $u, v \in X$,

$$\|Tu - Tv\| \leq k_1 \{ \|u - v\| + \|u - Tu\| + \|v - Tv\| + \|u - Tv\| + \|v - Tu\| \}.$$

Then, T has a fixed point by the approximation of Picard.

Definition 2.5. Let E be a uniformly Banach space and $T : E \rightarrow E$ a map for which there is a real constant $k_2 \in (0, 1/3)$ such that for each pair $u, v \in X$,

$$\|Tu - Tv\| \leq k_2 \left\{ \|u - v\| + \frac{\|u - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2} \right\}.$$

Then, T has a fixed point by the approximation of Picard.

Definition 2.6. Let E be a uniformly Banach space and $T : E \rightarrow E$ a given operator. Let $u_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers. The sequence $\{u_n\} \subset E$ defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, n = 0, 1, 2, \dots \quad (1)$$

is called the Mann iteration.

Definition 2.7. Let E be a uniformly Banach space and $T : E \rightarrow E$ a given operator. Let $u_0 \in E$ be arbitrary, $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$ a sequence of real numbers. The sequence $\{u_n\} \subset E$ defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T v_n, n = 0, 1, 2, \dots \quad (2)$$

$$v_n = (1 - \beta_n)u_n + \beta_n T u_n, n = 0, 1, 2, \dots \quad (3)$$

Then $\{u_n\}$ is called Ishikawa iteration.

Result 2.1. Berinde [2004] The condition of Mann and Ishikawa iteration for strong convergence are given below

- (a) Let K be a closed convex subset of a uniformly Banach space E and $T : K \rightarrow K$ as an operator satisfying contraction. Let $\{u_n\}$ be defined by Definition 2.6 and $x_0 \in K$, with $\{\alpha_n\} \in [0, 1]$ satisfying

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \quad (4)$$

Then, $\{u_n\}$ converges strongly to a fixed point.

- (b) Let K be a closed convex subset of a uniformly Banach space E and $T : K \rightarrow K$ as an operator satisfying the contraction. Let $\{u_n\}$ be defined by Definition 2.7 and $u_0 \in K$, with $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ satisfying

$$\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty. \quad (5)$$

Then, $\{u_n\}$ strongly converges to a fixed point. $\square$

3 Main Results

Theorem 3.1. Let K be a closed convex subset of a uniformly Banach space E and $T : K \rightarrow K$ an operator satisfying equation

$$\|Tu - Tv\| \leq k_1 \{\|u - v\| + \|u - Tu\| + \|v - Tv\| + \|u - Tv\| + \|v - Tu\|\}. \quad (6)$$

Let $\{u_n\}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\{\alpha_n\}$ satisfying equation $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}$ converges strongly to a fixed point of T .

Proof. Suppose T has a fixed point p in K . Consider $u, v \in K$ and T is an operator satisfies above equation,

$$\begin{aligned} \|Tu - Tv\| &\leq k_1\{\|u - v\| + \|u - Tu\| + \|v - Tv\| + \|u - Tv\| \\ &\quad + \|v - Tu\|\} \\ &= k_1\{2\|u - v\| + 2\|u - Tv\| + \|u - Tu\| + \|v - Tu\|\} \\ &\leq k_1\{2\|u - v\| + 2[\|u - Tu\| + \|Tu - Tv\|] + \|u - Tu\| \\ &\quad + \|v - Tu\|\} \\ (1 - 2k_1)\|Tu - Tv\| &\leq 2k_1\|u - v\| + 3k_1\|u - Tu\| + k_1\|v - Tu\| \\ \|Tu - Tv\| &\leq \frac{2k_1}{1 - 2k_1}\|u - v\| + \frac{3k_1}{1 - 2k_1}\|u - Tu\| + \frac{k_1}{1 - 2k_1}\|v - Tu\|. \end{aligned}$$

Take $\delta = \frac{k_1}{1 - 2k_1}$, then we have $\delta \in [0, 1)$, it result that the inequality

$$\|Tu - Tv\| \leq 2\delta\|u - v\| + 3\delta\|u - Tu\| + \delta\|v - Tu\| \forall u, v \in K. \quad (7)$$

Now let $\{u_n\}_{n=0}^{\infty}$ be Ishikawa iteration defined on Definition 2.7 and $u_0 \in K$ arbitrary then

$$\begin{aligned} \|u_{n+1} - p\| &= \|(1 - \alpha_n)u_n + \alpha_n Tv_n - (1 - \alpha_n + \alpha_n)p\| \\ &= \|(1 - \alpha_n)(u_n - p) + \alpha_n(Tv_n - p)\| \\ \|u_{n+1} - p\| &\leq (1 - \alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\|. \end{aligned} \quad (8)$$

In equation (7), put $u = p$ and $v = v_n$, we have

$$\begin{aligned} \|Tv_n - p\| &\leq 2\delta\|p - v_n\| + 3\delta\|p - Tp\| + \delta\|v_n - Tp\| \\ &= 2\delta\|p - v_n\| + \delta\|v_n - p\| \\ \|Tv_n - p\| &\leq 3\delta\|v_n - p\|. \end{aligned} \quad (9)$$

Furthermore, $\|v_n - p\| = \|(1 - \beta_n)u_n + \beta_n Tu_n - (1 - \beta_n + \beta_n)p\|$

$$\begin{aligned} &= \|(1 - \beta_n)(u_n - p) + \beta_n(Tu_n - p)\| \\ \|v_n - p\| &\leq (1 - \beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|. \end{aligned} \quad (10)$$

Again in equation (7), put $u = p$ and $v = u_n$, we get

$$\begin{aligned} \|Tu_n - p\| &\leq 2\delta\|p - u_n\| + 3\delta\|p - Tp\| + \delta\|u_n - p\| \\ &= (2\delta + \delta)\|u_n - p\| \end{aligned}$$

$$\|Tu_n - p\| \leq 3\delta\|u_n - p\|. \quad (11)$$

Using equation (9),(10),(11) in equation (8), we get

$$\begin{aligned} \|u_{n+1} - p\| &\leq (1 - \alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\| \\ &\leq (1 - \alpha_n)\|u_n - p\| + 3\alpha_n\delta\|v_n - p\| \\ &\leq (1 - \alpha_n)\|u_n - p\| + 3\alpha_n\delta[(1 - \beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|] \\ &= (1 - \alpha_n)\|u_n - p\| + 3\alpha_n\delta(1 - \beta_n)\|u_n - p\| + 3\alpha_n\delta\beta_n\|Tu_n - p\| \\ &\leq (1 - \alpha_n)\|u_n - p\| + 3\alpha_n\delta(1 - \beta_n)\|u_n - p\| + 9\alpha_n\delta^2\beta_n\|u_n - p\| \\ &= [1 - \alpha_n + 3\alpha_n\delta - 3\alpha_n\beta_n\delta + 9\alpha_n\delta^2\beta_n]\|u_n - p\| \\ &= [1 - \alpha_n(1 - 3\delta) - 3\alpha_n\beta_n\delta(1 - 3\delta)]\|u_n - p\| \\ &= [1 - (1 - 3\delta)\alpha_n(1 + 3\delta\beta_n)]\|u_n - p\| \end{aligned}$$

which by the inequality, $1 - (1 - 3\delta)\alpha_n(1 + 3\delta\beta_n) \leq 1 - (1 - 3\delta)^2\alpha_n$

$$\Rightarrow \|u_{n+1} - p\| \leq [1 - (1 - 3\delta)^2\alpha_n]\|u_n - p\|, n = 0, 1, 2, \dots \quad (12)$$

by equation (8), we obtain

$$\|u_{n+1} - p\| \leq \prod_{k=0}^n [1 - (1 - 3\delta)^2\alpha_k]\|u_0 - p\|. \quad (13)$$

Where, $\delta \in (0, 1)$, $\alpha_k, \beta_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ by result (a), we get

$\lim_{n \rightarrow \infty} \prod_{k=0}^n [1 - (1 - 3\delta)^2\alpha_k] = 0$. By equation (13) which implies

$\lim_{n \rightarrow \infty} \|u_{n+1} - p\| = 0$. Therefore, $\{u_n\}_{n=0}^{\infty}$ converges strongly to p . $\square$

Theorem 3.2. Let K be a closed convex subset of a uniformly Banach space E and $T : K \rightarrow K$ an operator satisfying equation

$$\|Tu - Tv\| \leq k_2\{\|u - v\| + \frac{\|u - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2}\}.$$

Let $\{u_n\}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ with $\{\alpha_n\}$ satisfying equation $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}$ converges strongly to a fixed point of T .

Proof. Consider T has a fixed point p in K . Consider $u, v \in K$ and T is an operator satisfies equation

$$\begin{aligned} \|Tu - Tv\| &\leq k_2\{\|u - v\| + \frac{\|u - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2}\} \\ &\leq k_2\{\|u - v\| + \frac{\|u - v\| + \|v - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2}\} \\ &= k_2\{\frac{3}{2}\|u - v\| + \|v - Tu\| + \frac{1}{2}\|v - Tv\| + \frac{\|u - Tv\|}{2}\} \\ (1 - k_2)\|Tu - Tv\| &\leq \frac{3}{2}k_2\{\|u - v\| + \|v - Tv\|\} + \frac{k_2}{2}\|u - Tv\| \\ \|Tu - Tv\| &\leq \frac{3k_2}{2(1 - k_2)}\{\|u - v\| + \|v - Tv\|\} + \frac{k_2}{2(1 - k_2)}\|u - Tv\| \end{aligned}$$

$$\text{Take } \delta = \frac{k_2}{1 - k_2} \in [0, 1). \|Tu - Tv\| \leq \frac{3}{2}\delta\{\|u - v\| + \|v - Tv\|\} + \frac{\delta}{2}\|u - Tv\|. \quad (14)$$

Now, let $\{u_n\}$ be Ishikawa iteration defined on Definition 2.7 and $u_0 \in K$ then,
 $\|u_{n+1} - p\| = \|(1 - \alpha_n)u_n + \alpha_n Tv_n - (1 - \alpha_n + \alpha_n)p\|$

$$\|u_{n+1} - p\| \leq (1 - \alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\| \quad (15)$$

In equation (14), put $v = p$ and $u = u_n$, $\|Tu_n - p\| \leq \frac{3\delta}{2}\|u_n - p\| + \frac{\delta}{2}\|u_n - p\|$

$$\|Tu_n - p\| \leq 2\delta\|u_n - p\| \quad (16)$$

In equation (14), put $u = v_n$ and $v = p$, $\|Tv_n - p\| \leq \frac{3\delta}{2}\|v_n - p\| + \frac{\delta}{2}\|v_n - p\|$

$$\|Tv_n - p\| \leq 2\delta\|v_n - p\| \quad (17)$$

$$\|v_n - p\| = \|(1 - \beta_n)u_n + \beta_n Tu_n - (1 - \beta_n + \beta_n)p\|$$

$$\|v_n - p\| \leq (1 - \beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\| \quad (18)$$

Using equation (16), (17), (18) in equation (15), we get

$$\begin{aligned} \|u_{n+1} - p\| &\leq (1 - \alpha_n) + 2\alpha_n\delta\|v_n - p\| \\ &\leq (1 - \alpha_n) + 2\delta\alpha_n[(1 - \beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|] \\ &\leq (1 - \alpha_n) + 2\delta\alpha_n(1 - \beta_n)\|u_n - p\| + 4\delta^2\alpha_n\beta_n\|u_n - p\| \\ &= [1 - \alpha_n(1 + 2\delta\beta_n)(1 - 2\delta)]\|u_n - p\| \end{aligned}$$

Which by inequality, $1 - \alpha_n(1 + 2\delta\beta_n)(1 - 2\delta) \leq 1 - (1 - 2\delta)^2\alpha_n$

$$\Rightarrow \|u_{n+1} - p\| \leq [1 - (1 - 2\delta)^2\alpha_n]\|u_n - p\| \quad (19)$$

By equation (19), we obtain

$$\|u_{n+1} - p\| \leq \prod_{k=0}^n [1 - (1 - 2\delta)^2\alpha_k]\|u_0 - p\| \quad (20)$$

where $\delta \in (0, 1)$, $\alpha_k, \beta_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ by result (a), we get,

$$\lim_{n \rightarrow \infty} \prod_{k=0}^n [1 - (1 - 2\delta)^2\alpha_k] = 0. \text{ By equation (20), } \Rightarrow \lim_{n \rightarrow \infty} \|u_{n+1} - p\| = 0.$$

Therefore $\{u_n\}_{n=0}^{\infty}$ converges strongly to p . $\square$

Corolary 3.1. Let K be a closed convex subset of a uniformly Banach space E and $T : K \rightarrow K$ an operator satisfying equation

$$\|Tu - Tv\| \leq \frac{k}{4}\{\|u - Tu\| + \|v - Tv\| + \|u - Tv\| + \|v - Tu\|\}.$$

Let $\{u_n\}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}$ satisfying equation $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}$ converges strongly to a fixed point of T .

Corollary 3.2. *Let E be an uniformly Banach space, K is a closed convex subset of E and $T : K \rightarrow K$ an operator satisfying equation*

$$\|Tu - Tv\| \leq k\|u - v\| \quad (21)$$

Let $\{u_n\}_{n=0}^{\infty}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .

Corollary 3.3. *Let E be an uniformly Banach space, K is a closed convex subset of E and $T : K \rightarrow K$ an operator satisfying equation*

$$\|Tu - Tv\| \leq \frac{k}{2} \{\|u - Tu\| + \|v - Tv\|\} \quad (22)$$

Let $\{u_n\}_{n=0}^{\infty}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .

4 Conclusions

In this work, we presented the result on strong convergence of fixed point of T . We developed different kinds of contractive conditions to prove strong convergence using Ishikawa iterative method in uniformly convex Banach space. Our main result may be vision for other authors using different contraction to prove several converging fixed point result.

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Elongation of Sets in Soft Lattice Topological Spaces

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Abstract

The aim of this paper, we investigate some Lattice sets such as soft lattice exterior, soft lattice interior, soft lattice boundary and soft lattice border sets in soft lattice topological spaces which are defined over a soft lattice L with a fixed set of parameter A and it is also a generalization of soft topological spaces. Further, we develop and continue the initial views of some soft lattice sets, which are deep-seated for further research on soft lattice topology and will consolidate the origin of the theory of soft topological spaces.

2020 AMS subject classifications: 54A05, 54A10. ¹

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1 Introduction

The concept of soft theory was first originated by Molodstov in 1999, which is deal with unpredictable problems meanwhile modeling results in engineering cases such as medical sciences, economics, etc., In 2003, Maji. et. al.[8] studied and discussed the fundamental ideas of soft theory. Following stage of soft set linked with netrosophic sets are introduced by Parimala Mani et. al.[9] in 2018 and Also, we introduced the new notion of neutrosophic complex $\alpha\Psi$ -connectedness in neutrosophic complex topological spaces and investigate some of its properties in 2022[5]. In 2019[13], several new generalizations of nano open sets be introduced and investigated by Nethaji, Ochanan.

The study of soft topological spaces (on short $\mathcal{S.T.S}$) is instated by Shabir and Naz[14] in 2011. They discussed $\mathcal{S.T}$ on the collection θ on soft set (on short $\mathcal{S.S}$) over U . Accordingly, they discussed fundamental notions of $\mathcal{S.T.S}$ such as soft open (on short $\mathcal{S.O}$), soft closed (on short $\mathcal{S.C}$), $\mathcal{S}$ closure, $\mathcal{S}$ neighborhood of a point, $\mathcal{S} T_i$ spaces, for ($i=1, 2, 3, 4$), $\mathcal{S}$ regular spaces, $\mathcal{S}$ normal spaces, and their specific features are also established. Therefore, in 2011[1], Naim Cagman, Serkan Karatas, and Serdar Enginoglu investigated a topology with $\mathcal{S.S}$ called $\mathcal{S.T}$ and its corresponding features. Then they present the foundation of the theory $\mathcal{S.T.S}$. The $\mathcal{S.T.S}$ may be the initial stage for the concepts of the soft mathematical opinion of structures which are the foundation of $\mathcal{S.S}$. theoretic operation.

From the concept of $\mathcal{S.S}$, the idea of soft lattices (on short $\mathcal{S.L}$) has arisen. In 2010[7], F. Li studied and defined this conviction of $\mathcal{S.L}$ and primary operations of results on $\mathcal{S.L}$. Additional, an application of $\mathcal{S.S}$ to lattices has executed by E. Kuppusamy in 2011. A different approach towards $\mathcal{S.L}$ can be seen in E. Kuppusamy apart from what F. Li has done. Further, the operation and the properties of $\mathcal{S.L}$ were studied by V. D. Jobish. et. al.[4] in 2013. Many theorems related to various types of unions, intersections, and complements including De Margon's Laws are obtained. In 2020[12], M. Parimala et. al explained the $nI\alpha g$ closed sets in nano ideal topological spaces with various prevailing closed sets.

Currently, topology depends toughly on the thoughts of the soft theory. Recently, $\mathcal{S.L.T.S}$ was first investigated by Sandhya. et. al.[11] in 2021 that are discussed throughout an $\mathcal{S.L}$ 'L' with a fixed set of parameters 'A' and it is also a generalization of $\mathcal{S.T.S}$. They detailed discussed the concept of Soft L - open (on short $\mathcal{S.L} - \mathcal{O}$), soft L - closed (on short $\mathcal{S.L} - \mathcal{C}$), $\mathcal{S.L}$ - closure, $\mathcal{S.L}$ - interior point, and $\mathcal{S.L}$ - neighborhood. In this paper, we continue investigating a soft L - interior (on short $\mathcal{S.L} - \mathcal{I}$), soft L - exterior (on short $\mathcal{S.L} - \mathcal{E}$), soft L - boundary (on short $\mathcal{S.L} - \mathcal{B}$), and soft L - border (on short $\mathcal{S.L} - \mathcal{Bor}$) which are basics for stimulating research on $\mathcal{S.T.S}$ and will build up the fountain of the theory of $\mathcal{S.T.S}$.

2 Preliminaries

Definition 2.1 (5,7). Let's take U be a whole set and A be a set parameters. A pair (F, A) , where F is a map from A to $\wp(U)$ is called a $\mathfrak{S}.\mathfrak{S}$ over U . Here, the $\mathfrak{S}.\mathfrak{S}$ is simply represented by f_A .

Example 2.1. Let say that there are 6 cars in the whole world

$U = \{\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3, \mathfrak{w}_4, \mathfrak{w}_5, \mathfrak{w}_6\}$ is the set of cars under regard and that $A = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ is a set of parameters denoted as colors. The $r_a, (a = 1, 2, 3, 4, 5)$ it means the parameters 'Red', 'Blue', 'Black', 'White', and 'Ash,' respectively.

Consider the mapping f_A given by 'cars' $(.)$, where $(.)$ is to be complete in by one of the parameters $r_a \in E$. For instance, $f_A(\rho_1)$ means 'Cars (Colors)'.

Suppose that $B = \{\rho_1, \rho_2, \rho_5\} \subseteq A$ and $f_A(\rho_1) = \{\mathfrak{w}_1, \mathfrak{w}_4\}$, $f_A(\rho_2) = U$, and $f_A(\rho_5) = \{\mathfrak{w}_2, \mathfrak{w}_4, \mathfrak{w}_5\}$ Then, we can view the $\mathfrak{S}.\mathfrak{S}$ F_A as consisting of the following collection of approximations:

$$F_A = \{(\rho_1, \{\mathfrak{w}_1, \mathfrak{w}_4\}), (\rho_2, U), (\rho_5, \{\mathfrak{w}_2, \mathfrak{w}_4, \mathfrak{w}_5\})\}.$$

Definition 2.2 (2,7). In two $\mathfrak{S}.\mathfrak{S}$ f_A, g_A over U , we say that

(i) f_A is a soft subset of g_A if

(a) $A \subseteq B$, and (b) $\forall \rho \in A, \lambda(\rho) = \mu(\rho)$ are equal to estimations.

(ii) f_A is soft equal set to g_A denoted by $f_A = g_A$ if $f_A \subseteq g_A$ and $g_A \subseteq f_A$

Definition 2.3 (7). Let $A = \{\rho_1, \dots, \rho_n\}$ be a parameters. The 'Not set of A ', denoted by ΓA is defined as $\Gamma A = \{\Gamma \rho_1, \dots, \Gamma \rho_n\}$, $\Gamma \rho_i$ means not $\rho_i \forall i = 1, 2, 3, \dots, n$.

Definition 2.4 (7,9). Complement of a $\mathfrak{S}.\mathfrak{S}$ f_A over U , represented by f'_A is defined as $f'_A = (F', \Gamma A)$, $F' : \Gamma A \longrightarrow \wp(U)$ such that (on short s.t) $F'(\Gamma \rho) = U - F(\rho)$, $\forall \Gamma \rho \in \Gamma A$.

Definition 2.5 (9). The relative complement of a $\mathfrak{S}.\mathfrak{S}$ f_A over U , stand for f_A^C is defined as $(f_A)^C = (F^C, A)$, $F^C : A \longrightarrow \wp(U)$ s.t $F^C(\rho) = U - F(\rho)$, $\forall \rho \in A$.

Definition 2.6 (7,9). Let f_A be a $\mathfrak{S}.\mathfrak{S}$ over U , then f_A is Null $\mathfrak{S}.\mathfrak{S}$ if $\forall \rho \in A, F(\rho) = \phi$ and is denoted by ϕ_A .
Let f_A be a $\mathfrak{S}.\mathfrak{S}$ over U , then f_A is absolute $\mathfrak{S}.\mathfrak{S}$ represented by U_A , if $\forall \rho \in A, F(\rho) = U$. Also, $U_A^C = \phi_A$ and $\phi_A^C = U_A$.

Definition 2.7 (2,7). Union of two $\mathfrak{S}.\mathfrak{S}$ f_A, g_B over U is the $\mathfrak{S}.\mathfrak{S}$ h_C ,

$$C = A \cup B \text{ and } \forall \rho \in C, \kappa(\rho) = \begin{cases} \lambda(\rho), & \text{if } \rho \in A - B \\ \mu(\rho), & \text{if } \rho \in B - A \\ \lambda(\rho) \cup \mu(\rho), & \text{if } \rho \in A \cap B \end{cases}$$

We write $f_A \cup g_B = h_C$.

Definition 2.8 (2,7). The intersection of two $\mathfrak{S}.\mathfrak{S}$ f_A, g_B over a whole set U is the $\mathfrak{S}.\mathfrak{S}$ h_C , here $C = A \cap B$ and $\forall \rho \in C, \kappa(\rho) = \lambda(\rho) \text{ or } \mu(\rho)$. We mark done $f_A \cap g_B = h_C$.

Definition 2.9 (1). Consider $\mathfrak{F}_A, \mathfrak{G}_A \in \mathfrak{S}.\mathfrak{S}(U, A)$. The soft symmetric difference of these sets is the $\mathfrak{S}.\mathfrak{S}$. $\mathfrak{H}_A \in \mathfrak{S}.\mathfrak{S}(U, A)$, here the map $\mathfrak{H} : A \rightarrow \wp(U)$ defined as follows:

$\mathfrak{H}(\rho) = ((f(\rho) \setminus g(\rho)) \cup ((g(\rho) \setminus (f(\rho)))$ for each $\rho \in A$. We mark down $\mathfrak{H}_A = \mathfrak{F}_A \Delta \mathfrak{G}_A$.

Definition 2.10 (3,6,10). A sublattice of a lattice L is a non-void subset of L that is a lattice with the same meet and join operation as L , ie., $\alpha, \beta \in L$ implies $\alpha \wedge \beta, \alpha \vee \beta \in L$.

Definition 2.11 (3,6,10). A Complete lattice L and A is the parameters of the $\mathfrak{S}.\mathfrak{L}$ over L . The triplet $M = (f, A, L)$, $f : A \rightarrow \wp(L)$ is $\mathfrak{S}.\mathfrak{L}$ if $f(\rho)$ is the sublattice of L for each $\rho \in A$. Then the $\mathfrak{S}.\mathfrak{L}$ is represented by f_A^L .

Definition 2.12 (10). Two $\mathfrak{S}.\mathfrak{L}$. f_A^L and g_A^L over L its difference is denoted by $h_A^L = f_A^L \setminus g_A^L$, is stated as $\mathfrak{H}(\rho) = ((f(\rho) \setminus g(\rho)) \forall \rho \in A$.

Definition 2.13 (10). Let us consider L be any complete lattice and A be the non void set of parameters. Let θ contains complete members, uniquely complemented $\mathfrak{S}.\mathfrak{L}$. over L , then θ is $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}$, then the condition hold:

- (i) $\phi_A, L_A \in \theta$.
- (ii) $\bigcup_{a \in n} \eta_a \in \theta, \forall \{ \eta_a : a \in n \} \subseteq \theta$
- (iii) $\eta_1 \cap \eta_2 \in \theta, \forall \eta_1, \eta_2 \in \theta$.

Then the triplet (L, θ, A) is called a $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}.\mathfrak{S}$. (soft L – space or soft L – topological space) over L . The members of θ are called soft lattice open sets in L . Also, a soft lattice (f_A^L) is called soft lattice closed if the relative complement $(f_A^L)^C$ belongs to θ .

3 Extension of $\mathfrak{S.L}$ - sets

Definition 3.1. In $\mathfrak{S.L.T.S}$, the $\mathfrak{S.L} - \mathfrak{I}$ of (f_A^L) is the union of all $\mathfrak{S.L} - \mathfrak{D}$ sets contained in f_A^L denoted by $(f_A^L)^\circ$.

i.e., $(f_A^L)^\circ = \bigcup \{g_A^L : g_A^L \in \theta \text{ and } g_A^L \subseteq f_A^L\}$.

Theorem 3.1. Let (L, θ, A) be a $\mathfrak{S.L.T.S}$ over L and f_A^L, g_A^L are $\mathfrak{S.L}$ over L . Then,

- (i) $\phi_A^\circ = \phi_A$ and $L_A = L_A^\circ$
- (ii) $(f_A^L)^\circ \subseteq (f_A^L)$
- (iii) f_A^L is a $\mathfrak{S.L} - \mathfrak{D}$ set $\iff (f_A^L)^\circ = f_A^L$
- (iv) $((f_A^L)^\circ)^\circ = (f_A^L)^\circ$
- (v) $f_A^L \subseteq g_A^L \Rightarrow (f_A^L)^\circ \subseteq (g_A^L)^\circ$
- (vi) $(f_A^L)^\circ \cap (g_A^L)^\circ = (f_A^L \cap g_A^L)^\circ$
- (vii) $(f_A^L)^\circ \cup (g_A^L)^\circ \subseteq (f_A^L \cup g_A^L)^\circ$

Proof

Results (i), (ii) are trival.

(iii) If (f_A^L) is $\mathfrak{S.L} - \mathfrak{D}$ set, (f_A^L) is itself a $\mathfrak{S.L} - \mathfrak{D}$ set contained in (f_A^L) . Since, $(f_A^L)^\circ$ is the largest $\mathfrak{S.L} - \mathfrak{D}$ set contained in (f_A^L) , $(f_A^L) = (f_A^L)^\circ$.
Conversely, Suppose that $(f_A^L) = (f_A^L)^\circ$. Since $(f_A^L)^\circ$ is a $\mathfrak{S.L} - \mathfrak{D}$ set, so (f_A^L) is $\mathfrak{S.L} - \mathfrak{D}$ set over L .

(iv) since $(f_A^L)^\circ$ is $\mathfrak{S.L} - \mathfrak{D}$ set, by (iii) $((f_A^L)^\circ)^\circ = (f_A^L)^\circ$.

(v) suppose that $(f_A^L) \subseteq (g_A^L)$. Since, $(f_A^L)^\circ \subseteq (f_A^L) \subseteq (g_A^L)$. $(f_A^L)^\circ$ is a $\mathfrak{S.L} - \mathfrak{D}$ subset of (g_A^L) , so by the definition of $(g_A^L)^\circ$, $(f_A^L)^\circ \subseteq (g_A^L)^\circ$.

(vi) we have $(f_A^L \cap g_A^L) \subseteq f_A^L$ and $(f_A^L \cap g_A^L) \subseteq g_A^L$. This implies (by v) $(f_A^L \cap g_A^L)^\circ \subseteq (f_A^L)^\circ$ and $(f_A^L \cap g_A^L)^\circ \subseteq (g_A^L)^\circ$ so that, $(f_A^L \cap g_A^L)^\circ \subseteq (f_A^L)^\circ \cap (g_A^L)^\circ$.

Also, since $(f_A^L)^\circ \subseteq f_A^L$ and $(g_A^L)^\circ \subseteq g_A^L$ implies

$(f_A^L)^\circ \cap (g_A^L)^\circ \subseteq (f_A^L \cap g_A^L)$ so that, $(f_A^L \cap g_A^L)^\circ$ is the largest $\mathfrak{S.L} - \mathfrak{D}$ subsets of $(f_A^L \cap g_A^L)$. Hence, $(f_A^L)^\circ \cap (g_A^L)^\circ \subseteq (f_A^L \cap g_A^L)^\circ$.

Thus, $(f_A^L)^\circ \cap (g_A^L)^\circ = (f_A^L \cap g_A^L)^\circ$.

- (vii) Since, $f_A^L \subseteq (f_A^L \cup g_A^L)$ and, $g_A^L \subseteq (f_A^L \cup g_A^L)$.
 So, by (v) $(f_A^L)^\circ \subseteq (f_A^L \cup g_A^L)^\circ$ and $(g_A^L)^\circ \subseteq (f_A^L \cup g_A^L)^\circ$. So that
 $(f_A^L)^\circ \cup (g_A^L)^\circ \subseteq (f_A^L \cup g_A^L)^\circ$.

Example 3.1. Now the given example to show that the statement of theorem 1(v) may be strict or equal, Let $L = \{S_{l_1}, S_{l_2}, S_{l_3}, S_{l_4}, S_{l_5}, S_{l_6}, S_{l_7}, S_{l_8}\}; A = \{\rho_1, \rho_2\};$
 $\theta = \{f_{1A}^L, f_{2A}^L, f_{3A}^L, f_{4A}^L, f_{5A}^L, L_A, \phi_A\}$

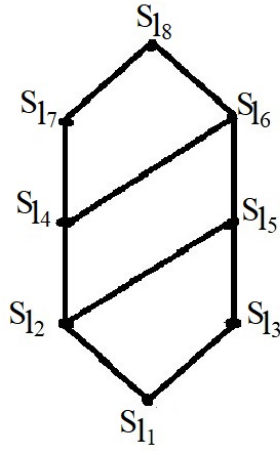


Figure 1: Complete lattice

$$\begin{aligned} f_{1A}^L &= \{(\rho_1, \{S_{l_4}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_3}, S_{l_6}\})\}, \\ f_{2A}^L &= \{(\rho_1, \{S_{l_6}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_4}\})\} \\ f_{3A}^L &= \{(\rho_1, \{S_{l_4}, S_{l_6}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}, S_{l_4}, S_{l_6}\})\}, \\ f_{4A}^L &= \{(\rho_1, \{S_{l_8}\}), (\rho_2, \phi)\} \\ \text{and } f_{5A}^L &= \{(\rho_1, \{S_{l_4}, S_{l_7}\}), (\rho_2, \{S_{l_3}, S_{l_6}\})\} \end{aligned}$$

For Equal Condition,

We choose any two $\mathfrak{S}, \mathfrak{L}$ from figure:1,

$$\begin{aligned} f_C^L &= \{(\rho_1, \{S_{l_6}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_4}\})\} \text{ and} \\ g_C^L &= \{(\rho_1, \{S_{l_1}, S_{l_6}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}, S_{l_4}, S_{l_6}\})\} \end{aligned}$$

$$(f_C^L)^\circ = f_{2A}^L \text{ and } (g_C^L)^\circ = f_{2A}^L.$$

Hence, $f_A^L \subset g_A^L$ implies $(f_A^L)^\circ = (g_A^L)^\circ$.

For inclusion condition,

We choose any two $\mathfrak{S.L}$ from figure:1,

$$\begin{aligned} f_D^L &= \{(\rho_1, \{S_{l_8}\}), (\rho_2, \{S_{l_3}, S_{l_6}\})\} \text{ and} \\ g_D^L &= \{(\rho_1, \{S_{l_4}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_3}, S_{l_6}\})\} \end{aligned}$$

$$(f_D^L)^\circ = f_{4A}^L \text{ and } (g_D^L)^\circ = f_{1A}^L.$$

$$\text{Hence, } f_A^L \subset g_A^L \text{ implies } (f_A^L)^\circ \subset (g_A^L)^\circ.$$

Example 3.2. Now the given example to show that the statement of theorem 1(vii) may be strict or equal, Let us consider the lattice and $\mathfrak{S.L.T}$ given in Example: 3.1

For inclusion Condition,

We choose any two $\mathfrak{S.L}$ from figure:1,

$$\begin{aligned} f_C^L &= \{(\rho_1, \{S_{l_6}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}, S_{l_6}\})\} \text{ and} \\ g_C^L &= \{(\rho_1, \{S_{l_4}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}, S_{l_4}, S_{l_6}\})\} \end{aligned}$$

$$(f_C^L)^\circ = f_{4A}^L \text{ and } (g_C^L)^\circ = f_{1A}^L, \text{ which implies } (f_C^L)^\circ \cup (g_C^L)^\circ = f_{1A}^L.$$

$$(f_C^L \cup g_C^L)^\circ \text{ is } f_{3A}^L.$$

$$\text{Hence, } (f_A^L)^\circ \cup (g_A^L)^\circ \subset (f_A^L \cup g_A^L)^\circ.$$

For equal condition,

We choose any two $\mathfrak{S.L}$ from figure:1,

$$\begin{aligned} f_C^L &= \{(\rho_1, \{S_{l_6}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_4}\})\} \text{ and} \\ g_C^L &= \{(\rho_1, \{S_{l_1}, S_{l_6}, S_{l_7}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}, S_{l_4}, S_{l_6}\})\} \end{aligned}$$

$$(f_C^L)^\circ = f_{2A}^L \text{ and } (g_C^L)^\circ = f_{2A}^L, \text{ which implies } (f_C^L)^\circ \cup (g_C^L)^\circ = f_{2A}^L.$$

$$\text{Hence, } (f_A^L)^\circ \cup (g_A^L)^\circ = (f_A^L \cup g_A^L)^\circ.$$

Definition 3.2. Let (L, θ, A) be a $\mathfrak{S.L.T.S}$ over L , then the $\mathfrak{S.L} - \mathfrak{E}$ of $\mathfrak{S.L}$ f_A^L is denoted by $(f_A^L)_\circ$ and is defined as $(f_A^L)_\circ = ((f_A^L)^C)^\circ$.

Theorem 3.2. Let f_A^L and g_A^L be $\mathfrak{S.L}$ of a $\mathfrak{S.L.T.S}$ (L, θ, A) . Then,

- (i) $(f_A^L \cup g_A^L)_\circ = (f_A^L)_\circ \cap (g_A^L)_\circ$.
- (ii) $(f_A^L)_\circ \cup (g_A^L)_\circ \subseteq (f_A^L \cap g_A^L)_\circ$.
- (iii) $f_A^L \subseteq g_A^L$ implies $(f_A^L)_\circ \supseteq (g_A^L)_\circ$.

Proof

- (i) $(f_A^L \cup g_A^L)_\circ = ((f_A^L \cup g_A^L)^C)^\circ = ((f_A^L)^C \cap (g_A^L)^C)^\circ = ((f_A^L)^C)^\circ \cap ((g_A^L)^C)^\circ = (f_A^L)_\circ \cap (g_A^L)_\circ$.
- (ii) $(f_A^L)_\circ \cup (g_A^L)_\circ = ((f_A^L)^C)^\circ \cup ((g_A^L)^C)^\circ \subseteq ((f_A^L)^C \cup (g_A^L)^C)^\circ = ((f_A^L \cap g_A^L)^C)^\circ = (f_A^L \cap g_A^L)_\circ$.
- (iii) $(g_A^L)_\circ = ((g_A^L)^C)^\circ \subseteq ((f_A^L)^C)^\circ = (f_A^L)_\circ$.

Example 3.3. Now the given example to show that the statement of theorem 2(ii) may be strict or equal, Let $L_A = \{S_{l_1}, S_{l_2}, S_{l_3}, S_{l_4}, S_{l_5}, S_{l_6}, S_{l_7}\}$; $A = \{\rho_1, \rho_2\}$; $\theta = \{f_{1A}^L, f_{2A}^L, f_{3A}^L, f_{4A}^L, L_A, \phi_A\}$

$f_{1A}^L = \{(\rho_1, \{S_{l_3}, S_{l_6}\}), (\rho_2, \{S_{l_4}, S_{l_5}\})\}$, $f_{2A}^L = \{(\rho_1, \{S_{l_6}\}), (\rho_2, \phi)\}$
 $f_{3A}^L = \{(\rho_1, \{S_{l_2}, S_{l_3}, S_{l_5}, S_{l_6}\}), (\rho_2, \{S_{l_4}, S_{l_5}, S_{l_6}\})\}$ and
 $f_{4A}^L = \{(\rho_1, \{S_{l_2}, S_{l_5}, S_{l_6}\}), (\rho_2, \{S_{l_6}\})\}$

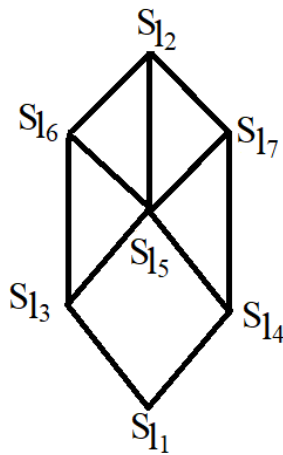


Figure 2: Complete lattice

For inclusion condition,

Now we take any two $\mathfrak{S}, \mathfrak{L}$ from the figure:2,

$$\begin{aligned} f_C^L &= \{(\rho_1, \{S_{l_2}, S_{l_6}\}), (\rho_2, \{S_{l_6}\})\} \text{ and} \\ g_C^L &= \{(\rho_1, \{S_{l_2}, S_{l_3}, S_{l_5}\}), (\rho_2, \{S_{l_4}, S_{l_5}\})\} \end{aligned}$$

Then, $(f_C^L \cap g_C^L) = \{(\rho_1, \{S_{l_2}\}), (\rho_2, \phi)\} \cdot (f_C^L)_\circ = \phi_A$ and $(g_C^L)_\circ = f_{2A}^L$, which implies $(f_C^L)_\circ \cup (g_C^L)_\circ = f_{2A}^L \cdot (f_C^L \cap g_C^L)_\circ$ is f_{1A}^L .

$$\text{Hence, } (f_A^L)_\circ \cup (g_A^L)_\circ \subset (f_A^L \cap g_A^L)_\circ.$$

For Equal condition,

Now we take any two $\mathfrak{S}, \mathfrak{L}$ from the figure:2,

$$\begin{aligned} f_C^L &= \{(\rho_1, \{S_{l_5}, S_{l_7}\}), (\rho_2, \{S_{l_4}, S_{l_7}\})\} \text{ and} \\ g_C^L &= \{(\rho_1, \{S_{l_3}, S_{l_5}, S_{l_7}\}), (\rho_2, \{S_{l_4}, S_{l_5}, S_{l_7}\})\} \\ \text{Then, } (f_C^L \cap g_C^L) &= \{(\rho_1, \{S_{l_5}, S_{l_7}\}), ((\rho_2, \{S_{l_4}, S_{l_7}\}))\}. \end{aligned}$$

$(f_C^L)_\circ = f_{2A}^L$ and $(g_C^L)_\circ = f_{2A}^L$, which implies $(f_C^L)_\circ \cup (g_C^L)_\circ = f_{2A}^L \cdot (f_C^L \cap g_C^L)_\circ$ is f_{2A}^L .

$$\text{Hence, } (f_A^L)_\circ \cup (g_A^L)_\circ = (f_A^L \cap g_A^L)_\circ.$$

Example 3.4. *Now the given example to show that the statement of theorem 2(iii) may be strict or equal, Let us consider the lattice and $\mathfrak{S}, \mathfrak{L}, \mathfrak{T}$ given in Example: 3.3*

For Equal condition,

Now we take any two $\mathfrak{S}, \mathfrak{L}$ from the figure:2,

$$\begin{aligned} f_D^L &= \{(\rho_1, \{S_{l_3}, S_{l_6}\}), (\rho_2, \{S_{l_4}, S_{l_5}, S_{l_6}\})\} \text{ and} \\ g_D^L &= \{(\rho_1, \{S_{l_1}, S_{l_3}, S_{l_5}, S_{l_6}\}), (\rho_2, \{S_{l_4}, S_{l_5}, S_{l_6}\})\} \end{aligned}$$

$$(f_D^L)_\circ = \phi_A \text{ and } (g_D^L)_\circ = \phi_A.$$

$$\text{Hence, } f_A^L \subseteq g_A^L \text{ implies } (f_A^L)_\circ = (g_A^L)_\circ.$$

For inclusion condition,

Now we take any two $\mathfrak{S}.\mathfrak{L}$ from the figure:2,

$$f_B^L = \{(\rho_1, \{S_{l_4}, S_{l_5}\}), (\rho_2, \{S_{l_2}, S_{l_3}, S_{l_6}, S_{l_7}\})\} \text{ and } \\ g_B^L = \{(\rho_1, \{S_{l_2}, S_{l_4}, S_{l_5}, S_{l_7}\}), (\rho_2, \{S_{l_1}, S_{l_2}, S_{l_3}, S_{l_5}, S_{l_6}, S_{l_7}\})\}$$

$$(f_B^L)_\circ = f_{1A}^L \text{ and } (g_B^L)_\circ = f_{2A}^L.$$

Hence, $f_A^L \subseteq g_A^L$ implies $(f_A^L)_\circ \supset (g_A^L)_\circ$.

Definition 3.3. In $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}.\mathfrak{S}$, then the $\mathfrak{S}.\mathfrak{L}$ - $\mathfrak{B}$ of $\mathfrak{S}.\mathfrak{L}$ f_A^L is denoted by $(f_A^L)^B$ and is defined as $(f_A^L)^B = \overline{f_A^L} \cap \overline{(f_A^L)^C}$.

Theorem 3.3. Let (L, θ, A) be a $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}.\mathfrak{S}$:

$$(i) (f_A^L)^B \cap (f_A^L)^\circ = f_\phi^L$$

$$(ii) (f_A^L)^B \cap (f_A^L)_\circ = f_\phi^L$$

Proof

$$(i) (f_A^L)^B \cap (f_A^L)^\circ = (\overline{f_A^L} \cap \overline{(f_A^L)^C}) \cap (f_A^L)^\circ \\ = \overline{f_A^L} \cap \overline{(f_A^L)^C} \cap (f_A^L)^\circ = f_\phi^L$$

$$(ii) (f_A^L)^B \cap (f_A^L)_\circ = \overline{f_A^L} \cap \overline{(f_A^L)^C} \cap (f_A^L)^C_\circ = \overline{f_A^L} \cap \overline{(f_A^L)^C} \cap ((f_A^L))_\circ^C \\ = f_\phi^L$$

Example 3.5. Now the given example for find the boundary, Let us consider the lattice and $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}$ given in Example: 3.1

Now we take any $\mathfrak{S}.\mathfrak{L}$ from the figure:1,

$$f_C^L = \{(\rho_1, \{S_{l_2}, S_{l_8}\}), (\rho_2, \{S_{l_1}, S_{l_3}\})\} \text{ Then, } (f_C^L)^B = (f_{4A}^L)^C$$

Definition 3.4. Let (L, θ, A) be a $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}.\mathfrak{S}$ over L , then the $\mathfrak{S}.\mathfrak{L}$ - $\mathfrak{Bor}$ of $\mathfrak{S}.\mathfrak{L}$ f_A^L is denoted by $(f_A^L)^\bullet$ and is defined as $(f_A^L)^\bullet = f_A^L - (f_A^L)_\circ$.

Theorem 3.4. Let (L, θ, A) be a $\mathfrak{S}.\mathfrak{L}.\mathfrak{T}.\mathfrak{S}$. Then the following hold:

- (i) $(f_A^L)^\bullet = A \cap \overline{(L_A - f_A^L)}$
- (ii) $(\phi_A^L)^\bullet = \phi_A^L$
- (iii) $(f_A^L)^\bullet \subseteq ((f_A^L)^\circ)^C$
- (iv) $(f_A^L)^\bullet \subseteq f_A^L \subseteq \overline{f_A^L}$

Proof

- (i) $f_A^L \cap ((f_A^L)^\circ)^C = f_A^L \cap \overline{(f_A^L)^C} = f_A^L \cap \overline{(L_A - f_A^L)}$
- (ii) $\phi_A^L \cap ((\phi_A^L)^\circ)^C = (\phi_A^L) \cap \overline{(\phi_A^L)^C} = \phi_A^L$
- (iii) $f_A^L - (f_A^L)^\circ = f_A^L \cap ((f_A^L)^\circ)^C \subseteq ((f_A^L)^\circ)^C$
- (iv) By definition of $(f_A^L)^\bullet$, $(f_A^L)^\bullet \subseteq f_A^L$.
we know that, $f_A^L \subset \overline{f_A^L}$ Therefore, $(f_A^L)^\bullet \subseteq f_A^L \subseteq \overline{f_A^L}$

Example 3.6. Now the given example to show that the statement of theorem 4(iv) may be strict or equal, Let us consider the lattice and $\mathfrak{S.L.T}$ given in Example: 3.3

We choose any two $\mathfrak{S.L}$ from the figure:2,

$$g_B^L = \{(\rho_1, \{S_{l_2}, S_{l_3}, S_{l_5}, S_{l_6}\}), (\rho_2, \{S_{l_4}, S_{l_5}, S_{l_6}\})\}$$

Now, the Closure of g_B^L is L_A , Then border of g_B^L , is ϕ_A

$$\text{Hence, } (g_B^L)^\bullet \subset g_B^L \subset \overline{g_B^L}.$$

4 Conclusions

In the present work, we defined and discussed some $\mathfrak{S.L}$ – sets of $\mathfrak{S.L.T.S}$. We extended some basic results relating to $\mathfrak{S.L} - \mathfrak{I}$, $\mathfrak{S.L} - \mathfrak{E}$, $\mathfrak{S.L} - \mathfrak{B}$, and $\mathfrak{S.L} - \mathfrak{Bor}$ of $\mathfrak{S.L.T.S}$. In the interior section, idempotent and monotonicity results are held. Formerly the intersection of the boundary and interior soft lattice gives the null set and the intersection of the boundary and exterior soft lattice should not give the non-empty soft sets. In end, this paper is the inception of a novel

structure. Further, we learned a few viewpoints, it will be needed to carry out a new seeking work to build future applications.

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Families of mappings satisfying a mixed implicit relation

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Abstract

A fixed point for a suitable map or operator is identical to the presence of a solution to a theoretical or real-world problem. As a result, fixed points are crucial in many fields of mathematics, science and engineering. The purpose of this paper is to prove unique common fixed point theorems for families of weakly compatible mappings. Given mappings satisfy common limit range property and a mixed implicit relation. Our results generalize, extend and improve the results of Imdad (2013) and Popa (2018). We provide an application for integral type contraction condition. An example is also mentioned to check the authenticity of our results.

Keywords: common fixed point, weakly compatible mappings, mixed implicit relation, almost altering distance, common limit range property.

2020 AMS subject classifications: 47H10, 54H25. ¹

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1 Introduction and Preliminaries

Fixed point theory is an important tool of modern mathematics as it helps to find a unique fixed point of multi-valued and single-valued mappings by restricting the condition of the domain of the function. It also helps to find the results of many differential as well as integral equations which can further be used to solve many industrial based problems. The most popular tool in fixed point theory is Banach contraction [6] principle which states that every contraction mapping on a complete metric space has a unique fixed point. Various authors have extended and generalized this principle in various directions. In 1976, Jungck [9] used the concept of commuting maps to prove a common fixed point theorem. Several authors have investigated various concepts of minimal commuting maps. A pair of self-mappings $(\mathcal{P}, \mathcal{Q})$ on a metric space $(\mathcal{X}, d)$ is said to be compatible [10] if $\lim_{n \rightarrow \infty} d(\mathcal{P}\mathcal{Q}u_n, \mathcal{Q}\mathcal{P}u_n) = 0$, whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}u_n = u$ for some $u \in \mathcal{X}$. A pair of self-mappings $(\mathcal{P}, \mathcal{Q})$ on a metric space $(\mathcal{X}, d)$ is called weakly compatible [11] if $\mathcal{P}$ and $\mathcal{Q}$ commute at their points of coincidence. Pant ([13], [14], [15]) initiated the study of common fixed points for non-compatible mappings. Further, Aamri and El-Moutawakil [1] introduced $(E.A)$ property as a generalization of non-compatible mappings. A pair of self-mappings $(\mathcal{P}, \mathcal{Q})$ on a metric space $(\mathcal{X}, d)$ is said to satisfy $(E.A)$ property [1] if there exists a sequence $\{u_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}u_n = u$, for some $u \in \mathcal{X}$.

In 2011, Sintunavarat and Kumam [22] introduced the concept of common limit range property.

Definition 1.1. [22] A pair of self-mappings $(\mathcal{P}, \mathcal{Q})$ on a metric space $(\mathcal{X}, d)$ is said to satisfy the common limit range property with respect to $\mathcal{Q}$, denoted by $CLR_{\mathcal{Q}}$, if there exists a sequence $\{u_n\}$ in $\mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}u_n = u, \text{ for some } u \in \mathcal{Q}(\mathcal{X}).$$

Thus one can note that the mappings $\mathcal{P}$ and $\mathcal{Q}$ satisfying property $(E.A)$ along with the closedness of the subspace $\mathcal{Q}(\mathcal{X})$ always have $CLR_{\mathcal{Q}}$ property with respect to $\mathcal{Q}$.

In 2013, Imdad *et al.* [8] extended the notion of common limit range property for pairs of self mappings.

Definition 1.2. [8] Two pairs of self-mappings $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{S}, \mathcal{T})$ on a metric space $(\mathcal{X}, d)$ are said to satisfy common limit range property with respect to $\mathcal{Q}$ and $\mathcal{T}$, denoted by $(CLR)_{(\mathcal{Q}, \mathcal{T})}$, if there exists two sequences $\{u_n\}$ and $\{v_n\}$ in $\mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}u_n = \lim_{n \rightarrow \infty} \mathcal{S}v_n = \lim_{n \rightarrow \infty} \mathcal{T}v_n = t,$$

where $t \in \mathcal{Q}(\mathcal{X}) \cap \mathcal{T}(\mathcal{X})$.

In 2018, Popa *et al.* [17] introduced a new type of limit range property as follows.

Definition 1.3. [17] Let $\mathcal{P}, \mathcal{Q}, \mathcal{T}$ be self mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q})$ is said to satisfy common limit range property with respect to $\mathcal{T}$ if there exists a sequence $\{u_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}u_n = u$, for some $u \in \mathcal{Q}(\mathcal{X}) \cap \mathcal{T}(\mathcal{X})$.

Now we extend the definition 1.3 for families of mappings.

Definition 1.4. Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_{2n}$ and $\mathcal{P}$ be self mappings of a metric space $(\mathcal{X}, d)$. The pair $(\mathcal{P}, \mathcal{Q}_1 \mathcal{Q}_3 \dots \mathcal{Q}_{2n-1})$ is said to satisfy common limit range property with respect to $\mathcal{Q}_2 \mathcal{Q}_4 \dots \mathcal{Q}_{2n}$ if there exists a sequence $\{u_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} \mathcal{P}u_n = \lim_{n \rightarrow \infty} \mathcal{Q}_1 \mathcal{Q}_3 \dots \mathcal{Q}_{2n-1} u_n = u$, for some $u \in \mathcal{Q}_1 \mathcal{Q}_3 \dots \mathcal{Q}_{2n-1}(\mathcal{X}) \cap \mathcal{Q}_2 \mathcal{Q}_4 \dots \mathcal{Q}_{2n}(\mathcal{X})$.

Remark 1.1. [17] Let $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{T}$ be self mappings of a metric space $(\mathcal{X}, d)$. If the pairs $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{S}, \mathcal{T})$ satisfy the common limit range property with respect to $\mathcal{Q}$ and $\mathcal{T}$, then $(\mathcal{P}, \mathcal{Q})$ satisfy the limit range property with respect to $\mathcal{T}$, but the converse does not hold.

Boyd and Wong [5] introduced ϕ contraction condition and generalized the Banach contraction principle using this contraction. A self mapping $\mathcal{P}$ on a complete metric space $(\mathcal{X}, d)$ is said to satisfy ϕ contraction if $d(\mathcal{P}\alpha, \mathcal{P}\beta) \leq \phi(d(\alpha, \beta))$, for all $\alpha, \beta \in \mathcal{X}$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. The theorems of existence of fixed points for self mappings in Hilbert spaces satisfying ϕ -weak contraction were studied by Alber and Guerre-Delabriere [3]. Further Rhoades [21] extended this concept in complete metric space. Some fixed point results are proved in [7], [8] and in other papers for mappings with common limit range property satisfying (ϕ, ψ) -weak contractive conditions.

The following theorem is proved in [8].

Theorem 1.1. [8] Let $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{T}$ be self mappings of a metric space $(\mathcal{X}, d)$ satisfying

$$\psi(d(\mathcal{P}x, \mathcal{Q}y)) \leq \psi(m(x, y)) - \phi(m(x, y)),$$

for all $x, y \in \mathcal{X}$ and for some ϕ, ψ ,

where $m(x, y) = \max\{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{S}x, \mathcal{P}x), d(\mathcal{T}y, \mathcal{Q}y), d(\mathcal{S}x, \mathcal{Q}y), d(\mathcal{T}y, \mathcal{P}x)\}$ and $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$ such that ϕ is a lower semi-continuous function and $\phi^{-1}(0) = 0$ and ψ is a non-decreasing continuous function with $\psi^{-1}(0) = 0$.

If the pairs $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{S}, \mathcal{T})$ satisfy the $(CLR)_{(\mathcal{S}, \mathcal{T})}$ property and are weakly compatible, then $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Definition 1.5. [12] An altering distance is a function $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying:

- (ψ_1) : ψ is increasing and continuous,
- (ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.6. [18] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is an almost altering distance if it satisfies:

- (ψ'_1) : ψ is continuous,
- (ψ'_2) : $\psi(t) = 0$ if and only if $t = 0$.

Example 1.1. Define a function $\psi : [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(t) = \begin{cases} 2t, & t \in [0, 1] \\ \frac{1}{1+t}, & t \in (1, \infty). \end{cases}$$

Here we note that every altering distance is an almost altering distance, but converse is not true.

Various authors have unified several common fixed point theorems by using implicit functions. In 2008, Ali and Imdad [2] introduced a new class of implicit functions.

Definition 1.7. [2] Let $\mathcal{F}$ be the family of lower semi-continuous functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ which are satisfying:

- (F_1) for all $u > 0$, $F(u, 0, u, 0, 0, u) > 0$;
- (F_2) for all $u > 0$, $F(u, 0, 0, u, u, 0) > 0$;
- (F_3) for all $u > 0$, $F(u, u, 0, 0, u, u) > 0$;

Definition 1.8. [17] Let $\mathcal{F}_D$ be the set of all lower semi-continuous functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ which are satisfying:

- (F_{1D}) for all $u > 0$, $F(u, 0, u, 0, 0, u) \geq 0$;
- (F_{2D}) for all $u > 0$, $F(u, 0, 0, u, u, 0) \geq 0$;
- (F_{3D}) for all $u > 0$, $F(u, u, 0, 0, u, u) \geq 0$;

Now we provide some examples in support of definition 1.8.

1. Let $F(u_1, \dots, u_6) = u_1 - t \max\{u_2, u_3, u_4, u_5, u_6\}$, where $t \in [0, 1]$.
2. Let $F(u_1, \dots, u_6) = u_1 - t \max\{u_2, u_3, u_4, \frac{u_5+u_6}{3}\}$, where $t \in [0, 1]$.
3. Let $F(u_1, \dots, u_6) = u_1 - \alpha \max\{u_2, u_3, u_4\} - \beta(u_5 + u_6)$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$.
4. Let $F(u_1, \dots, u_6) = u_1 - \alpha \max\{u_2, u_3, u_4, \frac{1}{2}(u_5 + u_6), \frac{u_3u_4}{1+u_2}, \frac{u_5u_6}{1+u_1}\}$, where $\alpha \in [0, 1]$.

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5. Let $F(u_1, \dots, u_6) = u_1 - \max\{cu_2, cu_3, cu_4, au_5 + bu_6\}$, where $c > 0$, $a, b \geq 0$ and $a + b + c \leq 1$.

Definition 1.9. [17] Let $\mathcal{G}_D$ be the set of all lower semi-continuous functions $G : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ such that $G(t_1, \dots, t_5) > 0$ if one of $t_1, \dots, t_5 > 0$.

The following functions belong to the set $\mathcal{G}_D$.

1. $G(t_1, \dots, t_5) = \max\{t_1, \dots, t_5\}$.
2. $G(t_1, \dots, t_5) = \max\{t_1, \frac{t_2+t_3}{2}, \frac{t_4+t_5}{2}\}$.
3. $G(t_1, \dots, t_5) = t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2$.
4. $G(t_1, \dots, t_5) = \frac{1}{t_1+t_2+t_3+t_4+t_5}$.

Definition 1.10. [17] A function $\phi(u_1, \dots, u_6, t_1, \dots, t_5) = F(u_1, \dots, u_6) + G(t_1, \dots, t_5)$ is called a mixed implicit relation.

The aim of this paper is to prove general fixed point theorems for families of weakly compatible mappings with common limit range property satisfying a mixed implicit relation. Our results generalize, extend and improve the results of Popa [17] and Imdad [8].

2 Main Results

In 2018, Popa *et al.* [17] proved the following theorem.

Theorem 2.1. [17] Let $(\mathcal{X}, d)$ be a metric space and P, Q, S and T be four self mappings on $\mathcal{X}$ satisfying

$$F(\psi(d(Px, Qy)), \psi(d(Sx, Ty)), \psi(d(Sx, Px)), \psi(d(Ty, Qy)), \psi(d(Sx, Qy)), \psi(d(Ty, Px))) + G(\psi(d(Sx, Ty)), \psi(d(Sx, Px)), \psi(d(Ty, Qy)), \psi(d(Sx, Qy)), \psi(d(Ty, Px))) \leq 0,$$

for all $x, y \in \mathcal{X}$, for some $F \in \mathcal{F}_D$, $G \in \mathcal{G}_D$ and ψ is an almost altering distance. If the pairs (P, S) and (Q, T) are weakly compatible and (P, S) and T satisfy $(CLR)_{(P,S)T}$ property, then P, Q, S and T have a unique common fixed point.

Now we extend the Theorem 2.1 for any even number of weakly compatible mappings.

Theorem 2.2. Let $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 be self mappings on a metric space $(\mathcal{X}, d)$, satisfying the following conditions:

$$(C1) \quad Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2,$$

$$Q_2Q_4(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})Q_2Q_4,$$

$\vdots$

$$Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2},$$

$$P_1(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})P_1,$$

$$P_1(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})P_1,$$

$\vdots$

$$P_1Q_{2n} = Q_{2n}P_1,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1,$$

$$Q_1Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1Q_3,$$

$\vdots$

$$Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3},$$

$$P_0(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})P_0,$$

$$P_0(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})P_0,$$

$\vdots$

$$P_0Q_{2n-1} = Q_{2n-1}P_0,$$

(C2) the pairs $(P_0, Q_1 \dots Q_{2n-1})$ and $(P_1, Q_2 \dots Q_{2n})$ are weakly compatible and

$(P_0, Q_1 \dots Q_{2n-1})$ and $Q_2 \dots Q_{2n}$ satisfy $(CLR)_{(P_0, Q_1 \dots Q_{2n-1})Q_2 \dots Q_{2n}}$ property,

(C3)

$$\begin{aligned} & F(\psi(d(P_0x, P_1y)), \psi(d(Q_1Q_3 \dots Q_{2n-1}x, Q_2Q_4 \dots Q_{2n}y)), \psi(d(Q_1Q_3 \dots Q_{2n-1}x, P_0x)), \\ & \quad \psi(d(Q_2Q_4 \dots Q_{2n}y, P_1y)), \psi(d(Q_1Q_3 \dots Q_{2n-1}x, P_1y)), \psi(d(Q_2Q_4 \dots Q_{2n}y, P_0x))) \\ & \quad + G(\psi(d(Q_1Q_3 \dots Q_{2n-1}x, Q_2Q_4 \dots Q_{2n}y)), \psi(d(Q_1Q_3 \dots Q_{2n-1}x, P_0x)), \\ & \quad \psi(d(Q_2Q_4 \dots Q_{2n}y, P_1y)), \psi(d(Q_1Q_3 \dots Q_{2n-1}x, P_1y)), \psi(d(Q_2Q_4 \dots Q_{2n}y, P_0x))) \leq 0, \end{aligned}$$

for all $x, y \in \mathcal{X}$, some $F \in \mathcal{F}_D$, $G \in \mathcal{G}_D$ and ψ is an almost altering distance. Then $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 have a unique common fixed point in $\mathcal{X}$.

Proof. Let $\mathcal{Q}'_1 = Q_1Q_3 \dots Q_{2n-1}$ and $\mathcal{Q}'_2 = Q_2Q_4 \dots Q_{2n}$. Since $(P_0, \mathcal{Q}'_1)$ and $\mathcal{Q}'_2$ satisfy $(CLR)_{(P_0, \mathcal{Q}'_1)\mathcal{Q}'_2}$ property, there exists a sequence $\{u_n\}$ in $\mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} P_0u_n = \lim_{n \rightarrow \infty} \mathcal{Q}'_1u_n = \lim_{n \rightarrow \infty} Q_1Q_3 \dots Q_{2n-1}u_n = z,$$

where $z \in \mathcal{Q}'_1(\mathcal{X}) \cap \mathcal{Q}'_2(\mathcal{X}) = Q_1 \dots Q_{2n-1}(\mathcal{X}) \cap Q_2 \dots Q_{2n}(\mathcal{X})$.

Since $z \in Q_2Q_4 \dots Q_{2n}(\mathcal{X})$, there exists $u \in \mathcal{X}$ such that $z = Q_2Q_4 \dots Q_{2n}u$. Using

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(C3) with $x = u_n$ and $y = u$, we get

$$\begin{aligned} & F(\psi(d(P_0u_n, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, Q_2Q_4\ldots Q_{2n}u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, P_0u_n)), \\ & \psi(d(Q_2Q_4\ldots Q_{2n}u, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, P_1u)), \psi(d(Q_2Q_4\ldots Q_{2n}u, P_0u_n))) \\ & + G(\psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, Q_2Q_4\ldots Q_{2n}u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, P_0u_n)), \\ & \psi(d(Q_2Q_4\ldots Q_{2n}u, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}u_n, P_1u)), \psi(d(Q_2Q_4\ldots Q_{2n}u, P_0u_n))) \leq 0. \end{aligned}$$

Taking limits as $n \rightarrow \infty$, we have

$$\begin{aligned} & F(\psi(d(z, P_1u)), 0, 0, \psi(d(z, P_1u)), \psi(d(z, P_1u)), 0) + G(0, 0, \psi(d(z, P_1u)), \\ & \psi(d(z, P_1u)), 0) \leq 0. \end{aligned}$$

If $d(z, P_1u) > 0$, then

$$G(0, 0, \psi(d(z, P_1u)), \psi(z, P_1u), 0) > 0,$$

which implies that

$$F(\psi(d(z, P_1u)), 0, 0, \psi(d(z, P_1u)), \psi(d(z, P_1u), 0) < 0,$$

a contradiction of (F_{2D}) . Hence $d(z, P_1u) = 0$ i.e., $z = P_1u = Q_2Q_4\ldots Q_{2n}u$. Since $(P_1, Q_2Q_4\ldots Q_{2n})$ is weakly compatible, we have

$$P_1z = P_1Q_2Q_4\ldots Q_{2n}u = Q_2Q_4\ldots Q_{2n}P_1u = Q_2Q_4\ldots Q_{2n}z.$$

Since $z \in Q_1Q_3\ldots Q_{2n-1}(\mathcal{X})$, which implies $z = Q_1Q_3\ldots Q_{2n-1}v$ for some $v \in \mathcal{X}$. On putting $x = v$ and $y = u$ in (C3), we have

$$\begin{aligned} & F(\psi(d(P_0v, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, Q_2Q_4\ldots Q_{2n}u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_0v)), \\ & \psi(d(Q_2Q_4\ldots Q_{2n}u, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_1u)), \psi(d(Q_2Q_4\ldots Q_{2n}u, P_0v))) \\ & + G(\psi(d(Q_1Q_3\ldots Q_{2n-1}v, Q_2Q_4\ldots Q_{2n}u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_0v)), \\ & \psi(d(Q_2Q_4\ldots Q_{2n}u, P_1u)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_1u)), \psi(d(Q_2Q_4\ldots Q_{2n}u, P_0v))) \leq 0. \end{aligned}$$

On simplification, we get

$$\begin{aligned} & F(\psi(d(P_0v, z)), 0, \psi(d(P_0v, z)), 0, 0, \psi(d(P_0v, z))) + G(0, \psi(d(P_0v, z)), 0, 0, \\ & \psi(d(P_0v, z))) \leq 0. \end{aligned}$$

If $d(P_0v, z) > 0$, then

$$G(0, \psi(d(P_0v, z)), 0, 0, \psi(d(P_0v, z))) > 0.$$

Therefore, we obtain

$$F(\psi(d(P_0v, z)), 0, \psi(d(P_0v, z)), 0, \psi(d(P_0v, z))) < 0,$$

a contradiction of (F_{1D}) . Hence $d(P_0v, z) = 0$, which implies that $z = P_0v = Q_1Q_3\ldots Q_{2n-1}v$. Since $(P_0, Q_1Q_3\ldots Q_{2n-1})$ is weakly compatible, we get

$$P_0z = P_0Q_1Q_3\ldots Q_{2n-1}v = Q_1Q_3\ldots Q_{2n-1}P_0v = Q_1Q_3\ldots Q_{2n-1}z.$$

Now, we prove that $z = P_1z$. On putting $x = v$ and $y = z$ in $(C3)$, we get

$$\begin{aligned} &F(\psi(d(P_0v, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, Q_2Q_4\ldots Q_{2n}z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_0v)), \\ &\psi(d(Q_2Q_4\ldots Q_{2n}z, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_1z)), \psi(d(Q_2Q_4\ldots Q_{2n}z, P_0v))) \\ &+ G(\psi(d(Q_1Q_3\ldots Q_{2n-1}v, Q_2Q_4\ldots Q_{2n}z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_0v)), \\ &\psi(d(Q_2Q_4\ldots Q_{2n}z, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}v, P_1z)), \psi(d(Q_2Q_4\ldots Q_{2n}z, P_0v))) \leq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &F(\psi(d(z, P_1z)), \psi(d(z, P_1z)), 0, 0, \psi(d(z, P_1z)), \psi(d(z, P_1z))) + G(\psi(d(z, P_1z)), 0, 0, \\ &\psi(d(P_1z, z)), \psi(d(z, P_1z))) \leq 0. \end{aligned}$$

If $d(z, P_1z) > 0$, then

$$G(\psi(d(z, P_1z)), 0, 0, \psi(d(P_1z, z)), \psi(d(z, P_1z))) > 0.$$

Thus from above, we get

$$F(\psi(d(z, P_1z)), \psi(d(z, P_1z)), 0, 0, \psi(d(z, P_1z)), \psi(d(z, P_1z))) < 0,$$

a contradiction of (F_{3D}) . Hence $d(z, P_1v) = 0$ i.e., $P_1z = z$ and hence $P_1z = Q_2Q_4\ldots Q_{2n}z = z$.

Further on putting $x = y = z$ in $(C3)$, we get

$$\begin{aligned} &F(\psi(d(P_0z, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}z, Q_2Q_4\ldots Q_{2n}z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}z, P_0z)), \\ &\psi(d(Q_2Q_4\ldots Q_{2n}z, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}z, P_1z)), \psi(d(Q_2Q_4\ldots Q_{2n}z, P_0z))) \\ &+ G(\psi(d(Q_1Q_3\ldots Q_{2n-1}z, Q_2Q_4\ldots Q_{2n}z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}z, P_0z)), \\ &\psi(d(Q_2Q_4\ldots Q_{2n}z, P_1z)), \psi(d(Q_1Q_3\ldots Q_{2n-1}z, P_1z)), \psi(d(Q_2Q_4\ldots Q_{2n}z, P_0z))) \leq 0. \end{aligned}$$

On simplification, we have

$$\begin{aligned} &F(\psi(d(P_0z, z)), \psi(d(P_0z, z)), 0, 0, \psi(d(P_0z, z)), \psi(d(P_0z, z))) + G(\psi(d(P_0z, z)), 0, \\ &0, \psi(d(P_0z, z)), \psi(d(P_0z, z))) \leq 0. \end{aligned}$$

Families of mappings satisfying a mixed implicit relation

If $d(P_0, z) > 0$, then

$$G(\psi(d(P_0z, z)), 0, 0, \psi(d(P_0z, z)), \psi(d(P_0z, z))) > 0,$$

which implies that

$$F(\psi(d(P_0z, z)), \psi(d(P_0z, z)), 0, 0, \psi(d(P_0z, z)), \psi(d(P_0z, z))) < 0,$$

a contradiction of (F_{3D}) . Hence $d(P_0z, z) = 0$ i.e., $P_0z = z$ and hence $P_0z = Q_1Q_3...Q_{2n-1}z = z$.

On putting $x = z$ and $y = Q_4...Q_{2n}z$ in $(C3)$ and using $(C1)$, $\mathcal{Q}'_1 = Q_1Q_3...Q_{2n-1}$ and $\mathcal{Q}'_2 = Q_2Q_4...Q_{2n}$, we get

$$\begin{aligned} &F(\psi(d(P_0z, P_1Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_1z, \mathcal{Q}'_2Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_1z, P_0z))), \\ &\psi(d(\mathcal{Q}'_2Q_4...Q_{2n}z, P_1Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_1z, P_1Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_2Q_4...Q_{2n}z, P_0z))) \\ &+ G(\psi(d(\mathcal{Q}'_1z, \mathcal{Q}'_2Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_1z, P_0z)), \psi(d(\mathcal{Q}'_2Q_4...Q_{2n}z, P_1Q_4...Q_{2n}z)), \\ &\psi(d(\mathcal{Q}'_1z, P_1Q_4...Q_{2n}z)), \psi(d(\mathcal{Q}'_2Q_4...Q_{2n}z, P_0z))) \leq 0. \end{aligned}$$

From this we get

$$\begin{aligned} &F(\psi(d(z, Q_4...Q_{2n}z)), \psi(d(z, Q_4...Q_{2n}z)), 0, 0, \psi(d(z, Q_4...Q_{2n}z)), \psi(d(Q_4...Q_{2n}z, z))) \\ &+ G(\psi(d(z, Q_4...Q_{2n}z)), 0, 0, \psi(d(z, Q_4...Q_{2n}z)), \psi(d(Q_4...Q_{2n}z, z))) \leq 0. \end{aligned}$$

If $(d(z, Q_4...Q_{2n}z)) > 0$ then

$$G(\psi(d(z, Q_4...Q_{2n}z)), 0, 0, \psi(d(z, Q_4...Q_{2n}z)), \psi(d(Q_4...Q_{2n}z, z))) > 0.$$

Therefore, we have

$$\begin{aligned} &F(\psi(d(z, Q_4...Q_{2n}z)), \psi(d(z, Q_4...Q_{2n}z)), 0, 0, \psi(d(z, Q_4...Q_{2n}z)), \\ &\psi(d(Q_4...Q_{2n}z, z))) < 0, \end{aligned}$$

a contradiction to (F_{3D}) . Hence $d(z, Q_4...Q_{2n}z) = 0$ i.e., $Q_4...Q_{2n}z = z$. Hence $Q_2Q_4...Q_{2n}z = Q_2z = z$. Continuing like this, we have

$$P_1z = Q_2z = Q_4z = ... = Q_{2n}z = z. \quad (1)$$

On putting $x = Q_3...Q_{2n-1}z$ and $y = z$ in $(C3)$ and using $(C1)$, $\mathcal{Q}'_1 = Q_1Q_3...Q_{2n-1}$ and $\mathcal{Q}'_2 = Q_2Q_4...Q_{2n}$, we get

$$\begin{aligned} &F(\psi(d(P_0Q_3...Q_{2n-1}z, P_1z)), \psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, \mathcal{Q}'_2z))), \\ &\psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, P_0Q_3...Q_{2n-1}z)), \psi(d(\mathcal{Q}'_2z, P_1z)), \\ &\psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, P_1z)), \psi(d(\mathcal{Q}'_2z, P_0Q_3...Q_{2n-1}z))) \\ &+ G(\psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, \mathcal{Q}'_2z)), \psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, P_0Q_3...Q_{2n-1}z)), \\ &\psi(d(\mathcal{Q}'_2z, P_1z)), \psi(d(\mathcal{Q}'_1Q_3...Q_{2n-1}z, P_1z)), \psi(d(\mathcal{Q}'_2z, P_0Q_3...Q_{2n-1}z))) \leq 0, \end{aligned}$$

which implies that

$$F(\psi(d(Q_3 \dots Q_{2n-1}z, z)), \psi(d(Q_3 \dots Q_{2n-1}z, z)), 0, 0, \psi(d(Q_3 \dots Q_{2n-1}z, z)), \\ \psi(d(z, Q_3 \dots Q_{2n-1}z))) + G(d(\psi(d(Q_3 \dots Q_{2n-1}z, z)), 0, 0, \psi(d(Q_3 \dots Q_{2n-1}z, z))), \\ \psi(d(z, Q_3 \dots Q_{2n-1}z))) \leq 0.$$

If $d(z, Q_3 \dots Q_{2n-1}z) > 0$ then

$$G(d(\psi(d(Q_3 \dots Q_{2n-1}z, z)), 0, 0, \psi(d(Q_3 \dots Q_{2n-1}z, z))), \psi(d(z, Q_3 \dots Q_{2n-1}z))) > 0.$$

Thus from above, we obtain

$$F(\psi(d(Q_3 \dots Q_{2n-1}z, z)), \psi(d(Q_3 \dots Q_{2n-1}z, z)), 0, 0, \psi(d(Q_3 \dots Q_{2n-1}z, z)), \\ \psi(d(z, Q_3 \dots Q_{2n-1}z))) < 0,$$

a contradiction to (F_{3D}) . Hence $d(z, Q_3 \dots Q_{2n-1}z) = 0$ i.e., $Q_3 \dots Q_{2n-1}z = z$. Hence $Q_1 Q_3 \dots Q_{2n-1}z = Q_1 z = z$. Continuing like this, we have

$$P_0 z = Q_1 z = Q_3 z = \dots = Q_{2n-1} z = z. \quad (2)$$

Hence from (1) and (2), we have

$$P_0 z = P_1 = Q_1 z = Q_2 z = Q_3 z = \dots = Q_{2n-1} z = Q_{2n} z = z.$$

Therefore, z is a common fixed point of the given self mappings.

Uniqueness. Let w be another fixed point of the given mappings. Then $P_0 w = P_1 w = Q_1 w = Q_2 w = Q_3 w = \dots = Q_{2n} w = w$. Suppose that $z \neq w$. Putting $x = z$ and $y = w$ in condition $(C3)$, we have

$$F(\psi(d(z, w)), \psi(d(z, w)), 0, 0, \psi(d(z, w)), \psi(d(w, z))) \\ + G(\psi(d(z, w)), 0, 0, \psi(d(z, w)), \psi(d(w, z))) \leq 0.$$

If $d(z, w) > 0$, then

$$G(\psi(d(z, w)), 0, 0, \psi(d(z, w)), \psi(d(w, z))) > 0.$$

Therefore, we obtain

$$F(\psi(d(z, w)), \psi(d(z, w)), 0, 0, \psi(d(z, w)), \psi(d(w, z))) < 0,$$

a contradiction of (F_{3D}) . Hence $z = w$. Therefore, z is a unique common fixed point of the given mappings. $\square$

Now we prove a theorem for families of mappings.

Families of mappings satisfying a mixed implicit relation

Theorem 2.3. Let $\{S_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space $(\mathcal{X}, d)$. Suppose that there exists a fixed $\beta \in J$ such that:

$$(C4) \quad Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2, \\ Q_2Q_4(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})Q_2Q_4,$$

$\vdots$

$$Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2},$$

$$S_\beta(Q_4 \dots S_{2n}) = (S_4 \dots S_{2n})S_\beta,$$

$$S_\beta(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})S_\beta,$$

$\vdots$

$$S_\beta Q_{2n} = Q_{2n} S_\beta,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1,$$

$$Q_1Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1Q_3,$$

$\vdots$

$$Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3},$$

$$S_\alpha(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})S_\alpha,$$

$$S_\alpha(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})S_\alpha,$$

$\vdots$

$$S_\alpha S_{2n-1} = S_{2n-1} S_\alpha,$$

(C5) the pairs $(S_\alpha, Q_1 \dots Q_{2n-1})$ and $(S_\beta, Q_2 \dots Q_{2n})$ are weakly compatible and

$(S_\alpha, Q_1 \dots Q_{2n-1})$ and $Q_2 \dots Q_{2n}$ satisfy $(CLR)_{(S_\alpha, Q_1 \dots Q_{2n-1})Q_2 \dots Q_{2n}}$ property,

(C6)

$$F(\psi(d(S_\alpha x, S_\beta y)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)), \\ \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)), \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x))) \\ + G(\psi(d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)), \\ \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)), \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x))) \leq 0,$$

for all $x, y \in \mathcal{X}$ and some $F \in \mathcal{F}_D$, $G \in \mathcal{G}_D$ and ψ is an almost altering distance.

Then all S_α and Q_i have a unique common fixed point in $\mathcal{X}$.

Proof. Let S_{α_0} be a fixed element in $\{S_\alpha\}_{\alpha \in J}$. By Theorem 2.2 with $P_0 = S_\alpha$ and $P_1 = S_{\alpha_0}$ it follows that there exists some $u \in \mathcal{X}$ such that

$$S_\alpha u = S_{\alpha_0} u = Q_1 Q_3 \dots Q_{2n-1} u = Q_2 Q_4 \dots Q_{2n} u = u.$$

Let $\beta \in J$ be arbitrary. Then from (C6), we get

$$F(\psi(d(S_\alpha u, S_\beta u)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} u, Q_2 Q_4 \dots Q_{2n} u)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} u, S_\alpha u)), \\ \psi(d(Q_2 Q_4 \dots Q_{2n} u, S_\beta u)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} u, S_\beta u)), \psi(d(Q_2 Q_4 \dots Q_{2n} u, S_\alpha u))) \\ + G(\psi(d(Q_1 Q_3 \dots Q_{2n-1} u, Q_2 Q_4 \dots Q_{2n} u)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} u, S_\alpha u)), \\ \psi(d(Q_2 Q_4 \dots Q_{2n} u, S_\beta u)), \psi(d(Q_1 Q_3 \dots Q_{2n-1} u, S_\beta u)), \psi(d(Q_2 Q_4 \dots Q_{2n} u, S_\alpha u))) \leq 0.$$

Hence

$$F(\psi(d(u, S_\beta u)), \psi(d(u, u)), \psi(d(u, u)), \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), \psi(d(u, u))) \\ + G(\psi(d(u, u)), \psi(d(u, u)), \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), \psi(d(u, u))) \leq 0,$$

i.e.,

$$F(\psi(d(u, S_\beta u)), 0, 0, \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), 0) \\ + G(0, 0, \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), 0) \leq 0.$$

If $d(u, S_\beta u) > 0$, we get

$$G(0, 0, \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), 0) > 0,$$

which implies that

$$F(\psi(d(u, S_\beta u)), 0, 0, \psi(d(u, S_\beta u)), \psi(d(u, S_\beta u)), 0) < 0,$$

a contradiction by (F_{2D}) and hence $\psi(d(u, S_\beta u)) = 0$ i.e., $S_\beta u = u$ for each $\beta \in J$. Uniqueness follows easily. $\square$

If we take $\psi(t) = t$ in Theorem 2.2, we get

Theorem 2.4. Let $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 be self mappings on a metric space $(\mathcal{X}, d)$, satisfying conditions (C1), (C2) and the following condition:
(C7)

$$F((d(P_0x, P_1y)), (d(Q_1Q_3\dots Q_{2n-1}x, Q_2Q_4\dots Q_{2n}y)), (d(Q_1Q_3\dots Q_{2n-1}x, P_0x)), \\ (d(Q_2Q_4\dots Q_{2n}y, P_1y)), (d(Q_1Q_3\dots Q_{2n-1}x, P_1y)), (d(Q_2Q_4\dots Q_{2n}y, P_0x))) \\ + G(d(Q_1Q_3\dots Q_{2n-1}x, Q_2Q_4\dots Q_{2n}y), d(Q_1Q_3\dots Q_{2n-1}x, P_0x), d(Q_2Q_4\dots Q_{2n}y, P_1y), \\ d(Q_1Q_3\dots Q_{2n-1}x, P_1y), d(Q_2Q_4\dots Q_{2n}y, P_0x)) \leq 0,$$

for all $x, y \in \mathcal{X}$, some $F \in \mathcal{F}_D$, $G \in \mathcal{G}_D$ and ψ is an almost altering distance.
Then $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 have a unique common fixed point in $\mathcal{X}$.

If we take $\psi(t) = t$ in Theorem 2.3, we get

Theorem 2.5. Let $\{S_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space $(\mathcal{X}, d)$. Suppose that there exists a fixed $\beta \in J$ such that conditions (C4) and (C5) are satisfied. Moreover,
(C8)

$$F(d(S_\alpha x, S_\beta y), d(Q_1Q_3\dots Q_{2n-1}x, Q_2Q_4\dots Q_{2n}y), d(Q_1Q_3\dots Q_{2n-1}x, S_\alpha x), \\ (d(Q_2Q_4\dots Q_{2n}y, S_\beta y), d(Q_1Q_3\dots Q_{2n-1}x, S_\beta y), d(Q_2Q_4\dots Q_{2n}y, S_\alpha x)) \\ + G(d(Q_1Q_3\dots Q_{2n-1}x, Q_2Q_4\dots Q_{2n}y), d(Q_1Q_3\dots Q_{2n-1}x, S_\alpha x), \\ d(Q_2Q_4\dots Q_{2n}y, S_\beta y), d(Q_1Q_3\dots Q_{2n-1}x, S_\beta y), d(Q_2Q_4\dots Q_{2n}y, S_\alpha x)) \leq 0,$$

for all $x, y \in \mathcal{X}$ and some $F \in \mathcal{F}_D$, $G \in \mathcal{G}_D$ and ψ is an almost altering distance.
Then all S_α and Q_i have a unique common fixed point in $\mathcal{X}$.

Families of mappings satisfying a mixed implicit relation

Remark 2.1. (i). Let ψ and ϕ be as in Theorem 1.1. Then

$$F(u_1, \dots, u_6) = \psi(u_1) - \psi(m(x, y))$$

and

$$G(v_1, \dots, v_5) = \phi(m(x, y)).$$

Then $F(u, 0, u, 0, 0, u) = F(u, 0, 0, u, u, 0) = F(u, u, 0, 0, u, u) = 0$ and

$$G(v_1, \dots, v_5) = \phi(\max\{v_1, \dots, v_5\}) > 0, \text{ if one of } v_1, \dots, v_5 > 0.$$

Hence $F \in \mathcal{F}_D$ and $G \in \mathcal{G}_D$. Then by Theorem 2.4, we get a generalization and extension of Theorem 1.1 for any even number of weakly compatible mappings. Similarly, Theorem 2.5 is a generalization and extension of Theorem 1.1 for families of weakly compatible mappings.

(ii). Theorems 2.2 and 2.3 are extension of Theorem 2.1 for any even number of weakly compatible mappings and families of weakly compatible mappings respectively.

Now we give an example in support of our theorems.

Example 2.1. Let $\mathcal{X} = [0, 1]$ and d be usual metric on $\mathcal{X}$. Define

$$S_\alpha(x) = \frac{x^4}{1 + x^4} \text{ for each } \alpha \in J \text{ and all } x \in \mathcal{X},$$

$$Q_i(x) = x^{\sqrt[n]{4}} \text{ for each } i \in \{1, 2, \dots, 2n\} \text{ and all } x \in \mathcal{X}.$$

Then $Q_2 Q_4 \dots Q_{2n} x = x^4$, $Q_1 Q_3 \dots Q_{2n-1} x = x^4$.

The pairs $(S_\alpha, Q_1 \dots Q_{2n-1})$ and $(S_\beta, Q_2 \dots Q_{2n})$ are weakly compatible..

Define implicit function F such that

$$\text{Let } F(u_1, \dots, u_6) = u_1 - \frac{9}{10} \max\{u_2, u_3, u_4, u_5, u_6\}.$$

and

$$G(t_1, \dots, t_5) = \frac{1}{100(t_1 + t_2 + t_3 + t_4 + t_5)}.$$

Then $F \in \mathcal{F}_D$ and $G \in \mathcal{G}_D$. Thus all the conditions of Theorems 2.2 (for $\alpha = 0, 1$) and 2.3 are satisfied for $\psi(t) = t$ and 0 is the unique common fixed point of the mappings.

3 Application

In 2002, Branciari [4] obtained Banach contraction principle for mappings satisfying an integral type contraction condition. In the same way, we analyze Theorem 2.3 for mappings satisfying integral type contraction condition.

Lemma 3.1. [19] Let $r : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue measurable mapping which is summable on each compact subset of $[0, \infty)$ such that $\int_0^\infty r(t)dt > 0$, for $\epsilon > 0$. Then $\psi(t) = \int_0^t r(x)dx$ is an almost altering distance.

Theorem 3.1. Let $\{S_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space $(\mathcal{X}, d)$. Suppose that there exists a fixed $\beta \in J$ such that conditions (C4) and (C5) are satisfied. Moreover, (C12)

$$\begin{aligned} F & \left(\int_0^{d(S_\alpha x, S_\beta y)} r(t)dt, \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)} r(t)dt, \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)} r(t)dt, \right. \\ & \left. \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)} r(t)dt, \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)} r(t)dt, \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x)} r(t)dt \right) \\ & + G \left(\int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)} r(t)dt, \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)} r(t)dt, \right. \\ & \left. \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)} r(t)dt, \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)} r(t)dt, \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x)} r(t)dt \right) \leq 0, \end{aligned}$$

for all $x, y \in \mathcal{X}$ and some $F \in \mathcal{F}_D$ and $G \in \mathcal{G}_D$. Then all S_α and Q_i have a unique common fixed point in $\mathcal{X}$.

Proof. Let $\psi(t)$ be as in Lemma 3.1. Then

$$\begin{aligned} \psi(d(S_\alpha x, S_\beta y)) &= \int_0^{d(S_\alpha x, S_\beta y)} r(t)dt, \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)) = \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\alpha x)} r(t)dt, \\ \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)) &= \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, Q_2 Q_4 \dots Q_{2n} y)} r(t)dt, \\ \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)) &= \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\beta y)} r(t)dt, \\ \psi(d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)) &= \int_0^{d(Q_1 Q_3 \dots Q_{2n-1} x, S_\beta y)} r(t)dt, \\ \psi(d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x)) &= \int_0^{d(Q_2 Q_4 \dots Q_{2n} y, S_\alpha x)} r(t)dt. \end{aligned}$$

Hence the proof of Theorem 3.1 follows by Theorem 2.3. $\square$

4 Conclusions

In this paper, we have established unique common fixed point theorems for families of weakly compatible mappings satisfying common limit range property and a mixed implicit relation. Our results generalize, extend and improve the results of Imdad [8] and Popa [17]. We provide an application for integral type contraction condition. In the end, we conclude that theory of fixed points can be extended in metric space for some applications as well and that the analogue of many known results can also be obtained in this literature.

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General Fuzzy Graphs

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Abstract

Fuzzy graphs have many applications in database theory, neural networks, and decision-making problems. Literature survey shows that in the existing definition of fuzzy graph the membership value on links is always less than or equal to the minimum of membership values on the corresponding nodes. This restriction brings a difficulty to address some practical problems where the membership value on links does not depend on that of corresponding nodes. So there exists a gap in the literature on fuzzy graphs. In this article we have developed general fuzzy graph and general weak fuzzy graphs and have proved some properties of general weak fuzzy graphs.

Keywords: Fuzzy graphs, Strong fuzzy graphs, Weak fuzzy graphs, General fuzzy graphs, General weak fuzzy graphs.

2010 AMS subject classification: 03E72, 05C72

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1. Introduction

The phenomena of uncertainty in real-life situations are described by a mathematical framework by Zadeh in 1965. Azriel Rosenfeld introduced fuzzy graphs in 1975. M S Sunitha and Vijayakumar A defined complement of fuzzy graphs in 2002. A Nagoor Gani and Chandrasekaran introduced μ complement of a fuzzy graph in 2006. In 2017, T. Al-Hawary discussed certain classes of fuzzy graphs. The existing definition of fuzzy graph in the literature shows that the membership value on links is always less than or equal to the minimum of membership values on the corresponding nodes. This restriction brings difficulty to address some practical problems where the membership value on links does not depend on that of corresponding nodes. To fill this literature gap, in this article, we have developed general fuzzy graphs.

In this article, the second section discusses some existing definitions. In the third section, we introduce the new definitions-general fuzzy graph and general weak fuzzy graph. In the fourth section, the operations-union, c-complement and $|\mu|$ complement on general fuzzy graphs are defined. In the fifth section, some properties of general weak fuzzy graphs are proved.

2. Fuzzy Graphs

In this section, we recall some existing definitions of fuzzy graphs, strong fuzzy graphs, weak fuzzy graphs, and complete fuzzy graphs.

Definition 2.1: A fuzzy graph with V as the underlying set is a pair of functions $G:(\sigma, \mu)$ where $\sigma:V \rightarrow [0,1]$ is a fuzzy subset, $\mu:V \times V \rightarrow [0,1]$ is a fuzzy relation on the fuzzy subset σ such that for all $u, v \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ where $\wedge$ stands minimum.

The underlying crisp graph of a fuzzy graph $G:(\sigma, \mu)$ is denoted by $G^*(\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V / \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V / \mu(u, v) > 0\}$. Two nodes u and v are said to be neighbours if $\mu(u, v) > 0$.

Definition 2.2: A fuzzy graph $G:(\sigma, \mu)$ is a strong fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in \mu^*$.

Definition 2.3: A fuzzy graph $G:(\sigma, \mu)$ is a weak fuzzy graph if $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in \mu^*$.

Definition 2.4: A fuzzy graph $G:(\sigma, \mu)$ is a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in \sigma^*$

Definition 2.5: The complement of a fuzzy graph $G:(\sigma, \mu)$ is a fuzzy graph $\bar{G}:(\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all $u, v \in V$.

Definition 2.6: The μ -complement of a fuzzy graph $G:(\sigma, \mu)$ is a fuzzy graph $G^\mu:(\sigma, \mu^\mu)$ where μ^μ is defined as $\mu^\mu(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^\mu(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ if $\mu(u, v) > 0$ for all $u, v \in \sigma^*$.

3. General Fuzzy Graph

In this section, we introduce the relatively new definitions of general fuzzy graph and general weak fuzzy graph.

Definition 3.1: A general fuzzy graph with V as the underlying set is a pair of functions $G:(\sigma, \mu)$ where $\sigma: V \rightarrow [0, 1]$ is a fuzzy subset of V and $\mu: V \times V \rightarrow [0, 1]$, is a fuzzy subset of $V \times V$. To illustrate our definition, we provide the following example:

Example:3.1. The graph $G:(\sigma, \mu)$ is a general fuzzy graph.

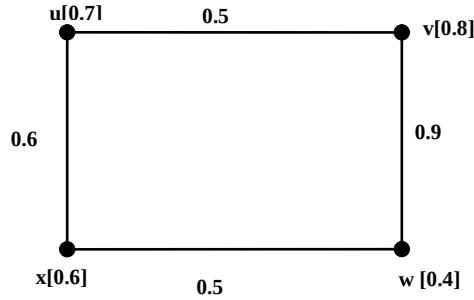


Figure 3.1, General fuzzy graph $G:(\sigma, \mu)$

Definition 3.2: A general weak fuzzy graph with V as the underlying set is a pair of functions $G:(\sigma, \mu)$ where $\sigma: V \rightarrow [0, 1]$ is a fuzzy subset of V and $\mu: V \times V \rightarrow [0, 1]$, is a fuzzy subset of $V \times V$ such that for all $(u, v) \in \mu^*$, $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$.

4. Some Operations on General Fuzzy Graphs

Union: Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two general fuzzy graphs with the underlying crisp graphs $G_1: (V_1, X_1)$ and $G_2: (V_2, X_2)$ respectively. The union $G: (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is defined by:

$(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u)$ if $u \in V_1 - V_2$, $(\sigma_1 \cup \sigma_2)(u) = \sigma_2(u)$ if $u \in V_2 - V_1$, and

$(\sigma_1 \cup \sigma_2)(u) = \max\{\sigma_1(u), \sigma_2(u)\}$ if $u \in V_1 \cap V_2$.

$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v)$ if $(u, v) \in X_1 - X_2$, $(\mu_1 \cup \mu_2)(u, v) = \mu_2(u, v)$ if $(u, v) \in X_2 - X_1$ and $(\mu_1 \cup \mu_2)(u, v) = \max\{\mu_1(u, v), \mu_2(u, v)\}$ if $(u, v) \in X_1 \cap X_2$.

c-complement: The c-complement of a general fuzzy graph $G: (\sigma, \mu)$ is a fuzzy graph $G^c: (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$, $\mu^c(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^c(u, v) = 1 - \mu(u, v)$, for all $u, v \in \sigma^*$.

Example:4.1. The c-complement of the general fuzzy graph $G: (\sigma, \mu)$ in example 3.1, $G^c: (\sigma^c, \mu^c)$ is represented in the following figure.

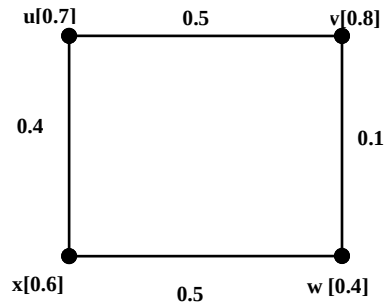


Figure 4.1, c-complement $G^c: (\sigma^c, \mu^c)$

$|\mu|$ complement: The $|\mu|$ complement of a general fuzzy graph $G: (\sigma, \mu)$ is a fuzzy graph $G^{|\mu|}: (\sigma, \mu^{|\mu|})$ where $\mu^{|\mu|}$ is defined as $\mu^{|\mu|}(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^{|\mu|}(u, v) = |\sigma(u) \wedge \sigma(v) - \mu(u, v)|$ if $\mu(u, v) > 0$ for all $u, v \in \sigma^*$.

Example:4.2. The $|\mu|$ complement of the general fuzzy graph $G: (\sigma, \mu)$ in example 3.1, $G^{|\mu|}: (\sigma, \mu^{|\mu|})$ is represented in the following figure.

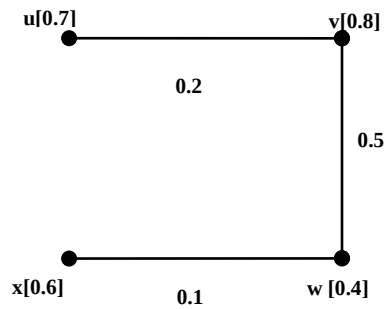


Figure 4.2, $|\mu|$ complement $G^{|\mu|}: (\sigma, \mu^{|\mu|})$

5. Some Properties of General Weak Fuzzy Graphs

In this section, we discuss some properties of general fuzzy weak graphs.

Theorem 5.1. General fuzzy graphs are closed under c-complementation and $|\mu|$ complementation.

The proof of this theorem directly follows from the definitions.

Theorem 5.2. The union of two general weak fuzzy graphs need not be a general weak fuzzy graph.

Proof. Let $F_1: (m_1, r_1)$ and $F_2: (m_2, r_2)$ be two general weak fuzzy graphs with the underlying crisp graphs $F_1^*: (N_1, X_1)$ and $F_2^*: (N_2, X_2)$ respectively where $N_1 = m_1^*$, $N_2 = m_2^*$, $X_1 = r_1^*$ and $X_2 = r_2^*$. The union $F: (m_1 \cup m_2, r_1 \cup r_2)$ is defined by

$$(m_1 \cup m_2)(a) = m_1(a) \text{ if } a \in N_1 - N_2.$$

$$(m_1 \cup m_2)(a) = m_2(a) \text{ if } a \in N_2 - N_1, \text{ and}$$

$$(m_1 \cup m_2)(a) = \max\{m_1(a), m_2(a)\} \text{ if } a \in N_1 \cap N_2.$$

$$(r_1 \cup r_2)(a, b) = r_1(a, b) \text{ if } (a, b) \in X_1 - X_2.$$

$$(r_1 \cup r_2)(a, b) = r_2(a, b) \text{ if } (a, b) \in X_2 - X_1, \text{ and}$$

$$(r_1 \cup r_2)(a, b) = \max\{r_1(a, b), r_2(a, b)\} \text{ if } (a, b) \in X_1 \cap X_2.$$

Case 1: When $(a, b) \in X_1 - X_2$

Since $F_1: (m_1, r_1)$ is a general weak fuzzy graph $(r_1 \cup r_2)(a, b) = r_1(a, b) \neq m_1(a) \wedge m_1(b)$. Since $m_2(a) = 0$ for all $a \in N_1 - N_2$, $(r_1 \cup r_2)(a, b) = r_1(a, b) \neq ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.

Case 2: When $(a, b) \in X_2 - X_1$.

Since $F_2: (m_2, r_2)$ is a general weak fuzzy graph, $(r_1 \cup r_2)(a, b) = r_2(a, b) \neq m_2(a) \wedge m_2(b)$. Since $m_1(a) = 0$ for all $a \in N_2 - N_1$, $(r_1 \cup r_2)(a, b) = r_2(a, b) \neq ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.

Case 3: When $(a, b) \in X_1 \cap X_2$

Suppose $m_2(a) > m_2(b)$ and $r_1(a, b) = m_1(a) \wedge m_1(b) = m_2(a) = r_2(a, b)$, then $(r_1 \cup r_2)(a, b) = r_1(a, b) = r_2(a, b)$. In this case $F: (m_1 \cup m_2, r_1 \cup r_2)$ is not a general weak fuzzy graph

Theorem 5.3. The $|\mu|$ complement of a general weak fuzzy graph is a general weak fuzzy graph.

Proof. Let $F^{|\mu|}: (m, r^{|\mu|})$ be the $|\mu|$ complement of a general weak fuzzy graph $F: (m, r)$, where $r^{|\mu|}$ is defined as

$$r^{|\mu|}(a, b) = |m(a) \wedge m(b) - r(a, b)| \text{ if } r(a, b) > 0 \text{ for all } a, b \in m^* \rightarrow (1)$$

$$\text{and } r^{|\mu|}(a, b) = 0 \text{ if } r(a, b) = 0 \rightarrow (2)$$

Since $F:(m, r)$ is a general weak fuzzy graph for all $(a, b) \in r^*$, $r(a, b) \neq m(a) \wedge m(b) \rightarrow (3)$

To prove that $F^{|\mu|}:(m, r^{|\mu|})$ is a general weak fuzzy graph, it is enough to prove that for all $(a, b) \in r^{|\mu|*}$, $r^{|\mu|}(a, b) \neq m(a) \wedge m(b)$

Case 1. Let $(a, b) \in r^{|\mu|*}$ such that $r(a, b) > 0$. Then by the equations (1) and (3)

$$r^{|\mu|}(a, b) = |m(a) \wedge m(b) - r(a, b)| \neq m(a) \wedge m(b)$$

Case 2. Let $(a, b) \in r^{|\mu|*}$ such that $r(a, b) = 0$. Then by the equation (2), $r^{|\mu|}(a, b) = 0$

Therefore, in both cases $r^{|\mu|}(a, b) \neq m(a) \wedge m(b)$ for all $(a, b) \in r^{|\mu|*}$.

Remark 5.4. If G is a general weak fuzzy graph then $(G^{|\mu|})^{|\mu|}$ need not be isomorphic to G .

Proof. A particular example is enough to prove this.

Consider $G: (\sigma, \mu)$

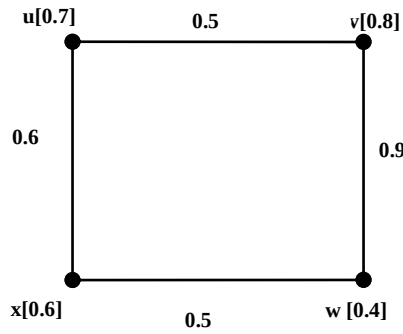


Figure 5.1, $G: (\sigma, \mu)$

$G^{|\mu|}: (\sigma, \mu^{|\mu|})$ is as follows

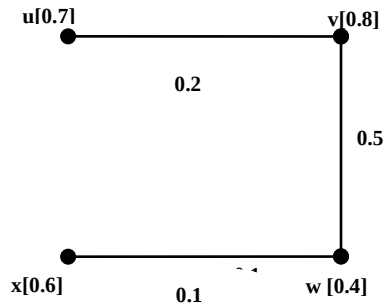


Figure 5.2, $G^{|\mu|}: (\sigma, \mu^{|\mu|})$

$(G^{|\mu|})^{|\mu|}$ is

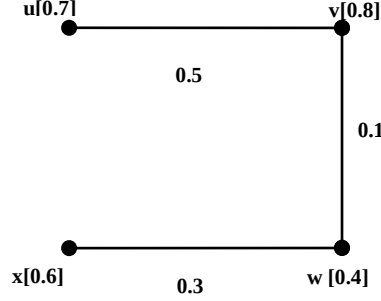


Figure 5.3, $(G^{|\mu|})^{|\mu|}$

This example shows that $(G^{|\mu|})^{|\mu|}$ need not be isomorphic to G .

Theorem 5.4. If $F: (m, r)$ is a general weak fuzzy graph such that $m(a) \wedge m(b) > r(a, b)$ for all $(a, b) \in r^*$, then $(F^{|\mu|})^{|\mu|} = F$.

Proof. Let $F^{|\mu|}: (m, r^{|\mu|})$ be the $|\mu|$ complement of $F: (m, r)$. Since $F: (m, r)$ is a general weak fuzzy graph, for all $(a, b) \in r^*$, $r(a, b) \neq m(a) \wedge m(b)$. We shall consider two cases.

Case 1. For all $(a, b) \in r^*$. As per definition of r^* , $r(a, b) > 0$. In this case $r^{|\mu|}(a, b) = |m(a) \wedge m(b) - r(a, b)|$. Now $m(a) \wedge m(b) > r(a, b)$ for all $(a, b) \in r^*$, implies that $(r^{|\mu|})^{|\mu|}(a, b) = |m(a) \wedge m(b) - r^{|\mu|}(a, b)| = r(a, b)$

Case 2. For all $(a, b) \notin r^*$. As per definition of r^* , $r(a, b) = 0$.

Here $r^{|\mu|}(a, b) = 0$ and $(r^{|\mu|})^{|\mu|}(a, b) = 0$. Therefore $(F^{|\mu|})^{|\mu|} = F$.

6 Conclusions

In this article we discussed the literature gap in existing research in fuzzy graph theory. To fill that gap, we have introduced general fuzzy graphs. The literature review shows that various authors describe the properties of strong fuzzy graph. But the studies on weak fuzzy graphs have not been given importance. In this article we defined general weak fuzzy graphs and proved some properties. We wish to extend our further studies on operations, algebraic properties and applications of general weak fuzzy graphs, weak fuzzy graphs, interval-valued weak fuzzy graphs, and various types of intuitionistic weak fuzzy graphs.

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Some fixed point results using (ψ, ϕ) -generalized almost weakly contractive maps in S-metric spaces

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Abstract

Fixed point theorems have been proved for various contractive conditions by several authors in the existing literature. In this article, we define an (ψ, ϕ) -generalized almost weakly contractive map in S-metric spaces and prove an existence and uniqueness of fixed point of such maps. And also we deduce some existing results as special cases of our result. Moreover, we give an example in support of the results.

Keywords: Fixed point; generalized almost weakly contractive map; S-metric space;

2020 AMS subject classifications: 47H10, 54H25¹

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1 Introduction

Fixed point technique is considered as one of the powerful tools to solve several problems occur in several fields like Computer science, Economics, Mathematics and its allied subjects. In the year 1906, M.Frechet [7] introduced metric spaces. Later, in the year 1922, Stefan Banach [4] proved a very famous theorem called "Banach Fixed Point Theorem". This theorem has been generalized in many directions by generalizing the underlying space or by viewing it as a common fixed point theorem along with other self maps. In the past few years, a number of generalizations of metric spaces like G -metric spaces, partial metric spaces and cone metric spaces were initiated. These generalizations are used to extend the scope of the study of fixed point theory. In 2012, Sedghi, Shobe and Aliouche [13] introduced S-metric spaces and studied some properties of these spaces. We observe that, every G-metric space need not be a S-metric space and vice-versa. For details, see Examples 2.1 and 2.2 in [5]. Generally, in proving fixed point results for a single self map, we utilize completeness and a contractive condition.

Nowadays, the study of fixed point theorems for self maps satisfying different contraction conditions is the center of rigorous research activities. In this direction, Dutta et al. [6] introduced (ψ, ϕ) -weakly contractive maps in 2008 and obtained some fixed point results for such contractions. Later, G.V.R. Babu et al. [1] introduced (ψ, ϕ) -almost weakly contractive maps in G-metric spaces in 2014. Fixed points of contractive maps on S-metric spaces were studied by several authors [2], [3] and [11]. Since then, several contractions have been considered for proving fixed point theorems.

The main purpose of this paper is to define an (ψ, ϕ) - generalized almost weakly contractive map in S-metric spaces and prove an existence and uniqueness of fixed point of such maps. Furthermore we deduce some results as corollaries to our result and provide an example to validate our result.

2 Preliminaries

Definition 2.1. [8] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be an altering distance function if it satisfies

- (i) ψ is continuous and non decreasing and
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Ψ .

We denote $\Phi = \{\phi : [0, \infty) \rightarrow [0, \infty) : (i) \phi \text{ is continuous and } (ii) \phi(t)=0 \text{ if and only if } t=0\}$.

In the following, Dutta and Choudhury [6] established the fixed points of (ψ, ϕ) -

weakly contractive maps in complete metric spaces.

Theorem 2.1. [6] Let (X, d) be a complete metric space and let $h: X \rightarrow X$ be a self-maps of X . If there exist $\psi, \phi \in \Psi$ such that

$$\psi(d(h\xi, h\vartheta)) \leq \psi(d(\xi, \vartheta)) - \phi(d(\xi, \vartheta)) \text{ for all } \xi, \vartheta \in X.$$

Then h has a unique fixed point.

Definition 2.2. [10] Let X be a non-empty set, $G: X^3 \rightarrow [0, \infty)$ be a function satisfying the following properties:

- (i) $G(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$,
- (ii) $G(\xi, \xi, \vartheta) > 0$ for all $\xi, \vartheta \in X$ with $\xi \neq \vartheta$,
- (iii) $G(\xi, \xi, \vartheta) \leq G(\xi, \vartheta, w)$ for all $\xi, \vartheta, w \in X$,
- (iv) $G(\xi, \vartheta, w) = G(\xi, w, \vartheta) = G(w, \xi, \vartheta) = \dots$ (symmetry in all three variables),
- (v) $G(\xi, \vartheta, w) \leq G(\xi, a, a) + G(a, \vartheta, w)$ for all $\xi, \vartheta, w, a \in X$.

Then the function G is called a generalized metric (G -metric) and the pair (X, G) is called a G -metric space.

Definition 2.3. [14] Let (X, G) be a G -metric space. A self mapping h of X is said to be weakly contractive if for all $\xi, \vartheta, w \in X$

$$G(h\xi, h\vartheta, hw) \leq G(\xi, \vartheta, w) - \psi(G(\xi, \vartheta, w))$$

where ψ is an altering distance function.

In 2012, Khandaqji, Al-Sharif and Al-Khaleel [9] proved the following for weakly contractive maps in G -metric spaces.

Theorem 2.2. [9] Let (X, G) be a complete G -metric space and $h: X \rightarrow X$ be a self map. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\begin{aligned} \psi(G(h\xi, h\vartheta, hw)) &\leq \psi(\max\{G(\xi, \vartheta, w), G(\xi, h\xi, h\xi), G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw), \\ &\quad \alpha G(h\xi, h\xi, \vartheta) + (1 - \alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi) \\ &\quad + (1 - \beta)G(\vartheta, h\vartheta, h\vartheta)\}) - \phi(\max\{G(\xi, \vartheta, w), G(\xi, h\xi, h\xi), \\ &\quad G(\vartheta, h\vartheta, h\vartheta), G(w, hw, hw), \alpha G(h\xi, h\xi, \vartheta) \\ &\quad + (1 - \alpha)G(h\vartheta, h\vartheta, w), \beta G(\xi, h\xi, h\xi) + (1 - \beta)G(\vartheta, h\vartheta, h\vartheta)\}) \end{aligned} \quad (1)$$

for all $\xi, \vartheta, w \in X$, where $\alpha, \beta \in (0, 1)$. Then h has a unique fixed point u (say) and h is G -continuous at u .

Definition 2.4. [13] Let a nonempty set X , then we say that a function $S: X^3 \rightarrow [0, \infty)$ is S -metric on X if:

- (S1) $S(\xi, \vartheta, w) > 0$ for all $\xi, \vartheta, w \in X$ with $\xi \neq \vartheta \neq w$,
- (S2) $S(\xi, \vartheta, w) = 0$ if $\xi = \vartheta = w$,
- (S3) $S(\xi, \vartheta, w) \leq [S(\xi, \xi, a) + S(\vartheta, \vartheta, a) + S(w, w, a)]$.

for all $\xi, \vartheta, w, a \in X$. Then (X, S) is called an S -metric space.

Example 2.1. [13] Let (X, d) be a metric space. Define $S: X^3 \rightarrow [0, \infty)$ by $S(\xi, \vartheta, w) = d(\xi, \vartheta) + d(\xi, w) + d(\vartheta, w)$ for all $\xi, \vartheta, w \in X$. Then S is an S -metric on X and S is called the S -metric induced by the metric d .

Example 2.2. [5] Let $X = \mathbb{R}$, the set of all real numbers and let $S(\xi, \vartheta, w) = |\vartheta + w - 2\xi| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X, S) is an S -metric space.

Example 2.3. [12] Let $X = \mathbb{R}$, the set of all real numbers and let $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X, S) is an S -metric space.

Example 2.4. Let $X = [0, 1]$ and we define $S: X^3 \rightarrow [0, \infty)$ by

$$S(\xi, \vartheta, w) = \begin{cases} 0 & \text{if } \xi = \vartheta = w \\ \max\{\xi, \vartheta, w\} & \text{otherwise} \end{cases}.$$

Then S is an S -metric on X .

The following lemmas are useful in our main results.

Lemma 2.1. [13] In an S -metric space, we have $S(\xi, \xi, \vartheta) = S(\vartheta, \vartheta, \xi)$.

Lemma 2.2. [5] In an S -metric space, we have

(i) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(\vartheta, \vartheta, w)$ and

(ii) $S(\xi, \xi, \vartheta) \leq 2S(\xi, \xi, w) + S(w, w, \vartheta)$.

Definition 2.5. [13] Let (X, S) be an S -metric space. We define the following:

(i) a sequence $\{\xi_n\} \in X$ converges to a point $\xi \in X$ if $S(\xi_n, \xi_n, \xi) \rightarrow 0$ as $n \rightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(\xi_n, \xi_n, \xi) < \epsilon$ and we denote it by $\lim_{n \rightarrow \infty} \xi_n = \xi$.

(ii) a sequence $\{\xi_n\} \in X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(\xi_n, \xi_n, \xi_m) < \epsilon$ for all $n, m \geq n_0$.

(iii) (X, S) is said to be complete if each Cauchy sequence in X is convergent.

Definition 2.6. Let (X, S) and (Y, S') be two S -metric spaces. Then a function $h: X \rightarrow Y$ is S -continuous at a point $\xi \in X$ if it is S -sequentially continuous at ξ , that is, whenever $\{\xi_n\}$ is S -convergent to ξ , we have $h(\xi_n)$ is S' -convergent to $h(\xi)$.

Lemma 2.3. [13] Let (X, S) be an S -metric space. If the sequences $\{\xi_n\}$ in X converges to ξ , then ξ is unique.

Lemma 2.4. [13] Let (X, S) be an S -metric space. If there exist sequences $\{\xi_n\}$ and $\{\vartheta_n\}$ in X such that $\lim_{n \rightarrow \infty} \xi_n = \xi$ and $\lim_{n \rightarrow \infty} \vartheta_n = \vartheta$, then $\lim_{n \rightarrow \infty} S(\xi_n, \xi_n, \vartheta_n) = S(\xi, \xi, \vartheta)$.

Some fixed point results in S -metric spaces

Definition 2.7. [13] Let (X, S) be an S -metric space. A map $h: X \rightarrow X$ is said to be an S -contraction if there exists a constant $0 \leq \lambda < 1$ such that

$$S(h(\xi), h(\xi), h(\vartheta)) \leq \lambda S(\xi, \xi, \vartheta) \text{ for all } \xi, \vartheta \in X.$$

We now introduce the following definition and support it with a subsequent example.

Definition 2.8. Let (X, S) be an S -metric space. A map $h: X \rightarrow X$ is called (ψ, ϕ) -generalized almost weakly contractive if it satisfies the inequality

$$\psi(S(h\xi, h\vartheta, hw)) \leq \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)) + L\theta(\xi, \vartheta, w) \quad (2)$$

for all $\xi, \vartheta, w \in X$, $\psi \in \Psi$, $\phi \in \Phi$ and $L \geq 0$, where

$$M(\xi, \vartheta, w) = \max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\},$$

$$\theta(\xi, \vartheta, w) = \min\{S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), S(w, w, hw), S(\xi, \xi, hw)\}.$$

Example 2.5. Let $X = [0, \frac{8}{7}]$ and we define $h: X \rightarrow X$ by

$$h\xi = \begin{cases} \frac{\xi}{10} & \text{if } \xi \in [0, 1] \\ \xi - \frac{4}{5} & \text{if } \xi \in (1, \frac{8}{7}] \end{cases}.$$

We define $S: X^3 \rightarrow [0, \infty)$ by $S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|$ for all $\xi, \vartheta, w \in X$. Then (X, S) is a complete S -metric space.

We now define functions $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(t) = t, \text{ for all } t \geq 0 \text{ and } \phi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0, 1] \\ \frac{t}{t+1} & \text{if } t \geq 1. \end{cases}.$$

We now show that h satisfies the inequality (2).

Case(i): Let $\xi, \vartheta, w \in [0, 1]$.

Without loss of generality, we assume that $\xi > \vartheta > w$.

$$S(h\xi, h\vartheta, hw) = S(\frac{\xi}{10}, \frac{\vartheta}{10}, \frac{w}{10}) = \frac{1}{10}(|\xi - w| + |\vartheta - w|) \text{ and}$$

$$S(\xi, \vartheta, w) = |\xi - w| + |\vartheta - w|.$$

sub case (i): If $|\xi - w| + |\vartheta - w| \in [0, 1]$.

In this case,

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= \frac{1}{10}(|\xi - w| + |\vartheta - w|) \leq \frac{1}{2}(|\xi - w| + |\vartheta - w|) \\ &= \frac{1}{2}S(\xi, \vartheta, w) \leq \frac{1}{2}M(\xi, \vartheta, w) \\ &= M(\xi, \vartheta, w) - \frac{1}{2}M(\xi, \vartheta, w) \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Sub case(ii): If $|\xi - \vartheta| + |\vartheta - w| \geq I$.

In this case,

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= \frac{1}{10}(|\xi - \vartheta| + |\vartheta - w|) \leq |\xi - \vartheta| + |\vartheta - w| - \frac{|\xi - \vartheta| + |\vartheta - w|}{1 + |\xi - \vartheta| + |\vartheta - w|} \\
 &= S(\xi, \vartheta, w) - \frac{S(\xi, \vartheta, w)}{1 + S(\xi, \vartheta, w)} \\
 &= \frac{(S(\xi, \vartheta, w))^2}{1 + S(\xi, \vartheta, w)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

Case(ii): Let $\xi, \vartheta, w \in (1, \frac{8}{7}]$.

Without loss of generality, we assume that $\xi > \vartheta > w$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\xi - \frac{4}{5}, \vartheta - \frac{4}{5}, w - \frac{4}{5}) = |\xi - w| + |\vartheta - w| \\
 &\leq \frac{2}{7} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
 &= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

Case(iii): Let $\vartheta, w \in [0, 1]$ and $\xi \in (1, \frac{8}{7}]$.

Without loss of generality, we assume that $\vartheta > w$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\xi - \frac{4}{5}, \frac{\vartheta}{10}, \frac{w}{10}) = |\xi - \frac{4}{5} - \frac{w}{10}| + |\frac{\vartheta}{10} - \frac{w}{10}| \\
 &= \xi - \frac{w}{10} - \frac{4}{5} + \frac{\vartheta - w}{10} = \xi + \frac{\vartheta}{10} - \frac{w}{5} - \frac{4}{5} \\
 &= \frac{31}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
 &= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

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Case(iv): Let $w \in [0,1]$ and $\xi, \vartheta \in (1, \frac{8}{7}]$.

Without loss of generality, we assume that $\xi > \vartheta$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\xi - \frac{4}{5}, \vartheta - \frac{4}{5}, \frac{w}{10}) = |\xi - \frac{4}{5} - \frac{w}{10}| + |\vartheta - \frac{4}{5} - \frac{w}{10}| \\
 &= \xi + \vartheta - \frac{w}{5} - \frac{8}{5} = \frac{12}{35} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
 &= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
 &= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

Case (v): Let $\xi, \vartheta \in [0,1]$ and $w \in (1, \frac{8}{7}]$.

Without loss of generality, we assume that $\xi > \vartheta$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \frac{\vartheta}{10}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\frac{\vartheta}{10} - w + \frac{4}{5}| \\
 &= |\frac{4}{5} - (w - \frac{\xi}{10})| + |\frac{4}{5} - (w - \frac{\vartheta}{10})| = w - \frac{\xi}{10} - \frac{4}{5} + w - \frac{\vartheta}{10} - \frac{4}{5} \\
 &= 2w - \frac{\xi + \vartheta}{10} - \frac{8}{5} = \frac{41}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} \\
 &= S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} \\
 &= \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

Case (vi): Let $\xi \in [0,1]$ and $w, \vartheta \in (1, \frac{8}{7}]$.

Without loss of generality, we assume that $w > \vartheta$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\frac{\xi}{10}, \vartheta - \frac{4}{5}, w - \frac{4}{5}) = |\frac{\xi}{10} - w + \frac{4}{5}| + |\vartheta - w| \\
 &= w - \frac{\xi}{10} - \frac{4}{5} + w - \vartheta = 2w - \frac{\xi}{10} - \frac{4}{5} - \vartheta
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{27}{70} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
 &= \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} \\
 &= M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

From all the above cases, we conclude that h is an (ψ, ϕ) -generalized almost weakly contraction map on X .

Lemma 2.5. [5] Let (X, S) be an S -metric space and $\{\xi_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} S_b(\xi_n, \xi_n, \xi_{n+1}) = 0$. If $\{\xi_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of natural numbers with $n_k > m_k > k$ such that $S(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) \geq \epsilon$, $S(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) < \epsilon$ and
 (i) $\lim_{k \rightarrow \infty} S_b(\xi_{m_k}, \xi_{m_k}, \xi_{n_k}) = \epsilon$. (ii) $\lim_{k \rightarrow \infty} S_b(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k}) = \epsilon$.
 (iii) $\lim_{k \rightarrow \infty} S_b(\xi_{m_k}, \xi_{m_k}, \xi_{n_k-1}) = \epsilon$. (iv) $\lim_{k \rightarrow \infty} S_b(\xi_{m_k-1}, \xi_{m_k-1}, \xi_{n_k-1}) = \epsilon$.

3 Main Results

Theorem 3.1. Let (X, S) be a complete S -metric space and $h: X \rightarrow X$ be a (ψ, ϕ) -generalized almost weakly contractive mapping. Then h has a unique fixed point in X .

Proof. Let $\xi_0 \in X$ be arbitrary. We define a sequence $\{\xi_n\}$ by $h\xi_n = \xi_{n+1}$, for $n = 0, 1, 2, \dots$.

If $\xi_n = \xi_{n+1}$, for some $n \in \mathbb{N}$, then ξ_n is a fixed point of h .

Suppose $\xi_n \neq \xi_{n+1}$, for all $n \in \mathbb{N}$.

Consider,

$$\begin{aligned}
 &\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) = \psi(S(h\xi_n, h\xi_n, h\xi_{n-1})) \\
 &\leq \psi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \\
 &\frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\}) \\
 &- \phi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \\
 &\frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\}) \\
 &+ L.\min\{S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), S(\xi_{n-1}, \xi_{n-1}, h\xi_n), S(\xi_n, \xi_n, h\xi_{n-1})\}
 \end{aligned}$$

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$$\begin{aligned}
 &= \psi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), \\
 &\frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\}) \\
 &- \phi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), \\
 &\frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\}) \\
 &+ L.\min\{S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_{n-1}, \xi_{n-1}, \xi_{n+1}), S(\xi_n, \xi_n, \xi_n)\} \\
 &= \psi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\}) - \phi(\max\{S(\xi_n, \xi_n, \xi_{n-1}), \\
 &S(\xi_n, \xi_n, \xi_{n+1})\}) + L.0
 \end{aligned}$$

If $\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\} = S(\xi_n, \xi_n, \xi_{n+1})$, then we get

$$\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) - \phi(S(\xi_{n+1}, \xi_{n+1}, \xi_n))$$

that is, $\phi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq 0$, which implies that $S(\xi_{n+1}, \xi_{n+1}, \xi_n) = 0$. Then we get $\xi_{n+1} = \xi_n$, which is a contradiction to our assumption that $\xi_n \neq \xi_{n+1}$, for each n .

Therefore, $\max\{S(\xi_n, \xi_n, \xi_{n-1}), S(\xi_n, \xi_n, \xi_{n+1})\} = S(\xi_n, \xi_n, \xi_{n-1})$, then we get

$$\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_n, \xi_n, \xi_{n-1})) - \phi(S(\xi_n, \xi_n, \xi_{n-1})) \quad (3)$$

that is $\psi(S(\xi_{n+1}, \xi_{n+1}, \xi_n)) \leq \psi(S(\xi_n, \xi_n, \xi_{n-1}))$

Therefore we get, $S(\xi_{n+1}, \xi_{n+1}, \xi_n) \leq S(\xi_n, \xi_n, \xi_{n-1})$, for all n and the sequence $\{S(\xi_{n+1}, \xi_{n+1}, \xi_n)\}$ is decreasing and bounded. So, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} S(\xi_{n+1}, \xi_{n+1}, \xi_n) = r.$$

Letting $n \rightarrow \infty$ in equation (3), we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$.

Hence,

$$\lim_{n \rightarrow \infty} S(\xi_{n+1}, \xi_{n+1}, \xi_n) = 0. \quad (4)$$

Now we prove that $\{\xi_n\}$ is a Cauchy sequence. If not, then there exists an $\epsilon > 0$ for which we can find subsequences $\{\xi_{m(k)}\}$ and $\{\xi_{n(k)}\}$ of $\{\xi_n\}$ and increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) \geq \epsilon \quad (5)$$

Then, we have

$$S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1}) < \epsilon \quad (6)$$

Now,

$$\begin{aligned} \epsilon &\leq S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) = S(\xi_{n(k)}, \xi_{n(k)}, \xi_{m(k)}) \\ &\leq 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1}) + S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1}) \\ &\leq \epsilon + 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1}) \quad (\text{Using equation 6}) \end{aligned}$$

Letting $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) = \epsilon. \quad (7)$$

Also,

$$\begin{aligned} S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)}) &\leq 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + S(\xi_{n(k)}, \xi_{n(k)}, \xi_{m(k)-1}) \\ &\leq 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + 2S(\xi_{n(k)}, \xi_{n(k)}, \xi_{n(k)-1}) \\ &\quad + S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}) \quad (8) \end{aligned}$$

and

$$\begin{aligned} S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}) &\leq 2S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}) + S(\xi_{n(k)-1}, \xi_{n(k)-1}, \xi_{m(k)}) \\ &= 2S(\xi_{m(k)}, \xi_{m(k)}, \xi_{m(k)-1}) + S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)-1}) \quad (9) \end{aligned}$$

Letting $k \rightarrow \infty$ in equation (9) and using equations (4), (6), (7) and (8) we get

$$\lim_{k \rightarrow \infty} S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}) = \epsilon \quad (10)$$

Setting $\xi = \xi_{m(k)-1}$, $y = \xi_{m(k)-1}$ and $z = \xi_{n(k)-1}$ in equation (2), we obtain

$$\begin{aligned} \psi(\epsilon) &\leq \psi(S(\xi_{m(k)}, \xi_{m(k)}, \xi_{n(k)})) = \psi(S(h\xi_{m(k)-1}, h\xi_{m(k)-1}, h\xi_{n(k)-1})) \\ &\leq \psi(\max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \\ &\quad S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}) \\ &\quad + S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})]\}) \\ &\quad - \phi(\max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \\ &\quad S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}) \\ &\quad + S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1})]\}) \end{aligned}$$

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$$\begin{aligned}
& + L.min\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{m(k)-1}), \\
& S(\xi_{n(k)-1}, \xi_{n(k)-1}, h\xi_{m(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, h\xi_{n(k)-1})\} \\
& \leq \psi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \\
& S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}) \\
& + S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})]\}) \\
& - \phi(max\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)-1}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \\
& S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \frac{1}{2}[S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}) \\
& + S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)})]\}) \\
& + L.min\{S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{m(k)}), \\
& S(\xi_{n(k)-1}, \xi_{n(k)-1}, \xi_{m(k)}), S(\xi_{m(k)-1}, \xi_{m(k)-1}, \xi_{n(k)})\}
\end{aligned}$$

Letting $k \rightarrow \infty$ and using equation (10) we get

$$\begin{aligned}
\psi(\epsilon) & \leq \psi(max\{\epsilon, 0, 0, 0\}) - \phi(max\{\epsilon, 0, 0, 0\}) + L.min\{0, 0, 0, \epsilon\} \\
\psi(\epsilon) & \leq \psi(\epsilon) - \phi(\epsilon) + L.0
\end{aligned}$$

This is a contradiction, since $\epsilon > 0$. This shows that $\{\xi_n\}$ is a Cauchy sequence in the complete S-metric space (X, S) . There exists $\kappa \in X$ such that $\{\xi_n\} \rightarrow \kappa$ as $n \rightarrow \infty$.

Now we prove that $h\kappa = \kappa$.

Put $\xi = \xi_n$, $\vartheta = \xi_n$ and $w = \kappa$ in equation (2), then we get

$$\begin{aligned}
\psi(S(\xi_{n+1}, \xi_{n+1}, f\kappa)) & = \psi(S(h\xi_n, h\xi_n, h\kappa)) \\
& \leq \psi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\}) \\
& - \phi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), \frac{1}{2}[S(\xi_n, \xi_n, h\xi_n) + S(\xi_n, \xi_n, h\xi_n)]\}) \\
& + L.min\{S(\xi_n, \xi_n, h\xi_n), S(\xi_n, \xi_n, h\xi_n), S(\kappa, \kappa, h\xi_n), S(\xi_n, \xi_n, h\kappa)\} \\
& = \psi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), \frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) \\
& + S(\xi_n, \xi_n, \xi_{n+1})]\}) - \phi(max\{S(\xi_n, \xi_n, \kappa), S(\xi_n, \xi_n, \xi_{n+1}), \\
& S(\xi_n, \xi_n, \xi_{n+1}), \frac{1}{2}[S(\xi_n, \xi_n, \xi_{n+1}) + S(\xi_n, \xi_n, \xi_{n+1})]\}) \\
& + L.min\{S(\xi_n, \xi_n, \xi_{n+1}), S(\xi_n, \xi_n, \xi_{n+1}), S(\kappa, \kappa, \xi_{n+1}), S(\xi_n, \xi_n, h\kappa)\}
\end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned}
\psi(S(\kappa, \kappa, h\kappa)) & \leq \psi(S(\kappa, \kappa, \kappa)) - \phi(S(\kappa, \kappa, \kappa)) + L.0 \\
\psi(S(\kappa, \kappa, h\kappa)) & \leq 0. \text{ So, we get } S(\kappa, \kappa, h\kappa) = 0.
\end{aligned}$$

Hence $h\kappa = \kappa$. That is κ is a fixed point of h .

To prove the uniqueness of κ , let j be a fixed point of h with $\kappa \neq j$.

Using equation (2), we consider

$$\begin{aligned} \psi(S(\kappa, \kappa, j)) &= \psi(S(h\kappa, h\kappa, hj)) \\ &\leq \psi(\max\{S(\kappa, \kappa, j), S(\kappa, \kappa, h\kappa), S(\kappa, \kappa, h\kappa), \frac{1}{2}[S(\kappa, \kappa, h\kappa) + S(\kappa, \kappa, h\kappa)]\}) \\ &\quad - \phi(\max\{S(\kappa, \kappa, j), S(\kappa, \kappa, h\kappa), S(\kappa, \kappa, h\kappa), \frac{1}{2}[S(\kappa, \kappa, h\kappa) + S(\kappa, \kappa, h\kappa)]\}) \\ &\quad + L.\min\{S(\kappa, \kappa, h\kappa), S(\kappa, \kappa, h\kappa), S(j, j, h\kappa), S(\kappa, \kappa, hj)\} \\ \text{That is, } \psi(S(\kappa, \kappa, j)) &\leq \psi(S(\kappa, \kappa, j)) - \phi(S(\kappa, \kappa, j)) \end{aligned}$$

is a contradiction, unless $S(\kappa, \kappa, j) = 0$. Hence we get $\kappa = j$.

This shows that the fixed point of h is unique. $\square$

If $L=0$ in the Theorem 3.1, then we get the following.

Corollary 3.1. *Let (X, S) be a complete S -metric space and $h: X \rightarrow X$ be a mapping. Suppose there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned} S(h\xi, h\vartheta, hw) &\leq \psi(\max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) \\ &\quad + S(\vartheta, \vartheta, h\xi)]\}) - \phi(\max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \\ &\quad \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}), \end{aligned}$$

for all $\xi, \vartheta, w \in X$. Then h has a unique fixed point κ in X .

If ψ is the identity map in the above Corollary (3.1), then we get the following.

Corollary 3.2. *Let (X, S) be a complete S -metric space and $h: X \rightarrow X$ be a mapping. Suppose there exist $\phi \in \Phi$ such that*

$$\begin{aligned} S(h\xi, h\vartheta, hw) &\leq \max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\} \\ &\quad - \phi(\max\{S(\xi, \vartheta, w), S(\xi, \xi, h\xi), S(\vartheta, \vartheta, h\vartheta), \frac{1}{2}[S(\xi, \xi, h\vartheta) + S(\vartheta, \vartheta, h\xi)]\}) \end{aligned}$$

for all $\xi, \vartheta, w \in X$. Then h has a unique fixed point κ in X .

The following example is in support of Theorem 3.1.

Example 3.1. *Let $X = [0, \frac{7}{6}]$. We define $S: X^3 \rightarrow [0, \infty)$ by $S(\xi, \vartheta, w) = \max\{|\xi - w|, |\vartheta - w|\}$, for all $\xi, \vartheta, w \in X$. Then S is an S -metric on X . We define $h: X \rightarrow X$ by*

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$$h\xi = \begin{cases} \frac{1}{2} & \text{if } \xi \in [0, 1] \\ \frac{4}{3} - \xi & \text{if } \xi \in (1, \frac{7}{6}] \end{cases}.$$

We define $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ by

$\psi(t) = t$, for all $t \geq 0$ and $\phi(t) = \frac{t}{1+t}$ for all $t \geq 0$.

We now show that h satisfies inequality (2).

Case(i) Let $\xi, \vartheta, w \in [0, 1]$.

Without loss of generality, we assume that $\xi > \vartheta > w$.

$S(h\xi, h\vartheta, hw) = S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$. Then trivially the inequality (2) holds.

Case(ii) Let $\xi, \vartheta, w \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $\xi > \vartheta > w$.

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{4}{3} - w) = \max\{|\frac{4}{3} - \xi - (\frac{4}{3} - w)|, |\frac{4}{3} - \vartheta - (\frac{4}{3} - w)|\} \\ &= \max\{|w - \xi|, |w - \vartheta|\} = \xi - w \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\ &\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\ &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Case(iii) Let $\vartheta, w \in [0, 1]$ and $\xi \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $\vartheta > w$.

$$\begin{aligned} S(h\xi, h\vartheta, hw) &= S(\frac{4}{3} - \xi, \frac{1}{2}, \frac{1}{2}) = \max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{1}{2} - \frac{1}{2}|\} \\ &= \xi - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\ &\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\ &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\ &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)). \end{aligned}$$

Case(iv) Let $w \in [0, 1]$ and $\xi, \vartheta \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $\vartheta > \xi$.

$$S(h\xi, h\vartheta, hw) = S(\frac{4}{3} - \xi, \frac{4}{3} - \vartheta, \frac{1}{2}) = \max\{|\frac{4}{3} - \xi - \frac{1}{2}|, |\frac{4}{3} - \vartheta - \frac{1}{2}|\}$$

$$\begin{aligned}
 &= \max\{|\frac{5}{6} - \xi|, |\frac{5}{6} - \vartheta|\} = \xi - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
 &\leq S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} = \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
 &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

Case(v) Let $\xi, \vartheta \in [0, 1]$ and $w \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $\xi > \vartheta$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= (\frac{1}{2}, \frac{1}{2}, \frac{4}{3} - w) = \max\{|\frac{1}{2} - (\frac{4}{3} - w)|, |\frac{1}{2} - (\frac{4}{3} - w)|\} \\
 &= w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
 &\leq S(\xi, \xi, h\xi) - \frac{S(\xi, \xi, h\xi)}{1 + S(\xi, \xi, h\xi)} = \frac{(S(\xi, \xi, h\xi))^2}{1 + S(\xi, \xi, h\xi)} \\
 &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

case(vi) Let $\xi \in [0, 1]$ and $\vartheta, w \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $w > \vartheta$.

$$\begin{aligned}
 S(h\xi, h\vartheta, hw) &= S(\frac{1}{2}, \frac{4}{3} - \vartheta, \frac{4}{3} - w) = \max\{|\frac{1}{2} - (\frac{4}{3} - w)|, |\frac{4}{3} - \vartheta - (\frac{4}{3} - w)|\} \\
 &= \max\{w - \frac{5}{6}, |w - \vartheta|\} = w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
 &= S(\vartheta, \vartheta, h\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} = \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
 &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

case(vii) Let $\vartheta \in [0, 1]$ and $\xi, w \in (1, \frac{7}{6}]$.

Without loss of generality, we assume that $w > \xi$.

$$S(h\xi, h\vartheta, hw) = S(\frac{4}{3} - \xi, \frac{1}{2}, \frac{4}{3} - w) = \max\{|\frac{4}{3} - \xi - (\frac{4}{3} - w)|, |\frac{1}{2} - (\frac{4}{3} - w)|\}$$

$$\begin{aligned}
 &= \max\{|w - \xi|, w - \frac{5}{6}\} = w - \frac{5}{6} \leq \frac{1}{6} \leq \frac{4}{15} = \frac{2}{3} - \frac{2}{5} \\
 &= S(\vartheta, \vartheta, hh\vartheta) - \frac{S(\vartheta, \vartheta, h\vartheta)}{1 + S(\vartheta, \vartheta, h\vartheta)} = \frac{(S(\vartheta, \vartheta, h\vartheta))^2}{1 + S(\vartheta, \vartheta, h\vartheta)} \\
 &\leq \frac{(M(\xi, \vartheta, w))^2}{1 + M(\xi, \vartheta, w)} = M(\xi, \vartheta, w) - \frac{M(\xi, \vartheta, w)}{1 + M(\xi, \vartheta, w)} \\
 &= \psi(M(\xi, \vartheta, w)) - \phi(M(\xi, \vartheta, w)).
 \end{aligned}$$

From all the above cases, we conclude that h is an (ψ, ϕ) -generalized almost weakly contraction map on X and $\frac{1}{2}$ is the unique fixed point of h .

4 Conclusion

In this paper, we establish an existence and uniqueness of a fixed point theorem for (ψ, ϕ) -generalized almost weakly contraction maps in S-metric spaces. As S-metric space is a generalization of metric space, our result in this article extends and improves the result of Khandaqji, Al-Sharif and Al-Khaleel [9] and also generalize several well-known comparable results in the literature. Further, the result in this paper can be extended to several spaces like S_b -metric space, partial S_b -metric spaces and other spaces.

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Transit in Corona product of graphs

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Abstract

For a graph $G(V, E)$, the transit of a vertex v is defined as the sum of the lengths of all geodesics with v as an internal vertex. This paper deals with the transit of vertices in Corona product of graphs. We obtain expressions for transit of an arbitrary vertex in the Corona product $G_1 \circ G_2$ of G_1 and G_2 . We also consider the cases where G_1 is a particular graph class. The expressions for transit of vertices in $G_1 \circ G_2$, with G_1 as path P_n , a cycle C_n , a star S_n , a complete graph K_n and a complete bipartite graph $K_{m,n}$ is established.

Keywords: Geodesic, Transit Index, Transit Identical, Corona product

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1 Introduction

Topological indices and centrality measures are graph invariant. Numerous studies have been carried out in these areas. The first notable topological index was the wiener index, named after Harry Wiener, a pioneer in chemical graph theory. It is defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph representing the non-hydrogen atoms in the molecule. Wiener made fundamental contributions to the study of topological indices and established a correlation between the Wiener index and boiling points (hence viscosity and surface tension) of the paraffin. He could establish relationships with many chemical properties of alkanes with the wiener index.

Centrality measures are a vital tool for understanding graphs. Each measure has its own definition of importance. Some are based on the degree of a vertex while others take the closeness to other vertices as the score of significance.

In the paper [Shimbel and Alfonso], they introduced the concept of the stress of a vertex. It is the number of shortest paths on which a vertex lies. This was further modified to produce measures of centrality. It found applications in social networking, for analyzing communication dynamics.

Keeping in mind the above two concepts, we introduced a new index, called the transit index of a graph. It considers the distances in the graph as well as the degree of vertices. In computing the stress of a vertex, we only take into account the number of shortest paths through it; the length of the paths is not considered. Be it in data transmission or in the measure of closeness, the length of the paths also matters. Hence, in the computation of transit we account for the number of shortest paths as well as their length.

Graph products are binary operations. Two graphs G_1 and G_2 are combined to produce a new graph H . In this paper, we study the transit of vertices in the Corona product of graphs. Information on individual graphs is used to compute the transit of vertices in their Corona products. Thus the transit of vertices in large graphs and networks, which can be viewed as Corona products of simple graphs, can be computed more efficiently.

2 Preliminaries

In this section, we come across certain definitions and terminologies employed in developing results in the latter sections. Throughout this paper we only consider simple connected and finite graphs.

Definition 2.1 (K.M.Reshmi and Pilakkat. Raji [2020]). *Let $v \in V$. Then the transit of v denoted by $T(v)$ is “the sum of the lengths of all shortest paths with v*

as an internal vertex” and the transit index of G denoted by $TI(G)$ is

$$TI(G) = \sum_{v \in V} T(v)$$

Lemma 2.2 (K.M.Reshmi and Pilakkat. Raji [2020]). $T(v) = 0$ iff $\langle N[v] \rangle$ is a clique.

Theorem 2.3 (K.M.Reshmi and Pilakkat. Raji [2020]). For a path P_n , Transit index is

$$TI(P_n) = \frac{n(n+1)(n^2 - 3n + 2)}{12}$$

Theorem 2.4. For a cycle, the transit of any vertex v is, $T(v) = \frac{(n^2-4)n}{24}$ and

$$i) TI(C_n) = \frac{n^2(n^2-4)}{24}, n \text{ even.}$$

$$ii) TI(C_n) = \frac{n(n^2-1)(n-3)}{24}, n \text{ odd}$$

Definition 2.5. Two vertices of a graph are called transit identical if the shortest paths passing through it are same in number and length.

We use the following terminologies.

The order of a graph G , denoted by $|G|$ is the number of vertices in $V(G)$. The distance between two vertices $u, v \in V$ is the length of any shortest $u - v$ path in G . A shortest path from u to v is also called a $u - v$ geodesic. The number of shortest $u - v$ paths is denoted by $\sigma(u, v)$ and the number of shortest $u - v$ path with ‘ a ’ as an internal vertex is denoted by $\sigma(u, v/a)$. It can be noted that a vertex ‘ a ’ lies on a shortest $u - v$ path iff $d(u, v) = d(u, a) + d(a, v)$. The number of shortest $u - v$ path with ‘ a ’ as an internal vertex can be computed as $\sigma(u, v/a) = \sigma(u, a) \times \sigma(a, v)$. Number of shortest paths in G with ‘ a ’ as an internal vertex is denoted by $\sigma_G(a)$. Clearly $\sigma_G(a) = \sum_{(u,v)} \sigma(u, v/a)$

3 Transit of vertices in Corona Product of graphs

3.1 Corona Product of Graphs

Definition 3.1. (Frucht and Harary [1970]) Let G_1 and G_2 be two graphs. The corona product $G_1 \circ G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 ; and by joining each vertex of the i -th copy of G_2 to the i -th vertex of G_1 , where $1 \leq i \leq |V(G_1)|$

Whenever we consider $G_1 \circ G_2$, we use the following notations.

1. G_2^i the i th copy of G_2 in $G_1 \circ G_2$
2. $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}, |E(G_1)| = m_1$
3. $V(G_2^i) = \{v_1^i, v_2^i, \dots, v_{n_2}^i\}, |E(G_2^i)| = m_2, \forall i$

Lemma 3.2. (Agnes [2015]) Let G_1 and G_2 be two arbitrary graphs. Then,

- $d_{G_1 \circ G_2}(u_i, u_p) = d_{G_1}(u_i, u_p), 0 \leq i, p \leq n_1$
- $d_{G_1 \circ G_2}(u_i, v_q^p) = d_{G_1}(u_i, u_p) + 1, 0 \leq i, p \leq n_1, 0 \leq q \leq n_2$
- $d_{G_1 \circ G_2}(v_j^i, v_q^p) = \begin{cases} d_{G_1}(u_i, u_p) + 2, & i \neq p \\ 1, & \text{if } i = p, v_j v_q \in E(G_2) \\ 2, & \text{if } i = p, v_j v_q \notin E(G_2) \end{cases}$

Proposition 3.3. For any two graphs G_1 and G_2 ,

1. $T_{G_2}(a) = 0$ iff $T_{G_1 \circ G_2}(a) = 0, a \in G_2$
2. $T_{G_2}(a) = T_{G_1 \circ G_2}(a)$, for $a \in G_2$ iff every shortest path in G_2 with 'a' as an internal vertex is of length 2.

Proof. Note that as the result 2 is obvious, we prove only 1. 1) $T_{G_2}(a) = 0 \iff \langle N_{G_2}[a] \rangle$ is a clique. $\iff \langle N_{G_1 \circ G_2}[a] \rangle$ is a clique $\iff T_{G_1 \circ G_2}(a) = 0$. $\square$

Proposition 3.4. Let G_1 and G_2 be arbitrary graphs,

1) For any u_p in G_1 ,

$$\sigma_{G_1 \circ G_2}(u_p) = (n_2 + 1) \left[(n_2 + 1) \sigma_{G_1}(u_p) + n_2 \sum_{p \neq k=1}^{n_1} \sigma_{G_1}(u_p, u_k) \right]$$

2) $\sigma_{G_1 \circ G_2}(v_k^i) = \text{Number of geodesic of length 2 in } G_2 \text{ with } v_k \text{ as an internal vertex.}$

Proof. 1) Let u_p be any vertex of G_1 . Every geodesic in G_1 with u_p as an internal vertex will be counted in $\sigma_{G_1 \circ G_2}(u_p)$.

For $k \neq l$, geodesics connecting $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$ will have $u_l - u_k$ geodesic as its part. Let P_1 be one of the $u_l - u_k$ geodesic with u_p as an internal vertex. Then P_1 will be part of a geodesic connecting vertices of $G_2^k \cup \{u_k\}$ to vertices of $G_2^l \cup \{u_l\}$. There will be

- n_2^2 geodesics connecting G_2^k to G_2^l ,
- n_2 geodesics connecting G_2^k to u_l and

- n_2 geodesics connecting G_2^l to u_k , with P_1 as its part. Hence for the pair of vertices (u_k, u_l) , there will be $(n_2^2 + 2n_2 + 1)\sigma_{G_1}(u_p)$ geodesics with u_p as an internal vertex.

The geodesics connecting vertices of G_2^p to other vertices of $G_1 \circ G_2$ will have $u_p - u_k$ geodesic as a part for some k . If P_2 is one of the $u_p - u_k$ geodesic, it will be part of

- n_2^2 geodesics connecting G_2^k to G_2^p and

- n_2 geodesics connecting G_2^p to u_k . Hence for every $u_p - u_k$ geodesic in G_1 there will be $\sigma_{G_1}(u_p, u_k)[n_2(n_2 + 1)]$ geodesics in $G_1 \circ G_2$ with u_p as an internal vertex. Considering every pair $u_k - u_p$ the result follows.

2) Since every vertex of G_2^i are joined to u_i , the maximum distance between vertices of G_2^i is 2. Hence the proof. $\square$

Next we find an expression for the transit of a vertex, u_p in $G_1 \circ G_2$, where G_1 and G_2 are arbitrary. Let (u_k, u_l) be a pair of vertices in G_1 such that $u_k - u_l$ geodesic has u_p as an internal vertex. Let $T_{kl}(u_p)$ denote the contribution to transit of u_p , due to geodesic connecting vertices of $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$. Also we denote the contribution of vertices in G_2^p to $T(u_p)$ by $T_p(u_p)$.

Lemma 3.5. For arbitrary graphs G_1 and G_2 ,

$$T_{kl}(u_p) = \sigma_{G_1}(u_k, u_l/u_p) \left[(n_2 + 1)^2 d(u_k, u_l) + 2n_2(n_2 + 1) \right]$$

Proof. Table 1 gives the length and number of geodesics through u_p

Vertices connected	Length	Number
G_2^k to G_2^l	$2 + d(u_k, u_l)$	$n_2^2 \sigma(u_k, u_l/u_p)$
u_k to G_2^l	$1 + d(u_k, u_l)$	$n_2 \sigma(u_k, u_l/u_p)$
G_2^k to u_l	$1 + d(u_k, u_l)$	$n_2 \sigma(u_k, u_l/u_p)$
u_k to u_l	$d(u_k, u_l)$	$\sigma(u_k, u_l/u_p)$

Table 1: Table detailing geodesics through u_p

The result follows. $\square$

Lemma 3.6. $T_p(u_p) = \sum_{p \neq k=1}^{n_1} \left[\sigma_{G_1}(u_p, u_k) [n_2(n_2 + 1)d(u_p, u_k) + n_2(1 + 2n_2)] \right] + 2 \left[\binom{n_2}{2} - m_2 \right]$

Proof. Table 2 gives the contribution of geodesics through u_p to $T_p(u_p)$. Considering every vertex $u_k, k \neq p$, the result follows. $\square$

Theorem 3.7. 1) $T(u_p) = T_p(u_p) + \sum_{kl} T_{kl}(u_p)$

2) $T(v_i^p) = 2 \times \text{number of geodesics in } G_2 \text{ of length 2 through } v_i.$

Vertices connected	Length	Number
u_k to G_2^p	$1 + d(u_k, u_p)$	$n_2 \sigma(u_k, u_p)$
G_2^k to G_2^p	$2 + d(u_k, u_p)$	$n_2^2 \sigma(u_k, u_p)$
G_2^p to G_2^p	2	$\binom{n_2}{2} - m_2$

Table 2: Geodesics through u_p

Proof. 1) Geodesics through u_p are either considered in $T_p(u_p)$ or in $T_{kl}(u_p)$. Hence the result is evident.

2) Follows from Proposition 3.4. $\square$

In the remaining sections we consider G_2 as arbitrary, while G_1 is replaced by various graph classes like $P_n, C_n, K_n, K_{m,n}$ and S_{m+1}

3.2 Path Graphs

Let P_n be the path graph with vertices $1, 2, \dots, n$. We give an expression for transit of k using Theorem 3.7 in $P_n \circ G_2$.

Theorem 3.8.

$$T(k) = \frac{(k-1)(n_2+1)(n-k)}{2} \left[(n_2+1)(n+1) + 4n_2 \right] +$$

$$n_2(n_2+1) \left[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2} \right] + n_2(2n_2+1)(n-1) + 2 \left[\binom{n_2}{2} - m_2 \right]$$

Proof. Let $1 \leq l < k < m \leq n$. Since G_1 is a path, we have $\sigma_{G_1}(l, m/k) = 1$. Hence $T_{lm}(k) = (n_2+1)^2(m-l) + 2n_2(n_2+1)$

$$\begin{aligned} \therefore \sum_{l,m} T_{lm}(k) &= (n_2+1)^2 \sum_{l=1}^{k-1} \sum_{m=k-1}^n (m-l) + \sum_{l=1}^{k-1} \sum_{m=k-1}^n 2n_2(n_2+1) \\ &= (n_2+1)^2 T_{G_1}(k) + (k-1)(n-k-1)2n_2(n_2+1) \\ &= (n_2+1)^2 \frac{(n+1)(k-1)(n-k)}{2} + (k-1)(n-k-1)2n_2(n_2+1) \\ &= \frac{(k-1)(n_2+1)(n-k)}{2} \left[(n_2+1)(n+1) + 4n_2 \right] \end{aligned}$$

In a similar manner we compute $T_k(k)$

$$\begin{aligned}
 T_k(k) &= \sum_{k \neq i=1}^n \left[(d(k, i) + 1)n_2 + (d(k, i) + 2)n_2^2 \right] \\
 &\quad + 2 \left[\binom{n_2}{2} - m_2 \right] \\
 &= (n_2 + n_2^2) \left[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2} \right] + n_2(2n_2+1)(n-1) \\
 &\quad + 2 \left[\binom{n_2}{2} - m_2 \right]
 \end{aligned}$$

The result follows. $\square$

In the following examples we compute transit for the vertices in various corona product of P_n . From Theorem 3.8, we have

$$\begin{aligned}
 T(k) &= \frac{(k-1)(n_2+1)(n-k)}{2} \left[(n_2+1)(n+1) + 4n_2 \right] \\
 &+ n_2(n_2+1) \left[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2} \right] + n_2(2n_2+1)(n-1) + 2 \left[\binom{n_2}{2} - m_2 \right] \\
 &= T_1 + T_2, \text{ say where } T_2 = 2 \left[\binom{n_2}{2} - m_2 \right]
 \end{aligned}$$

Examples

1. $G_2 = P_m$. Here $n_2 = m, m_2 = m-1$. Hence $T(k) = T_1 + (m-1)(m-2)$.
For pendant vertices of P_m^i , the transit is 0 and 2 for others, $\forall i$.
2. $G_2 = C_m$. Here $n_2 = m_2 = m$. For every vertex in C_m^i , there exist only one geodesic of length 2 through it.
Here $T(k) = T_1 + m(m-3)$ and $T(v_k^i) = 2, \forall k, i$.
3. $G_2 = K_m$. Then $n_2 = m, m_2 = \binom{m}{2}$.
Thus, $T(k) = T_1$ and $T(v_k^i) = 0, \forall k, i$.
4. $G_2 = S_m$. Here $n_2 = m, m_2 = m-1$. Hence $T(k) = T_1 + (m-1)(m-2)$.
 $T(v_k^i) = 0$, for pendant vertices and for central vertex of $S - m$, $T(v_k^i) = (m-1)(m-2)$
5. $G_2 = K_{l_1, l_2}$. $n_2 = m = l_1 + l_2$ and $m = l_1 l_2$
 $T(k) = T_1 + (m-1)m - 2l_1 l_2$ and $T(v_k^i) = T_{K_{l_1, l_2}}(v_k)$

3.3 Cycle

In this section we consider G_1 to be a cycle. We have already seen that transit of vertices in cycles with order $2n$ and $2n + 1$ are the same. Hence we consider $G_1 = C_{2n_1+1}$. We represent the vertices by $0, 1, \dots, 2n_1$. Also every vertex in the cycle being transit identical, it is enough we compute the transit for n_1 .

Theorem 3.9. *If a is any vertex of the cycle C_{2n} or C_{2n+1} , its transit in the corona product $C_{2n} \circ G_2$ or $C_{2n+1} \circ G_2$ is given by $T(a) = \frac{(n_2+1)(n_1-1)n_1}{3} [n_2n_1 + 4n_2 + n_1 + 1] + n_2n_1 [(n_2 + 1)(n_1 + 1) + 2(1 + 2n_2)] + 2 \left[\binom{n_2}{2} - m_2 \right]$*

Proof. For any k, l we know that $\sigma_{G_1}(k, l/n_1) = 1$.

$$\begin{aligned} T_{k,l}(n_1) &= \left[(n_2 + 1)^2 d(k, l) + 2n_2(n_2 + 1) \right] \\ \therefore \sum_{k,l} T_{k,l}(n_1) &= (n_2 + 1)^2 \sum_{k,l} d(k, l) + \frac{n_1(n_1 - 1)}{2} 2n_2(n_2 + 1) \\ &= (n_2 + 1)^2 T_{G_1}(n_1) + \frac{n_1(n_1 - 1)}{2} 2n_2(n_2 + 1) \\ &= (n_2 + 1)^2 \frac{(n^2 - 1)n}{24} + \frac{n_1(n_1 - 1)}{2} 2n_2(n_2 + 1) \\ &= \frac{(n_2 + 1)(n_1 - 1)n_1}{3} [n_2n_1 + 4n_2 + n_1 + 1] \end{aligned}$$

Next we compute $T_{n_1}(n_1)$

$$\begin{aligned} T_{n_1}(n_1) &= \sum_{n_1 \neq i=0}^{2n_1} \left[n_2(n_2 + 1)d(n, i) + n_2(1 + 2n_2) \right] \\ &\quad + 2 \left[\binom{n_2}{2} - m_2 \right], \sigma_{G_1}(n_1, i) \text{ being } 1 \\ &= n_2n_1 [(n_2 + 1)(n_1 + 1) + 2(1 + 2n_2)] + 2 \left[\binom{n_2}{2} - m_2 \right] \end{aligned}$$

And $T_{G_1 \circ G_2}(n_1) = \sum_{k,l} T_{k,l}(n_1) + T_{n_1}(n_1)$. Hence the proof. $\square$

3.4 Star

Let $G_1 = S_{n+1}$. In a star there are n pendant vertices and one central vertex. All pendant vertices are transit identical. Hence we need to compute transit of one of the pendant vertex and the central vertex in $S_{n+1} \circ G_2$. Let us name the vertices as $1, 2, \dots, n + 1$, where $n + 1$ is the central vertex.

Theorem 3.10. In $S_{n+1} \circ G_2$, $T(n+1) = n[(n-1)(n_2+1)(2n_2+1) + n_2(3n_2+2)] + 2\left[\binom{n_2}{2} - m_2\right]$ and $T(i) = n_2[(n_2+1)(2n-1) + n(2n_2+1)] + 2\left[\binom{n_2}{2} - m_2\right]$, $i \neq n+1$

Proof. Consider $n+1$. We have $\sigma(k, l/(n+1)) = 1$ and $d(k, l) = 2$

$$\text{Thus } T_{k,l}(n+1) = 2(n_2+1)(2n_2+1) \quad (1)$$

$$\therefore \sum_{k,l} T_{k,l}(n+1) = \binom{n}{2} 2(n_2+1)(2n_2+1) \quad (2)$$

$$= n(n-1)(n_2+1)(2n_2+1) \quad (3)$$

While computing $T_{n+1}(n+1)$, we see that $\sigma_{S_{n+1}}(n+1, i) = 1$ and $d(n+1, i) = 1, \forall i$. Thus we get $T_{n+1}(n+1) = nn_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$, which completes the computation for $T(n+1)$

Now consider the vertex $i \neq n+1$. It can easily be verified that $\sigma(k, l/i) = 0, \forall k, l$. Hence $\sum_{k,l} T_{k,l}(i) = 0$. for a fixed i , $\sigma(i, k) = 1 \forall k$ and $d(i, n+1) = 1$ and $d(i, k) = 2, k \neq n+1$

$\therefore T_i(i) = n_2[4nn_2 + 3n - n_2 - 1] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the proof. $\square$

3.5 Complete graph and Complete bipartite graph

Theorem 3.11. In the corona product $K_n \circ G_2$, the transit of any vertex of K_n is $(n-1)n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. Since every vertex of K_n is transit identical, we consider one of them. Let u_i be any vertex of K_n .

$$\sigma(u_k, u_l/u_i) = 0 \implies \sum T_{k,l}(u_i) = 0$$

Again $\sigma(u_i, u_k) = 1, \forall k \neq i$ and $d(u_i, u_k) = 1$

$\therefore T_i(u_i) = \sum_{k \neq i} [n_2(n_2+1) + n_2(2n_2+1)] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the result. $\square$

Next we consider a complete bipartite graph K_{l_1, l_2} with bipartition V_1, V_2 . Let $V_1 = a_1, a_2, \dots, a_{l_1}$ and $V_2 = b_1, b_2, \dots, b_{l_2}$. Then all a_i are transit identical. Similarly all b_i are also transit identical. Computation of $T(a_i)$ and $T(b_i)$ are similar. Hence we compute $T(a_i)$ only.

Theorem 3.12. In $K_{l_1, l_2} \circ G_2$, the transit of a_i , $T(a_i) = \binom{l_2}{2} 2(n_2+1)(2n_2+1) + l_2 n_2(4n_2+3)(l_1-1) + l_2 n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. The shortest path in K_{l_1, l_2} through a_i are those connecting vertices of V_2 . $\therefore T_{k,l}(a_i) = \sigma_{K_{l_1, l_2}}(b_k, b_l/a_i) [(n_2+1)^2 d(u_k, u_l) + 2n_2(n_2+1)]$. Thus $\sum_{k,l} T_{k,l}(a_i) = \binom{l_2}{2} 2(n_2+1)(2n_2+1)$.

While computing $T_i(a_i)$, we see that vertices in V_1 and V_2 behaves differently. Hence we split the summation as follows.

$$\begin{aligned} T_i(a_i) &= \sum_{a_j} \sigma(a_i, a_j) \left[n_2(n_2 + 1)d(a_i, a_j) + n_2(2n_2 + 1) \right] \\ &+ \sum_{b_j} \sigma(a_i, b_j) \left[n_2(n_2 + 1)d(a_i, b_j) + n_2(2n_2 + 1) \right] + 2 \left[\binom{n_2}{2} - m_2 \right] \\ &= l_2 n_2 (4n_2 + 3)(l_1 - 1) + l_2 n_2 (3n_2 + 2) + 2 \left[\binom{n_2}{2} - m_2 \right] \end{aligned}$$

□

4 Conclusion

In this paper, we first considered arbitrary graphs G_1 and G_2 . We could give an expression for the transit of vertices in their corona product. This result was applied to compute the transit of vertices in $G_1/circG_2$, where G_1 refers to a particular graph. It should be noted that we could express the transit of vertices in $G_1 \circ G_2$ in terms of individual graph parameters. Thus, the computation of transit in huge networks, which are Corona products, is now much easier.

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Transit in Corona product of graphs

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Connected end anti-fuzzy equitable dominating set in anti-fuzzy graphs

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Abstract

In this paper, the notion of connected end anti-fuzzy equitable dominating set of an anti-fuzzy graph is discussed. The connected end anti-fuzzy equitable domination number for some standard graphs are obtained. The relation between anti-fuzzy equitable domination number, end anti-fuzzy equitable domination number and connected end anti-fuzzy equitable domination number are established. Theorems related to these parameters are stated and proved.

Keywords: dominating set; end anti-fuzzy equitable dominating set; connected end anti-fuzzy equitable dominating set

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1. Introduction

A graph is an advantageous method for representing the data including connection between objects. The objects are represented by nodes and relations by arcs. Whenever there is uncertainty or vagueness in the description of items or in its connections or in both, it is common that we have to plan an anti-fuzzy graph model. A fuzzy set, as a superset of a crisp set, owes its origin to the work of Zadeh [14] in 1965 that has been introduced to deal with uncertainty. M. Akram [1] defined the concept of anti-fuzzy graph structures in 2012. A. Somasundaram and S.Somasundaram [12] presented several types of domination parameters such as independent domination, total domination, connected domination and domination in Cartesian product and composition of fuzzy graphs. R.Muthuraj and A. Sasireka [7] introduced domination in anti-fuzzy graphs. The concept of equitable domination in graphs was introduced by Swaminathan and Dharmalingam [13]. The end equitable domination number in graph has introduced by J.H.Hattingh and M.H.Henning [8]. Further results were extended by Murthy and Puttaswamy [10]. Some works in extension of fuzzy graphs can be found in [5, 6, 9]. S.Firthous Fatima and K.Janofer [2, 3, 4] introduced the concept of anti-fuzzy equitable dominating set, connected anti-fuzzy equitable dominating set and end anti-fuzzy equitable dominating set of an anti-fuzzy graphs. In this paper, the connected end anti-fuzzy equitable domination set of an anti-fuzzy graph is introduced. The connected end anti-fuzzy equitable domination number of an anti-fuzzy graphs is also obtained.

2. Preliminaries

Definition 2.1[1] A fuzzy graph is said to be an anti-fuzzy graph with a pair of functions $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$, where for all $u, v \in V$, we have $\mu(u, v) \geq \sigma(u) \vee \sigma(v)$ and it is denoted by $G_{AF}(\sigma, \mu)$ or $G(\sigma, \mu)$.

Definition 2.2 [1] The order p and size q of an anti-fuzzy graph $G_{AF} = (\sigma, \mu)$ are defined to be $p = \sum_{u \in V} \sigma(u)$ and $q = \sum_{uv \in E} \mu(u, v)$. It is denoted by $O(G)$ and $S(G)$.

Note 2.1 In all the examples, σ is chosen suitably and the function μ considered as reflexive and symmetric and G is an undirected anti-fuzzy graph.

Definition 2.3 An anti-fuzzy graph G_{AF} is said to be bipartite if the vertex set V can be partitioned into two sets σ_1 on V_1 and σ_2 on V_2 such that $\mu(v_1, v_2) = 0$ if $(v_1, v_2) \in V_1 \times V_1$ or $(v_1, v_2) \in V_2 \times V_2$.

Definition 2.4 A bipartite anti-fuzzy graph G_{AF} is said to be complete bipartite anti-fuzzy graph if $\mu(v_1, v_2) = \sigma(v_1) \vee \sigma(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$ and is denoted by K_{σ_1, σ_2} .

Definition 2.5 A path in an anti-fuzzy graph G_{AF} is a sequence of distinct vertices $u_0, u_1, u_2, \dots, u_n$ such that $\mu(u_{i-1}, u_i) = \sigma(u_{i-1}) \vee \sigma(u_i)$, $1 \leq i \leq n$, $n > 0$ is called the length of the path. The path in an anti-fuzzy graph is called an anti-fuzzy cycle if $u_0 = u_n$, $n \geq 3$.

Definition 2.6 An anti-fuzzy graph G_{AF} is said to be cyclic if it contains at least one anti-fuzzy cycle, otherwise it is called acyclic.

Definition 2.7 An anti-fuzzy graph G_{AF} is said to be connected if there exists at least one path between every pair of vertices. A connected acyclic anti-fuzzy graph is said to be an anti-fuzzy tree.

Definition 2.8 [4] Let G_{AF} be an anti-fuzzy graph and let $u, v \in V$. If $\mu(u, v) = \sigma(u) \vee \sigma(v)$ then u dominates v (or v dominates u) in G_{AF} . A set $D \subseteq V$ is said to be a dominating set of an anti-fuzzy graph G_{AF} if for every vertex $v \in V - D$ there exists $u \in D$ such that u dominates v .

Definition 2.9 [4] A dominating set D of an anti-fuzzy graph G_{AF} is called a minimal dominating set if there is no dominating set D' such that $D' \subset D$.

Definition 2.10 [5] The maximum scalar cardinality taken over all minimal dominating set is called anti-fuzzy domination number of an anti-fuzzy graph G_{AF} and is denoted by $\gamma_{AFG}^d(G_{AF})$.

Definition 2.11 [2] Let G_{AF} be an anti-fuzzy graph. Let v_1 and v_2 be two vertices of G_{AF} . A subset D of V is called a anti-fuzzy equitable dominating set if every $v_2 \in V - D$ there exist a vertex $v_1 \in D$ such that $v_1 v_2 \in E$ and $|d(v_1) - d(v_2)| \leq 1$ where $d(v_1)$ denotes the degree of vertex v_1 and $d(v_2)$ denotes the degree of vertex v_2 with $\mu(v_1, v_2) = \sigma(v_1) \vee \sigma(v_2)$.

Definition 2.12 [2] An anti-fuzzy equitable dominating set D of an anti-fuzzy graph G_{AF} is called a minimal anti-fuzzy equitable dominating set if there is no anti-fuzzy equitable dominating set D' such that $D' \subset D$. The maximum scalar cardinality taken over all minimal anti-fuzzy equitable dominating set is called anti-fuzzy equitable domination number and is denoted by γ_{AFG}^{ed} .

Definition 2.13 [4] An anti-fuzzy equitable dominating set S of a connected anti-fuzzy graph G_{AF} is called the end anti-fuzzy equitable dominating set if S contains all the terminal vertices.

Definition 2.14 [4] The maximum scalar cardinality taken over all minimal end anti-fuzzy equitable dominating set is called end anti-fuzzy equitable domination number of G_{AF} and it is denoted by γ_{AFG}^{eed} .

Definition 2.15 If for each $x \in V - S$ there exist a vertex $y \in S$ such that $xy \in E(G_{AF})$ and either one of the vertex x or y is with degree k and other vertex is with degree $k + 1$, then G_{AF} is called a bi-regular anti-fuzzy graph.

3. Connected end anti-fuzzy equitable dominating set

Definition 3.1 An end anti-fuzzy equitable dominating set S of an anti-fuzzy graph G_{AF} is called the connected end anti-fuzzy equitable dominating set (CEAFED-set) if induced anti-fuzzy subgraph $\langle S \rangle$ is connected.

Definition 3.2 The maximum scalar cardinality taken over all minimal connected end anti-fuzzy equitable dominating set is called connected end anti-fuzzy equitable domination number of G_{AF} and it is denoted by γ_{AFG}^{ceed} .

Example 3.3

Consider the following anti-fuzzy graph G_{AF} ,

In the anti-fuzzy graph G_{AF} , given in figure 3.1, the minimal connected end anti-fuzzy equitable dominating sets are $S_1 = \{v_3, v_4, v_5, v_6, v_7\}$ and $S_2 = \{v_2, v_3, v_4, v_5, v_6\}$.
 The scalar cardinality of $S_1 = |\{v_3, v_4, v_5, v_6, v_7\}| = 0.4 + 0.2 + 0.3 + 0.6 + 0.6 = 2.1$
 The scalar cardinality of $S_2 = |\{v_2, v_3, v_4, v_5, v_6\}| = 0.5 + 0.4 + 0.2 + 0.3 + 0.6 = 2$
 $\gamma_{AFG}^{ceed} = \max\{|S_1|, |S_2|\} = \max\{2.1, 2\} = 2.1$

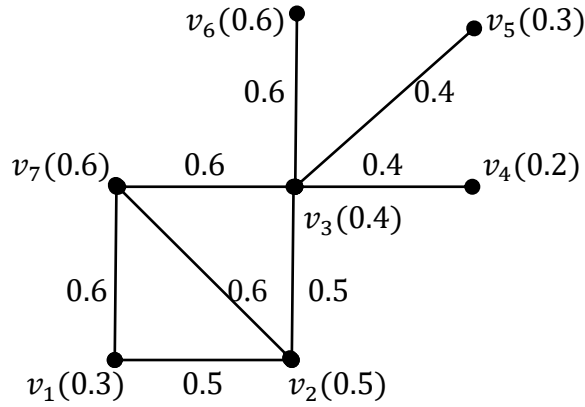


Figure 3.1: Anti-fuzzy graph G_{AF}

Therefore, connected end anti-fuzzy equitable domination number is $\gamma_{AFG}^{ceed} = 2.1$ corresponding to the connected end anti-fuzzy equitable dominating set S_1 .

Theorem 3.4

Let G_{AF} be any connected anti-fuzzy graph. Then

$$\gamma_{AFG}^d(G_{AF}) \leq \gamma_{AFG}^{ed}(G_{AF}) \leq \gamma_{AFG}^{eed}(G_{AF}) \leq \gamma_{AFG}^{ceed}(G_{AF}).$$

Proof:

Let G_{AF} be any connected anti-fuzzy graph.

Let $S \subseteq V(G_{AF})$ be any γ_{AFG}^{ceed} -set in G_{AF} .

Then obviously, S is also an end anti-fuzzy equitable dominating set in G .

$$\text{Therefore, } \gamma_{AFG}^{eed}(G_{AF}) = |S| \leq \gamma_{AFG}^{ceed}(G_{AF}) \quad (1)$$

Suppose let S' be any γ_{AFG}^{ed} -set of G_{AF} .

By definition of end anti-fuzzy equitable dominating set, S' is also an anti-fuzzy equitable dominating set of G_{AF} .

$$\text{Therefore, } \gamma_{AFG}^{ed}(G_{AF}) = |S'| \leq \gamma_{AFG}^{eed}(G_{AF}) \quad (2)$$

We know that, every anti-fuzzy equitable dominating set is an anti-fuzzy dominating set. Therefore, $\gamma_{AFG}^d(G_{AF}) \leq \gamma_{AFG}^{ed}(G_{AF}) \quad (3)$

Hence from (1), (2) and (3), we have

$$\gamma_{AFG}^d(G_{AF}) \leq \gamma_{AFG}^{ed}(G_{AF}) \leq \gamma_{AFG}^{eed}(G_{AF}) \leq \gamma_{AFG}^{ceed}(G_{AF}).$$

Remarks 3.5

The equality of theorem 3.4 can be hold when the anti-fuzzy graph G has no isolated vertices. For example, anti-fuzzy cycle and complete anti-fuzzy graphs can hold the equality condition.

Theorem 3.6

For any connected anti-fuzzy graph G_{AF} , $\gamma_{AFG}^{ced}(G_{AF}) \leq \gamma_{AFG}^{ceed}(G_{AF})$.

Proof:

Let $S \subseteq V$ be the minimum connected anti-fuzzy equitable dominating set of a connected anti-fuzzy graph G_{AF} . Then S is an anti-fuzzy dominating set of G and the induced anti-fuzzy subgraph $\langle S \rangle$ is connected.

Therefore, S is also a connected anti-fuzzy dominating set.

Clearly, any connected end anti-fuzzy equitable dominating set is also connected equitable dominating set.

Hence, $\gamma_{AFG}^{ced}(G_{AF}) \leq \gamma_{AFG}^{ceed}(G_{AF})$.

Theorem 3.7

For any k -regular anti-fuzzy graph for $k > 1$ then $\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF})$

Proof:

Let us assume that G_{AF} be a k -regular anti-fuzzy graph.

Then each vertex of G_{AF} has a same degree k .

Let S be the minimal connected anti-fuzzy equitable dominating set of G_{AF}

Then cardinality of $S = \gamma_{AFG}^{ced}(G_{AF})$.

If $u \in V - S$ then S is connected anti-fuzzy equitable dominating set, then there exists $v \in S$ and uv be the effective edge, also $d(u) = d(v) = k$.

Therefore $|d(u) - d(v)| = |k - k| = 0 \leq 1$.

Hence S is a connected end anti-fuzzy equitable dominating set of G such that

$$\gamma_{AFG}^{ced}(G_{AF}) \geq \gamma_{AFG}^{ceed}(G_{AF}) \quad (1)$$

$$\text{By theorem 3.6, we have } \gamma_{AFG}^{ced}(G_{AF}) \leq \gamma_{AFG}^{ceed}(G_{AF}). \quad (2)$$

Hence, from (1) and (2),

$$\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF})$$

Corollary 3.8

Let G_{AF} be $(k, k + 1)$ bi-regular anti-fuzzy graph. Then, $\gamma_{AFG}^{ced}(G) = \gamma_{AFG}^{ceed}(G)$.

Proof:

By theorem 3.6, $\gamma_{AFG}^{ced}(G) \leq \gamma_{AFG}^{ceed}(G)$. Now, let S be minimum connected end anti-fuzzy equitable set of $(k, k + 1)$ bi-regular anti-fuzzy graph. By the definition, the connected end anti-fuzzy equitable dominating set S is also an anti-fuzzy equitable

dominating set and $\langle S \rangle$ is connected, since G is $(k, k + 1)$ bi-regular anti-fuzzy graph. Therefore, S is also a connected end anti-fuzzy equitable dominating set.

Theorem 3.9

Let G_{AF} be an anti-fuzzy graph with n vertices then $\gamma_{AFG}^{ceed}(G) = \gamma_{AFG}^{eed}(G)$ if and only if G_{AF} has no end vertex and there is atleast one vertex $v \in V$ adjacent to $(n - 1)$ vertices in G_{AF} .

Proof:

Let G_{AF} be an connected anti-fuzzy graph with n vertices and without end vertex. Then, every vertex adjacent to atleast two vertices and there exists a one vertex say $u \in V(G_{AF})$ adjacent to $(n - 1)$ vertices, then the set $S = \{u\}$ is connected end anti-fuzzy equitable dominating set G .

By theorem 3.4, we get $\gamma_{AFG}^{eed}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF})$.

Conversely, suppose G_{AF} is connected anti-fuzzy graph and $\gamma_{AFG}^{ceed}(G_{AF}) = \gamma_{AFG}^{eed}(G_{AF})$ then G has no end vertex and there is $S = \{v\}$ which is connected end anti-fuzzy equitable dominating set. Therefore atleast any one vertex adjacent to $(n - 1)$ vertices in G .

Corollary 3.10

Let G_{AF} be an anti-fuzzy cycle with order p then $\gamma_{AFG}^{ceed}(G_{AF}) = p - \max \{ \min_{uv \in E} \{ \sigma(u), \sigma(v) \} \}$.

Proof:

Since $\gamma_{AFG}^{ced}(G_{AF}) = p - \max \{ \min_{uv \in E} \{ \sigma(u), \sigma(v) \} \}$ and $S = V - \{u, v\}$ is any subset of the vertices on the anti-fuzzy cycle G_{AF} such that u and v are adjacent vertices. Clearly S is connected end anti-fuzzy equitable set of G it means $\gamma_{AFG}^{ceed}(G_{AF}) \leq p - \max \{ \min_{uv \in E} \{ \sigma(u), \sigma(v) \} \}$ and by the theorem 3.6, we have, $p - \max \{ \min_{uv \in E} \{ \sigma(u), \sigma(v) \} \} = \gamma_{AFG}^{ceed}(G_{AF})$.

Hence $\gamma_{AFG}^{ceed}(G) = p - \max \{ \min_{uv \in E} \{ \sigma(u), \sigma(v) \} \}$.

Theorem 3.11

For any complete bipartite anti-fuzzy graph then

$$\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF}) = \begin{cases} \max_{u \in V_1} \{ \sigma(u) \} + \max_{v \in V_2} \{ \sigma(v) \}, & |d(u) + d(v)| \leq 1 \\ p + q, & |d(u) + d(v)| > 1 \end{cases}$$

Proof:

Case (1) : If $G \cong K_{m,n}$ and $|d(u) + d(v)| > 1$ for all $u \in V_1$ and $v \in V_2$ then the anti-fuzzy graph G_{AF} is totally anti-fuzzy equitable disconnected.

Therefore, $\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF}) = p + q$.

Case (2): If $G \cong K_{m,n}$ and $|d(u) + d(v)| \leq 1$ then if V_1 and V_2 be the partite sets of G_{AF} , be selecting one vertex $u \in V_1$ and $v \in V_2$ then $S = \{u, v\}$ is connected end anti-fuzzy equitable dominating set.

$$\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF}) \leq \max_{u \in V_1} \sigma(u) + \max_{v \in V_2} \sigma(v),$$

$$\text{but } \gamma_{AFG}^{ceed}(G_{AF}) \neq \max_{u \in V} \sigma(u).$$

$$\gamma_{AFG}^{ced}(G_{AF}) = \gamma_{AFG}^{ceed}(G_{AF}) = \max_{u \in V_1} \{\sigma(u)\} + \max_{v \in V_2} \{\sigma(v)\}.$$

Theorem 3.12

Let G_{AF} be connected anti-fuzzy graph with order m and N be the set of all end anti-fuzzy vertices of G_{AF} then $\gamma_{AFG}^{ceed}(G_{AF}) \geq |N| + \max \{\sigma(x)\}$, where $x \in V$ which is not an end anti-fuzzy vertex.

Proof:

Clearly if G_{AF} is connected anti-fuzzy graph with order m .

Let N be the set of all end anti-fuzzy vertices of G_{AF} . Let S be any connected end anti-fuzzy equitable dominating set of G_{AF} then all the end vertices and also supporting vertices must along to end anti-fuzzy equitable dominating set.

Hence $\gamma_{AFG}^{ceed}(G_{AF}) \geq |N| + \max \{\sigma(x)\}$.

4 Conclusions

A mathematical model helps to accomplish the problem in a complex situation. The possible solution is to convert the problem into a graph model. Anti-fuzzy graph theory has been used to model many decision making problems in uncertain situations. It have numerous applications in modern science in technology, computer science, especially in the field of information theory, neural network, cluster analysis, diagnosis and control theory etc., In this paper, the connected end anti-fuzzy equitable dominating set of an anti-fuzzy graph is defined. The relation between anti-fuzzy equitable domination number, end anti-fuzzy equitable domination number and connected end anti-fuzzy equitable domination number are established. Theorems related to these parameters are established. In future, we are going to establish these types of parameters in edge dominating sets of anti-fuzzy graphs.

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Minimal Reinhard Zumkeller divisor cordial graphs

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Abstract

In this paper, the notion of minimal Reinhard Zumkeller divisor cordial labeling has been introduced. Let $G = (V, E)$ be a simple graph and $\gamma : V(G) \rightarrow \text{minimum } \{2^i \times 3, 2^{j+1} \times 5, 2^{k+1} \times 7, 2^l \times 3 \times 5, 2^m \times 3 \times 7 \text{ where } i, j, k, l, m \geq 1\}$ be an injection such that the sum of the cardinality of exponent of $\gamma(V(G))$ should be equal to the order of the graph G . For each edge uv , assign the label 1 if $\gamma(u)|\gamma(v)$ or $\gamma(v)|\gamma(u)$ where $\gamma(u)$ and $\gamma(v)$ are Zumkeller numbers and the label 0 if $\gamma(u) \nmid \gamma(v)$ and also if $|e\gamma^l(0) - e\gamma^l(1)| \leq 1$ then γ is called minimal Reinhard Zumkeller divisor cordial labeling. This paper elucidates how the Zumkeller number, which is the generalization of the perfect number, goes along with the divisibility concept of the number theory and the cordial labeling technique. It also probes the existence of minimal Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete bipartite, complete graph K_n for $n < 17$, tadpole graph $T_{n,k}$ for all values of n and k .

Keywords: Zumkeller graph, divisor cordial labeling, Zumkeller divisor cordial graph.

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1 Introduction

Graphs regarded here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G . Most of the graph labeling methods trace their origin to the one introduced by Rosa [1967]. A graph labeling is an assignment of integer to the vertices or edges or both subject to certain conditions. Labelled graph has many branch out applications such as coding theory, missile guidance, X-ray, crystallography analysis, communication network addressing systems, astronomy, radar, circuit design, database management etc., The concept of cordial labeling was introduced by Cahit [1987]. Varatharajan et al. [2011] introduced divisor cordial labeling. If the sum of all the proper positive divisors of a positive integer is equal to the number, then the number is called perfect number. Generalizing the concept of perfect numbers R.H.Zumkeller defined a new type of number as a Zumkeller number. Peng and Rao [2013] established several results and conjectures on Zumkeller numbers. The notion of Zumkeller labeling of some cycle related graphs was investigated by Balamurugan et al. [2014]. Murali et al. [2017] proved results about Zumkeller cordial labeling of cycle related graphs. Shahbaz and Mahmood [2020] proved that Zumkeller number is either a super totient or a hyper totient number. Graph labeling has a potent communication between the number theory and graph network. The idea behind this work fosters us to develop a graph labeling technique called minimal Reinhard Zumkeller divisor cordial labeling by pooling the divisor cordial graph labeling technique and characteristics of Zumkeller number which is one of the engrossing parts of the number theory. Because of the existence of diverse γ vertex labeling design for some graph structure, minimal condition is emphasized. As the work has been focused on minimization condition, Zumkeller number chosen for the concept depicted herein is a sequence of least even Zumkeller numbers. An added reason for not using sequence of odd Zumkeller numbers is mainly due to point up the minimum sequence of Zumkeller numbers. Mahanta et al. [2020] stated that 945 is the smallest odd Zumkeller number. In this paper we discuss the existence of minimal Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete bipartite, complete graph K_n for $n < 17$, tadpole graph $T_{n,k}$ for all values of n and k .

2 Preliminaries

Definition 2.1. *Varatharajan et al. [2011] Let $G = (V, E)$ be a simple graph and $f : V \rightarrow \{1, 2, \dots, |V|\}$ be a bijection. For each edge uv , assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u) \nmid f(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.*

1. $e_{f^l}(0)$ is the number of edges of the graph G having label 0 under f^l
2. $e_{f^l}(1)$ is the number of edges of the graph G having label 1 under f^l

Definition 2.2. Peng and Rao [2013] A positive integer n is said to be a Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number n is a partition $\{A, B\}$ of the set of positive divisors of n so that each of A and B sums to the same value.

Proposition 2.3. Peng and Rao [2013] For any prime $p \neq 2$ and positive integer k with $p \leq 2^{k+1} - 1$, the number $2^k p$ is a Zumkeller number.

Fact 2.4. Peng and Rao [2013] Let the prime factorization of an even Zumkeller number n be $2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ where k is a positive integer. Then at least one of k_i must be odd.

Definition 2.5. Balamurugan et al. [2014] A simple graph $G = (V, E)$, where V is vertex set and E is edge set of G is said to admit a Zumkeller labeling if there exists an injective function $f : V \rightarrow N$ such that $f^* : E \rightarrow N$ defined as $f^*(xy) = f(x)f(y)$ is a Zumkeller number for $xy \in E; x, y \in V$. The labelled graph G is called as a Zumkeller graph.

Definition 2.6. Murali et al. [2017] Let $G = (V, E)$ be a graph. An injective function $f : V \rightarrow N$ is said to be a Zumkeller cordial labeling of the graph G if there exists an induced function $f^* : E \rightarrow \{0, 1\}$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions

1. For every $xy \in E$, $f^*(xy) = \begin{cases} 1, & f(x)f(y) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise} \end{cases}$
2. $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$

Definition 2.7. Murali et al. [2017] A graph $G = (V, E)$ which admits a Zumkeller cordial labeling is called a Zumkeller cordial graph.

3 Main Results

The vertex labeling γ mention in the definition 3.1 is defined by using Proposition 2.3 and fact 2.4.

Definition 3.1. Let $G = (V, E)$ be a simple graph and $\gamma : V(G) \rightarrow \text{minimum } \{2^i \times 3, 2^{j+1} \times 5, 2^{k+1} \times 7, 2^l \times 3 \times 5, 2^m \times 3 \times 7 \text{ where } i, j, k, l, m \geq 1\}$ be an injection satisfying any one of the following conditions $|i| + |j| = |V(G)|$ or $|i| + |k| = |V(G)|$ or $|i| + |l| = |V(G)|$ or $|i| + |m| = |V(G)|$ or $|i| + |j| + |k| =$

$|V(G)|$ or $|i|+|j|+|l| = |V(G)|$ or $|i|+|j|+|m| = |V(G)|$ or $|i|+|j|+|k|+|l| = |V(G)|$ or $|i|+|j|+|k|+|m| = |V(G)|$ or $|i|+|j|+|k|+|l|+|m| = |V(G)|$ i.e., the sum of the cardinality of exponent of $\gamma(V(G))$ should be equal to the order of the graph G . For each edge uv , assign the label 1 if $\gamma(u) \mid \gamma(v)$ or $\gamma(v) \mid \gamma(u)$ where $\gamma(u)$ and $\gamma(v)$ are Zumkeller numbers and the label 0 if $\gamma(u) \nmid \gamma(v)$ and also if $|e\gamma^l(0) - e\gamma^l(1)| \leq 1$ then γ is called minimal Reinhard Zumkeller divisor cordial labeling. A graph with a minimal Reinhard Zumkeller divisor cordial labeling is called a minimal Reinhard Zumkeller divisor cordial graph.

Theorem 3.2. *The path P_n is a minimal Reinhard Zumkeller divisor cordial when $n \equiv 0, 1 \pmod{2}$*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path P_n . Label those consecutive adjacent vertices in the order as $2^i \times 3$ and $2^l \times 3 \times 5$ where $1 \leq i \leq \frac{n}{2}$ and $1 \leq l \leq \frac{n}{2}$ for the path having even number of vertices and for the path having odd number of vertices $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq l \leq \frac{n-1}{2}$ and also $|i| + |l| = |V(G)|$. If $2^i \times 3 \mid 2^l \times 3 \times 5$ then the consecutive adjacent vertices contribute 1 to each edge and if $2^i \times 3 \nmid 2^l \times 3 \times 5$ then the consecutive adjacent vertices contribute 0 to each edge. Thus $e\gamma^l(1) = \frac{n}{2}$ and $e\gamma^l(0) = \frac{n-2}{2}$ if n is even and $e\gamma^l(1) = e\gamma^l(0) = \frac{n-1}{2}$ if n is odd. Hence $|e\gamma^l(0) - e\gamma^l(1)| \leq 1$. Thus P_n is a minimal Reinhard Zumkeller divisor cordial graph. $\square$

Theorem 3.3. *The cycle C_n is a minimal Reinhard Zumkeller divisor cordial when $n \equiv 1 \pmod{2}$, $n \geq 3$, $n \in \mathbb{N}$*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n . By making use of the similar pattern described as for path, the cycle of odd order is investigated as a minimal Reinhard Zumkeller divisor cordial graph. $\square$

Theorem 3.4. *The cycle C_n admits a minimal Reinhard Zumkeller divisor cordial when the vertex v_n is labelled with $2 \times 3 \times 7$ where $n \equiv 0 \pmod{2}$, $n \geq 4$, $n \in \mathbb{N}$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n . Label the vertex v_n with $2 \times 3 \times 7$ and appertain with the similar pattern described as for path for the remaining vertices results in a minimal Reinhard Zumkeller divisor cordial labeling for the cycle graph. $\square$

Theorem 3.5. *The Wheel graph $W_n = K_1 + C_n$ is a minimal Reinhard Zumkeller divisor cordial*

Proof. Let v_o be the center vertex of W_n and label the center vertex as $2^2 \times 5$

Case 1. n is even.

Label the vertices $v_1, \dots, v_n$ of C_n as $2^i \times 3$ where $1 \leq i \leq n$ such that $\gcd((20, \gamma(v_1)), (20, \gamma(v_2)), \dots, (20, \gamma(v_n))) = 1$ and $\gcd(\gamma(v_g), \gamma(v_{g+1})) > 1$ where $1 \leq g \leq n-1$ and also $|i| + |j| = n+1 = n+1 = |V(W_n)|$.

We observe that,

$$e\gamma^l(0) = e\gamma^l(1) = n. \quad (1)$$

Case 2. n is odd.

Label the center vertex v_o as $2^2 \times 5$ and label $v_1, \dots, v_n$ of C_n with the same labeling design mentioned in case 1

Here also,

$$e\gamma^l(0) = e\gamma^l(1) = n \quad (2)$$

Hence, from (1) and (2) we get that $|e\gamma^l(0) - e\gamma^l(1)| = \begin{cases} 0 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \square$

Theorem 3.6. *The star graph $K_{1,s}$ is a minimal Reinhard Zumkeller divisor cordial labeling when $s \equiv 0, 1 \pmod{2}$*

Proof. Let $G = K_{1,s}$ be the star graph with vertex set $V(G) = \{v_0 \cup \{v_g : 1 \leq g \leq s\}\}$ where v_0 is a center vertex and v_g 's are pendant vertices and an edge set $E(G) = \{e_g = v_0 v_g : 1 \leq g \leq s\}$. Here we notice that the order of the graph $|V(G)| = s + 1$ and the size of the graph $|E(G)| = s$.

Case 1. $s \equiv 1 \pmod{2}$

Assume $\gamma(v_0) = 2 \times 3$ which is a Zumkeller number.

The pendant vertices contribute 1 to its adjacent edges are labelled as follows.

$\gamma(v_i) = 2^{i+1} \times 3$ for $1 \leq i \leq \frac{s-1}{2} + 1$ and the pendant vertices contributes 0 to its adjacent edges are labelled as follows $\gamma(v_j) = 2^{j+1} \times 5$ for $1 \leq j \leq \frac{s-1}{2}$. And also $|i| + |j| = s + 1$

Case 2. $s \equiv 0 \pmod{2}$

Assume $\gamma(v_0) = 2 \times 3$ which is a Zumkeller number.

The pendant vertices contribute 1 to its adjacent edges are labelled as follows.

$\gamma(v_i) = 2^{i+1} \times 3$ for $1 \leq i \leq \frac{s}{2}$ and the pendant vertices contributes 0 to its adjacent edges are labelled as follows $\gamma(v_j) = 2^{j+1} \times 5$ for $1 \leq j \leq \frac{s}{2}$. And also $|i| + |j| = s + 1$

Hence from cases 1 and 2, we get that $e\gamma^l(0) = \frac{s+1}{2}$ and $e\gamma^l(1) = \frac{s-1}{2}$ when m is odd and $e\gamma^l(0) = e\gamma^l(1) = \frac{s}{2}$ when s is even.

$$\text{Hence } |e\gamma^l(0) - e\gamma^l(1)| = \begin{cases} 0 & \text{if } s \text{ is even} \\ 1 & \text{if } s \text{ is odd} \end{cases}$$

Thus $|e\gamma^l(0) - e\gamma^l(1)| \leq 1$. Hence, $k_{1,s}$ is a minimal Reinhard Zumkeller divisor cordial. $\square$

Theorem 3.7. *The complete bipartite graph $K_{x,z}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of $x, y \geq 2$*

Proof. Let $V = V_1 \cup V_2$ be the bipartition of $K_{x,z}$ such that $V_1 = \{v_1, v_2, \dots, v_x\}$ and $V_2 = \{w_1, w_2, \dots, w_z\}$. The order of the complete bipartite graph $K_{x,z}$ is $x + z = f$.

Case 1. $x = z$ where x and z are even

Obviously, there are $\frac{f}{2}$ vertices in V_1 and $\frac{f}{2}$ vertices in V_2 . Then label $\frac{f}{4}$ vertices out of $\frac{f}{2}$ vertices as $2^i \times 3$, where $1 \leq i \leq \frac{f}{2}$ and the remaining $\frac{f}{4}$ vertices get the label as $2^{j+1} \times 5$, where $1 \leq j \leq \frac{f}{2}$. Label $\frac{f}{4}$ vertices in V_2 as $2^i \times 3$, where $\frac{f}{2} + 1 \leq i \leq z$ and the remaining $\frac{f}{4}$ vertices as $2^{j+1} \times 5$ where $\frac{f}{2} + 1 \leq j \leq z$. Then the cordiality condition $|e\gamma^l(0) - e\gamma^l(1)| = 0$.

Case 2. $x = z$, when x and z are odd.

Label $\frac{f}{2}$ vertices in V_1 as follows: Label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2}$ in V_1 as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f}{2}+1}{2}$ and label the remaining vertices $\frac{f}{2} - \frac{(\frac{f}{2}+1)}{2}$ vertices are labelled as $2^{i+1} \times 5$, where $1 \leq j \leq \frac{f}{2} - \frac{\frac{f}{2}+1}{2}$. Then label $\frac{f}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2}$ in V_2 as $2^i \times 3$, where $\frac{(\frac{f}{2}+1)}{2} + 1 \leq i \leq z + 1$ and label the remaining vertices $z - \frac{\frac{f}{2}+1}{2}$ in V_2 as $2^{j+1} \times 5$, where $\frac{f}{2} - \frac{(\frac{f}{2}+1)}{2} + 1 \leq i \leq z - 1$. Then the cordiality condition $|e\gamma^l(0) - e\gamma^l(1)| = 1$.

Case 3. $x < z$ and $x + z$ where $z = x + 1$, x is odd and z is even

There are $\frac{f+1}{2} - 1$ vertices in V_1 and $\frac{f+1}{2}$ vertices in V_2 . Label $\frac{f+1}{2} - 1$ vertices in V_1 as follows: Label $\frac{\frac{f+1}{2}}{2}$ vertices out of $\frac{f+1}{2} - 1$ in V_1 as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f+1}{2}}{2}$ and the remaining vertices $\frac{f+1}{2} - 1 - \frac{\frac{f+1}{2}}{2}$ are labelled as $2^{j+1} \times 5$, where $1 \leq j \leq \frac{f+1}{2} - 1 - \frac{\frac{f+1}{2}}{2}$. Label $\frac{f+1}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f+1}{2}}{2}$ vertices as $2^i \times 3$, where $\frac{\frac{f+1}{2}}{2} + 1 \leq i \leq z$ and label the left over vertices $\frac{f+1}{2} - \frac{\frac{f+1}{2}}{2}$ as $2^{j+1} \times 5$, where $\frac{f+1}{2} - \frac{\frac{f+1}{2}}{2} \leq j \leq z - 1$. Following the labeling pattern results in $|e\gamma^l(0) - e\gamma^l(1)| = 1$.

Case 4. $x < z$ and $z = x + 1$, where x is even and z is odd.

There are $\frac{f+1}{2} - 1$ vertices in V_1 and $\frac{f+1}{2}$ vertices in V_2 . Label $\frac{f+1}{2} - 1$ vertices in V_1 as follows: Label $\frac{\frac{f+1}{2}-1}{2}$ vertices as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f+1}{2}-1}{2}$ and label

$\frac{f+1}{2} - 1 - \frac{\frac{f+1}{2}-1}{2}$ vertices as $2^{j+1} \times 5$, where $1 \leq j \leq \frac{\frac{f+1}{2}-1}{2}$. Now proceed to label $\frac{f+1}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f+1}{2}-1}{2} + 1$ vertices as $2^i \times 3$, where $\frac{\frac{f+1}{2}-1}{2} + 1 \leq i \leq z$ and label the left over vertices $\frac{f+1}{2} - \frac{\frac{f+1}{2}-1}{2} + 1$ as $2^{j+1} \times 5$, where $\frac{f+1}{2} - \frac{\frac{f+1}{2}-1}{2} + 1 \leq j \leq z - 1$. Then the cordiality condition $|e\gamma^l(0) - e\gamma^l(1)| = 0$.

Case 5. $x > z$ and $x = z + 2$.

Obviously, there are $\frac{f}{2} + 1$ vertices in V_1 and $\frac{f}{2} - 1$ vertices in V_2 . Then label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2} + 1$ vertices as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f}{2}+1}{2}$ and the remaining $\frac{\frac{f}{2}+1}{2}$ vertices get the label as $2^{j+1} \times 5$, where $1 \leq j \leq \frac{\frac{f}{2}+1}{2}$. Likewise label $\frac{f}{2} - 2$ vertices out of $\frac{f}{2} - 1$ in V_2 as $2^i \times 3$, where $\frac{\frac{f}{2}+1}{2} + 1 \leq i \leq z + 1$ and the remaining $\frac{f}{2} - 1 - \frac{f}{2} - 2$ vertices as $2^{j+1} \times 5$ where $\frac{\frac{f}{2}+1}{2} + 1 \leq j \leq z + 1$. Then the cordiality condition $|e\gamma^l(0) - e\gamma^l(1)| = 0$.

Proceeding like this for all values of x and z , the cordiality condition is satisfied. Hence the complete bipartite is a minimal Reinhard Zumkeller divisor cordial graph. $\square$

Theorem 3.8. The tadpole $T_{n,k}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of n and k

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $w_1, w_2, \dots, w_k$ be the vertices of the path P_k . Let $T_{n,k}$ be the repercussion graph obtained by recognizing a vertex of cycle C_n to an end vertex of the path P_k . Then the order of $T_{n,k}$ graph is $|V(T_{n,k})| = n+k$ and the size of $T_{n,k}$ graph is $|E(T_{n,k})| = n+k$. Concatenate the pendant vertex of P_k to one of the vertices of C_n with an edge in

$$\text{such a way that } \begin{cases} v_{\frac{n+3}{2}=w_1} \text{ for } n \equiv 1 \pmod{2} \text{ if } \frac{n+3}{2} \text{ is even} \\ v_{\frac{n+3}{2}-1=w_1} \text{ for } n \equiv 1 \pmod{2} \text{ if } \frac{n+3}{2} \text{ is odd} \\ v_{\frac{n+2}{2}} = w_1 \text{ for } n \equiv 0 \pmod{2} \text{ if } \frac{n+2}{2} \text{ is even} \\ v_{\frac{n+2}{2}-1} = w_1 \text{ for } n \equiv 0 \pmod{2} \text{ if } \frac{n+2}{2} \text{ is odd} \end{cases}.$$

We contemplate the following cases. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n be labelled as follows:

Case 1. $n \equiv 1 \pmod{2}$ and $K = 1$.

Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n be labelled as follows:

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \text{ where } g \equiv 1 \pmod{2} \text{ and } 1 \leq i \leq \frac{n+1}{2} \\ 2^l \times 3 \times 5 \text{ where } g \equiv 0 \pmod{2} \text{ and } 1 \leq l \leq \frac{n+1}{2} - 1 \end{cases} \quad (3)$$

Let the vertices of the path P_h w_1, w_2 be labelled as follows.

$$\gamma\left(v_{\frac{n+3}{2}}\right) = \gamma(w_1) \quad (4)$$

$$\gamma(w_2) = 2^{\frac{n+1}{2}+1} \times 3 \quad (5)$$

In regards to the labeling designs (3),(4),(5), we get that $e_\gamma^\perp(0) = \frac{n+K}{2}$; $e_\gamma^\perp(1) = \frac{n+K}{2}$. Hence $|e_\gamma^\perp(0) - e_\gamma^\perp(1)| = 0$.

Case 2. $n \equiv 1 \pmod{2}$ and $k \geq 2$, where k is even

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \text{ where } g \equiv 1 \pmod{2} \text{ and } 1 \leq i \leq \frac{n+1}{2} \\ 2^l \times 3 \times 5 \text{ where } g \equiv 0 \pmod{2} \text{ and } 1 \leq l \leq \frac{n+1}{2} - 1 \end{cases} \quad (6)$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \text{ where } h \equiv 1 \pmod{2}, 3 \leq h \leq k+1 \text{ and } \frac{n+1}{2} + 1 \leq i \leq \frac{n+1+k}{2} \\ 2^l \times 3 \times 5 \text{ where } h \equiv 0 \pmod{2}, 2 \leq h \leq k \text{ and } \frac{n+1}{2} \leq l \leq \frac{n+1+k}{2} - 1 \end{cases} \quad (7)$$

Hence from (6) and (7), we get that $e_\gamma^\perp(1) = \frac{n+k+1}{2}$; $e_\gamma^\perp(0) = \frac{n+k-1}{2}$.

Hence $|e_\gamma^\perp(0) - e_\gamma^\perp(1)| = 1$.

Case 3. $n \equiv 0 \pmod{2}$ and $k \geq 1$ where k is odd

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \text{ where } g \equiv 1 \pmod{2} \text{ and } 1 \leq i \leq \frac{n}{2} \\ 2^l \times 3 \times 5 \text{ where } g \equiv 0 \pmod{2} \text{ and } 1 \leq l \leq \frac{n}{2} \end{cases} \quad (8)$$

Let the vertices of the path $P_h, w_1, w_2, \dots, w_h$ be labelled as follows,

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \text{ where } h \equiv 0 \pmod{2}, 2 \leq h \leq k+1 \text{ and } \frac{n}{2} + 1 \leq i \leq \frac{n+k+1}{2} \\ 2^l \times 3 \times 5 \text{ where } h \equiv 1 \pmod{2}, 3 \leq h \leq k \text{ and } \frac{n}{2} + 1 \leq l \leq \frac{n+k+1}{2} - 1 \end{cases} \quad (9)$$

In regards to the above labeling design (8) and (9), we get that $e_\gamma^\perp(1) = \frac{n+k-1}{2}$; $e_\gamma^\perp(0) = \frac{n+k+1}{2}$.

Hence $|e_\gamma^\perp(0) - e_\gamma^\perp(1)| = 1$.

Case 4. $n \equiv 1 \pmod{2}$ where $k \geq 1$ where k is odd

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \text{ where } g \equiv 1 \pmod{2} \text{ and } 1 \leq i \leq \frac{n+1}{2} \\ 2^l \times 3 \times 5 \text{ where } g \equiv 0 \pmod{2} \text{ and } 1 \leq l \leq \frac{n+1}{2} - 1 \end{cases} \quad (10)$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \text{ where } h \equiv 1 \pmod{2}, 3 \leq h \leq k \text{ and } \frac{n+1}{2} + 1 \leq i \leq \frac{n+k}{2} \\ 2^l \times 3 \times 5 \text{ where } h \equiv 0 \pmod{2}, 2 \leq h \leq k-1 \text{ and } \frac{n+1}{2} + 1 \leq l \leq \frac{n+k}{2} - 1 \end{cases} \quad (11)$$

$$\gamma(w_{k+1}) = 2^2 \times 5 \quad (12)$$

In regards to the labeling design (10), (11) and (12), we get that $e_\gamma^\perp(1) = \frac{n+k}{2}$;

$$e_\gamma^\perp(0) = \frac{n+k}{2}$$

Hence $|e_\gamma^\perp(0) - e_\gamma^\perp(1)| = 0$

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Case 5. $n \equiv 0 \pmod{2}$ where $K \geq 2$ and K is even

$$\gamma(v_g) = \begin{cases} 2^i \times 3 & \text{where } g \equiv 1 \pmod{2} \text{ and } 1 \leq i \leq \frac{n}{2} \\ 2^l \times 3 \times 5 & \text{where } g \equiv 0 \pmod{2} \text{ and } 1 \leq l \leq \frac{n}{2} \end{cases} \quad (13)$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3 & \text{where } h \equiv 0 \pmod{2}, 2 \leq h \leq k \text{ and } \frac{n}{2} + 1 \leq i \leq \frac{n+k}{2} \\ 2^l \times 3 \times 5 & \text{where } h \equiv 1 \pmod{2}, 3 \leq h \leq k-1 \text{ and } \frac{n}{2} + 1 \leq l \leq \frac{n+k}{2} - 1 \end{cases} \quad (14)$$

$$\gamma(w_{k+1}) = 2^2 \times 5 \quad (15)$$

In regards to the labeling design (13), (14) and (15), we get that $e_\gamma^\perp(1) = \frac{n+k}{2}$; $e_\gamma^\perp(0) = \frac{n+k}{2}$ Hence $|e_\gamma^\perp(0) - e_\gamma^\perp(1)| = 0$

Hence from all cases we get that the tadpole $T_{n,k}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of n and k . $\square$

Theorem 3.9. *The complete graph K_n is a minimal Reinhard Zumkeller divisor cordial if $n \leq 16$*

Proof. Obviously K_1, K_2, K_3 are minimal Reinhard Zumkeller divisor cordial graph. The following table 1 brings forth a minimal Reinhard Zumkeller divisor cordial labeling of K_n for $4 \leq n < 17$

Order of K_n	Vertex labels	Cordiality Condition
4	6,12,20,24	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$
5	6,12,28,30,60	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$
6	6,12,24,28,30,60	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
7	6,12,24,28,30,48,60	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
8	6,12,24,28,30,48,60,96	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$
9	6,12,24,28,30,48,60,96,168	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$
10	6,12,24,28,30,48,60,96,168,192	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
11	6,12,24,28,30,48,60,84,96,168,192	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
12	6,12,24,28,30,48,60,84,96,120,168,192	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$
13	6,12,24,28,30,48,60,84,96,120,168,192,384	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
14	6,12,24,28,30,48,60,84,96,120,168,192,240,384	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
15	6,12,24,28,30,48,60,84,96,120,168,192,240,384,768	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 1$
16	6,12,24,28,30,48,60,84,96,120,168,192,240,336,384,768	$e_\gamma^\perp(0) - e_\gamma^\perp(1) = 0$

Table 1: Minimal Reinhard Zumkeller divisor cordial labeling of K_n for $4 \leq n < 17$

K_{17} is not Reinhard Zumkeller divisor cordial graph. Since by following the labeling pattern of K_{16} the vertex labels to be selected for the vertex v_{17} must be

anyone of them: 480 or 1536 or 672 or by choosing 20 instead of 28 from the above labeling pattern then the cordiality condition is $|e\gamma^l(1) - e\gamma^l(0)| = 69 - 67 = 2$ and $|e\gamma^l(0) - e\gamma^l(1)| = 69 - 67 = 2$ respectively. Since the labeling pattern for each complete graph K_n follows the labeling pattern of its predecessor, for all higher order complete graphs the cordiality condition increases by 1 for each $n \geq 17$. $\square$

4 Discussion

For the notion of minimal Reinhard Zumkeller divisor cordial labeling, this effort has produced several fresh findings. In order to create a minimal Reinhard Zumkeller divisor cordial graph and introduce a new element to the labeling pattern of various graph structures, the traits of the Zumkeller number are unified with the divisor cordial graph labeling technique. The results that are established in this paper are amalgamated and motivated us to get into the conclusion that for every connected minimal Reinhard Zumkeller divisor cordial graph G , $\gamma(u) \equiv 0(mod 6)$ for some vertex $u \in V(G)$. Deriving similar results for other graph families is an open problem.

5 Conclusions

In the present investigation, minimal Reinhard Zumkeller divisor cordial labeling has been introduced and probed for the existence of Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete graph K_n for $n < 17$, complete bipartite and tadpole graph $T_{n,k}$ for all values of n and k . In future research work, we will develop findings to construct dense minimal Reinhard Zumkeller divisor cordial graphs, book graphs with polygonal pages, generalized Petersen graphs, wheel graphs and product related graphs.

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Detour Global Domination for Degree Splitting Graphs of Some Graphs

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Abstract

In this paper, we introduced the new concept detour global domination number for degree splitting graph of standard graphs. A set S is called a *detour global dominating set* of $G = (V, E)$ if S is both detour and global dominating set of G . The *detour global domination number* is the minimum cardinality of a detour global dominating set in G . Let $V(G)$ be $S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$, where $1 \leq i \leq t$. The *degree splitting graph* $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining w_i to each vertex of S_i for $i = 1, 2, \dots, t$. In this article we recollect the concept of degree splitting graph of a graph and we produced some results based on the detour global domination number for degree splitting graph of path graph, cycle graph, star graph, bistar graph, complete bipartite graph and complete graph. **Keywords:** Detour set, Dominating set, Detour Domination, Global Domination, Detour Global Domination, Degree Splitting graphs.

2020 AMS subject classifications: 05C12, 05C69.¹

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to West[1]. For vertices x and y in a connected graph G , the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G [2]. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G . For $S \subseteq V$, $I_D[S] = \cup_{x, y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set* [3].

A set $S \subseteq V(G)$ in a graph G is a *dominating set* of G if for every vertex v in $V - S$, there exists a vertex $u \in S$ such that v is adjacent to u . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [4]. The complement $\overline{G}$ of a graph G also has $V(G)$ as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G . A set $S \subseteq V(G)$ is called a *global dominating set* of G if it is a dominating set of both G and $\overline{G}$ [5].

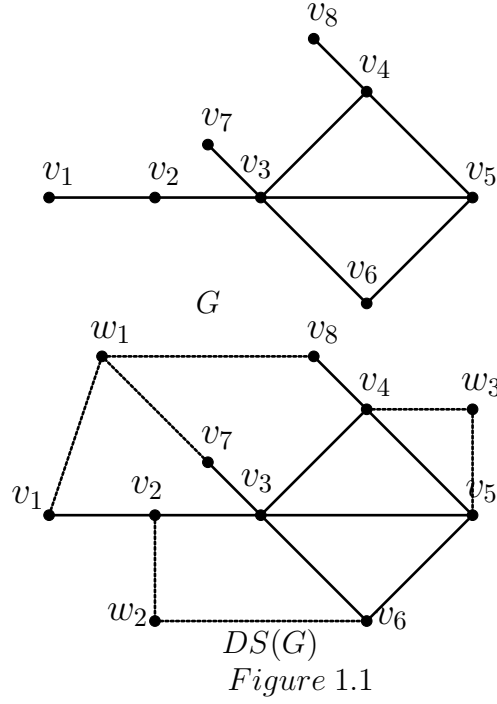
A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called an *end vertex* or a *pendant vertex*. A vertex that is adjacent to a pendant vertex is called a *support vertex*.

Definition 1.1. Let $G = (V, E)$ be a connected graph with atleast two vertices. A set $S \subseteq V(G)$ is said to be a *detour global dominating set* of G if S is both detour and global dominating set of G . The *detour global domination number*, denoted by $\bar{\gamma}_d(G)$ is the minimum cardinality of a detour global dominating set of G and the detour global dominating set with cardinality $\bar{\gamma}_d(G)$ is called the $\bar{\gamma}_d$ -set of G or $\bar{\gamma}_d(G)$ -set.[6]

In [7], R. Ponraj and S. Somasundaram have initiated a study on degree splitting graph $DS(G)$ of a graph G which is defined as follows:

Definition 1.2. Let $G = (V, E)$ be a graph with $V(G) = S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$, where $1 \leq i \leq t$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining w_i to each vertex of S_i for $i = 1, 2, \dots, t$.

Example 1.1. In Figure 1.1, a graph G and its degree splitting graph $DS(G)$ are shown.



Here, $S_1 = \{v_1, v_7, v_8\}$, $S_2 = \{v_2, v_6\}$, $S_3 = \{v_4, v_5\}$ and $T = \{v_3\}$.

Remark 1.1. If $V(G) = \cup S_i, 1 \leq i \leq t$, then $T = \phi$.

2 Some basic results

In this section, we recall some basic results of detour global domination number of a graph which will be used throughout the paper.

Theorem 2.1. [6] Every isolated vertex of G belongs to every detour global dominating set of G .

Theorem 2.2. [6] Every full vertex of a connected graph G of order p belongs to every detour global dominating set of G .

Theorem 2.3. [6] For the path graph P_p , ($p \geq 4$), $\bar{\gamma}_d(P_p) = \gamma_d(P_p) = \lceil \frac{p+2}{3} \rceil$.

Theorem 2.4. [6] For any star graph $K_{1,n-1}$, ($n \geq 2$), $\bar{\gamma}_d(K_{1,n-1}) = n$.

Theorem 2.5. [6] For any bistar graph $B_{m,n}$, $m, n \geq 1$, $\bar{\gamma}_d(B_{m,n}) = m + n$.

Theorem 2.6. [6] For $m, n \geq 2$, $\bar{\gamma}_d(K_{m,n}) = 2$.

3 Degree Splitting graphs of known graphs and their Detour Global Domination number

Let us find detour global domination number of degree splitting graph $DS(G)$ of the graphs path, cycle, star, bistar, complete bipartite and complete graph.

Theorem 3.1. For any integer $n \geq 3$, $\bar{\gamma}_d(DS(P_n)) = 2$.

Proof. Let $v_1v_2 \cdots v_n$ be the path P_n with partitions $S_1 = \{v_2, v_3, \cdots, v_{n-1}\}$ and $S_2 = \{v_1, v_n\}$. To obtain $DS(P_3)$ from P_3 we add x , which corresponds to S_2 also P_3 is isomorphic to C_4 and to obtain $DS(P_n)$ for $n \geq 4$ we add x_1 and x_2 , which corresponds to S_1 and S_2 , respectively. As a result, $V(DS(P_3)) = \{x, v_1, v_2, v_3\}$ and $V(DS(P_n)) = \{x_1, x_2, v_1, v_2, \cdots, v_n\}$, where $|V(DS(P_n))| = n + 2$ for $n \geq 4$.

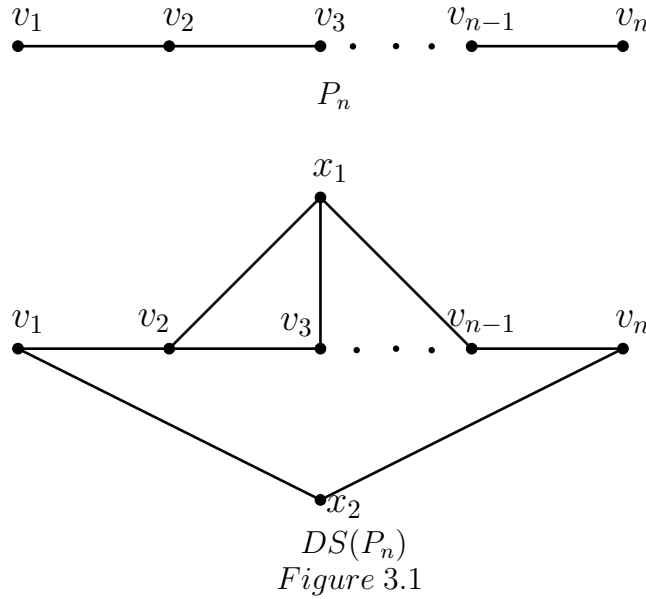


Figure 3.1

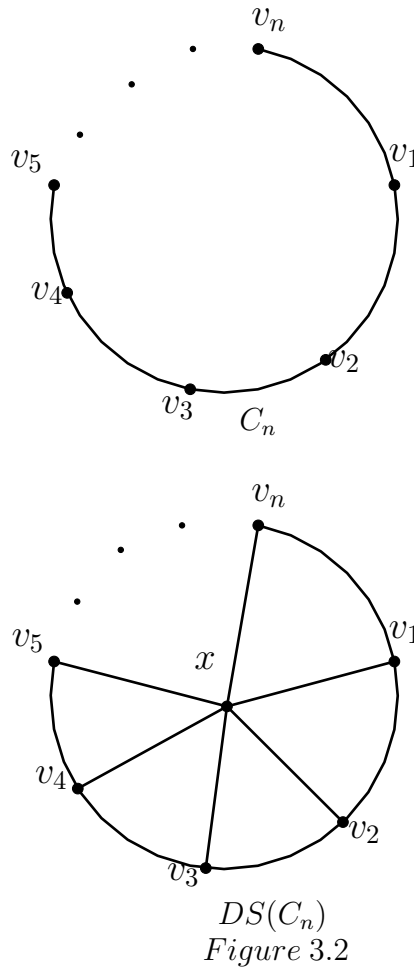
Consider $I_D[x, v_1]$ for $n = 3$, which has only one $x - v_1$ detour path of length 3 that contains all the vertices of $DS(P_3)$. As a result, $S = \{x, v_1\}$ is a minimum cardinality detour set. Also, x dominates S_2 and v_2 is dominated by v_1 . Thus, S is a minimum detour dominating set of $DS(P_3)$. Moreover in $\overline{DS(P_3)}$, x dominates v_2 and v_3 is dominated by v_1 . Hence, S is a minimum detour global dominating set of $DS(P_3)$ and so $\bar{\gamma}_d(DS(P_3)) = 2$.

Consider $I_D[x_1, x_2]$ for $n \geq 4$, which has two distinct $x_1 - x_2$ detour path of length n that contains all the vertices of $DS(P_n)$. As a result, $S = \{x_1, x_2\}$ is a minimum cardinality detour set. Also, x_1 dominates S_2 and x_2 dominates

S_1 . Thus, S is a detour dominating set of $DS(P_n)$. Moreover in $\overline{DS(P_n)}$, x_1 dominates S_1 and x_2 dominates S_2 and hence S is a detour global dominating set $DS(P_n)$. Hence, we conclude that for $n \geq 3$, $\bar{\gamma}_d(DS(P_n)) = 2$. $\square$

Theorem 3.2. For any integer $n \geq 3$, $\bar{\gamma}_d(DS(C_n)) = 3$.

Proof. Let $v_1v_2 \cdots v_n$ be the cycle C_n . To obtain $DS(C_n)$ for $n \geq 3$ we add a vertex x which is adjacent to every vertices in C_n . As a result, $V(DS(C_n)) = \{x, v_1, v_2, \dots, v_n\}$, where $|V(DS(C_n))| = n + 1$ for $n \geq 3$. Clearly, $DS(C_n)$ is isomorphic to the wheel graph W_n .

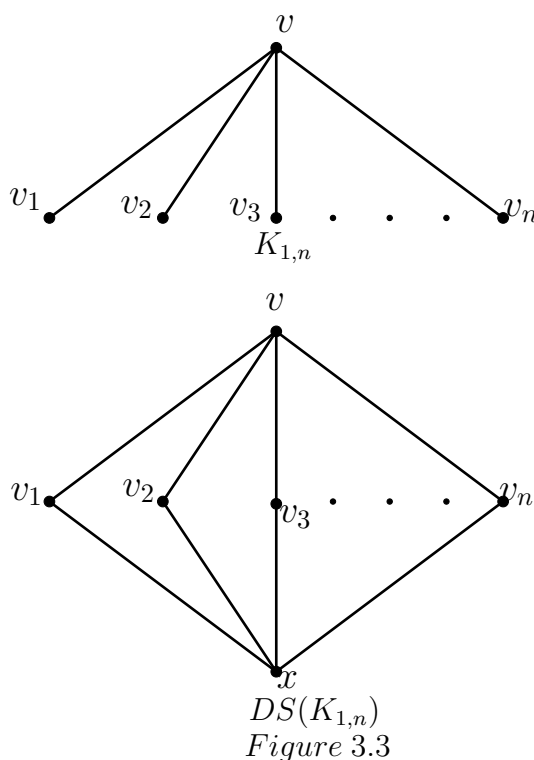


Since, $DS(C_n)$ is a connected graph, then $\bar{\gamma}_d(DS(C_n)) \geq 2$. Consider the set $S = \{x, v_i\}$, where $1 \leq i \leq n$. Clearly, the $x - v_i$ detour path of $DS(C_n)$ contains all the vertices of $DS(C_n)$ and also, x dominates every vertices of C_n . Hence, S is a minimum detour dominating set of $DS(C_n)$. But in $\overline{DS(C_n)}$, the neighbours of v_i are not dominated by any vertices of S . So, choose v_{i+1} from

$\overline{DS(C_n)}$ then v_i, v_{i+1} and x dominates every vertices from $\overline{DS(C_n)}$. Therefore, $S_1 = S \cup \{v_{i+1}\} = \{v, v_i, v_{i+1}\}$ is a minimum detour global dominating set of $DS(C_n)$ and hence $\bar{\gamma}_d(DS(C_n)) = 3$. $\square$

Theorem 3.3. For any integer $n \geq 2$, $\bar{\gamma}_d(DS(K_{1,n})) = 2$.

Proof. Let $v_1, v_2, \dots, v_{n-1}$ are the end vertices and v is the full vertex of the star $K_{1,n-1}$ and x be the corresponding vertex which is added to obtain the graph $DS(K_{1,n})$. Then $V(DS(K_{1,n})) = \{v, v_1, v_2, \dots, v_n, x\}$. Clearly, $|V(DS(K_{1,n-1}))| = n + 2$.



Since, $DS(K_{1,n})$ is connected, then $2 \leq \bar{\gamma}_d(DS(K_{1,n})) \leq n + 1$. Consider $S = \{v, v_i\}$ for some $i, 1 \leq i \leq n$. Then there are $n - 1$ detour path which travels between v and v_i that includes all the vertices of $DS(K_{1,n})$. Therefore, $S = \{v, v_i\}$ is a detour set of minimum cardinality. Moreover, v dominates every v_i and v itself in $DS(K_{1,n})$ and v_i dominates x in $DS(K_{1,n})$ for $1 \leq i \leq n$. Now consider in $\overline{DS(K_{1,n})}$ where v dominates x and v_i dominates all other v_j for $1 \leq j \neq i \leq n$. This concludes that S is a minimum detour global dominating set of $DS(K_{1,n})$ and hence $\bar{\gamma}_d(DS(K_{1,n})) = 2$. $\square$

Theorem 3.4. For the bistar graph $B_{m,n}$, $\bar{\gamma}_d(DS(B_{m,n})) = 2$.

Proof. Consider the bistar graph $B_{m,n}$ with $V(B_{m,n}) = \{u, v, u_i, v_j / 1 \leq i \leq m; 1 \leq j \leq n\}$. Here, u_i and v_j are the vertices adjacent with u and v respectively. Let x_1 and x_2 be the corresponding vertices which are added to obtain $DS(B_{m,n})$. Then $V(DS(B_{m,n})) = \{u, v, u_i, v_j, x_1, x_2 / 1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(S'(B_{m,n}))| = m + n + 4$.

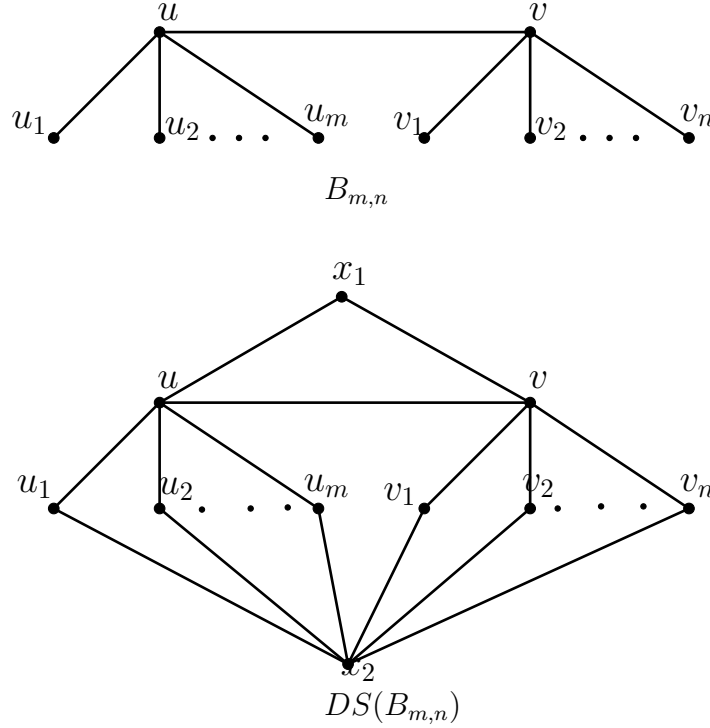


Figure 3.4

Since G is a connected graph, $\bar{\gamma}_d(DS(B_{m,n})) \geq 2$. Consider $I_D[x_1, x_2]$ which has $m + n$ transversal detour path of length four between x_1 and x_2 which include all the vertices of $DS(B_{m,n})$. Therefore, $S = \{x_1, x_2\}$ is a detour set of minimum cardinality. Also, x_1 dominates u and v and x_2 dominates all u_i and v_j for $1 \leq i \leq m$ and $1 \leq j \leq n$ in $DS(B_{m,n})$. It follows that S is a detour dominating set of $DS(B_{m,n})$. Moreover, in $\overline{DS(B_{m,n})}$, x_2 dominates x_1, u and v ; x_1 dominates $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$. Clearly, S is a minimum detour global dominating set of $DS(B_{m,n})$ and hence $\bar{\gamma}_d(DS(B_{m,n})) = 2$. $\square$

Theorem 3.5. For any integer $m, n \geq 2$, $\bar{\gamma}_d(DS(K_{m,n})) = \begin{cases} 3 & \text{if } m = n \\ 2 & \text{if } m \neq n \end{cases}$

Proof. Consider $K_{m,n}$ with $V(K_{m,n}) = \{u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$ with partition $V_1 = \{v_1, v_2, \dots, v_m\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$. Now we consider the following two cases.

Case (1) $m = n$

In this case each vertex is of same degree and so let x be the added vertex which is adjacent to every u_i and v_j , $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus, we obtain the graph $DS(K_{m,n})$. Then $V(DS(K_{m,n})) = \{u_i, v_j, x / 1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(DS(K_{m,n}))| = m + n + 1$.

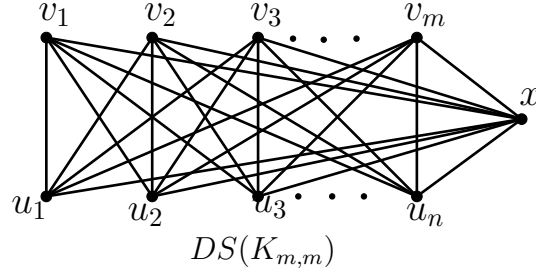


Figure 3.5

Consider the set $S = \{v_i, u_j\}$ for some i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then $I_D[v_i, u_j] = V(DS(K_{m,n}))$ and also v_i dominates every u_j and x ; u_j dominates every v_i . Hence, $S = \{v_i, u_j\}$ is a minimum detour dominating set. Now consider $\overline{DS(K_{m,n})}$, where x is an isolated vertex and not dominated by any vertex of S . This shows that S is not a detour global dominating set of $DS(K_{m,n})$. Hence, we include x in S such that $S = \{u_i, v_j, x\}$ is a detour global dominating set of $DS(K_{m,n})$ for $1 \leq i \leq m; 1 \leq j \leq n$. Therefore, for $m = n$, $\bar{\gamma}_d(DS(K_{m,n})) = 3$.

Case (2) $m \neq n$

In this case each vertex u_i is of same degree and each vertex v_j is of same degree where $\deg(u_i) \neq \deg(v_j)$, $1 \leq i \leq m; 1 \leq j \leq n$ so let x_1 and x_2 be the added vertex where x_1 is adjacent to every u_i and x_2 is adjacent to every v_j . Thus, we obtain the graph $DS(K_{m,n})$. Then $V(DS(K_{m,n})) = \{u_i, v_j, x_1, x_2 / 1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(DS(K_{m,n}))| = m + n + 2$.

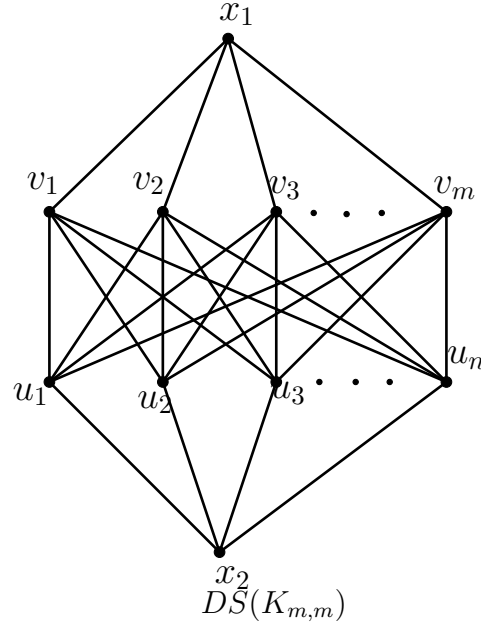


Figure 3.6

Consider the set $S = \{x_1, x_2\}$, where $I_D[x_1, x_2] = V(DS(K_{m,n}))$ and also x_1 dominates every v_i and x_2 dominates every u_j . Hence, $S = \{x_1, x_2\}$ is a minimum detour dominating set. Now consider $\overline{DS}(K_{m,n})$, where x_1 dominates every u_i for $1 \leq i \leq n$ and x_2 dominates every v_j for $1 \leq j \leq n$. Clearly, $S = \{x_1, x_2\}$ is a minimum detour global dominating set and so for $m \neq n$, $\bar{\gamma}_d(DS(K_{m,n})) = 2$. $\square$

Theorem 3.6. For any integer $n \geq 2$, $\bar{\gamma}_d(DS(K_n)) = n$.

Proof. Here, $DS(K_n)$ is isomorphic to K_{n+1} . We know that all the vertices are isolated vertices in the complement graph of $DS(K_n)$. Therefore, the detour global dominating set must contain all the vertices of $DS(K_n)$ and so, for $n \geq 2$, $\bar{\gamma}_d(DS(K_n)) = n$. $\square$

4 Conclusion

Inspired by the global dominating set and detour set we introduce the detour global dominating set for degree splitting graph. We have determined the detour global domination number for degree splitting graph of path graph, cycle graph, star graph, bistar graph, complete bipartite graph and complete graph. Furthermore our results are also justified with suitable examples. The detour global domination number can also be obtained for many more graphs.

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Application of homotopy to the ageing process of human bone

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Abstract

This article presents the ageing process of human bone which can play a major role in the structure of the human body from the concept of homotopy in algebraic topology. This article is about applications. Hence this title. In 2019, William Obeng-Denteh et al., Wrote the literature on the application of homotopy within the framework of algebraic topology on the ageing of the human body. We applied this application, which was generally created for the human body to the ageing of the human bone. The result was good. So we have solved this problem in the literature topologically. Also described by the Cartesian function. As a result of the literature, we have explained the assumption that bone age increases as human age increases with the use of homotopy. The structure of the human bone, which is precisely connected is considered here to be topologically equivalent to a cylinder [9]. The process of continuous ageing bone is considered to be a family of homotopy based on its functions. The study discusses the algebraic topology of homotopes through the homotopy of stable functions of the human bone from infancy to old age.

Keywords: Homomorphism, Homotopy, Homology, Chain complex, Topological Space.

2020 AMS subject classification: 51A10, 55P15, 55R40, 55U15, 18F60.

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1. Introduction

A lot of research articles have been published using the homotopy application. In particular, HPM also known as the homotopy perturbation method, plays in Engineering, Applied maths, Biology, Physics, Quantum Physics and real-world problems. It only began to expand in the 18th and 19th centuries. Homotopy analysis method for solving the biological population sample in 2011 used by Anas. A. M. Arafa et al., [2]. In 2012 F. Guerrero et al., as real-world applications of how smoking has evolved in Spain [4]. In 2019 William Obeng-Denteh et al., published a research paper application of homotopy to the ageing process of the human body [9]. Based on this, in this paper, we have described the ageing process of bone in the human body with homotopy application in algebraic topology.

The growth of human body organs or body growth occurs as a result of genetic, hormonal, dietary and other environmental factors. Bones play an important role in supporting the growing or developing organs. What is our condition if there are no bones? At birth, a baby has about 350 bones in its body and as the baby grows there is a large bone formed by the combination of tiny little bones found in many parts of the body. Physiological research reports that this results in a change in the number of bones reduced to 206. The development of human body organs from infancy to old age is considered a developmental change that does not only aggravate the appearance of the body. The internal organs of the body also continue to mature. The elegant but complex skeletal shape of the human bone is described by the Cartesian functions of $\alpha = S^1 \times I$ and $H, H': \alpha \rightarrow \alpha$ [9]. The bone shape of the human body at an early stage. That is the bone shape of the newborn is called $\alpha = S^1 \times I$ homotopy. We will explore the continuous changes between this childhood bone and the bone seen in old age using the properties of homotopy in mathematics. Especially in algebraic topology. For example, vector calculus is used to demonstrate the relationship between an integral line around a simple closure. Homotopy is used to define the surface area of an unaltered $a(\alpha) = 0$ human bone that can be calculated in algebraic topology where a is attached and closed. Thus, if the surface of a bone α is attached and closed $a(\alpha) = 0$. A few developed variations have been derived from algebraic topology. They reflect the connecting properties of spaces or objects.

More complex spatial biological systems are described by topological collection techniques. Spaces are objects connected in mathematics that have been considered for the introduction of topology. Homotopy theory is used to describe the ageing process of unchanged human bone. It offers numerous applications when Fredholm equations use homotopy analysis on integrated equations. The functions of the values of the given t and α parameters are given by selecting the appropriate value for the time parameters t

Appication of homotopy to the ageing process of human bone

of the process functions of the ageing bones. In which $t \in [0,1]$. In this study the growth or age of human bone is compared with the age of the bone is the same as the age of the body. The homotopy $H_t(\alpha)$ from $H(\alpha)$ to $H'(\alpha)$ such that $H_0(\alpha) = H(\alpha)$ and $H_N(\alpha) = H'(\alpha)$ is currently the same age as bone. That is the age of humans is one. As the age of bone grows and it is functions change. This is what we call homotopy. This is because continuous transformations take place from one function $H(\alpha)$ to another function $H'(\alpha)$. Thus, we can take $t \in [a,n]$. where $[n > a]$. If we consider the first age as a function, we can refer to it as $H_0(\alpha) = H(\alpha)$ and the old or mature age as $H_N(\alpha) = H'(\alpha)$. The bone undergoes a series of changes from $H_a(\alpha)$ to $H_n(\alpha)$. In this shape, the bone develops as homotopy. That is $H_a(\alpha) = H(\alpha)$ to $H_n(\alpha) = H'(\alpha)$ as the bone age expanse to n . Hence the interval $t \in [a,n]$ the human bone α is closed and connected surface at each change in the variable the age of the human bone α growth from one level to another. For instance, if $a = H_0(\alpha) = H(\alpha) \in \alpha$ and $c = H_1(\alpha) = H'(\alpha) \in \alpha$ then a and c are connected by a line that is a homotopy path $H_t(\alpha)$ given by a continuous map $H: [0,1] \rightarrow \alpha$ in α .

2. Homotopy

Definition 2.1. [7] let A, B be topological spaces and H, H' be continuous maps and $H, H': A \rightarrow B$ homotopy from H to H' is continuous function $M: A \times [0,1] \rightarrow B$; satisfying, $M(\alpha, 0) = H(\alpha)$ and $M(\alpha, 1) = H'(\alpha)$ for all $\alpha \in A$. if such a homotopy exists then H is homotopic to H' and it is denoted by $H \simeq H'$.

Definition 2.2. [7] Two paths H and H' . mapping the interval $I = [0,1]$ into A , are said to be path homotopy. If they have the same primary point α_0 and the same endpoint α_1 and if there is a continuous map $M: I \times I \rightarrow A$. Such that $M(u, 0) = H(u)$ and $M(u, 1) = H'(u)$. $M(0, t) = \alpha_0$ and $M(1, t) = \alpha_1$ for each $t, u \in I$. Then M is a path homotopy between H and H' . If H is pate homotopic to H' then notation $H \simeq_p H'$.

Definition 2.3. [5] Let A, B be topological spaces and the mapping $: A \rightarrow B$. A continuous real-valued function $H(\alpha)$, $\alpha \geq 0$ at any point α_0 in α . If for each neighbourhood $N(\alpha_0)$. Then there exists a neighbourhood $D(\alpha_0)$, such that $H(D) \subseteq N$.

Definition 2.4. [11] Let A, B be topological spaces. Let $H: B \rightarrow \mathbb{R}$. We say that H is continuous at β . If for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \beta| < \delta$ then $|H(\alpha) - H(\beta)| < \epsilon$ where $\alpha, \beta \in B$. A function $H: A \rightarrow B$ is said to be continuous if for each open subset W of B , the set $H^{-1}(W)$ is an open subset of space A .

Definition 2.5. [8] Let A, B be topological spaces. Then the function $H: A \rightarrow B$ and H' are continuous functions from A to B . Then H is homotopic to H' . If there is a continuous family of functions, $H_t: A \rightarrow B$ for $0 \leq t \leq 1$. Then the following conditions are satisfied;

- a. $H_0 = H$
- b. $H_1 = H'$
- c. $H_t(\alpha)$ is continuous both as a function $\alpha \in A$ and $t \in [0, 1]$

Theorem 2.6. [7] Let $H: B \rightarrow \mathbb{R}$ and $H': A \rightarrow \mathbb{R}$ be two functions. Suppose H and H' are continuous at β , such that $H(B) \subset A$ then $(H' \circ H): B \rightarrow \mathbb{R}$ is continuous at β .

Theorem 2.7. [1] A topological space A is a path connected if any two points A could be connected by a line. For instance, if $a = H(0) \in A$ and $c = H(1) \in A$ then a, c are connected by a line H given by a continuous map $H: [0, 1] \rightarrow A$ in a topological space A .

3. Homology

The fundamental group $\pi_1(x)$ is especially good for low-dimensional spaces. Because it's concerning loops. The definition of objects in the 2 dimensions expresses itself to the maximum. For example, when x is a CW complex then $\pi_1(x)$ depends only on the 2- skeleton of x . Homotopy is the best way to differentiate and construct all dimensions and spheres. However, these high-dimensional homotopy groups have some drawbacks. These are a bit difficult to calculate. There is an alternative to this. It is homology groups. Yes. Homology is a commutative alternative to homotopy. The calculation or definition of homology groups is less explicit than the calculation or definition of homotopy groups. The chain complex is the algebraic structure of the abelian groups that form the image of each homomorphism and the sequence of homomorphisms between each group added to the next kernel homotopy is related to the chain complex.

Definition 3.1. [3] A chain complex $\mathcal{C}$ is a sequence of additive abelian groups and homomorphism's $\rightarrow \mathcal{C}_{p+1} \xrightarrow{\mathfrak{d}_{p+1}} \mathcal{C}_p \xrightarrow{\mathfrak{d}_p} \mathcal{C}_{p-1} \rightarrow \dots \dots \rightarrow \mathcal{C}_0 \xrightarrow{\mathfrak{d}_0} 0$. The elements of $\mathcal{C}_p$ are called p -chains and these maps are boundary maps. Such that composition of 2 successive homomorphism's zero. That is, $\mathfrak{d}_p \circ \mathfrak{d}_{p+1} = 0$ for each p . Such a sequence is called a chain complex.

Definition 3.2. [3] The homology group of the chain complex is the quotient group $\mathcal{H}_p = \text{Ker } \mathfrak{d}_p / \text{Im } \mathfrak{d}_{p+1}$. That is, $\mathcal{H}_p(\mathcal{C}) = \mathfrak{Z}_p(\mathcal{C}) / \mathfrak{B}_p(\mathcal{C})$

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Here, $\mathcal{H}_p$ is the p^{th} homology group of $\mathcal{C}$.

Elements of $Ker \mathfrak{d}_p$ are called cycles.

Elements of $Im \mathfrak{d}_{p+1}$ are called boundaries.

Elements of $\mathcal{H}_p$ are cosets of $Im \mathfrak{d}_{p+1}$ called homology classes.

4. Description of ageing human bone function

The inherent properties of the human body can be explored topologically. That is various factors such as the functions, development and transformation of different body parts can be explored using the properties of algebraic topology. We can now compare the ageing functions of bone with the properties of homotopy in algebraic topology at intervals ranging from the age of human bone to the age of onset. The time interval is considered to be $t \in [0, 1]$ and the human bone A . The importance of algebraic invariants such as homotopy that reflect the connectivity of the bone. The structure of human bone $\alpha = S^1 \times I$ and let $\alpha \in A$ and $t \in T$ define the growth of the bone and the age of the bone respectively. Although we take the total duration of human bone to be from early age to final age. We cannot accurately determine the final bone age. That is, we must assume that the functions of the bone will last as long as man exists. So $t = N$ is the final age of the bone.

The topological shape of the bone at the beginning $H_t(\alpha)$ undergoes various changes every year. That is, $H_0(\alpha), H_1(\alpha), H_2(\alpha), \dots, H_N(\alpha)$. So, the full-scale development of this topological shape of the human bone can be referred to as $H_0(\alpha) = H(\alpha)$ and $H_t(\alpha) = H'(\alpha)$. From this, the age of the human which is the duration of the bone is determined as $t \in [0, 1]$. The total period time of a bone is divided into two functions H, H' and the total bone life of a human being is assumed to be N . We know the definition of path homotopy. The paths specified in it are given the starting point $H(0, t) = \alpha_0$ and the endpoint $H(1, t) = \alpha_1$ which have the same starting point and ending point even after the paths have undergone various changes and transformations. And the relationship between the two is an equivalence relation. Here it's called path homotopy. Similarly, the homotopic and equivalence relationship between $H_0(\alpha)$ bone in childhood and $H_N(\alpha)$ bone in old age is similar. We can extend the N-period in which the closed attached human bones $\alpha = S^1 \times I$ divided into two functions H and H' . Defined by the equivalence functions of homotopy $H, I, J, K, \dots, H': \alpha \rightarrow \alpha$ on the whole interval $I = [0, N]$.

The interval $t = [0, 1]$ indicates the initial age and height or growth of the bone. Since it satisfies the initial condition $H_0(\alpha) = H(\alpha)$ and $H_1(\alpha) = I(\alpha)$ The intervals also change with increasing age.

That is $t = [1, 2]$ there is a continuous function $H_1(\alpha) = I(\alpha)$ and $H_2(\alpha) = J(\alpha)$,

Similarly, $t = [2, 3]$ there is a continuous function $H_2(\alpha) = J(\alpha)$ and $H_3(\alpha) = K(\alpha)$

$t = [3, 4]$ there is continuous function $H_3(\alpha) = L(\alpha)$ and $H_4(\alpha) = M(\alpha)$ etc., these functions are satisfied for the homotopy $H_t(\alpha)$ from $H(\alpha)$ to $H'(\alpha)$. The total lifespan of human bone is $(1 + 2 + 3 + \dots \dots N)$ for every function we have $H(\alpha)$, $I(\alpha)$, $J(\alpha)$, $K(\alpha)$, $\dots \dots \dots$, $H'(\alpha)$ are continuous functions from α to α . And these are homotopy to each other. Strictly speaking, $H(\alpha)$ is homotopic to $I(\alpha)$, $I(\alpha)$ is homotopic to $J(\alpha)$, $J(\alpha)$ is homotopic to $K(\alpha)$ etc. if there is homotopy $H_1(\alpha)$ where $t = [0, N]$ from $H(\alpha)$ to $H'(\alpha)$. Such that; $H_0(\alpha) = H(\alpha)$, $H_1(\alpha) = I(\alpha)$, $H_2(\alpha) = J(\alpha)$, $H_3(\alpha) = K(\alpha)$, $\dots$, $H_N(\alpha) = H'(\alpha)$

These are all continuous but undergo many changes at different intervals $I = [0, N]$. Here we see information about the age of the bone found in the body with homotopy. Homotopic functions have been shown to change the shape and age of the bone $t > 0$ each year. That is the function $H_{t=0}(\alpha)$. No matter how many changes occur in bone function or growth, they all occur within a period of $t \in [0, N]$. To be clear, $t = [0, 1]$ in the first year, $t = [1, 2]$ in the second year, etc., if the homotopy $H_t(\alpha)$ is denoted by $H_0(\alpha)$ and $H_1(\alpha)$. That is, the age of human bone increased from $H_0(\alpha)$ to $H_1(\alpha)$ followed by $H_2(\alpha)$ etc., in this case, if we consider this series of functions available to us in homotopy $H: \alpha \rightarrow \alpha$ as a common element it will be available as $H_t(\alpha) = t\alpha$ where $t \in [0, N]$. That is, $H_0(\alpha) = (0.\alpha) = 0$ represents the initial age of the bone. $H_1(\alpha) = (1.\alpha) = \alpha$, $H_2(\alpha) = (2.\alpha) = 2\alpha$, $H_3(\alpha) = (3.\alpha) = 3\alpha$, etc., where $t = 0, 1, 2, \dots \dots$ all are represents the age of the bone.

All of these continuous activities show different stages of bone age and development. The increasing sequence of the function $H_t(\alpha)$ of the bone $\alpha = S^1 \times I$ provides the chains of the bone. In which human bone X is a 2-cell complex and each subsequent growth of each bone forms 2 chains which can be referred to as $\Delta_2(\alpha)$. Thus with each age of the bone, an additional 2 chain complexes are formed. That as the bone ages many 2 chains are formed. These 2 chains are all referred to as $\Delta_2(\alpha) = \sigma$, where σ is a positive integer. Bone α is a closed structure. Hence the set of all closed bones is called the kernel $\xi_2(\alpha) = \sigma$. At this point, the boundary of cell 2 is zero. That is $\pi_2(\alpha) = 0$ Because any 2 successive homomorphisms then the composition to each other we get the zero map. So the homotopy groups $\eta_2(\alpha) = \frac{\xi_2(\alpha)}{\pi_2(\alpha)}$, where $\eta_2(\alpha)$ characterizes the connected 2-cell at each human age level. The homological

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value of the topologically unchanged $\eta_2(\alpha)$ is the closed and attached bone in the body it changes with age. So the homology theory gives the sequence of α abelian groups of bone H and the sequence of $H: \alpha \rightarrow \alpha$ homomorphisms $H_2: \eta_2(\alpha) \rightarrow \eta_2(\alpha)$ for the image that follows it. All the maps we have are continuous and homomorphism, the composition of two homotopic functions is homotopic. Therefore homotopic function is obtained by assembling two or more homotopic functions in this ageing process of human bone.

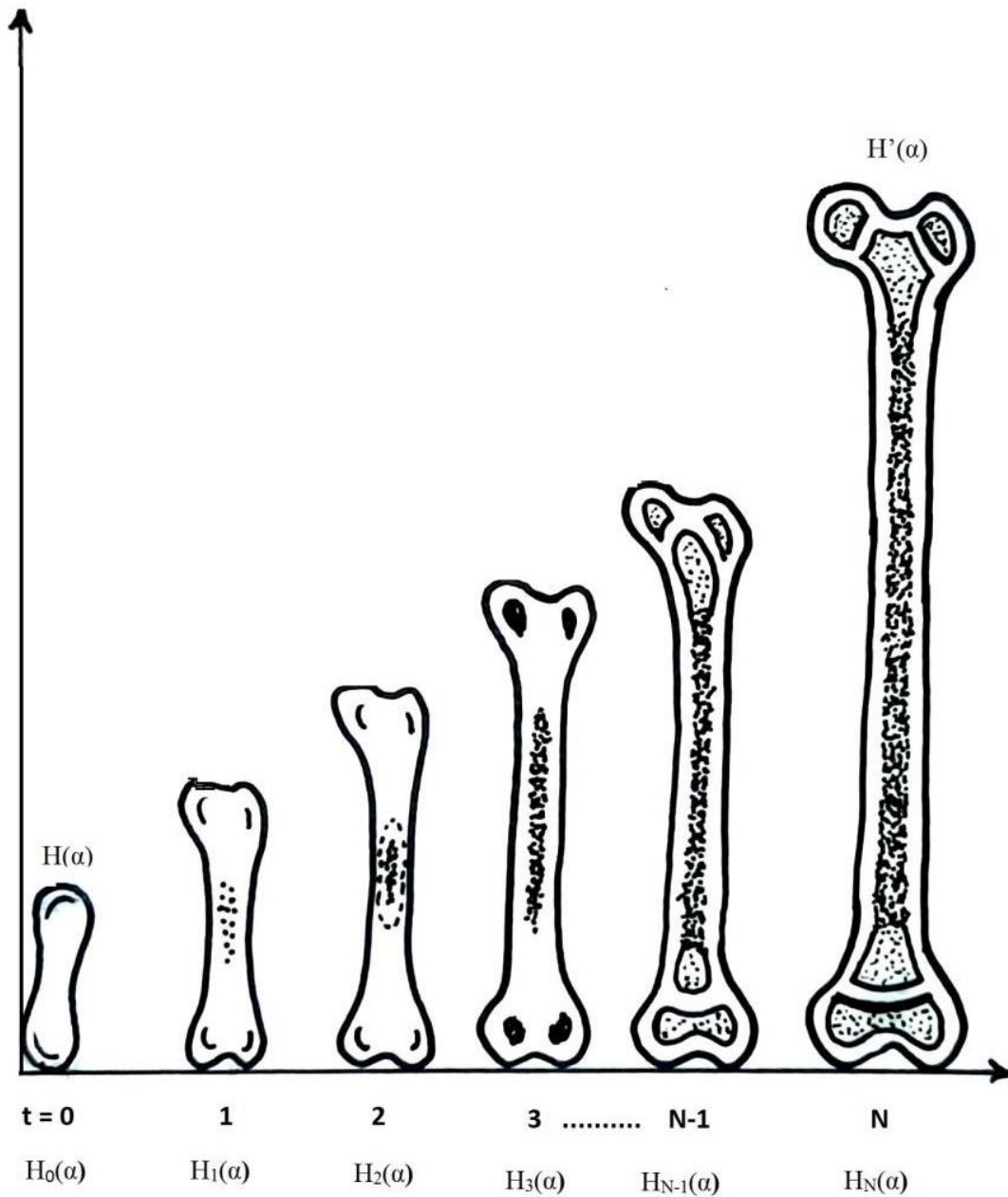


Figure 1. The shape of the bone N-years human from $H(\alpha)$ to $H'(\alpha)$

Remark 4.1. Let H, I, J be the maps and α be the space.

Proposition 4.2. If $H, I: \alpha \rightarrow \alpha$ and $I, J: \alpha \rightarrow \alpha$ are two continuous functions of the body. Show that $H \sim I, I \sim J$ then $(I \circ H) \sim (J \circ I)$.

Theorem 4.3. Prove that $H_1, H_2: P \rightarrow T$ and $H_3, H_4: T \rightarrow \Sigma$ be continuous and homotopic. Then $(H_3 \circ H_1), (H_4 \circ H_2): P \rightarrow \Sigma$ be continuous and homotopic.

5. Conclusions

In this article, we talked about the ageing process of human bone. This is because the application of homotopy, which is generally accepted for the human body is to know whether it corresponds to a particular body organ. This paper is designed to stabilize human bone growth in the event of physical and to state its function and structure. The age of man increases as the time variable t increases. The age of the bone also increases with age. It also describes the continuous process of $H(\alpha)$ in the ageing process. The concepts of homotopy were very helpful in accessing the ageing function of the human bone. We add the basic definition of homotopy to the definition of homology in the literature to justify the publication of the paper. As a result of the literature, we have explained with the applications of homotopy that bone age increases as human age increases. Since we have included the basic definition of homotopy, the interval is t . So here the ageing process is considered by choosing suitable values for the time parameters t of the functions $H_t(\alpha)$. Hence $H_t(\alpha)$ increases from $H_0(\alpha) = H(\alpha)$ to $H_n(\alpha) = H'(\alpha)$, $t \in [a, n]$. The homology theory is slightly incorporated here because the human bone $\alpha = S^1 \times I$ is denoted as a 2-cell complex and bone growths are also 2 chains. In the future, the development of homotopy will play an important role in the research of biologically interacting substances or elements that change over time.

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Weak and weak* I^K -convergence in normed spaces

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Abstract

The main object of this paper is to study the concept of weak I^K -convergence, a generalization of weak I^* -convergence of sequences in a normed space, introducing the idea of weak* I^K -convergence of sequences of functionals where I, K are two ideals on $\mathbb{N}$, the set of all positive integers. Also we study the ideas of weak I^K and weak* I^K -limit points to investigate the properties in the same space.

Keywords: Weak I^K -convergence, Weak* I^K -convergence, Condition $AP(I, K)$, Weak I^K -limit points, Weak* I^K -limit points.

2020 AMS subject classifications: 40A35; 40H05. ¹

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1 Introduction

The idea of statistical convergence, an extended form of ordinary convergence, based on the concept of natural density of subsets of $\mathbb{N}$, was introduced independently by Steinhaus [27] and by Fast [15]. Over the years and under different forms of statistical convergence turned out to be one of the most active research areas in the summability theory after the remarkable works of Friday [16, 17] and Salat [21, 26]. In 2000, Cannor et al.[10] introduced the idea of weak statistical convergence which has been used to characterize Banach spaces with separable duals. Last few years some basic properties of this concept were studied by many authors in [8, 24]. Recently the concept of weak* statistical convergence of sequence of functionals has been given by Bala [1].

In 2001, Kostyrko et al. [18] extended the idea of statistical convergence into I and I^* -convergence which depends on the structure of the ideals I of $\mathbb{N}$. The mutual relation between I and I^* -convergence was given in [19] using the condition AP of the ideals.(such ideals are often called P-ideals [2]). Later many works on ideal convergence have been done in [3, 4, 5, 11, 12, 13]. In 2010, Pehlivan et al.[25] introduced the idea of weak I and I^* -convergence in a normed space and using the condition AP, they established a relation between such types convergence. In 2012, Bhardwaj et al. [9] extended the idea of weak* statistical convergence to weak* ideal convergence of sequence of functionals and shew that the ideas of weak ideal convergence and weak* ideal convergence are identical in a reflexive Banach space.

In 2010, Macaj et al. [22] introduced the idea of I^K -convergence which is a generalization of all types of I^* -convergence. They have shown that if the ideal I has additive property with respect to an another ideal K (i.e. if condition $AP(I, K)$ holds) then I -convergence implies I^K -convergence. Very recently more results and applications of I^K -convergence have been carried out [23, 7, 10]. It seems therefore reasonable to think if we extend the idea of weak and weak* convergence using double ideals in a normed space and in that case we intend to investigate how far several basic properties are affected.

In our paper, we have studied the idea of weak I^K -convergence of sequences which is a generalization of weak I^* -convergent sequences as defined in [25] and we have presented an interrelation between weak I and weak I^K -convergence using the condition $AP(I, K)$. Next we have introduced the concept of weak* I^K -convergence for a sequence of functionals and observed that the ideas of weak and weak* I^K -convergence of sequences of functionals are same in a reflexive Banach space. In the last section of this paper we have discussed the notion of weak I^K -limit point of sequences and weak* I^K -limit points of sequences of functionals. Since the importance of the notion of weak and weak* convergence in functional analysis is very significant, we have realized that the ideas of weak and weak* I^K -

convergence in a normed space give more general frame for functional analysis to study summability theory as well.

2 Basic definitions and notations

Throughout the paper, we use $\mathbb{N}$ to denote the set of all positive integers and X for a normed linear space and X^* for dual of X . First recall that a subset A of $\mathbb{N}$ is said to have natural density $d(A)$ if $d(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_A(k)$, provided the limit exists where χ_A is characteristic function of $A \subset \mathbb{N}$.

Definition 2.1. [15] A sequence $\{x_n\}$ in X is said to be statistically convergent to l if for every $\epsilon > 0$ the set $K(\epsilon) = \{k \in \mathbb{N} : \|x_k - l\| \geq \epsilon\}$ has natural density zero.

Definition 2.2. [10] Let X be a normed linear space then a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be weak statistically convergent to $x \in X$ provided that for any $f \in X^*$, the sequence $\{f(x_n - x)\}_{n \in \mathbb{N}}$ is statistically convergent to 0. In this case we write $w\text{-st-}\lim_{n \rightarrow \infty} x_n = x$.

Definition 2.3. Let S be a non empty set and a class $I \subset 2^S$ of subset of S is said to an ideal if

(i) $A, B \in I$ implies $A \cup B \in I$ and (ii) $A \in I, B \subset A$ implies $B \in I$.

I is said nontrivial ideal if $S \notin I$ and $I \neq \{\emptyset\}$. In view of condition (ii) $\emptyset \in I$. If $I \subsetneq 2^S$ we say that I is proper ideal on S . A nontrivial ideal I is said admissible if it contains all the singletons of S . A nontrivial ideal I is said non-admissible if it is not admissible.

Definition 2.4. Let F be a class of subsets of non-empty set S . Then F is said to be a filter in S if

(i) $\emptyset \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F, A \subset B$ implies $B \in F$.

If I is a non-trivial on a non-void set S then $F = F(I) = \{A \subset S : S \setminus A \in I\}$ is clearly a filter on S and conversely. Again $F(I)$ is said associated filter with respect to ideal I .

Definition 2.5. [18] A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be I -convergent to x if for any $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\} \in I$. In this case we write $I\text{-}\lim_{n \rightarrow \infty} x_n = x$.

Definition 2.6. [25] A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be weak I -convergent to $x \in X$ if for any $\epsilon > 0$ and for any $f \in X^*$ the set $A(f, \epsilon) = \{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \in I$. In this case we write $w-I-\lim_{n \rightarrow \infty} x_n = x$.

Note 2.1. It is easy to observe that weak I -limit of a weak I -convergent sequence is unique and moreover for an admissible ideal I , weak convergence implies weak I -convergence with the same limit point but converse part is not true which has been shown in paper [25] by an interesting examples.

Note 2.2. It is obvious that if two ideals I_1, I_2 on $\mathbb{N}$ such that $I_1 \subseteq I_2$ then for a sequence $\{x_n\}$ $w-I_1-\lim x_n = x$ implies $w-I_2-\lim x_n = x$.

3 Weak I^K -convergence

We have already mentioned that our aim to generalize the notion of weak I^* -convergence of sequences. We need to modify this definition introduced in [25].

Definition 3.1. (cf.[25]) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be weak I^* -convergent to $x \in X$ if there exists a set $M \in F(I)$ such that the sequence $\{y_n\}_{n \in \mathbb{N}} \subset X$ defined by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}$$

is weak-convergent to x . we denote it by the notation $w-I^*-\lim x_n = x$.

Definition 3.2. (cf.[22]) Let I, K be two ideals on the set $\mathbb{N}$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to be weak I^K -convergent to $x \in X$ if there exists a set $M \in F(I)$ such that the sequence $\{y_n\}_{n \in \mathbb{N}} \subset X$ defined by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}$$

is weak K -convergent to x . we denote it by the notation $w-I^K-\lim x_n = x$.

Remark 3.1. We can give an equivalent definition of weak- I^K -convergence in the following way: if there exists an $M \in F(I)$ such that the sequence $\{x_n\}_{n \in M}$ is weak- $K|_M$ -convergent to x where $K|_M = \{B \cap M : B \in K\}$.

Lemma 3.1. If I and K are ideals on $\mathbb{N}$, the set of all positive integers and if $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $w-K-\lim \{x_n\} = x$, then $w-I^K-\lim \{x_n\} = x$.

The proof follows from the definition of weak K -convergence taking $M = \mathbb{N} \in F(I)$ and $y_n = x_n$.

Proposition 3.1. *Let X be normed space and I be an ideal on $\mathbb{N}$. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is weak I^I -convergent to x if and only if it is weak I -convergent to x .*

Proof. Let $\{x_n\}$ be weak I^I -convergent to x then there exists an $M \in F(I)$ such that the sequence $\{x_n\}_{n \in M}$ is weak- $I|_M$ -convergent to x . So there exists $G \in F(I)$ such that $\{n \in \mathbb{N} : |f(x_n) - f(x)| < \epsilon\} \cap M = G \cap M$. Clearly $G \cap M \in F(I)$ and $\{n \in \mathbb{N} : |f(x_n) - f(x)| < \epsilon\} \supseteq G \cap M$. Therefore $\{n \in \mathbb{N} : |f(x_n) - f(x)| < \epsilon\} \in F(I)$ i.e. $\{x_n\}$ is weak I -convergent to x .

Converse part follows from Lemma 3.1 taking $K = I$. $\square$

Proposition 3.2. *Let X be a normed space and I, I_1, K and K_1 be ideals on $\mathbb{N}$ such that $I \subseteq I_1$ and $K \subseteq K_1$. Then for any sequence $\{x_n\}_{n \in \mathbb{N}}$, we have*

(i) $w-I^K\text{-}\lim x_n = x \Rightarrow w-I_1^K\text{-}\lim x_n = x$ and

(ii) $w-I^K\text{-}\lim x_n = x \Rightarrow w-I^{K_1}\text{-}\lim x_n = x$.

Proof. (i) Now as $w-I^K\text{-}\lim x_n = x$ so there exists an $M \in F(I)$ such that the sequence $\{x_n\}_{n \in M}$ is weak- $K|_M$ -convergent to x where $K|_M = \{B \cap M : B \in K\}$. Here $M \in F(I) \subseteq F(I_1)$ as $I \subseteq I_1$. So obviously $w-I_1^K\text{-}\lim x_n = x$.

(ii) Again $w-I^K\text{-}\lim x_n = x$ then there exists a set $M \in F(I)$ such that the $\{y_n\} \in X$ given by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}$$

is weak K -convergent to x . Since $K \subseteq K_1$ and from the Note 2.2 we get $\{y_n\}$ is weak K -convergent to x . Hence $w-I^{K_1}\text{-}\lim x_n = x$. $\square$

Theorem 3.1. *Let I and K be ideals on $\mathbb{N}$ and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X then*

(i) $w-I\text{-}\lim x_n = x \Rightarrow w-I^K\text{-}\lim x_n = x$ if $I \subseteq K$. (ii) $w-I^K\text{-}\lim x_n = x \Rightarrow w-I\text{-}\lim x_n = x$ if $K \subseteq I$.

Proof. (i) Since $\{x_n\}$ is weak I -convergent to $x \in X$ then for any $\epsilon > 0$ and $f \in X^*$ the set $A(f, \epsilon) = \{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \in I$. Again $I \subseteq K$ so $A(f, \epsilon) \in K$. Therefore the sequence $\{x_n\}$ is weak K -convergent to x . So from the Lemma 3.1 we get $\{x_n\}$ is weak I^K -convergent to x .

(ii) Now $w-I^K\text{-}\lim x_n = x$ then there exists a set $M \in F(I)$ such that the sequence $\{y_n\}$ given by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ x & \text{if } n \notin M \end{cases}$$

is weak K -convergent to x . So $A(f, \epsilon) = \{n \in \mathbb{N} : |f(y_n) - f(x)| \geq \epsilon\} = \{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \cap M \in K \subseteq I$. Consequently $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \subseteq (\mathbb{N} \setminus M) \cup A(f, \epsilon) \in I$. So $w-I\text{-}\lim x_n = x$. $\square$

Note 3.1. If $K \not\subset I$ and $I \not\subset K$ then none of these implications in Theorem 3.1 may not be true. To support this we cite an example which is weak- I^K -convergence but not weak- I -convergence.

Example 3.1. Let I and K be two ideals on $\mathbb{N}$ such that $K \not\subset I$ and $I \not\subset K$, but $K \cap I \neq \emptyset$. Let $x, y \in X, x \neq y$ and consider a set $M \in K \setminus I$. Let us now consider a sequence $\{x_n\}$ with define by

$$x_n = \begin{cases} x & \text{if } n \notin M \\ y & \text{if } n \in M \end{cases}$$

then for every $\epsilon > 0$ and $f \in X^*$ we get $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \subset M \in K$. So $w - K - \lim x_n = x$. But, since $x - y \neq \theta$ so from Hahn Banach theorem there exist a $f \in X^*$ such that $f(x - y) = \|x - y\|$. Choose an $\epsilon = \frac{\|x - y\|}{2}$. then $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} = \{n \in M : |f(x_n) - f(x)| \geq \epsilon\} \cup \{n \in \mathbb{N} \setminus M : |f(x_n) - f(x)| \geq \epsilon\} = \{n \in M : |f(y) - f(x)| \geq \frac{\|x - y\|}{2}\} = \{n \in M : \|x - y\| \geq \frac{\|x - y\|}{2}\} M \notin I$. So $w - I - \lim x_n \neq x$.

Note 3.2. Consider any two ideals I and K on $\mathbb{N}$ then we can construct a new ideal $I \vee K = \{A \cup B : A \in I, B \in K\}$ containing both I, K . The dual filter of $I \vee K$ is $F(I \vee K) = \{G \cap H : G \in F(I), H \in F(K)\}$, when $I \vee K$ is non-trivial. It should be noted that if $I \vee K$ is non-trivial ideal and I, K are proper subsets of $I \vee K$ then both I and K are non-trivial. But converse part may or may not be true always. To establish this, following examples are given.

Example 3.2. Let the two sets $P = \{5p : p \in \mathbb{N}\}$ and $S = \{5s - 1 : s \in \mathbb{N}\}$ now it is clear that $2^P, 2^S$ and $2^P \vee 2^S$ all ideals are non-trivial on $\mathbb{N}$.

Example 3.3. Now let P be set of all odd integers and S be set of all even integers. Then $I = 2^P, K = 2^S$ both are non-trivial on the whole set $\mathbb{N}$ but $I \vee K$ is not a non-trivial ideal on $\mathbb{N}$.

Theorem 3.2. If $I \vee K$ is non-trivial ideal on $\mathbb{N}$ and X is normed space then weak I^K -limit of a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is unique.

Proof. If possible let sequence $\{x_n\}_{n \in \mathbb{N}}$ has two distinct weak I^K -limits say x and y . Since $x \neq y$ i.e. $(x - y) \neq \theta$ then by a consequence of Hahn Banach theorem there exists f such that $f(x - y) = \|x - y\| \neq \theta$ then $f(x) \neq f(y)$ and let $\epsilon = \frac{|f(x) - f(y)|}{3} > 0$. Since $\{x_n\}_{n \in \mathbb{N}}$ has weak I^K -limit x then there exists a set $A_1 \in F(I)$ such that the $\{y_n\} \in X$ given by

$$y_n = \begin{cases} x_n & \text{if } n \in A_1 \\ x & \text{if } n \notin A_1 \end{cases}$$

is weak K -convergent to x . So, $\{n \in \mathbb{N} : |f(y_n) - f(x)| \geq \epsilon\} \in K$ i.e. $\{n \in \mathbb{N} : |f(y_n) - f(x)| < \epsilon\} \in F(K)$ which implies that $\{n \in A_1 : |f(y_n) - f(x)| < \epsilon\} \cup \{n \in \mathbb{N} \setminus A_1 : |f(y_n) - f(x)| < \epsilon\} \in F(K)$ i.e. $(\mathbb{N} \setminus A_1) \cup \{n \in A_1 : |f(y_n) - f(x)| < \epsilon\} \in F(K)$ so $A_1 \setminus B_1 \in K$ where $B_1 = \{n \in A_1 : |f(x_n) - f(x)| < \epsilon\}$. Similarly as $\{x_n\}$ has weak I^K -limit y , so there exists a set $A_2 \in F(I)$ such that $A_2 \setminus B_2 \in K$ where $B_2 = \{n \in A_2 : |f(x_n) - f(y)| < \epsilon\}$. So, $(A_1 \setminus B_1) \cup (A_2 \setminus B_2) \in K$ then $(A_1 \cap A_2) \cap (B_1 \cap B_2)^c \subset (A_1 \cap B_1^c) \cup (A_2 \cap B_2^c) \in K$. Thus $(A_1 \cap A_2) \cap (B_1 \cap B_2)^c \in K$ i.e. $(A_1 \cap A_2) \setminus (B_1 \cap B_2) \in K$. Now by our construction we get $B_1 \cap B_2 = \phi$. For if $B_1 \cap B_2 \neq \phi$, let $n \in B_1 \cap B_2$ then $|f(x_n) - f(x)| < \epsilon$ and $|f(x_n) - f(y)| < \epsilon$. Therefore, $3\epsilon = |f(x) - f(y)| \leq |f(x) - f(x_n)| + |f(x_n) - f(y)| < 2\epsilon$, which is a contradiction. So $A_1 \cap A_2 \in K$ i.e. $\mathbb{N} \setminus (A_1 \cap A_2) \in F(K) \longrightarrow (i)$. Since $A_1, A_2 \in F(I)$ so $A_1 \cap A_2 \in F(I) \longrightarrow (ii)$. Since $I \vee K$ is non-trivial so the dual filter $F(I \vee K)$ exists. Now from (i) and (ii) we get $\phi \in F(I \vee K)$, which is a contradiction. Hence the weak I^K -limit is unique. $\square$

Theorem 3.3. *Let X be normed space and I, K be two ideals on $\mathbb{N}$. A sequence $\{x_n\}_{n \in \mathbb{N}} \in X$ is weak I^K -convergent to x if and only if it is weak $(I \vee K)^K$ -convergent to x .*

Proof. Suppose that $\{x_n\}$ is weak I^K -convergent to x then there exists an $M \in F(I)$ such that the sequence $\{x_n\}_{n \in M}$ is weak- $K|_M$ -convergent to x . Since $M \in F(I)$ so it is clear that $M \in F(I \vee K)$. Therefore $\{x_n\}$ is also weak $(I \vee K)^K$ -convergent to x .

Conversely, let $\{x_n\}$ is weak $(I \vee K)^K$ -convergent to x then there exists an $M \in F(I \vee K)$ such that the sequence $\{x_n\}_{n \in M}$ is weak- $K|_M$ -convergent to x . So for any $\epsilon (> 0)$ and for every $f \in X^*$ there exists $G \in F(K)$ such that $A(f, \epsilon) \cap M = G \cap M$ where $A(f, \epsilon) = \{n \in \mathbb{N} : |f(x_n) - f(x)| < \epsilon\}$. Since $M \in F(I \vee K)$ then $M = M_1 \cap M_2$ for some $M_1 \in F(I)$ and $M_2 \in F(K)$. Now we have $A(f, \epsilon) \cap M_1 \supseteq A(f, \epsilon) \cap M = (G \cap M_2) \cap M_1$. Since $G \cap M_2 \in F(K)$, this shows that $A(f, \epsilon) \cap M_1 \in F(K|_{M_1})$ i.e. $\{x_n\}$ is weak I^K -convergent to x . $\square$

In the rest of this section, using additive property of ideals we will investigate the relationship between weak- I and I^K -convergence. Now we recall the definition of K -pseudo intersection and then AP(I, K)-condition.

Definition 3.3. [21] *Let K be an ideal on $\mathbb{N}$. We denote $A \subset_K B$ whenever $A \setminus B \in K$. If $A \subset_K B$ and $B \subset_K A$ then we denote $A \sim_K B$. Clearly $A \sim_K B \Leftrightarrow A \triangle B \in K$.*

If $A \subset_K A_n$ holds for each $n \in \mathbb{N}$ then we say that a set A is K -pseudo intersection of a system $\{A_n : n \in \mathbb{N}\}$.

Definition 3.4. [21] Let I, K be ideals on the set X . We say that I has additive property with respect to K or that the condition $AP(I, K)$ holds if any one of the equivalent condition of following holds: (a) For every sequence $(A_n)_{n \in \mathbb{N}}$ of sets from I there is $A \in I$ such that $A_n \subset_K A$ for every n 's.
 (b) Any sequence $(F_n)_{n \in \mathbb{N}}$ of sets from $F(I)$ has K -pseudo intersection in $F(I)$.
 (c) For every sequence $(A_n)_{n \in \mathbb{N}}$ of sets from the ideal I there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j \in \mathbb{N}} B_j \in I$.
 (d) For every sequence of mutually disjoint sets $(A_n)_{n \in \mathbb{N}} \subset I$ there exists a sequence $(B_n)_{n \in \mathbb{N}} \subset I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j \in \mathbb{N}} B_j \in I$.
 (e) For every non-decreasing sequence $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \dots$ of sets from $I \exists$ a sequence $(B_n)_{n \in \mathbb{N}} \subset I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j \in \mathbb{N}} B_j \in I$.
 (f) In the Boolean algebra $2^S/K$ the ideal I corresponds to a σ -directed subset, i.e. every countable subset has an upper bound.

Note that the proof of the conditions (a) to (f) in the definition 3.4 are equivalent has been given in [21][Lemma 3.9]. Above definition is reformulation of the definition given below:

Definition 3.5. [14] Let I, K be ideals on the non-empty set S . We say that I has additive property with respect to K or that the condition $AP(I, K)$ holds if for every sequence of pairwise disjoint sets $A_n \in I$, there exists a sequence $B_n \in I$ such that $A_n \triangle B_n \in K$ for each n and $\cup_{n \in \mathbb{N}} B_n \in I$.

Theorem 3.4. If the condition $AP(I, K)$ holds then weak- I -convergence implies weak- I^K -convergence, where I, K are two ideals on $\mathbb{N}$.

Proof. Let $\{x_n\}$ be weak I -convergent sequence to $x \in X$. Let $f \in X^*$ and choose a sequence of rationals $\{\epsilon_i : i \in \mathbb{N}\}$ so that $\{(f(x) - \epsilon_i, f(x) + \epsilon_i) : i \in \mathbb{N}\}$ be a countable base for $\mathbb{R}$ at the point $f(x)$. By weak I -convergence of $\{x_n\}$ we have $B_i = \{n : |f(x_n) - f(x)| < \epsilon_i\} \in F(I)$ for each i , thus by definition 3.4(b) there exists a set $A \in F(I)$ with $A \subset_K B_i$ i.e. $A \setminus B_i \in K$ for all i 's. Now it suffices to show that the sequence $\{y_n\} \in X$ given by

$$y_n = \begin{cases} x_n & \text{if } n \in A \\ x & \text{if } n \notin A \end{cases}$$

is weak K -convergent to x . Now $\{n \in \mathbb{N} : |f(y_n) - f(x)| < \epsilon_i\} = \{n \in A : |f(y_n) - f(x)| < \epsilon_i\} \cup \{n \in \mathbb{N} \setminus A : |f(y_n) - f(x)| < \epsilon_i\} = (\mathbb{N} \setminus A) \cup \{n \in A : |f(x_n) - f(x)| < \epsilon_i\} = (\mathbb{N} \setminus A) \cup (B_i \cap A) = \mathbb{N} \setminus (A \setminus B_i)$. As $A \setminus B_i \in K$ then $\mathbb{N} \setminus (A \setminus B_i) \in F(K)$. Thus $\{n \in \mathbb{N} : |f(y_n) - f(x)| < \epsilon_i\} \in F(K)$ for each i and every $f \in X^*$. Thus $\{y_n\}$ is weak K -convergent to x . Hence $\{x_n\}$ is weak I^K -convergent to x . $\square$

Theorem 3.5. *Let I, K be ideals on $\mathbb{N}$. If for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X weak I -convergence implies weak I^K -convergence then the condition $AP(I, K)$ holds.*

Proof. Let $\{y_n\}$ be a sequence in X which is weak I -convergent to x . Since X is first countable and $f(x)$ is not isolated point in $\mathbb{R}$ then there exists a sequence $\{z_n\}$ of points from $X \setminus \{x\}$ which weak convergent to x . Let $\{A_n : n \in \mathbb{N}\}$ be a system of mutually disjoint sets from I . Let us define a sequence $\{x_n\}$ as

$$x_n = \begin{cases} z_j & \text{if } n \in A_j \\ y_n & \text{if } n \notin \cup A_j \end{cases}$$

Let $f \in X^*$ be arbitrary. Now $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \subset \{n \in \mathbb{N} : |f(y_n) - f(x)| \geq \epsilon\} \cup \cup_{j=1}^n A_j$ implies $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \in I$. This shows that $\{x_n\}$ is weak I -convergent to x . By our assumption this implies $\{x_n\}$ is weak I^K -convergent to x i.e. there exists a set $M \in F(I)$ such that $\{x_n\}_{n \in M}$ is weak $K|_M$ -convergent to x i.e. $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \cap M = A \cap M$ for some $A \in K$. This implies that $\{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \cap M \in K$. Let us define $B_i = A_i \setminus M$ we have $\cup_{i \in \mathbb{N}} B_i \subseteq \mathbb{N} \setminus M \in I$. At the same time, for the set $B_i \triangle A_i = A_i \cap M$ we have $A_i \cap M \subseteq \{n \in \mathbb{N} : |f(x_n) - f(x)| \geq \epsilon\} \cap M$ for any $\epsilon > 0$. Consequently $B_i \triangle A_i \in K$. Hence the condition $AP(I, K)$ holds. $\square$

4 Weak* I^K -convergence

In this section, Following Bala [1] and Bhardwaj et al. [9], now we introduce the concept of weak* I^K -convergence of sequence of functionals and present some result.

Definition 4.1. [9] *A sequence $\{f_n\}_{n \in \mathbb{N}}$ in X^* is said to be weak* I -convergent to $f \in X^*$ if for any $\epsilon > 0$ and for each $x \in X$ the set $A(x, \epsilon) = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\} \in I$. In this case we write $w^*I\text{-}\lim_{n \rightarrow \infty} f_n = f$.*

Definition 4.2. *A sequence $\{f_n\}_{n \in \mathbb{N}}$ in X^* is said to be weak* I^* -convergent to $f \in X^*$ if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \in F(I)$ such that $\lim_{k \rightarrow \infty} f_{m_k}(x) = f(x)$ for each $x \in X$. In this case we write $w^*I^*\text{-}\lim_{n \rightarrow \infty} f_n = f$.*

Theorem 4.1. *Let X be a normed space and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in X^* . If $\{f_n\}$ is weak* I^* -convergent to $f \in X^*$ then it is weak* I -convergent to f .*

Proof. By assumption, there exists a set $H \in I$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $\lim_{k \rightarrow \infty} f_{m_k}(x) = f(x)$ for each $x \in X$. Now let $\epsilon > 0$ and for this there exists an $N(\epsilon, x) \in \mathbb{N}$ such that $|f_{m_k}(x) - f(x)| < \epsilon$ for each $k >$

$N(\epsilon, x)$. Then we have $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\} \subset H \cup m_1, m_2, \dots, m_{N(\epsilon, x)}$. Since I is an admissible ideal so right-hand side of the above relation belongs to I . Hence the result. $\square$

Remark 4.1. We can reformulate the definition 4.2 in the following way: if there exists a set $M \in F(I)$ such that the sequence $\{g_n\} \in X^*$ given by

$$g_n = \begin{cases} f_n & \text{if } n \in M \\ f & \text{if } n \notin M \end{cases}$$

is weak* convergent to f .

Definition 4.3. Let X be a normed space with a separable dual X^* and I, K be two ideals on $\mathbb{N}$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in X^* is said to be weak* I^K -convergent to $f \in X^*$ if there exists a set $M \in F(I)$ such that the sequence $\{g_n\} \in X^*$ given by

$$g_n = \begin{cases} f_n & \text{if } n \in M \\ f & \text{if } n \notin M \end{cases}$$

is weak* K -convergent to f and we write $w^*-I^K-\lim_{n \rightarrow \infty} f_n = f$

Theorem 4.2. If $I \vee K$ is a non-trivial ideal on $\mathbb{N}$ and X is normed space with dual X^* then weak* I^K -limit of a sequence $\{f_n\}_{n \in \mathbb{N}}$ in X^* is unique.

The proof is parallel to proof of Theorem 3.2 with slight modification.

Theorem 4.3. Let X be a normed space. If a sequence $\{f_k\}$ in X^* is weak I^K -convergent to $f \in X^*$ then it is weak* I^K -convergent.

Proof. By our assumption, $w-I^K-\lim f_k = f$ then there exists a set $M \in F(I)$ such that the sequence $\{g_k\} \in X^*$ given by

$$g_k = \begin{cases} f_k & \text{if } k \in M \\ f & \text{if } k \notin M \end{cases}$$

is weak K -convergent to f . Then for every $h \in X^{**}$ and $\epsilon > 0$, we have $\{k : |h(g_k) - h(f)| \geq \epsilon\} \in K$. Let $x \in X$ and $F_x = C(x)$ where $C : X \rightarrow X^{**}$ is the canonical mapping we have $F_x(g_k) = g_k(x)$ and $F_x(f) = f(x)$ for every $x \in X$. So in particular for each $x \in X, \{k : |F_x(g_k) - F_x(f)| \geq \epsilon\} \in K$ i.e. $\{k : |g_k(x) - f(x)| \geq \epsilon\} \in K$. So the sequence $\{g_k\}$ is weak* K -convergent to f . Hence the result. $\square$

Theorem 4.4. Let X be a reflexive normed space with dual X^* . If a sequence $\{f_k\}$ in X^* is weak* I^K -convergent to $f \in X^*$ then it is weak I^K -convergent to f .

Weak and weak* I^K -convergence in normed spaces

Proof. By our assumption, $w^* - I^K - \lim f_k = f$. So there exists a set $M \in F(I)$ such that the sequence $\{g_k\} \in X^*$ given by

$$g_k = \begin{cases} f_k & \text{if } k \in M \\ f & \text{if } k \notin M \end{cases}$$

is weak* K -convergent to f . Then for each $x \in X$ and $\epsilon(> 0)$ the set $\{k \in \mathbb{N} : |g_k(x) - f(x)| \geq \epsilon\} \in K$. Let $F \in X^{**}$ then $F = C(x_0)$ for some $x_0 \in X$ where $C : X \rightarrow X^{**}$ is the canonical mapping. We have in particular $\{k \in \mathbb{N} : |g_k(x_0) - f(x_0)| \geq \epsilon\} \in K$ since $F(g_k) = g_k(x_0)$ and $F(f) = f(x_0)$. We have $\{k \in \mathbb{N} : |F(g_k) - F(f)| \geq \epsilon\} \in K$ for each $\epsilon(> 0)$ and $F \in X^{**}$. So the sequence $\{g_k\}$ is weak K -convergent to f . Hence the result. $\square$

5 Weak and weak* I^K -limit points

In this last part, we introduce weak and weak* I^K -limit points of sequences and sequence of functionals respectively. First we define weak I -limit point of a sequence.

Definition 5.1. (cf. [17]) Let X be a normed space and a sequence $\{x_n\}$ be a sequence in X . Then $y \in X$ is called an weak I -limit point of $\{x_n\}$ if there exists a set $M \notin I$ such that the sequence $\{y_n\}_{n \in \mathbb{N}} \in X$ defined by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ y & \text{if } n \notin M \end{cases}$$

is weak-convergent to y .

Definition 5.2. Let X be a normed space and I, K be two ideals on $\mathbb{N}$. Then $y \in X$ is called an weak I^K -limit point of a sequence $\{x_n\}$ if there exists a set $M \notin I, K$ such that the sequence $\{y_n\}_{n \in \mathbb{N}} \in X$ defined by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ y & \text{if } n \notin M \end{cases}$$

is weak K -convergent to y .

We denote $I(L_w)$ and $I^K(L_w)$ the collection of all weak I and weak I^K -limit points of $x_n \in X$.

Theorem 5.1. If K is an admissible ideal and $K \subset I$ then $I(L_w) \subset I^K(L_w)$.

Proof. Let $y \in I(L_w)$, so there exists a set $M \notin I$ such that the sequence $\{y_n\}$ give by

$$y_n = \begin{cases} x_n & \text{if } n \in M \\ y & \text{if } n \notin M \end{cases}$$

is weak-convergent to y . Then the sequence of scalars $\{f(y_n)\}$ converges to $f(y)$ for all $f \in X^*$ i.e. $\{n : |f(y_n) - f(y)| \geq \epsilon\}$ is a finite set. So $\{n : |f(y_n) - f(y)| \geq \epsilon\} \in K$ as K is an admissible ideal. Therefore $\{y_n\}$ is weak K -convergent sequence. Again $M \notin I$ and $K \subset I$, so $M \notin I, K$. Thus y is weak I^K -limit point of x_n . Hence the theorem. $\square$

In the similar way we can set the definition of weak* I^K -limit points for the sequence of functionals.

Definition 5.3. Let X be a normed space with its dual X^* and $\{f_n\}$ be a sequence in X^* . Then $h \in X^*$ is called an weak* I -limit point of $\{f_n\}$ if there exists a set $M \notin I$ such that the sequence $\{g_n\}_{n \in \mathbb{N}} \in X^*$ defined by

$$g_n = \begin{cases} f_n & \text{if } n \in M \\ h & \text{if } n \notin M \end{cases}$$

is weak*-convergent to h .

Definition 5.4. Let X be a normed space with its dual X^* and I, K be two ideals on $\mathbb{N}$. Then $h \in X^*$ is called an weak* I^K -limit point of $\{f_n\} \subset X^*$ if there exists a set $M \notin I, K$ such that the sequence $\{g_n\}_{n \in \mathbb{N}} \in X^*$ defined by

$$g_n = \begin{cases} f_n & \text{if } n \in M \\ h & \text{if } n \notin M \end{cases}$$

is weak* K -convergent to h .

We denote $I(L_w^*)$ and $I^K(L_w^*)$ the collection of all weak* I and I^K -limit points of the sequence $f_n \in X^*$.

Theorem 5.2. If K is an admissible ideal and $K \subset I$ then $I(L_w^*) \subset I^K(L_w^*)$.

The proof is parallel to proof of the Theorem 5.1.

Theorem 5.3. Let X be a normed space with its dual X^* . If $h \in X^*$ be weak I^K -limit point of a sequence $\{f_n\} \subset X^*$ then h is also weak* I^K -limit point.

Proof. Let y be weak I^K -limit point of $\{f_n\} \in X^*$ then there exists a set $M \notin I$ such that the sequence $\{g_n\}_{n \in \mathbb{N}} \in X^*$ defined by

$$g_n = \begin{cases} f_n & \text{if } n \in M \\ h & \text{if } n \notin M \end{cases}$$

is weak K -convergent to y . Again by Theorem 4.3 we get $\{g_n\}$ is weak* K -convergent to h . Hence h is weak* I^K -limit point. $\square$

Remark 5.1. *By the theorem 4.4 we get weak* K -convergence implies weak K -convergence when X is reflexive normed space. Therefore converse of above theorem holds when X is a reflexive normed space.*

6 Conclusions

The study of weak and weak* convergence plays an important role in functional analysis. So investigations on weak and weak* convergence using single ideal or double ideals take place a great position in the study of summability theory. In [9] and [1] weak and weak* convergence have been studied using single ideal respectively. So it quite natural to investigate whether such these results hold for double ideals. So it would be worthwhile to consider extending the notion of weak and weak* convergence by incorporating double ideals in a normed space, and to examine how various fundamental properties are impacted as a result. In case of double ideals, weak I -convergence implies weak I^K -convergence if the modified AP condition i.e. $AP(I, K)$ condition holds where as in case of single ideal AP condition is sufficient to hold this result. It is observed citing suitable example that weak I^K -convergence may not imply weak I -convergence in general although weak I^* -convergence always imply weak I -convergence. Besides, it is verified that where X is a reflexive Banach space the idea of weak I^K -convergence and weak* I^K -convergence coincide. Due to the inherent importance of concept of weak and weak* convergence in function analysis, we recognize that the concept of weak and weak* convergence via double ideals in normed spaces would provide a more comprehensive framework for function analysis. This extension would offer a broader scope and enhance our understanding of convergence in normed spaces.

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Bi-amalgamated algebra with (n, p) -weakly clean like properties

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Abstract

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let K and K' be two ideals of B and C , respectively such that $f^{-1}(K) = g^{-1}(K')$. In this paper, we give a characterization for the bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) (denoted by $A \bowtie^{f,g} (K, K')$) to be a (n, p) -weakly clean ring.

Keywords: (n, p) -clean ring ; (n, p) - weakly clean ring; bi-amalgamation algebra along ideals.

2020 AMS subject classifications: 16N40, 16U40, 16S99, 16U60. ¹

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1 Introduction

Throughout this paper all rings are commutative with identity elements. Let A and B be two rings with unity, let K be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. In D'Anna et al. [2009], the authors introduced and studied the new ring structure the following subring of $A \times B$:

$$A \bowtie^f K := \{(a, f(a) + k) \mid a \in A, k \in K\}$$

called *the amalgamation of A with B along K with respect to f* . This new ring structure construction is a generalization of *the amalgamated duplication of a ring along an ideal*. The amalgamated duplication of a ring along an ideal was introduced and studied in (D'Anna [2006], D'Anna and Fontana [2007]). In [D'Anna et al., 2009, Section 4], the authors studied the amalgamation can be in the frame of pullback constructions and also the basic properties of this construction (e.g., characterizations for $A \bowtie^f K$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Let $\alpha : A \rightarrow C$, $\beta : A \rightarrow C$ and $f : A \rightarrow B$ be ring homomorphisms. In D'Anna et al. [2009], the authors studied amalgamated algebras within the frame of pullback $\alpha \times \beta$ such that $\alpha = \beta \circ f$ [D'Anna et al., 2009, Proposition 4.2 and 4.4]. In this motivation, the authors created the new constructions, called bi-amalgamated algebras which arise as pullbacks $\alpha \times \beta$ such that the following diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \alpha \\ C & \xrightarrow{\beta} & D \end{array}$$

is commutative with $\alpha \circ \pi_B(\alpha \times \beta) = \alpha \circ f(A)$, where π_B denotes the canonical projection of $B \times C$ over B . Namely, let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and let K and K' be two ideals of B and C , respectively, such that $f^{-1}(K) = g^{-1}(K')$. The bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (K, K') := \{(f(a) + k, g(a) + k') \mid a \in A, (k, k') \in K \times K'\}$$

Following Kabbaj et al. [2013], the above definition was introduced by and studied by Kabbaj, Louartiti and Tamekkante.

Following Nicholson [1977], an element a in a ring A is called a clean if a is a sum of a unit and an idempotent in A . A ring is clean if all its elements are clean.

Clean rings were initially developed by Nicholson [1977], as a natural class of rings which have the exchange property.

This paper aims at studying the transfer of the notion of n -clean rings, (n, p) -weakly clean rings to bi-amalgamation of algebra along ideals.

We denote by $U(A)$, set of all unit elements of A .

2 (n, p) -clean ring

We start with definition of (n, p) -clean rings.

Definition 2.1. (Chen and Qua [2014]) An element $a \in A$ is said to be (n, p) -clean if $a = u_1 + \dots + u_n + x$ for some unit $u_i \in A$ ($i = 1, \dots, n$) and $x^p = x$, where x is called p -potent element. The ring A is said to be (n, p) -clean if all of its elements are (n, p) -clean.

Remark 2.1. Every clean rings are $(1, 2)$ -clean rings and n -clean rings are $(n, 2)$ -clean rings.

Following Chhiti [2018], the above definition is studied the authors in 2018. Now, we start with following example of $(2, 2)$ -clean ring.

Example 2.1. Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_4 \times \mathbb{Z}_4$ and $C := \mathbb{Z}_2$ and let $J := 0 \times \mathbb{Z}_4$ and $J' := \mathbb{Z}_2$ are ideals of B and C respectively. Consider the map $f : A \longrightarrow B$ is defined by $f(a) = (a, 0)$ for all $a \in A$ and the map $g : A \longrightarrow C$ is defined by $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Hence, $A \bowtie^{f,g} (J, J') = \left\{ ((0, 0), 0), ((0, 1), 0), ((0, 2), 0), ((0, 3), 0), ((2, 0), 0), ((2, 1), 0), ((2, 2), 0), ((2, 3), 0), ((1, 0), 1), ((1, 1), 1), ((1, 2), 1), ((1, 3), 1), ((3, 0), 1), ((3, 1), 1), ((3, 2), 1), ((3, 3), 1) \right\}$ is a 2-clean ring. Therefore, by the remark 2.1 $A \bowtie^{f,g} (J, J')$ is $(2, 2)$ -clean ring.

Proposition 2.1. The class of (n, p) -clean ring is closed under homomorphic images.

Proof. The proof is straightforward. $\square$

Proposition 2.2. If $A \bowtie^{f,g} (K, K')$ is a (n, p) -clean ring then $f(A) + K$ and $g(A) + K'$ are (n, p) -clean rings.

Proof. Clearly, by Proposition 2.1 (n, p) -clean ring is a (n, p) -clean ring. Thus, in view of Kabbaj et al. [2013] [Proposition 4.1], we have the following isomorphism of rings $\frac{A \bowtie^{f,g} (K, K')}{0 \times K'} \cong f(A) + K$ and $\frac{A \bowtie^{f,g} (K, K')}{K \times 0} \cong g(A) + K'$. Hence, $f(A) + K$ and $g(A) + K'$ are (n, p) -clean rings. $\square$

Definition 2.2. A ring is called uniquely (n, p) -clean ring if each element in A can be written as unique way.

Theorem 2.1. Assume that A is (n, p) -clean ring and $\frac{f(A) + K}{K}$ and $\frac{g(A) + K'}{K'}$ are uniquely n -clean rings. Then $A \bowtie^{f,g} (K, K')$ is (n, p) -clean ring if and only if $f(A) + K$ and $g(A) + K'$ are (n, p) -clean rings.

Proof. If $A \bowtie^{f,g} (K, K')$ is a (n, p) -clean ring, then so $f(A) + K$ and $g(A) + K'$ by Proposition 2.1. Conversely, assume that $f(A) + K$ and $g(A) + K'$ are (n, p) -clean rings. Since A is a (n, p) -clean ring, we can write $a = u_1 + \dots + u_n + x$ for some unit $u_i \in A$ ($i = 1, \dots, n$) and $x^p = x$. On the other hand, since $f(A) + K$ is a (n, p) -clean ring, $f(a) + k = (f(x_1) + k_1) + \dots + (f(x_n) + k_n) + f(p) + k^*$ with $f(x_i) + k_i$ ($i = 1, \dots, n$) and $f(p) + k^*$ are respectively units and $(f(p) + k^*)^p = f(p) + k^*$ element of $f(A) + K$. It is clear that $\overline{f(x_1)} = \overline{f(x_1) + k_1}$ (resp., $\overline{f(u_1)}), \dots, \overline{f(x_n)} = \overline{f(x_n) + k_n}$ (resp., $\overline{f(u_n)}$) and $\overline{f(p)} = \overline{f(p) + k^*}$ (resp., $\overline{f(x)}$), are respectively units and an p -potent element of $\frac{f(A) + K}{K}$, and we have $\overline{f(a)} = \overline{f(u_1)} + \dots + \overline{f(u_n)} + \overline{f(p)} = \overline{f(x_1)} + \dots + \overline{f(x_n)} + \overline{f(x)}$. Thus, $\overline{f(u_1)} = \overline{f(x_1)}, \dots, \overline{f(u_n)} = \overline{f(x_n)}$ and $\overline{f(p)} = \overline{f(x)}$ since $\frac{f(A) + K}{K}$ is an uniquely (n, p) -clean ring. Consider $k_1^*, \dots, k_n^*, k_l^* \in K$ such that $f(x_1) = f(u_1) + k_1^*, \dots, f(x_n) = f(u_n) + k_n^*$ and $f(x) = f(p) + k_l^*$ and also since $g(A) + K'$ is a (n, p) -clean ring, $g(a) + k' = (g(x'_1) + k'_1) + \dots + (g(x'_n) + k'_n) + g(p') + k'^*$ with $g(x'_i) + k'_i$ ($i = 1, \dots, n$) and $g(p') + k'^*$ are respectively units and p -potent element of $g(A) + K'$. It is clear that $\overline{g(x'_1)} = \overline{g(x'_1) + k'_1}$ (resp., $\overline{g(u_1)}), \dots, \overline{g(x'_n)} = \overline{g(x'_n) + k'_n}$ (resp., $\overline{g(u_n)}$) and $\overline{g(p')} = \overline{g(p') + k'^*}$ (resp., $\overline{g(x)}$), are respectively units and an p -potent element of $\frac{g(A) + K'}{K'}$, and we have $\overline{g(a)} = \overline{g(u_1)} + \dots + \overline{g(u_n)} + \overline{g(p)} = \overline{g(x'_1)} + \dots + \overline{g(x'_n)} + \overline{g(p')}$. Thus, $\overline{g(u_1)} = \overline{g(x'_1)}, \dots, \overline{g(u_n)} = \overline{g(x'_n)}$ and $\overline{g(x)} = \overline{g(p')}$ since $\frac{g(A) + K'}{K'}$ is an uniquely (n, p) -clean ring. Consider $k'_1, \dots, k'_n, k'_l \in K$ such that $g(x'_1) = g(u_1) + k'_1, \dots, g(x'_n) = g(u_n) + k'_n$ and $g(y') = g(p) + k'_l$. We have $(f(a) + k, g(a) + k') = \{(f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) + (f(p) + k_l^* + k^*), (g(u_1) + k_1'^* + k'_1) + \dots + (g(u_n) + k_n'^* + k'_n) + (g(p') + k_l'^* + k'^*)\} = (f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k'_1) + \dots + (f(u_n) + k_n^* + k_n, g(u_n) + k_n'^* + k'_n) + (f(p) + k_l^* + k^*, g(p') + k_l'^* + k'^*)$. It is clear that $(f(p) + k_l^* + k^*, g(p') + k_l'^* + k'^*)$ is an p -potent in $A \bowtie^{f,g} (K, K')$. Hence, we have only to prove that $(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k'_1)$ is invertible in $A \bowtie^{f,g} (K, K')$. Since $f(u_1) + k_1^* + k_1$ is invertible in $f(A) + K$, there exists an element $f(\beta) + k_0$ such that $(f(u_1) + k_1^* + k_1)(f(\beta) + k_0) = 1$. Thus $\overline{f(u_1)f(\beta)} = \overline{1}$. Then $\overline{f(\beta)} = \overline{f(u_1^{-1})}$. So there exists $k_0^* \in K$ such that $f(\beta) = f(u_1^{-1}) + k_0^*$. Similarly, $g(u_1) + k_1'^* + k'_1$ is invertible in $g(A) + K'$. Hence, $(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k'_1)(f(u_1^{-1}) + k_0^* + k_0, g(u_1^{-1}) + k_0'^* + k'_0) =$

$(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1')(f(\beta) + k_0, g(\gamma) + k_0') = (1, 1)$. Therefore, $(f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1')$ is invertible in $A \bowtie^{f,g} (K, K')$. Similarly, each term are invertible. This completes the proof. $\square$

Proposition 2.3. *Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two surjective ring homomorphisms, let K and K' be two ideals of B and C respectively such that $f^{-1}(K) = g^{-1}(K') = I_0$ and let A is a (n, p) -clean ring and A/I_0 is an uniquely (n, p) -clean ring. Then $A \bowtie^{f,g} (K, K')$ is a (n, p) -clean ring.*

Proof. It is clear that B and C are (n, p) -clean rings. So, since $f(A) + K = B$ and $g(A) + K' = C$, we conclude that $A \bowtie^{f,g} (K, K')$ is a (n, p) -clean ring by Theorem 2.1. $\square$

Let A be a commutative ring with identity and let M be a unitary A -module. The idealization of M in A (or trivial extension of A by M) is the commutative ring $A \propto M = \{(a, m) | a \in R, m \in M\}$ under the usual addition and the multiplication defined as $(a_1 m_1)(a_2 m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1)$ for all $(a_1, m_1), (a_2, m_2) \in A \propto M$.

Theorem 2.2. *Consider n and p two positive integers ($p \geq 2$). Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms. Assume that A is (n, p) -clean ring. Let K be an ideal of B such that $f(u) + k$ is invertible (in B) for each $u \in U(A)$ and $k \in K$ and K' be an ideal of C such that $g(u) + k'$ is invertible (in C) for each $u \in U(A)$ and $k' \in K'$. Then $A \bowtie^{f,g} (K, K')$ is a (n, p) -clean ring if and only if $f(A) + K$ and $g(A) + K'$ are (n, p) -clean ring.*

Proof. In light of Proposition 2.1, homomorphic image of (n, p) -clean ring is (n, p) -clean ring. Thus, in view of Kabbaj et al. [2013] [Proposition 4.1], we have the following isomorphism of rings $\frac{A \bowtie^{f,g} (K, K')}{0 \times K'} \cong f(A) + K$ and $\frac{A \bowtie^{f,g} (K, K')}{K \times 0} \cong g(A) + K'$. Hence, $f(A) + K$ and $g(A) + K'$ are (n, p) -rings. Conversely, we assume that A is (n, p) -clean and K be an ideal of B such that $f(u) + k$ is invertible (in B) and K' be an ideal of C such that $g(u) + k'$ is invertible (in C). Then there exist $v_1 \in B$ such that $(f(u_1) + k)v_1 = 1$ and there exist $v_2 \in C$ such that $(g(u_1) + k')v_2 = 1$. Hence, $(f(u_1) + k)(f(u_1^{-1}) - v_1 f(u_1^{-1})k) = f(u_1)f(u_1^{-1}) + k f(u_1^{-1}) - (f(u_1) + k)v_1 f(u_1^{-1})k = 1 + k f(u_1^{-1}) - f(u_1^{-1})k = 1$ and $(g(u_1) + k')(g(u_1^{-1}) - v_2 g(u_1^{-1})k') = g(u_1)g(u_1^{-1}) + k' g(u_1^{-1}) - (g(u_1) + k')v_2 g(u_1^{-1})k' = 1 + k' g(u_1^{-1}) - g(u_1^{-1})k' = 1$. Thus, $(f(u_1) + k, g(u_1) + k')$ is invertible in $A \bowtie^{f,g} (K, K')$. Hence, $(f(a) + k, g(a) + k') = (f(u_1 + u_2 + \dots + u_n + x) + k, g(u_1 + u_2 + \dots + u_n + x) + k') = (f(u_1) + k, g(u_1) + k') + (f(u_2), g(u_2)) + \dots + (f(u_n), g(u_n)) + (f(x), g(x))$, where $(f(u_1) + k, g(u_1) + k') \in U(A \propto M)$, $(f(u_i), g(u_i)) \in U(A \propto M)$ ($i = 2, 3, \dots, n$) and $(f(x), g(x))^p = (f(x), g(x))$. Consequently, $A \bowtie^{f,g} (K, K')$ is (n, p) -clean ring. $\square$

3 (n, p) -weakly clean ring

Now we introduce the new class of ring.

Definition 3.1. A ring A is called (n, p) -weakly clean ring if $a = u_1 + \dots + u_n + x$ or $a = u_1 + \dots + u_n - x$ for some unit $u_i \in A (i = 1, \dots, n)$ and $x^p = x$, where x is p -potent element.

If the above representation is unique, we say that A is uniquely (n, p) -weakly clean ring.

Note that (n, p) -clean ring is weakly (n, p) -clean ring. In this section we study the tranfer of (n, p) -weakly clean ring property to the ring $A \bowtie^{f,g} (K, K')$ is defined above. We establishes necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be (n, p) -weakly clean.

Proposition 3.1. The class of (n, p) -weakly clean is closed under homomorphic images.

Proof. The proof is straightforward. $\square$

Proposition 3.2. A ring A is called uniquely (n, p) -weakly clean ring if the representation of a (n, p) -weakly clean element in a unique way.

Our first main result gives a necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be (n, p) -weakly clean ring.

Theorem 3.1. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms, let K and K' be two ideals of B and C respectively such that $f^{-1}(K) = g^{-1}(K') = I_0$. Assume that the following conditions hold:

a) A is a (n, p) -weakly clean ring and A/I_0 is an uniquely (n, p) -weakly clean ring.

b) $f(A) + K$ and $g(A) + K'$ are (n, p) -weakly clean rings and atmost one of them is not a (n, p) -clean ring. Then $A \bowtie^{f,g} (K, K')$ is a (n, p) -weakly clean ring.

Proof. Without loss of generality, we assume that $f(A) + K$ is a n -weakly clean ring and $g(A) + K'$ is a n -clean ring. Let $a \in A$. Then a can be written as $a = u_1 + \dots + u_n + x$ or $a = u_1 + \dots + u_n - x$ for some $u_i \in A (i = 1, \dots, n)$ and p -potent $x \in A$. Since A is (n, p) -weakly clean ring and since $f(A) + K$ is a (n, p) -weakly clean ring, $f(a) + k = (f(x_1) + k_1) + \dots + (f(x_n) + k_n) + (f(p) + k^*)$ or $f(a) + k = (f(x_1) + k_1) + \dots + (f(x_n) + k_n) - (f(p) + k^*)$ with $f(x_i) + k_i (i = 1, \dots, n)$ and $f(p) + k^*$ are respectively units and p -potent element of $f(A) + K$. Therefore, $f(a) = f(u_1) + \dots + f(u_n) + \overline{f(x)}$ or $f(a) = f(u_1) + \dots + f(u_n) - \overline{f(x)}$. Then, in $f(A) + K/K$ we have: $\overline{f(a)} = \overline{f(u_1)} + \dots + \overline{f(u_n)} + \overline{f(x)}$

or $\overline{f(a)} = \overline{f(u_1)} + \dots + \overline{f(u_n)} - \overline{f(x)}$. It is clear that $\overline{f(x_1)} = \overline{f(x_1) + k_1}$ (resp., $\overline{f(u_1)}, \dots, \overline{f(x_n)} = \overline{f(x_n) + k_n}$ (resp., $\overline{f(u_n)}$) and $\overline{f(p)} = \overline{f(p) + k^*}$ (resp., $\overline{f(x)}$), are respectively units and p -potent element of $\frac{f(A) + K}{K}$, and we have $\overline{f(a)} = \overline{f(u_1)} + \dots + \overline{f(u_n)} + \overline{f(x)} = \overline{f(x_1)} + \dots + \overline{f(x_n)} + \overline{f(p)}$ or $\overline{f(a)} = \overline{f(u_1)} + \dots + \overline{f(u_n)} - \overline{f(x)} = \overline{f(x_1)} + \dots + \overline{f(x_n)} - \overline{f(p)}$. Thus, $\overline{f(u_1)} = \overline{f(x_1)}, \dots, \overline{f(u_n)} = \overline{f(x_n)}$ and $\overline{f(x)} = \overline{f(p)}$ since $\frac{f(A) + K}{K}$ is an uniquely (n, p) -weakly clean ring. Consider $k_1^*, \dots, k_n^*, k_l^* \in K$ such that $f(x_1) = f(u_1) + k_1^*, \dots, f(x_n) = f(u_n) + k_n^*$ and $f(p) = f(x) + k_l^*$. Hence, $f(a) + j = (f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) + (f(x) + k_l^* + k^*)$ or $f(a) + j = (f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) - (f(x) + k_l^* + k^*)$. Thus, using the same technique of the previous part $g(a) + k' = (g(u_1) + k_1'^* + k_1') + \dots + (g(u_n) + k_n'^* + k_n') + (g(p) + k_l'^* + k'^*)$ or $g(a) + k' = (g(u_1) + k_1'^* + k_1') + \dots + (g(u_n) + k_n'^* + k_n') - (g(p) + k_l'^* + k'^*)$ since $g(A) + K'/K' \cong A/I_0$ is an uniquely n -weakly clean ring. This implies that $(f(a) + k, g(a) + k') = \{(f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) + (f(x) + k_l^* + k^*), (g(u_1) + k_1'^* + k_1') + \dots + (g(u_n) + k_n'^* + k_n') + (g(x) + k_l'^* + k'^*)\} = (f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1') + \dots + (f(u_n) + k_n^* + k_n, g(u_n) + k_n'^* + k_n') + (f(x) + k_l^* + k^*, g(x) + k_l'^* + k'^*)$. Now, the same argument follows from Theorem 2.1 in the remaining case, $f(a) + j = (f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) - (f(x) + k_l^* + k^*)$. Let $g(a) + k' = (g(u_1) + k_1'^* + k_1') + \dots + (g(u_n) + k_n'^* + k_n') - (g(x) + k_l'^* + k'^*)$. We have $(f(a) + k, g(a) + k') = \{(f(u_1) + k_1^* + k_1) + \dots + (f(u_n) + k_n^* + k_n) + (f(x) + k_l^* + k^*), (g(u_1) + k_1'^* + k_1') + \dots + (g(u_n) + k_n'^* + k_n') + (g(x) + k_l'^* + k'^*)\} = (f(u_1) + k_1^* + k_1, g(u_1) + k_1'^* + k_1') + \dots + (f(u_n) + k_n^* + k_n, g(u_n) + k_n'^* + k_n') - (f(x) + k_l^* + k^*, g(x) + k_l'^* + k'^*)$. In all cases, $(f(a) + k, g(a) + k')$ is a (n, p) -weakly clean elements of $A \bowtie^{f,g} (J, J')$. This completes the proof. $\square$

4 Conclusions

Through the above we have studied the characterization for the bi-amalgamation of A with (B, C) along (K, K') with respect to (f, g) to be a (n, p) -clean ring along with an example. Further, we have studied the necessary and sufficient conditions for $A \bowtie^{f,g} (K, K')$ to be a (n, p) -weakly clean ring.

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Relatively Prime Inverse Domination on Line Graph

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Abstract

Let G be non-trivial graph. A subset D of the vertex set $V(G)$ of a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D . The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$. If $V - D$ contains a dominating set S of G , then S is called an inverse dominating set with respect to D . In an inverse dominating set S , every pair of vertices u and v in S such that $(deg(u), deg(v)) = 1$, then S is called relatively prime inverse dominating set. The minimum cardinality of a relatively prime inverse dominating set is called relatively prime inverse dominating number and is denoted by $\gamma_{rp}^{-1}(G)$. In this paper we find relatively prime inverse dominating number of some jump graphs.

Keywords: Domination number, Inverse domination number, Relatively prime domination number.

2020 AMS subject classifications: 05C69, 05C76 ¹

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1 Introduction

By a graph, we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to the book by Chartrand and Lesniak [1]. All graphs in this paper are assumed to be non-trivial. In a graph $G = (V, E)$, the degree of a vertex v is defined to be the number of edges incident with v and is denoted by $\deg(v)$. A set D of vertices of graph G is said to be a dominating set if every vertex in $V - D$ is adjacent to a vertex in D . A dominating set D is said to be a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality of a dominating set of a graph G is called the domination number of G and is denoted by $\gamma(G)$. Kulli V. R. et al. introduced the concept of inverse domination in graphs [8]. Let D be a minimum dominating set of G . If $V - D$ contains a dominating set S , then S is called an inverse dominating set of G with respect to D . The inverse domination number $\gamma^{-1}(S)$ is the minimum cardinality taken over all the minimal inverse dominating set of G . The Jewel graph J_n is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$ [7]. Bistar $B_{m,n}$ is the graph obtained by joining the center vertices of star graphs $K_{1,m}$ and $K_{1,n}$ by an edge. The vertex set of $B_{m,n}$ is $\{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ where u, v are apex vertices and u_i, v_j are pendent vertices. The edge set of $B_{m,n}$ is $\{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $|V(B_{m,n})| = m + n + 2, |E(B_{m,n})| = m + n + 1$ [2]. A spider graph is a tree with at most one vertex of degree greater than 2[2]. Let P_n be a path graph with n vertices. The Comb graph is defined as $P_n \odot K_1$. It has $2n$ vertices and $2n - 1$ edges[3]. A wounded spider graph is a graph obtained by subdividing at most $n - 1$ edges of a star $K_{1,n}$. The wounded spider includes K_1 , the star $K_{1,n-1}$ [9]. A set $S \subseteq V$ is said to be relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices u and v in S such that $(\deg(u), \deg(v)) = 1$. The minimum cardinality of a relatively prime dominating set of a graph G is called the relatively prime domination number of G and is denoted by $\gamma_{rpd}(G)$ [5]. The purpose of this paper is to study about the concept of relatively prime inverse domination on line graphs.

Definition 1.1. [6] Let D be a minimum dominating set of a graph G . If $V - D$ contains a dominating set S of G , then S is called an inverse dominating set with respect to D . If every pair of vertices u and v in S such that $(\deg(u), \deg(v)) = 1$, then S is called relatively prime inverse dominating set. The minimum cardinality of a relatively prime inverse dominating set is called a relatively prime inverse domination number and is denoted by $\gamma_{rp}^{-1}(G)$. If the relatively prime inverse dominating set is absent, then $\gamma_{rp}^{-1}(G) = 0$.

Definition 1.2. [4] A line graph $L(G)$ of a simple graph G is obtained by associ-

ating a vertex with each edge of the graph and connecting two vertices with an edge if only if the corresponding edges of G have a vertex in common.

Example 1.1. Consider the graphs G and $L(G)$ which are given Figure 1. Clearly $\{e_1, e_4\}$ is a minimum dominating set of $L(G)$ and $\{e_2, e_5\}$ is a corresponding minimum inverse dominating set of $L(G)$ and $(\deg(e_1), \deg(e_4)) = (4, 3) = 1$ and so $\gamma_{rp}^{-1}L(G) = 2$.

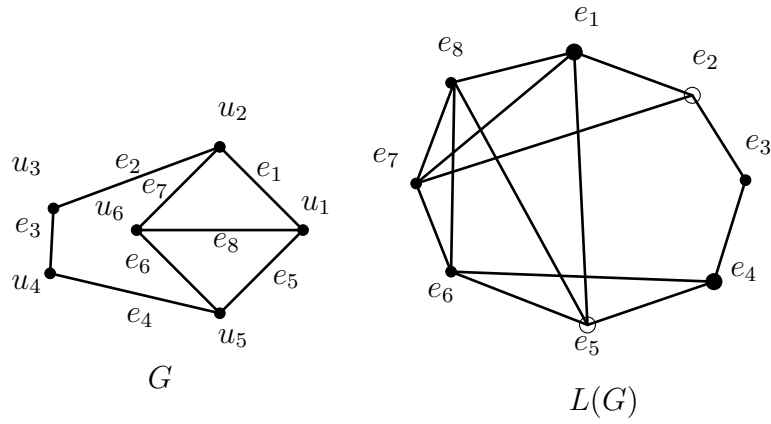


Figure 1: $G, L(G)$

We use the following theorem:

Theorem 1.1. [8] For a path P_n , $\gamma_{rp}^{-1}(P_n) = \begin{cases} 2 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n = 6, 7 \\ 0 & \text{otherwise} \end{cases}$

2 Relatively prime inverse domination on line graph

Theorem 2.1. For the spider graph $K_{1,n,n}$, $\gamma_{rp}^{-1}(L(K_{1,n,n})) = n$.

Proof. Let v be the centre vertex and the end vertices of $K_{1,n}$ be $v_1, v_2, \dots, v_n$. Let $u_1, u_2, \dots, u_n$ represent the vertices connected with $v_1, v_2, \dots, v_n$, respectively. The resulting graph is the spider graph $K_{1,n,n}$ with vertex set $V(K_{1,n,n}) = \{v, v_i, v'_i : 1 \leq i \leq n\}$ and $E(K_{1,n,n}) = \{vv_i, v_iv'_i : 1 \leq i \leq n\}$. Clearly, $\deg(v) = n$, $\deg(v_i) = 2$, and $\deg(v'_i) = 1, 1 \leq i \leq n$. Let the line graph of the graph $K_{1,n,n}$ be $L(K_{1,n,n})$. Denote the edges vv_i by e_i and $v_iv'_i$ by e'_i . Clearly $V(L(K_{1,n,n})) = \{e_i, e'_i : 1 \leq i \leq n\}$ and $E(L(K_{1,n,n})) = \{e_ie_j, e_ie'_i : 1 \leq i \neq j \leq n\}$. Let D be a minimum dominating set of $L(K_{1,n,n})$ and S be a corresponding minimum inverse dominating set of $L(K_{1,n,n})$. Although $L(K_{1,n,n})$ contains n end

vertices, any minimum dominating set of $L(K_{1,n,n})$ must include at least n vertices of $L(K_{1,n,n})$. Clearly, $D = \{e_i : 1 \leq i \leq n\}$ is a minimum dominating set and $S = \{e'_i : 1 \leq i \leq n\}$ is the corresponding minimum inverse dominating set of $L(K_{1,n,n})$. Since $\deg(e_i) = \deg(e'_j) = 1$ for $1 \leq i \neq j \leq n$, S is a minimum relatively prime inverse dominating set of $L(K_{1,n,n})$. As a result, $\gamma_{rp}^{-1}(L(K_{1,n,n})) = n$. $\square$

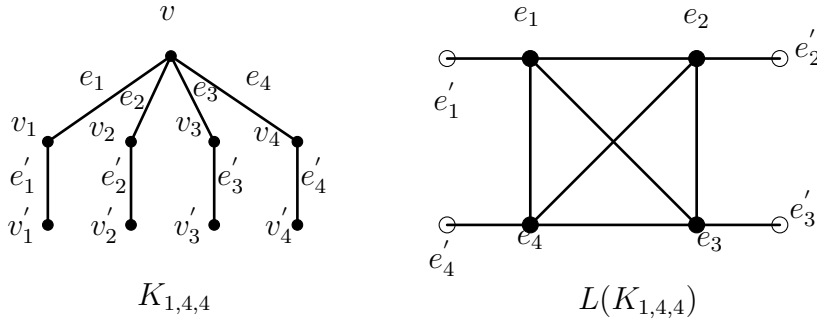


Figure 2: $K_{1,4,4}$, $L(K_{1,4,4})$

Theorem 2.2. For the wounded spider graph $K_{1,n,s}$, $\gamma_{rp}^{-1}(L(G)) = s + 1$ where $s < n$.

Proof. Let v be the centre vertex and $v_1, v_2, \dots, v_n$ be the end vertices of $K_{1,n}$. Attach $u_1, u_2, \dots, u_s$ to $v_1, v_2, \dots, v_n$ as appropriate where $s < n$. The resulting graph is the wounded spider graph $K_{1,n,s}$ with vertex set $V(K_{1,n,s}) = \{v, v_i, u_j : 1 \leq i \leq n, 1 \leq j \leq s\}$ and $E(K_{1,n,s}) = \{vv_i, v_j u_j : 1 \leq i \leq n, 1 \leq j \leq s\}$. Clearly in $K_{1,n,s}$, $\deg(v) = n$, $\deg(v_i) = 2$, $1 \leq i \leq s$, $\deg(v_k) = 1$, $s+1 \leq k \leq n$ and $\deg(u_i) = 1$, $1 \leq i \leq s$. Let the line graph of the graph $K_{1,n,s}$ be $L(K_{1,n,s})$ where we denote the edge vv_i by e_i and $v_j u_j$ by e'_j , $1 \leq i \leq n$, $1 \leq j \leq s$. Clearly, $V(L(K_{1,n,s})) = \{e_i, e'_j : 1 \leq i \leq n, 1 \leq j \leq s\}$ and $E(L(K_{1,n,s})) = \{e_i e_k, e_j e'_j : 1 \leq i \neq k \leq n, 1 \leq j \leq s\}$. Also in $L(K_{1,n,s})$, $\deg(e_j) = n$, $\deg(e'_j) = 1$ and $\deg(e_i) = n - 1$, $1 \leq j \leq s$ and $s + 1 \leq i \leq n$. Let D be a minimum dominating set of $L(K_{1,n,s})$ and S be a minimum inverse dominating set of $L(K_{1,n,s})$ with respect to D . Since $L(K_{1,n,s})$ contains s end vertices, any minimum dominating set of $L(K_{1,n,s})$ must include at least s vertices of $L(K_{1,n,s})$. Clearly, $D = \{e_j : 1 \leq j \leq s\}$ and $S = \{e_n, e'_j : 1 \leq j \leq s\}$ is a corresponding minimum inverse dominating set of $L(K_{1,n,s})$. Since the degree sequence of vertices in S is $(n, 1, 1, \dots, 1)$, S is a minimum relatively prime inverse dominating set of $L(K_{1,n,s})$ and hence $\gamma_{rp}^{-1}(L(K_{1,n,s})) = s + 1$. $\square$

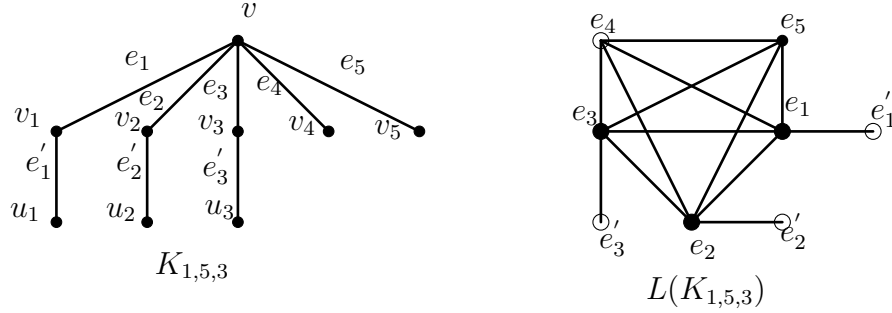


Figure 3: $K_{1,5,3}$, $L(K_{1,5,3})$

Theorem 2.3. For the jewel graph J_n , $\gamma_{rp}^{-1}(L(J_n)) = 2$ if $n \geq 1$.

Proof. Consider a 4-cycle $xwyux$. Join x and y . Now adding n new vertices $v_i, 1 \leq i \leq n$. Join v_i with u and $w, 1 \leq i \leq n$. The resulting graph is the jewel graph J_n with vertex set $V(J_n) = \{x, y, u, w, v_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{e_i, e'_j, e''_j : 1 \leq i \leq 5, 1 \leq j \leq n\}$, where $e_1 = xw, e_2 = wy, e_3 = yu, e_4 = ux, e_5 = xy, e'_j = uv_j, e''_j = wv_j$. Let the line graph of J_n be $L(J_n)$ where $V(L(J_n)) = E(J_n) = \{e_i, e'_j, e''_j : 1 \leq i \leq 5, 1 \leq j \leq n\}$ and $E(L(J_n)) = \{e_i e_{i+1}, e_1 e_4, e_5 e_j, e'_j e_i, e'_j e'_k, e''_j e_m, e''_j e''_p : 1 \leq i \leq 3, 1 \leq j \leq k, p \leq n, 3 \leq l \leq 4, 1 \leq m \leq 2\}, i \neq k$. Let D be a minimum dominating set of $L(J_n)$ and S be a corresponding minimum inverse dominating set. In $L(J_n)$, the number of vertices is $2n + 5$ and the maximum degree is $2n - 1$ and so any minimum dominating set contains at least two vertices. Now e_1 is adjacent to all vertices except e_3 and $e'_i, 1 \leq i \leq n$; e'_1 is adjacent to e_3, e_4, e'_1 and $e'_i, 2 \leq i \leq n$. Hence $D = \{e_1, e'_1\}$ is a minimum dominating set of $L(J_n)$. Clearly, $S = \{e_3, e''_1\} \subseteq V - D$ is also a minimum dominating set of $L(J_n)$. Hence, S is a minimum inverse dominating set of $L(J_n)$. In $L(J_n)$, $\deg(e_3) = n + 3$, $\deg(e''_1) = n + 2$ and therefore $(\deg(e_3), \deg(e''_1)) = (n + 3, n + 2) = 1$. This implies that S is a minimum relatively prime inverse dominating set of $L(J_n)$ and so $\gamma_{rp}^{-1}(L(J_n)) = 2$. $\square$

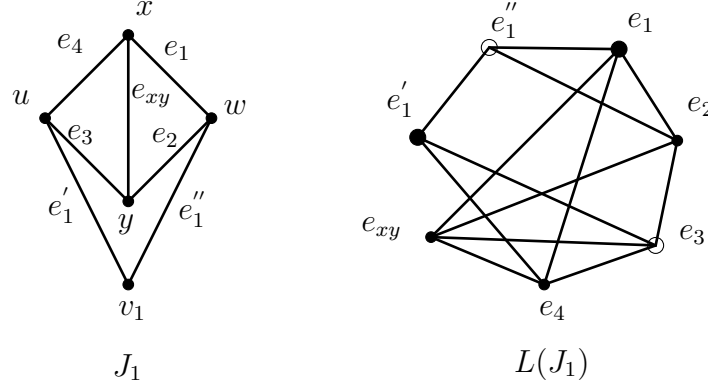


Figure 4: $J_1, L(J_1)$

Theorem 2.4. For the bistar graph $B_{m,n}$, $\gamma_{rp}^{-1}(L(B_{m,n})) = \begin{cases} 2 & \text{if } (m, n) = 1 \\ 0 & \text{otherwise} \end{cases}$.

Proof. A bistar graph $B_{m,n}$ consists of two star graphs $K_{1,m}$ and $K_{1,n}$ having center vertices u_0 and v_0 respectively. Join u_0 and v_0 with an edge. The resulting graph is a bistar graph $B_{m,n}$ with the vertex set $V(B_{m,n}) = \{u_i, v_j : 0 \leq i \leq m, 0 \leq j \leq n\}$ and edge set $E(B_{m,n}) = \{u_0u_i, v_0v_j, u_0v_0 : 1 \leq i \leq m, 1 \leq j \leq n\}$. Let the line graph of $B_{m,n}$ be $L(B_{m,n})$ with the vertex set $V(L(B_{m,n})) = \{e_0, e_i, e'_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ where $e_0 = u_0v_0, e_i = u_0u_i, e'_j = v_0v_j$ and edge set $E(L(B_{m,n})) = \{e_0e_i, e_0e'_j, e_ie_k, e'_je'_l : 1 \leq i \neq k \leq m, 1 \leq j \neq l \leq n\}$. Clearly in $L(B_{m,n})$, $\deg(e_0) = m + n, \deg(e'_i) = m$ and $\deg(e'_j) = n, 1 \leq i \leq m, 1 \leq j \leq n$. Let D be a minimum dominating set of $L(B_{m,n})$ and S be a corresponding minimum inverse dominating set of $L(B_{m,n})$. In $L(B_{m,n})$, the vertex e_0 dominates all other vertices and so the unique minimum dominating set of is $D = \{e_0\}$. In $V - D$, each e_x dominates e_0 and all other $e_i, 1 \leq i \leq m$ and $i \neq x$ and also each e'_y dominates all other $e'_j, j \neq y$ and $1 \leq j \leq n$. Hence a minimum inverse dominating set $S = \{e_x, e'_y\}$ for some x, y where $1 \leq x \leq m, 1 \leq y \leq n$. Now in $L(B_{m,n})$, $(\deg(e_x), \deg(e'_y)) = (m, n)$. This implies that S is a minimum relatively prime inverse dominating set if and only if $(m, n) = 1$. Hence the proof. $\square$

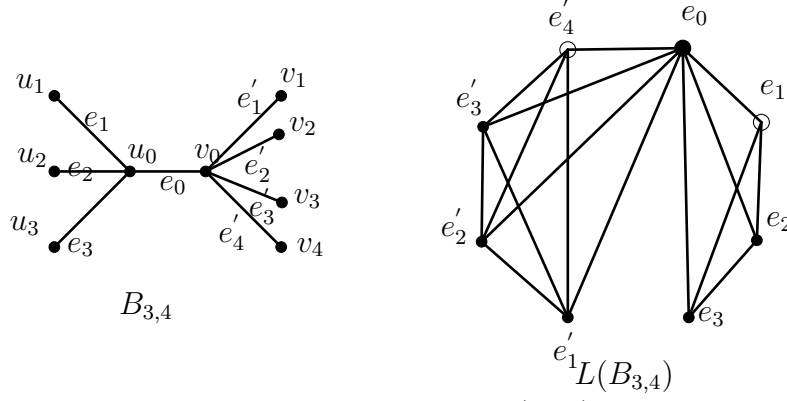


Figure 5: $B_{3,4}, L(B_{3,4})$

Theorem 2.5. For the comb graph $P_n \odot K_1$,

$$\gamma_{rp}^{-1}(L(P_n \odot K_1)) = \begin{cases} 2 & \text{if } n = 2, 3 \\ 3 & \text{if } n = 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Consider the path $P_n = v_1 v_2 \dots v_n$. For $1 \leq i \leq n$, add vertex u_i which is adjacent to v_i . The resulting graph $G = P_n \odot K_1$ is a comb graph with vertex set $V(G) = \{v_i, u_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{e_i, e'_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ where $e_i = v_i v_{i+1}, e'_j = v_j u_j, 1 \leq i \leq n-1, 1 \leq j \leq n$. Let the line graph of comb graph G be $L(G)$ where the vertex set $V(L(G)) = E(G) = \{e_i, e'_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ and edge set $E(L(G)) = \{e_i e_{i+1}, e_j e'_{j+1}, e_i e'_j : 1 \leq i \leq n-2, 1 \leq j \leq n-1\}$. Clearly in $L(G)$, $\deg(e_i) = 4, 2 \leq i \leq n-2, \deg(e_1) = \deg(e_{n-1}) = 3, \deg(e'_j) = 2, 2 \leq j \leq n-1$ and $\deg(e_i) = \deg(e'_n) = 1$. Let D be a minimum dominating set of $L(G)$ and S be a corresponding minimum inverse dominating set of $L(G)$. Now we consider the following five cases.

Case 1. $n = 2$

Then $L(G)$ is P_3 . By Theorem 1.1, $\gamma_{rp}^{-1}(L(G)) = 2$.

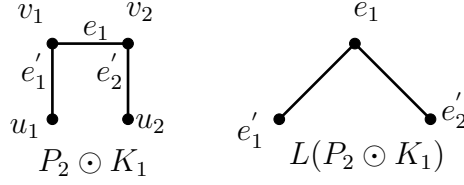


Figure 6: $P_2 \odot K_1, L(P_2 \odot K_1)$

Case 2. $n = 3$

In $L(G)$, e_1 is adjacent to all vertices except e'_3 , e'_3 is adjacent to e_2 only. Hence, $D = \{e_1, e'_3\}$ is a minimum dominating set of $L(G)$ and a corresponding minimum inverse dominating set $S = \{e_2, e'_1\}$. In $L(G)$, $(deg(e_2), deg(e'_1)) = (3, 1) = 1$. This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 2$.

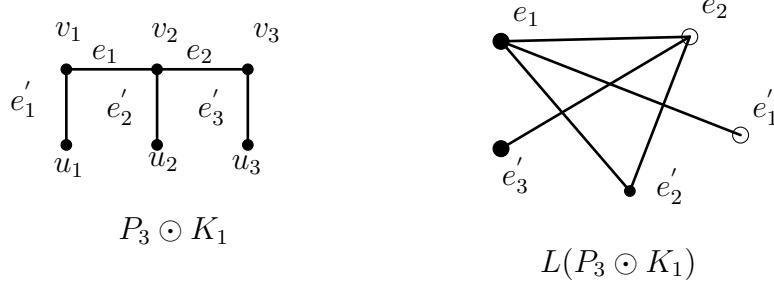


Figure 7: $P_3 \odot K_1, L(P_3 \odot K_1)$

Case 3. $n = 4$

In $L(G)$, e_1 is adjacent to all vertices except e_3 and $e'_i, i = 3, 4$; e'_i is adjacent to all vertices except $e_1, e'_i, i = 1, 2$. Hence, $D = \{e_1, e_3\}$ is a minimum dominating set of $L(G)$ and a corresponding minimum inverse dominating set $S = \{e_2, e'_1, e'_4\}$. The degree sequence vertices in S is $(4, 1, 1)$. This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 3$.

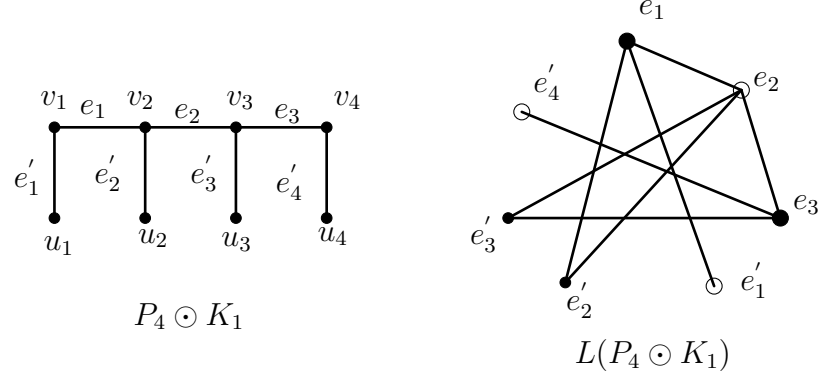


Figure 8: $P_4 \odot K_1$, $L(P_4 \odot K_1)$

Case 4. $n = 5$

In $L(G)$, e_1 is adjacent to all vertices except $e_i, i = 3, 4, e'_1, e'_4, e'_5$; e_3 is adjacent to all vertices except e_1, e'_1, e'_2, e'_5 ; e'_5 is adjacent to all vertices except e_4 . Hence, $D = \{e_1, e_3, e'_5\}$ is a minimum dominating set and a corresponding minimum inverse dominating $S = \{e_2, e_4, e'_1\}$. In $L(G)$, the degree sequence of vertices in S is $(4, 3, 1)$. This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 3$.

Case 5. $n \geq 6$

The degree sequence of $L(G)$ is $\{4, 4, \dots, 4(n-3)\text{times}, 3, 3, 2, 2, \dots, 2(n-2)\text{times}, 1, 1\}$. Any minimum dominating set must contain at least four vertices and so any minimum inverse dominating set S as at least four vertices of different degrees and thereby, there exists a pair of vertices (x, y) in S such that $(deg(x), deg(y)) = 2$ or 4 . Hence, $\gamma_{rp}^{-1}(L(G)) = 0$.

Thus the theorem following five cases. □

3 Conclusion

Inspired by inverse dominating set and relatively prime dominating set, we introduce the relatively prime inverse domination number on line graph. We have determined the relatively prime inverse domination on line graph of some standard graphs like spider graph, wounded spider graph, jewel graph, bistar graph, and comb graph. Furthermore our results are also justified with suitable examples.

The relatively prime inverse domination number can be obtained for many more graphs.

Acknowledgements

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Mathematical modelling and application of reduced differential transform method for river pollution

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Abstract

This paper presents the mathematical model of pollutant transport in a river. To effectively find the analytical solution of the advection-diffusion equation under various forms of suitable initial conditions, the reduced differential transform method (RDTM) is used. Three different initial concentration function cases, including rational, exponential, and power, are analyzed for the present model. A 2D and 3D visual comparison of the solutions obtained for each case is also shown. This article discusses the sufficient condition for convergence of the reduced differential transform approach to solving non-linear differential equations. The convergence results for the concentration functions in each case are briefly described. The present method is highly effective and more efficient in solving real-world problems. For all cases, the amount of phosphate pollutant concentration at various distances and time levels has been examined using numerical and graphical representations. While analyzing actual world problems, the current study demonstrates its effectiveness.

Keywords: Pollutant transport equation; Reduced differential transform method; Convergence; River pollution

2020 AMS subject classifications: 35A22, 35C10, 35G05¹

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1 Introduction

Accidents involving environmental contamination are common in the process plant, particularly in sectors like the chemical sector, manufacture of agricultural chemicals, natural gas extraction, etc [Li et al., 2009]. Organic materials and heavy metals are frequent and harmful contaminants in water pollution incidents, and examples of organic materials include benzene, naphthalene, phenol, anthracene, alcohol, etc. As a typical hydrocarbon, benzene is poisonous, and cancer-causing [Nomura et al., 2019] It may be inhaled or absorbed via the skin. Because of its low vapor pressure, benzene may be easily spread through the air. Exposure to benzene in the past is a common factor in developing leukemia [Jiang et al., 2018, Tsuji et al., 2018, Meszaros et al., 2017]. In many places, industrial or household human activity-related water contamination is a serious issue [Tchobanoglous et al., 1991]. The contamination of water sources is responsible for the deaths of over 25 million people annually. Models to manage and forecast water quality are vital. When analyzing river water quality, numerous aspects must be addressed, including dissolved oxygen, nitrates, chlorides, phosphates, suspended particles, environmental hormones, and chemical oxygen demand, such as heavy metals and bacteria. Agricultural pollution may degrade surface, and groundwater [Knight et al., 2000].

To satisfy its many needs, society relies heavily on river water, one of the few abundant sources of freshwater [Shi et al., 2019]. To ensure an undisturbed freshwater supply, specific water quality requirements along the rivers must be maintained [Chen et al., 2016]. Agricultural non-point source pollution (ANPSP), caused by the use of agrochemicals in farming, significantly impacts water quality and aquatic ecosystems [Bryan and Kandulu, 2011, Borges et al., 2017]. In 1925, the well-known model of Streeter and Phelps characterized the equilibrium of dissolved oxygen in rivers, marking the beginning of the era of mathematical water quality models. Since then, there have been many updates to this model [Streeter, 1925, James, 1978]. Weighted discretizations and the two-dimensional modified equation method solved the linear, constant coefficient advection-diffusion equation. The modified equivalent equation determines one- and multi-dimensional finite difference method accuracy [Noye and Tan, 1989]. The Eulerian-Lagrangian localized adjoint method (ELLAM) solves the nonlinear Buckley-Leverlea equation, which has degenerate diffusion and sharpening near-shock solutions [Dahle et al., 1995]. It is thought that a suitable strategy for identifying and evaluating the production of nutrients produced by management scenarios, which may aid in project prioritization and improve water quality, is extensive modeling of the surface water using tried-and-true techniques. They should be simulated as management scenarios before implementing plans to evaluate their effectiveness

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[Fakouri et al., 2019].

Numerous research on the impact of various water management strategies on water quality and quantity have been carried out using multiple models and experimental techniques in diverse places of different sizes with varying objectives. They used a MIKE11 pattern on the Pasikhan River and simulated nitrate and phosphate contaminant concentration. The effects of dumping waste water and draining water into rivers are significant and impact the river's water quality. In addition, Kerich assessed the chemicals used in the Ahero Irrigation Scheme and provided many recommendations to enhance the quality of water retrieved from the drainage canals. For this reason, the most efficient means of purifying water for human consumption in the area were biodegradable chemicals for pest and herbicide management and bio-sand filters [Kerich, 2020]. Groundwater quality in the Blinaja River basin was also investigated by Çadraku using irrigation water quality criteria. According to the findings, the groundwater in the research region is of sufficient quality for irrigating the crops. As well as addressing surface water issues, specific recommendations were made for preserving groundwater quality [Çadraku, 2021]. To solve the advection-dispersion equation (ADE) in rivers backward in time and a one-dimensional domain for different pollution loading patterns, an unique analytical approach was devised using the quasi-reversibility (QR) technique and the Fourier transform tool. To avoid the issue being ill-posed during the inverse solution process, a stability factor is added to the initial transport equation in this approach [Permanoon et al., 2022]. Mass movement is regulated by the molecular diffusion of solutes between mobile and static water in aquifers like the Chalk, which have long diffusion path lengths [Bibby, 1981].

In this paper, we formulate a one-dimensional mathematical model of pollutant transport. The governing equation is a 1D advection-diffusion equation solved by the reduced differential transform method (RDTM). This method requires an initial condition. To generate the initial condition, we have used the concentration of the river Khobistskali for PO_4 pollutant component [Tsuji et al., 2018]. Also, we have discussed the convergence of analytic solutions obtained by RDTM.

Section 2 covers the mathematical formulation of this problem. Section 3 contains the fundamental ideas behind the reduced differential transform method. In section 4, The process for achieving the convergence of the analytic series solution given by RDTM has been discussed. Section 5 includes the numerical outcomes and the convergence of the method for its effectiveness. 2D and 3D plots show a visual representation of the obtained solutions. Section 6 provides a summary of the conclusion.

2 Mathematical Formulation of the problem

Reaction, diffusion, advection, absorption, and sedimentation all have a role in transporting the pollutant material farther downstream. Variables such as the kind of pollution, its physicochemical characteristics, flow characteristics, and the surrounding environment all have a role. Thus, parameters linked to the flow of pollution are prioritized above those relating to the pollutant's nature [Kim and Chapra, 1997].

A linear partial differential equation, the mass transfer equation, is often utilized in research on water, soil, petroleum, the living environment, and several engineering subjects. A linear parabolic partial differential equation, the aforementioned equation is of the first and second orders in terms of time and space, respectively. In the one-dimensional domain (along the river length), the general form of this equation under unstable and non-uniform flow regimes is as follows [Amiri et al., 2021].

$$A \frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left(A D_x \frac{\partial c}{\partial x} \right) - A v \frac{\partial c}{\partial x} - A k c + A f \quad (1)$$

where c =Pollutant concentration, D_x =Diffusion coefficient along the x-direction, v = Mean flow velocity, k = Coefficient of non-conservation, A = Flow area, f = Source term, x = Distance from starting point of domain, t = Time dimension.

For the entire study, the area of the cross-section of the river is considered a constant. To analyze this problem, we have used the concentration of pollutant substance PO_4 of river Khobistskali. Here Khobistskali river's length is 44800 m. Using past collected data, at time $t = 0$ (any fixed time), the concentration is assumed as a rational, exponential, and power form, and the following values of parameters are used to obtain the solutions. $D_x = 0.55 \frac{\text{m}^2}{\text{sec}}$, $v = 0.534930 \frac{\text{m}}{\text{sec}}$, $k = 0 \frac{1}{\text{sec}}$ and $f = 0$ [Kachiashvili et al., 2007].

Goodness of fit				
Curve fitting	SSE	R-square	Adjusted R-square	RMSE
Rational	3.89E-06	0.9884	0.9876	0.0003662
Exponential	1.62E-05	0.9515	0.9499	0.0007354
Power	5.55E-06	0.9834	0.9823	0.0004376

Table 1: Statistical indices of initial function

Graphical representations of curve fitting for the initial condition are shown in Figures 1, 2, and 3. Table 1 lists the goodness of fit values for various initial conditions.

Hence this problem is studied for three following different cases:

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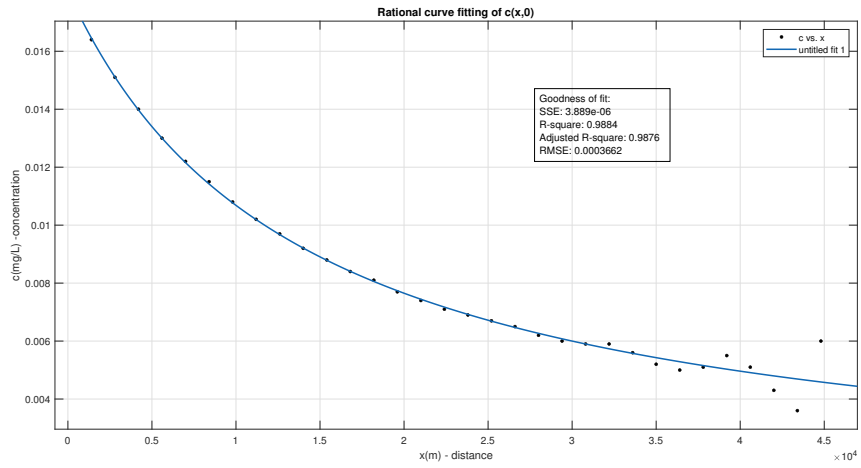


Figure 1: Rational curve fitting of $c(x, 0)$.

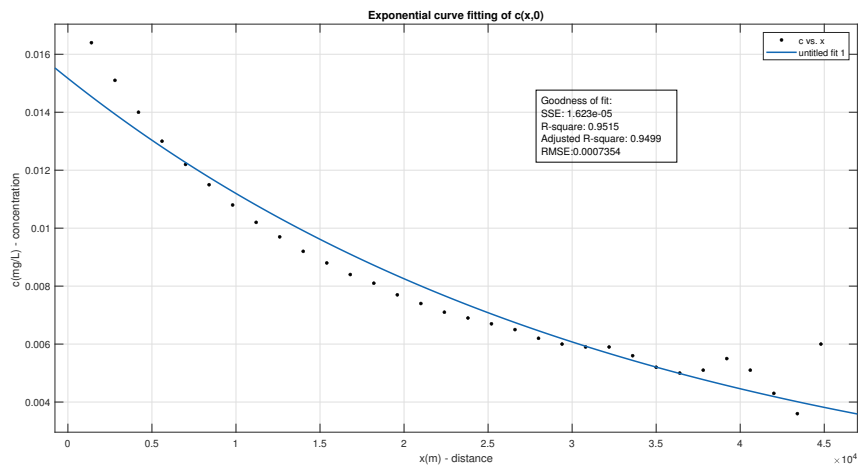


Figure 2: Exponential curve fitting of $c(x, 0)$.

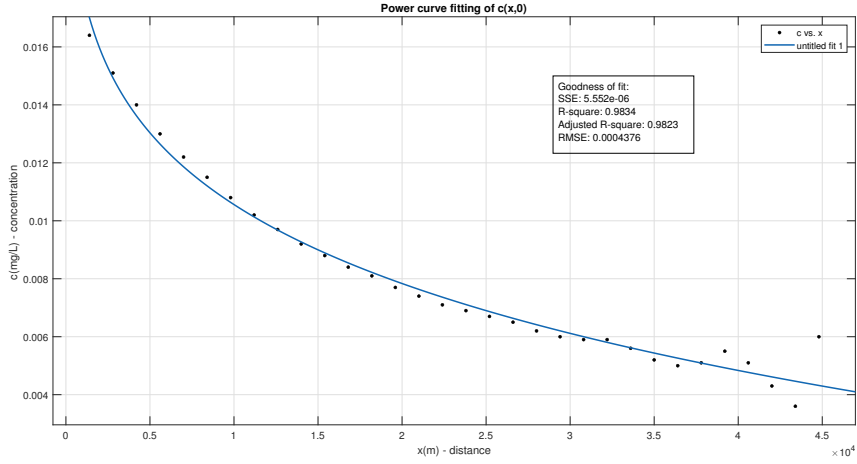


Figure 3: Power curve fitting of $c(x, 0)$.

Case-1 Rational initial function

For this case,

$$c(x, 0) = \frac{0.0004008x + 251.4}{x + 1.39e + 04} \quad (2)$$

with

$$c(44800, t) = 5.846e - 13t^2 + 3.463e - 08t + 0.004597 \quad (3)$$

Case-2 Exponential initial function

For this case,

$$c(x, 0) = 0.01533e^{-3e-05x} - 0.0001573 \quad (4)$$

with

$$c(44800, t) = 6.887e - 13t^2 + 6.167e - 08t + 0.003846 \quad (5)$$

Case-3 Power initial function

For this case,

$$c(x, 0) = -0.007562x^{0.1384} + 0.03762 \quad (6)$$

with

$$c(44800, t) = 6.846e - 08t + 0.004247 \quad (7)$$

3 Reduced Differential Transform Method

Let $b(\omega, \tau)$ be a two-variable function. Suppose $b(\omega, \tau)$ is written as $b(\omega, \tau) = f(\omega)g(\tau)$. $b(\omega, \tau)$ can be represented as the following using the features of the differential transform:

$$b(\omega, \tau) = \sum_{i=0}^{\infty} F_{(i)} \omega^i \sum_{j=0}^{\infty} G_{(j)} \tau^j = \sum_{k=0}^{\infty} B_k(\omega) \tau^k \quad (8)$$

where $B_k(\omega)$ is referred to as the τ -dimensional spectrum function of $b(\omega, \tau)$.

$$B_m(\omega) = \frac{1}{m!} \left[\frac{\partial^m}{\partial \tau^m} b(\omega, \tau) \right]_{\tau=0} \quad (9)$$

The original function is denoted by the lowercase $[b(\omega, \tau)]$, whereas the altered function is denoted by the capital $[B(\omega, \tau)]$. The way to define the differential inverse transform of $B_k(\omega)$ is as follows:

$$b(\omega, \tau) = \sum_{m=0}^{\infty} B_m(\omega) \tau^m \quad (10)$$

From Equations (9) and (10), we get

$$b(\omega, \tau) = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\partial^m}{\partial \tau^m} b(\omega, \tau) \right]_{\tau=0} \tau^m \quad (11)$$

Let us consider the following nonlinear PDE, to understand the basic concept of RDTM.

$$Tb(\omega, \tau) + Pb(\omega, \tau) + Ob(\omega, \tau) = f(\omega, \tau) \quad (12)$$

with initial condition $b(\omega, 0) = \eta(\omega)$
, where $T = \frac{\partial}{\partial \tau}$, $Pb(\omega, \tau)$ is a linear term that has partial derivatives, while $Ob(\omega, \tau)$ is a non-linear term, and $f(\omega, \tau)$ is a source term [Al-Amr, 2014].
By applying the transform on equation (12), we get

$$(m+1)B_{m+1}(\omega) = F_m(\omega) - PB_m(\omega) - OB_m(\omega) \quad (13)$$

where $B_m(\omega), F_m(\omega), PB_m(\omega)$ and $OB_m(\omega)$ are transform of $b(\omega, \tau), f(\omega, \tau), Pb(\omega, \tau)$

and $Ob(\omega, \tau)$ respectively. We are able to write this down based on the initial condition.

$$B_0(\omega) = \eta(\omega) \quad (14)$$

From equations (13) and (14), we get the values of $B_m(\omega)$. After that, an approximation solution is produced by carrying out an inverse transformation on the set of values $\{B_m(\omega)\}_{m=0}^n$. This transformation yields an approximation solution as

$$\tilde{b}_n(\omega, \tau) = \sum_{m=0}^n B_m(\omega) \tau^m \quad (15)$$

where n is the order of approximation answer. Consequently, the exact solution is given by [Al-Amr, 2014],

$$b(\omega, \tau) = \lim_{n \rightarrow \infty} \tilde{b}_n(\omega, \tau) \quad (16)$$

Function	Transformation
$b(\omega, \tau)$	$B_m(\omega) = \frac{1}{m!} \left[\frac{\partial^m}{\partial \tau^m} b(\omega, \tau) \right]_{\tau=0}$
$\alpha f(\omega, \tau) \pm \beta g(\omega, \tau)$	$\alpha F_m(\omega) + \beta G_m(\omega)$
$\omega^k \tau^n$	$\omega^k \delta(m - n)$
$\omega^k \tau^n b(\omega, \tau)$	$\omega^k B_{m-n}(\omega)$
$l(\omega, \tau) = f(\omega, \tau)g(\omega, \tau)$	$L_m(\omega) = \sum_{r=0}^m F_r(\omega) G_{m-r}(\omega)$
$\frac{\partial^r}{\partial \tau^r} b(\omega, \tau)$	$\frac{(m+r)!}{m!} B_{m+r}(\omega)$
$\frac{\partial}{\partial \omega} b(\omega, \tau)$	$\frac{\partial}{\partial \omega} B_m(\omega)$

Table 2: Transform Table[Al-Amr, 2014, Keskin and Oturanc, 2010, Srivastava et al., 2014]

4 Convergence of RDTM

To understand the convergence, Let us consider the solution of equation (13) in power series form as follow

$$b(\omega, \tau) = \sum_{n=0}^{\infty} B_n(\omega) \tau^n = \sum_{n=0}^{\infty} B_n \tau^n \quad (17)$$

which is obtained by equation (16) [Moosavi Noori and Taghizadeh, 2021, Saeed and Mustafa, 2017].

Theorem 4.1. *If $\sum_{n=0}^{\infty} B_n \tau^n$ is given series,*

$$[1] \exists 0 < \beta < 1 \ni \frac{\|B_{n+1}\|}{\|B_n\|} \leq \beta \Rightarrow \text{series is convergent.}$$

$$[2] \exists \beta > 1 \ni \frac{\|B_{n+1}\|}{\|B_n\|} \geq \beta \Rightarrow \text{series is divergent.}$$

Proof. Let $(C[l], \|\cdot\|)$ represent the Banach space that contains all continuous functions on l that satisfy the norm $\|\cdot\|$. Also, let's suppose that $\|B_0(\omega)\| \leq M$, where M is an integer in the positive range. Let $\{\delta_n\}_{n=0}^{\infty}$ be a partial sum

$$\delta_n = B_0 + B_1 + B_2 + \dots + B_n$$

If we can prove that $\{\delta_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach Space, then we can conclude that the sequence of partial sum is convergent in Banach Space. As a result, at this point, we shall demonstrate that the series of partial sums follows the Cauchy sequence. We take

$$\|\delta_{n+1} - \delta_n\| = \|B_{n+1}\| \leq \beta \|B_n\| \leq \dots \leq \beta^{n+1} \|B_0\| \leq \beta^{n+1} M$$

Therefore, $\forall n, m \in \mathbb{N}, n \geq m$, We have

$$\begin{aligned} \|\delta_n - \delta_m\| &= \|(\delta_n - \delta_{n-1}) + (\delta_{n-1} - \delta_{n-2}) + \dots + (\delta_{m+1} - \delta_m)\| \\ &\leq \|\delta_n - \delta_{n-1}\| + \|\delta_{n-1} - \delta_{n-2}\| + \dots + \|\delta_{m+1} - \delta_m\| \\ &\leq \frac{1-\beta^{n-m}}{1-\beta} \beta^{m+1} \|B_0\| \end{aligned}$$

Now here $0 < \beta < 1$, We obtain

$$\lim_{n,m \rightarrow \infty} \|\delta_n - \delta_m\| = 0$$

Hence $\{\delta_n\}_{n=0}^{\infty}$ is Cauchy sequence in Banach Space. Therefore, given series is convergent.

5 Result and discussion

We take three cases of initial function in rational, exponential and power form and obtain three solutions using reduced differential transform method.

Case-1 Rational initial function

Solving equation (1) by RDTM with initial condition (2), we get

$$c(x, t) = M_0 + M_1 t + M_2 t^2 + M_3 t^3 + M_4 t^4 + \dots$$

where

$$M_0 = \frac{0.0004008x+251.4}{x+1.39e+04}$$

$$M_1 = \frac{0.0270(4863x + 67605700)}{(x+13900)^3}$$

$$M_2 = \frac{8.9236e-06 \left(x^2 + 219193889400x + 1523735609830000 \right)}{(x+13900)^5}$$

$$M_3 = \frac{9.8159e-10 \left(x^3 + 1599028070073300x^2 + 22233066963300870000x + 103043693991250631000000 \right)}{(x+13900)^7}$$

$$M_4 = \frac{1.0798e-13 \left(186421425071787x^4 + 10368864699446257200x^3 + 216270792211342627620000x^2 + 2004850662624966111612000000x + 6969433288195869001126700000000 \right)}{(x+13900)^9}$$

Here,

$$\frac{\|M_1\|}{\|M_0\|} = 3.4116405e-05 < 1, \quad \frac{\|M_2\|}{\|M_1\|} = 3.4972143e-05 < 1,$$

$$\frac{\|M_3\|}{\|M_2\|} = 3.4976844e-05 < 1, \quad \frac{\|M_4\|}{\|M_3\|} = 3.4981545e-05 < 1$$

Therefore, the solution function $c(x, t)$ is convergent.

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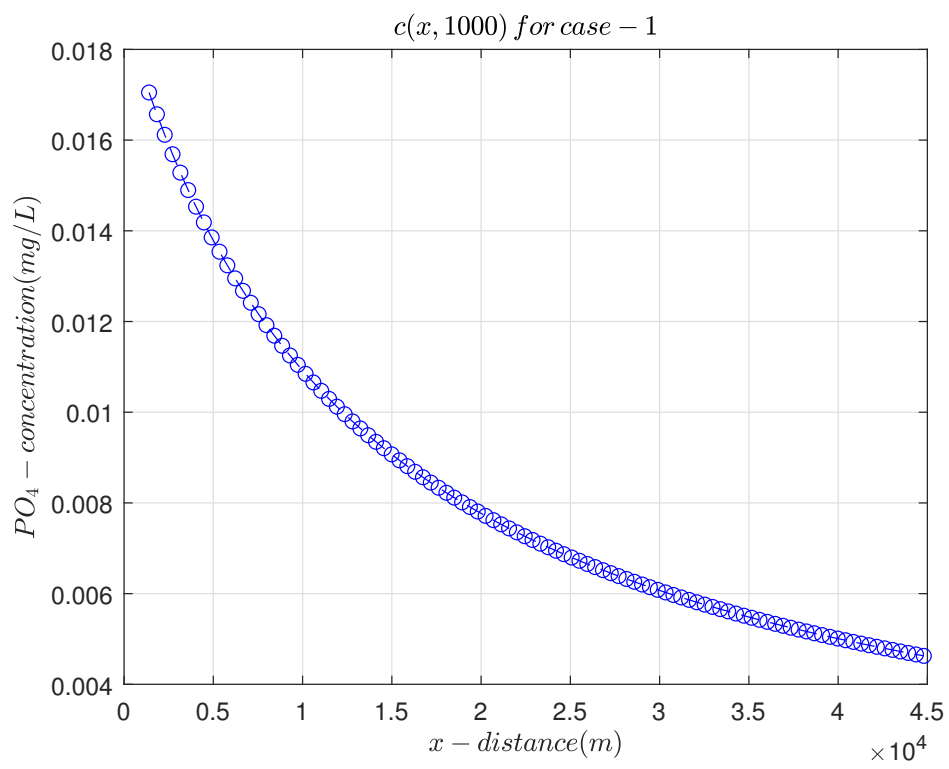


Figure 4: $c(x, t)$ at $t = 1000$ sec

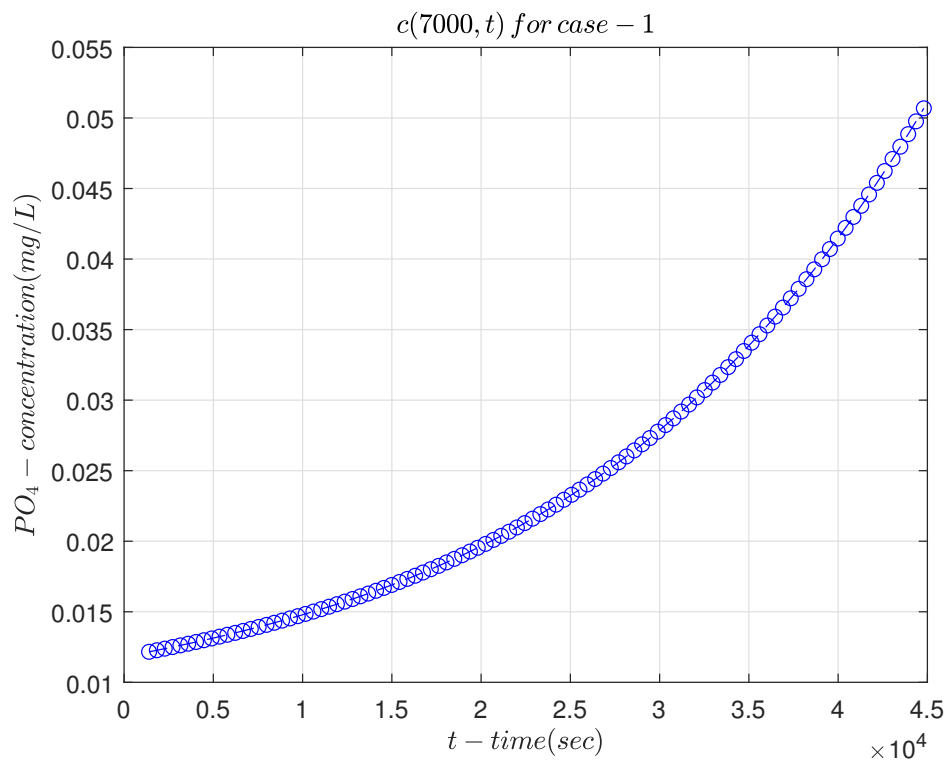


Figure 5: $c(x, t)$ at $x = 7000\text{ m}$

*Mathematical modelling and application of reduced differential transform
method for river pollution*

Two-dimensional representations of the concentration function for case-1 are provided in figures 4, and 5, respectively, for the fixed values of ($t = 1000sec$), and ($x = 7000m$). As shown in figure 4, we can see that the value of concentration decreases as the length (x) variable increases. In figure 5, we can see that the value of concentration rises as the passage of time (t) increases.

x(m)\t(sec)	4500	9000	13500	18000	22500	27000	31500	36000
1400	0.019467	0.023775	0.030123	0.039479	0.053046	0.072264	0.098812	0.134601
5740	0.014666	0.016968	0.020065	0.024264	0.029941	0.03754	0.047573	0.060618
10080	0.011796	0.013224	0.015034	0.017353	0.020329	0.024138	0.028979	0.035078
14420	0.009888	0.010858	0.01204	0.013494	0.015289	0.017506	0.020238	0.023587
18760	0.008527	0.009229	0.01006	0.011051	0.01224	0.013668	0.015383	0.017437
23100	0.007507	0.008038	0.008654	0.009372	0.010213	0.011201	0.012363	0.013728
27440	0.006715	0.007131	0.007605	0.008148	0.008772	0.009493	0.010326	0.01129
31780	0.006082	0.006416	0.006792	0.007217	0.007698	0.008246	0.00887	0.009581
36120	0.005564	0.005839	0.006144	0.006485	0.006868	0.007297	0.00778	0.008325
40460	0.005133	0.005362	0.005616	0.005896	0.006206	0.006551	0.006936	0.007365
44800	0.004768	0.004963	0.005176	0.00541	0.005667	0.00595	0.006264	0.00661

Table 3: $c(x, t)$ for Case-1

The values of the concentration function for case-1 are shown in table 3.

Case-2 Exponential initial function

Solving equation (1) by RDTM with initial condition (4), we get

$$c(x, t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + \dots$$

where

$$A_0 = 0.01533e^{-3e-05x} - 0.0001573$$

$$A_1 = 2.4602e - 07 e^{-\frac{3x}{100000}}$$

$$A_2 = 1.9741e - 12 e^{-\frac{3x}{100000}}$$

$$A_3 = 1.0561e - 17 e^{-\frac{3x}{100000}}$$

$$A_4 = 4.2370e - 23 e^{-\frac{3x}{100000}}$$

Here,

$$\frac{\|A_1\|}{\|A_0\|} = 1.6221987e - 05 < 1, \quad \frac{\|A_2\|}{\|A_1\|} = 8.0241975e - 06 < 1,$$

$$\frac{\|A_3\|}{\|A_2\|} = 5.3494650e - 06 < 1, \frac{\|A_4\|}{\|A_3\|} = 4.0120987e - 06 < 1$$

Therefore, the solution function $c(x, t)$ is convergent. Figures 6 and 7 illustrate,

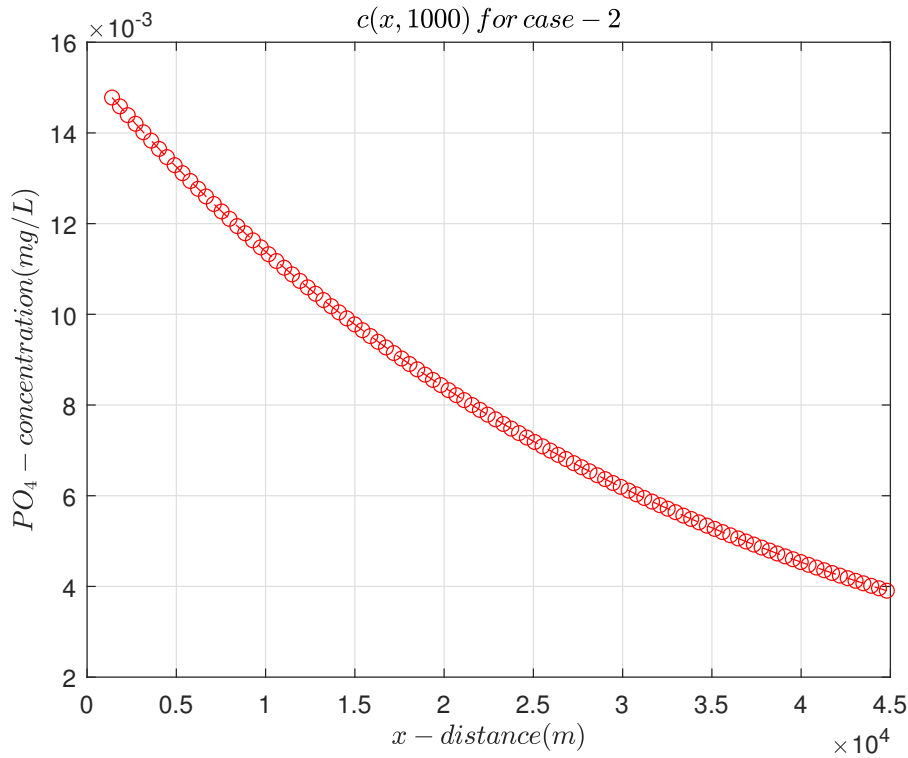


Figure 6: $c(x, t)$ at $t = 1000 \text{ sec}$

respectively, two-dimensional representations of the concentration function for case-2 with the fixed values of ($t = 1000 \text{ sec}$) and ($x = 7000 \text{ m}$). As shown in figure 6, the concentration value falls as the distance (x) variable rises. It is clear from graph 7 that when time (t) grows, so does the value of concentration.

The values of a concentration function for case-2 are presented in table 4.

Case-3 Power initial function

Solving equation (1) by RDTM with initial condition (6), we get

$$c(x, t) = P_0 + P_1t + P_2t^2 + P_3t^3 + P_4t^4 + \dots$$

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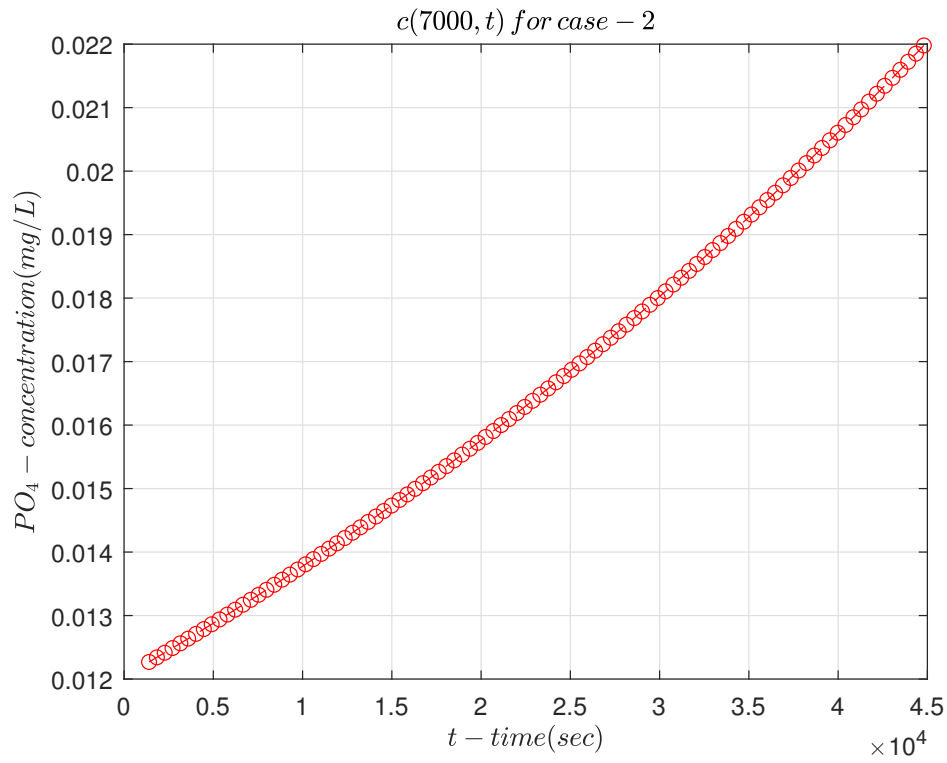


Figure 7: $c(x, t)$ at $x = 7000\text{ m}$

$x(m) \backslash t(sec)$	4500	9000	13500	18000	22500	27000	31500	36000
1400	0.015643	0.016826	0.018098	0.019465	0.020934	0.022512	0.024208	0.026029
5740	0.013714	0.014753	0.015869	0.01707	0.018359	0.019745	0.021233	0.022832
10080	0.012021	0.012933	0.013913	0.014966	0.016099	0.017315	0.018622	0.020025
14420	0.010534	0.011335	0.012195	0.01312	0.014114	0.015182	0.016329	0.017561
18760	0.009229	0.009932	0.010687	0.011499	0.012372	0.01331	0.014317	0.015398
23100	0.008083	0.0087	0.009363	0.010076	0.010842	0.011665	0.01255	0.013499
27440	0.007077	0.007619	0.008201	0.008827	0.0095	0.010222	0.010998	0.011832
31780	0.006194	0.006669	0.007181	0.00773	0.008321	0.008955	0.009637	0.010368
36120	0.005418	0.005836	0.006285	0.006767	0.007286	0.007843	0.008441	0.009083
40460	0.004738	0.005104	0.005498	0.005922	0.006377	0.006866	0.007391	0.007955
44800	0.00414	0.004462	0.004808	0.00518	0.005579	0.006009	0.00647	0.006965

Table 4: $c(x, t)$ for Case-2

where

$$\begin{aligned}
 P_0 &= -0.007562x^{0.1384} + 0.03762 \\
 P_1 &= \frac{3.4537e-07(1621x+1436)}{x^{2327/1250}} \\
 P_2 &= \frac{1.2902e-04x^2+4.9388e-04x+7.2656e-04}{x^{4827/1250}} \\
 P_3 &= \frac{4.2826e-05x^3+3.7801e-04x^2+0.0015x+0.0025}{x^{7327/1250}} \\
 P_4 &= \frac{1.6389e-05x^4+2.6028e-04x^3+0.0020x^2+0.0078x+0.0138}{x^{9827/1250}}
 \end{aligned}$$

Here,

$$\frac{\|P_1\|}{\|P_0\|} = 6.4108463e-05 < 1, \frac{\|P_2\|}{\|P_1\|} = 0.0001649 < 1,$$

$$\frac{\|P_3\|}{\|P_2\|} = 0.0002379 < 1, \frac{\|P_4\|}{\|P_3\|} = 0.0002747 < 1$$

Therefore, the solution function $c(x, t)$ is convergent. Figures 8 and 9 are two-dimensional representations of the concentration function for case-3 with fixed parameters ($t = 1000sec$) and ($x = 7000m$), respectively. As shown in figure 8, as the distance (x) variable increases, the concentration value decreases. Graph 9 demonstrates that as time (t) increases, so does the value of concentration. The

x(m)\t(sec)	4500	9000	13500	18000	22500	27000	31500	36000
1400	0.034284	0.149754	0.560422	1.579016	3.633997	7.269557	13.14562	22.03783
5740	0.014373	0.017404	0.02281	0.032238	0.047824	0.072202	0.108497	0.160329
10080	0.011539	0.012841	0.014609	0.017063	0.020482	0.025201	0.031609	0.040152
14420	0.00987	0.010713	0.011738	0.013011	0.014609	0.016626	0.019168	0.022356
18760	0.00866	0.009291	0.010017	0.010867	0.011871	0.013066	0.014494	0.016203
23100	0.007701	0.008209	0.008775	0.009415	0.010143	0.010977	0.011937	0.013046
27440	0.006902	0.007329	0.007796	0.008311	0.008883	0.009521	0.010237	0.011043
31780	0.006215	0.006584	0.006983	0.007416	0.007888	0.008405	0.008974	0.009603
36120	0.00561	0.005937	0.006285	0.006659	0.007062	0.007498	0.007971	0.008486
40460	0.005069	0.005362	0.005673	0.006003	0.006355	0.006732	0.007137	0.007573
44800	0.004579	0.004845	0.005126	0.005422	0.005735	0.006068	0.006423	0.006801

Table 5: $c(x, t)$ for Case-3

values of a concentration function for case-3 are shown in table 5.

Figure 10 give the 3D graphical comparison of solution obtained for all cases. Also figures 11 and 12 give the 2D graphical comparison of solution obtained for all cases for $t = 3000 sec$ and $x = 40000 m$ respectively.

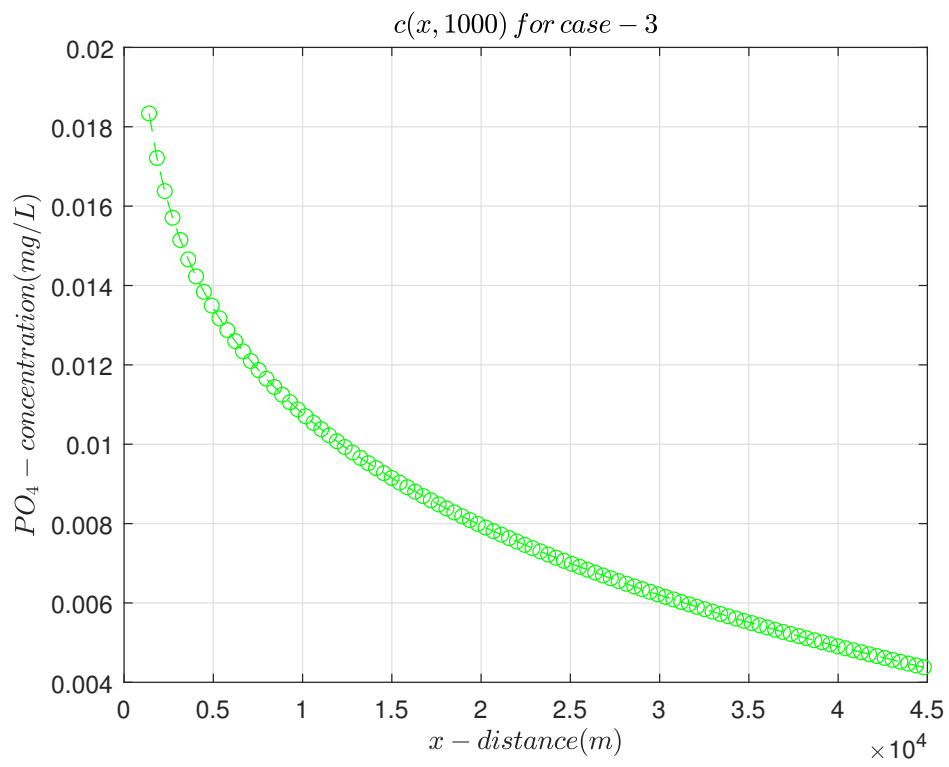


Figure 8: $c(x, t)$ at $t = 1000$ sec

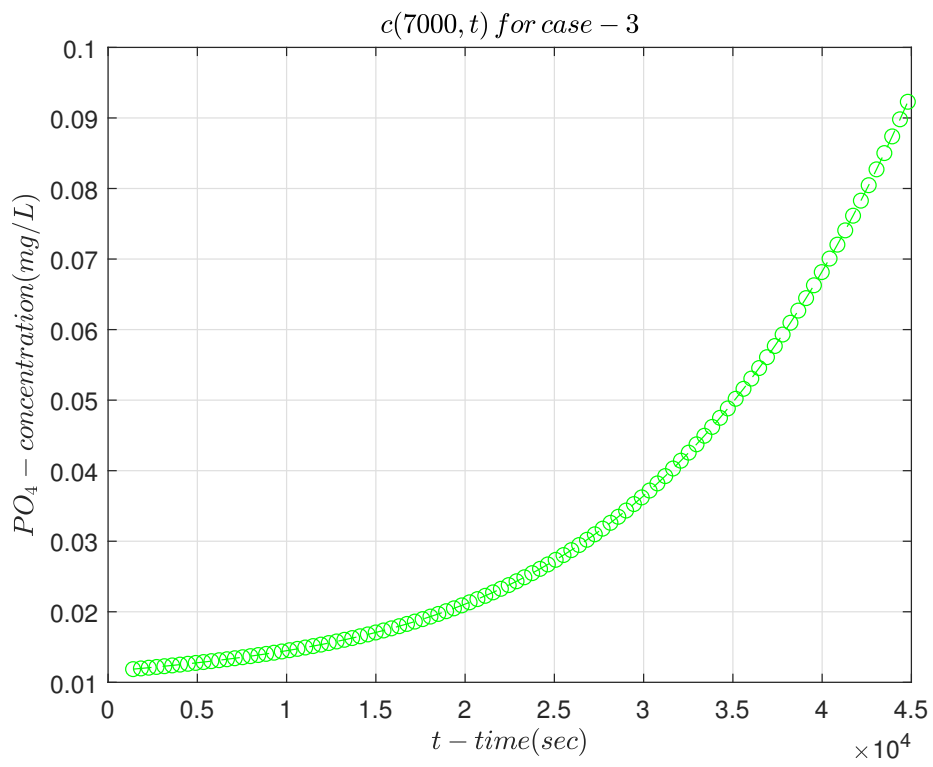


Figure 9: $c(x, t)$ at $x = 7000 \text{ m}$

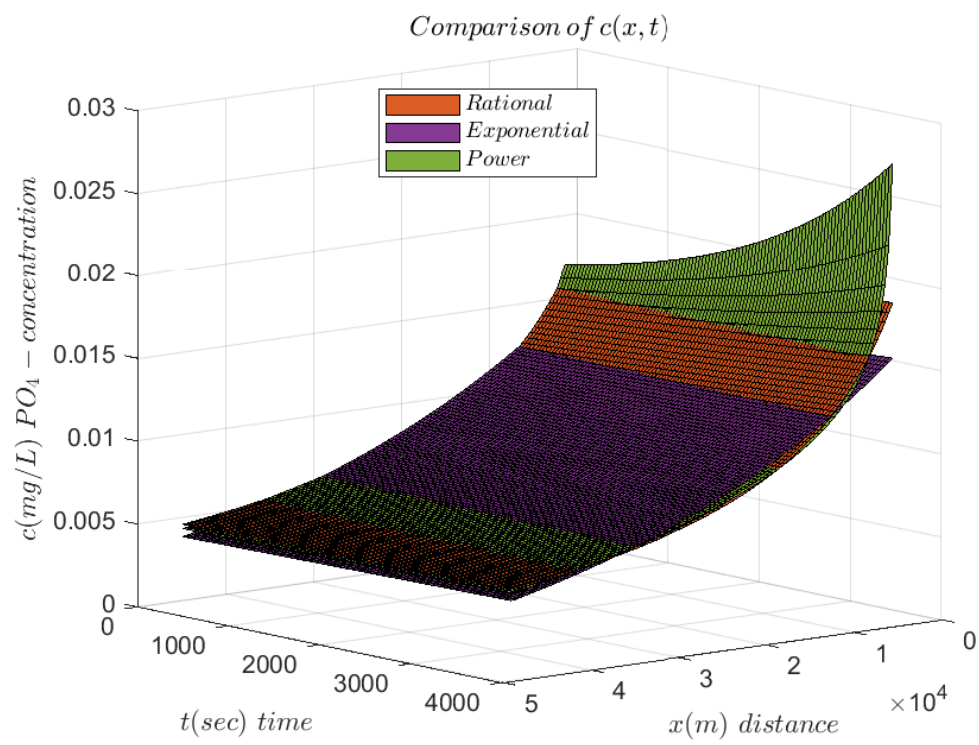


Figure 10: 3D Comparison of $c(x, t)$

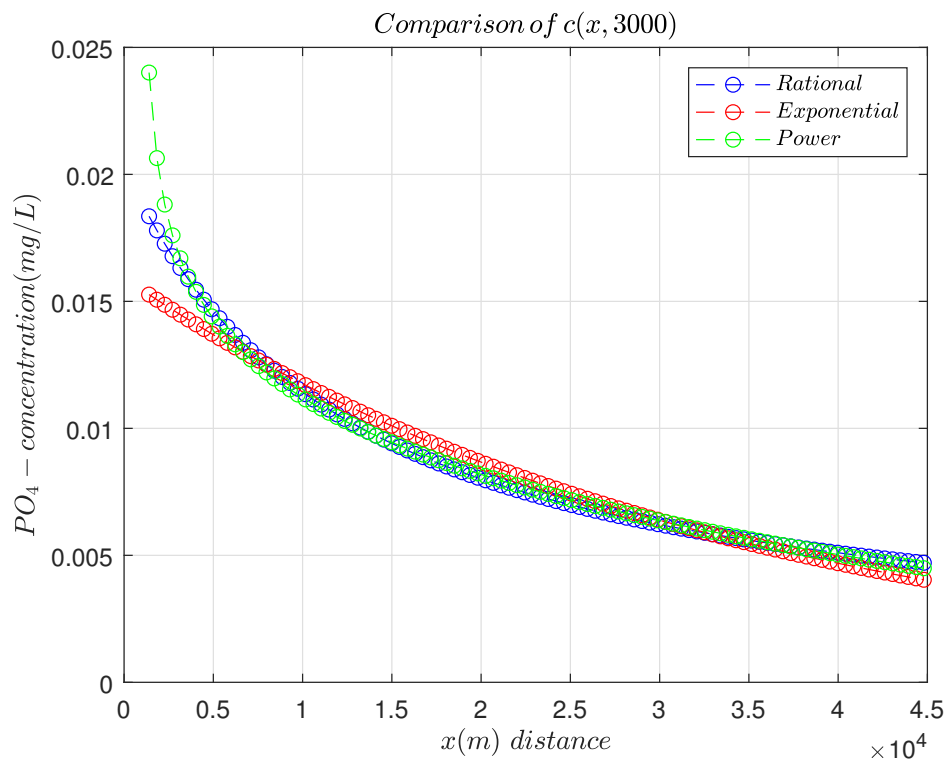


Figure 11: 2D Comparison of $c(x, 3000)$

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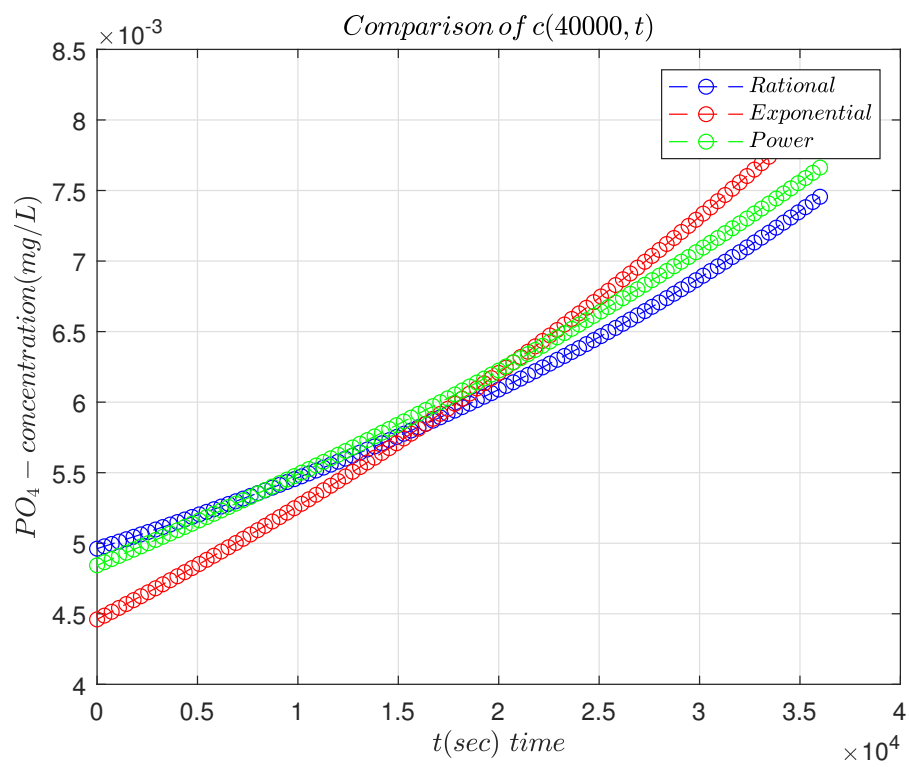


Figure 12: 2D Comparison of $c(40000, t)$

6 Conclusion

In this paper, we outlined the main components of a mathematical model that various ways to predict chemical concentrations in rivers due to pollutant discharges. We get initial condition from old collected data in different form like rational, exponential and power. We attain three different solutions from different form of initial conditions. We conclude that concentration rises as time(t) rises. Concentration decreases as length(x) increases. The fundamental benefit of the RDTM is that it offers the user a quick converging power series form with neatly calculated terms that contains an analytical approximation, and in many situations, an exact solution. There is no discretization or unavoidable presumptions while using RDTM. Sometimes RDTM is superior to other techniques (DTM). Furthermore, if highly polluted regions can be located physically, the research framework may provide a useful strategy for more economical watershed management. According to the findings, phosphate levels along the river fall when fertiliser usage is reduced. Among the management options, the use of less fertiliser greatly reduces river pollution. The current work provides a helpful tool for a scholar to compare and contrast the performance of different models and yields intriguing and practical outcomes. It may be possible to extend this research to consider the two-dimensional advection-diffusion equation.

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Door spaces on N - Topology

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Abstract

In this article, we explore the idea of door space on N -topological space. Here, we discuss which door spaces in this space are related with $N\tau$ submaximal. The equivalent conditions shows how it connects a N -topological property. Also, we derive various door spaces using separation axioms and discuss the characteristics of such door spaces. We take a strong form of open set in N -topological space and introduce a new door space called $N\tau\beta$ - door space. In addition, we analyze $N\tau\beta$ -door space and discuss the relationship between a $N\tau\beta$ -locally closed set and $N\tau$ -closed set.

Keywords: N -topology; door space; sub-maximal; $N\tau_d\beta$ -open sets.

2020 AMS subject classifications: 54A05, 54A10, 54C05 ¹

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1 Introduction

In 1955, J.L. Kelly [2], introduced the term 'door' in classical topology. He investigated the relationship between the different topological spaces and door spaces. McCartan [5] found three types of door spaces and established the concepts of connected door spaces and maximally connected. Muckenhaupt and Williams [6] concluded that there exists a non-zero Borel measure in every connected door space. Mathew [4]enquired about hyper-connected door space and proposed its related concepts. In 1983, Monsef et al.[1] initiated the study of β -open sets and β -continuity in a topological space.

This article explores the idea of door space in N - topological space. Here, we introduce $N\tau$ -door space and discuss which door spaces are related with $N\tau$ submaximal. The equivalent conditions shows how it connects with $N\tau$ frontier, a N -topological property. Also, we derive various door spaces using separation axioms and discuss the characteristics of such door spaces. We take a strong form of open set in N -topological space and introduce a new door space called $N\tau\beta$ - door space . Finally, we introduce $N\tau\beta$ - locally closed sets and its properties which are all the essential tool for the future development of this concept.

2 $N\tau$ - door spaces

In this section, we establish various door spaces in N -topological space and explicit its properties. Moreover, discussed about the sub maximal concept and analyze its characterization in this space.

Definition 2.1. A $(P, N\tau)$ - space is called Hausdorff space of N topological space if for every given pair of different points $z, q \in P$, there exists $R, S \in N\tau O(P)$ such that $z \in R, q \in S, R \cap S = \phi$ and is denoted by $N\tau$ - Hausdorff space.

Definition 2.2. A $(P, N\tau)$ - space is called semi - hausdorff space of N topological space if for every given pair of different points $z, q \in P$, there exists $R, S \in N\tau SO(P)$ such that $z \in R, q \in S, R \cap S = \phi$ and is denoted by $N\tau$ - semi hausdorff space.

Theorem 2.1. Each $N\tau$ hausdorff space of $(P, N\tau)$ is $N\tau$ - semi hausdorff space.

Proof. Suppose $(P, N\tau)$ is a $N\tau$ hausdorff space then there exists two disjoint points which can be isolated by disjoint $N\tau$ open set. Since each $N\tau$ open set is a $N\tau$ - semi open then we may conclude that each $N\tau$ hausdorff space of $(P, N\tau)$ is $N\tau$ - semi hausdorff space. $\square$

Remark 2.1. The inverse of the theorem 5.2.3 need not be true which is shown below.

Let $P = \{l, m, n, o, p\}$. For $N = 3$, consider $\tau_1 = \{\phi, \{l\}, \{l, p\}, \{l, m, o, p\}, P\}$, $\tau_2 = \{\phi, \{l, m, o\}, \{m, o\}, P\}$ and $\tau_3 = \{\phi, \{m, o, p\}, \{l, m, o, p\}, P\}$.

Then $3\tau = \{\phi, \{l\}, \{p\}, \{l, p\}, \{m, o\}, \{l, m, o\}, \{m, o, p\}, \{l, m, o, p\}, P\}$. Here $\{l, n\}$ is 3τ - semi open set but not 3τ - open.

Definition 2.3. If every subset of $(P, N\tau)$ is either $N\tau$ - open or $N\tau$ -closed then $(P, N\tau)$ is called as $N\tau$ - door space.

Example 2.1. Let $S = \{x, y, z\}$. For $N = 2$, $\tau_1 = \{\phi, \{z\}, \{x, z\}, S\}$. Then $2\tau O(S) = \{\phi, \{x\}, \{z\}, \{x, z\}, S\}$, $2\tau C(S) = \{\phi, \{y, z\}, \{x, y\}, \{y\}, S\}$. Therefore S is a 2τ - door space.

Theorem 2.2. Every subspace G of a $N\tau$ - door space is $N\tau$ - door space.

Proof. Let $(P, N\tau)$ be a door space. Let $G \subseteq P$ and $U \in G$. Since P is a $N\tau$ door space then U is either $N\tau$ open or $N\tau$ closed in P and hence in G . Therefore G is also a $N\tau$ door space. $\square$

Definition 2.4. A subset S of $(P, N\tau)$ is $N\tau$ - dense if $N\tau - cl(S) = P$.

Definition 2.5. A N -topological space $(P, N\tau)$ is sub maximal if every $N\tau$ - dense subset of P is $N\tau O(P)$ and is denoted as $N\tau$ - sub maximal.

Theorem 2.3. Every door space $(P, N\tau)$ is $N\tau$ - sub maximal.

Proof. Let $(P, N\tau)$ be a door space and $V \subset P$ be a $N\tau$ dense. If V is not a $N\tau$ open then it is $N\tau$ closed since P is a $N\tau$ door space. Now $V = N\tau cl(V) = P$ and V is $N\tau$ open. Hence P is $N\tau$ - sub maximal. $\square$

Theorem 2.4. Every subspace L of a sub maximal space $(P, N\tau)$ is again a $N\tau$ - sub maximal.

Proof. Let A be a $N\tau$ dense subset of L . Then $N\tau cl(A) \cap L = L$ and so $L \subset N\tau cl(A)$. Since $A \cup (P - N\tau cl(A))$ is $N\tau$ dense in P then it is an $N\tau$ open subset of P . Hence $S \cap (A \cup (P - N\tau cl(A))) = A$ is $N\tau$ - open in L or equivalently L is $N\tau$ - submaximal. $\square$

Theorem 2.5. In $(P, N\tau)$, the following conditions are equivalent

1. P is $N\tau$ - submaximal.
2. For any $A \subset P$, the subspace $N\tau - Fr(A) = N\tau - cl(A) - N\tau - int(A) = N\tau - cl(A) \cap N\tau - cl(P - A)$ is discrete.

Proof. (1) $\implies$ (2) Let $y \in N\tau Fr(A)$. Since A is $N\tau$ dense in $N\tau cl(A)$ then so is $A \cup \{y\}$. Since $N\tau cl(A)$ is sub maximal according to theorem 5.2.11 then $A \cup \{y\} = N\tau cl(A) \cap U$ where U is $N\tau$ open in P . In the same way it can be seen that $A^c \cup \{y\} = N\tau cl(A^c) \cap V$ where V is $N\tau$ open in P . Thus $\{y\} = (A \cup \{y\}) \cap (A^c \cup \{y\}) = N\tau cl(A) \cap N\tau cl(P - A) \cap U \cap V$. Hence $\{y\}$ is $N\tau$ open in $N\tau Fr(A)$ and so $N\tau Fr(A)$ is discrete.

(2) $\implies$ (1) Let A be a $N\tau$ dense in P . By assumption, $N\tau cl(A) - N\tau int(A) = P - N\tau int(A)$ is discrete and thus $A - N\tau int(A)$ is its $N\tau$ - open subset. Hence $A - N\tau int(A) = (P - N\tau int(A)) \cap U$ where U is $N\tau$ open in P . Thus $A - N\tau int(A) \subset U$ and so $A - N\tau int(A) \subset U - N\tau int(A)$. For the reverse inclusion if $y \in U - N\tau int(A)$ then $y \in (P - N\tau int(A)) \cap U = A - N\tau int(A)$. This shows that $A - N\tau int(A) = U - N\tau int(A)$ and hence $A = U \cup N\tau int(A)$. Thus A is $N\tau$ open in P . $\square$

Theorem 2.6. In $(P, N\tau)$, the following conditions are equivalent

1. P is $N\tau$ - submaximal.
2. Every $N\tau$ - pre open subset of P is $N\tau$ - open.

Proof. (1) $\implies$ (2) Let A be a $N\tau$ pre open in P . Then $A \subset N\tau int(N\tau cl(A))$. Since A is $N\tau$ dense in $N\tau cl(A)$ and $N\tau cl(A)$ is $N\tau$ sub maximal according to theorem 5.2.11 then A is $N\tau$ open in $N\tau cl(A)$. Thus A is $N\tau$ open in $N\tau int(N\tau cl(A))$. Since $N\tau int(N\tau cl(A))$ is $N\tau$ open in P then we may conclude that A is $N\tau$ open in P .

(2) $\implies$ (1) Let A be a $N\tau$ dense in P . Since $A \subset P = N\tau int(P) = N\tau int(N\tau cl(P))$ then A is $N\tau$ - pre open and by assumption, $N\tau$ open. This shows that P is $N\tau$ - sub maximal. $\square$

Theorem 2.7. If $(P, N\tau)$ is submaximal and $U \subset P$ then U is $N\tau$ - open iff it is the intersection of a $N\tau$ - dense and $N\tau$ - regular open [3].

Proof. It is enough to prove that for every $N\tau$ open set U , we have $U = D \cap V$ where D is $N\tau$ dense and V is $N\tau$ regular open since the reverse inclusion is trivial. Clearly $U \subset N\tau int(N\tau cl(U))$. Thus $U = N\tau cl(U) - (N\tau cl(U) - U) = N\tau cl(U) \cap (P - (N\tau cl(U) - U)) = N\tau int(N\tau cl(U)) \cap (U \cup P - N\tau cl(U))$ where $N\tau int(N\tau cl(U)) = V$ is $N\tau$ - regular open and $U \cup (P - N\tau cl(U)) = D$ is $N\tau$ - dense. $\square$

Theorem 2.8. The homeomorphic image of $N\tau$ - door space is a $N\sigma$ - door space.

Proof. Consider $(Z, N\tau)$ and $(Q, N\sigma)$ are door spaces and $\eta: Z \rightarrow Q$ be a homeomorphism. Let $U \subseteq Q$. Consider $\eta^{-1}(U) \subseteq Z$. Since Z is $N\tau$ - door space then η^{-1} is either $N\tau$ - open or $N\tau$ - closed in Z . Now $\eta(\eta^{-1}(U)) = U$ and U is either $N\sigma$ - open or $N\sigma$ - closed in Q . $\square$

Definition 2.6. In $(P, N\tau)$, if every subset of P is either $N\tau$ semi-open or $N\tau$ semi-closed then $(P, N\tau)$ is semi-door space and is denoted by $N\tau$ semi-door space

Theorem 2.9. The homeomorphic image of $N\tau$ semi-door space is a $N\sigma$ semi-door space.

Proof. Let $(Z, N\tau)$ and $(Q, N\tau)$ are N - topological spaces and $(Z, N\tau)$ be $N\tau$ semi-door space. A mapping $\eta: Z \rightarrow Q$ be a homeomorphism. Let $P \subset Q$. Consider $\eta^{-1}(P) \subset Z$, since Z is $N\tau$ semi - door space then $\eta^{-1}(P)$ is either $N\tau$ semi -open or $N\tau$ semi - closed. So $\eta(\eta^{-1}(P)) = P$ is either $N\tau$ - semi open or $N\tau$ - semi closed. Hence Q is a $N\tau$ - semi door space. $\square$

Theorem 2.10. A $N\tau$ - clopen subspace of a $N\tau$ semi-door space is $N\tau$ semi-door space.

Proof. Let $(Z, N\tau)$ be semi-door space. Let Q be a $N\tau$ -clopen subset of Z . Let $A \subseteq Q$ and $A \subseteq Z$. Since Z is a $N\tau$ semi-door space, then A is either $N\tau$ semi open or $N\tau$ semi closed in Z . Since Q is $N\tau$ open and $N\tau$ closed, then A is either $N\tau$ semi open or $N\tau$ semi closed in Q . Hence Q is $N\tau$ semi-door space. $\square$

Theorem 2.11. If $(P, N\tau)$ be a door space and if $z \in P$, S is a $N\tau$ - neighbourhood of z , then $S - z \in N\tau$ and $S \in N\tau$.

Proof. Let S be an neighbourhood of a point v and if v is $N\tau \text{int}(S)$ then it is enough to prove that $S - v$ is $N\tau$ open. If we assume $S - v$ is not $N\tau$ - open then $P - (S - v) = (P - S) \cup v$ should be $N\tau$ open. This contradicts that $v = S \cap ((P - S) \cup v)$ should be $N\tau$ - open. $\square$

Definition 2.7. In $(P, N\tau)$, if every two disjoint points in P can be isolated by disjoint $N\tau$ open sets then $(P, N\tau)$ is a hausdorff door space and is denoted by $N\tau$ -hausdorff door space.

Definition 2.8. In $(P, N\tau)$, if every two disjoint points in P can be isolated by disjoint $N\tau$ semi open sets then $(P, N\tau)$ is a semi - hausdorff door space and is denoted by $N\tau$ semi-hausdorff door space.

Definition 2.9. A $N\tau$ - semi door space is said to be hausdorff semi-door space of $(P, N\tau)$ if a given pair of different points $r, s \in P$, there exist $M, N \in N\tau O(P)$ such that $r \in M$, $s \in N$, $M \cap N = \phi$ and is denoted by $N\tau$ -hausdorff semi door space.

Proposition 2.1. Every $N\tau$ - hausdorff door space is $N\tau$ semi-hausdorff door space.

Proposition 2.2. *If (P, N_τ) be hausdorff door space and $z \in P$ then $r \in N_\tau SO(P) \iff r \in N_\tau$.*

Proposition 2.3. *If (P, N_τ) be hausdorff door space and $Z, Q \in (P, N_\tau)$. If $z \in N_\tau SO(P)$ and $Q \in N_\tau$ then $Z \cap Q \in N_\tau SO(P)$.*

Theorem 2.12. *N_τ - semi-hausdorff door space has atmost one limit point.*

Proof. Consider (P, N_τ) be a hausdorff space. Let a, b are distinct limit points in P . Since P is N_τ - semi-hausdorff, $\exists G, H \in SO(P) : a \in G, b \in H$ and $G \cap H = \phi$. Since P is N_τ - door space then $U = \{G - \{a\}\} \cup \{b\}$ is either N_τ - open or N_τ - closed. Suppose if it is N_τ - open then by theorem 5.2.19 $U \cap H = \{b\}$ is N_τ - semi open and hence by theorem 5.2.20 $U \cap H$ is N_τ - open. Otherwise U^c is N_τ - open and $U^c \cap G = \{a\}$ is N_τ - semi open and N_τ - open by propositions 5.2.24 and 5.2.25. Hence at least one of the two point will be isolated in P and by contradiction the result is proved. $\square$

Proposition 2.4. *A N_τ - hausdorff semi-door space has atmost one limit point.*

Proof. Proof is similar as discussed in the previous result. $\square$

3 $N_\tau\beta$ -door space and locally $N_\tau\beta$ -closed set

In this section, we introduce and analyze $N_\tau\beta$ - door space and $N_\tau\beta$ - locally closed sets.

Definition 3.1. *A subset U of P is said to be locally N_τ -closed set if $U = R \cap S$ where R is a N_τ - closed in P . The set of all locally N_τ - closed sets are denoted by $LN_\tau\text{-Cl}(P)$*

Definition 3.2. *A subset U of P is said to be locally $N_\tau\beta$ -closed set if $U = R \cap S$ where R is a $N_\tau\beta$ - open and S is $N_\tau\beta$ -closed in P . The set of all locally $N_\tau\beta$ - closed sets are denoted by $LN_\tau\beta\text{-Cl}(P)$*

Definition 3.3. *If every subset of (P, N_τ) is either $N_\tau\beta$ -open or $N_\tau\beta$ - closed then (P, N_τ) is called $N_\tau\beta$ - door space and signified by $N_\tau\beta_d$.*

Example 3.1. *Let $S = \{x, y, z\}$. For $N = 2$, $\tau_1 = \{\phi, S\}$, $\tau_2 = \{\phi, \{x, y\}, S\}$. Then $2_\tau O(S) = \{\phi, \{x, y\}, S\}$, $2_\tau \text{Cl}(S) = \{\phi, \{z\}, S\}$. Here S is a $2_\tau\beta$ - door space. $LN_\tau\text{-Cl}(S) = \{\phi, \{x, y\}, \{z\}, S\}$ and $LN_\tau\beta\text{-Cl}(P) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{z, x\}, S\}$.*

Remark 3.1. *From the example 5.3.4, we get*

1. Every locally $N\tau$ -closed set is locally $N\tau\beta$ - closed set but its converse need not be true.
2. Every $N\tau$ -door space is a $N\tau\beta_d$ but its converse need not be true.

Remark 3.2. If $(P, N\tau)$ is $N\tau\beta$ - door space then $LN\tau\beta\text{-Cl}(P) = \wp(P)$.

Theorem 3.1. Let $G \subseteq P$. Then the following are equivalent:

1. G is locally $N\tau\beta$ - closed sets
2. $G = F \cap N\tau\text{-cl}(G)$ for some $N\tau\beta$ -open set F .

Proof. (1) $\implies$ (2) Let $G \in LN\tau\beta\text{cl}(P)$. Now there will be a $N\tau\beta$ - open set F and a $N\tau\beta$ - closed subset E such that $G = F \cap E$ since $G \subseteq F$ and $G \subseteq N\tau\beta\text{cl}(P)$ then $N\tau\beta\text{cl}(P) \subseteq E$. Hence $F \cap N\tau\beta\text{cl}(P) \subseteq F \cap E = G$. Therefore $G = F \cap N\tau\beta\text{cl}(G)$. This proves (1) $\implies$ (2) .
 (2) $\implies$ (1) By definition, $N\tau\beta\text{cl}(G)$ is $N\tau\beta$ - closed. $\therefore G = F \cap LN\tau\beta\text{cl}(G) \in N\tau\beta\text{LC}(P)$. $\square$

4 Conclusion

In this paper, we introduced the idea of door spaces in N -topological space. Their structural properties have been discussed and emphasized. Some of the important results arrived through illustrated examples. The importance is to analyse the relationship with other N - topological properties. So, We investigated sub-maximal concepts and locally closed sets through N -topology. In addition, we introduce $N\tau\beta$ -door space and discuss the relationship between a $N\tau\beta$ -locally closed set and $N\tau$ -closed set. With the help of these locally closed sets it can be extend to introduce locally continuous maps in this topological space. These prime ideas can open the future scope of this concept and extended to other research areas of topology such as Fuzzy topology, Digital topology, and so on.

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p -clean properties in amalgamated rings

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Abstract

Let A be a ring. Then A is called p -clean ring if each element in A express as the sum of an idempotent and pure element. Let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . The amalgamation of A with B along J with respect to f is a new ring structure introduced and studied by Anna et al. in 2009. This construction is a generalization of the amalgamated duplication of a ring along an ideal and other classical constructions such as the $A + XB[X]$ and $A + XB[[X]]$ constructions. In this paper, the transfer of the notion of p -clean rings to the amalgamation of rings along ideal is studied. In particular, the necessary and sufficient conditions for amalgamation to be a p -clean ring are studied.

Keywords: pure element, amalgamation ring, p -clean ring

2020 AMS subject classifications: 13A15, 13D05. ¹

1 Introduction

Throughout this paper all rings are commutative with identity. Let A and B be two rings with unity, let J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. Anna et al. [2009] introduced and studied the new ring structure of the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

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called *the amalgamation of A with B along J with respect to f* . This new ring structure construction is a generalization of *the amalgamated duplication of a ring along an ideal*. The amalgamated duplication of a ring along an ideal was introduced and studied in [Anna [2006], Anna and Fontana [2007]]. In [Anna et al. [2009], Section 4], the authors studied the amalgamation can be in the frame of pullback constructions and also the basic properties of this construction [e.g., characterizations for $A \bowtie^f K$ to be a Noetherian ring, an integral domain, a reduced ring] and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Aruldoss et al. [2022] and Vijayanand and Selvaraj [2023] studied some rings and module characterized via amalgamation construction.

The notion of regular element was first introduced by von Neumann [1936], where an element $a \in A$ is called regular if there exists $b \in A$ such that $a = aba$. A ring A is called regular ring if each element in A is regular. Many other authors interested in studying regular rings, for example see Vas [2010] and Wardayani et al. [2020]. Let the set of regular elements in A be denoted by $Reg(A)$ that is $Reg(A) = \{r \in A; r = rsr, \text{ for some } s \in A\}$. The concept of clean ring first introduced by Nicholson [1977], where the ring A is called clean ring if for each $a \in A$ there exist $e \in Id(A)$ and $u \in U(A)$ such that $a = e + u$. Many other authors interested in studying clean rings, for example see Chen and Cui [2008] and Ashrafi and Nasibi [2013]. Ashrafi and Nasibi [2013], introduced the concept of r -clean ring, where the ring A is called r -clean ring if for each $a \in A$ there exist $e \in Id(A)$ and $r \in Reg(A)$ such that $a = e + r$. Many authors have studied r -clean rings such as Andari [2018] and Sharma and Singh [2018]. Let A be a ring, then an element $p \in A$ is called pure element if there exists $q \in A$ such that $p = pq$ [Majidinya et al. [2016]] and the set of pure elements in A write $Pu(A) = \{p \in A : p = pq, \text{ for some } q \in A\}$. The concept of von Neumann local ring was studied by Anderson and Badawi [2012], where a ring A is called von Neumann local ring if for each $a \in A$ we have either $a \in Reg(A)$ or $1 - a \in Reg(A)$. Mohammed et al. [2021] studied the class of rings namely rings in which each element express as the sum of an idempotent and pure.

This paper aims at studying the transfer of the notion of p -clean rings to the amalgamation of rings along ideal. In particular, we study the necessary and sufficient condition for amalgamation to be a p -clean ring.

We denote by $U(A)$, $Id(A)$, $Nilp(R)$ and $Pu(A)$, the set of unit elements, the set of idempotents, the set of nilpotent elements and set of all pure elements of A , respectively.

2 p -clean ring

We start with the following definition.

Definition 2.1 (Mohammed et al. [2021]). An element $a \in A$ is called *p-clean* if there exist $e \in Id(A)$ and $p \in Pu(A)$ such that $a = e + p$. A ring A is called *p-clean ring* if each element in A is *p-clean element*.

Proposition 2.1 (Mohammed et al. [2021]).

- (i) The class of *p-clean* is closed under homomorphic images.
- (ii) If I is an ideal of a *p-clean ring* A , then A/I is a *p-clean ring*.

Proposition 2.2. The ring A is *p-clean* if and only if the ring $A[[x]]$ of formal series over A is *p-clean*.

Proof. If $A[[x]]$ is *p-clean*, then it follows by the isomorphism $A \cong A[[x]]/(x)$ and by Proposition 2.1 (i) that A is *p-clean*.

Conversely, suppose that A is *p-clean*. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in A[[x]]$, then $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$. Since A is *p-clean*, then $a_0 = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$. So

$$f(x) = p + e + a_1 x + a_2 x^2 + \dots \in Id(A[[x]]),$$

with $p \in Pu(A) \subseteq Pu(A[[x]])$ and $e + a_1 x + a_2 x^2 + \dots \in Id(A[[x]])$. Thus, $A[[x]]$ is a *p-clean ring*. $\square$

Proposition 2.3. If A is a *p-clean ring*, then $A[\{X_\alpha\}]$ is a *p-clean ring*.

Proof. Let $f(X) \in A[\{X_\alpha\}]$. Then $f = f_0 + f'$, where $f_0 \in A$ and $f' \in (\{X_\alpha\})$. Since $f_0 \in A$, we can write $f_0 = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$. Now

$$f = p + e + f' = (p + f') + e,$$

where $p + f' \in Pu(A[\{X_\alpha\}])$ and $e \in Id(A) \subseteq Id(A[\{X_\alpha\}])$. Hence, $A[\{X_\alpha\}]$ is *p-clean*. $\square$

Let A be a ring and M an A -module. The trivial extension of A by M is the ring $A \rtimes M = \{(a, m) : a \in A, m \in M\}$ under coordinatewise addition and an adjusted multiplication defined by $(a, m)(a', m') = (aa', am' + a'm)$ for all $a, a' \in A, m, m' \in M$.

Theorem 2.1. Let A be a ring and M an A -module. Then $A \rtimes M$ is *p-clean* if and only if A is *p-clean*.

Proof. Note that $A \cong (A \rtimes M)/(\{0\} \times M)$ is a homomorphic image of $A \rtimes M$. Hence if $A \rtimes M$ is *p-clean*, so by Proposition 2.1 (i), A is *p-clean*.

Conversely, suppose that A is *p-clean*. Recall that $1_{A \rtimes M} = (1, 0)$ and observe that if $p \in Pu(A)$, then $(p, m) \in Pu(A \rtimes M)$ for each $m \in M$ and if $e \in Id(A)$, then $(e, 0)^2 = (e^2, 0) = (e, 0)$ in $A \rtimes M$. Hence, if $a \in A$ with $a = p + e$, where $p \in Pu(A)$ and $e \in Id(A)$, then for $m \in M$, $(a, m) = (p + e, m) = (p, m) + (e, 0)$, where $(p, m) \in Pu(A \rtimes M)$ and $(e, 0) \in Id(A \rtimes M)$. Hence, $A \rtimes M$ is *p-clean*. $\square$

Lemma 2.1. *Let $f = \sum_{i=0}^n a_i x^i \in A[x]$ be a pure element. Then a_0 is pure and a_i is nilpotent for each $i > 0$.*

Proof. Since f is a pure element, there exists $g = \sum_{i=0}^n b_i x^i \in A[x]$ such that $f = fg$. Therefore, $a_0 = a_0 b_0$ and so a_0 is pure. Now let P be a prime ideal of A . Then $(A/P)[x]$ is an integral domain. Define $\phi : A[x] \rightarrow (A/P)[x]$ by $\phi(\sum_{i=0}^k a_i x^i) = \sum_{i=0}^k (a_i + P)x^i$. Clearly, ϕ is an epimorphism. Then we have $\phi(f)\phi(g) = \phi(fg) = \phi(f)$. So $\deg(\phi(f)\phi(g)) = \deg(\phi(f))$. Thus, $\deg(\phi(f)) + \deg(\phi(g)) = \deg(\phi(f))$. Therefore $\deg(\phi(f)) = 0$. Hence, $a_i \in P$ for $i = 1, \dots, n$. Since P is arbitrary, a_i is nilpotent for each $i > 0$. $\square$

Theorem 2.2. *$A[x]$ is not p -clean.*

Proof. Let $f = \sum_{i=0}^n a_i x^i \in A[x]$. Suppose f is p -clean. Then $f = p + e$, where $p \in Pu(A[x])$ and $e \in Id(A[x])$. Since idempotents of A are exactly that of $A[x]$, $p = f - e$ is pure. Hence by Lemma 2.1, the element 1 should be nilpotent, which is a contradiction. $\square$

Theorem 2.3. *For every ring A , we have the following statements:*

- (i) *If e is a central idempotent element of A and eAe and $(1 - e)A(1 - e)$ are both p -clean, then so is A ;*
- (ii) *Let $e_1, e_2, \dots, e_n$ be orthogonal central idempotents with $e_1 + e_2 + \dots + e_n = 1$. Then $e_i Re_i$ is p -clean for each i , if and only if so is A .*
- (iii) *If A is p -clean, then so is the matrix ring $M_n(A)$ for any $n > 1$.*

Proof. (i) For convenience, write $\bar{e} = 1 - e$ for each $e \in Id(A)$. We use the Pierce decomposition of A : we have

$$A = eAe \oplus eA\bar{e} \oplus \bar{e}Ae + \bar{e}A\bar{e}.$$

Since $e, \bar{e}$ are central, we have

$$A = eAe \oplus \bar{e}A\bar{e} \cong \begin{pmatrix} eAe & 0 \\ 0 & \bar{e}A\bar{e} \end{pmatrix}.$$

So each matrix $B \in A$ is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a \in eAe, b \in \bar{e}A\bar{e}$.

By hypothesis, a and b are p -clean. Then $a = p_1 + e_1, b = p_2 + e_2$, where $p_1 \in Pu(eAe) \subseteq Pu(A), p_2 \in Pu(\bar{e}A\bar{e}) \subseteq Pu(A), e_1 \in Id(eAe) \subseteq Id(A), e_2 \in Id(\bar{e}A\bar{e}) \subseteq Id(A)$. So

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} p_1 + e_1 & 0 \\ 0 & p_2 + e_2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}.$$

Since p_1 and p_2 are pure elements of A , there exist q_1, q_2 in A such that $p_1 = p_1 q_1$ and $p_2 = p_2 q_2$. Therefore, we have

$$\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} p_1 q_1 & 0 \\ 0 & p_2 q_2 \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

So $\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$ is pure. Since $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ is an idempotent, we have A is p -clean.

(ii) One direction of (ii) follows from (i) by induction. The other direction follows from Proposition 2.1(ii).

(iii) Follows from (ii). □

3 p -clean properties in amalgamated ring

Proposition 3.1. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . If $A \bowtie^f J$ is p -clean ring, then A and $f(A) + J$ are p -clean rings.*

Proof. Define $p_A : A \bowtie^f J \rightarrow A$ by $p_A(a, f(a) + k) = a$ and $p_B : A \bowtie^f J \rightarrow B$ by $p_B(a, f(a) + k) = f(a) + k$. Then $A \bowtie^f J / (\{0\} \times J) \cong A$ and $A \bowtie^f J / (f^{(-1)}(J) \times \{0\}) \cong f(A) + J$. Since every homomorphic image of p -clean ring is p -clean, A and $f(A) + J$ are p -clean rings. □

The converse of the above Proposition is not true.

Proposition 3.2. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . Assume that $(f(A) + J)/J$ is uniquely p -clean and B is an Integral domain. Then $A \bowtie^f J$ is p -clean ring if and only if A and $f(A) + J$ are p -clean rings.*

Proof. By Proposition 3.1, $A \bowtie^f J$ is p -clean ring implies A and $f(A) + J$ are p -clean rings. Conversely, assume that A and $f(A) + J$ are p -clean rings. Since A is p -clean, we can write $a = e + p$ with $e \in Id(A), p \in Pu(A)$. Similarly, since $f(A) + J$ is p -clean, we can write $f(a) + j = f(e_1) + j_1 + f(p_1) + j_2$ with $f(e_1) + j_1$ is an idempotent element and $f(p_1) + j_2$ is a pure element. Clearly, $f(e_1) = \overline{f(e_1) + j_1}$ (resp., $\overline{f(e)}$) and $f(p_1) = \overline{f(p_1) + j_2}$ (resp., $\overline{f(p)}$) are respectively idempotent and pure element of $(f(A) + J)/J$. Then we have $\overline{f(a)} = \overline{f(e)} + \overline{f(p)} = \overline{f(e_1) + j_1} + \overline{f(p_1) + j_2}$. Since $(f(A) + J)/J$ is uniquely p -clean, $\overline{f(e)} = \overline{f(e_1)}$ and $\overline{f(p)} = \overline{f(p_1)}$. Consider $j'_1, j'_2 \in J$ such that $f(e_1) = f(e) + j'_1$ and $f(p_1) = f(p) + j'_2$. Then $(a, f(a) + j) = (e + p, f(e_1) + j_1 + f(p_1) + j_2) = (e, f(e) + j'_1 + j_1) + (p, f(p) + j'_2 + j_2)$. Clearly, $(e, f(e) + j'_1 + j_1)$ is an idempotent element of $A \bowtie^f J$. Since $f(p) + j'_2 + j_2$ is pure in $f(a) + J$, there exists

an element $f(\alpha_0) + j_0$ such that $f(p) + j'_2 + j_2 = (f(p) + j'_2 + j_2)(f(\alpha_0) + j_0)$. Since $p = pq$ for some $q \in A$, we have $f(p)f(q) = f(p) = f(p)f(\alpha_0)$. Since B is an integral domain, $f(q) = f(\alpha_0)$. This implies $f(\alpha_0) = f(q) + j'_0$ and hence $f(\alpha_0) + j_0 = f(q) + j'_0 + j_0$. Therefore, $f(p) + j'_2 + j_2 = (f(p) + j'_2 + j_2)(f(q) + j'_0 + j_0)$. Hence, $(p, f(p) + j'_2 + j_2) = (pq, (f(p) + j'_2 + j_2)(f(q) + j'_0 + j_0)) = (p, (f(p) + j'_2 + j_2))(q, (f(q) + j'_0 + j_0))$. Therefore, $(p, f(p) + j'_2 + j_2)$ is a pure element in $A \bowtie^f J$. Hence, $A \bowtie^f J$ is p -clean. $\square$

Remark 3.1. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B .

1. If $B = J$, then $A \bowtie^f B$ is p -clean if and only if A and B are p -clean since $A \bowtie^f J = A \times B$
2. If $f^{-1}(J) = 0$, then by [Anna et al. [2009], Proposition 5.1(3)], $A \bowtie^f J$ is p -clean if and only if $f(A) + J$ is p -clean.

Corollary 3.1. Let A be a ring and I an ideal such that A/I is uniquely p -clean. Then $A \bowtie^f I$ is p -clean if and only if A is p -clean.

Theorem 3.1. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B such that $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Then $A \bowtie^f J$ is p -clean if and only if A is p -clean.

Proof. $A \bowtie^f J$ is p -clean implies A is p -clean by Proposition 3.1. Conversely, assume that A is p -clean and $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Since A is p -clean, $a = e + p$, where e and p are idempotent and pure elements, respectively in A . Since p is pure in A , then there exists $q \in B$ such that $p = pq$. Therefore $(p, f(p) + j)(q, f(q) + j) = (pq, (f(p) + j)(f(q) + j)) = (pq, f(p)f(q) + j) = (p, f(p) + j)$. Thus, $(p, f(p) + j)$ is pure in $A \bowtie^f J$. Hence, $(a, f(p) + j) = (e, f(e)) + (p, f(p) + j)$ is a sum of idempotent and pure elements in $A \bowtie^f J$. Therefore, $A \bowtie^f J$ is p -clean. $\square$

Theorem 3.2. Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . Set $\overline{A} = A/Nilp(A)$, $\overline{B} = B/Nilp(B)$. $\pi : B \rightarrow \overline{B}$, the canonical projection and $\overline{J} = \pi(J)$. Consider a ring homomorphism $\overline{f} : \overline{A} \rightarrow \overline{B}$ defined by $\overline{f}(\overline{a}) = \overline{f(a)}$. Then $A \bowtie^f J$ is p -clean (resp., uniquely p -clean) if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is p -clean (resp., uniquely p -clean)

Proof. Clearly $\overline{f}$ is well defined and ring homomorphism. Consider the map $\chi : (A \bowtie^f J)/Nilp(A \bowtie^f J) \rightarrow \overline{A} \bowtie^{\overline{f}} \overline{J}$ defined by $\chi((a, f(a + j))) = (\overline{a}, \overline{f}(\overline{a + j}))$. If $(a, f(a + j)) = (b, f(b + j'))$, then $(a - b, f(a - b) + j - j') \in Nilp(A \bowtie^f J)$. Therefore, $a - b \in Nilp(A)$ and $j - j' \in Nilp(B)$. Then $\overline{a} = \overline{b}$ and $j = j'$. Hence χ is well defined. We can easily check that χ is a ring homomorphism.

Moreover, $(\bar{a}, \bar{f}(\bar{a} + \bar{j})) = (0, 0)$ implies that $a \in \text{Nilp}(A)$ and $j \in \text{Nilp}(B)$. Consequently, $(a, f(a) + j) \in \text{Nilp}(A \bowtie^f J)$. Hence, $(a, f(a + j)) = (0, 0)$. This implies that χ is injective. Clearly, by the construction, χ is surjective. Hence, χ is an isomorphism. $\square$

Proposition 3.3. *Let $f : A \rightarrow B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e of B . Then $A \bowtie^f (e)$ is p -clean if and only if A and $f(A) + (e)$ are p -clean. In particular, if e is an element of A , then $A \bowtie^f (e)$ is p -clean if and only if A is p -clean.*

Proof. By Theorem 3.1, $A \bowtie^f (e)$ is p -clean implies A and $f(A) + (e)$ are p -clean.

Conversely, assume that A and $f(A) + (e)$ are p -clean. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ with $a \in A$ and $r \in B$. Since A is p -clean, there exist an idempotent element v and pure element p in A such that $a = v + p$. Also, since $f(A) + (e)$ is p -clean, there exist an idempotent element v' and pure element p' in $f(A) + (e)$ such that $f(a) + re = v' + p'$. We have $(a, f(a) + re) = (v, f(v) + (v' - f(v))e) + (p, f(p) + (p' - f(p))e)$. On the other hand, $[f(v) + (v' - f(v))e]^2 = [f(v)(1 - e) + v'e]^2 = f(v)(1 - e) + v'e = f(v) + (v' - f(v))e$ and $[f(p) + (p' - f(p))e][f(q) + (q' - f(q))e] = [f(p)(1 - e) + p'e][f(q)(1 - e) + q'e] = f(pq)(1 - e) + p'q'e = f(p)(1 - e) + p'e = f(p) + (p' - f(p))e$. Then $(v, f(v) + (v' - f(v))e)$ and $(p, f(p) + (p' - f(p))e)$ are respectively idempotent and pure element in $A \bowtie^f (e)$. Hence $A \bowtie^f (e)$ is p -clean. Finally, if $A = B$ and $f = \text{id}_A$, then $A \bowtie^f (e) = A \bowtie (e)$ and $f(A) + (e) = A$. Then A is p -clean. $\square$

Theorem 3.3. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . $f(p) + j$ is pure (in B) for each $p \in \text{Pu}(A)$ and $j \in J$. Then $A \bowtie^f J$ is a von Neumann local ring if and only if A is a von Neumann local ring.*

Proof. Note in first that a commutative ring is von Neumann local ring if and only if it is an indecomposable p -clean ring (that is a p -clean ring where $\{0, 1\}$ is the set of all idempotent elements) by [Mohammed et al. [2021], Theorem 1.8]. Assume that $A \bowtie^f J$ is a von Neumann local ring. Then $A \bowtie^f J$ is an indecomposable p -clean ring. Then A must be p -clean ring. Also if $e \in \text{Id}(A)$, then $(e, f(e)) \in \text{Id}(A \bowtie^f J) = \{(0, 0), (1, 1)\}$. Then $\text{Id}(A) = \{0, 1\}$. This implies that A is an indecomposable p -clean ring, and so A is von Neumann local ring. Conversely, assume that A is a von Neumann local ring. Again by [Mohammed et al. [2021], Theorem 1.8], A is p -clean ring. Also by Theorem 3.1, $A \bowtie^f J$ is p -clean. On the other hand, by [Chhiti et al. [2015], Lemma 2.5], $\text{Id}(A \bowtie^f J) = \{(e, f(e)) | e \in \text{Id}(A)\} = \{(0, 0), (1, 1)\}$. Thus $A \bowtie^f J$ is an indecomposable p -clean ring. This implies that $A \bowtie^f J$ is a von Neumann local ring. $\square$

Corolary 3.2. *Let $f : A \rightarrow B$ be a ring homomorphism and J an ideal of B . If $f(p) + j$ is pure (in B) for each $p \in Pu(A)$ and $j \in J$. Then the following are equivalent: (i) $A \bowtie^f J$ is a von Neumann local and uniquely p -clean ring. (ii) A is a von Neumann local and uniquely p -clean ring. In particular, if A is a ring and J an ideal of B , then $A \bowtie J$ is a von Neumann local and uniquely p -clean ring if and only if A is a von Neumann local and uniquely p -clean ring.*

Proof. From Proposition 3.1 and Theorem 3.3, $A \bowtie^f J$ is a von Neumann local and uniquely p -clean ring implies that A is a von Neumann local and uniquely p -clean ring.

The converse is immediate. □

Proposition 3.4. *Let $f : A \rightarrow B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e . Then $A \bowtie^f (e)$ is p -clean ring if and only if A and $f(A) + (e)$ are p -clean ring. In particular, if e is an idempotent element of A , then $A \bowtie (e)$ is p -clean ring if and only if A is p -clean ring.*

Proof. Let A and $f(A) + (e)$ are p -clean ring. We show that $A \bowtie^f (e)$ is p -clean ring. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ with $a \in A$ and $r \in B$. Since A and $f(A) + (e)$ are p -clean, there exist p and v (resp., p' and v') in A (resp., $f(A) + (e)$) which are respectively pure and idempotent element such that $a = p + v$ and $f(a) = re = p' + v'$. We have $(a, f(a) + re) = (p, f(p) + (p' - f(p)e) + (v, f(v) + (v' - f(v)e)))$. On the other hand, $[f(p) + (p' - f(p)e)][f(q) + (q' - f(q)e)] = [f(p)(1 - e) + p'e][f(q)(1 - e) + q'e] = f(pq)(1 - e) + p'q'e = f(p)(1 - e) + p'e = f(p) + (p' - f(p)e)$. Also $[f(v) + (v' - f(v)e)]^2 = [f(v)(1 - e) + v'e]^2 = f(v)(1 - e) + v'e = f(v) + (v' - f(v)e)$. Then $(p, f(p) + (p' - f(p)e))$ and $(v, f(v) + (v' - f(v)e))$ are respectively pure and idempotent in $A \bowtie^f (e)$. Consequently, $A \bowtie^f (e)$ is p -clean as desired. Finally, if $A = B$ and $f = id_A$, then $A \bowtie^f (e) = A \bowtie (e)$ and $f(A) + (e) = A$. Thus, the particular case is obvious. □

4 Conclusion

This paper studies the transfer of the notion of p -clean rings to the amalgamation of rings along ideal. In particular, the necessary and sufficient conditions for amalgamation to be a p -clean ring are studied. This study will further help in studying the properties of other ring structure such as the $A + XB[X]$ and $A + XB[[X]]$ constructions. Furthermore, this work provides an answer to the question of when $A \bowtie^f J$ is a von Neumann local ring. In future, there is a scope to study the generalization of amalgamated ring namely bi-amalgamation ring with p -clean properties.

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Υ_G -operator in grill N -topology

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Abstract

In 2017, Lellis Thivagar et.al [4] introduced a closure operator $N\tau_G$ - cl by using the local function Φ_G in grill N -topology. In this article, we introduce a new operator Υ_G in the same topological space. We study the properties of this new operator which helps us to derive a few equivalent expressions and a characterizing condition, in terms of Υ_G . Then a suitability condition for a grill in N -topological space X is formulated. Also, we discuss the characterizing condition for the discussed suitability condition. In addition, we introduce and study $\hat{\Upsilon}_G$ -sets and utilize the Υ_G -operator to define a generalized open set and their properties.

Keywords: Grill N -topology, N -topology suitable for grill, relatively G -dense, anti-co dense.

2020 AMS subject classifications: 54A05, 54A99, 54C10. ¹

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1 Introduction

The grill concept is a powerful supporting tool, like nets and filters, in dealing with many a topological concept quite effectively. The idea of grill on a topological space was first introduced by Coquet [1]. Later Cattopadhyay and Thorn [2] proved the grills are always unions of ultra-filters. Further Roy and Mukherjee [6] established typical topology associated with grill on a topological space (X, τ) . Lellis thivagar et.al [4] initiated the concept of grill in N -topological space and a topology $N\tau_G$ was introduced in terms of an operator Φ_G , constructed rather naturally from a grill G on a N -topological space (X, τ) .

In this paper, we endeavour for an investigation in grill-associated N -topology with a new orientation. we introduce a new operator Υ_G , defined in terms of the previously introduced operator Φ_G , as a kind of dual of Φ_G . We study the properties of this new operator which helps us to derive equivalent expressions for the operator Υ_G and a characterizing condition, in terms of Υ_G . Also, a suitability condition is formed for a grill in N -topological space. Also, we discuss the characterizing condition for the discussed suitability condition. Finally, we introduce and study $\hat{\Upsilon}_G$ -sets and utilize the Υ_G -operator to define a generalized open set and their properties.

2 Prerequisites

In this section we recollect some definitions and results which are beneficial in the sequent. By a space X , we mean a grill N -topological space $(X, N\tau, G)$ with N -topology $N\tau$ and grill G on X on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. [4] A non-empty collection G of non empty subsets of a N - topological space $(X, N\tau)$ is called a grill on X if,

- (i) $A \in G$ and $A \subseteq B \subseteq X \implies B \in G$,
- (ii) $A, B \subseteq X$ and $A \cup B \in G \implies A \in G$ or $B \in G$.

Then a N -topological space $(X, N\tau)$ together with a grill G is called a grill N -topological space and it is denoted by $(X, N\tau, G)$.

Remark 2.2. [4] In $(X, N\tau)$, the following are true.

- (i) The grill $G = \mathcal{P}(A) - \{\emptyset\}$ is maximal grill in any N -topological space $(X, N\tau)$
- (ii) The grill $G = \{X\}$ is the minimal grill in any N -topological space $(X, N\tau)$.

Definition 2.3. [4] Let $(X, N\tau, G)$ be a grill N -topological space and for each $A \subseteq X$, the operator $\Phi_G(A, N\tau) = \{x \in X \mid A \cap U \in G, \forall U \in N\tau(x)\}$ is called the local function associated with the grill G and the N -topology $N\tau$. It is denoted as $\Phi_G(A)$. For any point x of a N -topological space $(X, N\tau)$, $N\tau(x)$ means the collection of all $N\tau$ -open sets containing x .

Theorem 2.4. [4] Let $(X, N\tau)$ be a N -topological space. Then the following are true.

- (i) If G is any grill on X , then Φ_G is an increasing function in the sense that $A \subseteq B$ implies $\Phi_G(A, N\tau) \subseteq \Phi_G(B, N\tau)$.
- (ii) If G_1 and G_2 are two grills on X with $G_1 \subseteq G_2$, then $\Phi_{G_1}(A, N\tau) \subseteq \Phi_{G_2}(A, N\tau)$ for all $A \subseteq X$.
- (iii) For any grill G on X and if $A \notin G$, then $\Phi_G(A, N\tau) = \emptyset$.

Theorem 2.5. [4] Let $(X, N\tau, G)$ be a grill N -topological space. Then for all $A, B \subseteq X$.

- (i) $\Phi_G(A \cup B) \supseteq \Phi_G(A) \cup \Phi_G(B)$
- (ii) $\Phi_G(\Phi_G(A)) \subseteq \Phi_G(A) = N\tau\text{-cl}(\Phi_G(A)) \subseteq N\tau\text{-cl}(A)$.

Theorem 2.6. [4] If G is a grill on a N -topological space $(X, N\tau)$ with $N\tau - \{\emptyset\} \subseteq G$, then for all $U \in N\tau$, $U \subseteq \Phi_G(U)$.

Lemma 2.7. [4] For any grill G on a N -topological space $(X, N\tau)$ and any $A, B \subseteq X$, $\Phi_G(A) - \Phi_G(B) = \Phi_G(A - B) - \Phi_G(B)$.

Corollary 2.8. [4] Let G is a grill on a N -topological space $(X, N\tau)$ and suppose $A, B \subseteq X$ with $B \notin G$. Then $\Phi_G(A \cup B) = \Phi_G(A) = \Phi_G(A - B)$.

Proposition 2.9. [4] Corresponding to a grill G on a N -topological space $(X, N\tau)$, the operator $N\tau_G\text{-cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by $N\tau_G\text{-cl}(A) = A \cup \Phi_G(A)$, for all $A \subseteq X$, satisfies Kuratowski's closure axioms and also there exists a unique topology $N\tau_G = \{U \subseteq X \mid N\tau_G\text{-cl}(U^c) = U^c\}$ which is finer than $N\tau$.

Proposition 2.10. [4] In a grill N -topological space $(X, N\tau, G)$, $N\tau \subseteq \beta(G, N\tau) \subseteq N\tau_G$ and in particular if $G = \mathcal{P}(X) - \{\emptyset\}$ then $N\tau = \beta(G, N\tau) = N\tau_G$.

Proposition 2.11. [4] In a grill N -topological space $(X, N\tau, G)$ and $A \subseteq X$ such that $A \subseteq \Phi_G(A)$, then $N\tau\text{-cl}(A) = N\tau_G\text{-cl}(A) = N\tau\text{-cl}(\Phi_G(A)) = \Phi_G(A)$.

Definition 2.12. [9] In $(X, N\tau)$, $S \subseteq X$ then S is $N\tau$ -dense if $N\tau\text{-cl}(S) = X$.

3 Υ_G -operator via grills

In this section, we suggest a new operator is called $\Upsilon_G(A, N\tau)$ (upsilon) in grill N -topological space, and take up some basic associated results. Throughout this section $(X, N\tau, G)$ denotes a grill N -topological space.

Definition 3.1. Let G be a grill on a N -topological space $(X, N\tau)$. We define a map $\Upsilon_G : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, given by $\Upsilon_G(A, N\tau) = X - \Phi_G(X - A)$ for any $A \subseteq X$. We shall simply write $\Upsilon_G(A)$, assuming that the grill G under consideration is understood.

Remark 3.2. It follows from theorem 2.5 (ii) that Υ_G is open in $(X, N\tau)$ for any subset A of X . Thus Υ_G treated as a mapping from $\mathcal{P}(X)$ to $N\tau$.

Remark 3.3. In view of Theorem 2.4(ii) it turns out that for two grills G_1 and G_2 on X , $G_1 \subseteq G_2 \implies \Upsilon_{G_1}(A) \supseteq \Upsilon_{G_2}(A)$. But for a given grill G on X , Υ_G is increasing in the sense that whenever $A \subseteq B \subseteq X$, then $\Upsilon_G(A) \subseteq \Upsilon_G(B)$. This is again an immediate consequence of theorem 2.4 (i); however it may so happen that $\Upsilon_G(A) \subseteq \Upsilon_G(B)$ even if $A \not\subseteq B$. The following is an example to justify our contention.

Example 3.4. Let $N = 3$ and $X = \{s, t, u\}$ and consider $\tau_1 O(X) = \{\emptyset, X, \{t, u\}\}$, $\tau_2 O(X) = \{\emptyset, X, \{s\}\}$ and $\tau_3 O(X) = \{\emptyset, X, \{s\}, \{t, u\}\}$. Then $3\tau O(X) = \{\emptyset, X, \{s\}, \{t, u\}\}$ is a tri topology and consider the grill $G = \{\{s\}, \{u\}, \{s, u\}, \{s, t\}, X\}$. Thus $(X, 3\tau, G)$ is a grill tri topological space on X . Now, $\Phi_G(\{s\}) = \{s\}$ and $\Phi_G(\{t\}) = \emptyset$. Then $\Upsilon_G(\{t, u\}) = X - \Phi_G(\{s\}) = \{t, u\}$ and $\Upsilon_G(\{s, u\}) = X - \Phi_G(\{t\}) = X$. Thus $\Upsilon_G(\{t, u\}) \subseteq \Upsilon_G(\{s, u\})$ although $\{t, u\} \not\subseteq \{s, u\}$.

Theorem 3.5. Let $(X, N\tau, G)$ be a grill N -topological space. Then the following statements are true:

Υ_G -operator in grill N-topology

- (i) If $S \in N\tau_G$ then $S \subseteq \Upsilon_G(S)$.
- (ii) If $S, T \subseteq X$ then $\Upsilon_G(S \cap T) = \Upsilon_G(S) \cap \Upsilon_G(T)$.
- (iii) If $S \subseteq X$ and $S \notin G$, then $\Upsilon_G(S) = X - \Phi_G(X)$.
- (iv) If $S, T \subseteq X$ with $T \notin G$, then $\Upsilon_G(S) = \Upsilon_G(S - T) = \Upsilon_G(S \cup T)$.
- (v) If $S, T \subseteq X$ with $(S - T) \cup (T - S) \notin G$, then $\Upsilon_G(S) = \Upsilon_G(T)$.

Proof. (i) In fact, $S \in N\tau_G \implies \Phi_G(X - S) \subseteq X - S$ by result 2.9, $S \subseteq X - \Phi_G(X - S) = \Upsilon_G(S)$.
(ii) $\Upsilon_G(S \cap T) = X - \Phi_G(X - (S \cap T)) = X - \Phi_G[(X - S) \cup \Phi_G(X - T)] = [X - \Phi_G[(X - S)]] \cap \Phi_G[(X - T)] = \Upsilon_G(S) \cap \Upsilon_G(T)$.
(iii) $\Upsilon_G(S) = X - \Phi_G(X - S) = X - [\Phi_G(X - S) - \Phi_G(S)] = X - [\Phi_G(X) - \Phi_G(S)] = X - \Phi_G(X)$.
(iv) $\Upsilon_G(S - T) = X - \Phi_G((X - S) \cup T) = X - [\Phi_G(X - S) \cup \Phi_G(T)] = X - \Phi_G(X - S) = X - \Upsilon_G(S)$.
(v) Let $(S - T) \cup (T - S) \notin G$ so that $S - T, T - S \notin G$. Then by using corollary 2.8 we Have $\Upsilon_G(S) = \Upsilon_G((T - (T - S)) \cup (S - T)) = \Upsilon_G(T - (T - S)) = \Upsilon_G(T)$. □

Remark 3.6. From (ii) of the above theorem we see that the operator Υ_G is distributive over finite intersection. That is not necessarily true for finite union which is shown below.

Example 3.7. Let $N = 2$ and $X = \{1, 2, 3\}$ and consider $\tau_1 O(X) = \{\emptyset, X, \{1, 2\}\}$, $\tau_2 O(X) = \{\emptyset, X\}$. Then $2\tau O(X) = \{\emptyset, X, \{1, 2\}\}$. Consider the grill $G = \{\{1\}, \{1, 2\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}$. Now $\Phi_G(\{1, 3\}) = \{1, 2, 3\} = X = \Phi_G\{2, 3\}$ and $\Phi_G(\{3\}) = \Phi$. Then $\Upsilon_G(\{1\}) = X - \Phi_G(\{2, 3\}) = \emptyset$, $\Upsilon_G(\{2\}) = X - \Phi_G(\{1, 3\}) = \emptyset$ and $\Upsilon_G(\{1, 2\}) = X - \Upsilon_G(\{3\}) = X$. Thus $\Upsilon_G(\{1\}) \cup \Upsilon_G(\{2\}) \neq \Upsilon_G(\{1, 2\})$.

Next we derive two equivalent expressions for $\Upsilon_G(A)$ where $A \subseteq (X, N\tau)$.

Theorem 3.8. In $(X, N\tau, G)$, Let $A \subseteq X$. Then the following statements are true:

- (i) $\Upsilon_G(A) = \{x \in X : \exists V \in N\tau(x) \text{ such that } V - A \notin G\}$.
- (ii) Let $\Upsilon_G(A) = \cup\{V \in N\tau : V - A \notin G\}$.

Proof. (i) $x \in \Upsilon_G(A)$ iff $x \notin \Phi_G(X - A) \iff$ there exist $V \in N\tau(x)$ such that $V - A = V \cap (X - A) \notin G \iff x \in \text{R.H.S.}$

(ii) Let $A^\# = \cup\{V \in N\tau : V - A \notin G\}$. Now $x \in A^\#$ then there exists $V \in N\tau$ with $x \in V$ such that $V - A \notin G$ which implies that there exist $V \in N\tau(x)$ such that $V - A \notin G$. Thus by (i), $x \in \Upsilon_G(A)$. From this it is clear that $\Upsilon_G(A) \subseteq A^\#$. $\square$

Remark 3.9. Let $G = \mathcal{P}(x) - \{\emptyset\}$, then by theorem 3.8 (ii) $\Upsilon_G(A) = \cup\{U \in N\tau : U - A = \emptyset\} = \cup\{U \in N\tau : U \subseteq A\} = N\tau\text{-int}(A)$, for any space $(X, N\tau)$.

Corolary 3.10. Let $(X, N\tau, G)$ be a grill N -topological space and $A \subseteq X$. Then $A \cap \Upsilon_G(A) = N\tau_G\text{-int}A$

Proof. We have, $x \in A \cap \Upsilon_G(A) \implies x \in A$ and $x \in \Upsilon_G(A) \implies x \in A$ and there exist $V \in N\tau(x)$ such that $V - A \notin G$ (by theorem 3.8 (i)) which implies $V - (V - A)$ is a $N\tau_G$ -open neighbourhood of x such that $V - (V - A) \subseteq A \implies x \in A$. Again $x \in N\tau_G\text{-int}A$ implies there exist a $N\tau_G$ -open neighbourhood $U - B$ of x , where $U \in N\tau$ and $B \notin G$, such that $x \in U - B \subseteq A \implies U - A \subseteq B$ and $U - A \notin G$. So by theorem 3.8 (i) $x \in \Upsilon_G(A)$. Thus $x \in A \cap \Upsilon_G(A) = N\tau_G\text{-int}A$. $\square$

Theorem 3.11. Let $(X, N\tau, G)$ be a grill N -topological space and if $A \in N\tau$ then $\Upsilon_G(A) = \cup\{S \in N\tau : S \Delta A \notin G\}$.

Proof. Let $A^\# = \cup\{S \in N\tau : S \Delta A \notin G\}$. Then by Theorem 3.8 (ii), $A^\# \subseteq \Upsilon_G(A)$. Now, $x \in \Upsilon_G(A)$ which implies there exist $S \in N\tau(x)$ such that $S - A \notin G$ (by theorem 3.8 (i)). Let $V = S \cup A \in N\tau$. Then $V \Delta A = S - A \notin G$ and $x \in V \in N\tau$. Thus $x \in A^\#$. $\square$

From the result so far, we arrive at the following simple and alternative description of the topology $N\tau_G$ in terms of our introduced operator.

Theorem 3.12. If $(X, N\tau, G)$ is a grill N -topological space then $N\tau_G = \{S \subseteq X, S \subseteq \Upsilon_G(S)\}$.

Proof. $T = \{S \subseteq X : S \subseteq \Upsilon_G(S)\}$. In fact, $\emptyset \subseteq \Upsilon_G(\emptyset) \implies \emptyset \in T$. $\Upsilon_G(X) = X - \Phi_G(X - X) = X - \Phi_G(\{\emptyset\}) = X \in T$. Now $A_1, A_2 \in T$ then $A_1 \subseteq \Upsilon_G(A_1)$ and $A_2 \subseteq \Upsilon_G(A_2)$ which implies $A_1 \cap A_2 \subseteq \Upsilon_G(A_1) \cap \Upsilon_G(A_2) =$

$\Upsilon_G(A_1 \cap A_2)$ (by theorem 3.5(ii)). Again $\{A_i : i \in \Lambda\} \in T$ which implies $A_i \subseteq \Upsilon_G(A_i)$ for each $i \in \Lambda$. This implies $A_i \subseteq \Upsilon_G(\cup_{i \in \Lambda} A_i)$ for each $i \in \Lambda$ (by remark 3.3) hence $\cup_{i \in \Lambda} A_i \subseteq \Upsilon_G(\cup_{i \in \Lambda} A_i)$ this implies that $\cup_{i \in \Lambda} A_i \in T$. We will show that $T = N\tau_G$. Indeed, $V \in N\tau_G$, then $V \in \Upsilon_G(V)$ (by theorem 3.5 (i)). This implies that $V \in T$.

Conversely, $A \in T \implies A \subseteq \Upsilon_G(A)$ this implies $A = A \cap \Upsilon_G(A) = N\tau_G\text{-int} A$ (Remark 3.9) which implies that $A \in N\tau_G$. $\square$

4 Suitable for a grill N-topology

In this segment, we intend to do some investigations in respect of $N\tau_G$, along with certain applications, under the assumption of such a suitability conditions imposed on the concerned grills. Throughout this section $(X, N\tau, G)$ denotes a grill N-topological space.

Definition 4.1. Let $(X, N\tau, G)$ be a grill N-topological space and $N\tau$ is called suitable for the grill G if $A - \Phi_G(A) \notin G$, for all $A \subseteq X$.

Example 4.2. Let $X = \{1, 2, 3\}$. For $N = 3$, consider $\tau_1 = \{\emptyset, X, \{1\}\}$, $\tau_2 = \{\emptyset, X, \{2\}\}$, $\tau_3 = \{\emptyset, X, \{1, 2\}\}$ and $3\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, X\}$. Let $G = \{\{1\}, \{1, 2\}, \{1, 3\}, X\}$. For every $A \subseteq X$, $A - \Phi_G(A) \notin G$. Hence 3τ is suitable for G .

Theorem 4.3. In a grill N-topological space $(X, N\tau, G)$, the following are equivalent:

- (i) $N\tau$ is suitable for the grill G .
- (ii) For any $N\tau_G$ -closed subset A of X , $A - \Phi_G(A) \notin G$.
- (iii) For any $A \subseteq X$ and each $x \in A$, there corresponds some $U \in N\tau(x)$ with $U \cap A \notin G$, it follows that $A \notin G$.
- (iv) $A \subseteq X$ and $A \cap \Phi_G(A) = \Phi \implies A \notin G$.

Proof.

- (i) $\implies$ (ii) It is obvious.

(ii) $\implies$ (iii) Let $A \subseteq X$ and suppose for every $x \in A$ there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Then $x \notin \Phi_G(A)$ so that $A \cap \Phi_G(A) = \emptyset$. Now as $A \cup \Phi_G(A)$ is $N\tau_G$ -closed, by (ii) we have $(A \cup \Phi_G(A)) - \Phi_G(A \cup \Phi_G(A)) \notin G$. That is $(A \cup \Phi_G(A)) - \Phi_G(A) \cup \Phi_G(\Phi_G(A)) \notin G \implies (A \cup \Phi_G(A)) - \Phi_G(A) \notin G$ (by theorem 2.5 (i)) that is $A \notin G$.

(iii) $\implies$ (iv) If $A \subseteq X$ and $A \cap \Phi_G(A) = \emptyset$ then $A \subseteq X - \Phi_G(A)$. Let $x \in A$. Then $x \notin \Phi_G(A)$ implies there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Then by (iii), $A \notin G$.

(iv) $\implies$ (i) Let $A \subseteq X$. We first claim that $(A - \Phi_G(A)) \cap \Phi_G(A - \Phi_G(A)) = \emptyset$. In fact $x \in (A - \Phi_G(A)) \cap \Phi_G(A - \Phi_G(A)) \implies x \in A - \Phi_G(A) \implies x \in A$ and $x \notin \Phi_G(A)$ implies there exist $U \in N\tau(x)$ such that $U \cap A \notin G$. Now $U \cap (A - \Phi_G(A)) \subseteq U \cap A \notin G \implies x \notin \Phi_G(A - \Phi_G(A))$, which is contradiction. Hence by (iv), $A - \Phi_G(A) \notin G$. $\square$

Now we derive, in term of the Υ_G , a characterizing condition for a N -topology $N\tau$ to be suitable for a grill G on a N -topological space X .

Theorem 4.4. *Let G be a grill on a N -topological space X then $N\tau$ is suitable for G iff $\Upsilon_G(A) - A \notin G$ for any $A \subseteq X$.*

Proof. Let $N\tau$ be suitable for G and $A \subseteq X$, We first observe that $x \in \Upsilon_G(A) - A$ iff $\Upsilon_G(A)$ and $x \notin A$ iff there exists $U \in N\tau(x)$ such that $x \in U - A \notin G$. Thus to each $x \in \Upsilon_G(A) - A$, there exist $U \in N\tau(x)$ such that $U \cap (\Upsilon_G(A) - A) \notin G$. As $N\tau$ is suitable for G , by theorem 4.2 we have $\Upsilon_G(A) - A \notin G$. Conversely, let $A \subseteq X$, suppose that to each $x \in A$ there corresponds some $U \in N\tau(x)$ such that $U \cap A \notin G$. We need to show by virtue of theorem 4.2 that $A \notin G$. Now, by theorem 3.8 (i) we have, $\Upsilon_G(X - A) = \{x \in X: \text{there exists } U \in N\tau(x) \text{ such that } U - (X - A) \notin G\} = \{x \in X: \text{there exists } U \in N\tau(x) \text{ such that } U \cap A \notin G\}$. Thus $A \subseteq \Upsilon_G(X - A)$ and hence $A = \Upsilon_G(X - A) \cap A = \Upsilon_G(X - A) - (X - A) \notin G$. $\square$

Corolary 4.5. *If the N -topology $N\tau$ of a space X is suitable for a grill G on X , then Υ_G is an idempotent operator i.e., $\Upsilon_G(\Upsilon_G(A)) = \Upsilon_G(A)$ for any $A \subseteq X$.*

Proof. Since Υ_G is $N\tau$ -open in X , hence $\Upsilon_G(A) \in N\tau$ for any $A \subseteq X$ and so $\Upsilon_G(A) \in N\tau_G$. Hence by theorem 3.5(i), $\Upsilon_G(A) \subseteq \Upsilon_G(\Upsilon_G(A))$ for any

$A \subseteq X$. Also $N\tau$ is suitable for G , so $\Upsilon_G(A) \subseteq \Upsilon_G(A \cup B)$ for some $B \notin G$. Thus $\Upsilon_G(\Upsilon_G(A)) \subseteq \Upsilon_G(A \cup B) = \Upsilon_G(A)$. $\square$

Theorem 4.6. *In $(X, N\tau, G)$, $N\tau$ is suitable for G . Let $A \subseteq X$ and V be a non-null open set such that $V \subseteq \Phi_G(A) \cap \Upsilon_G(A)$. Then $V - A \notin G$ and $V \cap A \in G$.*

Proof. $V \subseteq \Phi_G(A) \cap \Upsilon_G(A) \implies V \subseteq \Upsilon_G(A) \implies V - A \subseteq \Upsilon_G(A) - A \notin G$ by theorem 4.3 we get $V - A \notin G$. Again $V \subseteq \Phi_G(A)$ and $V \neq \emptyset \implies V \cap A \in G$. $\square$

Theorem 4.7. *In $(X, N\tau, G)$, the following assertions are similar:*

- (i) $N\tau - \{\emptyset\} \subseteq G$
- (ii) $\Upsilon_G(\emptyset) = \emptyset$
- (iii) *If A is $N\tau$ -closed then $\Upsilon_G(A) - A = \emptyset$*
- (iv) *If $A \subseteq X$ then $N\tau\text{-int}(N\tau\text{-cl}(A)) = \Upsilon_G(N\tau\text{-int}(N\tau\text{-cl}(A)))$*
- (v) *If A is $N\tau$ -regular open in X then $A = \Upsilon_G(A)$*
- (vi) *If $V \in N\tau$ then $\Upsilon_G(V) \subseteq N\tau\text{-int}(N\tau\text{-cl}(V)) \subseteq \Phi_G(V)$*

Proof. (i) $\implies$ (ii) : $\Upsilon_G(\emptyset) = \cup\{V \in N\tau : V - \emptyset \notin G\}$ by theorem 3.8 (ii) $\Upsilon_G(\emptyset) = \cup\{V \in N\tau : V \notin G\} = \emptyset$.

(ii) $\implies$ (iii) : Let $x \in \Upsilon_G(A) - A$ then there exists $V \in N\tau(x)$ such that $x \in V - A \notin G$. Since A is $N\tau$ -closed, we obtain $x \in V - A \in \{U \in N\tau : U \notin G\}$, a contradiction to $\Upsilon_G(\emptyset) = \emptyset$.

(iii) $\implies$ (iv) : Since $N\tau\text{-int}(N\tau\text{-cl}(A))$ is $N\tau$ -open, by theorem 3.5 (i) we get $N\tau\text{-int}(N\tau\text{-cl}(A)) \subseteq \Upsilon_G(N\tau\text{-int}(N\tau\text{-cl}(A)))$. Again using (iii), we get $\Upsilon_G(N\tau\text{-cl}(A)) \subseteq N\tau\text{-cl}(A)$. By remark 3.2, $\Upsilon_G(N\tau\text{-cl}(A)) = N\tau\text{-int}(\Upsilon_G(N\tau\text{-cl}(A))) \subseteq N\tau\text{-int}(N\tau\text{-cl}(A))$. Since $\Upsilon_G(N\tau\text{-int}(N\tau\text{-cl}(A))) \subseteq \Upsilon_G(N\tau\text{-cl}(A)) \subseteq N\tau\text{-int}(N\tau\text{-cl}(A))$. Thus $N\tau\text{-int}(N\tau\text{-cl}(A)) = \Upsilon_G(N\tau\text{-int}(N\tau\text{-cl}(A)))$.

(iv) $\implies$ (v): It is trivial.

(v) $\implies$ (vi): Let $V \in N\tau$. By assumption, $\emptyset = \Upsilon_G(\emptyset) = \cup\{V \in N\tau : U \notin G\}$. By theorem 3.8(ii) we obtain $N\tau - \{\emptyset\} \subseteq G$. Then by theorem 2.6 we get $V \subseteq \Phi_G(V)$ and hence by Proposition 2.11, we have $\Phi_G(V) = N\tau\text{-cl}(V)$. Now, $V \subseteq N\tau\text{-int}(N\tau\text{-cl}(V)) \subseteq N\tau\text{-cl}(V) = \Phi_G(V)$. Since $\Upsilon_G(V) \subseteq \Upsilon_G(N\tau\text{-int}(N\tau\text{-cl}(V))) = N\tau\text{-int}(N\tau\text{-cl}(V)) \subseteq \Phi_G(V)$.

(vi) $\implies$ (i): If $V \in N\tau - G$, by theorem 3.5 (i), $V \subseteq \Upsilon_G(V) \subseteq \Phi_G(V) = \emptyset$. That is $V = \emptyset$ by theorem 2.4(i). $\square$

5 $\hat{\Upsilon}_G$ -sets

In this segment, we discuss about a new open set $\hat{\Upsilon}_G$ in grill N -topological space and investigate some of its properties.

Definition 5.1. In $(X, N\tau, G)$ a subset S of X is called a $\hat{\Upsilon}_G$ -set if $S \subseteq N\tau\text{-cl}(\Upsilon_G(S))$. The group of all $\hat{\Upsilon}_G$ -sets in $(X, N\tau, G)$ is signify by $\hat{\Upsilon}_G(X, N\tau)$.

Proposition 5.2. If $\{S_\alpha : \alpha \in \Delta\}$ is a group of nonempty $\hat{\Upsilon}_G$ -sets in $(X, N\tau, G)$, then $\cup_{\alpha \in \Delta} S_\alpha \in \hat{\Upsilon}_G(X, N\tau)$.

Proof. For each $\alpha \in \Delta$, $S_\alpha \subseteq N\tau\text{-cl}(\Upsilon_G(S_\alpha)) \subseteq N\tau\text{-cl}(\Upsilon_G(\cup_{\alpha \in \Delta} S_\alpha))$ This implies that $\cup_{\alpha \in \Delta} S_\alpha \subseteq N\tau\text{-cl}(\Upsilon_G(\cup_{\alpha \in \Delta} S_\alpha))$. Thus $\cup_{\alpha \in \Delta} S_\alpha \in \hat{\Upsilon}_G(X, N\tau)$. $\square$

Remark 5.3. The intersection of two $\hat{\Upsilon}_G$ -sets need not be a $\hat{\Upsilon}_G$ -set and it is shown in the following example.

Example 5.4. Let $X = \{1, 2, 3, 4\}$. Let $N = 3$. Consider $\tau_1 = \{\emptyset, X, \{2, 3\}\}$, $\tau_2 = \{\emptyset, X, \{1, 2, 3\}\}$ and $\tau_3 = \{\emptyset, X, \{1\}, \{1, 2, 3\}\}$. Then $3\tau = \{\emptyset, X, \{1\}, \{1, 2, 3\}, \{2, 3\}\}$ and the grill $G = \{\{1\}, \{2\}, \{1, 3\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}$. Let $A = \{1, 4\}$ and $B = \{2, 3, 4\}$ are $\hat{\Upsilon}_G$ -sets but $A \cap B$ is not a $\hat{\Upsilon}_G$ -set. For Let $A = \{1, 4\}$, $\Phi_G(X - A) = \{2, 3, 4\}$ and $\hat{\Upsilon}_G(A) = \{1\}$. Hence $A \subseteq N\tau\text{-cl}(\hat{\Upsilon}_G(A))$ implies that A is $\hat{\Upsilon}_G$ -set. For $B = \{2, 3, 4\}$, $\Phi_G(X - B) = \{1, 4\}$ and $\hat{\Upsilon}_G(B) = \{2, 3\}$. Hence $B \subseteq N\tau\text{-cl}(\hat{\Upsilon}_G(B))$ implies that B is $\hat{\Upsilon}_G$ -set. On the other hand, since $A \cap B = \{4\}$, $\Phi_G(X - (A \cap B)) = X$ and $\hat{\Upsilon}_G(B) = \emptyset$. Hence $A \cap B \not\subseteq N\tau\text{-cl}(\hat{\Upsilon}_G(A \cap B))$ implies that $A \cap B$ is not a $\hat{\Upsilon}_G$ -set.

Remark 5.5. The intersection of an $N\tau\alpha$ -set and $\hat{\Upsilon}_G$ -set is a $\hat{\Upsilon}_G$ -set.

Corolary 5.6. In $(X, N\tau, G)$, if for any $S \in N\tau$ then $S \subseteq \Upsilon_G(S)$.

Theorem 5.7. Let $A \in \hat{\Upsilon}_G(X, N\tau)$ on $(X, N\tau, G)$. If $U \in N\tau^\alpha$ then $U \cap A \in \hat{\Upsilon}_G(X, N\tau)$.

Proof. Assume that A is $N\tau$ -open for every $A \subseteq X$, $G \cap N\tau\text{-cl}(A) \subseteq N\tau\text{-cl}(G \cap A)$. Let $U \in N\tau^\alpha$ and $A \in \hat{\Upsilon}_G(X, N\tau)$. By corollary 5.6, we have $U \cap A \subseteq$

$N\tau\text{-int}(N\tau\text{-cl}(N\tau\text{-int}(U))) \cap N\tau\text{-cl}(\Upsilon_G(A)) \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(U))) \cap N\tau\text{-cl}(\Upsilon_G(A)) \subseteq N\tau\text{-cl}(N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(U))) \cap \Upsilon_G(A) = N\tau\text{-cl}(N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(U) \cap \Upsilon_G(A)))) = N\tau\text{-cl}(\Upsilon_G(U) \cap \Upsilon_G(A)) = N\tau\text{-cl}(\Upsilon_G(U \cap A))$. Hence $U \cap A \in \widehat{\Upsilon}_G(X, N\tau)$. $\square$

Corollary 5.8. *Let $A \in \widehat{\Upsilon}_G(X, N\tau)$ on $(X, N\tau, G)$. If $U \in N\tau$ then $U \cap A \in \widehat{\Upsilon}_G(X, N\tau)$.*

Definition 5.9. *In $(X, N\tau, G)$, if for every relatively nonempty open set $L \cap K$, $L \in N\tau$ and $(L \cap K) \cap E \in G$ then the set E is relatively G -dense in a set K .*

Next we prove a necessary and sufficient condition for $A \notin \widehat{\Upsilon}_G(X, N\tau)$.

Theorem 5.10. *A set $A \notin \widehat{\Upsilon}_G(X, N\tau)$ if and only if there exists $x \in A$ such that there is a neighbourhood $V_x \in N\tau(x)$ for which $X - A$ is relatively $N\tau_G$ -dense in V_x .*

Proof. Let $A \notin \widehat{\Upsilon}_G(X, N\tau)$. We are to show that there exists $x \in A$ and a neighbourhood $V_x \in N\tau(x)$ satisfying that $X - A$ is relatively G -dense in V_x . Since $A \not\subseteq N\tau\text{-cl}(\Upsilon_G(A))$, there exists $x \in X$ such that $x \in A$ but $x \notin N\tau\text{-cl}(\Upsilon_G(A))$. Hence there exists a neighbourhood $V_x \in N\tau(x)$ such that $V_x \cap \Upsilon_G(A) = \emptyset$. This implies that $V_x \cap (X - \Phi_G(X - A)) = \emptyset$ and hence $V_x \subseteq \Phi_G(X - A)$. Let U be any non empty open set in V_x . Since $V_x \subseteq \Phi_G(X - A)$, therefore $U \cap (X - A) \in G$ which implies that $(X - A)$ is relatively G -dense in V_x . Converse is obvious. $\square$

Definition 5.11. *A space $(X, N\tau, G)$ is said to be anti-co-dense grill if $N\tau\text{-}\{\emptyset\} \subseteq G$*

Theorem 5.12. *Let G be a anti-co dense grill on a space $(X, N\tau, G)$. Then $SO(X, N\tau_G) = \widehat{\Upsilon}_G(X, N\tau)$.*

Definition 5.13. *A set $A \subseteq X$ in $(X, N\tau, G)$, A is called a Υ_A -set if $A \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(A)))$. The collection Υ_A -sets in $(X, N\tau, G)$ is denoted by $N\tau^A$. From Definitions of 5.1 and 5.13 it follows that $N\tau^A \subseteq \widehat{\Upsilon}_G(X, N\tau)$. The collection $N\tau^A$ forms a topology finer than $N\tau$.*

Theorem 5.14. *Let G be a anti-co dense grill on $(X, N\tau, G)$. Then the collection $N\tau^A = \{A \subseteq X : A \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(A)))\}$ forms a N -topology on X .*

Proof. (i) It is observed that $\emptyset \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(\emptyset)))$ and $X \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(X)))$, and thus $\emptyset$ and $X \in N\tau^\alpha$.

(ii) Let $\{A_\alpha : \alpha \in \Delta\} \subseteq N\tau^A$, then $\Upsilon_G(A_\alpha) \subseteq \Upsilon_G(\cup A_\alpha)$ for every $\alpha \in \Delta$. Thus $A_\alpha \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(A_\alpha))) \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(\cup A_\alpha)))$ for every $\alpha \in \Delta$, which implies that $\cup A_\alpha \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(\cup A_\alpha)))$. Therefore, $\cup A_\alpha \in N\tau^A$.

(iii) Let $A, B \in N\tau^A$. Since $\Upsilon_G(A)$ is open in $(X, N\tau)$, we obtain $A \cap B \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(A))) \cap N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(B))) = N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G(A) \cap \Upsilon_G(B)))$. Therefore $A \cap B \subseteq N\tau\text{-cl}(N\tau\text{-int}(\Upsilon_G(A \cap B)))$. $\square$

Proposition 5.15. *Let $(X, N\tau, G)$ be a grill N -topological space. Then $\Upsilon_G(A) \neq \emptyset$ if and only if A contains a nonempty $N\tau_G$ -interior.*

Corolary 5.16. *Let $(X, N\tau, G)$ be a grill N -topological space. Then $\{x\} \in \widehat{\Upsilon}_G(X, N\tau)$ if and only if $\{x\} \in N\tau^A$.*

Proof. Let $\{x\} \in \widehat{\Upsilon}_G(X, N\tau)$, therefore by proposition 5.15, $\{x\}$ is open in $(X, N\tau, G)$. Since $\{x\} \subseteq \Upsilon_G(\{x\})$ and $\Upsilon_G(\{x\})$ is $N\tau$ -open in $(X, N\tau)$, therefore $\{x\} \subseteq N\tau\text{-int}(N\tau\text{-cl}(\Upsilon_G\{x\}))$. $\square$

6 Conclusion

In this paper, we introduced a new operator in grill N -topological space, using this operator some important properties and equivalent expressions are derived. We arrived a topology $N\tau_G$ using the introduced operator. In addition, suitability condition of a grill with the N -topological space X is formulated. We discuss the characterizing condition for a N -topology to be a suitable for a grill G on X . Also we introduce and study $\widehat{\Upsilon}_G$ -sets and utilize the Υ_G -operator to define a generalized open set and their properties. This concept can be extended to other applicable research areas of topology such as Nano topology, Fuzzy topology, Intuitionistic topology, Digital topology and so on.

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Edge coloring in complement of bipolar fuzzy graphs

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Abstract

Graph theory is rapidly moving into the mainstream of mathematics because of its applications in diverse fields which include chemistry, bio-chemistry, electrical engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling). Graph coloring is one of the most important concepts in graph theory and is used in many real time applications like Job scheduling, Aircraft scheduling, computer network security, Map coloring, Automatic channel allocation for small wireless local area networks. Two types of coloring namely vertex coloring and edge coloring are usually associated with any graph. In this paper, we analyze edge coloring of the complement bipolar fuzzy graphs using concept of as bipolar fuzzy numbers through the α – cuts of bipolar fuzzy graphs. For different values of α – cuts which depend on edge and vertex membership value of the graph, we will get different graph and different chromatic number.

Keywords: complement bipolar fuzzy graph; edge coloring; chromatic number; α – cut of BFG.

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1. Introduction

In 1994, Zhang [13] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1,1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0,1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1,0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar fuzzy information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several direction. Akram[1] introduced the concept of bipolar fuzzy graphs and defined different operations on it. Sunil Mathew, M.S.Sunitha and A.Vijaya kumar[8] introduced the concept of complement bipolar fuzzy graphs in 2014 and discussed about some connectivity concepts in bipolar fuzzy graphs. As an advancement coloring of fuzzy graph was outlined by authors Eslahchi and Onagh[4] in 2004, and later developed by them a fuzzy vertex coloring in 2006. This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy set by M.Ananthanarayanan and S.Lavanya [2]. The concept of chromatic number of fuzzy graphs was introduced by Munoz.S[6]. Anjali and Sunitha [3] developed algorithms to the chromatic number of fuzzy graphs. Types of Paths and Strong Cycle Connectivity in Bipolar Fuzzy Graphs discussed by S. Y.Mohamed and Subashini[11,12] A.Tahmasbpour and R.A.Borzooei[9] introduced the concept of chromatic number of bipolar fuzzy graphs. In this paper, we determine edge chromatic number of Complement bipolar fuzzy graphs using α –cut value.

2. Preliminaries

Definition 2.1

By a bipolar fuzzy graph, we mean a pair $G = (A, B)$ where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on E such that $\mu_B^P(xy) \leq \min(\mu_A^P(x), \mu_A^P(y))$ and $\mu_B^N(xy) \geq \max(\mu_A^N(x), \mu_A^N(y))$ for all $(x, y) \in E$.

Definition 2.2

We call A the bipolar fuzzy vertex set of V , B the bipolar fuzzy edge set of E respectively. Note that B is symmetric bipolar fuzzy relation on A . we use the notation xy for an element of E . Thus, $G = (A, B)$ is a bipolar fuzzy graph of $G^* = (V, E)$ if $\mu_B^P(xy) \leq \min(\mu_A^P(x), \mu_A^P(y))$ and $\mu_B^N(xy) \geq \max(\mu_A^N(x), \mu_A^N(y))$ for all $xy \in E$. $H = (\alpha, \beta)$ (where $\alpha = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subset of a set A and $\beta = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on B) is called a partial bipolar fuzzy subgraph of G if $\alpha \leq A$ and $\beta \leq B$. We call $H = (\alpha, \beta)$ a spanning bipolar fuzzy subgraph of $G = (A, B)$ if $\alpha = A$.

Definition 2.3

An arc $\left(\left(\mu_A^P(u), \mu_A^N(u) \right), \left(\mu_A^P(v), \mu_A^N(v) \right) \right)$ of G is called m -strong if $\mu_B^P(u, v) = \min(\mu_A^P(u), \mu_A^P(v))$ and $\mu_B^N(u, v) = \max(\mu_A^N(u), \mu_A^N(v))$. Suppose $G : (A, B)$ be a bipolar fuzzy graph. The complement of G is denoted as $\bar{G} : (\bar{A}, \bar{B})$ where $\bar{A} = A$ and $\bar{\mu}_B^P(xy) = \min(\mu_A^P(x), \mu_A^P(y)) - \mu_B^P(xy)$ and $\bar{\mu}_B^N(xy) = \max(\mu_A^N(x), \mu_A^N(y)) - \mu_B^N(xy)$.

Definition 2.4

α – cut set of Bipolar fuzzy set A is denoted as A_α is made up of members whose positive membership is not less than α and negative membership is not greater than α . $A_\alpha^P = \{x \in X, \mu_A^P(x) \geq \alpha\}$ and $A_\alpha^N = \{x \in X, \mu_A^N(x) \leq \alpha\}$ where α – cut set of fuzzy set is crisp set. The α – cut of BFG defined as $G_\alpha = (V_\alpha, E_\alpha)$ where $V_\alpha = \{v \in V / \sigma \geq \alpha\}$ and $E_\alpha = \{e \in E / \mu \geq \alpha\}$.

3. Complement of bipolar fuzzy graph

Complement of FG has been defined by Moderson[5]. Complement of a BFG $G : (\sigma, \mu)$ as a BFG $\bar{G} : (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\mu_B^P(xy) = 0$ if $\mu(xy) > 0$, $\mu_B^N(xy) = 0$ if $\mu(xy) < 0$ and $\mu_B^P(xy) = \min[\mu_A^P(x), \mu_A^P(y)]$, $\mu_B^N(xy) = \max[\mu_A^N(x), \mu_A^N(y)]$ otherwise. From the definition $\bar{G}$ is a BFG even if G is not and $\bar{\bar{G}} = G$ if and only if G is m -strong BFG. Also, automorphism graph of G and $\bar{G}$ are not identical. But there is some drawbacks in the definition of complement of a BFG mentioned above. In Fig 3.3, $\bar{\bar{G}} \neq G$ and note that they are identical provided G is m -strong BFG.

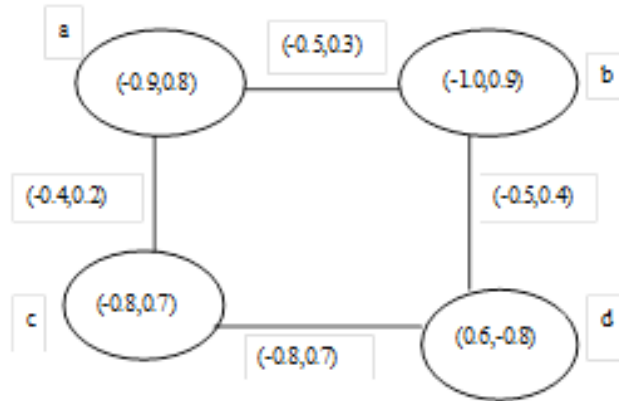


Figure 3.1: Bipolar fuzzy graph G

Now the Complement of a BFG $G : (\sigma, \mu)$ is the BFG $\bar{G} : (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\mu_B^P(xy) = \min[\mu_A^P(x), \mu_A^P(y)] - \mu_B^P(xy)$ and $\mu_B^N(xy) = \max[\mu_A^N(x), \mu_A^N(y)] - \mu_B^N(xy)$

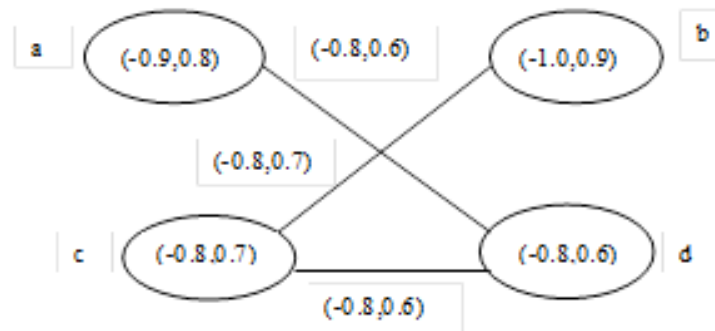


Figure 3.2: Complement of bipolar fuzzy graph $\bar{G}$

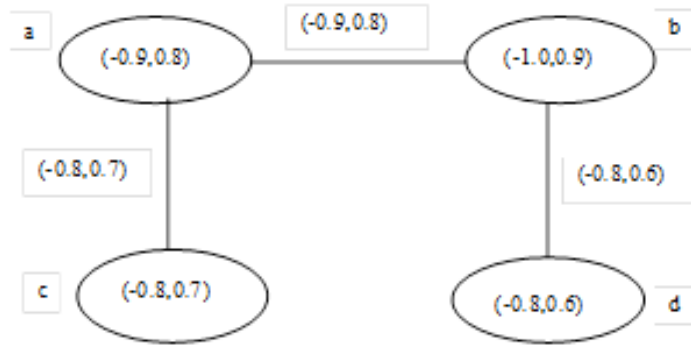


Figure 3.3: Complement of complement bipolar fuzzy graph $\bar{\bar{G}}$

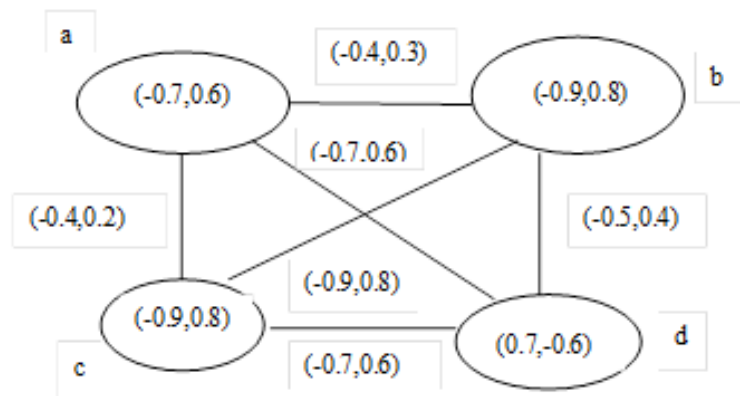


Figure 3.4: Bipolar fuzzy graph G

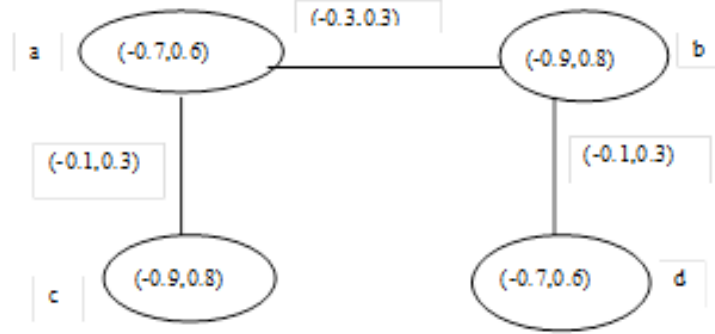


Figure 3.5: Complement of bipolar fuzzy graph $\bar{G}$

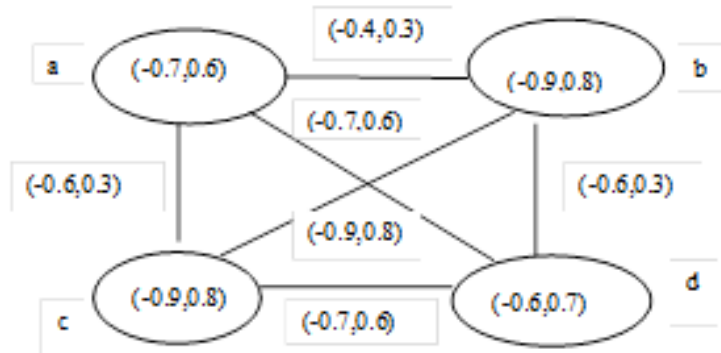


Figure 3.6: Complement of complement bipolar fuzzy graphs

Now $\bar{\sigma} = \sigma$ and $\mu_{\bar{B}}^P(xy) = \min [\mu_A^P(x), \mu_A^P(y)] - \mu_B^P(xy)$ and $\mu_{\bar{B}}^N(xy) = \max [\mu_A^N(x), \mu_A^N(y)] - \mu_B^N(xy)$.

Hence $\bar{\bar{G}} = G$. This shows that complement of complement BFG is a BFG.

4. Edge coloring in complement of BFG

We find all the different membership value of vertices and edges in the complement of a BFG. This membership value will work as a cut of this complement BFG. Depend upon the values of α – cut we find different types of Bipolar fuzzy sets for the same complement BFG. Then we color all the edges of the complement BFG so that no incident edges will not get the same color and find the minimum number of color will need to color the complement BFG is known as chromatic number.

For solving this problem we have the calculation into three cases. In first case we take a BFG (G) which have 5 vertices and 5 edges. All the vertices and edges have membership value. In second case we find the complement of this graph (G_1). In third case we define the edge coloring function to color the complement BFG.

Case 1:

Consider a BFG which have five vertices a, b, c, d, e and corresponding membership values $\{(-0.8, 0.9), (-0.8, 0.75), (-0.85, 0.95), (-0.85, 0.95), (-0.8, 0.9)\}$. Graph consist of five edges e_1, e_2, e_3, e_4, e_5 with their corresponding membership value $\{(-0.8, 0.75), (-0.8, 0.9), (-0.6, 0.85), (-0.85, 0.95), (-0.4, 0.6)\}$ corresponding BFG is shown in Fig 4.1.



Figure 4.1: Bipolar fuzzy graph G

Case 2.

We find the complement of a BFG.

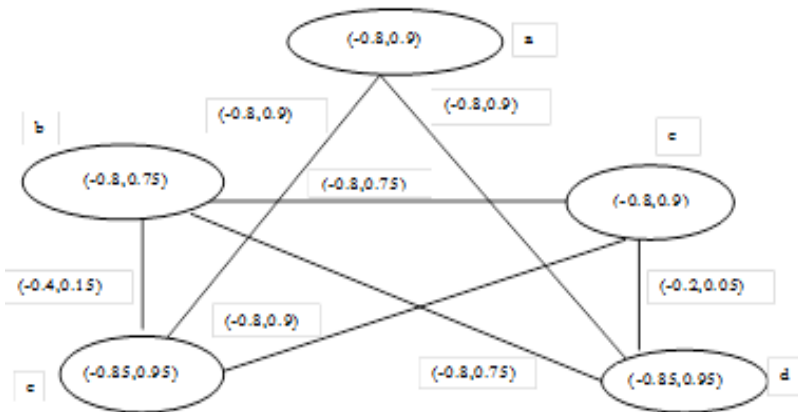


Figure 4.2: Complement of bipolar fuzzy graph $\bar{G}$

Case 3.

Given a BFG $G = (V, E)$ its edge chromatic number is bipolar fuzzy number $\chi(G) = \{(X_\alpha, \alpha)\}$ where X_α is the edge chromatic number of G_α where α values are the

Edge coloring in complement of bipolar fuzzy graphs

different membership value of vertex and edge of graph G . In this BFG there are five α -cuts. There are $\{(-0.2, 0.05), (-0.4, 0.15), (-0.8, 0.75), (-0.8, 0.9), (-0.85, 0.95)\}$. For every value of α , we find graph G_α and find its fuzzy edge chromatic number.

For $\alpha = (-0.2, 0.05)$, BFG $G = (\sigma, \mu)$ where $\sigma = \{(-0.8, 0.9), (-0.8, 0.75), (-0.85, 0.95), (-0.85, 0.95), (-0.8, 0.9)\}$.

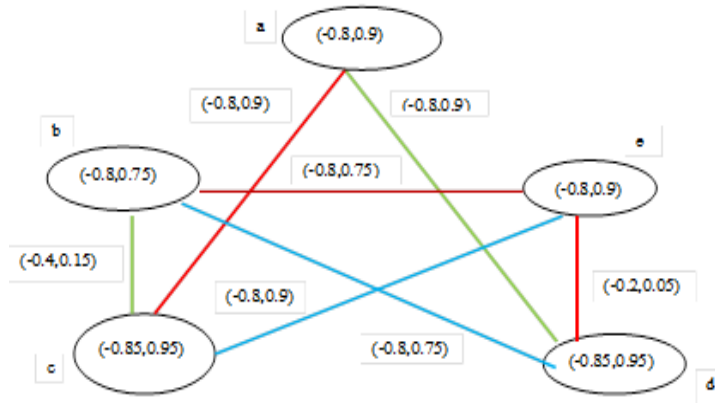


Figure 4.3: $\chi(-0.2, 0.05) = 4$

For α -cut $(-0.2, 0.05)$ we find the graph $G_{(-0.2, 0.05)}$ (Figure 4.3). Then we proper color all the edges of this graph and the chromatic number of this graph is 4.

For $\alpha = (-0.4, 0.15)$ BFG $G = (\sigma, \mu)$ where $\sigma = \{(-0.8, 0.9), (-0.8, 0.75), (-0.85, 0.95), (-0.85, 0.95), (-0.8, 0.9)\}$

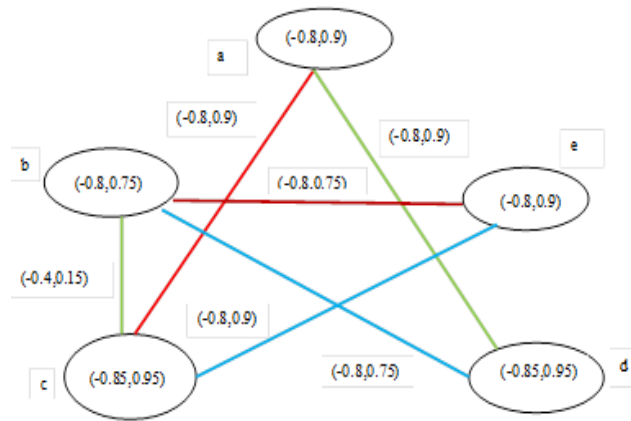


Figure 4.4: $\chi(-0.4, 0.15) = 4$

For α -cut value $(-0.4, 0.15)$ we find the graph $G_{(-0.4, 0.15)}$ (Figure 4.4). Then we proper color all the edges of this graph and the chromatic number of this graph is

4(Four). For $\alpha = (-0.8, 0.75)$ BFG $G = (\sigma, \mu)$ where $\sigma = \{(-0.8, 0.9), (-0.8, 0.75), (-0.85, 0.95), (-0.85, 0.95), (-0.8, 0.9)\}$.

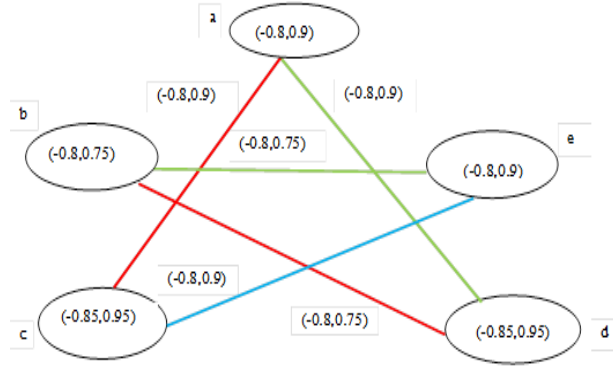


Figure 4.5: $\chi(-0.8, 0.75) = 3$

Now for α – cut value $(-0.8, 0.75)$, we find the graph $G_{(-0.8, 0.75)}$ (Figure 4.5). Then we proper color all the edges of this graph and the chromatic number of this graph is 3.

For $\alpha = (-0.8, 0.9)$ BFG $G = (\sigma, \mu)$ where $\sigma = \{(-0.8, 0.9), (-0.85, 0.95), (-0.85, 0.95), (-0.8, 0.9)\}$.

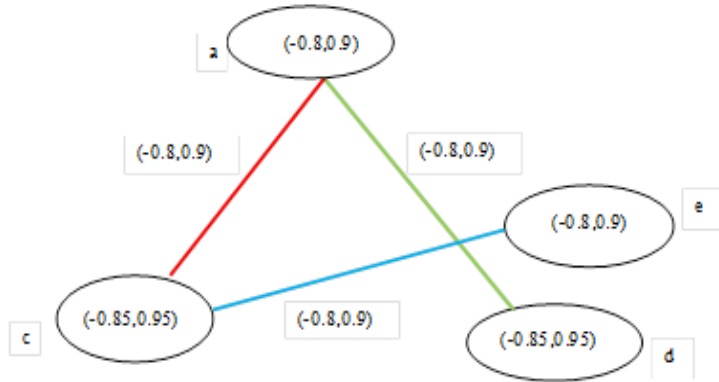


Figure 4.6: $\chi(-0.8, 0.9) = 2$

For α – cut value $(-0.8, 0.9)$ we find the graph $G_{(-0.8, 0.9)}$ (Figure 4.6). Then we proper color all the edges of this graph and the chromatic number of this graph is 2.

For $\alpha = (-0.85, 0.95)$, BFG $G = (\sigma, \mu)$ where $\sigma = \{(-0.85, 0.95)\}$ and

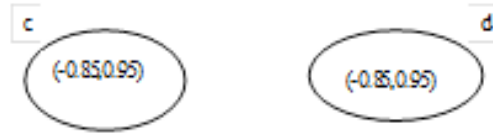


Figure 4.7: $\chi(-0.8, 0.9) = 0$

For α – cut value $(-0.85, 0.95)$, we find the graph $G_{(-0.85, 0.95)}$ (Figure 4.7). Then we proper color all the edges of this graph and the chromatic number of this graph is 0.

In the above example five crisp graph $G_\alpha = (V_\alpha, E_\alpha)$ are obtained by considering different values of α . Now for the edge chromatic number χ_α for any α , it can be shown that the chromatic number of BFG is $\chi(G) =$

$$\{(4, -0.2, 0.05), (4, -0.4, 0.15), (3, -0.8, 0.75), (2, -0.8, 0.9), (0, -0.85, 0.95)\}.$$

4 Conclusions

In this paper, we have found the complement bipolar fuzzy graph and color all the edges of that complement BFG through the α – cuts. Also, We observed that edge chromatic number depends on α – cut value.

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Analytic Hierarchy Process for Classes of Economic Behavior in the Context of Intertemporal Choices

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Giacomo di Tollo[‡]

Abstract

Due to the major crises of the past decade, the need to introduce consumer profiling operations to protect individuals from committing superficial business transactions has been realized in 2014/65/EU Directive. The present paper investigates how consumer behavioral attitudes influence the decision-making process so that the choice results, from a normative point of view, non-rational in the context of intertemporal choice. In addition, the particular focus by European institutions on closing the gender gap in the economic and financial sector motivated this research to enrich the analysis with gender assessments. The study of the relationship between cognitive characteristics and consumer decision-making are deepened with a multidisciplinary approach involving mathematics, behavioral finance, temperament theory and multi-criteria analysis. The results of an experimental investigation confirm that there is not a better temperament or a more adept gender in economic and financial choices.

Keywords: analytic hierarchy process, behavioral anomalies, economic behavior, gender analysis, impatience, intertemporal choice, personalized strategies.

2010 AMS subject classification: 91B08.[‡]

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1. Introduction

Theoretical and empirical investigation of the mechanisms that guide decision-making has become necessary to improve the functioning of economic and financial markets. The Directive 2014/65/EU of the European Parliament and of the Council of 15 May 2014 on markets in financial instruments stresses, due to shortcomings in the functioning and transparency of financial markets, the need to protect investors and to regulate financial instruments. Therefore, following the major crises of the last decade, the creation of investor profiles was necessary to protect savers and to offer products that fit the client profile. In addition to customer profiling, the EU aims to promote equality between men and women in all social, economic, and political activities.

Hebert Simon (1982) was the first to point out that individuals have limited cognitive resources that unconsciously lead to a simplification of the decision-making problem. Assuming that the decision-maker is rationally limited, research is turning to the personalization of decision-making strategies. In this regard, the study and analysis of the errors that characterize the dynamics of individual decisions must be conducted through personality theory. In addition, studies of personal financial skills have revealed an important gender gap (Goldsmith and Goldsmith, 2006) both in terms of approach with respect to investment strategies (Barber and Odean, 2001) and knowledge of financial topics (Hira and Mugenda, 2000). To improve the quality of women's choices and to reduce the gap, it is necessary to accept the existence of the many differences between the male and female genders and to investigate their reasons and consequences (Fonseca et al., 2012).

The present paper proposes a practical application of the multidisciplinary approach presented in Martino and Ventre (2022) addressed to two complementary purposes: the first purpose is to understand the extent to which emotional, cognitive and informational factors, which intervene during the evaluation and selection phase of an alternative, weigh on the economic decision-making process; the second aim is to investigate the effects and differences between the genders to promote the implementation of strategies aimed at closing the gender gap in all areas such as employment, social, political and economic. The disciplines involved in the present work are: mathematical finance for quantifying consumer cognitive and emotional biases; temperament theory for projecting characteristics of individuals into identifying classes; personalized finance for explaining relationships between bias and behavioral attitudes; and multicriteria analysis for quantitatively comparing class and gender characteristics. The multi-criteria decision support technique used is the Analytical Hierarchical Process (AHP) (Saaty, 1980) because it allows to compare elements difficult to compare directly, as shown in Ventre et al. (2022, c) and Ventre et al. (2023). In fact, its mathematical structure makes it possible to break down the analysis of choice into all the components that weigh on the decision. The originality of the present paper consists of an extension of Ventre et al. (2022, c) with gender analysis and in giving it an applicable context of the multidisciplinary approach used.

The analysis conducted by AHP proves that it is not possible to define an optimal strategy for everyone because of different behavioral characteristics. Therefore, consultants and automated devices that provide digital advice must consider the importance of the decision-maker's personality to whom service is provided. With respect to gender analysis, the AHP makes it clear that there is no privileged gender but there are only different attitudes, mostly complementary inclinations dictated by the known "*distance between Mars and Venus*" (Del Giudice et al., 2012).

The paper is structured as follows: in Section 2, a literature review about gender

analysis will be briefly exposed; in Section 3 the main results about anomalies in intertemporal choice will be shown and discussed in terms of emotional impulses; in Section 4, the structure of the AHP will be constructed with reference to Keirse's temperament theory (1998); in Section 5 the experimental part aimed at constructing the weight of anomalies for each temperament, both in general and by gender, will be implemented; finally, there will be a discussion section and a conclusion section with ideas for future development.

2. Gender differences

Many differences between the behavior of men and women were observed in various domains leading to the hypothesis that they were caused by different preferences between the genders (Croson and Gneezy, 2009). Most theoretical and empirical work analyses gender difference with a comparison between the environment in which the individual develops (Eagly and Wood, 1999) and evolutionary origins (Geary, 2010). In this regard, two main theories can be pointed out: the sociocultural theory (Eagly and Wood, 1999), which focuses the reasons for gender difference on a historical division of labour between men and women; the cognitive theory of social learning (Bussey and Bandura, 1999), on the other hand, which focuses gender differences with children's internalization of existing social norms.

Even in the workplace, gender discrimination is fueled by stereotypes and organizational factors. The study conducted by Bobbit-Zeher (2011) proves how the combination of cultural, structural and organizational factors, together with gender stereotypes, can have a strong impact on gender discrimination. For example, with respect to financial knowledge, Chen and Volpe's study (2002) proved that women are actually even less interested in learning about financial topics and that, also because of social stereotypes, female students are less oriented than their male peers towards financial studies (Hawash et al., 2020). These differences could lead to women being more exposed to financial problems. However, the studies in the literature on social preferences remain incomplete if the interaction between gender and intra-gender variability is not considered (Thoni and Volk, 2021). For this reason, gender differences in preferences have been investigated to deepen social and economic outcomes (Croson and Gneezy, 2009) and they are a good starting point for understanding gender differences in many areas such as education, the labor market and financial decision-making (Buser et al., 2014). The first differences are found in social preferences that express how individuals interact with others. The study conducted by Kamas and Preston (2015) not only proves that gender differences in economic behavior can be deepened and understood by social preferences, but also proves important findings. First, women tend to avoid inequality and this aversion to inequity could be linked to the fact that in trust games they send less. In addition, women are more inclined to charitable activities and equal pay, a preference that can be explained by inequity aversion and lower self-confidence. Some studies also prove that women are more altruistic than males, less present-oriented and reliable (Horn et al., 2022).

A determining factor for social preferences is also age as proven by a study conducted on Swedish and Austrian children and adolescents (Martinsson et al., 2011). Risk preferences are another of the most studied factors for which it is possible to conclude that men are more risk-prone than women. This conclusion confirms previous results of an economic nature (Eckel and Grossman, 2008) and psychological nature (Brody, 2022). In this regard, studies focusing on the portfolio selection problem have also provided direct evidence of gender differences. For example, Graham et al. (2002) have proven

that men invest in pension planning in a less conservative way than women do, and in the domain of financial risk, single women are less risk prone than single man (Sunden et al., 1998; Finucane et al., 2013). A study conducted by Jianakoplos (2002) on gender differences in risk preferences emphasizes that the many differences must be observed and considered by advisors to make appropriate recommendations as risk appetite has a decisive impact on the construction of an asset allocation. Precisely because risk tolerance is a key element in determining appropriate financial advice, some studies have focused on factors that have a strong impact on measurement. The study conducted by Marinelli et al. (2017) on 2,374 investors proves a relationship between low literacy, income, and economic behavior with inconsistency among the most used risk tolerance metrics. Therefore, the analysis of the gender difference in this respect must also incorporate the factors of the individual subjective structure. This gender gap is very evident, and many explanations have been proposed. A first explanation is referred to the concept of "*risk as a feeling*" developed by Loewenstein (2001) referring to the reaction that instinctively develops in a risky decision-making context and from a psychological point of view, women feel emotions more intensely than men. Moreover, in emotional terms, the difference between males and females is not only the intensity of the emotions they feel, but also the emotions felt in the same situation are different (Grossman and Wood, 1993). In this regard, Lerner et al. (2003) have shown that different emotions lead to different attitudes, explaining the different approach to risk. The bias of overconfidence is the second reason that could explain the gender differences because even if overconfidence is shared by both, men are more overconfident than women (Lundeberg, 1994). In particular, Niederle and Vesterlund (2007) have shown that men's more insistent overconfidence than women's leads to men being more convinced of their victories so that everything is seen as a competition in which the belief to win is greater and consequently so is the risk attitude. The way in which the risk situation is interpreted can also be another determining factor in investigating the underlying causes of these differences. While men perceive a risky situation as a challenge, women perceive the context as a threat (Arch, 1993). This implies that the gender difference does not consist of different abilities with respect to the decision-making environment but rather of different motivations driving both genders. At the same time, factors that may be motivating for men may be harmful to women (Block, 1983). Confirming the hypothesis that the gender gap is not a lack of aptitude are numerous studies looking at a sample of professionals for whom the difference between men and women is largely small and almost insignificant (Dwyer et al., 2002; Master and Maier, 1988).

Biological differences, such as different testosterone concentrations, also play an important role in this context (Sapienza et al., 2009). The studies outlined so far emphasize that to promote gender equality and achieve the main goals proposed by the European Union, it is not necessary to reduce diversity but to turn gender inequality into a strength for any sector. Zimmer (1988) suggests that theories that neutralize the gender gap, such as tokenism used to create the illusion of inclusiveness (Hogg and Vaughan, 2008), have a limitation that may even hinder inclusion and gender equality. In fact, in roles where approach and behavior are essential, inattention to gender differences can generate employment bottlenecks. Organizations can therefore reap many benefits from diversity. Indeed, not only does diversity enable them to reach different markets and customers, but it also promotes innovation and better performance (Nair and Vohra, 2015) some empirical evidence shows that diversity has its advantages (Herring, 2009). In addition, a recent study conducted by Shrader et al. (2020) also proves a relationship between the percentage of women at the general management level and better financial performance. In conclusion, to promote gender equality, the classic factors used to argue

gender difference, such as biological, intellectual, emotional, and behavioral aspect, should not be regarded with a negative connotation as caused by prejudice against women (Benschop and Verloo, 2011). Fletcher (2001), with an original and alternative point of view, conceptualizes the main characteristics of the female gender (empathy, vulnerability, connection, relational skills) as strengths and not weaknesses. From a strategic point of view, bridging the gap requires the use of continuous innovation processes (Benschop and Verloo, 2011), based on the combination of gender mainstreaming and intersectionality. Intersectionality refers to the evaluation of each element or trait of an individual as the intrinsic union of all the elements that characterize her identity (Palczewski and DeFrancisco, 2014). However, Holvino (2010) noted that intersectionality is not enough acknowledged by organization studies and gender change projects in organizations must consider the implications of other factors as class, ethnicity, sexuality, and nation.

3. Intertemporal choice and impatience

Most of the decision-making problems we face throughout our lives can be traced back to the typology of intertemporal choices, i.e., those whose consequences manifest only over time. This argument is particularly complex mainly for two reasons: the first one is that, deciding to get a result in the future necessarily implies renouncing to something in the present (Noor, 2011); the second aspect concerns the difficulty of keeping the coherence constant in future objectives (Sayman and Onculer, 2009). Intertemporal choices are studied in various fields such as economics and psychology Skylark et al.(2021) and Read et al. (2003) state that “*the goal of research into intertemporal choice is to understand how these choices are made, and how they should be made*”. The Discounted Utility Model (Samuelson, 1937,1952) assumes that the decision-making process of individuals is based on associating a level of utility to each alternative and then choosing the alternative with greater utility. The model introduced by Samuelson has some weaknesses when it refers to the effective decision-making behavior. With the term *anomalies*, we refer to those empirical results that are difficult to *rationalize*. To achieve the objective of customer profiling, the concept of investor's impatience will be used to measure the weight of anomalies. In the context of intertemporal choice, the impatience is defined as “*the amount of money that the agent is willing to lose in exchange for anticipating the availability of a \$1 reward*” (Cruz Rambaud and Muñoz Torrecillas, 2016). This analysis is inspired by the experimental method proposed by Ventre et al. (2022, b) in which, after demonstrating a correspondence between the anomaly and the alteration of the degree of impatience, the authors investigate the emotional drives of choice using the value of the hyperbolic factor (Rohde, 2010). Comparing the impatience of different discount functions, a relationship emerges between the speed with which the function varies and the degree of investor impatience: the faster the discount function decreases, the greater the associated impatience (Cruz Rambaud and Muñoz Torrecillas, 2016). Prelec (2004) proved that a decreasing degree of impatience has a significant effect on the selection of suboptimal choices and that variation in impatience underlying the anomalies in the Discounted Utility Model can be seen as a reflection of the decision-maker's bounded rationality. The originality of the present work lies in the fact that the analysis is not limited to the study of abnormal attitudes but deepens their properties by investigating behavioral characteristics in a context of gender to enrich the motivations presented about gender gap and financial anomalies. In practice, by quantifying the weight of emotional factors driving decision-making, it is possible to measure the different impact of cognitive and

emotional biases between men and women. For these reasons, the analysis submitted by Ventre et al. (2022, b) of anomalies will be enriched by behavioral classification and gender analysis.

3.1 Economic model

From an operational perspective, assuming that the decision-makers' preferences are continuous, monotonous, and impatient, the study of intertemporal choices is based on the Model of Discounted Utility (Samuelson 1937, 1952).

Fixed $X = R^n$ the set of available results and $T = R^+$ the set of points of time :

- The N-pla $(x_0, t_0; \dots; x_n, t_n) \in (X \times T)^n$ indicates an intertemporal prospectus: if the prospectus is accepted, the decision-maker will receive the result x_i at the time t_i and a null result for each t_j with $j \neq i$;
- The intertemporal utility defined on the prospectus is given by $\sum_{i=0}^n f(t_i)U(x_i)$ where $U(x_i)$ is the cardinal utility function associated with the consumption of the outcome x_i at the time t_i and $f(t_i)$ is the individual discount function.

The time lag between the decision and review of the result creates a risk due to uncertainty. The individual perception of this indeterminacy is enclosed in the discount rate $\rho(t) = -\frac{f'(t)}{f(t)}$ that indicates “the proportional variation of f in a standard period” (Read et al., 2003). Empirical evidence demonstrates that people violate the discounted-utility theory systematically (Rao and Li, 2011). In fact, the first formulation of the model hypothesized a constant discount rate over time but, due to the systematic deviation from the exponential model predictions, alternative functions were formulated to provide a more accurate empirical description. The main difference between the linear, exponential, and hyperbolic models in Table 1 is that, while exponential and linear functions present a constant discount rate for all future periods, hyperbolic models manifest a discount rate varying over time.

Linear	$f(t) = 1 - rt$		
Exponential	$f(t) = \left(\frac{1}{1 + \rho}\right)^t$		
Hyperbolic	Generalized	Quasi-hyperbolic	One parameter
	$f(t) = (1 + \alpha t)^{-\frac{\beta}{\alpha}}$	$f(t) = \beta \left(\frac{1}{1 + \rho}\right)^t$	$f(t) = \left(\frac{1}{1 + \rho t}\right)$

Table 1. Common discount functions in intertemporal choice reported by Read et al. (2003).

To understand the consequences of this aspect from a rational choice perspective, this study considers two outcomes: the first one is minor and anticipated (SS), and the second one is major and delayed (LL). A constant discount rate implies that, whatever the individual's choice, her preference remains stable over time in line with the profile of a

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rational investor assumed by classical finance. Empirically, however, since the imminent results appear disproportionately attractive, the decision-maker reverses her preferences, making the choice inconsistent as shown in Figure 1. In fact, with the term *temporal inconsistency*, we refer to a situation in which a subject varies her preferences over time.

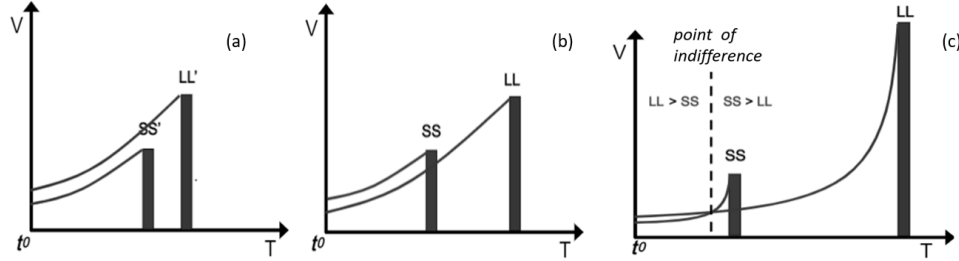


Figure 1. Comparison between SS and LL. The images (a) and (b) describe different situations in which choices are temporally consistent, i.e., preferences remain constant over time. The image (c) shows instead as a variation of the course of the curve involves an inversion of the preferences and therefore temporally inconsistent choices.

Prelec (2004), analyzing the psychological characteristics underlying inconsistency, proved equivalence between the selection of dominated prospects, i.e. those outcomes that are not optimal from any temporal viewpoint, and a degree of impatience that decreases over time. In particular:

Definition 1. The preference $\geq$ shows a Decreasing Impatience (DI) if $\forall \sigma > 0, y > x > 0, (x, s) \sim (y, t)$ we have $(x, s + \sigma) \leq (y, t + \sigma)$ (strictly decreasing if $(x, s + \sigma) < (y, t + \sigma)$).

Definition 2. An intertemporal prospect (x, s) dominates (y, t) with $s < t$, i.e. $(x, s) \gg (y, t)$, if $(x, s) \geq (y, s)$ and $(x, s) > (x, t)$.

Prelec (2004), studying dominance violation on two-step decision problems, describes the following attitudes, shown in Figure 2 and Figure 3:

Definition 3. The preference $\geq$ allows a sophisticated two-step violation of dominance at the time (t, s, r, τ) if the relationship is DI and $\exists x, y$:

today (0): $(x, t + \tau) \gg (x, r) \geq (y, s + \tau)$

tomorrow (τ): $(x, t) < (y, s)$

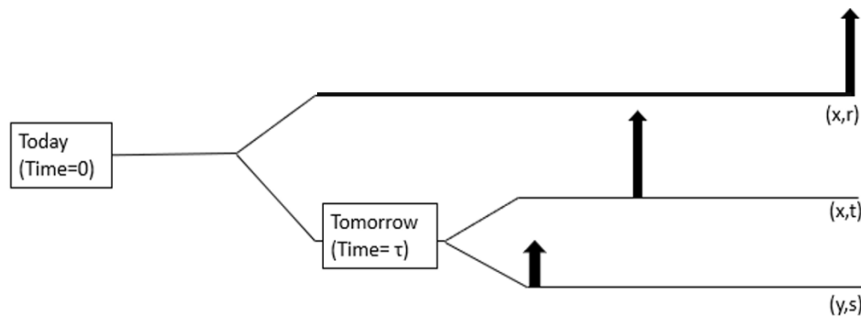


Figure 2. Sophisticated Decision-Maker (DM) predicts the selection of the lower outcome at time τ and commits in advance to the best available outcome at time 0 (Prelec 2004).

Definition 4. The preference $\geq$ allows a naïve two-step violation of dominance at the time (t, s, r, τ) if the relationship is DI and $\exists x, y$:

today (0): $(y, s+\tau) \geq (x, r) \gg (x, t+\tau)$

tomorrow (τ): $(y, s) < (x, t)$

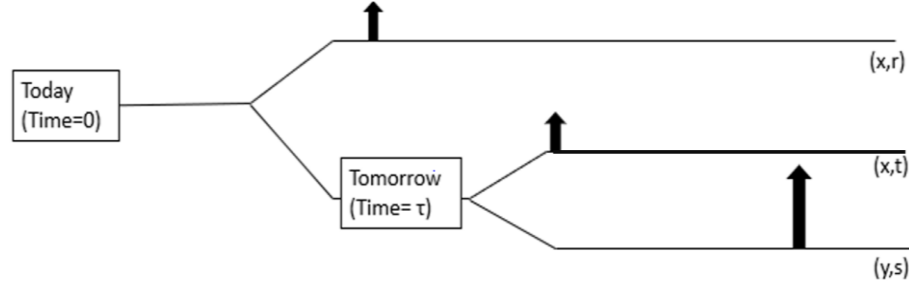


Figure 3. Naïve Decision-Maker (DM) at time 0 the DM prefers the most considerable available outcome at time τ , but his preferences change, and the choice will fall on the lower outcome (Prelec, 2004).

Such attitudes are not only related to a greater degree of decrease in impatience but also to a greater convexity of the discount function. Applying the previous definitions to infinitesimal intervals, it is possible to prove that the greater the degree of DI, the more significant the difference between the rates of impatience and temporal preference. The latter means that the decrease in impatience quantitatively represents the difference between preferring an event to occur and preferring an event to occur sooner. Moreover, the expression that Prelec (2004) proposes for calculating the degree of DI is equivalent to the speed with which the discount rate varies over time, determining a link between the subjective perception of time and impatience. In particular, the more distorted this perception is, the more hyperbolic will be the trend of the discount rate. A direct relationship between subjective perception of time and temporal inconsistency has been proven in Ventre and Martino (2022) and Ventre et al. (2022, a). The tool we exploit in the experimental phase to calculate the degree of DI is the hyperbolic factor formalized by Rohde (2010).

Definition 5. Consider $(x, s) \sim (y, t)$ $(x, s + \sigma) \sim (y, t + \tau)$ $s < t$, $x < y$, $\tau > 0$. The hyperbolic factor is:

$$H(s, t, y, \tau) = \frac{\tau - \sigma}{t\sigma - s\tau}$$

Theorem 1. A pair of indifference can be constructed as follows:

Step I. Fix $y \neq 0$ and fix s, t, τ with $s < t$, and $\tau > 0$;

Step II. Find x such that $(x, s) \sim (y, t)$;

Step III. Find σ such that $(x, s+\sigma) \sim (y, t+\tau)$.

3.2 Anomalies and decreasing in impatience

By virtue of the link between inconsistency and decreased impatience, it is possible to relate anomalies in the discounted utility model and the cognitive mechanisms responsible. The cognitive elements that influence our attitude are behavioral biases, and they can be of emotional or cognitive nature: the former is due to factors of the emotional sphere; the latter, instead, are “shortcuts” that the decision-maker creates in his mind during the interpretation and processing of information. Jordan and Rand (2018)

emphasized the importance of character in the decision-making process in a broader sense. Given that behavioral anomalies are proper to the nature of decision-makers, even assuming a reasonable and rational investor, it is always possible to analyze anomalous situations from a mathematical viewpoint. In the literature, researchers have identified various types of anomalies; the latter can be related to investors' emotional impulses through the concept of impatience by comparing the preferences associated with a hyperbolic discount and those associated with an exponential discount. The delay effect (Ventre and Ventre, 2012) consists in the inversion of the preferences due to the increase of the period between the decision and the reception of the result. For this anomaly, it is possible to prove that the speed with which the function of a hyperbolic discount is decreasing and, therefore, the impatience of the investor is intense according to the period of evaluation of the results. In particular, by Ventre et al. (2022, b):

Theorem 2. If $(x, s) \sim (y, t) \Rightarrow (x, s + h) < (y, t + h)$ holds, then the rate of variation of the discount function $f(t)$ is decreasing.

The interval effect (Read et al., 2003; Read and Roelofsma, 2003) is the anomaly for which the applied discount rate decreases as the length of the interval increases. Therefore, the phenomenon can be studied as a direct consequence of the subjective perception of time whose mathematical proof is in section 4.2 in Ventre et al. (2022, b).

The preferences of individuals are also conditioned by the size of the outcome. The anomaly in question, defined magnitude effect (Benzion et al., 1989), consists in an inversion of preferences due to the size of the results and from a psychological vantage point can be explained considering that the more the figures are important, the more the patience of the investor increases. The investor's attitude is determined by the fact that smaller figures are associated with immediate consumption while larger figures are associated with a future investment idea. It can be proved that the reversal of the preferences in question is related to the difference of the figures considered and not to their ratio, i.e. (Ventre et al., 2022, b):

Theorem 3. Let x and y , with $x < y$, and X and Y , with $x < X < Y$, such that $\frac{U(y)}{U(x)} = \frac{U(Y)}{U(X)}$. If $(x, s) \sim (y, t)$, then $(X, s) < (Y, t)$ if, and only if, $U(y) - U(x) < U(Y) - U(X)$.

Finally, the sign effect (Loewenstein and Thaler, 1989), also called gain-loss asymmetry, consists of the tendency of individuals to anticipate losses more than gains of the same magnitude. The first approach to studying the anomaly in question is the emotional loss aversion bias for which the disutility associated with a loss is greater than the utility associated with a gain of the same amount. The discount rate is, therefore, steeper in the context of a loss because the investor's haste toward payment is greater. The second approach is based on Prospect Theory (Kahneman and Tversky, 2013). In this regard, the anomaly can be justified by replacing the instantaneous cardinal utility function with the utility function proposed by the theory concerned as proven by Ventre et al. (2022, b):

Theorem 4. If the utility of a result is calculated as $V(x)$, where $\varepsilon > 1$ (according to the prospectus theory) and U is concave ($U'' < 0$), then, for every $0 < x < y$ and $s < t$, $(x, s) \sim (y, t)$ implies $(-x, s) > (-y, t)$.

3. The AHP model applied to economic behavior

To implement an AHP, it is first necessary to define a structure representing the problem. Our aim is to define strategies that best fit the investor's profile. The first level representing the general objective, i.e. "*selection of the best strategy*" will be followed by behavioral criteria taken by Keirsey's temperament theory. Keirsey believed that individuals can be divided into four major temperaments, each with specific behavioral characteristics. Pompian (2012) overlaid Keirsey's four temperaments with four types of behavioral investors through behavioral finance studies with the goal of relating cognitive biases to an individual's attitude in the financial market. Table 2 shows the relationship that exists between the four Keirsey's temperaments and the BIT.

Keirsey's temperaments	Artisan	Guardian	Idealist	Rational
Types of behavioral investors	Accumulators	Conservative	Followers	Independent

Table 2. Relationship between Keirsey's temperaments and types of behavioral investors

Since the financial personality is related to temperament and each behavioral investor is characterized by specific behavioral biases, the third level is composed of the anomalies of the discounted utility model studied in Ventre et al. (2022, b). The last level has as vertices the investment strategies assuming that a strategy can be viewed as combinations of the domains in which the anomalies develop. The final structure is shown in Figure 4.

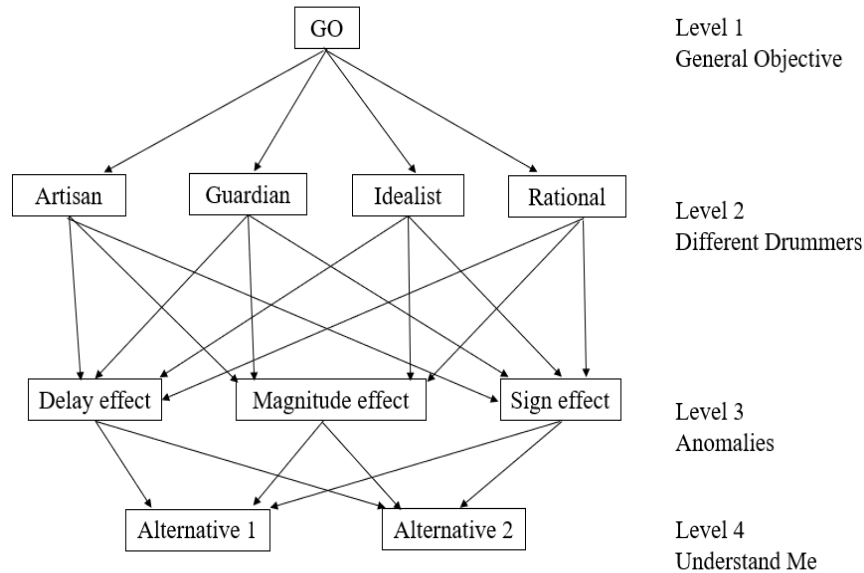


Figure 4. AHP structure based on four different levels. The present paper aims to determine the weights of Level 3 compared with Level 2.

"*Different Drummers*" and "*Understand Me*" refer to the book "Please Understand Me II. Temperament Character Intelligence" (Keirsey, 1998) of which was used The Keirsey FourTypes Sorter test for the determination of temperament. The quiz, consisting

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of sixteen questions with four alternatives to be ordered by decreasing preference, provides us with the intensity of the four temperaments in each individual. The obtained scores, observing that the main temperament corresponds to the minimum value, allow calculating the weights of level 2 with the following pairwise comparison matrix reported in Table 3.

GO	Artisan	Guardian	Idealist	Rational
Artisan	1	G/A	I/A	R/A
Guardian	A/G	1	I/G	R/G
Idealist	A/I	G/I	1	R/I
Rational	A/R	G/R	I/R	1

Table 3. Pairwise comparison matrix for the weights of GO with respect to each temperament. The score of each temperament is given by The Keirsey Four Types Sorter test (Keirsey, 1998, p.348).

In addition to empirically testing the relationships in section 3, this study tries to estimate the relative importance of anomalies with respect to higher-level items. To accomplish this, assuming that the degree of decrease in impatience reflects the non-rationality underlying the preference reversal phenomenon, the questionnaire is created by constructing the indifference pairs given in Theorem 1. The hyperbolic factor provides us with a measure of the altered degree of impatience for each anomaly that makes up level 3. For each trait that composes in level 2, the median of the hyperbolic factors is calculated based on the main temperament of each individual. In this way, for each main trait, the matrix of the pairwise comparison shown in the Table 4 is obtained. In the Table 4, $H(s, t, y, \tau)$ denotes the hyperbolic factor in a prospectus in which s , t , y and τ are fixed. The respondent is free to choose for each question the values of x and σ obtaining the indifference pairs $(x, s) \sim (y, t)$ and $(x, s + \sigma) \sim (y, t + \tau)$ identified by the Theorem 1.

Dominant Trait	Delay effect	Magnitude effect	Sign effect
Delay effect	1	$\frac{\text{MedianH}(0,6,500,12)}{\text{MedianH}(0,6,50,12)}$	$\frac{\text{MedianH}(0,6,500,12)}{\text{MedianH}(0,6,-500,12)}$
Magnitude effect	$\frac{\text{MedianH}(0,6,50,12)}{\text{MedianH}(0,6,500,12)}$	1	$\frac{\text{MedianH}(0,6,50,12)}{\text{MedianH}(0,6,-500,12)}$
Sign effect	$\frac{\text{MedianH}(0,6,-500,12)}{\text{MedianH}(0,6,500,12)}$	$\frac{\text{MedianH}(0,6,-500,12)}{\text{MedianH}(0,6,50,12)}$	1

Table 4. Pairwise comparison matrix for the weights of the temperaments with respect to the individual anomalies. Underlying this construction is the hypothesis that the degree of decrease in impatience represents the non-rationality underlying the inconsistency.

4. Results

To analyze the relationships among the four temperaments we consider the median of the hyperbolic factors of individuals according to the dominant trait. The median is recommended because the data exhibit high variability. In fact, it is possible to prove that the maximum reference value that can be obtained on the basis of the data set is $H(s, t, y, \tau) = 66.50$ while the lower value is not defined. Before proceeding with the presentation of the main results of the experimental phase, in Table 5 we report the distribution of the sample.

Temperament	Rational	Idealist	Guardian	Artisan
Distribution	38.5%	26.9%	19.2%	15.4%

Table 5. Distribution of the sample. The quiz administration sample consisted of 52 persons, ages ranging from 18 years old and 65 years old, of which 48.08% were women.

Table 6 shows the results for the hyperbolic factor $H(0,6,500, 12)$. The maximum degree of decrease in impatience indicates strong impulsiveness. Artisans, in general, are more focused on the short term and therefore prefer more immediate gratification. In fact, Artisans manifest the major median while Idealists, due to their general disinterestedness, stand out among all temperaments being the only ones not to realize in maximum value.

Main temperament	Minimum value and distribution	Maximum value and distribution	Median
Artisan	0.50 (12.50%)	66.50 (25.00%)	7.83
Guardian	0 (10%)	66.50 (20%)	2.83
Idealist	0 (7%)	28.40 (7.14%)	1.83
Rational	-0.02 (5%)	66.50% (25.00%)	1.83

Table 6. $H(0,6,500,12)$. Distribution and median for each temperament.

Table 7 shows the results for the hyperbolic factor where lesser digits are involved in the indifference pairs of Theorem 1. Comparing the values in Table 6 and Table 7, the distribution of the maximum value is doubled in the case of less important digits. In line with predictions, the medians are all greater than the factor $H(0,6,500,12)$. As for Artisans, their strong impulsiveness is even more evident with a distribution of the maximum value of 62.50%. The case of the Guardians is particularly interesting. In fact, this category presents the maximum value for the ratio $H(50)/H(500)$ which, from a cognitive perspective point of view, can be explained by the mental accounting bias for which money is categorized by subjective criteria. This is equivalent to saying that the prevalence of impulsiveness over self-control breeds inconsistency. Also, the Idealists realize $H=66.50$ unlike the case reported in Table 6. The Rationales present less inconsistency among the choices involving figures of different entity.

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Main temperament	Minimum value and distribution	Maximum value and distribution	Median
Artisan	2.69 (12.50%)	66.50 (62.50%)	28.4
Guardian	0 (10%)	66.50 (30%)	18.12
Idealist	0.06 (7.14%)	66.50 (35.71%)	4.83
Rational	-0.02 (5%)	66.50 (40.00%)	4.17

Table 7. H(0,6,50,12). Distribution and median for each temperament.

Table 8 shows the comparison between hyperbolic factors H(0,1,50,1) and H(0,6,50,12) with the purpose of understanding how much and how the applied discount varies with the length of the intervals involved. It is evident that the greater part of the variation of the degree of impatience happens in the first period confirming the hypothesis of a decreasing speed in the time. Moreover, as the width of the interval increases, the applied discount also increases. This outcome involves all temperaments.

	H (0,1,50,1)		H (0,6,50,6)	
Minimum value and distribution	0 (7.69%)		0 (3.85%)	
	M (25.00%)	F (75.00%)	M (50%)	F (50%)
Maximum value and distribution	32.33 (48.08%)		66.50 (40.38%)	
	M (64.00%)	F (36.00%)	M (61.90%)	F (38.10%)
Median	13.29		18.12	

Table 8. Comparison of H(0,1,50,1) and H(0,6,50,12).

Table 9 highlights how many individuals, just by varying the sign of the outcomes considered, exhibit different hyperbolic discounts. Analyzing the median for temperaments, particular attitudes characterize each category. The Artisans, for example, generally show a high value for the hyperbolic factor, underlining that the serenity with which they face the payment prevails over the displeasure of the payment itself. This is because Artisans are the most flexible with respect to losses. On the other hand, the Guardians have a different approach, as evidenced by the finding of the smallest maximum value of the entire experimental phase. This temperament suffers the payment because it is seen as a reduction of its assets. In fact, when investigating the timing with which individuals would prefer to act in a loss and gain situation, Guardians are the ones that most postpone losses among the four temperaments as shown in Figure 5.

Main temperament	Minimum value and distribution	Maximum value and distribution	Median H (-500)
Artisan	0.03 (25.00%)	66.50 (62.50%)	66.50
Guardian	0 (20.00%)	7.83 (20%)	2.83
Idealist	0 (7.14%)	66.50 (14.29%)	1.83
Rational	0 (5%)	66.50% (50.00%)	31.17

Table 9. H(0,6,-500,12).

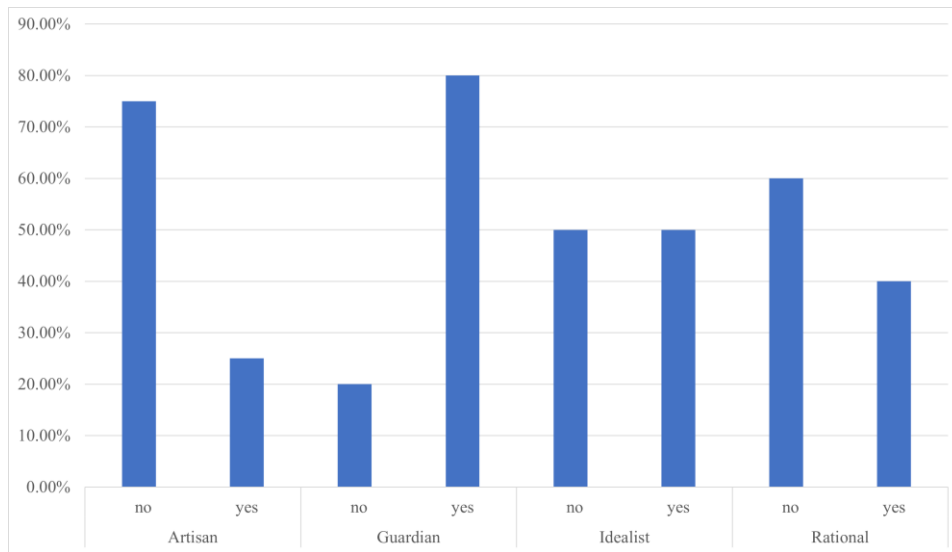


Figure 5. Distribution of individuals who defer payment. Guardians take an attitude that sets them apart from other temperaments in such a situation.

Figure 6 shows that Rationals despite being the least emotional are prone to particular distortions such as the overconfidence bias. In fact, Rationals present themselves as more irrational than the other temperaments in this circumstance in that, although they manifested a moderate hyperbolic factor, they were the individuals who would pay higher amounts than they had stated they would receive.

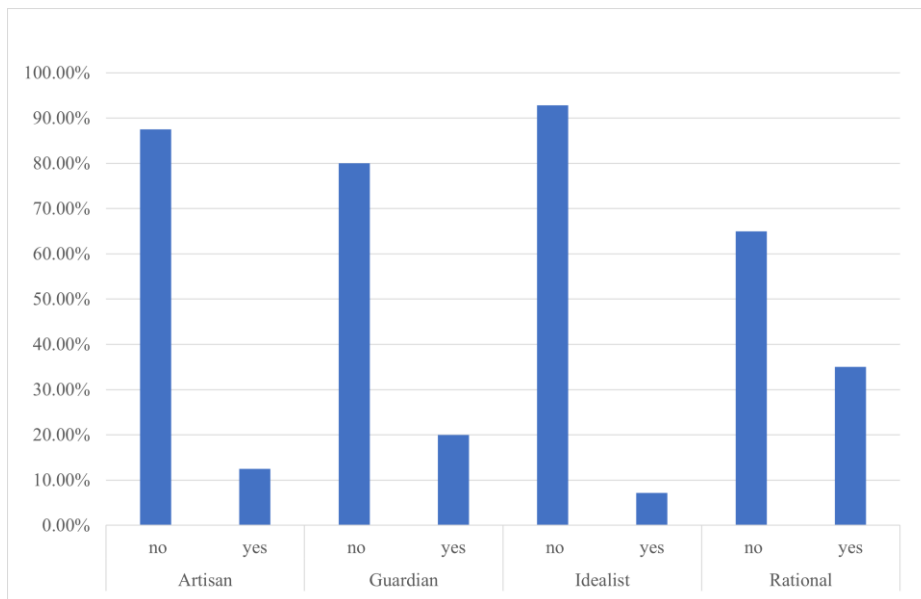


Figure 6. Distribution of individuals who would unconsciously increase the amount payable. For the bias of excess of confidence that characterizes them, the Rationales prefer to pay more than they would like to receive.

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The analysis ends by calculating pairwise comparison matrices for each temperament and determining the relative weight of the anomalies relative to the top level. The values obtained are shown in Figure 7.

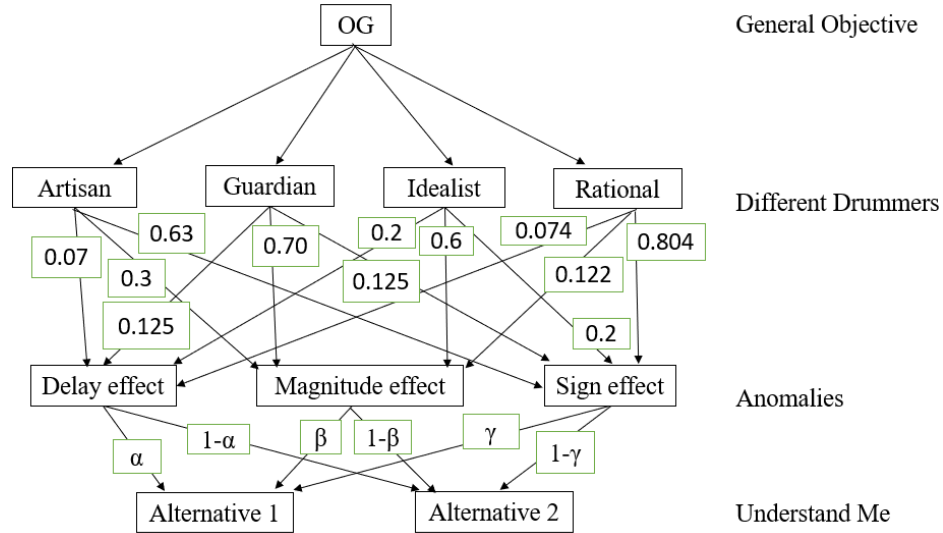


Figure 7. Final structure.

By coasting the pairwise comparison matrices for each temperament as shown in Table 4, the local priorities of the affected level were estimated. These priorities did not vary across individuals. Therefore, it could be a method to homogenize heterogeneous categories.

5.1 Results by genders

To investigate and deepen the differences between the genders, we repeat the analysis with the aim of creating two different AHP. In this way, it will be possible to quantify the strengths and weaknesses of both genders. The distribution of the sample with respect to gender (48.08% female, 51.92% male) is sufficient and this allows us to proceed with the analysis. The table 10 shows the medians of hyperbolic factors with respect to the genders.

	H(0,500,6,12)	H(0,50,6,12)	H(0,1,50,1)	H(0,-500,6,12)
Male	1.83	6.50	32.33	1.83
Female	7.83	28.41	3.00	6.50

Table 10. Medians of hyperbolic factors with respect to the genders.

The H(0,500,6,12) column in Table 10 shows that women are generally more impatient than men. In particular, the value corresponding to women is approximately 4.28 times greater than men. What has been said is equivalent to affirming that women are more prone to exhibit a dynamic inconsistency. This trait is still evident when we consider the magnitude effect. First, we observe that both genders undergo an important variation of the hyperbolic factor when smaller figures intervene, leading us to conclude that the magnitude effect has an equivalent impact on men and women. In fact, the ratio of the medians reported in column two of the Table 10 is 4.37, similar to the previous one, still highlighting the greater impatience of women. The last column of the Table 10,

relating to the median of $H(0, -500, 6, 12)$, again underlines a different approach in intertemporal choices between the genders, in this case with respect to losses. To investigate this result, the Figure 8 and the Figure 9 show who would pay more and who would postpone payment more according to gender.

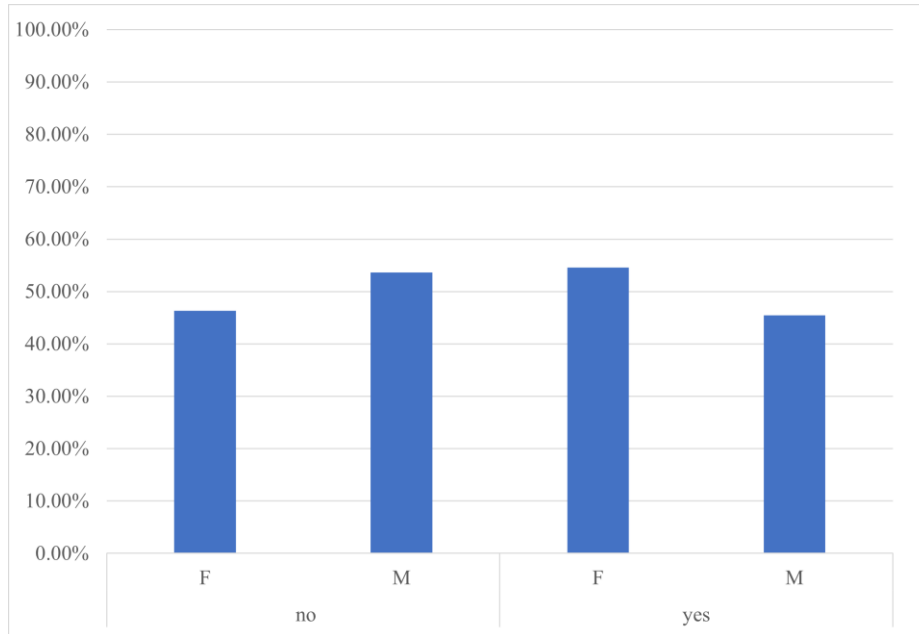


Figure 8. Distribution of individuals who would unconsciously increase the amount payable according to gender.

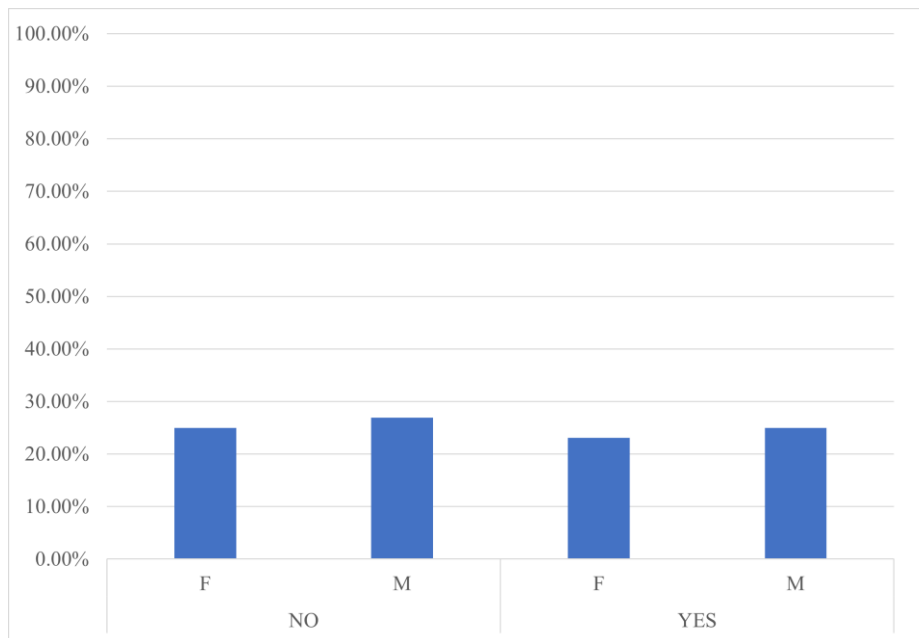


Figure 9. Distribution of individuals who defer payment according to gender.

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Women, in accordance with a lower aptitude for the financial environment, account for a higher percentage in Figure 8, while, with respect to Figure 9, the percentages are almost similar, and no further differences are evident. Finally, about the hyperbolic factor $H(0,1,50,1)$ it is possible to observe the only case in which men show greater impatience than women.

Although the gender distribution is uniform, we cannot be satisfied with our analysis after observing in section 5 many important differences between the four temperaments. In fact, we believe that the distribution of temperaments with respect to gender has a non-negligible impact on the medians shown in the Table 10. Table 11 shows the gender distribution in the four temperaments, whose inhomogeneity confirms our hypothesis.

	Artisan	Guardian	Idealist	Rational
Female	25.00%	70.00%	64.29%	35.00%
Male	75.00%	30.00%	35.71%	65.00%

Table 11. Distribution of gender with respect to each temperament.

To understand to what extent, the gender-temperament combination affected the gender analysis, we propose again an analysis of the medians of the four temperaments with respect to gender. Let's begin again with $H(0,500,6,12)$ for each temperament. The medians are reported in Table 12. It can be observed that the medians of the female gender are much higher than the respective medians of the male gender. For the male gender, in this case, we still find that the highest median is manifested by Artisans and the lowest by Guardians, as was the case with the general evaluation. For women, on the other hand, the highest median is manifested by Guardians and the lowest median by Idealists. This result underlines that gender difference has a great impact on the temperament of the Guardians in the delay effect.

	Artisan	Guardian	Idealist	Rational
Female	18.12	28.40	1.83	7.83
Male	5.83	0.17	0.83	1.83

Table 12. $H(0,6,500,12)$ - distribution and median for each temperament by genders.

To see if the attitude of Guardians is uniform with respect to anomalies, we proceed with the magnitude effect. From Table 13, which shows the medians for each temperament according to the gender of $H(0, 6, 500, 12)$, it is possible to say that one cannot generalize what happens for the delay effect. In fact, although the magnitude effect is confirmed by the fact that the values of the medians of $H(0, 6, 50, 12)$ are for the most part greater than the medians of $H(0, 6, 500, 12)$, it is evident that the financial behavior of the genders is different. The lack of uniformity in the survey confirms that each anomaly is driven by different psychological principles.

	Artisan	Guardian	Idealist	Rational
Female	28.40	28.40	7.83	1.83
Male	66.50	0.17	1.83	6.50

Table 13. $H(0,6,50,12)$ - distribution and median for each temperament by genders.

The first thing to note is the relationship between the temperament of the Guardians and the Artisans. In fact, while the female gender of these two temperaments has the highest median, the male gender shows totally opposite values: it is highest for Artisans and lowest for Guardians. Moreover, the Rational female gender shows a median that is about four times lower than the respective male temperament. Therefore, women are not always more impatient than men. To analyse the financial behaviour with respect to the length of the intervals involved, we compare $H(0,6,50,12)$ with $H(0,1,50,1)$. In Table 14 we can observe that the temperament of the Artisans is confirmed for both genders with a greater impulsiveness evident in the high degree of decreasing in impatience.

	Artisan	Guardian	Idealist	Rational
Female	22.81	3.00	3.00	13.29
Male	32.33	1.00	33.33	3.35

Table 14. $H(0,1,50,1)$ - distribution and median for each temperament by genders

Particularly interesting are the Idealists and the Rationales: the former, of the male gender, show a decrease in impatience equal to that of the Artisans; the latter, of the female gender, suffer a very strong impact from this type of anomaly as can be seen by the wide variation ranging from 1.83 to 13.29. However, it is clear to all that the change in the degree of impatience occurs in the first period of the prospectus. The behavior of the male Guardians stands out: emphasizes the cautious nature of this temperament.

From Table 15 it can be seen how strong the impact of the sign effect is on Guardians, the only temperament that shows a lower median of $H(-500)$ than $H(500)$. When analyzing the data with respect to gender, however, those who suffer most from the payment are women of this temperament, a result also shown in Figure 10. In fact, women, much more than men, tend to postpone payment.

	Artisan	Guardian	Idealist	Rational
Female	66.50	3.83	4.83	7.83
Male	47.50	0	8.18	65.00

Table 15. $H(0,6,-500,12)$ - distribution and median for each temperament by genders.

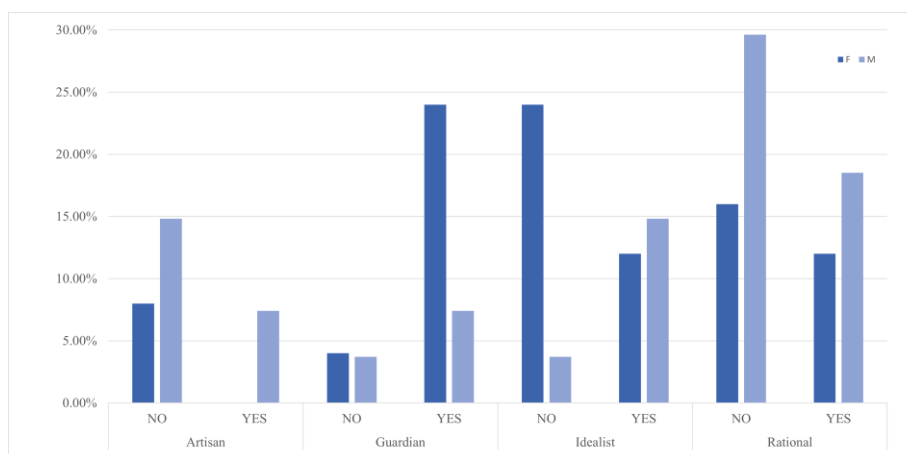


Figure 10. Distribution of individuals who defer payment divided by gender.

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To investigate how the overconfidence bias that characterized the Rationales in the general analysis in section 5 is distributed over gender, figure 6 has been enriched with the gender in Figure 11. It can be observed immediately that most of the Rationales who would pay more than they want to receive are about 60% male.

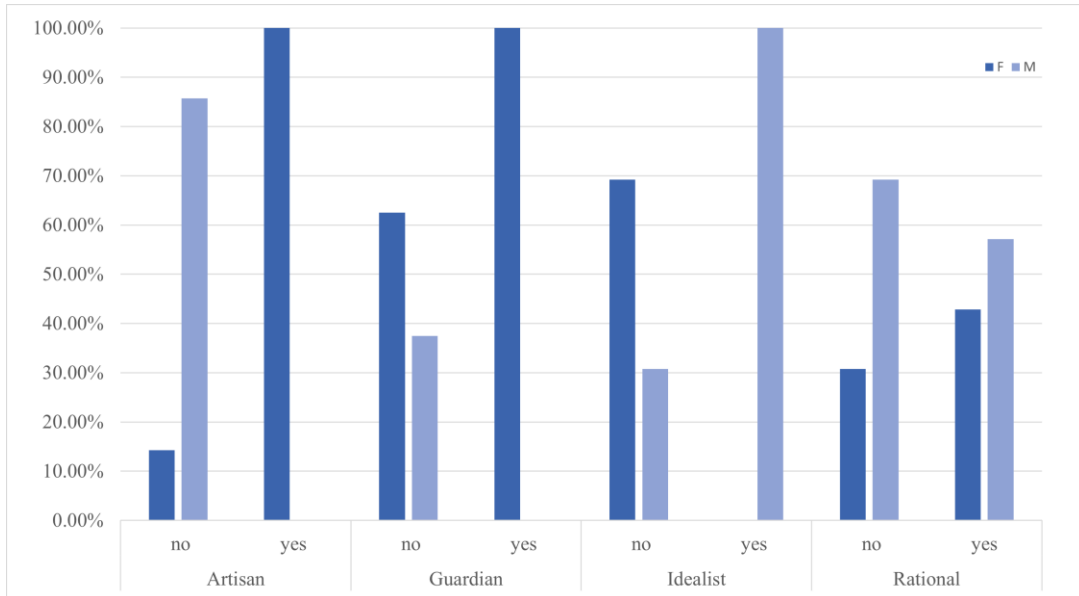


Figure 11. Distribution of individuals who would unconsciously increase the amount payable divided by gender.

Figure 12 and Figure 13 show the final structures of the female and male genders respectively.

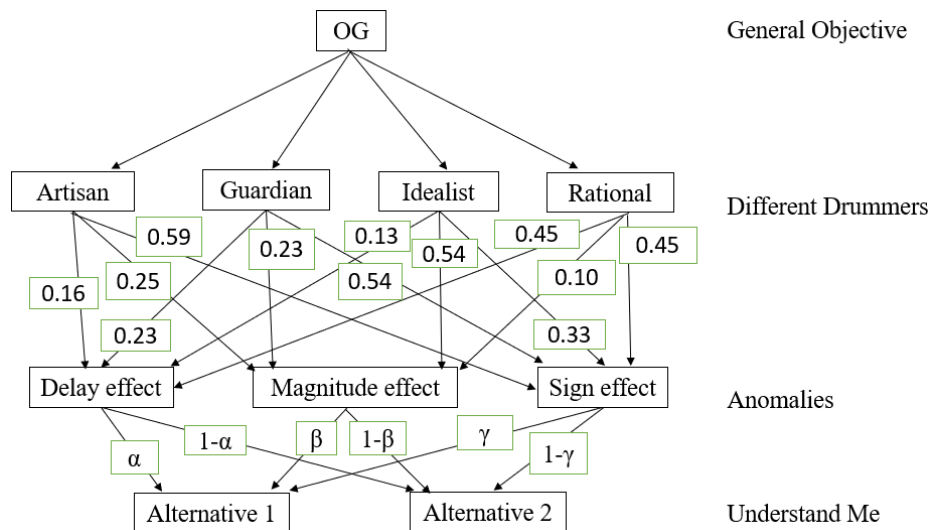


Figure 12. Final structure for female.

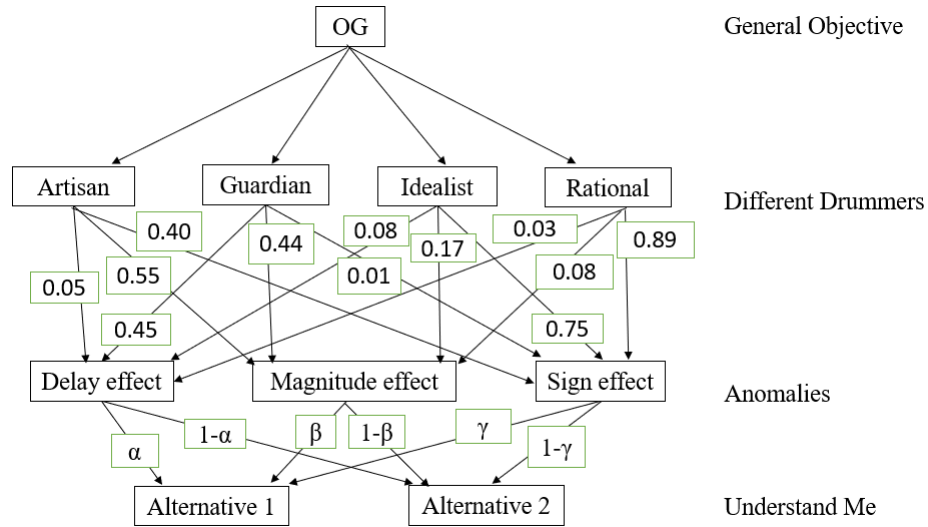


Figure 13. Final structure for male.

5. Discussion

This paper has applied AHP method in the context of intertemporal choices to decompose the economic behavior and to promote a better understanding of the dynamics of the decision-making. In particular, the applicability of the present research addresses to two necessary EU goals: customer profiling profiles and gender equality. The Directive 2014/65/EU (Art. 24 section 11 and Art. 25) emphasizes the need for information to understand whether a specific financial transaction meets the following objectives with respect to the client: matching the client's objectives and risk tolerance; a financial nature such that the client is able to bear the risks in relation to its investment objectives; a financial nature such that the client has the necessary knowledge to understand the transaction and the risks involved.

To address these needs, in this paper proposed a method to classify the investor by interweaving Keirsey's temperament theory and the concept of decreasing impatience in intertemporal choices. The anomalies of the Discounted Utility model studied in Ventre et al. (2022, b) are deepened and investigated from a behavioral point of view according to the temperament and gender of the respondents. The short section 2 on gender difference underlines the need to break down the decision-making process to understand the differences between men and women. In fact, to implement a strategy that aims at gender equality in any field and customer management, it is necessary to consider the diversity that characterizes men and women as a starting point. In section 3, the theoretical concepts necessary to understand the experimental framework were outlined, emphasizing that the decrease in impatience is a measure of the non-rationality of the decision-maker due to the emotional drives that develop during the evaluation and selection phase of an intertemporal prospectus. In this context, the theoretical results proven in Ventre et al. (2022, b) with respect to the interval effect, delay effect, magnitude effect and sign effect were recalled in section 3.2. After presenting the essential tools, the construction of the experiment is based on by the implementation of an Analytical Hierarchical Process. The mathematical structure of the AHP makes it possible to

decompose the decision-making process into all its elements and to explain their relationships in quantitative terms. Indeed, Figure 7 shows important differences between the four temperaments, confirming the need to design strategies that can adhere to the profile of the individual investor without ever generalizing their effectiveness. In fact, it immediately emerges that: the greatest weight obtained by AHP is shown by the Rationales with respect to the sign effect anomaly, with a value of 0.804; Guardians, have similar weights to the Rationales but with a greater influence of the magnitude effect; Guardians, on the other hand, have lower weights than the sign effect anomaly but are the temperament with the highest weight with respect to the magnitude effect; the Idealists, finally, is the temperament with the highest weight with respect to the delay effect.

The reported results can be further deepened and analyzed by considering not only the first behavioral trait but also the remaining ones. In fact, from the obvious differences between the temperaments we expect that a Rational with Guardian as second trait would behave differently towards the magnitude effect than a Rational with Artisan as second trait.

Gender analysis made it possible to further investigate the relationship between temperaments and anomalies. It emerges that there is not a better temperament or a more capable gender. In some temperaments, the male and female genders behave in almost opposite ways. For example, as can be seen for the Rationales with respect to the delay effect (0.45 for women and 0.03 for men), or for the Guardians with respect to the sign effect (0.54 for women and 0.01 for men). However, these results do not work against women. It is enough to look at the behavior of Artisans with respect to the magnitude effect (0.55 for men, 0.25 for women), or Idealists with respect to the sign effect (0.75 for men, 0.33 for women). The analysis presented and the quantification of differences in intertemporal preference can foster gender mainstreaming in the financial world through the inclusion of specific measures aimed toward women.

Finally, we stress that the structure of the AHP can be expanded with other non-negligible information, such as age and initial wealth. In this way it is possible to describe more and more defined classes of clients, describing the individual as the combination of multiple dimensions of identity and recalling the concept of intersections, allowing dominant and subordinate factors to be included at the same time (Atewologun et al., 2016)

With respect to the distribution of the data, the inhomogeneity of temperaments with respect to gender does not allow to draw definitive conclusions with respect to the combination of gender and temperament, but certainly highlights the importance of the subjective structures of the decision-maker during a choice. Moreover, during data collection, women were more reluctant to submit to the test, except for the temperament of the Guardians and Idealists. Therefore, in order not to distort the male-female distribution, it was difficult to obtain homogeneity in the temperaments. This attitude of refusal is certainly linked to a cultural gap, in which women feel less skilled and qualified in the field of finance than men. In this respect, various other factors such as age, social class and level of education play a role. Our work is an invitation to consider these elements to understand why it is necessary to define consumer classes. In fact, the analysis of the data did no more than experimentally confirm the requirements of the decree 2014/65/EU. Our work offers an original method that quantifies diversity and legitimizes anomaly.

6 Conclusions

The multidisciplinary approach used made it possible to conduct an original analysis on gender differences and consumers classification. Such attitudes have defined over the years a break between the theoretical model and actual actions of individuals. The basis of this discrepancy is the assumption of a rational investor, capable of making decisions always favoring her interests. Behavioral finance, which studies investor behavior in the financial market, describes "*special cases of deviation from rational action*" (Rubaltelli, 2006) through regular and systematic patterns of behavior for which the subject is rationally limited. Classical theories overlook the numerous distortions of individual decision-making arising from emotional drives and the complexity of the environment in which the choice is made. This is equivalent to state that non-rationality is part of human nature. Decision-makers do not always process information correctly and impartially and, as a result, make choices that worsen their condition.

From a practical point of view, consumer profiling operations have been introduced in European directives as written in Directive 2014/65/UE of the European Parliament and of the Council. The aim is to protect clients by designing financial products that correspond to individual profiles, client by client. To exploit the full potential of personalized advice, it is then necessary to know how to construct a strategic communication plan to improve client choices. The intersection between strategic customization and the understanding of gender differences is evident in different risk preferences and risk management. In fact, advisors consider the risk appetite index a key element for asset allocation modelling and the numerous empirical evidences presented in Section 2 underline the need to take into account the different risk perceptions between men and women in order to improve personalised finance services.

In addition, the analysis of gender differences using the intertemporal choice theory can also contribute to understanding important behavioral differences such as the propensity to depression, addictions and procrastination mechanisms. In fact, Prelec (2004) also proved that irrational degeneration of definition 3 and definition 4 lead to attitudes of procrastination and promotion of consumption, described in Figure 14 and Figure 15.

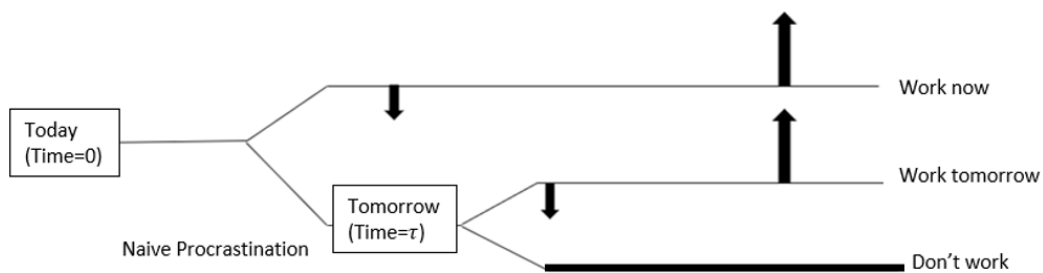


Figure 14. Procrastination: the procrastinating Decision-Maker prefers to do the work immediately rather than not do it at all but keep putting it off (Prelec 2004, p.221).

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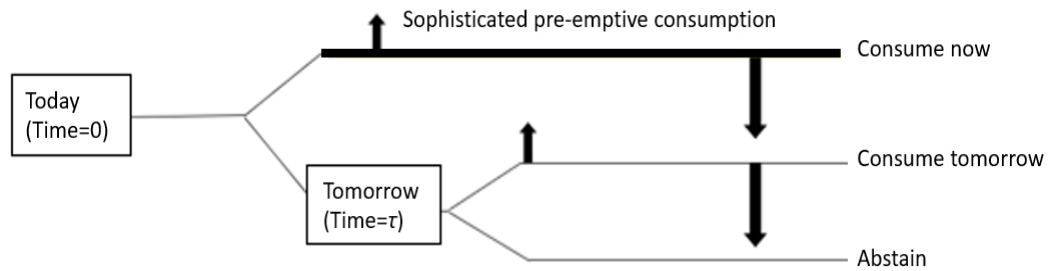


Figure 15. Pre-emptive consumption: the Decision-Maker anticipates the choice by deciding to consume today despite this being a less preferred option than abstaining (Prelec 2004, p.221).

The work presented can provide methodological support for studies that aim to understand gender differences in topics such as suicidal ideation, stress, alcohol use and depression (e.g., Rich et al., 1992). Understanding the differences that exist between individuals and accepting that each identifying element (such as gender, age, culture...) has a non-negligible impact on decision-making are necessary research processes to then to nudge the individual toward a better option using the same cognitive boundaries (Thaler and Sunstein, 2014). Nudges are intended to influence social behavior without significantly limiting freedom of choice. Various studies have shown that these interventions have positive results (Pe'er et al., 2019). Nudging has emerged as an efficient technique to improve the quality of individuals' choices to address this problem. The nudge technique is based on using behavioral biases to the individual's advantage through a variation in the decision-making environment. In this way, the decision-maker is nudged toward the best option without denying the freedom to choose the other alternatives. Such an intervention has been studied and implemented by Rubaltelli and Lotto (2021) to increase the pension funds of freelancers. To improve the nudging technique, two important observations must be considered. First, individuals' response to pushing can vary from subject to subject based on age, gender, and character; second, change necessarily involves an alteration in one's state. Therefore, a determining factor in optimizing interventions of this type is an analysis of the tendency to maintain one's status. In this regard, Gal (2006) refers to a "*psychological law of inertia*" to indicate, in the context of loss aversion, the tendency of individuals not to alter their habits. So, to optimize choice architecture, we need to move towards customized interventions. Describing anomalous attitudes in terms of inconsistency among decision-making processes involving choices defined over a short or long period allowed us to quantify the inconsistency of individuals' choices. The direct consequence is the possibility of being able to quantify how much behavioral bias weighs in individual decision-making. Decision theory makes it possible to implement a personalized choice architecture. Choice architecture plays the key role in implementing nudging. Indeed, if decision-makers are influential because they are rationally constrained, the decision-making environment can be organized to influence toward the best individual option. The methodology proposed in the present paper could be a framework to help the selection of the best choice architecture because behavioral differences show the necessity to diversify the strategies. The use of AHP allows for a construction of the decision context that fits the individual by modeling the structuring of options and the description of options based on the cognitive mechanisms involved. Our approach can be expanded in three different ways. First, one can consider sixteen temperaments instead of four. Keirsey divided the four temperaments into two categories, each with two variants by relating each group to

the sixteen types of the MBTI (Myers, 1980). This makes it possible to study the relationship between anomalies and personality in more details. The second development of our approach requires the inclusion within the structure of other levels that contain increasingly personal information such as age. Other anomalies in the discounted utility model such as delay-anticipation asymmetry (Loewenstein, 1988). and preference for sequences of outcomes (Loewenstein and Sicherman, 1991; Chapman, 2000) can also be included, too. Anticipation-delay asymmetry consists of the phenomenon whereby the discount function is steeper if an outcome is postponed than if it is anticipated. Loewenstein (1988) interpreted this anomaly through loss aversion, and it can therefore be quantified in terms of variation in the degree of impatience. Sequence preference indicates the situation in which a negative discounting of time prevails and is a phenomenon related to various psychological mechanisms. It would be interesting to investigate how the degree of decrease in impatience varies in these contexts. The third possible development is to deepen the analysis by studying relationships as a function of non-dominant traits. We believe that non-dominant traits also have a significant weight and this confirms the need to refer to a structure that can encompass all the nuances of the decision-maker.

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On the structure of composite odd integers and prime numbers

Giuseppe Buffoni*

Abstract

With the expression “structure of an odd integer” we mean the set of properties of the integer n which specifies the odd integer $2n + 1$ and brings about its behaviour. These properties of n , for both composite and prime numbers, are expounded in detail, together with their geometrical implications. In this context, a set, in a two dimensional space, where all the composite odd integers in $[2a + 1, 2n + 1]$ are localized, is illustrated.

Keywords: couples of divisors; no divisors.

2020 AMS subject classifications: 11A00, 11Y00.¹

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1 Introduction

Let $2n + 1$ be any composite odd integer, and $2i + 1, 2j + 1$ a couple of its divisors. In [1] it is pointed out that

$$2n + 1 = (2i + 1)(2j + 1) \iff n = k_{ij}, \quad (1)$$

where k_{ij} is the following symmetric form in i and j

$$k_{ij} = i + j + 2ij, \quad (i, j) \in \mathbb{N}. \quad (2)$$

Let $\mathbb{K} = \{k_{ij} | (i, j) \in \mathbb{N}\} \subset \mathbb{N}$. Since any odd integer is either a composite or a prime number, it follows that

$n \in \mathbb{K} \iff 2n + 1$ is a composite number, $n \in \mathbb{N} \setminus \mathbb{K} \iff 2n + 1$ is a prime number.

Some properties of the matrix $\{k_{ij}\}$ of finite order are reported in the appendix 1 of [1], where the relation $n = k_{ij}$ has been used to identify composite odd integers $2\nu + 1$, with $4 = k_{11} \leq \nu \leq n$, and having the same number of couples of divisors. The distributions of odd integers $\leq 2n + 1$ vs. the number of their couple of divisors have been computed up to $n \simeq 5 \cdot 10^7$.

The motivation for this work is just the reference [1], unique reference for this paper, and very useful for the analysis of the problem under study.

In section 2 we enlighten the basic properties of n for both composite odd numbers and primes. In section 3 their main geometrical implications are illustrated. In particular, the relation $n = k_{ij}$ for composite odd integers suggested to investigate the localization of the pairs (i, j) producing k_{ij} in a given interval. Thus, a set $\Omega(a, n)$ of points $(x, y) \in \mathbb{R}^2$ containing all the points with integer coordinates (i, j) such that $a \leq k_{ij} \leq n$ is defined.

2 The basic properties of n specifying the behaviour of $2n + 1$

For a pair (i, j) , identifying any possible couple of divisors of an odd integer $2n + 1$, we should have

$$n = k_{ij}, \quad (3)$$

where k_{ij} is defined in (2). Furthermore, any couple $(2i + 1, 2j + 1)$ of divisors of $2n + 1$, with $i \leq j$, must satisfy the following inequality

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$$2i + 1 \leq \sqrt{2n + 1} \leq 2j + 1 \quad (\text{equality holds iff } i = j).$$

It follows that

$$1 \leq i \leq I_n = \frac{1}{2}(-1 + \sqrt{2n + 1}) \leq j. \quad (4)$$

Let $\mathbb{I}_n = \{1, 2, \dots, i_n\}$, with $i_n = [I_n]$ =integer part of I_n .

The implications (1) state that the special form (2) of n is a necessary and sufficient condition in order for any odd integer $2n + 1$ to be a composite number. Moreover, if $2n + 1$ is a prime, then n cannot be put in the form (2). In the following theorem, given a generic n , necessary and sufficient conditions are given for either $n \in \mathbb{K}$ or $n \in \mathbb{N} \setminus \mathbb{K}$.

Theorem 2.1. *Let $n \in \mathbb{N}$. Then*

$$n \in \mathbb{K} \quad \text{iff} \quad \exists i \in \mathbb{I}_n : (2i + 1) \mid (n - i), \quad (5)$$

$$n \in \mathbb{N} \setminus \mathbb{K} \quad \text{iff} \quad \forall i \in \mathbb{I}_n : (2i + 1) \nmid (n - i). \quad (6)$$

Proof. A non implicit relation between i and j is obtained by making explicit the variable j in (3). In doing so, (3) is written in the form of a real homographic function of the integer variable i

$$y = \phi_n(i) = \frac{n - i}{2i + 1}, \quad i \in \mathbb{I}_n, \quad y \in \mathbb{R}. \quad (7)$$

If for some i we have $y \in \mathbb{N}$, then the pair (i, j) with $j = y = [y]$ identifies a couple of divisors of $2n + 1$. Thus, in this way we can identify all the couples of divisors of $2n + 1$, if they exist. The theses follow directly from (7). □

The i_n conditions assumed to have $n \in \mathbb{N} \setminus \mathbb{K}$ are not superabundant. Consequences of theorem 1 are the following two propositions.

If $n \in \mathbb{K}$, then n has a number of representations $n = k_{ij}$ just equal to the number of couples of divisors of $2n + 1$.

If $n \in \mathbb{N} \setminus \mathbb{K}$, then n can be expressed as

$$n = i + (2i + 1)q_i + r_i = k_{iq_i} + r_i,$$

with $q_i, r_i \in \mathbb{N}$ and $1 \leq r_i < 2i + 1, \forall i = 1, 2, \dots, i_n$.

Remark 2.1. (r_1) $\phi_n(i)$ is decreasing with i from $\phi_n(1) = (n - 1)/3$ to $\phi_n(i_n)$.

(r_2) Equation $y = \phi_n(i)$ is a simple algorithm to compute the couples of divisors of an integer $2n + 1$, if they exist. It can also be used as a primality test. The order of the number of operations is $\sqrt{n/2}$.

(r_3) By direct calculation it follows that

$$2n + 1 = 2k_{ij} + 1 = (i + j + 1)^2 - (j - i)^2.$$

This identity shows the well known fact that any composite odd integer may be written as a difference of two squares. Possibly in more than two ways, while for a prime only holds the decomposition $2n + 1 = (n + 1)^2 - n^2$.

(r_4) Let m_q be the sequence of integers defined recursively by

$$m_q = 2m_{q-1} + 1, \quad q = 1, 2, \dots, \quad m_0 = 0,$$

which implies that $m_q = 2^q - 1$. When q is a prime, m_q is a Mersenne number. Let $q = 2r$. Since $m_{2r} = 2^{2r} - 1 = (2^r + 1)(2^r - 1)$ is composite, we have that $m_{2r-1} \in \mathbb{K}, r = 2, 3, \dots$

3 Geometrical implications of the properties of n

The relation $n = k_{ij}$ can be rewritten as

$$n = i(j + 1) + (i + 1)j, \tag{8}$$

which can be interpreted as the determinant of the matrix

$$A = \begin{pmatrix} i & -j \\ i+1 & j+1 \end{pmatrix}.$$

Since $\det(A) = n$, n is the area of the parallelogram with the column vectors

$$u = (i, j + 1)^T, \quad v = (-j, j + 1)^T,$$

as two of its sides. This property holds $\forall (i, j)$ such that $k_{ij} = n$. In addition to the equality of determinants, no other relationships have been recognized between the matrices A related to the pairs (i, j) with $k_{ij} = n$.

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Moreover (8) shows that n is also the area of the union of the two rectangles of sides $i, (j + 1)$ and $(i + 1), j$.

Imagine that you are drawing, on a squared sheet, rectangles composed by an odd integer number of squares. The side of the squares is assumed as linear unit of measure, so that the rectangles have integer sides and area. Obviously, only integers $2n + 1$ with n satisfying (8) can represent rectangles. The number $2n + 1$ is the area of the rectangle of sides $(2i + 1), (2j + 1)$. Moreover, with the help of (8), we can show that this rectangle is the union of five rectangles: two with sides $i, j + 1$, two with sides $i + 1, j$, plus one unit square (Fig. 1).

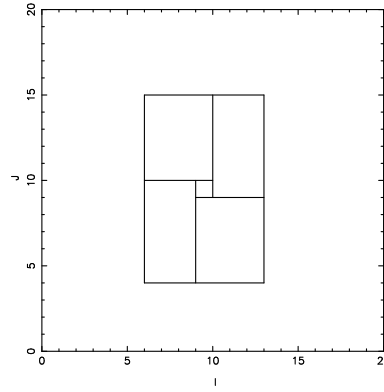


Figure 1: Decomposition of the rectangle of sides $2i + 1, 2j + 1$. $i = 3, j = 5$, $n = 3 \cdot (5 + 1) + (3 + 1) \cdot 5 = 38$, $2n + 1 = 2 \cdot 38 + 1 = 77$, $(2i + 1) \cdot (2j + 1) = 7 \cdot 11 = 77$

Thus we have shown that, when $n \in \mathbb{K}$, then n represents also the area of the union of two rectangles, and $2n + 1$ both the area of one rectangle and the union of five rectangles.

When $n \notin \mathbb{K}$, then n can be represented as $n = k_{iq_i} + r_i$, with $r_i \geq 1$, which leads to the following representation for $2n + 1$

$$2n + 1 = 2k_{iq_i} + 1 + 2r_i = (2i + 1)(2q_i + 1) + 2r_i, \quad 1 \leq r_i < 2i + 1.$$

Since $n \notin \mathbb{K}$, this expression can not be reduced to the product of two odd integers.

Let us now consider the problem of the localization of composite odd integers $2\nu + 1$, with $4 \leq a \leq \nu \leq n$. The discrete relation $n = k_{ij}$ is viewed in the continuous form as $n = \kappa(x, y) = x + y + 2xy$, $(x, y) \in \mathbb{R}^2$, and in explicit form it is written as

$$y = \phi_n(x) = \frac{n - x}{2x + 1}, \quad 1 \leq x \leq I_n. \quad (9)$$

$\phi_n(x)$ is decreasing with x from $\phi_n(1) = (n - 1)/3$ to $\phi_n(I_n) = I_n$.

Let the set $\Omega(a, n)$ be defined as

$$\Omega(a, n) = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq I_n, (x \leq y) \vee (a \leq \kappa(x, y) \leq n)\}. \quad (10)$$

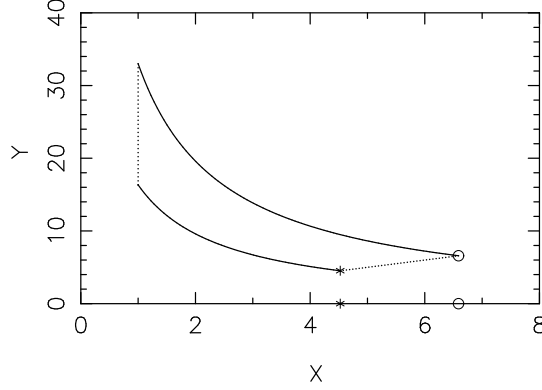


Figure 2: Set $\Omega(a, n)$: $n = 100$, $a = 50$. Continuous lines: $y = \phi_n(x)$ and $y = \phi_a(x)$; dotted lines: $x = 1$ and $y = x$; asterisk: points $(I_a, 0)$, (I_a, I_a) ; circle: points $(I_n, 0)$, (I_n, I_n) . Different scales for x and y .

It is a bounded, closed and convex set in a plane. From its implicit definition (10), it follows that $\Omega(a, n)$ (Fig. 2) can be defined explicitly by the following inequalities as the union of two sets

$$1 \leq x \leq I_a, \quad \phi_a(x) \leq y \leq \phi_n(x), \quad (11)$$

$$I_a < x \leq I_n, \quad x \leq y \leq \phi_n(x). \quad (12)$$

Both $\phi_a(x)$ and $\phi_n(x)$ are decreasing with x , and also their difference

$$\Delta_{n,a}(x) = \phi_n(x) - \phi_a(x) = \frac{n - a}{2x + 1}, \quad 1 \leq x \leq I_a. \quad (13)$$

Thus, when $n - a$ is small enough, we have that $\Delta_{n,a}(x) < 1$, and $\Omega(a, n)$ may contain at the most one integer point. Other details on this argument can be found in appendix.

4 Concluding remarks

It is worthy to draw a concluding remark. In other words, Theorem 2.1 states that any prime has just one parent: if a generic n satisfies (6) in Theorem 2.1, then it is the parent of $2n + 1$. Otherwise it is a composite number.

References

- [1] G. Buffoni. On odd integers and their couples of divisors. *Ratio Mathematica*, 40: 87-111, 2021.

Appendix

Consider the limit case $a = 4$ (Fig. 3, left), so that $I_a = 1$, $\phi_4(1) = 1$. Thus, the first set (11) consists of only one point $x = I_a = 1$. Since $\phi_a(I_a) = I_a$, (11) and (12) reduce to

$$1 \leq x \leq I_n, \quad x \leq y \leq \phi_n(x).$$

Consider the case a large, i.e. $n - a$ small (Fig. 3, right). Assume $n - a \ll a$. The difference $I_n - I_a$ may be written as

$$I_n - I_a = 0.5 \sqrt{2n+1} \left(1 - \sqrt{\frac{2a+1}{2n+1}}\right) = 0.5 \sqrt{2n+1} \left(1 - \sqrt{1 - \frac{2(n-a)}{2n+1}}\right).$$

From the assumption $n - a \ll a$ we have $2(n - a)/(2n + 1) \ll 1$, so that

$$I_n - I_a \simeq \frac{n - a}{2\sqrt{2n+1}} < 1. \quad (14)$$

Thus, the second set (12) reduces to a very small interval.

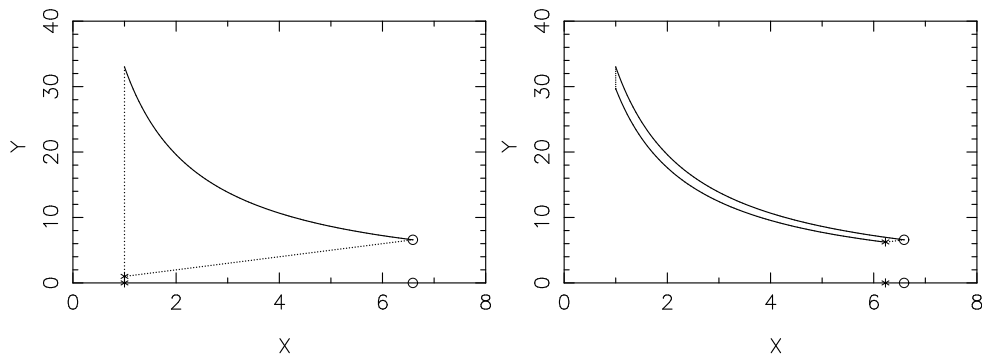


Figure 3: Set $\Omega(a, n)$: $n = 100$. Left: limit case $a = 4$; right: case $a = 90$, so that $I_n - I_a < 1$. Notes as in caption of Fig. 2

Consider now the set of points with integer coordinates. The set $\Omega^*(a, n)$ of the pairs (i, j) , such that $1 \leq i \leq j$, $a \leq k_{ij} \leq n$, defined by

$$\Omega^*(a, n) = \{(i, j) \in \mathbb{N} \mid i = 1, 2, \dots, I_n, (i \leq j) \vee (a \leq k_{ij} \leq n)\}, \quad (15)$$

is the set of the points $(x, y) \in \Omega(a, n)$ with x and y integer coordinates. Afterwards, it is straightforward to formulate the explicit definition of the pairs $(i, j) \in \Omega^*(a, n)$. Then, $\Omega^*(a, n)$ is again defined as the union of two sets

$$i = 1, 2, \dots, [I_a], \quad j = J(i), J(i) + 1, \dots, [\phi_n(i)], \quad (16)$$

where either $J(i) = [\phi_a(i)] + 1$ when $\phi_a(i) \notin \mathbb{N}$ or $J(i) = \phi_a(i)$ when $\phi_a(i) \in \mathbb{N}$, and

$$i = [I_a] + 1, \dots, [I_n], \quad j = i, i + 1, \dots, [\phi_n(i)]. \quad (17)$$

When $b - a$ is small enough, it follows that $I_n - I_a < 1$, so that either $[I_n] = [I_a]$ or $[I_n] = [I_a] + 1$. The second set (17) is missing when $[I_n] = [I_a]$, and consists of only one point when $[I_n] = [I_a] + 1$; thus (16) and (17) reduce to

$$i = 1, 2, \dots, [I_n], \quad j = J(i), J(i) + 1, \dots, [\phi_n(i)]. \quad (18)$$

If $\nu \in \{a, a + 1, \dots, n\} \setminus \{k_{ij} \mid (i, j) \in \Omega^*(a, n)\}$, then $2\nu + 1$ is a prime.

Predictions on the impact of the budget cap on Formula 1 teams' performance

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Abstract

This study aims at quantifying the impact of the budget on the performance of Formula 1 teams. Until recently, the budgets of Formula 1 teams varied; thus, competitive advantage was provided to those that consistently had greater funds than others. Understanding how dominant the effect of the budget is in the performance of a team will provide significant findings since a cost cap that aims at balancing the financial field, has been recently introduced. Prediction models regarding the team that performed better on a relatively lower budget will provide insights regarding which of them may thrive in the future.

Keywords: Formula 1; Cost cap; Performance; Efficiency; Drivers' effect; Regression models

Abbreviations

RP: Relative performance

RB: Relative budget

DR: Drivers' effect

EF: Efficiency ratio

2010 AMS subject classification: 91-00 General reference works (handbooks, dictionaries, bibliographies, etc.) pertaining to game theory, economics, and finance.[‡]

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1. Introduction

Formula One (F1) is an international motorsport that is considered the most competitive in single-seat motor racing. It is the pinnacle of innovation in the automotive industry due to its radical technological advances, research and development (R&D) programs, drivers' capabilities and teams' expertise. Until 2021, no financial restrictions on the expenditures of F1 teams were imposed by the governing association of the sport (Fédération Internationale de l'Automobile, FIA). FIA proposed the implementation of a \$70 million cost cap in 2010 to tackle the ramifications of the 2007-2009 economic crisis as three (3) constructors had withdrawn from the championship and the viability of F1 was at risk [Henderson et al., 2010]. Nonetheless, the endeavor did not thrive since the cap was considered very low and the subsequent F1 teams did not acquiesce to its application [Judde et al., 2013]. Instead, a Resource Restriction Agreement was established but it was not intended to operate as a budget cap [Straw, 2022]. For the first time in its history, the organizational structures and the F1 teams have agreed on implementing a budget cap beginning in 2021. The cap for 2021 and 2022 was set at \$145 million and \$140 million, respectively, and it is expected to be further reduced by \$5 million in 2023 [FIA, 2021].

According to FIA [2021], the purpose of the budget cap enforcement is threefold:

1. Enhance the sporting fairness of the championship;
2. Promote competition and equal opportunities for all the participating teams; and
3. Ensure the long-term financial sustainability of the F1 teams.

The two former issues pertain to the competitive balance that each sport (or motorsport) should feature to a certain degree. Competitive balance is key to the success of sports because the outcome becomes more unpredictable. Unpredictability is an important factor that drives spectators into a sport and makes it more popular [Rottenberg, 1956]. Formula 1 is a hybrid motorsport in which the performance of the team and the driver determine the outcome. Throughout its history, teams with higher budgets were able to dedicate more funds to R&D, development of structural components, designing new performance packages, or hiring skillful professionals compared to the teams with lower budgets. This resulted in an unbalanced field in which the outcome of the championship was (more or less) predetermined. Nonetheless, the recently introduced budget cap will significantly affect the top teams and allow the rest of the field to be competitive (at least in financial terms).

An interesting feature that will potentially allow greater competition is that the cost cap is representative of the initial budget of a midfield team. Therefore, it aims to allow less wealthy firms to operate without significant changes and make those with higher budgets significantly reduce their expenditures. In recent years, three teams have (reportedly) significantly more funds than the others:

1. Scuderia Ferrari F1
2. Mercedes AMG
3. Red Bull Racing.

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Therefore, the implications of the cap will primarily affect these three constructors. Nonetheless, the aforementioned teams will certainly experience a short-term advantage due to their existing facilities, staff and accumulated knowledge.

The budget cap involves all the operations of the F1 teams with some exceptions that are known as excluded costs. Those are reported in detail in the F1 Financial regulations [FIA, 2021] and the most prominent are summarized in **Table 1**. The budget cap's exemptions prove that its purpose is to impose financial restrictions that are solely associated with poor performance. It does not intend to affect the overall operations of an F1 team that are not associated with racing. A team can still spend any amount of money on non-F1 activities and promote its brand through marketing campaigns etc.

The present study aims at quantifying the historical relationship between the expenditures of the teams and the corresponding results in order to understand the actual impact of the budget on performance. The studied period will be 20 years, namely 2000-2019, and the accumulated data will refer to all the teams that participate in the current championship (2022). The initial goal of the study is to observe whether teams with higher budgets indeed performed better (as expected) than teams with lower budgets. However, a team's overall performance is also dependent on its staff efficiency and, eventually, on the performance of the drivers. Therefore, it is crucial to detect which of the teams participating in the championship were efficient, i.e., they achieved better results than teams with similar or even higher budgets. These teams are likely to thrive in the new era in which the budget cap will counterbalance the financial classes. The performance is assessed in terms of achievements in the sport and their budgets are derived via financial data.

F1 budget cap exemptions	
Travel and accommodation expenses	Marketing endeavors
F1 driver salaries	Property costs
Heritage activities	Corporate legal and taxation fees
Salaries of the highest-paid three (3) employees	All employees' bonuses and termination benefits
Depreciation or amortization losses due to revaluation of assets	Payables to FIA for entering the championship (new team) or for a driver's super license
Non-F1 activities	Any financial penalties
Power units' development, production and acquisition	Entertainment expenses for employees (max. expense \$1 million)
Human resources activities	Exchange rates loss or gains

Table 1. The most important F1 budget cap exemptions (FIA, 2021)

2. F1 organizational framework and teams

The F1 championship takes place under the auspices of the FIA. The entire organization of F1 comprises two additional parts, the Formula One Group, the group that has the commercial rights of the sport and controls various of its aspects (management, licenses, marketing, broadcasting, etc.) and the F1 teams. The Formula One Group was purchased by Liberty Media Corporation in 2017 for the amount of \$4.4 billion [BBC, 2016]. It is operated by the Formula One Management (FOM) which mainly controls the executive operations of the group. FOM also organizes the logistics, the traveling of staff and transport of equipment from one Grand Prix to the next.

There are currently ten (10) F1 teams that compete in the drivers and the constructors' championships which are listed below:

1. *Scuderia Ferrari (Ferrari)*
2. *Mercedes-AMG F1 (Mercedes)*
3. *Red Bull Racing (Red Bull)*
4. *McLaren Racing (McLaren)*
5. *Williams Racing (Williams)*
6. *Alpine BWT F1 (Alpine)*
7. *Alfa Romeo F1 (Alfa Romeo)*
8. *Scuderia Alpha Tauri (Alpha Tauri)*
9. *Aston Martin (Aston Martin)*
10. *Haas (Haas)*

The parentheses include the abbreviations that will be used for naming each team for the ease of the present study.

3. Data and methods

3.1 Methodology

Based on the theory of production functions [Heathfield, 1976], Duggal [2020] suggests that the performance of F1 teams (accumulation of points in the championship) is defined as a function of their budget, drivers' effect and team's fixed-effect. The latter will be referred to as the team's efficiency from now on. Therefore:

$$P = f(B, DR, EF) \quad (1)$$

Where P is the performance of each team and relates to the total points accumulated within a season, B is the annual budget, DR is the effect of the team's drivers and EF is the efficiency of the team.

To identify whether the performance of F1 teams can be approximated by the theoretical background, two regression models will be established for the corresponding data between 2000-2019. A hypothesis test on whether the budgets and performance are correlated will be conducted. This aims at establishing a direct correlation between performance and budgets. The analysis of variance (ANOVA) will be performed using the F-test to derive whether there is significant evidence that performance is impacted by

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the budgets. If this holds, a linear regression model between performance and budgets will be established:

$$P_{Y,j} = a + b \times B_{Y,j} + \varepsilon_{Y,j} \geq 0 \quad (2)$$

Where $P_{Y,j}$ is the performance of the team j at season Y , a is the performance intercept, $B_{Y,j}$ is the budget of the team j during season Y , b is the coefficient of a team's budget and $\varepsilon_{Y,j}$ is the residual of the team j in season Y .

Afterwards, a multivariable regression model that will account for the drivers' effect will also be employed and Equation [2] will be transformed into:

$$P_{Y,j} = a + b_1 \times B_{Y,j} + b_2 \times DR_{Y,j} + \varepsilon_{Y,j} \geq 0 \quad (3)$$

Where $DR_{Y,j}$ is the ranking of the drivers employed in team j during season Y , b_1 is the coefficient of a team's budget and b_2 is the coefficient of drivers' effect.

The 2nd model will provide more accurate results in terms of pure performance by incorporating the drivers' effect. A residual analysis based on the results of this model will be performed to assess the efficiency of F1 teams and construct a final prediction model accounting for the three (3) independent variables (B, DR and EF). Note that if a team's residuals are constantly positive, it proves that it consistently performed better than the average team with a similar budget and drivers' rating and vice versa.

3.2 Historical budgets and monetary inflation

To establish the regression models, data on the budgets of each team per season must be accumulated. One would suggest that obtaining this data would be a straightforward procedure. On the contrary, this does not hold. Unlike other sport organizations, F1 teams do not reveal their financial data and frequently disclose information [Sylt, 2020]. Herein, numerous information from journal papers, magazines, online sources and hardcopies were used to determine the budget of each team per year. In order to be comparable, the budgets have to be adjusted for the inflation rates during the studied period. This adjustment was made using the Consumer Price Index (CPI) which refers to the weighted average change of prices for a basket of goods and services [data from the U.S. Department of Labor Bureau of Labor Statistics, 2022] and is a measure of the inflation rate. The year 2020 was selected as reference. It is assumed that the utilization of annual inflations for the US economy is justified, as the budgets and expenses in F1 are expressed in USD. The adjusted budgets are presented in **Table 2** and depicted in **Figure 1**.

The absence of budget values in certain seasons observed in **Figure 1** and **Table 2** indicates that the corresponding team did not exist (at least in its current form) at the time. For example, Mercedes entered the championship in 2010; therefore no data before that year is recorded. It is pointed out that some teams have undergone minor changes (including re-branding) throughout the years but the organization remains practically the same, therefore, it is considered a cohesive unit for the purposes of the present study. For example, Renault F1 implemented certain organizational structure alterations in 2021 and changed its name to Alpine aiming to re-brand the team. Apart from some substitutions in staff and subsequent minor changes, the organization remained the same therefore, there is no incentive to consider Alpine a different team than Renault.

Team budgets (million \$) adjusted for inflation rates (Reference year: 2020)																					
F1 team/year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	Mean
Ferrari	360	415	639	589	584	576	521	508	498	387	377	299	448	444	469	456	402	417	422	439	462
Mercedes	-	-	-	-	-	-	-	-	-	-	291	247	287	327	436	510	388	423	412	429	375
Red Bull	-	-	-	-	-	185	257	315	198	212	236	236	422	383	491	511	315	294	319	338	314
McLaren	338	402	438	506	492	559	515	503	520	466	346	247	287	311	387	507	271	239	227	253	390
Williams	215	282	508	506	486	479	251	245	200	182	141	144	161	167	213	203	154	164	155	152	250
Alfa-Romeo	108	120	173	219	200	214	454	444	440	345	118	104	161	155	120	112	139	144	139	157	203
Alpine	138	263	298	360	353	382	415	405	473	357	165	184	233	250	207	152	190	205	196	212	272
Alpha Tauri	-	-	-	-	-	-	84	100	154	157	165	121	125	94	196	150	146	150	155	157	140
Aston Martin	-	-	-	-	-	-	-	-	146	133	126	138	179	111	153	142	132	133	124	157	139
Haas	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	146	137	134	152	142
<u>Seasons 2000-2001: Business F1, March and April 2003, Credit Suisse Formula 1 Office (hardcopy)</u> <u>Season 2002: F1 Racing Magazine 2003a (hardcopy)</u> <u>Season 2003: F1 Racing Magazine 2003b (hardcopy)</u> <u>Season 2004: F1 Racing Magazine 2004 (hardcopy)</u> <u>Season 2005: F1 Racing Magazine 2005 (hardcopy)</u> <u>Season 2006: F1 Racing Magazine 2006 (hardcopy)</u> <u>Season 2007: Wood (2008) in Bleacherreport.com</u> <u>Season 2008: Collantine (2008) in Racefans.net</u> <u>Season 2009: Forbes.com (2010)</u> <u>Season 2010: Autosport.com (2010)</u>											<u>Season 2011: Forbes.com (2012)</u> <u>Season 2012: Boxall-Legge (2019) in Autosport.com</u> <u>Season 2013: Forbes.com (2014)</u> <u>Season 2014: Forbes.com (2015)</u> <u>Season 2015: Mourão (2017)</u> <u>Season 2016: Rencken (2016) in Autosport.com</u> <u>Season 2017: Cooper (2018) in Motorsport.com & Rencken (2017) in Autosport.com</u> <u>Season 2018: Rencken (2018a) in Racefans.net & Rencken (2018b) in Racefans.net</u> <u>Season 2019: Rencken (2019) & Rencken (2020) in Racefans.net</u>										

Table 2. Budgets for the ten (10) F1 teams participating in the 2022 championship during 2000-2019 adjusted for inflation in 2020 (reference year).

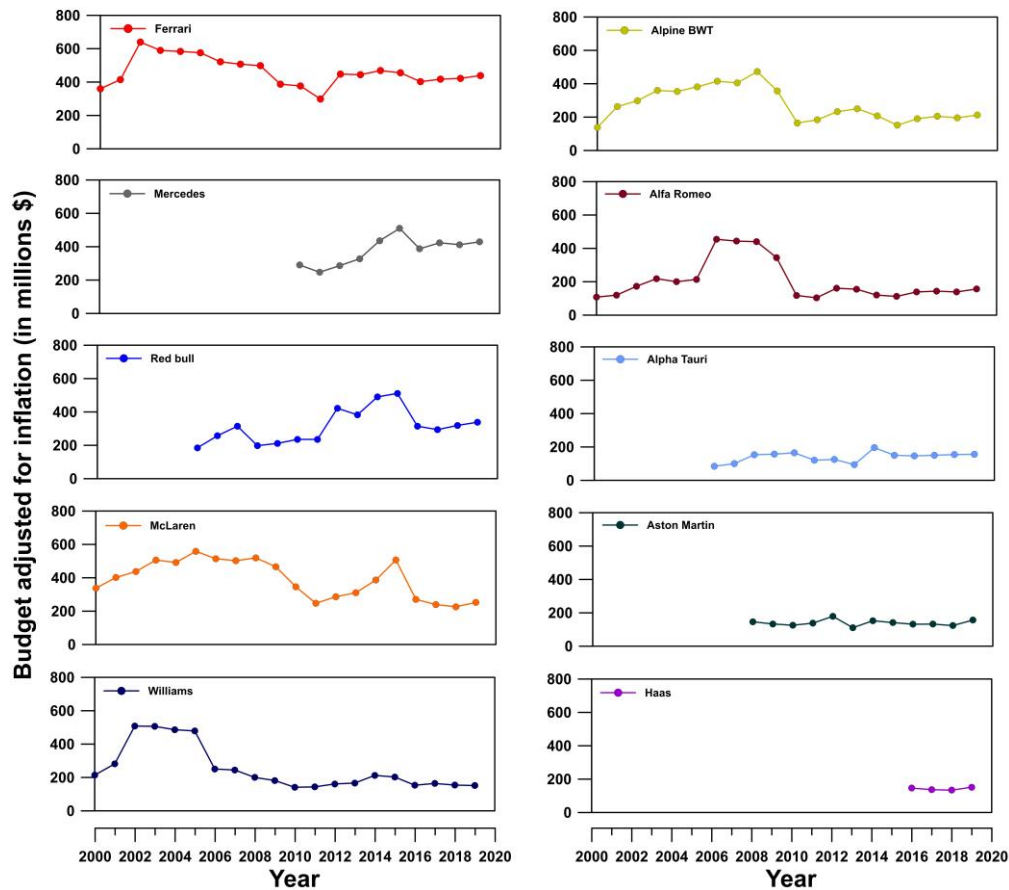


Figure 1. The budgets (adjusted for 2020 cumulative inflation rates) of each team participating in the 2022 championship from 2000-2019.

According to the data provided in **Table 2**, the teams can be classified based on their average adjusted annual budget as follows:

1. Ferrari;
2. McLaren;
3. Mercedes;
4. Red Bull;
5. Alpine;
6. Williams;
7. Alfa Romeo;
8. Alpha Tauri;
9. Aston Martin;
10. Haas

Nonetheless, the effect of the budget on the performance of an F1 team is a relative term meaning that a high or a low budget is always dependent on the corresponding budgets of subsequent teams. Hence, it was decided to use a relative measure of the teams' budget (RB), i.e.:

$$Relative\ Budget_{j,Y}(RB) = Budget_{j,Y} - Min(Budget_Y) \quad (4)$$

Where j represents the team and Y the season.

3.3 Performance evaluation

The performance of each F1 team is purely evaluated by its racing results, i.e., the collected points at the end of the season. The performance assessment would be a straightforward task if the reward system was constant throughout the period of interest (2000-2019). However, the point system has changed several times over the past two decades. In particular, the systems used between 2000-2002, 2003-2009, 2010-2018 and 2019 are different. Moreover, the number of races held each season has changed over the years. To overcome these difficulties in comparing the performance of F1 teams throughout the studied period, it was decided to acquire a relative measure, namely the ratio between the points scored to the maximum available points per season:

$$Relative\ Performance\ (RP) = \frac{TP_{j,Y}}{MAP_{j,Y}} \quad (5)$$

Where $TP_{j,Y}$ and $MAP_{j,Y}$ are the collected points of team j and the maximum available points in season Y, respectively. The relative performances per season for all the F1 teams are presented in **Table 3**. Note that any penalties imposed on F1 teams, that were irrelevant to performance, are not applied in this assessment.

During the studied period (2000-2019), 19 out of 20 constructors' championships were won by the teams that participate in the 2022 season. Ferrari has claimed 7 championships with a winning rate of 33.3%, Mercedes, participating in F1 since 2010, has won 6 championships and has the highest winning rate (63.3%), Red Bull has won 4 championships with a winning rate of 25% and Alpine has won 2 championships (winning rate of 9.5%). The average relative performance yields the following pecking order in terms of success:

1. Mercedes
2. Ferrari
3. Red Bull
4. McLaren
5. Alpine
6. Williams
7. Aston Martin
8. Alfa Romeo
9. Alpha Tauri
10. Haas

Boxplots with the finishing positions of each F1 team in the studied period are illustrated in **Figure 2**.

Predictions on the impact of the budget cap on Formula 1 teams' performance

Season	Scuderia Ferrari	Mercedes	Red Bull	McLaren	Williams	Alfa-Romeo	Alpine	Alpha Tauri	Aston Martin	Haas
2000	38	-	-	34	8	1	5	-	-	-
2001	40	-	-	23	18	5	2	-	-	-
2002	50	-	-	15	21	2	5	-	-	-
2003	25	-	-	23	23	3	14	-	-	-
2004	37	-	-	10	13	5	15	-	-	-
2005	13	-	5	25	9	3	26	-	-	-
2006	29	-	2	16	2	5	29	0	-	-
2007	31	-	4	33	5	15	8	1	-	-
2008	25	-	4	22	4	19	11	6	0	-
2009	11	-	24	11	5	6	4	1	2	-
2010	21	11	26	24	4	2	8	1	4	-
2011	20	9	34	26	0	2	4	2	4	-
2012	20	7	23	19	4	6	15	1	5	-
2013	18	19	31	6	0	3	16	2	4	-
2014	11	35	20	9	16	0	0	1	8	-
2015	22	37	10	1	13	2	4	3	7	-
2016	19	36	22	4	7	0	0	3	8	1
2017	26	33	18	2	4	0	3	3	9	2
2018	27	31	20	3	0	2	6	2	5	4
2019	24	35	19	7	0	3	4	4	3	1
Mean	24	26	17	15	7	4	9	2	5	2

Table 3. The relative performance (%) for each F1 team per season.

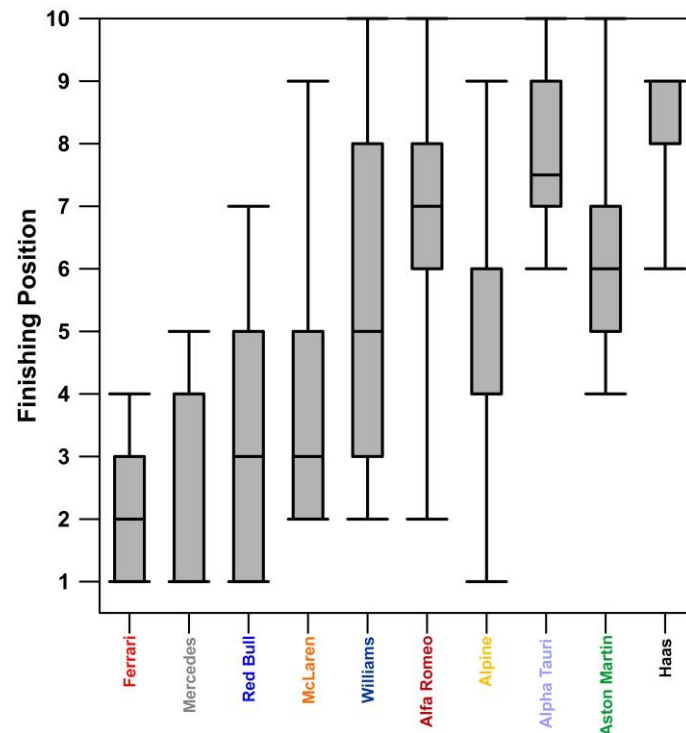


Figure 2. Boxplots of finishing position in the constructors' championship for each F1 team (2000-2019).

3.4 Quantification of the driver's effect on performance

F1 is a complex sport that involves the competition of both drivers and teams. The overall performance of a constructor is dependent on the performance of the team but also on the skills of the 2 drivers who participate in the championship. The most important component of winning lies in the performance of the car, but the extent of the driver effect is highly disputable [Bell et al., 2016]. Herein, the driver effect will be isolated since the purpose of the thesis is to study the performance of each F1 team. The rationale behind this lies in the fact that drivers are not a constant parameter when it comes to future performance predictions.

The quantification of the driver effect is a challenging task with limited studies found in the literature [Phillips, 2014; Bell et al., 2016; Duggal, 2020]. This is due to many parameters that must be taken under consideration in order to assess the performance of a driver and are summarized below:

1. A driver's performance is intercorrelated with the F1 car provided. For example, a driver in a winning car that finishes 5th performs worse than a driver in a non-competitive car that finishes 8th despite the former scoring more points.
2. There are certain parameters that affect the performance of a driver during the race but cannot be controlled by him. For instance, if a driver is involved in a crash that has no responsibility, his performance is unfairly devalued. The same applies when the team makes a strategic error.
3. The comparison of drivers' performance in different cars is inexpedient therefore, drivers are frequently compared to their teammates that drive the same car. Unfortunately, this approach also presents certain issues since the skills of the teammate are not always representative of the entire field.
4. There are non-racing parameters that affect the performance of the driver and cannot be readily quantified. These include his relationships with the team, communication with the staff and his motive to perform in the team's environment.

To overcome these issues herein, a database on F1 drivers' skills that utilizes data from 1950-2019 will be used to quantify the drivers' effect. This database was developed based on the TrueSkill ranking system [Herbrich et al., 2007], a Bayesian skill system established by Microsoft to assess the performance of video game players and distinguish the individual from team effects. According to the TrueSkill algorithm, a player's skills are represented by a normal distribution and characterized by a mean value and a variance based on their past performance. The algorithm also considers the corresponding effects of the teams participating in a game. The concept of separating individual and team effects is similar to the conditions found in F1; therefore, the TrueSkill system can be applied to quantify the skills of the F1 drivers. A detailed analysis of the mathematical formulas behind TrueSkill can be found in [Herbrich et al., 2007].

The drivers that participated in the championships during the studied period were classified according to their TrueSkill rating in 6 categories (Rank 1 represents the best drivers and Rank 6 the worst-performing ones), as presented in **Table 4**. An arbitrary point system is also established based on the ranking class of each driver with class I providing +5 points and class VI -5 points. Note that the ranking classes represent the average performance of the examined drivers and not their current form.

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The overall drivers' effect is the cumulative ranking of the team's drivers that participated in the season. If a team changed a driver during the year, a weighted average ranking is obtained based on the number of races that each of the corresponding drivers participated in. The maximum number of changes documented from one team in the studied period was 2, i.e., the maximum number of drivers for one chassis was 3. Therefore:

$$\overline{DR} = \frac{n_1}{N} \times DR_1 + \frac{n_2}{N} \times DR_2 + \frac{n_3}{N} \times DR_3 \quad (6)$$

Where $\overline{DR}$ is the weighted average driver effect, DR_1 , DR_2 , DR_3 the driver ranking of the 3 drivers that participated in the season; n_1 , n_2 , n_3 the races that each one drove; and N the total races.

Drivers	Ranking Class	Ranking Points
Schumacher M., Hamilton L.	I	+5
Vettel. S., Rosberg N., Alonso F., Raikkonen K., Verstappen M., Button J., Hakkinen M.	II	+3
Webber M., Barrichello R., Ricciardo D., Bottas V., Coulthard D., Massa F., Villeneuve J., Montoya J.P., Schumacher R.	III	+1
Fisichella G., Frentzen H-H., Kubica R., Perez S., Heidfeld N., Leclerc Ch., Kovalainen H., Trulli G., Hulkenberg N., di Resta P., Grosjean R., Sainz C., Magnussen K., Kvyat D., Gasly P., Ocon E., Albon A.	IV	-1
Wurz A., Salo M., Kobayashi K. Petrov V., Sutil A., Norris L., Nasr F. Stroll L., Alguersuari J., Buemi S., Nakajima K., Vergne J.E., Klien C., Senna B., Vandoorne S., De la Rosa P., Liuzzi V., Maldonado P., Piquet N. Jr., Pizzonia A., Speed S., Gutierrez E.	V	-3
Diniz P., Bourdais S., Giovinazzi A., Palmer J., Ericsson M., Wehrlein P., Hartley B., Russell G., Sirotkin S., Latifi N., Doornbos R.	VI	-5

Table 4. F1 drivers' skill ranking based on the TrueSkill ranking system (developed by Microsoft).

4. Results

4.1 Correlation of relative performance and relative budgets

Before taking into consideration the drivers' effect, a correlation between the relative budgets and the relative performance of F1 teams will be established using a linear regression model like the one presented in Equation (2). The simple linear regression

model yields an F-statistic of 165.9, meaning that the regression is statistically significant at any tested level. Therefore, it is confirmed that relative budgets and relative performance are correlated. The linear regression equation yields:

$$RP_{Y,j} = 2.76 + 0.059 \times RB_{Y,j} + \varepsilon_{Y,j}, R^2 = 0.52 \quad (7)$$

This correlation indicates that:

1. A team with the lowest budget during the season is expected to collect 2.76% of the available points and
2. The relative performance will increase by 0.059 (%) for every additional million \$ spent during the season.

The coefficient of determination indicates that 52% of the variation in RP can be explained by the variability of the RB in F1 for the studied period. The strength of the correlation is considered moderate. This proves that budgets affect the overall performance but the efficiency of the F1 teams and the drivers' effect also play important roles. In fact, having a correlation with a high R^2 would mean the performance only depends on the budget and that efficiency does not impact the outcome. The RP-RB data plot is illustrated in **Figure 3**.

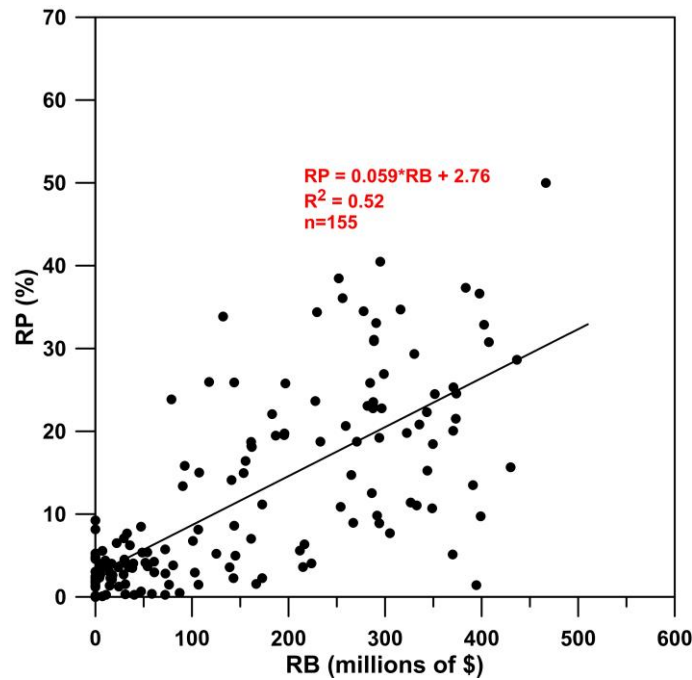


Figure 3. Correlation of relative performance (RP) and relative budgets (RB) for the 2000-2019 data using the simple linear regression model.

4.2 Correlation of relative performance, relative budgets and drivers' effect

To incorporate the drivers' effect (DR), a multivariable linear regression model is established. The constructed model yields an F-statistic of 123.4 and, therefore, the regression is statistically significant at any level. However, in order to observe whether

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this applies to both independent variables, the T-statistic is used. The T_{stats} for the DR and the RB variables are 6.27 and 5.86, respectively. The corresponding marginal levels of significance using the T_{stat} (p-values) are practically 0 ($p_{DR} = 3.6 \times 10^{-9}$ & $p_{RB} = 2.8 \times 10^{-8}$), hence there is strong evidence proving that both RB and DR are correlated with the RP. The multivariable linear regression model yields:

$$RP_{Y,j} = 5.981 + 0.0338 \times RB_{Y,j} + 1.136 \times DR_{Y,j} + \varepsilon_{Y,j} \geq 0, R_{adj}^2 = 0.61 \quad (8)$$

Where R_{adj}^2 is the adjusted coefficient of determination.

The derived model suggests that:

1. The predicted relative performance of a team with the lowest budget of the season (RB=0) that has a net driver effect (DR=0) is 5.981%;
2. An increase in the RB by one million \$ will increase the expected RP by 0.0338%
3. An increase/decrease in the team's drivers ranking by one will increase/decrease the overall relative performance by 1.136%.

The data plots between RP-RB and RP-DR given the multivariable regression model are depicted in **Figure 4**. A 3D plot of Equation [8] is depicted in **Figure 5**.

The relative budgets and drivers' effect were also examined in terms of collinearity using the variance inflationary factor [Marquardt, 1970]:

$$VIF = \frac{1}{1 - R_j} \quad (9)$$

Where R_j is the coefficient of determination between RB and DR (or DR and RB). VIF exceeding 5-10 indicates collinearity issues and the corresponding regression model should be used with caution [Berenson et al., 2020]. The corresponding VIF is equal to 1.92. This indicates that DR and RB are low-moderately correlated but not to an extent that makes the model unreliable.

The partial coefficients of determination $R_{DR/RB}$ and $R_{RB/DR}$ were 21% and 18%. Therefore:

1. An additional 21% of the variation in an F1 team performance is explained by the variation of the DR after the RB effect has been considered and;
2. An additional 18% of the variation in an F1 team performance is explained by the variation of the RB after the DR has been considered.

Those indicate that the additional contribution of the DR in the model, once the RP effect is included, is relatively low and vice versa. Hence, the RP of a team with a low RB will not be dramatically changed due to its drivers' abilities (a phenomenon that is reasonable and frequently observed in F1).

4.3 Multivariable regression residual analysis

The residuals derive from the multivariable regression model and allow for identifying which teams have been more efficient during the studied period. The residuals per season plots for each F1 team are presented in **Figures 6a-j** and descriptive statistics are documented in **Table 5**. Our analysis suggests that the teams that thrived during the studied period (namely, Ferrari, Red Bull and Mercedes) were also efficient (since their average residuals are positive), performing better than the average F1 team would, given

the same budget and drivers. This is an important finding proving that their dominant periods should not be attributed solely to their high budgets and the fact that they hired capable drivers. More generally, the assessment proves that it takes more than a high budget to be successful in Formula 1. The aforementioned teams have won 17/20 championships between 2000 and 2019 and are grouped among the most efficient ones. On the contrary, teams that had relatively high budgets and good drivers but did not succeed as much are considered inefficient (e.g., McLaren, Williams).

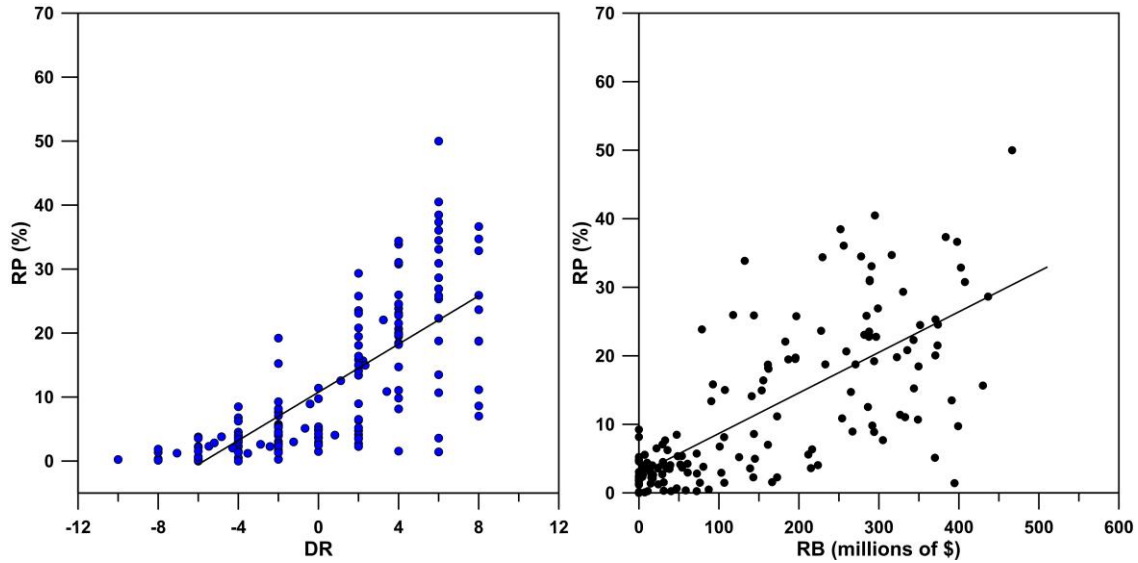


Figure 4. Correlation of relative performance-RP with drivers' effect-DR (left) and relative performance-RP with relative budget-RB (right) using the multivariable regression model.

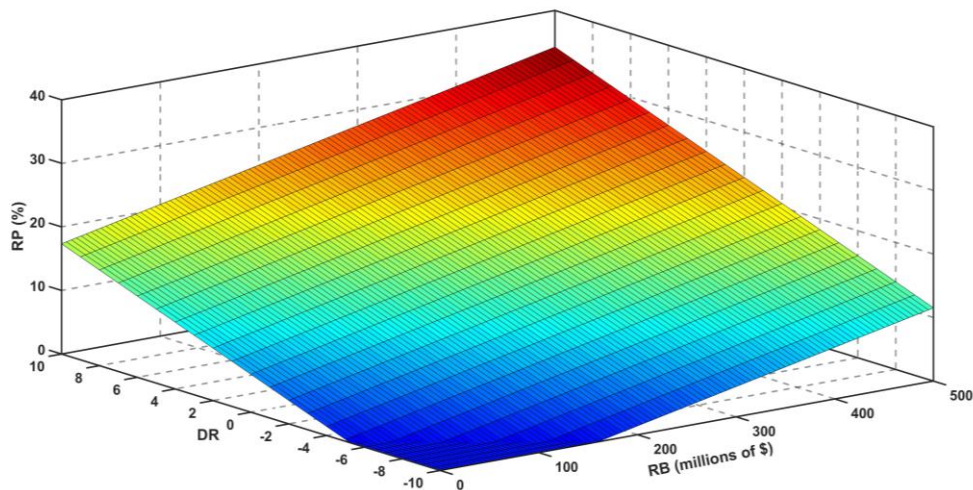
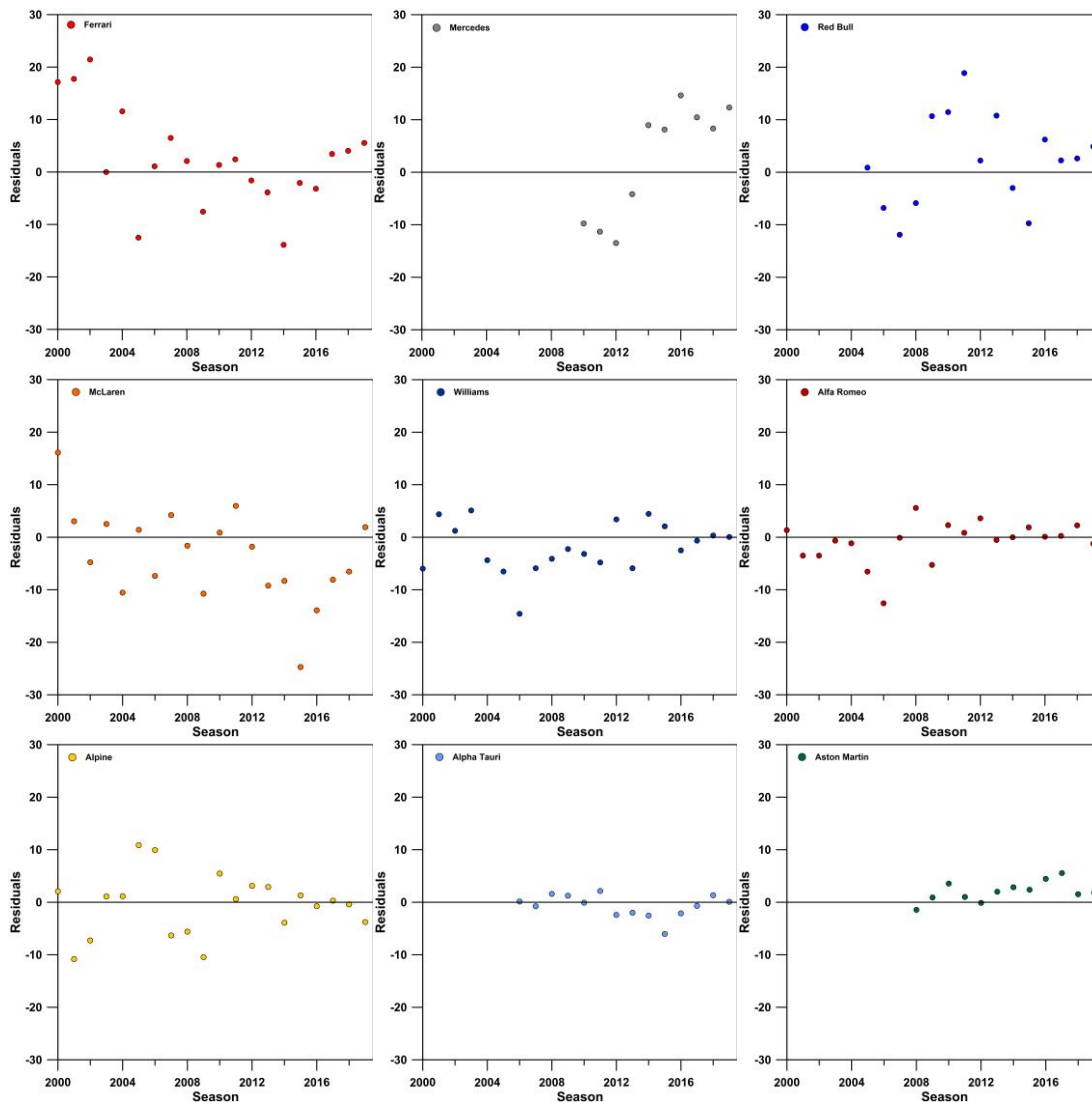


Figure 5. A 3D surface plot of the multiple regression model between the independent variables (relative budget-RB and drivers' effect-DR) and the predicted relative performance (RP).

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Statistics	Ferrari	Mercedes	Red Bull	McLaren	Williams	Alfa-Romeo	Alpine	Alpha Tauri	Aston Martin	Haas
Mean	2.48	2.41	2.24	-3.58	-1.99	-0.84	-0.53	-0.72	2.04	-1.04
Standard Error	2.0	3.4	2.2	2.0	1.1	0.9	1.3	0.6	0.6	0.6
Standard Deviation	9.2	10.8	8.6	8.7	4.8	4.0	5.8	2.2	1.9	1.2
Range	35.3	28.1	30.8	40.8	19.7	18.2	21.7	8.2	7.0	2.7
Minimum	-13.9	-13.5	-11.9	-24.7	-14.6	-12.6	-10.8	-6.0	-1.4	-2.4
Maximum	21.4	14.6	18.9	16.1	5.1	5.6	10.9	2.1	5.5	0.3
Seasons	20	10	15	20	20	20	20	14	12	4

Table 5. Descriptive statistics of the relative performance residuals for each F1 team (multivariable regression model).



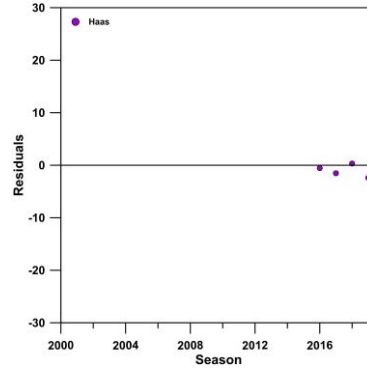


Figure 6. Residuals of relative performance (RP) per season (2000-2019) for a) Ferrari, b) Mercedes, c) Red Bull, d) McLaren, e) Williams, f) Alfa Romeo, g) Alpine, h) Alpha Tauri, i) Aston Martin and j) Haas.

Nonetheless, before further assessing the efficiency of each F1 team, a certain adjustment should be introduced. The aforementioned findings include all the data collected for F1 teams in the studied period. However, there are two types of teams involved in the F1 championship; those that existed before 2000 and those that formed after 2000 (Red Bull, Alpha Tauri, Aston Martin, Haas, Mercedes). The latter experience a competitive disadvantage in their initial seasons as they have to adjust to the sport. To remedy this, the data will be re-evaluated by excluding the first 3 seasons for the teams that entered the championship after 2000. This will allow the new teams to have a transition period to adjust to the sport. To be fair, this adjustment should also apply to newly formed teams in the early 2000s. The only team that was formed at that time is Alpine (as Renault) which bought the team of Benetton in 2000, hence it can be considered as a new team for 2000-2002 (3 seasons). This adjustment could not be applied to Haas since the team formed in 2016, therefore three out of four seasons would have to be excluded from the analyses. The mean residual RPs per team after these adjustments were implemented are documented in **Table 6**. Observe that the performance of Mercedes significantly increases to a residual of 8.37% while Red Bull's efficiency almost doubles. The efficiency of Alpine is also reversed presenting a positive mean residual. The only team that is negatively affected is Alpha Tauri which performed better in its initial seasons.

4.4 The efficiency ratio

To further understand the efficiency of the F1 teams, a relative measure should be employed. This measure must account for the differences in the expected relative performance of each team. The relative efficiency of each F1 team will be taken into consideration by introducing the efficiency ratio:

$$EF = \frac{ARP_j}{ARP_{e,j}} \quad (10)$$

$$ARP_{e,j} = ARP_j - \text{Mean Residual } RP_j \quad (11)$$

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Where ARP_j is the average relative performance of team j (**Table 6**) and $ARP_{e,j}$ is the expected average relative performance of team j given the multivariable regression model (Equation 8).

F1 team	Mean RP_{adj}
Ferrari	2.48
Mercedes	8.37
Red Bull	4.29
McLaren	-3.58
Williams	-1.99
Alfa Romeo	-0.84
Alpine	0.33
Alpha Tauri	-1.00
Aston Martin	2.38
Haas	-

Table 6. Average relative performance (RP) residuals adjusted for newly formed teams (multivariable regression model)

Following this rationale, the efficiency ratios for all teams are calculated and documented in **Table 7**. The efficiency ratio can be interpreted as follows: For every 1% of the RP points that the average F1 team with a specific budget and drivers would acquire, the studied team would convert it into its efficiency ratio (i.e., Ferrari would convert it into 1.11%, Mercedes into 1.35%, Red Bull into 1.26%, etc.). Observe that, in relative terms, Aston Martin is even more efficient than Mercedes (1.66%).

Conclusively, based on our analyses in the studied period, the teams can be classified in terms of relative efficiency among the following groups:

- Efficient teams: Ferrari, Mercedes, Red Bull, Aston Martin ($EF > 1.00$)
- Quasi-balanced teams: Alpine ($EF \approx 1.00$)
- Inefficient teams: McLaren, Williams, Alfa Romeo, Alpha Tauri, Haas. ($EF < 1.00$)

F1 team	Efficiency ratios
Ferrari	1.11
Mercedes	1.35
Red Bull	1.26
McLaren	0.82
Williams	0.80
Alfa Romeo	0.84
Alpine	1.03
Alpha Tauri	0.68
Aston Martin	1.66
Haas	0.69

Table 7. Efficiency ratios for each F1 team. Data from the three initial seasons for newly formed teams are excluded (except for Haas)

4.5 Establishment of a prediction model

Herein the efficiency ratios derived in the previous section will be incorporated into the multivariable regression model to establish a prediction model. Based on the efficiency ratios (**Table 7**) and the results from the multivariable regression model (Equation 8), the following prediction model is established:

$$RP_j = (5.981 + 0.0338 * RB_j + 1.136 * DR_j) * EF_j \quad (12)$$

Where EF_j is the efficiency ratio of each F1.

The new model has a higher adjusted coefficient of determination compared to the previous one, namely 70%. Therefore, 70% of the variation in RP can be explained by the variability of the RB, DR and EF. The relationship between the predicted versus the real RP of all the accumulated data is illustrated in **Figure 7**. The efficiency ratios regard the *average* relative efficiency of each F1 team and cannot account for the performance of each team on a seasonal basis. For example, observe the variability in the performance of each F1 team (**Table 3**). This cannot be predicted using a single coefficient but it is associated with more complex issues such as the initial conditions of each season, errors or success in the design of the car made in that season, the synergy of the staff, or even the instance of good or bad luck during races. It should not be neglected that F1 is fundamentally a sport and thus, unpredictable factors (e.g., fortune) play an important role in the overall performance of the contestants.

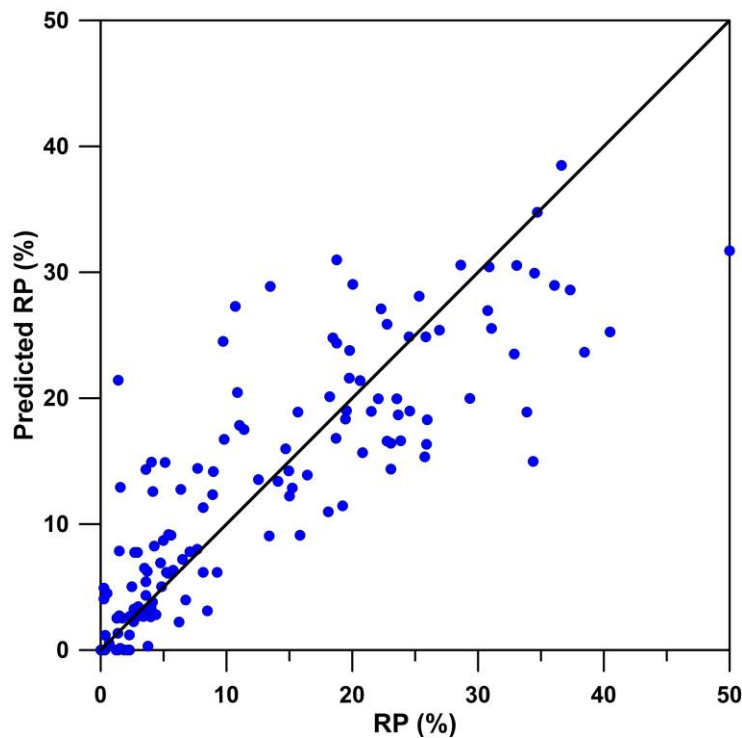


Figure 7. Predicted versus observed relative performance (RP) of all F1 teams given the established prediction model.

4.6 Future predictions

The future predictions presented herein are based on the two (2) following assumptions:

1. The budget cap will not be manipulated and will serve its purpose so that the teams' expenditures will be more balanced.
2. The F1 teams will maintain their historical efficiency as this was quantified through the efficiency ratios.

The fundamental difference between the studied period (2000-2019) and the future is that the effect of the budgets on performance will be mitigated. Still, some differences in the budgets are expected since not all teams will presumably reach the budget cap in the following years. According to G. Steiner, team principal of Haas, the differences will be reduced to "\$10-20 million" [Mitchell, 2022]. Therefore, it would be reasonable to deduce that the relative budget of an F1 team will not be greater than \$40 million from now on (2 times the amount that G. Steiner suggested).

Furthermore, the drivers' effect will be incorporated but the current form of the drivers participating in the 2022 championship cannot be derived from **Table 4**. That database includes the drivers' ranking up to 2019 and fits the studied period. However, young drivers who were considered rookies at that time have evolved and are currently performing very well, thus, the results based on that date would be unrealistic. According to their recent performance, the 2022 drivers' rankings are presented in **Table 8**. In addition, since the future drivers' lineups are unknown, it is essential to observe how the efficiency will affect the predicted RP if drivers with the same capabilities are assumed for all the teams. **Figures 8a** and **8b** illustrate the expected RP and RB correlation for all teams given the current drivers' ranking and assuming all the drivers are alike, respectively. In **Figure 8a**, the team with the highest expected performance is Mercedes since it has both a high driver's ranking (6) and an efficiency ratio (1.35). Red Bull and Ferrari follow in the 2nd and 3rd place. The efficiency of Aston Martin is not so evident since it currently features a driver's lineup with a moderate rating (-2). It may seem that the balance has not changed but the field is, in fact, much closer. The highest RPs predicted are less than 20% when the corresponding values during 2000-2019 reached up to 50% (Ferrari, 2002) with an average value of 32%. This indicates that the championship will be more competitive. Moreover, the data plot proves that the differences in the budgets will not make a significant variance and, thus, winning will be a matter of efficiency and drivers' skills. The predicted performance of the teams is much closer when all drivers are considered equivalent (**Figure 8b**). In that case, the championship is expected to be very unpredictable and only the details will be making the difference.

Finally, the historical RP of F1 teams from 2000 to 2019 and the predicted RP (with and without the drivers' effect) are illustrated in **Figure 9**. Observe that the advantage of the Ferrari, Mercedes and Red Bull is mitigated in the predicted RP model with the DR incorporated and it is almost eliminated once the DR is assumed to be equal for every team. This extreme case will entirely balance the field since all teams will have the same funds (more or less) to design and develop their car and the skills of their drivers will be equal. Therefore, the efficiency of each team will play the most important role.

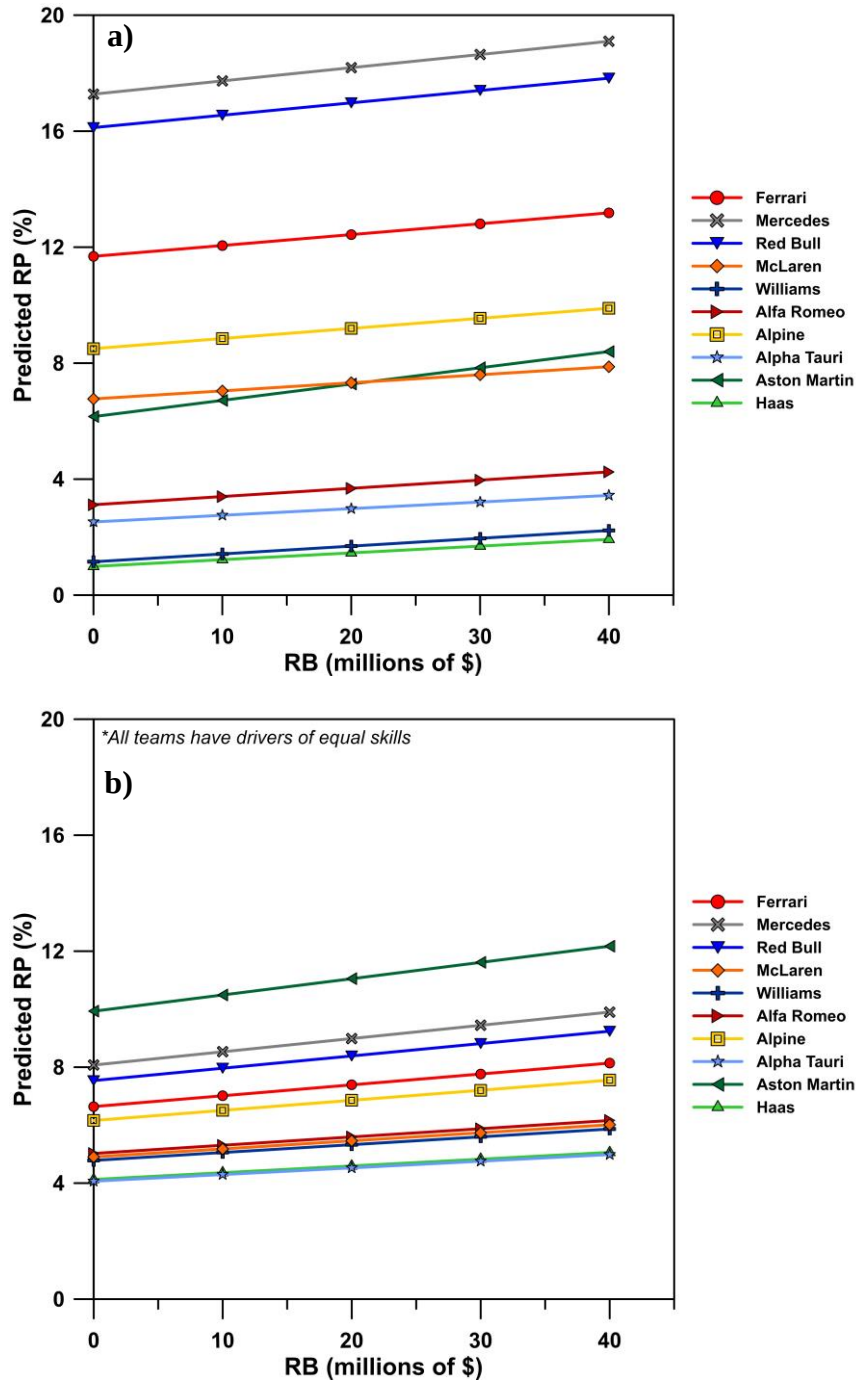


Figure 8. Predicted relative performance (RP) versus relative budget (RB) for all F1 teams given their historical efficiency and a) current drivers' lineup and b) assuming drivers of equal skills for all F1 teams.

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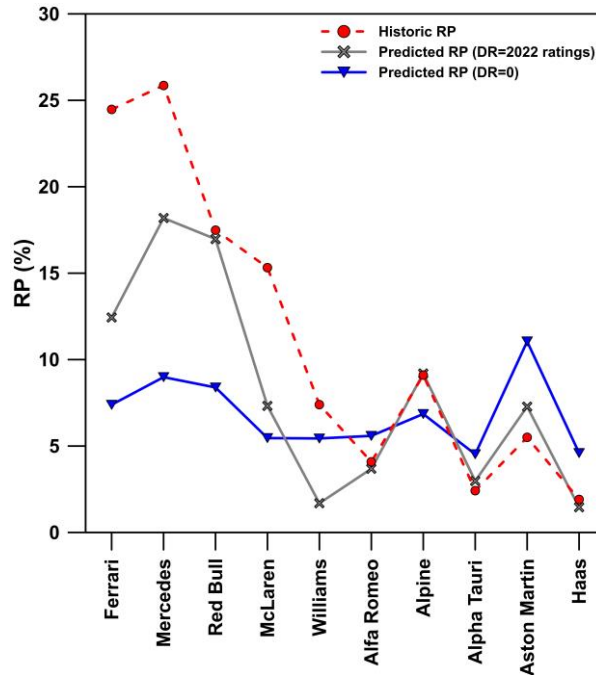


Figure 9. Average historical and predicted relative performance (RP) (with and without the drivers' effect) for all F1 teams participating in the 2022 championship.

5. Discussion

The present study showed that the budgets and performances are interconnected in F1. However, additional parameters are incorporated when it comes to the total output of an F1 team. Those were grouped into the team effects (efficiency) and drivers' effect (driver rating). Our prediction model indicates that the budget cap will eventually reduce the gap between top teams and the midfield. The fact that the top-3 teams (i.e., Ferrari, Mercedes and Red Bull) have also proved to be efficient (besides having higher budgets) and currently occupy the best drivers in F1 will certainly make the attempts of the subsequent teams more difficult but, not unachievable. Also, note that the aforementioned findings concern the long-term performance of the F1 teams. F1 is a motorsport that relies on past development and experience. Evidently, the top teams will still have an advantage in the early years of the budget cap enforcement due to past R&D, knowledge and experience.

Regarding the limitations of the present study, it should be noted that the performance assessment is a highly complex procedure that cannot be exactly predicted (as in all sports). Herein, its variability was explained adequately up to 70%, with the use of the three (3) main independent variables (budgets, drivers' effect and team efficiency) which are used to assess the historical performance of F1 teams. The subsequent 30% is due to numerous factors that cannot be readily quantified. An accident, a defective component, a bad strategic decision during a race, or a human error are all examples of incidents that will affect the performance of an F1 team but cannot be accounted for.

In addition, we evaluated the teams' performance over the past 2 decades, drew

conclusions on their current form and, based on these results, we established predictions for the future. The assumption made here is that the teams will continue to perform as they did during the studied period. It is unknown whether each team's efficiency will improve or deteriorate in the future.

Finally, it is also noted that the budgets for each season were derived from journal papers, reliable sources available online, or hard copies of F1 magazines. This was necessary since the F1 teams do not publish their financial statements and therefore, no financial information could be obtained from official sources.

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