

ISSN 1592-7415

eISSN 2282-8214

Volume n. 46 – 2023

# **RATIO MATHEMATICA**

**Journal of Mathematics, Statistics, and Applications**

**Special Issue:**

*Advance in Mathematics for Science and Engineering*

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# Influence of chemical reaction, radiation and heat source on separation of binary fluid mixture with Soret and Dufour effects on MHD mixed convective flow past a vertical plate

Krishnandan Verma\*

Sanjit Basfor

## Abstract

The motive of the current study is to analyse the consequences of Soret, Dufour and thermal radiation on chemically reactive MHD mixed convective flow on separation of binary fluid mixture. Numerical techniques are applied to obtain the results by using Matlab solver bvp4c. Results are acquired for velocity, temperature and concentration distributions for various parameters and are portrayed graphically. The study discloses some interesting results where different parameters like Soret number, Dufour number, chemical reaction parameter helps in restricting the temperature near the plate and at the same time raising the lightweight ingredient's concentration close to the plate, ultimately assisting the separation process. Numerical data are computed for coefficient of skin-friction coefficient, Local Nusselt number as well as Sherwood number to analyse the effects of dimensional shear stress at the surface of the plate, heat and mass transmission process.

**Keywords:** Binary fluid mixture; bvp4c; Dufour effect; MHD; Soret effect

**2020 AMS subject classifications:** 76D05; 76S05. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1040. ISSN: 1592-7415. eISSN: 2282-8214. ©Krishnandan Verma. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Separation of the constituents of a mixture is important in the field of chemical engineering. Hydrocarbon mixtures are separated and analyzed by chemist in laboratories whereas chemical engineers use the process of distillation. Often it is seen that some component, which is present in a binary fluid mixture in a very small amount, is very important in engineering and industrial use. The isotope of heavy water that is found in sea water is present in the ratio 1:5000. Though, the amount of the isotope is negligible but in atomic power plant, it is used as a coolant in nuclear reactor. Therefore, in a binary fluid composition the lightweight or the fewer constituents that is contained in a small amount can have many practical uses.

In certain engineering areas, e.g. nuclear reactors and operations that involve metals in liquefied form, the necessity of the process involving detachment of fluid mixtures play a remarkable role. In a given volume, the configuration of binary fluid mixture is indicated by  $C$ , which represent concentration as the fraction of the light weight and fewer ingredient's mass to the entire fluid mixture's mass, and the configuration of the heavy weight and significant ingredient is indicated by  $C' = 1 - C$ . Separation process involving liquid mixture is affected by pressure, temperature, and concentration gradients. The impact of pressure gradient on the separation of the liquid mixture is very less as compared to temperature gradient. Landau [1987] have given the expression for diffusion flux density as.

$$\vec{i} = -\rho D(\nabla C + K_p \nabla p + k_T \nabla T) \quad (1)$$

where  $D$ ,  $k_p D$  and  $k_T D$  represent diffusion coefficient, pressure diffusion coefficient and thermal diffusion coefficient respectively. Many researchers studied the separation process of binary fluid mixture on various geometries. Kim et al. [2007] studied theoretically instabilities in binary nanofluids with Soret and Dufour effects in convective conditions. Elhajjar et al. [2008] investigated the detachment of the constituents of a binary liquid combination in a pervious environment in a horizontal cavity using Rayleigh-Bernard configuration. Sharma and Singh [2008] carried out analytical study on the detachment of binary fluid mixture with Soret effect and baro-diffusion confined between two parallel disks. They found that the lighter constituent of the fluid is concentrated near the rotating disks for larger values of Reynolds number. Sharma and Singh [2010a] investigated separation of constituents of binary fluid mixture under the existence of strong radial magnetic field that was restricted in between two concentric cylinders rotating about a fixed axis. Sharma and Singh [2010b] investigated analytically the dissociation of binary fluid ingredients through a rotating disk with thermal diffusion in a weak magnetic environment. They found that the detachment of the binary

fluid ingredients was affected significantly by thermal diffusion and axial magnetic field. Sharma et al. [2014] studied numerically the consequences of Soret and diffusion thermo effect on the operation of detachment of constituents of a binary ingredient composition in pervious environment considering heat source and chemical reaction. Cates and Tjhung [2018] investigated theoretically binary fluid mixture with kinetics of phase separation and active emulsions.

Soret effect has an active role in the dissociation of binary fluid ingredients under the influence of temperature difference in a convection free fluid mixture accompanied by the transfer of heat. Many works have been reported in literature on Soret effect as well as its counter effect i.e. Dufour effect on heat and mass transmission process. Chamkha and Khaled [2000] analysed thermal transmission on hydromagnetic hybrid nanofluid flow between two parallel plates where one plate is stretchable while other is penetrable and both the plates and fluid rotates simultaneously under Ohmic heating and thermal effects. Sharma and Aich [2015] studied numerically thermal and mass transmission near stagnation point of a circular cylinder on a free convective flow with radiation and thermal diffusion effects. Ashraf et al. [2018] examined thermal and mass transmission past a semi-infinite erect plate on convective flow with Soret and Dufour effects. Verma et al. [2020a] probed numerically Soret and Dufour effects past a stretching sheet on MHD chemically reacting fluid and discovered that the fluid concentration increases near the sheet due to Soret effect while Dufour effect increases the fluid temperature. Verma et al. [2020b] studied numerically the action of thermal and mass transmission on MHD fluid flow in porous medium on Darcy-Forchheimer model on a rotating infinite disk. Krishna et al. [2021] investigated MHD convective heat generating/absorbing second grade rotating unsteady fluid flow past a semi-infinite pervious surface with ion slip and Hall Effects. They found that the ion slip and Hall parameter boosts the fluid velocity while the heat source parameter decreases the temperature profile. Other important research on heat and mass transmission are reported by Sharma [2011], Bhattacharyya et al. [2014], Rashidi et al. [2015], Reddy and Chamkha [2016], Ullah et al. [2017], Jain and Choudhary [2018], Verma et al. [2021] and Verma and Sharma [2022].

The above list of literature reviews motivate us to investigate numerically the separation of the lightweight ingredient of the binary fluid in the existence of radiation, heat source, Soret, Dufour and chemical effects over an upright plane surface inserted in a porous medium with Darcy-Boussinesq model. The current work will focus to examine the separation process numerically using bvp4c scheme, which is an inbuilt solver in Matlab that uses finite difference code to obtain the approximate solution. Numerical method in solving the current problem will enable us investigate the effects of different parameter deeply as bvp4c is able to obtain the results will ease for all the parameters which is sometimes difficult to attain by analytic method depending upon the complexity of the problem. The results

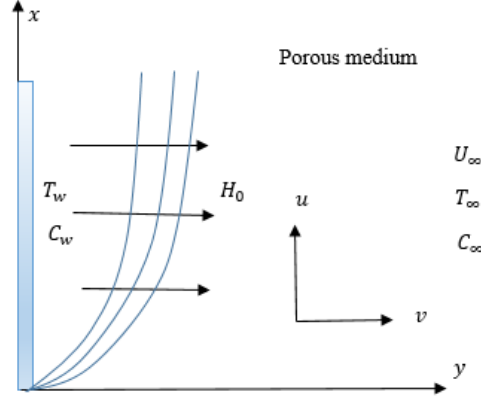


Figure 1: Geometric configuration

will be demonstrated by graphs for various parameters representing the changes in velocity, temperature and concentration of the fluid.

## 2 Construction of the Problem

Consider a stationary impenetrable vertical plate in a binary fluid mixture in a semi-infinite porous medium in a two-dimensional system. The  $x$ -axis is along-side the plate and  $y$ -axis perpendicular to it. The flow is steady and the fluid is Newtonian. An unvarying magnetic field of strength  $H_0$  is exerted in a direction normal to the plate. Let  $T_w$  be the constant wall temperature of the plate whereas  $T_\infty$  is the ambient fluid temperature and  $T_w > T_\infty$ . Let  $C$  be the light weight or the fewer ingredient's concentration of the binary fluid close to the surface and  $C_\infty$  outlying the plate.

The governing equation of the problem together with Boussinesq approximations are given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K} \left( 1 + \frac{KH_0^2 \sigma \mu_e^2}{\mu} \right) u + g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty), \quad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{Dk_T}{c_s c_p} \frac{\partial^2 C}{\partial y^2} + \frac{1}{\rho c_p} Q(T - T_\infty) - \frac{1}{c_p \rho} \frac{\partial q_r}{\partial y}, \quad (4)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} + \frac{Dk_T}{T_m} \frac{\partial^2 T}{\partial y^2} - k_1(C - C_\infty). \quad (5)$$

The border restrictions are

$$u = 0, v = 0, T = T_w, \frac{\partial C}{\partial y} + \frac{K_T}{T_m} \frac{\partial T}{\partial y} \text{ at } y = 0 \quad (6)$$

$$u \rightarrow U_\infty, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } y \rightarrow \infty \quad (7)$$

where  $v, g, \beta_T, \beta_c, K, \mu_e, \sigma, T, \alpha, D, k_T, c_s, c_p, \rho, Q, C, T_m$  and  $k_1$  represent kinematic viscosity, coefficient of thermal expansion, coefficient of concentration expansion, permeability, magnetic permeability, electrical conductivity, fluid temperature, thermal diffusivity, mass diffusion rate, heat diffusion ratio, concentration susceptibility, specific heat at constant pressure, density, heat source, concentration of the rarer constituent, mean fluid temperature and rate of chemical reaction respectively.

Heat flow by radiation is expressed as,

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y} \quad (8)$$

where  $T^4 = 4T_\infty^3 T - 3T_\infty^4$

Introducing stream function  $\psi$  given by  $u = \frac{\partial \psi}{\partial y}$  and  $v = -\frac{\partial \psi}{\partial x}$ , it is found that equation (2) is automatically satisfied.

The dimensionless variables given below are introduced to reduce equations (2)-(5) to non-dimensional form:

$$\eta = y \sqrt{\frac{U_\infty}{xv}}, \psi = \sqrt{vxU_\infty} f(\eta), \theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}, \phi(\eta) = \frac{C-C_\infty}{C_w-C_\infty}$$

The non-dimensional form of (2)-(5) are

$$f''' + \frac{1}{2} f f' + \lambda_1 \theta + \lambda_2 \phi - \frac{1}{DaRe} (1 + M^2) f' = 0, \quad (9)$$

$$(1 + \frac{4}{3}R) \theta'' + Pr f \theta' + D_f Pr \phi'' + Pr \delta \theta = 0, \quad (10)$$

$$\phi'' + \frac{1}{2} Sc f \phi' + Sr Sc \theta'' - Sc k_c \phi = 0. \quad (11)$$

where,

$G_r = \frac{g\beta_T(T_w-T_\infty)x^2}{v^2}, G_m = \frac{g\beta_C C_\infty x^2}{v^2}, \lambda_1 = \frac{Gr}{Re^2}, \lambda_2 = \frac{Gm}{Re^2}, Da = \frac{K}{x^2}, Re = \frac{U_\infty x}{v}, Pr = \frac{v}{\alpha}, Sc = \frac{v}{D}, Sr = \frac{DK_T(T_w-T_\infty)}{vT_m(C_w-C_\infty)}, D_f = \frac{DK_T(C_w-C_\infty)}{vc_s c_p(T_w-T_\infty)}, M = (\frac{K\sigma\mu_e^2 H_0^2}{v})^2, k_c = \frac{xk_1}{U_\infty}, R = \frac{4\sigma^* T_\infty^3}{k^* k}$  and  $\delta = \frac{Qx}{U_\infty}$  represents local Grashof number, local modified Grashof number, temperature buoyancy parameter, concentration buoyancy parameter, local Darcy number, local Reynolds number, Prandtl number, Schmidt number, Soret number, Dufour number, Magnetic parameter, Chemical reaction

parameter, Radiation parameter and heat generation parameter respectively. The dimensionless border restrictions are

$$f = 0, f' = 0, \theta = 1, \phi' + ScSr\theta' = 0, \text{ at } \eta = 0, \quad (12)$$

$$f' \rightarrow 1, \theta \rightarrow 0, \phi \rightarrow 0 \text{ as } \eta \rightarrow \infty \quad (13)$$

Local skin-friction coefficient, Nusselt number and Sherwood number are equivalent to  $f''(0)$ ,  $-\theta'(0)$  and  $\phi'(0)$  respectively in the current problem and their numerical values are calculated to analyse the heat and mass transfer process.

### 3 Results and Discussion

Matlab inbuilt solver bvp4c has been used to solve equations (9)-(11) with boundary restrictions (12) and (13). Numerical results are obtained for various parameters in the form of graph and tables using Matlab solver bvp4c. For computational purpose, the parameters are assigned the given values  $Sr = 0.4$ ,  $Sc = 0.22$ ,  $Pr = 0.71$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 0.1$ ,  $Da = 0.5$ ,  $Re = 400$ ,  $M = 1$ ,  $D_f = 0.15$ ,  $R = 0.5$ ,  $\delta = 0.2$  and  $k_c = 0.2$ . Numerical data of  $f''(0)$ ,  $-\theta'(0)$  and  $\phi'(0)$  are obtained for  $Sr$ ,  $D_f$  and  $M$  to gain some insight of skin friction coefficient, thermal and mass transmission rate.

It is observed from Fig. 2 and Fig. 3 that the velocity and temperature of the fluid decrease for larger values of Soret number. Soret number depends directly upon  $(T_w - T_\infty)$  and  $T_w > T_\infty$ . Therefore, increase in Soret number increases the temperature difference and hence heat diffuses away from the sheet which results in lowering of fluid temperature. From Fig. 4, it is spotted that the hike in Soret number leads to accumulation of larger concentration of the rarer constituent nearby the plate. The gathering of the light weight ingredient of the binary fluid is high in between  $0.5 \leq \eta \leq 3$ , so Soret effect contributes highly in the dissociation of the binary ingredients of the fluid by limiting the temperature and increasing the assemblage of the rarer constituent alongside the plate.

It is spotted from Fig. 5 and Fig. 6 that velocity as well as temperature decreases with the growth in the values of Dufour number. Fig. 7 disclose us that the hike in Dufour number decreases the concentration of the rarer constituent but the figure also informs us that the rarer constituent is accumulated near the surface i.e.,  $1 \leq \eta \leq 3.5$  thereby throwing the denser component away from surface of the plate. Thus Dufour number helps in the dissociation of the binary ingredients of the fluid.

The action of heat generation parameter on fluid velocity is illustrated in Fig. 8. It is perceived that the rise in heat generation parameter augments the fluid velocity. Heat generation parameter provides heat in the flow field because of which the



temperature increases as seen in the Fig. 9. Due to addition of heat by heat generation effect, the molecules of the fluid get excited thereby increasing the velocity of the binary fluid mixture. Fig. 10 reveals us that the rise in heat generation parameter raises the concentration of the lightweight constituent but the increase is quite negligible.

In Fig. 11 and Fig. 12. it is clearly perceived that temperature and concentration rises with the hike in radiation parameter. The rarer constituent of the fluid are concentrated near the plate i.e.,  $1 \leq \eta \leq 4$ , thus helping in the separation process. The velocity and temperature profiles due to chemical reaction parameter,  $k_c$  are illustrated in Fig. 13 and Fig. 14. There is a minor decrement in fluid velocity and temperature with the hike in the values of  $k_c$ . From Fig. 15, it is perceived that the concentration of the lightweight ingredient of the fluid hikes with the growth in the values of  $k_c$ . The rarer constituent is concentrated near the plate i.e.,  $1 \leq \eta \leq 4.5$ , thereby throwing the denser component far away from the plate ultimately aiding the separation process.

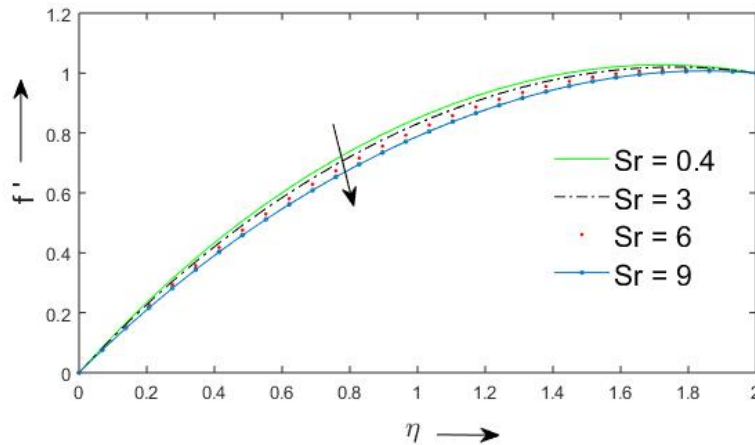


Figure 2: Distribution of velocity due to  $Sr$

## 4 Conclusions

The prime motive of the current study is to analyse Soret and Dufour effect-salong with different parameters on the dissociation of the binary fluid mixture. Industrial importance of separation of species of fluid mixture e.g., isotopes of heavy water that is present in a negligible amount in sea is very useful in nuclear reactor and so many researchers have reported several works on dissociation of

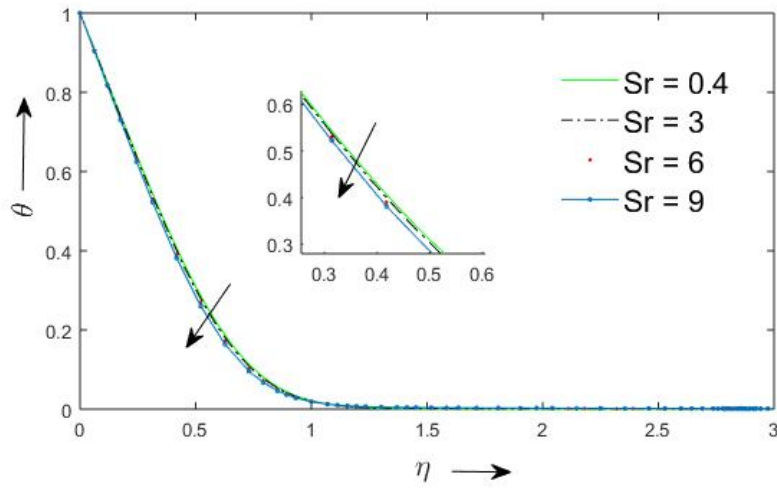


Figure 3: Changes in temperature as a consequence of  $Sr$

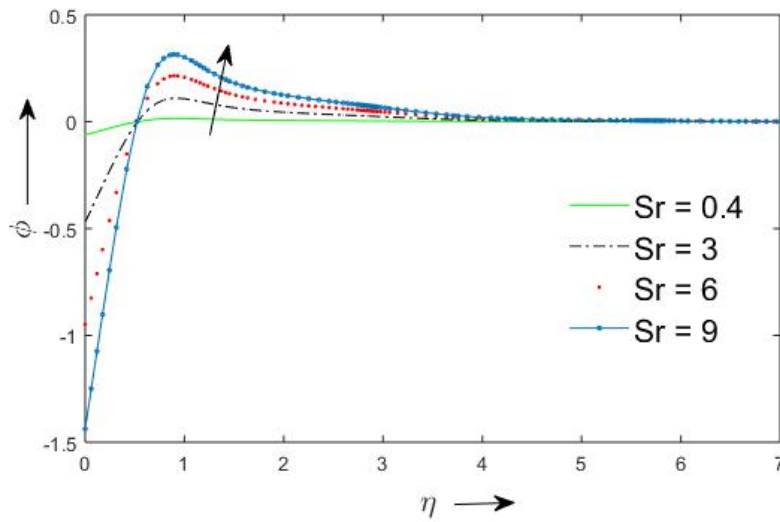


Figure 4: Changes in concentration as a consequence of  $Sr$

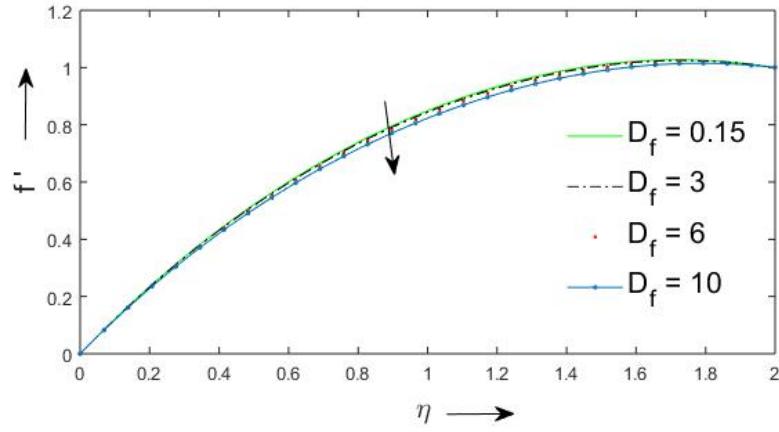


Figure 5: Distribution velocity due to  $D_f$

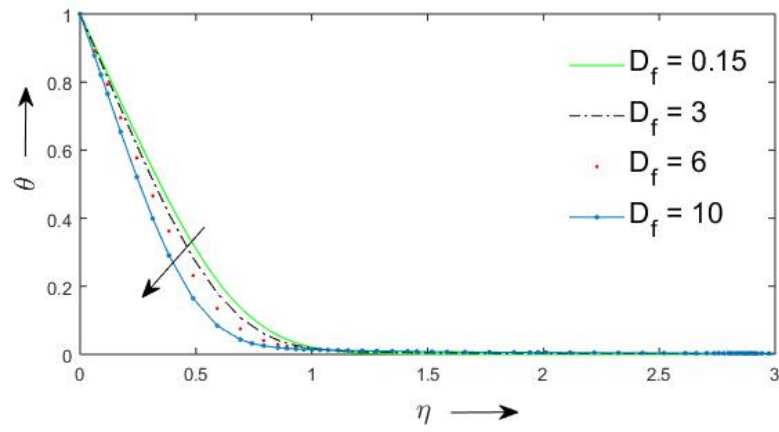


Figure 6: Distribution temperature due to  $D_f$

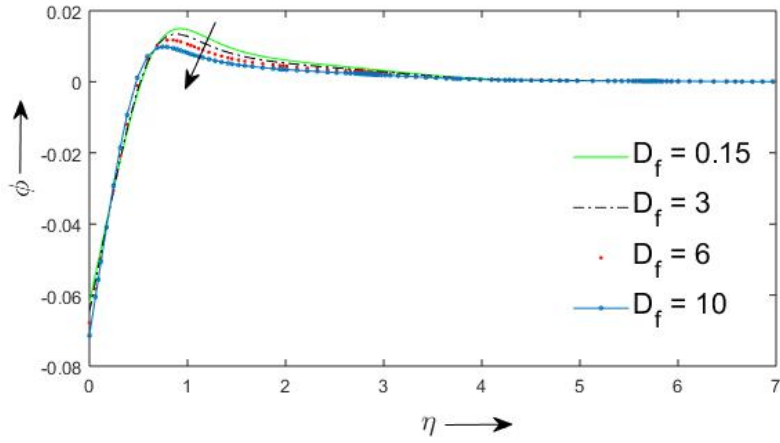


Figure 7: Distribution concentration due to  $D_f$

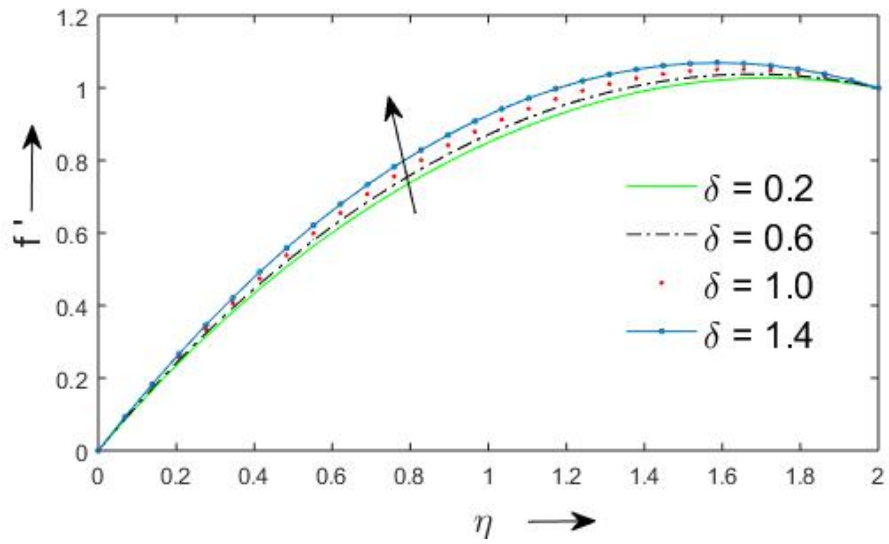


Figure 8: Distribution velocity due to  $\delta$

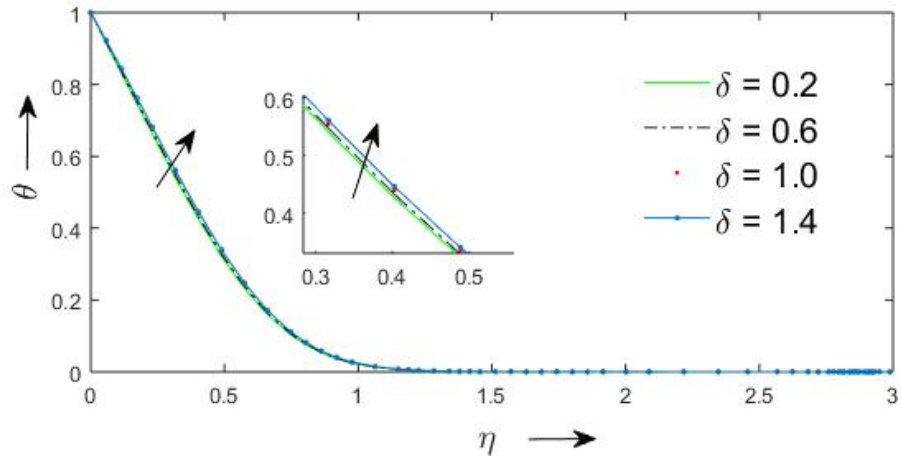


Figure 9: Distribution temperature due to  $\delta$

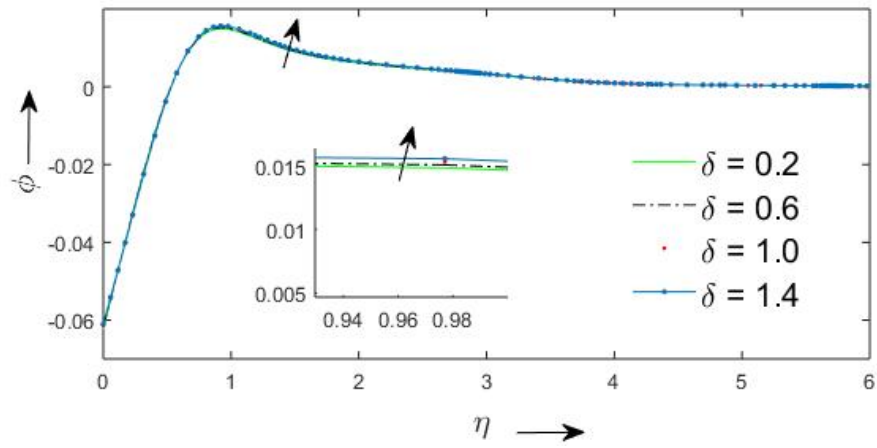


Figure 10: Distribution concentration due to  $\delta$

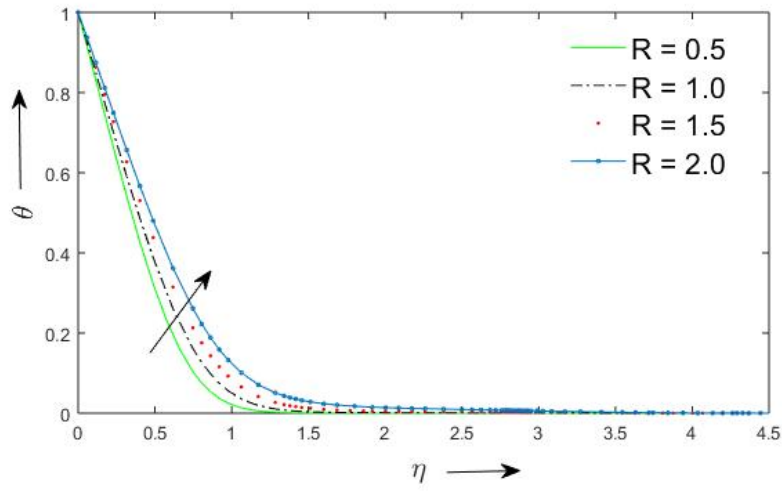


Figure 11: Changes in temperature as a consequence of  $R$

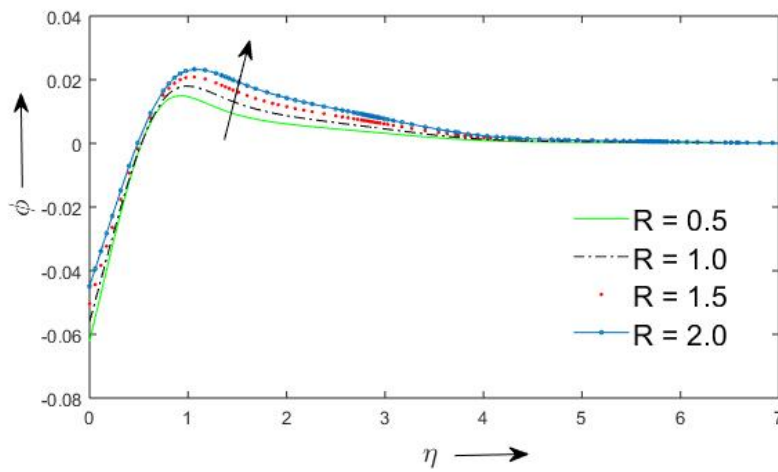


Figure 12: Changes in concentration as a consequence of  $R$

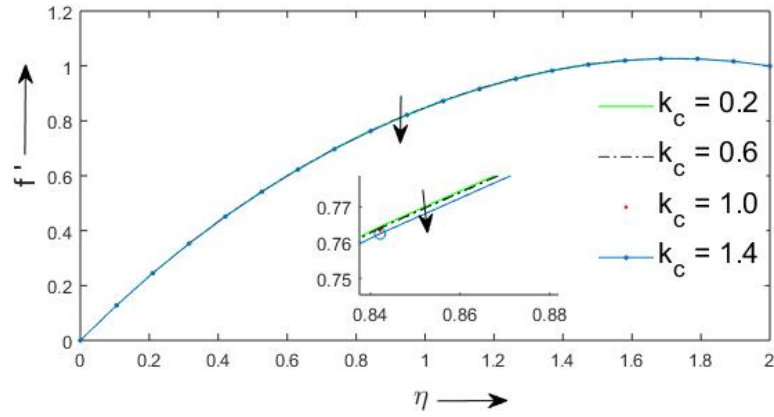


Figure 13: Distribution velocity due to  $k_c$

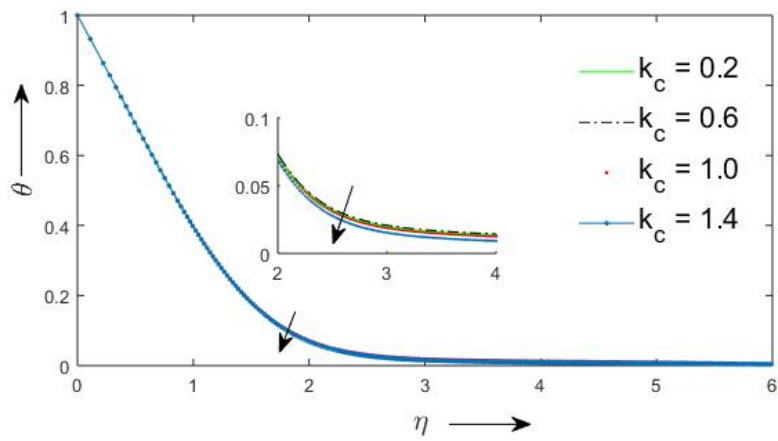


Figure 14: Distribution temperature due to  $k_c$

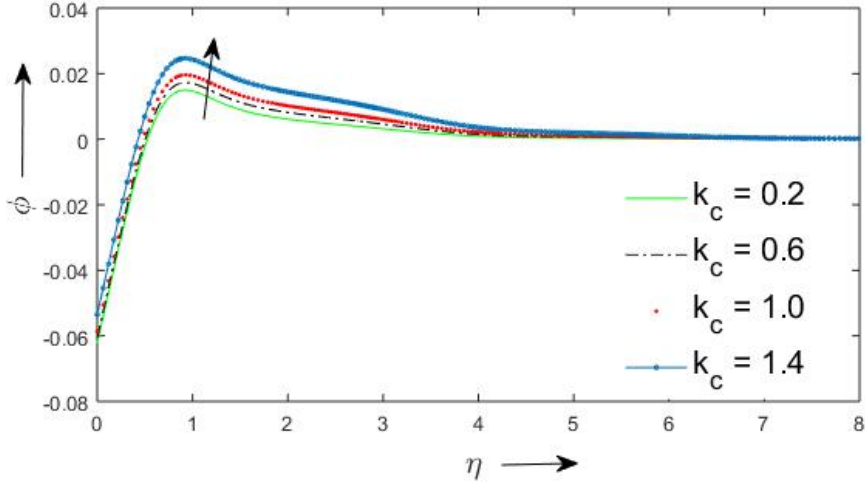


Figure 15: Distribution concentration due to  $k_c$

Table 1: Numerical data of  $f''(0)$ ,  $-\theta'(0)$  and  $\phi'(0)$  for  $Sr$ ,  $D_f$  and  $M$  when  $Sc = 0.22$ ,  $Pr = 0.71$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 0.1$ ,  $Da = 0.5$ ,  $Re = 400$ ,  $R = 0.5$ ,  $\delta = 0.2$  and  $k_c = 0.2$ .

| $Sr$ | $D_f$ | $M$ | $f''(0)$ | $-\theta'(0)$ | $\phi'(0)$ |
|------|-------|-----|----------|---------------|------------|
| 0.4  | 0.15  | 1   | 1.7398   | 0.5946        | 0.0523     |
| 3    | 0.15  | 1   | 1.7885   | 0.6215        | 0.4102     |
| 6    | 0.15  | 1   | 1.8400   | 0.6542        | 0.8638     |
| 9    | 0.15  | 1   | 1.9035   | 0.6880        | 1.3622     |
| 0.4  | 0.15  | 1   | 1.7398   | 0.5936        | 0.0523     |
| 0.4  | 3     | 1   | 1.6073   | 0.6257        | 0.0551     |
| 0.4  | 6     | 1   | 1.4951   | 0.6742        | 0.0593     |
| 0.4  | 10    | 1   | 1.2346   | 0.7298        | 0.0634     |
| 0.4  | 0.15  | 1   | 1.7398   | 0.5946        | 0.0523     |
| 0.4  | 0.15  | 4   | 1.7364   | 0.5950        | 0.0524     |
| 0.4  | 0.15  | 7   | 1.7304   | 0.5960        | 0.0524     |
| 0.4  | 0.15  | 10  | 1.7245   | 0.5977        | 0.0526     |



the fluid species using many analytical methods that are complicated, lengthy and time-consuming method used to obtain approximate solution. Numerical solution has provided a cost and time effective process to obtain the solution. The following problem can also be solved using other numerical method apart from bvp4c and can also be extended to different geometry like wedge, cone etc. The problem can also be extended using nanoparticles to see the necessary changes in heat transmission process which is an important future development of the current work. In the present work numerical solution presents some interesting results where different parameters help in restricting the temperature and raising the rarer ingredient's concentration nearby the plate that are highlighted below:

1. Heat generation parameter increases the fluid velocity while increase in Soret number, Dufour number and chemical reaction parameter decreases the fluid motion.
2. Temperature of the fluid decreases for increasing values of Soret number, Dufour number and chemical reaction parameter while heat generation and radiation parameter enhance the temperature of the fluid.
3. All the parameters help in raising the concentration of the lightweight ingredient of the binary fluid mixture near the plate and thus helping in the separation process by throwing the denser and heavier component away from the plate.
4. Soret number, Dufour number and magnetic parameter boost the heat transfer as well as the mass transfer rate of the fluid. Soret number increases the skin friction coefficient and increase in magnetic parameter and Dufour number decrease the skin friction coefficient.

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# Transversal eccentric domination in graphs

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## Abstract

Eccentricity of a vertex  $v$  is a maximum among the shortest distances between the vertex  $v$  and all other vertices. A set  $D$  is called eccentric dominating if every vertex in its complement has an eccentric vertex in the set  $D$ . A dominating set is transversal if the intersection of the set with all the minimum dominating sets is non-empty. Inspired by both the concepts we introduce transversal eccentric dominating (TED) set. An eccentric dominating set  $D$  is called a TED-set if it intersects with every minimum eccentric dominating set  $D'$ . We find the TED-number  $\gamma_{ted}(G)$  of family of graphs, theorems related to their properties are stated and proved.

**Keywords:** Eccentricity, TED-set, TED-number.

**2020 AMS subject classifications:** 05C69. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1043. ISSN: 1592-7415. eISSN: 2282-8214. ©Riyaz Ur Rehman et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

The classical queens problem in chess or the study of networks in electronics domination finds its application everywhere and plays a pivotal role in modern day science and technology. Domination is a vast arena in graph theory which is just not limited to adjacency between vertices belonging to the dominating set and its compliment. For a graph  $G(V, E)$ , a set  $S \subseteq V$  is said to be a dominating set, if every vertex in  $V-S$  is adjacent to some vertex in  $S$ . The domination number  $\gamma_d(G)$  of a graph  $G$  equals the minimum cardinality of an dominating set. There are many different invariants of domination. The concept of transversal domination in graphs was introduced by Nayaka S.R, Anwar Alwardi and Puttaswamy in 2018. A dominating set  $D$  which intersects every minimum dominating set in  $G$  is called a transversal dominating set. The minimum cardinality of a transversal dominating set is called the transversal domination number denoted by  $\gamma_{td}(G)$ . Geodesic being the shortest distance between any two vertices. The concept of shortest path has always intrigued the researchers in graph theory, operation research, computer science and other fields. There are many different types of distances in graphs, one such distance is eccentricity. The concept of eccentricity incorporated with a dominating set yields an eccentric dominating set. Eccentric domination was introduced by T. N. Janakiraman et al in 2010. The eccentricity  $e(v)$  of  $v$  is the distance to a vertex farthest from  $v$ . Thus,  $e(v) = \max d(u, v) : u \in V$ . For a vertex  $v$ , each vertex at a distance  $e(v)$  from  $v$  is an eccentric vertex. Eccentric set of a vertex  $v$  is defined as  $E(v) = \{u \in (G) : d(u, v) = e(v)\}$ . A set  $D \subseteq V(G)$  is an eccentric dominating set if  $D$  is a dominating set of  $G$  and for every  $v \in V - D$ , there exists at least one eccentric vertex of  $v$  in  $D$ . The eccentric domination number  $\gamma_{ed}(G)$  of a graph  $G$  equals the minimum cardinality of an eccentric dominating set. The main motive of this paper is to hybrid two different types of dominations and define a new domination variant. Inspired by this idea we combine transversal domination with eccentric domination. In this paper, we introduce transversal eccentric domination and calculate the TED-number of different graphs. Results related to TED-number of family of complete, star, path, cycle and wheel graphs are discussed. The upper TED-set, upper TED-number, lower TED-set and lower TED-number of different standard graphs are tabulated. For undefined terminologies refer the book graph theory by Frank Harary.

## 2 Transversal eccentric domination in graphs

**Definition 2.1.** An eccentric dominating (ED) set  $S \subseteq V(G)$  is called a transversal eccentric dominating set (TED-set) if it intersects with every minimum ED-set  $D'$  ie  $S \cap D' \neq \emptyset$ .

**Definition 2.2.** A TED-set  $S$  is called a minimal TED-set if no proper subset of  $S$  is TED-set.

**Definition 2.3.** The TED-number  $\gamma_{ted}(G)$  of a graph  $G$  is the minimum cardinality among the minimal TED-sets of  $G$ .

**Definition 2.4.** The upper TED-number  $\Gamma_{ted}(G)$  of a graph  $G$  is the maximum cardinality among the minimal TED-sets of  $G$ .

**Example 2.1.** .

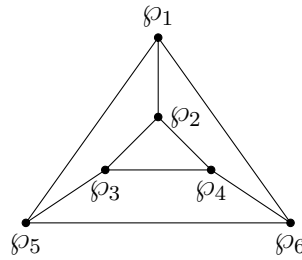


Figure 2.1: Graph  $G$

Consider the above example where the graph  $G$  consists of 6 vertices and 9 edges.

- (i) The dominating sets are  $\{\rho_1, \rho_2\}$ ,  $\{\rho_1, \rho_3\}$ ,  $\{\rho_1, \rho_4\}$ ,  $\{\rho_2, \rho_5\}$ ,  $\{\rho_2, \rho_6\}$ ,  $\{\rho_3, \rho_5\}$ ,  $\{\rho_3, \rho_6\}$ ,  $\{\rho_4, \rho_5\}$ ,  $\{\rho_4, \rho_6\}$ .
- (ii) The minimum ED-sets are  $\{\rho_1, \rho_2\}$ ,  $\{\rho_3, \rho_5\}$ ,  $\{\rho_4, \rho_6\}$ .
- (iii) The TED-sets are  $\{\rho_1, \rho_5, \rho_6\}$ ,  $\{\rho_2, \rho_3, \rho_4\}$ .

**Observation 2.1.** For any graph  $G$ ,

1.  $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{ted}(G) \leq \Gamma_{ted}(G)$ .
2.  $\gamma_{ted}(G) \leq n$  and  $\Gamma_{ted}(G) \leq n$ .
3.  $V(G)$  is also a TED-set.

**Theorem 2.1.** For complete graph  $K_n$ ,  $\gamma_{ted}(K_n) = n$ ,  $\forall n \geq 2$ .

**Proof:** Let  $V(K_n) = \{\rho_1, \rho_2, \dots, \rho_n\}$ . Since  $\deg(\rho_i) = n - 1 \forall \rho_i \in V(K_n)$  the eccentric vertex of  $\rho_i$  is given by  $E(\rho_i) = V - \{\rho_i\}$  and every single vertex dominates all other vertices. Since every vertex  $\rho_i \in V$  forms an ED-set of the form  $D_1 = \{\rho_1\}$ ,  $D_2 = \{\rho_2\}$ ,  $D_3 = \{\rho_3\}$ ,  $\dots$ ,  $D_n = \{\rho_n\}$ . The vertex set  $V$  is the only set which forms a TED-set, since  $V(K_n) \cap D_i \neq \emptyset$  where  $i = 1, 2, 3, \dots, n$  and  $D_i$  is any ED-set.

**Theorem 2.2.** For star graph  $S_n$ ,  $\gamma_{ted}(S_n) = 2 \forall n \geq 3$ .

**Proof:** Let  $V(S_n) = \{\wp_1, \dots, \wp_i, \dots, \wp_n\}$  where  $\deg(\wp_i) = n - 1$  where  $\wp_i$  is the central vertex and  $\deg(\wp_j) = 1$  where  $\wp_j$  is a pendant vertex of star graph  $S_n$ .  $E(\wp_i) = V - \{\wp_i\}$  and  $E(\wp_j) = V - \{\wp_i, \wp_j\}$ . The central vertex  $\wp_i$  forms a dominating set  $\{\wp_i\}$  but it is not an ED-set for any  $\wp_j \in V - D$ ,  $E(\wp_j) \not\subseteq D$ . But  $D = \{\wp_i, \wp_j\}$  forms an ED-set, then for  $S_3$  we have 3 ED-sets which forms the minimum ED-sets and for any star graph  $S_n$ ,  $\forall n \geq 4$ , we have  $(n - 1)$  ED-sets which forms the minimum ED-sets  $D_1 = \{\wp_i, \wp_1\}$ ,  $D_2 = \{\wp_i, \wp_2\}$ ,  $D_3 = \{\wp_i, \wp_3\}, \dots, D_n = \{\wp_i, \wp_n\}$ . Any minimum ED-set  $D = \{\wp_i, \wp_j\}$  also forms a TED-set, since  $D \cap \{\wp_i, \wp_j\} = \{\wp_i\} \neq \emptyset$ . Therefore  $\gamma_{ted}(S_n) = 2 \forall n \geq 3$ .

**Theorem 2.3.** For path graph  $P_n$ ,  $\gamma_{ted}(P_n) = \lfloor \frac{n+1}{3} \rfloor + 1, \forall n \geq 2$ .

**Proof:** Let the vertices of  $P_n$  be given by  $V(P_n) = \{\wp_1, \wp_2, \dots, \wp_n\}$ . Every path  $P_n$  contains two pendant vertices  $\{\wp_1, \wp_n\}$ . For any vertex  $\wp_i \in V(P_n)$  the eccentric vertex of  $\wp_i$  is  $E(\wp_i) = \{\wp_1\}$  or  $\{\wp_n\}$  where  $n$  is even. If  $n$  is odd then  $E(\wp_i) = \{\wp_1\}$  or  $\{\wp_n\}$  but if  $\wp_i$  is a vertex equidistant from both the pendant vertices then  $\wp_i = \wp_{\frac{n+1}{2}}$ ,  $E(\wp_{\frac{n+1}{2}}) = \{\wp_1, \wp_n\}$ . For any path  $P_n$ ,  $\lceil \frac{n}{3} \rceil$  set of vertices can dominate all the vertices of  $P_n$ . Similarly a set  $D$  whose cardinality is  $\lfloor \frac{n+1}{3} \rfloor + 1$  will eccentric dominate all the vertices of  $P_n$ . By the definition of TED-set, a set  $D$  should intersect all the minimum ED-set. An ED-set  $D$  will intersect all the minimum ED-sets. Therefore every minimum ED-set is a TED-set. Therefore  $\gamma_{ed}(P_n) = \gamma_{ted} = \lfloor \frac{n+1}{3} \rfloor + 1$

**Theorem 2.4.** For cycle graph  $C_n$  where  $n \geq 3$

$$\gamma_{ted}(C_n) = \begin{cases} 5, & \text{for } n = 8 \\ \lceil \frac{n+1}{3} \rceil + 1, & \text{otherwise} \end{cases}$$

**Proof: Case(i):** For  $C_8$ , the set  $D = \{\wp_i, \wp_j, \wp_k, \wp_l\}$  whose cardinality is  $\lceil \frac{n+1}{3} \rceil + 1 = 4$  does not form a TED-set which is an exception from case(i). Adding a vertex to  $D$  is of the form  $\{\wp_i, \wp_j, \wp_k, \wp_l, \wp_m\}$  whose cardinality is five will increasing the cardinality of  $D$ . Here every vertex in  $V(C_8) - D$  has an eccentric vertex in  $D$  and  $D$  is also dominating set which intersects all the minimum dominating sets of  $C_8$ . Therefore  $\gamma_{ted}(C_8) = 5$ .

**Case(ii):** For a cycle graph  $C_n$ , if  $n$  is even and  $n \neq 8$  then every vertex  $\wp_i \in V(C_n)$  has a unique eccentric vertex ie,  $E(\wp_i) = \{\wp_j \mid \wp_j \in V(C_n)\}$ .  $E(\wp_i)$  is at a distance of  $\frac{n}{2}$  edges from  $\wp_i$  for an even cycle. If  $n$  is odd then every vertex  $\wp_i$  has two eccentric vertices.  $E(\wp_i) = \{\wp_j, \wp_k \mid \wp_j, \wp_k \in V(C_n)\}$ .  $E(\wp_i)$  is at a distance of  $\lfloor \frac{n}{2} \rfloor$  edges from  $\wp_i$  for odd cycle. Every single vertex  $\wp_i$  can dominate itself and two vertices adjacent to it. Therefore for any cycle  $C_n$ ,  $\lceil \frac{n}{3} \rceil$  set of vertices forms the dominating set. Here we see that any set  $D = \{\wp_1, \wp_2, \dots, \wp_i\}$  which has the cardinality of the form  $\lceil \frac{n+1}{3} \rceil + 1$  forms a dominating set as well

as an ED-set. Since  $D$  whose cardinality is  $\lceil \frac{n+1}{3} \rceil + 1$  intersects every minimum ED-set of cardinality  $\gamma_{ed}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \lceil \frac{n}{3} \rceil \text{ or } \lceil \frac{n}{3} \rceil + 1, & \text{if } n \text{ is odd} \end{cases}$   $D$  forms a TED-set. Hence  $\gamma_{ted}(C_n) = \lceil \frac{n+1}{3} \rceil + 1$ .

**Theorem 2.5.** For wheel graph  $W_n$  where  $n \geq 4$ ,  $a \geq 1$

$$\gamma_{ted}(W_n) = \begin{cases} 3, & \text{for } n = (6a - 1), (6a) \text{ or } (6a + 1) \\ 4, & \text{for } n = (6a - 2), (6a + 2) \text{ or } (6a + 3) \end{cases}$$

**Proof: Case(i):** If  $n = 6a - 1$ ,  $6a$  and  $6a + 1$ , the wheel graphs are of the form  $W_5, W_6, W_7, W_{11}, W_{12}, W_{13}, W_{17}, W_{18}, W_{19}, \dots, W_{6a-1}, W_{6a}, W_{6a+1}$ . Let  $\wp_c$  be the central vertex of wheel graph,  $\deg(\wp_c) = n - 1$ . Therefore  $\wp_c$  has  $n - 1$  eccentric vertices,  $|E(\wp_c)| = n - 1$ . Let  $\wp_i$  be the non-central vertex,  $\deg(\wp_i) = 3$ . Then closed neighbourhood of  $\wp_i$  ie,  $N[\wp_i] = 4$ . Therefore  $\wp_i$  has  $n - 4$  eccentric vertices,  $|E(\wp_i)| = n - 4$ .  $D = \{\wp_c\}$  forms the only dominating set of cardinality one, but not an ED-set. Other than  $W_5$  and  $W_7$  every other wheel graph has an ED-set  $D = \{\wp_c, \wp_x, \wp_y\}$  where  $\wp_c, \wp_x, \wp_y \in V(W_n)$  forms an ED-set and for every  $v \in V(W_n) - D$  there exists a vertex  $\wp_c, \wp_x$  or  $\wp_y$  in  $D$  such that  $E(v) = \wp_c$  or  $\wp_x$  or  $\wp_y$  and  $D = \{\wp_c, \wp_x, \wp_y\}$  forms a TED-set, since  $D$  intersects every minimum ED-set. Therefore  $|D| = 3$ ,  $\gamma_{ted}(W_n) = 3$  for  $n = 6a - 1, 6a, 6a + 1$ .

**Case(ii):** If  $(6a - 2)$ ,  $(6a + 2)$  and  $(6a + 3)$ , then the wheel graphs are of the form  $W_4, W_8, W_9, W_{10}, W_{14}, W_{15}, W_{16}, \dots, W_{6a-2}, W_{6a+2}, W_{6a+3}$ . For  $W_4$ ,  $\gamma_{ted}(W_4) = 4$ . Since  $W_4$  is  $K_4$  which is complete graph (by theorem-2.1). Similar to case(i),  $\wp_c$  is the central vertex of wheel graph and  $\wp_j$  is the non-central vertex,  $|E(\wp_c)| = n - 1$  and  $|E(\wp_i)| = n - 4$ . Similar to case(i) the only unique dominating set  $D = \{\wp_c\}$  whose cardinality is one does not form an ED-set. But a set  $D = \{\wp_c, \wp_x, \wp_y\}$  containing three vertices forms an ED-set, since every vertex  $\wp_i \in V(W_n) - D$  has an eccentric vertex in  $D$  ie,  $E(\wp_i) = \wp_c, \wp_x$  or  $\wp_y$ . But  $D = \{\wp_c, \wp_x, \wp_y\}$  whose cardinality is three does not form a TED-set since it does not intersect every minimum ED-set. But an addition of vertex  $\wp_z$  to the same set gives us a set  $D = \{\wp_c, \wp_x, \wp_y, \wp_z\}$  whose cardinality is four forms an ED-set and it intersects every minimum ED-set of cardinality three, thus becoming TED-set. Therefore  $\gamma_{ted}(W_n) = 4$  for  $n = (6a - 2), (6a + 2)$  and  $(6a + 3)$ .

**Proposition 2.1.** For any graph  $G$ ,

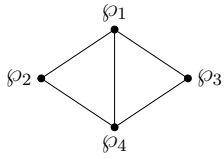
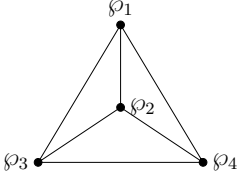
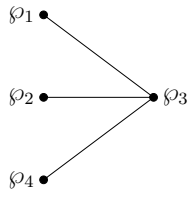
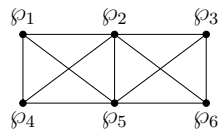
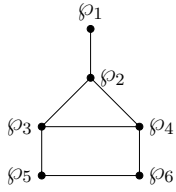
1.  $\gamma_{ted}(G) \geq \lfloor \frac{(2n-q)}{4} \rfloor$ .
2.  $\gamma_{ted}(G) \geq \frac{\text{diam}(G)+1}{3}$ .
3.  $\gamma_{ted}(G) \leq \lfloor \frac{p \Delta(G)}{\delta} \rfloor$ .



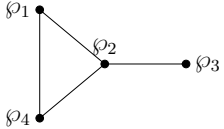
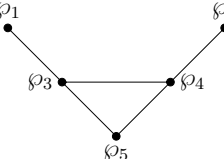
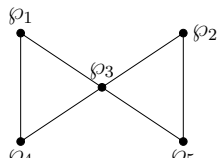
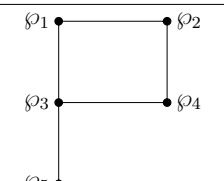
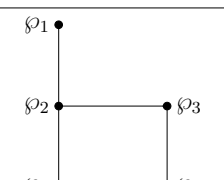
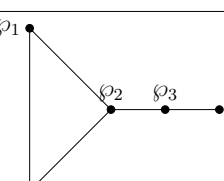
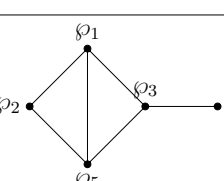
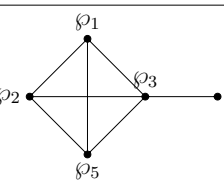
4.  $\gamma_{ted}(G) \geq \lceil \frac{p}{1+\Delta(G)} \rceil$ .

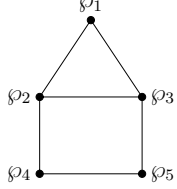
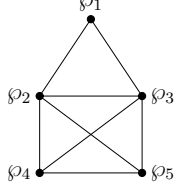
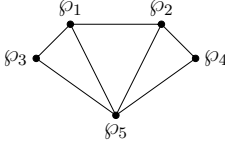
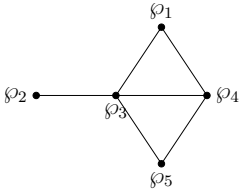
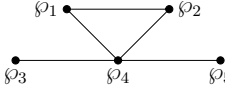
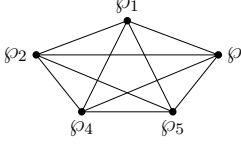
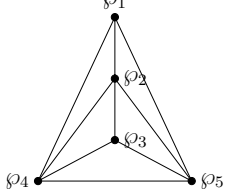
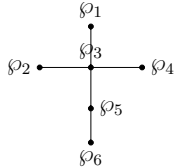
5.  $\gamma_{ted}(G) \leq \lceil n + \Delta(G) - \sqrt{2q} \rceil$ .

The transversal eccentric dominating set,  $\gamma_{ted}(G)$ , upper transversal eccentric dominating set and  $\Gamma_{ted}(G)$  of standard graphs are tabulated.

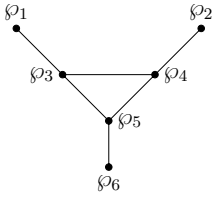
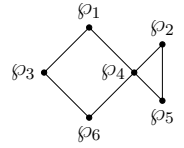
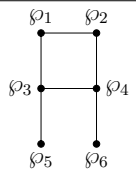
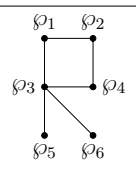
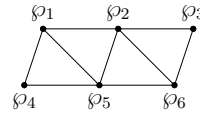
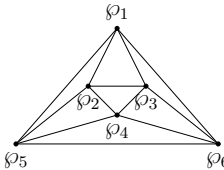
| Graph             | Figure                                                                              | D - Minimum TED set.<br>$ D  = \gamma_{ted}(G)$                                                                                                                                                      | $\gamma_{ted}(G)$ | S - Upper TED set.<br>$ S  = \Gamma_{ted}(G)$                                                                                                                                                        | $\Gamma_{ted}(G)$ |
|-------------------|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| Diamond graph     |    | $\{v_2, v_3\}$ .                                                                                                                                                                                     | 2                 | $\{v_2, v_3\}$ .                                                                                                                                                                                     | 2                 |
| Tetrahedral graph |   | $\{v_1, v_2, v_3, v_4\}$ .                                                                                                                                                                           | 4                 | $\{v_1, v_2, v_3, v_4\}$ .                                                                                                                                                                           | 4                 |
| Claw graph        |  | $\{v_1, v_3\}$ ,<br>$\{v_2, v_3\}$ ,<br>$\{v_3, v_4\}$ .                                                                                                                                             | 2                 | $\{v_1, v_2, v_4\}$ .                                                                                                                                                                                | 3                 |
| (2,3)-King graph  |  | $\{v_1, v_2, v_4\}$ ,<br>$\{v_1, v_3, v_4\}$ ,<br>$\{v_1, v_3, v_6\}$ ,<br>$\{v_1, v_4, v_5\}$ ,<br>$\{v_1, v_4, v_6\}$ ,<br>$\{v_2, v_3, v_6\}$ ,<br>$\{v_3, v_4, v_6\}$ ,<br>$\{v_3, v_5, v_6\}$ . | 3                 | $\{v_1, v_2, v_4\}$ ,<br>$\{v_1, v_3, v_4\}$ ,<br>$\{v_1, v_3, v_6\}$ ,<br>$\{v_1, v_4, v_5\}$ ,<br>$\{v_1, v_4, v_6\}$ ,<br>$\{v_2, v_3, v_6\}$ ,<br>$\{v_3, v_4, v_6\}$ ,<br>$\{v_3, v_5, v_6\}$ . | 3                 |
| Antenna graph     |  | $\{v_1, v_2, v_5\}$ ,<br>$\{v_1, v_2, v_6\}$ ,<br>$\{v_1, v_3, v_5\}$ ,<br>$\{v_1, v_3, v_6\}$ ,<br>$\{v_1, v_4, v_5\}$ ,<br>$\{v_1, v_4, v_6\}$ ,<br>$\{v_1, v_5, v_6\}$ ,<br>$\{v_2, v_5, v_6\}$ . | 3                 | $\{v_1, v_2, v_3, v_4\}$ .                                                                                                                                                                           | 4                 |

*Transversal eccentric domination in graphs*

| Graph                | Figure                                                                              | D - Minimum TED set.<br>$ D  = \gamma_{ted}(G)$                                                                                              | $\gamma_{ted}(G)$ | S - Upper TED set.<br>$ S  = \Gamma_{ted}(G)$                                                                                                | $\Gamma_{ted}(G)$ |
|----------------------|-------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------|-------------------|----------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| Paw graph            |    | $\{v_1, v_3\},$<br>$\{v_2, v_3\},$<br>$\{v_3, v_4\}.$                                                                                        | 2                 | $\{v_1, v_2, v_4\}.$                                                                                                                         | 3                 |
| Bull graph           |    | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_2, v_4\},$<br>$\{v_1, v_2, v_5\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_2, v_3, v_4\}.$                         | 3                 | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_2, v_4\},$<br>$\{v_1, v_2, v_5\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_2, v_3, v_4\}.$                         | 3                 |
| Butterfly graph      |    | $\{v_1, v_2, v_4\},$<br>$\{v_1, v_2, v_5\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_1, v_4, v_5\},$<br>$\{v_2, v_3, v_5\},$<br>$\{v_2, v_4, v_5\}.$ | 3                 | $\{v_1, v_2, v_4\},$<br>$\{v_1, v_2, v_5\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_1, v_4, v_5\},$<br>$\{v_2, v_3, v_5\},$<br>$\{v_2, v_4, v_5\}.$ | 3                 |
| Banner graph         |   | $\{v_2, v_5\}.$                                                                                                                              | 2                 | $\{v_2, v_5\}.$                                                                                                                              | 2                 |
| Fork graph           |  | $\{v_1, v_2, v_5\},$<br>$\{v_1, v_4, v_5\},$<br>$\{v_2, v_3, v_5\},$<br>$\{v_2, v_4, v_5\}.$                                                 | 3                 | $\{v_1, v_2, v_3, v_4\},$<br>$\{v_1, v_3, v_4, v_5\}.$                                                                                       | 4                 |
| (3,2)-Tadpole graph  |  | $\{v_1, v_4\},$<br>$\{v_4, v_5\}.$                                                                                                           | 2                 | $\{v_1, v_2, v_3, v_5\}.$                                                                                                                    | 4                 |
| Kite graph           |  | $\{v_2, v_4\}.$                                                                                                                              | 2                 | $\{v_1, v_2, v_3, v_5\}.$                                                                                                                    | 4                 |
| (4,1)-Lollipop graph |  | $\{v_1, v_4\},$<br>$\{v_2, v_4\},$<br>$\{v_3, v_4\},$<br>$\{v_4, v_5\}.$                                                                     | 2                 | $\{v_1, v_2, v_3, v_5\}.$                                                                                                                    | 4                 |

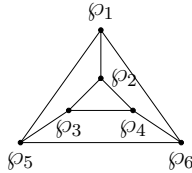
| Graph                                    | Figure                                                                              | D - Minimum<br>TED set.<br>$ D  = \gamma_{ted}(G)$                                                                                           | $\gamma_{ted}(G)$ | S - Upper<br>TED set.<br>$ S  = \Gamma_{ted}(G)$                                                                                             | $\Gamma_{ted}(G)$ |
|------------------------------------------|-------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------|-------------------|----------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| House<br>graph                           |    | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_4, v_5\},$<br>$\{v_2, v_3, v_4\},$<br>$\{v_2, v_3, v_5\},$<br>$\{v_2, v_4, v_5\},$<br>$\{v_3, v_4, v_5\}.$ | 3                 | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_4, v_5\},$<br>$\{v_2, v_3, v_4\},$<br>$\{v_2, v_3, v_5\},$<br>$\{v_2, v_4, v_5\},$<br>$\{v_3, v_4, v_5\}.$ | 3                 |
| House X<br>graph                         |    | $\{v_1, v_2\},$<br>$\{v_1, v_3\},$<br>$\{v_1, v_4\},$<br>$\{v_1, v_5\}.$                                                                     | 2                 | $\{v_2, v_3, v_4, v_5\}.$                                                                                                                    | 4                 |
| Gem<br>graph                             |   | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_2, v_4\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_1, v_3, v_5\},$<br>$\{v_2, v_3, v_4\},$<br>$\{v_2, v_4, v_5\}.$ | 3                 | $\{v_1, v_2, v_3\},$<br>$\{v_1, v_2, v_4\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_1, v_3, v_5\},$<br>$\{v_2, v_3, v_4\},$<br>$\{v_2, v_4, v_5\}.$ | 3                 |
| Dart<br>graph                            |  | $\{v_2, v_3\},$<br>$\{v_2, v_4\}.$                                                                                                           | 2                 | $\{v_1, v_2, v_5\},$<br>$\{v_1, v_3, v_4\},$<br>$\{v_3, v_4, v_5\}.$                                                                         | 3                 |
| Cricket<br>graph                         |  | $\{v_3, v_4\},$<br>$\{v_4, v_5\}.$                                                                                                           | 2                 | $\{v_1, v_2, v_4\},$<br>$\{v_1, v_3, v_5\},$<br>$\{v_2, v_3, v_5\}.$                                                                         | 3                 |
| Pentatope<br>graph                       |  | $\{v_1, v_2, v_3, v_4, v_5\}.$                                                                                                               | 5                 | $\{v_1, v_2, v_3, v_4, v_5\}.$                                                                                                               | 5                 |
| Johnson<br>solid<br>skeleton 12<br>graph |  | $\{v_1, v_3\}.$                                                                                                                              | 2                 | $\{v_1, v_2, v_4, v_5\},$<br>$\{v_2, v_3, v_4, v_5\}.$                                                                                       | 4                 |
| Cross<br>graph                           |  | $\{v_1, v_3, v_6\},$<br>$\{v_2, v_3, v_6\},$<br>$\{v_3, v_4, v_6\},$<br>$\{v_3, v_5, v_6\}.$                                                 | 3                 | $\{v_1, v_2, v_3, v_4, v_5\},$<br>$\{v_1, v_2, v_4, v_5, v_6\}.$                                                                             | 5                 |

## Transversal eccentric domination in graphs

| Graph              | Figure                                                                              | D - Minimum<br>TED set.<br>$ D  = \gamma_{ted}(G)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | $\gamma_{ted}(G)$ | S - Upper<br>TED set.<br>$ S  = \Gamma_{ted}(G)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | $\Gamma_{ted}(G)$ |
|--------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| Net graph          |    | $\{\rho_1, \rho_2, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_6\},$<br>$\{\rho_1, \rho_4, \rho_6\},$<br>$\{\rho_2, \rho_3, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                  | 3                 | $\{\rho_1, \rho_3, \rho_4, \rho_5\},$<br>$\{\rho_2, \rho_3, \rho_4, \rho_5\},$<br>$\{\rho_3, \rho_4, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                           | 4                 |
| Fish graph         |    | $\{\rho_2, \rho_3\},$<br>$\{\rho_3, \rho_5\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | 2                 | $\{\rho_1, \rho_2, \rho_4, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 5                 |
| A graph            |    | $\{\rho_1, \rho_5, \rho_6\},$<br>$\{\rho_2, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | 3                 | $\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                    | 5                 |
| R graph            |   | $\{\rho_1, \rho_2, \rho_3\},$<br>$\{\rho_2, \rho_3, \rho_4\},$<br>$\{\rho_2, \rho_3, \rho_5\},$<br>$\{\rho_2, \rho_3, \rho_6\},$<br>$\{\rho_2, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                 | 3                 | $\{\rho_1, \rho_3, \rho_4, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 5                 |
| 4-polynomial graph |  | $\{\rho_3, \rho_4\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 2                 | $\{\rho_1, \rho_2, \rho_3, \rho_5, \rho_6\},$<br>$\{\rho_1, \rho_2, \rho_4, \rho_5, \rho_6\}.$                                                                                                                                                                                                                                                                                                                                                                                                                                                    | 5                 |
| Octahedral graph   |  | $\{\rho_1, \rho_2, \rho_3, \rho_4\},$<br>$\{\rho_1, \rho_2, \rho_3, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_3, \rho_6\},$<br>$\{\rho_1, \rho_2, \rho_4, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_4, \rho_6\},$<br>$\{\rho_1, \rho_2, \rho_5, \rho_6\},$<br>$\{\rho_1, \rho_3, \rho_4, \rho_6\},$<br>$\{\rho_1, \rho_3, \rho_5, \rho_6\},$<br>$\{\rho_1, \rho_4, \rho_5, \rho_6\},$<br>$\{\rho_2, \rho_3, \rho_4, \rho_5\},$<br>$\{\rho_2, \rho_3, \rho_4, \rho_6\},$<br>$\{\rho_2, \rho_4, \rho_5, \rho_6\},$<br>$\{\rho_3, \rho_4, \rho_5, \rho_6\}.$ | 4                 | $\{\rho_1, \rho_2, \rho_3, \rho_4\},$<br>$\{\rho_1, \rho_2, \rho_3, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_3, \rho_6\},$<br>$\{\rho_1, \rho_2, \rho_4, \rho_5\},$<br>$\{\rho_1, \rho_2, \rho_4, \rho_6\},$<br>$\{\rho_1, \rho_2, \rho_5, \rho_6\},$<br>$\{\rho_1, \rho_3, \rho_4, \rho_6\},$<br>$\{\rho_1, \rho_3, \rho_5, \rho_6\},$<br>$\{\rho_1, \rho_4, \rho_5, \rho_6\},$<br>$\{\rho_2, \rho_3, \rho_4, \rho_5\},$<br>$\{\rho_2, \rho_3, \rho_4, \rho_6\},$<br>$\{\rho_2, \rho_4, \rho_5, \rho_6\},$<br>$\{\rho_3, \rho_4, \rho_5, \rho_6\}.$ | 4                 |

## 3 Conclusions

In this paper TED-set of a graph is defined. Theorems related to find the TED-number of different family of graphs are stated and proved. The upper and lower

| Graph         | Figure                                                                            | D - Minimum TED set.<br>$ D  = \gamma_{ted}(G)$ | $\gamma_{ted}(G)$ | S - Upper TED set.<br>$ S  = \Gamma_{ted}(G)$                                                                                                                              | $\Gamma_{ted}(G)$ |
|---------------|-----------------------------------------------------------------------------------|-------------------------------------------------|-------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| 3-prism graph |  | $\{v_1, v_5, v_6\},$<br>$\{v_2, v_3, v_4\}.$    | 3                 | $\{v_1, v_2, v_3, v_6\},$<br>$\{v_1, v_2, v_4, v_5\},$<br>$\{v_1, v_3, v_4, v_5\},$<br>$\{v_1, v_3, v_4, v_6\},$<br>$\{v_2, v_3, v_5, v_6\},$<br>$\{v_2, v_4, v_5, v_6\}.$ | 4                 |

TED-number along with their respective sets of different standard graphs are tabulated. In future the comparative study of TED-set with eccentric dominating set will be done. The properties of a TED-set related to graph operations such as union, intersection, join and product of graphs will be explored.

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## Some coupled fixed point theorems in complete $E$ – fuzzy metric spaces

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Magie Jose<sup>†</sup>

### Abstract

In this paper some coupled coincidence fixed point theorems are established for the mappings under  $\phi$ - contraction in complete generalized  $E$ - fuzzy metric spaces. Also give an example to validate the theorem and present an application of this theorem to integral equations.

**Keywords:** Fuzzy metric space,  $G$ -metric space,  $E$ -fuzzy metric space, Complete  $E$ -fuzzy metric space, coupled fixed point, coupled coincidence point,  $\phi$  map.

**2020 AMS subject classifications:** 03E72, 47H10, 54E20, 54H25. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1055. ISSN: 1592-7415. eISSN: 2282-8214. ©Shahana A R et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

L.A.Zadeh [1965] introduced fuzzy set theory. This concept was used in almost all branches of mathematics. George.A and P.Veeramani [1994] modified the notion of fuzzy metric space proposed by O.Kramosil and J.Michalek [1975]. Many authors investigated the idea of contractive mappings and constructed fixed point theorems on the basis of this fuzzy metric space.

The notion of Generalized metric space was investigated by Mustafa.Z and B.Sims [2006]. Many researchers proved existence and uniqueness of fixed point of mappings under some contractions in  $G$ -metric space. The concept of coupled fixed point for a function  $F : X^2 \rightarrow X$  was introduced by Bhaskar.T.G and Lakshmikantham.G.V [2006] and proved some coupled fixed point theorems in partially ordered metric space. Also they presented some applications of this in the existence and uniqueness of a solution of periodic boundary value problems. Lakshmikantham.V and Ćirić.Lj. [2009] gave the concept of coupled coincidence and coupled common fixed point for the mappings  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  and established some results. Sedghi et al. [2012] discussed a coupled fixed point theorem for contractions in fuzzy metric space and as a generalization of this result, Xin-Qi-Hu [2011] established a common fixed point theorem for mapping under  $\phi$ -contraction in fuzzy metric space. Many authors have studied coupled fixed point theorems in metric space, fuzzy metric space and  $G$ -metric space. Using the concept of  $G$ -metric space, Sukanya.K.P and Jose [2017] proposed  $E$ -fuzzy metric space and proved some fixed point theorems in this space.

The purpose of this paper is to establish coupled coincidence fixed point theorems for the mappings under  $\phi$ -contraction in complete  $E$ -fuzzy metric space. Also an example to illustrate the theorem is given and an application of this theorem to integral equations is presented.

## 2 Preliminaries

**Definition 2.1.** *B.Schweizer and A.Sklar [1983]: A binary operation  $*$  :  $[0, 1]^2 \rightarrow [0, 1]$  is a continuous  $t$ -norm if it satisfies the following:*

1. *commutative and associative,*
2. *continuous,*
3.  *$t * 1 = t$  for all  $t \in [0, 1]$ ,*
4.  *$t * s \leq t' * s'$  whenever  $t \leq t'$  and  $s \leq s'$  for all  $t, s, t', s' \in [0, 1]$ .*

**Definition 2.2.** Sukanya.K.P and Jose [2017] Let  $X$  is an arbitrary set and  $*$  is a continuous  $t$ -norm, a 3-tuple  $(X, E, *)$  is called  $E$ -fuzzy metric space if  $E$  is a fuzzy set on  $X^3 \times (0, \infty)$  satisfying the following:

1.  $E(\alpha, \beta, \gamma, t) > 0$  and  $E(\alpha, \alpha, \beta, t) \leq E(\alpha, \beta, \gamma, t)$  for all  $\alpha, \beta, \gamma \in X$  with  $\gamma \neq \beta$ ,
2.  $E(\alpha, \beta, \gamma, t) = 1$ , for all  $t > 0$  if and only if  $\alpha = \beta = \gamma$ ,
3.  $E(\alpha, \beta, \gamma, t) = E(p(\alpha, \beta, \gamma), t)$  (symmetry), where  $p$  is a permutation function,
4.  $E(\alpha, a, \gamma, t) * E(a, \beta, \gamma, s) \leq E(\alpha, \beta, \gamma, t + s)$ ,
5.  $E$  is continuous,  $\forall \alpha, \beta, \gamma, a \in X$  and  $t, s > 0$ .

**Definition 2.3.** Sukanya.K.P and Jose [2022] A sequence  $\{\alpha_n\}$  in  $(X, E, *)$  is said to be converges to a point  $\alpha \in X$  if  $E(\alpha_n, \alpha, \alpha, t) \rightarrow 1$  as  $n \rightarrow \infty$ .

**Remark 2.1.** Shahana.A.R and Jose [in press] A sequence  $\{\alpha_n\}$  in  $(X, E, *)$  is said to be a cauchy sequence if for each  $0 < r < 1$  and  $t > 0$  there exists  $k \in \mathbb{N}$  such that  $E(\alpha_m, \alpha_m, \alpha_n, t) > 1 - r$  for each  $m, n \geq k$ .

**Definition 2.4.** Sukanya.K.P and Jose [2022] A complete  $E$ -fuzzy metric space is the  $E$ -fuzzy metric space in which every cauchy sequence is convergent.

**Remark 2.2.** Shahana.A.R and Jose [in press] Let  $(X_1, E_1, *)$  and  $(X_2, E_2, *)$  be  $E$ -fuzzy metric spaces. A function  $f : X_1 \rightarrow X_2$  is continuous at a point  $\alpha \in X_1$  if and only if every convergent sequence  $\{\alpha_n\} \rightarrow \alpha$  in  $X_1$ ,  $\{f(\alpha_n)\} \rightarrow f(\alpha)$  in  $X_2$ .

**Definition 2.5.** Bhaskar.T.G and Lakshmikantham.G.V [2006] Let  $X$  be a nonempty set, an element  $(\alpha, \beta) \in X^2$  is called a coupled fixed point of a mapping  $F : X^2 \rightarrow X$  if  $F(\alpha, \beta) = \alpha$  and  $F(\beta, \alpha) = \beta$ .

**Definition 2.6.** Lakshmikantham.V and Ćirić.Lj. [2009] Let  $X$  be a non empty set, the element  $(\alpha, \beta) \in X^2$  is called a coupled coincidence point of the mappings  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  if  $F(\alpha, \beta) = g\alpha$  and  $F(\beta, \alpha) = g\beta$ .

**Definition 2.7.** J.X.Fang [2009] An element  $\alpha \in X$  is called common fixed point of the mappings  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  if  $F(\alpha, \alpha) = g\alpha = \alpha$ , then the  $F$  and  $g$ .

**Definition 2.8.** Lakshmikantham.V and Ćirić.Lj. [2009] The mappings  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  are said to be commutative if  $gF(\alpha, \beta) = F(g\alpha, g\beta)$ .



### 3 Coupled fixed point theorems in $E$ -fuzzy metric spaces

Let  $\phi$  be the function,  $\phi : (0, \infty) \rightarrow (0, \infty)$  such that  $\phi(t) < t$  and for any  $t > 0$ ,  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$  for any  $t > 0$ . Some coupled fixed point theorems for  $\phi$  contraction in complete  $E$ -fuzzy metric space are established in this section.

In this paper we denote,

$$[E(\alpha, \beta, \gamma, t)]^n = \underbrace{E(\alpha, \beta, \gamma, t) * E(\alpha, \beta, \gamma, t) * \dots * E(\alpha, \beta, \gamma, t)}_n \text{ for all } n \in N.$$

**Theorem 3.1.** Let  $(X, E, *)$  be a complete  $E$ -Fuzzy metric space. Let  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), \phi(t)) \geq E(g(\alpha), g(\mu), g(\gamma), t) * E(g(\beta), g(\eta), g(\omega), t)$ ,  $\forall \alpha, \beta, \mu, \eta, \gamma, \omega \in X$ . Suppose that  $F$  and  $g$  satisfy the following:

- $F(X^2) \subseteq g(X)$
- $g$  is continuous and commutes with  $F$

Then there exists a unique  $\alpha$  in  $X$  such that  $\alpha = g\alpha = F(\alpha, \alpha)$ .

**Proof.** Since  $F(X^2) \subseteq g(X)$ , we can construct two sequences  $\{g\alpha_n\}$  and  $\{g\beta_n\}$  in  $X$  such that  $g(\alpha_{n+1}) = F(\alpha_n, \beta_n)$ ,  $g(\beta_{n+1}) = F(\beta_n, \alpha_n) \forall n \geq 0$ . For any  $0 < r < 1$  there exists a  $0 < s < 1$  such that  $\underbrace{(1-s) * (1-s) * \dots * (1-s)}_k >$

$$(1-r) \forall k \in N.$$

We have

$$\lim_{t \rightarrow \infty} E(\alpha, \beta, \gamma, t) = 1 \quad \forall \alpha, \beta, \gamma \in X \quad (1)$$

Thus,

for any  $t > 0$  and  $s > 0$  there exists  $t_0 > 0$  such that  $E(g\alpha_0, g\alpha_1, g\alpha_2, t_0) > 1-s$ ,  $E(g\beta_0, g\beta_1, g\beta_2, t_0) > 1-s$  and  $E(g\alpha_2, g\alpha_1, g\alpha_0, t_0) > 1-s$ ,  $E(g\beta_2, g\beta_1, g\beta_0, t_0) > 1-s$ . For a  $\phi$  function there exists  $k \in N$  such that  $\sum_{n=k_0}^{\infty} \phi^n(t_0) < \frac{t}{2}$ .

$$\begin{aligned} E(g\alpha_1, g\alpha_2, g\alpha_3, \phi(t_0)) &= E(F(\alpha_0, \beta_0), F(\alpha_1, \beta_1), F(\alpha_2, \beta_2), \phi(t_0)) \\ &\geq E(g\alpha_0, g\alpha_1, g\alpha_2, t_0) * E(g\beta_0, g\beta_1, g\beta_2, t_0). \end{aligned}$$

Similarly,

$$E(g\beta_1, g\beta_2, g\beta_3, \phi(t_0)) \geq E(g\beta_0, g\beta_1, g\beta_2, t_0) * E(g\alpha_0, g\alpha_1, g\alpha_2, t_0).$$

Now,

$$\begin{aligned} E(g\alpha_2, g\alpha_3, g\alpha_4, \phi^2(t_0)) &= E(F(\alpha_1, \beta_1), F(\alpha_2, \beta_2), F(\alpha_3, \beta_3), \phi^2(t_0)) \\ &\geq E(g\alpha_1, g\alpha_2, g\alpha_3, \phi(t_0)) * E(g\beta_1, g\beta_2, g\beta_3, \phi(t_0)) \\ &\geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^2 * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^2. \end{aligned}$$

Similarly,

$$E(g\beta_2, g\beta_3, g\beta_4, \phi^2(t_0)) \geq [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^2 \\ * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^2.$$

Continuing this, we get

$$E(g\alpha_n, g\alpha_{n+1}, g\alpha_{n+2}, \phi^n(t_0)) \geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}} \\ * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{n-1}} \quad \text{and} \\ E(g\beta_n, g\beta_{n+1}, g\beta_{n+2}, \phi^n(t_0)) \geq [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{n-1}} \\ * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}}.$$

In the same way,

$$E(g\alpha_{n+2}, g\alpha_{n+1}, g\alpha_n, \phi^n(t_0)) \geq [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}} \\ * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}} \quad \text{and} \\ E(g\beta_{n+2}, g\beta_{n+1}, g\beta_n, \phi^n(t_0)) \geq [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}} \\ * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}}.$$

For  $m, n \in N$  with  $m > n \geq k_0$

$$E(g\alpha_m, g\alpha_m, g\alpha_n, t) \geq E(g\alpha_{n+1}, g\alpha_m, g\alpha_n, \frac{t}{2}) * E(g\alpha_m, g\alpha_{n+1}, g\alpha_n, \frac{t}{2}) \\ \geq E(g\alpha_{n+1}, g\alpha_m, g\alpha_n, \sum_{k=n}^{m-2} \phi^k(t_0)) * E(g\alpha_m, g\alpha_{n+1}, g\alpha_n, \sum_{k=n}^{m-2} \phi^k(t_0)) \\ \geq E(g\alpha_n, g\alpha_{n+1}, g\alpha_{n+2}, \phi^n(t_0)) * E(g\alpha_{n+1}, g\alpha_{n+2}, g\alpha_{n+3}, \phi^{n+1}(t_0)) * \dots \\ * E(g\alpha_{m-2}, g\alpha_{m-1}, g\alpha_m, \phi^{m-2}(t_0)) * E(g\alpha_{n+2}, g\alpha_{n+1}, g\alpha_n, \phi^n(t_0)) \\ * E(g\alpha_{n+3}, g\alpha_{n+2}, g\alpha_{n+1}, \phi^{n+1}(t_0)) * \dots \\ * E(g\alpha_m, g\alpha_{m-1}, g\alpha_{m-2}, \phi^{m-2}(t_0)) \\ \geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}} [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{n-1}} \\ * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^n} * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^n} \\ * \dots * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{m-3}} \\ * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{m-3}} * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}} \\ * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}} * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^n} \\ * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^n} * \dots * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{m-3}} \\ * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{m-3}} \\ \geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} \\ [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} \\ > \underbrace{(1-s) * (1-s) * \dots * (1-s)}_{2^{2(m-n-1)(m+n-4)}} \\ > (1-r).$$

Thus  $\{g\alpha_n\}$  is a cauchy sequence. Similarly we can show that  $\{g\beta_n\}$  is a cauchy sequence. Since  $E$  is a complete  $E$ -fuzzy metric space, we get  $\{g\alpha_n\}$  converges to some  $\alpha \in X$  and  $\{g\beta_n\}$  converges to some  $\beta \in X$ . Thus  $\lim_{n \rightarrow \infty} F(\alpha_n, \beta_n) = \lim_{n \rightarrow \infty} g(\alpha_n) = \alpha$  and  $\lim_{n \rightarrow \infty} F(\beta_n, \alpha_n) = \lim_{n \rightarrow \infty} g(\beta_n) = \beta$ .

As  $g$  is continuous, we have  $\{gg\alpha_n\}$  converges to  $g\alpha$  and  $\{gg\beta_n\}$  converges to  $g\beta$ .

Also since  $g$  and  $F$  commute, we have

$$\begin{aligned}
 E(gg\alpha_{n+1}, F(\alpha, \beta), F(\alpha, \beta), t) &= E(F(g\alpha_n, g\beta_n, F(\alpha, \beta), F(\alpha, \beta), t) \\
 &\geq E(F(g\alpha_n, g\beta_n), F(\alpha, \beta), F(\alpha, \beta), \phi(t)) \\
 &\geq E(gg\alpha_n, g\alpha, g\alpha, t) * E(gg\beta_n, g\beta, g\beta, t).
 \end{aligned}$$

Since  $E$  is continuous on its variable and letting  $n \rightarrow \infty$ , we get

$$E(g\alpha, F(\alpha, \beta), F(\alpha, \beta), t) \geq E(g\alpha, g\alpha, g\alpha, t) * E(g\beta, g\beta, g\beta, t).$$

Hence  $E(g\alpha, F(\alpha, \beta), F(\alpha, \beta), t) = 1$ . That is  $F(\alpha, \beta) = g\alpha$ .

Similarly we can show  $F(\beta, \alpha) = g\beta$ . Thus  $(\alpha, \beta)$  is a coupled coincidence point of  $F$  and  $g$ .

Now, for any  $r > 0$  there exists  $s > 0$  such that

$$\underbrace{(1-s) * (1-s) * \dots * (1-s)}_k > (1-r) \quad \forall k \in N.$$

Using (1), for any  $t > 0$  and  $s > 0$  there exists  $t_0 > 0$  such that  $E(g\alpha, \beta, \beta, t_0) > 1-s$  and  $E(g\beta, \alpha, \alpha, t_0) > 1-s$ . Since  $\phi^n(t_0) \rightarrow 0$  there exists  $k \in N$  such that  $\phi^n(t_0) < t \quad \forall n \geq k$ .

$$\begin{aligned}
 E(g\alpha, g\beta_{n+1}, g\beta_{n+1}, \phi(t_0)) &= E(F(\alpha, \beta), F(\beta_n, \alpha_n), F(\beta_n, \alpha_n), \phi(t_0)) \\
 &\geq E(g\alpha, g\beta_n, g\beta_n, t_0) * E(g\beta, g\alpha_n, g\alpha_n, t_0).
 \end{aligned}$$

As  $n \rightarrow \infty$

$$E(g\alpha, \beta, \beta, \phi(t_0)) \geq E(g\alpha, \beta, \beta, t_0) * E(g\beta, \alpha, \alpha, t_0).$$

Similarly,  $E(g\beta, \alpha, \alpha, \phi(t_0)) \geq E(g\beta, \alpha, \alpha, t_0) * E(g\alpha, \beta, \beta, t_0)$ .

$$E(g\alpha, \beta, \beta, \phi(t_0)) * E(g\beta, \alpha, \alpha, \phi(t_0)) \geq [E(g\alpha, \beta, \beta, t_0)]^2 * [E(g\beta, \alpha, \alpha, t_0)]^2.$$

Continuing this process, we get,  $\forall n \in N$

$$\begin{aligned}
 E(g\alpha, \beta, \beta, \phi^n(t_0)) * E(g\beta, \alpha, \alpha, \phi^n(t_0)) &\geq [E(g\alpha, \beta, \beta, t_0)]^{2^n} \\
 &* [E(g\beta, \alpha, \alpha, t_0)]^{2^n}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 E(g\alpha, \beta, \beta, t) * E(g\beta, \alpha, \alpha, t) &\geq E(g\alpha, \beta, \beta, \phi^k(t_0)) * E(g\beta, \alpha, \alpha, \phi^k(t_0)) \\
 &\geq [E(g\alpha, \beta, \beta, t_0)]^{2^k} * [E(g\beta, \alpha, \alpha, t_0)]^{2^k} \\
 &> \underbrace{(1-s) * (1-s) * \dots * (1-s)}_{2^{2k}} \\
 &> 1-r.
 \end{aligned}$$

Thus for any  $r > 0$ ,  $E(g\alpha, \beta, \beta, t) * E(g\beta, \alpha, \alpha, t) > 1-r \quad \forall t > 0$ .

Therefor  $g\alpha = \beta$  and  $g\beta = \alpha$ . Now for any  $r > 0$  there exists a  $s > 0$  such that

$$\underbrace{(1-s) * (1-s) * \dots * (1-s)}_k > (1-r) \quad \forall k \in N.$$

Using (1), for any  $t > 0$  and  $s > 0$  there exists  $t_0 > 0$  such that  $E(\alpha, \beta, \gamma, t_0) > 1-s$ . Since  $\phi^n(t_0) \rightarrow 0$  there exists  $k \in N$  such that  $\phi^n(t_0) < t \quad \forall n \geq K$ .

$$\begin{aligned}
 E(g\alpha_{n+1}, g\beta_{n+1}, g\beta_{n+1}, \phi(t_0)) &= E(F(\alpha_n, \beta_n), F(\beta_n, \alpha_n), F(\beta_n, \alpha_n), \phi(t_0)) \\
 &\geq E(g\alpha_n, g\beta_n, g\beta_n, t_0) * E(g\beta_n, g\alpha_n, g\alpha_n, t_0)
 \end{aligned}$$

As  $n \rightarrow \infty$ ,  $E(\alpha, \beta, \beta, \phi(t_0)) \geq E(\alpha, \beta, \beta, t_0) * E(\beta, \alpha, \alpha, t_0)$

Continuing this process, we get

$$E(\alpha, \beta, \beta, \phi^n(t_0)) \geq [E(\alpha, \beta, \beta, t_0)]^n * [E(\beta, \alpha, \alpha, t_0)]^n.$$

Thus

$$\begin{aligned} E(\alpha, \beta, \beta, t) &\geq E(\alpha, \beta, \beta, \phi^n(t_0)) \\ &\geq [E(\alpha, \beta, \beta, t_0)]^n * [E(\beta, \alpha, \alpha, t_0)]^n \\ &> \underbrace{(1-s) * (1-s) * \dots * (1-s)}_{2k} \\ &> 1-r. \end{aligned}$$

Hence for any  $r > 0$ ,  $E(\alpha, \beta, \beta, t) > 1-r \forall t$ , implies  $\alpha = \beta$ , hence  $g(\alpha) = F(\alpha, \alpha) = \alpha$ . Thus  $(\alpha, \alpha)$  is the unique common coupled fixed point of  $F$  and  $g$ . That is  $F$  and  $g$  have a unique common fixed point in  $X$ .  $\square$

**Corollary 3.1.** Let  $(X, E, *)$  be a complete  $E$ -Fuzzy metric space and  $F : X^2 \rightarrow X$  be a mapping such that  $E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), \phi(t)) \geq E(\alpha, \mu, \gamma, t) * E(\beta, \eta, \omega, t) \forall \alpha, \beta, \gamma, \mu, \eta, \omega \in X$  and  $t > 0$ . Then there exists a unique  $\alpha \in X$  such that  $\alpha = F(\alpha, \alpha)$ .

**Proof.** Take  $g$  as the identity map in Theorem(3.1), we get the result.  $\square$

**Corollary 3.2.** Let  $(X, E, *)$  be a complete  $E$ -Fuzzy metric space and  $F : X^2 \rightarrow X$  be a mapping such that  $E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), kt) \geq E(\alpha, \mu, \gamma, t) * E(\beta, \eta, \omega, t) \alpha, \beta, \gamma, \mu, \eta, \omega \in X$ ,  $0 < k < 1$  and  $t > 0$ . Then there exists a unique  $\alpha \in X$  such that  $\alpha = F(\alpha, \alpha)$ .

**Proof.** Define  $\phi(t) = kt$ . Then by Corollary(3.1) we get the result.  $\square$

**Example 3.1.** Let  $X = [0, 1]$ . Define  $t * s = ts \forall t, s \in X$  and  $E : X^3 \rightarrow [0, 1]$  as  $E(\alpha, \beta, \gamma, t) = [\exp(\frac{G(\alpha, \beta, \gamma)}{t})]^{-1}$  where  $G(\alpha, \beta, \gamma) = |\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|$ . Let  $\phi(t) = \frac{t}{6}$ . Define  $g : X \rightarrow X$  as  $g(\alpha) = \alpha$  and  $F : X^2 \rightarrow X$  as  $F(\alpha, \beta) = \frac{\alpha\beta}{6} \forall \alpha, \beta \in X$ .

Then there exists unique common fixed point of  $F$  and  $g$ , by Theorem 3.1.

$$\begin{aligned} G(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega)) &= |\frac{\alpha\beta}{6} - \frac{\mu\eta}{6}| + |\frac{\mu\eta}{6} - \frac{\gamma\omega}{6}| + |\frac{\gamma\omega}{6} - \frac{\alpha\beta}{6}| \\ &\leq \frac{1}{6}\{(|\alpha - \mu| + |\mu - \gamma| + |\gamma - \alpha|) + (|\beta - \eta| + |\eta - \omega| + |\omega - \beta|)\} \\ &\leq \frac{1}{6}\{G(g\alpha, g\mu, g\gamma) + G(g\beta, g\eta, g\omega)\} \end{aligned}$$

$$\begin{aligned} E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), \phi(t)) &= [\exp(\frac{G(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega))}{\phi(t)})]^{-1} \\ &\geq [\exp(\frac{\frac{1}{6}G(g\alpha, g\mu, g\gamma) + \frac{1}{6}G(g\beta, g\eta, g\omega)}{\frac{1}{6}})]^{-1} \\ &\geq [\exp(\frac{G(g\alpha, g\mu, g\gamma)}{t})]^{-1} * [\exp(\frac{G(g\beta, g\eta, g\omega)}{t})]^{-1} \\ &\geq E(g\alpha, g\mu, g\gamma, t) * E(g\beta, g\eta, g\omega, t). \end{aligned}$$

It satisfies Theorem(3.1).

**Theorem 3.2.** Let  $(X, E, *)$  be a complete  $E$ -fuzzy metric space and let  $F : X^2 \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $\forall \alpha, \beta, \gamma, \mu, \eta, \omega \in X$ ,  $E(F(\alpha, \beta), F(\mu, \eta), F(\omega, \gamma), \phi(t)) \geq \min\{E(g\alpha, g\mu, g\omega, t), E(g\beta, g\eta, g\gamma, t)\}$ . Assume that  $F(X^2) \subseteq g(X)$  and  $g$  is continuous and commutes with  $F$ . Then there exists unique  $\alpha \in X$  such that  $F(\alpha, \alpha) = g(\alpha) = \alpha$ .

*Proof.* The procedure of proof of this theorem is similar to that of theorem(3.1), except for the implementation of the condition given in this statement.  $\square$

## 4 Application

This section presents the existence and uniqueness for the solution of the following integral equation by using the theoretical results discussed in the previous section.

Let  $X = C([0, 1])$ . Define  $E : X^3 \times (0, \infty) \rightarrow (0, 1]$  as  $E(\alpha, \beta, \gamma, t') = [\exp(\frac{G(\alpha, \beta, \gamma)}{t'})]^{-1}$  where  $G : X^3 \rightarrow R$  is  $G(\alpha, \beta, \gamma) = \sup_{\epsilon \in [0, 1]} |\alpha(\epsilon) - \beta(\epsilon)| + \sup_{\epsilon \in [0, 1]} |\beta(\epsilon) - \gamma(\epsilon)| + \sup_{\epsilon \in [0, 1]} |\gamma(\epsilon) - \alpha(\epsilon)|$  and the  $t$ -norm  $*$  is  $t * s = ts$ . Then  $(X, E, *)$  is a complete  $E$ -fuzzy metric space. Let  $\phi : (0, \infty) \rightarrow (0, \infty)$  be defined as  $\phi(t') = rt'$ ,  $0 < r < 1$  and  $g : X \rightarrow X$  as  $g(\alpha) = \alpha$ . Consider the integral equation,

$$F(\alpha, \beta)(\epsilon) = f(\epsilon) + \int_0^1 k(\epsilon, \delta) h(\delta, \alpha(\delta), \beta(\delta)) d\delta, \epsilon \in [0, 1]$$

with the following conditions

1.  $h : [0, 1] \times (0, \infty)^2 \rightarrow (0, \infty)$  and  $f : [0, 1] \rightarrow (0, \infty)$  are continuous.
2. there exists  $0 < r < 1$  such that  $|h(\delta, \alpha, \beta) - h(\delta, \mu, \eta)| \leq r(|\alpha - \mu| + |\beta - \eta|)$   $\forall \alpha, \beta, \mu, \eta \in (0, \infty)$  and  $\forall \delta \in [0, 1]$ .
3.  $\forall \epsilon, \delta \in [0, 1], \sup_{\epsilon \in [0, 1]} \int_0^1 k(\epsilon, \delta) d\delta < 1$ .

Then by Theorem(3.1) there exists an unique solution of the integral equation. For

$$\begin{aligned} & G(F(\alpha, \beta)(\epsilon), F(\mu, \eta)(\epsilon), F(\gamma, \omega)(\epsilon)) \\ &= \sup_{\epsilon \in [0, 1]} |F(\alpha, \beta)(\epsilon) - F(\mu, \eta)(\epsilon)| + \sup_{\epsilon \in [0, 1]} |F(\mu, \eta)(\epsilon) - F(\gamma, \omega)(\epsilon)| \\ & \quad + \sup_{\epsilon \in [0, 1]} |F(\gamma, \omega)(\epsilon) - F(\alpha, \beta)(\epsilon)| \end{aligned}$$

$$\begin{aligned} & \leq \sup_{\epsilon \in [0, 1]} \left| \int_0^1 k(\epsilon, \delta) h(\delta, \alpha(\delta), \beta(\delta)) d\delta - \int_0^1 k(\epsilon, \delta) h(\delta, \mu(\delta), \eta(\delta)) d\delta \right| \\ & \quad + \sup_{\epsilon \in [0, 1]} \left| \int_0^1 k(\epsilon, \delta) h(\delta, \mu(\delta), \eta(\delta)) d\delta - \int_0^1 k(\epsilon, \delta) h(\delta, \gamma(\delta), \omega(\delta)) d\delta \right| \\ & \quad + \sup_{\epsilon \in [0, 1]} \left| \int_0^1 k(\epsilon, \delta) h(\delta, \gamma(\delta), \omega(\delta)) d\delta - \int_0^1 k(\epsilon, \delta) h(\delta, \alpha(\delta), \beta(\delta)) d\delta \right| \end{aligned}$$

$$\begin{aligned} & \leq \sup_{\epsilon \in [0, 1]} \left| \int_0^1 k(\epsilon, \delta) \{ |h(\delta, \alpha(\delta), \beta(\delta)) - h(\delta, \mu(\delta), \eta(\delta))| \} d\delta \right| \\ & \quad + \sup_{\epsilon \in [0, 1]} \left| \int_0^1 k(\epsilon, \delta) \{ |h(\delta, \mu(\delta), \eta(\delta)) - h(\delta, \gamma(\delta), \omega(\delta))| \} d\delta \right| \end{aligned}$$

### Some coupled fixed point theorems in complete E-fuzzy metric spaces

$$\begin{aligned}
& + \sup_{\epsilon \in [0,1]} \left| \int_0^1 k(\epsilon, \delta) \{h(\delta, \gamma(\delta), \omega(\delta)) - h(\delta, \alpha(\delta), \beta(\delta))\} d\delta \right| \\
& \leq \sup_{\epsilon \in [0,1]} \int_0^1 k(\epsilon, \delta) d\delta \{ \sup |\alpha(\epsilon) - \mu(\epsilon)| + \sup |\beta(\epsilon) - \eta(\epsilon)| + \sup |\mu(\epsilon) - \gamma(\epsilon)| \\
& \quad + \sup |\eta(\epsilon) - \omega(\epsilon)| + \sup |\gamma(\epsilon) - \alpha(\epsilon)| + \sup |\omega(\epsilon) - \beta(\epsilon)| \} \\
& \leq r \{G(\alpha, \mu, \gamma) + G(\beta, \eta, \omega)\}. \text{ Then}
\end{aligned}$$

$$\begin{aligned}
E(F(\alpha, \beta)(t), F(\mu, \eta)(t), F(\gamma, \omega)(t), \phi(t')) &= [exp^{\frac{G(F(\alpha, \beta)(t), F(\mu, \eta)(t), F(\gamma, \omega)(t))}{\phi(t')}}]^{-1} \\
&\geq [exp^{\frac{r\{G(\alpha, \mu, \gamma) + G(\beta, \eta, \omega)\}}{rt'}}]^{-1} \\
&\geq [exp^{\frac{G(\alpha, \mu, \gamma)}{t'}}]^{-1} [exp^{\frac{G(\alpha, \mu, \gamma)}{t'}}]^{-1} \\
&\geq E(\alpha, \mu, \gamma, t') * E(\beta, \eta, \omega, t').
\end{aligned}$$

Hence it satisfies Theorem(3.1) Thus there exists unique  $\alpha \in C[0, 1]$  as the solution of the given integral equation.

## 5 Conclusions

In this work, coupled coincidence fixed point theorems for the mappings under  $\phi$ -contraction in complete E-fuzzy metric space are established and applied these results to show the existence and uniqueness for the solution of the integral equations. These are the generalization of existing results in the literature of fuzzy metric spaces. Using different types of contractive conditions, we can develop coupled fixed point theorems in complete E-fuzzy metric spaces.

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## Some new odd prime graphs

S. Meena\*

G. Gajalakshmi<sup>†</sup>

### Abstract

For a graph  $G$ , a bijection  $f$  is called an odd prime labeling, if  $f$  from  $V$  to  $\{1, 3, 5, \dots, 2|V| - 1\}$  for each edge  $uv$  in  $G$  the greatest common divisor of the labels of end vertices  $(f(u), f(v))$  is one. In this paper we investigate the existence of odd prime labeling of some new classes of graphs and we prove that the graphs such as the  $Z - P_n$  graph, Fish graph, Umbrella graph, Cocount tree,  $F$ -tree,  $Y$ -tree and Double Sunflower graph are odd prime graphs.

**Keywords:** Odd prime graph,  $Z - P_n$  graph, Fish graph, Umbrella graph, Cocount tree,  $F$ -tree,  $Y$ -tree, Double Sunflower graph.

**2020 AMS subject classifications:** 05C78<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1056. ISSN: 1592-7415. eISSN: 2282-8214. ©S. Meena et al. This paper is published under the CC-BY licence agreement.



## 1 Introduction

In this paper by a graph  $G = \langle V(G), E(G) \rangle$  we mean a simple graph. For graph theoretical terminologies we refer J .A.Bondy and U .S. R.Murthy [1976].

A graph labeling is an assignment of integers to the vertices or edges or both subject to some constraints. For entire survey of graph labeling we refer Gallian [2009]. The concept of prime labeling was introduced by Roger Etringer and was studied further by many researchers Deretsky et al. [1991], Tout et al. [1982]. Meena and Vaithilingam [2013] proved that crown related graphs are prime graphs. Meena and Naveen [2018] investigated about the prime labeling of graphs related to bicyclic graphs. Prime labeling in the concept of graph operation was discussed by Meena and Kavitha [2015]. The Existence of odd prime labeling for some new classes of graph was discussed by Meena et al. [2021].

The notion of odd prime labeling was introduced by Prajapati and Shah [2018] and they have proved many researchers. We give some families of odd prime graphs and some necessary condition for a graph to be odd prime graph.

Motivated by this study, further studied by in this paper we investigate the existence of odd prime labeling of some new classes of odd prime graphs.

**Definition 1.1.** Let  $G = \langle V(G), E(G) \rangle$  be a graph. A bijection  $f : V(G) \rightarrow O_{|V|}$  is called an odd prime labeling if for each edge  $uv \in E$ , greatest common divisor  $\langle f(u), f(v) \rangle$  is 1. A graph is called an odd prime graph if it admits odd prime labeling.

Here  $O_{|V|} = \{1, 3, 5, \dots, 2|V| - n\}$

**Definition 1.2.**  $Z - P_n$  is a graph obtained from a pair of  $P_1$  and  $P_2$  of path of length  $n$  in which the  $i^{th}$  vertex of a path  $P_1$  is joined with  $(i + 1)^{th}$  vertex of a path  $P_2$ .

**Definition 1.3.** Fish graph is a graph obtained by attaching one of the vertex of  $K_3$  to any one of the vertex of  $C_n$ . It is denoted by  $C_n @ K_3$ .

**Definition 1.4.** For any integers  $m > 2, n > 1$ . An umbrella graph  $U(m, n)$  is the graph obtained by identifying the end vertex of path  $P_n$  with the central vertex of a Fan graph  $F_m$ .

**Definition 1.5.** Coconut tree graph is obtained by identifying the central vertex of  $K_{1,m}$  with a pendant vertex of the path  $P_n$ .

**Definition 1.6.**  $F$ - tree on  $n + 2$  vertices, denoted by  $F_n$  is obtained from a path  $P_n$  by attaching exactly two pendant vertices of the  $n - 1$  and  $n^{th}$  vertex of  $P_n$ .

**Definition 1.7.**  $Y$  - tree on  $n + 1$  vertices, denoted by  $Y_n$  is obtained from a path  $P_n$  by attaching a pendant vertex of the  $n^{th}$  vertex of  $P_n$ .

**Definition 1.8.** A double sunflower graph order  $n$ , denoted by  $DSF_n$ , is a graph obtained from the graph  $SF_n$  by inserting a new vertex  $C_i$  on each edges  $a_i a_{i+1}$  and adding edges for each  $i$ .

## 2 Main results

**Theorem 2.1.**  $Z - (P_n)$  is an odd prime graph for all integers  $n \geq 3$ .

*Proof.* Let  $G = Z - (P_n)$  be the graph

$$V(G) = \{u_i, v_i / 1 \leq i \leq n\}$$

$$E(G) = \{(u_i u_{i+1}), (v_i v_{i+1}) / 1 \leq i \leq n - 1\} \cup \{(v_i u_{i+1}) / 1 \leq i \leq n - 1\}$$

Now  $|V(G)| = 2n$  and  $|E(G)| = 3(n-1)$

Define a labeling  $f : V \rightarrow O_{2n}$  as follows

$$f(u_i) = 4i-5 \quad \text{for } 1 \leq i \leq n$$

$$f(v_i) = 4i+1 \quad \text{for } 1 \leq i \leq n$$

Clearly vertex labels are distinct.

For each  $e = uv \in E$ , if  $\gcd(f(u), f(v)) = 1$

$$(i) e = u_1 u_2, \gcd(f(u_1), f(u_2)) = \gcd(1, 3) = 1$$

$$(ii) e = u_i u_{i+1}, \gcd(f(u_i), f(u_{i+1})) = \gcd(4i-5, 4i-1) = 1 \quad \text{for } 1 \leq i \leq n$$

$$(iii) e = v_i v_{i+1}, \gcd(f(v_i), f(v_{i+1})) = \gcd(4i+1, 4i+5) = 1 \quad \text{for } 1 \leq i \leq n$$

$$(iv) e = v_i u_{i+1}, \gcd(f(v_i), f(u_{i+1})) = \gcd(4i+1, 4i-5) = 1 \quad \text{for } 1 \leq i \leq n$$

$$(v) e = v_{n-1} v_n, \gcd(f(v_{n-1}), f(v_n)) = \gcd(4n-3, 4n-1) = 1 \quad \text{for } 1 \leq i \leq n$$

Thus  $f$  admits odd prime labeling on  $Z - (P_n)$  and hence  $Z - (P_n)$  is an odd prime graph.  $\square$

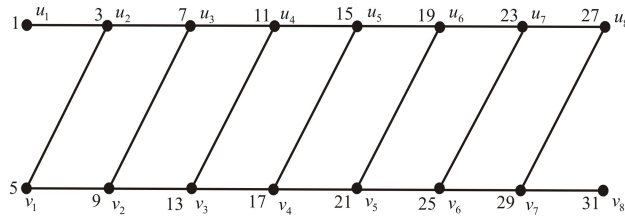


Figure 1:  $Z - (P_n)$  and its odd prime labeling

**Theorem 2.2.** Fish graph is an odd prime graph for  $n \geq 3$ .

*Proof.* Let  $G = C_n @ K_3$  be the graph

$$V(G) = \{u_i / 1 \leq i \leq n\} \cup \{(v_1 v_2)\}$$

$$E(G) = \{(u_i u_{i+1}) / 1 \leq i \leq n - 1\} \cup \{(u_1 u_n)\} \cup \{(u_1 v_i) / 1 \leq i \leq 2\} \cup \{(v_1, v_2)\}$$

Now  $|V(G)| = n + 2$  and  $|E(G)| = n + 3$

Define a labeling  $f : V \rightarrow O_{n+2}$  as follows.

$$f(u_1) = 1$$

$$f(v_1) = 3$$

$$f(v_2) = 5$$

$$f(u_i) = 2i + 3 \text{ for } 2 \leq i \leq n$$

Clearly vertex labels are distinct.

For each  $e = uv \in E$ , if  $\gcd(f(u), f(v)) = 1$

$$(i) e = v_1v_2, \gcd(f(v_1), f(v_2)) = \gcd(3, 5) = 1$$

$$(ii) e = u_1v_1, \gcd(f(u_1), f(v_1)) = \gcd(1, 3) = 1 \text{ for } 1 \leq i \leq n$$

$$(iii) e = u_1v_2, \gcd(f(u_1), f(v_2)) = \gcd(1, 5) = 1 \text{ for } 1 \leq i \leq n$$

$$(iv) e = u_iu_{i+1}, \gcd(f(u_i), f(u_{i+1})) = \gcd(2i + 3, 2i + 5) = 1 \text{ for } 2 \leq i \leq n - 1$$

$$(v) e = u_1u_2, \gcd(f(u_1), f(u_2)) = \gcd(1, f(u_2)) = 1$$

$$(vi) e = u_1u_n, \gcd(f(u_1), f(u_n)) = \gcd(1, f(u_n)) = 1$$

Thus  $f$  admits odd prime labeling on  $C_n @ K_3$  and hence Fish graph is an odd prime graph.  $\square$

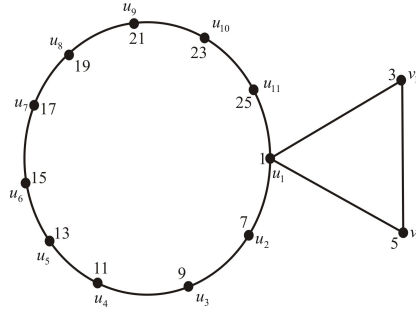


Figure 2: Fish graph  $C_n @ K_3$  and its odd prime labeling

**Theorem 2.3.** The Umbrella graph  $U(m, n)$  is an odd prime graph.

*Proof.* Consider the umbrella graph  $U(m, n)$  with vertex set.

$$V(U(m, n)) = \{x_i, y_i / 1 \leq i \leq m, 1 \leq i \leq n\}$$

$$E(U(m, n)) = \{x_i x_{i+1} / 1 \leq i \leq m - 1\}$$

$$\cup \{x_i y_1 / 1 \leq i \leq m\} \cup \{y_i y_{i+1} / 1 \leq i \leq n - 1\}$$

$$\text{Now } |V(U(m, n))| = m + n \text{ and } |E(U(m, n))| = 2m + n - 2$$

Define  $f : V \rightarrow O_{m+n}$  as follows.

$$f(x_i) = 2i + 1 \quad \text{for } 1 \leq i \leq m$$

$$f(y_1) = 1$$

$$f(y_i) = 2(i + m) - 1 \quad \text{for } 2 \leq i \leq n$$

Clearly vertex labels are distinct. With this labeling for each  $e = uv \in E$ , if  $\gcd(f(u), f(v)) = 1$ .

### Some new odd prime graphs

- (i)  $e = x_i x_{i+1}, \gcd(f(x_i), f(x_{i+1})) = \gcd(2i+1, 2i+3) = 1$  for  $1 \leq i \leq m-1$
- (ii)  $e = x_i y_1, \gcd(f(x_i), f(y_1)) = \gcd(2i+1, 1) = 1$  for  $1 \leq i \leq m$
- (iii)  $e = y_1 y_2, \gcd(f(y_1), f(y_2)) = \gcd(1, 2m+3) = 1$
- (iv)  $e = y_i y_{i+1}, \gcd(f(y_i), f(y_{i+1})) = \gcd(2(i+m)-1, 2(i+m)+1) = 1$  for  $2 \leq i \leq n-1$

as they are consecutive odd integers.

This  $f$  is a odd prime labeling on  $U(m, n)$  and hence it is an odd prime graph.  $\square$

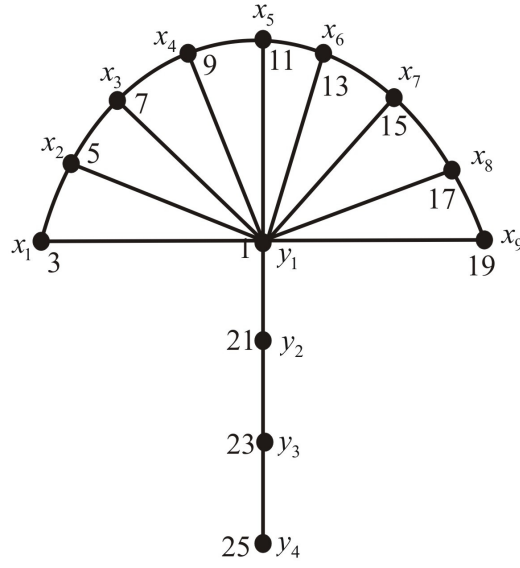


Figure 3:  $U(m, n)$  and its odd prime labeling

**Theorem 2.4.** *Cocount tree  $CT(m, n)$  is an odd prime graph.*

*Proof.*  $V(G) = \{u_i, v_i / 1 \leq i \leq m, 1 \leq i \leq n\}$  and edge set

$E(G) = \{u_i v_1 / 1 \leq i \leq m\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\}$

Now  $|V(CT(m, n))| = m + n$  and  $|E(CT(m, n))| = m + n - 1$

Define a labeling  $f : V(G) \rightarrow \{1, 3, 5, \dots, 2m + 2n - 1\}$  as follows

$f(u_i) = 2(n + i) - 1$  for  $1 \leq i \leq m$

$f(v_i) = 2i - 1$  for  $1 \leq i \leq n$

Clearly vertex labels are distinct.

For each  $e = uv \in E$ , if  $\gcd(f(u), f(v)) = 1$

(i)  $e = u_i v_1, \gcd(f(u_i), f(v_1)) = \gcd(2(n + i) - 1, 1) = 1$  for  $1 \leq i \leq m$ ;

(ii)  $e = v_i v_{i+1}, \gcd(f(v_i), f(v_{i+1})) = \gcd(2i - 1, 2i + 1) = 1$  for  $1 \leq j \leq m - 1$ .

Thus  $f$  admits odd prime labeling on  $CT(m, n)$  and hence  $CT(m, n)$  is an odd prime graph.  $\square$

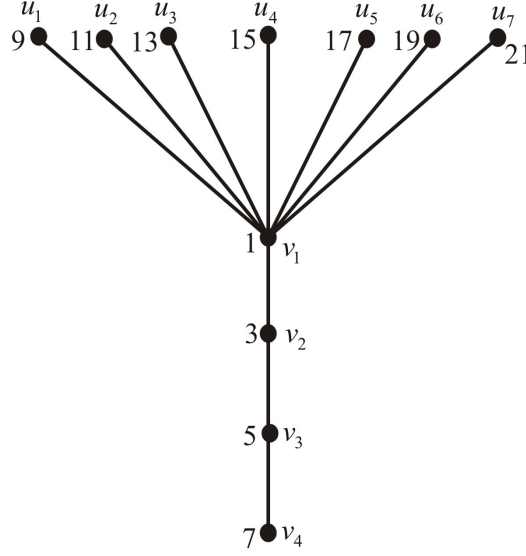


Figure 4:  $CT(m, n)$  and its odd prime labeling

**Theorem 2.5.** Let  $G$  be the graph obtained by identifying a pendant vertex of  $P_m$  with a leaf of  $K_{1,n}$  then  $G$  is an odd prime graph for all  $m$  and  $n$ .

*Proof.*  $V(G) = \{u, u_i, v_j / 1 \leq i \leq n, 2 \leq j \leq m\}$  and the edge set

$$E(G) = \{uv_i / 1 \leq i \leq n\} \cup \{v_j v_{j+1} / 2 \leq j \leq m-1\} \cup \{uv_2\}$$

Here  $u = v_1$

Now  $|V(G)| = m+n$  and  $|E(G)| = m+n-1$

Define a labeling  $f : V(G) \rightarrow \{1, 3, 5, \dots, 2m+2n-1\}$  as follows

$$f(u) = 1$$

$$f(u_i) = 2i+1 \quad \text{for } 1 \leq i \leq n$$

$$f(v_i) = 2(n+i)-1 \quad \text{for } 2 \leq i \leq n$$

Clearly the vertex labels are distinct.

For each  $e = uv \in E$  if  $\gcd(f(u), f(v)) = 1$

(i)  $e = uu_i, \gcd(f(u), f(u_i)) = \gcd(1, 2i+1) = 1$  for  $1 \leq i \leq n$ ;

(ii)  $e = uv_2, \gcd(f(u), f(v_2)) = \gcd(1, 2n+3) = 1$  for  $1 \leq i \leq n$ ;

(iii)  $e = v_i v_{i+1}, \gcd(f(v_i), f(v_{i+1})) = \gcd(2(n+i)-1, 2(n+i+1)-1) = 1$  for  $2 \leq i \leq n$ ;

Thus  $f$  admits odd prime labeling on  $G$  and hence  $G$  is an odd prime graph.  $\square$

*Some new odd prime graphs*

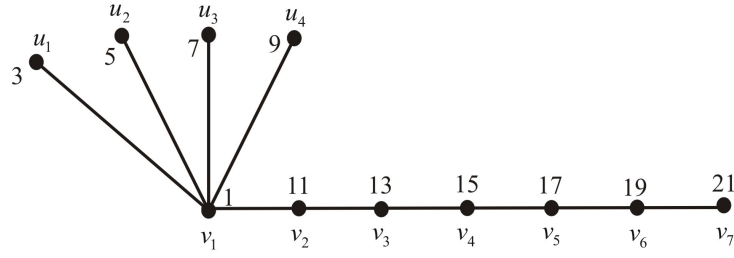


Figure 5:  $G$  and its odd prime labeling

**Theorem 2.6.**  $F$ - tree  $FP_n$   $n \geq 3$  is an odd prime graph.

*Proof.* Let  $V(G) = \{u, v, v_i, /1 \leq i \leq n-1\}$

$E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{uv_2, vv_1\}$

be the vertex set and edge set of  $FP_n$

Now  $|V(FP_n)| = n+2$  and  $|E(FP_n)| = n+1$

Define a labeling  $f : V(G) \rightarrow \{1, 3, 5, \dots, 2n+3\}$  as follows

$f(u) = 3$

$f(v) = 1$

$f(v_i) = 2i+3$  for  $1 \leq i \leq n$

Clearly vertex labels are distinct.

For each  $e = uv \in E$  if  $\gcd(f(u), f(v)) = 1$

(i)  $e = vv_1, \gcd(f(v), f(v_1)) = \gcd(1, 5) = 1$

(ii)  $e = uv_2, \gcd(f(u), f(v_2)) = \gcd(3, 7) = 1$

(iii)  $e = v_i v_{i+1}, \gcd(f(v_i), f(v_{i+1})) = \gcd(2i+3, 2i+5) = 1$

Thus  $f$  admits odd prime labeling on  $FP_n$  and hence  $F$ - tree  $FP_n$  is an odd prime graph.  $\square$

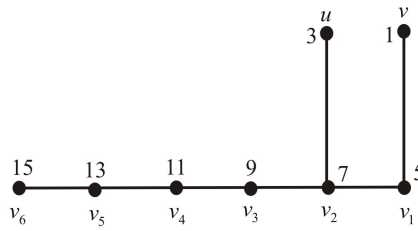


Figure 6:  $FP_n$  and its odd prime labeling

**Theorem 2.7.**  $Y$ - tree is an odd prime graph.

*Proof.* Let  $V(G) = \{uv_i, /1 \leq i \leq n\}$

$E(G) = \{v_i v_{i+1}, v_{n-1} u / 1 \leq i \leq n-1\}$  be the vertex set and edge set of  $y$ -tree

Now  $|V(G)| = n+1$  and  $|E(G)| = n$

Define a labeling  $f : V(G) \rightarrow \{1, 3, 5, \dots, 2n+1\}$  as follows

$$f(u) = 2n+1$$

$$f(v_i) = 2i-1 \text{ for } 1 \leq i \leq n$$

Clearly vertex labels are distinct.

For each  $e = uv \in E$  if  $\gcd(f(u), f(v)) = 1$

$$(i) e = uv_{n-1}, \gcd(f(u), f(v_{n-1})) = \gcd(2n+1, 2n-1) = 1$$

$$(iii) e = v_i v_{i+1}, \gcd(f(v_i), f(v_{i+1})) = \gcd(2i-1, 2i+1) = 1 \text{ for } 1 \leq i \leq n-1$$

Thus  $f$  admits odd prime labeling on  $Y$ -tree and hence  $Y$ -tree is an odd prime graph.  $\square$

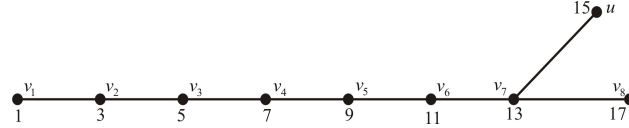


Figure 7:  $Y$ -tree and its odd prime labeling

**Theorem 2.8.** For any natural numbers  $k \geq 3$  graph  $DSF_n$  is an odd prime graph.

*Proof.* The vertex set and edge set of  $DSF_n$  of order  $K$  respectively are

$$V(DSF_n) = \{l_i, m_i, n_i / 1 \leq i \leq k\}$$

$$E(DSF_n) = \{l_i m_i, l_i n_i, m_i n_i, m_i l_{i+1}, l_k m_k, l_k n_k, l_1 n_k / 1 \leq i \leq n-1\}$$

$$\text{Now } |V(DSF_n)| = 3k \text{ and } |E(DSF_n)| = 5k$$

Define a labeling  $f : V(DSF_n) \rightarrow \{1, 3, 5, \dots, 6n-1\}$  as follows

$$f(l_i) = 6i-5 \text{ for } 1 \leq i \leq n$$

$$f(m_i) = 6i-3 \text{ for } 1 \leq i \leq n$$

$$f(n_i) = 6i-1 \text{ for } 1 \leq i \leq n$$

Clearly all the vertex labels are distinct. With this labeling for each  $e = uv \in E$  if  $\gcd(f(u), f(v)) = 1$

$$(i) e = l_i m_i, \gcd(f(l_i), f(m_i)) = \gcd(6i-5, 6i-3) = 1 \text{ for } 1 \leq i \leq n$$

$$(ii) e = l_i n_i, \gcd(f(l_i), f(n_i)) = \gcd(6i-5, 6i-1) = 1 \text{ for } 1 \leq i \leq n$$

$$(iii) e = m_i n_i, \gcd(f(m_i), f(n_i)) = \gcd(6i-3, 6i-1) = 1 \text{ for } 1 \leq i \leq n$$

$$(iv) e = m_i l_{i+1}, \gcd(f(m_i), f(l_{i+1})) = \gcd(6i-3, 6i-5) = 1 \text{ for } 1 \leq i \leq n-1$$

$$(v) e = m_k l_1, \gcd(f(m_k), f(l_1)) = \gcd(f(m_k), f(l_1)) = 1$$

Thus  $f$  is a odd prime labeling on  $DSF_n$  and hence  $DSF_n$  is an odd prime graph.  $\square$

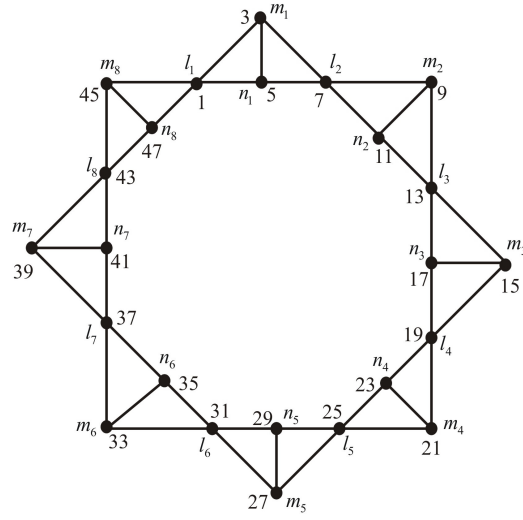


Figure 8:  $DSF_n$  and its odd prime labeling

### 3 Conclusions

Odd Prime labelings of various classes of graphs such as  $Z - P_n$  graph, Fish graph, Umbrella graph, Cocount tree,  $F$ -tree,  $Y$ -tree and Double Sunflower graph are investigated. To derive similar results for other graph families is an open area of research.

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# Prime labeling of $H$ - super subdivision of $Y$ -tree related graphs

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G. Gajalakshmi<sup>†</sup>

## Abstract

A graph  $G$  with  $p$  points is called a prime labeling, if it is possible to label the points  $x \in V$  with distinct labels  $f(x)$  from  $\{1, 2, \dots, p\}$  in such a way that for each line  $e = uv$   $\gcd(f(u), f(v)) = 1$ . In this paper we prove that some classes of graphs related to  $H$ - super subdivision of  $Y$ -tree are prime graphs.

**Keywords:** Prime labeling,  $Y$ - tree,  $H$ - graph,  $H$ -super subdivision graph, prime graph.

**2020 AMS subject classifications:** 05C78 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1057. ISSN: 1592-7415. eISSN: 2282-8214. ©S. Meena et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

In this paper, we consider finite, simple, undirected, nontrivial and connected graphs  $G = \langle V(G), E(G) \rangle$  where  $V, E$  is point set are line set. We refer Bondy and Murthy Bondy and Murthy [1976].

A graph labeling is an assignment the number to the points are lines or both subject to some constraints. For entire survey of graph labeling we refer Esakkiammal et al. [2016]. The idea of prime labeling was introduced by Roger Etringer and studied by many authors Deretsky et al. [1991], Tout et al. [1982].

The super subdivision of graph was defined by Sethuraman and Selvaraju in G.Sethuraman and P.Selvaraju [2001] and further studied by Esakkiammal et.al.Esakkiammal et al. [2016]. S.Meena et.al.Meena and J.Naveen [2016].Prime labeling of  $\mathcal{HSS}$  of  $\mathcal{Y}$ -tree related graphs.

**Definition 1.1.** Let  $\mathcal{G} = \langle V(\mathcal{G}), E(\mathcal{G}) \rangle$  be a graph with  $p$  points. A mapping  $\mathcal{G} : V(\mathcal{G}) \rightarrow \{1, 2, \dots, p\}$  is known as prime labeling if for every line  $e = uv \in E$ , greatest common divisor  $\langle f(u) \text{ and } f(v) \rangle$  is 1.

**Definition 1.2.** The tree on 6 points having two points of degree 3 is called a  $\mathcal{H}$ -graph. We consider a  $\mathcal{H}$ -graph got by adding a line between even degree points of two paths  $P_2$  and  $P'_2$  each of length two.

**Definition 1.3.** The graph  $\mathcal{H}$ -super sub-division of the graph is denote  $\mathcal{HSS}(\mathcal{G})$  is the graph got from  $\mathcal{G}$  by changing each line  $\mathcal{H}$ -graph so that end point of  $e_i$  are replaced by end point in  $P_2$  and end point  $P'_2$ .

**Definition 1.4.** A  $\mathcal{Y}$ - tree  $\mathcal{Y}_{m+1}$  ( $n \geq 2$ ) is a graph got from the path  $\mathcal{Q}_n$  by appending an line to a point of path  $\mathcal{Q}_n$  adjacent to an end point.

**Definition 1.5.** Let  $\mathcal{Y}_{m+1}$  be a  $\mathcal{Y}$ -tree ( $m \geq 2$ ) with  $m + 2$  points and  $m + 1$  lines. Let the points of  $\mathcal{Y}_{m+1}$  be  $v_1, v_2, v_3, \dots, v_m$ . The  $\mathcal{H}$ - super subdivision of  $\mathcal{Y}$ -tree  $\mathcal{HSS}(\mathcal{Y}_{m+1})$  is constructed from  $\mathcal{Y}_{m+1}$  by replacing each line by the  $\mathcal{H}$ - graph.

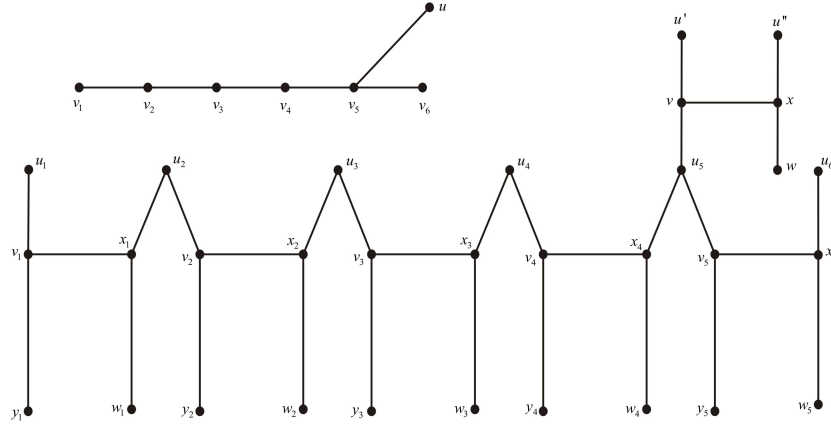


Figure 1:  $HSS(Y_{5+i})$

## 2 Main results

**Theorem 2.1.** *The graph  $\mathcal{HSS}(Y_{m+1})$  is a prime graph.*

**Proof:**

Let  $\mathcal{G} = \mathcal{HSS}(Y_{m+1})$  be the graph with point set

$$V(\mathcal{G}) = \{u_l, v_l, x_l, y_l, w_l / 1 \leq l \leq m\} \cup \{u', u'', v, x, w, u_{m+1}\}$$

$$E(\mathcal{G}) = \{(u_l v_l, v_l x_l, v_l y_l, w_l x_l, x_l u_{l+1} / 1 \leq l \leq m\} \cup \{(v x, v u', u'' x, x w)\}$$

Define a mapping  $\mathcal{G} : V(\mathcal{HSS}(\mathcal{Y}_{m+1})) \rightarrow \{1, 2, \dots, 5m + 6\}$

$$f(u_l) = 5l-4 \quad \text{for } 1 \leq l \leq m+1$$

$$f(v_l) = 5l-3 \quad \text{for } 1 \leq l \leq m \quad l \not\equiv 0 \pmod{3}$$

$$f(v_l) = 5l-2 \quad \text{for } 1 \leq l \leq m \quad l \equiv 0 \pmod{3}$$

$$f(x_l) = 5l \quad \text{for } 1 \leq l \leq m$$

$$f(y_l) = 5l-2 \quad \text{for } 1 \leq l \leq m \quad l \not\equiv 0 \pmod{3}$$

$$f(y_l) = 5l-3 \quad \text{for } 1 \leq l \leq m \quad l \equiv 0 \pmod{3}$$

$$f(w_l) = 5l-1 \quad \text{for } 1 \leq l \leq m$$

$$f(v) = 5(m+1)$$

$$f(x) = 5(m+1)-2$$

$$f(u') = 5(m+1)+1$$

$$f(u'') = 5(m+1)-1$$

$$f(w) = 5(m+1)-3$$

Clearly the point labels are different with this labeling for each  $e \in E$ .

greatest common divisor  $(f(u) \text{ and } f(v)) = 1$ .

Thus  $\mathcal{G}$  is a prime labeling. Hence  $\mathcal{HSS}(Y_{m+1})$  is a prime graph.

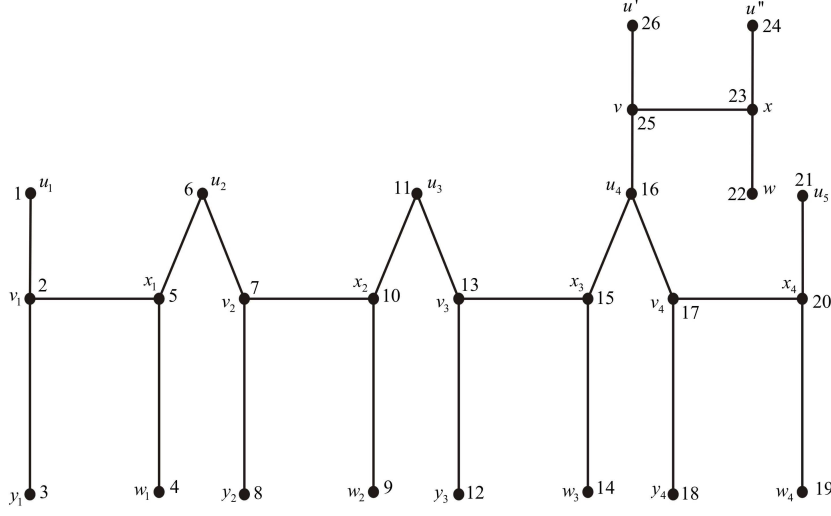


Figure 2:  $HSS(Y_{5+i})$

**Theorem 2.2.** *The  $HSS(\mathcal{Y}_{m+1})@K_2$  is a prime graph.*

**Proof:**

Let  $G = HSS(Y_{m+1})@K_2$  be the graph with point set

$$V(G) = \{u_l, v_l, x_l, y_l, w_l, r_l, p_l, q_l, s_l, t_l / 1 \leq l \leq m\}$$

$\cup \{u', u'', r, v, x, y, w, q, t', t''\}$  and the line set

$$E(G) = \{(u_l v_l, v_l y_l, x_l r_l, v_l x_l, v_l s_l, w_l q_l, y_l p_l, x_l w_l / 1 \leq l \leq m\}$$

$$\cup \{(u' v, u' t', v s, v x, x y, u_l t_l, u'' x, u'' t'', x w, w q, r_n r)\}$$

Define a mapping  $\mathcal{G} : V(HSS(\mathcal{Y}_{m+1})@K_2) \rightarrow \{1, 2, \dots, 10(m+1) + 1\}$

$$f(u_l) = 10l-9 \quad \text{for } 1 \leq l \leq m$$

$$f(v_l) = 10l-6 \quad \text{for } 1 \leq l \leq m-1 \quad l \not\equiv 0(mod 3)$$

$$f(v_l) = 10l-7 \quad \text{for } 1 \leq l \leq m \quad l \equiv 0(mod 3)$$

$$f(y_l) = 10l-5 \quad \text{for } 1 \leq l \leq m$$

$$f(w_l) = 10l-2 \quad \text{for } 1 \leq l \leq m$$

$$f(t_l) = 10l-8 \quad \text{for } 1 \leq l \leq m$$

$$f(s_l) = 10l-7 \quad \text{for } 1 \leq l \leq m-1 \quad l \not\equiv 0(mod 3)$$

$$f(s_l) = 10l-6 \quad \text{for } 1 \leq l \leq m \quad l \equiv 0(mod 3)$$

$$f(x_l) = 10l-1 \quad \text{for } 1 \leq l \leq m$$

$$f(r_l) = 10l \quad \text{for } 1 \leq l \leq m$$

$$f(p_l) = 10l-4 \quad \text{for } 1 \leq l \leq m$$

$$f(q_l) = 10l-3 \quad \text{for } 1 \leq l \leq m$$

$$f(u') = 10m+8$$

$$f(u'') = 10m+5$$

$$f(v) = 10m+9$$

*Prime labeling of H- super subdivision of Y-tree related graphs*

$$\begin{aligned}
 f(x) &= 10m+4 \\
 f(q) &= 10m+2 \\
 f(s) &= 10(m+1) \\
 f(y) &= 10m+7 \\
 f(t') &= 10(m+1)+1 \\
 f(t'') &= 10m+6 \\
 f(w) &= 10m+3 \\
 f(r) &= 10m+1
 \end{aligned}$$

Clearly the point labels are different with this labeling for each  $e \in E$ .

greatest common divisor ( $f(u)$  and  $f(v)$ ) = 1.

Thus  $\mathcal{G}$  is a prime labeling.

Hence  $\mathcal{HSS}(\mathcal{Y}_{m+1})@K_2$  is a prime graph.

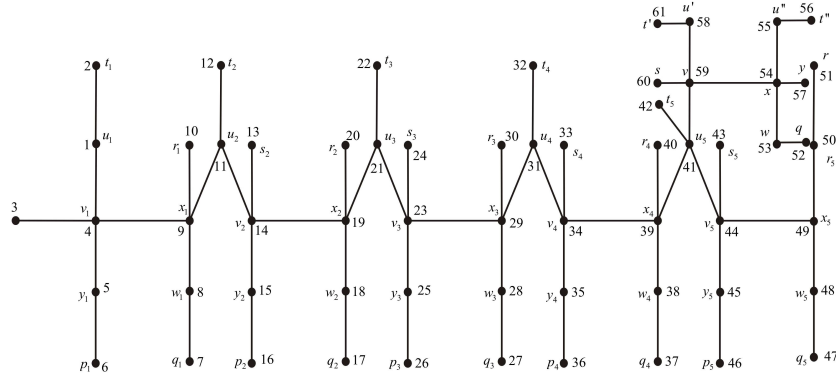


Figure 3:  $\mathcal{HSS}(\mathcal{Y}_{m+1})@K_2$

**Theorem 2.3.** *The  $\mathcal{HSS}(\mathcal{Y}_{m+1})@2K_2$  is prime graph.*

**Proof:**

Let  $\mathcal{G} = \mathcal{HSS}(\mathcal{Y}_{m+1})@2K_2$  be the graph with point set

$$\begin{aligned}
 V(\mathcal{G}) &= \{u_l, v_l, x_l, y_l, w_l, r_l, p_l, q_l, s_l, t_l, p'_l, q'_l, s'_l, t'_l / 1 \leq l \leq m\} \cup \\
 &\{u', u'', r, v, x, y, w, r_{11}, r_{12}, q_{11}, q_{12}, t'_{11}, t'_{12}, t_{11}, t_{12}, s_{11}, s_{12}, u_{n+1}, t_{n+1}t'_{m+1}\} \\
 &\text{and line set.}
 \end{aligned}$$

$$\begin{aligned}
 E(\mathcal{G}) &= \{(u_l v_l, v_l y_l, u_l t'_l, v_l s_l, v_l s'_l, y_l p_l, y_l p'_l, \\
 &v_l x_l, x_l r_l, x_l r'_l, x_l w_l, w_l q_l, w_l q'_l, x_l u_{l+1} / 1 \leq l \leq m\} \\
 &\cup \{(v x, u'' v, v s_{11}, v s_{12}, u'' t_{11}, u'' t_{12}, u' x, x w, x r_{11}, x r_{12}, w q_{11}, \\
 &w q_{12}, u_{m+1} t_{m+1}, u_{m+1} t'_{m+1}\}
 \end{aligned}$$

Define a mapping

$$\begin{aligned}
 \mathcal{G} : V(\mathcal{HSS}(\mathcal{Y}_{m+1})@2K_2) &\rightarrow \{1, 2, \dots, 15(m+1) + 3\} \\
 f(u_1) &= 5
 \end{aligned}$$

$$\begin{aligned}
f(u_l) &= 15l-14 && \text{for } 2 \leq l \leq m+1 \quad l \equiv 1(mod 2) \\
f(u_l) &= 15l-13 && \text{for } 2 \leq l \leq m+1 \quad l \equiv 0(mod 2) \\
f(x_l) &= 15l-1 && \text{for } 1 \leq l \leq m \\
f(v_l) &= 15l-10 && \text{for } 1 \leq l \leq m \\
f(v_1) &= 3 \\
f(y_l) &= 15l-8 && \text{for } 1 \leq l \leq m \quad l \equiv 1(mod 2) \\
f(y_l) &= 15l-7 && \text{for } 1 \leq l \leq m \quad l \equiv 0(mod 2) \\
f(w_l) &= 15l-4 && \text{for } 1 \leq l \leq m \\
f(r_l) &= 15l-2 && \text{for } 1 \leq l \leq m \\
f(r'_l) &= 15l && \text{for } 1 \leq l \leq m \\
f(t_l) &= 15l-13 && \text{for } 2 \leq l \leq m+1 \quad l \equiv 1(mod 2) \\
f(t_l) &= 15l-14 && \text{for } 2 \leq l \leq m+1 \quad l \equiv 0(mod 2) \\
f(t_1) &= 2 \\
f(p_l) &= 15l-7 && \text{for } 1 \leq l \leq m \quad l \equiv 1(mod 2) \\
f(p_l) &= 15l-8 && \text{for } 1 \leq l \leq m \quad l \equiv 0(mod 2) \\
f(p_1) &= 6 \\
f(p'_l) &= 15l-6 && \text{for } 1 \leq l \leq m \\
f(q_l) &= 15l-5 && \text{for } 1 \leq l \leq m \\
f(q'_l) &= 15l-3 && \text{for } 1 \leq l \leq m \\
f(s_1) &= 4 \\
f(s'_1) &= 8 \\
f(t'_l) &= 15l-12 && \text{for } 2 \leq l \leq m+1 \\
f(t'_l) &= 15(m+1)+3 \\
f(u') &= 15m+10 \\
f(u'') &= 15(m+1)-1 \\
f(v) &= 1 \\
f(x) &= 15m+4 \\
f(r_{11}) &= 15m+7 \\
f(r_{12}) &= 15+9 \\
f(q_{11}) &= 15m+5 \\
f(q_{12}) &= 15+6 \\
f(t'_{11}) &= 15(m+1)-3 \\
f(t'_{12}) &= 15(m+1)-4 \\
f(t_{11}) &= 15(m+1)-2 \\
f(t_{12}) &= 15(m+1) \\
f(s_{11}) &= 15(m+1)+1 \\
f(s_{12}) &= 15(m+1)+2
\end{aligned}$$

Clearly the point labels are different with this labeling for each  $e \in E$ .

greatest common divisor  $(f(u) \text{ and } f(v)) = 1$ .

Thus  $\mathcal{G}$  is a prime labeling. Hence  $\mathcal{HSS}(Y_{m+1}) @ 2K_2$  is a prime graph.

Prime labeling of  $H$ - super subdivision of  $Y$ -tree related graphs

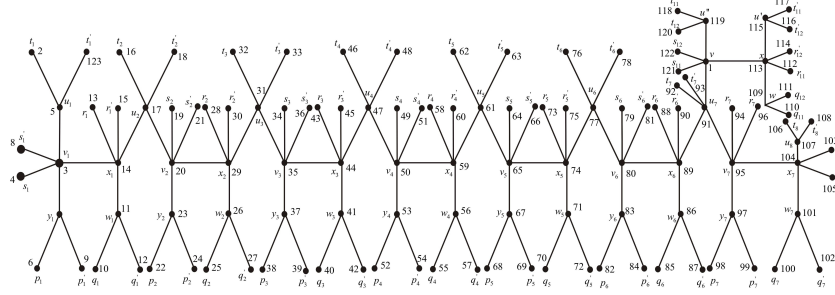


Figure 4:  $\mathcal{HSS}(\mathcal{Y}_{m+1}) @ 2K_2$

**Theorem 2.4.** *The  $\mathcal{HSS}(\mathcal{Y}_{m+1}) @ (K_3 \cup K_2)$  is a prime graph.*

**Proof:**

Let  $\mathcal{G} = \mathcal{HSS}(\mathcal{Y}_{n+1}) @ (K_3 \cup K_2)$  be the graph with point set  
 $V(\mathcal{G}) = \{u_l, v_l, x_l, y_l, s_l, s'_l, t_l, t'_l, z_l, p_l, p'_l, w_l, q_l, q'_l, r_l, r'_l, k_l, k'_l, l_l, l'_l$   
 $/ 1 \leq l \leq m\} \cup \{u, u', v, x, y, w, k, q_{11}, q'_{11}, t_{11}, t'_{12}, r_{11}, r_{12}, r, t'_{11}, t'_{12}, t, z,$   
 $s_{11}, s_{12}, s, t_{m+1}, t'_{m+1} z_{m+1}\}$  and the line set.

$E(\mathcal{G}) = \{u_l v_l, v_l y_l, u_l z_l, u_l t_l, u_l t'_l, v_l k_l, v_l s_l, v_l s'_l, y_l r_l, t_l t'_l, s_l s'_l, y_l p_l, y_l p'_l,$   
 $w_l r_l, w_l q'_l, x_l w_l, x_l k'_l, x_l l'_l, p_l p'_l, q_l q'_l, l_l l'_l, x_l u_{l+1} 1 \leq l \leq m\}$   
 $\cup \{v u', v s_{11}, v s_{12}, u' z, u' t_{11}, u' t_{12}, u' t, u' t'_{11},$

$u' t'_{12}, v s, v x, u' x, x r, x r_{11}, x r_{12}, x w, w k, w q_{11}, w q'_{11}, r_{11}, r_{12}, q_{11}, q_{12},$   
 $s_{11}, s_{12}, t_{11}, t_{12}, t'_{11} t'_{12}, t_{m+1} t'_{m+1}, u_{m+1} t_{m+1}, u_{m+1} t'_{m+1}, u_{n+1} z_{n+1}\}$

Define a mapping  $\mathcal{G} : V(\mathcal{HSS}(\mathcal{Y}_{m+1}) @ (K_3 \cup K_2)) \rightarrow \{1, 2, \dots, 20(m+1) + 3\}$

$f(u_l) = 20l-15$  for  $1 \leq l \leq m+1$   $l \not\equiv 0(mod 3)$   
 $f(u_l) = 20l-17$  for  $1 \leq l \leq m+1$   $l \equiv 0(mod 3)$   
 $f(v_l) = 20l-13$  for  $1 \leq l \leq m$   $l \not\equiv 0(mod 3)$   
 $f(v_l) = 20l-11$  for  $1 \leq l \leq m$   $l \equiv 0(mod 3)$   
 $f(x_l) = 20l-1$  for  $1 \leq l \leq m$   
 $f(y_l) = 20l-9$  for  $1 \leq l \leq m$   
 $f(w_l) = 20l-5$  for  $1 \leq l \leq m$   
 $f(s_l) = 20l-11$  for  $1 \leq l \leq m$   
 $f(t_l) = 20l-17$  for  $1 \leq l \leq m+1$   $l \not\equiv 0(mod 3)$   
 $f(t_l) = 20l-16$  for  $1 \leq l \leq m+1$   $l \equiv 0(mod 3)$   
 $f(t'_l) = 20l-16$  for  $1 \leq l \leq m+1$   $l \not\equiv 0(mod 3)$   
 $f(t'_l) = 20l-15$  for  $1 \leq l \leq m+1$   $l \equiv 0(mod 3)$   
 $f(z'_l) = 20l-18$  for  $1 \leq l \leq m$   
 $f(p_l) = 20l-8$  for  $1 \leq l \leq m$   
 $f(p'_l) = 20l-7$  for  $1 \leq l \leq m$   
 $f(q_l) = 20l-4$  for  $1 \leq l \leq m$



$$\begin{aligned}
 f(q'_l) &= 20l-3 && \text{for } 1 \leq l \leq m \\
 f(r_l) &= 20l-10 && \text{for } 1 \leq l \leq m \\
 f(r'_l) &= 20l-6 && \text{for } 1 \leq l \leq m \\
 f(k_l) &= 20l-14 && \text{for } 1 \leq l \leq m \\
 f(k'_l) &= 20l-12 && \text{for } 1 \leq l \leq m \\
 f(l_l) &= 20l && \text{for } 1 \leq l \leq m \\
 f(l'_l) &= 20l+1 && \text{for } 1 \leq l \leq m \\
 f(u') &= 20(m+1)-1 \\
 f(u) &= 20(m+1)-5 \\
 f(v) &= 1 \\
 f(x) &= 20m+11 \\
 f(w) &= 20m+7 \\
 f(k) &= 20m+6 \\
 f(q_{11}) &= 20m+8 \\
 f(q'_{11}) &= 20m+9 \\
 f(t_{11}) &= 20(m+1) \\
 f(t_{12}) &= 20(m+1)+1 \\
 f(r_{11}) &= 20m+12 \\
 f(r_{12}) &= 20m+13 \\
 f(r) &= 20m+10 \\
 f(t'_{11}) &= 20(m+1)-4 \\
 f(t'_{12}) &= 20(m+1)-3 \\
 f(t) &= 20(m+1)-6 \\
 f(z) &= 20(m+1)-2 \\
 f(s_1) &= 20(m+1)+2 \\
 f(s_2) &= 20(m+1)+3 \\
 f(s) &= 20(m+1)+4
 \end{aligned}$$

Clearly the point labels are different with this labeling for each  $e \in E$ .

greatest common divisor  $(f(u) \text{ and } f(v)) = 1$ .

Thus  $\mathcal{G}$  is a prime labeling.

Hence  $\mathcal{HSS}(\mathcal{Y}_{m+1}) @ (K_3 \cup K_2)$  is a prime graph.

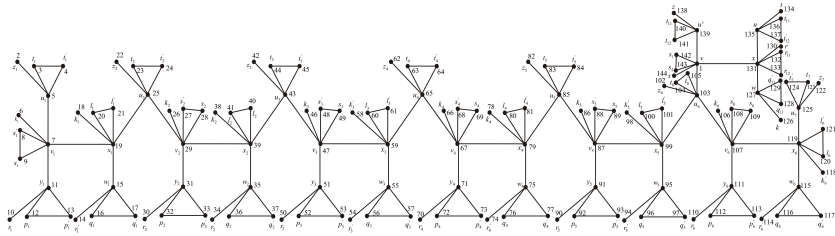


Figure 5:  $\mathcal{HSS}(\mathcal{Y}_{m+1}) @ (K_3 \cup K_2)$

### 3 Conclusions

Prime labeling of  $\mathcal{H}$ - Super Subdivision of  $\mathcal{Y}$ - tree related graphs.  $\mathcal{HSS}(\mathcal{Y}_{n+1})$ ,  $\mathcal{HSS}(\mathcal{Y}_{n+1})@K_2$ ,  $\mathcal{HSS}(\mathcal{Y}_{n+1})2K_2$ ,  $\mathcal{HSS}(\mathcal{Y}_{n+1})@(K_3 \cup K_2)$  are prime graphs.

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# IFG $^{\#}_{\alpha}$ -CS in intuitionistic fuzzy topological spaces

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## Abstract

The primary aim of this prospectus is to introduce and study the basic properties of Intuitionistic fuzzy generalized  $\#_{\alpha}$ -closed sets, Intuitionistic fuzzy generalized  $\#_{\alpha}$ -open sets. Here we, compare the  $\#_{\alpha}$ -closed sets with the existing closed sets with proper examples given.

**Keywords:** IFS, IFT, IFG $^{\#}_{\alpha}$ CS, IFG  $\#_{\alpha}$ OS.

**2020 AMS subject classifications:** 54A40.<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1058. ISSN: 1592-7415. eISSN: 2282-8214. ©Christy Jenifer J et al. This paper is published under the CC-BY licence agreement.

# 1 Introduction

Earlier  $g$   $\alpha$  closed sets has been introduced in general topology, fuzzy topology, supra topology, nano topology, we wish to introduce  $g^{\# \alpha}$  in intuitionistic fuzzy topological spaces. Following authors motivated me to further continue my research in intuitionistic fuzzy topology. A. Zadeh A.Zadeh [1965] initiated the concept of fuzzy sets in which the values are taken between 0 and 1. Further Atanassov [1986] established the idea of IFS by generalizing fuzzy sets, here similar to fuzzy topology values are taken between 0 and 1 but it is defined as membership value and non-membership value. Later on IFTS was initiated using the notion of IFSs which was proposed by Coker Coker [1997], using the membership and non-membership values it was applied in general topological axioms. In continuation of above we initiate IFG  $^{\# \alpha}$ -closed sets and IFG  $\alpha$ -open sets and establish its characterization and find the weaker and stronger forms of topology by comparing it to other existing sets and also find whether it satisfies the topological axioms as union and finite intersection properties

# 2 Preliminaries

In this segment, few basic definitions and results are reviewed.

**Definition 2.1.** *Thakur and Chaturvedi [2008]* Let  $A$  be an IFS in  $(X, \tau)$ , is proposed to be IFGCS if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is an IFOS in  $X$ .

Similarly, IF $\alpha$ GCS and IFG $\alpha$ CS Kalamani et al. [2012], , IFGSPCS Santhi and Jayanthi [2010], IFGSCS Santhi and Sakthivel [2009], IFGSRCS Anitha and Mohana [2018], were introduced. With the help of above closed sets we, initiate new set IFG $^{\# \alpha}$ - closed set.

# 3 IFG $^{\# \alpha}$ - closed sets

**Definition 3.1.** *An IFS  $C$  in  $(X, \tau)$  is proposed to be an IFG $^{\# \alpha}$ -closed set if  ${}_{\alpha}cl(C) \subseteq C$ , whenever  $C \subseteq U$  and  $U$  is an IFGOS in  $(E, \tau)$ . The family of all IFG $^{\# \alpha}$ CS of an IFTS  $(E, \tau)$  is defined by  $IFG^{\# \alpha}C(X)$ .*

**Example 3.1.** *Consider  $E = \{p, q\}$ ,  $\tau = \{0_{\sim}, J, 1_{\sim}\}$  is IFT on  $E$ , in that  $J = < e, (0.3, 0.2), (0.5, 0.6) >$ . In this the only  $\alpha$ -open sets are  $0_{\sim}, 1_{\sim}, J$ . At that point IFS,  $C = < e, (0.1, 0), (0.6, 0.8) >$  an IFG $^{\# \alpha}$ CS in  $(E, \tau)$ .*

**Theorem 3.1.** *Every IFCS is IFG $^{\# \alpha}$ CS, but reverse implication is not possible.*

*Proof.* Consider  $C$  is IFCS in  $(E, \tau)$ . Suppose an IFS,  $C \subseteq U$  where  $U$  is IFGOS. Considering  $\alpha cl(C) \subseteq cl(C)$  and  $C$  is an IFCS in  $E$ ,  $\alpha cl(C) \subseteq cl(C) = C \subseteq U$  and  $U$  is IFGOS. That is  $\alpha cl(C) \subseteq U$ . Consequently  $C$  is  $\mathcal{IFG}^\#_\alpha$ CS in  $E$ .  $\square$

**Example 3.2.** Let  $E = \{p, q\}$  and let  $\tau = \{0_\sim, J, 1_\sim\}$  is an IFT on  $E$ , where  $J = \langle e, (0.4, 0.1), (0.5, 0.6) \rangle$ . Let  $C = \langle e, (0.3, 0.1), (0.7, 0.9) \rangle$  be an IFS in  $E$ . Here  $C$  is an  $\mathcal{IFG}^\#_\alpha$ CS but not IFCS in  $(E, \tau)$ .

**Theorem 3.2.** Every  $IF_\alpha$ CS is  $\mathcal{IFG}^\#_\alpha$ CS but, reverse implication is not possible.

*Proof.* Consider  $C$  is a  $IF_\alpha$ CS in  $E$ . Suppose an IFS,  $C \subseteq U$ , where  $U$  is IFGOS. Considering  $C$  is  $IF_\alpha$ CS,  $\alpha cl(C) = C$ . Hence  $\alpha cl(C) \subseteq U$  once  $C \subseteq U$ ,  $U$  is IFGOS. Consequently  $\mathcal{IFG}^\#_\alpha$ CS in  $E$ .  $\square$

**Example 3.3.** Let  $E = \{p, q\}$  and let  $\tau = \{0_\sim, J, 1_\sim\}$  be IFT, we have  $J = \langle e, (0.2, 0.4), (0.6, 0.5) \rangle$ . Consider  $C = \langle e, (0.2, 0.4), (0.6, 0.5) \rangle$  be an IFS on  $E$ . Then  $C$  is  $\mathcal{IFG}^\#_\alpha$ CS but not  $IF_\alpha$ CS in  $(E, \tau)$ .

**Theorem 3.3.** Every IFRCS is  $\mathcal{IFG}^\#_\alpha$ CS but, reverse implication is not possible.

*Proof.* Consider  $C$  is an IFRCS. We know that  $C = cl(int(C))$  using definition. This signifies  $cl(C) = cl(int(C))$ . Consequently  $cl(C) = C$ . By which  $C$  is IFCS in  $E$ .  $C$  is an  $\mathcal{IFG}^\#_\alpha$ CS in  $E$ .  $\square$

**Example 3.4.** Consider  $E = \{p, q\}$ ,  $\tau = \{0_\sim, J, 1_\sim\}$  be IFT, we have  $J = \langle e, (0.4, 0.4), (0.5, 0.5) \rangle$ . Here an IFS,  $C = \langle e, (0.2, 0.2), (0.7, 0.8) \rangle$  is an  $\mathcal{IFG}^\#_\alpha$ CS but not IFRCS in  $(E, \tau)$ .

**Theorem 3.4.** Every  $\mathcal{IFG}^\#_\alpha$ CS is IFSGCS but, reverse implication is not possible.

*Proof.* Consider  $C$  an  $\mathcal{IFG}^\#_\alpha$ CS. Suppose an IFS,  $C \subseteq U$  where  $U$  is IFSO set. Since, every IFSO set is IFGO set and  $C$  be an  $\mathcal{IFG}^\#_\alpha$ CS. We have  $scl(C) \subseteq \alpha cl(C) \subseteq U$ . Therefore  $C$  is IFSGCS.  $\square$

**Example 3.5.** Let  $E = \{p, q\}$   $\tau = \{0_\sim, J, 1_\sim\}$  be IFT we have  $J = \langle e, (0.5, 0.7), (0.6, 0.7) \rangle$ . Let  $C = \langle e, (0.5, 0.6), (0.8, 0.9) \rangle$ . Then  $C$  is an IFSGCS but not  $\mathcal{IFG}^\#_\alpha$ CS in  $(E, \tau)$ .

**Theorem 3.5.** Every  $\mathcal{IFG}^\#_\alpha$ CS is IFGSCS but, reverse implication is not possible.

*Proof.* Consider  $C$  an  $\mathcal{IFG}^\#_\alpha$ CS in  $(E, \tau)$ . Suppose an IFS,  $C \subseteq U$  where  $U$  is IFOS. Since, every IFOS set is an IFGOS and  $A$  be an  $\mathcal{IFG}^\#_\alpha$ CS. We have  $scl(C) \subseteq \alpha cl(C) \subseteq U$ . Therefore,  $C$  is IFGSCS set.  $\square$

**Example 3.6.** Let  $E = \{p, q\}$ ,  $\tau = \{0_{\sim}, J, 1_{\sim}\}$  be IFT, we have  $J = \langle e, (0.1, 0.2), (0.6, 0.6) \rangle$ . Let,  $C = \langle e, (0, 0.2), (0.9, 0.6) \rangle$ . Here  $C$  is an IFGSCS but not an  $IFG^{\#}\alpha CS$ .

**Theorem 3.6.** Every  $IFG^{\#}\alpha CS$  is IFGSRCS in  $(E, \tau)$  but, reverse implication is not possible.

*Proof.* Consider  $C$  an  $IFG^{\#}\alpha CS$  in  $(E, \tau)$ . Suppose an IFS,  $C \subseteq U$  where  $U$  is an IFROS. Since, every IFROS set is an IFGSRCS and  $C$  be an  $IFG^{\#}\alpha CS$ . We have  $scl(C) \subseteq \alpha cl(C) \subseteq U$ . Therefore,  $C$  is IFGSRCS set.  $\square$

**Example 3.7.** Let  $E = \{p, q\}$ ,  $\tau = \{0_{\sim}, J, 1_{\sim}\}$ , here  $J = \langle e, (0.3, 0.6), (0.7, 0.4) \rangle$ . Consider,  $C = \langle e, (0.3, 0.4), (0.7, 0.6) \rangle$ . Here  $C$  is an IFGSRCS but not an  $IFG^{\#}\alpha CS$ .

**Remark 3.1.** For any two  $IFG^{\#}\alpha CS$  intersection is also  $IFG^{\#}\alpha CS$ .

*Proof.* Consider  $C$  and  $D$  any two  $IFG^{\#}\alpha CS$ . That is  $I\alpha cl(C) \subseteq G$ . Once  $C \subseteq G$  and  $G$  is IFGOS and is  $I\alpha cl(D) \subseteq G$  whenever  $D \subseteq G$  and  $G$  is IFGOS. Now,  $I\alpha cl(C \subseteq D) = I\alpha cl(C) \cap I\alpha cl(D) \subseteq G$ , where  $(C \cap D) \subseteq G$  and  $G$  is IFGOS. Thus, intersection of any two  $IFG^{\#}\alpha$ - closed set is  $IFG^{\#}\alpha CS$ .  $\square$

**Theorem 3.7.** Let  $(E, \tau)$  be IFTS. Then for every  $C \subseteq IFG^{\#}\alpha CS(E)$  and for every  $D \in IFS(E)$ ,  $C \subseteq D \subseteq \alpha cl(C)$  implies  $D \in IFG^{\#}\alpha CS(E)$ .

*Proof.* Consider IFS  $D \subseteq U$  and  $U$  be IFGOS, considering  $C \subseteq D$ ,  $C \subseteq U$  and  $C$  is  $IFG^{\#}\alpha CS$ ,  $\alpha cl(C) \subseteq U$ . By assumption,  $D \subseteq \alpha cl(C)$ ,  $\alpha cl(D) \subseteq \alpha cl(C) \subseteq U$ . Consequently  $\alpha cl(D) \subseteq U$ . Thus  $C$  is  $IFG^{\#}\alpha CS$  of  $E$ .  $\square$

**Theorem 3.8.** Consider  $E$  an IFTS. Then  $IFGO(E) = IFGC(E)$  if and only if every IFS in  $E$  an  $IFG^{\#}\alpha CS$  in  $E$ .

*Proof. Necessity:* Assume  $IFGO(E) = IFGC(E)$ . Consider  $C \subseteq G$ ,  $G$  an IFGOS. This signifies  $\alpha cl(C) \subseteq \alpha cl(G)$ . Considering  $G$  an IFGOS in  $E$ , by assumption  $G$  is IFGCS in  $E$ ,  $\alpha cl(C) \subseteq G$ . This signifies  $\alpha cl(C) \subseteq G$ . Consequently  $C$  is  $IFG^{\#}\alpha CS$  in  $E$ .

*Sufficiency:* Assume, every IFS is  $IFG^{\#}\alpha CS$ . Consider  $G \in IFO(E)$ , we have  $G \in IFGO(E)$  and so  $C \subseteq G$  also  $G$  is IFOS in  $E$ , by assumption  $\alpha cl(C) \subseteq G$ . That is  $G \in IFGC(E)$ . Accordingly  $IFGO(E) \subseteq IFGC(E)$ . Consider  $C \in IFGC(E)$ , we have  $C^c$  is IFGOS in  $E$ . But  $IFGO(E) \subseteq IFGC(E)$ . Consequently  $C^c \in IFGC(E)$ . Here  $C \in IFGO(E)$ . Consequently  $IFGC(E) \subseteq IFGO(E)$ . We know that  $IFGO(E) \subseteq IFGC(E)$ .  $\square$

**Theorem 3.9.** Let  $C$  be  $IFG^{\#}\alpha CS$  of  $E$ , then  $\alpha cl(C)$ - $C$  contains no non-empty IFGCS.

*Proof.* Suppose  $C$  is  $IFG^\#_\alpha$ CS of  $E$  and consider  $F$  to be non-empty IFGCS of  $E$ , and so  $F \subseteq \alpha cl(C) - C$ . We have  $A \subseteq E - F$ . Considering  $C$  is  $IFG^\#_\alpha$ CS and  $E - F$  is IFGOS, and so  $\alpha cl(C) \subseteq E - F$ . This signifies  $F \subseteq E - \alpha cl(C)$ . Also  $F \subseteq (E - \alpha cl(C)) \cap (\alpha cl(C) - C) \subseteq (E - \alpha cl(C)) \cap \alpha cl(C) = \phi$ . Consequently  $F$  is empty.  $\square$

**Theorem 3.10.** Let  $C \subseteq D \subseteq E$  and assume that  $C$  is  $IFG^\#_\alpha$ CS in  $E$  then  $C$  is an  $IFG^\#_\alpha$ CS relative to  $D$ .

*Proof.* Here we have,  $C \subseteq D \subseteq E$  also  $C$  an  $IFG^\#_\alpha$ CS. Considering  $C \subseteq D \cap F$  where  $F$  is IFGOS in  $E$ . Since  $C$  is an  $IFG^\#_\alpha$ CS in  $E$ ,  $C \subseteq F$  implies,  $\alpha cl(C) \subseteq F$ . It follows that  $D \cap \alpha cl(C) \subseteq D \cap F = F$ . Thus  $C$  is an  $IFG^\#_\alpha$ CS relative to  $D$ .  $\square$

## 4 $IFG^\#_\alpha$ -open sets

In this segment, we define and establish the idea of  $IFG^\#_\alpha$ -open sets (briefly  $IFG^\#_\alpha$ OS) in IFTS and establish its characterizations.

**Definition 4.1.** A subset  $B$  of IFTS  $E$  is proposed to be an  $IFG^\#_\alpha$ -open if  $A^c$  is  $IFG^\#_\alpha$ -open set.

**Theorem 4.1.** Consider  $E$  an IFTS we have,

- (i) . Every IF-open set is  $IFG^\#_\alpha$ OS.
- (ii) . Every  $IF_\alpha$ -open set is  $IFG^\#_\alpha$ OS.
- (iii) . Every IFR-open set is  $IFG^\#_\alpha$ OS.

*Proof.* Proof is obvious  $\square$

**Theorem 4.2.** Let  $(E, \tau)$  be the IFTS then,

- (i) Every  $IFG^\#_\alpha$ -open set is IFSGOS.
- (ii) Every  $IFG^\#_\alpha$ -open set is IFGSPOS.
- (iii) Every  $IFG^\#_\alpha$ -open set is IFGSOS.
- (iv) Every  $IFG^\#_\alpha$ -open set is IFGSROS.

*Proof.* Proof is obvious  $\square$

**Theorem 4.3.** An IFS  $C$  of IFTS  $E$  is  $IFG^\#_\alpha$ OS on the condition that  $D \subseteq \alpha int(C)$  at any moment  $D$  is IFGCS in  $E$  also  $D \subseteq F$ .

**Proof. Necessity:** Let  $C$  is  $IFG^\#_\alpha OS$  in  $E$ . Consider  $D$  be IFGCS in  $E$  also  $D \subseteq C$ . We have  $D^c$  is IFGOS in  $E$  in this extent  $C^c \subseteq D^c$ . Considering  $C^c$  is  $IFG^\#_\alpha CS$ , then  $\alpha cl(C^c) \subseteq D^c$ . So  $(\alpha cl(C))^c \subseteq D^c$ . Consequently  $D \subseteq \alpha cl(C)$ .  
**Sufficiency:** Consider  $D \subseteq \alpha int(C)$  at any moment  $D$  is IFGCS also  $D \subseteq C$ . We have  $C^c \subseteq D^c$  also  $D^c$  an IFGOS. By assumption,  $(\alpha cl(C))^c \subseteq D^c$ . Therefore  $C^c$  is  $IFG^\#_\alpha CS$  of  $E$ . Consequently  $C$  is  $IFG^\#_\alpha OS$ .  $\square$

**Theorem 4.4.** Consider  $E$  an IFTS. We have for all  $C \subseteq IFG^\#_\alpha OS$  also probably  $D \in IFS(E)$ ,  $\alpha int(C) \subseteq D \subseteq C$  signifies  $D \in IFG^\#_\alpha OS$ .

**Proof.** Here  $\alpha int(C) \subseteq D \subseteq C$  implies  $C^c \subseteq D^c \subseteq (\alpha int(C))^c$ . Consider  $D^c \subseteq F$  also  $F$  is IFGOS in  $E$ . Considering  $C^c \subseteq D^c$ ,  $C^c \subseteq F$ . Since  $C^c$  is  $IFG^\#_\alpha CS$ ,  $\alpha cl(C^c) \subseteq F$  and so  $D^c \subseteq (\alpha int(C))^c = \alpha cl(C^c)$ . Consequently  $\alpha cl(D^c) \subseteq \alpha cl(C^c) \subseteq F$ . Accordingly  $D^c$  is  $IFG^\#_\alpha CS$  in  $\mathcal{E}$ . This signifies  $D$  is  $IFG^\#_\alpha OS$  in  $E$ . Therefore  $D \in IFG^\#_\alpha CS$ .  $\square$

## 5 Conclusions

Here we have derived a new concept of closed set called  $IFG^\#_\alpha CS$ . We have proved that  $IFG^\#_\alpha CS$  is stronger than IFCS, IFCS, IFRCs. Also, IFGSCS, IFSGCS and IFGSRCS is stronger than  $IFG^\#_\alpha CS$ . Further, it satisfies the intersection axiom but it does not satisfy union property. Thus we can conclude that  $IFG^\#_\alpha CS$  does not form a topology. This concept can further be extended to continuous functions, irresolute functions, various forms of continuous functions such as completely continuous, perfectly continuous, contra continuous, almost continuous, slightly continuous and spaces can be introduced which are normal space and regular space, also connectedness can be described. Application can be done based on membership and non-membership values and find the MCDM problems.

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# Odd prime labeling for some arrow related graphs

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S. Meena<sup>†</sup>

## Abstract

In a graph  $\mathcal{G}$  a mapping  $g$  is known as odd prime labeling, if  $g$  is a bijection from  $\mathcal{V}$  to  $\{1, 3, 5, \dots, 2|\mathcal{V}| - 1\}$  satisfying the condition that for each line  $xy$  in  $\mathcal{G}$  the gcd of the labels of end points  $(g(x), g(y))$  is one. In this article we prove that some new arrow related graphs such as  $A_y^2, A_y^3, A_y^5$ , are all odd prime graphs. Also we prove that double arrow graphs,  $\mathcal{DA}_y^2$  and  $\mathcal{DA}_y^3$  are odd prime graphs.

**Keywords:** Prime graph, Odd prime graph, Arrow graphs.

**2020 AMS subject classifications:** 05C78<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1059. ISSN: 1592-7415. eISSN: 2282-8214. ©G. Gajalakshmi et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

In this article by a graph  $\mathcal{G} = \langle V(\mathcal{G}), E(\mathcal{G}) \rangle$  we mean a simple graph. For graph theoretical notations we refer J.A.Bondy and U.S. R.Murthy [1976] .

Graph labeling has been introduced in mid 1960. For entire survey of graph labeling we refer Gallian [2015].

The concept of prime labeling was established by Roger Entringer and was discussed in a article by Deretsky et al. [1991], Tout et al. [1982]. A graph  $\mathcal{G}$  of order  $p$  is known as prime graph if it's points can be labeled with distinct positive integers  $\{1, 2, 3, \dots, p\}$  such that the labels of any two adjacent points are relatively prime Meena and Vaithilingam [2013]. Meena and Kavitha [2014] investigated prime labeling for some butterfly related graphs. Meena et al. [2021] investigated odd prime labeling for some new classes of graph.

The notion of odd prime labeling was established by Prajapati and Shah [2018] and many researchers. Arrow graph was introduced by Kaneria et al. [2015]. Motivated by this study, in this article investigate the existence of odd prime labeling of some graphs related to arrow graphs.

**Definition 1.1.** Let  $\mathcal{H} = \langle \mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}) \rangle$  be a graph. A bijection  $g : \mathcal{V}(\mathcal{H}) \rightarrow O_{|V|}$  is known as odd prime labeling if for each line  $xy \in \mathcal{E}$ , greatest common divisor  $\langle g(x), g(y) \rangle = 1$ . A graph is known as odd prime graph if it admits odd prime labeling .

**Definition 1.2.** Let  $\mathcal{H}_1 = (P_1, Q_1)$  and  $\mathcal{H}_2 = (P_2, Q_2)$  be two graphs with  $P_1 \cap P_2 = \phi$ . The cartesian product  $\mathcal{H}_1 \times \mathcal{H}_2$  is defined as a graph having  $P = P_1 \times P_2$  and  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are adjacent if  $x_1 = y_1$  and  $x_2$  is adjacent to  $y_2$  in  $\mathcal{H}_2$  or  $x_1$  is adjacent to  $y_1$  in  $\mathcal{H}_1$  and  $x_2 = y_2$ . The cartesian product of two paths  $P_m$  and  $P_n$  denoted as  $P_m \times P_n$  is known as a grid graph on  $nm$  points and  $2nm - (n + m)$ .

**Definition 1.3.** In rectangular grid  $P_m \times P_n$  on  $mn$  points the  $n$  points  $v_{1,1}, v_{2,1}, v_{3,1} \dots v_{m,n}$  and points  $v_{1,n}, v_{2,n}, v_{3,n} \dots v_{m,n}$  are called an superior points from both the ends.

**Definition 1.4.** An arrow graph  $A_y^x$  with width  $x$  and length  $y$  is got by connecting a point  $v$  with superior points of  $P_x \times P_y$  by new edges from one end.

**Definition 1.5.** A double arrow graph  $\mathcal{DA}_y^x$  with width  $x$  and length  $y$  is got by connecting two points  $v$  and  $w$  with superior points of  $P_m \times P_y$  by  $x + x$  new edges from both the end.

## 2 Main Results

**Theorem 2.1.**  $A_y^2$  is an odd prime graph where  $y \geq 2$ .

*Proof.* Let  $\mathcal{G} = A_y^2$  be an arrow graph got by connecting a point  $g(u_0)$  with superior points of  $P_2 \times P_y$  by new lines.

$$\mathcal{V}(\mathcal{G}) = \{u_l/0 \leq l \leq y\} \cup \{v_l/1 \leq l \leq y\}$$

$$\mathcal{E}(\mathcal{G}) = \{u_l u_{l+1}/1 \leq l \leq y-1\} \cup \{u_0 v_1\} \cup \{u_0 u_1\} \\ \cup \{v_l v_{l+1}/1 \leq l \leq y-1\} \cup \{u_l v_l/1 \leq l \leq y\}.$$

Now  $|\mathcal{V}(\mathcal{G})| = 2y+1$  and  $|\mathcal{E}(\mathcal{G})| = 3y$

Define a Mapping  $f : \mathcal{V} \rightarrow O_{2y}$  as follows

$$g(u_0) = 1$$

$$g(u_l) = 4l - 1 \quad \text{for } 1 \leq l \leq y$$

$$g(v_l) = 4l + 1 \quad \text{for } 1 \leq l \leq y$$

Clearly point labels are distinct.

For each  $e \in E$ , if  $\gcd(g(u), g(v)) = 1$

$$(i) e = u_0 u_l, \gcd(g(u_0), g(u_l)) = \gcd(1, 3) = 1$$

$$(ii) e = u_0 v_1, \gcd(g(u_0), g(v_1)) = \gcd(1, 5) = 1$$

$$(iii) e = u_l v_l, \gcd(g(u_l), g(v_l)) = \gcd(4l - 1, 4l + 1) = 1 \\ \text{for } 1 \leq l \leq y$$

$$(iv) e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(4l - 1, 4l + 3) = 1 \\ \text{for } 1 \leq l \leq y - 1$$

$$(v) e = v_l v_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(4l + 1, 4l + 5) = 1 \\ \text{for } 1 \leq l \leq y - 1$$

Hence  $A_y^2$  is an odd prime graph. □

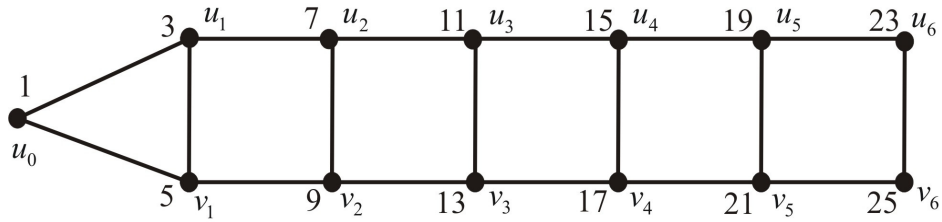


Figure 1: Arrow graph  $A_y^2$  and its odd prime labeling

**Theorem 2.2.**  $A_y^3$  is an odd prime graph where  $y \geq 2$ .

*Proof.* Let  $G = A_y^3$  be an arrow graph got by connecting a point  $g(u_0)$  with superior points of  $P_3 \times P_2$  by 3 new lines.

$$\mathcal{V}(\mathcal{G}) = \{u_l, v_l, w_l/1 \leq l \leq y\} \cup \{u_0\}$$

$$\mathcal{E}(\mathcal{G}) = \{u_l u_{l+1}, v_l v_{l+1}, w_l w_{l+1}/1 \leq l \leq y-1\} \cup \{v_l w_l, u_l v_l/1 \leq l \leq y\} \\ \cup \{u_0 u_1\} \cup \{u_0 v_1\} \cup \{u_0 w_1\}$$

Now  $|\mathcal{V}(\mathcal{G})| = 3y + 1$  and  $|\mathcal{E}(\mathcal{G})| = 5y - 1$

Define a mapping  $f : \mathcal{V} \rightarrow O_{2y}$  as follows

$$g(u_0) = 1$$

$$g(u_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(u_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(v_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(v_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(w_l) = 6l + 1 \quad \text{for } 1 \leq l \leq y$$

Clearly all the point labels are distinct. With this labeling for each  $e = uv \in E$  if

$$(i) e = u_0u_1, \gcd(g(u_0), g(u_1)) = \gcd(1, 3) = 1 \quad \text{for } 1 \leq l \leq y$$

$$(ii) e = u_0w_1, \gcd(g(u_0), g(w_1)) = \gcd(1, 7) = 1 \quad \text{for } 1 \leq l \leq y$$

$$(iii) e = u_0v_1, \gcd(g(u_0), g(v_1)) = \gcd(1, 5) = 1 \quad \text{for } 1 \leq l \leq y$$

$$(iv) e = u_lv_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y, l \not\equiv 0 \pmod{2}$$

$$(v) e = u_lv_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y, l \equiv 0 \pmod{2}$$

$$(vi) e = v_lw_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 1, 6l + 1) = 1 \quad \text{for } 1 \leq l \leq y, l \not\equiv 0 \pmod{2}$$

$$(vii) e = v_lw_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 1, 6l + 1) = 1 \quad \text{for } 1 \leq l \leq y, l \equiv 0 \pmod{2}$$

$$(viii) e = u_lu_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 3, 6l - 5) = 1 \quad \text{for } 1 \leq l \leq y, l \not\equiv 0 \pmod{2}$$

$$(ix) e = u_lu_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 1, 6l + 3) = 1 \quad \text{for } 1 \leq l \leq y, l \equiv 0 \pmod{2}$$

$$(x) e = v_lv_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y, l \equiv 0 \pmod{2}$$

$$(xi) e = v_lv_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(6l - 1, 6l - 3) = 1 \quad \text{for } 1 \leq l \leq y, l \not\equiv 0 \pmod{2}$$

$$(xii) e = w_lw_{l+1}, \gcd(g(w_l), g(w_{l+1})) = \gcd(6l + 1, 6l + 7) = 1 \quad \text{for } 1 \leq l \leq y$$

Hence  $A_y^3$  is an odd prime graph.  $\square$

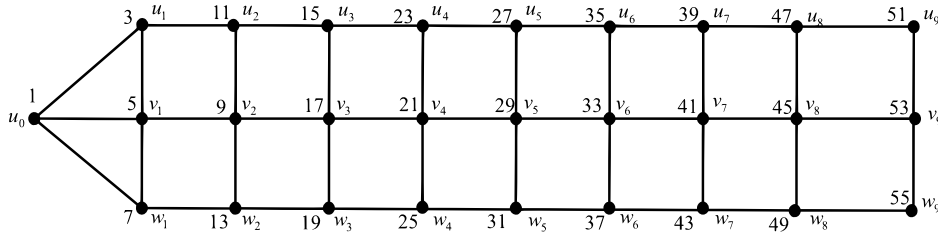


Figure 2: Arrow graph  $A_y^3$  and its odd prime labeling

**Theorem 2.3.**  $A_y^5$  is an odd prime graph where  $y \geq 5$ .

*Odd prime labeling for some arrow related graphs*

*Proof.* Let  $\mathcal{G} = A_y^5$  be an arrow graph got by connecting a point  $v$  with superior points  $P_5 \times P_y$  by 5 new lines.

$$\mathcal{V}(\mathcal{G}) = \{u_l, v_l, w_l / 1 \leq l \leq y\} \cup \{u_0\}$$

$$\mathcal{E}(\mathcal{G}) = \{u_l v_l, v_l w_l / 1 \leq l \leq y\} \cup \{(u_l u_{l+1}), (v_l v_{l+1}), (w_l w_{l+1}) / 1 \leq l \leq y-1\}$$

Now  $|\mathcal{V}(\mathcal{G})| = 5y + 1$  and  $|\mathcal{E}(\mathcal{G})| = 9y$

Define a mapping  $f : \mathcal{V} \rightarrow O_y$  as follows

$$g(u_0) = 1$$

$$g(u_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(u_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(v_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(v_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(w_l) = 6l + 1 \quad \text{for } 1 \leq l \leq y,$$

Clearly all the point labels are distinct. With this labeling for each  $e \in E$  if  $\gcd(g(u), g(v)) = 1$

$$(i) e = u_0 u_1, \gcd(g(u_0), g(u_1)) = \gcd(1, 3) = 1$$

$$(ii) e = u_0 u_{l+1}, \gcd(g(u_0), g(u_{l+1})) = \gcd(1, 6l - 3) = 1 \quad \text{for } 1 \leq l \leq y$$

$$(iii) e = u_l v_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y \\ l \not\equiv 0 \pmod{2}$$

$$(iv) e = u_l v_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 1, 6l - 3) = 1 \quad \text{for } 1 \leq l \leq y \\ l \equiv 0 \pmod{2}$$

$$(v) e = v_l w_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 1, 6l + 1) = 1 \quad \text{for } 1 \leq l \leq y \\ l \not\equiv 0 \pmod{2}$$

$$(vi) e = v_l w_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 3, 6l + 1) = 1 \quad \text{for } 1 \leq l \leq y \\ l \equiv 0 \pmod{2}$$

$$(vii) e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 3, 6l + 5) = 1 \text{ for } 1 \leq l \leq y \\ l \not\equiv 0 \pmod{2}$$

$$(viii) e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 1, 6l - 3) = 1 \text{ for } 1 \leq l \leq y \\ l \equiv 0 \pmod{2}$$

$$(ix) e = v_l v_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(6l - 1, 6l + 3) = 1 \text{ for } 1 \leq l \leq y \\ l \not\equiv 0 \pmod{2}$$

$$(x) e = v_l v_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(6l - 3, 6l + 5) = 1 \text{ for } 1 \leq l \leq y \\ l \equiv 0 \pmod{2}$$

$$(xi) e = w_l w_{l+1}, \gcd(g(w_l), g(w_{l+1})) = \gcd(6l + 1, 6l + 7) = 1 \text{ for } 1 \leq l \leq y$$

Hence  $A_y^5$  is an odd prime graph.  $\square$

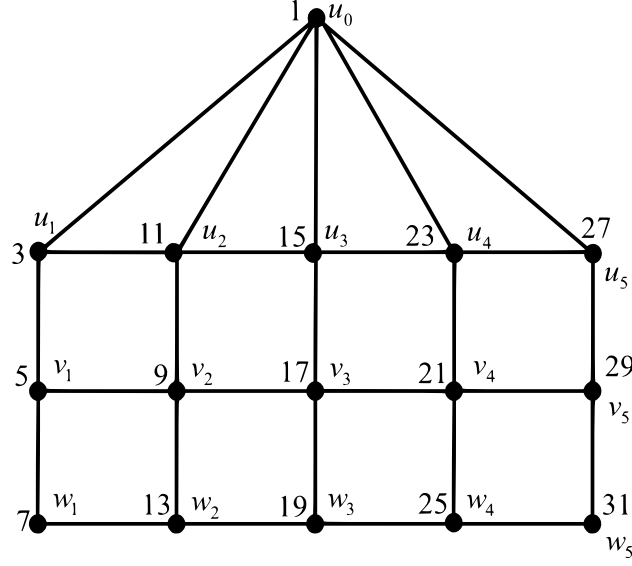


Figure 3: Arrow graph  $A_y^5$  and its odd prime labeling

**Theorem 2.4.**  $\mathcal{DA}_y^2$  is an odd prime graph where  $y \geq 2$ .

*Proof.* Let  $\mathcal{G} = \mathcal{DA}_y^2$  be a double arrow graph got by connecting two points  $u, v$  with superior points from both the ends of  $P_2 \times P_y$  by 2+2 new lines.

Let  $\mathcal{V}(\mathcal{G}) = \{u_l v_l / 1 \leq l \leq y\} \cup \{v, v_0\}$

$\mathcal{E}(\mathcal{G}) = \{(u_l u_{l+1}), (v_l v_{l+1}), 1 \leq l \leq y-1\} \cup \{v_l u_l / 1 \leq l \leq y\} \cup \{v v_1\} \cup \{v u_1\} \cup \{u_y v_0\} \cup \{v_y v_0\}$

Now  $|\mathcal{V}(\mathcal{G})| = 2y+2$  and  $|\mathcal{E}(\mathcal{G})| = 3y+4$

Define a mapping  $f : \mathcal{V} \rightarrow O_{2y}$  as follows

$g(v) = 1$

$g(u_i) = 4l - 1$  for  $1 \leq l \leq y$

$g(v_i) = 4l + 1$  for  $1 \leq l \leq y$

$g(v_0) = 4y + 3$

Clearly point labels are distinct.

For every  $e = uv \in E$ , if  $\gcd(g(u), g(v)) = 1$

(i)  $e = vu_1, \gcd(g(v), g(u_1)) = \gcd(1, 3) = 1$

(ii)  $e = vv_1, \gcd(g(v), g(v_1)) = \gcd(1, 5) = 1$

(iii)  $e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(4l - 1, 4l + 3) = 1$  for  $1 \leq l \leq y - 1$

(iv)  $e = v_l v_{l+1}, \gcd(g(v_l), g(v_{l+1})) = \gcd(4l + 1, 4l + 5) = 1$  for  $1 \leq l \leq y - 1$

(v)  $e = v_l u_l, \gcd(g(v_l), g(u_l)) = \gcd(4l + 1, 4l - 1) = 1$  for  $1 \leq l \leq y$

(vi)  $e = v_y w, \gcd(g(v_y), g(w)) = \gcd(4y + 1, 4y + 3) = 1$

(vii)  $e = u_y w, \gcd(g(u_y), g(w)) = \gcd(4y - 1, 4y + 3) = 1$

Hence  $\mathcal{DA}_y^2$  is an odd prime graph.  $\square$

*Odd prime labeling for some arrow related graphs*

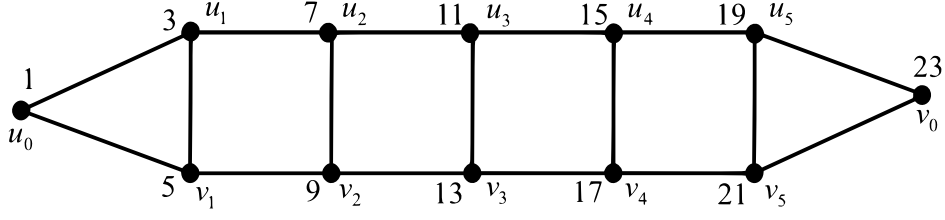


Figure 4: Arrow graph  $\mathcal{DA}_y^2$  and its odd prime labeling

**Theorem 2.5.**  $\mathcal{DA}_y^3$  is an odd prime graph where  $y \geq 3$ .

*Proof.* Let  $\mathcal{D} = \mathcal{DA}_y^3$  be an arrow graph got by connecting two point set  $u_0$  and  $z_0$  with superior pointss from both the ends of  $P_3 \times P_2$  by 3+3 new lines.

$$\mathcal{V}(\mathcal{G}) = \{u_l, v_l, w_l, / 1 \leq l \leq y\} \cup \{u_0\} \cup \{z_0\}$$

$$\mathcal{E}(\mathcal{G}) = \{u_l u_{l+1}, v_l v_{l+1}, w_l w_{l+1} / 1 \leq l \leq y-1\} \cup \{w_l v_l, v_l u_l / 1 \leq l \leq y\} \cup \{u_0 u_1, u_0 v_1, u_0 w_1, z_0 u_y, z_0 v_y, z_0 w_y\}$$

$$\text{Now } |\mathcal{V}(\mathcal{G})| = 3y + 2 \text{ and } |\mathcal{E}(\mathcal{G})| = 5y + 3$$

Define a mapping  $f : \mathcal{V} \rightarrow O_y$  as follows

$$g(u_0) = 1$$

$$g(u_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(u_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(v_l) = 6l - 1 \quad \text{for } 1 \leq l \leq y, l \text{ is odd}$$

$$g(v_l) = 6l - 3 \quad \text{for } 1 \leq l \leq y, l \text{ is even}$$

$$g(w_l) = 6l + 1 \quad \text{for } 1 \leq l \leq y$$

$$g(z_0) = 6y + 3 \quad \text{for } 1 \leq i \leq y$$

Clearly all the point values are different. With this labeling for each  $e \in E$  if

$$(i) e = u_0 u_1, \gcd(g(u_0), g(u_1)) = \gcd(1, 3) = 1$$

$$(ii) e = u_0 v_1, \gcd(g(u_0), g(v_1)) = \gcd(1, 5) = 1$$

$$(iii) e = u_0 w_1, \gcd(g(u_0), g(w_1)) = \gcd(1, 7) = 1$$

$$(iv) e = u_l v_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y \quad l \not\equiv 0 \pmod{2}$$

$$(v) e = u_l v_l, \gcd(g(u_l), g(v_l)) = \gcd(6l - 1, 6l - 3) = 1 \quad \text{for } 1 \leq l \leq y \quad l \equiv 0 \pmod{2};$$

$$(vi) e = v_l w_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 1, 6l + 1) = 1 \quad \text{for } 1 \leq l \leq y \quad l \not\equiv 0 \pmod{2}$$

$$(vii) e = v_l w_l, \gcd(g(v_l), g(w_l)) = \gcd(6l - 3, 6l - 1) = 1 \quad \text{for } 1 \leq l \leq y \quad l \equiv 0 \pmod{2};$$

$$(viii) e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 3, 6l + 5) = 1 \quad \text{for } 1 \leq l \leq y-1 \quad l \not\equiv 0 \pmod{2};$$

$$(ix) e = u_l u_{l+1}, \gcd(g(u_l), g(u_{l+1})) = \gcd(6l - 1, 6l + 3) = 1 \quad \text{for } 1 \leq l \leq y-1 \quad l \equiv 0 \pmod{2};$$

Hence  $\mathcal{DA}_y^3$  is an odd prime graph . □



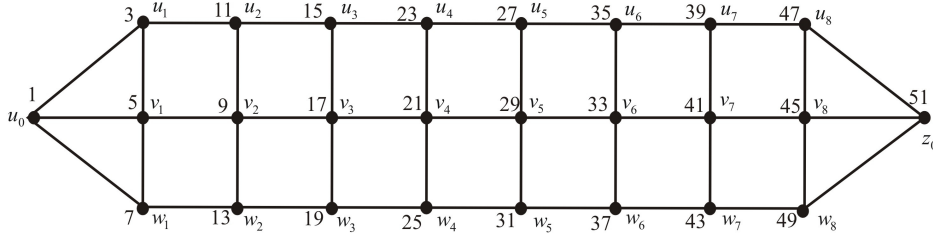


Figure 5: Arrow graph  $DA_n^3$  and its odd prime labeling

### 3 Conclusions

The odd Prime labeling of various classes of graphs such as  $A_y^2$  where  $y \in N$ ,  $A_y^3$ ,  $A_y^5$ , where  $y \geq 2$  are odd prime graph and double arrow graphs  $\mathcal{DA}_y^2, \mathcal{DA}_y^3$  are proved. To derive similar results for other graph families is an open area of research.

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*Odd prime labeling for some arrow related graphs*

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# Some operations on complex fuzzy graphs

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## Abstract

In this paper we discussed about some types of complex fuzzy graphs which is the extension of fuzzy graph. As the membership value of elements of fuzzy graph is in between 0 and 1. In complex fuzzy graph it will be extended to unit circle of complex plane. Therefore we have to consider the amplitude value as well as phase term value. We are considering complex fuzzy graphs in polar forms. Some operations on complex fuzzy graphs such as union, intersection, composition, and Cartesian products are introduced. Some basic theorems with respect to the above mentioned operations are proved with examples.

**2020 AMS subject classifications:** 05C62, 05C72, 05C76<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1060. ISSN: 1592-7415. eISSN: 2282-8214. ©N.Azhagendran et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

L.A.Zadeh [1965] gave the idea of fuzzy set which is similar to probability function. The idea of complex fuzzy set (CFS) was first given by Ramot et al. [2002]. Accordingly a CFS is an extension of the fuzzy set whose range is extended from  $[0,1]$  to a disc of unit radius in complex plane. Xueling et al. [2019] introduced some basic operations on complex fuzzy set. They have developed a new algorithm in signals and system by using complex fuzzy sets. After that some results have been given by P.Bhattacharya [1987] about fuzzy graphs. Thirunavukarasu et al. [2016] extended the fuzzy graph to complex fuzzy graph. Fuzzy graph were narrated by Mordeson and Peng [1994]. Nagoorgani and Latha [2015] gave some operations such as cartesian product, conjunction, disjunction in fuzzy graph. Shannon and Atanassov [1994] defined intuitionistic fuzzy graphs. After that many authors added their ideas to intuitionistic fuzzy graph. Naveed et al. [2019] introduced the complex intuitionistic fuzzy graphs with certain notions of union, join and composition. For Crisp graph we can refer Graph theory by Harary [1969].

In this paper, we introduced some types of complex fuzzy graphs with examples. Also some operations on complex fuzzy graphs such as union, intersection, composition and cartesian product with suitable examples.

## 2 Preliminaries

**Definition 2.1.** A complex fuzzy graph  $G_c = (\sigma_c, \mu_c)$  is defined on a graph  $G = (V, E)$  is a pair of complex functions  $\sigma_c : V \rightarrow r(z)e^{i\theta(z)}$ ,  $\mu_c : E \subseteq V \times V \rightarrow R(e)e^{i\phi(e)}$  such that  $\mu_c(z_1, z_2) = R(e)e^{i\phi(e)}$ , where  $R(e) \leq \min\{r(z_1), r(z_2)\}$  and  $\phi(e) \leq \min\{\theta(z_1), \theta(z_2)\}$  for all  $z_1, z_2 \in V$  and  $0 \leq r(z_1), r(z_2) \leq 1, 0 \leq \theta(z_1), \theta(z_2) \leq 2\pi$ .

**Example 2.1.** Consider the complex fuzzy graph  $G_c = (\sigma_c, \mu_c)$  where  $\sigma_c = \{z_1/0.2e^{i\pi}, z_2/0.5e^{i0.5\pi}, z_3/0.7\}$ ,  $\mu_c = \{(z_1, z_2)/0.1e^{i0.5\pi}, (z_1, z_3)/0.1, (z_2, z_3)/0.5\}$

**Definition 2.2.** A complex fuzzy graph  $\overline{G_c} = (\overline{\sigma_c}, \overline{\mu_c})$  is said to be a complement of CFG  $G_c$  if

- i)  $\overline{\sigma_c(z)} = \sigma_c(z)$  and
- ii)  $\overline{\mu_c(z_1, z_2)} = \overline{R(e)}e^{i\overline{\phi(e)}}$ , where  $\overline{R(e)} = \min\{r(z_1), r(z_2)\} - R(e)$  and  $\overline{\phi(e)} = \min\{\theta(z_1), \theta(z_2)\} - \phi(e), \forall z_1, z_2 \in V$ .

Some operations on complex fuzzy graphs

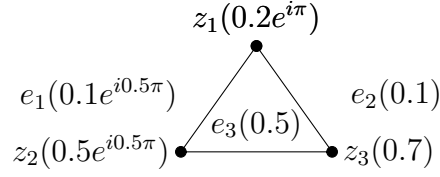


Figure 1: Complex fuzzy graph

**Example 2.2.** Consider the example 2.1, The complement of a CFG given in the figure 1 is given by

$$\overline{\sigma}_c = \{z_1/0.2e^{i\pi}, z_2/0.5e^{i0.5\pi}, z_3/0.7\}, \overline{\mu}_c = \{(z_1, z_2)/0.1, (z_1, z_3)/0.1\}$$

**Definition 2.3.** The order  $p$  and size  $q$  of a CFG  $G_c = (\sigma_c, \mu_c)$  defined on  $G = (V, E)$  are defined by

$$p = \sum_{z \in V} r(z) \cdot e^{i \sum_{z \in V} \theta(z)}; q = \sum_{e=(z_i, z_j) \in E} R(e) \cdot e^{i \sum_{e=(z_i, z_j) \in E} \phi(e)},$$

where  $R(e) \leq \min \{r(z_i), r(z_j)\}$  and  $\phi(e) \leq \min \{\theta(z_i), \theta(z_j)\}$  for all  $z_i, z_j \in V$ .

**Example 2.3.** Consider the example 2.1, the order of  $G_c$  is  $p = 1.4e^{i1.5\pi}$ , the size of  $G_c$  is  $q = 0.7e^{i0.5\pi}$

**Definition 2.4.** The degree of a vertex  $z_i$  in a CFG  $G_c = (\sigma_c, \mu_c)$  defined on  $G = (V, E)$  is defined by  $d(z_i) = \sum_{e=(z_i, z_j) \in \mu_c} R(e) \cdot e^{i \sum_{e=(z_i, z_j) \in \mu_c} \phi(e)}$  such that  $\mu_c(z_i, z_j) = R(e) \cdot e^{i\phi(e)}$ , for all  $z_j \in \sigma_c$ .

**Example 2.4.** Consider the example 2.1,  $d(z_1) = 0.2e^{i0.5\pi}$ ;  $d(z_2) = 0.6e^{i0.5\pi}$ ;  $d(z_3) = 0.6$

**Definition 2.5.** A CFG is said to be complete for every pair of vertices  $\mu_c(z_1, z_2) = R(e)e^{i\phi(e)}$  where  $R(e) = \min \{r(z_1), r(z_2)\}$  and  $\phi(e) = \min \{\theta(z_1), \theta(z_2)\}$  for all  $z_1, z_2 \in V$

**Example 2.5.** Let  $G_c = (\sigma_c, \mu_c)$  be a CFG, where  $\sigma_c = \{z_1/0.5e^{i0.7\pi}, z_2/0.8e^{i\pi}, z_3/0.6e^{i\pi}\}$   $\mu_c = \{(z_1, z_2)/0.5e^{i0.7\pi}, (z_1, z_3)/0.5e^{i0.7\pi}, (z_2, z_3)/0.6e^{i\pi}\}$

**Definition 2.6.** A CFG is regular if  $d(z_i) = d(z_j)$  for all  $z_i, z_j \in \sigma_c$ .

**Definition 2.7.** In a CFG  $G_c = (\sigma_c, \mu_c)$  for all  $z_i, z_j \in \sigma_c$ , the neighbourhood of  $z_i$  is defined by  $N(z_i) = \{z_j \in \sigma_c / (z_i, z_j) \in \mu_c\}$

**Definition 2.8.** A path  $P$  in a CFG  $G_c = (\sigma_c, \mu_c)$  is a sequence of distinct vertices  $z_0, z_1, z_2, \dots, z_n \in V$  (except possibly  $z_0$  and  $z_n$ ) such that  $\mu_c(z_{i-1}, z_i) = R(e_i)e^{i\phi(e_i)}$ ,  $R(e_i) > 0$ ,  $\phi(e_i) \geq 0$ ,  $i = 1, 2, \dots, n$ . Here  $n$  is called the length of the path. The consecutive pairs are called edges of the path. The strength of the path in a CFG is defined by  $\mu_c(z_{i-1}, z_i) = \min \{R(e_i)\} e^{i \min \phi(e_i)}$ ,  $i = 1, 2, 3, \dots, n$ . It is denoted by  $S(p)$ .

**Definition 2.9.** The strength of connectedness between two vertices  $z_i$  and  $z_j$  which is defined as the maximum amplitude and maximum phase term values of the strength of all paths between  $z_i$  and  $z_j$ . In symbol we denote it as  $\mu_c^\infty(z_i, z_j) = \text{CONN}_{G_c}(z_i, z_j)$ ,  $\mu_c^\infty(z_i, z_j) = T(e)e^{i\psi(e)}$ ,  $0 \leq T(e) \leq 1$ ,  $0 \leq \psi(e) \leq 2\pi$ , where  $T(e)$  is maximum amplitude value of all paths between  $z_i$  and  $z_j$  and  $\psi(e)$  is maximum phase term value of all paths between  $z_i$  and  $z_j$ .

For any arc  $(z_i, z_j)$ , if  $R(e) \geq T(e)$  and  $\phi(e) \geq \psi(e)$  then the arc  $(z_i, z_j)$  is said to be strong.

**Definition 2.10.** The strong degree of a vertex  $z$  in a CFG is defined by sum of membership values of strong arcs incident at  $z$ , and it is denoted by  $d_s(z)$ .

**Definition 2.11.** The strong neighbourhood of  $z_i$  in a CFG is defined by  $N_s(z_i) = \{z_j \in V / (z_i, z_j) \text{ is a strong arc}\}$ .

**Definition 2.12.** A CFG  $G_c = (\sigma_c, \mu_c)$  is said to be bipartite if the vertex  $\sigma_c$  can be partitioned into two non-empty sets  $\sigma_{c1}$  and  $\sigma_{c2}$  such that  $\mu_c(z_i, z_j) = 0$  if  $z_i, z_j \in \sigma_{c1}$  and  $z_i, z_j \in \sigma_{c2}$ .

**Definition 2.13.** A CFG  $G_c = (\sigma_c, \mu_c)$  is said to be complete bipartite if the vertex  $\sigma_c$  can be partitioned into two non-empty sets  $\sigma_{c1}$  and  $\sigma_{c2}$  such that  $\mu_c(z_i, z_j) = R(e).e^{i\phi(e)}$ , where  $R(e) = \min \{r(z_i), r(z_j)\}$ ,  $\phi(e) = \min \{\theta(z_i), \theta(z_j)\}$  for  $z_i \in \sigma_{c1}$  and  $z_j \in \sigma_{c2}$ .

**Definition 2.14.** A vertex  $z_i$  of a complex fuzzy graph  $G_c = (\sigma_c, \mu_c)$  is said to be an isolated vertex if  $\mu_c(z_i, z_j) = 0$ ,  $\forall z_j \in V - \{z_i\}$ ,  $(i.e) N(z_i) = \emptyset$ .

### 3 Operations on complex fuzzy graph

In this section the operations union, intersection, composition and cartesian product of CFGs are defined with examples. Some propositions based on the above operations are stated and proved.

**Definition 3.1.** Let the two complex fuzzy graphs  $G_{c1} = (\sigma_{c1}, \mu_{c1})$ , and  $G_{c2} = (\sigma_{c2}, \mu_{c2})$  defined on two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  respectively. Let  $G_{c1}$  be a pair of complex functions  $\sigma_{c1} : V_1 \rightarrow r_1(z)e^{i\theta_1(z)}$ ,  $\mu_{c1} : E_1 \subseteq$

### Some operations on complex fuzzy graphs

$V_1 \times V_1 \rightarrow R_1(e).e^{i\phi_1(e)}$  such that  $\mu_c(z_1, z_2) = R_1(e).e^{i\phi_1(e)}$  where  $R_1(e) \leq \min \{r_1(z_1), r_1(z_2)\}$ ,  $\phi_1(e) \leq \min \{\theta_1(z_1), \theta_1(z_2)\}$ . Also  $G_{c_2}$  is a pair of complex functions  $\sigma_{c_2} : V_2 \rightarrow r_2(z).e^{i\theta_2(z)}$ ,  $\mu_{c_2} : E_2 \subseteq V_2 \times V_2 \rightarrow R_2(e).e^{i\phi_2(e)}$  such that  $\mu_c(z_1, z_2) = R_2(e).e^{i\phi_2(e)}$  where  $R_2(e) \leq \min \{r_2(z_1), r_2(z_2)\}$ ,  $\phi_2(e) \leq \min \{\theta_2(z_1), \theta_2(z_2)\}$ . Then the union of two complex fuzzy graphs  $G_c = (\sigma_c = \sigma_{c_1} \cup \sigma_{c_2}, \mu_c = \mu_{c_1} \cup \mu_{c_2})$  on  $G = (V = V_1 \cup V_2, E = E_1 \cup E_2)$  is defined as follows

$$(i) \quad \sigma_c(z) = (\sigma_{c_1} \cup \sigma_{c_2})(z) = \begin{cases} r_1(z).e^{i\theta_1(z)}, \forall z \in V_1 \text{ and } z \notin V_2 \\ r_2(z).e^{i\theta_2(z)}, \forall z \in V_2 \text{ and } z \notin V_1 \\ \max \{r_1(z), r_2(z)\} e^{i \max \{\theta_1(z), \theta_2(z)\}}, z \in V_1 \cap V_2 \end{cases}$$

$$(ii) \quad \mu_c(z_1, z_2) = (\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) = \begin{cases} R_1(e).e^{i\phi_1(e)}, \forall (z_1, z_2) \in E_1 \text{ and } (z_1, z_2) \notin E_2 \\ R_2(e).e^{i\phi_2(e)}, \forall (z_1, z_2) \in E_2 \text{ and } (z_1, z_2) \notin E_1 \\ \max \{R_1(e), R_2(e)\} e^{i \max \{\phi_1(e), \phi_2(e)\}}, (z_1, z_2) \in E_1 \cap E_2 \end{cases}$$

**Proposition 3.1.** *Prove that the union of two complex fuzzy graph is also a complex fuzzy graph.*

*Proof.* Let  $G_{c_1}$  and  $G_{c_2}$  be two CFGs, then the union  $G_c = G_{c_1} \cup G_{c_2}$  is discussed in three different case. For vertices,

- i) Suppose that  $z \in V_1$  and  $z \notin V_2$  then
$$\begin{aligned} (\sigma_{c_1} \cup \sigma_{c_2})(z) &= \max \{r_1(z), r_2(z)\} .e^{i \max \{\theta_1(z), \theta_2(z)\}} \\ &= \max \{r_1(z), 0\} .e^{i \max \{\theta_1(z), 0\}} \\ &= r_1(z).e^{i\theta_1(z)}, \forall z \in V_1 \text{ and } z \notin V_2. \end{aligned}$$
- ii) Similarly we can prove, for  $z \notin V_1$  and  $z \in V_2$
- iii) Suppose that  $z \in V_1 \cap V_2$ , then
$$\begin{aligned} (\sigma_{c_1} \cup \sigma_{c_2})(z) &= \max \{\sigma_{c_1}, \sigma_{c_2}\} \\ &= \max \{r_1(z).e^{i\theta_1(z)}, r_2(z).e^{i\theta_2(z)}\} \\ &= \max \{r_1(z), r_2(z)\} .e^{i \max \{\theta_1(z), \theta_2(z)\}}, \forall z \in V_1 \cap V_2. \end{aligned}$$

For edges,

- i) Suppose that  $(z_1, z_2) \in E_1, (z_1, z_2) \notin E_2$ 

$$\begin{aligned} (\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) &= \max \{\mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2)\} \\ &\leq \max \{\min \{r_1(z_1), r_1(z_2), \min \{r_2(z_1), r_2(z_2)\}\}\} \\ &\quad e^{i \max \{\min \{\theta_1(z_1), \theta_1(z_2)\}, \min \{\theta_2(z_1), \theta_2(z_2)\}\}} \\ &= \max \{R_1(e), R_2(e)\} .e^{i \max \{\phi_1(e), \phi_2(e)\}} \\ &= \max \{R_1(e), 0\} .e^{i \max \{\phi_1(e), 0\}} \\ &= R_1(e).e^{i\phi_1(e)}, \forall (z_1, z_2) \in E_1, (z_1, z_2) \notin E_2 \end{aligned}$$

ii) Similarly, we can prove that,  $(z_1, z_2) \notin E_1, (z_1, z_2) \in E_2$

iii) Suppose that  $(z_1, z_2) \in E_1 \cap E_2$

$$\begin{aligned} (\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) &= \max \{ \mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2) \} \\ &\leq \max \{ \min \{ r_1(z_1), r_1(z_2) \}, \min \{ r_2(z_1), r_2(z_2) \} \} . \\ &e^{i \max \{ \min \{ \theta_1(z_1), \theta_1(z_2) \}, \min \{ \theta_2(z_1), \theta_2(z_2) \} \}} \\ &= \max \{ R_1(e), R_2(e) \} . e^{i \max \{ \phi_1(e), \phi_e(z) \}} \text{ for } (z_1, z_2) \in E_1 \cap E_2 \end{aligned}$$

Therefore the union of two CFGs is also a CFG.  $\square$

**Example 3.1.** Consider the two complex fuzzy graphs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$  and  $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$  given below  $\sigma_{c_1} = \{z_1/0.4e^{i0.8\pi}, z_2/0.7e^{i2\pi}, z_3/0.5e^{i1.2\pi}\}$ ,  $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i0.8\pi}, (z_2, z_3)/0.4e^{i\pi}\}$  and  $\sigma_{c_2} = \{z_1/0.5e^{i\pi}, z_3/0.5e^{i1.4\pi}, z_4/0.8e^{i2\pi}\}$ ,  $\mu_{c_2} = \{(z_1, z_4)/0.4e^{i0.6\pi}, (z_3, z_4)/0.5e^{i1.4\pi}\}$  Then the union  $G_c = (\sigma_{c_1} \cup \sigma_{c_2} = \sigma_c, \mu_{c_1} \cup \mu_{c_2} = \mu_c)$  on  $G = (V_1 \cup V_2, E_1 \cup E_2)$  can be written as

$$\begin{aligned} \sigma_c &= \{z_1/0.5e^{i\pi}, z_2/0.7e^{i2\pi}, z_3/0.5e^{i1.4\pi}, z_4/0.8e^{i2\pi}\} \\ \mu_c &= \{(z_1, z_2)/0.4e^{i0.8\pi}, (z_1, z_4)/0.4e^{i0.6\pi}, (z_3, z_4)/0.5e^{i1.4\pi}, (z_2, z_3)/0.4e^{i\pi}\} \end{aligned}$$

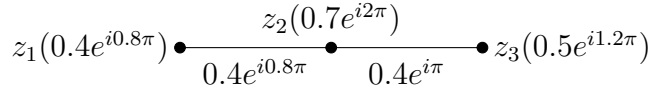


Figure 2:  $G_{c_1}$

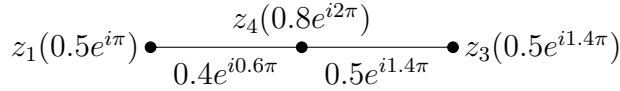


Figure 3:  $G_{c_2}$

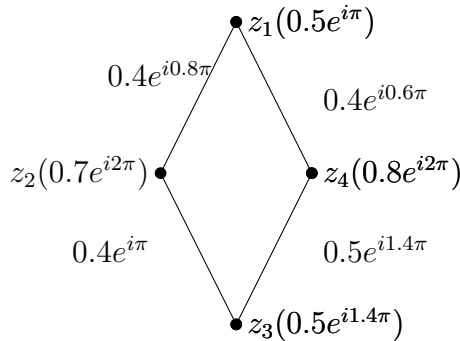


Figure 4:  $G_{c_1} \cup G_{c_2}$

**Remark 3.1.** Union of two strong CFG need not be a strong CFG.



### Some operations on complex fuzzy graphs

**Example 3.2.** Consider the Strong CFGs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$  and  $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$  where  $\sigma_{c_1} = \{z_1/0.4e^{i2\pi}, z_2/0.6e^{i\pi}\}$ ,  $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i\pi}\}$  and  $\sigma_{c_2} = \{z_1/0.8e^{i0.2\pi}, z_2/0.2e^{i\pi}\}$ ,  $\mu_{c_2} = \{(z_1, z_2)/0.2e^{i0.2\pi}\}$  then the union is given by  $\sigma_c = \{z_1/0.8e^{i2\pi}, z_2/0.6e^{i\pi}\}$ ,  $\mu_c = \{(z_1, z_2)/0.4e^{i0.2\pi}\}$ . However this is not a strong CFG.

**Definition 3.2.** Let the two complex fuzzy graphs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ , and  $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$  defined on two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  respectively. Let  $G_{c_1}$  be a pair of complex functions  $\sigma_{c_1} : V_1 \rightarrow r_1(z)e^{i\theta_1(z)}$ ,  $\mu_{c_1} : E_1 \subseteq V_1 \times V_1 \rightarrow R_1(e).e^{i\phi_1(e)}$  such that  $\mu_c(z_1, z_2) = R_1(e)e^{i\phi_1(e)}$  where  $R_1(e) \leq \min\{r_1(z_1), r_1(z_2)\}$ ,  $\phi_1(e) \leq \min\{\theta_1(z_1), \theta_1(z_2)\}$ . Also  $G_{c_2}$  is a pair of complex functions  $\sigma_{c_2} : V_2 \rightarrow r_2(z)e^{i\theta_2(z)}$ ,  $\mu_{c_2} : E_2 \subseteq V_2 \times V_2 \rightarrow R_2(e).e^{i\phi_2(e)}$  such that  $\mu_c(z_1, z_2) = R_2(e)e^{i\phi_2(e)}$  where  $R_2(e) \leq \min\{r_2(z_1), r_2(z_2)\}$ ,  $\phi_2(e) \leq \min\{\theta_2(z_1), \theta_2(z_2)\}$ . Then the intersection of two complex fuzzy graphs  $G_c = (\sigma_c = \sigma_{c_1} \cap \sigma_{c_2}, \mu_c = \mu_{c_1} \cap \mu_{c_2})$  is defined as follows

$$(i) \quad \sigma_c(z) = (\sigma_{c_1} \cap \sigma_{c_2})(z) = \min\{r_1(z), r_2(z)\} e^{i \min\{\theta_1(z), \theta_2(z)\}}, z \in V_1 \cap V_2$$

$$(ii) \quad \mu_c(z_1, z_2) = (\mu_{c_1} \cap \mu_{c_2})(z_1, z_2) = \begin{cases} \min\{R_1(e), R_2(e)\} e^{i \min\{\phi_1(e), \phi_2(e)\}}, (z_1, z_2) \in E_1 \cap E_2 \\ 0, \text{otherwise} \end{cases}$$

**Proposition 3.2.** Prove that the intersection of two complex fuzzy graphs is also a complex fuzzy graph.

*Proof.* Let  $G_{c_1}$  and  $G_{c_2}$  be two complex fuzzy graphs, then

$$i) \quad (\sigma_{c_1} \cap \sigma_{c_2})(z) = \min\{r_1(z), r_2(z)\} . e^{i \min\{\theta_1(z), \theta_2(z)\}}$$

*It is trivial.*

ii) For  $(z_1, z_2) \in E_1 \cap E_2$

$$\begin{aligned} (\mu_{c_1} \cap \mu_{c_2})(z_1, z_2) &= \min\{\mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2)\} . e^{i \min\{\phi_1(z), \phi_2(z)\}} \\ &\leq \min\{\min\{r_1(z_1), r_1(z_2)\}, \min\{r_2(z_1), r_2(z_2)\}\} . \\ &\quad e^{i \min\{\min\{\theta_1(z_1), \theta_1(z_2)\}, \min\{\theta_2(z_1), \theta_2(z_2)\}\}} \\ &= \min\{\min\{r_1(z_1), r_2(z_1)\}, \min\{r_1(z_2), r_2(z_2)\}\} . \\ &\quad e^{i \min\{\min\{\theta_1(z_1), \theta_1(z_2)\}, \min\{\theta_2(z_1), \theta_2(z_2)\}\}} \\ &= \min\{R_1(z), R_2(z)\} . e^{i \min\{\phi_1(z), \phi_2(z)\}} \end{aligned}$$

$$\begin{aligned} \text{where } R_1(z) &\leq \min\{r_1(z_1), r_2(z_1)\}, R_2(z) \leq \min\{r_1(z_2), r_2(z_2)\}, \\ \phi(z_1) &\leq \min\{\theta_1(z_1), \theta_2(z_1)\}, \\ \phi(z_2) &\leq \min\{\theta_2(z_1), \theta_2(z_2)\} \end{aligned}$$

Hence the  $G_{c_1} \cap G_{c_2}$  is a CFG. □

**Example 3.3.** Consider the two complex fuzzy graphs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$  where  $\sigma_{c_1} = \{z_1/0.5e^{i0.7\pi}, z_2/0.8, z_3/0.7e^{i1.5\pi}\}; \mu_{c_1} = \{(z_1, z_2)/0.5, (z_2, z_3)/0.7\}$  and  $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$  where  $\sigma_{c_2} = \{z_1/0.4e^{i2\pi}, z_2/0.6e^{i0.8\pi}, z_4/0.7e^{i\pi}\}; \mu_{c_2} = \{(z_1, z_4)/0.4e^{i\pi}, (z_1, z_2)/0.4e^{i0.8\pi}\}$ . Then the intersection of  $G_{c_1}$  and  $G_{c_2}$  on a pair  $G = (V_1 \cap V_2, E_1 \cap E_2)$  is given by  $V = \{z_1/0.4e^{i0.7\pi}, z_2/0.6\}, E = \{(z_1, z_2)/0.4\}$  where  $V = V_1 \cap V_2; E = E_1 \cap E_2$ .

**Remark 3.2.** Intersection of two strong CFGs is also a strong CFG.

**Definition 3.3.** Let the two complex fuzzy graphs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ , and  $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$  defined on two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  respectively. Let  $G_{c_1}$  be a pair of complex functions  $\sigma_{c_1} : V_1 \rightarrow r_1(a)e^{i\theta_1(a)}, \mu_{c_1} : E_1 \subseteq V_1 \times V_1 \rightarrow R_1(e).e^{i\phi_1(e)}$  such that  $\mu_{c_1}(a_1, a_2) = R_1(e)e^{i\phi_1(e)}$  where  $R_1(e) \leq \min\{r_1(a_1), r_1(a_2)\}, \phi_1(e) \leq \min\{\theta_1(a_1), \theta_1(a_2)\}$ . Also  $G_{c_2}$  is a pair of complex functions  $\sigma_{c_2} : V_2 \rightarrow r_2(a)e^{i\theta_2(a)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \rightarrow R_2(e).e^{i\phi_2(e)}$  such that  $\mu_{c_2}(a_1, a_2) = R_2(e)e^{i\phi_2(e)}$  where  $R_2(e) \leq \min\{r_2(a_1), r_2(a_2)\}, \phi_2(e) \leq \min\{\theta_2(a_1), \theta_2(a_2)\}$ . Then the composition of two complex fuzzy graphs is defined as follows

- (i)  $(\sigma_{c_1} \circ \sigma_{c_2})(a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\} e^{i \min\{\theta_1(a_1), \theta_2(a_2)\}}, \forall a_1, a_2 \in V = V_1 \cap V_2$
- (ii)  $(\mu_{c_1} \circ \mu_{c_2})((a, a_2), (a, b_2)) = \min\{r_1(a), R_2(a)\} e^{i \min\{\theta_1(a), \phi_2(a)\}}$   
where  $R_2(a) \leq \min\{r_2(a_2), r_2(b_2)\}; \phi_2(a) \leq \min\{\theta_2(a_2), \theta_2(b_2)\}, \forall a \in V_1, (a_2, b_2) \in E_2$
- (iii)  $(\mu_{c_1} \circ \mu_{c_2})((a_1, a), (b_1, a)) = \min\{R_1(a), r_2(a)\} e^{i \min\{\phi_1(a), \theta_2(a)\}}$   
where  $R_1(a) \leq \min\{r_1(a_1), r_1(b_1)\}; \phi_1(a) \leq \min\{\theta_1(a_1), \theta_1(b_1)\}, \forall a \in V_2, (a_1, b_1) \in E_1$
- (iv)  $(\mu_{c_1} \circ \mu_{c_2})((a_1, a_2), (b_1, b_2)) = \min\{r_2(a_2), r_2(b_2), R_1(a)\} . e^{i \min\{\theta_2(a_2), \theta_2(b_2), \phi_1(a)\}}, \forall a_2, b_2 \in V_2, a_2 \neq b_2, (a_1, b_1) \in E_1$ , where  $R_1(a) \leq \min\{r_1(a_1), r_1(b_1)\}; \phi_2(a) \leq \min\{\theta_1(a_1), \theta_1(b_1)\},$

**Proposition 3.3.** Prove that the composition of two complex fuzzy graphs is also a complex fuzzy graph.

*Proof.* Let  $G_{c_1}$  and  $G_{c_2}$  be two CFGs then we prove that  $G_{C_1} \circ G_{C_2}$  is a CFG.

- (i)  $(\sigma_{c_1} \circ \sigma_{c_2})(a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\} e^{i \min\{\theta_1(a_1), \theta_2(a_2)\}}, \forall a_1, a_2 \in V$ . It is trivial.

### Some operations on complex fuzzy graphs

- (ii)  $(\mu_{c_1} \circ \mu_{c_2})((a, a_2)(a, b_2)) = \min \{r_1(a), R_2(a)\} e^{i \min \{\theta_1(a), \theta_2(a)\}}$   
 $\leq \min \{r_1(a), \min \{r_2(a_2), r_2(b_2)\}\} e^{i \min \{\theta_1(a), \min \{\theta_2(a_2), \theta_2(b_2)\}\}}$   
 $= \min \{\min \{r_1(a), r_2(a_2)\}, \min \{r_1(a), r_2(b_2)\}\} .$   
 $e^{i \min \{\min \{\theta_1(a), \theta_2(a_2)\}, \min \{\theta_1(a), \theta_2(b_2)\}\}}$   
 $= \min \{r(a_1), r(b_1)\} e^{i \min \{\theta(a_1), \theta(b_1)\}}$  , where  $r(a_1) = \min \{r_1(a), r_2(a_2)\}$  ,  
 $r(b_1) = \min \{r_1(a), r_2(b_2)\}$  ,  $\theta(a_1) = \min \{\theta_1(a), \theta_2(a_2)\}$  ,  
 $\theta(b_1) = \min \{\theta_1(a), \theta_2(b_2)\}$  for all  $a \in V_1$  and  $(a_2, b_2) \in E_2$ .
- (iii)  $(\mu_{c_1} \circ \mu_{c_2})((a_1, b)(b_1, b)) = \min \{R_1(a), r_2(b)\} e^{i \min \{\theta_1(a), \theta_2(b)\}}$   
 $\leq \min \{\min \{r_1(a_1), r_1(b_1)\}, r_2(b)\} e^{i \min \{\min \{\theta_1(a_1), \theta_1(b_1)\}, \theta_2(b)\}}$   
 $= \min \{\min \{r_1(a_1), r_2(b)\}, \min \{r_1(b_1), r_2(b)\}\} .$   
 $e^{i \min \{\min \{\theta_1(a_1), \theta_2(b)\}, \min \{\theta_1(b_1), \theta_2(b)\}\}}$   
 $= \min \{r(a_2), r(b_2)\} e^{i \min \{\theta(a_2), \theta(b_2)\}}$  , where  $r(a_2) = \min \{r_1(a_1), r_2(b)\}$  ,  
 $r(b_2) = \min \{r_1(b_1), r_2(b)\}$  ,  $\theta(a_2) = \min \{\theta_1(a_1), \theta_2(b)\}$  ,  
 $\theta(b_2) = \min \{\theta_1(b_1), \theta_2(b)\}$  for all  $b \in V_2$  and  $(a_1, b_1) \in E_1$ .
- (iv) For all  $a_2, b_2 \in V_2, a_2 \neq b_2, (a_1, b_1) \in E_1$   
 $(\mu_{c_1} \circ \mu_{c_2})((a_1, a_2), (b_1, b_2)) = \min \{r_2(a_2), r_2(b_2), R_1(a)\} .$   
 $e^{i \min \{\theta_2(a_2), \theta_2(b_2), \phi(a)\}}$   
 $\leq \min \{r_2(a_2), r_2(b_2), \min \{r_1(a_1), r_1(b_1)\}\} .$   
 $e^{i \min \{\theta_2(a_2), \theta_2(b_2), \min \{\theta_1(a_1), \theta_1(b_1)\}\}}$   
 $= \min \{r_2(a_2), r_2(b_2), r_1(a_1), r_1(b_1)\} . e^{i \min \{\theta_2(a_2), \theta_2(b_2), \theta_1(a_1), \theta_1(b_1)\}}$   
 $= \min \{r(a), r(b)\} . e^{i \min \{\theta(a), \theta(b)\}}$  .  
where  $r(a) = \min \{r_1(a_1), r_2(a_2)\}$  ,  $r(b) = \min \{r_1(b_1), r_2(b_2)\}$   
 $\theta(a) = \min \{\theta_1(a_1), \theta_2(a_2)\}$  ,  $\theta(b) = \min \{\theta_1(b_1), \theta_2(b_2)\}$ .

Hence  $G_{C1} \circ G_{C2}$  is a CFG. □

**Example 3.4.** Consider the CFGs,  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$   
where  $\sigma_{c_1} = \{z_1/0.4e^{i0.7\pi}, z_2/0.6e^{i2\pi}\}$  ;  $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i0.5\pi}\}$  and  $G_{c_2} =$   
 $(\sigma_{c_2}, \mu_{c_2})$  where  $\sigma_{c_2} = \{z_3/0.5e^{i1.2\pi}, z_4/0.6e^{i0.7\pi}\}$  ;  $\mu_{c_2} = \{(z_3, z_4)/0.5e^{i0.7\pi}\}$  .  
Then the composition of  $G_{c_1}$  and  $G_{c_2}$  is given by  $G_c = (\sigma_c, \mu_c)$  where  $\sigma_c =$   
 $\{z_1z_3/0.4e^{i0.7\pi}, z_1z_4/0.4e^{i0.7\pi}, z_2z_3/0.5e^{i1.2\pi}, z_2z_4/0.6e^{i0.7\pi}\}$  ,  
 $\mu_c = \{(z_1z_3, z_1z_4)/0.4e^{i0.7\pi}, (z_1z_4, z_2z_4)/0.4e^{i0.5\pi}, (z_2z_3, z_2z_4)/0.5e^{i0.7\pi},$   
 $(z_1z_3, z_2z_3)/0.4e^{i0.5\pi}, (z_1z_3, z_2z_4)/0.4e^{i0.5\pi}, (z_1z_4, z_2z_3)/0.4e^{i0.5\pi}\}$

**Remark 3.3.** The Composition of two strong CFG is need not be a strong CFG.

**Definition 3.4.** Let the two complex fuzzy graphs  $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ , and  $G_{c_2} =$   
 $(\sigma_{c_2}, \mu_{c_2})$  defined on two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  respectively.  
Let  $G_{c_1}$  be a pair of complex functions  $\sigma_{c_1} : V_1 \rightarrow r_1(a)e^{i\theta_1(a)}, \mu_{c_1} : E_1 \subseteq$   
 $V_1 \times V_1 \rightarrow R_1(e).e^{i\phi_1(e)}$  such that  $\mu_c(a_1, a_2) = R_1(e)e^{i\phi_1(e)}$  where  $R_1(e) \leq$

$\min \{r_1(a_1), r_1(a_2)\}, \phi_1(e) \leq \min \{\theta_1(a_1), \theta_1(a_2)\}$ . Also  $G_{c_2}$  is a pair of complex functions  $\sigma_{c_2} : V_2 \rightarrow r_2(a)e^{i\theta_2(a)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \rightarrow R_2(e).e^{i\phi_2(e)}$  such that  $\mu_c(a_1, a_2) = R_2(e)e^{i\phi_2(e)}$  where  $R_2(e) \leq \min \{r_2(a_1), r_2(a_2)\}, \phi_2(e) \leq \min \{\theta_2(a_1), \theta_2(a_2)\}$ .

Then the Cartesian product of two complex fuzzy graphs is defined as follows

- (i)  $(\sigma_{c_1} \times \sigma_{c_2})(a_1, a_2) = \min \{r_1(a_1), r_2(a_2)\} e^{\min\{\theta_1(a_1), \theta_2(a_2)\}}, \text{ for } a_1, a_2 \in V$
- (ii)  $(\mu_{c_1} \times \mu_{c_2})((a, a_2), (a, b_2)) = \min \{r_1(a), R_2(a)\} .e^{\min\{\theta_1(a), \phi_2(a)\}}$   
where  $R_2(a) \leq \min \{r_2(a_2), r_2(b_2)\}; \phi_2(a) \leq \{\theta_2(a_2), \theta_2(b_2)\}$  for all  $a \in V_1$  and  $(a_2, b_2) \in E_2$
- (iii)  $(\mu_{c_1} \times \mu_{c_2})((a_1, a), (b_1, a)) = \min \{R_1(a), r_2(a)\} .e^{\min\{\phi_1(a), \theta_2(a)\}}$   
where  $R_1(a) \leq \min \{r_1(a_1), r_1(b_1)\}; \phi_2(a) \leq \{\theta_1(a_1), \theta_1(b_1)\}$  for all  $a \in V_2$  and  $(a_1, b_1) \in E_1$

**Proposition 3.4.** The cartesian product of two complex fuzzy graphs is also a complex fuzzy graph.

*Proof.* Let  $G_{c_1}$  and  $G_{c_2}$  be two complex fuzzy graphs, then we prove that  $G_{c_1} \times G_{c_2}$  is a CFG.

- (i)  $\sigma_{c_1} \times \sigma_{c_2}(a_1, a_2) = \min \{r_1(a_1), r_2(a_2)\} .e^{i \min\{\theta_1(a_1), \theta_2(a_2)\}}, \forall a_1, a_2 \in V_1 \times V_2$ . It is trivial, we have to verify the conditions only for  $E_1 \times E_2$
- (ii)  $(\mu_{c_1} \times \mu_{c_2})((a, a_2), (a, b_2)) = \min \{r_1(a), R_2(a)\} .e^{i \min\{\theta_1(a), \phi_2(a)\}}$   
 $\leq \min \{r_1(a), \min \{r_2(a_2), r_2(b_2)\}\} .e^{i \min\{\theta_1(a), \min\{\theta_2(a_2), \theta_2(b_2)\}\}}$   
 $= \min \{\min \{r_1(a), r_2(a_2)\}, \min \{r_1(a), r_2(b_2)\}\} .$   
 $e^{i \min\{\min\{\theta_1(a), \theta_2(a_2)\}, \min\{\theta_1(a), \theta_2(b_2)\}\}}$   
 $= \min \{r(a_1), r(b_1)\} .e^{i \min\{\theta(a_1), \theta(b_1)\}}$   
where  $r(a_1) = \min \{r_1(a), r_2(a_2)\}, r(b_1) = \min \{r_1(a), r_2(b_2)\},$   
 $\theta(a_1) = \min \{\theta_1(a), \theta_2(a_2)\}, \theta(b_1) = \min \{\theta_1(a), \theta_2(b_2)\}$   
for all  $a \in V_1, (a_2, b_2) \in E_2$
- (iii)  $(\mu_{c_1} \times \mu_{c_2})((a_1, a), (b_1, a)) = \min \{R_1(a), r_2(a)\} .e^{i \min\{\phi_1(a), \theta_2(a)\}}$   
 $\leq \min \{\min \{r_1(a_1), r_1(b_1)\}, r_2(a)\} .e^{i \min\{\min\{\theta_1(a_1), \theta_1(b_1)\}, \theta_2(a)\}}$   
 $= \min \{\min \{r_1(a_1), r_2(a)\}, \min \{r_1(b_1), r_2(a)\}\} .$   
 $e^{i \min\{\min\{\theta_1(a_1), \theta_2(a)\}, \min\{\theta_1(b_1), \theta_2(a)\}\}}$   
 $= \min \{r(a_2), r(b_2)\} .e^{i \min\{\theta(a_2), \theta(b_2)\}}$   
where  $r(a_2) = \min \{r_1(a_1), r_2(a)\}, r(b_2) = \min \{r_1(b_1), r_2(a)\},$   
 $\theta(a_2) = \min \{\theta_1(a_1), \theta_2(a)\}, \theta(b_2) = \min \{\theta_1(b_1), \theta_2(a)\}$   
for all  $a \in V_2, (a_1, b_1) \in E_1$

Hence  $G_{c_1} \times G_{c_2}$  is a CFG. □

### Some operations on complex fuzzy graphs

**Example 3.5.** Let  $G_{c_1}$  and  $G_{c_2}$  be two complex fuzzy graphs.

$\sigma_{c_1} = \{z_1/0.4e^{i2\pi}, z_2/0.5e^{i1.2\pi}\}$ ,  $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i1.2\pi}\}$ ,  
 $\sigma_{c_2} = \{z_3/0.5e^{i0.7\pi}, z_4/0.8e^{i\pi}, z_5/0.4e^{i1.7\pi}\}$ ,  
 $\mu_{c_2} = \{(z_3, z_4)/0.4e^{i0.7\pi}, (z_4, z_5)/0.4e^{i\pi}\}$ . Then the cartesian product is given by  
 $\sigma_{c_1} \times \sigma_{c_2} = \{z_1z_3/0.4e^{i0.7\pi}, z_1z_4/0.4e^{i\pi}, z_1z_5/0.4e^{i1.7\pi},$   
 $z_2z_3/0.5e^{i0.7\pi}, z_2z_4/0.5e^{i\pi}, z_2z_5/0.4e^{i1.2\pi}\}$   $\mu_{c_1} \times \mu_{c_2} = \{(z_1z_3, z_1z_4)/0.4e^{i0.7\pi},$   
 $(z_1z_4, z_1z_5)/0.4e^{i\pi}, (z_1z_3, z_2z_3)/0.4e^{i0.7\pi}, (z_2z_3, z_2z_4)/0.5e^{i0.7\pi},$   
 $(z_2z_4, z_2z_5)/0.4e^{i1.2\pi}, (z_1z_4, z_2z_4)/0.4e^{i\pi}, (z_1z_5, z_2z_5)/0.4e^{i1.2\pi}\}$

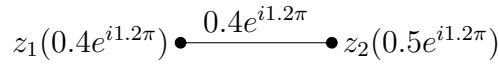


Figure 5:  $G_{c_1}$

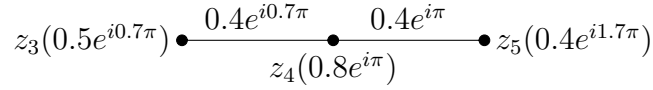


Figure 6:  $G_{c_2}$

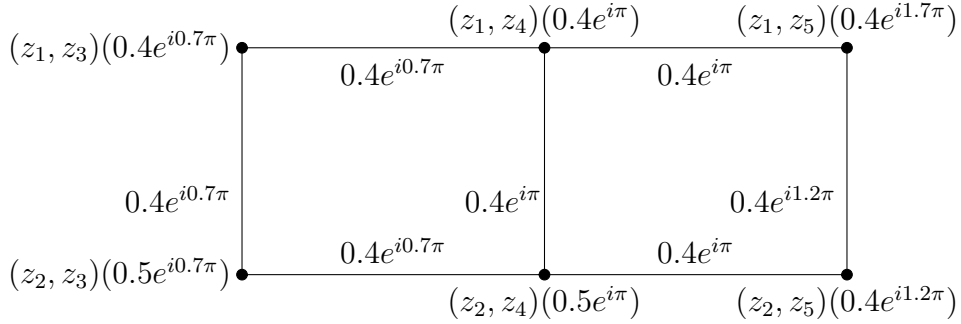


Figure 7:  $G_{c_1} \times G_{c_2}$

**Remark 3.4.** Cartesian product of two strong CFG is also a strong CFG.

## 4 Conclusions

In this paper, we discussed about some operations on complex fuzzy graphs such as union, intersection, composition and Cartesian product with examples.

Complex fuzzy graph is an extension of a fuzzy graph. We are working to extend the algorithms applied to complex fuzzy graphs are i) Domination on complex fuzzy graphs ii) Connectivity of complex fuzzy graphs and more operations on Complex fuzzy graphs

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# Degree of an edge and Platt number in signed networks

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## Abstract

Positive labelled edges play a vital role in network analysis. The degree of edges in signed graphs is introduced by giving importance to positive edges incident on the end vertices of that edge. The concept of Platt number of a graph, which is the sum of degrees of its edges, is extended to signed graphs based on the degree defined. Bounds of degree of an edge and Platt number in certain classes of signed graphs are determined. Some characterizations on Platt number of signed graphs are also established. A model to analyse social networks using degree of edges and Platt number is also proposed.

**Keywords:** Signed graph, positive edges, negative edges, networks, information diffusion, degree of an edge, Platt number

**2020 AMS subject classifications:** 54A40. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1061. ISSN: 1592-7415. eISSN: 2282-8214. ©Diviya K D et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Graph Theory is a branch of Mathematics that is extensively used to mathematically model social science which is discussed in Harary and Norman [1953]. According to Cartwright and Harary, in the paper Cartwright and Harary [1956], graphs can be used for representing relationships between persons whatever be the nature of relationships. The social relationships between individuals in society is not uniform. There may be a lot of variations in the type and intensity in the relationships between individuals. Friendship, like etc. are examples of positive relationships between two persons and enmity, indifference etc. are examples of negative relationships between them. In Graph theory, positive labelled edges can be used to represent the positive relationships and negative labelled edges can be used to represent negative relationships. If there does not exist any kind of relationships between persons, it can be represented by non adjacent vertices. Such graphs that are used to represent networks, especially in social science, are called signed graphs by Harary et al. [1965]. From Acharya [1986], Zaslavsky [2012], Anchuri and Magdon-Ismael [2012] and Tang et al. [2016] we can understand that signed graphs are well studied in literature because of their easiness to represent social systems in the real world.

A signed graph is a graph  $\sum(G, \sigma)$  with underlying graph  $G(V, E)$  and signature function  $\sigma : E \rightarrow \{1, -1\}$ . Degree of an edge in an unsigned graph is the number of edges incident on the end vertices of that edge excluding it. Degree of an edge plays a vital role in networks such as communication networks, transportation networks etc. as it indicates the influence of that edge (type of connection) in that particular network. In Platt [1947] the concept of Platt number is discussed and it is mainly intended to study molecular structures in Chemistry. Several characterizations related to Platt number of various graphs representing molecular structures are discussed in Belavadi and Mangam [2018] and Mahadevi et al. [2018]. Here we study these concepts in signed graphs.

Information diffusion in social networks is widely studied in literature which can be seen in Guille et al. [2013], Jalali et al. [2016] and Tselykh et al. [2020]. Various models also were proposed in literature to study information diffusion and influence maximisation in social networks which was discussed in Rui et al. [2018] and Li et al. [2019]. The motivation behind the study is that signed graphs can be effectively used to represent social networks as depicted in Acharya [1985], Gionis et al. [2020] and Sun et al. [2020]. Degree of an edge can be used to interpret the strength of influence of that particular edge which represents the specified relationship in the concerned network. Hence information diffusion is possible through the edge with maximum degree. Platt number can also be used to analyse the strength of connectedness of various networks.

In this paper, we extend the definition of degree of an edge and Platt number



to signed graphs by giving more importance to positive labelled edges incident on the end vertices of that edge. This is done because of the extensive importance of positive edges in networks especially in social networks. Further, we find bounds on degree of an edge and Platt number in signed graphs and certain classes of signed graphs related to social networks. We then present some characterizations on the Platt number of signed graphs. Finally we propose a model to analyse networks based on degree of edges and Platt number.

## **2 Degree of an edge in a signed graph**

In this section the definition of degree of an edge, discussed in Acharya et al. [2009], is extended to signed graphs and also some of its characterizations in signed graphs are investigated.

### **2.1 Definition**

If  $x \in V$ , then positive degree of  $x$  is defined as the number of positive edges incident at  $x$  and is denoted by  $d^+(x)$  and negative degree of  $x$  is defined as the number of negative edges incident at  $x$  and is denoted by  $d^-(x)$ . The degree of  $x$  is denoted by  $d(x)$  and is given by  $d(x) = d^+(x) + d^-(x)$  in Acharya et al. [2009].

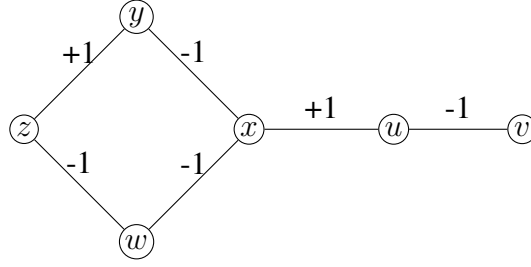
### **2.2 Definition(Degree of an edge)**

Let  $\sum(G, \sigma)$  be a signed graph with underlying graph  $G(V, E)$  and signature function  $\sigma : E \rightarrow \{1, -1\}$ . Let  $xy \in E$  where  $x, y \in V$ . The degree of an edge is defined as follows

$$d(xy) = \begin{cases} d^+(x) + d^+(y) - 1 & \text{if } \sigma(xy) = +1 \\ d^+(x) + d^+(y) & \text{if } \sigma(xy) = -1 \end{cases} \quad (1)$$

### **2.3 Example**

Consider the following signed graph.



Signed graph

By the definition,  $d(xy) = 2$ ,  $d(yz) = 1$ ,  $d(zw) = 1$ ,  $d(wx) = 1$ ,  $d(xu) = 1$  and  $d(uv) = 1$ .

## 2.4 Bounds on degree of an edge

Now, bounds on degree of an edge in certain classes of graphs are determined. It is obvious that the lower bound of degree of an edge in any class of graphs is 0. The upperbound of degree of an edge in certain classes of graphs are given below

| Class of graphs                     | Upperbound of edge degree $d(e)$ |
|-------------------------------------|----------------------------------|
| $P_n$ -Path graph                   | 3                                |
| $C_n$ -Cycle graph                  | 3                                |
| $W_n$ -Wheel graph                  | $n + 1$                          |
| $K_n$ -Complete graph               | $2n - 3$                         |
| $K_{p,q}$ -Complete bipartite graph | $p + q - 1$                      |
| $H_n$ -Helm graph                   | $n + 3$                          |
| $S_n$ -Sunflower graph              | $n + 4$                          |

Table 1: Upper Bound-Edge Degree

The following result is on the bounds of the degree of an edge in any signed graph

**Proposition 2.1.** *For a given signed graph  $\sum(G, \sigma)$  with underlying graph  $G(V, E)$  with order  $n \geq 3$ ,  $0 \leq d(e) \leq 2n - 3$*

*Proof.* Consider a connected signed graph  $\sum(G, \sigma)$ . By definition of degree of an edge it is clear that  $d(e) \geq 0$

In order to find the upperbound of degree of an edge consider an edge  $e = xy$  where  $x$  and  $y$  are vertices of maximum degree i.e.  $n - 1$  each and all edges on

$x$  and  $y$  are positive. In order to determine the maximum possible value of degree of an edge we assume that  $e$  is labelled positive.

Now  $d(e) = 2(n - 1) - 1 = 2n - 3$

Hence  $0 \leq d(e) \leq 2n - 3$  for any edge  $e$  in  $\Sigma(G, \sigma)$  □

### 3 Platt number of a signed graph

The concept of Platt number was discussed in Platt [1947] and it is defined as the sum of degrees of edges of a graph as in Mahadevi et al. [2018]. Here we extend the definition of Platt number to signed graphs.

Consider a signed graph  $\Sigma(G, \sigma)$  where  $G(V, E)$  is the underlying graph and  $\sigma$  is the signature function. The Platt number of  $\Sigma$  is given by

$$F(\Sigma) = \sum_{xy \in E(G)} d(xy) \quad (2)$$

#### 3.1 Example

Consider the graph given in example 2.3. Degree of edges are already calculated and its Platt number is given by

$$F(\Sigma) = d(xy) + d(yz) + d(zw) + d(wx) + d(xu) + d(uv) = 7$$

### 4 Bounds on Platt number

In this section bounds on Platt number of certain graph classes are proposed. From the definition of Platt number itself it is clear that its lowerbound is 0 and is attained in the graph in which all edges are labelled negative. The upperbound of Platt number of certain graph classes are given below.

| Class of graphs                      | Upperbound of Platt number |
|--------------------------------------|----------------------------|
| $P_n$ -Path graph                    | $3n - 5$                   |
| $C_n$ -Cycle graph                   | $3n$                       |
| $W_n$ -Wheel graph                   | $(n - 1)(n + 6)$           |
| $K_n$ -Complete graph                | $\frac{n(n-1)(2n-3)}{2}$   |
| $K_{p,q}$ -Complete bipartite graph  | $pq(p + q - 1)$            |
| $H_n$ -Helm graph                    | $n^2 + 14n$                |
| $S_n$ -Sunflower graph( $n \geq 5$ ) | $n^2 + 25n$                |

Table 2: Upperbound-Platt number

**Theorem 4.1.** In a connected signed graph  $\Sigma(G, \sigma)$  with underlying graph  $G(V, E)$  and order  $n \geq 2$ ,  $0 \leq F(\Sigma) \leq \frac{n(n-1)(2n-3)}{2}$

*Proof.* For any signed graph  $\Sigma(G, \sigma)$ ,  $F(\Sigma) \geq 0$  since  $d(e) \geq 0$  for any edge  $e$  in  $\Sigma$ .

Upperbound of the degree of an edge in a signed graph is  $2n - 3$  and the maximum number of edges in a signed graph of order  $n$  is  $\frac{n(n-1)}{2}$ . So consider the maximum possible case which is the signed graph with maximum number of edges for a given order  $n$  and with all edges are labelled positive. Here Platt number is  $\frac{n(n-1)(2n-3)}{2}$ .

Hence for any signed graph  $\Sigma(G, \sigma)$

$$0 \leq F(\Sigma) \leq \frac{n(n-1)(2n-3)}{2}$$

□

**Remark 4.1.** In the above theorem, the upper bound is attained by the complete graph with all edges labelled positive.

**Theorem 4.2.** For a connected signed graph  $\Sigma(G, \sigma)$  of size  $m \geq 2$ ,  $0 \leq F(\Sigma) \leq m^2$ .

In particular if all edges are signed positive, then  $3m - 2 \leq F(\Sigma) \leq m^2$

*Proof.* It is obvious that  $F(\Sigma) \geq 0$  for any connected signed graph  $\Sigma$ . Now for a connected graph of given size  $m \geq 2$ , Platt number takes its maximum value in the case of a signed graph in which all edges are labelled positive and each edge takes its maximum degree which is  $m$ . In this case  $F(\Sigma) = m^2$ . Hence  $0 \leq F(\Sigma) \leq m^2$  for any connected signed graph  $\Sigma$  of size  $m \geq 2$

In the case of a connected signed graph with all edges are labelled positive and size  $m \geq 2$ , degree of an edge takes its least value 2 which is by the edges of pendant vertices and for all other  $m - 2$  edges the least value is 3. In this case Platt number is  $3m - 2$ .

Hence  $3m - 2 \leq F(\Sigma) \leq m^2$  for a connected signed graph of order  $m \geq 2$  with all edges labelled positive. □

**Remark 4.2.** In the above theorem, the upper bound is attained by the graph  $K_{1,n-1}$  with all edges labelled positive and lower bound is attained by the path graph  $P_n$  with all edges labelled positive.

**Theorem 4.3.** For a signed tree  $T_n$  of order  $n \geq 3$ ,  $0 \leq F(T_n) \leq (n-1)^2$ . In particular if all edges are labelled positive, then  $3n - 5 \leq F(T_n) \leq (n-1)^2$ .

*Proof.* For a given signed tree on  $n$  vertices, the maximum degree for an edge is  $n - 1$  which is attained when all edges are labelled positive and the maximum number of edges is  $n - 1$ . In this case the Platt number is maximum and is given by  $(n - 1)^2$ . Hence  $0 \leq F(T_n) \leq (n - 1)^2$  for any signed tree  $T_n$  of order  $n \geq 3$ .

When all edges are labelled positive, the minimum value of degree of an edge in a tree is 2 and as in the above theorem it is taken by edges incident on pendant vertices and for all other  $n - 3$  edges, the minimum value is 3. In this case Platt number is given by  $F(T_n) = 3n - 5$ . Hence  $3n - 5 \leq F(T_n) \leq (n - 1)^2$  for any signed tree with all edges labelled positive.  $\square$

**Remark 4.3.** In the above theorem also, when all edges are labelled positive the upper bound is attained by the tree  $K_{1,n-1}$  and lower bound is attained by the tree  $P_n$ .

## 5 Network analysis model using degree of edges and Platt number

$$\begin{pmatrix} 0 & +1 & -1 & 0 & +1 & -1 & 0 & +1 & 0 & +1 \\ +1 & 0 & +1 & -1 & 0 & +1 & 0 & -1 & +1 & +1 \\ -1 & +1 & 0 & +1 & +1 & +1 & -1 & +1 & +1 & 0 \\ 0 & -1 & +1 & 0 & +1 & 0 & +1 & -1 & +1 & +1 \\ +1 & 0 & +1 & +1 & 0 & +1 & 0 & 0 & +1 & 0 \\ -1 & +1 & +1 & 0 & +1 & 0 & +1 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 & +1 & 0 & -1 & +1 & 0 \\ +1 & -1 & +1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ 0 & +1 & +1 & +1 & +1 & 0 & +1 & -1 & 0 & +1 \\ +1 & +1 & 0 & +1 & 0 & +1 & 0 & +1 & +1 & 0 \end{pmatrix}.$$

First we calculate degree of edges in order to analyse the network as degree of an edge can be considered as a measure of the influence of that particular relationship in the given network. Degree of edges are given in the following table. Note that, since  $d(xy) = d(yx)$  we have filled the upper triangular part of the table.

As we have mentioned in the introductory part of this paper, degree of an edge can be used to analyse social networks since degree of an edge indicates the influence of that particular relationship in the corresponding network. Information diffusion in social networks is possible at its maximum through the edges, which represent relationships, with maximum influence. So information diffusion is fast through the edge with maximum degree. In this section we propose a model to analyse a network and to find the most influential relationship in that network through an example. Consider a classroom consisting of ten students. Suppose a

|          | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| $v_1$    |       | 8     | 10    |       | 8     | 9     |       | 6     |       | 8        |
| $v_2$    |       |       | 10    | 10    |       | 9     |       | 8     | 10    | 9        |
| $v_3$    |       |       |       | 10    | 10    | 10    | 9     | 9     | 11    |          |
| $v_4$    |       |       |       |       | 9     |       | 7     | 7     | 10    | 9        |
| $v_5$    |       |       |       |       |       | 9     |       |       | 10    |          |
| $v_6$    |       |       |       |       |       |       | 7     | 8     |       | 9        |
| $v_7$    |       |       |       |       |       |       |       | 5     | 8     |          |
| $v_8$    |       |       |       |       |       |       |       |       | 8     | 7        |
| $v_9$    |       |       |       |       |       |       |       |       |       | 10       |
| $v_{10}$ |       |       |       |       |       |       |       |       |       |          |

Table 3: Degree of edges of the network

teacher needs to identify the most influential student or relationship in the class in order to implement new methods of teaching effectively in the class. First we model that class using signed graphs. Edges with signature function  $\sigma(e) = +1$  represent the friendship between students and those with  $\sigma(e) = -1$  represent the indifference between them and vertices without edges between them shows that there is neither a friendship nor an indifference between them.

Consider the following adjacency matrix of that particular signed graph which represents the above mentioned network of the class room.

From the above table it is clear that edge with maximum degree is  $v_3v_9$  and is 11. Hence we can conclude that the most influential relationship in this network is between  $v_3$  and  $v_9$ . So in order to make communications in that class more effective and fast or to implement new strategies effectively by the teacher it is more desirable to communicate it with either  $v_3$  or  $v_9$ .

Platt number of this graph is 287. We can compare different networks using this Platt number as Platt number indicates the intensity of the connectedness between individuals of that particular network. Moreover, by assigning degree of edges to corresponding edges, signed graphs can be converted to weighted graphs in which weight indicates the intensity of relationships between individuals.

## 6 Conclusions

In this paper the definition of degree of an edge and Platt number is extended to signed graphs. Some characterizations of degree of an edge and Platt number are proposed. A model to analyse networks based on the degree of edges and Platt number is also proposed. Further we intend to study Platt number in signed graphs by applying certain graph operations which are relevant in social networks. We

also intend to study networks by converting the corresponding signed graphs in to weighted graphs.

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# Algebraic coding theory using Pell equation

$$x^2 - 8y^2 = 1$$

Janaki G\*

Gowri Shankari A<sup>†</sup>

## Abstract

An interdisciplinary field with significant practical use is cryptography. The difficulty of specific mathematical computing tasks affects a public key cryptosystem's security. The technique for coding and decoding the messages was described in this work, utilising the solutions of Pell equation  $x^2 - 8y^2 = 1$  and matrix  $Q^{8*}$ . It was noted that messages can be turned into an even size that is then divided into slabs. Considering the information security has become a more serious issue in recent years, coding and decoding algorithms are essential in order to improve information security. In this article, a new matrix  $Q^{8*}$  and a decryption system based on the solutions of the Pell Equation  $x^2 - 8y^2 = 1$  are devised. The messages can be divided into even-sized slabs during encryption. This algorithm will not only improve information security but also has a high degree of accuracy.

**Keywords:** Pell equation; Encryption-decryption algorithm;  $Q^{8*}$  matrix; Cryptography

**2020 AMS subject classifications:** 11B37, 11C20, 11D09, 11T71.

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1062. ISSN: 1592-7415. eISSN: 2282-8214. ©Janaki G et al. This paper is published under the CC-BY licence agreement.

# 1 Introduction

Currently, almost all current coding methods can be supported by number theory Carmichael [1950], Disckson [1952], Trappe and Washington [2006]. Samuel F.B. Morse, an American inventor, developed the first coding technique that used just two symbols—a dot and a comma—to transmit the first cypher message in 1844 through an electric telegraph. He called it Morse code. Later, binary code encoding improved as a more secure method for encrypting messages, and it is still in use today Saranya and Janaki [2019], Janaki and Saranya [2020]. Binary code encoding divides each coded word into blocks of ones and zeros. The twentieth century saw important advancements in coding. It is common knowledge that the Fibonacci sequences are defined as  $F_{n+1} = F_n + F_{n-1}$  with the initial parameters  $F_0=0$  and  $F_1=1$ . The definition of the Fibonacci Q-matrix is  $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ , and its  $n$ th power is of the form Carmichael [1950], Disckson [1952], Trappe and Washington [2006]. Several approaches were used to study the Fibonacci coding principle. For instance, utilising the key variability notion in symmetric key algorithm and the Fibonacci Q-matrix, a novel method for safe information transmission through communication channels was developed. By using blocking matrices and Fibonacci numbers, a new cryptography algorithm was recently described in Berges [1981], Gould [1981]. Using the help of the Pell equation's solutions  $x^2 - 8y^2 = 1$  that were found in Tas et al. [2018], Sumeyra et al. [2019] and the new matrix  $Q^{8*}$ , we present new coding and decoding algorithms in this work. Also, this method's fundamental principle hinges on subdividing the message into a  $2 \times 2$  slab matrix. The security of information will be improved by using this technology, which also has a high degree of accuracy in data transmission via communication channels Saranya and Janaki [2019], Janaki and Saranya [2020].

The connection of recurrence for the Pell equation  $x^2 - 8y^2 = 1$  is

$$\begin{aligned} x_{n+1} &= 3x_n + 8y_n \\ y_{n+1} &= x_n + 3y_n \end{aligned}$$

For  $n \geq 1$ , where  $x_1 = 3, y_1 = 1$ .

It is noted that  $Q^{8*} = \begin{pmatrix} x_j & 8y_j \\ y_j & x_j \end{pmatrix}$

The primary concept is to turn the messages into order  $2 \times 2$  block matrices. The number of blocks and letter positions must be determined based on this. Then encrypted matrix  $P$  is obtained and  $Q^{8*}$  utilized for decryption.

## 2 Main results

### 2.1 Representations:

1.  $G$  – a message-to-be-sent matrix constituted by an even order.
2.  $G_j$  –  $j^{th}$  block of  $G$  whose size is 2.
3.  $g$  – Number of slabs  $G_j$  of  $G$ .
4.  $m = \begin{cases} 3 & \text{if } g \geq 3 \\ g & \text{if } g > 3 \end{cases}$
5.  $D_j = |G_j|$ .
6.  $G_j = \begin{pmatrix} g_{j1} & g_{j2} \\ g_{j3} & g_{j4} \end{pmatrix}$
7.  $P$  – encrypted matrix defined by  $P = (d_j \ g_{jk})_{k \in \{1,2,4\}}$
8.  $(Q^{8*})^m = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$
9.  $*$  – denotes the space between the words.

### 2.2 Assignment of alphabets:

|      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|
| A    | B    | C    | D    | E    | F    | G    | H    | I    |
| m    | m+1  | m+2  | m+3  | m+4  | m+5  | m+6  | m+7  | m+8  |
| J    | K    | L    | M    | N    | O    | P    | Q    | R    |
| m+9  | m+10 | m+11 | m+12 | m+13 | m+14 | m+15 | m+16 | m+17 |
| S    | T    | U    | V    | W    | X    | Y    | Z    | *    |
| m+18 | m+19 | m+20 | m+21 | m+22 | m+23 | m+24 | m+25 | m-1  |

### 2.3 Algorithm for encryption:

1. Create an even order matrix  $G$  for the provided message.
2. Separate  $G_j$  into slabs of size 2 and calculate  $g$ .
3. Selecting  $m$  using  $g$ .

4. Replace the alphabets with the provided numbers to get the elements of  $G_j$ .
5. Find  $D_j$ .
6. Create  $P$ .

## 2.4 Algorithm for decryption:

Using  $P$ , one must find  $G$ . The goal is to locate  $g_{j3}$  as  $P$  contains  $g_{j1}, g_{j2}, g_{j4}$ .

1. Identify  $(Q^{8*})^m$
2. Identify its constituents as  $q_{ij}$ 's.
3. Explore  $p_{j1} = q_{11}g_{j1} + q_{21}g_{j2}$
4. Explore  $p_{j2} = q_{12}g_{j1} + q_{22}g_{j2}$
5. Solve  $D_j = p_{j1}(q_{12}s_j + q_{22}g_{j4}) - p_{j2}(q_{11}s_j + q_{21}g_{j4})$  for  $s_j$ .
6. Replace  $s_j = g_{j3}$
7. Establish  $G_j$
8. Establish  $G$

## 2.5 Encryption decryption algorithm using $(x, y)$ such that $x^2 - 8y^2 = 1$

The following are a few Pell equation  $x^2 - 8y^2 = 1$  solutions:

| $j$   | 1 | 2  | 3  | 4   | 5    | 6     | 7      | 8      |
|-------|---|----|----|-----|------|-------|--------|--------|
| $x_j$ | 3 | 17 | 99 | 577 | 3363 | 19601 | 114243 | 665857 |
| $y_j$ | 1 | 6  | 35 | 204 | 1189 | 6930  | 40391  | 235416 |

### Case (i): $g=1$

**Example: 1** Take "ARMY" as the encrypted message.

**Encryption:**

$$G = \begin{pmatrix} A & R \\ M & Y \end{pmatrix}$$

Here there is only one block so  $g = 1$ .

This implies  $m = 3$ .

Thus  $G_1 = \begin{pmatrix} 3 & 20 \\ 15 & 27 \end{pmatrix}$  and so  $g_{11} = 3, g_{12} = 20, g_{13} = 15, g_{14} = 27$ .

$$\therefore D_1 = -219$$

$$P = (-219 \ 3 \ 20 \ 27)$$

**Decryption:**

$$(Q^{8^*})^3 = \begin{pmatrix} x_3 & 8y_3 \\ y_3 & x_3 \end{pmatrix} = \begin{pmatrix} 99 & 280 \\ 35 & 99 \end{pmatrix}$$

$$\Rightarrow q_{11} = 99, q_{12} = 280, q_{21} = 35, q_{22} = 99.$$

Apply  $p_{j1}$  and  $p_{j2}$ , one can get  $p_{11} = 997$  and  $p_{12} = 2820$ .

$$\begin{aligned} D_1 &= p_{11}(q_{12}s_1 + q_{22}g_{14}) - p_{12}(q_{11}s_1 + q_{21}g_{14}) \\ -219 &= 997(280s_1 + 2673) - 2820(99s_1 + 945) \\ \Rightarrow s_1 &= 15 \end{aligned}$$

Thus  $g_{13} = s_1 = 15$ .

$$\therefore G_1 = \begin{pmatrix} 3 & 20 \\ 15 & 27 \end{pmatrix} \text{ from } P$$

$$\text{Hence } G = \begin{pmatrix} A & R \\ M & Y \end{pmatrix}.$$

**Case (ii): g=4**

**Example: 2** Take “AMOUNT RECEIVED” as the encrypted message.

**Encryption:**

$$G = \begin{pmatrix} A & M & O & U \\ N & T & * & R \\ E & C & E & I \\ V & E & D & * \end{pmatrix}$$

Here there are four blocks so  $g = 4$ .

$$\therefore G_1 = \begin{pmatrix} A & M \\ N & T \end{pmatrix}, G_2 = \begin{pmatrix} O & U \\ * & R \end{pmatrix}, G_3 = \begin{pmatrix} E & C \\ V & E \end{pmatrix}, G_4 = \begin{pmatrix} E & I \\ D & * \end{pmatrix}$$

Choose  $m = 4$

$$\text{Thus } G_1 = \begin{pmatrix} 4 & 16 \\ 17 & 23 \end{pmatrix}, G_2 = \begin{pmatrix} 18 & 24 \\ 3 & 21 \end{pmatrix}, G_3 = \begin{pmatrix} 8 & 6 \\ 25 & 8 \end{pmatrix}, G_4 = \begin{pmatrix} 8 & 12 \\ 7 & 3 \end{pmatrix}$$

$$\text{Hence } D_1 = -180, D_2 = 306, D_3 = -86, D_4 = -60$$

$$P = \begin{pmatrix} -180 & 4 & 216 & 23 \\ 306 & 18 & 24 & 21 \\ -86 & 8 & 6 & 8 \\ -60 & 8 & 12 & 3 \end{pmatrix}$$

**Decryption:**

$$(Q^{8*})^4 = \begin{pmatrix} x_4 & 8y_4 \\ y_4 & x_4 \end{pmatrix} = \begin{pmatrix} 577 & 1632 \\ 204 & 577 \end{pmatrix}$$

$$\Rightarrow q_{11} = 577, q_{12} = 1632, q_{21} = 204, q_{22} = 577.$$

$\therefore$  One can find,

| $p_{11}$ | $p_{12}$ | $p_{21}$ | $p_{22}$ | $p_{31}$ | $p_{32}$ | $p_{41}$ | $p_{42}$ |
|----------|----------|----------|----------|----------|----------|----------|----------|
| 5572     | 15760    | 15282    | 43224    | 5840     | 16518    | 7064     | 19980    |

On solving the equation

$$D_j = p_{j1}(q_{12}s_j + q_{22}g_{j4}) - p_{j2}(q_{11}s_j + q_{21}g_{j4}),$$

one can get  $s_1 = 17, s_2 = 3, s_3 = 25, s_4 = 7$ .

Thus  $g_{13} = s_1 = 17, g_{23} = s_2 = 3, g_{33} = s_3 = 25, g_{43} = s_4 = 7$ .

Thus,

$$G_1 = \begin{pmatrix} 4 & 16 \\ 17 & 23 \end{pmatrix}, G_2 = \begin{pmatrix} 18 & 24 \\ 3 & 21 \end{pmatrix}, G_3 = \begin{pmatrix} 8 & 6 \\ 25 & 8 \end{pmatrix}, G_4 = \begin{pmatrix} 8 & 12 \\ 7 & 3 \end{pmatrix} \text{ from } P.$$

Hence

$$G = \begin{pmatrix} A & M & O & U \\ N & T & * & R \\ E & C & E & I \\ V & E & D & * \end{pmatrix}$$

### 3 Conclusions

The paper identifies a few particular cryptographic applications that can be utilised to demonstrate and comprehend the fundamental idea of the Fibonacci Q matrix. Fibonacci, often spelled Bonacci, Leonard of Pisa, or Leonardo Bigollo Pisano, was an Italian mathematician who was born in the Republic of Pisa. He is regarded as "the most talented Western mathematician of the Middle Ages" and had a significant impact on number theory. Moreover, Pell's equation refers to any Diophantine equation of the type  $x^2 - 8y^2 = 1$ , where  $n$  is a given positive non-square integer and integer solutions are sought for  $x$  and  $y$ .

In this work, using the solutions of the Pell equation  $x^2 - 8y^2 = 1$  and  $Q^{D^*}$  is defined. This contributes to the decryption algorithm. In encryption, the message must first be transformed into an even single matrix, then into slabs of size 2. The fact that the entries  $Q^{D^*}$  get bigger and bigger depends on the strict secrecy. One may search the decryption algorithm for the solutions of other well-known equations with the new matrix  $Q^{D^*}$ . On the basis of this scheme various more algorithms can be developed.

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# Bounds on fuzzy dominator chromatic number of fuzzy soft bipartite graphs

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## Abstract

An FSG  $G^S(T, V)$  fuzzy's soft dominator colouring (FSDC) is a suitable Fuzzy Soft Colouring (FSC) where every node of a colour group is dominated by a vertex of  $G^S(T, V)$ . In the current work, we characterize the sharp bounds for the Fuzzy Dominator Chromatic Number (FDCN) of fuzzy soft bipartite graphs and we present limits on the FDCN of fuzzy soft bipartite graph in terms of the  $\gamma^e(G^S(T, V))$ . Furthermore, we classify fuzzy soft bipartite graphs into three classes based on FDCN.

**Keywords:** Fuzzy Soft Bipartite Graph, Fuzzy dominator chromatic number, Fuzzy soft path, Fuzzy soft cycle, Strong arcs.

**2020 AMS subject classifications:** 05C72; 05C15. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1063. ISSN: 1592-7415. eISSN: 2282-8214. ©R. Jahir Hussain et al. This paper is published under the CC-BY licence agreement.

## **1 Introduction**

Fuzzy soft graphs are a useful mathematical tool for simulating the ambiguity of the actual world in view of parameters. Fuzzy soft graphs, which are often utilised in many different disciplines, including decision-making issues, combine fuzzy soft sets and the graph model. Scheduling problems are just one of many practical issues involving the allocation of scarce resources for which graph colouring is used as a model. Graph colouring also has an important place in discrete mathematics and combinatorial optimization.

To address ambiguous issues in the fields of engineering, social science, economics, medical research, and environment, Molodstov [1999] created the idea of soft set theory. Fuzzy soft sets, a blend of a fuzzy set and a soft set, were first introduced by P.K. Maji and Biswas [2001]. Thumkara and George introduced the idea of a soft graph in 2014. Rosenfeld first proposed the idea of fuzzy graph theory in 1975. Fuzzy soft graphs were independently introduced in 2015 by Mohinta and Samanta as well. Akram and Nawaz [2015] presented fuzzy soft graphs and examined their operations as well as several other graph theoretical ideas.

Domination is a fast growing area of graph theory study, and the numerous ways it is used to networks, distributed computers, social networks, and online graphs helps to explain why there is more interest in this subject. The dominator colouring problem in graphs was first described by Gera [2007]. An appropriate colouring of a graph  $G$  with the extra characteristic that each vertex in the graph dominates an entire class is known as a dominator colouring of  $G$ . The smallest number of colour classes in a graph's dominator colouring is known as the dominator chromatic number. We want to reduce the number of colour categories. In 2007, Gera [2007] explored the dominator chromatic number for the hypercube and more broadly for bipartite graphs. He also presented dominator colorings in bipartite graphs. Chellali and Maffary [2012] studied dominator colorings in some classes of graphs.

The idea of fuzzy dominator colouring in fuzzy graphs was created by Hussain and Fathima [2015]. They investigated the fuzzy dominator chromatic number for a number of fuzzy graphs and explored its boundaries. The FDCN of bipartite, middle, and subdivision fuzzy graphs was created by Hussain and Fathima [2015], and established its limits. Domination in fuzzy soft graphs was the subject of study done by Hussain and Hussain [2017].

Fuzzy dominator coloring applied to fuzzy soft graph yields fuzzy soft dominator coloring which concentrates on minimizing the number of color classes of FSG. A Soft, Fuzzy Dominator FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of  $G^S(T, V)$ . The proposed method concentrates on strength of connectedness, strong arc and strong neighbor of fuzzy soft bipartite graph in view of parameters from the existing method.

In this paper, we introduced Bounds of Fuzzy Soft Bipartite Graphs with Fuzzy Dominator Chromatic Number.

## 2 Preliminaries

**Definition 2.1.** Let  $R$  be a parameter set and  $T$  is a subset of  $R$ , Let  $V = \{x_1, x_2, x_3, \dots, x_n\}$  is a non-empty set. As well

(i)  $\alpha: T \rightarrow F(V)$  ( $V$ 's collection of all fuzzy subsets)

$e \dashv \alpha(e) = \alpha_e$  (say)

$\alpha_e: V \rightarrow [0, 1]$

$(T, \alpha):$  Fuzzy Soft Vertex

(ii)  $\beta: T \rightarrow F(V \times V)$  ( $V \times V$ 's collection of all fuzzy subsets)

$e \dashv \beta(e) = \beta_e$  (say)

$\beta_e: V \times V \rightarrow [0, 1]$

$(T, \beta):$  Fuzzy Soft Edge

followed by  $((T, \alpha), (T, \beta))$  iff  $\beta_e(x, y) \leq \alpha_e(x) \wedge \alpha_e(y)$  for all  $e$  in  $T$  and this FSGs are written by  $G^S(T, V)$ , it is referred to as a Fuzzy Soft Graph (FSG).

Additionally, a FSG is a parametrized family unit of fuzzy graphs.

**Definition 2.2.** A Path is intended to be a set of different points  $x_1, x_2, \dots, x_n$  in an FSG such that for all  $e$  in  $T$  and  $\beta_e(x_{i-1}, x_i) > 0$ , for all  $i = 1$  to  $n$ .

**Definition 2.3.** If an FSG  $G^S(T, V)$  contains more than 1 smallest arc,  $\forall e \in T$  it is referred to as a Fuzzy Soft Cycle.

**Definition 2.4.** A FSG  $G^S(T, V)$  is supposed to be a fuzzy soft bipartite if the node set  $V$  can be divided into 2 non-empty sets  $V_1^e$  and  $V_2^e$  such that  $V_1^e$  and  $V_2^e$  are fuzzy independent sets. These sets are called fuzzy bipartition of  $V$ , thus each efficient arc of FSG has one end in  $V_1^e$  and other end in  $V_2^e$ ,  $\forall e \in T$ .

**Definition 2.5.** A fuzzy soft bipartite graph is Complete if for an individual node  $V_1^e$ , each single node of  $V_2^e$  is an efficient neighbour,  $e \in T$ .

## 3 Fuzzy soft graphs with fuzzy dominator colouring

**Definition 3.1.** If  $\beta_e(x, y) = \beta_e^\infty(x, y)$ ,  $e \in T$  iff the arc  $(x, y)$  in FSG is said to be a strong arc, where  $\beta_e^\infty(x, y)$ ,  $e \in T$  is the maximum strength of all pathways among  $x$  and  $y$ .

**Definition 3.2.** A Soft, Fuzzy Dominator a FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of  $G^S(T, V)$ .

**Definition 3.3.** In a FSDC of FSG, an FDCN is the least number of colour groups, and it is denoted as  $\chi_{fd}^e(G^S(T, V))$ ,  $e \in T$ .

## 4 Bounds on fuzzy soft bipartite graphs with fuzzy dominator chromatic number

**Theorem 4.1.** Let  $G^S(T, V)$  be a connected fuzzy soft bipartite graph. Soon after  $2 \leq \chi_{fd}^e(G^S(T, V)) \leq \lfloor \frac{n}{2} \rfloor + 1$ ,  $e \in T$  and these limits are precise.

**Proof.** Take into account a connected fuzzy soft bipartite graph  $G^S(T, V)$ . Every acceptable fuzzy soft colour must also be present in every fuzzy soft dominator colour,  $\chi_f^e(G^S(T, V)) = 2$ , Hussain and Farhana [2020], as a result  $\chi_{fd}^e(G^S(T, V)) \geq 2$ ,  $e \in T$ .

To acquire the upper bound, let  $V_1^e$  and  $V_2^e$  be the 2 bipartite sets of  $G^S(T, V)$  with the condition  $|V_1^e| \leq |V_2^e|$ . After that, allocate colours  $1, 2, \dots, |V_1^e|$  to the nodes of  $V_1^e$  and colour  $|V_1^e| + 1$  to the nodes of  $V_2^e$ , is a least FSDC. Hence  $\chi_{fd}^e(G^S(T, V)) \leq |V_1^e| + 1$ , since  $|V_1^e| \leq \lfloor \frac{n}{2} \rfloor$ , which implies  $\chi_{fd}^e(G^S(T, V)) \leq \lfloor \frac{n}{2} \rfloor + 1$ ,  $e \in T$ .  $\square$

**Definition 4.1.** Consider a FSG  $G^S(T, V)$  with  $n$  nodes. Connect each node of  $G^S(T, V)$  onto any one of the  $n$  isolated nodes, where  $n$  is the total number of nodes in the FSG. The resultant graph is called Corona of  $G^S(T, V)$  and is symbolized as  $Cor(G^S(T, V)) = ((T, \alpha_1), (T, \beta_1))$  where

$$\alpha_1^e(u) = \alpha_e(u), u \in V, e \in T \text{ and}$$

$$\alpha_e(u) = \alpha_1^e(u) \in (0, 1], \text{ if } u \text{ is isolated, } e \in T.$$

$$\beta_1^e(u, v) = \beta_e(u, v) \in E, e \in T \text{ and}$$

$$\beta_1^e(u, v) = \alpha_1^e(u) \wedge \alpha_e(v), \text{ if } u \in V \text{ and } v \text{ is isolated, } e \in T.$$

**Remark:**

1. Obviously  $Cor(G^S(T, V))$  is a FSG if  $G^S(T, V)$  be a FSG.
2. If  $G^S(T, V)$  has 'n' nodes and 'e' edges then  $Cor(G^S(T, V))$  has  $2n$  nodes and  $e+n$  edges.
3.  $Cor(G^S(T, V))$  has  $k+n$  strong arcs if the no. of strong nodes in  $G^S(T, V)$  is 'k'. Now we discuss about the sharpness of lower bound and upper bound of Theorem:4.1. Since the lower bound is sharp, it may be inferred that the entire fuzzy soft bipartite graph's FDCN is 2. The subsequent theorem proves the sharpness of the upper bound.

**Theorem 4.2.** Having a 'n'-node fuzzy soft path  $G^S(T, V)$  and  $n'$  is the number of nodes in  $cor(G^S(T, V))$ . Then  $\chi_{fd}^e(cor(G^S(T, V))) = \lfloor \frac{n'}{2} \rfloor + 1$ ,  $e \in T$ .

**Proof.** Consider a fuzzy soft path having  $v_1, v_2, \dots, v_n$  as nodes. Noticeably  $G^S(T, V)$  is a fuzzy soft bipartite graph, Hussain and Farhana [2020] and so

$Cor(G^S(T, V))$  is also a fuzzy soft bipartite graph. This implies every arc is a strong arc in  $Cor(G^S(T, V))$ . The number of nodes in  $G^S(T, V)$  is  $n$ , then  $n' = 2n$  nodes in  $Cor(G^S(T, V))$ .

At this instant, we have to colour the nodes of  $Cor(G^S(T, V))$ . To obtain the least fuzzy soft dominator colouring, every node in minimum dominating set be full of distinctive colour. The nodes in  $G^S(T, V)$  is strong adjoining to just one node of  $Cor(G^S(T, V))$ . This implies that every node of  $G^S(T, V)$  is in a minimum dominating set. Therefore  $\gamma^e(Cor(G^S(T, V))) = n, e \in T$ .

Color  $1, 2, \dots, n$  is distributed to the nodes of  $G^S(T, V)$ , and colour  $n + 1$  is distributed to the left over nodes. This is the least FSDC of  $Cor(G^S(T, V))$ . Hence we prove  $\chi_{fd}^e(Cor(G^S(T, V))) = \lfloor \frac{n'}{2} \rfloor + 1, e \in T$  which is the upper bound and it is sharp.  $\square$

Now we prove that the values in the middle of 2 and  $\lfloor \frac{n'}{2} \rfloor + 1$  can be achieved as the FDCN of some fuzzy soft bipartite graphs of order  $n$ .

**Theorem 4.3.** *Let  $k$  be an integer with  $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$ . If so, a connected fuzzy soft bipartite graph  $G^S(T, V)$  with  $n$ - nodes and its FDCN is  $k$ .*

**Proof.** Consider a fuzzy soft path  $P_k^e = ((T, \alpha_1), (T, \beta_1))$  with  $k \geq 2$  nodes and it is assumed as  $v_1, v_2, \dots, v_k$ . Now put up a FSG from fuzzy soft path by extending  $k$  nodes  $u_1, u_2, \dots, u_k$  in order that  $u_i v_i \in E, (1 \leq i \leq k)$  and by adding  $n-2k$  nodes  $x_j$  to  $v_k$  ( $1 \leq j \leq n-2k$ ). We have  $n-2k \geq 1$  and  $k \geq 2$ , it follows that  $n \geq 5$ , where

$$\alpha^e(v_1) = \alpha_1^e(v_i), e \in T$$

$$\beta_e(u, v) = \beta_1^e(u, v), (u, v) \in E, e \in T$$

$$\beta_e(u_i, v_j) = \alpha_e(u_i) \wedge \alpha_e(v_j), \text{ for every } i, e \in T.$$

$$\beta_e(v_k, x_j) = \alpha_e(v_k) \wedge \alpha_e(x_j), \text{ for every } j, e \in T.$$

Clearly,  $G^S(T, V)$  is a fuzzy soft tree because there are no fuzzy soft cycles of odd length, implying that it is a fuzzy soft bipartite graph, Hussain and Farhana [2020]. At the moment, the sets  $\{v_1, v_2, \dots, v_k\}$  and  $\{u_1, u_2, \dots, u_{k-1}, v_k\}$  have the lowest dominance. As a result,  $\gamma^e(G^S(T, V)) = k, e \in T$ .

Since  $\gamma^e(G^S(T, V)) = k$ , we require no less than  $k$  distinctive colours for least FSDC. Allocate colours  $1, 2, \dots, k$  to  $v_1, v_2, \dots, v_k$  respectively and  $k+1$  more colour to  $u_1, u_2, \dots, u_k, x_1, x_2, \dots, x_{n-2k}$ . So each node has  $k+1$  colours offered in  $G^S(T, V)$  dominate no less than one colour group. At last, the colouring is least FSDC.

Alternatively, we allocate  $k$  unique colours to the nodes of the set  $\{u_1, u_2, \dots, u_{k-1}, v_k\}$  then we have a FSDC of ' $k$ ' colours, since we can colour the

remaining nodes using  $k$  colours. Thus  $\chi_{fd}^e(G^S(T, V)) = k \in T$ .  $\square$

We have a necessary condition for the upper bound.

**Theorem 4.4.** Define  $G^S(T, V)$  a soft fuzzy bipartite graph with double-edged  $(U, V)$ . Following this, corona of  $G^S(T, V)$  is a fuzzy soft bipartite graph whose  $\chi_{fd}^e$  is  $\lfloor \frac{|cor(G^S(T, V))|}{2} \rfloor + 1$ ,  $e \in T$  where  $|cor(G^S(T, V))|$  is the number of nodes in  $cor(G^S(T, V))$ .

**Proof.** Assume a soft fuzzy bipartite graph with bipartition  $(U, V)$  and  $|U| + |V|$  = number of nodes in  $G^S(T, V)$ . Label  $G^S(T, V)'$  be the corona of  $G^S(T, V)$ . Undoubtedly  $G^S(T, V)'$  is a fuzzy soft bipartite graph.

The points of soft fuzzy bipartite graph allocated with colours  $1, 2, \dots, |V|$  and  $1 + |V|$  to the left behind nodes, which is a FSDC of  $G^S(T, V)'$ . Seeing as each node of  $G^S(T, V)$  dominate itself and the left over nodes dominates its nearest nodes. This colouring is a least FSDC. This implies that each node in minimum dominating set gets an exclusive colour. Hence  $\chi_{fd}^e(cor(G^S(T, V))) = \lfloor \frac{|cor(G^S(T, V))|}{2} \rfloor + 1$ ,  $e \in T$ .

## 5 Bounds on fuzzy dominator chromatic number in terms of domination number

**Theorem 5.1.** Define a soft fuzzy bipartite graph  $G^S(T, V)$ . In that case  $\gamma^e(G^S(T, V)) \leq \chi_{fd}^e(G^S(T, V)) \leq 2 + \gamma^e(G^S(T, V))$ ,  $e \in T$ .

**Proof.** Allow  $G^S(T, V)$  to be a soft fuzzy bipartite graph and  $c$  to be the least FSDC with colours  $1, 2, \dots, \chi_{fd}^e(G^S(T, V))$ . For every colour group of  $G^S(T, V)$ , allow  $a_n$  be a point in the colour group  $n$  having  $1 \leq n \leq \chi_{fd}^e(G^S(T, V))$ ,  $e \in T$ . We have to prove that  $S = \{a_n : 1 \leq n \leq \chi_{fd}^e(G^S(T, V)), e \in T\}$  is a dominating set.

By the definition of FSDC, all nodes of  $G^S(T, V)$  will dominate all nodes of some colour class. Since  $S$  contains a node of each colour group, each node of  $G^S(T, V)$  dominate some node in  $S$ . As a result,  $S$  is a fuzzy dominating set, and every least FSDC of  $G^S(T, V)$  yields a fuzzy dominating set of  $G^S(T, V)$ . Hence  $\gamma^e(G^S(T, V)) \leq \chi_{fd}^e(G^S(T, V))$ ,  $e \in T$ .

We must now validate the upper bound. Because the FCN of fuzzy soft bipartite graph is 2, Hussain and Farhana [2020], we can colour the nodes of  $G^S(T, V)$  using 2 colours 1 and 2. Allot colours  $3, 4, \dots, \gamma^e(G^S(T, V)) + 2$  to the nodes of  $S$  and offer colours 1 and 2 to the lasting nodes of  $G^S(T, V)$  such that 2 strong adjoining

nodes be given dissimilar colours. This colouring fall out in fuzzy soft dominator colouring since it is an appropriate fuzzy soft colouring and each node in  $G^S(T, V)$  dominate all nodes of at least one colour group. Hence  $\chi_{fd}^e(G^S(T, V)) \leq 2 + \gamma^e(G^S(T, V))$ ,  $e \in T$ .  $\square$

**Observation:**

Fuzzy Soft Bipartite Graphs  $G^S(T, V)$  on the basis of the limits, may be divided into three categories on  $\chi_{fd}^e(G^S(T, V))$ .

1.  $G^S(T, V)$  is of CLASS 0 if  $\chi_{fd}^e(G^S(T, V)) = \gamma^e(G^S(T, V))$ ,  $e \in T$ .
2.  $G^S(T, V)$  is of CLASS 1 if  $\chi_{fd}^e(G^S(T, V)) = \gamma^e(G^S(T, V)) + 1$ ,  $e \in T$ .
3.  $G^S(T, V)$  is of CLASS 2 if  $\chi_{fd}^e(G^S(T, V)) = \gamma^e(G^S(T, V)) + 2$ ,  $e \in T$ .

**Example:CLASS 0**

Consider a fuzzy soft path of  $k \geq 2$  nodes. Now form a FSG as in **Theorem:4.3**. Then FSG is of CLASS 0.

**Example:CLASS 1**

Consider a complete fuzzy soft bipartite graph  $K_{1,n}$ . The FDCN of  $K_{1,n}$  is two and its  $\gamma^e(G^S(T, V))$  is one, therefore  $\chi_{fd}^e(G^S(T, V)) = \gamma^e(G^S(T, V)) + 1$ . Hence  $G^S(T, V)$  is of CLASS 1.

**Example:CLASS 2**

The fuzzy soft cycle of length  $n \geq 5$  has FSDC  $\lceil \frac{n}{3} \rceil + 2$  and its  $\gamma^e(G^S(T, V))$  is  $\lceil \frac{n}{3} \rceil$ , therefore  $\chi_{fd}^e(G^S(T, V)) = \gamma^e(G^S(T, V)) + 2$ . Hence  $G^S(T, V)$  is of CLASS 2.

## 6 Conclusions

In this work, we characterized the bounds on FDCN of fuzzy soft bipartite graphs and also presented bounds in respect of  $\gamma^e(G^S(T, V))$ . Additionally, based on the fuzzy dominator chromatic number (FDCN) that was found, the fuzzy soft bipartite graphs are divided into three categories as fuzzy soft path in CLASS 0, complete fuzzy soft bipartite graph in CLASS 1 and fuzzy soft cycle in CLASS 2. We suggest this study on a few distinct classes of fuzzy soft graphs.

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# Algorithm approaches for shortest path problem in an interval - valued triangular Pythagorean fuzzy network

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## Abstract

In this paper, it has been proposed to use Pythagorean fuzzy sets (PFS), is an extension of fuzzy sets (FSs), to address uncertainty in practical decision-making problems. Then the Shortest Path Problem (SPP) is a well-known network improvement issue with numerous practical applications. Then the shortest Path (SP) and the shortest distance (SD) in an Interval – valued Pythagorean fuzzy graph (I-VPFG) are found using a method in the current communication. Nodes and connections are crisp, while the edge weights are I-V triangular Pythagorean fuzzy numbers (I-VTPFN). Additionally, a numerical example has been used to demonstrate the suggested strategy.

**Keywords:** I - V Pythagorean fuzzy sets (I-VPFS), I – V Pythagorean fuzzy graph (I-VPFG), I – V triangular Pythagorean fuzzy number (I-VTPFN), ranking function (RF), I – V Pythagorean fuzzy shortest path problem (I-VPFSPP)

**2020 AMS subject classifications:** 05C72, 12D15, 03D45. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1064. ISSN: 1592-7415. eISSN: 2282-8214. ©M. Asim Basha et al. This paper is published under the CC-BY licence agreement.

## **1 Introduction**

The pair of sets  $(V, E)$  that make up a graph are  $V$ , which represents the set of vertices, and  $E$ , which represents the set of edges that connect the pair of vertices. Numerous domains, including computer, social and natural research, among others, use graph theory. Any mathematical, scientific, or engineering problem can be represented as a graph. Leonard Euler first introduced the idea of graph theory in the year 1736. In order to address the issue of the seven Konigsberge bridges constructed across the Pregel River of Prussia, he produced the first graph. The key factor contributing to graph theory's explosive growth is its application across a variety of disciplines. Even though graphs can be used to simulate events that happen in real-world problems, graph theory became interesting. Graphs are useful for studying these issues.

In the graphs are a very crucial model for networks. The SPPs are very useful and are used in various areas, applications for the road network, transportation, communication channel routing, and scheduling issues are a few examples. The weights of the edges in the basic network issue are taken to be actual numbers. After all, in most constructive applications, the bounds are generally not precise. Therefore, they can be considered fuzzy numbers (FN) in the real world.

Zadeh, 1965 extracted the idea of a Fuzzy Set (FS) as a way to identify uncertainty in 1965. As a generalisation of FSs, Zadeh introduced the idea of  $I - V$  fuzzy sets (I-VFS). Intuitionistic fuzzy sets (IFS), which Atanassov [1986] suggested in 1986, look more exactly at uncertainty dimension and give the chance to precisely model the issue based on the available data and consideration I-VFS was introduced three years later by Atanassov and Gargov [1989]. Then the definition of an I-V fuzzy graph (FG) was given by Kumar et al. [2009]. On Intuitionistic fuzzy graph (IFGs), Mohamed and Ali [2018] defined several new applications. Dubois and Prade [1980] were the first to investigate the SPP in an ambiguous and unreliable surroundings. They claim that while the length of the shortest way may be determined, the network may not contain a path that corresponds to it. Numerous academics have looked into the SPP in literature, each in a different way. Lin and Chen proposed the Fuzzy SP length in a network created using fuzzy linear programming. Furthermore, Yao and Lin [2003] created two distinct fuzzy SP network challenges, the first of which makes use of triangular fuzzy integers. Okada [2004] proposed an algorithm to complete the degree of incident for each arc in the network and introduce the similarity index among the sums of Fuzzy numbers (FN) by taking into consideration interactivity with FNs. Kumar and Kaur [2011] suggested one such approach for locating the SP in a network flow with imprecise arc lengths. A based on model the idea of dynamic scripting was presented by Karunambigai et al. [2007] et al. to locate the SP in IFGs. Additionally, Gani and Jabarulla [2010] created a technique for finding the IFSP in a network.

To handle the complex, imprecise, and ambiguity in challenges with effective decision-making, Yager [2014] introduced the concept of the PFSs, as a generalisation of the IFS. The key feature of the PF model is that it relaxes the action that the sum of its mem-ship degree and non-mem-ship degree i.e.,  $(0.5+0.4 \leq 1)$  with the squar sum of its mem-ship and non-mem-ship degree i.e.,  $(0.6^2 + 0.7^2 \leq 1)$ . Following Yager [2013] invention of the PFS, Xu and Zhang [2014] granted the mathematical form of the PFS as well as the concept of the PFN. Then the idea of a PFGs was proposed by Naz et al. [2013]. As a generalisation of PFSs, presented the idea of I-VPF sets Xindong and Young [2016]. The idea of an I-VPFG was suggested by Mohamed and Ali [2018]. In order to accomplish this, we first offer a mathematical formulation for SP issues where contradiction costs of arcs are expressed in terms of I-VPF numbers. Then, in order to construct a solution algorithm, we present the optimality in I-VPF networks. To compare the costs of several pathways, whose arc are expressed by I-VPF numbers, an improved score function (SF) is employed.

In this study, we present a novel method for finding the SP using I-VPFSs. Using the suggested approach, a decision maker can determine the SP and the SD between each node and the SN. The following is how the study is structured. Sec 2 goes over some fundamental ideas such as I-VTPFNs, arithmetic operations, and RFs. In sec 3, a algorithm for determining the SP and the SD in an I-VPFG is proposed. Sec 4 contains an example of determining the SP and SD between the SN and the DNs. The final section includes some closing remarks.

## 2 Preface

This chapter discusses some fundamental ideas, operations in mathematics, and rank order functions for I-VTPFNs are covered.

**Definition 2.1.** (I-VPFS)Xindong and Young [2016]

An I-VPFS  $\tilde{A}_{\tilde{P}}$ , in the global set  $Z'$  is defined by

$$\tilde{A}_{\tilde{P}} = \langle x, \check{\mu}_{\tilde{A}_{\tilde{P}}}(x), \check{\lambda}_{\tilde{A}_{\tilde{P}}}(x) \rangle : x \in Z'$$

where  $\check{\mu}_{\tilde{A}_{\tilde{P}}} = [\check{\mu}_{\tilde{A}_{\tilde{P}_L}}(x), \check{\mu}_{\tilde{A}_{\tilde{P}_U}}(x)] \subseteq [0, 1]$  and  $\check{\lambda}_{\tilde{A}_{\tilde{P}}} = [\check{\lambda}_{\tilde{A}_{\tilde{P}_L}}(x), \check{\lambda}_{\tilde{A}_{\tilde{P}_U}}(x)] \subseteq [0, 1]$  are interval numbers satisfied as  $0 \leq [\check{\mu}_{\tilde{A}_{\tilde{P}_U}}(x)]^2 + [\check{\lambda}_{\tilde{A}_{\tilde{P}_U}}(x)]^2 \leq 1$  for separate element  $x \in Z'$ . For each I-VPFS  $\tilde{A}_{\tilde{P}}$  and  $x \in X'$ ,

$$\begin{aligned} \tilde{\pi}_{\tilde{A}_{\tilde{P}}}(x) &= \left[ \pi'_{\tilde{A}_{\tilde{P}_L}}(x), \pi'_{\tilde{A}_{\tilde{P}_U}}(x) \right] \\ &= \left[ \sqrt{1 - [\check{\mu}_{\tilde{A}_{\tilde{P}_U}}(x)]^2 - [\check{\lambda}_{\tilde{A}_{\tilde{P}_U}}(x)]^2}, \sqrt{1 - [\check{\mu}_{\tilde{A}_{\tilde{P}_L}}(x)]^2 - [\check{\lambda}_{\tilde{A}_{\tilde{P}_L}}(x)]^2} \right] \end{aligned}$$

is called the hesitancy interval of  $x$  to  $\tilde{A}_{\tilde{P}}$ . For an I-VPFS  $\tilde{A}_{\tilde{P}}$  is defined the pair  $\langle [\check{\mu}_{\tilde{A}_{\tilde{P}_L}}(x), \check{\mu}_{\tilde{A}_{\tilde{P}_U}}(x)], [\check{\lambda}_{\tilde{A}_{\tilde{P}_L}}(x), \check{\lambda}_{\tilde{A}_{\tilde{P}_U}}(x)] \rangle$  is called an I-VPFN is denoted by  $\tilde{A}_{\tilde{P}} = \langle [\check{\mu}_{\tilde{A}_{\tilde{P}_L}}, \check{\mu}_{\tilde{A}_{\tilde{P}_U}}], [\check{\lambda}_{\tilde{A}_{\tilde{P}_L}}, \check{\lambda}_{\tilde{A}_{\tilde{P}_U}}] \rangle$ .

**Definition 2.2:-** (Interval - valued Pythagorean fuzzy graph) Mohamed and Ali [2018]

An I-VPFG with the underneath set  $v''$  is defined as a pair of  $G' = (\tilde{P}, \tilde{Q})$  where,

- (i) In the functions  $\check{\mu}_{\tilde{P}} : v'' \rightarrow [0, 1]$  and  $\check{\lambda}_{\tilde{P}} : v'' \rightarrow [0, 1]$  express the degree of membership and nonmembership element of the  $x' \in v''$ , respectively, such that  $0 \leq \check{\mu}_{\tilde{P}}(x') + \check{\lambda}_{\tilde{P}}(x') \leq 1, \forall x' \in v''$ .
- (ii) In the functions  $\check{\mu}_{\tilde{Q}} : E' \subseteq v'' \times v'' \rightarrow [0, 1]$ ,  $\check{\lambda}_{\tilde{Q}} : E' \subseteq v'' \times v'' \rightarrow [0, 1]$  is,

$$\begin{aligned} \check{\mu}_{\tilde{Q}_L}((x', y')) &\leq \min(\check{\mu}_{\tilde{P}_L}(x'), \check{\mu}_{\tilde{P}_L}(y')) \text{ and} \\ \check{\mu}_{\tilde{Q}_L}((x', y')) &\geq \max(\check{\lambda}_{\tilde{P}_L}(x'), \check{\lambda}_{\tilde{P}_L}(y')) \\ \check{\mu}_{\tilde{Q}_U}((x', y')) &\leq \min(\check{\mu}_{\tilde{P}_U}(x'), \check{\mu}_{\tilde{P}_U}(y')) \text{ and} \\ \check{\mu}_{\tilde{Q}_U}((x', y')) &\geq \max(\check{\lambda}_{\tilde{P}_U}(x'), \check{\lambda}_{\tilde{P}_U}(y')) \end{aligned}$$

such that,

$$0 \leq \check{\mu}_{\tilde{Q}_U}^2((x', y')) + \check{\lambda}_{\tilde{P}_U}^2((x', y')) \leq 1, \forall (x', y') \in E'.$$

**Definition 2.3** Kumar et al. [2015]

Let  $\tilde{A}_{\tilde{P}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \langle [\check{\mu}_{\tilde{A}_{\tilde{P}_L}}, \check{\mu}_{\tilde{A}_{\tilde{P}_U}}], [\check{\lambda}_{\tilde{A}_{\tilde{P}_L}}, \check{\lambda}_{\tilde{A}_{\tilde{P}_U}}] \rangle$  and  $\tilde{B}_{\tilde{P}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) \langle [\check{\mu}_{\tilde{B}_{\tilde{P}_L}}, \check{\mu}_{\tilde{B}_{\tilde{P}_U}}], [\check{\lambda}_{\tilde{B}_{\tilde{P}_L}}, \check{\lambda}_{\tilde{B}_{\tilde{P}_U}}] \rangle$  be two I-VPTFNs, Then defined as,

(i)

$$\begin{aligned} \tilde{A}_{\tilde{P}} \oplus \tilde{B}_{\tilde{P}} &= \langle \left( \sqrt{\tilde{a}_1^2 + \tilde{b}_1^2}, \sqrt{\tilde{a}_2^2 + \tilde{b}_2^2}, \sqrt{\tilde{a}_3^2 + \tilde{b}_3^2} \right) \\ &\quad \sqrt{\left( \check{\mu}_{\tilde{A}_{\tilde{P}_L}} \right)^2 + \left( \check{\mu}_{\tilde{B}_{\tilde{P}_L}} \right)^2 - \left( \check{\mu}_{\tilde{A}_{\tilde{P}_L}} \right)^2 \cdot \left( \check{\mu}_{\tilde{B}_{\tilde{P}_L}} \right)^2} \\ &\quad \times \sqrt{\left( \check{\mu}_{\tilde{A}_{\tilde{P}_U}} \right)^2 + \left( \check{\mu}_{\tilde{B}_{\tilde{P}_U}} \right)^2 - \left( \check{\mu}_{\tilde{A}_{\tilde{P}_U}} \right)^2 \cdot \left( \check{\mu}_{\tilde{B}_{\tilde{P}_U}} \right)^2} \\ &\quad \times \left[ \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_L}} \right) \cdot \left( \check{\lambda}_{\tilde{B}_{\tilde{P}_L}} \right), \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_U}} \right) \left( \check{\lambda}_{\tilde{B}_{\tilde{P}_U}} \right) \right] \rangle. \end{aligned}$$

**Definition 2.4** (Score and accuracy functions) Garg [2016]

The SF and the AF of any I-VPTFN

$$\tilde{A}_{\tilde{P}} = \left\langle (\tilde{a}, \tilde{b}, \tilde{c}) \left[ \check{\mu}_{\tilde{A}_{\tilde{P}_L}}, \check{\mu}_{\tilde{A}_{\tilde{P}_U}} \right] \cdot \left[ \check{\lambda}_{\tilde{A}_{\tilde{P}_L}}, \check{\lambda}_{\tilde{A}_{\tilde{P}_U}} \right] \right\rangle$$

are defined as,

$$S(\tilde{A}_{\tilde{P}}) = \frac{1}{2} \left[ \left( \check{\mu}_{\tilde{A}_{\tilde{P}_L}} \right)^2 + \left( \check{\mu}_{\tilde{A}_{\tilde{P}_U}} \right)^2 - \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_L}} \right)^2 - \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_U}} \right)^2 \right], \quad S(\tilde{A}_{\tilde{P}}) \in [-1, 1] \text{ and}$$

$$H(\tilde{A}_{\tilde{P}}) = \frac{1}{2} \left[ \left( \check{\mu}_{\tilde{A}_{\tilde{P}_L}} \right)^2 + \left( \check{\mu}_{\tilde{A}_{\tilde{P}_U}} \right)^2 + \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_L}} \right)^2 + \left( \check{\lambda}_{\tilde{A}_{\tilde{P}_U}} \right)^2 \right], \quad H(\tilde{A}_{\tilde{P}}) \in [0, 1].$$

**Remark 2.5** Kumar et al. [2015]

Let  $\tilde{A}_{\tilde{P}}$  and  $\tilde{B}_{\tilde{P}}$  be two I-VTPFVs the collection of real numbers. Then we define a method of ranking is,

(i) If  $S(\tilde{A}_{\tilde{P}}) > S(\tilde{B}_{\tilde{P}})$ , then  $(\tilde{A}_{\tilde{P}}) > (\tilde{B}_{\tilde{P}})$ , that is  $(\tilde{A}_{\tilde{P}})$  is sup to  $(\tilde{B}_{\tilde{P}})$ , denoted by  $(\tilde{A}_{\tilde{P}}) > (\tilde{B}_{\tilde{P}})$ .

(ii) If  $S(\tilde{A}_{\tilde{P}}) = S(\tilde{B}_{\tilde{P}})$  and  $H(\tilde{A}_{\tilde{P}}) > H(\tilde{B}_{\tilde{P}})$  then  $(\tilde{A}_{\tilde{P}}) > (\tilde{B}_{\tilde{P}})$ , that is  $(\tilde{A}_{\tilde{P}})$  is sup to  $(\tilde{B}_{\tilde{P}})$  denoted by  $(\tilde{A}_{\tilde{P}}) > (\tilde{B}_{\tilde{P}})$ .

**I-VTPF distance 2.6** Mohamed and Ali [2018]

Let  $F$  be any connected I-VPFG. For any path  $\{\tilde{P} : \tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n\}$  length of  $\tilde{P}$  is described as the weights added together.  $(\tilde{W}_i)$  those arcs in  $\tilde{P}$ , in which the weight among two adjacent I-VTriPFNs vertices are  $J(\tilde{P}') = \sum_{i=1}^n (\tilde{W}'_{i-1}, \tilde{W}'_i)$ . any two nodes  $u, v \in F$ , let  $\tilde{P} = \{\tilde{P}_i \text{ is a } u - v \text{ path, } i = 1, 2, 3, \dots\}$  The definition of the I-VTriPF distance is,  $I_{\tilde{P}}\{u, v\} = \min \left\{ J(\tilde{P}'_i) ; \tilde{P}, i = 1, 2, \dots, n \right\}$ .

### 3 Algorithm for SP

An algorithm is suggested in this section for the SP and SD of each node from the source node (SN) Okada [2004].

The following steps for algorithm are defined by

Step – 1: Surmise  $I_{\tilde{P}_i} = [0, 0, 0] ; [0, 0] [1, 1]$  and label the SN is,  $[0, 0, 0] ; [0, 0] [1, 1] \quad J(\tilde{G}) = \dots$ .

Step – 2: Calculate  $I_{\tilde{P}'_j} = \min\{I_{\tilde{P}'_i} \oplus I_{\tilde{P}_j} / i \in \tilde{G}_{\tilde{P}}(j)\}; j = 2, 3, \dots, n$ .

Step – 3: A minimum value appearing from step 2 and matching the unique value of  $i$ , the label node next  $j$  as  $I_{\tilde{P}_i}(x)$ . If the minimum value appears to be

equal to one of the values of  $i$ , then it indicates that there is just one remaining path between the SN as well as the node  $j$  because the separation between all pathways is  $I_{\widetilde{P}_i}$ , choose any value for  $i$ .

Step – 4: The DN is denoted by  $I_{\widetilde{P}_n}\gamma$ , then the SD between the SN and the DN is  $I_{\widetilde{P}_n}$ .

Step – 5: The DN is now marked  $I_{\widetilde{P}_n}\gamma$ . Check the marking of node  $\gamma$  to find the SP between the SN and the DN. Let it be  $I_{\widetilde{P}_n}\eta$ , now check the marking of node  $q$ , and so on. Repeat the process until node 1. The SP can be found by combining all nodes.

## 4 Numerical explanation

Consider an I-VPF – scaled graph, in Fig. 1, where the I-VTPFNs represents the distance between any two vertices. The problem is find to the SD and the path enclosed by SN and the DN in network.

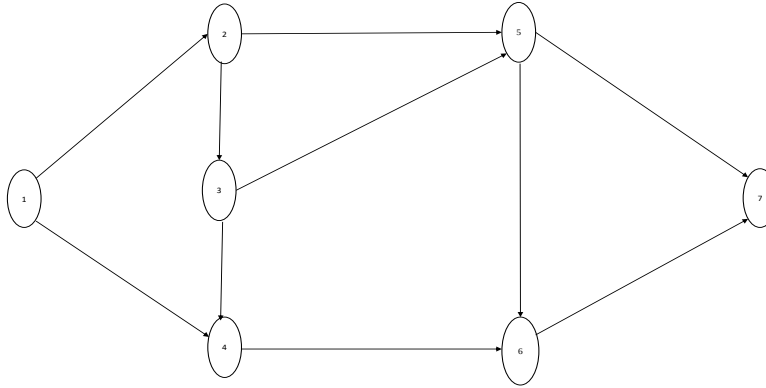


Figure 1: I-VTPF directed graph..

The below (fig 1) network shall be formed in each edges to evaluate the I-VTPFN as follows:

### Explanation

Let node 7 is the DN is defined as,  $n = 7$ .

Let us deal with the initial distance is  $\widetilde{d}_1'' = [0, 0, 0] ; [0, 0] [1, 1]$  and the SN [declared nod 1] as  $\left[ [0, 0, 0] ; [0, 0] [1, 1] , J \left( \widetilde{G} \right) = -- \right]$ , the values of  $\widetilde{d}_j'' ; j = 2, 3, 4, 5, 6, 7$  can be assigned the network terminology:

**Iteration – I** Let taken away from fig.1 PN 2 is node 1, so place the values from  $i = 1$  &  $j = 2$  appropriately in the St – 2 being the above algorithm.

| Edges | I-VTPF distance                           |
|-------|-------------------------------------------|
| 1 – 2 | $[0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8]$ |
| 1 – 4 | $[0.4, 0.5, 0.6] ; [0.5, 0.6] [0.7, 0.8]$ |
| 2 – 3 | $[0.3, 0.5, 0.7] ; [0.6, 0.7] [0.6, 0.7]$ |
| 2 – 5 | $[0.4, 0.5, 0.7] ; [0.4, 0.6] [0.7, 0.8]$ |
| 3 – 4 | $[0.3, 0.5, 0.6] ; [0.6, 0.7] [0.7, 0.7]$ |
| 3 – 5 | $[0.5, 0.5, 0.5] ; [0.5, 0.6] [0.6, 0.8]$ |
| 4 – 6 | $[0.4, 0.5, 0.6] ; [0.6, 0.7] [0.6, 0.8]$ |
| 5 – 6 | $[0.5, 0.6, 0.5] ; [0.5, 0.6] [0.6, 0.8]$ |
| 5 – 7 | $[0.4, 0.5, 0.7] ; [0.4, 0.7] [0.7, 0.7]$ |
| 6 – 7 | $[0.5, 0.6, 0.7] ; [0.5, 0.7] [0.6, 0.7]$ |

Table 1: I-VTPFNs edge weights

From the path for  $\widetilde{d''}_2$  is,  $\widetilde{d''}_2 = \min\{\widetilde{d''}_1 \oplus \widetilde{d''}_{12}\}$   
 $= \min\{[0, 0, 0] ; [0, 0] [1, 1] \oplus [0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8]\}$

$$\widetilde{d''}_2 = \min\{[0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8]\}$$

$$\widetilde{d''}_2 = \{[0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8]\}$$

From the above calculation, there is a min that corresponds to node 1, and then node 2 is considered to be  $\{[0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8] , J(\widetilde{G}) = 1\}$

**Iteration – II** The PN for node 3 is node 2, then the distance between the SN and node 3 according to the above procedure is

$$\widetilde{d''}_3 = [0.424264, 0.707107, 0.860233] ; [0.68, 0.820731] [0.36, 0.56]$$

and the node 3 is

$$\{[0.424264, 0.707107, 0.860233] ; [0.68, 0.820731] [0.36, 0.56] , J(\widetilde{G}) = 2\}$$

**Iteration – III** The PN nodes for node 4 are nodes 1 and 3, then the value of  $\widetilde{d''}_4$  is given,

$$\widetilde{d''}_4 = \min\{\widetilde{d''}_1 \oplus \widetilde{d''}_{14}, \widetilde{d''}_3 \oplus \widetilde{d''}_{34}\}$$

$$= \min\left\{\begin{array}{l} [0, 0, 0] ; [0, 0] [1, 1] \oplus [0.4, 0.5, 0.6] ; [0.5, 0.6] [0.7, 0.8] , \\ [0.424264, 0.707107, 0.860233] ; [0.68, 0.820731] [0.36, 0.56] \\ \oplus [0.3, 0.5, 0.6] ; [0.6, 0.7] [0.7, 0.7] \end{array}\right\}$$

$$= \min \left\{ \begin{array}{l} [0.5, 0.707107, 0.781025] ; [0.60828, 0.768375] [0.42, 0.64] , \\ [0.519645, 0.86602, 1.048809] ; [0.809899, 0.912968] [0.252, 0.396] \end{array} \right\}$$

Applying the ranking method (based on the SF) results in the following value for  $\widetilde{d''}_4$ .

$$[0.5, 0.707107, 0.781025] ; [0.60828, 0.768375] [0.42, 0.64]$$

and the term for  $\widetilde{d''}_4$  is

$$\left\{ [0.5, 0.707107, 0.781025] ; [0.60828, 0.768375] [0.42, 0.64] , J(\widetilde{G}) = 1 \right\}.$$

**Iteration – IV** The PNs for node 5 are nodes 2 & 3, then the value of  $\widetilde{d''}_5$  is

$$\widetilde{d''}_5 = \min \{ \widetilde{d''}_2 \oplus \widetilde{d''}_{25}, \widetilde{d''}_3 \oplus \widetilde{d''}_{35} \}$$

Using the rank order method, the value of  $\widetilde{d''}_5$  is

$$\widetilde{d''}_5 = [0.5, 0.707107, 0.860233] ; [0.542586, 0.768375] [0.42, 0.64]$$

$\widetilde{d''}_5$  and the name for  $\widetilde{d''}_5$  is,

$$\left\{ [0.5, 0.707107, 0.860233] ; [0.542586, 0.768375] [0.42, 0.64] , J(\widetilde{G}) = 2 \right\}.$$

**Iteration – V** The PN nodes for node 6 are nodes 4 and 5, then the value of  $\widetilde{d''}_6$  is

$$\widetilde{d''}_6 = \min \{ \widetilde{d''}_4 \oplus \widetilde{d''}_{46}, \widetilde{d''}_5 \oplus \widetilde{d''}_{56} \}$$

Using the ranking approach,  $\widetilde{d''}_6$  value is

$$\widetilde{d''}_6 = [0.707107, 0.92279, 0.994988] ; [0.686148, 0.858986] [0.252, 0.512] \text{ and}$$

its label is

$$\left\{ [0.707107, 0.92279, 0.994988] ; [0.686148, 0.858986] [0.252, 0.512] , J(\widetilde{G}) = 5 \right\}.$$

**Iteration – VI** The PN nodes for node 7 are nodes 5 and 6, then the value of  $\widetilde{d''}_7$  is

$$\widetilde{d''}_7 = \min \{ \widetilde{d''}_5 \oplus \widetilde{d''}_{57}, \widetilde{d''}_6 \oplus \widetilde{d''}_{67} \}$$

Using the ranking approach,  $\widetilde{d''}_7$  value is

$$\widetilde{d''}_7 = [0.640312, 0.86602, 1.109028] ; [0.809919, 0.931328] [0.252, 0.448] \text{ and}$$

its label is

$$\left\{ [0.640312, 0.86602, 1.109028] ; [0.809919, 0.931328] [0.252, 0.448] , J(\widetilde{G}) = 5 \right\}.$$

Path 1  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  7 is determined to be the SP from the computations above, and the distance between the SN and DNs is



$[0.640312, 0.86602, 1.109028] ; [0.809919, 0.931328] [0.252, 0.448]$ .

The following table lists the SP taken by all nodes from the SN using the suggested algorithm:

| Node | $I_{P_j}$                                               | $W$                                           |
|------|---------------------------------------------------------|-----------------------------------------------|
| 2    | $[0.3, 0.5, 0.5] ; [0.4, 0.6] [0.6, 0.8]$               | $1 \rightarrow 2$                             |
| 3    | $[0.424, 0.707, 0.860] ; [0.68, 0.821] [0.36, 0.56]$    | $1 \rightarrow 2 \rightarrow 3$               |
| 4    | $[0.3, 0.5, 0.7] ; [0.6, 0.7] [0.6, 0.7]$               | $1 \rightarrow 4$                             |
| 5    | $[0.5, 0.707, 0.860] ; [0.543, 0.768] [0.42, 0.64]$     | $1 \rightarrow 2 \rightarrow 5$               |
| 6    | $[0.707, 0.923, 0.995] ; [0.686, 0.859] [0.252, 0.512]$ | $1 \rightarrow 5 \rightarrow 6$               |
| 7    | $[0.640, 0.866, 1.109] ; [0.809, 0.931] [0.252, 0.448]$ | $1 \rightarrow 2 \rightarrow 5 \rightarrow 7$ |

Table 2: SP and SD

where  $W = SP$  between the SN and the  $j^{\text{th}}$  node

## 5 Conclusions

In a SPP, the SD and I-VPFSP provide useful information for making decisions. The distance function, which aids in determining the SP, has been defined for I-VTPFNs in this study. On a network with I-VTPF length, a new approach for resolving the I-VPFSP issue has been suggested. The ranking of the paths is a very helpful decision-making tool for selecting the best alternative path. The process of determining the SP has been adequately explained. Furthermore, a used to explain how the suggested algorithm can be implemented.

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# Strong interval – valued Pythagorean fuzzy soft graphs

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## Abstract

A Strong interval – valued Pythagorean fuzzy soft sets (SIVPFSS) an extending the theory of Interval-valued Pythagorean fuzzy soft set (IVPFSS). Then we Propose Strong interval valued Pythagorean fuzzy soft graphs (SIVPFSGs). We also present several different types of operations on Strong interval- valued Pythagorean fuzzy soft graphs and explore of their analysis.

**Keywords:** Strong Interval-valued Pythagorean fuzzy graph; Strong Interval-valued Pythagorean fuzzy soft graph;

**2020 AMS subject classifications:** 05C72, 06D72, 12D15. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1067. ISSN: 1592-7415. eISSN: 2282-8214. ©M. Mohammed Jabarulla et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Fuzzy set is a analytical imitation to grips the exciting and insufficient details. consider a differentiating that uncertainty is also independently, FS was continued to intuitionistic fuzzy set (IFS) by Atanassov and Gargov [1989]. If assigned a membership value  $\alpha$  and a non membership value  $\beta$  to the conditions, satisfying this results  $\alpha + \beta \leq 1$  and uncertainty elements,  $\gamma = 1 - \alpha - \beta$ . In decision-making problems, the membership value 0.7 and non membership value 0.4 for some information, then IF fails in this situation because  $0.7 + 0.4 > 1$ , but  $(0.7)^2 + (0.4)^2 \leq 1$ . To overcome this situation, the notion of Pythagorean fuzzy set (PFS) was satisfying the condition  $\alpha^2 + \beta^2 \leq 1$ . A PFS has more potential as compared to IFS is solving decision-making problems. The Pythagorean fuzzy number (PFG) was determinate by Zhang ( see S.Shahzadi and Akram [2020]). Zhang provided the Pythagorean fuzzy weighted averaging operator.

The theory of IVFS was introduced by Zadeh [1965] as a perpetuation of fuzzy sets. Because they present more adequate description for uncertainty, interval-valued fuzzy sets more useful than conventional fuzzy sets. Soft set theory was started by Molodstov [1999] for the parameterized point of view for uncertainty modeling and soft computing. The interpretation of IFSGs was given by Akram [2011]. The explanation of novel intuitionistic fuzzy soft multiple – decision-making methods of grips by Akram. Pythagorean fuzzy soft graphs with applications was proposed by S.Shahzadi and Akram [2020].The SIVPFSG is defined and some results on SIVPFSG are studied. Also explore of their analysis.

## 2 Preliminaries

**Definition 2.1.** An IVFSG over the set  $V$  is given by ordered 4 tuple  $\tilde{\xi} = (\xi^*, X, Y, A)$  such that

- (i)  $A$  is of parameters.
- (ii)  $(X, A)$  is an IVFSS over  $V$ .
- (iii)  $(Y, A)$  is an IVFSS over  $E$ .
- (iv)  $(X(e), Y(e))$  is an IVFSG for all  $e \in A$ .

That is,

$$\alpha_{Y(e)}^-(pq) \leq \min(\alpha_{X(e)}^-(p), \alpha_{X(e)}^-(q)) \text{ and } \alpha_{Y(e)}^+(pq) \leq \min(\alpha_{X(e)}^+(p), \alpha_{X(e)}^+(q)) \text{ for all } pq \in E.$$

We denote  $\xi^* = (V, E)$  a crisp graph  $H(e) = (X(e), Y(e))$  an IVFSG and  $\tilde{\xi} = (\xi^*, X, Y, A)$  an IVFSG.

**Definition 2.2.** An IVFSG over the set  $V$  is defined to be a pair  $\xi = (X, Y)$  where  
1) The conditions  $\widetilde{\alpha}_X : V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_X : V \rightarrow D[0, 1]$  denote the degree of

### Strong interval – valued Pythagorean fuzzy soft graphs

membership and non membership of the element  $p \in V$ . such that

$$0 \leq \widetilde{\alpha}_X(p) + \widetilde{\beta}_X(p) \leq 1 \forall (p, q) \in V.$$

2) The conditions  $\widetilde{\alpha}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  defined by

$$\begin{aligned} \alpha_{YL}^-(p, q) &\leq \min(\alpha_{XL}^-(p), \alpha_{XL}^-(q)) \text{ and } \beta_{YL}^-(p, q) \geq \max(\beta_{XL}^-(p), \beta_{XL}^-(q)), \\ \alpha_{YU}^+(p, q) &\leq \min(\alpha_{XU}^+(p), \alpha_{XU}^+(q)) \text{ and } \beta_{YU}^+(p, q) \geq \max(\beta_{XU}^+(p), \beta_{XU}^+(q)), \\ \text{such that } 0 &\leq \alpha_{YU}^2(p, q) + \beta_{YU}^2(p, q) \leq 1 \forall (p, q) \in E. \end{aligned}$$

We the notation  $pq$  for  $(p, q)$  an element of  $E$ .

**Definition 2.3.** An IVPFSG over the set  $V$  is given by  $\tilde{\xi} = (\xi^*, X, Y, A)$  such that

1) The conditions  $\widetilde{\alpha}_X : V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_X : V \rightarrow D[0, 1]$  standered for the degree of membership and non membership of the element  $p \in V$ . such that

$$0 \leq \widetilde{\alpha}_X(p, q) + \widetilde{\beta}_X(p, q) \leq 1 \forall (p, q) \in V.$$

2)(i)  $A$  is set of parameters

(ii)  $(X, A)$  is an IVPFSS over  $V$ .

(iii)  $(Y, A)$  is an IVPFSS over  $E$ .

(iv)  $(X(e), Y(e))$  is an IVPFSG for all  $e \in A$ .

The conditions  $\widetilde{\alpha}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  defined by

$$\begin{aligned} \alpha_{YU}^+(p, q) &\leq \min(\alpha_{XU}^+(p), \beta_{XU}^+(q)) \text{ and } \beta_{YU}^+(p, q) \geq \max(\beta_{XU}^+(p), \beta_{XU}^+(q)), \\ \alpha_{YL}^-(p, q) &\leq \min(\alpha_{XL}^-(p), \beta_{XL}^-(q)) \text{ and } \beta_{YL}^-(p, q) \geq \max(\beta_{XL}^-(p), \beta_{XL}^-(q)), \\ \text{such that } 0 &\leq \alpha_{YU}^2(p, q) + \beta_{YU}^2(p, q) \leq 1 \forall (p, q) \in E. \end{aligned}$$

## 3 Strong interval-valued Pythagorean fuzzy Graphs

**Definition 3.1.** An SIVPFSG over the set  $V$  is given by  $\tilde{\xi} = (\xi^*, X, Y, A)$  such that

1) The conditions  $\widetilde{\alpha}_X : V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_X : V \rightarrow D[0, 1]$  denote the degree of membership and non membership of the element  $x \in V$ . such that

$$0 \leq \widetilde{\alpha}_X(p, q) + \widetilde{\beta}_X(p, q) \leq 1 \forall (p, q) \in V.$$

2)(i)  $A$  is set of parameters

(ii)  $(X, A)$  is an SIVPFSS over  $V$ .

(iii)  $(Y, A)$  is an SIVPFSS over  $E$ .

(iv)  $(X(e), Y(e))$  is an SIVPFSG for all  $e \in A$ .

The conditions  $\widetilde{\alpha}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  and  $\widetilde{\beta}_Y : E \subseteq V \times V \rightarrow D[0, 1]$  defined by

$\alpha_{YU}^+(p, q) = \min(\alpha_{XU}^+(p), \beta_{XU}^+(q))$  and  $\beta_{YU}^+(p, q) = \max(\beta_{XU}^+(p), \beta_{XU}^+(q))$ ,  
 $\alpha_{YL}^-(p, q) = \min(\alpha_{XL}^-(p), \alpha_{XL}^-(q))$  and  $\beta_{YL}^-(p, q) = \max(\alpha_{XL}^-(p), \beta_{XL}^-(q))$ ,  
 such that  $0 \leq \alpha_{YU}^2(p, q) + \beta_{YU}^2(p, q) \leq 1 \forall (p, q) \in E$ .

**Example 3.1.** If  $\xi^* = (X, Y)$  is a simple graph with  $X = \{a, b, c, d\}$  and  $Y = \{ab, bc, cd, ad\}$ . Let  $A = \{e_1, e_2\}$  be a parameter set and  $(X, A)$  be an SIVPFSS  $V$  determine

$$X_1(e) = \left\{ \langle a, [0.3, 0.4][0.2, 0.7] \rangle, \langle b, [0.2, 0.5][0.3, 0.7] \rangle, \langle c, [0.1, 0.6][0.2, 0.5] \rangle, \right. \\ \left. \text{and } \langle d, [0.2, 0.7][0.3, 0.5] \rangle \right\}$$

$$X_2(e) = \left\{ \langle a, [0.2, 0.7][0.3, 0.5] \rangle, \langle b, [0.1, 0.6][0.2, 0.5] \rangle, \langle c, [0.3, 0.4][0.2, 0.7] \rangle \right\}$$

Take  $(Y, A)$  be an SIVPFSS  $E$  determine

$$Y_1(e) = \left\{ \langle ab, [0.2, 0.5][0.3, 0.7] \rangle, \langle bc, [0.1, 0.6][0.3, 0.7] \rangle, \langle ad, [0.2, 0.7][0.3, 0.7] \rangle, \right. \\ \left. \text{and } \langle cd, [0.1, 0.6][0.3, 0.5] \rangle \right\}$$

$$Y_2(e) = \left\{ \langle ab, [0.1, 0.5][0.4, 0.6] \rangle, \langle bc, [0.1, 0.4][0.4, 0.8] \rangle, \langle ac, [0.1, 0.3][0.4, 0.8] \rangle \right\}$$

It is clearly seen that  $H(e_1) = (X(e_1), Y(e_1))$  and  $H(e_2) = (X(e_2), Y(e_2))$  are SIVPFSSs comparable to the parameters  $e_1$  and  $e_2$  accordingly, by Figure 1. Hence  $\tilde{\xi} = (\xi^*, X, Y, A)$  SIVPFSSs.

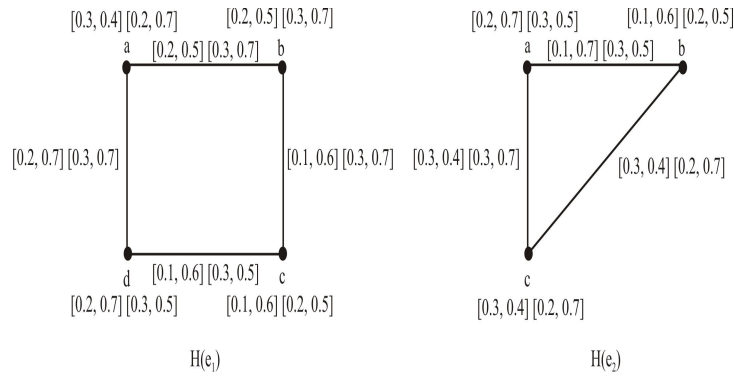


Figure 1: SIVPFSSs  $\tilde{G}$ .

**Definition 3.2.** If  $\tilde{\xi}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{\xi}_2 = (\xi_2^*, X_2, Y_2, B)$  be double SIVPFSSs of  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  accordingly. The cross product of  $\tilde{\xi}_1$  and

*Strong interval – valued Pythagorean fuzzy soft graphs*

$\tilde{\xi}_2$  is denoted by  $\tilde{\xi}_1 \times \tilde{\xi}_2 = (X_1 \times X_2, Y_1 \times Y_2)$  and is defined by

- 1)  $(\alpha_{X_1L} \times \alpha_{X_2L})(p_1, p_2) = \min(\alpha_{X_1L}(p_1), \alpha_{X_2L}(p_2)),$   
 $(\alpha_{X_1U} \times \alpha_{X_2U})(p_1, p_2) = \min(\alpha_{X_1U}(p_1), \alpha_{X_2U}(p_2)),$   
 $(\beta_{X_1L} \times \beta_{X_2L})(p_1, p_2) = \min(\beta_{X_1L}(p_1), \beta_{X_2L}(p_2)),$   
 $(\beta_{X_1U} \times \beta_{X_2U})(p_1, p_2) = \max(\beta_{X_1U}(p_1), \beta_{X_2U}(p_2)), \forall p_1 \in V_1, p_2 \in V_2.$
- 2)  $(\alpha_{Y_1L} \times \alpha_{Y_2L})(p, p_2)(p, q_2) = \min(\alpha_{Y_1L}(p), \alpha_{Y_2L}(p_2, q_2)),$   
 $(\alpha_{Y_1U} \times \alpha_{Y_2U})(p, p_2)(p, p_2) = \min(\alpha_{Y_1U}(p), \alpha_{Y_2U}(p_2, q_2)),$   
 $(\beta_{Y_1L} \times \beta_{Y_2L})(p, p_2)(p, q_2) = \max(\beta_{Y_1L}(p), \beta_{Y_2L}(p_2, q_2)),$   
 $(\beta_{Y_1U} \times \beta_{Y_2U})(p, p_2)(p, q_2) = \max(\beta_{Y_1U}(p), \beta_{Y_2U}(p_2, q_2)), \forall p \in V_1, p_2 q_2 \in E_2.$
- 3)  $(\alpha_{Y_1L} \times \alpha_{Y_2L})(p_1, r)(q_1, r) = \min(\alpha_{Y_1L}(p_1 q_1), \alpha_{Y_2L}(r)),$   
 $(\alpha_{Y_1U} \times \alpha_{Y_2U})(p_1, r)(q_1, r) = \min(\alpha_{Y_1U}(p_1 q_1), \alpha_{Y_2U}(r)),$   
 $(\beta_{Y_1L} \times \beta_{Y_2L})(p_1, r)(q_1, r) = \max(\beta_{Y_1L}(p_1 q_1), \beta_{Y_2L}(r)),$   
 $(\beta_{Y_1U} \times \beta_{Y_2U})(p_1, r)(q_1, r) = \max(\beta_{Y_1U}(p_1 q_1), \beta_{Y_2U}(r)), \forall r \in V_2, p_1 q_1 \in E_1.$

**Example 3.2.** Let Consider a graph  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  be two graphs such that  $X_1 = \{a_1, b_1, c_1, d_1\}$ ,  $Y_1 = \{a_1 b_1, c_1 d_1\}$  and  $X_2 = \{a_2, b_2, c_2, d_2\}$ ,  $Y_2 = \{a_2 b_2, c_2 d_2\}$ . Let  $A = e_1$  be a set of parameters and let  $(X_1, A)$  and  $(Y_1, A)$  be two SIVPFSSs over  $X_1$  and  $Y_1$  accordingly, defined by

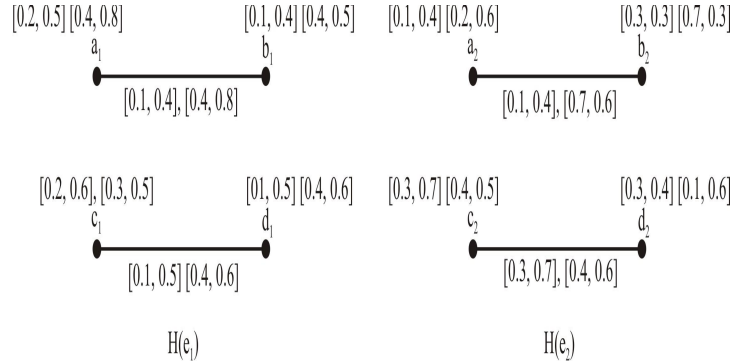


Figure 2: SIVPFSGs  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$ .

$$\begin{aligned}
 X_1(e) &= \left\{ \langle a_1 [0.2, 0.5] [0.4, 0.8] \rangle, \langle b_1 [0.1, 0.4] [0.4, 0.5] \rangle, \langle c_1 [0.2, 0.6] [0.3, 0.5] \rangle, \right. \\
 &\quad \left. \text{and } \langle d_1 [0.1, 0.5] [0.4, 0.6] \rangle \right\} \\
 Y_1(e) &= \left\{ \langle a_1 b_1 [0.1, 0.4] [0.4, 0.8] \rangle, \langle c_1 d_1 [0.1, 0.5] [0.4, 0.6] \rangle \right\}
 \end{aligned}$$

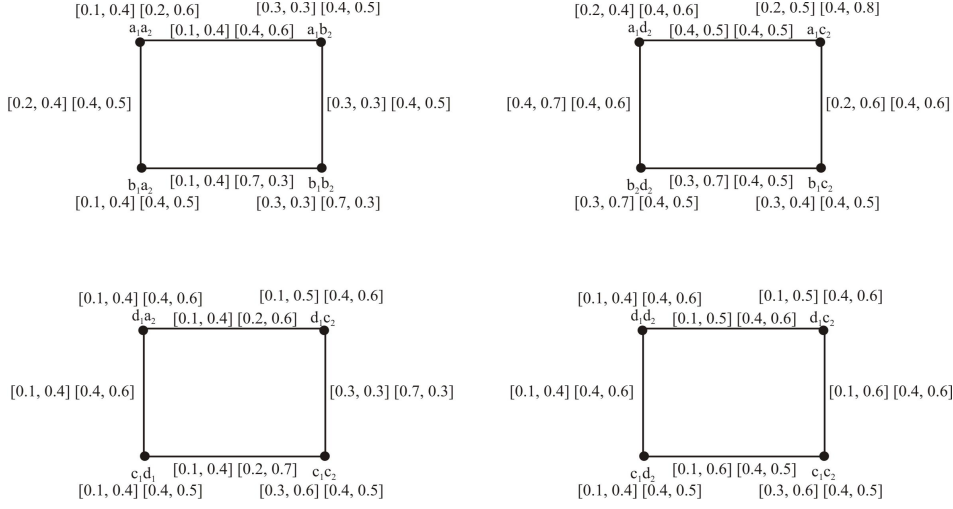


Figure 3: Cross product of  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$ .

Take  $B = e_2$  be a set of parameters and let  $(X_2, B)$  and  $(Y_2, B)$  be two SIVPFSSs over  $X_2$  and  $Y_2$  accordingly, Find out

$$X_2(e) = \left\{ \langle a_2[0.1, 0.4][0.2, 0.6] \rangle, \langle b_2[0.3, 0.3][0.7, 0.3] \rangle, \langle c_2[0.3, 0.7][0.4, 0.5] \rangle, \right. \\ \left. \text{and } \langle d_2[0.3, 0.4][0.1, 0.6] \rangle \right\}$$

$$Y_2(e) = \left\{ \langle a_2 b_2[0.1, 0.4][0.7, 0.6] \rangle, \langle c_2 d_2[0.3, 0.7][0.4, 0.6] \rangle \right\}$$

Clearly  $H(e_1) = (X(e_1), Y(e_1))$  and  $H(e_2) = (X(e_2), Y(e_2))$  are SIVPFSSGs. Hence  $\tilde{\xi}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{\xi}_2 = (\xi_2^*, X_2, Y_2, B)$  are SIVPFSSGs  $\xi_1^*$  and  $\xi_2^*$ , accordingly, as shown in the Figure 2.

**Definition 3.3.** If  $\tilde{\xi}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{\xi}_2 = (\xi_2^*, X_2, Y_2, B)$  be two SIVPFSSGs of  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  accordingly. The composition of  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  is standed by  $\tilde{\xi}_1 \circ \tilde{\xi}_2 = (X_1 \circ X_2, Y_1 \circ Y_2)$  and is defined by

- 1)  $(\alpha_{X_1 L} \circ \alpha_{X_2 L})(p_1, p_2) = \min(\alpha_{X_1 L}(p_1), \alpha_{X_2 L}(p_2)),$   
 $(\alpha_{X_1 U} \circ \alpha_{X_2 U})(p_1, p_2) = \min(\alpha_{X_1 U}(p_1), \alpha_{X_2 U}(p_2)),$   
 $(\beta_{X_1 L} \circ \beta_{X_2 L})(p_1, p_2) = \min(\beta_{X_1 L}(p_1), \beta_{X_2 L}(p_2)),$   
 $(\beta_{X_1 U} \circ \beta_{X_2 U})(p_1, p_2) = \max(\beta_{X_1 U}(p_1), \beta_{X_2 U}(p_2)), \forall p_1 \in V_1, p_2 \in V_2.$
- 2)  $(\alpha_{Y_1 L} \circ \alpha_{Y_2 L})(p, p_2) = \min(\alpha_{Y_1 L}(p), \alpha_{Y_2 L}(p_2, q_2)),$   
 $(\alpha_{Y_1 U} \circ \alpha_{Y_2 U})(p, p_2) = \min(\alpha_{Y_1 U}(p), \alpha_{Y_2 U}(p_2, q_2)),$   
 $(\beta_{Y_1 L} \circ \beta_{Y_2 L})(p, p_2) = \max(\beta_{Y_1 L}(p), \beta_{Y_2 L}(p_2, q_2)),$   
 $(\beta_{Y_1 U} \circ \beta_{Y_2 U})(p, p_2) = \max(\beta_{Y_1 U}(p), \beta_{Y_2 U}(p_2, q_2)), \forall p_1 \in V_1, p_2 q_2 \in E_2.$



3)  $(\alpha_{Y_1L} \circ \alpha_{Y_2L})(p_1, r)(q_1, r) = \min(\alpha_{Y_1L}(p_1q_1), \alpha_{Y_2L}(r)),$   
 $(\alpha_{Y_1U} \circ \alpha_{Y_2U})(p_1, r)(q_1, r) = \min(\alpha_{Y_1U}(p_1q_1), \alpha_{Y_2U}(r)),$   
 $(\beta_{Y_1L} \circ \beta_{Y_2L})(p_1, r)(q_1, r) = \max(\beta_{Y_1L}(p_1q_1), \beta_{Y_2L}(r)),$   
 $(\beta_{Y_1U} \circ \beta_{Y_2U})(p_1, r)(q_1, r) = \max(\beta_{Y_1U}(p_1q_1), \beta_{Y_2U}(r)), \forall r \in V_2, p_1q_1 \in E_1.$   
4)  $(\alpha_{Y_1L} \circ \alpha_{Y_2L})(p_1, p_2)(q_1, q_2) = \min(\alpha_{X_2L}(p_2), \alpha_{X_2L}(q_2), \alpha_{X_1L}(p_1, q_1)),$   
 $(\alpha_{Y_1U} \circ \alpha_{Y_2U})(p_1, p_2)(q_1, q_2) = \min(\alpha_{X_2U}(p_2), \alpha_{X_2U}(q_2), \alpha_{X_1U}(p_1, q_1)),$   
 $(\beta_{Y_1L} \circ \beta_{Y_2L})(p_1, p_2)(q_1, q_2) = \max(\beta_{X_2L}(p_2), \beta_{X_2L}(q_2), \beta_{Y_1L}(p_1, q_1)),$   
 $(\beta_{Y_1U} \circ \beta_{Y_2U})(p_1, p_2)(q_1, q_2) = \max(\beta_{X_2U}(p_2), \beta_{X_2U}(q_2), \beta_{Y_1U}(p_1, q_1)),$   
 $\forall (p_1, p_2)(q_1, q_2) \in E^\circ - E.$   
where  $E^\circ = E \cup \{(p_1, p_2)(q_1, q_2) | p_1q_1 \in E_1, p_2 \neq q_2\}.$

**Definition 3.4.** Let  $\tilde{\xi}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{\xi}_2 = (\xi_2^*, X_2, Y_2, B)$  be two SIVPF-SGs of  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  accordingly. If  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  is standed by  $\tilde{\xi}_1 \cup \tilde{\xi}_2 = (G^*, X, Y, A \cup B)$  where  $(X_1 \cup X_2, Y_1 \cup Y_2)$  and is replace

1) (i)  $(\alpha_{X_1L} \cup \alpha_{X_2L})(p) = \max(\alpha_{X_1L}(p), \alpha_{X_2L}(p))$  if  $p \in V_1 \cap V_2$   
 $(\alpha_{X_1U} \cup \alpha_{X_2U})(p) = \max(\alpha_{X_1U}(p), \alpha_{X_2U}(p))$  if  $p \in V_1 \cap V_2$   
(ii)  $(\beta_{X_1L} \cup \beta_{X_2L})(p) = \max(\beta_{X_1L}(p), \beta_{X_2L}(p))$  if  $p \in V_1 \cap V_2$   
 $(\beta_{X_1U} \cup \beta_{X_2U})(p) = \max(\beta_{X_1U}(p), \beta_{X_2U}(p))$  if  $p \in V_1 \cap V_2$   
2) (i)  $(\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q) = \max(\alpha_{X_1L}(p, q), \alpha_{X_2L}(p, q))$  if  $pq \in E_1 \cap E_2$   
 $(\alpha_{Y_1U} \cup \alpha_{Y_2U})(p, q) = \max(\alpha_{X_1U}(p, q), \alpha_{X_2U}(p, q))$  if  $pq \in E_1 \cap E_2$   
(ii)  $(\beta_{Y_1L} \cup \beta_{Y_2L})(p, q) = \max(\beta_{X_1L}(p, q), \beta_{X_2L}(p, q))$  if  $pq \in E_1 \cap E_2$   
 $(\beta_{Y_1U} \cup \beta_{Y_2U})(p, q) = \max(\beta_{X_1U}(p, q), \beta_{X_2U}(p, q))$  if  $pq \in E_1 \cap E_2$

**Definition 3.5.** Let  $\tilde{G}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{G}_2 = (\xi_2^*, X_2, Y_2, B)$  be two SIVPF-SGs of  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  accordingly. If  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  is standed by  $\tilde{\xi}_1 + \tilde{\xi}_2 = (\xi_1^*, X_1, Y_1, A + B)$ . Where  $\xi^* = (X_1 + X_2, Y_1 + Y_2)$  and is defined by

1)  $(\alpha_{X_1L} + \alpha_{X_2L})(p) = (\alpha_{X_1L} \cup \alpha_{X_2L})(p)$   
 $(\alpha_{X_1U} + \alpha_{X_2U})(p) = (\alpha_{X_1U} \cup \alpha_{X_2U})(p)$  if  $p \in V_1 \cup V_2$   
 $(\beta_{X_1L} + \beta_{X_2L})(p) = (\beta_{X_1L} \cup \beta_{X_2L})(p)$   
 $(\beta_{X_1U} + \beta_{X_2U})(p) = (\beta_{X_1U} \cup \beta_{X_2U})(p)$  if  $p \in V_1 \cup V_2$   
2)  $(\alpha_{Y_1L} + \alpha_{Y_2L})(p, q) = (\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q)$   
 $(\alpha_{Y_1U} + \alpha_{Y_2U})(p, q) = (\alpha_{Y_1U} \cup \alpha_{Y_2U})(p, q)$  if  $p \in E_1 \cap E_2$   
 $(\beta_{Y_1L} + \beta_{Y_2L})(p, q) = (\beta_{Y_1L} \cup \beta_{Y_2L})(p, q)$   
 $(\beta_{Y_1U} + \beta_{Y_2U})(p, q) = (\beta_{Y_1U} \cup \beta_{Y_2U})(p, q)$  if  $(p, q) \in E_1 \cap E_2.$   
3)  $(\alpha_{Y_1L} + \alpha_{Y_2L})(p, q) = \min(\alpha_{X_1L}(p), \alpha_{X_2L}(q))$   
 $(\alpha_{Y_1U} + \alpha_{Y_2U})(p, q) = \min(\alpha_{X_1U}(p), \alpha_{X_2U}(q))$   
 $(\beta_{Y_1L} + \beta_{Y_2L})(p, q) = \max(\beta_{X_1L}(p), \beta_{X_2L}(q))$   
 $(\beta_{Y_1U} + \beta_{Y_2U})(p, q) = \max(\beta_{X_1U}(p), \beta_{X_2U}(q))$  if  $pq \in E$

Where  $E$  is the set of all edges joining the vertices of  $V_1$  and  $V_2$ .

**Theorem 3.1.** If  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  are SIVPFSGs, then so is  $\tilde{\xi}_1 \times \tilde{\xi}_2$ .

**proof** Let  $\tilde{\xi}_1 = (\xi_1^*, X_1, Y_1, A)$  and  $\tilde{\xi}_2 = (\xi_1^*, X_1, Y_1, B)$  be two SIVPFSGs of simple graphs  $\xi_1^* = (X_1, Y_1)$  and  $\xi_2^* = (X_2, Y_2)$  accordingly. for all  $e_1 \in A$  and  $e_2 \in B$ , there are some results. Let  $\xi_1$  and  $\xi_2$  be SIVPFSGs  
Let  $E = \{(p, p_2)(p, q_2)/p \in V_1, p_2q_2 \in E_2\} \cup \{(p_1, r)(q_1, r)/r \in V_2, p_1q_1 \in E_1\}$ .  
Consider  $(p, p_2)(p, q_2) \in E$ , we have

$$\begin{aligned} &(\alpha_{Y_1L} \times \alpha_{Y_2L})(p, p_2)(p, q_2) = \min(\alpha_{X_1L}(p), \alpha_{Y_2L}(p_2q_2)) \\ &= \min(\alpha_{X_1L}(p), \alpha_{X_2L}(p_2) \cdot \alpha_{X_2L}(q_2)) \\ &= \min(\min(\alpha_{X_1L}(p), \alpha_{X_2L}(p_2)) \min(\alpha_{X_1L}(p), \alpha_{X_2L}(q_2))) \\ &(\alpha_{Y_1L} \times \alpha_{Y_2L})(p, p_2)(p, q_2) = \min((\alpha_{X_1L} \times \alpha_{X_2L})(p, p_2), (\alpha_{X_1L} \times \alpha_{X_2L})(p, q_2)) \end{aligned}$$

Similarly,

$$(\alpha_{Y_1U} \times \alpha_{Y_2U})(p, p_2)(p, q_2) = \min((\alpha_{X_1U} \times \alpha_{X_2U})(p, p_2), (\alpha_{Y_1U} \times \alpha_{Y_2U})(p, q_2))$$

Now,

$$(\beta_{Y_1L} \times \beta_{Y_2L})(p, p_2)(p, q_2) = \max((\beta_{X_1L} \times \beta_{X_2L})(p, p_2), (\beta_{X_1L} \times \beta_{X_2L})(p, q_2))$$

Similarly,

$$(\beta_{Y_1U} \times \beta_{Y_2U})(p, p_2)(p, q_2) = \max((\beta_{X_1U} \times \beta_{X_2U})(p, p_2), (\beta_{X_1U} \times \beta_{X_2U})(p, q_2))$$

Consider,  $(p_1, r)(q_1, r) \in E$ , we have

$$\begin{aligned} &(\alpha_{Y_1L} \times \alpha_{Y_2L})(p_1, r)(q_1, r) = \min(\alpha_{Y_1L}(p_1q_1), (\alpha_{X_2L}(r))) \\ &= \min(\alpha_{X_1L}(p_1), \alpha_{X_2L}(q_1) \cdot \alpha_{X_2L}(r)) \\ &= \min(\min(\alpha_{X_1L}(p_1), \alpha_{X_2L}(r)) \min(\alpha_{X_1L}(q_1), \alpha_{X_2L}(r))) \end{aligned}$$

$$(\alpha_{Y_1L} \times \alpha_{Y_2L})(p_1, r)(q_1, r) = \min((\alpha_{X_1L} \times \alpha_{X_2L})(p_1, r), (\alpha_{X_1L} \times \alpha_{X_2L})(q_1, r))$$

Similarly,

$$(\alpha_{Y_1U} \times \alpha_{Y_2U})(p_1, r)(q_1, r) = \min((\alpha_{X_1U} \times \alpha_{X_2U})(p_1, r), (\alpha_{X_1U} \times \alpha_{X_2U})(q_1, r))$$

Now,

$$(\beta_{Y_1L} \times \beta_{Y_1U})(p_1, r)(q_1, r) = \max((\beta_{X_1L} \times \beta_{X_2L})(p_1, r), (\beta_{X_1L} \times \beta_{X_2L})(q_1, r))$$

Similarly,

$$(\beta_{Y_1U} \times \beta_{Y_2U})(p_1, r)(q_1, r) = \max((\beta_{X_1U} \times \beta_{X_2U})(p_1, r), (\beta_{X_1U} \times \beta_{X_2U})(q_1, r))$$

Hence  $\xi_1 \times \xi_2$  is an SIVPFSGs.

**Theorem 3.2.** If  $\tilde{\xi}_1[\tilde{\xi}_2]$  be SIVPFSGs  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  of  $\xi_1^*$  and  $\xi_2^*$  is an SIVPFSGs.

**Proof** Take  $(p, p_2)(p, q_2) \in E$ , we get

$$\begin{aligned} & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p, p_2)(p, q_2)) = \min((\alpha_{X_1L}(p), \alpha_{Y_2L}(p_2q_2)) \\ & = \min(\alpha_{X_1L}(p), \alpha_{X_2L}(p_2), \alpha_{X_2L}(q_2)) \\ & = \min(\min(\alpha_{X_1L}(p), \alpha_{X_2L}(p_2)), \min(\alpha_{X_1L}(p), \alpha_{X_2L}(q_2))) \\ & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p, p_2)(p, q_2) = \min(\alpha_{X_1L} \circ \alpha_{X_2L})(p, p_2), (\alpha_{X_1L} \circ \alpha_{X_2L})(p, q_2)). \end{aligned}$$

Similarly,

$$(\alpha_{Y_1U} \circ \alpha_{Y_2U})((p, p_2)(p, q_2) = \min(\alpha_{X_1U} \circ \alpha_{X_2U})(p, p_2), (\alpha_{X_1U} \circ \alpha_{X_2U})(p, q_2))$$

Consider  $(p_1, r)(q_1, r) \in E$ ,

$$\begin{aligned} & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p_1, r)(q_1, r)) = \min(\alpha_{Y_1L}(p_1, q_1), \alpha_{X_2L}(r)) \\ & = \min(\alpha_{X_1L}(p_1), \alpha_{X_1L}(q_1), \alpha_{X_2L}(r)) \\ & = \min(\min(\alpha_{X_1L}(p_1), \alpha_{X_2L}(r)), \min(\alpha_{X_1L}(q_1), \alpha_{X_2L}(r))) \\ & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p_1, r)(q_1, r)) = \min(\alpha_{X_1L} \circ \alpha_{X_2L})(p_1, r), (\alpha_{X_1L} \circ \alpha_{X_2L})(q_1, r)) \end{aligned}$$

Similarly,

$$(\alpha_{X_1U} \circ \alpha_{X_2U})((p_1, r)(q_1, r) = \min(\alpha_{X_1U} \circ \alpha_{X_2U})(p_1, r), (\alpha_{X_1U} \circ \alpha_{X_2U})(q_1, r))$$

Consider  $(p_1, p_2)(q_1, q_2) \in E$ ,

$$\begin{aligned} & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p_1, p_2)(q_1, q_2)) = \min(\alpha_{X_2L}(p_2), \alpha_{X_2L}(q_2), \alpha_{Y_1L}(p_1q_1)) \\ & = \min(\alpha_{X_2L}(p_2), \alpha_{X_2L}(q_2), \min(\alpha_{X_1L}(p_1), \alpha_{X_1L}(q_1))) \\ & = \min(\min(\alpha_{X_1L}(p_1), \alpha_{X_2L}(p_2)), \min(\alpha_{X_1L}(q_1), \alpha_{X_2L}(q_2))) \\ & (\alpha_{Y_1L} \circ \alpha_{Y_2L})((p_1, p_2)(q_1, q_2)) = \min(\alpha_{X_1L} \circ \alpha_{X_2L})(p_1, p_2), (\alpha_{X_1L} \circ \alpha_{X_2L})(q_1, q_2)) \end{aligned}$$

Hence  $\tilde{\xi}_1[\tilde{\xi}_2]$  be SIVPFSG .

**Theorem 3.3.** If  $\tilde{\xi}_1 \cup \tilde{\xi}_2$  be SIVPFSGs  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  of  $\xi_1^*$  and  $\xi_2^*$  is an SIVPFSGs.

**Proof** Take  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  be the SIVPFSGs of  $\xi_1^*$  and  $\xi_2^*$  accordingly. Since all conditions for  $X_1 \cup X_2$  are obviously satisfied. It is enough to verify the conditions for  $Y_1 \cup Y_2$ , Consider  $(p, q) \in E_1 \cup E_2$ . Then

$$\begin{aligned} & (\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q) = \max(\alpha_{Y_1L}(p, q), \alpha_{Y_2L}(p, q)) \\ & = \max(\min(\alpha_{X_1L}(p), \alpha_{X_1L}(q)), (\min(\alpha_{X_2L}(p), \alpha_{X_2L}(q))) \\ & = \min(\max(\alpha_{X_1L}(p), \alpha_{X_2L}(p)), (\max(\alpha_{X_1L}(q), \alpha_{X_2L}(q)))) \\ & = \min((\alpha_{Y_1L} \cup \alpha_{Y_2L})(p), (\alpha_{Y_1L} \cup \alpha_{Y_2L})(q)) \\ & (\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q) = \min((\alpha_{Y_1L} \cup \alpha_{Y_2L})(p), (\alpha_{Y_1L} \cup \alpha_{Y_2L})(q)). \end{aligned}$$

Similarly,

$$(\alpha_{Y_1U} \cup \alpha_{Y_2U})(p, q) = \min((\alpha_{Y_1U} \cup \alpha_{Y_2U})(p), (\alpha_{Y_1U} \cup \alpha_{Y_2U})(q))$$

If  $(x, y) \in E_1$  and  $(x, y) \notin E_2$ ,

$$\begin{aligned} (\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q) &= \min((\alpha_{Y_1L} \cup \alpha_{Y_2L})(p), (\alpha_{Y_1L} \cup \alpha_{Y_2L})(q)) \\ (\alpha_{Y_1U} \cup \alpha_{Y_2U})(p, q) &= \min((\alpha_{Y_1U} \cup \alpha_{Y_2U})(p), (\alpha_{Y_1U} \cup \alpha_{Y_2U})(q)). \end{aligned}$$

If  $(p, q) \in E_2$  and  $(p, q) \in E_1$ ,

$$\begin{aligned} (\alpha_{Y_1L} \cup \alpha_{Y_2L})(p, q) &= \min((\alpha_{Y_1L} \cup \alpha_{Y_2L})(p), (\alpha_{Y_1L} \cup \alpha_{Y_2L})(q)) \\ (\alpha_{Y_1U} \cup \alpha_{Y_2U})(p, q) &= \min((\alpha_{Y_1U} \cup \alpha_{Y_2U})(p), (\alpha_{Y_1U} \cup \alpha_{Y_2U})(q)). \end{aligned}$$

**Theorem 3.4.** If  $\tilde{\xi}_1 + \tilde{\xi}_2$  be SIVPFSGs  $\tilde{\xi}_1$  and  $\tilde{\xi}_2$  of  $\xi_1^*$  and  $\xi_2^*$  is an SIVPFSGs.

**Proof** Take  $\tilde{\xi}_1 + \tilde{\xi}_2$  be the SIVPFSGs of  $\xi_1^*$  and  $\xi_2^*$  accordingly. , it is enough to find that  $\tilde{\xi}_1 + \tilde{\xi}_2 = (X_1 + X_2, Y_1 + Y_2)$  is an SIVPFSGs. Then Let  $(p, q) \in E$

$$\begin{aligned} (\alpha_{Y_1L} + \alpha_{Y_2L})(p, q) &= \min(\alpha_{X_1L}(p), \alpha_{X_2L}(q)) \\ &= \min((\alpha_{X_1L} \cup \alpha_{X_2L})(p), ((\alpha_{X_1L} \cup \alpha_{X_2L})(q))) \\ (\alpha_{Y_1L} + \alpha_{Y_2L})(p, q) &= \min((\alpha_{X_1L} + \alpha_{X_2L})(p), ((\alpha_{X_1L} + \alpha_{X_2L})(q))). \end{aligned}$$

Similarly,

$$(\alpha_{Y_1U} + \alpha_{Y_2U})(p, q) = \min((\alpha_{X_1U} + \alpha_{X_2U})(p), ((\alpha_{X_1U} + \alpha_{X_2U})(q))).$$

## 4 Conclusions

Graph theory is a very helpful mathematical tool for tackling challenging issues in a variety of disciplines. The IVPFSs model is appropriate for modeling issues involving uncertainty and inconsistent data when human understanding and evaluation are required. In contrast to IVFS models, IVIFS models, and, IVPFS models provide systems with sensitivity, flexibility, and conformance. SIVPFSGs are a novel idea that is introduced in this work. We also defined for the Cartesian product as well as some information about its composition on SIVPFSGs. We plan to use this data to create some algorithms and models shortly soon.

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*Strong interval – valued Pythagorean fuzzy soft graphs*

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# Domination in $m$ – polar soft fuzzy graphs

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## Abstract

In this article, we have introduced dominating set, minimal dominating set, independent dominating set, maximal independent dominating set in  $m$  – polar soft fuzzy graphs. We proved theorems and also some properties of dominating set in  $m$  – polar soft fuzzy graphs.

**Keywords:** Dominating set, Independent dominating set, Maximal independent dominating set in  $m$  – polar soft fuzzy graphs.

**2020 AMS subject classifications:** 03E72, 05C72. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1070. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.



[illegible]

$\forall v \in V \setminus \{u\}$  in  $\tilde{H}_{P,V}(e) \forall e \in P$ . so that  $N^{hd}(u) = \phi$ . That means an isolated vertex does not dominate any other vertex in  $\tilde{G}_{P,V}$ .

**Example 2.1.** Consider a 3-psf-graph  $\tilde{G}_{P,V}$ .

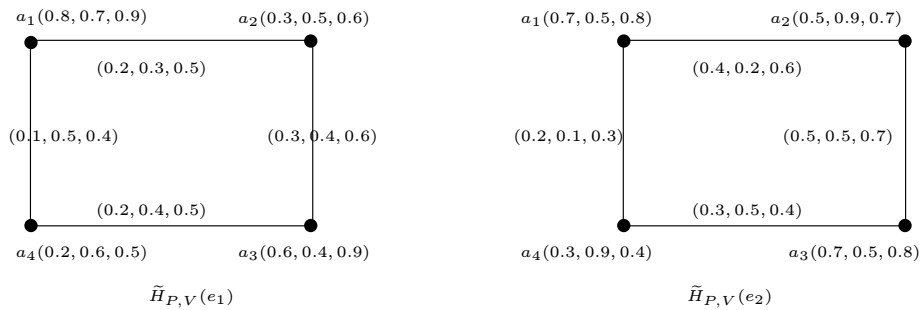


Fig.1. Effective edges of 3-psf-graph  $\tilde{G}_{P,V} = \{\tilde{H}_{P,V}(e_1), \tilde{H}_{P,V}(e_2)\}$ .

In this example,  $a_2a_3$  and  $a_3a_4$  are effective edges. Also  $N^{hd}(a_1) = \{\phi\}$ ,  $N^{hd}(a_2) = \{a_3\}$ ,  $N^{hd}(a_3) = \{a_4a_2\}$ ,  $N^{hd}(a_4) = \{a_3\}$ . Here  $a_1$  is an isolated vertex.

**Definition 2.3.** Let  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$  be an  $m$ -psf-graph. For any two vertices  $u, v \in V$ , we call  $u$  dominates  $v$  in  $\tilde{G}_{P,V}$  if

[illegible]

in  $\tilde{H}_{P,V}(e) \forall e \in P$  and  $\forall u, v \in V$ .

**Definition 2.4.** Consider a subset  $\tilde{\mathcal{S}}$  of  $V$  in an  $m$ -psf-graph  $\tilde{G}_{P,V}$ . The cardinality of  $\tilde{\mathcal{S}}$  is defined as,

$$|\tilde{\mathcal{S}}| = \sum_{e \in P} (\sum_{u \in \tilde{\mathcal{S}}} \tilde{\rho}_e(u)) \text{ in } \tilde{H}_{P,V}(e) \forall e \in P.$$



**Definition 2.5.** A subset  $\tilde{S}$  of  $V$  is called a dominating set of an  $m$ -psf-graph  $\tilde{G}_{P,V}$  if for every vertex  $u \in V \setminus \tilde{S}$  then  $\exists v \in \tilde{S}$  such that  $u$  dominates  $v$  in  $\tilde{H}_{P,V}(e) \forall e \in P$ . The domination number  $\gamma(\tilde{G}_{P,V})$  means the infimum cardinality of all dominating set in  $\tilde{G}_{P,V}$  and  $\gamma(\tilde{G}_{P,V}) = \min_{\tilde{S} \subseteq V} \sum_{e \in P} (\sum_{u \in \tilde{S}} \tilde{\rho}_e(u))$ .

**Definition 2.6.** A dominating set  $\tilde{S}$  is called a minimal dominating set of  $m$ -psf-graph  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$  if for any  $a \in \tilde{S}$ ,  $\tilde{S} \setminus \{a\}$  is not a dominating set in  $\tilde{H}_{P,V}(e) \forall e \in P$ .

**Definition 2.7.** Lower and upper domination number of an  $m$ -psf-graph  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$  is denoted by  $\gamma(\tilde{G}_{P,V})$  and  $\Gamma(\tilde{G}_{P,V})$ , respectively, and defined by infimum cardinality and supremum cardinality of all minimal dominating set of that  $m$ -psf-graph, respectively.

**Example 2.2.** Consider a 3-psf-graph  $\tilde{G}_{P,V}$ .

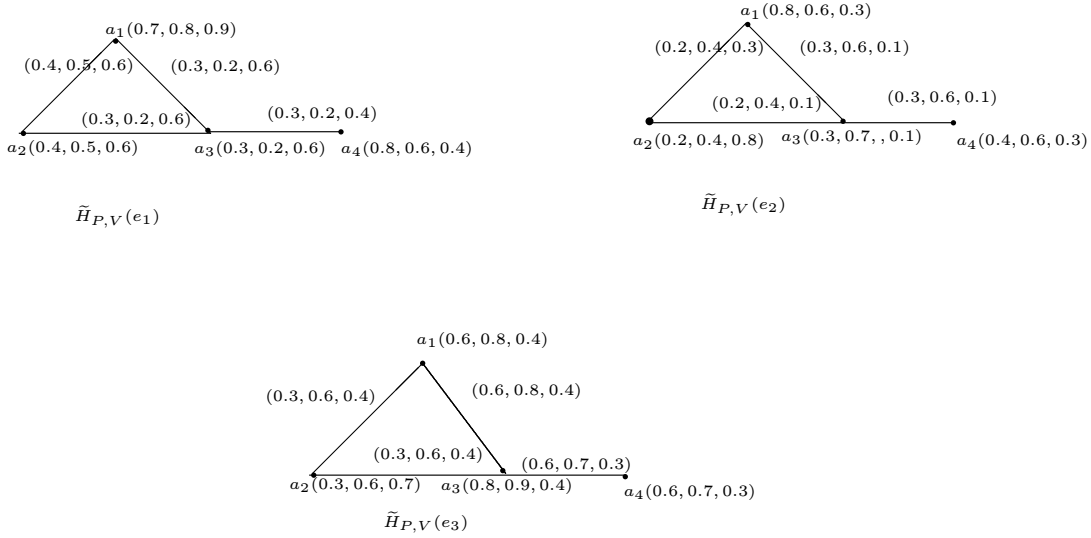


Fig.2. Minimal dominating set of 3-psf-graph

In Fig.2. Here, the minimal dominating sets in each parameterized graph is  $\tilde{S}_1 = \{a_1, a_4\}, \tilde{S}_2 = \{a_2, a_4\}, \tilde{S}_3 = \{a_3\}$ .

**Theorem 2.1.** A dominating set  $\tilde{S}$  is minimal if and only if one of the below mentioned criteria holds.

- (1)  $N^{hd}(a) \cap \tilde{S} = \phi$ .
- (2) There is a vertex  $b \in V \setminus \tilde{S}$  such that  $N^{hd}(b) \cap \tilde{S} = \{a\}$ , for each  $a \in \tilde{S}$ .

*Proof.* For a minimal dominating set  $\tilde{S}$  of a 3-psf-graph  $\tilde{G}_{P,V}$ , for every vertex  $a \in \tilde{S}$ ,  $\tilde{S} \setminus \{a\}$  is not dominating set and so then  $\exists b \in V \setminus (\tilde{S} \setminus \{a\})$  which is not dominated by any vertex in  $\tilde{S} \setminus \{a\}$  of  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . If  $a = b$  then  $N^{hd}(a) \subseteq V \setminus \tilde{S} \Rightarrow N^{hd}(a) \cap \tilde{S} = \phi$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . If  $a \neq b$ , then  $b$  is not dominated by  $\tilde{S} \setminus \{a\}$  but is dominated by  $\tilde{S}$ , i.e.,  $b$  is dominated only by  $a$  in  $\tilde{S}$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ .  $\therefore N^{hd}(b) \cap \tilde{S} = \{a\}$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Hence  $N^{hd}(b) \cap \tilde{S} = \{a\}$  in 3-psf-graph  $\tilde{G}_{P,V}$ .

Conversely, let  $\tilde{S}$  holds one of the two given criterias. If  $\tilde{S}$  is not minimal dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Then  $\exists$  a vertex  $a \in \tilde{S}$  such that  $\tilde{S} \setminus \{a\}$  is a dominating set in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . Thus  $a$  is dominated by atleast one vertex in  $\tilde{S} \setminus \{a\}$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . Then  $N^{hd}(a) \not\subseteq \tilde{S} \setminus \{a\}$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . Hence condition (1) does not hold. Also, If  $\tilde{S} \setminus \{a\}$  is a dominating set in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . Then every vertex  $b$  in  $V \setminus (\tilde{S} \setminus \{a\})$  is dominated by at least one vertex in  $\tilde{S} \setminus \{a\}$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . So  $N^{hd}(b) \cap \tilde{S} \neq \{a\}$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . hence condition (2) does not hold. This leads to a  $\Rightarrow \Leftarrow$ .  $\therefore \tilde{S}$  must be a minimal dominating set in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . Hence  $\tilde{S}$  must be a minimal dominating set in 3-psf-graph  $\tilde{G}_{P,V}$ .  $\square$

**Theorem 2.2.** Let  $\tilde{G}_{P,V} = ((P, \tilde{\rho}), (P, \tilde{\mu}))$  be a 3-psf-graph without isolated vertices. If  $\tilde{S}$  is a minimal dominating set of  $\tilde{G}_{P,V}$  then  $V \setminus \tilde{S}$  is a dominating set of  $\tilde{G}_{P,V}$ .

*Proof.* Let  $\tilde{S}$  be a minimal dominating set and  $a \in \tilde{S}$ . Since  $\tilde{G}_{P,V}$  has no isolated vertices  $\exists$  a vertex  $b \in N^{hd}(a)$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Utilization of the same content similar to the proof given for Theorem 2.1, we get that  $b$  in  $V \setminus \tilde{S}$ . Thus every element of  $\tilde{S}$  is dominated by some element of  $V \setminus \tilde{S}$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Consequently  $V \setminus \tilde{S}$  is a dominating set of  $\tilde{G}_{P,V}$ .  $\square$

**Theorem 2.3.** Superset of a dominating set in  $\tilde{G}_{P,V}$  is a dominating set.

*Proof.* Assume that  $\tilde{S}$  is a dominating set in  $\tilde{G}_{P,V}$ . Then all the vertices in  $V \setminus \tilde{S}$  are adjacent to some element of  $\tilde{S}$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Let  $b \in V \setminus \tilde{S}$ . Then  $\tilde{S} \cup \{b\}$  is a dominating set, Since all the vertices in  $(V \setminus \tilde{S}) \setminus \{b\}$  adjacent to some elements of  $\tilde{S} \cup \{b\}$ . Since  $b$  is arbitrary, any superset of a dominating set is a dominating set.  $\square$

**Remark 2.1.** Subset of a dominating set in  $\tilde{G}_{P,V}$  need not to be dominating set.

**Example 2.3.** Consider a 3-psf-graph.

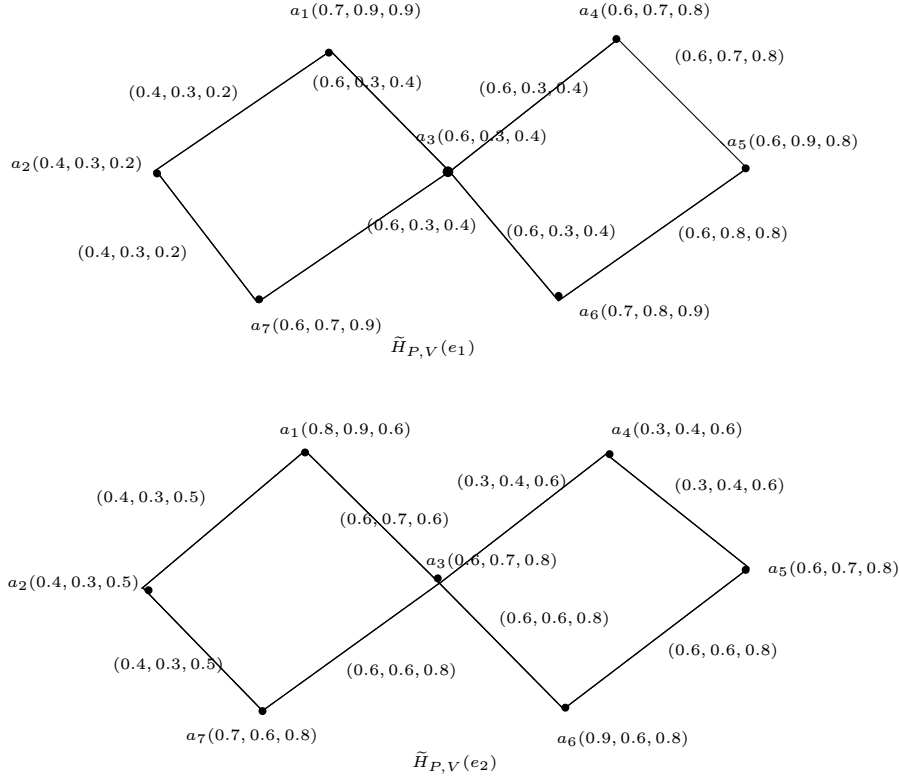


Fig.3. 3-psf-graphs

Here each parameterized graph is  $\tilde{\mathcal{S}} = \{a_2, a_3, a_5\}$  and  $\tilde{\mathcal{S}} \cup \{b\} = \{a_1, a_2, a_3, a_5\}$   $\tilde{\mathcal{S}} \setminus \{b\} = \{a_2, a_5\}$ . Here  $\{a_1, a_2, a_3, a_5\}$  is a dominating set. But  $\{a_2, a_5\}$  is not a dominating set.

**Definition 2.8.** A set  $\tilde{\mathcal{S}} \subseteq V$  of an  $m$ -psf-graph  $\tilde{G}_{P,V}$  is called an independent set if

$$\begin{aligned} \tilde{\mu}_e x_1(uv) &< (\tilde{\rho}_e x_1(u) \wedge \tilde{\rho}_e x_1(v)) \\ \tilde{\mu}_e x_2(uv) &< (\tilde{\rho}_e x_2(u) \wedge \tilde{\rho}_e x_2(v)) \\ &\dots\dots\dots \\ \tilde{\mu}_e x_m(uv) &< (\tilde{\rho}_e x_m(u) \wedge \tilde{\rho}_e x_m(v)) \end{aligned}$$

in  $\tilde{H}_{P,V}(e) \forall e \in P$  and for all  $u, v \in \tilde{\mathcal{S}}$ .

**Definition 2.9.** An independent set  $\tilde{\mathcal{S}}$  of an  $m$ -psf-graph  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$  is said to be maximal independent set if for every vertex  $u \in V \setminus \tilde{\mathcal{S}}$ , the set  $\tilde{\mathcal{S}} \cup \{u\}$  is not independent in  $\tilde{H}_{P,V}(e) \forall e \in P$ .

**Definition 2.10.** Lower and upper independence number of an  $m$ -psf-graph  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$  is denoted by  $i(\tilde{G}_{P,V})$  and  $I(\tilde{G}_{P,V})$ , respectively, and defined by infimum cardinality and supremum cardinality among all the maximum independent set of that  $m$ -psf-graph, respectively.

**Example 2.4.** Consider a 3-psf-graph  $\tilde{G}_{P,V}$ .

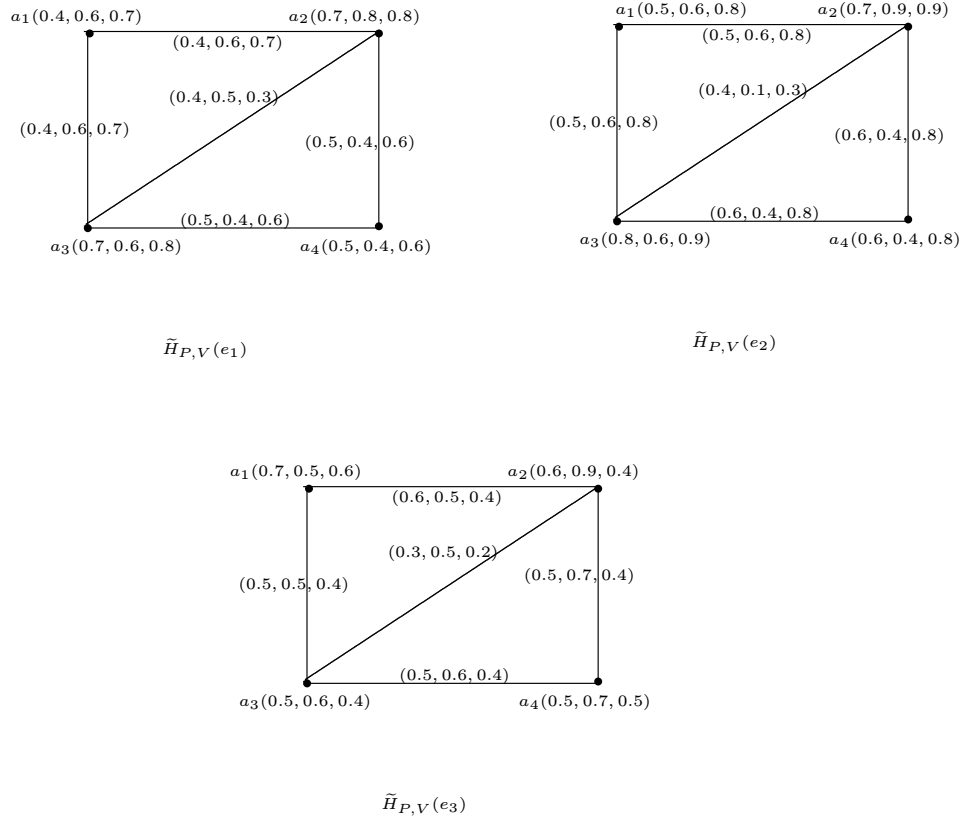


Fig.4. Independent set of a 3-psf-graphs

In this example, Here  $\{a_2, a_3\}$  is an independent set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, 3$ . Similarly, other independent sets in each parameterized graph is  $\{\{a_3, a_4\}, \{a_2, a_3, a_4\}, \{a_2, a_4\}, \{a_1, a_4\}\}$ . The maximal independent set in each parameterized graph is  $\{\{a_1, a_4\}, \{a_2, a_3, a_4\}\}$ . Here  $\{a_1, a_4\}$  is a maximal independent set of  $\tilde{G}_{P,V}$  with infimum cardinality and  $i(\tilde{G}_{P,V}) = (3.2, 3.2, 4.0)$ . Also here  $\{a_2, a_3, a_4\}$  is maximal independent set of  $\tilde{G}_{P,V}$  with supremum cardinality and  $I(\tilde{G}_{P,V}) = (5.6, 5.9, 6.1)$ .

**Proposition 2.1.** For any 3-psf-graph  $\tilde{G}_{P,V} = (G^*, \tilde{\rho}, \tilde{\mu}, P)$ ,  $\gamma(\tilde{G}_{P,V}) \leq i(\tilde{G}_{P,V})$ .

**Example 2.5.** Consider a 3-psf-graph.

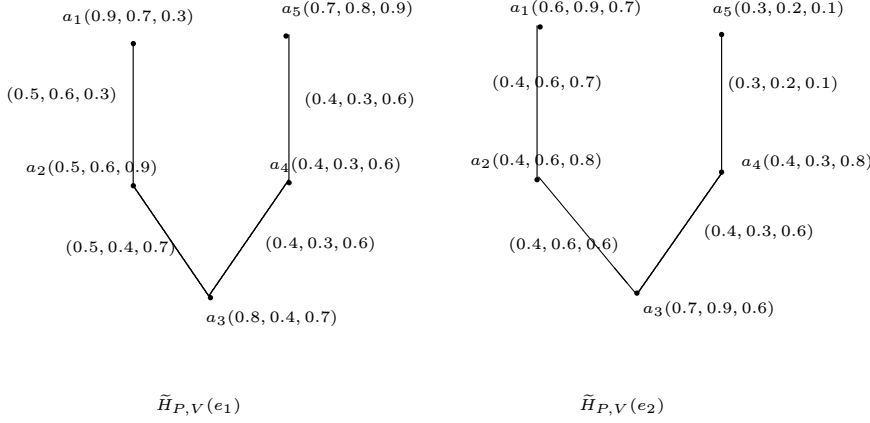


Fig.5. Independent set of a 3-psf-graphs

In Fig.5. that minimum dominating set of a 3-psf-graph  $\tilde{G}_{P,V}$  is  $\{a_2, a_4\}$  and the maximal independent set of  $\tilde{G}_{P,V}$  is  $\{a_1, a_3, a_5\}$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2$ . Also here  $\gamma(\tilde{G}_{P,V}) = (1.7, 1.8, 3.1)$  and  $i(\tilde{G}_{P,V}) = (4.0, 3.9, 3.3)$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2$ . Clearly,  $\gamma(\tilde{G}_{P,V}) \leq i(\tilde{G}_{P,V})$ .

**Theorem 2.4.** A set  $\tilde{\mathcal{S}} \subseteq V$  is a maximal independent set of a 3-psf-graph  $\tilde{G}_{P,V}$  if and only if it is independent and dominating set.

*Proof.* Assume that  $\tilde{\mathcal{S}}$  is a maximal independent set of  $\tilde{G}_{P,V}$ . Then for each vertex  $a \in V \setminus \tilde{\mathcal{S}}$ , the set  $\tilde{\mathcal{S}} \cup \{a\}$  is not independent set in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . In this manner for every vertex  $a \in V \setminus \tilde{\mathcal{S}}$ , then  $\exists$  vertex  $b \in \tilde{\mathcal{S}}$  such that  $b$  dominates  $a$  in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Hence  $\tilde{\mathcal{S}}$  is a dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Therefore  $\tilde{\mathcal{S}}$  is both independent and dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Hence  $\tilde{\mathcal{S}}$  is both independent and dominating set in  $\tilde{G}_{P,V}$ .

Conversely if we suppose that  $\tilde{\mathcal{S}}$  is not maximal independent set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Then  $\exists$  a vertex  $a \in V \setminus \tilde{\mathcal{S}}$ , such that  $\tilde{\mathcal{S}} \cup \{a\}$  is independent set. Thus there  $\nexists$  any vertex  $b$  in  $\tilde{\mathcal{S}}$  which dominates  $a$  in  $\tilde{H}_{P,V}(e_i)$  for  $i = 1, 2, \dots, n$ . So  $\tilde{\mathcal{S}}$  is not a dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Which  $\Rightarrow \Leftarrow$  to the choice of  $\tilde{\mathcal{S}}$ . Accordingly  $\tilde{\mathcal{S}}$  is a maximal independent set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Hence  $\tilde{\mathcal{S}}$  is a maximal independent set of  $\tilde{G}_{P,V}$ .  $\square$

**Theorem 2.5.** In a 3-psf-graph  $\tilde{G}_{P,V} = ((P, \tilde{\mu}), (P, \tilde{\rho}))$ , every maximal independent set is a minimal dominating set

*Proof.* For a maximal independent  $\tilde{S}$  of  $\tilde{G}_{P,V}$ . By Theorem 2.4,  $\tilde{S}$  is dominating set in  $\tilde{G}_{P,V}$ . If we consider  $\tilde{S}$  being not minimal dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Then  $\exists$  atleast one vertex  $a \in \tilde{S}$  so that  $\tilde{S} \setminus \{a\}$  is a dominating set that means  $\tilde{S} \setminus \{a\}$  dominates  $V \setminus (\tilde{S} \setminus \{a\})$ . Thus, there exists atleast one vertex in  $\tilde{S}$  which dominates  $a$ . This contradict our assumption. Therefore  $\tilde{S}$  is a minimal dominating set in  $\tilde{H}_{P,V}(e_i) \forall e_i \in P$  for  $i = 1, 2, \dots, n$ . Hence  $\tilde{S}$  is a minimal dominating set in  $G_{P,V}$ .  $\square$

### 3 Conclusions

The multiplicity of applications and vast range of domination parameters that can be defined make domination theory research attractive. The terms dominating set, independent set, dominance number etc. are used in this study that proposed the  $m$ -polar soft fuzzy graphs, and several intriguing findings have been demonstrated. In a similar situation, future studies can define and examine additional domination parameters.

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## Anti Q-M-fuzzy normal subgroups

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### Abstract

The fuzzy set has been applied in wide area by many researchers. We define the concept of anti-homomorphism in Q-fuzzy subgroups and Q-fuzzy normal subgroups and establish some result in this research article and develop some theory of anti-homomorphism in Q-fuzzy subgroups, normal subgroups and also extend results on Q-fuzzy abelian subgroup and Q-fuzzy normal subgroup. Many research scholars completed their research in field of fuzzy subgroup, anti fuzzy subgroup, Q-fuzzy subgroup, anti Q-fuzzy subgroup, homomorphism, anti homomorphism etc.

**Keywords:** Q-M-Fuzzy subgroup, Q-M-Fuzzy Normal Subgroups, Anti Q-M-Fuzzy Normal Subgroups, group Q-M-Homomorphism and group anti Q-M-Homomorphism.

**AMS Subject Classification:** 03E72, 03E75, 08A72 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1071. ISSN: 1592-7415. eISSN: 2282-8214. ©S.Palaniyandi et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

According to Zadeh.L.A Rosenfeld [1971] was introduce fuzzy sets. It has subsequently been employed in a variety of scientific domains, including engineering, social science, medicine, and pure and applied mathematics. Rosenfeld developed the concept of fuzzy subgroups Asaad [1991]. Biswas.R proposed anti-fuzzy subgroups in Biswas [1990]. Solairaju.A and Nagarajan.R pioneered the structure of Q-fuzzy groups Palaniappan and Muthuraj [2004]. Jacobson.N was the first to use the words M-group and M-subgroup in Jacobson [1951]. In this study, we present and discuss the concepts of group Q- M homomorphism, group anti Q- M homomorphism, and anti Q- M-fuzzy normal subgroup of M-group.

## 2 Preliminaries

**Definition 2.1.** Let  $X \neq \phi$ . Let  $X$  is fuzzy  $\theta \subseteq \theta : X \in [0, 1]$ .

**Definition 2.2.** Zadeh [1965] A fuzzy  $\subseteq \theta \leq N$ .  
It is satisfying the axioms,

- (i)  $\theta(\alpha\beta) \geq \text{lower}\{\theta(\alpha), \theta(\beta)\}$
- (ii)  $\theta(\alpha^{-1}) = \theta(\alpha), \forall \alpha, \beta \in N$ .

**Definition 2.3.** Biswas [1990] A fuzzy  $\subseteq \theta$  of a group  $G$  is said to be anti fuzzy  $\subseteq G$  if it is satisfying the following conditions,

- (i)  $\theta(uv) \leq \text{lower}\{\theta(u), \theta(v)\}$
- (ii)  $\theta(u^{-1}) = \theta(u), \forall u, v \in G$ .

**Definition 2.4.** Biswas [1990] Let  $G: M$  and  $\theta \subseteq G$ . Then  $\theta$  is called M-fuzzy  $\subseteq G$  if  $\forall u \in G$  and  $m \in M$ , then  $\theta(mx) \leq \theta(u)$

**Definition 2.5.** Jacobson [1951] A Q-fuzzy set  $\theta$  is Q - fuzzy  $\leq G$  if  $\forall u, v \in G$ , and  $\rho \in Q$

- (i)  $\theta(uv, \rho) \geq \text{lower}\{\theta(u, \rho), \theta(v, \rho)\}$
- (ii)  $\theta(u^{-1}, \rho) = \theta(u, \rho)$

**Definition 2.6.** Solairaju and Nagarajan [2009] Let fuzzy  $\lambda \subseteq X$ . For  $t \in [0, 1] \subseteq$   
 $\theta$  is denoted by  $[\theta_t = \{u \in U : \theta_\lambda(u) \geq t\}]$

**Definition 2.7.** Sithar Selvam et al. [2014] A Q-fuzzy set  $\theta$  is called  $Q \leq G$  if  $\forall u, v \in G$ , and  $\rho \in Q$  in anti Q fuzzy.



$$(i) \theta(uv, \rho) \leq \text{lower}\{\theta(u, \rho), \theta(v, \rho)\}$$

$$(ii) \theta(u^{-1}, \rho) = \theta(u, \rho)$$

**Definition 2.8.** Sithar Selvam et al. [2014] An antifuzzy normal  $Q \leq G$ . Then  $G \nearrow \theta$  of  $G$  if  $\forall x, y \in G$  and  $\rho \in Q$ ,  $\theta(uyx^{-1}, \rho) = \theta(v, \rho)$ .

### 3 Anti Q-M- fuzzy normal subgroups and its level subsets

**Definition 3.1.** Let  $\theta$  be anti fuzzy  $Q - M - \leq M - \text{group}G$ , then  $\theta \leq M(G)$  if  $\forall u, v \in G, \rho \in Q$ , and  $m \in M$  such that

$$\theta(m(uvu^{-1}), \rho) = \theta(m(v), \rho) \text{ (or) } \theta(m(uv), \rho) = \theta(m(vu), \rho).$$

**Definition 3.2.** Let  $\theta$  be anti fuzzy  $Q - M \leq M - \text{group}G$ . For any  $t \in [0, 1]$ , the subset  $\theta_t$  is defined by  $\theta_t = \{u \in G, \rho \in Q, m \in M \mid \theta(m(u), \rho) \leq t\}$  and it is the subset of  $\theta$ .

**Theorem 3.1.** If  $\theta$  is a fuzzy  $Q - M - \subseteq$  of a  $M$ -group  $G$ , then  $\theta$  is an anti fuzzy  $Q - M - \leq M - \text{group}G$  iff the level subset  $\theta_t, t \in [0, 1]$  is subgroup of  $M$ -group  $G$ .

*Proof.* Let us assume that  $\theta$  is an anti fuzzy  $Q - M - \leq M - \text{group}G$ .

The level subset  $\theta_t = \{u \in G, \rho \in Q, m \in M \mid \theta(m(u), \rho) \leq t, t \in [0, 1]\}$ .

Let  $u, v \in \theta_t$ , then  $\theta(mu, \rho) \leq t$  and  $\theta(mv, \rho) \leq t$

Now

$$\begin{aligned} \theta(m(uy^{-1}), \rho) &\leq \text{upper}\{\theta(mu, \rho), \theta(m(v^{-1}), \rho)\} \\ &= \text{upper}\{\theta(mu, \rho), \theta(mv, \rho)\} \\ &\leq \text{upper}\{t, t\} \end{aligned}$$

Thus

$$\theta(m(uv^{-1}), \rho) \leq t$$

Hence  $xy^{-1} \in \theta_t$ . Therefore  $\theta_t \leq M(G)$ .

Conversely, Let  $\theta_t$  be a subgroup of a  $M$ -group  $G$ .

Let  $u, v \in \theta_t$ . Then  $\theta(mu, \rho) \leq t$  and  $\theta(mv, \rho) \leq t$ .

$$\begin{aligned} \Rightarrow \theta(m(uv^{-1}), \rho) &\leq t, \text{ Because } \{uv^{-1} \in \theta_t\} \\ &= \text{upper}\{t, t\} \\ &= \text{upper}\{\theta(mu, \rho), \theta(mv, \rho)\} \end{aligned}$$

Therefore

$$\theta(m(uv^{-1}), \rho) \leq \text{upper}\{\theta(mu, \rho), \theta(mv, \rho)\}$$

Hence  $\theta$  is an anti fuzzy  $Q \leq M - \text{group}G$ .  $\square$

**Definition 3.3.** Let  $\theta$  be a anti fuzzy  $Q - M \leq m - \text{group}G$ . The set  $N(\theta)$  is defined by  $N(\theta) = \{\alpha \in G \mid \theta(m(\alpha u a^{-1}), \rho) = \theta(mu, \rho) \forall u \in G \text{ and } \rho \in Q, m \in M. \text{ and it is called an anti fuzzy } Q\text{-}M\text{-normalizer of } \theta.$

**Theorem 3.2.** If  $\theta$  is a fuzzy  $Q - M \leq M - \text{group}G$ . Then  $\theta$  is an anti fuzzy  $Q - M - \text{fuzzy} \leq M - \text{group}G$  iff the level subsets  $\theta_t, t \in [0, 1] \leq M(G)$ .

*Proof.* Let us assume that  $\theta \leq Q - M - \text{antifuzzynormalsubgroup of a } M(G)$  and the level subsets  $\theta_t, t \in [0, 1]$  is a subgroup of a  $M$ -group  $G$ .

We take  $u \in G$  and  $\alpha \in \theta_t$ , then  $\theta(ma, \rho) \leq t$

Now  $\theta(m(\alpha x a^{-1}), \rho) = \theta(ma, \rho) \leq t$ .

Since  $\theta$  is an anti fuzzy normal  $Q\text{-}M \leq M(G)$ ,  $\theta(m(uau^{-1}), \rho) \leq t$

Therefore  $u\alpha u^{-1} \in \theta_t$ , hence  $\theta_t \leq M(G)$ .  $\square$

**Theorem 3.3.** If  $\theta$  is an  $\leq Q - M - \text{antifuzzynormalsubgroup of a } M(G)$  Then

(i)  $N(\theta) \leq M(G)$ .

(ii)  $\theta$  is an normal anti fuzzy  $-Q\text{-}M \leq$  iff  $N(\theta) = G$ .

(iii)  $\theta$  is an normal fuzzy anti  $-Q - M \leq N(\theta)$ .

*Proof.* Let  $\alpha, \beta \in N(\theta)$ . (i) Then  $\theta(m(\alpha u a^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M$  and  $\theta(m(\beta x \beta^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M$ . Now

$$\begin{aligned} \theta(m(\alpha \beta u (\alpha \beta)^{-1}), \rho) &= \theta(m(\alpha \beta u \beta^{-1} \alpha^{-1}), \rho) \\ &= \theta(m(\beta u \beta^{-1}), \rho) \\ &= \theta(mu, \rho) \end{aligned}$$

Then we have,  $\theta(m(\alpha \beta u (\alpha \beta)^{-1}), \rho) = \theta(mu, \rho)$

$\Rightarrow \alpha \beta \in N(\theta)$

Therefore  $N(\theta) \leq M(G)$ .

(ii) We know that

$$N\theta \subseteq G, \tag{1}$$

$\theta$  is an normal fuzzy anti- $Q - M \leq G$ .

Let  $\alpha \in G$ , then  $\theta(m(\alpha u \alpha^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M$ .

Then

$$\alpha \in N(\theta) \Rightarrow G \subseteq N(\theta) \tag{2}$$

### Anti Q-M-Fuzzy Normal Subgroups

From (1)&(2), we get

$$N(\theta) = G$$

Conversely, assume that  $N(\theta) = G$

We have,  $\theta(m(\alpha x \alpha^{-1}), \rho) = \theta(mx, \rho) \forall \alpha, x \in G, \rho \in Q, m \in M$ .

Therefore  $\theta$  is an fuzzy normal anti  $Q - M \leq M(G)$ .

(iii) Let  $\theta$  be an fuzzy normal anti  $Q - M \leq M$

We take  $\alpha \in G$ , then we have

$$\theta(m(\alpha u \alpha^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M.$$

Therefore  $\alpha \in N(\theta) \Rightarrow G \subseteq N(\theta)$ .

Hence  $\theta$  is an fuzzy normal anti  $Q - M \leq N(\theta)$  □

**Theorem 3.4.** Let  $\theta$  be an fuzzy normal anti  $Q - M \leq M(G)$ , then  $h\theta h^{-1}$  is also an fuzzy normal anti  $Q - M \leq M(G) \forall h \in G, \rho \in Q, m \in M$ .

*Proof.* Given  $\theta \leq M(G) \leq M(G)$

$$\begin{aligned} (i) (h\theta h^{-1})(m(uv), \rho) &= \theta(m(h^{-1}(uv)h), \rho) \\ &= \theta(m(h^{-1}(uhh^{-1}v)h), \rho) \\ &= \theta(m((h^{-1}uh)(h^{-1}vh)), \rho) \\ &\leq \text{upper}\{\theta(m(h^{-1}uh), \rho), \theta(m(h^{-1}vh), \rho)\} \\ &\leq \text{upper}\{h\theta h^{-1}(mu, \rho), h\theta h^{-1}(mv, \rho)\} \end{aligned}$$

$\forall u, v \in G, \rho \in Q$  and  $m \in M$ .

$$\begin{aligned} (ii) h\theta h^{-1}(mu, \rho) &= \theta(m(h^{-1}uh), \rho) \\ &= \theta(m(h^{-1}uh)^{-1}, \rho) \\ &= \theta(m(h^{-1}u^{-1}h), \rho) \\ &= h\theta h^{-1}(mu^{-1}, \rho) \end{aligned}$$

$\forall u, v \in G, \rho \in Q, m \in M$ .

Therefore  $h\theta h^{-1}$  is an fuzzy anti  $Q - M \leq M(G)$ . □

**Theorem 3.5.** Let  $\theta$  fuzzy anti  $Q - M \leq M(G)$ , then  $h\theta h^{-1}$  fuzzy anti  $Q - M \leq M(G), \forall h \in G, \rho \in Q, m \in M$ .

*Proof.* Given  $\theta$  is an anti-Q-M-fuzzy normal subgroup of M-group G.

Then  $h\theta h^{-1} \leq G$ .

Now

$$\begin{aligned} h\theta h^{-1}(m(uvu^{-1}), \rho) &= \theta(m(h^{-1}(uvu^{-1})h), \rho) \\ &= \theta(m(uvu^{-1}), \rho) \\ &= \theta(mv, \rho) \\ &= \theta(m(hvh^{-1}), \rho) \\ &= h\theta h^{-1}(mv, \rho) \end{aligned}$$

Therefore  $h\theta h^{-1}$  is also an fuzzy normal anti  $Q - M \leq M(G)$ .  $\square$

**Theorem 3.6.** *The disjoint two fuzzy normal anti  $Q - M \leq M(G)$  is also an anti fuzzy anti  $Q - M \leq M(G)G$ .*

*Proof.* Let  $\alpha$  and  $\beta$  be two anti-Q-M-fuzzy subgroups of a M-group G. Then

$$\begin{aligned} (\alpha\beta)(m(uv^{-1}), \rho) &= \text{lower}\{\alpha(m(uv^{-1}), \rho), \beta(m(uv^{-1}), \rho)\} \\ &\leq \text{lower}\{\text{upper}\{\alpha(c(u), \rho), \alpha(m(v^{-1}), \rho)\} \\ &\quad , \text{upper}\{\beta(c(u), \rho), \beta(m(v^{-1}), \rho)\}\} \\ &\leq \text{lower}\{\text{upper}\{\alpha(cu, \rho), \alpha(mv, \rho)\}, \\ &\quad \text{upper}\{\beta(cu, \rho), \beta(mv, \rho)\}\} \\ &\leq \text{upper}\{\text{lower}\{\alpha(cu, \rho), \alpha(mv, \rho)\}, \\ &\quad \text{lower}\{\beta(cu, \rho), \beta(mv, \rho)\}\} \end{aligned}$$

Therefore  $\{(\alpha\beta)(c(uv^{-1}), \rho)\} \leq \text{upper}\{(\alpha\beta)(cu, \rho), (\alpha\beta)(cv, \rho)\}$

Hence  $\alpha\beta$  is an fuzzy normal anti  $Q - M \leq M(G)$ .  $\square$

**Theorem 3.7.** *If C and D are an anti fuzzy normal  $Q-M \leq M(G)$ . Then  $A \cap B$  is anti fuzzy normal  $Q-M \leq M(G)$ .*

*Proof.* For any  $x, y \in G, q \in Q, m \in M$  We have

$$\begin{aligned} (C \cap D)(m(xyx^{-1}), q) &= \text{upper}\{C(m(xyx^{-1}), q), D(m(xyx^{-1}), q)\} \\ &= \text{upper}\{C(my, q), D(my, q)\} \\ &= (C \cap D)(my, q) \end{aligned}$$

Hence  $C \cap D$  is an anti fuzzy normal  $Q-M \leq M(G)$ .  $\square$

## 4 Group Q-M- homomorphism and group anti Q-M- homomorphism

**Definition 4.1.** *The function  $f : G \times Q \rightarrow H \times Q$  is homomorphism group Q-M*

(i)  $f : G \rightarrow H$  is a homomorphism group and

(ii)  $f(m(uv), \rho) = (f(mv).f(mu), \rho) \forall u, v \in G, \rho \in Q, m \in M$ . where G and H are M-groups.

**Definition 4.2.** *The function  $f : G \times Q \rightarrow H \times Q$  is anti homomorphism group of Q-M if*

### Anti Q-M-Fuzzy Normal Subgroups

(i)  $f : G \rightarrow H$  is homomorphism group

(ii)  $f(m(uv), \rho) = (f(mx).f(mv), \rho) \forall u, v \in G, \rho \in Q, m \in M$ .

**Theorem 4.1.** *If the function  $f : G \times Q \rightarrow H \times Q$  is a group anti Q-M-homomorphism*

(i) *If  $\theta$  is an anti Q-M-fuzzy normal subgroup of  $H$ , then  $f^{-1}(\theta)$  is an fuzzy normal anti  $Q - M \leq M(G)$ .*

(ii) *If  $f$  is an epimorphism and  $\theta$  is an fuzzy normal anti  $Q - M \leq M(G)$  then  $f(\theta)$  is an anti normal fuzzy  $Q-M \leq H$ . Where  $G$  and  $H$  are  $M$ -groups.*

*Proof.* (i) Given the function  $f : G \times Q \rightarrow H \times Q$  is a group anti-Q-M-homomorphism and  $\theta$  is an anti normal fuzzy  $Q-M \leq H$ .

For all  $u, v \in G, \rho \in Q, m \in M$  we have,

$$\begin{aligned} f^{-1}(\theta)(m(uvu^{-1}), \rho) &= \theta(f(m(uvu^{-1})), \rho) \\ &= \theta(fm(u^{-1}).f(mv).f(mu), \rho) \\ &= \theta(f(mv), \rho) \\ &= f^{-1}(mv, \rho) \end{aligned}$$

Hence  $f^{-1}(\theta)$  is an fuzzy normal anti  $Q - M \leq M(G)$ .

(ii) Given  $\theta$  is an fuzzy normal anti  $Q - M \leq M(G)$ . Then  $f(\theta)$  is an anti Q-M-fuzzy subgroup of  $H$ .

For any  $\alpha, \beta \in H$ , we have

$$\begin{aligned} f(\theta)(m(\alpha\beta\alpha^{-1}), \rho) &= \inf \theta(mv, \rho) = \inf \theta(m(uvu^{-1}), \rho) \\ f(v) &= \alpha\beta\alpha^{-1} = \inf \theta(mv, \rho) \\ f(u) = a, f(v) = \beta &= f(\theta)(mb, \rho) \quad (\text{Since } f \text{ is an epimorphism}) \end{aligned}$$

Therefore  $f(\theta)$  is an fuzzy normal anti  $Q - M \leq H$ . □

**Definition 4.3.** *Let  $A$  and  $B$  be two fuzzy anti  $Q - M \leq M(G)$ . The Product of  $A$  and  $B$  is defined by  $AB(m(u), \rho) = \inf_{upper} (A(mv, \rho), v = u, B(mz, \rho))u \in G, \rho \in Q, m \in M$ .*

**Theorem 4.2.** *If  $A$  and  $B$  are fuzzy normal anti  $Q - M \leq M(G)$ , then  $AB$  is an fuzzy normal anti  $Q - M \leq G$ .*

*Proof.* Given A and B are two fuzzy normal anti  $Q - M \leq M(G)$ .

$$\begin{aligned}
 (i) \quad AB(m(uv), \rho) &= \inf \text{upper}\{A(m(u_1y_1), \rho), B(m(u_2y_2), \rho)\} \\
 &\quad \text{where } u = u_1y_1 \text{ and } v = u_2y_2 \\
 &\leq \inf \text{upper}\{\text{upper}\{A(mu_1, \rho), A(mv_1, \rho)\}, \\
 &\quad \text{upper}\{B(mu_2, \rho), B(mv_2, \rho)\}\} \\
 &\leq \text{upper}\{\inf \text{upper}\{A(mu_1, \rho), A(mv_1, \rho)\}, \\
 &\quad \inf \text{upper}\{B(mu_2, \rho), B(mv_2, \rho)\}\} \\
 \text{i.e., } AB(m(uv), \rho) &\leq \text{upper}\{AB(m(u_1y_1), \rho), AB(m(u_2y_2), \rho)\}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad AB(m(u^{-1}), \rho) &= \inf \text{upper}\{B(m(z^{-1}), \rho), A(m(v^{-1}), \rho)\} \\
 &\quad \text{where } (vz)^{-1} = u^{-1} \\
 &= \inf \text{upper}\{B(mz, \rho), A(mv, \rho)\} \\
 &= \inf \text{upper}\{A(mv, \rho), B(mv, \rho)\} \\
 &= AB(mv, \rho). \\
 AB(mv^{-1}, \rho) &= AB(mv, \rho)
 \end{aligned}$$

Hence AB is anti fuzzy normal  $Q-M \leq M(G)$ . □

## 5 Conclusions

In this research article, we gave some results of anti  $Q-M$ -fuzzy normal subgroup, Group  $Q-M$  homomorphism and Group anti  $Q-M$  homomorphism. This article used to further research in fuzzy algebra.

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*Anti Q-M-Fuzzy Normal Subgroups*

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# A dynamical analysis of a mathematical model on type-2 diabetic from obesity

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## Abstract

A model for type-2 diabetic from obesity with two control variables diet with physical activity and medication is formulated. The disease free and endemic equilibrium points of the model are obtained. The existence of optimal controls is verified through Pontryagin's maximum principle. The local stability is analyzed using Routh-Hurwitz criteria. The global stability is studied using Lyapunov function. The parameters are chosen based on the female population in India. The aim of this research is to construct a model for type-2 diabetic from obesity using parameters based on the female population in India. We have introduced two control variables as diet with physical activity and medication. The positive endemic equilibrium is obtained. The local and global stability of the model are analyzed with some specific conditions. Numerical simulations are carried out to exhibit the flow of variables with controls. Our study mainly highlights the awareness of metabolic risk by healthy diet, physical activities and medications.

**Keywords:** Obesity, Diabetic, Pontryagin's maximum principle, Lyapunov function, Stability.

**2020 AMS subject classifications:** 34D20, 34D23, 81T80 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1072. ISSN: 1592-7415. eISSN: 2282-8214. ©G. Gnana Priya et al. This paper is published under the CC-BY licence agreement.



## **1 Introduction**

Obesity and Diabetes are two epidemics that are wreaking havoc around the world. Obesity puts you at a much increased chance of having diabetes throughout the course of your life. Obesity is more than a cosmetic issue. It is a long-term medical condition that can progress to diabetes. Obese people account for over 90 percent of people with type 2 diabetes which is characterised by the body's failure to regulate and control blood sugar levels. Obesity allows diabetes to take up residence in the body. Obesity increases the risk of developing diabetes which is characterised by an excess of glucose in the bloodstream. Obesity is linked to a higher risk of non communicable diseases such as diabetes, heart disease and hypertension. Obesity is one of the primary causes of early death and 4.7 million deaths worldwide was recorded in 2017 due to obesity. The key to understanding the link between obesity and diabetes is to look at insulin resistance. Physical inactivity, overeating, heredity, medications and psychological factors are the most prevalent causes of obesity. Type 2 diabetes has no known treatment however, it can be in control. Inactivity, family history, high blood pressure and obese are the most common causes of type 2 diabetics. Obesity reduction is a significant objective for the treatment and prevention of type 2 diabetes. Many authors studied a mathematical models for obesity and diabetic. A Boutayed Boutayed et al. [2010] developed a mathematical model for the burdern of diabetes and its complications. Wiam Boutayeb et al. [2014] introduced a mathematical model for the impact of obesity on predisposed people to type 2 diabetes. Nikhil Kumar et al. [2017] studied a diabetsity an epidemic with its causes, prevention and control with special focus on dietary regime. Miguel A. Ortega Ortega et al. [2020] illustrated a technique for type 2 diabetes mellitus associated with obesity. Abdelfatah Kouidere K.Abdelfatah et al. [2020] developed a mathematical model for the dynamics of population of diabetics and its complications with effect of behavioral factors. Wellars Banzi et al. [2021] studied a mathematical model for Glucose-Insulin System and test of abnormalities of type 2 diabetic patients. In this paper, we formulated and anlaysed a mathematical model for type-2 diabetic from obesity with control variables.

## **2 Formulation of model**

The system of Ordinary Differential Equation represents the Obesity and Diabetic model as follows::

$$\begin{aligned}
 \frac{dS}{dt} &= \omega N - (\beta + \mu)S \\
 \frac{dO}{dt} &= \beta S - (\alpha + \mu + u)O \\
 \frac{dD}{dt} &= \alpha O - (\mu + u + v)D \\
 \frac{dR}{dt} &= (u + v)D + uO - \mu R
 \end{aligned} \tag{1}$$

With the initial conditions as  $S(t), O(t), D(t), R(t) \geq 0$ . Also  $\beta, \mu, \omega, \alpha, u, v > 0$ .

Where  $S(t), O(t), D(t), R(t)$  are the Susceptible, Infected by Obesity, Infected by Diabetic and Recovery state respectively and  $\omega$  - Average birth rate of female,  $\mu$  - Average death rate of female,  $\beta$  - Obesity rate,  $\alpha$  - Diabetic rate,  $N(t)$  - Female Population. The control variables are  $u$  - Diet with Physical Activity and  $v$  - Medication.

The following figure shows the considered Obesity and Diabetic model:

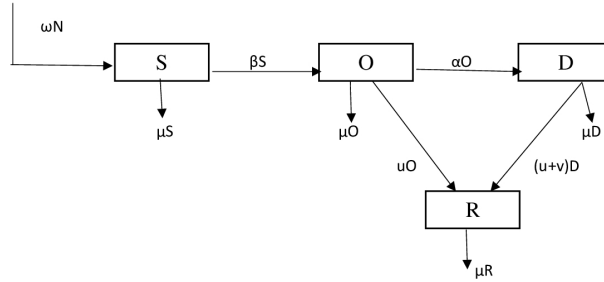


Figure 1: Obesity and Diabetic model

### 3 Equilibrium analysis

The steady states are  $G_0(0, 0, 0, 0)$ ,  $G_1(\bar{S}, 0, 0, 0)$ ,  $G_2(\tilde{S}, \tilde{O}, 0, 0)$ ,  $G_3(S', O', D', 0)$  and  $G_4(S^*, O^*, D^*, R^*)$ .

Case(1):  $G_0(0, 0, 0, 0)$  exists always.

Case(2): In  $G_1(\bar{S}, 0, 0, 0)$ ,

Let  $\bar{S}$  be the positive solution of  $\frac{dS}{dt} = 0$

From (1),

$$\bar{S} = \frac{\omega N}{(\beta + \mu)} > 0$$

$$\bar{S} = \frac{\omega N}{(\beta + \mu)} > 0$$

$$\therefore G_1(\bar{S}, 0, 0, 0) = \left( \frac{\omega N}{(\beta + \mu)}, 0, 0, 0 \right)$$

Case(3): In  $G_2(\tilde{S}, \tilde{O}, 0, 0)$ ,

Let  $\tilde{S}, \tilde{O}$  be the positive solutions of  $\frac{dS}{dt} = 0, \frac{dO}{dt} = 0$ .

From (1),

$$\tilde{S} = \frac{\omega N}{(\beta + \mu)} > 0$$

$$\tilde{O} = \frac{\beta \omega N}{(\beta + \mu)(\alpha + \mu + u)} > 0$$

$$\therefore G_2(\tilde{S}, \tilde{O}, 0, 0) = G_2\left(\frac{\omega N}{(\beta + \mu)}, \frac{\beta \omega N}{(\beta + \mu)(\alpha + \mu + u)}, 0, 0\right)$$

Case(4): In  $G_3(S', O', D', 0)$ ,

Let  $S', O'$  and  $D'$  be the positive solutions of  $\frac{dS}{dt} = 0, \frac{dO}{dt} = 0, \frac{dD}{dt} = 0$

From (1),

$$S' = \frac{\omega N}{(\beta + \mu)}$$

$$O' = \frac{\beta \omega N}{(\beta + \mu)(\alpha + \mu + u)}$$

$$D' = \frac{\alpha \beta \omega N}{(\beta + \mu)(\alpha + \mu + u)(\mu + u + v)}$$

$$\therefore G_3(S', O', D', 0) = \left( \frac{\omega N}{\beta + \mu}, \frac{\beta \omega N}{(\beta + \mu)(\alpha + \mu + u)}, \frac{\alpha \beta \omega N}{(\beta + \mu)(\alpha + \mu + u)(\mu + u + v)}, 0 \right)$$

Case(5): For  $G_4(S^*, O^*, D^*, R^*)$ ,

Let  $S^*, O^*, D^*$  and  $R^*$  be the positive solutions of  $\frac{dS}{dt} = 0, \frac{dO}{dt} = 0, \frac{dD}{dt} = 0, \frac{dR}{dt} = 0$

From (1),

$$S^* = \omega N \left( \frac{1}{\beta + \mu} \right)$$

$$O^* = \omega N \left( \frac{\beta}{(\alpha + \mu + u)(\beta + \mu)} \right)$$

$$D^* = \omega N \left( \frac{\alpha\beta}{(\alpha + \mu + u)(\beta + \mu)(\mu + u + v)} \right)$$

$$R^* = \omega N \left[ \frac{1}{\mu} \left( \frac{\alpha\beta(u + v)}{(\alpha + \mu + u)(\beta + \mu)(\mu + u + v)} + \frac{\beta u}{(\alpha + \mu + u)(\beta + \mu)} \right) \right]$$

$\therefore$  the diabetes equilibrium is given by

$$G_4(S^*, O^*, D^*, R^*)$$

$$= (\omega N \left( \frac{1}{\beta + \mu} \right), \omega N \left( \frac{\beta}{(\alpha + \mu + u)(\beta + \mu)} \right),$$

$$\omega N \left( \frac{\alpha\beta}{(\alpha + \mu + u)(\beta + \mu)(\mu + u + v)} \right),$$

$$\omega N \left[ \frac{1}{\mu} \left( \frac{\alpha\beta(u + v)}{(\alpha + \mu + u)(\beta + \mu)(\mu + u + v)} + \frac{\beta u}{(\alpha + \mu + u)(\beta + \mu)} \right) \right])$$

## 4 Pontryagin's maximum principle

We compute the optimal controls of the control variables by using Pontryagin's maximum principle. We prove the following four properties :

I: The set of controls and state variables that correspond to them is non empty.

II: U is a closed and convex control set.

III: The state system's R.H.S is bounded by a linear function in the state and control variables.

IV: The integrand of the objective functional is convex on U and is bounded below by  $k_2 + k_1|U|^\eta$ , with  $k_1 > 0$ ,  $k_2 > 0$ ,  $\eta > 1$ .

**To Prove I:**

From (1),

$$\begin{aligned} F_1 &= \omega N - (\beta + \mu)S \\ F_2 &= \beta S - (\alpha + \mu + u)O \\ F_3 &= \alpha O - (\mu + u + v)D \\ F_4 &= (u + v)D + uO - \mu R \end{aligned} \tag{2}$$

From (2)

$$\begin{aligned} \left| \frac{\partial F_1}{\partial S} \right| &< \infty, \left| \frac{\partial F_1}{\partial O} \right| < \infty, \left| \frac{\partial F_1}{\partial D} \right| < \infty, \left| \frac{\partial F_1}{\partial R} \right| < \infty \\ \left| \frac{\partial F_2}{\partial S} \right| &< \infty, \left| \frac{\partial F_2}{\partial O} \right| < \infty, \left| \frac{\partial F_2}{\partial D} \right| < \infty, \left| \frac{\partial F_2}{\partial R} \right| < \infty \\ \left| \frac{\partial F_3}{\partial S} \right| &< \infty, \left| \frac{\partial F_3}{\partial O} \right| < \infty, \left| \frac{\partial F_3}{\partial D} \right| < \infty, \left| \frac{\partial F_3}{\partial R} \right| < \infty \\ \left| \frac{\partial F_4}{\partial S} \right| &< \infty, \left| \frac{\partial F_4}{\partial O} \right| < \infty, \left| \frac{\partial F_4}{\partial D} \right| < \infty, \left| \frac{\partial F_4}{\partial R} \right| < \infty \end{aligned}$$

As it is continuous and bounded then there exist a unique solution of (1) by Existence and uniqueness theorem.

Hence, I is satisfied.

**To prove II:**

The control set is given by,

$$U = \{(u(t), v(t)) / 0 \leq u \leq 1, 0 \leq v \leq 1, t \in [0, 1]\}$$

U is closed, from our definition.

Let  $u, v \in U$ ,  $\alpha \in [0, 1]$

Then  $\alpha u + (1 - \alpha)v \geq 0$

As  $u \leq 1$  and  $v \leq 1$ ,

$0 \leq \alpha u + (1 - \alpha)v \leq (1 - \alpha)$ , for all  $u, v \in U$

Hence II is satisfied.

**To prove III:**

From (2),

$$F_1 \leq \omega N$$

$$F_2 \leq \beta S - uO$$

$$F_3 \leq \alpha O - (u + v)D$$

$$F_4 \leq (u + v)D + uO$$

It can be rewritten as follows:

$$\bar{F}(t, \bar{X}, U) \leq \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} S \\ O \\ D \\ R \end{bmatrix} + \begin{bmatrix} 0 \\ -O \\ -D \\ D + O \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Which can be written as the linear combination of controls

$$\begin{aligned} |\bar{F}(t, \bar{X}, U)| &\leq \|\bar{m}\| |\bar{X}| + \|D + O\| |u(t), v(t)| \\ &\leq K [|\bar{X}| + |u(t), v(t)|] \end{aligned}$$

As  $\bar{S}, \bar{O}, \bar{D}$  are bounded and  $K$  is upper bound.

$\therefore$  III is satisfied.

**To Prove IV:**

The control variables and state variables are non negative  $u, v \in U$ .

$u, v$  is convex and closed.

Then from (3),

$$J = AI(t) + \frac{\omega_1 u^2 + \omega_2 v^2}{2}$$

$$\begin{aligned} &\geq -AI(t) + \frac{\omega}{2}(u^2 + v^2) \\ &= -k_2 + k_1(u, v)^2 \end{aligned}$$

where  $\omega = \omega_1 + \omega_2$ ,  $k_1 = \frac{\omega}{2}$ .

Here  $k_2 > 0$ ,  $k_1 > 0$  and  $\eta = 2 > 1$

Hence IV is satisfied.

Hence there exists an optimal control variables which is found in the following section.

## 5 Existence of optimal control

The objective functional is defined as in (2) as follows:

$$J = \min AI(t) + \frac{1}{2} \int_0^T (\omega_1 u^2 + \omega_2 v^2) dt \quad (3)$$

Where  $AI(t)$  is the number of individuals,  $\omega_1, \omega_2$  are the weight parameters for the cost of diet with physical activity and the cost of medication respectively.

From previous section, our system (1) converts into a problem of minimizing.

An Hamiltonian  $H$  is defined as,

$$\begin{aligned} H(S, O, D, R) = & AI(t) + \frac{\omega_1 u^2}{2} + \frac{\omega_2 v^2}{2} + \lambda_S (\omega N - (\beta + \mu)S) \\ & + \lambda_O (\beta S - (\alpha + \mu + u)O) + \lambda_D (\alpha O - (\mu + u + v)D) \\ & + \lambda_R ((u + v)D + uO - \mu R) \end{aligned} \quad (4)$$

The adjoint system is given by

$$\begin{aligned} \frac{d\lambda_S}{dt} &= \lambda_S(\beta + \mu) - \lambda_O\beta \\ \frac{d\lambda_O}{dt} &= \lambda_O(\alpha + \mu + u) - \alpha\lambda_D - \lambda_R u \\ \frac{d\lambda_D}{dt} &= \lambda_D(\mu + u + v) - \lambda_R(u + v) \\ \frac{d\lambda_R}{dt} &= \lambda_R\mu \end{aligned}$$

with the final condition  $\lambda_{S(T)} = \lambda_{O(T)} = \lambda_{D(T)} = \lambda_{R(T)} = 0$  for free problem.

The optimality is found from  $\frac{\partial H}{\partial u} = 0$  and  $\frac{\partial H}{\partial v} = 0$ .

$\therefore$  The optimal controls are,

$$u^* = \min \left( 1, \max \left( 0, \frac{D(\lambda_D - \lambda_R) - O\lambda_R}{\omega_1} \right) \right), v^* = \min \left( 1, \max \left( 0, \frac{D(\lambda_D - \lambda_R)}{\omega_2} \right) \right)$$

Here  $u^*, v^*$  are positive when  $\lambda_D > \lambda_R$  and  $D(\lambda_D - \lambda_R) > O\lambda_R$ .

## 6 Local stability

The Jacobian matrix for the system (1) is

$$\begin{pmatrix} -(\beta + \mu) & 0 & 0 & \delta \\ \beta & -(\alpha + \mu + u) & 0 & 0 \\ 0 & \alpha & -(\mu + u + v) & 0 \\ 0 & u & u + v & -\mu \end{pmatrix} \quad (5)$$

At the interior equilibrium (5) becomes

$$\begin{pmatrix} -\frac{\omega N}{S} & 0 & 0 & 0 \\ \beta & -\frac{\beta S}{O} & 0 & 0 \\ 0 & \alpha & -\frac{\alpha O}{D} & 0 \\ 0 & u & u + v & -\frac{(u+v)D+uO}{R} \end{pmatrix} \quad (6)$$

The characteristic equation of (6) is given by

$$\begin{vmatrix} -\frac{\omega N}{S} - \lambda & 0 & 0 & 0 \\ \beta & -\frac{\beta S}{O} - \lambda & 0 & 0 \\ 0 & \alpha & -\frac{\alpha O}{D} - \lambda & 0 \\ 0 & u & (u + v) & -\frac{(u+v)D+uO}{R} - \lambda \end{vmatrix} = 0$$

$$\begin{aligned} & \lambda^4 + \left( \frac{\omega N}{S} + \frac{\beta S}{O} + \frac{\alpha S}{D} + \frac{uO}{R} + \frac{(u+v)}{R} \right) \lambda^3 \\ & + \left( \frac{\omega N \beta}{O} + \frac{(u+v)D\omega N}{SR} + \frac{uO\omega N}{SR} + \frac{\omega N \alpha}{D} + \frac{(u+v)D\beta S}{R} + \frac{u\beta S}{R} + \frac{\beta \alpha S^2}{DO} \right. \\ & + \left. \frac{(u+v)\alpha S}{R} + \frac{uO\alpha S}{RD} \right) \lambda^2 + \left( \frac{(u+v)\alpha \omega N D \beta}{R} + \frac{u\beta \omega N}{R} + \frac{\omega N \beta \alpha S}{DO} \right. \\ & + \left. \frac{(u+v)\alpha \omega N}{R} + \frac{\omega N u O \alpha}{RD} + \frac{(u+v)\alpha \beta S^2}{RO} + \frac{u\alpha \beta S^2}{RD} \right) \lambda \\ & + \frac{(u+v)\alpha \beta \omega N S}{RO} + \frac{u\alpha \beta \omega N S}{RD} = 0 \end{aligned} \quad (7)$$

Comparing (4) with  $S^4 + AS^3 + BS^2 + CS + D = 0$   
where

$$\begin{aligned} A &= \frac{\omega N}{S} + \frac{\beta S}{O} + \frac{\alpha S}{D} + \frac{uO}{R} + \frac{(u+v)}{R}, \\ B &= \frac{\omega N\beta}{O} + \frac{(u+v)D\omega N}{SR} + \frac{uO\omega N}{SR} + \frac{\omega N\alpha}{D} + \frac{(u+v)D\beta S}{R} \\ &\quad + \frac{u\beta S}{R} + \frac{\beta\alpha S^2}{DO} + \frac{(u+v)\alpha S}{R} + \frac{uO\alpha S}{RD}, \\ C &= \frac{(u+v)\alpha\omega ND\beta}{R} + \frac{u\beta\omega N}{R} + \frac{\omega N\beta\alpha S}{DO} + \frac{(u+v)\alpha\omega N}{R} \\ &\quad + \frac{\omega NuO\alpha}{RD} + \frac{(u+v)\alpha\beta S^2}{RO} + \frac{u\alpha\beta S^2}{RD}, \\ D &= \frac{(u+v)\alpha\beta\omega NS}{RO} + \frac{u\alpha\beta\omega NS}{RD} \end{aligned}$$

Here  $A > 0$ ;  $D > 0$ ;  $AB - C > 0$ ;  $C(AB - C) - A^2D > 0$  when  $1 - S > 0$  and the system (1) is locally stable by the Routh-Hurwitz criteria.

## 7 Global stability

We establish the Lyapunov function given below,

$$\begin{aligned} V(S, O, D, R) &= \left( (S - S^*) - S^* \ln \frac{S}{S^*} \right) + l_1 \left( (O - O^*) - O^* \ln \frac{O}{O^*} \right) \\ &\quad + l_2 \left( (D - D^*) - D^* \ln \frac{D}{D^*} \right) + l_3 \left( (R - R^*) - R^* \ln \frac{R}{R^*} \right) \end{aligned} \quad (8)$$

Differentiate (8) with respect to t,

$$\frac{dV}{dt} = \left( \frac{S - S^*}{S} \right) \frac{dS}{dt} + \left( \frac{O - O^*}{O} \right) \frac{dO}{dt} + \left( \frac{D - D^*}{D} \right) \frac{dD}{dt} + \left( \frac{R - R^*}{R} \right) \frac{dR}{dt}$$

From the model equations (1),

$$\begin{aligned} \frac{dV}{dt} &= \left( \frac{S - S^*}{S} \right) (\omega N - (\beta + \mu)S) + \left( \frac{O - O^*}{O} \right) (\beta S - (\alpha + \mu + u)O) \\ &\quad + \left( \frac{D - D^*}{D} \right) (\alpha O - (\mu + u + v)D) + \left( \frac{R - R^*}{R} \right) ((u + v)D + uO - \mu R) \\ &= (S - S^*) \left( \frac{\omega N}{S} - (\beta + \mu) \right) + l_1 (O - O^*) \left( \frac{\beta S}{O} - (\alpha + \mu + u) \right) \end{aligned}$$



$$+l_2(D - D^*) \left( \frac{\alpha O}{D} - (\mu + u + v) \right) + l_3(R - R^*) \left( \frac{(u + v)D + uO}{R} - \mu \right)$$

At  $(S^*, O^*, D^*, R^*)$ , we have

$$\begin{aligned} \frac{dV}{dt} &= (S - S^*) \left[ \frac{\omega N}{S} - \left( \frac{\omega N}{S^*} \right) \right] + l_1(O - O^*) \left[ \frac{\beta S}{O} - \left( \frac{\beta S^*}{O^*} \right) \right] \\ &+ l_2(D - D^*) \left[ \frac{\alpha O}{D} - \left( \frac{\alpha O^*}{D^*} \right) \right] + l_3(R - R^*) \left[ (u + v) \left( \frac{D + uO}{R} - \left( \frac{(u + v)D^* + uO^*}{R^*} \right) \right) \right] \\ &= (S - S^*) \omega N \left( \frac{1}{S} - \frac{1}{S^*} \right) + l_1(O - O^*) \beta \left( \frac{S}{O} - \frac{S^*}{O^*} \right) \\ &+ l_2(D - D^*) \alpha \left( \frac{O}{D} - \frac{O^*}{D^*} \right) + l_3(R - R^*) \left[ (u + v) \left( \frac{D}{R} - \frac{D^*}{R^*} \right) + u \left( \frac{O}{R} - \frac{O^*}{R^*} \right) \right] \end{aligned}$$

Choosing  $l_1 = \frac{1}{\beta}, l_2 = \frac{1}{\alpha}, l_3 = \frac{1}{(u+v)u}$

$$\begin{aligned} \frac{dV}{dt} &= (S - S^*) \omega N \left( \frac{1}{S} - \frac{1}{S^*} \right) + \frac{(O - O^*) \beta}{\beta} \left( \frac{S}{O} - \frac{S^*}{O^*} \right) + \frac{(D - D^*) \alpha}{\alpha} \\ &\left( \frac{O}{D} - \frac{O^*}{D^*} \right) + \frac{(R - R^*)}{(u + v)u} \left( \frac{(u + v)D}{R} - \frac{D^*}{R^*} \right) + \frac{(R - R^*)}{(u + v)u} \left( u \left( \frac{D}{R} - \frac{D^*}{R^*} \right) \right) \\ &= -\frac{\omega N(S - S^*)^2}{SS^*} + \frac{1}{OO^*} [SOO^* - S^*O^2 - SO^*2 + S^*O^*O] \\ &+ \frac{1}{DD^*} [OD^*D - D^2O^* - OD^*2 + D^*O^*D] + \frac{1}{uRR^*} [DR^*R - R^2D^* - R^*2D \\ &+ R^*RD^*] + \frac{1}{(u + v)RR^*} [DR^*R - R^2D^* - R^*2D + R^*RD^*] \\ &= -(S - S^*) \omega N \left( \frac{S - S^*}{SS^*} \right) + (O - O^*) \left( \frac{SO^* - S^*O}{OO^*} \right) + (D - D^*) \\ &\left( \frac{OD^* - DO^*}{DD^*} \right) + \frac{R - R^*}{u} \left( \frac{DR^* - RD^*}{RR^*} \right) + \frac{R - R^*}{u + v} \left( \frac{DR^* - RD^*}{RR^*} \right) \\ &= \frac{-\omega N(S - S^*)^2}{SS^*} + \left( S - \frac{SO^*}{O} - \frac{S^*O}{O^*} + S^* \right) + \left( O - \frac{OD^*}{D} + \frac{DO^*}{D^*} + O^* \right) \\ &+ \frac{1}{u} \left( R - \frac{DR^*}{R} + \frac{D^*R}{R^*} + R^* \right) + \frac{1}{u + v} \left( R - \frac{DR^*}{R} + \frac{D^*R}{R^*} + R^* \right) \end{aligned}$$

$\therefore \frac{dV}{dt} < 0$ , when  $\frac{R}{S} < \frac{R^*}{S^*}, \frac{S}{O} < \frac{S^*}{O^*}, \frac{O}{D} < \frac{O^*}{D^*}, \frac{D}{R} < \frac{D^*}{R^*}, \frac{O}{R} < \frac{O^*}{R^*}$

Here the system (1) is globally asymptotically stable by Lyapunov theorem.

## 8 Numerical simulations

We have choose the values of parameters as  $\omega = 0.0234$ ,  $\mu = 0.008$ ,  $\alpha = 0.2792$ ,  $\beta = 0.23$  based on a female population.

Figure (2) shows the flow of variables with control parameters. The obesity rate

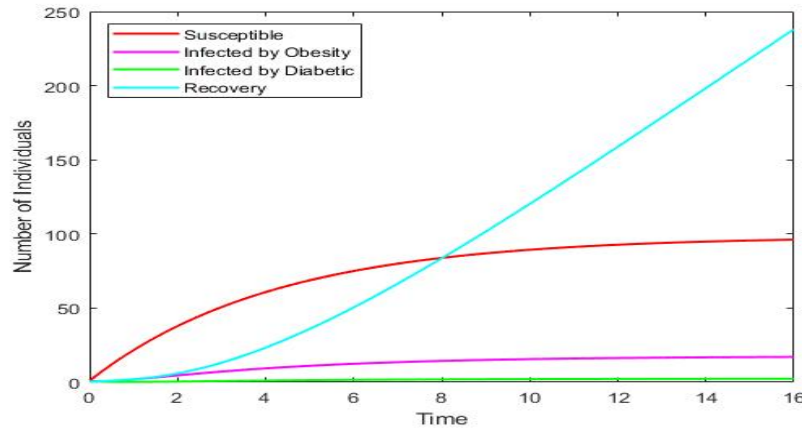


Figure 2: Obesity and Diabetic with control parameters

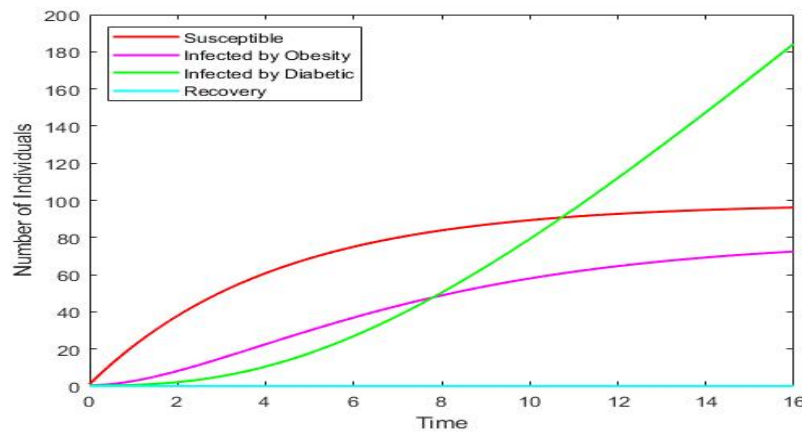


Figure 3: Obesity and Diabetic without control parameters

and diabetic rate are stable with high recovery rate with respect to time . Figure (3) shows the flow of variables without control parameters. The diabetic rate is high due to no recovery rate.

Figure (4) shows the flow of individuals of susceptible class for various values of  $\beta$ . If  $\beta$  values are increasing, the number of individuals of susceptible class move

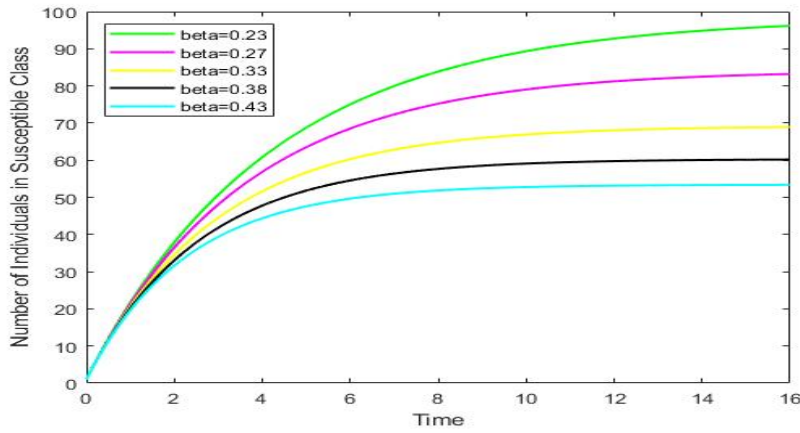


Figure 4: Flow of individuals in Susceptible Class for various values of  $\beta$

towards the infected by obesity class which becomes stable with respect to time.

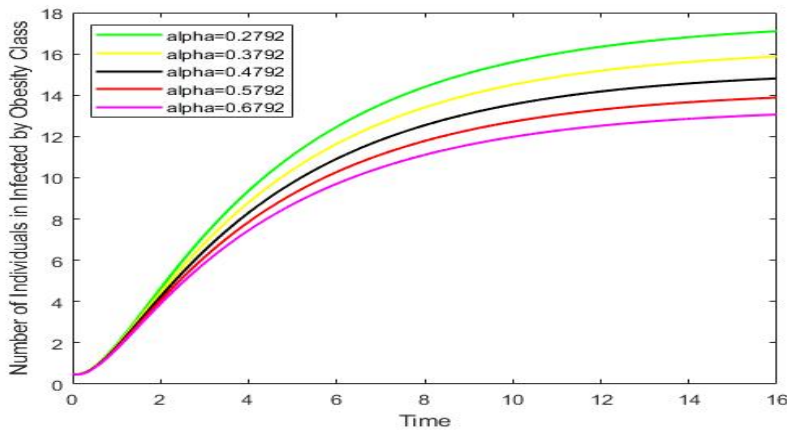


Figure 5: Infected by Obesity class for different values of  $\alpha$  with control parameters

Figure(5) and figure(6) show the flow of individuals of infected by obesity class for various values of  $\alpha$  with and without control parameters respectively. If we use the control parameters by giving different values to  $\alpha$ , then the number of individuals in obesity class will be stable with respect to time. Otherwise it will increase the diabetic rate. While comparing, figures (5) and (6) the obesity rate of figure (5) is higher than the obesity rate of figure(6). So we can understand that the obesity rate can be managed with control parameters. Which gives that the

# *A dynamical analysis of a mathematical model on type-2 diabetic*

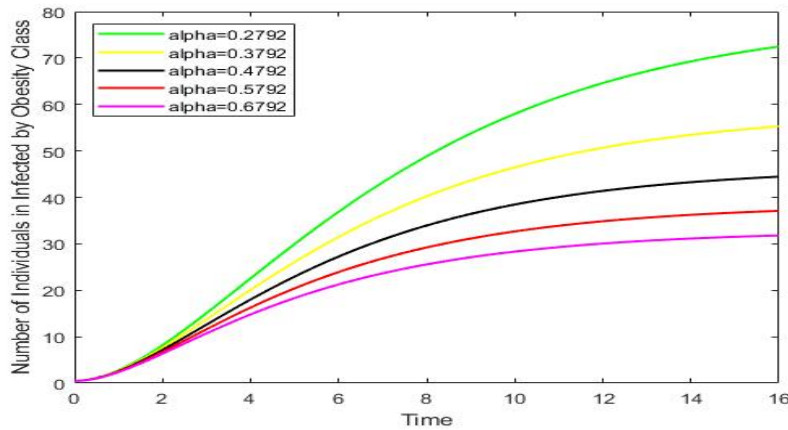


Figure 6: Infected by Obesity class for different values of  $\alpha$  without control parameters

obesity rate is controlled with the control parameters.

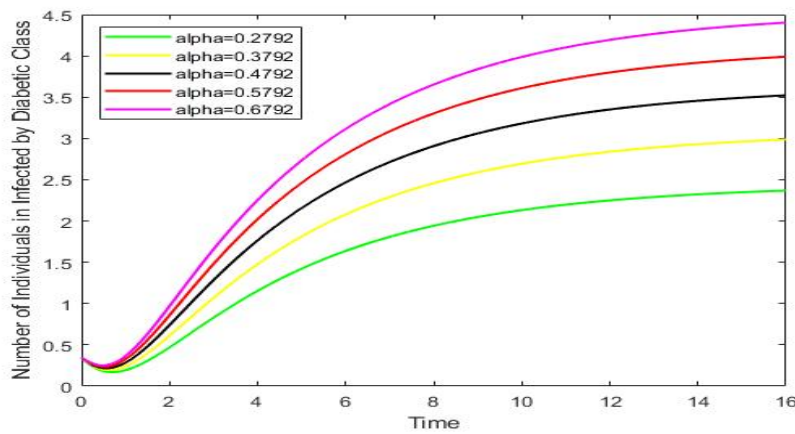


Figure 7: Infected by Diabetic class for different values of  $\alpha$  with control parameters

Figure(7) and figure(8) show the flow of individuals in infected by diabetic class for various values of  $\alpha$  with and without control parameters respectively. If we use the control parameters by giving different values to  $\alpha$ , then the number of individuals in diabetic class will be stable with respect to time. Otherwise it will reduce the recovery rate. While comparing, figures (7) and (8) the diabetic rate of figure (7) is higher than the diabetic rate of figure(8). So we can understand that

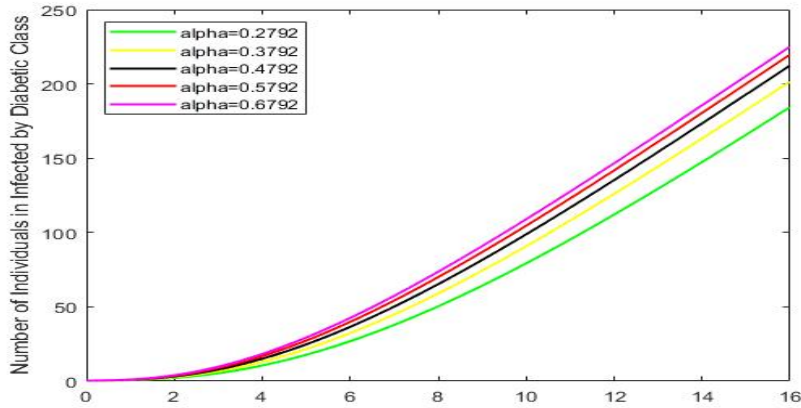


Figure 8: Infected by Diabetic class for different values of  $\alpha$  without control parameters

the diabetic rate can be managed with control parameters. Which gives that the diabetic rate is controlled with the control parameters.

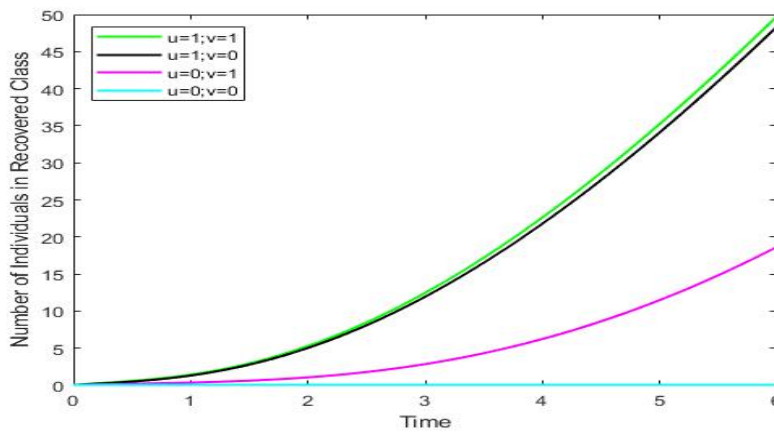


Figure 9: Recovered class for various values of control parameters

Figure(9) shows the flow of individuals in recovered state for the various values of control parameters. We can observe the following cases:

Case 1 : The recovery rate increases when both the control parameters exist.

Case 2 : The recovery rate is slightly decrease when the control parameter diet with physical activity is involved.

Casse 3 : The recovery rate decreases from case 2, when the control parameter medication is involved.

Case 4 : There is no recovery rate, when no control parameters are involved.

Hence, both the control parameters are required to control the obesity and diabetic rates and to increase the recovery rate.

## 9 Conclusions

This paper presents a mathematical model for type 2 diabetic from obesity with controls diet with physical activity and medication. By the Pontryagin's maximum principle, the optimal levels of two control variables are attained with some specific conditions. An optimal control approach is proposed in order to reduce the burden of obesity to diabetic. The positive equilibrium points have been found. The system is locally asymptotically stable with a condition on susceptible population. The system is globally asymptotically state in the identified parametric domain. The numerical simulations show the effectiveness of proposed control strategies. In the numerical simulations, we can observe that the flow of diabetic state was dropped down, when the two control parameters are involved. We can also observe that the obesity rate can be managed at moderate levels when the controls parameters are at optimal levels. Hence the two introduced control parameters healthy diet with physical activity and medication are effectively reduce the rates of obesity and diabetic and increase the recovery rate.

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# Anti-homomorphism in Q-fuzzy subgroups and normal subgroups

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S.Palaniyandi<sup>†</sup>

## Abstract

The fuzzy set has been applied in wide area by many researchers. We define the concept of anti-homomorphism in Q-fuzzy subgroups and Q-fuzzy normal subgroups and establish some result in this research article and develop some theory of anti-homomorphism in Q-fuzzy subgroups, normal subgroups and also extend results on Q-fuzzy abelian subgroup and Q-fuzzy normal subgroup. Many researchers have explored the fuzzy set extensively. We propose the notion of anti-homomorphism in Q-fuzzy subgroups and normal subgroups. It is establish some findings in this study article and build the theory of anti-homomorphism in Q-fuzzy subgroups, normal subgroups. It is also extend results on Q-fuzzy abelian subgroup and Q-fuzzy normal subgroup.

**Keywords:** Fuzzy, subgroup, Q-fuzzy, fuzzy abelian, fuzzy normal subgroup, anti-homomorphism,.

**AMS Subject Classification:** 03E72, 03E75, 08A72 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1073. ISSN: 1592-7415. eISSN: 2282-8214. ©R.Jahir Hussain et al. This paper is published under the CC-BY licence agreement.



## 1 Introduction

Zadeh L.A. Zadeh [1965] introduced the fuzzy set concept. Numerous scholars have used the fuzzy set in several different contexts. Fuzzy subgroups are first discussed by Rosenfeld [1971]. Biswas.R Biswas [1990] was introduced, the anti-fuzzy subgroups. The novel structure of Q-fuzzy subgroups was introduced by Solairaju.A and Nagarajan.R Solairaju and Nagarajan [2009]. Fuzzy subgroups and fuzzy homomorphisms were defined by Choudhury, F.P., Chakraborty, A. B, and Khare Choudhury et al. [1988] Sheik Anti-homomorphism in fuzzy subgroups was defined by Abdullah A. and Jeyaraman K. Sheik Abdullah and Jeyaraman [2010]. In this study, we demonstrate various results and define the notion of anti-homomorphism in Q-fuzzy subgroups and fuzzy normal subgroups.

## 2 Preliminaries

**Definition 2.1.** Zadeh [1965] A function of fuzzy subset  $\delta \neq S$  is  $\delta : S \rightarrow [0, 1]$ .

**Definition 2.2.** Rosenfeld [1971] A fuzzy subset  $\delta$  of a group  $J = J$  (fuzzy subgroup) if it is satisfying the following conditions,

- (i)  $\delta(\varrho) \geq \min\{\delta(\varrho), \delta(\gamma)\}$
- (ii)  $\delta(\varrho^{-1}) = \delta(\varrho), \forall \varrho, \gamma \in J$ .

**Definition 2.3.** Solairaju and Nagarajan [2009] A Q-fuzzy set  $\delta = J$  if  $\forall \varrho, \gamma \in J$ , and  $\kappa \in Q$

- (i)  $\delta(\varrho\gamma, \kappa) \geq \min\{\delta(\varrho, \kappa), \delta(\gamma, \kappa)\}$
- (ii)  $\delta(\varrho^{-1}, \kappa) = \delta(\varrho, \kappa)$

**Definition 2.4.** Zadeh [1965]  $\nu \subseteq S$  (fuzzy subset of a set) . For  $\beta \in [0, 1]$ , the level subset of  $\delta$  is defined by

$$\delta_\beta = \{e \in S : \delta_\nu(\varrho) \geq \beta\}$$

**Definition 2.5.** Solairaju and Nagarajan [2009]  $\nu \subseteq S$ . For  $\beta \in [0, 1]$ , the set  $\delta_\beta = \{e \in S, \kappa \in Q : \delta_\nu(\varrho, \kappa) \geq \beta\}$  is called a  $Q \subseteq \delta$ .

**Definition 2.6.** Palaniappan and Muthuraj [2004] Consider  $\delta < J$ . The fuzzy subgroup  $\delta$  is said to be fuzzy normal subgroup if  $\delta(\varrho\gamma) = \delta(\gamma\varrho), \forall \varrho, \gamma \in J$ .

**Definition 2.7.** Palaniappan and Muthuraj [2004] A fuzzy subgroup  $\delta$  of a group  $J$  is a Q-fuzzy normal subgroup if  $\delta(\varrho\gamma, \kappa) = \delta(\gamma\varrho, \kappa), \forall \varrho, \gamma \in J$ , and  $\kappa \in Q$ .

**Definition 2.8.** Choudhury et al. [1988] Let  $(J_1, \bullet)$  and  $J_2, \bullet$  be the function  $g : J_1 \rightarrow J_2$  is called a group homomorphism if  $g(\varrho\gamma) = g(\varrho) \cdot g(\gamma), \forall \varrho, \gamma \in J_1$ .

**Definition 2.9.** Sheik Abdullah and Jeyaraman [2010] Let  $(J_1, \bullet)$  and  $J_2, \bullet$  be the function  $g : J_1 \rightarrow J_2$  is called a group anti homomorphism if  $g(\varrho\gamma) = g(\gamma) \cdot g(\varrho), \forall \varrho, \gamma \in J_1$ .

**Definition 2.10.** Sheik Abdullah and Jeyaraman [2010] Let  $g : J_1 \rightarrow J_2$  is called anti automorphism if  $g(\varrho\gamma) = g(f) \cdot g(\varrho) \forall \varrho, \gamma \in J_1$ .

**Definition 2.11.** Sheik Abdullah and Jeyaraman [2010] The function  $\delta$  is a fuzzy characteristic subgroup of a group  $J$  if  $\delta(h(\varrho)) = \delta(\varrho)$ .

### 3 Some results On Q -fuzzy subgroups in anti- homomorphism

**Theorem 3.1.** Let  $g : J \rightarrow J^*$  be an anti-homomorphism, if  $\delta^*$  is a  $Q < J^*$ . Then  $g^{-1}(\delta^*)$  is a  $Q < J$ .

*Proof.* Let  $e, \gamma \in J$ . Then

$$\begin{aligned} g^{-1}(\delta^*)(\varrho\gamma, \kappa) &= \delta^*(h(\varrho\gamma, \kappa)) \\ &= \delta^*\{g(\gamma, \kappa) \cdot g(\varrho, \kappa)\} \\ &\geq \min\{\delta^*(h(\gamma, \kappa)), \delta^*(h(\varrho, \kappa))\} \\ &= \min\{(h^{-1}\delta^*)(\gamma, \kappa), (h^{-1}\delta^*)(\varrho, \kappa)\} \end{aligned} \quad (1)$$

and

$$\begin{aligned} g^{-1}(\delta^*)(\varrho^{-1}, \kappa) &= \delta^*(h(\varrho^{-1}, \kappa)) \\ &= \delta^*(h(\varrho, \kappa)) \\ &= g^{-1}(\delta^*)(\varrho, \kappa) \end{aligned} \quad (2)$$

From (1) and (2),  $g^{-1}(\delta^*)$  is a  $Q < J$ . □

**Theorem 3.2.** If  $\delta$  is a  $Q < J$  and  $g : S \rightarrow S^*$  is an anti-homomorphism, then  $g^{-1}(\delta)$  is a Q-fuzzy normal subgroup of  $S^*$ .

*Proof.* For every  $\varrho, \gamma \in S$ . We get,

$$\begin{aligned} g^{-1}(\delta)(\varrho, \kappa) &= \delta(h(\varrho, \kappa)) \\ &= \delta\{g(\gamma, \kappa), g(\varrho, \kappa)\} \\ &= \delta(h(\gamma\varrho, \kappa)) \\ &= g^{-1}(\delta)(\gamma\varrho, \kappa) \end{aligned}$$

Hence  $g^{-1}(\delta)$  is a  $Q < J$ . □

**Theorem 3.3.** A fuzzy characteristic subgroup of a  $Q < Q$ . It is a fuzzy normal subgroup.

*Proof.* Given  $g$  is an anti automorphism of  $S$ . For all  $\varrho, \gamma \in S$ , and  $\kappa \in Q$ . Then

$$g(\varrho\gamma) = g(\gamma) \cdot g(\varrho), \text{ for every } \varrho, \gamma \in S$$

Now,

$$\begin{aligned} \delta(\varrho\gamma, \kappa) &= \delta(h(\varrho\gamma, \kappa)) \\ &= \delta\{g(\gamma, \kappa), g(\varrho, \kappa)\} \end{aligned}$$

Since  $\delta$  is a characteristics  $Q < S$ .

Then  $\delta(\varrho\gamma, \kappa) = \delta\{g(\gamma, \kappa), g(\varrho, \kappa)\}$

Since  $g$  is anti automorphism of  $S$ ,

$$\begin{aligned} \delta(\varrho\gamma, \kappa) &= \delta(h(\gamma\varrho, \kappa)) \\ &= \delta(\gamma\varrho, \kappa), \text{ for all } \varrho, \gamma \in S, \text{ and } \kappa \in Q \end{aligned}$$

$\delta$  is a characteristics  $Q < S$ .

Hence  $\delta$  is a  $Q < S$ . □

**Definition 3.1.** The  $Q$ -fuzzy subgroup  $\delta$  of a group  $S$  is called a  $Q$ -fuzzy abelian subgroup of  $S$  if  $H = \{\varrho \in S : \delta(\varrho, \kappa) = \delta(i, \kappa)\}, \forall \varrho, \gamma \in S, \text{ and } \kappa \in Q$ .

**Theorem 3.4.** The commutative property satisfies all anti-homomorphism pre-images of a  $Q$ -fuzzy commutative subgroup.

*Proof.* If  $\delta$  is a  $Q < S$ .

To prove  $\delta$  is a  $Q = [S, S]$

Let us consider  $\nu$  is a  $Q = [S^*, S^*]$  ( $Q$  fuzzy commutative subgroup of  $S^*$ ).

Since  $\delta$  is a  $Q = [S^*, S^*]$  and  $\nu$  is  $Q = [S^*, S^*]$

Then  $V = \{\gamma \in S^*, \kappa \in Q : \nu(\gamma, \kappa) = \nu(i^*, \kappa)\}$  is a  $Q = [S^*, S^*]$ , where  $i^*$  is the identify element of  $S^*$ .

Let  $T = \{\varrho \in S, \kappa \in Q : \delta(\varrho, \kappa) = \delta(i, \kappa)\}$  where  $i$  is the identify element of  $S$ .

Take  $\varrho, \gamma \in T$  this implies  $\varrho\gamma \in T \subseteq S$ . Then

$$\begin{aligned} \delta(\varrho\gamma, \kappa) &= \delta(i, \kappa) \\ \nu(h(\varrho\gamma, \kappa)) &= \nu(h(i, \kappa)) \\ &= \nu(i^*, \kappa) \\ \nu\{g(\gamma, \kappa) \cdot g(\varrho, \kappa)\} &= \nu(i^*, \kappa) \end{aligned}$$

### Anti-homomorphism in $Q$ -fuzzy subgroups and normal subgroups

Since  $g(\gamma, \kappa) \cdot g(\varrho, \kappa) \in V$  and  $V$  is abelian,

$$\begin{aligned} g(f, \kappa) \cdot g(\varrho, \kappa) &= g(\varrho, \kappa) \cdot g(\gamma, \kappa) \\ \nu(h(\gamma, \kappa) \cdot g(\varrho, \kappa)) &= \nu(h(\varrho, \kappa) \cdot g(\gamma, \kappa)) \\ \nu(\gamma(\gamma\varrho, \kappa)) &= \nu(\gamma(\varrho\gamma, \kappa)), \text{ since } g \text{ is anti-homomorphism.} \\ \delta(\varrho\gamma, \kappa) &= \delta(\gamma\varrho, \kappa) \\ \delta(i, \kappa) &= \delta(\gamma\varrho, \kappa) \end{aligned}$$

i.e  $(\gamma\varrho, \kappa) = \delta(i, \kappa)$ , this implies  $\gamma\varrho \in T$  and  $\kappa \in Q$ .

For all  $\varrho, \gamma \in T$ ,  $\varrho\gamma \in T$  and  $\gamma\varrho \in T$ .

This implies  $\varrho\gamma = \gamma\varrho$

$T$  satisfies commutative. Therefore  $\delta$  satisfies commutative property.  $\square$

**Theorem 3.5.** *Anti-homomorphism image of a  $Q$ -fuzzy commutative subgroup is also satisfies commutative level.*

*Proof.* Let  $\nu$  be a  $Q$ -fuzzy subgroup of  $S^*$ .

To prove:  $\nu$  is a  $Q$ -fuzzy commutative subgroup of  $S^*$ .

Let  $g$  be an anti-homomorphism from  $S$  to  $S^*$ .

Since  $\delta$  is a  $Q=[S, S]$ .

Then  $T = \{\varrho \text{ in } S, \kappa \in Q : \delta(\varrho, \kappa) = \delta(i, \kappa)\}$  is an commutative  $Q$ -fuzzy subgroup of  $S^*$  where  $i$  is the identity element of  $S$ .

Let  $\nu$  be the  $Q$ -fuzzy subgroup of  $S^*$ .

Let  $V = \{x \in S^*, \kappa \in Q : \nu(\varrho, \kappa) = \delta(i^*, \kappa)\}$  where  $i^*$  is the identity element  $S^*$ .

Let  $e, \gamma \text{ in } V \subseteq S^*$

$$\begin{aligned} \nu(\varrho\gamma, \kappa) &= \nu(i^*, \kappa) \\ \text{Sup } \delta(r, \kappa) &= \text{Sup } \delta(r, \kappa) \\ r \in g^{-1}(\varrho\gamma) \quad , \quad r \in g^{-1}(i^*) \\ \delta(\varrho\gamma, \kappa) &= \delta(i, \kappa) \end{aligned}$$

Then  $\varrho\gamma \in T$  and  $T$  is an commutative  $Q$ -fuzzy subgroup.

$$\begin{aligned} (\varrho\gamma, \kappa) &= (\gamma\varrho, \kappa) \\ \delta(\varrho\gamma, \kappa) &= \delta(\gamma\varrho, \kappa) \\ \text{Sup } \delta(r, \kappa) &= \text{Sup } \delta(r, \kappa) \\ r \in g^{-1}(\varrho\gamma) \quad , \quad r \in g^{-1}(\gamma\varrho) \\ \nu(\varrho\gamma, \kappa) &= \nu(\gamma\varrho, \kappa) \\ \nu(i^*, \kappa) &= \nu(\gamma\varrho, \kappa) \end{aligned}$$

That is  $\nu(\varrho\gamma, \kappa) = \nu(i^*, \kappa)$ , this implies  $\gamma\varrho \in V$  and  $\kappa \in Q$ .

For all  $\varrho, \gamma \in V$ ,  $\varrho\gamma \in V$

This implies  $\gamma\varrho \in V$  and  $\varrho\gamma = \gamma\varrho$ .

Then  $V = [S^*, S^*]$  (commutative subgroup of  $S^*$ ).

Therefore  $\nu = [S^*, S^*]$ . Where  $Q$  fuzzy commutative subgroup of  $S^*$ .  $\square$

## 4 Conclusions

Many results can be found from the research article. But, in this paper we found few concepts of anti-homomorphism in Q-fuzzy subgroups. Further this paper used to developing the concept of Q-fuzzy abelian subgroup. There are so many concepts can be availed by future research work.

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# Theoretical analysis on the growth kinetics of SARS-CoV (within host)

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## Abstract

A mathematical model is investigated to analyze the biological interactions between the immune system and SARS-CoV (within host). Homotopy Perturbation Method is executed to obtain an analytical solution to the non-linear system of ordinary differential equations. Graphical illustration to these solutions is also presented. The reliability and the simplicity of the aforementioned method is studied through the comparison between the numerical and graphical results. This comparison aids the better understanding of the disease dynamics and also the establishment of probable strategies for the treatment of COVID-19.

**Keywords:** Mathematical Modeling, COVID-19, Non -linear initial value problem, Homotopy Perturbation Method.

**2020 AMS subject classifications:** Find your subject classification at <https://mathscinet.ams.org/msnhtml/msc2020.pdf>.<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1074. ISSN: 1592-7415. eISSN: 2282-8214. ©Pavithra Sivasamy et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

The World Health Organization declared the SARS-CoV as a potential threat to the human society Sharma et al. [2020] Ramadan and Shaib [2019] Ouassou et al. [2020]. Since then, researchers have been working really hard on various strategies to control the disease Ahmed et al. [2020] Nie et al. [2020]. A key factor to control the disease is by studying the disease severity and progression in the host. However, the quantitative analysis of the growth kinetics of the virus has not been able to study the severity in the host. It is therefore important to analyze the interactions within the host between SARS-CoV and the immune system Du and Yuan [2020] Hernandez-Vargas and Velasco-Hernandez [2020] Wang et al. [2020]. And this can be achieved by developing a mathematical model as they serve as a tool for characterizing the disease dynamics and in the forecast of severity of the disease. A mathematical model is therefore developed by S.M.E.K Chowdhury et.al Chowdhury et al. [2022] in the context of immune surveillance. We study this model analytically. The goal of our work is to derive a closed form analytical solution for the COVID-19 model for the susceptible, infected, recovered and exposed. This is done by implementing the technique of Homotopy Perturbation Method [HPM] He [1999] He [2004a] He [2004b] He [2004c] which is the coupling of homotopy and perturbation technique. The derived analytical results are beneficial in two fronts: first, it would contribute to a better comprehension of the disease dynamics which assist in designing effective treatment strategies and secondly, would enable the scientist in modifying the COVID-19 model to study the correlation between various model parameters and their impacts. This approach in its implementation is novel to this non-linear system of COVID-19 model.

## 2 Model formulation

The mathematical framework of the developed model is built on the interplay between the lymphocytes and virus particles within the host. Taking into account this interaction the model is proposed as Chowdhury et al. [2022]

$$\frac{dE}{dt} = a_1 - a_2 E - a_3 EV \quad (1)$$

$$\frac{dE^i}{dt} = a_3 EV - b_1 E^i \quad (2)$$

$$\frac{dV}{dt} = c_1 E^i - c_2 V - c_3 VL \quad (3)$$

$$\frac{dN}{dt} = d_1 - d_2 N \quad (4)$$

| Parameters | Physical Meaning                                        |
|------------|---------------------------------------------------------|
| $a_1$      | Regeneration rate of epithelial cells                   |
| $a_2$      | Rate at which epithelial cells die                      |
| $a_3$      | Rate of infection of epithelial cells                   |
| $b_1$      | Rate at which the infected epithelial cells die         |
| $c_1$      | Production rate of virus from infected epithelial cells |
| $c_2$      | Rate at which the virus dies                            |
| $e_1$      | Rate of proliferation of T-lymphocytes                  |
| $e_2$      | Rate at which the T-lymphocytes die                     |
| $e_3$      | Regeneration rate of T-lymphocytes                      |
| $c_3$      | Rate at which the Natural Killer cells kills the virus  |
| $d_1$      | External influx rate of Natural Killer cells            |
| $d_2$      | Rate at which the Natural Killer cells die              |
| $c_4$      | Rate at which the T-lymphocytes kills the virus         |
| $n$        | Half saturation constant of T-lymphocytes               |

Table 1: Nomenclature

$$\frac{dL}{dt} = \frac{e_1 LV}{(n + V)} - e_2 L + e_3 \quad (5)$$

Where denotes the count of susceptible epithelial cells  $E(t)$  , infected epithelial cells  $E^i(t)$ , viral load  $V(t)$  , natural killer cells  $N(t)$  and T-lymphocytes  $L(t)$  respectively. The production rate of the virus is  $c_1$  where as the infection rate is denoted by  $a_3$  . the natural death rate of susceptible epithelial cells, infected epithelial cells, viral load , natural killer cells and T-lymphocytes are  $a_2, b_1, c_2, d_2, e-2$  respectively.  $a_1$  denotes the regeneration rate of the epithelial cells. The degeneration of virus particles occur when they are interacted with natural killer cells and T-lymphocytes at the rate  $d_1$  and  $a_3$  respectively. The NK cell's constant external source is denoted as  $a_2$  . refers to the T-lymphocytes natural recruitment rate  $c_4$  . In the presence of virus particles, the T-lymphocytes proliferate at a rate  $n$ .

### 3 Homotopy perturbation method

Epidemiological modeling of the diseases using nonlinear dynamical equations gives deeper insights into the behavioral patterns and the transmission dynamics of the disease. As solutions to these non-linear problems are a bit more complex, a substantial amount of work has been dedicated by researchers in developing a solution method to these models. Such methods include ADM , VIM , HAM , HPM He [1999]He [2004c]He [2005]Abbasbandy [2006]Rafei and Ganji [2006]Yıldırım and Öziş [2007]Sivasamy and Kumar [2021] etc. In this paper,



we execute HPM in finding an analytical solution to the COVID-19 model. The advantage of HPM over other method is its ability to reduce a complex non-linear problem into a series of linear equation and finding the solution the same.

### 3.1 Basic idea of homotopy perturbation method [HPM]

Consider the following function

$$D_o(u) - f(r) = 0, r \in \Omega \quad (6)$$

with the boundary conditions as

$$B_o(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \quad (7)$$

Where  $D_0$  is a general differential operator,  $B_0$  is a boundary operator,  $f(r)$  is a known analytical function and  $\Gamma$  is the boundary of the domain  $\Omega$ . In general, the operator  $D_0$  can be divided into a linear part  $L$  and a non-linear part  $N$ . We can rewrite the eqn (6) as

$$L(u) + N(u) - f(r) = 0 \quad (8)$$

We now construct a homotopy as  $v(r, p) : \rightarrow \Omega \times [0, 1] \times \Re$  by the homotopy technique which statisfies

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[D_0 - f(r)] = 0 \quad (9)$$

$$H(v, p) = L(v) - L(v_0) + p[N(v) - f(r)] = 0 \quad (10)$$

Here  $p$  is the embedding parameter and belongs to the interval  $[0, 1]$  and the initial approximation of eqn.(6) is  $u_0$  which satisfies the boundary condition. Now,eqn (9) and (10) leads to

$$H(v, 0) = L(v) - L(u_0) = 0 \quad (11)$$

$$H(v, 1) = D_0 - f(r) = 0 \quad (12)$$

Setting  $p=0$  makes the eqns. (9) and (10) as linear and seeting  $p=1$  makes them non-linear.. This process is presented as  $L(v) - L(u_0) = 0$  to  $D_0 - f(r) = 0$ . We use  $p$ , the embedding parameter as a small parameter and assume that the solutions of eqns. (9) and (10) can be written as a power series in  $p$ :

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (13)$$

Fixing  $p=1$  leads to the approximation of eqn (6) as

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (14)$$

This is the basic idea of the HPM.

## 4 Analytical solution to the COVID-19 model using homotopy perturbation method

$$(1-p) \left( \frac{dE}{dt} = a_1 - a_2 E \right) + p \left( \frac{dE}{dt} = a_1 - a_2 E - a_3 EV \right) = 0 \quad (15)$$

$$(1-p) \left( \frac{dE^i}{dt} + b_1 E^i \right) = p \left( \frac{dE^i}{dt} = a_3 EV - b_1 E^i \right) = \quad (16)$$

$$(1-p) \left( \frac{dV}{dt} + c_2 V \right) + p \left( \frac{dV}{dt} = c_1 E^i - c_2 V - c_3 VL \right) = 0 \quad (17)$$

$$(1-p) \left( \frac{dL}{dt} + e_2 L - e_3 \right) + p \left( \frac{dL}{dt} = \frac{e_1 LV}{(n+V)} - e_2 L + e_3 \right) = 0 \quad (18)$$

Supposing the approximate solutions of Eq. (1-5) have the form

$$E = E_0 + pE_1 + p^2E_2 + \dots \quad (19)$$

$$E^i = E_0^i + pE_1^i + p^2E_2^i + \dots \quad (20)$$

$$V = V_0 + pV_1 + p^2V_2 + \dots \quad (21)$$

$$L = L_0 + pL_1 + p^2L_2 + \dots \quad (22)$$

Substituting the Eq. (15-18) respectively into Eq. (1-5)

$$\begin{aligned} (1-p) \left( \frac{d(E_0 + pE_1 + p^2E_2 + \dots)}{dt} - a_1 + a_2(E_0 + pE_1 + p^2E_2 + \dots) \right) + \\ p \left( \frac{d(E_0 + pE_1 + p^2E_2 + \dots)}{dt} - a_1 + a_2(E_0 + pE_1 + p^2E_2 + \dots) \right. \\ \left. + a_3(E_0 + pE_1 + p^2E_2 + \dots)(V_0 + pV_1 + p^2V_2 + \dots) \right) = 0 \end{aligned} \quad (23)$$

$$\begin{aligned} (1-p) \left( \frac{d(E_0^i + pE_1^i + p^2E_2^i + \dots)}{dt} + b_1(E_0^i + pE_1^i + p^2E_2^i + \dots) \right) + \\ p \left( \frac{d(E_0^i + pE_1^i + p^2E_2^i + \dots)}{dt} - a_3(E_0 + pE_1 + p^2E_2 + \dots)(V_0 + pV_1 + p^2V_2 + \dots) \right. \\ \left. - b_1(E_0^i + pE_1^i + p^2E_2^i + \dots) \right) = 0 \end{aligned} \quad (24)$$

$$\begin{aligned}
 (1-p) & \left( \frac{d(V = V_0 + pV_1 + p^2V_2 + \dots)}{dt} + c_2(V = V_0 + pV_1 + p^2V_2 + \dots) \right) + \\
 p & \left( \left( \frac{d(V = V_0 + pV_1 + p^2V_2 + \dots)}{dt} - c_1(E_0^i + pE_1^i + p^2E_2^i + \dots) \right. \right. \\
 & \quad \left. \left. - c_2(V = V_0 + pV_1 + p^2V_2 + \dots) \right. \right. \\
 & \quad \left. \left. - c_3(V = V_0 + pV_1 + p^2V_2 + \dots)(L_0 + pL_1 + p^2L_2 + \dots) \right) = 0 \quad (25)
 \end{aligned}$$

$$\begin{aligned}
 (1-p) & \left( \frac{d(L_0 + pL_1 + p^2L_2 + \dots)}{dt} + e_2(L_0 + pL_1 + p^2L_2 + \dots) - e_3 \right) + \\
 p & \left( \left( \frac{dL}{dt} = \frac{e_1(L_0 + pL_1 + p^2L_2 + \dots)(V_0 + pV_1 + p^2V_2 + \dots)}{(n + (V_0 + pV_1 + p^2V_2 + \dots))} \right. \right. \\
 & \quad \left. \left. - e_2(L_0 + pL_1 + p^2L_2 + \dots) + e_3 \right) = 0 \quad (26)
 \end{aligned}$$

Equating the terms of Eq (23-26) with the identical powers of  $p$ , we obtain

$$p^0 : \frac{dE_0}{dt} + a_2E - a_1 = 0 \quad (27)$$

$$p^0 : \frac{dE_0^i}{dt} + b_1E_0^i = 0 \quad (28)$$

$$p^0 : \frac{dV_0}{dt} + c_2V_0 = 0 \quad (29)$$

$$p^0 : \frac{dL_0}{dt} + e_2L_0 - e_3 = 0 \quad (30)$$

$$\frac{dE_1}{dt} + a_2E_1 + a_3E_0V_0 = 0 \quad (31)$$

$$\frac{dE_1^i}{dt} - a_3E_0V_0 + b_1E_0^i = 0 \quad (32)$$

$$\frac{dV_1}{dt} - c_1E_0^i - c_2V_1 + c_3V_0L_0 = 0 \quad (33)$$

$$\frac{dL}{dt} - \frac{e_1L_0V_0}{(n + V_i)} + e_2L_1 = 0 \quad (34)$$

Number of susceptible epithelial cells are given by

$$\begin{aligned}
 E(t) = \frac{a_1}{a_2} + (E_i - \frac{a_1}{a_2})e^{-a_2t} - \frac{a_3a_1V_ie^{-c_2t}}{a_2(a_2 - c_2)} + \frac{a_3(E_i - \frac{a_1}{a_2})V_ie^{(-a_2+c_2)t}}{c_2} \\
 + \frac{a_3a_1V_ie^{-a_2t}}{a_2(a_2 - c_2)} - \frac{a_3(E_i - \frac{a_1}{a_2})V_ie^{-a_2t}}{c_2} \quad (35)
 \end{aligned}$$

Infected epithelial cells are given by

$$E^i(t) = E^i(t)e^{b_1t} + \frac{a_3a_1V_ie^{-c_2t}}{a_2(b_1 - c_2)} + \frac{a_3(E_i - \frac{a_1}{a_2})V_ie^{(-a_2-c_2)t}}{b_1 - a_2 - c_2} + \frac{a_3a_1V_ie^{-b_1t}}{a_2(b_1 - c_2)} - \frac{a_3(E_i - \frac{a_1}{a_2})V_ie^{-b_1t}}{b_1 - a_2 - c_2} \quad (36)$$

Viral load cells are given by

$$V(t) = V_i(t)e^{-c_2t} + \frac{c_1E_ie^{-b_1t}}{c_2b_1} + (\frac{c_3d_1V_i}{d_2} - \frac{c_4e_3V_i}{e_2})te^{-c_2t} + \frac{c_3(N_i - \frac{d_1}{d_2})V_ie^{-(d_2+c_2)t}}{d_2} - \frac{c_1E_ie^{-c_2t}}{c_2(b_1 - c_2)} - \frac{c_3(N_i - \frac{d_1}{d_2})V_ie^{(-c_2)t}}{d_2} + \frac{c_3(N_i - \frac{d_1}{d_2})L_ie^{-(e_2+c_2)t}}{e_2} - \frac{c_3(N_i - \frac{d_1}{d_2})L_ie^{(-e_2)t}}{e_2} \quad (37)$$

Natural killer cells are given by

$$N(t) = \frac{d_1}{d_2} - \left(N_i - \frac{d_1}{d_2}\right)e^{-d_2t} \quad (38)$$

T-lymphocytes cells are given by

$$L(t) = \frac{e_3}{e_2} + (L_i - \frac{e_3}{e_2})e^{-e_2t} + \frac{e_3e_1V_ie^{-c_2t}}{e_3(n + V_i)(-c_2 + e_2)} - \frac{e_1(L_i - \frac{e_3}{e_2})V_ie^{(-a_2+e_2)t}}{c_2(n + V_i)} - \frac{e_3e_1V_ie^{-e_2t}}{e_3(n + V_i)(-c_2 + e_2)} + \frac{e_1(L_i - \frac{e_3}{e_2})V_ie^{(-e_2+e_2)t}}{c_2(n + V_i)} \quad (39)$$

## 5 Numerical simulation

An analytical expression for the time-dependent non-linear COVID-19 model is derived by executing the method of Homotopy Perturbation for the equations (1-6). The numerical solutions to these equations are obtained using MATLAB software. In order to check the efficiency of HPM in solving the COVID-19 model, comparison between the numerical and analytical results are carried out which is illustrated in the figures 1-12.

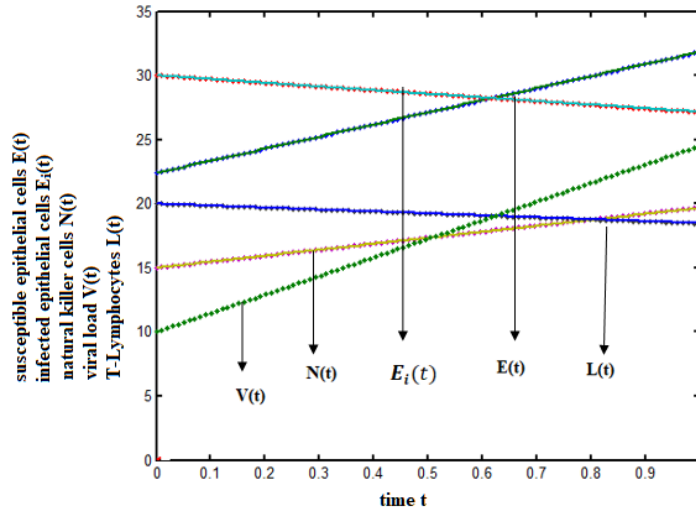


Figure 1: Illustration of numerical and analytical results for the populations of susceptible epithelial cells , infected epithelial cells , Viral Load , Natural Killer Cells and T-lymphocytes against time t.

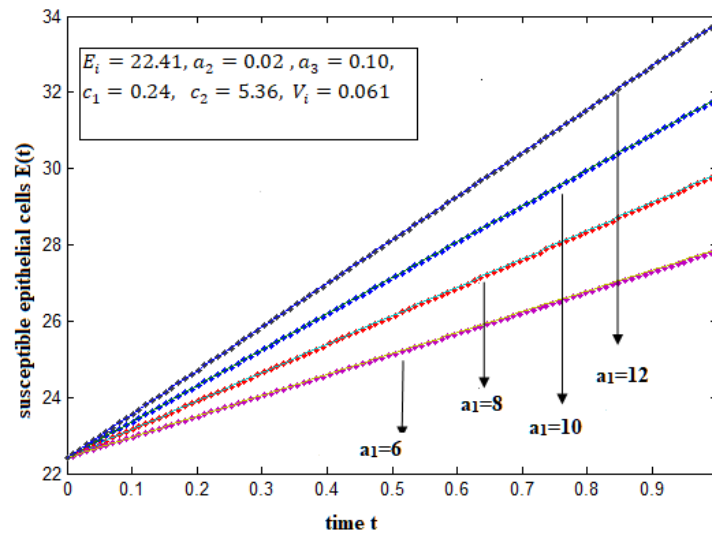


Figure 2: Illustration of analytical and graphical results for the population of susceptible against time t.

*Theoretical analysis on the growth kinetics of SARS-CoV (within host)*

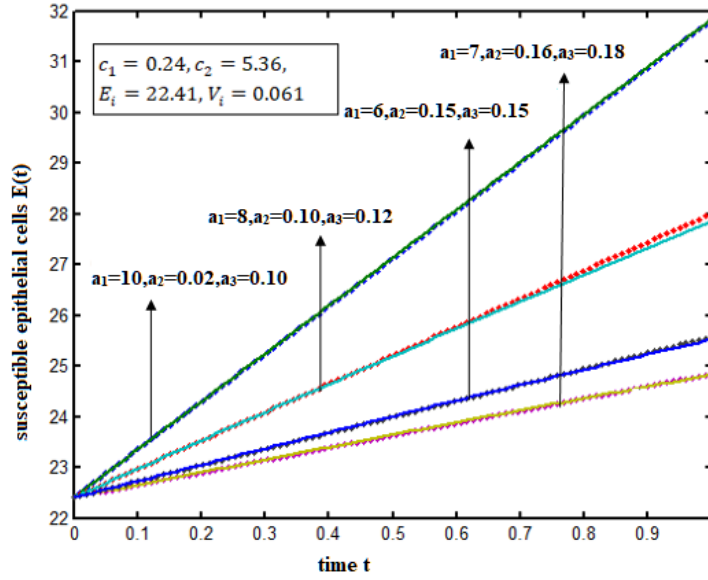


Figure 3: Illustration of analytical and graphical results for the population of susceptible against time  $t$ .

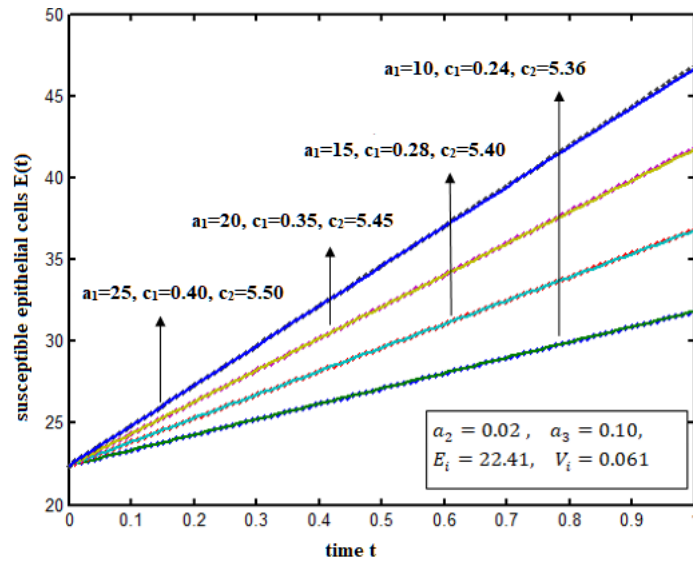


Figure 4: Illustration of analytical and graphical results for the population of susceptible against time  $t$ .

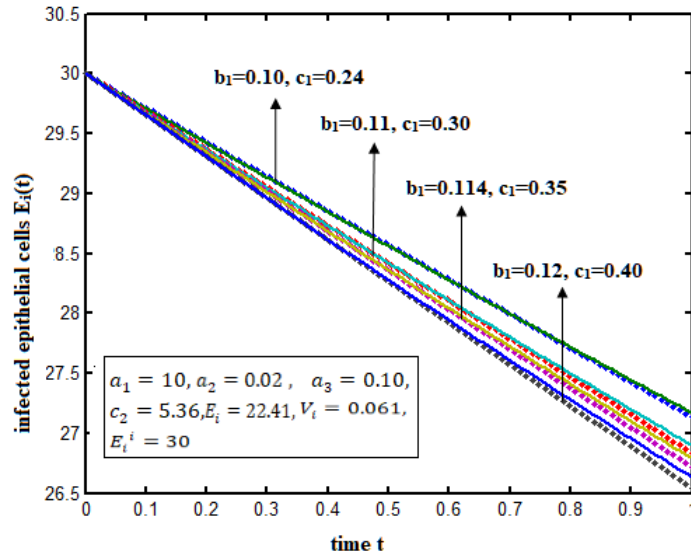


Figure 5: Illustration of analytical and graphical results for the population of susceptible against time  $t$ .

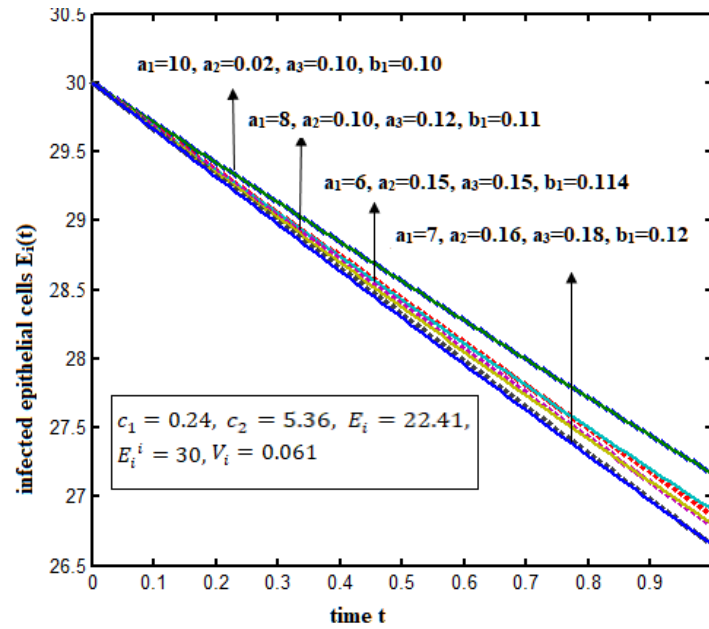


Figure 6: Illustration of analytical and graphical results for the population of infected epithelial against time  $t$ .

*Theoretical analysis on the growth kinetics of SARS-CoV (within host)*

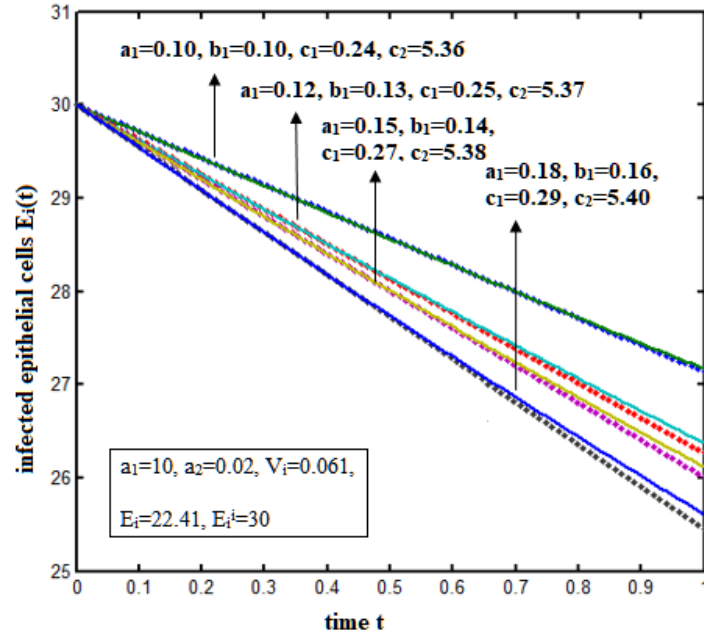


Figure 7: Illustration of analytical and graphical results for the population of infected epithelial against time  $t$ .

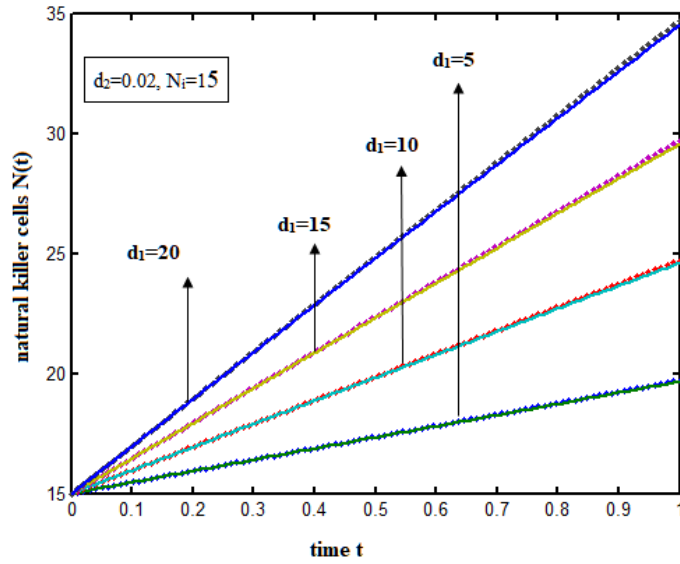


Figure 8: Illustration of analytical and graphical results for the population of natural killer cells against time  $t$ .



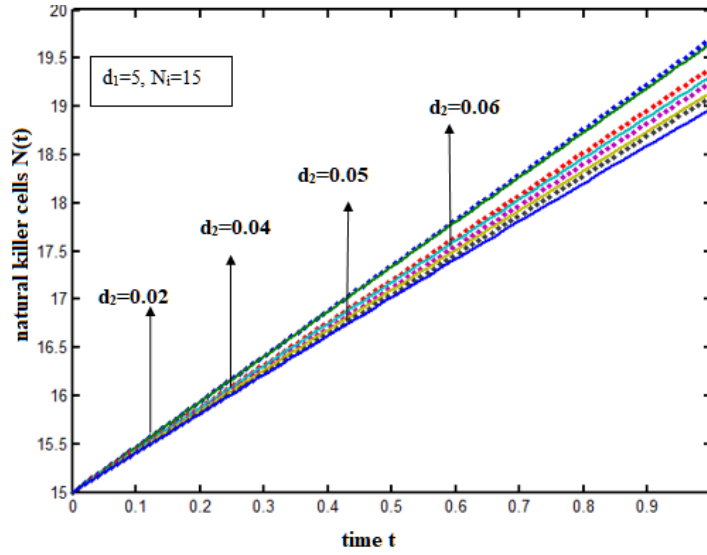


Figure 9: Illustration of analytical and graphical results for the population of natural killer cells against time  $t$ .

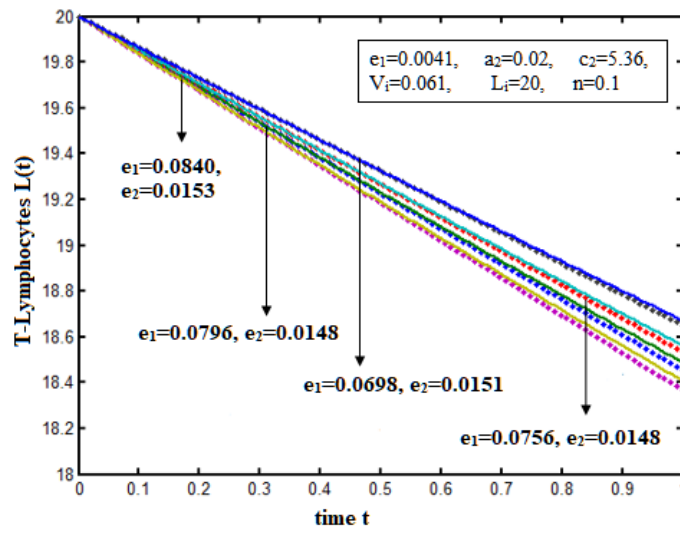


Figure 10: Illustration of analytical and graphical results for the population of T-Lymphocytes against time  $t$ .

*Theoretical analysis on the growth kinetics of SARS-CoV (within host)*

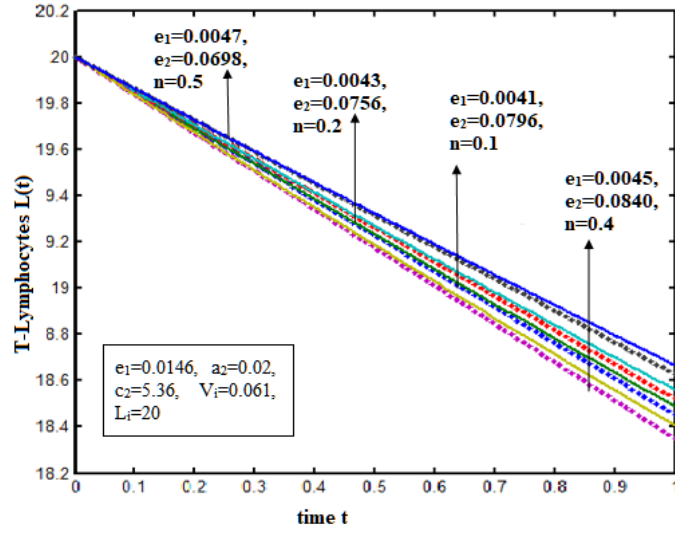


Figure 11: Illustration of analytical and graphical results for the population of T-lymphocytes against time  $t$ .

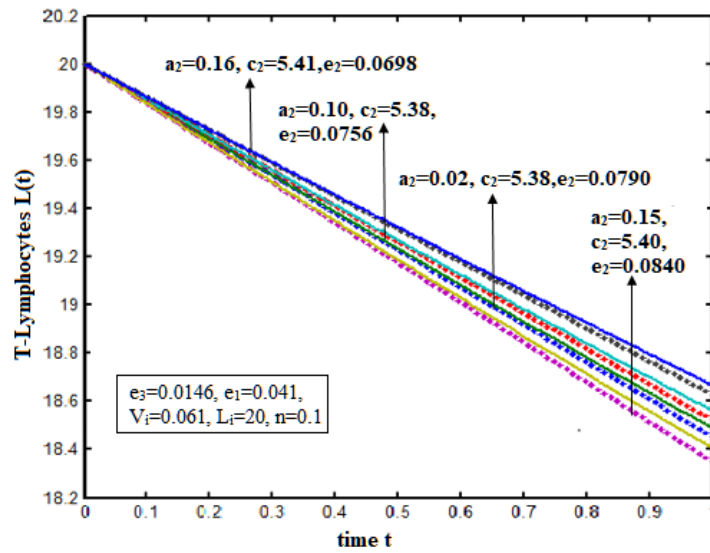


Figure 12: Illustration of analytical and graphical results for the population of T-lymphocytes against time  $t$ .

## 5.1 Results and discussion

Figure 1 illustrates the comparison between the analytical and numerical results for the populations of susceptible epithelial cells, infected epithelial cells, Viral Load, Natural Killer Cells and T-lymphocytes against time  $t$  for the parameter values  $n = 0.1, d_1 = 5, a_1 = 10, a_2 = 0.02, a_3 = 0.10, b_1 = 0.10, c_1 = 0.24, c_2 = 5.36, c_3 = 0.231, c_4 = 0.431, e_1 = 0.0041, e_2 = 0.0796, e_3 = 0.0146, d_2 = 0.02$ . Figure 2-4 presents the plot of susceptible epithelial cells against time  $t$ . Figure 5-7 presents the plot of infected epithelial cells against time  $t$ . Figure 8-9 indicated the plot of Natural Killer Cells against time  $t$ . Figure 10-12 represents the plot of T-lymphocytes against time  $t$ . Figure 2 depicts that the ratio of susceptible cells increases with time as the regeneration rate increases. Figure 3 indicates that the growth of epithelial cells is exponential when the rate of infection is higher. Figure 4 presents that even when the death of the infected cells is higher the growth of susceptible cells increases as there is an increase in the production rate of infected cells. Figure 5 depicts that there is a steady decline in the rate of infected cells as the death rate of the infected cells increases. Figure 6 presents that the rate of growth of infected cells decreases when the rate of infection of the epithelial cells is lower and also a steady decline in the death rate of infected epithelial cells. Figure 7 indicates that the growth rate of the infected epithelial cells also depends on the death rate of the virus. Figure 8 and 9 represents that rate of natural killer cells increases in spite of the infection when there is an increase in external influx. Figure 10-12 presents that proliferation rate, regeneration rate and the death rate of the T-lymphocytes has an impact on the growth of T-lymphocytes.

## 6 Conclusions

A theoretical model outlining the interactions between the SARS-Cov and the immune system within the host has been investigated by combining the functions of NK cells and T-lymphocytes. Homotopy Perturbation method is executed to solve the non-linear equations and a closed form analytical solution is obtained. Graphical illustration of the analytical and numerical solution is performed. A sound agreement between these results is noted. This comparison shows that HPM is an effective approach for solving such models.

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# Properties of nano generalized pre c-interior in a nano topological space.

Padmavathi P\*

## Abstract

The aim of this paper is to introduce and study the properties the nano generalized pre c- interior of a set such as nano generalized pre c-border and nano generalized pre c-exterior in a nano topological space.

**Keywords:** Nano generalized pre c-border, Nano generalized pre c-exterior.

**2020 AMS subject classifications:** 06F20, 06F15, 20Cxx. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on March, 2023. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1075. ISSN: 1592-7415. eISSN: 2282-8214. ©P.Padmavathi. This paper is published under the CC-BY licence agreement.

## 1 Introduction

The concept of generalized-semi closed sets to characterize the S-normality axiom was introduced by S.P.Arya et.al. The semi-generalized mappings and generalized-semi mappings were studied. In 2013 , Govindappa Navalagi investigated some of the regularity axioms, normality axioms and continuous functions through gs-open sets and sg-open sets. Also, Govindappa Navalagi continued the study of gs-continuous and sg-continuous functions to introduce the new notions like generalized semiclosure and generalized semi-interior operators. Lellis Thivagar [1] obtained the notion of nano topology and he studied the various forms of nano sets, their closures and interiors and their homeomorphisms Lellis Thivagar et al introduced nano topological space with respect to a subset of a Universe which is defined in terms of approximations and boundary region. In this paper, I have introduced the properties of nano generalized pre c-interior in a nano topological space.

## 2 Preliminaries

**Definition 2.1.** [3] Let  $\mathfrak{S}$  be a non empty finite set of objects called the universe and  $\mathfrak{R}$  be an equivalence relation on  $\mathfrak{S}$  named as indiscernibility relation. Then  $\mathfrak{S}$  is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair  $(\mathfrak{S}, \mathfrak{R})$  is said to be approximation space. Let  $\mathfrak{N} \subseteq \mathfrak{S}$ . Then

- (i) The lower approximation of  $\mathfrak{N}$  with respect to  $\mathfrak{R}$  is the set of all objects, which can be for certain classified as  $\mathfrak{N}$  with respect to  $\mathfrak{R}$  and it is denoted by  $\gamma_{\mathfrak{R}}(\mathfrak{N})$ .  $\gamma_{\mathfrak{R}}(\mathfrak{N}) = \mathfrak{S}_{x \in \mathfrak{S}} \mathfrak{R}(x) : \mathfrak{R}(x) \subseteq \mathfrak{N}$  by  $\gamma_{\mathfrak{R}}(\mathfrak{N})$ . where  $\mathfrak{R}(\mathfrak{N})$  denotes the equivalence class determined.
- (ii) The upper approximation of  $\mathfrak{N}$  with respect to  $\mathfrak{R}$  is the set of all objects which can be possibly classified as  $\mathfrak{N}$  with respect to  $\mathfrak{R}$  and it is denoted by  $\tau_{\mathfrak{R}}(\mathfrak{N})$ .  $\tau_{\mathfrak{R}}(\mathfrak{N}) = \mathfrak{S}_{x \in \mathfrak{S}} \mathfrak{R}(x) : \mathfrak{R}(x) \cap \mathfrak{N} \neq \emptyset$ .
- (iii) The boundary region of  $\mathfrak{N}$  with respect to  $\mathfrak{R}$  is the set of all objects which can be classified neither as  $\mathfrak{N}$  nor as not- $\mathfrak{N}$  with respect to  $\mathfrak{R}$  and it is denoted by  $B_{\mathfrak{R}}(\mathfrak{N})$ .  $B_{\mathfrak{R}}(\mathfrak{N}) = \tau_{\mathfrak{R}}(\mathfrak{N}) - \gamma_{\mathfrak{R}}(\mathfrak{N})$ .

**Proposition 2.1.** [3] If  $(\mathfrak{S}, \mathfrak{R})$  is an approximation space and  $\mathfrak{N}, Y \subseteq \mathfrak{S}$ , then

1.  $\gamma_{\mathfrak{R}}(\mathfrak{N}) \subseteq \mathfrak{N} \subseteq \tau_{\mathfrak{R}}(\mathfrak{N})$
2.  $\gamma_{\mathfrak{R}}(\emptyset) = \tau_{\mathfrak{R}}(\mathfrak{N}) = \emptyset$

3.  $\gamma_{\mathfrak{R}}(U) = \tau_{\mathfrak{R}}(\mathfrak{S}) = \mathfrak{S}$
4.  $\tau_{\mathfrak{R}}(\mathfrak{N} \cup Y) = \tau_{\mathfrak{R}}(\mathfrak{N}) \cup \tau_{\mathfrak{R}}(Y)$
5.  $\tau_{\mathfrak{R}}(\mathfrak{N} \cap Y) \subseteq \tau_{\mathfrak{R}}(\mathfrak{N}) \cap \tau_{\mathfrak{R}}(Y)$
6.  $\gamma_{\mathfrak{R}}(\mathfrak{N} \cup Y) \supseteq \gamma_{\mathfrak{R}}(\mathfrak{N}) \cup \gamma_{\mathfrak{R}}(Y)$
7.  $\gamma_{\mathfrak{R}}(\mathfrak{N} \cap Y) = \gamma_{\mathfrak{R}}(\mathfrak{N}) \cap \gamma_{\mathfrak{R}}(Y)$
8.  $\gamma_{\mathfrak{R}}(\mathfrak{N}) \subseteq \gamma_{\mathfrak{R}}(Y)$  and  $\tau_{\mathfrak{R}}(\mathfrak{N}) \subseteq \tau_{\mathfrak{R}}(Y)$ , whenever  $\mathfrak{N} \subseteq Y$ .
9.  $\tau_{\mathfrak{R}}(\mathfrak{N}^c) = [\gamma_{\mathfrak{R}}(\mathfrak{N})]^c$  and  $\gamma_{\mathfrak{R}}(\mathfrak{N}^c) = [\tau_{\mathfrak{R}}(\mathfrak{N})]^c$
10.  $\tau_{\mathfrak{R}}[\tau_{\mathfrak{R}}(\mathfrak{N})] = \gamma_{\mathfrak{R}}[\tau_{\mathfrak{R}}(\mathfrak{N})] = \tau_{\mathfrak{R}}(\mathfrak{N})$
11.  $\gamma_{\mathfrak{R}}[\gamma_{\mathfrak{R}}(\mathfrak{N})] = \tau_{\mathfrak{R}}[\gamma_{\mathfrak{R}}(\mathfrak{N})] = \gamma_{\mathfrak{R}}(\mathfrak{N})$

**Definition 2.2.** [1] Let  $\mathfrak{S}$  be the universe,  $\mathfrak{R}$  be an equivalence relation on  $\mathfrak{S}$  and  $r_{\mathfrak{R}}(\mathfrak{N}) = \{\mathfrak{S}, \phi, \gamma_{\mathfrak{R}}(\mathfrak{N}), \tau_{\mathfrak{R}}(\mathfrak{N}), B_{\mathfrak{R}}(\mathfrak{N})\}$  where  $\mathfrak{N} \subseteq \mathfrak{S}$ . Then  $r_{\mathfrak{R}}(\mathfrak{N})$  satisfies the following axioms.

1.  $\mathfrak{S}$  and  $\phi \in r_{\mathfrak{R}}(\mathfrak{N})$ .
2. The union of all the elements of any sub-collection of  $r_{\mathfrak{R}}(\mathfrak{N})$  is in  $r_{\mathfrak{R}}(\mathfrak{N})$ .
3. The intersection of the elements of any finite sub collection of  $r_{\mathfrak{R}}(\mathfrak{N})$  is in  $r_{\mathfrak{R}}(\mathfrak{N})$ . Then  $r_{\mathfrak{R}}(\mathfrak{N})$  is a topology on  $\mathfrak{S}$  called the nano topology on  $\mathfrak{S}$  with respect to  $\mathfrak{N}$ . The elements of  $r_{\mathfrak{R}}(\mathfrak{N})$  are called as nano open sets in  $\mathfrak{S}$  and  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is called as a nano topological space. The complement of the nano open sets are called nano closed sets.

**Definition 2.3.** [1] If  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is a nano topological space with respect to  $\mathfrak{N}$ , where  $\mathfrak{N} \subseteq \mathfrak{S}$  and if  $A \subseteq \mathfrak{S}$ , then

1. The nano interior of  $A$  is defined as the union of all nano open subsets contained in  $A$  and is denoted by  $Nint(A)$ . That is  $Nint(A)$  is the largest nano open subset of  $A$ .
2. The nano closure of  $A$  is defined as the intersection of all nano closed sets containing  $A$  and is denoted by  $Ncl(A)$ . That is  $Ncl(A)$  is the smallest nano closed set containing  $A$ .

**Definition 2.4.** [2] A subset  $A$  of a nano topological space  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is called a nano generalized pre c-closed set (briefly Ngpc-closed set) if  $Npcl(A) \subseteq G$  whenever  $A \subseteq G$  and  $G$  is nano c-set.

The complement of a Ngpc-closed set is called Ngpc-open set.



**Definition 2.5.** [2] The Nano generalized pre  $c$ -interior of a set  $A$  in  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is defined as the union of all  $Ngpc$ -open sets of  $U$  contained in  $A$  and it is denoted by  $Ngpc - int(A)$ . That is  $Ngpc - int(A)$  is the largest  $Ngpc$ -open subset of  $A$ .

**Definition 2.6.** [2] The Nano generalized pre  $c$ -closure of a set  $A$  in  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is defined as the intersection of all  $Ngpc$ -closed sets of  $U$  containing  $A$  and it is denoted by  $Ngpc - cl(A)$ . That is  $Ngpc - cl(A)$  is the smallest  $Ngpc$ -closed superset of  $A$  in  $IU$ .

**Remark 2.1.** [2]

1. A subset  $A$  of  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is  $Ngpc$ -open if and only if  $Ngpc - int(A) = A$ .
2. A subset  $A$  of  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is  $Ngpc$ -closed if and only if  $Ngpc - cl(A) = A$ .

**Theorem 2.1.** [2] Let  $A$  and  $B$  be subsets of  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$ . Then

1.  $Ngpc - int(\mathfrak{S}) = \mathfrak{S}$  and  $Ngpc - int(\phi) = \phi$ .
2.  $Ngpc - int(A) \subset A$ .
3. If  $B$  is any  $Ngpc$ -open set contained in  $A$ , then  $B \subset Ngpc - int(A)$ .
4. If  $A \subset B$  then  $Ngpc - int(A) \subseteq Ngpc - int(B)$ .
5.  $Ngpc - int(Ngpc - int(A)) = Ngpc - int(A)$ .

**Theorem 2.2.** [2] If  $A$  and  $B$  are subsets of  $\mathfrak{S}$ , then the following statements are true.

1.  $Ngpc - int(A) \cup Ngpc - int(B) \subset Ngpc - int(A \cup B)$ .
2.  $Ngpc - int(A \cap B) = Ngpc - int(A) \cap Ngpc - int(B)$ .

**Theorem 2.3.** [2] If  $A$  is a subset of  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$ , then  $Nint(A) \subset Ngpc - int(A)$ .

**Theorem 2.4.** [2] For the subsets  $A$  and  $B$  of  $\mathfrak{S}$ , the following statements are true.

1.  $\mathfrak{S} - Ngpc - cl(A) \subset Ngpc - cl(\mathfrak{S} - A)$ .
2. If  $A$  is  $Ngpc$ -closed then  $Ngpc - cl(A) - Ngpc - cl(B) \subset Ngpc - cl(A - B)$ .

### 3 Properties of nano generalized pre c-interior

In this section the nano generalized pre c-border and nano generalized pre c-exterior of a set are defined in terms of nano generalized pre c-interior and some of their properties are derived.

**Definition 3.1.** The nano generalized pre c-border of a set  $A$  in  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is defined as  $A - Ngpc - int(A)$  and it is denoted by  $Ngpc - Bd(A)$ .

**Definition 3.2.** The nano generalized pre c-exterior of a set  $A$  in  $(\mathfrak{S}, r_{\mathfrak{R}}(\mathfrak{N}))$  is defined as  $Ngpc - int(\mathfrak{S} - A)$  and it is denoted by  $Ngpc - ext(A)$ .

**Example 3.1.** Let  $\mathfrak{S} = \{a, b, c, d\}$  with  $\mathfrak{S}/\mathfrak{R} = \{\{a\}, \{b\}, \{c, d\}\}$  and  $\mathfrak{N} = \{b, d\}$ . Then  $r_{\mathfrak{R}}(\mathfrak{N}) = \{\mathfrak{S}, \phi, \{b\}, \{c, d\}, \{b, c, d\}\}$  is a nano topology on  $U$  with respect to  $\mathfrak{N}$ . The complement of  $r_{\mathfrak{R}}(\mathfrak{N})$  is given by  $r_C(\mathfrak{N}) = \{U, \phi, \{a\}, \{a, b\}, \{a, c, d\}$ . Ngpc-closed sets are  $\{\phi, \mathfrak{S}, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ . Ngpc-open sets are  $\phi, \mathfrak{S}, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}$ . Here  $Ngpc - int(\{a\}) = \phi$ ,  $Ngpc - int(\{b\}) = \{b\}$ ,  $Ngpc - int(\{a, c, d\}) = \{c, d\}$ ,  $Ngpc - int(\{c, d\}) = \{c, d\}$  and  $Ngpc - int(\{a, b, d\}) = \{a, b, d\}$ . Then  $Ngpc - Bd(\{a\}) = \{a\}$ ,  $Ngpc - Bd(\{b\}) = \phi$ ,  $Ngpc - Bd(\{a, b, d\}) = \phi$  and  $Ngpc - Bd(\{a, c, d\}) = \{a\}$ .  $Ngpc - ext(\{a\}) = \{b, c, d\}$ ,  $Ngpc - ext(\{b\}) = \{c, d\}$ ,  $Ngpc - ext(\{a, b\}) = \{c, d\}$  and  $Ngpc - ext(\{a, c, d\}) = \{b\}$ .

**Theorem 3.1.** For a subset  $A$  of  $\mathfrak{S}$  the following statements hold.

1.  $Ngpc - Bd(\phi) = Ngpc - Bd(\mathfrak{S}) = \phi$ .
2.  $Ngpc - Bd(A) \subset NBd(A)$ .
3.  $A = Ngpc - int(A) \cup Ngpc - Bd(A)$ .
4.  $Ngpc - int(A) \cap Ngpc - Bd(A) = \phi$ .
5.  $Ngpc - int(A) = A - Ngpc - Bd(A)$ .
6.  $Ngpc - int(Ngpc - Bd(A)) = Ngpc - Bd(Ngpc - int(A)) = \phi$ .
7.  $A$  is Ngpc-open if and only if  $Ngpc - Bd(A) = \phi$ .
8.  $Ngpc - Bd(Ngpc - Bd(A)) = Ngpc - Bd(A)$ .

*Proof.* 1. The proof is an immediate consequence of definition (3.1).

2. Let  $x \in Ngpc - Bd(A)$ .  $\Rightarrow x \in A - Ngpc - int(A)$ . By theorem (2.3),  $Nint(A) \subset Ngpc - int(A) \Rightarrow A - Ngpc - int(A) \subset A - Nint(A)$ .  
Hence  $x \in A - Ngpc - int(A) \Rightarrow x \in A - Nint(A)$ .  
 $\Rightarrow x \in NBd(A)$ . Therefore  $Ngpc - Bd(A) \subset NBd(A)$ .
3.  $Ngpc - int(A) \cup Ngpc - Bd(A) = Ngpc - int(A) \cup (A - Ngpc - int(A)) = A$ .
4.  $Ngpc - int(A) \cap Ngpc - Bd(A) = Ngpc - int(A) \cap (A - Ngpc - int(A)) = \phi$ .
5. The proof directly follows from definition (3.1).
6. Let  $x \in Ngpc - int(Ngpc - Bd(A))$ . Then  $x \in Ngpc - Bd(A)$  as  $Ngpc - Bd(A) \subset A$ .  
Also  $x \in Ngpc - int(Ngpc - Bd(A)) \subset Ngpc - int(A)R$ .  
Therefore  $x \in Ngpc - int(A) \cap Ngpc - Bd(A)$  which is a contradiction to (d).  
Hence  $Ngpc - int(Ngpc - Bd(A)) = \phi$ .
7. By result (2.8),  $A$  is  $Ngpc - open \Leftrightarrow Ngpc - int(A) = A \Leftrightarrow A - Ngpc - int(A) = \phi$   
 $\Leftrightarrow Ngpc - Bd(A) = \phi$ . (by definition (3.1))
8. In definition (3.1) let  $A = Ngpc - Bd(A)$ .  
Then  $Ngpc - Bd(Ngpc - Bd(A)) = Ngpc - Bd(A) - Ngpc - int(Ngpc - Bd(A)) = Ngpc - Bd(A) - \phi$   
 $= Ngpc - Bd(A)$ . (Using (6)).

□

**Theorem 3.2.** For the subsets  $A$  and  $B$  of  $\mathfrak{S}$  the following statements hold.

1.  $Ngpc - ext(\phi) = \mathfrak{S}$  and  $Ngpc - ext(\mathfrak{S}) = \phi$ .
2.  $Next(A)Ngpc - ext(A)$ .
3. If  $A \subset B$  then  $Ngpc - ext(B) \subset Ngpc - ext(A)$ .
4.  $Ngpc - ext(A)$  is  $Ngpc - open$ .
5.  $Ngpc - ext(A) = \mathfrak{S} - Ngpc - cl(A)$ .
6.  $A$  is  $Ngpc - closed$  if and only if  $Ngpc - ext(A) = \mathfrak{S} - A$ .
7.  $Ngpc - ext(Ngpc - ext(A)) = Ngpc - int(Ngpc - cl(A))$

*Properties of nano generalized pre c-interior in a nano topological space*

8.  $Ngpc - ext(Ngpc - ext(A)) = Ngpc - ext(Ngpc - int(\mathfrak{S} - A)) = Ngpc - ext(\mathfrak{S} - Ngpc - cl(A)).$
9.  $Ngpc - ext(A \cup B) \subset Ngpc - ext(A) \cup Ngpc - ext(B).$
10.  $Ngpc - ext(A \cup B) = Ngpc - ext(A) \cap Ngpc - ext(B).$
11.  $Ngpc - ext(A) \cap Ngpc - ext(B) \subset Ngpc - ext(A \cap B).$

*Proof.* 1. The proof is immediate from definition (3.2).

2.  $N_{ext}(A) \subset Ngpc - ext(A)$  follows from theorem (2.3).
3. If  $A \subset B$  then  $\mathfrak{S} - B \subset \mathfrak{S} - A$ . By (iv) of theorem (2.2),  $Ngpc - int(\mathfrak{S} - B) \subset Ngpc - int(\mathfrak{S} - A)$ . Hence  $Ngpc - ext(B) \subset Ngpc - ext(A)$ .
4. Consider  $Ngpc - int(Ngpc - ext(A)) = Ngpc - int(Ngpc - int(\mathfrak{S} - A)) = Ngpc - int(\mathfrak{S} - A) = Ngpc - ext(A)$ . (by (v) of theorem (2.9)) By remark (2.1),  $Ngpc - ext(A)$  is  $Ngpc$ -open.
5.  $Ngpc - ext(A) = Ngpc - int(\mathfrak{S} - A) = \mathfrak{S} - Ngpc - cl(A)$ . (from (ii) of theorem (2.4)).
6. By remark (2.1),  $A$  is  $Ngpc$ -closed  $\Leftrightarrow Ngpc - cl(A) = A \Leftrightarrow \mathfrak{S} - Ngpc - cl(A) = \mathfrak{S} - A - Ngpc - int(\mathfrak{S} - A) = \mathfrak{S} - A \Leftrightarrow Ngpc - ext(A) = \mathfrak{S} - A$ .
7. In definition let  $A = Ngpc - ext(A)$ . Then  $Ngpc - ext(Ngpc - ext(A)) = Ngpc - int(\mathfrak{S} - Ngpc - ext(A)) = Ngpc - int(Ngpc - cl(A))$ . (Using (5)).
8. It follows from definition (3.2) and (5).
9. We know that  $A \subset A \cup B$  and  $B \subset A \cup B$ . From (c)  $Ngpc - ext(A \cup B) \subset Ngpc - ext(B)$  and  $Ngpc - ext(A \cup B) \subset Ngpc - ext(A)$ . Hence  $Ngpc - ext(A \cup B) \subset Ngpc - ext(A) \cap Ngpc - ext(B)$ .
10.  $Ngpc - ext(A \cup B) = Ngpc - int(\mathfrak{S} - (A \cup B))$ . (by definition (3.2))  
 $= Ngpc - int((\mathfrak{S} - A) \cap (\mathfrak{S} - B))$ .  
 $= Ngpc - int(\mathfrak{S} - A) \cap Ngpc - int(\mathfrak{S} - B)$ . (by (ii) of theorem (2.4))  
 $= Ngpc - ext(A) \cap Ngpc - ext(B)$ .  
Hence  $Ngpc - ext(A \cup B) = Ngpc - ext(A) \cap Ngpc - ext(B)$ .

11. We know that  $A \cap B \subset A$  and  $A \cap B \subset B$ . From (c)  $Ngpc - ext(A) \subset Ngpc - ext(A \cap B)$  and  $Ngpc - ext(B) \subset Ngpc - ext(A \cap B)$ . Hence  $Ngpc - ext(A) \cap Ngpc - ext(B) \subset Ngpc - ext(A \cap B)$ . □

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# Transversal core of intuitionistic fuzzy $k$ -partite hypergraphs

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## Abstract

In graph theory, a transversal is a set of vertices incident to every edge in a graph but in Intuitionistic Fuzzy  $k$ -Partite Hypergraph(IF $k$ -PHG), the transversal is a hyperedge which cuts every hyperedges. In this article, Intuitionistic Fuzzy Transversal(IFT), minimal IFT, locally minimal IFT, IFTC(Intuitionistic Fuzzy Transversal Core) of IF $k$ -PHG has been defined. It has been proved that every IF $k$ -PHG has a nonempty IFT. Also few of the properties relating to the transversal of IF $k$ -PHG were discussed.

**Keywords:** IFT; minimal IFT; locally minimal IFT; IFTC of IF $k$ -PHG.

**2020 AMS subject classifications:** 34K36, 57Q65, 05C65, 93B20.

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1076. ISSN: 1592-7415. eISSN: 2282-8214. ©Myithili K.K et. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Euler was the first author who found graph theory in 1736. The Graph theoretical approach is widely used to solve numerous issues in different areas like computer science, optimization, algebra and number theory. As an application part, the concept of graph has been extended to hypergraph, an edge with more than one or two vertices. An idea of graph and hypergraph was popularized by Berge [1976] in 1976. Fuzzy graph and fuzzy hypergraph concepts are developed by the authors in J.N.Mordeson and Nair [2000]. In K.T.Atanassov [1999], the author wrote ideas of Intuitionistic Fuzzy Sets(IFS). According to K.T.Atanassov [2002], K.T.Atanassov [2012], the researcher putforth ideas of intuitionistic fuzzy relations and cartesian products are defined.

In Myithili and Keerthika [2020a], the authors proposed the notion of  $k$ -partite hyperedges in IFHG(Intuitionistic Fuzzy Hypergraphs). Certain operations like Union, Intersection, Ringsum, Cartesian Product were discussed in Myithili and Keerthika [2020b] . It has numerous application problems in decision-making. In Myithili and Parvathi [2015], Myithili and Parvathi [2016], Myithili et al. [2014] transversals and its properties on intuitionistic fuzzy directed hypergraphs were discussed.

The authors in Goetschel [1995], Goetschel [1998], Goetschel et al. [1996] initiated the concepts like fuzzy transversal and fuzzy coloring in fuzzy hypergraph. In this article an attempt has been made to analyze the transversal and its related properties in IF $k$ -PHGs.

## 2 Symbolic representation

MNMV-membership and non-membership values

FSV-Finite set of vertices

FIFS-family of intuitionistic fuzzy subsets

IFH-intuitionistic fuzzy hyperedge

ONV-Open Neighborhood of the vertex

CNV-Closed Neighborhood of the vertex

$\aleph = (\vee, \Xi, \psi)$  - Intuitionistic fuzzy(IF)  $k$ -partite hypergraph with edge set  $\Xi$ , vertex set  $\vee$  and  $k$ -partite hyperedge  $\psi$

$h(\aleph)$  - Height of IF $k$ -PHG

$F_k(\aleph)$  - Fundamental sequence (FS) of IF $k$ -PHG

$c(\aleph)$  - Core set(CS) of IF $k$ -PHG

$I_k(\aleph)$  - Induced fundamental sequence(IFS) of IF $k$ -PHG

- $\aleph^{(a_i, b_i)}$  -  $(a_i, b_i)$ -level of IF $k$ -PHG  
 $(a_i, b_i)$  - Edge membership(EM) and non-membership values(ENMV)  
 $\mathcal{TR}(\aleph)$  - minimal intuitionistic fuzzy transversal(MIFT) of IF $k$ -PHG

### 3 Preliminaries

**Definition 3.1.** *Myithili and Keerthika [2020a] The IF $k$ -PHG  $\aleph$  is an ordered triple  $\aleph = (\vee, \Xi, \psi)$  where,*

- $\vee = \{g_1, g_2, g_3, \dots, g_n\}$  is a FSV,
- $\Xi = \{\Xi_1, \Xi_2, \Xi_3, \dots, \Xi_m\}$  is a FIFS of  $\vee$ ,
- $\Xi_j = \{(g_i, \omega_j(g_i), \nu_j(g_i)) : \omega_j(g_i), \nu_j(g_i) \geq 0, \omega_j(g_i) + \nu_j(g_i) \leq 1\}, 1 \leq j \leq m$ ,
- $\Xi_j \neq \emptyset, 1 \leq j \leq m$ ,
- $\bigcup_j \text{supp}(\Xi_j) = \vee, 1 \leq j \leq m$ .

For all  $g_i \in \Xi \exists k$  - disjoint sets  $\psi_i, i = 1, 2, \dots, k$  and no two vertices in the same set are adjacent such that  $\Xi_k = \bigcap_{i=1}^k \psi_i = \emptyset$

**Definition 3.2.** *Myithili and Keerthika [2020a] Let an IF $k$ -PHG be  $\aleph = (\vee, \Xi, \psi)$ . The height of IF $k$ -PHG is defined by  $h(\aleph) = \{\max(\min(\omega_{k_{ij}})), \max(\max(\nu_{k_{ij}}))\}$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Also  $\omega_{k_{ij}}$  and  $\nu_{k_{ij}}$  are MNMV of the  $k$ -partite hyperedge  $\psi_{ij}$ .*

**Definition 3.3.** *Myithili and Keerthika [2020a] Let  $\aleph$  be an IF $k$ -PHG. Suppose  $\psi_j, \psi_k \in \psi$  and  $0 < \delta, \varepsilon \leq 1$ . The  $(\delta, \varepsilon)$ -level is defined by  $(\psi_j, \psi_k)^{(\delta, \varepsilon)} = \{g_i \in \vee \mid \min(\omega_{k_{ij}}^\delta(g_i)) \geq \delta, \max(\nu_{k_{ij}}^\varepsilon(g_i)) \leq \varepsilon\}$ .*

**Definition 3.4.** *Myithili and Keerthika [2020a] Let  $\aleph$  be IF $k$ -PHG,  $\aleph^{a_i, b_i} = \langle \vee^{a_i, b_i}, \Xi^{a_i, b_i} \rangle$  be the  $(a_i, b_i)$ -level of  $\aleph$ . The sequence of real numbers  $\{a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k\} \ni 0 \leq a_i \leq h_\omega(\aleph)$  and  $0 \leq b_i \leq h_\nu(\aleph)$ , satisfies:*

- (i) If  $a_1 < \delta \leq 1$  &  $0 \leq \varepsilon < b_1$  then  $\psi^{\delta, \varepsilon} = \emptyset$ ,
- (ii) If  $a_{i+1} \leq \delta \leq a_i; b_i \leq \varepsilon \leq b_{i+1}$  then  $\psi^{\delta, \varepsilon} = \psi^{a_i, b_i}$ ,
- (iii)  $\psi^{a_i, b_i} \sqsubset \psi^{a_{i+1}, b_{i+1}}$  is fundamental sequence of IF $k$ -PHG and it is denoted as  $F_k(\aleph)$ .

**Definition 3.5.** *Myithili and Keerthika [2020a] Let  $c(\aleph) = \{\aleph^{a_1, b_1}, \aleph^{a_2, b_2}, \dots, \aleph^{a_k, b_k}\}$  be core set of  $\aleph$ . The analogous set of  $(a_i, b_i)$ -level hypergraphs is  $\aleph^{a_1, b_1} \subset \aleph^{a_2, b_2} \subset \dots \subset \aleph^{a_k, b_k}$  is said to be  $\aleph$ -IFS and it is denoted by  $I_k(\aleph)$ . The  $(a_k, b_k)$ -level is known as support level of  $\aleph$ .  $\aleph^{a_k, b_k}$  is known as the support of  $\aleph$ .*

**Definition 3.6.** *Myithili and Keerthika [2020a] Let  $\aleph = (\vee, \Xi, \psi)$  &  $\aleph' = (\vee', \Xi', \psi')$  are IF $k$ -PHGs,  $\aleph$  is known as partial IF $k$ -PHG of  $\aleph'$ , if*



$$\psi' = \begin{cases} \min(\text{supp}(\omega_{k_{ij}})) \mid \omega_{k_{ij}} \in \psi' \\ \max(\text{supp}(\nu_{k_{ij}})) \mid \nu_{k_{ij}} \in \psi' \end{cases}$$

the partial IFk-PHG generated by  $\psi'$  and is represented as  $\aleph \subseteq \aleph'$ . Also, if  $\aleph \subseteq \aleph'$  and  $\aleph \neq \aleph'$  exists then  $\aleph \subset \aleph'$ .

**Definition 3.7.** Myithili and Keerthika [2020a] Let  $\aleph$  be the IFk-PHG,  $c(\aleph) = \{\aleph^{a_1, b_1}, \aleph^{a_2, b_2}, \dots, \aleph^{a_k, b_k}\}$ .  $\aleph$  is called as ordered if  $c(\aleph)$  is ordered (i.e)  $\aleph^{a_1, b_1} \subset \aleph^{a_2, b_2} \subset \dots \subset \aleph^{a_k, b_k}$ . The IFk-PHG is known as simply ordered if  $\{\aleph^{a_i, b_i} \mid i = 1, 2, \dots, k\}$  is simply ordered, (i.e) if it is ordered and if  $\psi \in \aleph^{a_{i+1}, b_{i+1}} \setminus \aleph^{a_i, b_i}$  then  $\psi \not\subseteq \aleph^{a_i, b_i}$ .

## 4 Main results

**Definition 4.1.** Consider an IFk-PHG  $\aleph$ . An IFT  $\mathcal{T}$  of IFk-PHG is an IF subset of  $\vee$  with  $\mathcal{T}^{(\psi_j, \psi_k)} \cap \mathcal{A}^{(\psi_j, \psi_k)} \neq \emptyset$  for each  $\mathcal{A} \in \psi$  where  $\psi_j = \min(\omega_{k_{ij}})$  and  $\psi_k = \max(\nu_{k_{ij}}) \forall 1 \leq i \leq m, 1 \leq j \leq n$ . Also  $\omega_{k_{ij}}$  and  $\nu_{k_{ij}}$  is the MNMV of  $k^{th}$  partition of  $j^{th}$  edge in  $i^{th}$  vertex.

**Definition 4.2.** A minimal IFT  $\mathcal{T}$  for IFk-PHG be a transversal of  $\aleph$ , which satisfies the condition that if  $\mathcal{T}_1 \subset \mathcal{T}$ , then  $\mathcal{T}_1$  is not IFT of  $\aleph$ .

**Note:** The set of minimal IFT of IFk-PHG is denoted as  $\mathcal{TR}(\aleph)$ . Always  $\mathcal{TR}(\aleph) \neq \emptyset$ .

**Example 4.1.** An IFH (intuitionistic fuzzy hypergraph) with  $\vee = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ ,  $\Xi = \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$  has been considered.

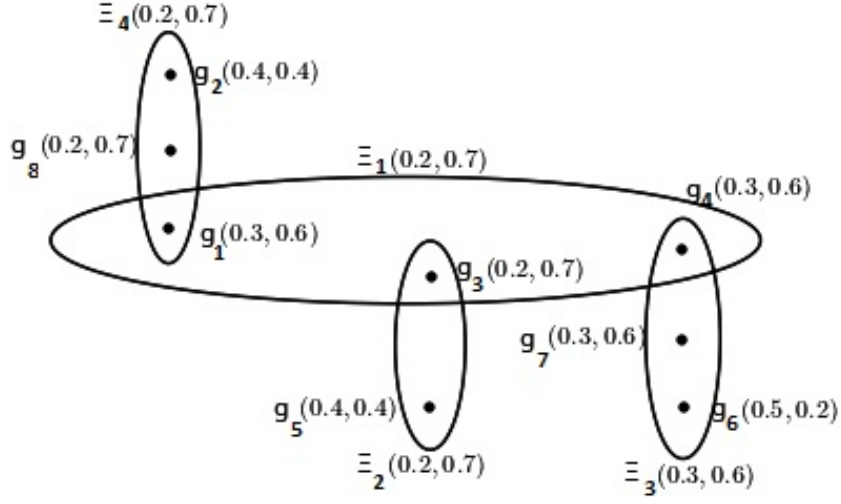


Figure 1: Intuitionistic Fuzzy Hypergraph

Using the above figure we can construct an IF $k$ -PHG  $\mathfrak{H}$ , with  $\psi = \{\psi_1, \psi_2, \psi_3\}$  disjoint hyperedges which are represented below as incidence matrix

$$\begin{array}{c}
 \psi_1 \quad \psi_2 \quad \psi_3 \\
 \begin{array}{c}
 \mathbf{g}_1 \\
 \mathbf{g}_2 \\
 \mathbf{g}_3 \\
 \mathbf{g}_4 \\
 \mathbf{g}_5 \\
 \mathbf{g}_6 \\
 \mathbf{g}_7 \\
 \mathbf{g}_8
 \end{array}
 \begin{pmatrix}
 \langle 0.3, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0.4, 0.4 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0.2, 0.7 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.6 \rangle \\
 \langle 0.4, 0.4 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0.5, 0.2 \rangle & \langle 0, 1 \rangle \\
 \langle 0.3, 0.6 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.7 \rangle
 \end{pmatrix}
 \end{array}$$

The minimal IFT of IF $k$ -PHG is attained as follows,

$$\begin{array}{c}
 \mathcal{T}_1 \qquad \mathcal{T}_2 \\
 \begin{array}{l}
 \mathbf{g}_1 \\
 \mathbf{g}_2 \\
 \mathbf{g}_3 \\
 \mathbf{g}_4 \\
 \mathbf{g}_5 \\
 \mathbf{g}_6 \\
 \mathbf{g}_7 \\
 \mathbf{g}_8
 \end{array}
 \begin{pmatrix}
 \langle 0.3, 0.6 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0.2, 0.7 \rangle & \langle 0.2, 0.7 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
 \langle 0, 1 \rangle & \langle 0.3, 0.6 \rangle \\
 \langle 0.2, 0.7 \rangle & \langle 0.2, 0.7 \rangle
 \end{pmatrix}
 \end{array}$$

The corresponding graph is shown below.

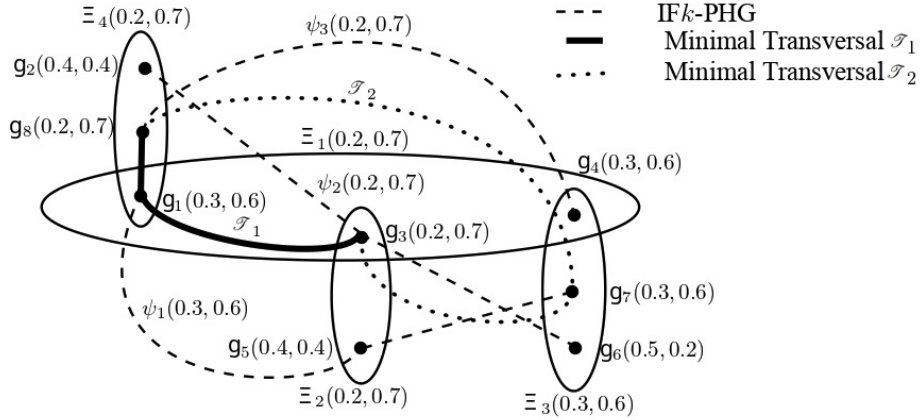


Figure 2:  $\aleph$  and minimal IFT of  $\aleph$

**Definition 4.3.** If  $\mathcal{T}$  is an IFS with  $\mathcal{T}^{(a_i, b_i)}$  as minimal IFT (MIFT) of  $\aleph^{(a_i, b_i)}$  for each  $(a_i, b_i) \in (0, 1)$  then  $\mathcal{T}$  is called as locally minimal IFT (LMIFT) of IFk-PHG. The set containing LMIFT of IFk-PHG is written as  $\mathcal{T}\mathcal{R}^*(\aleph)$

**Theorem 4.1.** If  $\mathcal{T}$  is an IFT of  $\aleph$  then  $h(\mathcal{T}) \geq h(\psi_j)$  for  $\psi_j \in \psi$ . Also, if  $\mathcal{T}$  is the minimal IFT of IFk-PHG, then

## Transversal Core of Intuitionistic Fuzzy $k$ -Partite Hypergraphs

$$h(\mathcal{T}) = \{ \max(\min(\omega_{k_{ij}})), \max(\max(\nu_{k_{ij}})) \mid \omega_{k_{ij}}, \nu_{k_{ij}} \in \psi \} = h(\aleph).$$

**Theorem 4.2.** Every IF $k$ -PHG has a nonempty IFT.

**Note:** Every IFT of IF $k$ -PHG contains a MIFT.

**Theorem 4.3.** If  $\mathcal{T}' \in \mathcal{TR}(\aleph)$  and for every  $g \in \vee$ ,  $\mathcal{T}'(g) \in F_k(\aleph)$ , then  $F_k(\mathcal{TR}(\aleph)) \subseteq F_k(\aleph)$ .

**Theorem 4.4.**  $\mathcal{TR}(\aleph)$  is sectionally elementary.

**Proof.** Let  $F_k(\mathcal{TR}(\mathcal{H})) = a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k$ . Assume that  $\mathcal{T}' \in \mathcal{TR}(\mathcal{H})$  and some  $\delta, \varepsilon \in (a_i, b_i)$  such that  $\mathcal{T}'^{(a_i, b_i)} \subset \mathcal{T}'^{(\delta, \varepsilon)}$ . Since  $\mathcal{TR}(\aleph^{a_i, b_i}) = \mathcal{TR}(\aleph^{\delta, \varepsilon})$ ,  $\exists$  some  $\mathcal{A} \in \mathcal{TR}(\mathcal{H}) \ni \mathcal{A}^{a_i, b_i} = \mathcal{T}'^{\delta, \varepsilon}$ . Then  $\mathcal{T}'^{\delta, \varepsilon} \subset \mathcal{A}^{a_i, b_i}$  implies the IFS  $\vee(g_i)$  defined by

$$\vee(g_i) = \begin{cases} (\delta, \varepsilon) & \text{if } x \in \mathcal{A}^{a_i, b_i} \setminus \mathcal{T}'^{a_i, b_i} \\ \mathcal{A}(g_i) & \text{Otherwise} \end{cases}$$

is an IFT of IF $k$ -PHG. Here  $\vee < \mathcal{A}$ , implies the contradiction of minimality (CM) of  $\mathcal{A}$ .

**Theorem 4.5.** For every  $\mathcal{A} \in \mathcal{TR}(\aleph)$ ,  $\mathcal{A}^{a_1, b_1}$  is a minimal IFT of  $\aleph^{a_1, b_1}$ .

**Proof.** For any IF $k$ -PHG  $\aleph = (\vee, \Xi, \psi)$ , consider a minimal IFT  $\mathcal{T}$  of  $\aleph^{a_1, b_1}$  such that  $\mathcal{T} \subset \mathcal{A}^{a_1, b_1}$ .

Define the IFS  $\vee(g_i)$  where

$$\vee(g_i) = \begin{cases} (a_2, b_2) & \text{if } x \in \mathcal{A}^{a_1, b_1} \setminus \mathcal{T} \\ \mathcal{A}(g_i) & \text{Otherwise} \end{cases}$$

By the above theorem,  $\vee$  is an IFT of IF $k$ -PHG, CM of  $\mathcal{A}$ .

**Definition 4.4.** Let  $\aleph$  be IF $k$ -PHG. The Intuitionistic Fuzzy Transversal Core (IFTC) of  $\aleph$  is  $\aleph' = (\vee', \Xi', \psi')$  with the following condition that

- (i)  $\min \mathcal{TR}(\aleph) = \min \mathcal{TR}(\aleph')$ ,
- (ii)  $\aleph' = \cup \min \mathcal{TR}(\aleph)$ ,
- (iii)  $\psi \setminus \psi'$  is exactly the set containing vertices of  $\aleph$  which does not belong to  $\mathcal{TR}(\aleph)$ , where  $\psi'$  is the remaining hyperedge set, after deleting hyperedges that are correctly contained in another hyperedge.

The remarks of the statement is,

- (i) For any IF $k$ -PHG without spike hyperedges,  $\exists$  transversal core which are always unique.
- (ii) The definition also holds good for IF $k$ -PHGs with spike (a hyperedge with single vertex) hyperedges.

**Definition 4.5.** In IFk-PHG, the ONV  $g_i$  is the set containing adjacent vertices of  $g_i$  except itself in a  $k$ -partite hyperedge and is denoted as  $N_k(g_i)$ .

**Example 4.2.** Consider an IFk-PHG with  $\mathcal{V} = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ ,  
 $\Xi = \{\Xi_1, \Xi_2, \Xi_3\}$  where,  
 $\Xi_1 = \{g_1 \langle 0.5, 0.2 \rangle, g_2 \langle 0.3, 0.4 \rangle, g_3 \langle 0.6, 0.3 \rangle\}$ ,  
 $\Xi_2 = \{g_2 \langle 0.3, 0.4 \rangle, g_4 \langle 0.2, 0.5 \rangle, g_5 \langle 0.3, 0.4 \rangle\}$ ,  
 $\Xi_3 = \{g_3 \langle 0.6, 0.3 \rangle, g_6 \langle 0.4, 0.3 \rangle, g_7 \langle 0.1, 0.7 \rangle\}$  with  
 $\psi_1 = \{g_1 \langle 0.5, 0.2 \rangle, g_4 \langle 0.2, 0.5 \rangle, g_7 \langle 0.1, 0.7 \rangle\}$ ,  $\psi_2 = \{g_2 \langle 0.3, 0.4 \rangle, g_6 \langle 0.4, 0.3 \rangle\}$ ,  
 $\psi_3 = \{g_3 \langle 0.6, 0.3 \rangle, g_5 \langle 0.3, 0.4 \rangle\}$   
 Here  $g_1$  and  $g_7$  are the ONV  $g_4$  in  $\psi_1$ .

**Definition 4.6.** In IFk-PHG, the CNV  $g_i$  is the set containing adjacent vertices of  $g_i$  including the vertex in a  $k$ -partite hyperedge and is denoted as  $N_k[g_i]$ .

**Example 4.3.** From the above example it is clear that the Closed Neighborhood of the vertex  $g_3$  is  $g_3$  and  $g_5$  in  $\psi_3$ .

**Theorem 4.6.** In  $\aleph$ , the following claims are related

- (i)  $\mathcal{T}$  is an IFT of IFk-PHG,
- (ii)  $\mathcal{T}^{a_i, b_i} \cap \mathcal{A}^{a_i, b_i} \neq \emptyset$ , for all IFH  $\mathcal{A} \in \psi$  and every  $(a_i, b_i)$  with  $0 < a_i \leq h_\omega(\aleph)$ ,  $0 < b_i \leq h_\nu(\aleph)$ ,
- (iii)  $\mathcal{T}^{a_i, b_i}$  is an IFT of  $\aleph^{a_i, b_i}$ , for each  $(a_i, b_i)$  with  $0 < a_i \leq \delta$ ,  $0 < b_i \leq \varepsilon$ .

**Proof.** From the definition, "A minimal IFT  $\mathcal{T}$  for IFk-PHG is a transversal of  $\aleph$ , which satisfies the property that if  $\mathcal{T}_1 \subset \mathcal{T}$ , then  $\mathcal{T}_1$  is not an IFT of  $\aleph$ " the result is immediate.

**Theorem 4.7.** For a simple IFk-PHG,  $\mathcal{TR}(\mathcal{TR}(\aleph)) = \aleph$ .

**Theorem 4.8.** For any IFk-PHG,  $\mathcal{TR}(\mathcal{TR}(\aleph)) \subseteq \aleph$ .

**Proof.** From definition 4.4,  $\exists$  a  $\aleph'$  (partial hypergraph) of a simple IFk-PHG  $\ni \mathcal{TR}(\aleph') = \mathcal{TR}(\aleph)$ . From Theorem 4.7,  $\mathcal{TR}(\mathcal{TR}(\aleph)) = \mathcal{TR}(\mathcal{TR}(\aleph'))$  implies  $\aleph' \subseteq \aleph$ .

**Theorem 4.9.** Let  $\aleph$  be an IFk-PHG and suppose that  $\mathcal{T} \in \mathcal{TR}(\aleph)$ . If  $\aleph' \subseteq \text{supp}(\mathcal{T}) \subseteq \aleph$ , then  $\exists$  a hyperedge of IFk-PHG  $\mathcal{A}$ ,  $(a_i, b_i) \in \mathcal{A}$  represents the MNMV of  $\mathcal{A} \ni$

- (i)  $(a_i, b_i) = h(\mathcal{A}) = h(\mathcal{T}^{a_i, b_i}) > 0$ ,
- (ii)  $\mathcal{T}_{h(\mathcal{A})} \cap \mathcal{A}_{h(\mathcal{A})} = \aleph$ .

**Proof.** Let  $0 < h(\mathcal{T}^{a_i, b_i}) \leq 1$  and  $\psi'$  be the set of all IF  $k$ -partite hyperedges where  $h(\tau^{a_i, b_i}) \geq h(\mathcal{T}^{a_i, b_i})$  for each  $\tau \in \psi'$ .

### Transversal Core of Intuitionistic Fuzzy $k$ -Partite Hypergraphs

Since  $\mathcal{T}^{a_i, b_i}$  is an IFT of  $\aleph^{a_i, b_i}$  and  $\aleph' \subseteq \mathcal{T}^{a_i, b_i}$  is nonempty. Further, for each  $\tau \in \psi'$ ,  $h(\tau) \geq h(\tau^{a_i, b_i}) \geq h(\mathcal{T}^{a_i, b_i})$  is true. Also, assume that  $\mathcal{T}^{a_i, b_i}$  is the MIFT, then for all  $\tau \in \psi'$ ,  $h(\tau) > h(\mathcal{T}^{a_i, b_i})$  and  $\exists \aleph_\tau \neq \aleph$  with  $\aleph_\tau \in \tau_{h(\tau)} \cap \mathcal{T}_{h(\tau)}$ . Define an IF $k$ -PHG  $\aleph_1 \ni$

$$\aleph_1(U) = \begin{cases} \mathcal{T}(U) \text{ whenever } U \neq \aleph', \\ \min(h(\mathcal{A})/h(\mathcal{A}') < h(\mathcal{T}^{a_i, b_i})), \max(h(\mathcal{A})/h(\mathcal{A}') < h(\mathcal{T}^{a_i, b_i})) \\ \text{whenever } U = \aleph' \end{cases}$$

Hence  $\aleph_1$  is an IFT of IF $k$ -PHG and  $h(\aleph_1^{a_i, b_i}) < h(\mathcal{T}^{a_i, b_i})$ , It does not meet the basic requirement of  $\mathcal{T}$ . For each  $\tau \in \psi'$  satisfies the first part of the theorem 4.9 and has  $\aleph_\tau$  which is not in  $\aleph$  with  $\aleph_\tau \in \tau_{h(\tau)} \cap \mathcal{T}_{h(\tau)}$ . The procedure is repeated, and the argument of (i) provides a contradiction and bringing close to the proof.

**Theorem 4.10.** *Let  $\aleph$  be an IF $k$ -PHG. Then,  $\exists \mathcal{T} \in \mathcal{TR}(\aleph)$  with  $\aleph' \subseteq \text{supp}(\mathcal{T}) \subseteq \aleph$ , if and only if for  $\mathcal{A} \in \psi$  it meets the following requirements:*

- (i)  $(a_i, b_i) = h(\mathcal{A})$ ,
- (ii) The level cut  $(a_j, b_j)$  of  $h(\mathcal{A}')$  is not a subhypergraph of the level cut  $(a_i, b_i)$  of  $h(\mathcal{A})$ , for all  $\mathcal{A}' \in \psi$  with  $h(\mathcal{A}') > h(\mathcal{A})$ ,
- (iii) The level cut  $(a_i, b_i)$  of  $h(\mathcal{A})$  does not contain any other hyperedge of  $\aleph_{h(\mathcal{A})}$ , where  $(a_i, b_i)$  denotes MNMV of  $\mathcal{A}$ .

**Proof.** Necessary Part:

(i) Let  $\mathcal{T} \in \mathcal{TR}(\aleph)$  and  $0 < h(\mathcal{T}^{a_i, b_i}) \leq 1$ . Condition (i) is followed from Theorem 4.9.

(ii) Suppose that for each  $\mathcal{A}$  satisfying (i)  $\exists \mathcal{A}' \in \psi \ni h(\mathcal{A}') > h(\mathcal{A})$  and  $\mathcal{A}'_{h(\mathcal{A}')} \subseteq \mathcal{A}_{h(\mathcal{A})}$ , then  $\exists U \neq \aleph'$ , with  $U \in \mathcal{A}'_{h(\mathcal{A}')} \cap \mathcal{T}_{h(\mathcal{A}')} \subseteq \mathcal{A}_{h(\mathcal{A})} \cap \mathcal{T}_{h(\mathcal{A})}$  which differs from the concept of Theorem 4.9.

(iii) Assume for each  $\mathcal{A}$  satisfying (i) and (ii) then  $\exists \mathcal{A}' \in \psi$  so that  $\emptyset \neq \mathcal{A}'_{h(\mathcal{A}')} \subset \mathcal{A}_{h(\mathcal{A})}$ . Since  $\mathcal{A}'_{h(\mathcal{A}')} \neq \emptyset$  and by (ii), we have  $h(\mathcal{A}') = h(\mathcal{A}) = (a_i, b_i)$ .

If  $(a_j, b_j) = h(\mathcal{A}')$  and  $\mathcal{A}'' \in \psi$  such that  $\emptyset \neq \mathcal{A}''_{h(\mathcal{A}')} \subset \mathcal{A}'_{h(\mathcal{A}')} \subset \mathcal{A}_{h(\mathcal{A})}$ . The process is continued and the chain ends finitely, without loss of abstraction assume  $(a_i, b_i) < h(\mathcal{A})$ . But,  $\exists U \neq \aleph' \ni U \in \mathcal{A}'_{h(\mathcal{A}')} \cap \mathcal{T}_{h(\mathcal{A}')} \subseteq \mathcal{A}_{h(\mathcal{A})} \cap \mathcal{T}_{h(\mathcal{A})}$ , which contradicts Theorem 4.9.

Sufficient Part:

Let  $\mathcal{A} \in \psi$  satisfy the condition (i), (ii) and (iii). By condition (i), the process is trivial. By condition (ii) and (iii)  $\exists U \in \mathcal{A}'_{h(\mathcal{A}')} \setminus \mathcal{A}_{h(\mathcal{A})}$  for every  $\mathcal{A}' \in \psi \ni \mathcal{A}' \neq \mathcal{A}$  and  $h(\mathcal{A}') \geq h(\mathcal{A})$ . Let  $\vee_{\mathcal{A}}$  be the set of all vertices of  $\aleph \ni \vee_{\mathcal{A}} \cap \mathcal{A}_{h(\mathcal{A})} = \emptyset$ .

The initial sequence of transversals are constructed. So  $\tau_s \subseteq \vee$  for all  $s$ ,

$1 \leq s < i$  and  $\tau_i \subseteq \bigvee \mathcal{A} \cup \bigvee_i$ . Hence,  $\bigvee_i \in \tau_i$  for each  $i$ . The process is terminated till it reaches a minimal IFT with  $(a_i, b_i) = h(\mathcal{A}) = h(\mathcal{T}^{a_i, b_i})$ .

**Theorem 4.11.** *Let  $\aleph$  be an IFk-PHG with  $F_k(\aleph) = \{a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k\}$  so that  $0 \leq a_i \leq h_\omega(\aleph)$ ,  $0 \leq b_i \leq h_\nu(\aleph)$ . Also,  $\aleph^{a_i, b_i} \subseteq \mathcal{A}'$ , be the elementary IFk-PHG if and only if  $h(\mathcal{A}') = (a_i, b_i)$  and  $\text{supp}(\mathcal{A}')$  is a hyperedge of  $\aleph^{a_i, b_i}$ . Then  $\mathcal{TR}(\mathcal{TR}(\aleph))$  is the partial IFk-PHG of  $\aleph^{a_i, b_i}$ .*

**Proof.** From Theorem 4.5 and by the construction of minimal IFT, the  $(a_i, b_i)$ -level hypergraph of  $\mathcal{TR}(\aleph)$  is  $\mathcal{TR}(\aleph^{a_i, b_i})$  which means that  $(\mathcal{TR}(\aleph))^{a_i, b_i} = \mathcal{TR}(\aleph^{a_i, b_i})$ . Let  $\tau$  belongs to  $\mathcal{TR}(\mathcal{TR}(\aleph))$ . From Theorem 4.9,  $h(\tau(\bigvee_i)) > 0$ , this implies that  $\exists \mathcal{T} \in \mathcal{TR}(\aleph)$  with  $h(\tau(\bigvee_i)) = h(\mathcal{T})$ . From Theorem 4.1,  $h(\mathcal{T}) = (\max(\min(\omega_{k_{ij}})), \max(\max(\nu_{k_{ij}}))) = h(\aleph)$  for all minimal IFT  $\mathcal{T}$ . Hence  $\tau$  is elementary with  $h(a_i, b_i)$ . Since  $\text{supp}(\tau) = \tau^{a_i, b_i}$ , Theorem 4.5 suggest that  $\text{supp}(\tau)$  is the minimal IFT of  $(\mathcal{TR}(\aleph))^{a_i, b_i}$ . It is obvious that  $\text{supp}(\tau)$  is a hyperedge of  $\aleph^{a_i, b_i}$ . Hence  $\tau$  is a hyperedge of  $\aleph^{a_i, b_i}$ .

**Theorem 4.12.** *Let  $\aleph$  be an IFk-PHG with  $\aleph^{a_i, b_i}$  is a simple. Then  $\mathcal{TR}(\mathcal{TR}(\aleph)) = \aleph^{a_i, b_i}$ .*

**Proof.** By the above theorem,  $\mathcal{TR}(\mathcal{TR}(\aleph)) \subseteq \aleph^{a_i, b_i}$ . Let  $\tau$  be an elementary with  $h(\mathcal{T}) = (a_i, b_i)$  and  $\text{supp}(\tau) \in \aleph^{a_i, b_i}$ . By Theorem 4.11,  $\text{supp}(\tau)$  is a minimal IFT of  $(\mathcal{TR}(\aleph))^{a_i, b_i}$ . Since each minimal IFT of  $\mathcal{TR}(\aleph)$  is elementary by definition of minimal IFT the process ends at  $(a_i, b_i)$ -level and  $\tau \in \mathcal{TR}(\mathcal{TR}(\aleph))$ .

Hence  $\aleph^{a_i, b_i} \subseteq \mathcal{TR}(\mathcal{TR}(\aleph))$  which implies  $\aleph^{a_i, b_i} = \mathcal{TR}(\mathcal{TR}(\aleph))$ .

## 5 Conclusion

In this article, some interesting concepts like IFT, minimal IFT, locally minimal IFT and IFTC of IFk-PHG were discussed. It is important to note that IFTC exists for both spike and non-spike intuitionistic fuzzy  $k$ -partite hyperedges. In future, the authors planned to work on Robotics with multi-task concept as an application part of IFk-PHG.

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# On fz- domination number of fuzzy graphs

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## Abstract

Given a fuzzy graph  $\mathcal{G} = (V, \mu, \sigma)$ , the fz- domination number,  $\gamma_{fz}(\mathcal{G})$ , is the least scalar cardinality of an fz- dominating set of  $\mathcal{G}$ . In this article, we examine several features of fz-domination number of fuzzy graphs as a result of various fuzzy graph operations. We find bounds for the fz-domination number of a few graph products and look at the requirements for the sharpness of these bounds.

**Keywords:** fuzzy graph; fz-dominating sets; fz-domination number; graph operations

**2020 AMS subject classifications:**05C69, 05C72. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1078. ISSN: 1592-7415. eISSN: 2282-8214. ©Lekh A et al. This paper is published under the CC-BY licence agreement.

# 1 Introduction

Since the initial introduction of fuzzy graphs by Rosenfeld [1975], a large number of researchers have studied the subject. The notion of domination in fuzzy graphs was first proposed by Somasundaram and Somasundaram [1998]. Somasundaram [2005], Gani and Chandrasekaran [2006], Manjusha and Sunitha [2015], Bhutani and Sathikala [2016] also studied domination in fuzzy graphs. Mordeson and Chang-Shyh [1994] developed operations of fuzzy graphs that are comparable to those in crisp graphs.

Different variations of domination in fuzzy graphs found in literature do not consider the situations where we need to take all the non-zero edges incident at a vertex into consideration. These definitions use either the effective or the strong edges of the fuzzy graph. But our model of fuzzy domination in fuzzy graphs [2022] takes into account all the non-zero edges incident at a vertex, even if they are small in strength. Further most variations of domination in fuzzy graphs found in literature do not consider the fuzzy subsets of the vertex set, instead considered the crisp subsets of the fuzzy vertex set. But while considering fuzzy graphs and their subset problems it is more apt to consider fuzzy subsets of the vertex set than their crisp subsets. By taking all these into consideration we defined fz-domination in fuzz graphs [2022].

We, Lekha and Parvathy [2022] developed fz-domination in fuzzy graphs, which coincides with fractional domination in crisp graphs presented by Hedetniemi and Wimer [1987] and explored by Hedetniemi and Mynhardt [1990]. In this article, we examine the effects of several graph operations on fz-domination.

For basic definitions, terminology and notation in fuzzy graphs we refer to Mordeson and Nair [2000].

**Definition 1.1.** (Lekha and Parvathy [2022]). Given a fuzzy graph  $\mathcal{G} = (V, \mu, \sigma)$ , a fuzzy subset  $\mu'$  of  $\mu$  is defined as an fz-dominating set of  $\mathcal{G}$ , if for every  $v \in V$ ,

$$\mu'(v) + \sum_{x \in V} \left( \sigma(x, v) \wedge \mu'(x) \right) \geq \mu(v).$$

A fuzzy subset  $\mu'$  is a minimal fz-dominating set, if  $\mu'' \subset \mu'$  is not an fz-dominating set.

**Definition 1.2.** (Lekha and Parvathy [2022]) Fuzzy domination number or fz-domination number of  $\mathcal{G}$ , denoted by  $\gamma_{fz}(\mathcal{G})$ , is defined as

$$\gamma_{fz}(\mathcal{G}) = \min \{ |\mu'| : \mu' \text{ is a minimal fz-dominating set of } \mathcal{G} \}$$

**Example 1.1.** For  $\mathcal{F} = (\mu, \sigma)$  shown in Fig. 1,

$$\mu_1 = \{(x, 0.1), (y, 0.5), (z, 0.2)\}$$

*On fz- domination number of fuzzy graphs*

$$\mu_2 = \{(x, 0.6), (y, 0), (z, 0.6)\}$$

$$\mu_3 = \{(x, 0.4), (y, 0.2), (z, 0.4)\}$$

$$\mu_4 = \{(x, 0.5), (y, 0.1), (z, 0.5)\}$$

are all minimal fz-dominating sets of  $\mathcal{F}$ .  $\mu_1$  is a minimum fz-dominating set and  $\gamma_{fz}(\mathcal{F}) = 0.8$ .

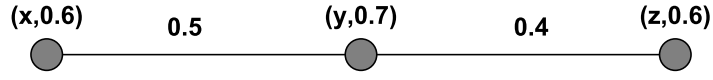


Figure 1: Fuzzy graph,  $\mathcal{F}$

## 2 fz- domination in union of fuzzy graphs

Let  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ .  $\mathcal{G} \cup \mathcal{H} = (V, \mu, \sigma)$  where

$$V = V_1 \cup V_2$$

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu_1(u) \vee \mu_2(u) \quad \text{if } u \in V_1 \cap V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u, v) &= \sigma_1(u, v) \quad \text{if } u \in V_1 \setminus V_2, v \in V_1 \\ &= \sigma_2(u, v) \quad \text{if } u \in V_2 \setminus V_1, v \in V_2 \\ &= \sigma_1(u, v) \vee \sigma_2(u, v) \quad \text{if } u, v \in V_1 \cap V_2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The following theorem gives a general upper bound for the fz-domination number of union of two fuzzy graphs.

**Theorem 2.1.** *For any two non- trivial fuzzy graphs  $\mathcal{G}$  and  $\mathcal{H}$ ,*

$$\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$$

*Proof.* Consider the fuzzy graphs  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ . Let  $\mu'_1$  and  $\mu'_2$  be the minimum fz-dominating sets of  $\mathcal{G}$  and  $\mathcal{H}$  respectively. Let the fuzzy subset  $\mu'$  of  $V$  be defined by

$$\begin{aligned}\mu'(u) &= \mu'_1(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu'_2(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu'_1(u) \vee \mu'_2(u) \quad \text{if } u \in V_1 \cap V_2\end{aligned}$$

Now let  $v \in V$ .

**Case (i)**

If  $v \in V_1 \setminus V_2$ , then

$$\begin{aligned}\mu(v) &= \mu_1(v) \\ &\leq \left( \mu'_1(v) + \sum_{x \in V_1} \sigma_1(x, v) \wedge \mu'_1(x) \right) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x).\end{aligned}$$

**Case (ii)**

If  $v \in V_2 \setminus V_1$ , then

$$\begin{aligned}\mu(v) &= \mu_2(v) \\ &\leq \left( \mu'_2(v) + \sum_{x \in V_2} \sigma_2(x, v) \wedge \mu'_2(x) \right) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x).\end{aligned}$$

**Case (iii)**

If  $v \in V_1 \cap V_2$

$$\begin{aligned}\mu(v) &= \mu_1(v) \vee \mu_2(v) \\ &\leq \left( \mu'_1(v) + \sum_{x \in V_1} \sigma_1(x, v) \wedge \mu'_1(x) \right) \vee \left( \mu'_2(v) + \sum_{x \in V_2} \sigma_2(x, v) \wedge \mu'_2(x) \right) \\ &\leq (\mu'_1(v) \vee \mu'_2(v)) + \left( \sum_{x \in V_1 \setminus V_2} (\sigma_1(x, v) \wedge \mu'_1(x)) + \sum_{x \in V_2 \setminus V_1} (\sigma_2(x, v) \wedge \mu'_2(x)) \right. \\ &\quad \left. + \sum_{x \in V_1 \cap V_2} (\sigma_1(x, v) \vee \sigma_2(x, v)) \wedge (\mu'_1(x) \vee \mu'_2(x)) \right) \\ &\leq \mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x).\end{aligned}$$

Thus  $\mu'$  is an fz-dominating set of  $\mathcal{G} \cup \mathcal{H}$  and  $\mu'(v) \leq \mu'_1(v) + \mu'_2(v)$ .

Hence  $|\mu'| \leq |\mu'_1| + |\mu'_2|$ .

Thus,  $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$ . □

**Remark 2.1.** Obviously equality holds in the above theorem if the vertex sets of  $\mathcal{G}$  and  $\mathcal{H}$  are disjoint. The following example shows that equality may hold even if they are not disjoint. For the graphs in Fig. 2,  $\gamma_{fz}(\mathcal{G}) = 0.5$ ,  $\gamma_{fz}(\mathcal{H}) = 0.6$  and  $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = 1.1$  so that  $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$ .

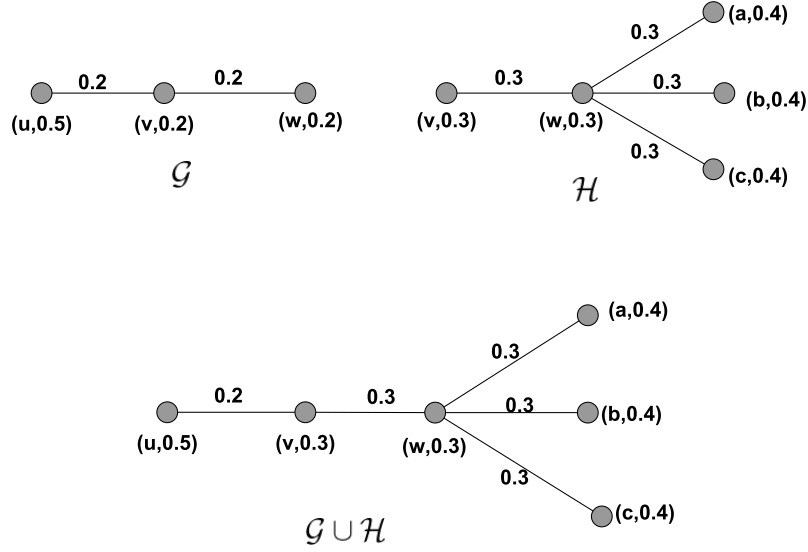


Figure 2: Fuzzy Graphs  $\mathcal{G}$ ,  $\mathcal{H}$  and  $\mathcal{G} \cup \mathcal{H}$

### 3 fz- domination in join of fuzzy graphs

Let  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$  whose vertex sets are disjoint. The join  $\mathcal{G} + \mathcal{H}$  is defined by  $\mathcal{G} + \mathcal{H} = (V, \mu, \sigma)$  where  $V = V_1 \cup V_2$ ,

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u, v) &= \sigma_1(u, v) \quad \text{if } u, v \in V_1 \\ &= \sigma_2(u, v) \quad \text{if } u, v \in V_2 \\ &= \mu_1(u) \wedge \mu_2(v) \quad \text{if } u \in V_1 \text{ and } v \in V_2 \end{aligned}$$

**Theorem 3.1.** For any two non- trivial fuzzy graphs  $\mathcal{G}$  and  $\mathcal{H}$  whose vertex sets are disjoint,

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

*Proof.* Let  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$  be two fuzzy graphs such that  $V_1 \cap V_2 = \phi$ . Let  $\gamma_{fz}(\mathcal{G}) \geq \gamma_{fz}(\mathcal{H})$  and let  $\mu'_1$  be a minimum fz-dominating set of  $\mathcal{G}$ .

Define  $\mu' \subset \mu$  by

$$\begin{aligned}\mu'(u) &= \mu'_1(u) \quad \text{if } u \in \mathcal{G} \\ &= 0 \quad \text{if } u \in \mathcal{H}\end{aligned}$$

Let  $m$  be such that  $m = \max\{\mu_2(u); u \in \mathcal{H}\}$ .

Now  $m \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$  implies that  $\mu'$  is an fz-dominating set of  $\mathcal{G} + \mathcal{H}$ .

Hence,

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

□

In the following discussion  $M$ ,  $m_1$  and  $m_2$  denote the maximum membership value of a vertex in  $\mathcal{G} + \mathcal{H}$ ,  $\mathcal{G}$  and  $\mathcal{H}$  respectively.

**Observation 3.1.** *It is possible that*

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \min\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

For example, if  $M \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$ , then  $\mu' \subset \mu$  defined by

$$\begin{aligned}\mu'(u) &= \mu'_2(u) \quad \text{if } u \in \mathcal{H} \\ &= 0 \quad \text{if } u \in \mathcal{G}\end{aligned}$$

is an fz-dominating set of  $\mathcal{G} + \mathcal{H}$  and hence  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(\mathcal{H})$ . Here equality occurs if  $M = \gamma_{fz}(\mathcal{H})$ . The following example shows that strict inequality can also occur in this relation.

**Example 3.1.** Consider the fuzzy graphs  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  and  $\mathcal{G}_1 + \mathcal{G}_2$  given in Fig.3.

Here  $\gamma_{fz}(\mathcal{G}_1) = 1.6$ ,  $\gamma_{fz}(\mathcal{G}_2) = 1$ ,  $M = 0.8$ ,  $\gamma_{fz}(\mathcal{G}_1 + \mathcal{G}_2) = 0.9$  so that

$$\gamma_{fz}(\mathcal{G}_1 + \mathcal{G}_2) < \gamma_{fz}(\mathcal{G}_2)$$

**Observation 3.2.** *If  $\gamma_{fz}(\mathcal{H}) \leq M \leq |\mu_2|$ , then  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) = M$ .*

**Claim:** Define  $\mu''_2 \supset \mu'_2$  in  $\mathcal{H}$  such that  $|\mu''_2| = M$ . Then,  $\mu''_2$  is an fz-dominating set of  $\mathcal{G} + \mathcal{H}$ . Hence  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq M$ . Also, since there is a vertex of membership value  $M$  in  $\mathcal{G} + \mathcal{H}$ , we get  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) = M$ .

**Observation 3.3.** *If  $m_1 \leq |\mu_2|$  and  $m_2 \leq |\mu_1|$ , then  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq m_1 + m_2$ .*

On fz- domination number of fuzzy graphs

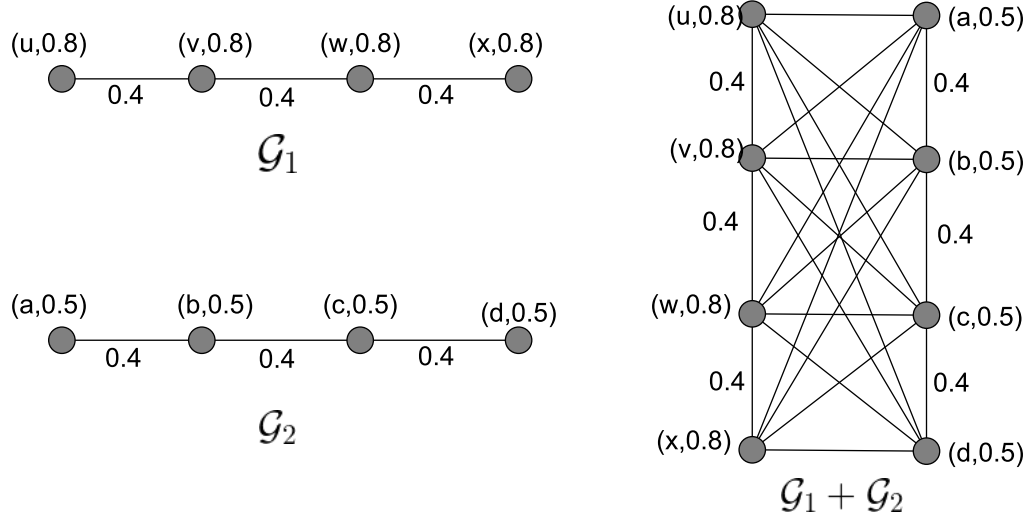


Figure 3: Fuzzy Graphs  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  and  $\mathcal{G}_1 + \mathcal{G}_2$

**Claim:** Define  $\mu'_1 \subset \mu_1$  in  $\mathcal{G}$  such that  $|\mu'_1| = m_2$  and  $\mu'_2 \subset \mu_2$  in  $\mathcal{H}$  such that  $|\mu'_2| = m_1$ . Now  $\mu'$  defined by

$$\begin{aligned} \mu'(u) &= \mu'_1(u) \quad \text{if } u \in \mathcal{G} \\ &= \mu'_2(u) \quad \text{if } u \in \mathcal{H} \end{aligned}$$

is an fz- dominating set in  $\mathcal{G} + \mathcal{H}$ . Hence  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq m_1 + m_2$ .

**Observation 3.4.** If  $|\mu_2| \leq M$ , then  $\mu' \subset \mu$  in  $\mathcal{G} + \mathcal{H}$  defined by

$$\begin{aligned} \mu'(u) &= \mu_2(u) \quad \text{if } u \in \mathcal{H} \\ &= \max\{0, \mu'_1(u) - |\mu_2|\} \quad \text{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in  $\mathcal{G} + \mathcal{H}$ . Then,  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) - n|\mu_2|$  where  $n$  is the number of vertices  $u \in \mathcal{G}$  having  $\mu_1(u) \geq |\mu_2|$ .

**Observation 3.5.** If  $\gamma_{fz}(\mathcal{H}) \leq M \leq \gamma_{fz}(\mathcal{G})$ , then  $\mu' \subset \mu$  in  $\mathcal{G} + \mathcal{H}$  defined by

$$\begin{aligned} \mu'(u) &= \mu'_2(u) \quad \text{if } u \in \mathcal{H} \\ &= \max\{0, \mu'_1(u) - \gamma_{fz}(\mathcal{H})\} \quad \text{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in  $\mathcal{G} + \mathcal{H}$ . Then,  $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) - n\gamma_{fz}(\mathcal{H})$  where  $n$  is the number of vertices  $u \in \mathcal{G}$  having  $\mu_1(u) \geq \gamma_{fz}(\mathcal{H})$ .

## 4 fz- domination in corona of fuzzy graphs

The corona of  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{K}_1 = (u, \mu_2(u))$  is the fuzzy graph  $\mathcal{G} \circ \mathcal{K}_1$  obtained by attaching a copy of  $\mathcal{K}_1$  to each vertex  $v_i \in V_1$  such that  $\sigma(v_i, u_i) = \mu_1(v_i) \wedge \mu_2(u_i)$  where  $u_i$  represents the vertex in the copy of  $\mathcal{K}_1$  corresponding to  $v_i \in V_1$ .

**Observation 4.1.** *The following two results are obvious.*

1.  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$
2.  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$

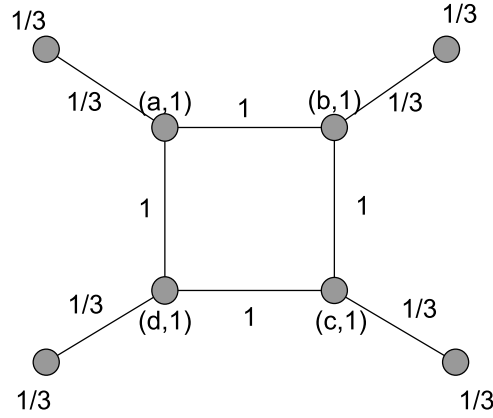


Figure 4: Fuzzy graph  $\mathcal{G} \circ \mathcal{K}_1$

**Remark 4.1.** *The example below shows that equality may occur in observation 4.1(a).*

Consider  $\mathcal{G} \circ \mathcal{K}_1$  in figure 4.  $\mu' = \{(a, \frac{1}{3}), (b, \frac{1}{3}), (c, \frac{1}{3}), (d, \frac{1}{3})\}$  is a minimum fz-dominating set of  $\mathcal{G}$  and  $\gamma_{fz}(\mathcal{G}) = \frac{4}{3}$ .  $\mu'$  is an fz-dominating set of  $\mathcal{G} \circ \mathcal{K}_1$  also. Therefore,  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \frac{4}{3} = \gamma_{fz}(\mathcal{G})$ .

On the other hand from observation 4.1(a),  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$ .

Thus we get  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = \gamma_{fz}(\mathcal{G})$ .

**Theorem 4.1.**  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$  where  $n = |V_1|$ .

*Proof.* Let  $\mu$  be the fuzzy subset of  $\mathcal{G} \circ \mathcal{K}_1$  and  $\mu'_1$  be a minimum fz-dominating set of  $\mathcal{G}$ . Let  $\mu' \subset \mu$  be such that

$$\begin{aligned} \mu'(v) &= \mu'_1(v) & \text{if } v \in V_1 \\ &= \mu_2(v) & \text{otherwise} \end{aligned}$$



*On fz- domination number of fuzzy graphs*

Then  $\mu'$  is an fz-dominating set of  $\mathcal{G} \circ \mathcal{K}_1$  and

$$\begin{aligned} |\mu'| &= |\mu'_1| + n|\mu_2| \\ &= \gamma_{fz}(\mathcal{G}) + n\mu_2(u) \end{aligned}$$

Therefore  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$  □

**Theorem 4.2.** *If  $\mu_2(u) \geq \mu_1(v)$  for all  $v \in V_1$ , then*

$$\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$$

*Proof.* It is clear that  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$

Consider  $\mu'$  where

$$\begin{aligned} \mu'(v) &= 0 \quad \text{if } v \in V_1 \\ &= \mu_2(v) \quad \text{if } v = u \end{aligned}$$

Then,  $\mu'$  is fz-dominating set of  $\mathcal{G} \circ \mathcal{K}_1$ .

Therefore,  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq |\mu'| = n\mu_2(u)$ .

Hence  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$  □

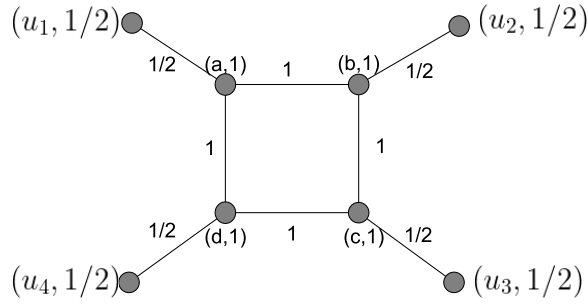


Figure 5:  $\mathcal{G}' \circ \mathcal{K}'_1$

**Remark 4.2.** *The condition  $\mu_2(u) \geq \mu_1(v)$  for all  $v \in V_1$  is not necessary to get  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$ . For example, consider  $\mathcal{G}' \circ \mathcal{K}'_1$  given in figure 5.*

*Here,  $\mu_2(u) < \mu_1(v)$  for all  $v \in V_1$ . Now  $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$  implies that  $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) \geq 2$ .*

*Also  $\mu' = \{(a, \frac{1}{2}), (b, \frac{1}{2}), (c, \frac{1}{2}), (d, \frac{1}{2})\}$  is fz-dominating set of  $\mathcal{G}' \circ \mathcal{K}'_1$ .*

*Hence  $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) = 2 = n\mu_2(u)$*

## 5 fz- domination in Cartesian product

Let  $\mathcal{G}_1 = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{G}_2 = (V_2, \mu_2, \sigma_2)$ . The Cartesian product is the fuzzy graph  $\mathcal{G}_1 \square \mathcal{G}_2 = (V, \mu_1 \times \mu_2, \sigma_1 \times \sigma_2)$  where  $V = V_1 \times V_2$ ,

$$(\mu_1 \times \mu_2)(a, b) = \mu_1(a) \wedge \mu_2(b)$$

and

$$\begin{aligned} (\sigma_1 \times \sigma_2)((a_1, b_1), (a_2, b_2)) &= \mu_1(a_1) \wedge \sigma_2(b_1, b_2) \quad \text{if } a_1 = a_2 \\ &= \sigma_1(a_1, a_2) \wedge \mu_2(b_1) \quad \text{if } b_1 = b_2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

**Theorem 5.1.** For any two nontrivial fuzzy graphs  $\mathcal{G}$  and  $\mathcal{H}$ ,

$$\gamma_{fz}(\mathcal{G} \square \mathcal{H}) \leq \min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\},$$

$m, n$  are the number of vertices with nonzero membership values in  $\mathcal{G}$  and  $\mathcal{H}$  respectively.

*Proof.* Let  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$  where

$$V_1 = \{(u_1, \mu_1(u_1)), (u_2, \mu_1(u_2)), \dots, (u_m, \mu_1(u_m))\}$$

and

$$V_2 = \{(v_1, \mu_2(v_1)), (v_2, \mu_2(v_2)), \dots, (v_n, \mu_2(v_n))\}$$

$\mathcal{G} \square \mathcal{H} = (V, \mu, \sigma)$  where  $V = V_1 \times V_2$ ,  $\mu(u, v) = \mu_1(u) \wedge \mu_2(v)$  and

$$\begin{aligned} \sigma((u_i, v_j), (u'_i, v'_j)) &= \sigma_1(u_i, u'_i) \quad \text{if } v_j = v'_j \\ &= \sigma_2(v_j, v'_j) \quad \text{if } u_i = u'_i \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Let  $\mathcal{G}_j$  denotes the fuzzy sub-graph of  $\mathcal{G} \square \mathcal{H}$  induced by  $V_1 \times v_j \subset V_1 \times V_2$ . Then,

$$V(\mathcal{G}_j) = \{(u_1, v_j), (u_2, v_j), \dots, (u_m, v_j)\}$$

$$\mu(u_i, v_j) = \mu_1(u_i) \wedge \mu_2(v_j) \leq \mu_1(u_i)$$

and

$$\sigma((u_i, v_j), (u'_i, v_j)) = \min\{\sigma_1(u_i, u'_i), \mu_2(v_j)\} \leq \sigma(u_i, u'_i)$$

**Claim:**  $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$ . Define  $\mu'_j$  on  $\mathcal{G}_j$  as  $\mu'_j(u_i, v_j) = \mu'_1(u_i) \wedge \mu_2(v_j)$ . Consider  $(u_i, v_j) \in \mathcal{G}_j$ .  $\mu'_1$  is an fz-dominatng set of  $\mathcal{G}$  implies that

*On fz- domination number of fuzzy graphs*

$$\mu_1(u_i) \leq \mu'_1(u_i) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu'_1(u_k).$$

Hence,

$$\begin{aligned} \mu_1(u_i) \wedge \mu_2(v_j) &\leq \mu'_1(u_i) \wedge \mu_2(v_j) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu'_1(u_k) \wedge \mu_2(v_j) \\ &\leq \mu'_j(u_i, v_j) + \sum_{u_k \in \mathcal{G}} \sigma((u_k, v_j), (u_i, v_j)) \wedge \mu'_j(u_k, v_j) \end{aligned}$$

That is,

$$\mu(u_i, v_j) \leq \mu'_j(u_i, v_j) + \sum_{u_k \in \mathcal{G}} (\sigma((u_k, v_j), (u_i, v_j)) \wedge \mu'_j(u_k, v_j))$$

Thus we get  $\mu'_j$  is an fz- dominating set of  $\mathcal{G}_j$  for  $j = 1, 2, \dots, n$

Also  $|\mu'_j| \leq |\mu'_1|$  shows that  $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$  for  $j = 1, 2, \dots, n$

Hence

$$\gamma_{fz}(\mathcal{G} \square \mathcal{H}) \leq n\gamma_{fz}(\mathcal{G})$$

Similarly,

$$\gamma_{fz}(\mathcal{G} \square \mathcal{H}) \leq m\gamma_{fz}(\mathcal{H})$$

Thus we get,

$$\gamma_{fz}(\mathcal{G} \square \mathcal{H}) \leq \min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\}$$

□

In the previous theorem, equality might hold. For  $\mathcal{G}, \mathcal{H}$  and  $\mathcal{G} \square \mathcal{H}$  given in Fig. 6,  $\gamma_{fz}(\mathcal{G}) = 0.2$ ,  $\gamma_{fz}(\mathcal{H}) = 0.2$  and  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) = 0.4$  so that  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) = \min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\}$

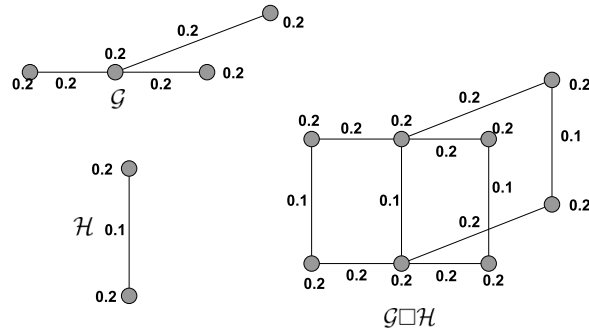


Figure 6: Fuzzy graph  $\mathcal{G}, \mathcal{H}$  and  $\mathcal{G} \square \mathcal{H}$

V. G. Vizing presented the following conjecture regarding the Cartesian product of crisp graphs in 1968.

$$\gamma(G \square H) \geq \gamma(G)\gamma(H), \quad \text{for every pair of finite crisp graphs } G \text{ and } H.$$

Possibly the most significant unsolved issue in the field of domination theory is Vizing's conjecture. Here, we investigate the applicability of Vizing's like inequality to fz-dominantion in fuzzy graphs.

Vizing's conjecture is said to be satisfied by a fuzzy graph  $\mathcal{G}$ , if  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) \geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$  for every fuzzy graph  $\mathcal{H}$ .

**Definition 5.1.** A fuzzy graph  $\mathcal{H} = (V_1, \mu_1, \sigma_1)$  is known as a partial fuzzy subgraph of  $\mathcal{G} = (V, \mu, \sigma)$  induced by  $V_1$  if  $V_1 \subset V$ ,  $\mu_1(u) = \mu(u)$  if  $u \in V_1$ , 0 otherwise and  $\sigma_1(u, v) = \sigma(u, v) \wedge \mu(u) \wedge \mu(v)$  for all  $u, v \in V$ .

**Definition 5.2.** The spanning fuzzy subgraph of  $\mathcal{G} = (V, \mu, \sigma)$  is the partial fuzzy subgraph  $\mathcal{G}' = (V_1, \mu', \sigma')$  where  $V = V_1$  and  $\mu = \mu'$

If  $\mathcal{G}'$  is a spanning fuzzy subgraph of the fuzzy graph  $\mathcal{G}$ , then  $\gamma_{fz}(\mathcal{G}') \geq \gamma_{fz}(\mathcal{G})$ .

**Theorem 5.2.** If  $\mathcal{G}$  satisfies Vizing's Conjecture and  $\mathcal{G}'$  is a spanning fuzzy subgraph of  $\mathcal{G}$  such that  $\gamma_{fz}(\mathcal{G}') = \gamma_{fz}(\mathcal{G})$ , then  $\mathcal{G}'$  also satisfies Vizing's Conjecture.

*Proof.*  $\mathcal{G}' \square \mathcal{H}$  is a spanning fuzzy subgraph of  $\mathcal{G} \square \mathcal{H}$  for every fuzzy graph  $\mathcal{H}$ . Hence

$$\begin{aligned} \gamma_{fz}(\mathcal{G}' \square \mathcal{H}) &\geq \gamma_{fz}(\mathcal{G} \square \mathcal{H}) \\ &\geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H}) = \gamma_{fz}(\mathcal{G}')\gamma_{fz}(\mathcal{H}) \end{aligned}$$

□

The example below illustrates that in general this inequality does not hold for fz-dominantion in fuzzy graphs.

**Example 5.1.** Consider the fuzzy graphs  $\mathcal{G} = (V_1, \mu_1, \sigma_1)$  and  $\mathcal{H} = (V_2, \mu_2, \sigma_2)$  given in figure 7. For  $\mathcal{G}$ ,  $V_1 = \{(a, 1), (b, 1), (c, 1)\}$ ,  $\sigma_1(a, b) = \sigma_1(b, c) = 1$ ,  $\sigma_1(a, c) = 0$ . For  $\mathcal{H}$ ,  $V_2 = \{(u, 1), (v, 1), (w, 1)\}$ ,  $\sigma_2(u, v) = \sigma_2(v, w) = 1$ ,  $\sigma_2(u, w) = 0$ .  $\mu'_1 = \{(a, 0.8), (b, 0.6), (c, 0.8)\}$  is a minimum fz-dominating set of  $\mathcal{G}$ .

Hence  $\gamma_{fz}(\mathcal{G}) = 2.2$ . Similarly  $\gamma_{fz}(\mathcal{H}) = 2.2$

Now  $\mu' = \{((a, u), 0.6), ((a, v), 0.4), ((a, w), 0.6), ((b, u), 0.4), ((b, v), 0.2), ((b, w), 0.4), ((c, u), 0.6), ((c, v), 0.4), ((c, w), 0.6)\}$  is a minimum fz-dominating set of  $\mathcal{G} \square \mathcal{H}$ . Hence  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) = 4.2$

Here

$$\gamma_{fz}(\mathcal{G} \square \mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$$

There are fuzzy graphs for which

1.  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$
2.  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) = \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$
3.  $\gamma_{fz}(\mathcal{G} \square \mathcal{H}) > \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$

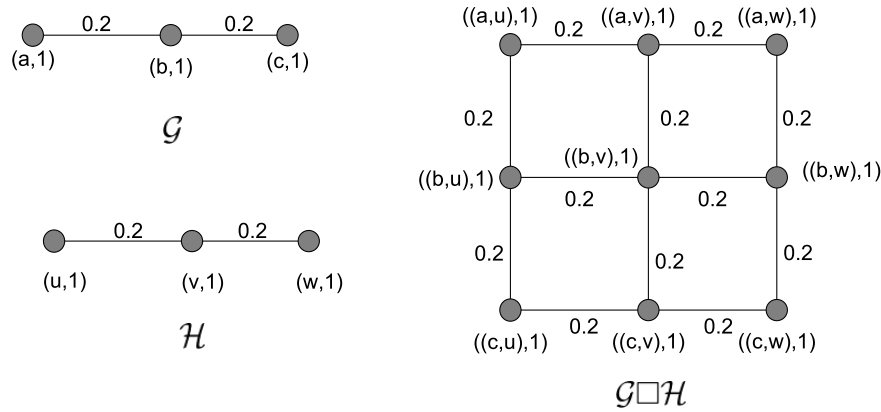


Figure 7: Fuzzy Graphs  $\mathcal{G}$ ,  $\mathcal{H}$  and  $\mathcal{G} \square \mathcal{H}$

## 6 Conclusions

Graph operations are techniques for creating new graphs from ones that already exist, and they are crucial in the design and analysis of large networks. In this article, we investigate various characteristics of the fz-domination number of fuzzy graphs under the influence of some graph operations. It is possible to derive bounds for the fz-domination number of the union, join, corona, and Cartesian product of fuzzy graphs. Through examples, the sharpness of these bounds are demonstrated and the factors that contribute to the sharpness are examined.

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# Super fuzzy matrix of inverse in $k^{th}$ Order

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## Abstract

In normal matrix and fuzzy matrix working well but sometimes fail to work for classical model problems. Because, it is not satisfied the consistency conditions and other parameter. Our proposed method to satisfy the all condition including consistency and perform well compared to the other existing models. Unexpected event modelling is a affluent area of study in fuzzy matrix (FM) modelling. Every fuzzy matrix may be shown as a multi-dimensional concept, but standard matrices cannot achieve this without the proper scale. To solve this issue, a certain kind of classical fuzzy matrix is required. In this study, the idea of an inverse of a  $k$ -regular fuzzy matrix is introduced, and some of its key characteristics are listed. As a result, the same regularity indicator is used to describe a matrix. Investigation is also done on the relationship between the regular,  $k$ -regular, and consistency of fuzzy matrices powers.

**Keywords:** inverse;  $k$ -regular; super fuzzy matrix; Fuzzy matrix; regular

**2020 AMS subject classifications:** 03E72, 15A09, 35D10, 39B42, 58J52.<sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1079. ISSN: 1592-7415. eISSN: 2282-8214. ©R.Deepa et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Each component of a matrix known as a boolean matrix has a value of 0 or 1. Basic values for a fuzzy matrix fall between  $[0, 1]$ . By Kim and Roush, the idea of fuzzy matrix sections was introduced. In all auothers contributed significantly to fuzzy matrix augmentation. Later, utilising modified regular inverses of the coefficient matrix, Zheng and Wang developed the widely used  $m \times n$  fuzzy linear machine and the inconsistent fuzzy linear system. Using matrix-modified common inverse theory, Abbasbandy .S and . M [2005] investigated the minimal response of the overall twin fuzzy linear device. We provide a unique way for finding the inverse of a fantastic fuzzy matrix by changing a well-known idea in this research D and H [1978],P [1963], Ravi.J and et. al. [2022], Kandasamy W.B.V. and llanthenral K. [2007], M.Z. and E.G [1994], M.G [1977], K and B [2006], B and K [2006].

The super fuzzy matrix's initial and core notions are covered in Section 2 along with an illustration. The method of our suggested notion is presented in Section 3. The main findings, theorems, and numerical examples of super fuzzy inverse matrices of order  $k$  are presented in Section 4, and the conclusion is derived in Section 5.

## 2 Preliminaries

noindent The core concepts and notations of a super fuzzy matrix will be described in this section. This article concentrates on the same old number ordering, underlying maxmin (min max) operations, and fuzzy matrices in super (SFM) with  $[0,1]$  support. By (SF) $m \times n$  and (SF) $n$ , respectively, all fuzzy matrices in super of order  $m \times n$  and  $n \times n$  are represented. Space created by the row (or) column is indicated by the letters  $R(A)$  or  $C(A)$  ( $A$ ), Kaufmann and Gupta [1985], J.B [1983], .

**Definition 2.1.** Deepa.R and Sundararajan.P [2020] *A matrix containing entries that fall between  $[0, 1]$  is referred to as a fuzzy matrix. In the case of matrices, we have a fuzzy related different matrix.*

**Definition 2.2.** Deepa.R and Sundararajan.P [2020] *Let us deem  $C$  fuzzy matrix*

$$C = \begin{bmatrix} C_{11} & C_{12} & C \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

The fuzzy submatrices  $C_{11}$  and  $C_{21}$  have the same number of columns because they are fuzzy submatrices, along with  $C_{12}$ ,  $C_{13}$ ,  $C_{21}$ ,  $C_{22}$ , and  $C_{23}$ . The fuzzy matrices  $C_{13}$  and  $C_{23}$ , as well as the fuzzy submatrices  $C_{12}$  and  $C_{22}$ , all have



### Super fuzzy matrix of inverse in $k^{th}$ order

equal columns. The second index of the fuzzy sub-matrices reveals this. The number of rows in fuzzy submatrices  $C_{11}$ ,  $C_{12}$ , and  $C_{13}$  is the same. There are exactly the same number of rows in the  $C_{21}$ ,  $C_{22}$ , and  $C_{23}$  fuzzy submatrices. as a result, we currently possess  $C$  generic super fuzzy matrix.

$$C = \begin{bmatrix} C_{11} & C_{12} \dots & C_{1n} \\ C_{21} & C_{22} \dots & C_{2n} \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ C_{m1} & C_{m2} \dots & C_{mn} \end{bmatrix}$$

where  $C_{ij}$ 's  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

**Definition 2.3.** *The modified regular inverse of matrix solution of a positive set of equations can be found by altering the regular inverse of a non-singular matrix. The modified regular inverse of any (potentially square) matrix with complex components may be found using any method. It is used here and in other programmes to immediately solve linear matrix issues as well as to get an expression for the principal idempotent components of the matrix.*

## 3 Methodology

The  $k^{th}$  regular super fuzzy matrix is examined in this section.

$$\text{Definition 3.1. Let } C = \begin{bmatrix} c_{11} & c_{12} \dots & c_{1n} \\ c_{21} & c_{22} \dots & c_{2n} \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ c_{m1} & c_{m2} \dots & c_{mn} \end{bmatrix}$$

*It is a matrix of fuzzy that is  $C^k$  each  $a_{ij}$  is  $0 \leq a_{ij} \leq 1$ . The exact strength of each ingredient is examined here. It is the  $G$ -name. inverse's.*

$$C^k = \begin{bmatrix} 1 - kc_{11} & 1 - kc_{12} \dots & 1 - kc_{1n} \\ 1 - kc_{21} & 1 - kc_{22} \dots & 1 - kc_{2n} \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ 1 - kc_{m1} & 1 - kc_{m2} \dots & 1 - kc_{mn} \end{bmatrix}$$

Here,  $k$  is the smallest positive integer (Based on Probability). It is possible to multiply two fuzzy matrices without requiring that they both be fuzzy. Every entry that is positive is changed to a 1, and every entry that is negative is changed to a 0 or a tendency toward zero.

**Definition 3.2.** The Properties of Super fuzzy matrix is,

$$(A^k)^k = A$$

$$\begin{aligned} AxA^k &= A^k, \\ (A+B)^k &= A^k + B^k, \\ (\lambda A)^k &= \lambda A^k, \\ (BA)^k &= A^k B^k, \\ AA^k &= 0 \text{ implies } A=0. \end{aligned}$$

**Definition 3.3.** The condition of the supper fuzzy matrix are given below,

$$\begin{aligned} AXA &= A \\ XAX &= X \\ (AX)^k &= AX \\ (XA)^k &= XA \\ XX^k A^k &= X \\ XAA^k &= A^k \\ BA^k AA^k &= A^k \\ X = XX^k A^k = XX^k A^k AY = XAY = XAA^k Y^k Y = A^k Y^k Y = Y. \end{aligned}$$

## 4 Result & application

The  $C^k$  of super fuzzy matrix met all of the requirements for convergence.

**Theorem 4.1.** Let  $A$  be the inverse of the  $k$  by KRSFM ( $k$  regular super fuzzy matrix), with non-zero rows representing the norm. If  $A$  meets the maximin conditions for the matrix equation  $ASA=A$  for some super fuzzy matrix  $S$ , then  $C$  is  $k$  regular.

**Proof.** The non-zero rows of the inverse of the KRSFM of  $C$  serve as the standard basis in this case. If  $SB=Q$ , then  $Q$ 's rows are permutations of  $C$ 's rows. Then,  $X$  is an idempotent of KRSFM, with the same row space as  $C$  and non-zero  $Z$  rows functioning as a standard basis..

Since the standard foundation is different for each  $S$ ,  $B=GQ$ .

Then

$$CS^T C = S Q S^T S Q = S Q Q = S Q = C^k$$

*Super fuzzy matrix of inverse in  $k^{th}$  order*

$$\Rightarrow CSC = C^k.$$

So, C is k regular.

**Theorem 4.2.** Let  $C, D \in (SF)_{n \times m}$  and  $k=p$  be two inverse of p regular super fuzzy matrix. If C is p regular, then we prove that

$$\text{Hence } D^* = \sum x_{ij} A^*.$$

$$\Rightarrow D = XCp$$

$$\Rightarrow D = XCC'C \text{ (since } CC'C = Cp)$$

$$\Rightarrow D = DC'Cp$$

$$\Rightarrow D = CC'C$$

$$\Rightarrow D = CC'CY.$$

$$\Rightarrow D = CpC'D.$$

**Theorem 4.3.** For  $C \in (SF)_n$  and for any  $G^- \in (SF)_n$ , if  $C^k X = C^k G^-$ , where X is a  $\{1^K, 3^K\}$  inverse of C then,  $G^-$  is a  $\{1^K, 3^K\}$  inverse of C.

**Proof.** Since X is a  $\{1^K, 3^K\}$   
 $\Rightarrow C^k X C = C^k$  and  $(C^k X)^T = C^k X$ .

The Post is multiplied by A on both sides of  $C^k X = C^k G^-$ ,  $C^k G^- C = C^k X C = C^k$ .

$$(C^k G^-)^T = (C^k X)^T = C^k X = C^k G^-.$$

Hence  $G^-$  is a  $\{1^K, 3^K\}$  inverse of C.

**Theorem 4.4.** For  $C \in (SF)_n$ , X is a  $\{1^P, 3^P\}$  inverse of C and  $H^-$  is a  $\{1^P, 3^P\}$  inverse of C then,  $C^p X = C^p H^-$ .

**Proof.** Since X is a  $\{1^P, 3^P\}$  inverse of C.

$$\Rightarrow C^p X C = C^p \text{ and } (C^p X)^T = C^p X.$$

$H^-$  is a  $\{1^P, 3^P\}$  inverse of C,

$$\Rightarrow C H^- C^p = C^p \text{ and } (C H^-)^T = C H^-.$$

$$\begin{aligned} C^p H^- &= (C^p X C) H^- = (C^p X)^T (C H^-) = (C^p X)^T (C H^-)^T = X^T (C^T)^p (H^-)^T C^T \\ &= X^T (C H^- C^p)^T = X^T (C^p)^T = (C^p X)^T = C^p X. \end{aligned}$$

**Theorem 4.5.** For  $D \in (SF)_n$ , if  $D^T D$  is D right KRSFM and  $R(D^p) \subseteq R(D^T D)^p$  then D has a  $\{1^P, 3^P\}$  inverse. In particular for  $p=1$ ,  $U = (D^T D)^- D^T$  is a  $\{1, 3\}$  inverse of D.

**Proof.** let  $(D^T D)^p (D^T D)^- (D^T D) = (D^T D)^p$

Since  $R(D^p) \subseteq R((D^T D)^p)$ ,

$$D^p = X(D^T D)^p \text{ for some } X \in (SF)_n \text{ Dnd tape } U = (D^T D)^- D^T$$

$$D^p U D = (D^p)(U D) = (X(D^T D)^p)((D^T D)^- D^T D)$$

$$= X((D^T D)^p (D^T D)^- (D^T D))$$

$$= X(D^T D)^p = (D^p).$$

$$TDpe V = (D^T D)^- (D^p)^T.$$

$$\begin{aligned}
 D^p V &= (D^p) V \\
 &= (X(D^T D)^p)((D^T D)^-(D^p)^T) \\
 &= X(D^T D)^p(D^T D)^-(D^T D)^p D^T \\
 &= X(D^T D)^p(D^T D)^-(D^T D)(D^T D)^{p-1} X^T \\
 &= X(D^T D)^p(D^T D)^{p-1} X^T \\
 &= X(D^T D)^{2p-1} X^T = (X(D^T D)^{2p-1} X^T)^T = (D^p V)^T.
 \end{aligned}$$

Hence D has  $\{1^P, 3^P\}$  inverse.

In particular for  $p=1$ ,  $Y = (D^T D)^- D^T$  is  $\{1, 3\}$  inverse of D.

**Theorem 4.6.** Let  $C \in (SF)_n$  be a right KRSFM and  $R(C^T C)^k \subseteq R(C^k)$  then  $C^T C$  has a  $\{3^K\}$  inverse.

**Proof.** Since,  $C^k X C = C^k$ .

Since  $R((C^T C)^k) \subseteq R(C^k)$ ,  $(C^T C)^k = Z C^k$  for some  $Z \in (SF)_n$  and take  $Y = X C$ .

$$\begin{aligned}
 (C^T C)^k Y &= (Z C^k)(X C) = Z(C^k X C) = Z C^k = (C^T C)^k = ((C^T C)^k)^T \\
 &= ((C^T C)^k Y)^T
 \end{aligned}$$

Hence  $C^T C$  has a  $\{3^K\}$  inverse.

The findings for the other associated fuzzy matrices improve, and the theorem is fulfilled. While some techniques (regular and inverse) meet some conditions, our proposed KRSFM meets all of them (regular & inverse).

**Example 4.1.** Let us consider IKSFM

$$A = \begin{bmatrix} 0.5 & 0.7 & 0.6 \\ 0.3 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.1 \end{bmatrix}$$

The rows of A are separate and make up the foundation. IKSFPM is a form that

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that  $APA = A$ . Now for IKSFM

$$R = \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}$$

There for  $RA = A$ . So g inverse of A sis

$$PR = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.1 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}$$

*Super fuzzy matrix of inverse in  $k^{th}$  order*

Which is satisfy the relation  $AXA=A$ .

**Example 4.2.** Let  $A = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix}$  and

$$B = \begin{bmatrix} \langle 0.6, 0.2 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.7, 0.1 \rangle & \langle 0.4, 0.4 \rangle \end{bmatrix}$$

For which  $B=BA^T A$  holds.

**Example 4.3.** Let IKSFM is,

$$C = \begin{bmatrix} \langle 0.8, 0.2 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.3, 0.2 \rangle \\ \langle 0.4, 0.2 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.3, 0.2 \rangle \\ \langle 0.6, 0.2 \rangle & \langle 0.6, 0.2 \rangle & \langle 0.8, 0.2 \rangle \end{bmatrix}$$

$$D = \begin{bmatrix} \langle 0.8, 0.2 \rangle & \langle 0.4, 0.3 \rangle & \langle 0.2, 0.2 \rangle \\ \langle 0.4, 0.3 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.2, 0.2 \rangle \\ \langle 0.4, 0.2 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.6, 0.2 \rangle \end{bmatrix}$$

$$E = \begin{bmatrix} \langle 0.7, 0.2 \rangle & \langle 0.4, 0.3 \rangle & \langle 0.3, 0.3 \rangle \\ \langle 0.4, 0.3 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.4, 0.2 \rangle \\ \langle 0.4, 0.2 \rangle & \langle 0.6, 0.2 \rangle & \langle 0.8, 0.2 \rangle \end{bmatrix}$$

Be two of its g inverse of A.

Then  $X = CDE = \begin{bmatrix} \langle 0.8, 0.2 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.3, 0.2 \rangle \\ \langle 0.4, 0.2 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.3, 0.2 \rangle \\ \langle 0.6, 0.2 \rangle & \langle 0.6, 0.2 \rangle & \langle 0.8, 0.2 \rangle \end{bmatrix}$

Forth above X,  $CXC=C$  and  $XCX=X$  holds. So X is a semi inverse of the IKSFM.

**Example 4.4.** Let us consider the symmetric IKSFM

$$A = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix}$$

Now,  $A^2$  is,

$$A^2 = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix} \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix}$$

$= A$

The above matrix is symmetric and idempotent.

**Example 4.5.** Let us consider the symmetric IKSFM

$$A = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0 \rangle & \langle 0, 0.4 \rangle \end{bmatrix}$$

Now,  $A^2$  is,

$$B^2 = \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0 \rangle & \langle 0, 0.4 \rangle \end{bmatrix} \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0 \rangle & \langle 0, 0.4 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.4, 0.2 \rangle & \langle 0.4, 0.5 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.3, 0.4 \rangle \end{bmatrix}$$

$\neq B$

The above matrix is not idempotent.

## 5 Conclusions

The main conclusions of the study may be presented in a short Conclusions section, which may stand alone or form a subsection of a Discussion or Results. In this paper, the inverse of the KRSFM is used to provide several innovative proposals and theorems. The original machine is replaced with matrix coefficient A by two distinct n m matrix equation systems. As a result, FM must resolve the issue. The k-regular, the regularity of the fuzzy matrix powers, and the relationship between each regular are all examples of regularities. Our approach addresses any issues that need the inverse of k regular super fuzzy matrices. A appropriate theorem is also shown. In the next essay, we will look at a variety of fundamental properties as well as computer vision applications. In our proposed classical model to proved, it is a consistent. Also in future, we are trying this concept in neural networking system and IOT.

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*Super fuzzy matrix of inverse in  $k^{th}$  order*

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## Some results on range labeling

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R. Jahir Hussain<sup>†</sup>

### Abstract

Let  $G=(V, E)$  be a graph with finite and simple  $V$ . Let  $V(G)$  and  $E(G)$  be the vertex set and edge set of  $G$  respectively. A Range labeling of a graph  $G$  is an injective function.  $f: V(G) \rightarrow \{1, 2, \dots\}$  such that the edge labeling  $\alpha^*: E(G) \rightarrow \{1, 2, \dots\}$  is defined by  $\alpha^*(e) = \max\{f(u), f(v)\} - \min\{f(u), f(v)\}$ . A graph which admits such labeling is called a range graph. In this paper the range labeling is introduced and range labeling for a some trees as for star tree, spider tree and Banana tree are calculated.

**Keywords:** Graph; Labeling; Graceful labeling; Range labeling; Star trees; Spider trees; Banana trees.

**2020 AMS subject classifications:** 05C78 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1080 ISSN: 1592-7415. eISSN: 2282-8214. ©J. Senthamizh Selvan et al. This paper is published under the CC-BY licence agreement.



## 1 Introduction

Graph labeling is an assignment of set of integers to the set of vertices, edges or both based on certain conditions, In 1967 Rosa was first introduced the graph labeling. A graph labeling are useful family of mathematical models applied in many areas such as radar, missile guidance, radio frequency modulation and many more. A graph labeling is one topic in graph theory. So many kind of graph labeling among others: magic labeling, graceful labeling Afsana Ahmed munia [2014], o - Edge magic labeling, 1- Edge magic labeling, mean labeling, and etc. Every year and updated survey comes about various types of labeling by J. A. Gallian. From the survey, various types of labeling analyzed and introduced a new type of labeling called Range Labeling Moinin Al Aziz Md [2014]. In this paper are to proof that some trees namely Star tree, Spider tree, and Banana tree admit range labeling.

This article assumed for all graphs are finite and simple. The graph  $G = (\Psi, \tau)$  where  $\Psi(G)$  set of vertices and  $\tau(G)$  set of edges. A labeling is a most one of the part of the graph theory, R.Uma [2012]. A labeling is the assignment of labels, traditionary defined by integers to edges or vertices or both of vertices and edges Moinin Al Aziz Md [2014].

The origin of labeling is Rosa by 1967. R. Jahir Hussain [2022] and Afsana Ahmed munia [2014] was First developed Range labeling in 2022. The article Range labeling apply for star trees, spider trees, Banana tress A. N.Mohamed [2013].

## 2 Preliminary

**Definition 2.1.** *R.Uma [2012] A graceful labeling of a graph  $G$  is a vertex labeling  $\alpha : \Psi \rightarrow [0, m]$  such that  $\alpha$  is injective and induced mapping*

$$\alpha(\tau) = |\alpha(\Psi_k) - \alpha(\Psi_{k+1})|, \text{ for every } \Psi_k \Psi_{k+1} \in \tau(G).$$

*Assigns different labels to different edges of  $G$ . The differences  $|\alpha(\Psi_k) - \alpha(\Psi_{k+1})|$  is labeled weight of the edges  $\Psi_k \Psi_{k+1}$ . A graph  $G$  is called graceful.*

**Definition 2.2.** *Let  $G = (\Psi, \tau)$  be a graph with  $n$  vertices. A bijection on  $\alpha : \Psi \rightarrow \{1, 3, 6, 10, 15, \dots, \frac{n^2+n}{2}\}$  is called a range labeling if for each edge  $\tau$  is distinct and  $\tau$  is defined by  $\alpha^*(\tau) = \text{Maximum value } (\Psi_k, \Psi_{k+1}) - \text{Minimum value } (\Psi_k, \Psi_{k+1})$ .*

**Definition 2.3.** *A tree for  $l$ -internal vertex and  $k$  edges is called star  $S_1, K$  that appear to be complete by bipartite graph  $K_1, K$ .*

**Definition 2.4.** A Spider tree with atmost one vertex of degree greater than 2.

**Definition 2.5.** An  $(m, t)$ -Banana tree is a graph attained by attaching  $l$ -edge of all  $m$  copies of an  $t$ -star graph for a  $l$ -root vertex is different from each stars.

### 3 Main results

**Theorem 3.1.** All star tree take a range labeling.

**Proof.** Let  $G = (\Psi, \tau)$ , be a graph for  $\Psi_1$  is an interval vertex, 10 edges.

Consider,  $\alpha : \Psi \rightarrow \{1, 3, 6, 10, 15, \dots, \frac{n^2+n}{2}\}$

$\alpha^*(\tau) = \text{Maximum value } (\Psi_k, \Psi_{k+1}) - \text{Minimum value } (\Psi_k, \Psi_{k+1})$

$\alpha^*(\tau_1) = \text{Maximum value } (\Psi_1, \Psi_2) - \text{Minimum value } (\Psi_1, \Psi_2)$

if  $\Psi_1$  is a maximum value,  $\Psi_2$  is a minimum value.

$\alpha^*(\tau_1) = (\Psi_1 - \Psi_2).$

$= 1$  is an integer.

Suppose  $\Psi_2$  is a maximum value,  $\Psi_1$  is a minimum value.

$\alpha^*(\tau_1) = (\Psi_2 - \Psi_1).$

$= 1$  is an integer.

$\alpha^*(\tau_2) = \text{Maximum value } (\Psi_1, \Psi_3) - \text{Minimum value } (\Psi_1, \Psi_3)$

if  $\Psi_1$  is a maximum value,  $\Psi_3$  is a minimum value.

$\alpha^*(\tau_2) = (\Psi_1 - \Psi_3).$

$= 1$  is an integer.

Suppose  $\Psi_3$  is a maximum value,  $\Psi_1$  is a minimum value.

$\alpha^*(\tau_2) = (\Psi_3 - \Psi_1).$

$= 1$  is an integer.

$\alpha^*(\tau_9) = \text{Maximum value } (\Psi_1, \Psi_{10}) - \text{Minimum value } (\Psi_1, \Psi_{10})$

if  $\Psi_1$  is a maximum value,  $\Psi_{10}$  is a minimum value.

$\alpha^*(\tau_9) = (\Psi_1 - \Psi_{10}).$

$= 1$  is an integer.

Suppose  $\Psi_{10}$  is a maximum value,  $\Psi_1$  is a minimum value.

$\alpha^*(\tau_9) = (\Psi_{10} - \Psi_1).$

$= 1$  is an integer.

$\alpha^*(\tau_{10}) = \text{Maximum value } (\Psi_1, \Psi_{11}) - \text{Minimum value } (\Psi_1, \Psi_{11})$

if  $\Psi_1$  is a maximum value,  $\Psi_{11}$  is a minimum value.

$\alpha^*(\tau_{10}) = (\Psi_1 - \Psi_{11}).$

$= 1$  is an integer.

Suppose  $\Psi_{11}$  is a maximum value,  $\Psi_1$  is a minimum value.

$\alpha^*(\tau_{10}) = (\Psi_{11} - \Psi_1).$

$= 1$  is an integer.

Hence, Every star tree is a received range labeling

Therefore, Any star tree is a Range graph.

**Example 3.1.**

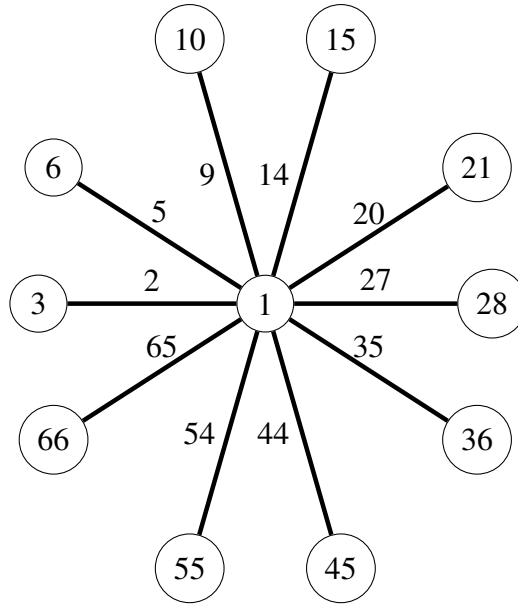


Fig.1. Range Labelling for star tree  $S_{10}$ .

**Theorem 3.2.** Every Spider tree take a range labeling.

**Proof.** Let  $G = (\Psi, \tau)$ , be a graph.

Let  $\alpha : \Psi \rightarrow \{1, 3, 6, 10, 15, \dots, \frac{n^2+n}{2}\}$

A spider tree for atmost 1-node of degree greater than 2 and this node is said to be section node and is denoted by  $\Psi_0$ . A stage of a spider graph is a path from the section node to edge of the tree. The edge is denoted by  $\Psi_1, \Psi_2, \Psi_3$ .

$$\alpha^*(\tau) = \text{Maximum value } (\Psi_k, \Psi_{k+1}) - \text{Minimum value } (\Psi_k, \Psi_{k+1})$$

$$\alpha^*(\tau_1) = \text{Maximum value } (\Psi_0, \Psi_1) - \text{Minimum value } (\Psi_0, \Psi_1)$$

if  $\Psi_0$  is a maximum value,  $\Psi_1$  is a minimum value.

$$\alpha^*(\tau_1) = (\Psi_0 - \Psi_1).$$

$= 1$  is an integer.

Suppose  $\Psi_1$  is a maximum value,  $\Psi_0$  is a minimum value.

$$\alpha^*(\tau_1) = (\Psi_1 - \Psi_0).$$

$= 1$  is an integer.

*Some results on range labeling*

$$\alpha^*(\tau_2) = \text{Maximum value } (\Psi_0, \Psi_2) - \text{Minimum value } (\Psi_0, \Psi_2)$$

if  $\Psi_0$  is a maximum value,  $\Psi_2$  is a minimum value.

$$\alpha^*(\tau_2) = (\Psi_0 - \Psi_2).$$

= 1 is an integer.

Suppose  $\Psi_2$  is a maximum value,  $\Psi_0$  is a minimum value.

$$\alpha^*(\tau_2) = (\Psi_2 - \Psi_0).$$

= 1 is an integer.

$$\alpha^*(\tau_3) = \text{Maximum value } (\Psi_0, \Psi_3) - \text{Minimum value } (\Psi_0, \Psi_3)$$

if  $\Psi_0$  is a maximum value,  $\Psi_3$  is a minimum value.

$$\alpha^*(\tau_3) = (\Psi_0 - \Psi_3).$$

= 1 is an integer.

Suppose  $\Psi_3$  is a maximum value,  $\Psi_0$  is a minimum value.

$$\alpha^*(\tau_3) = (\Psi_3 - \Psi_0).$$

= 1 is an integer.

So, All spider tree accepted range labeling

Thus, Every spider tree is a range graph.

**Example 3.2.**

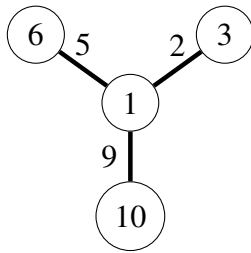


Fig.2. Range Labelling for Spider Tree.

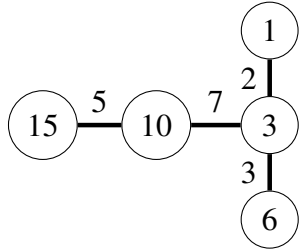


Fig.3. Range Labelling for Spider Tree.

**Example 3.3.**

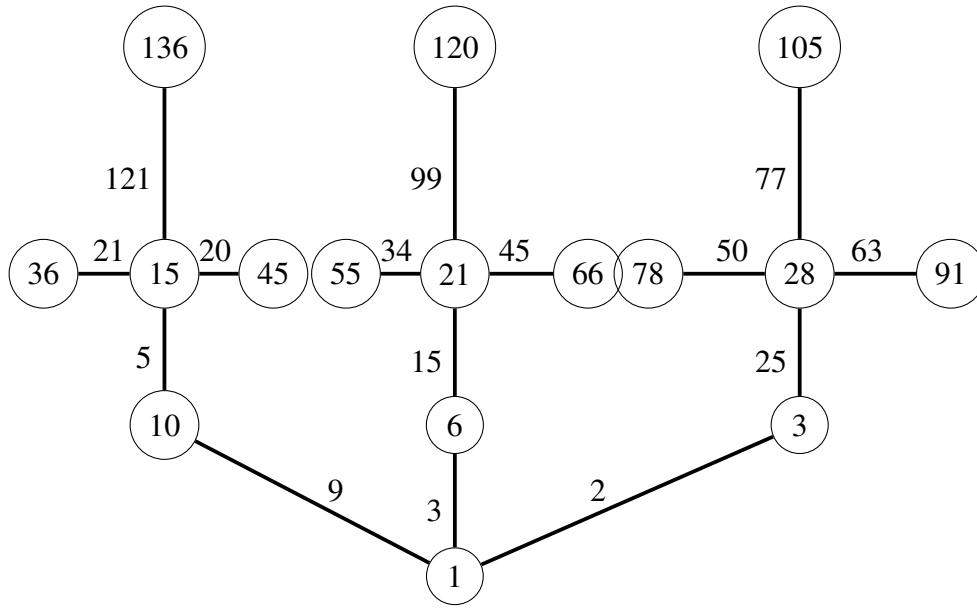


Fig.1. Range Labelling for Banana tree B(3,5).

**Theorem 3.3.** All Banana tree take a range labeling.

**Proof.** Let  $G = (\Psi, \tau)$ , be a graph.

Let  $\alpha : \Psi \rightarrow \{1, 3, 6, 10, 15, \dots, \frac{n^2+n}{2}\}$

$B(3, 5)$  is a Banana tree by attaching 1-edge of every 3 copies  $(\Psi_1, \Psi_2, \Psi_3)$  of an 5-star  $(\Psi_1, \Psi_2, \Psi_3)$  graph for 1-root vertex is different from every stars.

$\alpha^*(\tau) = \text{Maximum value } (\Psi_k, \Psi_{k+1}) - \text{Minimum value } (\Psi_k, \Psi_{k+1})$

$\alpha^*(\tau_1) = \text{Maximum value } (\Psi_0, \Psi_1) - \text{Minimum value } (\Psi_0, \Psi_1)$

if  $\Psi_0$  is a maximum value,  $\Psi_1$  is a minimum value.

$\alpha^*(\tau_1) = (\Psi_0 - \Psi_1).$

$= 1$  is an integer.

Suppose  $\Psi_1$  is a maximum value,  $\Psi_0$  is a minimum value.

$\alpha^*(\tau_1) = (\Psi_1 - \Psi_0).$

$= 1$  is an integer.

$\alpha^*(\tau_2) = \text{Maximum value } (\Psi_0, \Psi_2) - \text{Minimum value } (\Psi_0, \Psi_2)$

if  $\Psi_2$  is a maximum value,  $\Psi_0$  is a minimum value.

$\alpha^*(\tau_2) = (\Psi_2 - \Psi_0).$

$= 1$  is an integer.

Suppose  $\Psi_0$  is a maximum value,  $\Psi_2$  is a minimum value.

$\alpha^*(\tau_2) = (\Psi_0 - \Psi_2).$

$= 1$  is an integer.

$\alpha^*(\tau_3) = \text{Maximum value } (\Psi_0, \Psi_3) - \text{Minimum value } (\Psi_0, \Psi_3)$

*Some results on range labeling*

if  $\Psi_0$  is a maximum value,  $\Psi_3$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_3) &= (\Psi_0 - \Psi_3). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_3$  is a maximum value,  $\Psi_0$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_3) &= (\Psi_3 - \Psi_0). \\ &= 1 \text{ is an integer.}\end{aligned}$$

$$\alpha^*(\tau_4) = \text{Maximum value } (\Psi_1, \Psi_{11}) - \text{Minimum value } (\Psi_1, \Psi_{11})$$

if  $\Psi_1$  is a maximum value,  $\Psi_{11}$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_4) &= (\Psi_1 - \Psi_{11}). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_{11}$  is a maximum value,  $\Psi_1$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_4) &= (\Psi_{11} - \Psi_1). \\ &= 1 \text{ is an integer.}\end{aligned}$$

$$\alpha^*(\tau_7) = \text{Maximum value } (\Psi_1, \Psi_{14}) - \text{Minimum value } (\Psi_1, \Psi_{14})$$

if  $\Psi_1$  is a maximum value,  $\Psi_{14}$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_7) &= (\Psi_1 - \Psi_{14}). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_{14}$  is a maximum value,  $\Psi_1$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_7) &= (\Psi_{14} - \Psi_1). \\ &= 1 \text{ is an integer.}\end{aligned}$$

$$\alpha^*(\tau_8) = \text{Maximum value } (\Psi_2, \Psi_{21}) - \text{Minimum value } (\Psi_2, \Psi_{21})$$

if  $\Psi_2$  is a maximum value,  $\Psi_{21}$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_8) &= (\Psi_2 - \Psi_{21}). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_{21}$  is a maximum value,  $\Psi_2$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_8) &= (\Psi_{21} - \Psi_2). \\ &= 1 \text{ is an integer.}\end{aligned}$$

$$\alpha^*(\tau_{11}) = \text{Maximum value } (\Psi_2, \Psi_{24}) - \text{Minimum value } (\Psi_2, \Psi_{24})$$

if  $\Psi_2$  is a maximum value,  $\Psi_{24}$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_{11}) &= (\Psi_2 - \Psi_{24}). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_{24}$  is a maximum value,  $\Psi_2$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_{11}) &= (\Psi_{24} - \Psi_2). \\ &= 1 \text{ is an integer.}\end{aligned}$$

$$\alpha^*(\tau_{12}) = \text{Maximum value } (\Psi_3, \Psi_{31}) - \text{Minimum value } (\Psi_3, \Psi_{31})$$

if  $\Psi_3$  is a maximum value,  $\Psi_{31}$  is a minimum value.

$$\begin{aligned}\alpha^*(\tau_{12}) &= (\Psi_3 - \Psi_{31}). \\ &= 1 \text{ is an integer.}\end{aligned}$$

Suppose  $\Psi_{31}$  is a maximum value,  $\Psi_3$  is a minimum value.

$$\alpha^*(\tau_{12}) = (\Psi_{31} - \Psi_3).$$

$= 1$  is an integer.

$$\alpha^*(\tau_{15}) = \text{Maximum value } (\Psi_3, \Psi_{34}) - \text{Minimum value } (\Psi_3, \Psi_{34})$$

if  $\Psi_3$  is a maximum value,  $\Psi_{34}$  is a minimum value.

$$\alpha^*(\tau_{15}) = (\Psi_3 - \Psi_{34}).$$

$= 1$  is an integer.

Suppose  $\Psi_{34}$  is a maximum value,  $\Psi_3$  is a minimum value.

$$\alpha^*(\tau_{15}) = (\Psi_{34} - \Psi_3).$$

$= 1$  is an integer.

Hence, Every Banana tree received range labeling.

Therefore, All Banana tree is a range labeling.

## 4 Conclusions

In this article discussed for some trees received range labeling, so this trees star tree, spider tree, banana tree is also a range graph. Further more Analysis for this labeling apply for some special graphs.

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# Stability of domination in graphs

Reeja Kuriakose\*

Parvathy K S<sup>†</sup>

## Abstract

The stability of dominating sets in Graphs is introduced and studied, in this paper. Here  $D$  is a dominating set of Graph  $G$ . In this paper the vertices of  $D$  and vertices of  $V - D$  are called donors and acceptors respectively. For a vertex  $u$  in  $D$ , let  $\psi_D(u)$  denote the number  $|N(u) \cap (V - D)|$ . The donor instability or simply  $d$ -instability  $d_{inst}^D(e)$  of an edge  $e$  connecting two donor vertices  $v$  and  $u$  is  $|\psi_D(u) - \psi_D(v)|$ . The  $d$ -instability of  $D$ ,  $\psi_d(D)$  is the sum of  $d$ -instabilities of all edges connecting vertices in  $D$ . For a vertex  $u$  not in  $D$ , let  $\phi_D(u)$  denote the number  $|N(u) \cap D|$ . The Acceptor Instability or simply  $a$ -instability  $a_{inst}^D(e)$  of an edge  $e$  connecting two acceptor vertices  $u$  and  $v$  is  $|\phi_D(u) - \phi_D(v)|$ . The  $a$ -instability of  $D$ ,  $\phi_a(D)$  is the sum of  $a$ -instabilities of all edges connecting vertices in  $V - D$ . The dominating set  $D$  is  $d$ -stable if  $\psi_d(D) = 0$  and  $a$ -stable if  $\phi_a(D) = 0$ .  $D$  is stable, if  $\psi_d(D) = 0$  and  $\phi_a(D) = 0$ . Given a non negative integer  $\alpha$ ,  $D$  is  $\alpha$ - $d$ -stable, if  $d_{inst}^D(e) \leq \alpha$  for any edge  $e$  connecting two donor vertices and  $D$  is  $\alpha$ - $a$ -stable, if  $a_{inst}^D(e) \leq \alpha$  for any edge  $e$  connecting two acceptor vertices. Here we study  $\alpha$ -stability number of graphs for non negative integer  $\alpha$ .

**Keywords:** Domination number; stable domination

**2020 Mathematics Subject Classification:** 05C69: <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1081. ISSN: 1592-7415. eISSN: 2282-8214. ©Reeja Kuriakose et al. This paper is published under the CC-BY licence agreement.



# 1 Introduction

Graphs considered here are the simple undirected graphs. The study of domination in graphs started in 1850 with a problem of finding the minimum number of queens that are needed to place on a Chess board such that each field not occupied by queen can be attacked by atleast one. In 1962, Ore was the first in publishing about domination in graphs. Many researchers including Ernie Cockayne, Michael Henning have contributed much to this field. A beautiful survey of results in domination is given in fundamentals of domination in graphs by W.Haynes et al. [1998]. The theory of domination has applications in the study of facility location problems, social networking and so on. Several types of domination number such as perfect domination number, connected domination number, have been defined and studied.

In this paper the elements of a dominating set are called donors and the other vertices are called acceptors. A subset  $D$  of vertices in a social network graph with the condition that each member in  $D$  dominates almost equally many members in  $V - D$ , or that each member in  $V - D$  is dominated by almost equally many members in  $D$ , or both, plays a key role. This concept of equitable domination in graphs was defined and studied by A. Anitha, S. Arumugam and Mustapha Chelali. It is referred in the paper Anitha et al. [2011]. In social network problems related to marketing, banking and others the instability affects the system when adjacent acceptors are dominated by unequal number of donors or adjacent donors dominates unequal number of acceptors. This situation become worse when the instability is large. Motivated from this idea the concept of  $\alpha$ -stability of dominating sets is being introduced.

## 2 $\alpha$ -d stable domination

**Definition 2.1.** Let  $D$  be a dominating set. For a vertex  $u$  in  $D$  let  $\psi_D(u) = |N(u) \cap (V - D)|$ . The donor instability or  $d$ -instability of an edge  $e$  connecting two donor vertices  $u$  and  $v$ ,  $d_{inst}^D(e) = |\psi_D(u) - \psi_D(v)|$ . Let  $D \subset V$ , the  $d$ -instability of  $D$ , is the sum of  $d$ -instabilities of all edges connecting vertices in  $D$ ,  $\psi_d(D) = \sum_{e \in G[D]} d_{inst}^D(e)$ .

**Definition 2.2.** Let  $D$  be a dominating set. Given a non negative integer  $\alpha$ ,  $D$  is an  $\alpha$ - $d$ -stable dominating set, if  $d_{inst}^D(e) \leq \alpha$  for any edge  $e$  connecting two donor vertices. Cardinality of a minimum  $\alpha$ - $d$ -stable dominating set is  $\alpha$ - $d$ -stable domination number and it is denoted by  $\gamma_d^\alpha(G)$ .

**Definition 2.3.** A dominating set  $D$  is  $d$ -stable if  $\psi_d(D) = 0$ . Cardinality of a minimum  $d$ -stable dominating set is  $d$ -stable domination number and it is denoted

by  $\gamma_d^0(G)$ .

**Observation 2.1.** If  $\alpha \geq \beta$ , then  $\gamma(G) \leq \gamma_d^\alpha(G) \leq \gamma_d^\beta(G)$ .

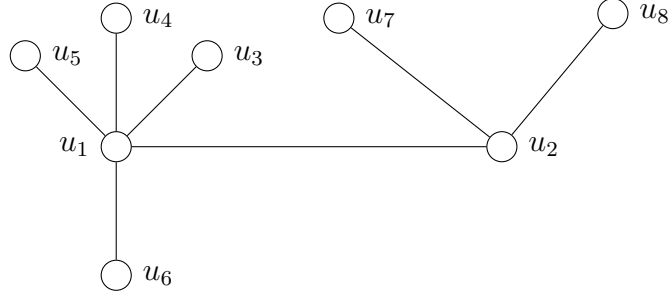


Figure 1: Graph with  $\gamma(G) \leq \gamma_d^\alpha(G) \leq \gamma_d^\beta(G)$

**Example 2.1.** In Figure 1,  $D = \{u_1, u_2\}$  is the minimum dominating set.  $\psi_D(u_1) = 4$  and  $\psi_D(u_2) = 2$ . And  $d_{inst}^D(u_1 u_2) = 2$ . Hence  $D$  is a minimum 2-d-stable dominating set. And for  $\alpha \geq 2$ ,  $\gamma_d^\alpha(G) = 2$ .

If  $S = \{u_1, u_7, u_8\}$ ,  $\psi_D(u_1) = 3$ ,  $\psi_D(u_7) = 0$  and  $\psi_D(u_8) = 0$ . None of them are adjacent. Hence  $\gamma_d^1(G) = \gamma_d^0(G) = 3$ .

**Observation 2.2.** Property of being  $\alpha$ -d-stable dominating set is neither super-hereditary nor hereditary.

**Theorem 2.1.** An  $\alpha$ -d-stable dominating set  $D$  is a minimal  $\alpha$ -d-stable dominating set if and only if for each vertex  $v$  in  $D$  one of the following conditions holds

1.  $v$  is an isolate of  $D$ .
2.  $v$  has a private neighbour  $u$  in  $V - D$ .
3. There exist two adjacent vertices  $u_1$  and  $u_2$  different from  $v$  in  $D$ ,  $u_1$  adjacent to  $v$ ,  $u_2$  not adjacent to  $v$  and  $\psi_D(u_1) = \psi_D(u_2) + \alpha$ .

*Proof.* If an  $\alpha$ -d-stable dominating set  $D$  is minimal, then  $D$  is an  $\alpha$ -d-stable dominating set and for each vertex  $v$  in  $D$ ,  $D - \{v\}$  is not an  $\alpha$ -d-stable dominating set. This means that some vertex  $u$  in  $(V - D) \cup \{v\}$  is not dominated by  $D - \{v\}$  or there exist two adjacent vertices  $u_1$  and  $u_2$  different from  $v$  in  $D$  with  $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$  but  $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$ .

Now if some vertex  $u$  in  $(V - D) \cup \{v\}$  is not dominated by any vertex in  $D - \{v\}$ , either  $u = v$ , means  $v$  is an isolate of  $D$  or  $u \in V - D$ . If  $u$  is not

dominated by  $D - \{v\}$ , then  $u$  is adjacent only to vertex  $v$  in  $D$ . ie,  $v$  has a private neighbour  $u$  in  $V - D$ .

If  $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$  and  $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$ , let  $\alpha = 0$ , then  $\psi_D(u_1) = \psi_D(u_2)$  and  $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| = \alpha + 1$ . Assume  $\psi_{D-\{v\}}(u_1) > \psi_{D-\{v\}}(u_2)$ . Then  $u_1$  is adjacent to  $v$  but  $u_2$  is not adjacent to  $v$  and  $\psi_D(u_1) = \psi_D(u_2) + \alpha$ . If  $\alpha > 0$ , then assume  $\psi_D(u_1) > \psi_D(u_2)$ . Then  $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) = \alpha + 1$ . Then  $u_1$  is adjacent to  $v$  but  $u_2$  is not adjacent to  $v$  and  $\psi_D(u_1) = \psi_D(u_2) + \alpha$ .

Conversely, suppose that  $D$  is an  $\alpha$ - $d$ -stable dominating set and for each vertex  $v \in D$ , one of the three statements holds. We show that  $D$  is a minimal  $\alpha$ - $d$ -stable dominating set. If  $D$  is not a minimal  $\alpha$ - $d$ -stable dominating set, then there exists a vertex  $v \in D$  such that  $D - \{v\}$  is an  $\alpha$ - $d$ -stable dominating set. Then each vertex  $u$  in  $(V - D) \cup \{v\}$  is adjacent with atleast one vertex in  $D - \{v\}$ . Then  $v$  is not an isolate of  $D$  and condition 1 does not hold. And  $v$  has no private neighbour in  $V - D$  and condition 2 does not hold.  $D - \{v\}$  is an  $\alpha$ - $d$ -stable dominating set implies for any two adjacent vertices  $u_1$  and  $u_2$  in  $D - \{v\}$ ,  $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) \leq \alpha$ . Hence condition 3 does not hold. Hence  $D$  is a minimal  $\alpha$ - $d$ -stable dominating set.  $\square$

**Observation 2.3.** For non negative integer  $\alpha$ ,  $\gamma_d^\alpha(G) = 1 \iff \gamma(G) = 1$ .

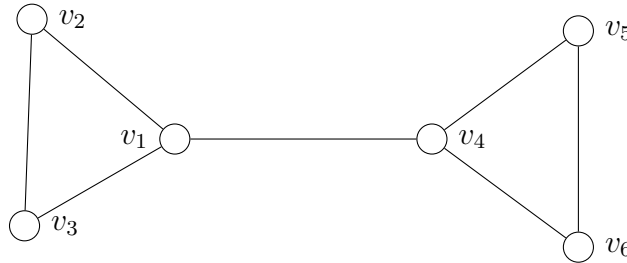


Figure 2: Graph with  $\gamma_d^\alpha(G) = 1$

**Theorem 2.2.** For a graph  $G$  and non negative integer  $\alpha$ ,  $\beta_o(G) \geq \gamma_d^\alpha(G)$ .

*Proof.* Let  $S$  be a maximum independent set. Then, every vertex in  $V - S$  is adjacent with atleast one vertex in  $S$ . Thus  $S$  is a dominating set.  $S$  is an independent set. It follows that  $S$  is a  $d$ -stable dominating set. Hence,  $\beta_o(G) \geq \gamma_d^\alpha(G)$ . For the graph in figure 2,  $\gamma_d^\alpha(G) = 2 = \beta_o(G)$ . Hence the bound is sharp.  $\square$

**Theorem 2.3.** If  $D$  is an  $\alpha$ - $d$ -stable dominating set of a graph  $G$  and  $u$  and  $v$  are adjacent vertices in  $D$  with  $d(v) = d(u) + k + \alpha$ ,  $k \in \mathbb{Z}^+$ , then  $D$  contains atleast  $k$  elements from  $(N[v] - N[u])$ .

### Stability of domination in graphs

*Proof.* If  $D$  is an  $\alpha$ - $d$ -stable dominating set of a graph  $G$  and  $u$  and  $v$  are adjacent vertices in  $D$  with  $d(v) = d(u) + k + \alpha$ ,  $k \in \mathbb{Z}^+$ , then  $|\psi_D(v) - \psi_D(u)| \leq \alpha$ .

Hence  $|N[v] \cap (V - D)| \leq |N[u] \cap (V - D)| + \alpha$ . Thus,  $d(v) - d(u) \leq |(N[v] - N[u]) \cap D| + \alpha$ .

Hence  $D$  contains atleast  $k$  elements from  $(N[v] - N[u])$ .  $\square$

**Corollary 2.1.** *If  $D$  is a  $d$ -stable dominating set of a graph  $G$  and  $u$  and  $v$  are adjacent vertices in  $D$  with  $d(v) > d(u)$ , then  $D$  contains atleast  $d(v) - d(u)$  elements from  $(N[v] - N[u])$ .*

**Corollary 2.2.** *If  $u$  is a pendant vertex adjacent to  $v$ ,  $D$  is a  $d$ -stable dominating set and  $u, v \in D$ , then  $N[v] \subset D$ .*

**Theorem 2.4.** *For any non negative integer  $\beta$ , there exist graph  $G$  with  $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G) \dots > \gamma_d^\beta(G)$*

*Proof.* Take  $k = \beta + 1$

Construct  $G$  as follows,

Step 1:- Let  $H$  be the complete graph with vertex set  $\{a_1, a_2, \dots, a_k\}$ .

Step 2:- Let  $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,i+1}\}$  for  $i = 1, 2, \dots, k$ .

Form  $G$  by joining each vertices in  $A_i$  with  $a_i$  in  $H$  for  $i = 1, 2, \dots, k$ .

Let  $D$  be a  $d$ -stable dominating set,  $B_i = \{a_i, a_{i,1}, a_{i,2}, \dots, a_{i,i+1}\}$  and  $C_i = B_i \cap D$ .

If  $a_r, a_s \in D$  with  $r < s$  then by corollary 2.1,  $D$  contains  $s - r$  elements from  $A_s$  and hence by corollary 2.2,  $N[a_s] \subset D$ . Thus  $B_s \subset D$ . Thus  $C_i = B_i$  for all  $i = 1, 2, \dots, k$ .

Hence  $D = V(G)$  or  $D = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{s-1} \cup A_{s+1} \cup \dots \cup A_{k-1} \cup A_k \cup \{a_s\}$ . Hence we take  $D = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{s-1} \cup A_{s+1} \cup \dots \cup A_{k-1} \cup A_k \cup \{a_s\}$ .

Thus  $|C_i| = i + 1$  for  $i \neq s$  and  $|C_s| = 1$ . Then,

$$\begin{aligned} |D| &= 2 + 3 + \dots + (s) + 1 + (s + 2) + \dots + (k + 1) \\ &= \frac{(k+1)(k+2)}{2} - (s + 1). \end{aligned}$$

Hence  $|D|$  is minimum when  $\frac{(k+2)(k+1)}{2} - (s + 1)$  is minimum. That is when  $s = k$ . And  $D = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{k-1} \cup \{a_k\}$  will form a  $d$ -stable dominating set with  $|D| = \frac{(k+2)(k+1)}{2} - (k + 1)$ .

Hence  $\gamma_d^0(G) = \frac{(k+2)(k+1)}{2} - (k + 1) = \frac{(k)(k+1)}{2}$ .

Similarly,

$$\begin{aligned} \gamma_d^0(G) &= \frac{(k)(k+1)}{2} \\ \gamma_d^1(G) &= \frac{k(k-1)}{2} + 1 \\ &\vdots \\ \gamma_d^\beta(G) &= \frac{(k-\beta+1)(k-\beta)}{2} + \beta. \end{aligned}$$

Figure 3 illustrates the graph with  $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)$ .  $\square$

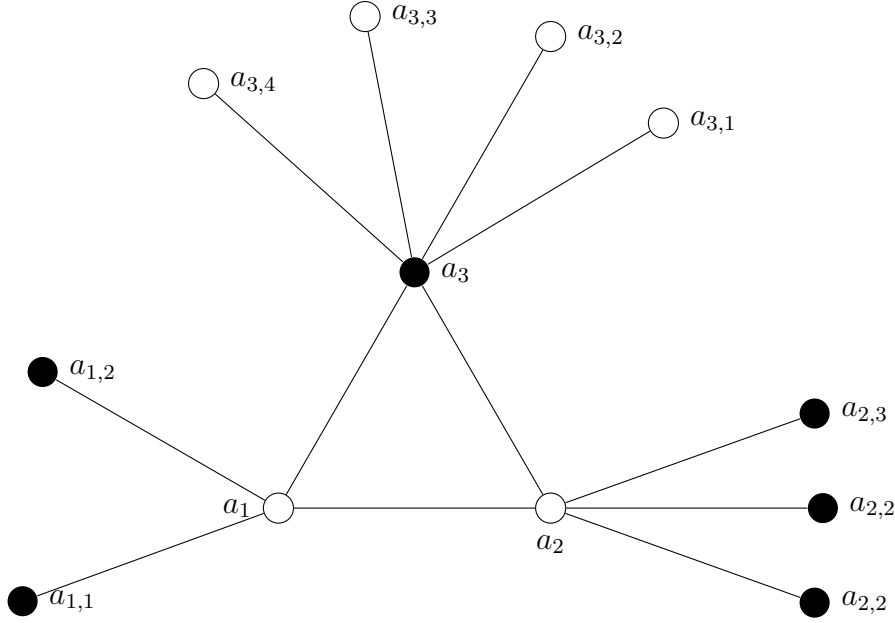


Figure 3: Graph with  $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)$

### 3 $\alpha$ -a-stable domination

**Definition 3.1.** Let  $D$  be a dominating set. For a vertex  $u$  not in  $D$ , let  $\phi_D(u) = |N(u) \cap D|$ . The Acceptor Instability or  $a$ -instability of an edge  $e$  connecting two acceptor vertices  $u$  and  $v$  is,  $a_{inst}^D(e) = |\phi_D(u) - \phi_D(v)|$ . The  $a$ -instability of  $D$ ,  $\phi_a(D)$  is the sum of  $a$ -instabilities of all edges connecting vertices in  $V - D$ ,  $\phi_a(D) = \sum_{e \in G[V-D]} a_{inst}^D(e)$ .

**Definition 3.2.** Let  $D$  be a dominating set. Given a non negative integer  $\alpha$ ,  $D$  is an  $\alpha$ - $a$ -stable dominating set, if  $a_{inst}^D(e) \leq \alpha$  for any edge  $e$  connecting two acceptor vertices. Cardinality of a minimum  $\alpha$ - $a$ -stable dominating set is  $\alpha$ - $a$ -stable domination number and denoted by  $\gamma_a^\alpha(G)$ .

**Definition 3.3.** The dominating set  $D$  is  $a$ -stable if  $\phi_a(D) = 0$ . Minimum cardinality of an  $a$ -stable dominating set is  $a$ -stable domination number and denoted by  $\gamma_a^0(G)$ .

**Observation 3.1.** If  $\alpha \geq \beta$ , then  $\gamma(G) \leq \gamma_a^\alpha(G) \leq \gamma_a^\beta(G)$

**Example 3.1.** In Figure 1,  $D = \{u_1, u_2\}$  is the minimum dominating set.  $\phi_D(u_3) = \phi_D(u_4) = \phi_D(u_5) = \phi_D(u_6) = \phi_D(u_7) = \phi_D(u_8) = 1$ . Hence  $D$  is a minimum  $a$ -stable dominating set and  $\gamma_a^\alpha(G) = 2$  for all non negative integer  $\alpha$ .

**Observation 3.2.** *Property of being  $\alpha$ -a-stable dominating set is neither super-hereditary nor hereditary.*

**Theorem 3.1.** *An  $\alpha$ -a-stable dominating set  $D$  is a minimal  $\alpha$ -a-stable dominating set if and only if for each vertex  $v$  in  $D$  one of the following conditions holds*

1.  $v$  is an isolate of  $D$ .
2.  $v$  has a private neighbour  $u$  in  $V - D$ .
3. There exist two adjacent vertices  $u_1$  and  $u_2$  in  $V - D$ ,  $u_1$  adjacent to  $v$ ,  $u_2$  not adjacent to  $v$  and  $\phi_D(u_2) = \phi_D(u_1) + \alpha$ .

*Proof.* If an  $\alpha$ -a-stable dominating set  $D$  is minimal then  $D$  is an  $\alpha$ -a-stable dominating set and for each vertex  $v$  in  $D$ ,  $D - \{v\}$  is not an  $\alpha$ -a-stable dominating set. This means that some vertex  $u$  in  $(V - D) \cup \{v\}$  is not dominated by  $D - \{v\}$  or there exist two adjacent vertices  $u_1$  and  $u_2$  in  $V - D$  with  $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$  but  $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$ .

Now if some vertex  $u$  in  $(V - D) \cup \{v\}$  is not dominated by any vertex in  $D - \{v\}$ , either  $u = v$ , means  $v$  is an isolate of  $D$  or  $u \in V - D$ . If  $u$  is not dominated by  $D - \{v\}$ , then  $u$  is adjacent only to vertex  $v$  in  $D$ . ie,  $v$  has a private neighbour  $u$  in  $V - D$ .

If  $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$  and  $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$ , let  $\alpha = 0$ , then  $\phi_D(u_1) = \phi_D(u_2)$  and  $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| = \alpha + 1$ . Assume  $\phi_{D-\{v\}}(u_2) > \phi_{D-\{v\}}(u_1)$ . Then  $u_1$  is adjacent to  $v$  but  $u_2$  is not adjacent to  $v$  and  $\phi_D(u_2) = \phi_D(u_1) + \alpha$ . If  $\alpha > 0$ , then assume  $\phi_D(u_2) > \phi_D(u_1)$ . Then  $\phi_D(u_2) - \phi_D(u_1) = \alpha$  and  $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) = \alpha + 1$ . Then  $u_1$  is adjacent to  $v$  but  $u_2$  is not adjacent to  $v$  and  $\phi_D(u_2) = \phi_D(u_1) + \alpha$ .

Conversely, suppose that  $D$  is an  $\alpha$ -a-stable dominating set and for each vertex  $v \in D$ , one of the three statements holds. We show that  $D$  is a minimal  $\alpha$ -a-stable dominating set. If  $D$  is not a minimal  $\alpha$ -a-stable dominating set, then there exists a vertex  $v \in D$  such that  $D - \{v\}$  is an  $\alpha$ -a-stable dominating set. Then each vertex  $u$  in  $(V - D) \cup \{v\}$  is adjacent with atleast one vertex in  $D - \{v\}$ . Then  $v$  is not an isolate of  $D$  and condition 1 does not hold. And  $v$  has no private neighbour in  $V - D$  and condition 2 does not hold. If  $D - \{v\}$  is an  $\alpha$ -a-stable dominating set then for any adjacent vertices  $u_1$  and  $u_2$  in  $(V - D) \cup \{v\}$ ,  $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) \leq \alpha$ . Hence condition 3 does not hold. Hence  $D$  is a minimal  $\alpha$ -a-stable dominating set.  $\square$

**Observation 3.3.** *For non negative integer  $\alpha$ ,  $\gamma_a^\alpha(G) = 1 \iff \gamma(G) = 1$*

**Theorem 3.2.** *For  $\alpha \geq 1$ ,  $\gamma_a^\alpha(G) = 2 \iff \gamma(G) = 2$*

*Proof.* If  $\gamma(G) = 2$ , then for a minimum dominating set  $D$ ,  $|D| = 2$

$$|D| = 2 \Rightarrow \phi_D(v) = 1 \quad \text{or} \quad \phi_D(v) = 2 \quad \forall v \in V - D$$

$$\Rightarrow |\phi_D(v_1) - \phi_D(v_2)| \leq 1, \quad \forall v_1, v_2 \in V - D$$

$$\Rightarrow \gamma_a^\alpha(G) = 2$$

Conversely, if  $\gamma_a^\alpha(G) = 2$ , then  $\gamma(G) \neq 1$ . If  $D$  is a minimum  $\alpha$ -a- stable dominating set, then  $|D| = 2$ , and  $D$  is a dominating set. Thus,  $\gamma(G) = 2$   $\square$

**Theorem 3.3.** For any graph  $G$  and non negative integer  $\alpha$ ,  $\gamma_a^\alpha(G) \leq \gamma_p(G)$ . And this bound is sharp.

*Proof.* If  $D$  is a  $\gamma_p$ - set, then  $|N[v] \cap D| = 1, \forall v \in (V - D)$ . Hence  $\phi_D(v) = 1$ , for all  $v \in (V - D)$ . And so  $D$  is an  $\alpha$ -stable dominating set. Thus every perfect dominating set is an  $\alpha$ -stable dominating set. Hence,  $\gamma_a^\alpha(G) \leq \gamma_p(G)$ . For  $G = P_{3n}$ ,  $\gamma_p(G) = n = \gamma_a^\alpha(G)$ . So we can see that the bound is sharp.  $\square$

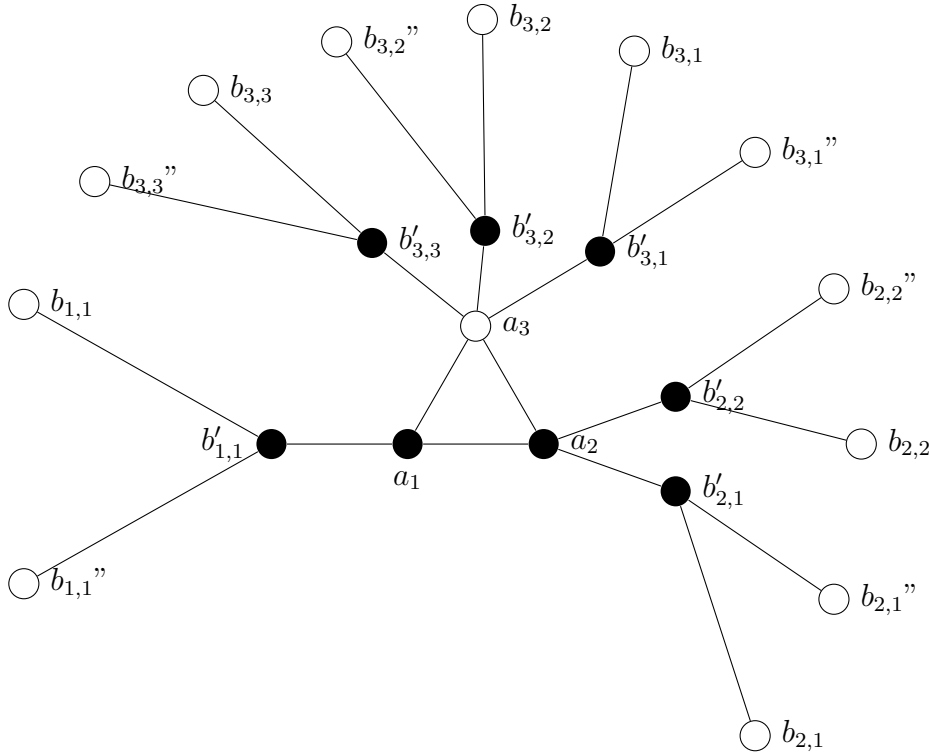


Figure 4: Graph with  $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)$

**Theorem 3.4.** For any positive integer  $\beta$ , there exist graph  $G$  with  $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G) \dots > \gamma_a^\beta(G)$ .

*Proof.* Let  $k = \beta + 1$ . Construct  $G$  as follows

Step 1: Let  $H$  be the complete graph with vertex set  $\{a_1, a_2, \dots, a_k\}$

Step 2: For each  $i$  take  $i$  copies of  $P_3$  with vertex set  $\{b_{i,j}, b'_{i,j}, b_{i,j}''\}$  for  $j = 1, 2, \dots, i$  and join  $b'_{i,j}$  with  $a_i$  for each  $j = 1, 2, \dots, i$ .

Let  $A_i^j = \{b_{i,j}, b'_{i,j}, b_{i,j}''\}$  for  $j = 1, 2, \dots, i$  and  $A_i = \{a_i\} \cup \bigcup_{j=1}^i \{b_{i,j}, b'_{i,j}, b_{i,j}''\}$  for all  $i \in \{1, 2, \dots, k\}$ .

Let  $D$  be an  $a$ -stable dominating set. If  $a_i \in D$  then  $|A_i \cap D| \geq i + 1$ .

Let  $r$  be the smallest integer such that  $a_r \notin D$ . Then  $|A_r \cap D| \geq r$ .

If  $s > r$  and  $a_s \notin D$ , since  $\gamma_{inst}^a(a_r, a_s) = 0$ ,  $b'_{sj} \in D$  for atmost  $r$  values of  $j$ . And if there exist  $j$  for which  $b'_{sj} \notin D$  then  $b_{sj}, b_{sj}'' \in D$ .

$$\begin{aligned} \implies |A_s \cap D| &\geq r + 2(s - r) \\ &= 2s - r \geq s + 1 \end{aligned}$$

$$\begin{aligned} \text{Hence, } |D| &\geq |A_1 \cap D| + |A_2 \cap D| + |A_3 \cap D| + \dots + |A_{r-1} \cap D| + \\ &\quad |A_r \cap D| + |A_{r+1} \cap D| + \dots + |A_k \cap D| \\ &\geq 2 + 3 + \dots + (r - 1 + 1) + r + (r + 2) + \dots + (k + 1) . \\ &= 1 + 2 + \dots + k + k - 1 \\ &= \frac{k(k+1)}{2} + (k - 1) \end{aligned}$$

Thus,  $\gamma_d^0(G) \geq \frac{k(k+1)}{2} + (k - 1)$ .

And  $D' = \{a_1, \dots, a_{k-1}\} \cup \bigcup_{i=1}^k \{b'_{i1}, b'_{i2}, \dots, b'_{ii}\}$  is an  $a$ -stable dominating set with  $|D'| = \frac{k(k+1)}{2} + (k - 1)$ .

Hence,  $\gamma_a^0(G) \leq \frac{k(k+1)}{2} + (k - 1)$

Thus,  $\gamma_a^0(G) = \frac{k(k+1)}{2} + (k - 1)$

Similarly,

$$\begin{aligned} \gamma_a^1(G) &= \frac{k(k+1)}{2} + (k - 2) \\ \gamma_a^2(G) &= \frac{k(k+1)}{2} + (k - 3) \\ \gamma_a^3(G) &= \frac{k(k+1)}{2} + (k - 4) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \gamma_a^{\beta-1}(G) &= \frac{k(k+1)}{2} + 1 \\ \gamma_a^\beta(G) &= \frac{k(k+1)}{2} \end{aligned}$$

Figure 4 illustrates the graph with  $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)$ . □

## 4 $\alpha$ -stable domination

**Definition 4.1.** A dominating set  $D$  is stable, if  $\psi_a(D) = 0$  and  $\phi_a(D) = 0$ . Minimum cardinality of a stable dominating set is called stable domination number and denoted by  $\gamma^0(G)$ .



**Definition 4.2.** If a dominating set  $D$  is an  $\alpha$ -d-stable dominating set and  $\alpha$ -a-stable dominating set, then  $D$  is called an  $\alpha$ -stable dominating set and cardinality of a minimum  $\alpha$ -stable dominating set is defined as  $\alpha$ -stable domination number and denoted by  $\gamma^\alpha(G)$

**Observation 4.1.** If a minimum  $\alpha$ -a-stable dominating set is an  $\alpha$ -d-stable dominating set, then  $\gamma^\alpha(G) = \gamma_a^\alpha(G)$ . And if a minimum  $\alpha$ -d-stable dominating set is an  $\alpha$ -a-stable dominating set, then  $\gamma^\alpha(G) = \gamma_d^\alpha(G)$ .

**Observation 4.2.** If  $\alpha \geq \beta$ , then  $\gamma(G) \leq \gamma^\alpha(G) \leq \gamma^\beta(G)$ .

**Definition 4.3.** Minimum  $\alpha$  so that  $\gamma^\alpha(G) = \gamma(G)$  is called stable dominating index and denoted by  $I_{sd}(G)$ .

**Example 4.1.** In figure 1, the minimum d-stable dominating set  $\{u_1, u_7, u_8\}$  is an  $\alpha$ -stable dominating set. Hence  $\{u_1, u_7, u_8\}$  is a minimum stable dominating set and  $\gamma^0(G) = 3$ .

A minimum 1-d-stable dominating set  $\{u_1, u_7, u_8\}$  form a 1-a-stable dominating set and hence  $\{u_1, u_7, u_8\}$  is a minimum 1-stable dominating set and  $\gamma^1(G) = 3$ . And minimum dominating set  $\{u_1, u_2\}$  is a 2-a-stable dominating set and a 2-d-stable dominating set.  $\{u_1, u_2\}$  form a minimum 2-stable dominating set. Hence,  $\gamma^2(G) = 2$ .

And  $\forall \alpha \geq 2$ ,  $\gamma^\alpha(G) = 2 = \gamma(G)$ . Hence,  $I_{sd}(G) = 2$ .

Compliment of a minimum  $\alpha$ -stable dominating set need not be an  $\alpha$ -stable dominating set. In graph figure 1,  $\{u_1, u_2, u_3\}$  is a minimum 1-stable dominating set but its compliment is not a 1-stable dominating set.

**Observation 4.3.** Property of being  $\alpha$ -stable dominating set is neither superhereditary nor hereditary.

**Theorem 4.1.** For any graph  $G$  and for any non-negative integer  $\alpha$ ,  $\gamma(G) = 1 \iff \gamma^\alpha(G) = 1$ .

*Proof.* If  $\gamma(G) = 1$ , then the single vertex set  $\{v\}$  which dominates all vertices of  $G$ , is an  $\alpha$ -d-stable dominating set and an  $\alpha$ -a-stable dominating set. Then  $\gamma^\alpha(G) = 1$ . Also any  $\alpha$ -stable dominating set is a dominating set. So, if  $\gamma^\alpha(G) = 1$  then  $\gamma(G) = 1$ .  $\square$

**Lemma 4.1.** For any graph  $G$  and for any non negative integer  $\alpha$ ,  $\gamma^\alpha(G) = n \iff G = \overline{K_n}$ .

*Proof.* If  $G \neq \overline{K_n}$ , there is atleast one vertex  $v$  with  $d(v) \geq 1$ . Then  $V - \{v\}$  is an  $\alpha$ -stable dominating set. This means that  $\gamma^\alpha(G) \leq n - 1$ . Hence if  $\gamma^\alpha(G) = n$ , then  $G = \overline{K_n}$ . If  $G = \overline{K_n}$ , then  $\gamma^\alpha(G) = n$  trivially.  $\square$

**Theorem 4.2.** For a graph  $G$  with  $\delta(G) \geq 1$ ,  $\gamma^\alpha(G) \leq n - 1$ .

*Proof.* From lemma 3.9 it is clear that  $\gamma^\alpha(G) \leq n - 1$ .  $\square$

**Theorem 4.3.** For any non negative integer  $\alpha$ ,  $\gamma^\alpha(G) = \gamma(G)$  for the following Graphs

- Complete graph  $K_n$
- Path  $P_n$
- Cycle  $C_n$
- Wheel graph  $W_n$
- Helm graph  $H_n$

*Proof.* In these graphs minimum dominating set  $D$ , form an  $\alpha$ -stable dominating set. Hence  $\alpha$ -stable domination number is same as its domination number.  $\square$

**Theorem 4.4.** Let  $G$  and  $H$  be two graphs of order  $n_1$  and  $n_2$ , then for any non-negative integer  $\alpha$ ,

- $\gamma_a^\alpha(G \square H) \leq \min\{n_1 \gamma_a^\alpha(H), n_2 \gamma_a^\alpha(G)\}$
- $\gamma_d^\alpha(G \square H) \leq \min\{n_1 \gamma_d^\alpha(H), n_2 \gamma_d^\alpha(G)\}$
- $\gamma^\alpha(G \square H) \leq \min\{n_1 \gamma^\alpha(H), n_2 \gamma^\alpha(G)\}$ .

*Proof.* Let  $S_H$  be a minimum  $\alpha$ -a-stable dominating set of  $H$ . Let us see that  $S = V(G) \times S_H$  is an  $\alpha$ -a-stable dominating set of  $G \square H$ . If  $(u, v) \in (V(G) \times V(H)) - S$ . Then  $(u, v)$  is adjacent to atleast one vertex in  $S$ . And if  $(u_1, v_1) \in (V(G) \times V(H)) - S$  and  $(u_2, v_2) \in (V(G) \times V(H)) - S$  and  $(u_1, v_1)$  adjacent to

$$\begin{aligned}
 \phi_S(u_1, v_1) &= |\{(u, v) \in S : (u_1, v_1) \text{ adjacent to } (u, v)\}| \\
 &= |\{(u, v) \in S : u_1 = u \text{ and } v_1 \text{ adjacent to } v\} \cup \\
 &\quad \{(u, v) \in S : u_1 \text{ adjacent to } u \text{ and } v_1 = v\}| \\
 &= |\{(u, v) \in S : u_1 = u \text{ and } v_1 \text{ adjacent to } v\}| \\
 &= \phi_{S_H}(v_1). \\
 \phi_S(u_2, v_2) &= \phi_{S_H}(v_2). \text{ And } |\phi_S(u_1, v_1) - \phi_S(u_2, v_2)| = |\phi_{S_H}(v_1) - \phi_{S_H}(v_2)| \leq \alpha. \\
 &\text{Hence } S \text{ is an } \alpha\text{-a-stable dominating set of } G \square H.
 \end{aligned}$$

Similarly, if  $S_G$  is a minimum  $\alpha$ - $a$ -stable dominating set of  $G$ , then  $S_G \times V(H)$  is an  $\alpha$ - $a$ -stable dominating set of  $G \square H$ .

Thus,  $\gamma_a^\alpha(G \square H) \leq \min\{n_1 \gamma_a^\alpha(H), n_2 \gamma_a^\alpha(G)\}$ .

Let  $S_H$  be a minimum  $\alpha$ - $d$ -stable dominating set of  $H$ . Let  $S = V(G) \times S_H$ . If  $(u, v) \in (V(G) \times V(H)) - S$ , then  $(u, v)$  is adjacent to atleast one vertex in  $S$ . And if  $(u_1, v_1) \in S$  and  $(u_2, v_2) \in S$  and  $(u_1, v_1)$  adjacent to  $(u_2, v_2)$ . Then,

$$\begin{aligned} & \psi_S(u_1, v_1) \\ &= |\{(u, v) \in (V(G) \times V(H)) - S : (u_1, v_1) \text{ adjacent to } (u, v)\}| \\ &= |\{(u, v) \in (V(G) \times V(H)) - S : u_1 = u \text{ and } v_1 \text{ adjacent to } v\} \cup \\ & \quad \{(u, v) \in (V(G) \times V(H)) - S : u_1 \text{ adjacent to } u \text{ and } v_1 = v\}| \\ &= |\{(u, v) \in (V(G) \times V(H)) - S : u_1 = u \text{ and } v_1 \text{ adjacent to } v\}| \\ &= \psi_{S_H}(v_1). \end{aligned}$$

Similarly  $\psi_S(u_2, v_2) = \psi_{S_H}(v_2)$ .

Thus

$$\begin{aligned} |\psi_S(u_1, v_1) - \psi_S(u_2, v_2)| &= |\psi_{S_H}(v_1) - \psi_{S_H}(v_2)| \\ &\leq \alpha \end{aligned}$$

Thus  $S$  is an  $\alpha$ - $d$ -stable dominating set of  $G \square H$ .

Similarly, if  $S_G$  is a minimum  $\alpha$ - $d$ -stable dominating set of  $G$ ,  $S_G \times V(H)$  is an  $\alpha$ - $d$ -stable dominating set of  $G \square H$ . Thus,

$$\gamma_d^\alpha(G \square H) \leq \min\{n_1 \gamma_d^\alpha(H), n_2 \gamma_d^\alpha(G)\}.$$

Hence,  $\gamma^\alpha(G \square H) \leq \min\{n_1 \gamma^\alpha(H), n_2 \gamma^\alpha(G)\}$ . □

**Remark 4.1.** The bound in theorem 3.12 is attained if  $G = K_n$  and  $H = K_2$ ; because  $\gamma^\alpha(K_n \square K_2) = 2 = \min\{2\gamma^\alpha(K_n), n\gamma^\alpha(K_2)\}$ .

**Theorem 4.5.** For any two graphs  $G$  and  $K$  and non negative integer  $\alpha$ ,  $\alpha$ -stable domination number of its corona,  $\gamma^\alpha(G \circ K) = |V(G)|$ .

*Proof.* Let  $\{v_1, v_2, \dots, v_n\}$  be the vertices of  $G$  and  $\{u_1, u_2, \dots, u_m\}$  be the vertices of  $K$ .  $K_i$  be the  $i^{th}$  copy  $K$  in  $G \circ K$ . To make sure that each vertex of  $K_i$  is dominated, we need atleast one vertex of  $K_i$  or  $v_i$ . Thus the dominating set contains atleast  $n$  vertices. Let  $D = \{v_1, v_2, \dots, v_n\}$ . Then each vertex of  $V - D$  is adjacent with exactly one vertex of  $D$  and each vertex of  $D$  dominates exactly  $m$  vertices in  $V - D$ . And so  $\psi_D(v) = m, \forall v \in D$  and  $\phi_D(v) = 1, \forall v \in V - D$ . Therefore,  $|\phi_D(v_1) - \phi_D(v_2)| = 0, \forall v_1, v_2 \in V - D$  and  $|\psi_D(v_1) - \psi_D(v_2)| = 0, \forall v_1, v_2 \in D$ . And so  $D$  is a minimum  $\alpha$ -stable dominating set. Thus  $\gamma^\alpha(G \circ K) = n = |V(G)|$ . □

## 5 Conclusions

Motivated from this idea the concept of  $\alpha$ -stability of dominating sets is being introduced. The stability of dominating sets in Graphs is introduced and studied, in this paper. In social network problems related to marketing, banking and others the instability affects the system when adjacent acceptors are dominated by unequal number of donors or adjacent donors dominates unequal number of acceptors. Several types of domination number have been defined and studied. This situation become worse when the instability is large. Here we study  $\alpha$ - stability number of graphs for non negative integer  $\alpha$ . Instability of domination in Graphs the concept of stability of domination in Graphs are defined. The open problems are Characterize the graphs with  $I_{sd}(G) = 0$ . Characterize the graphs for which the compliment of minimum  $\alpha$ -stable dominating set is an  $\alpha$ -stable dominating set. Develop an algorithm to find the  $\alpha$ -stable domination number of a graph. In future we extended this topic.

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# Superior eccentric domination polynomial

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## Abstract

Superior distance involves the path which travels through the closed neighbourhood of both the vertices and the shortest path between them. This unique distance led to the advent of superior dominating sets and superior eccentric dominating sets. The former has a superior neighbour in its complement for every vertex in itself and the latter has a superior eccentric vertex in itself for every vertex in its complement. The domination polynomials discuss the idea of total number of dominating sets and dominating sets of specific cardinality. This inspired us to conceptualise the idea of superior eccentric domination polynomial. In this paper, we introduce the superior eccentric domination polynomial  $SED(G, \phi) = \sum_{l=\gamma_{sed}(G)}^{\beta} |sed(G, l)| \phi^l$  where  $|sed(G, l)|$  is the number of all distinct superior eccentric dominating sets with cardinality  $l$  and  $\gamma_{sed}(G)$  is superior eccentric domination number. We find  $SED(G)$  for family of wheel graphs and different standard graphs. The correlation between the coefficients of different  $SED$  polynomials are stated and proved. The motivation for this paper is to find a domination polynomial using distance concept in graphs. Eccentricity is a distance and eccentric dominating set was already existing.

**Keywords:** Superior distance, superior eccentricity, superior eccentric domination polynomial.

**2020 AMS subject classifications:** 05C69, 11B83, 05C12. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1082. ISSN: 1592-7415. eISSN: 2282-8214. ©Tejaskumar R et al. This paper is published under the CC-BY licence agreement.

# 1 Introduction

The shortest path between any two vertices is known as geodesic. The concept of distance in graphs always yields to cater the needs of applications in technology. There are many variants of distances in graphs. The shortest, longest, path involving degree of vertices and chords. Kathiresan et al. [2007] introduced the superior distance in graphs. Let the path  $D_{pq} = N[p] \cup N[q]$ . The shortest superior distance between  $p$  to  $q$  is  $d_D(p, q)$ . Superior eccentricity of  $e_D(p) = \max\{d_D(q, p) : p, q \in V\}$ . Superior neighbour of  $p$  is  $d_D(p) = \min\{d_D(p, q) : q \in V - \{p\}\}$ . A vertex  $p(q)$  is a superior neighbour of  $p$  if  $d_D(p, q) = d_D(p)$ . The superior distance involves the shortest path between two vertices which travels through all their closed neighbourhoods.

Using the superior distance the same authors Kathiresan and Marimuthu [2008] introduced the superior domination(SD-set). A set  $\subseteq V$  is called a SD-set if every vertex in  $S$  has superior neighbour in  $S$ -D. The SD-number is the cardinality of the minimum SD-set, denoted by  $\gamma_{Sed}(G)$ . The superior eccentric vertex  $p$  is given by  $d_D(p, q) = e_D(p)$ . Here the adjacency between the vertices in  $S$  and its compliment is not mandatory. Bhanumathi and Abhirami [2017] introduced the superior eccentric domination(SED-set) in graphs. A set  $\subseteq V$  is an SED-set if every vertex in  $S$ -D has a superior eccentric vertex in  $S$ . The SED-number is the cardinality of the minimum SED-set, denoted by  $\gamma_{Sed}(G)$ . Along with being a superior dominating set if the same set has a superior eccentric vertex in itself for every vertex in the compliment of  $S$  then it becomes a superior eccentric dominating set. These two conditions play a vital role in the formation of a SED-set. Alikhani and Peng [2009] conceptualized the idea of domination polynomial, a domination polynomial consists of a coefficients which gives the number of dominating sets and the power of the variable denotes the cardinality of the dominating set which varies between one and the vertex cardinality of graph. They discussed and proved certain properties which speaks of the correlation between the dominating sets.

The motivation for this paper is to find a domination polynomial using distance concept in graphs. Eccentricity is a distance and eccentric dominating set was already existing. Inspired by this work Ismayil and Tejaskumar [2020] introduced the eccentric domination polynomial. The eccentric dominating polynomial gives the idea about the number of eccentric dominating sets with different cardinality and the symmetry in the coefficients of their polynomials were discussed and proved. Superior distance, superior eccentric domination existed but there was a gap in the literature we did not have a formula which could find the total number of SED in a graph or a SED of specific cardinality, henceforth the same authors Ismayil and Tejaskumar extended the idea of domination polynomial to superior eccentric domination polynomial. In this paper, we discuss the concept of superior eccentric domination polynomial with an apt example, this concept was mainly in-

roduced to find the total number of SED-sets of any graph. We found the formula which yields a SED polynomial for the family of wheel graphs which helps us to easily find the total number of SED-sets of any cardinality at a given point of time for a wheel graph. We obtain the formulas and discuss the corelation between the coefficients of different SED polynomials. We tabulate the SED polynomials and their roots of different standard graphs. For all the undefined terminologies refer the book Graph Theory by Harary [2001].

## 2 Superior eccentric domination polynomial

**Definition 2.1.** The superior eccentric domination polynomial  $SED(G, \phi) = \sum_{l=\gamma_{sed}(G)}^{\beta} |sed(G, l)| \phi^l$  where  $|sed(G, l)|$  is the number of distinct superior eccentric dominating sets (SED-sets) with cardinality  $l$ ,  $\beta \in \mathbb{N}$  and  $\gamma_{sed}(G)$  is superior eccentric domination number.

**Example 2.1.** .

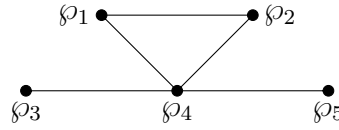


Figure 1: Cricket graph

| Vertex   | Superior eccentricity | Superior eccentric vertex $e_D(\phi)$ |
|----------|-----------------------|---------------------------------------|
| $\phi_1$ | 2                     | $\phi_2$                              |
| $\phi_2$ | 2                     | $\phi_1$                              |
| $\phi_3$ | 2                     | $\phi_5$                              |
| $\phi_4$ | 6                     | $\phi_1, \phi_2, \phi_3, \phi_5$      |
| $\phi_5$ | 2                     | $\phi_3$                              |

Here we see the cricket graph has a SED-set  $\{\phi_4\}$  of cardinality 1,  $\{\phi_1, \phi_4\}$ ,  $\{\phi_2, \phi_4\}$ ,  $\{\phi_3, \phi_4\}$ ,  $\{\phi_4, \phi_5\}$  SED sets of cardinality 2,  $\{\phi_1, \phi_2, \phi_4\}$ ,  $\{\phi_1, \phi_3, \phi_4\}$ ,  $\{\phi_1, \phi_4, \phi_5\}$ ,  $\{\phi_2, \phi_3, \phi_4\}$ ,  $\{\phi_2, \phi_4, \phi_5\}$ ,  $\{\phi_3, \phi_4, \phi_5\}$  SED sets of cardinality 3,  $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ ,  $\{\phi_1, \phi_2, \phi_4, \phi_5\}$ ,  $\{\phi_1, \phi_3, \phi_4, \phi_5\}$ ,  $\{\phi_2, \phi_3, \phi_4, \phi_5\}$  SED sets of cardinality 4 and  $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$  SED sets of cardinality 5. Therefore  $SED(G, \phi) = \phi^5 + 4\phi^4 + 6\phi^3 + 4\phi^2 + \phi$ .

### 3 Superior eccentric domination polynomial of wheel graph

**Definition 3.1.** Superior eccentric domination polynomial of a wheel graph  $W_\beta$  is given by  $SED(W_\beta, \phi) = \sum_{l=\gamma_{sed}(W_\beta)}^{\beta} |sed(W_\beta, l)| \phi^l$  where  $|sed(W_\beta, l)|$  is the number of distinct SED-sets with cardinality  $l$  and  $\gamma_{sed}(W_\beta)$  is SED-number of wheel.

**Observation 3.1.** .

1.  $SED(W_\beta, \phi) = (1 + \phi)^\beta - 1$ , for  $\beta = 4, 5$ .
2.  $SED(W_\beta, \phi) = (1 + \phi)^\beta$ , for  $\beta = 6$ .

**Theorem 3.1.** For a wheel graph  $W_\beta$  of order  $\beta$ ,

1.  $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$  where  $l \leq \beta$  and  $\beta \geq 7$ .
2.  $SED(W_\beta, \phi) = \phi SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi)$ .
3.  $SED(W_\beta, \phi) = \phi(\phi + 1)^{\beta-1}$ , for all  $\beta \geq 7$ .

**Proof:** Let  $V(W_\beta) = \{\wp_1, \wp_2, \dots, \wp_\beta\}$ .

1. Since  $|sed(W_\beta, l)| = {}^{\beta-1}C_{l-1}$ ,  $|sed(W_\beta, l-1)| = {}^{\beta-2}C_{l-2}$  and  $|sed(W_{\beta-1}, l)| = {}^{\beta-2}C_{l-1}$ . But  ${}^{\beta-1}C_{l-1} = {}^{\beta-2}C_{l-2} + {}^{\beta-2}C_{l-1}$ .  
Therefore  $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$ .
2. By theorem-wheelTHM01 – (1) we have  $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$ .  
When  $l = 1$ ,

$$\begin{aligned} |sed(W_\beta, 1)| &= |sed(W_{\beta-1}, 0)| + |sed(W_{\beta-1}, 1)| \\ \implies \phi, |sed(W_\beta, 1)| &= \phi, |sed(W_{\beta-1}, 0)| + \phi, |sed(W_{\beta-1}, 1)|. \end{aligned}$$

When  $l = 2$ ,

$$\begin{aligned} |sed(W_\beta, 2)| &= |sed(W_{\beta-1}, 1)| + |sed(W_{\beta-1}, 2)| \\ \implies \phi^2 |sed(W_\beta, 2)| &= \phi^2 |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 2)|. \end{aligned}$$

When  $l = 3$ ,

$$\begin{aligned} |sed(W_\beta, 3)| &= |sed(W_{\beta-1}, 2)| + |sed(W_{\beta-1}, 3)| \\ \implies \phi^3 |sed(W_\beta, 3)| &= \phi^3 |sed(W_{\beta-1}, 2)| + \phi^3 |sed(W_{\beta-1}, 3)|. \end{aligned}$$



### Superior eccentric domination polynomial

When  $l = 4$ ,

$$\begin{aligned} |sed(W_\beta, 4)| &= |sed(W_{\beta-1}, 3)| + |sed(W_{\beta-1}, 4)| \\ \implies \phi^4 |sed(W_\beta, 4)| &= \phi^4 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 4)|. \end{aligned}$$

$\vdots$

When  $l = \beta - 1$ ,

$$\begin{aligned} |sed(W_\beta, \beta - 1)| &= |sed(W_{\beta-1}, \beta - 2)| \\ &\quad + |sed(W_{\beta-1}, \beta - 1)| \\ \implies \phi^{\beta-1} |sed(W_\beta, \beta - 1)| &= \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 2)| \\ &\quad + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 1)|. \end{aligned}$$

When  $l = \beta$ ,

$$\begin{aligned} |sed(W_\beta, \beta)| &= |sed(W_{\beta-1}, \beta - 1)| + |sed(W_{\beta-1}, \beta)| \\ \implies \phi^\beta |sed(W_\beta, \beta)| &= \phi^\beta |sed(W_{\beta-1}, \beta - 1)| + \phi^\beta |sed(W_{\beta-1}, \beta)|. \end{aligned}$$

Therefore  $\phi |sed(W_\beta, 1)| + \phi^2 |sed(W_\beta, 2)| + \phi^3 |sed(W_\beta, 3)| + \phi^4 |sed(W_\beta, 4)| + \dots + \phi^{\beta-1} |sed(W_\beta, \beta - 1)| + \phi^\beta |sed(W_\beta, \beta)|$   
 $= \phi |sed(W_{\beta-1}, 0)| + \phi |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 2)| + \phi^3 |sed(W_{\beta-1}, 1)| + \phi^3 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 4)| + \dots + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 2)| + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 1)| + \phi^\beta |sed(W_{\beta-1}, \beta - 1)| + \phi^\beta |sed(W_{\beta-1}, \beta)|.$   
 $= \phi |sed(W_{\beta-1}, 0)| + \phi^2 |sed(W_{\beta-1}, 1)| + \phi^3 |sed(W_{\beta-1}, 1)| + \phi^4 |sed(W_{\beta-1}, 3)| + \dots + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 2)| + \phi^\beta |sed(W_{\beta-1}, \beta - 1)| + \phi |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 2)| + \phi^3 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 4)| + \dots + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 1)| + \phi^\beta |sed(W_{\beta-1}, \beta)|$   
 $= \phi [\phi |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 2)| + \phi^3 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 4)| + \dots + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 1)|] + \phi |sed(W_{\beta-1}, 1)| + \phi^2 |sed(W_{\beta-1}, 2)| + \phi^3 |sed(W_{\beta-1}, 3)| + \phi^4 |sed(W_{\beta-1}, 4)| + \dots + \phi^{\beta-1} |sed(W_{\beta-1}, \beta - 1)|$

Since  $|sed(W_{\beta-1}, 0)| = |sed(W_{\beta-1}, \beta)| = 0$ .

$$= \phi \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| \phi^l + \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| \phi^l.$$

$$SED(W_\beta, \phi) = \phi SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi).$$

1. By mathematical induction (MI).

It is true for  $\beta = 7$ .

$$\begin{aligned}
 SED(W_{\beta-1}, \phi) &= \phi(\phi + 1)^{7-1} \\
 &= \phi(\phi + 1)^6 \\
 &= \phi(\phi + 1)^3(\phi + 1)^3 \\
 &= \phi^7 + 6\phi^6 + 15\phi^5 + 20\phi^4 + 15\phi^3 + 6\phi^2 + 1.
 \end{aligned}$$

Assume it is true  $\forall \mathbb{N}$  less than  $\beta'$ .

$$SED(W_{\beta}, \phi) = \phi(1 + \phi)^{(\beta-1)-1} = \phi(1 + \phi)^{\beta-2}$$

$$\begin{aligned}
 \text{For } \beta', SED(W_{\beta}, \phi) &= \phi SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi) \\
 \text{using theorem - 3.1 - (2)} \\
 &= \phi[\phi(\phi + 1)^{\beta-2}] + \phi(\phi + 1)^{\beta-2} \\
 &= \phi(\phi + 1)^{\beta-1}
 \end{aligned}$$

$\therefore$  Proved  $\forall \beta'$ .

**Table:**  $|sed(W_{\beta}, l)|$  is the number of superior eccentric dominating sets of  $W_{\beta}$  with cardinality  $l$  where  $1 \leq l \leq 15$ .

| $\beta \backslash l$ | 1 | 2  | 3  | 4   | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12  | 13 | 14 | 15 |
|----------------------|---|----|----|-----|------|------|------|------|------|------|------|-----|----|----|----|
| 1                    | 0 |    |    |     |      |      |      |      |      |      |      |     |    |    |    |
| 2                    | 0 | 0  |    |     |      |      |      |      |      |      |      |     |    |    |    |
| 3                    | 0 | 0  | 0  |     |      |      |      |      |      |      |      |     |    |    |    |
| 4                    | 4 | 6  | 4  | 1   |      |      |      |      |      |      |      |     |    |    |    |
| 5                    | 5 | 10 | 10 | 5   | 1    |      |      |      |      |      |      |     |    |    |    |
| 6                    | 1 | 5  | 10 | 10  | 5    | 1    |      |      |      |      |      |     |    |    |    |
| 7                    | 1 | 6  | 15 | 20  | 15   | 6    | 1    |      |      |      |      |     |    |    |    |
| 8                    | 1 | 7  | 21 | 35  | 35   | 21   | 7    | 1    |      |      |      |     |    |    |    |
| 9                    | 1 | 8  | 28 | 56  | 70   | 56   | 28   | 8    | 1    |      |      |     |    |    |    |
| 10                   | 1 | 9  | 36 | 84  | 126  | 126  | 84   | 36   | 9    | 1    |      |     |    |    |    |
| 11                   | 1 | 10 | 45 | 120 | 210  | 252  | 210  | 120  | 45   | 10   | 1    |     |    |    |    |
| 12                   | 1 | 11 | 55 | 165 | 330  | 462  | 462  | 330  | 165  | 55   | 11   | 1   |    |    |    |
| 13                   | 1 | 12 | 66 | 220 | 495  | 792  | 924  | 792  | 495  | 220  | 66   | 12  | 1  |    |    |
| 14                   | 1 | 13 | 78 | 286 | 715  | 1287 | 1716 | 1716 | 1287 | 715  | 286  | 78  | 13 | 1  |    |
| 15                   | 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3423 | 3003 | 2002 | 1001 | 364 | 91 | 14 | 1  |

**Theorem 3.2.** The following properties for the co-efficients of  $SED(W_{\beta}, \phi)$  holds.

1.  $|sed(W_{\beta}, 1)| = 1$  for all  $\beta \geq 6$ .
2.  $|sed(W_{\beta}, \beta)| = 1, \forall \beta \geq 4$ .
3.  $|sed(W_{\beta}, \beta - 1)| = \beta - 1, \forall \beta \geq 6$ .

*Superior eccentric domination polynomial*

4.  $|sed(W_\beta, \beta - 2)| = \frac{(\beta-1)(\beta-2)}{2}, \forall \beta \geq 6.$
5.  $|sed(W_\beta, \beta - 3)| = \frac{(\beta-1)(\beta-2)(\beta-3)}{6}, \forall \beta \geq 6.$
6.  $|sed(W_\beta, \beta - 4)| = \frac{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}{24}, \forall \beta \geq 6.$
7.  $|sed(W_\beta, l)| = |sed(W_\beta, \beta - l + 1)|, \forall \beta \geq 6.$
8. *If  $SED_\beta = \sum_{l=1}^{\beta} |sed(W_\beta, l)|, \forall \beta \geq 6$ , then  $SED_\beta = 2(SED_{\beta-1}), \forall \beta \geq 7.$*
9.  $SED_\beta = \text{Total number of SED-sets in } W_\beta = 2^{\beta-1}, \forall \beta \geq 6.$

**Proof:**

1. Let  $V(W_\beta) = \{\wp_1, \wp_2, \wp_3, \dots, \wp_\beta\}$ . In a wheel graph  $W_\beta$  all the vertices form a superior neighbour of central vertex  $\wp_1$  except itself. Therefore the only set with single cardinality  $D = \{\wp_1\}$  forms the superior eccentric dominating set of wheel graph  $W_\beta$  where  $\beta \geq 6$ .
2. The vertex set  $V(W_\beta)$  forms the superior eccentric dominating set  $|sed(W_\beta, \beta)| = 1$  for all  $\beta \geq 4$ .
3. By MI on ' $\beta'$ '.  
For  $\beta = 6, |sed(W_6, 6 - 1)| = |sed(W_6, 5)| = 5$ .  
Assume it is true  $\forall \mathbb{N}$  less than ' $\beta'$ '.  
For ' $\beta'$ ', By theorem-3.1-(2) and 3.2-(2)

$$\begin{aligned}
 |sed(W_\beta, \beta - 1)| &= |sed(W_{\beta-1}, \beta - 2)| + |sed(W_{\beta-1}, \beta - 1)| \\
 &= (\beta - 2) + 1 \\
 &= \beta - 1
 \end{aligned}$$

$\therefore$  Proved  $\forall \beta'$ .

4. By MI on ' $\beta'$ '.  
For  $\beta = 6, |sed(W_6, 6 - 2)| = |sed(W_6, 4)| = 10$ .  
For  $\beta = 7, |sed(W_7, 7 - 2)| = |sed(W_7, 5)| = 15$ .  
Assume it is true  $\forall \mathbb{N}$  less than ' $\beta'$ '.

For ' $\beta$ ', By theorem-3.1 and 3.2-(3)

$$\begin{aligned}
 |sed(W_\beta, \beta - 2)| &= |sed(W_{\beta-1}, \beta - 3)| + |sed(W_{\beta-1}, \beta - 2)| \\
 &= \frac{(\beta - 2)(\beta - 3)}{2} + (\beta - 2) \\
 &= \frac{(\beta - 2)(\beta - 3) + 2(\beta - 2)}{2} \\
 &= \frac{(\beta - 2)(\beta - 3 + 2)}{2} \\
 &= \frac{(\beta - 2)(\beta - 1)}{2}
 \end{aligned}$$

$\therefore$  Proved  $\forall$  ' $\beta$ '.

5. By MI on ' $\beta$ '.

For  $\beta = 6$ ,  $|sed(W_6, 6 - 3)| = |sed(W_6, 3)| = 10$ .

For  $\beta = 7$ ,  $|sed(W_7, 7 - 3)| = |sed(W_7, 4)| = 20$ .

Assume it is true  $\forall$   $\mathbb{N}$  less than ' $\beta$ '.

For ' $\beta$ ', By theorem-3.1 and 3.2-(4)

$$\begin{aligned}
 |sed(W_\beta, \beta - 3)| &= |sed(W_{\beta-1}, \beta - 4)| + |sed(W_{\beta-1}, \beta - 3)| \\
 &= \frac{(\beta - 2)(\beta - 3)(\beta - 4)}{6} + \frac{(\beta - 2)(\beta - 3)}{2} \\
 &= \frac{(\beta - 2)(\beta - 3)(\beta - 4 + 3)}{6} \\
 &= \frac{(\beta - 1)(\beta - 2)(\beta - 3)}{6}
 \end{aligned}$$

$\therefore$  Proved  $\forall$  ' $\beta$ '.

6. By MI on ' $\beta$ '.

The result is true for  $\beta = 6$ ,  $|sed(W_6, 6 - 4)| = |sed(W_6, 2)| = 5$ .

For  $\beta = 7$ ,  $|sed(W_7, 7 - 4)| = |sed(W_7, 3)| = 15$ .

Assume it is true  $\forall$ ,  $\mathbb{N} < \beta$ .

*Superior eccentric domination polynomial*

For  $'\beta'$ , By theorem-3.1 and 3.2-(5)

$$\begin{aligned}
 |sed(W_\beta, n-4)| &= |sed(W_{\beta-1}, \beta-5)| + |sed(W_{\beta-1}, \beta-4)| \\
 &= \frac{(\beta-2)(\beta-3)(\beta-4)(\beta-5)}{24} \\
 &\quad + \frac{(\beta-2)(\beta-3)(\beta-4)}{6} \\
 &= \frac{(\beta-2)(\beta-3)(\beta-4)(\beta-5+4)}{24} \\
 &= \frac{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}{24}
 \end{aligned}$$

$\therefore$  Proved  $\forall '\beta'$ .

7. By MI on  $'\beta'$ . The result is true for  $\beta = 6$ .

$$\begin{aligned}
 |sed(W_6, 2)| &= |sed(W_6, 6-2+1)| = |sed(W_6, 5)| = 5 \\
 |sed(W_7, 3)| &= |sed(W_7, 7-3+1)| = |sed(W_7, 4)| = 20.
 \end{aligned}$$

Assume it is true  $\forall \mathbb{N}$  less than  $'\beta'$ . For  $'\beta'$ , by theorem-3.1 we have

$$\begin{aligned}
 |sed(W_\beta, l)| &= |sed(W_{\beta-l}, l-1)| + |sed(W_{\beta-1}, l)| \\
 &= |sed(W_{\beta-1}, (\beta-1-(l-1)+1))| \\
 &\quad + |sed(W_{\beta-1}, (\beta-1-(l)+1))| \\
 &= |sed(W_{\beta-1}, (\beta-1-l+1+1))| \\
 &\quad + |sed(W_{\beta-1}, (\beta-1-l+1))| \\
 &= |sed(W_{\beta-1}, (\beta-l+1))| \\
 &\quad + |sed(W_{\beta-1}, (\beta-l))| \\
 &= |sed(W_\beta, (\beta-l+1))|
 \end{aligned}$$

$\therefore$  Proved  $\forall '\beta'$ .

8.  $SED_\beta = \sum_{l=1}^{\beta} |sed(W_\beta, l)|$

By theorem-3.1 we have

$$\begin{aligned}
 SED_\beta &= \sum_{l=1}^{\beta} [|sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|] \\
 &= \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| + \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| \\
 &= SED_{\beta-1} + SED_{\beta-1} \\
 SED_\beta &= 2(SED_{\beta-1})
 \end{aligned}$$

9. By MI on ' $\beta$ '. When  $\beta = 6$ ,  
 $SED_6 = 2^{6-1} = 2^5 = 32$ .  
 $SED_7 = 2^{7-1} = 2^6 = 64$ .  
 Assume it is true  $\forall \mathbb{N}$  less than ' $\beta$ '.  
 $SED_{\beta-1} = 2^{\beta-1-1} = 2^{\beta-2}$   
 For ' $\beta$ ',

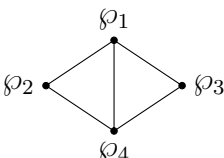
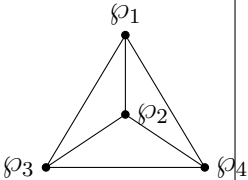
$$\begin{aligned} SED_{\beta} &= 2[SED_{\beta-1}] \quad \text{from theorem - 3.2 - (8)} \\ &= 2[2^{\beta-2}] \\ &= 2^{\beta-1} \end{aligned}$$

$\therefore$  Proved  $\forall \beta$ .  
 Hence the theorem.

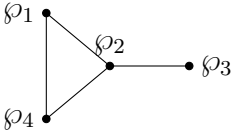
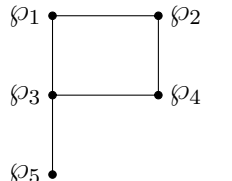
**Remark 3.1. .**

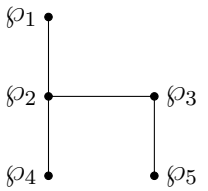
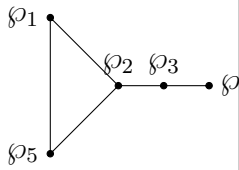
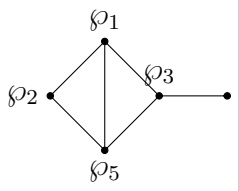
1. For any graph  $G$  0 is one of the root of every  $SED(G, \phi)$ .
2. A graph with more than 3 pendant vertices has at least 2 real roots.

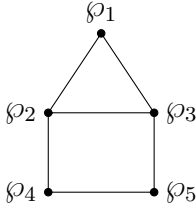
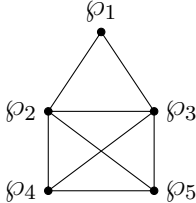
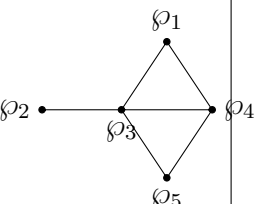
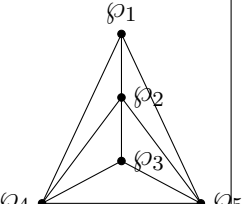
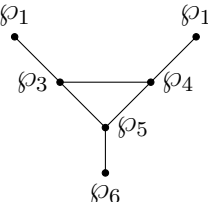
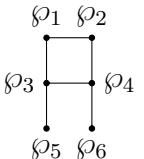
$SED(G, \phi)$  of different standard graphs and their roots are tabulated below:

| Graph             | Figure                                                                              | Superior eccentric domination polynomial<br>$SED(G, \phi)$ | Roots                                                                                                  |
|-------------------|-------------------------------------------------------------------------------------|------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| Diamond graph     |  | $\phi^4 + 4\phi^3 + 5\phi^2$                               | $\phi_1 = 0,$<br>$\phi_2 = -0.4563,$<br>$\phi_3 = -1.7718 + 1.1151i,$<br>$\phi_4 = -1.7718 - 1.1151i.$ |
| Tetrahedral graph |  | $\phi^4 + 4\phi^3 + 6\phi^2 + 4\phi$                       | $\phi_1 = 0,$<br>$\phi_2 = -2,$<br>$\phi_3 = -1 + i,$<br>$\phi_4 = -1 - i.$                            |

*Superior eccentric domination polynomial*

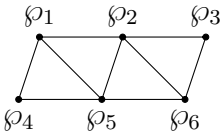
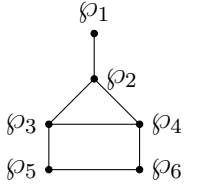
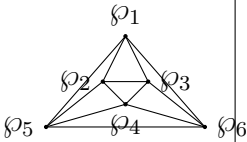
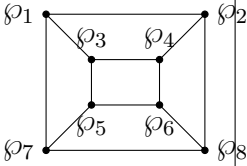
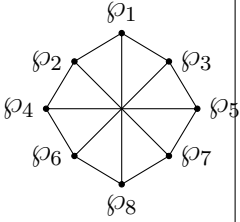
| Graph        | Figure                                                                            | Superior eccentric domination polynomial<br>$SED(G, \phi)$ | Roots                                                                                                  |
|--------------|-----------------------------------------------------------------------------------|------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| Paw graph    |  | $\phi^4 + 3\phi^3 + 3\phi^2 + \phi$                        | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -1.$                                    |
| Banner graph |  | $\phi^5 + 4\phi^4 + 5\phi^3 + 3\phi^2$                     | $\phi_1 = 0,$<br>$\phi_2 = -2.4656,$<br>$\phi_3 = -0.7672 + 0.7926i,$<br>$\phi_4 = -0.7672 - 0.7926i.$ |

| Graph               | Figure                                                                              | Superior eccentric domination polynomial<br>$SED(G, \phi)$ | Roots                                                                                                                                        |
|---------------------|-------------------------------------------------------------------------------------|------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------|
| Fork graph          |  | $\phi^5 + 4\phi^4 + 5\phi^3 + 2\phi^2$                     | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -2.$                                                                          |
| (3,2)-Tadpole graph |  | $\phi^5 + 4\phi^4 + 3\phi^3 + \phi^2$                      | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -2.$                                                                          |
| Kite graph          |  | $\phi^5 + 5\phi^4 + 6\phi^3 + 2\phi^2$                     | $\phi_1 = 0,$<br>$\phi_2 = -0.378 + 0.1877i,$<br>$\phi_3 = -0.378 - 0.1877i,$<br>$\phi_4 = -2.122 + 1.0538i,$<br>$\phi_5 = -2.122 - 1.053i.$ |

| Graph                           | Figure                                                                              | Superior eccentric domination polynomial<br>$SED(G, \phi)$ | Roots                                                                                                                                             |
|---------------------------------|-------------------------------------------------------------------------------------|------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|
| House graph                     |    | $\phi^5 + 5\phi^4 + 6\phi^3$                               | $\phi_1 = 0,$<br>$\phi_2 = -0.3076 + 0.3182i,$<br>$\phi_3 = -0.3076 - 0.3182i,$<br>$\phi_4 = -2.1924 + 0.5479i,$<br>$\phi_5 = -2.1924 - 0.5479i.$ |
| House X graph                   |   | $\phi^5 + 5\phi^4 + 6\phi^3$                               | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -0.382,$<br>$\phi_5 = -2.618.$                                                     |
| Dart graph                      |  | $\phi^5 + 4\phi^4 + 6\phi^3 + 3\phi^2$                     | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i,$<br>$\phi_4 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i.$               |
| Johnson solid skeleton 12 graph |  | $\phi^5 + 5\phi^4 + 10\phi^3 + 10\phi^2 + 5\phi$           | $\phi_1 = 0,$<br>$\phi_2 = -0.691 + 0.9511i,$<br>$\phi_3 = -0.691 - 0.9511i,$<br>$\phi_4 = -1.809 + 0.5878i,$<br>$\phi_5 = -1.809 - 0.5878i.$     |
| Net graph                       |  | $\phi^6 + 3\phi^5 + 3\phi^4 + \phi^3$                      | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -1.$                                                                               |
| A graph                         |  | $\phi^6 + 4\phi^5 + 6\phi^4 + 4\phi^3 + \phi^2$            | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -1,$<br>$\phi_4 = -1,$<br>$\phi_5 = -1.$                                                             |



*Superior eccentric domination polynomial*

| Graph                | Figure                                                                              | Superior eccentric domination polynomial<br>$SED(G, \phi)$                | Roots                                                                                                                                                                                                               |
|----------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 4-polynomial graph 1 |    | $\phi^6 + 4\phi^5 + 4\phi^4$                                              | $\phi_1 = 0,$<br>$\phi_2 = -2.$                                                                                                                                                                                     |
| Antenna graph        |    | $\phi^6 + 4\phi^5 + 3\phi^4 + \phi^3$                                     | $\phi_1 = 0,$<br>$\phi_2 = -1,$<br>$\phi_3 = -2,$<br>$\phi_4 = -1.$                                                                                                                                                 |
| Octahedral graph     |   | $\phi^6 + 6\phi^5 + 15\phi^4 + 20\phi^3 + 15\phi^2 + 6\phi$               | $\phi_1 = 0,$<br>$\phi_2 = -2,$<br>$\phi_3 = -0.5 + 0.866i,$<br>$\phi_4 = -0.5 - 0.866i,$<br>$\phi_5 = -1.5 + 0.866i,$<br>$\phi_6 = -1.5 - 0.866i.$                                                                 |
| Cubical graph        |  | $\phi^8 + 8\phi^7 + 28\phi^6 + 56\phi^5 + 68\phi^4 + 48\phi^3 + 16\phi^2$ | $\phi_1 = 0,$<br>$\phi_2 = -0.6714 + 0.5756i,$<br>$\phi_3 = -0.6714 - 0.5756i,$<br>$\phi_4 = -0.8352 + 1.4854i,$<br>$\phi_5 = -0.8352 - 1.4854i,$<br>$\phi_6 = -2.4934 + 0.9097i,$<br>$\phi_7 = -2.4934 - 0.9097i.$ |
| Wagner graph         |  | $\phi^8 + 8\phi^7 + 24\phi^6 + 32\phi^5 + 16\phi^4$                       | $\phi_1 = 0,$<br>$\phi_2 = -2,$<br>$\phi_3 = -2,$<br>$\phi_4 = -2,$<br>$\phi_5 = -2.$                                                                                                                               |

## 4 Conclusions

In this paper SED polynomial for a graph was defined. Formula to find the SED polynomials of family of wheel graphs were stated and proved. Corelation between the coefficients of SED polynomials were discussed. The SED polynomial and its roots for different standard graphs are tabulated. In the course of future work the graphs can be classified based on the roots of SED polynomials. The similarproperties of coefficients based on similar roots for different standard graphs can be discussed. The same concept can be extended to other domination parameter.

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# Linear predictor and autocorrelation for noisy and delayed digital signal

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Jothilakshmi R<sup>†</sup>

## Abstract

This paper deals with the association between the linear prediction and digital signal modeling and ends up with the suitable ways to predict the signal by considering a stationary signal  $y_n$ . The linear prediction of signal modeling based on the finite past and the solutions are arrived in a recursive manner. Further we analyzed the wiener filter along with spectral theorem and autocorrelation in terms of predictive analysis. This estimates the gap function along with delay and noise. The delayed signal's properties are analyzed like causal, stability and applied these into optimum filtering. Finally the predicted error is compared with linear predictor and Wiener filter. Then transfer function is applied to estimate the interval function and gap function along with delay.

**Keywords:** Linear Predictor, Weiner filter, gapped function, delay, autocorrelation, orthogonal.

**2020 AMS subject classifications:** 39 A10, 39 A45. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1083. ISSN: 1592-7415. eISSN: 2282-8214. ©G.Vinu Priya et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Due to recent developments in Digital signal processing and communication technology, and its subfields of spectrum estimation, real-time adaptive signal processing and prediction algorithms comes into an attention of many researchers Chen et al. [2006], Dogariu et al. [2021] and Gland and Oudjane [2003]. The unified extension of this digital signal processing is developed in terms of analysis of geometrical point of view, linear estimator and applied various algorithms like Gram-Schmidt orthogonalizations, lattice realizations and so on. Further these concepts deals with the autoregressive extensions and singular autocorrelation matrices and their sinusoidal representations Mao et al. [2017]. This motivates us to proceed further with linear prediction Makhoul [1975] and Pituk [2004].

This paper is organized as follows. Section II focuses the linear prediction and digital signal modeling Welch et al. [2006]. Section III applies the autoregressive models into the prediction coefficients. In section IV Linear Predictions and Levinson's Formula are applied in the random input signal. Finally, section V concludes the paper.

## 2 Linear predictor

The linear prediction and digital signal modeling and ends up with the suitable ways to predict the signal by considering a stationary signal  $y_n$ . This rules the signal pattern as follows

$$S_{yy}(z) = \sigma_e^2 B(z)B(z^{-1})\epsilon_n \rightarrow (B(z)) \rightarrow y_n \quad (1)$$

Here  $B(z)$  be any filter as bounded,  $\epsilon_n$  be a sequence of noise term by spectral factoring theorem. Let  $R_{yy}(k)$  be the autocorrelation of  $y_n$ :

$$R_{yy}(k) = E[y_n + ky_n]$$

This is used to predict the present value through the past values by using  $Y_{n-1} = \{y_i, -\infty < i \leq n-1\}$ . If  $y_1(n) = y_{n-1}$ , then the linear prediction is identified and compared with the optimum Wiener filtering and estimated the signal  $y_1(n)$ . Now we identify  $Y_1(z) = z^{-1}Y(z)$  with the spectral value  $B(z)$ . Now define the optimum filter  $H(z)$  as

$$H(z) = \frac{1}{\sigma_e^2 B(z)} \left[ \frac{\sigma_e^2 B(z)B(z^{-1})}{B(z^{-1})} \right], \quad (2)$$

### Linear predictor and autocorrelation for noisy and delayed digital signal

Here  $B(z)$  be a causal and stable filter, and extend the causal and stale filter as  $zB(z)$  is then

$$zB(z) = z(b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \dots)$$

The optimum filter  $H(z)$  is then

$$H(z) = z[1 - \frac{1}{B(z)}] \quad (3)$$

$$y_n \rightarrow (z^{-1}) \rightarrow y_{n-1} \xrightarrow{H(z)} \hat{y}_{n/n-1}$$

This filter output is  $y_{n-1}$  and the consequent output is predicted by  $\hat{y}_{n/n-1}$ . The predicted error is defined as  $\epsilon_n$ . In the figure 1, the indicator line separates the

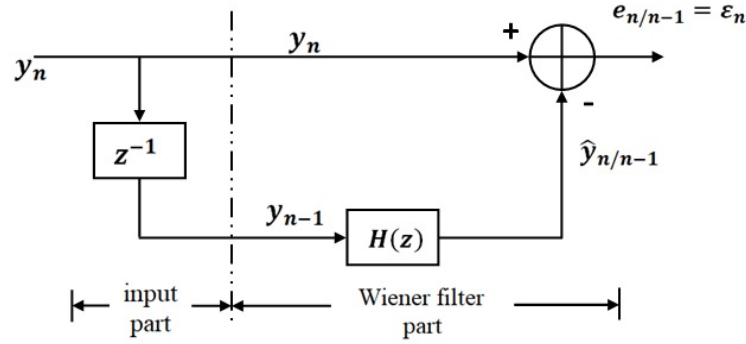


Figure 1: Error Predictor through Wiener Filter

linear predictor part and Wiener filter part in the signal error prediction [2, 7]. Apply the reduction equation (1) in terms of the predicted error filter  $A(z)$  as,

$$S_{yy}(z) = \frac{\sigma_\epsilon^2}{A(z)A(z^{-1})} \quad (4)$$

and

$$A(z)S_{yy}(z) = \frac{\sigma_\epsilon^2}{A(z^{-1})} \quad (5)$$

this follows that

$$S_{ey}(z) = A(z)S_{yy}(z) \quad (6)$$

furthermore

$$R_{\epsilon y}(k) = E[\epsilon_n y_{n-k}] = \sum_{i=0}^{\infty} a_i R_{yy}(k-i) \quad (7)$$

which is recognized by the interval function [3]. Now construct  $\epsilon_n$  from the orthogonal complement of  $Y_{n-1} = y_{n-k}, k = 1, 2, \dots$ , and hence  $y_{n-k}$  is orthogonal to all  $k = 1, 2, \dots$ . Therefore, the equation (7) implies

$$R_{\epsilon y}(k) = E[\epsilon_n y_{n-k}] = \sum_{i=0}^{\infty} a_i R_{yy}(k-i) = 0 \quad (8)$$

This result follows from the  $z$ -domain equation of (6) and interval function. Applying the symmetry property in (7) provided  $k = 0$  and we get

$$\sigma_{\epsilon}^2 = E[\epsilon_n^2] = E[\epsilon_n y_n] = R_{yy}(0) + a_1 R_{yy}(1) + a_2 R_{yy}(2) + \dots \quad (9)$$

Combined the equations (8) and (9),

$$\sum_{i=0}^{\infty} a_i R_{yy}(k-i) = \sigma_{\epsilon}^2 \delta(k), k \geq 0 \quad (10)$$

This normal equation is extended with the parameters  $\{a_1, a_2, \dots, \sigma_{\epsilon}^2\}$  based on the output signal  $y_n$  and this is computed with  $R_{yy}(k)$ .

### 3 Autoregressive models

In general, the prediction coefficients are infinite since the predictor is predicated on the infinite past. When  $y_n$  is autoregressive, then the signal model  $B(z)$  is defined as

$$B(z) = \frac{1}{(1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_p z^{-p})} \quad (11)$$

This shows that the prediction filter is polynomial

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_p z^{-p} \quad (12)$$

The output function  $y_n$  is defined for uncorrelated sequence  $\epsilon_n$ , we get

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} + \dots + a_p y_{n-p} = \epsilon_n \quad (13)$$

further optimum prediction of  $y_n$  is written like

$$\hat{y}_{n/n-1} = -[a_1 y_{n-1} + a_2 y_{n-2} + \dots + a_p y_{n-p}] \quad (14)$$

Here most effective prediction of  $y_n$  is calculated based on the past  $p$  samples. The infinite set of equations (10) or (11) remains valid and the primary  $p + 1$  samples coefficients  $\{1, a_1, a_2, \dots, a_p\}$  are nonzero [8, 14]. These primary past samples are a part of the equation (11) and these samples are enough to define the parameters of  $\{a_1, a_2, \dots, a_p; \sigma_\epsilon^2\}$ :

$$\begin{bmatrix} R_{yy}(0) & R_{yy}(1) & \dots & R_{yy}(p) \\ R_{yy}(1) & R_{yy}(0) & \dots & R_{yy}(p-1) \\ R_{yy}(2) & R_{yy}(1) & \dots & R_{yy}(p-2) \\ \vdots & \vdots & \vdots & \vdots \\ R_{yy}(p) & R_{yy}(p-1) & \dots & R_{yy}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (15)$$

These equations are solved efficiently through Levinson's algorithm and this algorithm needs  $O(p^2)$  operations and  $O(p)$  memory locations.  $O(p^3)$  and  $O(p^2)$  which is necessary to calculate the inverse of the autocorrelation matrix  $R_{yy}$ . The parameters  $\{a_1, a_2, \dots, a_p; \sigma_\epsilon^2\}$  completely determines  $y_n$ . By considering  $z = e^{j\omega}$  in the equation (5) we determine

$$S_{yy}(\omega) = \frac{\sigma_\epsilon^2}{|A(\omega)|^2} = \frac{\sigma_\epsilon^2}{|1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega} + \dots + a_p e^{-j\omega p}|} \quad (16)$$

The normal equations (16) build is used to approximate and estimates the parameters  $\{a_1, a_2, \dots, a_p; \sigma_\epsilon^2\}$ . There are many various ways to extract the estimates and the parameters. Here are the few methods

1. Yule-Walker methodology
2. Variance methodology and
3. Burg's methodology.

Autocorrelations  $R_{yy}(k)$  of equation (16) is written based on the Yule-Walker methodology, is

$$R_{yy}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} y_{n+k} y_n \quad (17)$$

The primary  $p + 1$  changes are required in (16) as like  $p \leq N - 1$  based on the parameters  $\{\hat{a}_1, \hat{a}_2, \dots, \hat{a}_p; \hat{\sigma}_\epsilon^2\}$ . This represents the block of  $N$  samples and filter

parameters (i.e.  $p + 1$ ). To synthesize the random samples, variance  $\hat{\sigma}_\epsilon^2$  would be generated and pass through the generator filter whose coefficients are calculated like,

$$\hat{B}(Z) = \frac{1}{\hat{A}(Z)} = \frac{1}{|1 + \hat{a}_1 z^{-1} + \hat{a}_2 z^{-2} + \dots + \hat{a}_p z^{-p}|^2} \quad (18)$$

## 4 Linear predictions and Levinson's Formula

In this section, we come across that if the autoregressive random input signal is of order  $p$ , then the optimum linear predictor reduces to a predictor of order  $p$ . A geometrical method to perceive this property could be extended in to the projection of  $y_n$  onto the topological subspace based on the output signal  $\{y_{n-i}, 1 \leq i < \infty\}$  and the same could be reduced based on past samples; i.e.  $\{y_{n-i}, 1 \leq i \leq p\}$ . This generates the output function  $y_n$ .

Consider a stationary series (based on time)  $y_n$  with the autocorrelation function  $R(k) = E[y_{n+k}y_n]$ . For any given  $p$ , the output function takes the following new form Consider a stationary series (based on time)  $y_n$  with the autocorrelation function  $R(k) = E[y_{n+k}y_n]$ . For any given  $p$ , the output function takes the following new form

$$\hat{y}_n = -[a_1 y_{n-1} + a_2 y_{n-2} + \dots + a_p y_{n-p}] \quad (19)$$

The prediction coefficients are chosen to reduce the mean square error as

$$\varepsilon = E[e_n^2] \quad (20)$$

where  $e_n$  is the predicted error and define  $e_n$  as follows

$$e_n = y_n - \hat{y}_n = y_n + a_1 y_{n-1} + a_2 y_{n-2} + \dots + a_p y_{n-p} \quad (21)$$

$$E[e_n y_{n-i}] = 0, \quad (22)$$

By substituting (21) in the equation (22), we get  $p$  linear equations

$$\sum_{j=0}^p a_j E[y_{n-j} y_{n-i}] = \sum_{j=0}^p R(i-j) a_j = 0 \quad (23)$$

By (22), we found the reduced value as

$$\sigma_\epsilon^2 = E[e_n y_n] \quad (24)$$



Equations (23) and (24) may be combined into the matrix equation like  $(p + 1) \times (p + 1)$ ,

$$\begin{bmatrix} R_{yy}(0) & R_{yy}(1) & \cdots & R_{yy}(p) \\ R_{yy}(1) & R_{yy}(0) & \cdots & R_{yy}(p-1) \\ R_{yy}(2) & R_{yy}(1) & \cdots & R_{yy}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{yy}(p) & R_{yy}(p-1) & \cdots & R_{yy}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (25)$$

which is identical for equation (16) for the autoregressive case.

It was necessary to connect the order of the predictor associate with the previous one. Hence the lower order optimum predictors also are calculated. Consider the gap function as

$$g_p(k) = E\left[\left(\sum_{i=0}^p a_{pi} y_{n-i}\right) y_{n-k}\right] = \sum_{i=0}^p a_{pi} R(k-i) \quad (26)$$

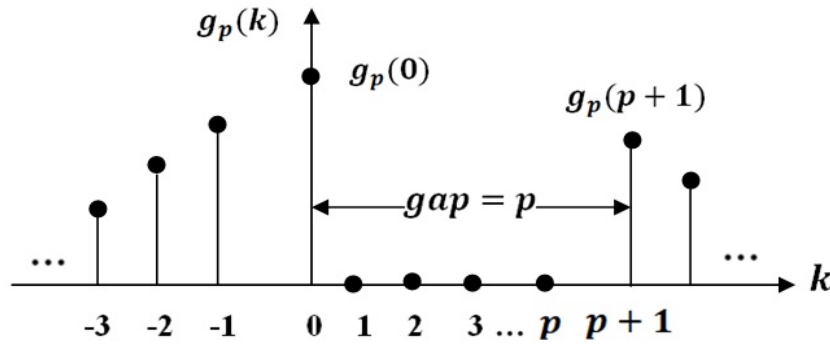


Figure 2: Gap conditions for the delay

These gap conditions are an equivalent because of the orthogonal equations (22) which is illustrated in figure 2. Utilizing  $g_p(k)$  construct a new function with space  $g_{p+1}(k)$  from the past  $p + 1$  hence we get,  $g_p(k) \rightarrow g_p(-k)$ . A delay of  $(p + 1)$  time can be realigned and illustrated in the following figure. This shows the minimum of  $p$  and chosen the parameter  $\gamma_{p+1}$  and  $g_{p+1}(k)$  adds an additional delay which deviates the length  $p + 1$  are illustrated in the figure 3.

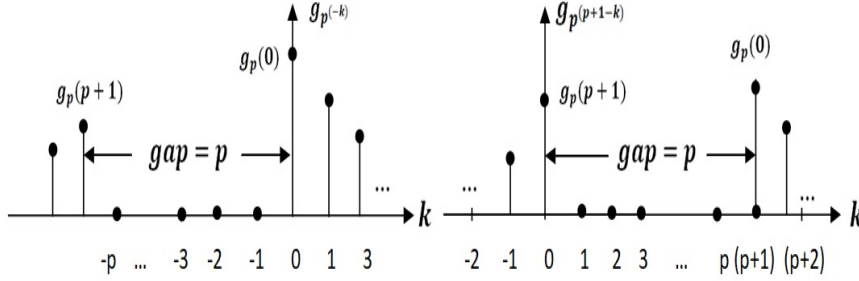


Figure 3: Gap (delay) conditions for  $g_p(k)$  and  $g_p(p+1-k)$

## 5 Conclusions

In this paper, linear prediction is predicted the present value through the past values. The linear prediction of signal modeling related to finite past and the solutions are arrived in a recursive manner. Further we analyzed the wiener filter along with spectral theorem and autocorrelation in terms of predictive analysis. This estimates the gap function along with delay and noise. There will be an indicator line which separates the linear predictor part and Wiener filter part in the signal error prediction. This normal equation is extended with the signal parameters based on the output signal  $y_n$  and this is computed with  $R_{yy}(k)$ . Then the infinite matrix equation is reduced to a finite form and, moreover, the  $R_{yy}(k)$  is obviously measurable. Finally the predicted error is compared with linear predictor and Wiener filter. Then transfer function is applied to estimate the interval function and gap function along with delay. Finally the gapped function  $g_p(k)$  and  $g_p(p+1-k)$  possess the same value.

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# Wheat crop yield forecasting using various regression models

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## Abstract

The prediction of crop yield, particularly paddy production is a challenging task and researchers are familiar with forecasting the paddy yield using statistical methods, but they have struggled to do so with greater accuracy for a variety of factors. Therefore, machine learning methods such as Elastic Net, Ridge Regression, Lasso and Polynomial Regression are demonstrated to predict and forecast the wheat yield accurately for all India-level data. Assessment metrics such as coefficient of determination ( $R^2$ ), Root Mean Square Error (RMSE), Mean Absolute Error (MAE), Mean Squared Error (MSE) and Mean Absolute Percentage Error (MAPE) are used to evaluate the performance of each developed model. Finally, while evaluating the prediction accuracy using evaluation metrics, the performance of the Polynomial Regression model is shown to be high when compared to other models that are already accessible from various research in the literature.

**Keywords:** Elastic Net; Ridge Regression; Lasso Regression; Polynomial Regression; Ordinary Least Squares; forecast

**2020 AMS subject classifications:** 62J05, 62J07, 62-04, 62-06. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1085. ISSN: 1592-7415. eISSN: 2282-8214. ©Shakila C V et al. This paper is published under the CC-BY licence agreement.

## **1 Introduction**

Agriculture has been the economic backbone of many nations. There are more than 118.9 million farmers in India, and as the population expands, there will be a demand for food. As a result, we need new methods to produce more food products in a shorter amount of time. However, since agriculture is not a profitable industry, not many people choose it as a career. Bhosale et al. [2018]. Agriculture has always been recognized as a vital and great culture that India has traditionally practiced. In the past, people used the land where they lived and made crop choices based on the local weather and conditions. However, due to the greenhouse effect and changes in climatic conditions, farmers today cannot predict when it will rain, snow, or whether there will be water available for their crops, among other things. With this serious issues and demand, it is important to estimate sustainable agricultural production using a system that can accurately measure crop conditions, crop type, and yield. Freie et al. [1999].

There are a few approaches to dealing with constructing the suitable improvement in the agricultural industry. There are several methods to approach these issues by utilizing some of the most significant technological advancements. One of the greatest and simplest technologies we can employ is AI-based and machine learning prediction principles. Furthermore, recently developed machine learning (ML) algorithms are more capable than statistical techniques to find yield estimations.

Artificial neural networks, Decision trees, Regression analysis, Clustering, Bayesian networks, Time series analysis, and Markov chain models are just a few of the mathematical and statistical techniques used in machine learning (ML) approaches for crop prediction. Due to the availability of multiple data from various sources to expose hidden information, the use of these machine learning techniques in crop production demonstrates even more significant advantages.

## **2 Literature survey**

Machine learning addresses problems when the relationship between information and yield variables is uncertain or difficult to comprehend (Shaik et al. [1999] and Veenadhari et al. [2014]). Unlike traditional measuring techniques, machine learning explicitly displays the data whose trade-mark is beneficial to conduct modeling of complex and non-linear practices, such as a capacity for crop output forecasting (Praveen and Rama [2019] and Kumar et al. [2019]). Using supervised learning, the majority of machine learning algorithms are successfully used to predict crop yields (Praveen et al. [2017] and RaviKumar et al. [2019]). The preparation measures will continue until the model attained the desired level of

accuracy on the preparation data.

The majority of research in the past have developed statistical agricultural production prediction models using multiple linear regressions (MLRs) (Rai et al. [2013]; Kumar et al. [2014]; Dhekale et al. [2014]).

Das et al. [2017] has studied about the Statistical approaches for feature selection or feature extraction, such as Least Absolute Shrinkage and Selection Operator (LASSO), Stepwise Multiple Linear Regression (SMLR) or Elastic Net (ENET) method, can be utilized to address these issues. Yousefi et al. [2015] discussed to forecast the output energy of rice production in Iran, several researchers used the polynomial and radial basis function kernels of support vector regression (SVR). Paidipati et al. [2021] developed a model using SVR Approach with Various Non-Linear Patterns for Forecasting Rice Cultivation in India.

There are few studies comparing the accuracy of feature selection, feature extraction, and both approaches combined for agricultural yield forecasting. With the following objectives: (i) to develop overall crop yield prediction models using various multivariate models; and (ii) to assess the analytical performance of the developed models, our study has found scope to develop and select a statistical forecasting model for rice using various regression techniques for the India level. Elastic Net Regression, Ridge Regression, Lasso Regression, and Polynomial Regression are some of the techniques we employed to construct this work. There are many comparable projects on the market, but what sets our project apart from the competition is how we've integrated Python with machine learning to cut down on the number of lines of code and production costs while still producing accurate results (Pramod et al. [2019]; Tutun et al. [2016]).

## **3 Material and methods**

### **3.1 Ridge regression**

In Ridge regression  $\mu$  is a penalty term and that penalty function is equal to the squared root of the coefficient. The square of the coefficients magnitude corresponds to the  $L_2$  term. To regulate that penalty term, we additionally incorporate the coefficient  $\mu$ . In this instance, if  $\mu$  is zero, the formula is the fundamental OLS; however, if  $\mu$  is more than zero, a constraint will be added to the coefficient. This constraint makes the quantity of the coefficient tend towards zero as we raise the amount of  $\mu$ . This results in a tradeoff between smaller variance and increased bias.

$$L_R = \arg.\min_{\hat{\alpha}} \left( \|Y - \alpha * X\|^2 + \mu * \|\alpha\|^2 \right).$$

where  $\mu$  is regularization penalty.

Because it never reaches a coefficient of zero but only minimises it, ridge regression lowers a model's complexity without lowering the number of variables. As a result, this model is unable to achieve feature reduction.

### 3.2 Lasso regression

Least Absolute Shrinkage and Selection Operator is short for lasso regression. It extends the cost function's penalty term. The whole sum of the coefficients is represented by this phrase. When the value of the coefficients increases from 0 to 1, this term penalises, causing the model to lower the value of variables in order to minimise loss. While lasso regression usually makes the value of the coefficient to absolute zero, ridge regression never does.

$$L_{lasso} = \arg.\min_{\alpha} (\|Y - \alpha * X\|^2 + \mu * \|\alpha\|_1) .$$

With various data types, Lasso occasionally has difficulties. If the number of predictors (p) is more than the number of observations, Lasso will choose at most n predictors as non-zero even if all of the predictors are significant (n). The LASSO regression method chooses one of the highly collinear variables at random when there are two or more, which is bad for data interpretation.

### 3.3 Elastic net

To address the drawbacks of Ridge and Lasso regression, ? formed an elastic net regression. In general, ridge regression performs best with highly correlated variables, whereas Lasso regression performs well with less correlated variables. However, there are many models that represent a significant number of variables but lack information on attributes like correlation. Ridge regressions and Lasso are not very helpful in these circumstances. To get away from this problem, the function is estimated using ENR since it takes into account the consequences of both Lasso and Ridge regressions.  $L_1$  and  $L_2$  norms can be used to define the Lasso and ridge regression penalties, respectively. For accurate prediction, ENR take into account the  $L_1$  and  $L_2$  penalties by the following equations.

$$L_{ENR} = \arg.\min_{\alpha} \sum_i (Y_i - \alpha X_i)^2 + \beta^1 \sum_{k=1}^1 |\alpha_k| + \beta^2 \sum_{k=1}^1 \alpha_k^2 .$$

where  $\|L_1\| = \beta^1 \sum_{k=1}^1 |\alpha_k|$  and  $\|L_2\| = \beta^2 \sum_{k=1}^1 \alpha_k^2$ .  $L_1$  is sum of the weights and  $L_2$  is the sum of the square of the weights.

### 3.4 Polynomial regression

In polynomial regression, a kind of linear regression, the relationship between the random variable  $x$  and the dependent variables  $y$  is represented as a  $n^{th}$ -degree polynomial. Polynomial regression is used to fit a nonlinear relationship between the value of  $x$  and the corresponding dependent mean of  $y$ , denoted by the notation  $E(y|x)$ .

Here is a polynomial regression model's general equation.

$$L = s_0 + s_1x_1 + s_2x_1^2 + s_3x_1^3 + \dots + s_nx_1^n$$

Some correlations may be curvy, according to a researcher's hypothesis. Such scenarios will undoubtedly have a polynomial term. The assumption in common multiple linear regression analysis is that every independent variable is independent of every other independent variable. In the case of polynomial regression models, this assumption is incorrect.

### 3.5 Model validation

#### 3.5.1 Mean absolute percentage error (MAPE)

To determine the mean absolute percentage error (MAPE), the absolute error for each period is subtracted from predicted values then as follows the procedures.

$$MAPE = \frac{100}{n} \sum_{t=1}^n \left| \frac{Predicted_i - Actual_i}{Actual_i} \right|$$

## 4 Results and discussion

### 4.1 Dataset

Directorate Of Economics and Statistics Department of Agricultural and Farmers Welfare, and Government of India provided the time series data of wheat yield at India level (1966 to 2017). The research used characteristics like Area Under Cultivation (Thousand / Hectares), Production (Thousand / Tons), and Yield (KG / Hectare) to evaluate data from all of India. Elastic Net, Ridge Regression, Lasso Regression, and Polynomial Regression were constructed and compared to determine the best-fit model.

### 4.2 Overview of wheat parameter statistics

The Elastic Net, Ridge Regression, Lasso Regression, and Polynomial Regression models were separately applied to the wheat yield data to investigate



Table 1: Statistical Measures

| Measures | Elastic Net Regression | Lasso Regression | Ridge Regression | Polynomial Regression |
|----------|------------------------|------------------|------------------|-----------------------|
| $R^2$    | 0.9252                 | 0.9252           | 0.9252           | 0.9688                |
| MSE      | 38964456.45            | 40910114.34      | 42898765.78      | 17083450.77           |
| RMSE     | 5996.1034              | 6396.1015        | 6876.3675        | 4133.2131             |
| MAE      | 4894.87                | 5194.88          | 5794.86          | 3302.65               |
| MAPE     | 20.435                 | 55.405           | 69.203           | 8.232                 |

their connection, and the effectiveness of each model was evaluated using MSE,  $R^2$ , and MAPE.

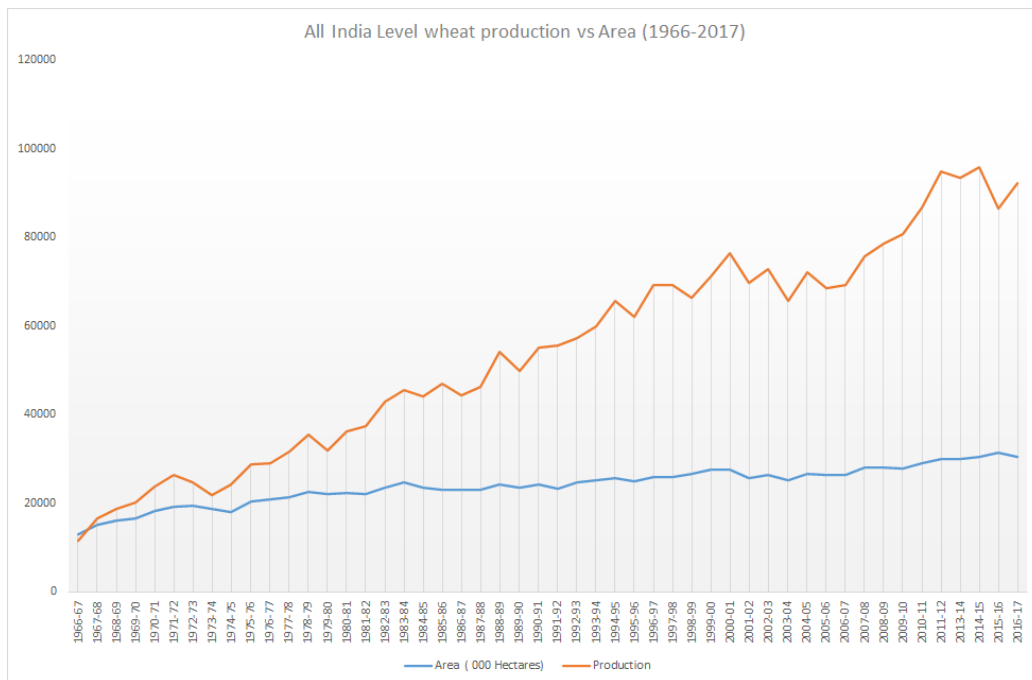


Figure 1: All India Level wheat production vs Area (1966-2017)

Figure 1 shows that the all-India level wheat production wise Area. The stronger correlation and the error higher form will be regarded as the most effective method for predicting agricultural production (kg/acre). The statistical approach's results are shown in the first case  $R^2$  values were verified among all the particular regression models (Table 1).

### 4.3 Regression models

Numerous research showed that machine learning techniques could forecast wheat production. For a consistent wheat yield, it is necessary to increase prediction accuracy. To ensure a consistent wheat production, it is necessary to improve prediction accuracy. The accuracy of the suggested Elastic Net Regression, Ridge Regression, Lasso Regression, and Polynomial Regression for wheat yield prediction is assessed using the  $R^2$ , RMSE, MAE, MSE, and MAPE metrics, as was previously stated.

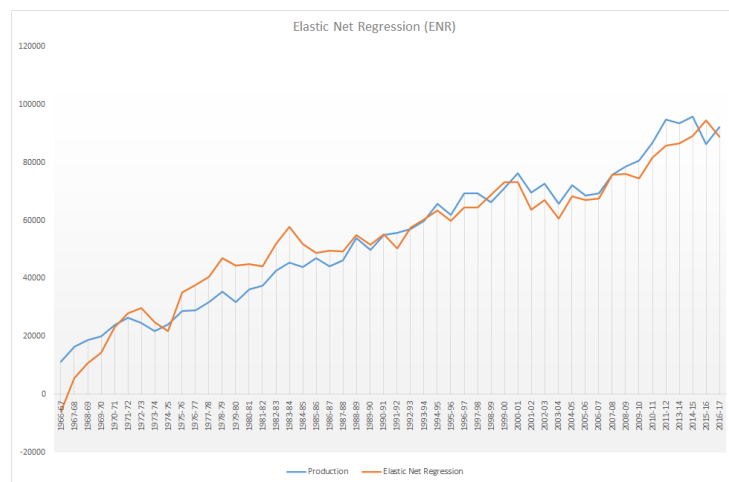


Figure 2: Forecasting using Elastic Net Regression

Analysing the data by using Elastic Net Regression, we have got the values of  $R^2$ , MSE, RMSE, MAE, and MAPE of around 0.9252, 38964456.45, 5996.1034, 4894.87, and 20.435, respectively, the forecasting utilizing elastic net regression is shown in Fig.2 Here 92% of data are used to fit the model.

Fig. 3 shows the Forecasting using the Ridge regression, which is evaluation metrics with values of  $R^2$ , MSE, RMSE, MAE, and MAPE of about 0.9252, 42898765.78, 6876.3675, 5794.86 and 69.203 respectively. Here 92% of data are used to fit the model. And the MAPE values is 69.2 which large in model fitting.

By using Lasso Regression model the values of  $R^2$ , MSE, RMSE, MAE, and MAPE of around 0.9251, 40910114.34, 6396.10, 5194.88 and 55.405, respectively, the forecasting utilizing Lasso regression is shown in Fig 4. Here also 92% of data are used to fit the model. But the MAPE values are huge as 55% for the model fit.

In Fig. 5 forecasting the wheat yield data with Polynomial regression, estimate the  $R^2$ , MSE, RMSE, MAE, and MAPE of around is 0.9687, 17083450.77,

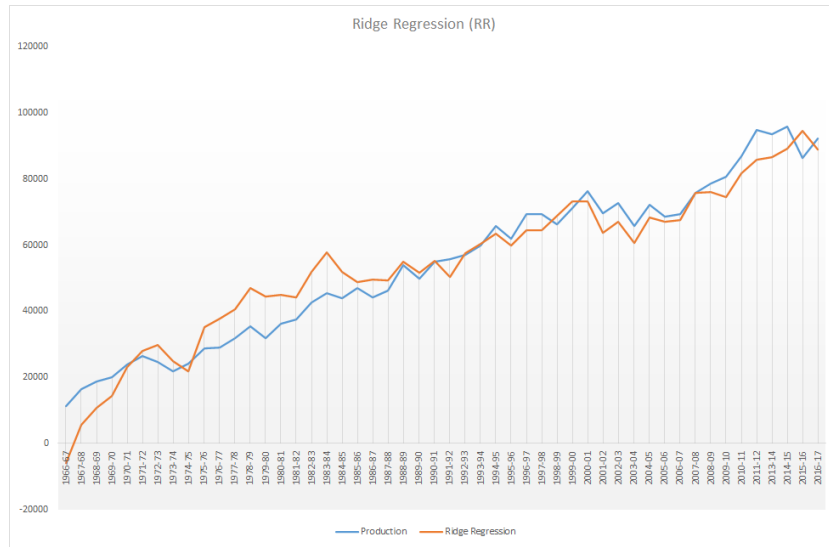


Figure 3: Forecasting using Ridge Regression

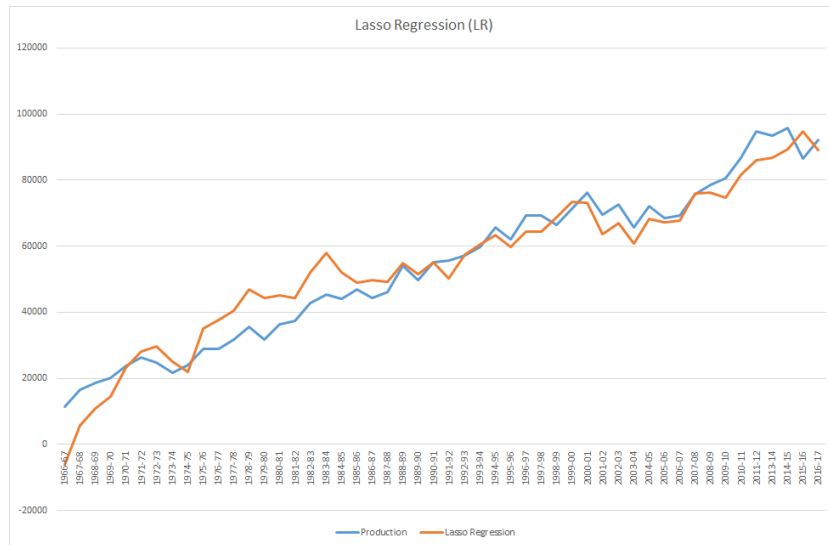


Figure 4: Forecasting using Lasso Regression

4133.213, 3302.674 and 8.23 respectively. Comparing the all above various measure value the  $R^2$  is 96% data using to fit the model. Mean-while, the MAPE values is 8.23 which is highly acceptably accurate level.

Fig. 6 shows the forecasting of India-wide level wheat crop production using various regression models. The accuracy of the Polynomial regression model exhibits superior scale than other chosen machine learning models, as like regression

## Wheat crop yield forecasting using various regression models

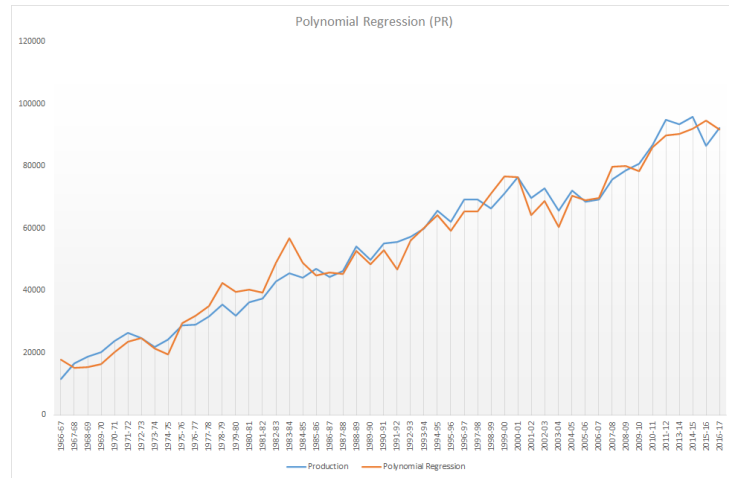


Figure 5: Forecasting using Polynomial Regression

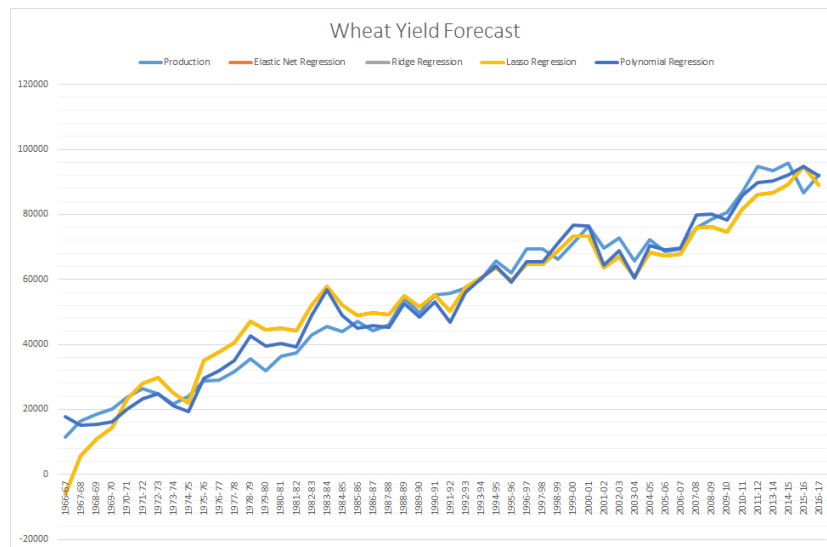


Figure 6: Forecasting using Various Regression Models.

models.

## 5 Conclusions

Statistical and machine learning methods are used to predict agricultural yield. The statistical analysis of various Regression approaches and machine learning, specifically Elastic Net Regression, Ridge Regression, Lasso Regression, and Polynomial Regression are among the techniques that are evaluated to acquire

higher accurate crop yield forecast. To evaluate the level of accuracy of the various methods, model performance measures are updated. The main findings are drawn from the results seen:

- Assessment metrics such as coefficient of determination ( $R^2$ ), Root Mean Square Error (RMSE), Mean Absolute Error (MAE), Mean Squared Error (MSE) and Mean Absolute Percentage Error (MAPE) are used to evaluate the performance of each developed model.
- The obtained study showed that the Polynomial Regression method produces superior evaluation metrics with values of  $R^2$ , MSE, RMSE, MAE, and MAPE of about 0.9687, 17083450.77, 4133.21, 3302.67, and 8.2326 respectively.
- The  $R^2$  metrics of the Polynomial regression is 4.498 percent better than other existing models from other regression models.
- Hence, its improved performance metrics, the suggested machine learning algorithm in particular, Polynomial Regression reduces the risk factor for Wheat yield.

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# BIBO stability and decomposition analysis of signals and system with convolution techniques

C. B. Sumathi \*

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## Abstract

In this paper control system's stability is arrived based on Bounded Input Bounded Output (BIBO) when bounded input is given in the form of discrete values. The control system allows the state estimation constraints to reach the convergence even when fluctuations in the parameters of the input system occur. To overcome this DTFT (Discrete Time Fourier Transform) is used when the signal is completely absolutely summable. Stability of the LTI (Linear time invariant) system is showed and is depending on the absolute summable of their impulse response. Simultaneously for continuous signal the stability occurs if it is absolutely integrable. In addition to that the linearity and time-invariance properties are discussed. This provide a new way to decompose the periodic signals into Fourier series by convolving the fundamental signals. Continuous and discrete time signals are focused in this paper to get linear time invariant system (LTI) through complex exponentials. Finally filtering techniques were used to eliminate the noisy frequency component in a signal.

**Keywords:** Stability, DTFT, CTFT, Dirichlet conditions.

**2020 AMS subject classifications:** 39A12, 39A30, 39A60. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1086. ISSN: 1592-7415. eISSN: 2282-8214. ©C. B. Sumathi et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Difference equations are described the evolution of applied mathematics phenomenon of latest technology from artificial intelligence to IoT. Higher order difference equations and time invariant systems are focusing more real world problems in the diverse fields like communication technology Srinivasan et al. [2022]. When we classify these difference equations, each category of classification are solving specific engineering and technological problems R.P.Agarwal [2000], Thamvichai and T.Bose [2002]. Various methodologies are used to resolve error reduction and quality enhancement techniques. In such cases analytical solutions are possible to address particular specified problems Kayar and KaymakçalanBistriz [2022].

To analyze these difference equations finite difference methods can be utilized. In particular when the errors are magnified, then the difference equations will be unstable and for smaller components these equation guarantees the stability Camouzis and Ladas [2008], Liu1 et al. [2022]. When the larger components are involved, tri-diagonal system (based on Crank-Nicolson Method) will be compared with other explicit and implicit methods with higher derivatives Camouziss and G.Ladas [2010], Liu [2008].

Time Delay is one of the reasons for instability which appears in dynamical systems such as biological systems, chemical systems, communication systems, nuclear systems, electrical systems, etc., and it is one of the key performances in these systems Bose [1995], Kaczorek [2011], Oppenheim et al. [2009]. As the signal is impulsive, it goes to infinity at any time and hence, the system is unstable even when an input is bounded but an output is infinite.

Bounded Signal is a signal which is having a finite value at all instants of time. In general a signal is bounded if it has finite value  $M > 0$ , and the signal does not exceed  $M$ , i.e.  $|y(n)| \leq M, \forall n \in Z$  for discrete-time signals.

This paper is structured as follows: Section II provides the fundamental concepts of LTI systems in time as well as frequency domain. The systems represents through linear time invariant difference equations are discussed in section III. Section IV dealt with the development of Fourier series in difference equations with time dependent variable. Section V depicts the Filtering Techniques Related to LTI systems that change the shape of the input signal. Finally section VI concludes the paper.



## 2 LTI systems

### 2.1 Time domain conditions

**Theorem 2.1.1** (Sufficient condition) : The BIBO stability of discrete time LTI system and its impulse response is absolutely summable i.e.  $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ .  
Proof : Consider the following By convolution,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad (1)$$

Then

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[n-k]x[k] \right| \quad (2)$$

Applying triangular inequality

$$|y[n]| \leq \|x\|_{\infty} \sum_{k=-\infty}^{\infty} |h[k]| \quad (3)$$

Therefore  $h[n]$  is absolutely summable.

**Theorem 2.1.2** (sufficient condition of continuous time) : The BIBO stability of continuous time LTI is  $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ .

**Proof :** Consider the output function

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-T)h(T)dT \right|, \quad (4)$$

$$\begin{aligned} &\leq \int_{-\infty}^{\infty} M|h(T)|dT \\ &= M \int_{-\infty}^{\infty} |h(T)|dT \end{aligned} \quad (5)$$

Hence the proof.

### 2.2 LTI systems - frequency domain conditions

#### Discrete Time signal

In general, for BIBO stability a unit circle in the  $Z$ -plane must contain all the

poles of a system. The condition for stability can be obtained by the above time domain condition

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |h[n]| |e^{-jwn}|$$

**Continuous Time signal** In the continuous case, Laplace transform must include the imaginary axis. For BIBO stability  $s$ -plane must contain all the poles of a system. The condition for stability can be obtained by the above condition

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} |h(t)| |e^{-j\omega t}| dt \quad (6)$$

$$= \int_{-\infty}^{\infty} |h(t)| (e^{-\sigma t}) dt \quad (7)$$

where  $s = \sigma + j\omega$  and  $Re(s) = \sigma = 0$

In addition to above condition, the continuous time signals is convergence if it encounter the following Dirichlet conditions

- if the interval is finite then  $x$  is of bounded variation ,
- for all finite number of points,  $x$  is continuous, if the interval is finite.

These conditions are satisfied by the periodic interval. Assume that

$$\sum_{m=0}^{\infty} a^m = \frac{1}{(1-a)} \quad (8)$$

where  $|a| < 1$ . On multiplying  $1 - a$  both side we get

$$\sum_{m=0}^{\infty} a^m - a \sum_{m=0}^{\infty} a^m = 0^{\infty} a^m = 1$$

$a^0 = 1$  , since  $|a| < 1$ , the sums converge. As an example, if  $h(t) = a^t u(t)$  for every  $t \in R$ , where  $a > 0$ . Since the integral is infinite if  $a \geq 1$ , it is unstable and it is finite if  $0 < a < 1$ , and thus

$$\int_0^{\infty} a^t dt = \frac{-1}{\ln a}$$

Therefore, if  $0 < a < 1$ , the system becomes stable.

### **3 Representation of systems via LTI difference equations**

The output form of difference equation of discrete time system is defined like Haung and P. M.Knopf [2012]

$$y[n] = y[n - 1] + x[n]$$

If the system  $h[n]$  is LTI system then

$$y[n] = x[n] * h[n] \quad (9)$$

In causal system, the impulse response is zero for all  $t < 0, n < 0$  in both discrete and continuous system respectively Alzabut et al. [2021], Oppenheim et al. [2009], Thamvichai and T.Bose [2002]. For example consider the following system,

$$y[n] - \frac{1}{2}y[n - 1] = x[n]$$

If  $x[n] = \delta[n]$ , then we have

$$y[0] = 1$$

$$y[1] = \frac{1}{2}$$

$$y[2] = \frac{1}{4}$$

$$\dots$$

$$y[n] = \left(\frac{1}{2}\right)^n$$

Then

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

### **4 Development of Fourier series in different equations**

In this section we calculate the transform signals through Fourier series Hu [2011], Bistritz [2004]. We consider some simple transformations with time variable. The output function  $y(t)$  is

$$y(t) = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

The input signal  $x(t) = \cos w_0 t$ , where  $w_0 > 0$ , and then

$$x(t) = \frac{1}{2}e^{jw_0 t} - \frac{1}{2}e^{-jw_0 t}$$

Filtering is used to modify or eliminate some frequency components in the discrete signals. Consider

$$H(e^{jw}) = \frac{1}{(1 - ae^{(-jw)})}$$

Now define  $y(t)$  as

$$y(t) = e^{(-at)} \int_0^1 e^{a\tau} d\tau$$
$$y(t) = \frac{1}{a}[1 - e^{(-at)}]$$

Here  $x(\tau)$  and  $h(t - \tau)$  does not overlap, and hence  $y(t) = 0$ .

## 5 Conclusions

This paper analyzed the stability of the BIBO system in time as well as frequency domain. Based on the state of time domain, a linear time invariant system is stable only if its impulse response is absolutely stable. In addition, it is focused the decomposition of signals into LTI system through suitable examples. The development of these techniques has been used in implementation of time-varying convolution filters. The signals are convolved to produce the linear time invariant system and non overlapping systems.

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# Intuitionistic Robust fuzzy matrix for the diagnosis of stress, anxiety and hypertension

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## Abstract

The mathematical model given here attempts to improve precision in the diagnosis of stress, anxiety, and hypertension using Intuitionistic robust fuzzy matrix (IRFM). In practice, the imprecise nature of medical documentation and the uncertainty of patient information frequently do not provide the appropriate level of confidence in the diagnosis. To that purpose, a novel method based on distinct fuzzy matrices and fuzzy relations is devised, which makes use of the capabilities of fuzzy logic in describing, understanding, and exploiting facts and information that are unclear and lack clarity. With the assistance of 30 doctors, a medical knowledge base is created during the procedure. The model obtained 95.55% accuracy in the diagnosis, demonstrating its utility.

**Keywords:** Fuzzy logic; fuzzy matrices; max-min principles; Robust; Intuitionistics.

**2020 AMS subject classifications:** 03B52, 62F35, 70H30, 20N25 <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1087. ISSN: 1592-7415. eISSN: 2282-8214. ©K.Revathi et al. This paper is published under the CC-BY licence agreement.

## **1 Introduction**

The computer scientist Zadeh was the first to utilize fuzzy logic as a scientific notion. Medicine is one subject where the use of fuzzy logic was recognized as early as the mid-1970s. The ambiguity encountered during the illness diagnostic process has repeatedly been the subject of application of fuzzy set theory in this sector. The greatest and most helpful explanations of clinical symptoms frequently include language concepts that are inevitably ambiguous Adlassing [1986].

Fuzzy mathematics seeks to derive exact meaning from erroneous information by capturing ambiguity in real life. It is a valuable tool for decision-making systems. Using the fuzzy set paradigm, several ways have been created to imitate medical diagnostic procedures. The basic principle behind a medical diagnosis is to associate a patient's symptoms or signals with potential illnesses based on the expert's medical knowledge. According to Sanchez [1979], Meenakshi and M [2011], Elizabeth and Sujatha.L [2013], Ravi.J [2022] and Raich and Dalal [2009] is approach demonstrates the doctor's medical skill as a blurry association between symptoms and illnesses. The method to medical diagnostics by employing an interval-valued fuzzy matrix as a representation. The another approach of medical diagnosis by utilising a triangular fuzzy membership matrix representation. Saravanan and Prasanna [2016] presented a fuzzy matrices application for use in medicine that utilised the idea network and concept matrices. Raich and Dalal [2009] employed fuzzy matrices for the first time in diabetes research.

In this study, we employed IRFM to improve medical diagnostic accuracy Klir and Yuan [1995]. This mathematical model produces diagnoses based on the expertise and experience of the 30 doctors. We created a differentiating strategy for studying indication relationships for diagnosis, which can be described as non-symptom indication non-occurrence indication conformability indication Gupta [1976]. Our method is reasonably accurate, as evidenced by a 95% confidence between the genuine diagnoses supplied by physicians and the diagnostic conclusion produced by our algorithm.

## **2 Research method**

This section shows the consists of the following elements:

### **2.1 Medical terms:**

The expertise of physicians is depicted as a hazy relationship between symptoms and illnesses. There are two kinds of hazy relationships between symptoms and diseases:

*Intuitionistic robust fuzzy matrix for the diagnosis of stress, anxiety and hypertension*

(i). **An incidence relation**  $r_i$

It describes the possibility or proclivity of a symptom manifesting when a specific condition is present, i.e., how frequently does the symptom I occur with disease p.

(ii). **A comfortability relation**  $r_j$

It denotes how well the symptoms differentiate between diseases, or how strongly symptom I verifies disease j.

The distinction between occurrence and conformability is significant because, whereas a symptom is present in many diseases, its occurrence and conformability differ depending on the condition. The above-mentioned correlations were discovered using medical records from an expert.

## 2.2 Intuitionistic robust fuzzy matrices (IRFM)

Then the membership function of A and B is defined as,

$$A \blacksquare B = \left\{ \max \left[ \min \left( \frac{\sigma_{ij}(r_1) + \sigma_{jk}(r_1)}{2} \right), \max \left[ \min \left( \frac{s_{ij}(r_2) + s_{jk}(r_2)}{2} \right) \right] \right] \right\}$$

If  $s_{ij}(r_2) = s_{jk}(r_2) = 0$  for all I,j,k then,

$$A \blacksquare B = \left\{ \max \left[ \min \left( \frac{\sigma_{ij}(r_1) + \sigma_{jk}(r_1)}{2} \right), 0 \right] \right\}$$

$$A \blacksquare B = \left\{ \max \left[ \min \left( \frac{a_1 + b_1}{2} \right), \left( \frac{a_2 + b_2}{2} \right), \left( \frac{a_3 + b_3}{2} \right), \dots, \left( \frac{a_n + b_n}{2} \right) \right], 0 \right\}$$

Let D represent a grouping of specific illnesses, S represent a grouping of symptoms, and P represent a grouping of persons in need of a diagnosis. According to the fuzzy relation  $r_s(p, s)$  (where pm, sn) membership grades, the set PS is supposed to indicate the degree to which the symptom s is present in the patient p. On the set S, D, the fuzzy relation  $r_i(s, d) = r_i(m, n)$  is defined, where sS, dD denotes the recurrence frequency of symptoms s with sickness d. The same set defines the fuzzy relation  $r_j$ , which shows how strongly a symptom (s) predicts the presence of an illness (d). A fuzzy relation  $r_j$  is also formed on the same set, where  $r_j(p, d) = r_2(n, p)$  denotes the strength with which a symptom (s) supports the occurrence of an illness (d). Using relations  $r_s$ ,  $r_i$ , and  $r_j$ , the following four alternative indicator relations supplied on set  $P \times D$  are computed:

(i). Relationship between the two  $r_1 = r_s \blacksquare r_i$

(ii). Conformability indication relation  $r_2 = r_s \blacksquare r_j$



$$r_s = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} p_1 \\ p_2 \end{matrix} & \begin{bmatrix} 0.2 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.1 \end{bmatrix} \end{matrix} \quad r_1 = \begin{matrix} & d_1 & d_2 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 0.1 & 0.3 \\ 0.7 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{matrix}$$

$$R_1 = \begin{matrix} & d_1 & d_2 \\ \begin{matrix} p_1 \\ p_2 \end{matrix} & \begin{bmatrix} 0.2 & 0.15 \\ 0.1 & 0.15 \end{bmatrix} \end{matrix}$$

(iii). Non-occurrence Occurrence indication relation indication relation  $r_3 = r_s \blacksquare (1 - r_i)$

(iv). Non-symptom indication relation  $r_4 = r_i \blacksquare (1 - r_j)$

The max-min product of fuzzy matrices is used to compose fuzzy relations.

#### Example 1

Let  $D = \{d_1, d_2\}$ ,  $S = \{s_1, s_2, s_3\}$ ,  $P = \{p_1, p_2\}$

Let  $r_l = r_s \blacksquare r_i$  So for every  $i=1, 2$  and  $j= 1, 2$ ,  $r_1(p_i, d_j) = \max\{\min\{R_s(p_i, s), r_1(s, d_j)\} \mid s \in S\}$

#### Example 2

Let  $r_1(p_1, d_1) = \max\{\min\{0.2, 0.1\}/2, \min\{0.4, 0.7\}/2, \min\{0.1, 0.2\}/2\} = \max\{0.05, 0.2, 0.15\} = 0.2$ . Similarly,  $r_1(p_i, d_j)$  for every  $i$  and  $j$ . Hence, we get Other indication relations are calculated in the same way.

## 2.3 Hypertension overview

A blood pressure reading of 120/80 mmHg or less is considered normal. You may aim to keep your blood pressure in a healthy range every day, regardless of your age. One of the most dangerous elements of high blood pressure is that you may be unaware of it. In reality, one-third of persons with high blood pressure are unaware of their condition. This is because indications of high blood pressure are uncommon until blood pressure is quite high. Regular blood pressure checks are the best way to discover if your blood pressure is too high. You may also monitor your blood pressure at home. This is especially important if someone in your family has high blood pressure. I regularly suffer from headaches. Headaches and nosebleeds are not always signs of high blood pressure. This is feasible when blood pressure increases above 180/120, as it does during a hypertensive crisis.

### 2.3.1 There are two kinds of hypertension.

(i). **The primary (essential) hypertension** The most patient instances of high blood pressure have no known cause. Primary (essential) hypertension is a form of high blood pressure that appears gradually over time.

(ii). **Secondary hypertension** Some people develop high blood pressure as a result of a more serious condition. Secondary hypertension, a more severe variant of primary hypertension, is a kind of high blood pressure that develops suddenly. Secondary hypertension can be caused by a variety of medical conditions and medications, including obstructive sleep apnea, kidney disease, adrenal cancer, and thyroid problems. Since birth, your body has had certain blood vessel anomalies. Drugs include birth control pills, allergy and cold treatments, decongestants, over-the-counter pain relievers, and some prescription prescriptions. Illicit drugs include amphetamines and cocaine.

**Disease 1: Heart attack or stroke or Heart failure.**

Atherosclerosis, or artery hardening and thickening, can lead to heart attacks, strokes, and other complications. To pump blood against the increased pressure in your veins, your heart must work harder. As a result, the walls of the heart's pumping chamber thicken (left ventricular hypertrophy) (left ventricular hypertrophy). Heart failure may occur if the developing muscle is unable to pump enough blood to meet your body's demands.

**Disease 2: Aneurysm is a weakening and narrowing of the blood arteries in your kidneys.**

High blood pressure can weaken and bulge your blood vessels, resulting in an aneurysm. A ruptured aneurysm can endanger one's life.

**Disease 3: Eye blood vessel enlargement, constriction, or rupture; metabolic syndrome; memory or cognitive problems; dementia**

As a result, several organs may be unable to operate effectively. This might result in vision loss. This syndrome is characterised by a larger waist, greater triglyceride levels, lower HDL cholesterol (the "good" cholesterol), higher blood pressure, and higher insulin levels. All of these disorders raise your chances of acquiring diabetes, heart disease, or stroke. Uncontrolled high blood pressure may impair your ability to think, remember, and learn. Patients with high blood pressure have difficulty grasping and recalling their ideas. One type of dementia is caused by clogged or restricted arteries, which limit the quantity of blood that can reach the brain. A stroke can cause vascular dementia by cutting off blood supply to the brain.

**2.3.2 Symptoms of Severe High Blood Pressure**

If your blood pressure is really high, you should be aware of the following symptoms:

**Symptom  $S_1$ :** Severe headaches

**Symptom  $S_2$ :** Nosebleed

**Symptom  $S_3$ :** Fatigue or confusion

**Symptom S<sub>4</sub>:** Vision problems

**Symptom S<sub>5</sub>:** Chest pain

**Symptom S<sub>6</sub>:** Difficulty breathing

**Symptom S<sub>7</sub>:** Irregular heartbeat

**Symptom S<sub>8</sub>:** Blood in the urine

**Symptom S<sub>9</sub>:** Pounding in your chest, neck, or ears.

People sometimes feel that other symptoms may be related to high blood pressure, however it is possible that they are not:

**Symptom S<sub>10</sub>:** Dizziness

**Symptom S<sub>11</sub>:** Nervousness

**Symptom S<sub>12</sub>:** Sweating

**Symptom S<sub>13</sub>:** Trouble sleeping

**Symptom S<sub>14</sub>:** Facial flushing

S represents the collection of all symptoms.

$S = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13}, S_{14}\}$

Let D represent the whole collection of illnesses,  $D = \{d_1, d_2, d_3\}$ .

Let P represent the precise universal set of all Patients,  $D = \{p_1, p_2, p_3\}$ .

### 3 Experimental study

As a case study, we picked a serious condition with three types of symptoms, such as hypertension. These ailments were chosen because, according to WHO (World Health Organization) figures, they are the most frequent in India. According to WHO data from 2016, the adult prevalence of hypertension in India is 24%. Following the selection of the three illnesses, we chose 14 symptoms that are commonly found in people with those ailments. Then, with the help of medical specialists, we constructed a medical knowledge base. We looked at the prevalence of a symptom in a linked sickness and the confirmation of a disease from a connected symptom as we built the database. The phrase "symptom emergence in the setting of the associated sickness" is related to the question "How frequently does a symptom appear in the presence of a disease?" When a certain sickness is present, the average or frequency of symptom presentation is high. It alludes to the question, "To what degree do symptoms strongly confirm disease d?" A symptom's discriminating power in confirming the presence of an illness is defined by its ability to confirm the presence of a disease based on its own observations. We distributed blank charts (Appendix) to 30 randomly selected specialist doctors in Namakkal district, Tamilnadu, India, in order to build a knowledge base that includes the presence of a symptom in the linked illness and confirmation of a disease based on a specific symptom. We asked the clinicians to keep track of how many patients with the relevant disease exhibited each symptom out of 100.

*Intuitionistic robust fuzzy matrix for the diagnosis of stress, anxiety and hypertension*

For example, if a doctor checks 100 diabetic patients and finds "severe thirst" in 80 of them, he or she must document 90 cases of excessive thirst in hypertension. To determine the presence of a disease, we asked the doctors to specify the percentage of the time that they detect the relevant symptom in the patient. These statistics should be multiplied by 100 such that they fall between [0, 1] and indicate an 80% chance of anaemia. As shown in table 1, we calculated the mean of the information obtained to estimate the data's central tendency. We awarded membership ratings to each of the 14 fuzzification symptoms based on the severity or frequency of the patient's condition. The following are the grades:

**Table 1: A patient's symptoms according to their frequency or intensity**

| Sr. No. | Symptom                                | Cont.sever | Cont.mild | Occa. | Rare | No   |
|---------|----------------------------------------|------------|-----------|-------|------|------|
| 1.      | Severe headaches                       | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 2.      | Nosebleed                              | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 3.      | Fatigue or confusion                   | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 4.      | Vision problems                        | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 5.      | Chest pain                             | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 6.      | Difficulty breathing                   | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 7.      | Irregular heartbeat                    | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 8.      | Blood in the urine                     | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 9.      | Pounding in your chest, neck, or ears. | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 10.     | Dizziness                              | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 11.     | Nervousness                            | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 12.     | Sweating                               | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 13.     | Trouble sleeping                       | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |
| 14.     | Facial flushing                        | 1.0        | 0.75      | 0.50  | 0.25 | 0.00 |

D denotes "disease 1, disease 2, disease 3," and S denotes "set of 14 symptoms, s1, s2,..., s14." Our diagnostic technique will be demonstrated using three hypothetical instances. As a result,  $P = p_1, p_2, \text{ and } p_3$ , and we've built a fuzzy relation  $r_s$  on the set PS in which membership grades  $r_s(p, s)$  (where  $p_P, s_S$  reflect the severity or frequency of symptoms identified in these three people) suggest:

**Table 2: Expert knowledge-base obtained in a robust manner**

| SSr. No. | Symptom                               | Disease 1   |             | Disease 2   |             | Disease 3   |             |
|----------|---------------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|
|          |                                       | i           | j           | i           | j           | i           | j           |
| 1.       | Severe headaches                      | <b>0.16</b> | 0.07        | <b>0.50</b> | 0.37        | <b>0.51</b> | 0.41        |
| 2.       | Nosebleed                             | <b>0.20</b> | 0.10        | <b>0.89</b> | 0.80        | <b>0.09</b> | 0.05        |
| 3.       | Fatigue or confusion                  | <b>0.86</b> | 0.74        | <b>0.08</b> | 0.05        | 0.05        | <b>0.07</b> |
| 4.       | Vision problems                       | <b>0.47</b> | 0.36        | <b>0.17</b> | 0.11        | <b>0.07</b> | 0.03        |
| 5.       | Chest pain                            | <b>0.39</b> | 0.38        | <b>0.16</b> | 0.11        | 0.08        | <b>0.09</b> |
| 6.       | Difficulty breathing                  | <b>0.02</b> | 0.02        | <b>0.49</b> | 0.37        | <b>0.80</b> | 0.70        |
| 7.       | Irregular heartbeat                   | <b>0.52</b> | 0.45        | <b>0.25</b> | 0.19        | <b>0.37</b> | 0.27        |
| 8.       | Blood in the urine                    | <b>0.16</b> | 0.07        | <b>0.50</b> | 0.37        | <b>0.51</b> | 0.41        |
| 9.       | Pounding in your chest, neck or ears. | <b>0.71</b> | 0.64        | <b>0.85</b> | 0.75        | <b>0.33</b> | 0.22        |
| 10.      | Dizziness                             | 0.17        | <b>0.19</b> | <b>0.52</b> | 0.43        | <b>0.03</b> | 0.00        |
| 11.      | Nervousness                           | <b>0.79</b> | 0.66        | <b>0.60</b> | 0.54        | 0.13        | <b>0.79</b> |
| 12.      | Sweating                              | <b>0.19</b> | 0.15        | <b>0.39</b> | 0.27        | <b>0.04</b> | 0.03        |
| 13.      | Trouble sleeping                      | <b>0.34</b> | 0.24        | 0.61        | <b>0.64</b> | <b>0.10</b> | 0.07        |
| 14.      | Facial flushing                       | <b>0.15</b> | 0.05        | <b>0.55</b> | 0.54        | <b>0.04</b> | 0.02        |

*Note: i = Occurrence of a symptom & j = Confirmation of a disease*

**Table 3: The presence of a disease is confirmed by the symptoms**

| Sr. No. | SYMPTOMS                               | P <sub>1</sub> | P <sub>2</sub> | P <sub>3</sub> |
|---------|----------------------------------------|----------------|----------------|----------------|
| 1.      | Severe headaches                       | 0.51           | 0.15           | 0.00           |
| 2.      | Nosebleed                              | <b>1.00</b>    | 0.00           | 0.51           |
| 3.      | Fatigue or confusion                   | 0.15           | <b>1.00</b>    | <b>1.00</b>    |
| 4.      | Vision problems                        | 0.00           | 0.15           | 0.51           |
| 5.      | Chest pain                             | <b>1.00</b>    | 0.51           | 0.00           |
| 6.      | Difficulty breathing                   | 0.00           | <b>1.00</b>    | 0.26           |
| 7.      | Irregular heartbeat                    | 0.00           | 0.00           | 0.00           |
| 8.      | Blood in the urine                     | 0.00           | 0.15           | 0.51           |
| 9.      | Pounding in your chest, neck, or ears. | 0.51           | 0.26           | <b>1.00</b>    |
| 10.     | Dizziness                              | 0.15           | 0.00           | 0.00           |
| 11.     | Nervousness                            | <b>1.00</b>    | 0.00           | 0.51           |
| 12.     | Sweating                               | 0.26           | 0.00           | 0.00           |
| 13.     | Trouble sleeping                       | 0.51           | 0.00           | <b>1.00</b>    |
| 14.     | Facial flushing                        | <b>1.00</b>    | 0.51           | 0.15           |

*Intuitionistic robust fuzzy matrix for the diagnosis of stress, anxiety and hypertension*

$$\begin{aligned}
 R_1 &= R_s \circ R_i \\
 R_1 &= \begin{matrix} & \begin{matrix} D_1 & D_2 & D_3 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \begin{bmatrix} 0.51 & 0.50 & 0.16 \\ 0.10 & 0.08 & 0.02 \\ 0.08 & 0.37 & 0.05 \end{bmatrix} \end{matrix} \\
 R_2 &= R_s \circ R_j \\
 R_2 &= \begin{matrix} & \begin{matrix} D_1 & D_2 & D_3 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \begin{bmatrix} 0.57 & 0.52 & 0.58 \\ 0.55 & 0.56 & 0.53 \\ 0.52 & 0.59 & 0.53 \end{bmatrix} \end{matrix} \\
 R_3 &= R_j \circ (1 - R_i) \\
 R_3 &= \begin{matrix} & \begin{matrix} D_1 & D_2 & D_3 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \begin{bmatrix} 0.65 & 0.66 & 0.76 \\ 0.76 & 0.66 & 0.65 \\ 0.65 & 0.76 & 0.65 \end{bmatrix} \end{matrix}
 \end{aligned}$$

**Table 4: The existence of illness is confirmed by the symptoms ‘d’**

| SSr. No. | Symptom                                | $r_i$     |           |           | $r_j$     |           |           |
|----------|----------------------------------------|-----------|-----------|-----------|-----------|-----------|-----------|
|          |                                        | Disease 1 | Disease 2 | Disease 3 | Disease 1 | Disease 2 | Disease 3 |
| 1.       | Severe headaches                       | 0.16      | 0.50      | 0.51      | 0.07      | 0.37      | 0.41      |
| 2.       | Nosebleed                              | 0.20      | 0.89      | 0.09      | 0.10      | 0.80      | 0.05      |
| 3.       | Fatigue or confusion                   | 0.86      | 0.08      | 0.07      | 0.74      | 0.05      | 0.05      |
| 4.       | Vision problems                        | 0.47      | 0.17      | 0.07      | 0.36      | 0.11      | 0.03      |
| 5.       | Chest pain                             | 0.39      | 0.16      | 0.09      | 0.38      | 0.11      | 0.08      |
| 6.       | Difficulty breathing                   | 0.02      | 0.49      | 0.80      | 0.02      | 0.37      | 0.70      |
| 7.       | Irregular heart-beat                   | 0.52      | 0.25      | 0.37      | 0.45      | 0.19      | 0.27      |
| 8.       | Blood in the urine                     | 0.16      | 0.50      | 0.51      | 0.07      | 0.37      | 0.41      |
| 9.       | Pounding in your chest, neck, or ears. | 0.71      | 0.85      | 0.33      | 0.64      | 0.75      | 0.22      |
| 10.      | Dizziness                              | 0.19      | 0.52      | 0.03      | 0.17      | 0.43      | 0.00      |
| 11.      | Nervousness                            | 0.79      | 0.60      | 0.79      | 0.66      | 0.54      | 0.13      |
| 12.      | Sweating                               | 0.19      | 0.39      | 0.04      | 0.15      | 0.27      | 0.03      |
| 13.      | Trouble sleeping                       | 0.34      | 0.64      | 0.10      | 0.24      | 0.61      | 0.07      |
| 14.      | Facial flushing                        | 0.15      | 0.55      | 0.04      | 0.05      | 0.54      | 0.02      |

Using the relations  $R_s$ ,  $R_i$ , and  $R_j$ , we developed four alternative indicator relations, namely  $R_1$ ,  $R_2$ ,  $R_3$ , and  $R_4$ .

We may get a number of diagnostic conclusions from these four indicator associations. For example, if  $(p, d) = 1$ , we can validate patient  $p$ 's diagnosis of disease  $d$ . Despite the fact that none of our four patients had this condition, it appears to suggest that disease  $d_1$  is strongly confirmed for patient  $p_1$ , sickness  $d_2$  is highly confirmed for patient  $p_3$ , and disease  $d^3$  has a 90% probability of occurring for patient  $p_2$ .

Using the standards described above, we may conclude that patient  $p_1$  has high blood pressure, patient  $p_3$  has hypertension, and patient  $p_2$  has anaemia and mild hypertension. We utilised our programme to diagnose 50 cases using this method, and we compared our findings to physician diagnoses. In 43 of 50 instances of diabetes, anaemia, and hypertension, our diagnosis and the doctors' diagnosis were identical. The degree of accuracy of the model may be calculated as follows:  
**Finding of Accuracy** = [(The number of accurate data)/ (The number of total data)]\*100

$$= (86/90)*100 = 95.55\%$$

As a consequence, the Chi-square test, often known as the "goodness of fit test," was used to check that our mathematical model was statistically sound. Before utilising the Chi-square test, we assumed that the medical professionals' diagnoses were completely correct.

$H_0$ : The medical professionals' and our diagnoses are identical.

$H_1$ : Our diagnosis differs from that of the physicians.

Acceptance level ( $\alpha$ ) = 0.05 & DF= 2 at Critical value= 5.991

**Table 5: Contingency Table for Chi- square test**

|                     | Observed | Expected |
|---------------------|----------|----------|
| <b>Stress</b>       | 50       | 50       |
| Anxiety             | 47       | 50       |
| <b>Hypertension</b> | 46       | 50       |

The Chi-square test calculation formula is as follows:  $\chi^2 = \sum [(O - E)^2 / E]$  where O is observed frequency, E is expected frequency.

1. where E is the expected frequency and O is the actual frequency
2. Using the contingency table, we calculated the chi-square (??) as follows:  
 $(\chi^2)$  as ,

$$\chi^2 = ((50-50)^2/50) + ((47-50)^2/50) + (46-50)^2/50) \\ = 0.50.$$

### *Intuitionistic robust fuzzy matrix for the diagnosis of stress, anxiety and hypertension*

Because 0.50 is greater than 5.991, or the "2" critical value, the null hypothesis  $H_0$  is accepted. We got to the conclusion that the physicians' and our diagnosis are similar.

## **4 Conclusions**

Fuzzy logic can be used to improve the precision and verification of medical diagnosis. The diagnosis offered by the "max-min" composition of fuzzy relations built using the "mean" of the physicians' data matches to the physicians' diagnosis. According to the chi-square test, which measures dependability, the proposed mathematical model gets a reliability score of 95.55%. This study's innovative technique tries to quantify the core principle behind the diagnostic procedure. This technique distinguishes itself by thoroughly investigating the connections between the indicators, particularly the "non-symptom indication-non-occurrence indication-conformability indication."

This method of preserving specialised information allows patients and general practitioners to access it. While it does not replace a doctor's diagnosis, it does help to reinforce it. The diagnostic implications of this mathematical model might be considered as a second opinion to the doctor's diagnosis. This mathematical approach is mostly employed to improve diagnostic accuracy.

The main flaw of the proposed mathematical model is that it will be off-target if a patient supplies incorrect inputs, resulting in incorrect diagnostic findings. As more doctor data is collected, software and an Android app will be created in the future. The development of new algorithms to improve accuracy is one of the future goals.

In recent trends they are given for 95 percent of accurate results. But our proposed concept given more than 95 percent accuracy compared to other existing model. Also our proposed method performed all direction (360°). In future work we are trying to write the coding with the help of Python and analyze the data for the betterment of the concept.

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# Solving fuzzy linear programming problems by using the fuzzy exponential barrier method

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## Abstract

In order to resolve the fuzzy linear programming problem, the fuzzy exponential barrier approach is the major strategy employed in this article. To overcome the problems with fuzzy linear programming, this method uses an algorithm. In this concept, a fuzzy inequality constraint is produced since the objective functions are convex. numerical examples are provided.

**Keywords:** fuzzy exponential barrier function, fuzzy exponential barrier convergence, fuzzy optimality solution.

**2020 AMS subject classifications:** 54E20, 54H25. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022.Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1088. ISSN: 1592-7415. eISSN: 2282-8214. ©A.Nagoor gani et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Fuzzy set theory has aided mathematical modelling and control theory. Unsolved decision-making problems can be applied in real-world situations using a variety of techniques. The fuzzy set theory developed by A. et al. [2012] has presented in 1965. Fuzzy linear programming was first established in 1997 by Tanaka and Asai [1984], and fuzzy numbers were published in Dubois [1983]. A function was proposed by Hsieh [1999] to cope with fuzzy arithmetical operations. Using a technique that divides, adds, and multiplies fuzzy integers in addition to subtracting and adding them. The fuzzy linear programming problem, as formulated by Mahadevi et al. [2009], A. and Yogarani [2021] and Fiacco and V. [1990] Zimmermann [1978] likewise has a duality. The process of defuzzifying involves converting fuzzy values into clear crisp values. Since a few years ago, these techniques have been thoroughly researched and applied in fuzzy systems Mahadevi et al. [2006]. A representative value from a given set, according to some characters, was the main objective of these processes. The defuzzification method establishes a link between all fuzzy sets and all real numbers. Given the high cost of an infeasible solution, Moengin and Parwadi [2011] et al. Fiacco and Anthony [1976] developed the exponential barrier function technique. The fuzzy exponential barrier approach is a different way to solve fuzzy linear programming problems. The fuzzy exponential barrier technique includes a fuzzy objective function that specifies a severe penalty for violating the fuzzy requirements in order to approximate fuzzy constraints. a fuzzy barrier function in fuzziness optimization problems that has a value that grows exponentially with the separation from the viable zone. The fuzzy exponential barrier parameter, a positive decreasing parameter used in this process, establishes how close to the original fuzzy unconstrained problem is. In section 2 of this study, some fundamental ideas of fuzzy set theory and the algebraic operation of triangular fuzzy numbers are provided. A fuzzy exponential barrier technique is used to build an algorithm in section 3 to tackle the problem of fuzzy linear programming problem. Section 4 includes an example.

## 2 Preliminaries

### 2.1 : Fuzzy Set

If  $\widetilde{M} = \{(x, \mu_{\widetilde{M}}(x)) : x \in M, \mu_{\widetilde{M}}(x) \in [0, 1]\}$  defines a fuzzy set  $\widetilde{M}$  The pair  $(x, \mu_{\widetilde{M}}(x))$  that makes up the membership function has two elements:  $x$ , which is a member of the classical set  $\widetilde{M}$ , and the  $\mu_{\widetilde{M}}(x)$  belongs to the interval  $[0, 1]$ .

### 2.2 : Triangular fuzzy number

The triangular fuzzy number is the fuzzy set  $\widetilde{M} = (M_1, M_2, M_3), M_1 \leq M_2 \leq$

*Solving fuzzy linear programming problems by using the fuzzy exponential barrier method*

$M_3 \in R$ , if the membership function of  $\widetilde{M}$  is defined by

$$\mu_{\widetilde{M}}(y) = \begin{cases} \frac{y-M_1}{M_2-M_1}, & M_1 \leq y \leq M_2 \\ \frac{m_3-y}{m_3-M_2}, & M_2 \leq y \leq M_3 \\ 0, & otherwise \end{cases}$$

### 3 : Fuzzy exponential barrier method

Primal-dual of the fuzzy linear programming problems

$$\max \widetilde{Z} = \widetilde{f}^t(\widetilde{y})$$

subject to  $M_j \widetilde{y} \geq \widetilde{N}_k, j = 1, 2$ .

Where  $M_1 \in R^{m \times n}, \widetilde{f}, \widetilde{y} \in R^n, \widetilde{N}_k \in R^m$

$$\widetilde{f}^t = (f_1, f_2, f_3), \widetilde{N} = (n_1, n_2, n_3) \quad (1)$$

$$\min \widetilde{Z} = \widetilde{N}(\widetilde{y})$$

$$M_j^T(\widetilde{x}) \leq \widetilde{f}, M_2 = M_1^T \in R^{m \times n}, \widetilde{N}_k^T, \widetilde{f}^t = \widetilde{f},$$

We may consider the rank of the matrix  $M_1$  without losing generality. The triangular fuzzy number is used to specify the fuzzy components in the preceding problem. Consider that every problem has at least one feasible solution (1)

The fuzzy exponential barrier function is considered by

$\widetilde{E}(\widetilde{y}, \gamma) : R^n \rightarrow R$  for every scalar  $\delta > 0$  as follows.

$$\widetilde{E}(\widetilde{y}, \gamma) = \widetilde{f}^t(\widetilde{y}) - \gamma \sum_{i=1}^m e^{(B_i(\widetilde{y}))^\beta} \quad (2)$$

Consider  $\widetilde{E} : R^n \rightarrow (-\infty, \infty)$  is a function given by

$$B_i(\widetilde{y}) = M_j \widetilde{y} - \widetilde{N}_k \leq 0, M_j \widetilde{y} - \widetilde{N}_k = 0 \text{ for all } j,$$

$$M_j \widetilde{y} - \widetilde{N}_k > 0, M_j \widetilde{y} - \widetilde{N}_k \neq 0 \text{ for all } j.$$

#### 3.1 : Fuzzy exponential barrier lemma

This lemma is based on the local and global behaviour of the unconstrained maximizer of the fuzzy exponential barrier function. Statement: An exponential barrier parameter  $\gamma^k$  is a fuzzy sequence that increased in size. The fuzzy linear programming problems and fuzzy exponential barrier function then for each k condition are below.

$$(i). \widetilde{E}(\widetilde{y}^{k+1}, \gamma^{k+1}) \geq \widetilde{E}(\widetilde{y}^k, \gamma^k)$$

$$(ii). \widetilde{E}(\widetilde{y}^k \leq \widetilde{E}(\widetilde{y}^{k+1})$$

$$(iii). \widetilde{f}^t(\widetilde{y}^k) \geq \widetilde{f}^t(\widetilde{y}^{k+1})$$

$$(iv). \widetilde{f}^t(\widetilde{y}^*) \leq \widetilde{E}(\widetilde{y}^k, \gamma^k) \leq \widetilde{f}^t(\widetilde{y}^k)$$

**Proof :**

$$\begin{aligned}
 (i). \tilde{E}(\tilde{y}^{k+1}, \gamma^{k+1}) &= \tilde{f}^t(\tilde{y}^{k+1}) - \gamma^{k+1} \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^{k+1} - \tilde{N}_k))} \geq \tilde{f}^t(\tilde{y}^k) \\
 &\quad - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \\
 &= \tilde{E}(\tilde{x}^k, \gamma^k)
 \end{aligned}$$

$$(ii). \tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^{k+1} - \tilde{N}_k))} \leq \tilde{f}^t(\tilde{y}^{k+1}) \gamma^{k+1} \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^{k+1} - \tilde{N}_k))} \quad (3)$$

$$\tilde{f}^t(\tilde{y}^{k+1}) - \gamma^{k+1} \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^{k+1} - \tilde{N}_k))} \leq \tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \quad (4)$$

The inequalities adding (3) and (4), we get

$$\begin{aligned}
 &\tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} + \\
 &\tilde{f}^t(\tilde{y}^{k+1}) - \sum_{i=1}^m e^{(\beta^{k+1}(M_j \tilde{y}^k - \tilde{N}_k))} \leq \tilde{f}^t(\tilde{y}^{k+1}) - \sum_{i=1}^m e^{(\beta^{k+1}(M_j \tilde{y}^{k+1} - \tilde{N}_k))} \\
 &\quad + \tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \dots (\tilde{f}^t(\tilde{y}^k) \\
 &\quad + \tilde{f}^t(\tilde{y}^{k+1})) - (\tilde{f}^t(\tilde{y}^{k+1}) \\
 &\quad + (\tilde{f}^t(\tilde{y}^k)) + \gamma^k \sum_{i=1}^m e^{(\beta^k(M_j \tilde{y}^k - \tilde{N}_k))} \\
 &\quad + \gamma^{k+1} \sum_{i=1}^m e^{(\beta^{k+1}(M_j \tilde{y}^k - \tilde{N}_k))} \\
 &\leq \gamma^k \sum_{i=1}^m e^{(\beta^k(M_j \tilde{y}^k - \tilde{N}_k))} \\
 &\quad + \gamma^{k+1} \sum_{i=1}^m e^{(\beta^{k+1}(M_j \tilde{y}^k - \tilde{N}_k))}.
 \end{aligned}$$

However  $(\tilde{f}^t(\tilde{y}^k) + \tilde{f}^t(\tilde{y}^{k+1})) - (\tilde{f}^t(\tilde{x}^{k+1}) + \tilde{f}^t(\tilde{x}^k)) = 0$ .

Then  $(\gamma^k + \gamma^{k+1}) \gamma^k \sum_{i=1}^m e^{(\beta^k(M_j \tilde{y}^k - \tilde{N}_k))} \geq (\gamma^k + \gamma^{k+1}) \sum_{i=1}^m e^{(\beta^{k+1}(M_j \tilde{y}^k - \tilde{N}_k))}$

*Solving fuzzy linear programming problems by using the fuzzy exponential barrier method*

Since  $\delta^k \leq \delta^{k+1}$ ,  $\tilde{E}(\tilde{y}^{k+1}) \geq \tilde{E}(\tilde{y}^k)$ .

(iii). From the proof of (i), it can be obtained that

$$\tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \geq \tilde{f}^t(\tilde{y}^{k+1}) - \gamma^{k+1} \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^{k+1} - \tilde{N}_k))}.$$

$$\tilde{E}(\tilde{y}^k) \leq \tilde{E}(\tilde{y}^{k+1}). \text{ Then } \tilde{f}^t(\tilde{y}^{k+1}) \geq \tilde{f}^t(\tilde{y}^k).$$

(iv). From the proof of (ii)  $\sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \leq \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))}$ ,  $\tilde{f}^t(\tilde{y}^k) \geq \tilde{f}^t(\tilde{y}^{k+1})$ .

$$\tilde{f}^t(\tilde{y}^*) \geq \tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} = \tilde{E}(\tilde{y}^k, \gamma^k) \geq \tilde{f}^t(\tilde{y}^k).$$

### 3.2 : A Fuzzy exponential barrier convergence theorem

Suppose that the fuzzy linear programming problem and the fuzzy exponential barrier functions are both continuous functions. In a fuzzy linear programming problem, an increasing series of positive fuzzy exponential barrier parameters  $k$  is necessary such that  $\{\gamma^k\} \rightarrow \infty, k \rightarrow \infty$  to  $\tilde{y}$  of  $\tilde{z}$  there is an optimal solution. If the limit point  $\tilde{y}$  is within the boundaries of the range of  $\tilde{z}$ , then it exists.

**Proof :**

Let  $\tilde{y}$  be a limit point of  $\{\tilde{y}^k\}$

$$\tilde{E}(\tilde{y}^k, \gamma^k) = \tilde{f}^t(\tilde{y}^k) - \gamma^k \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \leq \tilde{f}^t(\tilde{y}^*)$$

$$\lim_{k \rightarrow \infty} \tilde{f}^t(\tilde{y}^k) = \tilde{f}^t(\tilde{y}), \lim_{k \rightarrow \infty} e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))} \leq 0$$

From the lemma 3.2 (iv) we get,

$$\lim_{k \rightarrow \infty} \tilde{E}(\tilde{y}^k, \gamma^k) = \tilde{f}^t(\tilde{y}^*)$$

$\tilde{y}$  is feasible.

#### 3.2 fuzzy exponential barrier function algorithm

1. Acknowledge the fuzzy objective function and constraints of the problem and rephrase the problem standard forms to reflect them. write  $\max \tilde{z} = \tilde{f}^t(\tilde{y})$ . Subject to  $M_j \tilde{y}^k - \tilde{N}_k \leq 0$ .

2. The fuzzy exponential barrier function defined by

$$\tilde{E}(\tilde{y}, \tilde{\gamma}) = \tilde{f}^t(\tilde{y}) - \gamma \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))}.$$

### 3.3. Maxinequality constraints with the fuzzy exponential barrier functions

$$\max \tilde{E}(\tilde{y}, \tilde{\gamma}) = \tilde{f}^t(\tilde{y}) - \gamma \sum_{i=1}^m e^{(\beta(M_j \tilde{y}^k - \tilde{N}_k))}.$$

4.First-order needed conditions for optimality  $\gamma \rightarrow \infty$ , which are used to find the best solution to the specified fuzzy linear programming problems.

5.Find  $\tilde{E}(\tilde{y}^k, \tilde{\gamma}^k) = \max_{x \geq 0} \tilde{E}(\tilde{y}, \tilde{\gamma})$ , then minimize  $\tilde{y}^k$  and  $\gamma = 1, k = 1, 2, \dots, k = I$  then stop.

Alternatively, can continue to step 5.

The fuzzy exponential barrier approach employs the same procedure as the earlier algorithm for the dual fuzzy linear programming problems.

### 4. Numerical example The primal fuzzy linear programming problem

$$\begin{aligned} \min \tilde{z} &= (3.75, 4, 4.25)\tilde{y}_1 + (2.75, 3, 3.25)\tilde{y}_2 \\ 2\tilde{y}_1 + 3\tilde{y}_2 &\geq (5.75, 6, 6.25), \\ 4\tilde{y}_1 + \tilde{y}_2 &\geq (3.75, 4, 4.25). \end{aligned}$$

**Solution :** Figure 1: Finding an infeasible solution for primal fuzzy linear programming problem Using the fuzzy exponential barrier function algorithm we get, The fuzzy exponential barrier method is given by

$$\tilde{E}(\tilde{y}, \gamma) = ((3.75, 4, 4.25)\tilde{y}_1 + (2.75, 3, 3.25)\tilde{y}_2 - \gamma \sum_{i=1}^m e^{(B_i(y))^\beta}$$

According to this strategy, you can convert fuzzy linear programming problems into a standard form of unconstrained problems.

$$\tilde{y}_1 = (0.45, 0.8, 1.15) - \frac{1.2104}{\gamma}, \tilde{y}_2 = (0.0727, 0.6, 1.1273) - \frac{1.3951}{\gamma}$$

$$\tilde{x}_1 = \frac{0.0060}{\gamma} + (0.35, 0.60, 0.85), \tilde{x}_2 = (0.9518, 1.60, 2.25) + \frac{0.0180}{\gamma},$$

*Solving fuzzy linear programming problems by using the fuzzy exponential barrier method*

For different values of  $\gamma^j, j \rightarrow \infty$  the optimal values of  $\tilde{y}_1(\gamma), \tilde{y}_2(\gamma)$  are calculated as listed in Tables (1) and (2) we get,

Table1: Primal fuzzy linear programming problem solution( $i$ )

| No | $\delta^k$ | $\tilde{y}_1$          | $\tilde{y}_2$           |
|----|------------|------------------------|-------------------------|
| 1  | 10         | (0.3290,0.6790,1.0290) | (-0.0668,0.4605,0.9878) |
| 2  | $10^2$     | (0.4379,0.7879,1.1379) | (0.0587,0.5860,1.1133)  |
| 3  | $10^3$     | (0.4488,0.7988,1.1488) | (0.0713,0.5986,1.1259)  |
| 4  | $10^4$     | (0.4499,0.7999,1.1499) | (0.0726,0.5999,1.1272)  |
| 5  | $10^5$     | (0.45,0.8,1.15)        | (0.0727,0.6,1.1273)     |

Table 2: Dual fuzzy linear programming problem solution( $i$ )

| No | $\gamma^j$ | $\tilde{y}_1$          | $\tilde{y}_2$          |
|----|------------|------------------------|------------------------|
| 1  | 10         | (0.3506,0.6006,0.8506) | (0.9518,1.6018,2.2518) |
| 2  | $10^2$     | (0.3501,0.6001,0.8501) | (0.9502,1.6002,2.2502) |
| 3  | $10^3$     | (0.3500,0.600,0.8500)  | (0.9500,1.600,2.2500)  |
| 4  | $10^4$     | (0.3500,0.600,0.8500)  | (0.9500,1.600,2.2500)  |
| 5  | $10^5$     | (0.3500,0.600,0.8500)  | (0.9500,1.600,2.2500)  |

The optimal value of the given problem (1) can be obtained by

$$\tilde{y}_1 = (0.45, 0.8, 1.15), \tilde{y}_2 = (0.0727, 0.6, 1.1273), \min \tilde{z} = (2.9908, 7.2, 11.4092).$$

The corresponding given problem of the optimal values for  $\tilde{x}_1, \tilde{x}_2$  are to be obtained by

$$\tilde{x}_1 = (0.35, 0.6, 0.85), \tilde{x}_2 = (0.95, 1.6, 2.25), \max \tilde{z} = (4.25, 7.2, 10.15).$$

### 3 Conclusions

This article outlines a method for solving the primal-dual fuzzy linear programming problem more effectively by combining the fuzzy exponential barrier function and the fuzzy exponential barrier parameter. The table for the primal-dual problem demonstrates how quickly the best solution can be reached using the fuzzy exponential barrier primal-dual algorithm we developed when the Fuzzy Exponential barrier parameter is used.



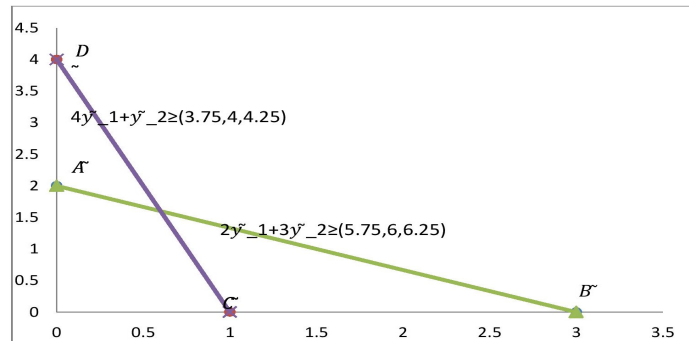


Figure 1: Finding an infeasible solution for primal fuzzy linear programming problem

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*Solving fuzzy linear programming problems by using the fuzzy exponential barrier method*

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# Robust Fuzzy Graph

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## Abstract

The idea of fuzzy graphs is crucial for dealing with uncertainty in day-to-day living situations. In this work, we have a present a work fuzzy graph associates with robustness model that's easy, flexible, and simple. With this model, we have a tendency to address the examination planning downside below uncertainty supported. Randomness and unclearness are two very separate types of uncertainty in knowledge. This study bridges resilient fuzzy graphs and addresses every type of uncertainty in higher cognitive processes. In this paper we introduce a type of fuzzy graph relating with robustness is called robust fuzzy graph and some of its properties. Robust fuzzy graph becomes a best tool in evidence theory for calculating belief functions, plausibility functions, spanning functions etc.

**Keywords:** Robust; fuzzy graph; Belief measure; Plausibility measure; spanning.

**2020 AMS subject classifications:** 93B51, 03E72, 15A09, 58J52. <sup>1</sup>

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<sup>1</sup>Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1089. ISSN: 1592-7415. eISSN: 2282-8214. ©A.Sudha et al. This paper is published under the CC-BY licence agreement.

## 1 Introduction

Graphs are commonly thought to be nothing more than relational representations. A graph can be used to express information about item connections. Edges represent connections, whereas vertices represent things. It is common to need to develop a "fuzzy graph model" when the description of an object, its relationships, or both are imprecise.

Particularly in the fields of cluster analysis, neural networks, computer networks, pattern recognition, decision making, and professional systems, the use of fuzzy connections is pervasive and essential. Every one of them has a fuzzy graph as its fundamental mathematical structure.

A graph is a symmetric binary relation on a nonempty set  $V$ , as we are well aware. A fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. In 1973, A.Kaufmann [1965] initially proposed the idea of a fuzzy graph, which was based on Zadeh's fuzzy relations Zadeh [1980], Zadeh [1968]. Goguen is recognised with creating the idea of fuzzy graphs and taking into account fuzzy relations on fuzzy sets in 1968, however. Ravi.J et.al.,Ravi [2022] discovered several connection notions in fuzzy graphs at the same time.

There are few high-quality papers on fuzzy graphs. According to our data, a strength-associated degreed fuzzy graph and a degreed fuzzy graph have never been combined in a programming task. The works M. Blue and Puckett [2002] and D.E.Goldberg [1989], address programming flaws A.Lim and Wang [2005], Sunitha [2001], Sunitha and Vijayakumar [2002], Wang and Xu [2013] and Y´añez and Ramirez [2003] contain works on fuzzy graph colouring. The structure of the paper is presented below. The section 2 briefly reviews a few essential definitions of fuzzy graphs. Section 3 describes the resilient fuzzy graph model and Section 4 concludes our discussion.

## 2 Preliminaries

**Definition 2.1.** *The two functions:  $Q \rightarrow [0,1]$  and  $: Q \times Q \rightarrow [0,1]$  and a non-empty set  $V$ . such that the fuzzy graph  $G = (Q)$  is formed for each  $x, y$  in  $V$  by  $\rho(.) \leq \mu(.) \wedge \mu(.)$ . Fuzzy edge set and fuzzy vertex set for  $G$  are represented by and, respectively.*

**Definition 2.2.** *Let  $G = (\mu, \rho)$  to have  $\rho^* = (u, v, w)$  and  $x$ . Let  $(u, v) = 0.2$ ,  $(v, w) = 0.2$ ,  $(w, x) = 0.3$ ,  $(x, u) = 0.5$ , and  $(u, w) = 0.4$ .*

*Due to the presence of the two weakest arcs in  $G$ , arcs  $(u, v)$  and  $(v, w)$ ,  $C1 = u, v, w, x, u$  is a fuzzy cycle, but  $C2 = u, w, x, u$  is not..*

**Definition 2.3.** *Assume that the graph  $G=(\mu, \rho)$  has more fuzziness. The word*

## Robust fuzzy graph

"connectedness strength" refers to the total strength of all paths linking two vertices,  $x$  and  $y$ , and is denoted by the sign  $CONN_G(.)$ . The strongest  $x$ - $y$  route is one in which  $P = CONN_G(.)$ .

**Definition 2.4.** This is a definition of a mathematical object. If  $CONN_G(.) > 0$  for each pair of  $(.)$  in  $\rho^*$ , an  $f$ -graph  $G = (.)$  is connected. When  $G$  is not present, components are maximally connected fuzzy graphs. If  $G$  is linked, any two vertices can be connected by a route. The arc  $(x,y)$  of a fuzzy graph  $G$  is considered normal if and only if  $(.) = CONN_G(.) > 0$ .

**Definition 2.5.** Based on judgments of plausibility and belief, the evidence hypothesis is developed. The function  $Bel.: (.) \rightarrow [0,1]$  represents a belief measure for a finite universal set  $X$  such that  $Bel(.) = 0$ ,  $Bel(.) = 1$ , and  $Bel(.)_j - jk Bel(.)_j - jk Bel(.)_j + \dots + (-1)^{n+1} j Bel(.) \subseteq X$ . According to the data that is now available,  $Bel$  understands for each  $A(.)$  the degree of confidence that a certain member of  $(.)$  belongs to the set  $A$ . Belief measures are superadditive when  $(.)$  is infinite and continuous above..

**Definition 2.6.** A plausibility measure is a function  $P_I.: (.) \rightarrow [0,1]$  such that  $P_I(.) = 0$ ,  $P_I(.) = 1$ , and  $P_I(.) \leq \sum (.) - \sum < (.) \cup (.) + \dots + (-1)^{n+1} \sum (.) \subseteq X$ .

**Definition 2.7.** A function  $m: (.) \rightarrow [0,1]$  that has the properties  $m(.) = 0$  and  $m(.) = 1$  characterises belief and plausibility measurements. The term "basic probability assignment" refers to this function..

The value  $m(.)$  for each  $A$  indicates the strength of the argument that a specific member of  $(.)$ , whose categorization in terms of relevant attributes is lacking, belongs to the set  $A(.)$ .

## 3 Robust fuzzy graph (RFG)

The  $k^{th}$  regular super fuzzy matrix is examined in this section.

**Definition 3.1.** Let  $\{X\}$  and  $A \subseteq Q \neq P(.)$  and two functions  $m: Q \rightarrow [0,1]$  and  $: Q \times Q \rightarrow [0,1]$   $A, B \in Q$ ,  $(.) \in \delta$ , whenever  $A \subseteq B$  and  $(.) = m(.)$   $m$  are present, constitute a resilient fuzzy graph  $(.)$ . Also  $\sum_{. \in Q} (.) = 1$ . RFG is represented by the formula  $G = (.)$  where  $m$  is referred to as the assignment function and is referred to as the edge function.

**Definition 3.2.** If  $n(.) = m(.)$  for all  $A$  and  $(.) = (.)$  for all  $(.)$  such that the FG,  $H = (.)$  is referred to as a partial RFG of  $G = (.)$ . If  $P \forall$ ,  $n(.) = m(.)$ ,  $(.) = (.)$  for any  $(.)$  such that the FG,  $H = (.)$  is referred to as a RFG of  $G = (.)$  induced by  $P$ .

**Definition 3.3.** The vertex in a RFG which is adjacent from every other vertex is called complete vertex.

**Definition 3.4.** Consider the robust fuzzy matrix  $MG=(.)$  of  $G=(.)$ , where  $mAB=(.,.) \subseteq (.,.)$   $0 \leq h$ . The matrix  $M^{G^k}$  such that  $M^{G^k} = M^{G^{k+1}}$ , where  $k$  is a positive integer is called the evidence reachability matrix of  $G$  denoted by  $R_G=(r_{AB})$ .

**Definition 3.5.** Two vertices  $A$  and  $B$  of a RFG is said to be mutually disconnected if there is neither an edge  $(.,.)$  nor  $(.,.)$ . The vertices  $A$  and  $B$  are mutually disconnected. The Belief and plausibility. The measures of a complete vertex are always one.

**Theorem 3.1.** Fuzzy graphs (FG) include robust fuzzy graphs (RFG).

**Proof.** Proof is evident from the definition of RFG.

**Theorem 3.2.** Number of vertices of a RFG corresponding to a crispest  $X$  with  $n$  elements is,  $2n-1$ .

**Proof.** Let  $G=(m,\delta)$  be the RFG. Then by the definition,  $V=\emptyset \setminus \psi$  is the vertex set and vertices is  $2n-1$ .

**Theorem 3.3.** Number of edges of a robust fuzzy graph corresponding to a crisp set with  $n$  elements is  $(\sum^{-1}(-1))_{-1} + (\sum^{-2}(-2))_{-2} + \dots + n$ .

**Proof.** Proof is the consequence of set theory.

Let us start with singleton sets. Every vertex corresponding to singleton sets is adjacent to all the vertices corresponding to their supersets - 2-element sets, 3-element sets etc.

$\{.\} \rightarrow \{1,m\}, \{1,m,n\}, \{1,m,n,o\}, \dots$

This can be done in  $\sum^{-1}(-1) = nCn-1$

So the total cases corresponding to singleton sets is  $(\sum^{-1}(-1))_{-1}$

The vertices corresponding to 2-element sets is adjacent to all vertices corresponding to all vertices corresponding to their super sets - 3-element sets, 4-element sets etc.

$\{.\} \rightarrow \{1,m\}, \{1,m,n\}, \{1,m,n,o\}, \dots$   $\square$

**Theorem 3.4.** Robust fuzzy graph is complete.

**Proof.** A complete fuzzy graph (CFG) is a FG,  $G=(.)$  such that  $\delta(.,.)=m(.,.) \wedge m(.,.)$  for all  $(.,.)$ . So by definition every RFG is complete.  $\square$

**Theorem 3.5.** There does not exist an edge  $(.,.)$ , such that  $\delta(.,.)=1$  in a RFG,  $G=(.,.)$ .

**Proof.** If possible, suppose that there exist an edge  $(.)$  such that  $\delta(.) = 1$   
 $\Rightarrow m(.) = m(.) = 1$  by the definition of RFG.  
 $\Rightarrow \sum(.) \neq 1$ , a contradiction.  $\square$

**Theorem 3.6.** *The partial RFs and RFs of an RFG need not be a RFG*

**Proof.** For a partial RFG and RFG,  $\sum(.)$  equal to 1.  
 But  $\sum(.) \leq 1$ .  $\square$

**Theorem 3.7.** *The partial RFs of an RFG is an RFG if and only if  $m(.) = n(.)$  for all A and  $\tau(.,.) = \delta(.,.)$  for all  $(.)$ .*

**Proof.** By definition  $\sum(.) = 1$ .  
 Let  $H=(.)$  be a partial RFs of G.  
 For a partial RFs,  $H=(n, \tau)$  of  $G=(.,)$ ,  $n(.) \leq m(.)$  for all A and  $\tau(.,.) \leq \delta(.,.)$ .  
 But  $\sum(.) \neq 1$  if  $n(.) < m(.)$ . So  $n(.) = m(.)$  for all A which implies  $\tau(.,.) = \delta(.,.)$  for all  $(.,.)$ .  
 Converse is obvious.  $\square$

**Theorem 3.8.** *Maximum length of a path P in a RFG with n vertices is n-1.*

**Proof.** Consider a RFG with n vertices  $A_0, A_1, \dots, A_n$ . Start from an arbitrary vertex  $A_i$ . Since there are only n-1 vertices remaining, choose a vertex  $A_j$  such that  $\rho(A_i, A_j) > 0$ ,  $A_i \subseteq A_j$ ,  $i \neq j$ . Similarly choose a vertex  $A_r$  from the remaining n-2 such that  $\rho(A_j, A_r) > 0$ ,  $A_j \subseteq A_r$ , and so on. Since there are only n distinct vertices the process must terminate at a vertex  $A_p$  such that,  $\rho(A_{p-1}, A_p) > 0$ ,  $A_{p-1} \subseteq A_p$ ,  $p < n$ .  
 We get the sequence  $A_i, A_j, \dots, A_p$  which is of length less than n and equal to n-1 only if every vertex is ordered by the relation  $\subseteq$ .  $\square$

**Theorem 3.9.** *RFG does not contain cycles and so fuzzy cycles.*

**Proof.** Since in a RFG  $G = (.,)$ , for all  $(.,) \in V$ ,  $(.,) \in \delta$  whenever  $(.) \subseteq (.)$  there will not be an edge  $(.,.)$  and so a cycle.  $\square$

**Theorem 3.10.** *The RFG is always disconnected.*

**Proof.** For  $X = \{.,.\}$  there does not exist a path between x and y.  $\square$

## 4 Conclusions

In this paper we introduce new type of fuzzy graph called robust fuzzy graph. We find some of its properties like completeness, paths, connectivity etc. We also present fuzzy graph's application in robust theory for finding belief measure

plausibility measure etc in uncertain situations. We can calculate belief measure using RFG as  $\text{Bel}(\cdot) = m(\cdot) + \sum (\cdot)$ ,  $(\cdot, \cdot)$  is an edge; where  $m(\cdot)$  is the degree of the vertex  $(\cdot)$  in RFG. Our proposed method RFG perform is well even it is uncertainty situation. In future this concept is applying in computer vision concept.

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TABLE OF CONTENTS

SI: ADVANCES IN MATHEMATICS FOR SCIENCE AND ENGINEERING

|                                                 |                                                              |         |
|-------------------------------------------------|--------------------------------------------------------------|---------|
| Verma                                           | INFLUENCE OF CHEMICAL REACTION, RADIATION AND HEAT SOURCE... | 7-23    |
| Rehman A, Ismayil                               | TRANSVERSAL ECCENTRIC DOMINATION IN GRAPHS                   | 24-33   |
| A R, Jose                                       | SOME COUPLED FIXED POINT THEOREMS IN COMPLETE E-FUZZY...     | 34-43   |
| S, G                                            | PRIME LABELING OF H- SUPER SUBDIVISION OF Y-TREE RELATED...  | 54-62   |
| S, G                                            | SOME NEW ODD PRIME GRAPHS                                    | 44-53   |
| J, V                                            | IFG#A-CS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES          | 63-69   |
| G, S                                            | ODD PRIME LABELING FOR SOME ARROW RELATED GRAPHS             | 70-78   |
| N, A                                            | SOME OPERATIONS ON COMPLEX FUZZY GRAPHS                      | 79-90   |
| K D, Kishore                                    | DEGREE OF AN EDGE AND PLATT NUMBER IN SIGNED NETWORKS        | 91-100  |
| G, A                                            | ALGEBRAIC CODING THEORY USING PELL EQUATION $X^2 - 8Y^2 = 1$ | 101-108 |
| R, M                                            | BOUNDS ON FUZZY DOMINATOR CHROMATIC NUMBER OF FUZZY SOFT...  | 109-116 |
| M, M                                            | ALGORITHM APPROACHES FOR SHORTEST PATH PROBLEM IN AN...      | 117-126 |
| Jabarulla Mohamed, Rajamanickam                 | STRONG INTERVAL – VALUED PYTHAGOREAN FUZZY SOFT GRAPHS       | 127-137 |
| Ramkumar, Sridevi                               | DOMINATION IN M– POLAR SOFT FUZZY GRAPHS                     | 138-146 |
| Jahir Hussain, Palaniyandi                      | ANTI-HOMOMORPHISM IN Q-FUZZY SUBGROUPS AND NORMAL SUBGROUPS  | 147-155 |
| Gnana Priya, Sabarmathi, Naga Soundarya Lakshmi | A DYNAMICAL ANALYSIS OF A MATHEMATICAL MODEL ON TYPE-2...    | 156-171 |
| Palaniyandi, Jahir Hussain                      | ANTI Q-M-FUZZY NORMAL SUBGROUPS                              | 172-177 |
| Sivasamy, Ramesh Kumar                          | THEORETICAL ANALYSIS ON THE GROWTH KINETICS OF SARS-COV...   | 178-193 |
| Padmavathi                                      | PROPERTIES OF NANO GENERALIZED PRE C-INTERIOR IN A NANO...   | 194-201 |
| Myithili, Keerthika                             | TRANSVERSAL CORE OF INTUITIONISTIC FUZZY K-PARTITE...        | 202-212 |
| Lekha, Parvathy                                 | ON FZ- DOMINATION NUMBER OF FUZZY GRAPHS                     | 213-226 |
| Deepa, Sundararajan                             | SUPER FUZZY MATRIX OF INVERSE IN KTH ORDER                   | 227-235 |
| Senthamizh Selvan, Jahir Hussain                | SOME RESULTS ON RANGE LABELING                               | 236-243 |
| Kuriakose, Parvathy                             | STABILITY OF DOMINATION IN GRAPHS                            | 244-256 |
| Tejaskumar, Mohamed Ismayil                     | SUPERIOR ECCENTRIC DOMINATION POLYNOMIAL                     | 257-270 |
| Vinu Priya, Jothilakshmi                        | LINEAR PREDICTOR AND AUTOCORRELATION FOR NOISY AND...        | 271-279 |
| Shakila, Khadar Babu                            | WHEAT CROP YIELD FORECASTING USING VARIOUS REGRESSION MODELS | 280-290 |
| Sumathi, Jothilakshmi                           | BIBO STABILITY AND DECOMPOSITION ANALYSIS OF SIGNALS AND...  | 291-297 |
| Revathi, Sundararajan                           | INTUITIONISTIC ROBUST FUZZY MATRIX FOR THE DIAGNOSIS OF...   | 298-309 |
| Nagoor gani, Yogarani                           | SOLVING FUZZY LINEAR PROGRAMMING PROBLEMS BY USING THE...    | 310-318 |
| Sudha, Sundararajan                             | ROBUST FUZZY GRAPH                                           | 319-324 |