

ISSN 1592-7415

eISSN 2282-8214

Volume n. 43 – 2022

RATIO MATHEMATICA

Journal of Mathematics, Statistics, and Applications

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A.P.A.V.

Accademia Piceno – Aprutina dei Velati in Teramo, Italy

RATIO MATHEMATICA
Journal of Mathematics, Statistics, and Applications

ISSN 1592-7415 - eISSN 2282-8214

Ratio Mathematica is an International, double peer-reviewed, open-access journal published every six months (June-December).

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Publisher: APAV - Accademia Piceno-Aprutina dei Velati in Teramo, Italy

Tax Code 92036140678, **Vat Id. Number** 02184450688

Registered office: Contrada Salice, 28 - 87023 - Diamante (CS)

Operational headquarters: c / o Hotel Cristina - via Pietrarossa, 24 - Diamante (CS)

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Analysis of Finite Population Stochastic modeling with State-Dependent Arrival and Service Facilities

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Abstract

This paper investigates a stock-dependent arrival process(SDAP) and queue-dependent service process(QDSP) in the stochastic queueing-inventory system(SQIS). The arriving units in the system generated from the finite source population. The arrival process holds the properties of quasi-random process and its intensity rate is defined based on the two-component demand rate(TCDR). The customers departure time is exponentially distributed. The concepts of non-SDAP and SDAP, non-QDSP and QDSP are to be generalized. The inventory system may have the perishable quality of the products. It adopts the (s, Q) reordering policy whenever the replenishment is required. Further, the join probability distribution of a Markov process is derived and necessary system performance measures are computed. The comparative discussion is presented to improve the quality of this model.

Keywords: Stock-dependent arrival process; Non-stock dependent arrival process; Two component demand rate; Finite population; queue-dependent service process

2020 AMS subject classifications: 90C15, 60G07. ¹

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¹Received on February 9, 2022. Accepted on December 1, 2022. Published on December 30, 2022. doi:

1 Introduction

As the continuous increment in the population growth, the demand of the people such as foods, cloths and public service etc are also skyrocketed in this century. So the analysis of queueing-inventory management is inevitable in this current situation, especially in that food items related queueing-inventory systems. The results can be applied effectively in queue and inventory management systems and the optimum cost will increase the economy also. This model consists of a perishable inventory system with queue-dependent service rate and customers from a finite source. Due to the decay of the food items after some certain period, its must to reorder it at some fixed inventory level to avoid economical loss. The arrival rate of customer is dependent on the number of items available in the shops. Also service rate is dependent on the mood of server. The following illustration will give the exact model idea.

In the shopping malls and theaters, some stores are used to sell snack items for the customers. The possibility of customer occurrence is dependent on inventory level and also service rate can be dependent on the queue length. Here the population size is finite, since the only possible customers to purchase the snacks are from theater or the mall. Though there are many shops are available, most of the customers are interested to purchase from the shop where many items are available comparing to the shops where less items are available. Here the customer arrival depends on the inventory level. When a small queue is formed in the shop, server may serve slowly, speaking with someone or using mobile phone. Also server would provide a quick service if the queue length is big, in order to decrease the customer loss. Though the formed queue is bigger, customer won't leave the queue as the service rate is high. This realistic situation motivates the author to develop the proposed model. In this model, stocks are replenished according to (s, Q) policy.

2 Literature review

As many researchers show their interest of research on the Stochastic Queueing Inventory modeling, this area has developed enormously in certain period. Queueing inventory systems with retrial is in-fusible in all walk of life. Reshmi and Jose studied the Queueing inventory model, considering perishable items and customer retrial on Reshmi and Jose [2019]. Periyasamy considered a finite population perishable inventory system where server is looking for the customers from the orbit to provide service after completing service to each primary customer on Periyasamy [2017]. Berman and Sapna Berman and Sapna [2002] discussed the rate of optimal service with perishable inventory in which instantaneous reordering policy was assumed. Considering the negative

exponential rate for the life time of stocks, Kalpagam and Arivagam Kalpakam and Arivarignan [1988] analyzed the (s, S) inventory system in which stock one is evicted from the inventory whenever the demand or failure of item occurs. Sangeetha investigated the production optimal control of production time of perishable inventory system with finite source in order to get the minimal total cost on N. Sangeetha and Arivarignan [2015].

Alfred introduced the concept of occurring demand rate depends upon the stock level in the inventory system on Alfares [2007] and determined the total cost by variable holding cost assuming holding cost per unit item to be a monotonically increasing function of spending time in the storage. Diana Tom Varghese and Dhanya Shajin Varghese and Shajin [2018] studied the state dependent demand on the continuous review M/M/1/S inventory model. K. Venkata Subbaiah et al. K. Venkata Subbaiah and Satyanarayana [2004] developed the perishable inventory model with stock dependent demand rate. Rathod and Bhathawala Rathod and Bhathawala [2013] analyzed the inventory system with stock dependent demand having variable holding cost and shortages. The effect of demand rate depending on stock level was discussed through the proposed logistical growth model of Tsoularis Tsoularis [2014]. A shortage free inventory model with stock dependent demand was analyzed by Datta and Pal on Datta and Pal [1990]. Sudhir Kumar Sahu et al. Sudhir Kumar Sahu and Sahoo [2008] developed an inventory system with stock dependent demand rate and constant deterioration with the possibilities of partial or complete backlog and without it. Shib Sankar Sana Sana and Sankar [2010] proposed an EOQ model for the perishable inventory item with discount rate and the demand depending on stock level. Mandal Mandal and S. [1989] derived an inventory system with consumption rate depending on stock level.

For analyzing the local area management, Falin and Artalejo Falin and Artalejo [1998] proposed a retrial queue with finite source customer. Shophia Lawrence et al. A. Shophia Lawrence and Arivarignan [2013] discussed the perishable queueing-inventory system with demands from finite homogeneous source. Attahiru Sule Alfaa and Sapna Isotupa ? discussed an M/PH/k retrial queue with the finite source. K. Jeganathan K. Jeganathan and Vigneshwaran [2015] analyzed the perishable inventory system with the possibility of server interruption and the multiple server vacation and customer is provided service only when customer level reaches to a particular N and no customer is left behind the system after service started. Jeganathan Jeganathan [2015] discussed finite source inventory system with an additional service for some customers which is called bonus service. Artalejo and Lopez-Herrero investigated retrial queue involving finite population with an BSDE approach on Artalejo and Lopez-Herrero [2012]. Sivakumar analyzed the perishable inventory system with retrial demand from finite source without service on Sivakumar [2009].

Shanthikumar and Yao Shanthikumar and Yao [1988] studied the upper and lower bounds on a closed queuing network with the queue dependent service rate. Menich ronald Ronald [1987] derived the optimal of shortest queue routing to the queue depen-

dent service station considering a general Markovian system. Avhishek Chatterjee et al. Avhishek Chatterjee and varshney lav [2017] studied the information-theoretic limit of reliable information processing using queue dependent service facility. Jeganathan et al.[2021] proposed a finite inventory single server system and analyzed the queue dependent service rate.

Though the large number of researches have been done in this area, there is a research gap in analyzing stock dependent demand rate on the finite source queueing inventory system with the queue dependent service rate and retrial customer. As the demand rate depending upon stock level on the inventory and service rate depends on the queue length, this model simulates the realistic situation.

3 Model Developing

3.1 Mathematical Formulation of the model

This model deals the state dependent arrival and queue dependent service processes in a single server Markovian queueing-inventory system(SSMQIS) in a finite source environment. The system holds maximum of S units of inventory product in its storage place. It allows the customers to buy the product from a finite source, N only. It admits the arriving customers into the finite waiting hall of size N . There is only two possible choice of a customer such that they must be either free or in the waiting hall at any time.

The appearance of arrival process generates a output process called quasi-random process; that is, the probability that any particular customer generates a request for demand in any interval $(t, t + dt)$ is $\theta_r dt + o(dt)$ ($r \in B_0^S$) as $dt \rightarrow 0$ if the customer is free at time t and zero if the customer is in waiting hall at time t , independently of the behavior of any other customers. The arrival process of any individual customer is non-homogeneous, since the generation of arrivals must dependent upon the current stock level of the system. This non-homogeneous arrival streams come under the category of state dependent arrival stream. Next, the service pattern is processed following a first come and first serve(FCFS) service discipline. The service time of any customer at time t is non-homogeneous and exponentially distributed. That is, μ_s ($s \in B_1^N$) is the service rate of an individual at any time. This service process comes under the category of state dependent service processes. After each service completion, there will be one unit dropped in the storage place.

The stored products in the system does not have any guarantee about its life time till it will be sold. It may have the deteriorating quality. So this deterioration process follows exponential distribution and have the intensity rate $r\alpha_1$ where $r \in B_1^S$. The service and deterioration processes cause the depletion of an inventory product unit by unit. At one fine stage, the current number of product in the storage system will reach the predetermined value s . As and when the maximum inventory level reduced to s or

less than s , then the replenishment process will be triggered immediately. Each time there are $Q = (S - s)$ items will be replaced whenever the reorder required. This policy is known as (s, Q) reordering policy and this processing time is exponentially distributed with an intensity rate α .

The defined arrival and service rates are ordered, $\theta_0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_S$ and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N$ as an increasing manner. When the case $\theta_1 = \theta_2 = \dots = \theta_S = \theta$, the considered model comes under the category of two component demand rate. That is the arrival rate is homogeneous in the positive stock period and during the stock out period, it is θ_0 . When the case $\mu_1 = \mu_2 = \dots = \mu_N = \mu$ means that the service rate become homogeneous.

- Remark 3.1.**
- For a numerical computation θ_r can be defined by $\theta_r r^{\beta_1}$, $0 < \beta_1 \leq 1$ and $r \in B_1^S$.
 - For a numerical computation μ_s can be defined by $\mu_s s^{\beta_2}$, $0 \leq \beta_2 \leq 1$ and $s \in B_1^N$.
 - The case $\beta_1 = 0$ and $\beta_2 = 0$ explores the result of non-stock dependent arrival process and non-queue dependent service process of the proposed model.

4 Analytical Discussion of the Model

Let $\{ (R_1(t), R_2(t)) ; t \geq 0 \}$ be a stochastic process having state space $\{ (r_1, r_2) : r_1 \in B_0^S \text{ and } r_2 \in B_0^N \}$ satisfies the Markov process, where $R_1(t)$ denotes the level of inventory at time t and $R_2(t)$ denotes the number of customers in the orbit at time t . The transition from any state (r_1, r_2) to other state (r'_1, r'_2) at any interval is denoted by $P((r_1, r_2), (r'_1, r'_2))$. Any y items in the inventory perish alone at the rate of $r_1 \gamma$ and the occurrence of primary demand is $(N - r_2) \theta_{r_1}$ from any one of the sources $(N - r_2)$. Hence, the probability of transition is

$P((r_1, r_2), (r_1 - 1, r_2)) = r_1 \alpha_1$ $r_1 \in B_1^S$ and $r_2 \in B_0^K$. Since the service rate is queue dependent service rate $P((r_1, r_2), (r_1 - 1, r_2 - 1)) = \mu_{r_2}$ where $r_1 \in B_1^S$ and $r_2 \in B_1^K$. If the arrival rate is dependent on inventory, arriving customers enter into the waiting hall. So the probability of the transition from the state (r_1, r_2) to the state $(r_1, r_2 + 1)$ is $P((r_1, r_2), (r_1, r_2 + 1)) = (K - r_2) \theta_{r_1}$ where $r_1 \in B_0^S$, $r_2 \in B_0^{K-1}$. When Q items are ordered, the probability of transition from the state (r_1, r_2) to state $(r_1 + Q, r_2)$ for all r_2 and $r_1 \in B_0^S$ is given by $P((r_1, r_2), (r_1 + Q, z)) = \alpha$. The rate of other transitions is zero. The sum of each row of this matrix should be zero. Hence, the diagonal entry is multiplied by a negative sign after summing all the entries from the row.

All the possible transitions are given below.

$$R((r_1, r_2), (r'_1, r'_2)) = \begin{cases} r_1 \alpha_1, & r'_1 = r_1 - 1, r_1 \in B_1^S, \\ \mu_{r_2}, & r'_2 = r_2, r_2 \in B_0^K, \\ (K - r_2) \theta_{r_1}, & r'_1 = r_1 - 1, r_1 \in B_1^S, \\ & r'_2 = r_2 - 1, r_2 \in B_1^K, \\ \alpha, & r'_1 = r_1, r_1 \in B_0^S, \\ & r'_2 = r_2 + 1, r_1 \in B_0^{K-1}, \\ & r'_1 = r_1 + Q, r_1 \in B_0^s, \\ & r'_2 = r_2, r_2 \in B_0^K, \\ -(\bar{\delta}_{K,r_2}(K - r_2) \theta_{r_1} + \alpha), & r'_1 = r_1, r_1 \in B_0^0, \\ & r'_2 = r_2, r_2 \in B_0^K, \\ -(\bar{\delta}_{K,r_2}(K - r_2) \theta_{r_1} + \alpha + \bar{\delta}_{0,r_2} \mu_{r_2} + r_1 \alpha_1), & r'_1 = r_1, r_1 \in B_1^s, \\ & r'_2 = r_2, r_2 \in B_0^K, \\ -(\bar{\delta}_{K,r_2}(K - r_2) \theta_{r_1} + \bar{\delta}_{0,r_2} \mu_{r_2} + r_1 \alpha_1), & r'_1 = r_1, r_1 \in B_{s+1}^S, \\ & r'_2 = r_2, r_2 \in B_0^K, \\ 0, & \text{Otherwise.} \end{cases}$$

The block partitioned matrices of the proposed model is structured as follows:

$$R = \begin{cases} L_y, & r'_1 = r_1, \quad r_1 \in B_0^S, \\ M_y, & r'_1 = r_1 - 1, \quad r_1 \in B_1^S, \\ N, & y' = Q + r_1 \quad r_1 \in B_0^s, \\ 0, & \text{Otherwise.} \end{cases}$$

For $r_1 \in B_1^S$,

$$M_{r_1} = \begin{cases} r_1 \alpha_1, & r'_2 = r_2, \quad r_2 \in B_0^K, \\ \mu_{r_2} & r'_2 = r_2 - 1, \quad r_2 \in B_1^K, \\ 0, & \text{Otherwise.} \end{cases}$$

For $r_1 \in B_0^s$,

$$N = \begin{cases} \alpha, & r'_2 = r_2, r_2 \in B_0^K, \\ 0, & \text{Otherwise.} \end{cases}$$

For $r_1 = 0$,

$$L_{r_1} = \begin{cases} (K - r_2) \theta_0, & r'_2 = r_2 + 1, \quad r_2 \in B_0^{K-1}, \\ -((K - r_2) \theta_0 + \alpha), & r'_2 = r_2, \quad r_2 \in B_0^{K-1}, \\ -\alpha, & r'_2 = r_2, \quad r_2 \in B_K^K, \\ 0, & \text{Otherwise.} \end{cases}$$

For $r_1 \in B_1^s$,

$$L_{r_1} = \begin{cases} (K - r_2)\theta_{r_1}, & r'_2 = r_2 + 1, \quad r_2 \in B_0^{K-1}, \\ -(\bar{\delta}_{K,r_2}(K - r_2)\theta_{r_1} + \alpha + \bar{\delta}_{0,r_2}\mu_{r_2} + r_1\alpha_1), & r'_2 = r_2, \quad r_2 \in B_0^K, \\ 0, & \text{Otherwise.} \end{cases}$$

For $r_1 \in B_{s+1}^S$,

$$L_{r_1} = \begin{cases} (K - r_2)\theta_{r_1}, & r'_2 = r_2 + 1, \quad r_2 \in B_0^{K-1}, \\ -(\bar{\delta}_{K,r_2}(K - r_2)\theta_{r_1} + \bar{\delta}_{0,r_2}\mu_{r_2} + r_1\alpha_1), & r'_2 = r_2, \quad r_2 \in B_0^K, \\ 0, & \text{Otherwise.} \end{cases}$$

4.1 Steady state analysis

The structure of the homogeneous Markov process $\{(R_1(t), R_2(t); t \geq 0)\}$ with finite state space indicates that it is irreducible. Hence, the limiting distribution is

$$\xi^{(r_1, r_2)} = \lim_{t \rightarrow \infty} Pr \{(R_1(t) = r_1, R_2(t) = r_2) | (R_1(0), R_2(0))\}$$

$$\text{Let } \xi = (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(S)})$$

where each $\xi^{(r_1)} = (\xi^{(r_1,0)}, \xi^{(r_1,1)}, \dots, \xi^{(r_1,K)})$ for $r_1 \in B_0^S$ which satisfies

$$\xi P = \mathbf{0} \text{ and } \xi \mathbf{e} = 1 \quad (1)$$

From the above we get the following equation

$$\xi^{(r_1)} L_{r_1} + \xi^{(r_1+1)} M_{r_1+1} = \mathbf{0} \quad r_1 \in B_0^{Q-1}, \quad (2)$$

$$\xi^{(r_1)} L_{r_1} + \xi^{(r_1+1)} M_{r_1+1} + \xi^{(r_1-Q)} N = \mathbf{0} \quad r_1 = Q \quad (3)$$

$$\xi^{(r_1)} L_{r_1} + \xi^{(r_1+1)} M_{r_1+1} + \xi^{(r_1-Q)} N = \mathbf{0} \quad r_1 \in B_{Q+1}^{S-1}, \quad (4)$$

$$\xi^{(r_1)} L_{r_1} + \xi^{(r_1-Q)} N = \mathbf{0} \quad r_1 = S \quad (5)$$

Except the $r_1 = Q$ case, solving other equations recursively, we get,

$$\xi^{(r_1)} = \xi^{(Q)} \Delta_{r_1}, \quad r_1 \in B_0^S,$$

where

$$\Delta_i = \begin{cases} (-1)^{(Q-r_1)} (M_Q M_{Q-1} \dots M_{r_1+1}) (L_{Q-1}^{-1} L_{Q-2}^{-1} \dots L_{r_1}^{-1}), & r_1 \in B_0^{Q-1}, \\ I & r_1 = Q, \\ (-1)^{2Q+1-r_1} \sum_{j=0}^{C-r_1} (M_Q M_{Q-1} \dots M_{s+1-j}) (L_{Q-1}^{-1} L_{Q-2}^{-1} \dots L_{s-j}^{-1}) N L_{S-j}^{-1} & \\ (M_{S-j} M_{S-j-1} \dots M_{r_1+1}) (L_{S-j-1}^{-1} L_{S-j-2}^{-1} \dots L_{r_1}^{-1}) & r_1 \in B_{Q+1}^S, \end{cases}$$

$\xi^{(Q)}$ can be yield solving

$$\xi^{(Q)} \left[(-1)^{2Q+1-r_1} \sum_{j=0}^{S-r_1} \left[(M_Q L_{Q-1}^{-1} M_{Q-1} \dots M_{s+1-j} L_{s-j}^{-1}) N L_{S-j}^{-1} (M_{S-j} L_{S-j-1}^{-1} M_{S-j-1} \dots \dots M_{Q+2} L_{Q+1}^{-1}) \right] M_{Q+1} + L_Q + (-1)^Q M_Q L_{Q-1}^{-1} M_{Q-1} \dots M_1 L_0^{-1} N \right] = \mathbf{0}$$

$$\xi^{(Q)} \left[\sum_{r_1=Q+1}^S \left((-1)^{2Q-r_1+1} \sum_{j=0}^{S-r_1} \left[(M_Q L_{Q-1}^{-1} M_{Q-1} \dots M_{s+1-j} L_{s-j}^{-1}) N L_{S-j}^{-1} (M_{S-j} L_{S-j-1}^{-1} M_{S-j-1} \dots M_{r_1+1} L_{r_1}^{-1}) \right] M_{Q+1} \right) + \sum_{r_1=0}^{Q-1} ((-1)^{Q-r_1} M_Q L_{Q-1}^{-1} \dots M_{r_1+1} L_{r_1}^{-1}) + I \right] \mathbf{e} = 1.$$

5 System Performance Measures

To make a detailed investigation of the proposed model, some significant system characteristics are to be computed as follows:

1. Expected present stock level $E[psl] = \sum_{r_1=1}^S \sum_{r_2=0}^K r_1 \xi^{(r_1, r_2)}$.
2. Expected reorder level $E[reorder] = \sum_{r_2=1}^K \mu_{r_2} \xi^{(s+1, r_2)} + \sum_{r_2=0}^K (s+1) \alpha_1 \xi^{((s+1), r_2)}$.
3. Expected perishable rate $E[perishable] = \sum_{r_1=1}^S \sum_{r_2=0}^K r_1 \alpha_1 \xi^{(r_1, r_2)}$.
4. Expected number of customers in the waiting hall $E[CWH] = \sum_{r_0=1}^S \sum_{r_2=1}^K r_2 \xi^{(r_1, r_2)}$.
5. Expected number of customers enter into the waiting hall $E[CEWH] = \sum_{r_1=0}^S \sum_{r_2=0}^{K-1} (K - r_2) \theta_{r_1} \xi^{(r_1, r_2)}$.
6. Expected waiting time of a customer in the waiting hall $E[WT] = \frac{E[CWH]}{E[CEWH]}$
7. Probability that the server is busy $P(busy) = \sum_{r_0=1}^S \sum_{r_2=1}^K \xi^{(r_1, r_2)}$.
8. Probability that the server is idle $P(idle) = 1 - P(busy)$
9. The total expected cost value of the proposed model is defined as $TCV = c_a E[psl] + c_b E[reorder] + c_c E[perishable] + c_d E[WT]$

where

c_a – Holding cost per unit, c_b – Setup cost per unit, c_c – Perishable cost per unit and c_d – Waiting cost per customer.

Table 1: TCV for the case of SDAP and QDSP

s	7	8	9	10	11	12	13
S							
58	2.805974	2.805623	2.806103	2.807504	2.809919	2.813445	2.818190
59	2.806534	2.805602	2.805445	2.806146	2.807788	2.810462	2.814264
60	2.807988	2.806522	2.805778	2.805835	2.806769	2.808661	2.811598
61	2.810287	2.808327	2.807044	2.806509	2.806792	2.807966	2.810111
62	2.813383	2.810970	2.809190	2.808110	2.807794	2.808309	2.809726
63	2.817234	2.814403	2.812165	2.810584	2.809717	2.809625	2.810374
64	2.821800	2.818583	2.815924	2.813880	2.812506	2.811856	2.811991

6 Simulation Analysis

In this section, the optimum cost analysis, monotonic behavior of some system characteristics are to be discussed by the numerical illustrations. This will be helpful to deliver a effective decision making polices for every inventory business tycoons. For knowing such curious results of our proposed model, we need to fix the value of the parameters and the cost values such that $\theta = 5, \theta_0 = 2, \mu = 9, \alpha = 0.9, \gamma = 0.07, \beta_1 = 0.5, \beta_2 = 0.5, S = 61, s = 10, N = 10, c_a = 0.05, c_b = 0.9, c_c = 0.1$, and $c_d = 7$.

Example 6.1. Optimum cost analysis

This example briefly investigate the minimum optimal TCV for the category of both arrival and service processes of homogeneous and non-homogeneous cases as shown in Table (1)-(2). In Table (1), $S \in B_{58}^{64}$ and $s \in B_7^{13}$ are used to find the minimal optimum TCV under the case of discussion between SDAP and QDSP. In this case, the $TCV^* = 2.805445$ and corresponding optimum $S^* = 59$ and $s^* = 9$ are obtained.

Next, the output values of the case non-SDAP and non-QDSP are given in Table (2). Here, $S \in B_{57}^{63}$ and $s \in B_{10}^{16}$ are varied to get an optimum TCV. In this case, the $TCV^* = 8.966159$ and corresponding optimum $S^* = 60$ and $s^* = 13$ are obtained.

As we expected due to the assumption of the proposed model, the case non-SDAP and non- QDSP have a higher TCV^* than the SDAP and QDSP case. That is, the minimal optimum TCV obtained in the case of SDAP and QDSP. Hence the arrival and service rates influence the cost value become a minimum one.

$$k1 = 0, k2 = 0$$

$$k1 = 0, k2 = 0.6$$

Example 6.2. The variation of TCV under the parameter variation

In this example, we describe the path of TCV with each parameter considered in the model. In such a way, the major objective of this example is discussed with the

Table 2: TCV for the case of non-SDAP and non-QDSP

s	10	11	12	13	14	15	16
S							
57	8.986034	8.976046	8.970983	8.970903	8.975952	8.986375	9.002534
58	8.985814	8.975121	8.969166	8.967979	8.971672	8.980449	8.994615
59	8.986685	8.975367	8.968613	8.966432	8.968907	8.976202	8.988575
60	8.988575	8.976700	8.969232	8.966159	8.967537	8.973497	8.984257
61	8.991414	8.979048	8.970942	8.967068	8.967457	8.972214	8.981518
62	8.995143	8.982342	8.973667	8.969073	8.968572	8.972242	8.980232
63	8.999705	8.986521	8.977338	8.972097	8.970793	8.973483	8.980284

scaling factors β_1 and β_2 , because they are deciding factors whether the arrival and service processes are non-SDAP and non-QDSP or not respectively. In Table (3), the scaling factor β_1 increases the total cost if it is increasing. That is, β_1 increases means, the arriving customers in the system is increased. Subsequently, the sales of number of product in the inventory is raised. So the management is often ready to store or making reorder for their requirement. These jobs cause the increase of total cost. The same characteristics are holds the parameter θ . Simultaneously, when we are focusing the another scaling factor β_2 , more interestingly it reduces the TCV. If β_2 increases means that the service time of an individual become reduced. So the number of customers leaves the system after a successful service completion of them is increasing. This helps to reduce the mean service time of a customer. So this is the reason for TCV is reduced if β_2 increases. If β_2 and μ are directly proportional to each other μ holds the same behavior as β_2 . Then the perishable parameter α_1 affects the item life time. If α_1 raises, the number of current stock level starts falling down. If it happens, the management has to store more number of products which cause the extra expenditure to maintain the system. So this expenditure cause the increase of total cost. Finally, the reorder intensity rate α minimize the total cost when it is increasing. The successive mean reorder time reduced means the number of available product of the system become positive. Therefore, the service completion will be done as soon as possible. Hence, all the parameters involved in Table (3) and Table (4) are satisfies their own properties.

Example 6.3. Graphical Analysis

- The scaling factors β_1 and β_2 shows the increasing/decreasing path due to its SDAP and non-SDAP, QDSP and non-QDSP. We observe that $0.2 \leq \beta_1 \leq 1$ the β_2 curves deviation is high and $0.5 \leq \beta_2 \leq 1.0$ the β_2 curves deviation is low for all $\beta_1 \in (0.2, 1)$.

- The graph of expected waiting time is shown in Figure (2) when β_1 and β_2 are varying together. Here, the deviation of β_2 curves coincides with the characteristics as we said in Figure (1).

Table 3: The variation of TCV under the parameter variation

β_2	θ	α_1	β_1	0			0.5			1		
			μ	7	9	11	7	9	11	7	9	11
			α									
0	4.5	0.05	0.70	11.08	8.92	7.55	44.64	35.40	29.67	136.60	104.43	85.33
			0.90	10.83	8.66	7.28	44.78	35.11	29.13	145.45	109.22	87.88
			1.10	10.71	8.52	7.14	45.25	35.24	29.04	154.70	114.88	91.46
		0.07	0.70	11.18	9.02	7.64	44.31	35.24	29.60	132.94	102.29	83.98
			0.90	10.92	8.74	7.36	44.40	34.91	29.03	141.43	106.85	86.36
			1.10	10.79	8.60	7.21	44.86	35.02	28.92	150.45	112.34	89.81
		0.09	0.70	11.29	9.12	7.73	44.01	35.10	29.54	129.67	100.36	82.73
			0.90	11.01	8.82	7.43	44.06	34.73	28.93	137.82	104.70	84.96
			1.10	10.87	8.68	7.28	44.50	34.82	28.80	146.61	110.02	88.30
	5	0.05	0.70	11.21	9.06	7.69	44.81	35.59	29.87	137.00	104.88	85.82
			0.90	10.95	8.79	7.41	44.93	35.28	29.31	145.74	109.58	88.28
			1.10	10.83	8.65	7.27	45.38	35.38	29.20	154.89	115.16	91.78
		0.07	0.70	11.31	9.15	7.78	44.48	35.43	29.80	133.34	102.75	84.46
			0.90	11.04	8.87	7.49	44.55	35.08	29.20	141.73	107.21	86.76
			1.10	10.91	8.73	7.34	44.98	35.16	29.08	150.65	112.62	90.14
		0.09	0.70	11.42	9.25	7.86	44.19	35.29	29.74	130.08	100.82	83.22
			0.90	11.13	8.95	7.57	44.21	34.90	29.11	138.12	105.06	85.36
			1.10	10.99	8.80	7.41	44.63	34.96	28.96	146.81	110.30	88.62
	5.5	0.05	0.70	11.31	9.17	7.80	44.95	35.74	30.03	137.33	105.26	86.22
			0.90	11.06	8.89	7.52	45.04	35.41	29.45	145.97	109.87	88.61
			1.10	10.93	8.76	7.38	45.48	35.50	29.33	155.04	115.37	92.04
		0.07	0.70	11.42	9.26	7.89	44.62	35.59	29.96	133.68	103.13	84.87
			0.90	11.14	8.98	7.60	44.67	35.21	29.35	141.96	107.50	87.09
			1.10	11.01	8.83	7.45	45.08	35.28	29.21	150.80	112.84	90.40
		0.09	0.70	11.52	9.36	7.97	44.33	35.45	29.91	130.41	101.20	83.63
			0.90	11.23	9.06	7.67	44.33	35.03	29.26	138.36	105.35	85.69
			1.10	11.09	8.91	7.52	44.73	35.08	29.10	146.97	110.52	88.89
0.5	4.5	0.05	0.70	2.02	1.83	1.70	5.58	4.87	4.41	18.73	16.13	14.48
			0.90	2.04	1.85	1.73	5.31	4.61	4.16	17.71	15.08	13.43
			1.10	2.06	1.88	1.76	5.16	4.48	4.03	17.18	14.51	12.85
		0.07	0.70	2.06	1.87	1.74	5.640	4.92	4.46	18.78	16.19	14.54
			0.90	2.08	1.89	1.76	5.36	4.66	4.21	17.75	15.13	13.48
			1.10	2.10	1.92	1.80	5.22	4.53	4.08	17.21	14.55	12.89
		0.09	0.70	2.10	1.90	1.77	5.69	4.98	4.50	18.83	16.25	14.59
			0.90	2.12	1.93	1.80	5.42	4.71	4.25	17.79	15.17	13.52
			1.10	2.15	1.96	1.84	5.27	4.58	4.12	17.23	14.59	12.93
	5	0.05	0.70	2.02	1.83	1.70	5.60	4.89	4.44	18.80	16.21	14.56
			0.90	2.04	1.85	1.73	5.33	4.63	4.18	17.773	15.15	13.50
			1.10	2.06	1.88	1.76	5.18	4.50	4.05	17.22	14.57	12.91
		0.07	0.70	2.06	1.86	1.73	5.66	4.95	4.48	18.85	16.27	14.62
			0.90	2.08	1.89	1.76	5.38	4.68	4.23	17.80	15.19	13.55
			1.10	2.11	1.92	1.80	5.24	4.55	4.10	17.25	14.61	12.95

Table 4: The variation of TCV under the parameter variation

β_2	θ	α_1	β_1	0			0.5			1		
			μ	7	9	11	7	9	11	7	9	11
			α									
		0.09	0.70	2.106	1.904	1.769	5.719	5.003	4.532	18.908	16.327	14.678
			0.90	2.125	1.931	1.801	5.441	4.737	4.276	17.845	15.242	13.596
			1.10	2.154	1.967	1.843	5.294	4.600	4.148	17.283	14.649	12.999
	5.5	0.05	0.70	2.029	1.832	1.699	5.618	4.918	4.460	18.860	16.278	14.637
			0.90	2.046	1.855	1.727	5.347	4.656	4.205	17.818	15.205	13.562
			1.10	2.071	1.888	1.765	5.202	4.519	4.076	17.266	14.616	12.965
		0.07	0.70	2.068	1.868	1.733	5.678	4.970	4.506	18.913	16.335	14.693
			0.90	2.087	1.893	1.763	5.403	4.706	4.250	17.854	15.250	13.609
			1.10	2.114	1.928	1.803	5.256	4.568	4.121	17.293	14.655	13.008
		0.09	0.70	2.107	1.903	1.766	5.737	5.022	4.552	18.966	16.392	14.749
			0.90	2.127	1.931	1.798	5.457	4.755	4.294	17.891	15.296	13.656
			1.10	2.156	1.967	1.840	5.309	4.617	4.165	17.321	14.695	13.051
1.0	4.5	0.05	0.70	1.021	0.981	0.957	1.290	1.190	1.124	3.460	3.200	3.026
			0.90	1.124	1.086	1.064	1.329	1.228	1.160	3.231	2.976	2.805
			1.10	1.207	1.173	1.153	1.382	1.282	1.215	3.129	2.882	2.716
		0.07	0.70	1.045	1.004	0.979	1.313	1.210	1.142	3.489	3.224	3.047
			0.90	1.152	1.113	1.090	1.355	1.251	1.181	3.261	3.001	2.827
			1.10	1.239	1.203	1.183	1.410	1.307	1.239	3.160	2.908	2.739
		0.09	0.70	1.069	1.027	1.002	1.335	1.230	1.161	3.518	3.248	3.068
			0.90	1.179	1.140	1.116	1.380	1.274	1.202	3.290	3.026	2.849
			1.10	1.270	1.233	1.212	1.438	1.333	1.262	3.191	2.935	2.763
	5	0.05	0.70	0.998	0.955	0.929	1.283	1.182	1.114	3.469	3.209	3.036
			0.90	1.101	1.060	1.035	1.322	1.219	1.150	3.239	2.985	2.814
			1.10	1.185	1.147	1.125	1.374	1.272	1.204	3.136	2.890	2.725
		0.07	0.70	1.022	0.978	0.951	1.306	1.202	1.132	3.498	3.234	3.057
			0.90	1.129	1.086	1.061	1.347	1.241	1.170	3.269	3.010	2.836
			1.10	1.216	1.177	1.154	1.402	1.297	1.227	3.167	2.916	2.748
		0.09	0.70	1.045	1.000	0.973	1.328	1.221	1.150	3.527	3.258	3.078
			0.90	1.156	1.113	1.087	1.372	1.264	1.191	3.298	3.035	2.858
			1.10	1.247	1.207	1.183	1.429	1.322	1.250	3.198	2.942	2.771
	5.5	0.05	0.70	0.978	0.932	0.904	1.277	1.175	1.106	3.477	3.218	3.045
			0.90	1.081	1.037	1.010	1.315	1.211	1.141	3.246	2.992	2.822
			1.10	1.166	1.125	1.100	1.367	1.264	1.194	3.143	2.897	2.732
		0.07	0.70	1.001	0.955	0.926	1.300	1.194	1.124	3.506	3.242	3.066
			0.90	1.108	1.063	1.036	1.340	1.233	1.161	3.276	3.017	2.844
			1.10	1.197	1.154	1.129	1.395	1.289	1.217	3.174	2.923	2.755
		0.09	0.70	1.024	0.977	0.947	1.322	1.214	1.142	3.534	3.266	3.086
			0.90	1.135	1.089	1.060	1.365	1.256	1.181	3.305	3.042	2.866
			1.10	1.227	1.183	1.157	1.423	1.314	1.240	3.204	2.949	2.778

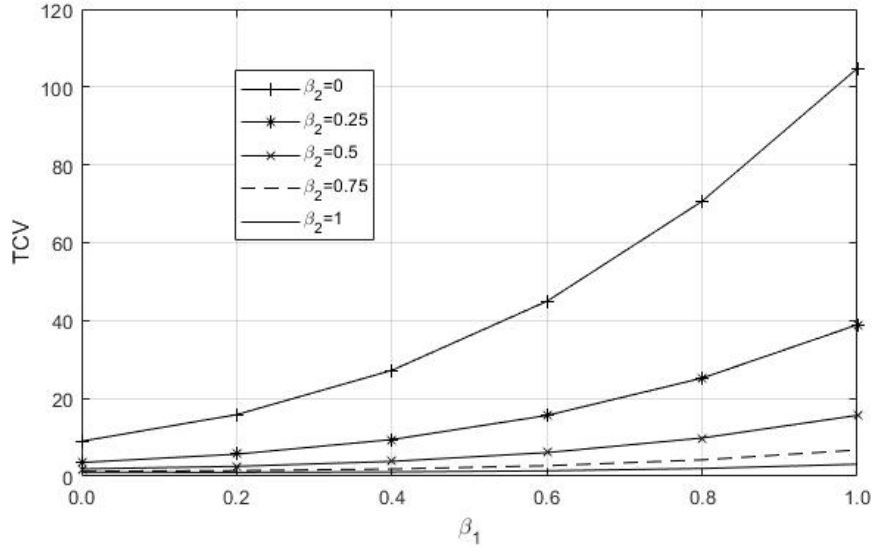


Figure 1: Impact of TCV on β_1 vs β_2

- Figure (3) explores expected present stock level of the system on the combination of β_1 vs β_2 . Both β_1 vs β_2 reduce the $E[psl]$ when they are increasing. Here, the deviation of β_2 curves is high when $0.2 \leq \beta_1 \leq 1$.
- The parameters α and α_1 are affects the $E[WT]$ as shown in Figure 4. In this graph, the beta curve has the high deviation with themselves and low deviation with α_1 .
- The $E[WT]$ is shown in Figure 5 if θ and μ are increasing together. The θ curves are decreasing when μ is increasing and it means that the increased service rate cause less mean service time of an individual. Therefore, the $E[WT]$ is decreased. If θ and μ are inversely proportional each other, θ reacts against μ .
- The average waiting time of a customer is discussed for the case of θ vs K in Figure 6 and μ vs K in Figure 7. As we have enough discussion about θ and μ on the $E[WT]$, we shall move to analyses the impact of K . When the number of finite source population is increases, for the $E[WT]$, the θ curve will be straight line.

Example 6.4. Impact of $E[psl]$, $E[reorder]$ and $E[perishable]$ with the parameter variation

This example describes the important system performance measures, $E[psl]$, $E[reorder]$ and $E[perishable]$ are to be discussed with the parameter analysis of α , α_1 , θ , μ and β_2 as shown in Table 5-7. As per the scaling factor, $\beta_2 = 0$, $\beta_2 = 0.5$ and $\beta_2 = 1$ are to be explored in Table 5, 6 and 7 respectively. If we increase the reorder rate, the expected present stock level increases. For every replenishment, there are Q items replaced as it reaches the system. So it makes the $E[psl]$ is increasing when it is increase. When α

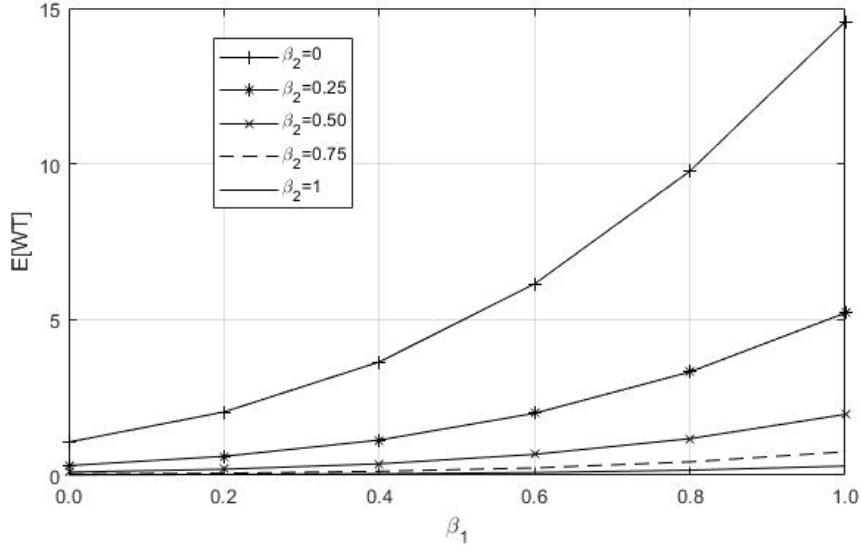


Figure 2: Impact of $E[WT]$ on β_1 vs β_2

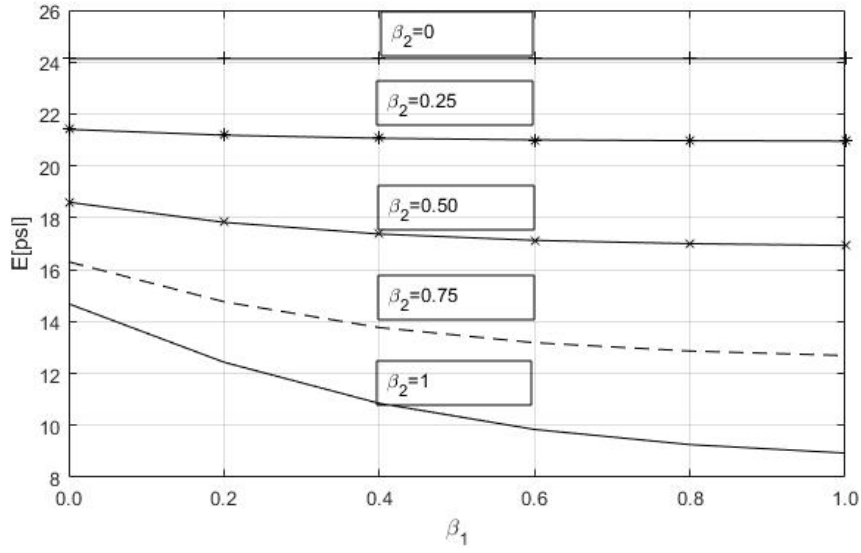


Figure 3: Impact of $E[psl]$ on β_1 vs β_2

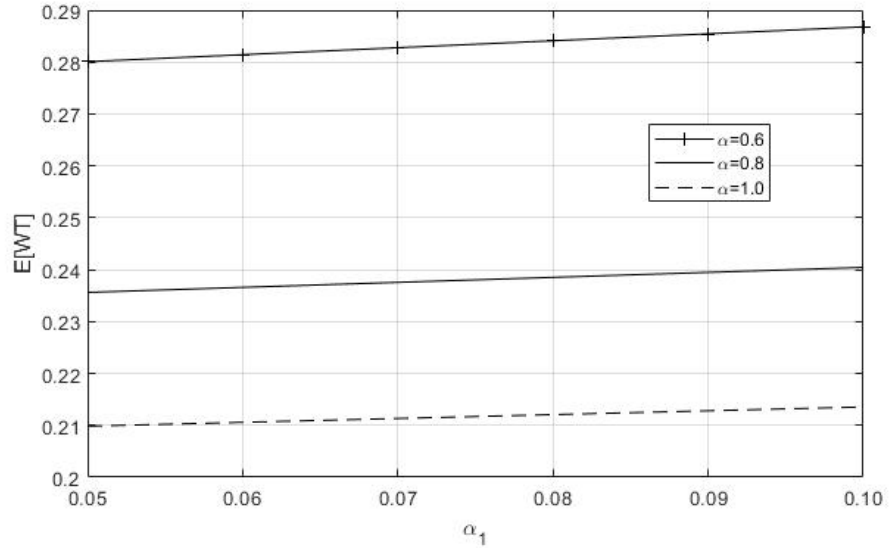


Figure 4: Impact of $E[WT]$ on α vs α_1

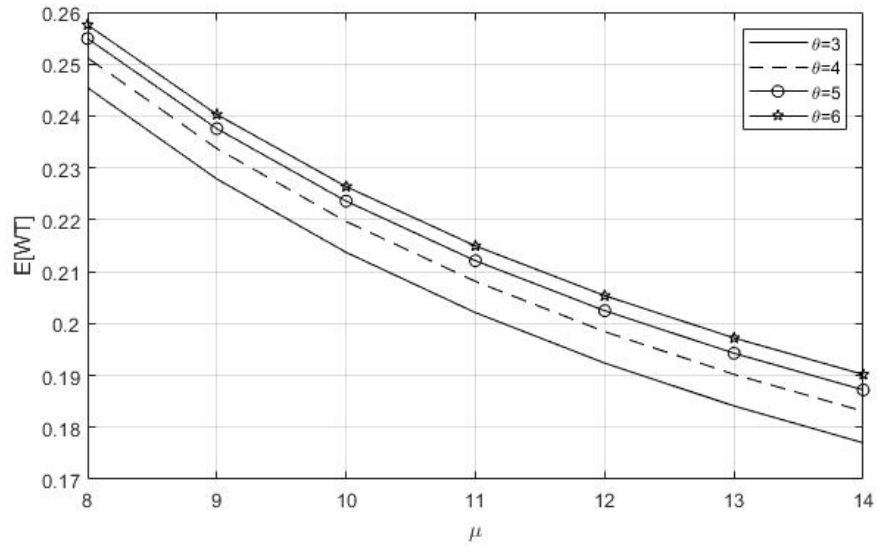


Figure 5: Impact of $E[WT]$ on μ vs θ

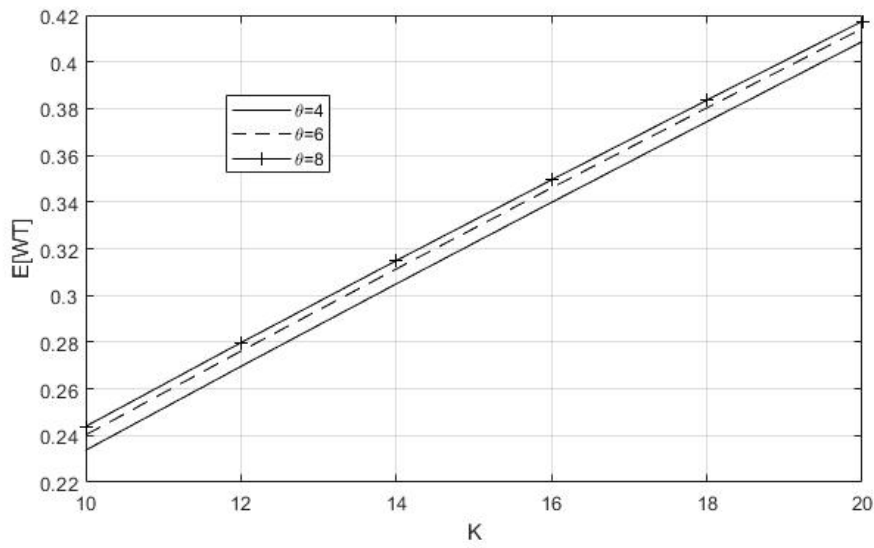


Figure 6: Impact of Impact of $E[WT]$ on K vs θ

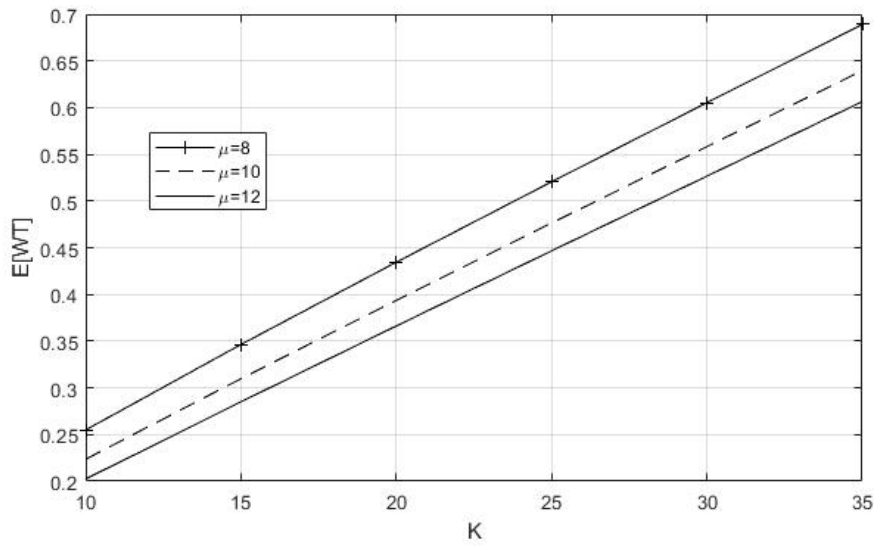


Figure 7: Impact of $E[WT]$ on μ vs K

is increasing, for the value of $\mu = 7$ $E[\text{reorder}]$ behaves first increasing and then decreasing but at $\mu = 11$ it is increasing only. Further, for both μ , the $E[\text{perishable}]$ will increase. Here, the perishable quality of the products depends on the number of present stock level of the system.

The parameter α_1 affects the $E[\text{psl}]$ to fall down if it is increasing. Perishable products starts deterioration process depending the existing current stock level. So it is decreased. Since the items in the inventory storage system are perished, the system requires more number of products to provides the sales service. Hence the expected reorder level is increased. Here the raise of a perishable rate obviously influence the increase of $E[\text{perishable}]$. Then the parameter θ changes the $E[\text{psl}]$ and $E[\text{reorder}]$ by direct variation where as with $E[\text{perishable}]$ it varies by indirect variation. For every arrival, there will be an unit in the system getting down when they leave the system. To fulfill such required number of items, there must be a reorder needed. Since the inventory reduces by the more sales, there must be less number of items remaining in the inventory storage place. This cause the $E[\text{perishable}]$ become less.

More interestingly, as we predicted earlier, the intensity rate μ is inversely proportional to each $E[\text{psl}]$, $E[\text{perishable}]$. If we increase μ , each of them starts falling down. If mean service time of individual customer too short, number of inventory falls down fast and less inventory requires more reorder and less number of perishable items.

Table 5: Impact of E[psl], E[reorder and E[perishable] with the parameter variation

β_2	θ	α_1		E[stock]		E[reorder]		E[perishable]	
			μ	7	11	7	11	7	11
			α						
0	4.5	0.05	0.70	24.9645	22.6523	0.3136	0.4386	1.2482	1.1326
			0.90	26.3157	24.3173	0.3138	0.4543	1.3158	1.2159
			1.10	27.2531	25.5090	0.3089	0.4602	1.3627	1.2755
		0.07	0.70	24.4147	22.2220	0.3277	0.4503	1.7090	1.5555
			0.90	25.8047	23.9103	0.3293	0.4679	1.8063	1.6737
			1.10	26.7722	25.1230	0.3252	0.4751	1.8741	1.7586
		0.09	0.70	23.9075	21.8168	0.3408	0.4614	2.1517	1.9635
			0.90	25.3337	23.5268	0.3439	0.4809	2.2800	2.1174
			1.10	26.3299	24.7591	0.3408	0.4893	2.3697	2.2283
	5	0.05	0.70	24.9645	22.6523	0.3188	0.4503	1.2482	1.1326
			0.90	26.3157	24.3173	0.3191	0.4665	1.3158	1.2159
			1.10	27.2531	25.5090	0.3142	0.4726	1.3627	1.2755
		0.07	0.70	24.4147	22.2220	0.3330	0.4621	1.7090	1.5555
			0.90	25.8047	23.9103	0.3347	0.4803	1.8063	1.6737
			1.10	26.7722	25.1230	0.3307	0.4877	1.8741	1.7586
		0.09	0.70	23.9075	21.8168	0.3463	0.4733	2.1517	1.9635
			0.90	25.3337	23.5268	0.3495	0.4934	2.2800	2.1174
			1.10	26.3299	24.7591	0.3464	0.5022	2.3697	2.2283
	5.5	0.05	0.70	24.9645	22.6523	0.3211	0.4553	1.2482	1.1326
			0.90	26.3157	24.3173	0.3214	0.4718	1.3158	1.2159
			1.10	27.2531	25.5090	0.3165	0.4780	1.3627	1.2755
		0.07	0.70	24.4147	22.2220	0.3353	0.4672	1.7090	1.5555
			0.90	25.8047	23.9103	0.3371	0.4856	1.8063	1.6737
			1.10	26.7722	25.1230	0.3330	0.4932	1.8741	1.7586
		0.09	0.70	23.9075	21.8168	0.3486	0.4785	2.1517	1.9635
			0.90	25.3337	23.5268	0.3519	0.4988	2.2800	2.1174
			1.10	26.3299	24.7591	0.3488	0.5078	2.3697	2.2283

Table 6: Impact of E[psl], E[reorder and E[perishable] with the parameter variation

β_2	θ	α_1		E[stock]		E[reorder]		E[perishable]	
			μ	7	11	7	11	7	11
			α						
0.5	4.5	0.05	0.70	18.5347	15.4582	0.2337	0.2938	0.9267	0.7729
			0.90	20.5232	17.5264	0.2533	0.3280	1.0262	0.8763
			1.10	22.0246	19.1546	0.2662	0.3532	1.1012	0.9577
		0.07	0.70	18.2647	15.2755	0.2384	0.2973	1.2785	1.0693
			0.90	20.2574	17.3398	0.2589	0.3323	1.4180	1.2138
			1.10	21.7661	18.9683	0.2725	0.3582	1.5236	1.3278
		0.09	0.70	18.0051	15.0982	0.2430	0.3007	1.6205	1.3588
			0.90	20.0014	17.1583	0.2644	0.3366	1.8001	1.5442
			1.10	21.5169	18.7869	0.2787	0.3631	1.9365	1.6908
	5	0.05	0.70	18.2833	15.0507	0.2366	0.2977	0.9142	0.7525
			0.90	20.2883	17.1217	0.2573	0.3339	1.0144	0.8561
			1.10	21.8085	18.7631	0.2712	0.3609	1.0904	0.9382
		0.07	0.70	18.0201	14.8771	0.2412	0.3009	1.2614	1.0414
			0.90	20.0281	16.9432	0.2628	0.3380	1.4020	1.1860
			1.10	21.5549	18.5840	0.2773	0.3657	1.5088	1.3009
		0.09	0.70	17.7668	14.7084	0.2456	0.3041	1.5990	1.3238
			0.90	19.7774	16.7695	0.2681	0.3420	1.7800	1.5093
			1.10	21.3101	18.4095	0.2834	0.3705	1.9179	1.6569
	5.5	0.05	0.70	18.1713	14.8689	0.2379	0.2993	0.9086	0.7434
			0.90	20.1830	16.9399	0.2591	0.3364	1.0092	0.8470
			1.10	21.7112	18.5861	0.2733	0.3642	1.0856	0.9293
		0.07	0.70	17.9110	14.6992	0.2424	0.3025	1.2538	1.0289
			0.90	19.9254	16.7649	0.2644	0.3404	1.3948	1.1735
			1.10	21.4596	18.4102	0.2794	0.3689	1.5022	1.2887
		0.09	0.70	17.6605	14.5342	0.2468	0.3056	1.5894	1.3081
			0.90	19.6770	16.5946	0.2697	0.3443	1.7709	1.4935
			1.10	21.2168	18.2387	0.2854	0.3736	1.9095	1.6415

Table 7: Impact of E[psl], E[reorder and E[perishable] with the parameter variation

β_2	θ	α_1		E[stock]		E[reorder]		E[perishable]	
			μ	7	11	7	11	7	11
			α						
1	4.5	0.05	0.70	12.2895	10.2694	0.1562	0.1954	0.6145	0.5135
			0.90	14.2804	12.1335	0.1792	0.2281	0.7140	0.6067
			1.10	15.9157	13.7102	0.1973	0.2549	0.7958	0.6855
		0.07	0.70	12.1776	10.1919	0.1586	0.1974	0.8524	0.7134
			0.90	14.1615	12.0488	0.1822	0.2306	0.9913	0.8434
			1.10	15.7933	13.6209	0.2007	0.2578	1.1055	0.9535
		0.09	0.70	12.0681	10.1157	0.1610	0.1994	1.0861	0.9104
			0.90	14.0450	11.9654	0.1851	0.2330	1.2640	1.0769
			1.10	15.6733	13.5331	0.2041	0.2606	1.4106	1.2180
	5	0.05	0.70	11.2674	9.0422	0.1473	0.1796	0.5634	0.4521
			0.90	13.2040	10.7934	0.1709	0.2123	0.6602	0.5397
			1.10	14.8221	12.3059	0.1899	0.2400	0.7411	0.6153
		0.07	0.70	11.1733	8.9823	0.1494	0.1812	0.7821	0.6288
			0.90	13.1024	10.7265	0.1734	0.2144	0.9172	0.7509
			1.10	14.7161	12.2343	0.1928	0.2424	1.0301	0.8564
		0.09	0.70	11.0811	8.9232	0.1514	0.1828	0.9973	0.8031
			0.90	13.0026	10.6606	0.1759	0.2164	1.1702	0.9594
			1.10	14.6120	12.1636	0.1958	0.2448	1.3151	1.0947
	5.5	0.05	0.70	10.7714	8.4426	0.1426	0.1709	0.5386	0.4221
			0.90	12.6732	10.1258	0.1662	0.2033	0.6337	0.5063
			1.10	14.2751	11.5941	0.1856	0.2311	0.7138	0.5797
		0.07	0.70	10.6852	8.3903	0.1445	0.1723	0.7480	0.5873
			0.90	12.5794	10.0669	0.1686	0.2051	0.8806	0.7047
			1.10	14.1766	11.5305	0.1883	0.2332	0.9924	0.8071
		0.09	0.70	10.6007	8.3388	0.1464	0.1738	0.9541	0.7505
			0.90	12.4872	10.0088	0.1709	0.2069	1.1239	0.9008
			1.10	14.0798	11.4676	0.1910	0.2354	1.2672	1.0321

7 Conclusion

The finite source population is considered to explore the non-SDAP and non-QDSP in the SQIS. The generalization of homogeneous and non-homogeneous arrival and service processes are given in the steady state of the model. Also, the comparative discussion is made in the numerical investigations. The illustrations given in the examples enhance minimum optimal total cost for the QDSP category. The SDAP will increase the number of units arriving to the inventory system. This increased units of arrival will produce the more sales of the inventory. When we are focusing the development of the inventory business, the first step has to be initialized to attract the customers towards the system. For such process, displayed stock level will assure the increase of customers in the inventory system. And Maintaining the sufficient current stock level will play the crucial role for the development of an inventory system. Simultaneously, the QDSR contribute the reduce of waiting time of a customer in the system. If a management provides some polices to reduce the waiting time of an individual in the servicing system, the customers are impressed and they will come to the same system often. In such a way that, the proposed model will applicable in a economically.

Acknowledgement

The second author is supported by the fund for improvement of Science and Technology Infrastructure (FIST) of DST (SR/FST/MSI-115/2016). The authors thank the Editor and Reviewers of the Ratio Mathematica for their constant support towards the successful completion of this work.

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Application of bipolar Pythagorean fuzzy regular α generalized closed soft matrices in decision-making problems

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Abstract

The main focus of this paper is to introduce bipolar Pythagorean fuzzy soft matrix topology and define bipolar Pythagorean fuzzy regular α generalized closed soft matrix and study some properties over the soft matrices. We propose the Moora strategy in bipolar Pythagorean fuzzy set environment to handle multiple criteria decision-making problems. Finally, one numerical example illustrates an application of bipolar Pythagorean fuzzy Moora method and analyses the results of different values of the decision-making weights of criteria on ranking order of the alternatives.

Keywords: Bipolar Pythagorean fuzzy soft set, bipolar Pythagorean fuzzy soft matrix, bipolar Pythagorean fuzzy soft matrix topology, bipolar Pythagorean fuzzy regular α generalized closed soft matrix, bipolar Pythagorean fuzzy Moora method.

2010 subject classification: 03B52, 15B15, 90C70.

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¹Received on March 17, 2022. Accepted on August 11, 2022. Published on December 30, 2022. doi:10.23755/rm.v43i0.733. ISSN: 1592-7415. eISSN: 2282-8214. ©Nithiyapriya et al. This paper is published under the CC-BY licence agreement.

1. Introduction

Zadeh [3] was the first to introduce the theory of fuzzy sets. In 1999 Molodstov [1] introduced a novel mathematical tool for dealing with the uncertainties, called soft set, which is free from the inadequacy of parametrization and established various results based on this. Maji et al. [5] studied the notion of soft sets, fuzzy soft sets and intuitionistic fuzzy soft set. Yang and Chenli [16] initiated fuzzy soft matrix by introducing matrix representation of fuzzy soft sets and applied it to problems associated with decision making problem. Munir Abdul Khalik Al Khafaji and Majd Hamid Mahmood [11] introduced the fuzzy soft matrix topology. Decision making is one of the most significant and useful thing in our day-to-day life. But with the growing complexities of the systems it is difficult to the decision maker for making a correct decision under the environments.

In this paper, we propose a new hybrid bipolar Pythagorean fuzzy regular α generalized closed soft matrix. Further, we discuss some properties and definitions with relevant examples. Multi-objective optimization method on the basis of ratio analysis was introduced by Brauers and Zavadskas [9] in 2006. Here we apply Moora method in bipolar Pythagorean fuzzy environment. The basic idea of bipolar Pythagorean fuzzy Moora method is to calculate the overall performance of each alternative as the difference between the sums of its normalized performances which belongs to cost and benefit criteria.

2. Preliminaries

Definition 2.1. Let $U = \{c_1, c_2, c_3, \dots, c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, e_3, \dots, e_n\}$. Let $A \subseteq E$ and (F, A) be a fuzzy soft set in the fuzzy soft class (U, E) . A fuzzy soft set (F, A) in matrix form is denoted by $A_{m \times n} = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]_{i=1, 2, \dots, m, j=1, 2, 3, \dots, n}$

where $a_{ij} = \begin{cases} \mu_j(c_i) & \text{if } e_j \in A \\ 0 & \text{if } e_j \notin A \end{cases}$, $\mu_j(c_i)$ represents the membership of c_i in the fuzzy set.

Definition 2.2. A Pythagorean fuzzy matrix (PFM) is a matrix of pairs $A = [(a_{ij}, a_{ij}')]_{i,j}$ of nonnegative real numbers, $a_{ij}, a_{ij}' \in [0, 1]$, satisfying $a_{ij}^2 + a_{ij}'^2 \leq 1$ for all i, j .

Definition 2.3. A bipolar fuzzy matrix (BFM) $A = [(a_{ij})]_{l \times m} \in \mathcal{M}_{l \times m}$, then $a_{ij} = (-a_{ij}^n, a_{ij}^p)$, where $a_{ij}^n \in [-1, 0]$ and $a_{ij}^p \in [0, 1]$ are the negative and positive membership values of the element a_{ij} respectively.

Definition 2.4. A bipolar Pythagorean fuzzy matrix (BPFM) of order $(l \times m)$ is denoted by $\mathcal{M}_{l \times m}$ and that of order $(m \times m)$, that is square BPFM is denoted by $\mathcal{M}_{m \times m}$. We conclude that the matrix $A = [((-a_{ij}^n, a_{ij}^p - a_{ij}'^n, a_{ij}'^p)]$, where $a_{ij}^n, a_{ij}^p, a_{ij}'^n, a_{ij}'^p \in [-1, 1]$ are the negative and positive membership values of the element a_{ij}, a_{ij}' respectively.

Also satisfies $0 \leq (a_{ij}^n)^2 + (a_{ij}^p)^2 \leq 1$ and $0 \leq (a_{ij}'^n)^2 + (a_{ij}'^p)^2 \leq 1$ for all i, j .

Definition 2.5. Let $U = \{c_1, c_2, c_3, \dots, c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, e_3, \dots, e_n\}$. Let $A \subseteq E$ and (F, A) be a bipolar Pythagorean fuzzy soft set in the fuzzy soft class (U, E) . Then bipolar Pythagorean fuzzy soft set (F, A) in a matrix form as $A_{m \times n} = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$ $i=1, 2, \dots, m, j=1, 2, 3, \dots, n$.

$$\text{Where } a_{ij} = \begin{cases} \mu_j^+(c_i), \vartheta_j^+(c_i), \mu_j^-(c_i), \vartheta_j^-(c_i) & \text{if } e_j \in A \\ (0, 1, 0, -1) & \text{if } e_j \notin A \end{cases}$$

$\mu_j^+(c_i)$ represents the positive membership of c_i in the bipolar Pythagorean fuzzy set $F(e_i)$.

$\vartheta_j^+(c_i)$ represents the non-membership of c_i in the bipolar Pythagorean fuzzy set $F(e_i)$.

$\mu_j^-(c_i)$ represents the negative membership of c_i in the bipolar Pythagorean fuzzy set $F(e_i)$.

$\vartheta_j^-(c_i)$ represents the negative non-membership of c_i in the bipolar Pythagorean fuzzy set $F(e_i)$.

Definition 2.6. Some operations on BPF soft matrix: Various standard operations over two bipolar Pythagorean fuzzy soft matrices $A = [\mu_j^{A+}(c_i), \vartheta_j^{A+}(c_i), \mu_j^{A-}(c_i), \vartheta_j^{A-}(c_i)]$ and

$B = [\mu_j^{B+}(c_i), \vartheta_j^{B+}(c_i), \mu_j^{B-}(c_i), \vartheta_j^{B-}(c_i)] \in \text{BPFSM}_{m \times n}$ can be defined as follows:

- (i) $A^c = [(\vartheta_j^{A+}(c_i), \mu_j^{A+}(c_i), \vartheta_j^{A-}(c_i), \mu_j^{A-}(c_i))] \text{ for all } i \text{ and } j.$
- (ii) $A \cup B = [(\max(\mu_{ij}^{A+}, \mu_{ij}^{B+}), \min(\vartheta_{ij}^{A+}, \vartheta_{ij}^{B+}), \min(\mu_{ij}^{A-}, \mu_{ij}^{B-}), \max(\vartheta_{ij}^{A-}, \vartheta_{ij}^{B-}))]$
- (iii) $A \cap B = [(\min(\mu_{ij}^{A+}, \mu_{ij}^{B+}), \max(\vartheta_{ij}^{A+}, \vartheta_{ij}^{B+}), \max(\mu_{ij}^{A-}, \mu_{ij}^{B-}), \min(\vartheta_{ij}^{A-}, \vartheta_{ij}^{B-}))]$
- (iv) $A * B = (\max \min(\mu_{ij}^{A+}, \mu_{ij}^{B+}), \min \max(\vartheta_{ij}^{A+}, \vartheta_{ij}^{B+}), \min \max(\mu_{ij}^{A-}, \mu_{ij}^{B-}), \max \min(\vartheta_{ij}^{A-}, \vartheta_{ij}^{B-}))$ for all i, j .

Definition 2.7. Fuzzy soft matrix topology: Let X be a nonempty set, E be a set of parameters, let $\tau_{\mathcal{M}}$ be a collection of FSM-sets, $A \subset U$, if $\tau_{\mathcal{M}}$ satisfies the following axioms:

- (i) $\phi_{\mathcal{M}}, U_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$
- (ii) Arbitrary union of any members of FSM-sets in $\tau_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$
- (iii) Finite intersection of any two FSM-sets in $\tau_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$.

Then $\tau_{\mathcal{M}}$ is said to be fuzzy soft matrix topology. The triplet $(X, \tau_{\mathcal{M}}, E)$ is called FSM topological space over X , the members of $\tau_{\mathcal{M}}$ are called fuzzy soft open matrices and their complements are called fuzzy soft closed matrices.

Definition 2.8. Let $(X, \tau_{\mathcal{M}}, E)$ be a FSM topological space and $P = \{x, \mu_A^+(x), \mu_A^-(x), \vartheta_A^+(x), \vartheta_A^-(x) : x \in X\}$ be a fuzzy soft set over X . A fuzzy soft interior and fuzzy soft closure of P is defined by:

- (i) $Fint(P_{\mathcal{M}}) = \cup \{G_{\mathcal{M}} / G_{\mathcal{M}} \text{ is a FOM in } (X, \tau_{\mathcal{M}}) \text{ and } G_{\mathcal{M}} \subseteq P_{\mathcal{M}}\}$
- (ii) $Fcl(P_{\mathcal{M}}) = \cap \{K_{\mathcal{M}} / K_{\mathcal{M}} \text{ is a FCM in } (X, \tau_{\mathcal{M}}) \text{ and } P_{\mathcal{M}} \subseteq K_{\mathcal{M}}\}$

It is clear that

- a) $Fint(P_{\mathcal{M}})$ is the biggest fuzzy soft open matrix contained in $P_{\mathcal{M}}$.
- b) $Fcl(P_{\mathcal{M}})$ is the smallest fuzzy soft closed matrix containing $P_{\mathcal{M}}$.

In this paper we denote,

BPF- Bipolar Pythagorean Fuzzy

BPFSM- Bipolar Pythagorean fuzzy soft matrix

BPFSMT- Bipolar Pythagorean fuzzy soft matrix topology

BPFR α GCSM- Bipolar Pythagorean fuzzy regular α generalized closed soft matrix.

3. Bipolar Pythagorean fuzzy soft matrix topology

Definition 3.1. Let X be a nonempty set, E be a set of parameters, let $\tau_{\mathcal{M}}$ be a collection of BPFSM-sets, $A \subset U$, if $\tau_{\mathcal{M}}$ satisfies the following axioms:

- (i) $\phi_{\mathcal{M}}, U_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$.
- (ii) Arbitrary union of any members of BPFSM-sets in $\tau_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$.
- (iii) Finite intersection of any two BPFSM sets in $\tau_{\mathcal{M}}$ belongs to $\tau_{\mathcal{M}}$.

Then $\tau_{\mathcal{M}}$ is said to be BPF soft matrix topology. The triplet $(X, \tau_{\mathcal{M}}, E)$ is called BPFSM topological space over X , the members of $\tau_{\mathcal{M}}$ are called BPF soft open matrices and their complements are called BPF soft closed matrices.

Definition 3.2. Let $(X, \tau_{\mathcal{M}}, E)$ be a BPFSM topological space and $P = \{x, \mu_A^+(x), \mu_A^-(x), \vartheta_A^+(x), \vartheta_A^-(x) : x \in X\}$ be a BPF soft set over X . Then the bipolar Pythagorean fuzzy soft interior, bipolar Pythagorean fuzzy soft closure of P is defined by:

- (i) $BPFint(P_{\mathcal{M}}) = \cup \{G_{\mathcal{M}} / G_{\mathcal{M}} \text{ is a BPFOM in } (X, \tau_{\mathcal{M}}) \text{ and } G_{\mathcal{M}} \subseteq P_{\mathcal{M}}\}$
- (ii) $BPFcl(P_{\mathcal{M}}) = \cap \{K_{\mathcal{M}} / K_{\mathcal{M}} \text{ is a BPFCM in } (X, \tau_{\mathcal{M}}) \text{ and } P_{\mathcal{M}} \subseteq K_{\mathcal{M}}\}$

It is clear that

- a) $BPFint(P_{\mathcal{M}})$ is the biggest bipolar Pythagorean fuzzy soft open matrix contained in $P_{\mathcal{M}}$.
- b) $BPFcl(P_{\mathcal{M}})$ is the smallest bipolar Pythagorean fuzzy soft closed matrix containing $P_{\mathcal{M}}$.

Example 3.1. Let $X = \{a, b\}$ and $\tau_{\mathcal{M}} = \{\phi_{\mathcal{M}}, A_{\mathcal{M}}, B_{\mathcal{M}}, C_{\mathcal{M}}, U_{\mathcal{M}}\}$ is a BPF soft matrix topology on X , where

$$\begin{aligned} A_{\mathcal{M}} &= \begin{bmatrix} (a, 0.3, 0.8, -0.4, -0.9) & (b, 0.2, 0.7, -0.5, -0.8) \\ (a, 0.5, 0.6, -0.5, -0.7) & (b, 0.4, 0.5, -0.6, -0.8) \end{bmatrix} \\ B_{\mathcal{M}} &= \begin{bmatrix} (a, 0.6, 0.8, -0.4, -0.9) & (b, 0.2, 0.3, -0.3, -0.8) \\ (a, 0.5, 0.7, -0.1, -0.7) & (b, 0.4, 0.6, -0.9, -0.8) \end{bmatrix} \\ C_{\mathcal{M}} &= \begin{bmatrix} (a, 0.3, 0.8, -0.1, -0.9) & (b, 0.7, 0.4, -0.5, -0.8) \\ (a, 0.5, 0.6, -0.6, -0.7) & (b, 0.4, 0.1, -0.9, -0.8) \end{bmatrix} \end{aligned}$$

Definition 3.3. Let $(X, \tau_{\mathcal{M}}, E)$ be a BPF SMTS and $A_{\mathcal{M}}, B_{\mathcal{M}}$ be two bipolar Pythagorean fuzzy soft matrices in X . Then bipolar Pythagorean fuzzy soft interior holds the following properties:

- a) $\text{int}(A_{\mathcal{M}}) \subseteq A_{\mathcal{M}}$
- b) $A_{\mathcal{M}} \subseteq B_{\mathcal{M}} \Rightarrow \text{int}(A_{\mathcal{M}}) \subseteq \text{int}(B_{\mathcal{M}})$
- c) $\text{int}(\text{int}(A_{\mathcal{M}})) = \text{int}(A_{\mathcal{M}})$
- d) $\text{int}(A_{\mathcal{M}} \cap B_{\mathcal{M}}) = \text{int}(A_{\mathcal{M}}) \cap \text{int}(B_{\mathcal{M}})$
- e) $\text{int}(\phi_{\mathcal{M}}) = \phi_{\mathcal{M}}$
- f) $\text{int}(U_{\mathcal{M}}) = U_{\mathcal{M}}$
- g) $\text{int}(A_{\mathcal{M}} \cup B_{\mathcal{M}}) \supseteq \text{int}(A_{\mathcal{M}}) \cup \text{int}(B_{\mathcal{M}})$

Definition 3.4. Let $(X, \tau_{\mathcal{M}}, E)$ be a BPFTS and $A_{\mathcal{M}}, B_{\mathcal{M}}$ be two bipolar pythagorean fuzzy soft matrices in X . Then bipolar pythagorean fuzzy soft closure holds the following properties:

- a) $A_{\mathcal{M}} \subseteq \text{cl}(A_{\mathcal{M}})$
- b) $A_{\mathcal{M}} \subseteq B_{\mathcal{M}} \Rightarrow \text{cl}(A_{\mathcal{M}}) \subseteq \text{cl}(B_{\mathcal{M}})$
- c) $\text{cl}(\text{cl}(A_{\mathcal{M}})) = \text{cl}(A_{\mathcal{M}})$
- d) $\text{cl}(A_{\mathcal{M}} \cup B_{\mathcal{M}}) = \text{cl}(A_{\mathcal{M}}) \cup \text{cl}(B_{\mathcal{M}})$
- e) $\text{cl}(\phi_{\mathcal{M}}) = \phi_{\mathcal{M}}$
- f) $\text{cl}(U_{\mathcal{M}}) = U_{\mathcal{M}}$
- g) $\text{cl}(A_{\mathcal{M}} \cap B_{\mathcal{M}}) \subseteq \text{cl}(A_{\mathcal{M}}) \cap \text{cl}(B_{\mathcal{M}})$.

4. Bipolar Pythagorean fuzzy regular α generalized closed soft matrix

Definition 4.1 Let X be a nonempty set, E be a set of parameters. A BPF soft matrix $A_{\mathcal{M}}$ of a BPF soft topological space $(X, \tau_{\mathcal{M}}, E)$ is called a BPFR α G closed soft matrix if $\alpha \text{cl}(A_{\mathcal{M}}) \subseteq (U_R)_{\mathcal{M}}$ whenever $A_{\mathcal{M}} \subseteq (U_R)_{\mathcal{M}}$, $(U_R)_{\mathcal{M}}$ is a BPF regular open soft matrices in $(X, \tau_{\mathcal{M}}, E)$.

Example 4.1. Let $X = \{a, b\}$ and $\tau_{\mathcal{M}} = \{\phi_{\mathcal{M}}, T_{\mathcal{M}}, U_{\mathcal{M}}\}$ be a BPF SMT on X , where $\phi_{\mathcal{M}} = \begin{bmatrix} (0, 1, 0, -1) & (0, 1, 0, -1) \\ (0, 1, 0, -1) & (0, 1, 0, -1) \end{bmatrix}$,

$$T_{\mathcal{M}} = \begin{bmatrix} (0.3, 0.8, -0.4, -0.9) & (0.2, 0.7, -0.5, -0.8) \\ (0.5, 0.6, -0.5, -0.7) & (0.4, 0.5, -0.6, -0.8) \end{bmatrix},$$

$$U_{\mathcal{M}} = \begin{bmatrix} (1, 0, -1, 0) & (1, 0, -1, 0) \\ (1, 0, -1, 0) & (1, 0, -1, 0) \end{bmatrix}.$$

Consider the BPFsM $A_{\mathcal{M}}$ where $A_{\mathcal{M}} = \begin{bmatrix} (0.5, 0.4, -0.6, -0.5) & (0.4, 0.3, -0.7, -0.6) \\ (0.6, 0.6, -0.7, -0.7) & (0.5, 0.5, -0.8, -0.8) \end{bmatrix}$.

Now $\alpha cl(A_{\mathcal{M}}) = T_{\mathcal{M}}^c \subseteq (U_R)_{\mathcal{M}}$ whenever $A_{\mathcal{M}} \subseteq (U_R)_{\mathcal{M}}$ and $(U_R)_{\mathcal{M}}$ is BPF regular open soft matrix. Therefore, $A_{\mathcal{M}}$ is a BPF α G closed soft matrix.

Definition 4.2 Let X be a nonempty set, E be a set of parameters. A BPF soft matrix $A_{\mathcal{M}}$ of a BPF soft topological space $(X, \tau_{\mathcal{M}}, E)$ is called BPF α G open soft matrix if $\alpha cl(A_{\mathcal{M}}) \supseteq (U_R)_{\mathcal{M}}$ whenever $A_{\mathcal{M}} \supseteq (U_R)_{\mathcal{M}}$, $(U_R)_{\mathcal{M}}$ is a BPF regular closed soft matrices in $(X, \tau_{\mathcal{M}}, E)$.

5. Application of BPF Moora method in multi criterion decision making problem

In this section, we present a multi attribute decision making application of the BPF α G open soft matrix and BPF soft topology using bipolar Pythagorean fuzzy Moora (BPF- Moora) method. In this paper, we present BPF- Moora method by solving MCDM problems that are equipped with bipolar Pythagorean fuzzy information. We illustrate our proposed method with a numerical example.

Algorithm for BPF-Moora method:

Step 1: Construct the BPF open soft matrix which are BPF α G open soft matrix in BPF soft matrix topological space.

Step 2: Write the decision matrix for positive and negative information and provide weightage for positive and negative information where $W_p, W_N \in [0, 1]$.

Step 3: Normalize the decision matrix by using the formula $Nx_{ij} = \frac{x_{ij}}{\sqrt{\sum_{j=1}^n x_{ij}^2}}$ where $i=1, 2, 3, \dots, m, j=1, 2, 3, \dots, n$.

Step 4: Calculate the combined weighted normalized decision matrix by using the following relation $WNM^+ = [NM^+][W_p]^T$ and $WNM^- = [NM^-][W_N]^T$ and the values of benefit and non-benefit criteria are used to compute the reference points R_p^*, R_p^{**}, R_N^* and R_N^{**} using the following equations

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$$R_p^* = R_p^{**} = (\text{sum of benefit Criteria}) - (\text{sum of non benefit criteria})$$

$$R_N^* = R_N^{**} = (\text{sum of benefit Criteria}) - (\text{sum of non benefit criteria})$$

Step 5: Defuzzify the benefit and non- benefit criteria by using the reference points R_p^*, R_p^{**}, R_N^* and R_N^{**} by means of the following equations

$$M_R^+ = (R_p^*)^2 - (R_p^{**})^2 \text{ and } M_R^- = (R_N^*)^2 - (R_N^{**})^2$$

Step 6: Compute the value of M_R , where $M_R = M_R^+ - M_R^-$

Step 7: Obtain the ranking of alternatives with highest value as rank 1 and lowest as last rank.

Example: Tractors are available in a wide range of options to suit specific tasks and requirements. Tractors offer advantages on small farms as well as in regular lawn and garden work. Modern tractors are used for ploughing, tilling and planting fields in addition to routine lawn care, landscape maintenance, moving or spreading fertilizer and clearing bushes. We consider a problem to select agriculture machinery. Especially agriculture required a lot of machinery part, among that the major one is a tractor. The idea of bipolar Pythagorean fuzzy regular α generalized open soft matrix is to be applied in BPF Moora method and we select a best tractor.

Let $T = \{T_1, T_2, T_3, T_4\}$ be the tractors with the following criteria, $S = \{s_1 = \text{safety of operator when maneuvering}, s_2 = \text{customer service from supplier}, s_3 = \text{rated power}, s_4 = \text{initial cost of tractor}\}$. Out of these parameters, let s_1, s_3, s_4 be the beneficial criteria and s_2 be the non- beneficial criteria. Our aim is to calculate the ranking of tractors by using bipolar Pythagorean fuzzy regular alpha generalized open soft matrix using Moora method.

Step 1: Construct the BPF soft matrix which is BPFR α G soft matrix in BPF soft matrix topological space.

DM

$$= \begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{matrix} \begin{bmatrix} (0.5 & 0.3 & -0.7 & -0.5) & (0.3 & 0.3 & -0.2 & -0.7) & (0.5 & 0.4 & -0.6 & -0.5) & (0.4 & 0.4 & -0.6 & -0.6) \\ (0.3 & 0.7 & -0.5 & -0.7) & (0.1 & 0.4 & -0.6 & -0.4) & (0.6 & 0.6 & -0.7 & -0.7) & (0.5 & 0.5 & -0.6 & -0.6) \\ (0.6 & 0.3 & -0.8 & -0.3) & (0.2 & 0.7 & -0.3 & -0.9) & (0.3 & 0.7 & -0.4 & -0.8) & (0.4 & 0.8 & -0.3 & -0.7) \\ (0.5 & 0.4 & -0.6 & -0.6) & (0.3 & 0.4 & -0.3 & -0.8) & (0.9 & 0.1 & -0.9 & -0.2) & (0.8 & 0.1 & -0.8 & -0.1) \end{bmatrix}$$

Step 2: Decision matrix for positive and negative information.

$$DM^+ = \begin{bmatrix} (0.5 & 0.3) & (0.3 & 0.3) & (0.5 & 0.4) & (0.4 & 0.4) \\ (0.3 & 0.7) & (0.1 & 0.4) & (0.6 & 0.6) & (0.5 & 0.5) \\ (0.6 & 0.3) & (0.2 & 0.7) & (0.3 & 0.7) & (0.4 & 0.8) \\ (0.5 & 0.4) & (0.3 & 0.4) & (0.9 & 0.1) & (0.8 & 0.1) \end{bmatrix}$$

$$DM^- = \begin{bmatrix} (-0.7 & -0.5) & (-0.2 & -0.7) & (-0.6 & -0.5) & (-0.6 & -0.6) \\ (-0.5 & -0.7) & (-0.6 & -0.4) & (-0.7 & -0.7) & (-0.6 & -0.6) \\ (-0.8 & -0.3) & (-0.3 & -0.9) & (-0.4 & -0.8) & (-0.3 & -0.7) \\ (-0.6 & -0.6) & (-0.3 & -0.8) & (-0.9 & -0.2) & (-0.8 & -0.1) \end{bmatrix}$$

Weightage for positive and negative information

$$W_p = [(0.3 \quad 0.2 \quad 0.3 \quad 0.2)]$$

$$W_N = [(0.2 \quad 0.15 \quad 0.25 \quad 0.4)]$$

Step 3: Normalized decision matrix for positive and negative information.

$$NM^+ = \begin{bmatrix} (0.513 & 0.329) & (0.626 & 0.316) & (0.407 & 0.396) & (0.363 & 0.388) \\ (0.308 & 0.768) & (0.208 & 0.421) & (0.488 & 0.594) & (0.454 & 0.485) \\ (0.616 & 0.329) & (0.417 & 0.738) & (0.244 & 0.693) & (0.363 & 0.777) \\ (0.513 & 0.439) & (0.626 & 0.421) & (0.732 & 0.099) & (0.727 & 0.097) \end{bmatrix}$$

$$NM^- = \begin{bmatrix} (-0.534 & -0.458) & (-0.262 & -0.483) & (-0.444 & -0.419) & (-0.498 & -0.543) \\ (-0.381 & -0.642) & (-0.788 & -0.276) & (-0.518 & -0.587) & (-0.498 & -0.543) \\ (-0.610 & -0.275) & (-0.394 & -0.621) & (-0.296 & -0.671) & (-0.249 & -0.634) \\ (-0.458 & -0.550) & (-0.394 & -0.552) & (-0.667 & -0.167) & (-0.664 & -0.0905) \end{bmatrix}$$

Step 4: Calculate the weighted normalized decision matrix for positive and negative information and in this decision-making problem s_1, s_3, s_4 are benefit criteria and s_2 is the non-benefit criteria. The reference points are calculated using the formula.

$$R_p^* = R_p^{**} = (\text{sum of benefit criteria}) - (\text{sum of non benefit criteria})$$

$$= (S1 + S3 + S4) - (S2)$$

	S1	S2	S3	S4	(R_p^*, R_p^{**})
T_1	(0.153 0.098)	(0.125 0.063)	(0.122 0.118)	(0.072 0.077)	(0.222 0.230)
T_2	(0.092 0.230)	(0.041 0.084)	(0.146 0.178)	(0.090 0.097)	(0.287 0.421)
T_3	(0.184 0.098)	(0.083 0.147)	(0.073 0.207)	(0.072 0.155)	(0.246 0.313)
T_4	(0.153 0.131)	(0.125 0.084)	(0.219 0.029)	(0.145 0.019)	(0.392 0.095)

Table 1: Positive reference point

$$R_N^* = R_N^{**} = (\text{sum of benefit criteria}) - (\text{sum of non benefit criteria})$$

$$= (S1 + S3 + S4) - (S2)$$

	S1	S2	S3	S4	(R_N^*, R_N^{**})
T_1	(-0.106 -0.091)	(-0.039 -0.072)	(-0.111 -0.104)	(-0.199 -0.217)	(-0.377 -0.340)
T_2	(-0.076 -0.128)	(-0.118 -0.041)	(-0.129 -0.146)	(-0.199 -0.217)	(-0.286 -0.450)

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T_3	(-0.122 -0.055)	(-0.059 -0.093)	(-0.074 -0.167)	(-0.099 -0.253)	(-0.236 -0.382)
T_4	(-0.091 -0.11)	(-0.059 -0.082)	(-0.166 -0.041)	(-0.265 -0.036)	(-0.463 -0.105)

Table 2: Negative reference point

Step 5: Compute M_R^+ and M_R^- by using the formulas follows:

$$M_R^+ = (R_p^*)^2 - (R_p^{**})^2 \text{ and } M_R^- = (R_N^*)^2 - (R_N^{**})^2$$

	$M_R^+ = (R_p^*)^2 - (R_p^{**})^2$	$M_R^- = (R_N^*)^2 - (R_N^{**})^2$
M_{R1}^+	-0.0036	+0.0265
M_{R2}^+	-0.0949	-0.1207
M_{R3}^+	-0.0376	-0.0902
M_{R4}^+	+0.1446	+0.2033

Table 3: Defuzzification

Step 6: Calculate $M_R = M_R^+ - M_R^-$ for all alternatives and rank according to the descending order.

M_R	Ranking
-0.0301	3
+0.0258	2
+0.0527	1
-0.0587	4

Table 4: Ranking

Step 7: The ranking order is $T_3 > T_2 > T_1 > T_4$.

Hence T_3 is the first choice. The next choice will be T_2 followed by T_1 and T_4 .

6. Conclusion

In this paper, we combine the concept of bipolar Pythagorean fuzzy soft set and matrix to introduce the concept of bipolar Pythagorean fuzzy soft matrix topology. Moreover, we discuss some properties of BPFS interior and BPFS closure. Finally, we apply BPFR α GCSM on a decision making problem by using BPF Moora method. In future

work, it would be motivating to apply BPF Moora to device MCDM problems, for example robot selection, project selection.

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Controlling measles transmission dynamics with optimal control analysis

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Abstract

In this paper, a deterministic model for the transmission dynamics of measles infection with two doses of vaccination and isolation is studied. The disease-free equilibrium state and basic reproduction number, R_0 , of the model are computed. The sensitivity analysis of the model parameters is carried out using the Latin Hypercube Sampling (LHS) scheme in order to ascertain the parameters that contribute to the spread of measles in the population. The result of the sensitivity analysis shows that transmission rates, vaccination rates and isolation of the infected persons in the prodromal stage are significant parameters to be targeted for the eradication of measles infection. Based on the result of sensitivity analysis, an optimal control model with nutritional support as a control is developed. The analysis of optimal control model is carried out using Pontryagin's maximum principle to identify the optimal control strategies to be adopted by public health practitioners and policy health makers in curtailing the spread of measles infection. The result of the optimal control analysis via numerical simulations revealed that combined timely implementation of correct administration of the two doses of vaccination, isolation of infected persons in the prodromal stage and mass distribution of nutritional support would curtail the measles disease outbreak in the population. However, in a situation where there is a limited facility to isolate the infected persons in the prodromal stage, the combined implementation of mass distribution of nutritional support and administration of the two doses of vaccination will still eradicate measles infection in the population.

Keywords: Measles; Nutritional Support; Vaccination; Isolation; Sensitivity Analysis; Pairwise Comparison; Optimal Control Analysis.

2010 AMS subject classification: 49K15, 49K40, 90C31, 34D20, 34C60.

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§ Received on March 29, 2022. Accepted on June 12, 2022. Published on September 25, 2022.

doi:10.23755/rm.v43i0.742. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Measles is a viral infectious disease caused by a single-stranded RNA virus that belongs to the group of Morbilliviruses of the Paramyxoviridae family. It is a seasonal disease that occurs mostly during the dry season in tropical zones where it is endemic and it peaks during late winter and early spring in temperate zones [24]. Non-immune people are infected via direct contact with the nasal and oral secretions or inhaling the aerosol droplets of an infected person. 90% of non-immune people are exposed to an infective and have the chance of being infected with measles disease [24, 27]. Measles has an average of 10 - 12 days incubation from exposure to the prodromal stage [28], which spans for seven days of the infection period, after which the infective recovers with lifelong immunity against the disease. The symptoms of measles are based on different stages of the disease. Prodromal stage symptoms include high fever, runny nose (coryza), cough, and red eyes (conjunctivitis) that lasts 2 to 4 days with a range of 1- 7 days, while the rash stage symptoms occur a few days after the initial symptoms. The rash stage can lead to fatal complications or death if not treated early [10,12].

Annually, measles affects up to 20 million people worldwide and most cases are from Africa and Asia [24]. According to a report from World Health Organization and United State Center for Diseases Control and Prevention (CDC) [44, 45], 869,770 infection cases with 207,500 deaths of Measles were recorded globally in 2019, making it the highest number since the 1996 outbreak and has 50% increment as of 2016. For sub-Saharan Africa, about 134,200 measles deaths were recorded in 2015, while Nigeria recorded a significant increase of 28,400 cases in 2019 compared with 5,067 cases in 2018. Despite the cases dropping to 9,316 in 2020, the confirmed cases of measles remain high, and the case fatality is yet to be eased anytime there is an outbreak. This implies a need for comprehensive efforts and intervention strategies to reduce the menace of measles is crucial. Therefore, it is necessary to examine the optimal strategy that can be implemented to control measles disease in high-burden countries.

According to WHO, the two major strategies to eradicate measles are vaccine and treatment [21, 23]. Isolation of infected people is also important in preventing further spread of the disease but increasing population immunity through vaccination remains the most effective way to prevent outbreaks of measles in a community [22]. The vaccination is mainly based on MMR (measles, mumps, and rubella) and MMRV (measles, mumps, rubella, and varicella) vaccines. These vaccines are about 95% effective as they globally prevent 4.5 million deaths yearly [16]. There are two doses of MMR vaccine. The first dose produces 90 to 95% immunity to measles while the second dose produces a stronger immunity for those that do not respond to the first dose [17]. Among the childhood vaccine-preventable diseases, measles causes the most deaths in children. Measles outbreaks is prevented in a community if 90% to 95% of children are vaccinated.

Mathematical models of infectious diseases are useful in studying transmission dynamics of diseases, testing theories, planning, implementing, evaluating and comparing various control programs that will prevent the further spread of diseases and their epidemics. A notable number of mathematical models have been elaborated and applied to infectious diseases like measles [4, 7, 9, 18, 25]. Some authors, like [8, 10, 19],

developed an SEIR model of measles where testing and diagnosis therapy was incorporated as in [10] at the latent period. Authors [15, 20] considered the effect of supplemental immunization activities as an optimal policy for measles using an age-stratified compartmental model. Stephen *et al.* [17] revealed that the spread of measles disease largely depends on the contact rates with infected people within a population and the disease dies out in the population if the proportion of the population that is immune exceeds the herd immunity level. Vaccination is considered in the autonomous models [3, 4, 11, 13, 19, 29] as constant parameter or compartment for vaccinated people, while in [27-31], it is examined as a time-dependent control function to determine the optimal vaccination strategy that can be implemented to control measles in high-burden countries. Although [30 – 35] considered optimal control of vaccination for measles but the effect of two doses of vaccination and nutritional support are not examined. However, the authors [3, 37, 40] examined the effect of two doses of vaccination and isolation on measles disease as constant parameters without nutritional support impact and optimal control analysis. As advised by WHO [46], it is important to consider the effect of two doses of vaccination and nutritional support on measles transmission dynamics, which forms the study's motivation. This involves modification of the model by [3] to investigate the impact of the two doses of vaccination, nutritional support and isolation on measles dynamics using sensitivity analysis and optimal control analysis approaches. This will help to provide the mathematical analysis of the possible control strategy (vaccine and nutritional support) that will help the public health practitioners to achieve the best strategy for the prevention and control of the spread of measles in community.

The rest of the paper is organized as follows: Section 2 is the model formulation for measles with constant control measures. The model analysis is discussed in Section 3 which includes sensitivity analysis. We obtained the optimal control of the formulated model in Section 4. In Section 5, we carried out numerical simulation to verify some analytic results and their discussion, while Section 6 is the conclusion.

2. Model formulation

A deterministic model for measles disease is presented by modifying the model by Aldila and Asrianti [3]. The total population at any time (t) denoted by $N(t)$, is subdivided into Susceptible persons, $S(t)$, Exposed persons, $E(t)$, Infected persons in prodromal stage, $P(t)$, Infected persons in rash stage, $I(t)$, Isolated persons, $J(t)$, 1st dose of vaccinated persons, $V_1(t)$, 2nd dose of vaccinated persons, $V_2(t)$, and Recovered persons, $R(t)$ such that

$$N = S + E + P + I + J + V_1 + V_2 + R. \quad (1)$$

The susceptible persons, $S(t)$, decreases when they come in contact with the infected persons at a force of infection, λ_1 and become exposed person or by vaccination with 1st dose vaccine at a rate, ε_1 . The persons vaccinated with 1st dose vaccine may be

infected at a force of infection, λ_2 since vaccine is not 100% efficacy. The force of infections, λ_1 and λ_2 , are given by

$$\left. \begin{aligned} \lambda_1 &= \frac{\beta_1(P+n_1I+n_2J)}{N} \\ \lambda_2 &= \frac{\beta_2(P+n_1I+n_2J)}{N} \end{aligned} \right\} \quad (2)$$

where β_1 and β_2 are the transmission rate for the susceptible and 1st dose vaccinated persons respectively, n_1 and n_2 are the parameters that reduce the infectivity of the infected persons in the rash stage and isolated persons respectively.

The 1st dose of vaccinated persons, $V_1(t)$ receive 2nd dose vaccine at a rate, ε_2 and achieve immunity at the rate, σ . The exposed persons, $E(t)$, becomes infected persons in the prodromal stage, $P(t)$ at a rate, k after incubation period of measles disease. The infected persons in the prodromal stage, $P(t)$, then progress to rash stage at a rate, α while some are isolated for further treatment at a rate, φ_1 or they recovered at a rate, δ_1 . In a similar way, the infected persons at the rash stage, $I(t)$, are isolated at a rate, φ_2 or recovered from measles at a rate, δ_2 . Meanwhile, the isolated persons recovered at a rate, δ_3 . It is assumed that all the subpopulations experience natural death at a rate, μ and the subpopulations, $I(t)$ and $J(t)$ may die of measles disease at d_1 and d_2 respectively.

The model description and details of the model parameters are presented in Figure 1 and Table 1. respectively.

With Figure 1 and Table 1, the transition within subpopulations are expressed by the following system of first order differential equations;

$$\left. \begin{aligned} \frac{dS}{dt} &= \Lambda - \lambda_1 S - (\varepsilon_1 + \mu)S, \\ \frac{dV_1}{dt} &= \varepsilon_1 S - (\lambda_2 + \varepsilon_2 + \mu)V_1, \\ \frac{dV_2}{dt} &= \varepsilon_2 V_1 - (\mu + \sigma)V_2, \\ \frac{dE}{dt} &= \lambda_1 S + \lambda_2 V_1 - (\mu + k)E, \\ \frac{dP}{dt} &= kE - (\alpha + \mu + \varphi_1 + \delta_1)P, \\ \frac{dI}{dt} &= \alpha P - (\varphi_2 + \delta_2 + \mu + d_1)I, \\ \frac{dJ}{dt} &= \varphi_1 P + \varphi_2 I - (\delta_3 + \mu + d_2)J, \\ \frac{dR}{dt} &= \sigma V_2 + \delta_1 P + \delta_2 I + \delta_3 J - \mu R \end{aligned} \right\} \quad (3)$$

Controlling measles transmission dynamics with optimal control analysis

where $\lambda_1 = \frac{\beta_1(P+n_1I+n_2J)}{N}$, $\lambda_2 = \frac{\beta_2(P+n_1I+n_2J)}{N}$ and the initial conditions, $S(0) > 0, V_1(0) \geq 0, V_2(0) \geq 0, E(0) \geq 0, P(0) \geq 0, I(0) \geq 0, J(0) \geq 0, R(0) \geq 0$. The model parameters are assumed to be nonnegative except recruitment rate, Λ , that is strictly positive.

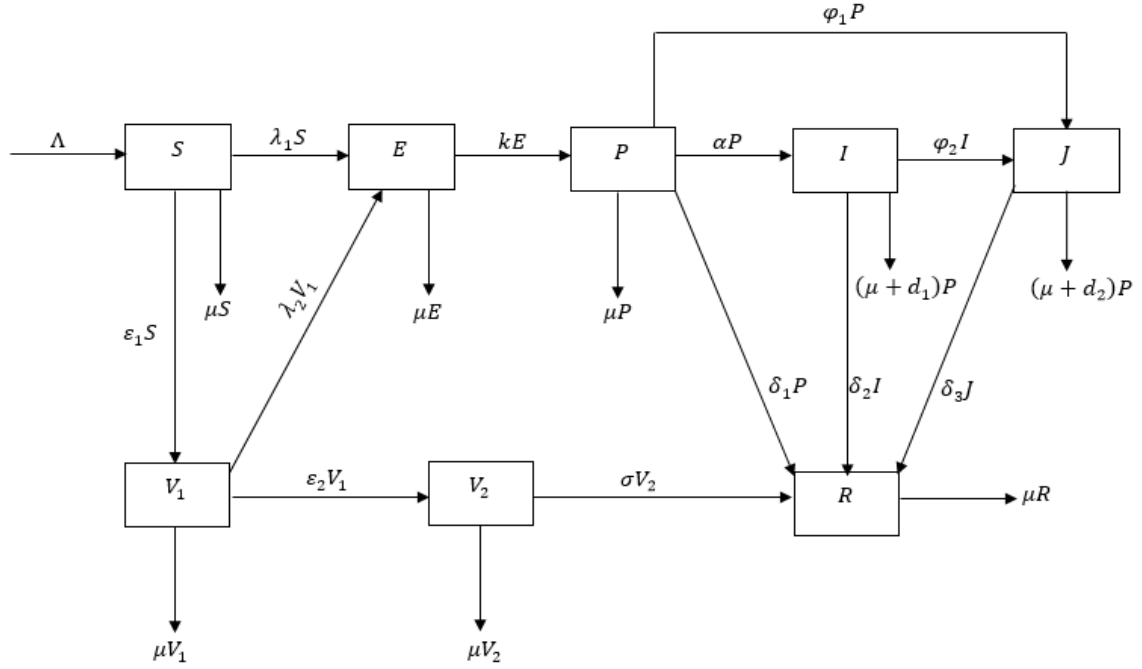


Figure 1. Model flow diagram for transmission dynamics of measles disease.

Table 1. Parameters and their descriptions

Parameters	Parameters Description	(Ranges)Nominal values	Sources
Λ	Recruitment rate	($-$)2000	[1]
β_1	Transmission rate for $S(t)$	(0.0004 – 0.5)0.6	Assumed
β_2	Transmission rate for $V(t)$	(0.0003 – 0.4)0.5	Assumed
ε_1	Vaccination rate of first dose vaccine	(0.01 – 0.95)0.6	[38]
ε_2	Vaccination rate of second dose vaccine	(0.01 – 0.95)0.01	Assumed
n_1	Infectivity reduction rate of $I(t)$	($-$)0.1	[3]
n_2	Infectivity reduction rate of $J(t)$	($-$)0.01	[3]
μ	Natural death rate	($-$) 1/65.365	[3]
k	Progression rate from $E(t)$ to $P(t)$	($-$)0.09	[1]
δ_1	Recovery rate for $P(t)$ class	($-$)0.2	[3]
δ_2	Recovery rate for $I(t)$ class	($-$)0.06	Assumed
δ_3	Recovery rate for $J(t)$ class	($-$)0.3121	[36]
d_1	Disease-related death rate for $I(t)$ class	($-$)0.125	[39]
d_2	Disease-related death rate for $J(t)$ class	($-$)0.1	[1]
α	Progression rate from $P(t)$ to $I(t)$	($-$)0.003	Assumed
ω_1	Isolation rate for $P(t)$ class	(0.0001 – 0.05)0.01	[1]
ω_2	Isolation rate for $I(t)$ class	(0.001 – 0.5)0.001	Assumed
σ	Recovery rate due to 2 nd dose of vaccine	($-$)0.01	[38]

3. Model analysis

Here, the system (3) is establish to show well-posedness of the model that makes biological sense. This is done by proving the existence of nonnegative solution and its boundedness given initial solutions of the model.

Theorem 1. With the initial solutions, $S(0) > 0$, $V_1(0) \geq 0$, $V_2(0) \geq 0$, $E(0) \geq 0$, $P(0) \geq 0$, $I(0) \geq 0$, $J(0) \geq 0$, $R(0) \geq 0$, the model equation (3) has non-negative solutions for all time, $t > 0$.

Proof. Let $t_1 = \sup\{t > 0: S(0) > 0, V_1(0) \geq 0, V_2(0) \geq 0, E(0) \geq 0, P(0) \geq 0, I(0) \geq 0, J(0) \geq 0, R(0) \geq 0\} \in [0, t]$.

From the first equation of system (3), we have

$$\frac{dS}{dt} = \Lambda - \lambda_1 S - \varepsilon_1 S - \mu S \geq -(\varepsilon_1 + \mu + \lambda_1)S.$$

Applying the method of integrating factor with initial condition, $S(0)$, we have

$$S(t) \geq S(0) \exp \left\{ - \int_0^t (\varepsilon_1 + \mu + \lambda_1) t_1 \right\} > 0$$

which is always positive for $t > 0$.

In similar way, $V_1(t) > 0, V_2(t) > 0, E(t) > 0, P(t) > 0, I(t) > 0, J(t) > 0, R(t) > 0$ for $t > 0$. This means that the solution set $\{S(t), V_1(t), V_2(t), E(t), P(t), I(t), J(t), R(t)\}$ of the system (3) is non-negative for all $t > 0$.

To show the boundedness of the solutions of the system (3), we state and prove feasible region of the system (3).

Theorem 2. The solutions of system (3) are contained in the feasible region, $\Omega = \{(S, V_1, V_2, E, P, I, J, R) \in \mathfrak{R}_+^8: N \leq \frac{\Lambda}{\mu}\}$ with the non-negative initial conditions.

Proof. To obtain the total population, $N(t)$, we sum up the equations of system (3) to yields

$$\frac{dN}{dt} = \Lambda - \mu N - d_1 I - d_2 J \leq \Lambda - \mu N. \quad (4)$$

Applying Gronwall's inequality with the initial condition, $N(0) = N_0$ in equation (4) gives

$$N(t) \leq \frac{\Lambda}{\mu} + \left[N_0 - \frac{\Lambda}{\mu} \right] e^{-\mu t}. \quad (5)$$

If $N_0 > (<) \frac{\Lambda}{\mu}$, the total population, N , tends to $\frac{\Lambda}{\mu}$ as $t \rightarrow \infty$. Thus, in either case, the total population, $N(t) \rightarrow \frac{\Lambda}{\mu}$ as $t \rightarrow \infty$ in (5). Hence, the solution set of system (3) will enter the feasible region, Ω that is positively invariant.

3.1 Existence of disease-free equilibrium state and basic reproduction number

Disease-free equilibrium state occurs when there is no infection in the population, that is when the infected state variables are zero.

Solving simultaneously at equilibrium state, $\frac{dS}{dt} = 0$, $\frac{dV_1}{dt} = 0$, $\frac{dV_2}{dt} = 0$, $\frac{dE}{dt} = 0$, $\frac{dP}{dt} = 0$, $\frac{dI}{dt} = 0$, $\frac{dJ}{dt} = 0$, $\frac{dR}{dt} = 0$ of the system (3) gives the disease-free equilibrium state,

$$E_0 = (S^0, V_1^0, V_2^0, E^0, P^0, I^0, J^0, R^0) \\ = \left(\frac{\Lambda}{(\varepsilon_1 + \mu)}, \frac{\varepsilon_1 \Lambda}{f(\varepsilon_1 + \mu)}, \frac{\varepsilon_1 \varepsilon_2 \Lambda}{f(\sigma + \mu)(\varepsilon_1 + \mu)}, 0, 0, 0, 0, \frac{\sigma \varepsilon_1 \varepsilon_2 \Lambda}{\mu f(\sigma + \mu)(\varepsilon_1 + \mu)} \right) \quad (6)$$

with $f = \varepsilon_2 + \mu$.

Basic Reproduction Number, R_0

The basic reproduction number is a threshold quantity that determines the persistence and eradication of the infectious disease in the population, making it the most important quantity in infectious disease epidemiology. It is defined as the mean number of persons infected when a single infective is introduced into a wholly susceptible population [6]. R_0 is computed using the next-generation matrix approach [6].

Following the approach in [6], the rate of new infection, $\mathcal{F}_i$, and the rate of transitional terms, $\mathcal{V}_i$, in compartment i , of the system (3) are given as

$$\mathcal{F}_i = \begin{pmatrix} \frac{\beta_1(P+n_1I+n_2J)S}{N} + \frac{\beta_2(P+n_1I+n_2J)V_1}{N} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathcal{V}_i = \begin{pmatrix} gE \\ -kE + hP \\ -\alpha P + pI \\ -\varphi_1 I - \varphi_2 P + qJ \end{pmatrix},$$

where $i = 1, \dots, 4$ is the number of infected compartments and

$$g = (\mu + k), h = (\alpha + \varphi_1 + \delta_1 + \mu), p = (\varphi_2 + \delta_2 + \mu + d_1), q = (\delta_3 + \mu + d_2). \quad (7)$$

Taking the partial derivative of $\mathcal{F}_i$ and $\mathcal{V}_i$ with respect to E, P, I and J at DFE, E_0 , we have respective Jacobian matrices

$$F = \begin{pmatrix} 0 & \frac{(\beta_1 S^0 + \beta_2 V_1^0)}{N^0} & \frac{n_1(\beta_1 S^0 + \beta_2 V_1^0)}{N^0} & \frac{n_2(\beta_1 S^0 + \beta_2 V_1^0)}{N^0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ V = \begin{pmatrix} g & 0 & 0 & 0 \\ -k & h & 0 & 0 \\ 0 & -\alpha & p & 0 \\ 0 & -\varphi_1 & -\varphi_2 & q \end{pmatrix},$$

where

$$N^0 = S^0 + V_1^0 + V_2^0 + E^0 + P^0 + I^0 + J^0 + R^0 = \frac{\Lambda}{\mu}.$$

The inverse of V is given as

$$V^{-1} = \begin{pmatrix} \frac{1}{g} & 0 & 0 & 0 \\ \frac{k}{gh} & \frac{1}{h} & 0 & 0 \\ \frac{\alpha k}{ghp} & \frac{\alpha}{hp} & \frac{1}{p} & 0 \\ \frac{k(\alpha\varphi_2+p\varphi_1)}{ghpq} & \frac{(\alpha\varphi_2+p\varphi_1)}{hpq} & \frac{\varphi_2}{pq} & \frac{1}{q} \end{pmatrix}.$$

With definition of basic reproduction number, R_0 , as the spectral radius of matrix, FV^{-1} , we have

$$R_0 = \frac{(\beta_1 S^0 + \beta_2 V_1^0)}{N^0} \left[\frac{k}{gh} + \frac{\alpha k n_1}{ghp} + \frac{k n_2 (\alpha\varphi_2 + p\varphi_1)}{ghpq} \right].$$

With the substitution of $S^0 = \frac{\Lambda}{\varepsilon_1 + \mu}$, $V_1^0 = \frac{\varepsilon_1 \Lambda}{f(\varepsilon_1 + \mu)}$ and $N^0 = \frac{\Lambda}{\mu}$, we have

$$R_0 = \frac{\mu(\beta_1 f + \beta_2 \varepsilon_1)}{f(\varepsilon_1 + \mu)} \left[\frac{k}{gh} + \frac{\alpha k n_1}{ghp} + \frac{k n_2 (\alpha\varphi_2 + p\varphi_1)}{ghpq} \right].$$

By the virtue of next-generation matrix approach [6], the disease-free equilibrium, E_0 , of system (3) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. This means that the measles infection will die out in the population if $R_0 < 1$ while it will persist in the population when $R_0 > 1$.

3.2 Sensitivity analysis

Sensitivity analysis plays an important role in examining the effect, influence and contribution of the parameters of a mathematical model to the model output. To know the type of intervention strategies to adopt in reducing the transmission and prevalence of any disease, sensitivity analysis is carried-out to determine the biological significance of the model parameters in relation to the reproduction number, R_0 . We adopt the Latin Hypercube Sampling (LHS) scheme used by [41,42,43] with the Partial Rank Correlation Coefficients (PRCCs) procedure to assess the biological implications of each input parameter to the output parameter, the disease threshold, R_0 . This type of approach provides numerical results that enable us to explore the entire parameter space simultaneously, thereby producing an unbiased selection of the parameter values. The

sign (positive or negative) of the PRCCs indicates the precise strength of the relationship between the input variables (parameters of the model) and the output variable, R_0 in this case. It also provides an insight to the degree of monotonicity between the parameters of the model and R_0 . Thus, comparing the values of PRCC enabled us to directly evaluate the impact of the model parameters on R_0 .

Figure 2 shows the PRCCs for some important parameters of the model. The parameters β_1 and β_2 have positive PRCCs meaning increasing their values increase R_0 , which in return increase the spread of measles infection in the population. Whereas, the parameters ε_1 , ε_2 , ω_1 and ω_2 with negative PRCCs reduce the value of R_0 when they are increased. They have the capacity of ameliorate the spread of measles infection in the population, which leads to the eradication of the disease in the population. However, the parameter ω_2 has a small magnitude of PRCC that is non-monotonically related to R_0 but it can still produce a change in the transmission dynamics of measles infection. In other to identify the model parameters that are significant in curtailing or enhancing the spread of measles disease, the Fisher Transformation is applied to the PRCCs to compute the p-values of each of the model parameters as used in [42]. This is shown in Table 2. It is observed in Table 2 that the parameters, $(\beta_1, \beta_2, \varepsilon_1, \varepsilon_2, \omega_1)$ have p-values that are significant while the parameter, ω_2 , has an insignificance p-value. This is further shown in Figure 3 as scatterplots for R_0 against some model parameters. From Figure 3, it is observed that the parameters $(\beta_1, \beta_2, \varepsilon_1, \varepsilon_2, \omega_1)$ have a significant impact on R_0 than ω_2 .

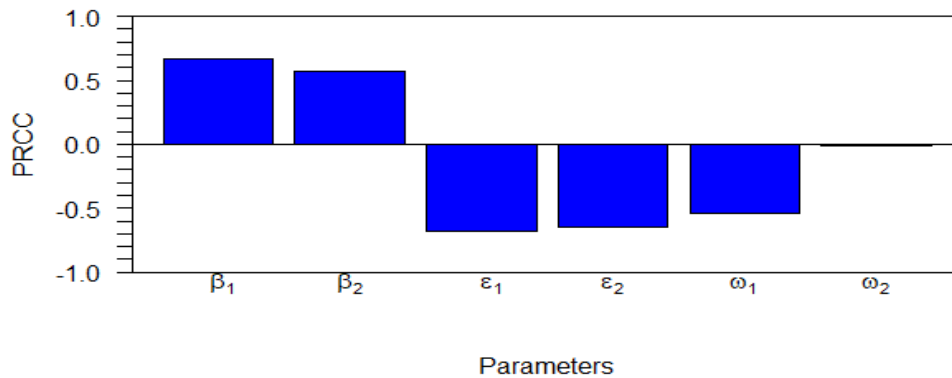


Figure 2. Tornado plot for some significant model parameters.

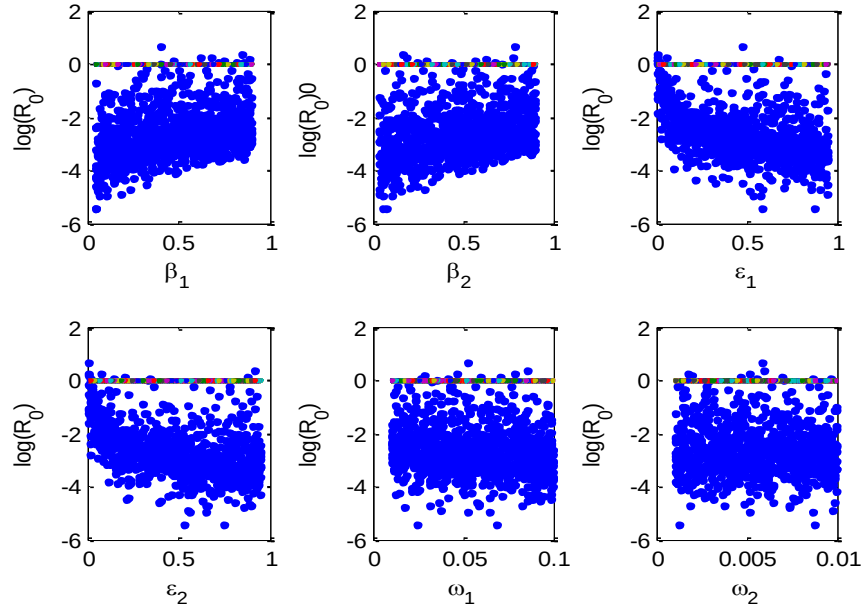


Figure 3. Monte Carlo simulations for some important parameters of the model generated using the parameter values in Table 1. In each simulation run, 1000 randomly selected parameters were used.

Table 2. Parameter PRCC significance (unadjusted p-value)

Parameter	PRCC	p-value	Keep
β_1	0.65564141	0.0000	TRUE
β_2	0.57787305	0.0000	TRUE
ϵ_1	-0.67496222	0.0000	TRUE
ϵ_2	-0.65937154	0.0000	TRUE
ω_1	-0.54171573	0.0000	TRUE
ω_2	-0.01142558	0.7049	FASLE

Table 3. Pairwise PRCC Comparisons (unadjusted p-values)

	β_1	β_2	ε_1	ε_2	ω_1
β_1		0.00506	0	0	0
β_2			0	0	0
ε_1				0.5314	2.048×10^{-6}
ε_2					3.743×10^{-5}
ω_1					

Table 4. Pairwise PRCC Comparisons (FDR Adjusted p-values)

	β_1	β_2	ε_1	ε_2	ω_1
β_1		0.005622	0	0	0
β_2			0	0	0
ε_1				0.5314	2.926×10^{-6}
ε_2					4.679×10^{-5}
ω_1					

Table 5. Parameters different after FDR adjustment?

	β_1	β_2	ε_1	ε_2	ω_1
β_1		TRUE	TRUE	TRUE	TRUE
β_2			TRUE	TRUE	TRUE
ε_1				FASLE	TRUE
ε_2					TRUE
ω_1					

Tables 3 and 4 show the pairwise comparison of the important parameters of the model, whose p-values are less than 0.05, in other to establish if there exist any difference between the processes describing the compared parameters. The results of the pairwise PRCC comparison for the unadjusted p-values and the false discovery rate (FDR) adjusted p-values are presented in Table 3 and Table 4, respectively. With the FDR adjusted p-values in Table 4, we present the parameters different in Table 5. If the p-values of the compared pair of significant parameters are less than 0.05, we say that they

are different (TRUE); otherwise not different (FALSE). We also noted from Table 5, that apart from $\epsilon_1 - \epsilon_2$ pair, all other pairs of parameters are significantly different.

The parameters $\beta_1, \beta_2, \epsilon_1, \epsilon_2, \omega_1$ play a vital role in the eradication of the measles disease. The spread of measles infection will reduce drastically if the value of R_0 is less than a unity ($R_0 < 1$), which implies reducing the values of β_1 and β_2 as well as increasing the values of $\epsilon_1, \epsilon_2, \omega_1$. This establishes that isolating the infected individuals in the prodromal stage and minimizing contact with infected persons (both at the prodromal and rash stage) will eradicate the spread of measles in the population. Also, increasing the rate of correct administration of vaccines (first and second dose) will go a long way in reducing the number of infected individuals as many susceptible people will be protected by vaccination thereby minimize the spread of measles before and during the epidemic.

4. Optimal control analysis

Optimal control has been extensively applied as a strategy in controlling many epidemic outbreaks. The main idea of applying the optimal control to measles epidemics is to choose among the available strategies, the most suitable and effective strategies that will reduce the measles infection rate to a minimum level while optimizing the cost of deploying these strategies [26]. In terms of measles epidemics, such strategy can include therapies, vaccines, isolation and educational campaigns [5].

Based on the result of the sensitivity analysis, the functions, $u_1(t), u_2(t), u_3(t), u_4(t)$, are considered as time-dependent control functions where $u_1(t)$ is mass distribution of nutrition (supplement) support that reduces the transmission rates, $u_2(t)$ is the first dose vaccination control, $u_3(t)$ is the second dose vaccination control and $u_4(t)$ is isolation of infected people in the prodromal stage. The nutritional support is to boost the immune system of the body. Thus, the optimal control model of the system (3) is given by

$$\left. \begin{aligned} \frac{dS}{dt} &= \Lambda - \frac{(1-u_1(t))\beta_1(P+n_1I+n_2J)S}{N} - \epsilon_1 u_2(t)S - \mu S, \\ \frac{dV_1}{dt} &= \epsilon_1 u_2(t)S - \frac{(1-u_1(t))(1-u_2(t))\beta_2(P+n_1I+n_2J)V_1}{N} - \epsilon_2 u_3(t)V_1 - \mu V_1, \\ \frac{dV_2}{dt} &= \epsilon_2 u_3(t)V_1 - (\mu + \sigma)V_2, \\ \frac{dE}{dt} &= \frac{(1-u_1(t))\beta_1(P+n_1I+n_2J)S}{N} + \frac{(1-u_1(t))(1-u_2(t))\beta_2(P+n_1I+n_2J)V_1}{N} - (\mu + k)E, \\ \frac{dP}{dt} &= kE - (\alpha + \mu + \varphi_1 + u_4(t) + \delta_1)P, \\ \frac{dI}{dt} &= \alpha P - (\varphi_2 + \delta_2 + \mu + d_1)I, \\ \frac{dJ}{dt} &= \varphi_1 P + u_4(t)P + \varphi_2 I - (\delta_3 + \mu + d_2)J, \\ \frac{dR}{dt} &= \sigma V_2 + \delta_2 I + \delta_1 P + \delta_3 J - \mu R. \end{aligned} \right\} \quad (8)$$

These control functions are bounded, Lebesgue integrable functions that satisfy $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 0.95$, $0 \leq u_3 \leq 0.95$ and $0 \leq u_4 \leq 1$ with assumption that the highest vaccination coverage will be 95%.

The goal is to minimize the number of infected people $(P(t), I(t), J(t))$ and maximize the number of susceptible people $S(t)$ while minimizing the cost of controls. Therefore, the objective function is given as

$$\Gamma(u_1, u_2, u_3, u_4) = \int_0^{t_f} \left(bP + cI + dJ + \frac{1}{2} \sum_{i=1}^4 m_i u_i^2(t) \right) dt \quad (9)$$

and is subject to equation (8) with the initial conditions of the system (3).

In equation (9), the constants, $b, c, d, m_1, m_2, m_3, m_4$, are positive weights to balance the size of the terms attached with them and t_f is the final time to implement the controls, $u_1(t), u_2(t), u_3(t), u_4(t)$. The terms, bP, cI, dJ are the cost related to reducing the number of infected people, (P, I, J) such as cost of the mass distribution of nutrition (supplement) support, isolation and vaccination, first dose and second dose vaccination at the due time.

We seek optimal controls $u_1^*, u_2^*, u_3^*, u_4^*$ such that

$$\Gamma(u_1^*, u_2^*, u_3^*, u_4^*) = \min\{\Gamma(u_1, u_2, u_3, u_4) | u_1, u_2, u_3, u_4 \in U\}. \quad (10)$$

With the application of Pontryagin's maximum principle [14], the equations (8) and (9) are converted into a problem of minimizing pointwise a Hamiltonian, H with respect to u_1, u_2, u_3, u_4 . This is given by

$$\begin{aligned} H = & bP + cI + eJ + \frac{m_1 u_1^2(t)}{2} + \frac{m_2 u_2^2(t)}{2} + \frac{m_3 u_3^2(t)}{2} + \frac{m_4 u_4^2(t)}{2} + \zeta_1 \left(\Lambda - \right. \\ & \left. \frac{(1-u_1(t))\beta_1(P+n_1I+n_2J)S}{N} - \varepsilon_1 u_2(t)S - \mu S \right) + \zeta_2 \left(\varepsilon_1 u_2(t)S - \right. \\ & \left. \frac{(1-u_1(t))(1-u_2(t))\beta_2(P+n_1I+n_2J)V_1}{N} - \varepsilon_2 u_3(t)V_1 - \mu V_1 \right) + \zeta_3 (\varepsilon_2 u_3(t)V_1 - (\mu + \sigma)V_2) + \\ & \zeta_4 \left(\frac{(1-u_1(t))\beta_1(P+n_1I+n_2J)S}{N} + \frac{(1-u_1(t))(1-u_2(t))\beta_2(P+n_1I+n_2J)V_1}{N} - (\mu + k)E \right) + \zeta_5 (kE - \\ & (\alpha + \mu + \varphi_1 + u_4(t) + \delta_1)P) + \zeta_6 (\alpha P - (\varphi_2 + \delta_2 + \mu + d_1)I) + \zeta_7 (\varphi_1 P + \\ & u_4(t)P + \varphi_2 I - (\delta_3 + \mu + d_2)J) + \zeta_8 (\sigma V_2 + \delta_2 I + \delta_1 P + \delta_3 J - \mu R) \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8$ as respective adjoint variables for the state variables, $S, V_1, V_2, E, P, I, J, R$.

The system of adjoint variables are derived by taking the partial derivative of H with respect to each of their corresponding state variables. This is given by

$$\left. \begin{aligned}
 \frac{d\zeta_1}{dt} &= -\frac{\partial H}{\partial S} = a + A \left(1 - \frac{S}{N}\right) (\zeta_1 - \zeta_4) + BV_1(\zeta_4 - \zeta_2) \\
 &\quad + \zeta_1\mu + \varepsilon_1 u_2(t)(\zeta_1 - \zeta_2), \\
 \frac{d\zeta_2}{dt} &= -\frac{\partial H}{\partial V_1} = BN \left(1 - \frac{S}{N}\right) (\zeta_2 - \zeta_4) + \frac{AS}{N} (\zeta_4 - \zeta_1) + \zeta_2\mu \\
 &\quad + \varepsilon_2 u_3(t)(\zeta_2 - \zeta_3), \\
 \frac{d\zeta_3}{dt} &= -\frac{\partial H}{\partial V_2} = BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1) + \sigma(\zeta_3 - \zeta_8) + \zeta_3\mu, \\
 \frac{d\zeta_4}{dt} &= -\frac{\partial H}{\partial E} = BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1) + (\zeta_4 - \zeta_5)k + \zeta_4\mu, \\
 \frac{d\zeta_5}{dt} &= -\frac{\partial H}{\partial P} = -b + BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1) + (\zeta_5 - \zeta_8)\delta_1 \\
 &\quad + \frac{(1-u_1(t))(1-u_2(t))\beta_2 V_1}{N} (\zeta_2 - \zeta_4) + (\zeta_5 - \zeta_7)(\varphi_1 + u_4(t)) \\
 &\quad + \frac{(1-u_1(t))\beta_1 S}{N} (\zeta_1 - \zeta_4) + (\zeta_5 - \zeta_6)\alpha + \zeta_5\mu, \\
 \frac{d\zeta_6}{dt} &= -\frac{\partial H}{\partial I} = -c + BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1) + \zeta_6(\mu + d_1) \\
 &\quad + \frac{(1-u_1(t))(1-u_2(t))n_1\beta_2 V_1}{N} (\zeta_2 - \zeta_4) + (\zeta_6 - \zeta_7)\varphi_2 \\
 &\quad + \frac{(1-u_1(t))n_1\beta_1 S}{N} (\zeta_1 - \zeta_4) + (\zeta_6 - \zeta_8)\delta_2, \\
 \frac{d\zeta_7}{dt} &= -\frac{\partial H}{\partial J} = -d + BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1) + \zeta_7(\mu + d_2) \\
 &\quad + \frac{(1-u_1(t))(1-u_2(t))n_2\beta_2 V_1}{N} (\zeta_2 - \zeta_4) + \zeta_8\mu + (\zeta_7 - \zeta_8)\delta_3 \\
 &\quad + \frac{(1-u_1(t))n_2\beta_1 S}{N} (\zeta_1 - \zeta_4), \\
 \frac{d\zeta_8}{dt} &= -\frac{\partial H}{\partial R} = BV_1(\zeta_4 - \zeta_2) + \frac{AS}{N} (\zeta_4 - \zeta_1),
 \end{aligned} \right\} \quad (11)$$

where $A = \frac{(1-u_1(t))\beta_1(P+n_1I+n_2J)}{N}$ and $B = \frac{(1-u_1(t))(1-u_2(t))\beta_2(P+n_1I+n_2J)}{N^2}$ with transversality conditions

$$\zeta_1(t_f) = \zeta_2(t_f) = \zeta_3(t_f) = \zeta_4(t_f) = \zeta_5(t_f) = \zeta_6(t_f) = \zeta_7(t_f) = \zeta_8(t_f) = 0. \quad (12)$$

Furthermore, the respective controls, $u_1^*, u_2^*, u_3^*, u_4^*$ are obtained by solving $\frac{\partial H}{\partial u_1} = 0$, $\frac{\partial H}{\partial u_2} = 0$, $\frac{\partial H}{\partial u_3} = 0$, $\frac{\partial H}{\partial u_4} = 0$ and these are given by

$$\begin{aligned}
 u_1^* &= \frac{(P+n_1I+n_2J)(\beta_1 S(\zeta_4 - \zeta_1) + (1-u_2(t))\beta_2 V_1(\zeta_4 - \zeta_2))}{m_1 N}, \\
 u_2^* &= \frac{(P+n_1I+n_2J)(1-u_1(t))\beta_2 V_1(\zeta_4 - \zeta_2)}{m_1 N} + \frac{(\zeta_1 - \zeta_2)\varepsilon_1 S}{m_2},
 \end{aligned}$$

$$u_3^* = \frac{(\zeta_2 - \zeta_3)\varepsilon_2 V_1}{m_3}, \quad u_4^* = \frac{(\zeta_5 - \zeta_7)P}{m_4}.$$

With the controls $u_1^*, u_2^*, u_3^*, u_4^*$, the optimality condition is given by

$$\left. \begin{aligned} u_1^{opt} &= \max\{0, \min(1, u_1^*)\}, \\ u_2^{opt} &= \max\{0, \min(0.95, u_2^*)\}, \\ u_3^{opt} &= \max\{0, \min(0.95, u_3^*)\}, \\ u_4^{opt} &= \max\{0, \min(1, u_4^*)\}. \end{aligned} \right\} \quad (13)$$

The optimality system consists of the state system (8), the adjoint system (11) with initial conditions of (3) and transversality condition (12) together with the characterization of the optimality condition (13).

The restrictions of obtaining the uniqueness of the optimal control based on the length of time follow the technique in [2, 9, 14].

5. Numerical simulations

In this section, the solutions of the optimality system are solved numerically using the forward and backward fourth-order Runge-Kutta method that is coded in MatLab software. The parameter values in Table 1, $b = c = d = 100$, $m_1 = 10000$, $m_2 = 2000$, $m_3 = 2000$, $m_4 = 5000$ and the initial conditions, $S(0) = 200000$, $V_1(0) = 2000$, $V_2(0) = 1800$, $E(0) = 80$, $P(0) = 60$, $I(0) = 100$, $J(0) = 20$, $R(0) = 10000$ are used for the numerical simulations purpose.

Controlling measles transmission dynamics with optimal control analysis

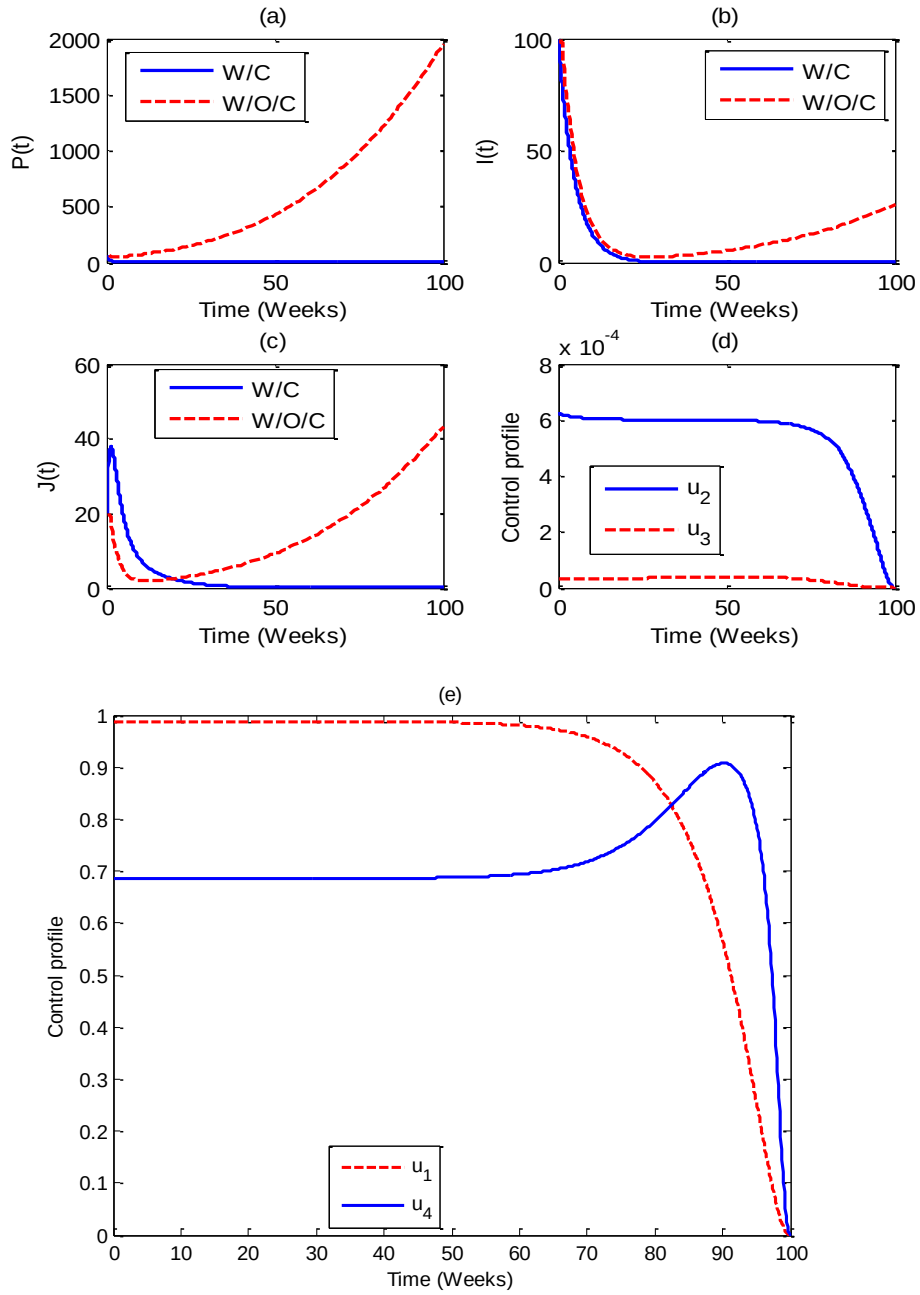


Figure 4. The population dynamics of (a) infected persons in the prodromal stage, $P(t)$, (b) infected persons in the rash stage, $I(t)$, (c) Isolated persons, $J(t)$, (d) Controls, u_2 , u_3 , and (e) Controls, u_1 , u_4 . Here, W/C means “With Control” while W/O/C means “Without Control”.

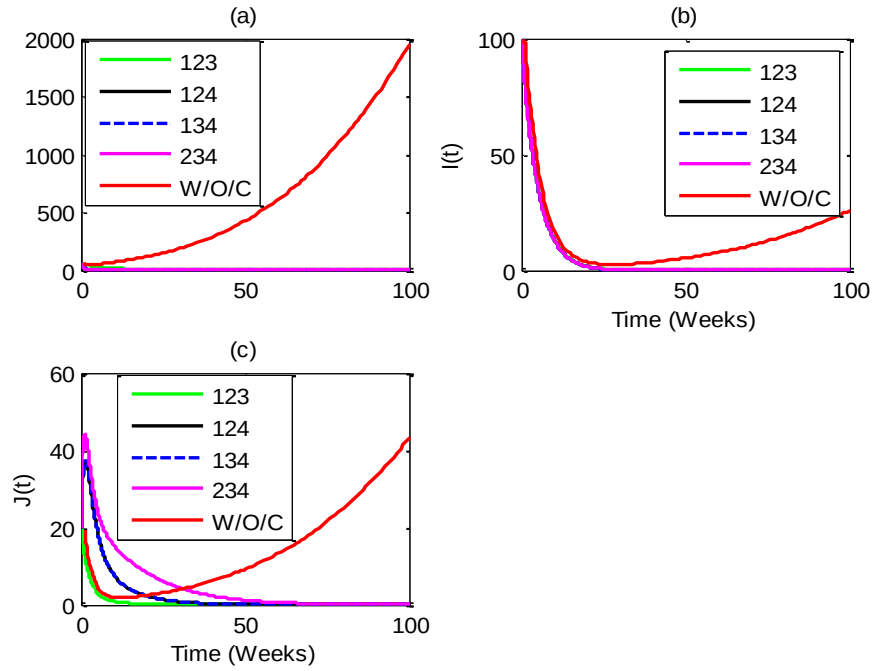


Figure 5. The population dynamics of (a) infected persons in the prodromal stage, $P(t)$, (b) infected persons in the rash stage, $I(t)$, (c) Isolated persons, $J(t)$ when triple controls are implemented together. Here, the numbers 1,2,3, and 4, are subscripts of the control functions, u_1, u_2, u_3, u_4 while W/O/C means “Without Control”.

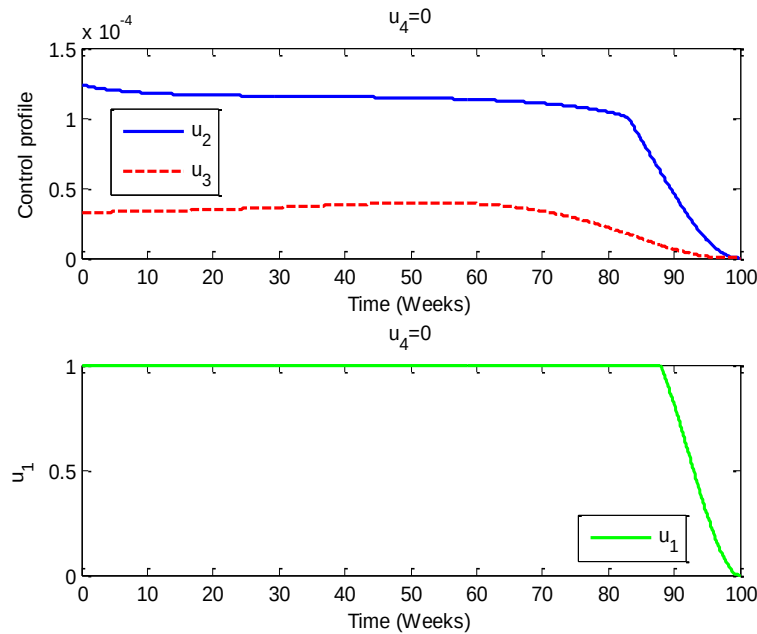


Figure 6. Control profile for implementation of triple optimal controls.

5.1 Discussion

Figure 4 shows the population dynamics of infected persons in the prodromal stage, $P(t)$, infected persons in the rash stage, $I(t)$, isolated persons, $J(t)$, for with and without control and the control profile. According to Figure 4a, with control measures, measles-free population is achieved for $P(t)$ population immediately, and thus reducing the number of persons moving to the infected people in the rash stage, while Figure 4b, the number of infected persons in the rash stage, $I(t)$, decrease to zero within 25 weeks with control measures in place. Also, Figure 4c reveals that with the implementation of the control measures, there is a sharp increase in the number of isolated persons before decreasing to zero by 25 weeks and achieves measles-free population. The control profile for achieving the result in Figures 4a-4c are display in Figures 4d and 4e. They show that the upper bounds for u_1, u_2, u_3 are 9.9, 6.1×10^{-4} and 2×10^{-5} , respectively, where the control, u_3 maintains a bound of 6.9 for 50 weeks before it gradually increases to 0.9 as at 90 weeks. For without control measures, Figures 4a – 4c show the endemicity of the measles infection in the population.

In Figure 5, the implementation of triple control measures for the dynamics of the infected compartments ($P(t), I(t), J(t)$) are evaluated. We observed in Figures 5a -5b that simultaneous implementation of any triple control measures reduce the number of infected persons ($P(t), I(t)$) in the population as they achieve a measles-free population within a short time while for isolated persons, $J(t)$, the combine implementation of u_1, u_2, u_3 (123) yields a faster and better result in achieving a measles-free population compared with other combinations (see Figure 5c). With this, it implies that combined implementation of control measures, u_1, u_2, u_3 (123), reduces the number of infected persons in the prodromal stage, $P(t)$, rash stage, $I(t)$ and the isolated persons, ($J(t)$) compare with any other combination of control measures. The control profile for triple control measures are display in Figure 6. The control profile when $u_4 = 0$ is the combined implementation of u_1, u_2, u_3 (123) that gives the best result in Figure 5, which indicates that mass distribution of nutritional (supplement) support, administration of first and second dose vaccine control measures have much effect on controlling measles in the population. To achieved this, u_1 maintains an upper bound that declines after 85 weeks, whereas u_2 maintains a bound of 1.2×10^{-4} that decreases gradually till 85 weeks where it declines to the final time. For u_3 , it starts with a bound of 3.2×10^{-5} that slightly increases to 4.0×10^{-5} at 70 weeks before declining to the final time.

The numerical simulations imply that combined implementation of mass distribution of nutritional support, complete vaccination with the first and second dose of vaccine and isolation of infected persons in the prodromal stage will help eradicate the spread of measles in the population. This is in agreement with the sensitivity analysis result and the results in [3]. This indicates that implementation of control measures will help prevent the spread of measles infection in the population. However, if there are limited facilities to isolate the infected persons in the prodromal stage, the triple control measures, mass distribution of nutritional support and complete vaccination with first and second doses of vaccine will reduce the spread of measles infection as fewer people will

be infected and thus help the health practitioners achieve the best strategy in the control of the spread of measles in the community.

6. Conclusion

In this paper, an autonomous system for the transmission dynamics of measles disease involving isolated persons and two doses of vaccination is formulated. The model disease-free equilibrium and basic reproduction number (R_0) are computed. The sensitivity analysis of the basic reproduction number, which includes the pairwise comparison and scatterplots of important parameters of the basic reproduction number is carried out using Latin Hypercube Sampling (LHS) scheme. LHS scheme is also used to compute and compare the values of Partial Rank Correlation Coefficients (PRCCs) of the model parameters. The result of the sensitivity analysis indicates that transmission rates, first and second-dose vaccination rates and isolation rate of the infected persons in the prodromal stage are significant parameters in eradicating the spread of measles infection. Furthermore, the optimal control model of the autonomous system is developed and analysed with four control measures, mass distribution of nutritional support, administration of first and second dose vaccination and isolation of infected persons in the prodromal stage. From the numerical simulations, we found out that the combined implementation of the four control measures achieves a measles-free population on time than without control measures. However, if there are limited facilities to isolated the infected persons in the prodromal stage, the triple control measures, mass distribution of nutritional support and complete vaccination with first and second doses of vaccine will reduce the spread of measles infection as fewer people will be infected and thus help the health practitioners achieve the best strategy in the control of the spread of measles in the community.

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Weaker Forms of Nano Irresolute and Its Contra Functions

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Abstract

In this paper the concept of some weaker forms of irresolute and contra irresolute functions in Nano Topological spaces are studied and its related characteristics are discussed. Also we introduced the notion called contra nano alpha irresolute function, contra nano semi irresolute function, contra nano pre irresolute function and its properties are examined. Finally, we have revealed some applications related to recent scenario of online teaching and COVID-19 which can be expressed as nano irresolute functions and contra irresolute functions respectively.

Keywords: Ns-irresolute function, Np-irresolute function, contra N_α -irresolute function, contra Ns-irresolute function, contra Np-irresolute function.

2020 AMS subject classifications: 54B05 ¹

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¹Received on April 18, 2022. Accepted on August 25, 2022. Published on September 25, 2022.

doi: 10.23755/rm.v43i0.764. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.

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1 Introduction

In 1982 Pawlak [Pawlak, 1982] investigated about approximate operations, equality and inclusion on sets. In [Crossley and Hildebrand, 1972], irresolute functions was introduced and analysed by Crossley and Hildebrand in topological spaces. Weak and Strong forms of irresolute functions in topology were discussed by Maio and Noiri [Maio and Noiri, 1988]. The conception of nano-topology was initiated by Lellis Thivagar [Thivagar and Richard, 2013b],[M. Lellis Thivagar and Richard, 2013] and [M. Lellis Thivagar and Devi, 2017]. Also in [Thivagar and Richard, 2013a], nano continuous functions, nano interior and nano closure was look over by Lellis and Carmel Richard. Bhuvaneshwari and Ezhilarasi[Bhuvaneshwari and Ezhilarasi, 2016] introduced irresolute maps and semi-generalized irresolute maps in nano topological spaces. New functions called Ns-irresolute and Np-irresolute functions are originated and look into its behaviour in this article. Further the notions called contra $N\alpha$ -irresolute function, contra Ns-irresolute function, contra Np-irresolute function were introduced and examined their properties. Throughout this article we use the notation NTS, N-open, $N\alpha$ -open, Ns-open, Np-open, $N\alpha$ -continuous, Ns-continuous, Np-continuous for "Nano Topological spaces, Nano open, Nano α -open, Nano semi-open, Nano Pre-open sets, Nano α -continuous, Nano semi-continuous, Nano pre-continuous" respectively. Similar notation is used for their respective closed sets.

2 Nano Irresolute Functions

Definition 2.1. Let U_1 and U_2 be NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is called

1. Ns-irresolute if $h^{-1}(S)$ is Ns-open set in U_1 for each Ns-open set S in U_2 ,
2. Np-irresolute if $h^{-1}(S)$ is Np-open set in U_1 for each Np-open set S in U_2 .

Example 2.1. Take $U_1 = \{w, x, y, z\}$ with $U_1/R = \{\{x, z\}, \{y, w\}\}$ and $X = \{x, z\}$. Then $\tau_R(X) = \{U_1, \phi, \{x, z\}\}$. Let $U_2 = \{q, r, s, t\}$ with $U_2/R' = \{\{q\}, \{r, s\}, \{t\}\}$ and $Y = \{q, t\}$. Then $\tau_{R'}(Y) = \{U_2, \phi, \{q, t\}\}$. We define $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(x) = q$, $h(y) = r$, $h(z) = t$, $h(w) = s$. Then the inverse image of any Ns-open in U_2 is Ns-open in U_1 and the inverse image of any Np-open in U_2 is Np-open in U_1 . Therefore h is Ns-irresolute and Np-irresolute.

Theorem 2.1. Let U_1 and U_2 be the NTS with reference to $\tau_R(X)$ and $\tau_{R'}(Y)$ and $h : U_1 \rightarrow U_2$ be a mapping. Then the statements given below are equivalent.

1. h is $N\alpha$ -irresolute.

2. $h^{-1}(S)$ is $N\alpha$ -closed in U_1 , for each $N\alpha$ -closed set S in U_2 .
3. $h(N\alpha cl(S)) \subseteq N\alpha cl(h(S))$ for each $S \subseteq U_1$.
4. $N\alpha cl(h^{-1}(S)) \subseteq h^{-1}(N\alpha cl(S))$ for each $S \subseteq U_2$.
5. $h^{-1}(N\alpha int(S)) \subseteq (N\alpha int(h^{-1}(S)))$ for each $S \subseteq U_2$.
6. h is $N\alpha$ -irresolute for each $x \in U_1$.

Proof. (i) $\implies$ (ii). It is obvious.

(ii) $\implies$ (iii). Let $S \subseteq U_1$. Then, $N\alpha cl(h(S))$ is a $N\alpha$ -closed set of U_2 . By (ii), $h^{-1}(N\alpha cl(h(S)))$ is a $N\alpha$ -closed set in U_1 and $N\alpha cl(S) \subseteq N\alpha cl(h^{-1}h(S)) \subseteq N\alpha cl(h^{-1}(N\alpha cl(h(S)))) = h^{-1}(N\alpha cl(h(S)))$. So $h(N\alpha cl(h(S))) \subseteq N\alpha cl(h(S))$.

(iii) $\implies$ (iv). Let S be a subset of U_2 . By (iii) $h(N\alpha cl(h^{-1}(S))) \subseteq N\alpha cl(hh^{-1}(S)) \subseteq N\alpha cl(S)$. So $N\alpha cl(h^{-1}(S)) \subseteq h^{-1}h(N\alpha cl(h^{-1}(S))) \subseteq h^{-1}(N\alpha cl(S))$.

(iv) $\implies$ (v). Let S be a subset of U_2 . By (iv), $h^{-1}(N\alpha cl(U_2 - S)) \supseteq N\alpha cl(h^{-1}(U_2 - S)) = N\alpha cl(U_1 - h^{-1}(S))$. Since $U_1 - N\alpha cl(U_1 - S) = N\alpha int(S)$, then $h^{-1}(N\alpha int(S)) = h^{-1}(U_2 - N\alpha cl(U_2 - S)) = U_1 - h^{-1}(N\alpha cl(U_2 - S)) \subseteq U_1 - N\alpha cl(U_1 - h^{-1}(S)) = N\alpha int(h^{-1}(S))$.

(v) $\implies$ (vi). Let S be a $N\alpha$ -open set of U_2 , then $S = N\alpha int(S)$. By (v), $h^{-1}(S) = h^{-1}(N\alpha int(S)) \subseteq N\alpha int(h^{-1}(S)) \subseteq h^{-1}(S)$. So, $h^{-1}(S) = N\alpha int(h^{-1}(S))$. Thus, $h^{-1}(S)$ is a $N\alpha$ -open set of U . Therefore h is $N\alpha$ -irresolute.

(i) $\implies$ (vi). Let h be $N\alpha$ -irresolute, $x \in U_1$ and any $N\alpha$ -open set S of U_2 , such that $h(x) \in S$. Then $x \in h^{-1}(S) = N\alpha int(h^{-1}(S))$. Let $B = h^{-1}(S)$, then B is a $N\alpha$ -open set of U_1 and so $h(B) = hh^{-1}(S) \subseteq S$. Thus h is $N\alpha$ -irresolute for each $x \in U_1$.

(vi) $\implies$ (i). Let S be a $N\alpha$ -open set of U_2 , $x \in h^{-1}(S)$. Then $h(x) \in S$. By hypothesis there exists a $N\alpha$ -open set B of U_1 such that $x \in B$ and $h(B) \subseteq S$. Hence $x \in B \subseteq h^{-1}(h(B)) \subseteq h^{-1}(S)$ and $x \in B = N\alpha int(B) \subseteq N\alpha int(h^{-1}(S))$. So, $h^{-1}(S) \subseteq N\alpha int(h^{-1}(S))$. Hence $h^{-1}(S) = N\alpha int(h^{-1}(S))$. Thus h is $N\alpha$ -irresolute. $\square$

Theorem 2.2. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$ and $h : U_1 \rightarrow U_2$ be a 1-1 and onto function. Then h is $N\alpha$ -irresolute iff $N\alpha int(h(S)) \subseteq h(N\alpha int(S))$ for each $S \subseteq U_1$.

Proof. Let S be any subset of U_1 . By Theorem 2.1 and since h is 1-1 and onto, $h^{-1}(N\alpha int(h(S))) \subseteq N\alpha int(h^{-1}(h(S))) = N\alpha int(S)$. So, $hh^{-1}(N\alpha int(h(S))) \subseteq h(N\alpha int(S))$. Thus $N\alpha int(h(S)) \subseteq h(N\alpha int(S))$.

Conversely, Let S be a $N\alpha$ -open set of U_2 . Then $S = N\alpha int(S)$. By hypothesis, $h(N\alpha int(h^{-1}(S))) \supseteq N\alpha int(h(h^{-1}(S))) = N\alpha int(S) = S$. Thus we get $h^{-1}h(N\alpha int(h^{-1}(S))) \supseteq h^{-1}(S)$. Since h is 1-1 and onto, $N\alpha int(h^{-1}(S)) = h^{-1}h(N\alpha int(h^{-1}(S))) \supseteq h^{-1}(S)$. Hence $h^{-1}(S) = N\alpha int(h^{-1}(S))$. So $h^{-1}(S)$ is $N\alpha$ -open set of U . Thus h is $N\alpha$ -irresolute. $\square$

Lemma 2.1. Let U_1 be a NTS with respect to $\tau_R(X)$ then

1. $N\alpha cl(S) \subseteq Ncl(S)$ for every subset S of U_1 ,
2. $Ncl(S) = N\alpha cl(S)$ for every α -open subset S of U_1 .

Theorem 2.3. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$ and $h : U_1 \rightarrow U_2$ be a $N\alpha$ -irresolute. Then $Ncl(h^{-1}(S)) \subseteq h^{-1}(Ncl(S))$ for every $S \subseteq U_2$.

Proof. Let S be any N -open subset of U_2 . Since h is $N\alpha$ -irresolute and $N\alpha cl(h^{-1}(S))$ is equal to $Ncl(h^{-1}(S))$. By Theorem 2.1, $N\alpha cl(h^{-1}(S)) \subseteq h^{-1}(N\alpha cl(S))$ and by Lemma 2.1 $h^{-1}(N\alpha cl(S)) \subseteq h^{-1}(Ncl(S))$. Then $N\alpha cl(h^{-1}(S)) \subseteq h^{-1}(Ncl(S))$. Therefore $Ncl(h^{-1}(S)) \subseteq h^{-1}(Ncl(S))$. $\square$

Theorem 2.4. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is a Ns -irresolute iff for each Ns -closed subset $h^{-1}(S)$ is Ns -closed in U_1 .

Proof. If h is Ns -irresolute, then $h^{-1}(B)$ is Ns -open in U_1 for each Ns -open set $B \subseteq U_2$. If S is any Ns -closed subset of U_2 , then $U_2 - S$ is Ns -open. Thus $h^{-1}(U_2 - S)$ is Ns -open in U_1 , but $h^{-1}(U_2 - S) = U_1 - h^{-1}(S)$ so that $h^{-1}(S)$ is Ns -closed in U_1 . Conversely, if for all Ns -closed set $S \subseteq U_2$, $h^{-1}(S)$ is Ns -closed in U_1 and if B is any Ns -open subset of U_2 , then $U_2 - B$ is Ns -closed. Also $h^{-1}(U_2 - B) = U_1 - h^{-1}(B)$ which is Ns -closed in U_1 . Therefore $h^{-1}(B)$ is Ns -open set in U_1 . Hence h is Ns -irresolute. $\square$

Theorem 2.5. If $h : U_1 \rightarrow U_2$ and $g : U_2 \rightarrow U_3$ is Ns -irresolute(Np -irresolute) then $g \circ h : U_1 \rightarrow U_3$ is Ns -irresolute(Np -irresolute).

Proof. (i) If $A \subseteq U_3$ is Ns -open(Np -open), then $g^{-1}(A)$ is Ns -open(Np -open) set in U_2 because g is Ns -irresolute(Np -irresolute). Consequently since h is Ns -irresolute(Np -irresolute), $h^{-1}(g^{-1}(A)) = (g \circ h)^{-1}(A)$ is Ns -open(Np -open) set in U_1 . Hence $g \circ h$ is Ns -irresolute(Np -irresolute).

Theorem 2.6. If $h : U_1 \rightarrow U_2$ is $N\alpha$ -irresolute(Ns -irresolute, Np -irresolute) and $g : U_2 \rightarrow U_3$ is $N\alpha$ -continuous(Ns -continuous, Np -continuous) then $g \circ h : U_1 \rightarrow U_3$ is $N\alpha$ -continuous(Ns -continuous, Np -continuous).

Proof. Let $S \subseteq U_3$ is N -open. Since g is $N\alpha$ -continuous(Ns -continuous, Np -continuous), $g^{-1}(S)$ is $N\alpha$ -open(Ns -open, Np -open) set in U_2 . Consequently since h is $N\alpha$ -irresolute(Ns -irresolute, Np -irresolute), $h^{-1}(g^{-1}(S)) = (g \circ h)^{-1}(S)$ is $N\alpha$ -open(Ns -open, Np -open) set in U_1 . Hence $g \circ h$ is $N\alpha$ -continuous(Ns -continuous, Np -continuous). $\square$

Theorem 2.7. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. A function $h : U_1 \rightarrow U_2$ is

1. Ns -irresolute and Np -irresolute then h is $N\alpha$ -irresolute,
2. $N\alpha$ -continuous iff it is Ns -continuous and Np -continuous.

Proof. It is obvious.

3 Nano Contra Irresolute Functions

Here we introduce contra irresolute functions and its characteristics are discussed. The notations used are $NC\alpha$ -open, NCs -open, NCp -open for "Nano contra α -open, Nano contra semi-open, Nano contra pre-open functions" respectively.

Definition 3.1. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is said to be

1. $NC\alpha$ -open if $h(S)$ is $N\alpha$ -closed in U_2 for each N -open S in U_1 ,
2. NCs -open if $h(S)$ is Ns -closed in U_2 for each N -open S in U_1 ,
3. NCp -open if $h(S)$ is Np -closed in U_2 for each N -open S in U_1 .

Example 3.1. 1. Let $U_1 = \{j, k, l\}$ with $U_1/R = \{\{l\}, \{j, k\}\}$ and $X = \{k, l\}$. Then $\tau_R(X) = \{U_1, \phi, \{l\}, \{j, k\}\}$. Let $U_2 = \{x, y, z\}$, $U_2/R' = \{\{y\}, \{x, z\}\}$ and $Y = \{y, z\}$. Subsequently $\tau_{R'}(Y) = \{U_2, \phi, \{y\}, \{x, z\}\}$. We label $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(j) = x, h(k) = z, h(l) = y$. Subsequently $h(S)$ is $N\alpha$ -closed in U_2 for every N -open set S in U_1 . Hence h is $NC\alpha$ -open.

2. Let $U_1 = \{j, k, l, m\}$ with $U_1/R = \{\{j\}, \{l\}, \{k, m\}\}$ and $X = \{j, k\}$. Subsequently $\tau_R(X) = \{U_1, \phi, \{j\}, \{k, m\}, \{j, k, m\}\}$. Let $U_2 = \{p, q, r, s\}$ with $U_2/R' = \{\{p\}, \{s\}, \{q, r\}\}$ and $Y = \{p, r\}$. Subsequently $\tau_{R'}(Y) = \{U_2, \phi, \{p\}, \{q, r\}, \{p, q, r\}\}$. We label $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(j) = s, h(k) = r, h(l) = p, h(m) = q$. Then $h(S)$ is Ns -closed in U_2 for every N -open set S in U_1 . Hence h is NCs -open.

3. Let $U_1 = \{j, k, l, m\}$ with $U_1/R = \{\{l\}, \{m\}, \{j, k\}\}$ and $X = \{j, l\}$. Subsequently $\tau_R(X) = \{U_1, \phi, \{l\}, \{j, k\}, \{j, k, l\}\}$. Let $U_2 = \{p, q, r, s\}$ with $U_2/R' = \{\{q\}, \{r\}, \{p, s\}\}$ and $Y = \{p, r\}$. Subsequently $\tau_{R'}(Y) = \{U_2, \phi, \{r\}, \{p, s\}, \{p, r, s\}\}$. We define $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(j) = q, h(k) = s, h(l) = p, h(m) = r$. Then $h(S)$ is Np -closed in U_2 for every N -open set S in U_1 . Hence h is NCp -open.

Definition 3.2. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is said to be $CN\alpha$ -irresolute(CNs -irresolute, CNp -irresolute) if $h^{-1}(S)$ is $N\alpha$ -closed(Ns -closed, Np -closed) set in U_1 for every $N\alpha$ -open set(Ns -open, Np -open) in U_2 .

Example 3.2. 1. Let $U_1 = \{j, k, l, m\}$ with $U_1/R = \{\{j\}, \{k\}, \{l\}, \{m\}\}$ and $X = \{j\}$. Then $\tau_R(X) = \{U_1, \phi, \{j\}\}$. Let $U_2 = \{w, x, y, z\}$ with $U_2/R' = \{\{w\}, \{x\}, \{y\}, \{z\}\}$ and $Y = \{x, y, z\}$. Then $\tau_{R'}(Y) = \{U_2, \phi, \{x, y, z\}\}$. We label $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(j) = w, h(k) = x, h(l) = y, h(m) = z$. Then $h^{-1}(S)$ is Ns -closed in U_1 for every Ns -open set S in U_2 . Therefore h is $CN\alpha$ -irresolute and CNs -irresolute.

2. Let $U_1 = \{p, q, r\}$ with $U_1/R = \{\{p\}, \{q, r\}\}$ and $X = \{q, r\}$. Then $\tau_R(X) = \{U_1, \phi, \{q, r\}\}$. Let $U_2 = \{j, k, l\}$ with $U_2/R' = \{\{j\}, \{k, l\}\}$ and $Y = \{j\}$. Then $\tau_{R'}(Y) = \{U_2, \phi, \{j\}\}$. We define $h : (U_1, \tau_R(X)) \rightarrow (U_2, \tau_{R'}(Y))$ as $h(p) = j, h(q) = k, h(r) = l$. Then $h^{-1}(S)$ is Np -closed in U_1 for every Np -open set S in U_2 . So h is CNp -irresolute.

Theorem 3.1. Consider U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is $CN\alpha$ -irresolute iff for each $N\alpha$ -closed subset S of U_2 , $h^{-1}(S)$ is $N\alpha$ -open in U_1 .

Proof. If h is $CN\alpha$ -irresolute, then for each $N\alpha$ -open subset B in U_2 , $h^{-1}(B)$ is $N\alpha$ -closed in U_1 . If S is any $N\alpha$ -closed subset in U_2 , then $U_2 - S$ is $N\alpha$ -open. Thus $h^{-1}(U_2 - S)$ is $N\alpha$ -closed but $h^{-1}(U_2 - S) = U_1 - h^{-1}(S)$ so that $h^{-1}(S)$ is $N\alpha$ -open in U_1 .

Conversely, if, for all $N\alpha$ -closed subsets S of U_2 , $h^{-1}(S)$ is $N\alpha$ -open in U_1 and if B is any $N\alpha$ -open subset of U_2 , then $U_2 - B$ is $N\alpha$ -closed. Also $h^{-1}(U_2 - B) = U_1 - h^{-1}(B)$ is $N\alpha$ -open. Thus $h^{-1}(B)$ is $N\alpha$ -closed in U_1 . Hence h is $CN\alpha$ -irresolute. $\square$

Corolary 3.1. Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is CNs -irresolute(CNp -irresolute) if and only if for each Ns -closed subset(Np -closed subset) S of U_2 , $h^{-1}(S)$ is Ns -open(Np -open) in U_1 .

Theorem 3.2. If the functions $h : U_1 \rightarrow U_2$ and $g : U_2 \rightarrow U_3$ are $CN\alpha$ -irresolute then $goh : U_1 \rightarrow U_3$ is $N\alpha$ -irresolute.

Proof. If $S \subseteq U_3$ is $N\alpha$ -open, then $g^{-1}(S)$ is $N\alpha$ -closed in U_2 because g is $CN\alpha$ -irresolute. Consequently since h is $CN\alpha$ -irresolute, $h^{-1}(g^{-1}(S)) = (goh)^{-1}(S)$ is $N\alpha$ -open set in U_1 , by corollary 4.6. Hence goh is $N\alpha$ -irresolute. $\square$

Corolary 3.2. If the functions $h : U_1 \rightarrow U_2$ and $g : U_2 \rightarrow U_3$ are CNs -irresolute (CNp -irresolute) then $goh : U_1 \rightarrow U_3$ is Ns -irresolute(Np -irresolute).

Theorem 3.3. *If the function $h : U_1 \rightarrow U_2$ is $CN\alpha$ -irresolute and the function $g : U_2 \rightarrow U_3$ is $NC\alpha$ -continuous then $g \circ h : U_1 \rightarrow U_3$ is $N\alpha$ -continuous.*

Proof. Let $S \subseteq U_3$ is N -open. Since g is $NC\alpha$ -continuous, $g^{-1}(S)$ is $N\alpha$ -closed in U_2 . Consequently since h is $CN\alpha$ -irresolute, $h^{-1}(g^{-1}(S)) = (g \circ h)^{-1}(S)$ is $N\alpha$ -open set in U_1 , by theorem 4.5. Hence $g \circ h$ is $N\alpha$ -continuous. $\square$

Corollary 3.3. *If the function $h : U_1 \rightarrow U_2$ is CNs -irresolute(CNp -irresolute) and the function $g : U_2 \rightarrow U_3$ is NCs -continuous(NCp -continuous) then $g \circ h : U_1 \rightarrow U_3$ is Ns -continuous(Np -continuous).*

Theorem 3.4. *Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is CNs -irresolute and CNp -irresolute then h is $CN\alpha$ -irresolute.*

Proof. It is obvious.

Theorem 3.5. *Let U_1 and U_2 be the NTS with respect to $\tau_R(X)$ and $\tau_{R'}(Y)$. Then $h : U_1 \rightarrow U_2$ is $CN\alpha$ -irresolute then it is $NC\alpha$ -continuous.*

Proof. Consider the N -open set $T \subseteq U_2$. Which implies T is a $N\alpha$ -open set in U_2 . But h is $CN\alpha$ -irresolute So $h^{-1}(T)$ is a $N\alpha$ -closed set in U_1 . It shows that h is $NC\alpha$ -continuous function. $\square$

4 Applications

Finally, we discuss the application of nano irresolute functions and its contra functions.

Example 4.1. *Advances in technology and some pandemic situations allow students to study entirely online. Consider the impact of e-learning on students characteristics, as a function of, the innovative strategies used in online teaching. Let us consider some of the strategies used in online teaching are powerpoint presentation (P), videos (V), mind map (M), Online Quiz (Q), Group discussion (G) and its impact on students characteristics are Intellectually curious (I), Good time management (T), Self-driven (S), Enhanced Communication skills (C). Let $U_1 = \{P, V, M, Q, G\}$ be the universe of the innovative strategies used in online teaching with $U_1/R = \{\{P, V\}, \{M, G\}, \{Q\}\}$ and $X_1 = \{P, Q\}$. Subsequently $\tau_R(X_1) = \{U_1, \phi, \{Q\}, \{P, V\}, \{P, V, Q\}\}$. Let $U_2 = \{I, T, S, C\}$ be the universe on students characteristics with $U_2/R' = \{\{I, S\}, \{T, C\}\}$ and $X_2 = \{T, C\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{T, C\}\}$. We define $h : (U_1, \tau_R(X_1)) \rightarrow (U_2, \tau_{R'}(X_2))$ as $h(P) = C$, $h(V) = C$, $h(M) = I$, $h(Q) = T$ and $h(G) = S$. Then for every $N\alpha$ -open set in U_2 , inverse image is $N\alpha$ -open set in U_1 and also for every Ns -open set in U_2 , inverse image is Ns -open set in U_1 . Hence h is $N\alpha$ -irresolute*

and Ns -irresolute. Thus, the impact of e-learning on students characteristics, as a function of the innovative strategies used in online teaching, are $N\alpha$ -irresolute and Ns -irresolute function.

Example 4.2. The main cause of illness is the infectious diseases. However, some initial precautions may help to prevent infections. If not, it leads to serious medical conditions and sometimes to death. Consider the precautionary measures to be adopted to prevent affecting from COVID-19, as a function of, its symptoms. Let the symptoms of COVID-19 are Dry cough (K), Fever (F), Shortness of Breath (B), Loss of Taste/Smell (L) and the precautionary measures to be adopted are Sanitizing (S), Social distancing (D), Wearing mask (M), Boosting Immunity power (I). Let $U_1 = \{K, F, B, L\}$ be the universe of symptoms of COVID-19 with $U_1/R = \{\{K\}, \{F\}, \{B\}, \{L\}\}$ and $X_1 = \{K\}$. Then $\tau_R(X_1) = \{U_1, \phi, \{K\}\}$. Let $U_2 = \{S, D, M, I\}$ be the universe of the precautionary measures to be adopted with $U_2/R' = \{\{S\}, \{D\}, \{M\}, \{I\}\}$ and $X_2 = \{D, M, I\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{D, M, I\}\}$. We define $h : (U_1, \tau_R(X_1)) \rightarrow (U_2, \tau_{R'}(X_2))$ as $h(K) = S$, $h(F) = D$, $h(B) = M$ and $h(L) = I$. Then for every $N\alpha$ -open set in U_2 , inverse image is $N\alpha$ -closed set in U_1 and also for every Ns -open set in U_2 , inverse image is Ns -closed set in U_1 . Thus h is contra $N\alpha$ -irresolute and contra Ns -irresolute. Thus, the precautionary measures to be adopted to prevent affecting from COVID-19, as a function of its symptoms, are contra $N\alpha$ -irresolute and contra Ns -irresolute function.

5 Conclusions

Through the above discussions we have summarized the conceptualization of irresolute functions and contra irresolute functions in NTS along with examples. Further, We have revealed some applications related to current scenario of online teaching and COVID-19 which can be expressed as nano irresolute functions and contra irresolute functions respectively. Thus these notions can be applied in many real time situations.

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A Common Fixed Point Theorem For Three Weakly Compatible Selfmaps Of A S-metric Space

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Abstract

Fixed point theorems were established by using contractive conditions. In this paper we prove a common fixed point theorem for three weakly compatible selfmaps of a S-metric space by utilizing a contractive condition of rational type. Further we deduce a common fixed point theorem for two weakly compatible selfmaps of a S-metric space.

Keywords: S-metric space; Fixed point; Weakly compatible mappings; Associated sequence of a point relative to three selfmaps.

2020 AMS subject classifications: 54H25, 47H10. ¹

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¹Received on April 21, 2022. Accepted on September 1, 2022. Published on October 1, 2022.
doi: 10.23755/rm.v43i0.767. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Fixed point theory is an important branch of non-linear analysis due to its application potential. In proving fixed point theorems, we use completeness, continuity, convergence and various other topological aspects. Banach's Contraction Principle or Banach's fixed point theorem is one of the most important results in nonlinear analysis. This theorem has been generalized in many directions by generalizing the underlying space or by viewing it as a common fixed point theorem along with other selfmaps.

In the past few years, a number of generalizations of metric spaces like G -metric spaces, partial metric spaces and cone metric spaces were initiated. These generalizations were used to extend the scope of the study of fixed point theory. Recently one more generalization, namely S -metric spaces, was introduced by Sedghi S , Shobe N, Aliouche.A [2012]. Among all generalizations, S-metric spaces evinced a lot of interest in many researchers as they unified, extended, generalized and refined several existing results onto these S -metric spaces.

Commutativity plays an important role in proving common fixed point theorems. As it is a stronger requirement, Sessa [1982] introduced the notion of weakly commuting maps as a generalization of commuting maps. Afterwards the idea of compatibility was introduced by G. Jungck [1986]. Later on Jungck and Rhoades [1998] introduced the notion of weakly compatibility as a generalization of compatibility. They also proved that compatible mappings are weakly compatible but not conversely.

In this paper we establish a common fixed point theorem for three weakly compatible selfmaps of a S-metric space using a contractive condition of rational type. Our theorem which is established in the framework of S-metric spaces generalizes the theorem of Sumit Chandok [2018] which is proved in metric space.

Now we recall some basic definitions required in the sequel in section 2 and establish main results in section 3.

2 Preliminaries

We now recollect the essential definitions which are useful for our discussion.

Definition 2.1. *Let Y be a nonempty set. A function*

$S : Y^3 \rightarrow [0, \infty)$ is said to be S - metric if it satisfies the following conditions for each $\beta_1, \beta_2, \beta_3, \beta_4 \in Y$

- (i) $S(\beta_1, \beta_2, \beta_3) \geq 0$,*
- (ii) $S(\beta_1, \beta_2, \beta_3) = 0$ if and only if $\beta_1 = \beta_2 = \beta_3$,*
- (iii) $S(\beta_1, \beta_2, \beta_3) \leq S(\beta_1, \beta_1, \beta_4) + S(\beta_2, \beta_2, \beta_4) + S(\beta_3, \beta_3, \beta_4)$.*

Then (Y, S) is said to be a S-metric space.

A Common Fixed Point Theorem For Three Weakly Compatible Selfmaps Of A S-metric Space

Example 2.1. Let $Y = \mathbb{R}$ and $S : \mathbb{R}^3 \rightarrow [0, \infty)$ be defined by $S(\beta_1, \beta_2, \beta_3) = |\beta_2 + \beta_3 - 2\beta_1| + |\beta_2 - \beta_3|$ for $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, then (Y, S) is a S-metric space.

Remark 2.1. It is shown in a S-metric space that $S(\beta_1, \beta_1, \beta_2) = S(\beta_2, \beta_2, \beta_1)$ for all $\beta_1, \beta_2 \in Y$.

Definition 2.2. Let (Y, S) be an S-metric space. A sequence $\{t_n\}$ in Y said to convergent, if there is a $t \in Y$ such that $S(t_n, t_n, t) \rightarrow 0$; that is for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(t_n, t_n, t) < \epsilon$ and we denote this by $\lim_{n \rightarrow \infty} t_n = t$.

Definition 2.3. Suppose (Y, S) is an S-metric space. A sequence $\{t_n\}$ in Y is called a Cauchy sequence if to each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(t_n, t_n, t) < \epsilon$ for each $n, m \geq n_0$.

Definition 2.4. Let (Y, S) be an S-metric space, If there exists sequences $\{t_n\}$ and $\{u_n\}$ such that $\lim_{n \rightarrow \infty} t_n = t$ and $\lim_{n \rightarrow \infty} u_n = u$ then $\lim_{n \rightarrow \infty} S(t_n, t_n, u_n) = S(t, t, u)$, then we say that $S(t, u, v)$ is continuous in t and u .

Definition 2.5. Suppose ϕ and ψ self maps of a S-metric space (Y, S) such that for every sequence $\{t_n\}$ in Y with $\lim_{n \rightarrow \infty} \psi t_n = \lim_{n \rightarrow \infty} \phi t_n = t$ for some $t \in X$ we have $\lim_{n \rightarrow \infty} S(\psi \phi t_n, \psi \phi t_n, \phi \psi t_n) = 0$, then ψ and ϕ are called compatible mappings.

Definition 2.6. In a S-metric space (Y, S) , two selfmaps ϕ and ψ of Y are said to be weakly compatible if $\phi \psi t = \psi \phi t$ whenever $\phi t = \psi t$ for $t \in Y$.

Definition 2.7. If ψ , μ and ϕ are self maps of a non empty set Y such that $\psi(Y) \subseteq \phi(Y)$, and $\mu(Y) \subseteq \phi(Y)$ then for any $t_0 \in Y$, if $\{t_n\}$ is a sequence in Y such that $\phi t_{2n+1} = \psi t_{2n}$ and $\phi t_{2n+2} = \mu t_{2n+1}$ for $n \geq 1$ then $\{t_n\}$ is called an associated sequence of t_0 relative to three selfmaps ψ , μ and ϕ .

3 Main Theorem

We now state our main theorem of the section.

Theorem 3.1. Let P be a subset of a S-metric space (Y, S) , ψ , μ and ϕ are three selfmaps of P such that

- (i) $\psi(Y) \cup \mu(Y) \subseteq \phi(Y)$ and $(\phi(P), S)$ is complete.

(ii)

$$\begin{aligned} & S(\mu y_1, \mu y_1, \psi y_2) \\ & \leq k_1 \left\{ \frac{S(\phi y_1, \phi y_1, \mu y_1) \cdot S(\phi y_2, \phi y_2, \psi y_2)}{S(\phi y_1, \phi y_1, \phi y_2) + S(\phi y_1, \phi y_1, \psi y_2) + S(\phi y_2, \psi y_2, \mu y_1)} \right\} \\ & \quad + k_2 S(\phi y_1, \phi y_1, \phi y_2) \end{aligned}$$

for every $y_1, y_2 \in P$ and $k_1, k_2 \in [0, 1)$ with $2k_1 + k_2 < 1$.

(iii) The pairs (ϕ, ψ) and (ϕ, μ) are weakly compatible.

Then ψ, μ and ϕ have a unique common fixed point.

Proof. Let t_0 be a point in Y . Since $\psi(Y) \cup \mu(Y) \subseteq \phi(Y)$, we obtain an associated sequence $\{t_n\}$ in Y such that

$$\phi t_{2n+1} = \mu t_{2n}, \phi t_{2n+2} = \psi t_{2n+1}.$$

From the condition (ii) of Theorem 3.1 we have,

$$\begin{aligned} & S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2}) \\ & = S(\mu t_{2n}, \mu t_{2n}, \psi t_{2n+1}) \\ & \leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \mu t_{2n}) \cdot S(\phi t_{2n+1}, \phi t_{2n+1}, \psi t_{2n+1})}{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) + S(\phi t_{2n}, \phi t_{2n}, \psi t_{2n+1}) + S(\phi t_{2n+1}, \phi t_{2n+1}, \mu t_{2n})} \right] \\ & \quad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\ & \leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \cdot S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2})}{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) + S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+2}) + S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+1})} \right] \\ & \quad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\ & \leq 2k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \cdot S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2})}{S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+1})} \right] + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\ & \leq (2k_1 + k_2) S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}). \end{aligned}$$

Similarly, we can prove

$$S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \leq (2k_1 + k_2) S(\phi t_{2n-1}, \phi t_{2n-1}, \phi t_{2n})$$

Therefore

$$\begin{aligned} S(\phi t_n, \phi t_n, \phi t_{n+1}) & \leq (2k_1 + k_2) S(\phi t_{n-1}, \phi t_{n-1}, \phi t_n) \\ & \leq (2k_1 + k_2)^2 S(\phi t_{n-2}, \phi t_{n-2}, \phi t_{n-1}) \\ & \leq (2k_1 + k_2)^3 S(\phi t_{n-3}, \phi t_{n-3}, \phi t_{n-2}) \\ & \dots \quad \dots \quad \dots \\ & \dots \quad \dots \quad \dots \\ & \leq (2k_1 + k_2)^n S(\phi t_0, \phi t_0, \phi t_1) \rightarrow 0, \end{aligned}$$

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since $(2k_1 + k_2)^n \rightarrow 0$ as $n \rightarrow \infty$.

Now we claim that $\{\phi t_n\}$ is a Cauchy sequence.

For any $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned}
 S(\phi t_n, \phi t_n, \phi t_m) &\leq 2[S(\phi t_n, \phi t_n, \phi t_{n+1}) + S(\phi t_{n+1}, \phi t_{n+1}, \phi t_{n+2}) + \cdots \\
 &\quad + S(\phi t_{m-1}, \phi t_{m-1}, \phi t_m)] \\
 &\leq 2c^n S(\phi t_0, \phi t_0, \phi t_1) + c^{n+1} S(\phi t_0, \phi t_0, \phi t_1) + \cdots \\
 &\quad + c^m S(\phi t_0, \phi t_0, \phi t_1) \\
 &\leq 2c^n (1 + c + c^2 + \cdots + c^{m-n}) S(\phi t_0, \phi t_0, \phi t_1) \\
 &\leq 2c^n \frac{1 - c^{m-n}}{1 - c} S(\phi t_0, \phi t_0, \phi t_1) \\
 &\leq 2 \frac{c^n}{1 - c} S(\phi t_0, \phi t_0, \phi t_1) \rightarrow 0,
 \end{aligned}$$

since $c < 1$ then $c^n \rightarrow 0$ as $n \rightarrow \infty$.

Therefore $\{\phi t_n\}$ is a Cauchy sequence in X .

Since $(\phi(P), S)$ is complete, there is a $t \in P$ such that $\phi t_n \rightarrow \phi t$ as $n \rightarrow \infty$.

We now prove that t is a point of coincidence of μ, ψ and ϕ .

From the condition (ii) of Theorem 3.1 we have,

$$\begin{aligned}
 &S(\phi t_{2n+1}, \phi t_{2n+1}, \psi t) \\
 &= S(\mu t_{2n}, \mu t_{2n}, \psi t) \\
 &\leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \mu t_{2n}) \cdot S(\phi t, \phi t, \psi t)}{S(\phi t_{2n}, \phi t_{2n}, \phi t) + S(\phi t_{2n}, \phi t_{2n}, \psi t) + S(\phi t, \phi t, \mu t_{2n})} \right] \\
 &\quad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t) \\
 &\leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \cdot S(\phi t, \phi t, \psi t)}{S(\phi t_{2n}, \phi t_{2n}, \phi t) + S(\phi t_{2n}, \phi t_{2n}, \psi t) + S(\phi t, \phi t, \phi t_{2n+1})} \right] \\
 &\quad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t),
 \end{aligned}$$

which gives $S(\phi t, \phi t, \psi t) = 0$ as $n \rightarrow \infty$ and hence $\phi t = \psi t$.

Also we have,

$$\begin{aligned}
 &S(\mu t, \mu t, \phi t) \\
 &= S(\mu t, \mu t, \psi t) \\
 &\leq k_1 \left[\frac{S(\phi t, \phi t, \mu t) \cdot S(\phi t, \phi t, \psi t)}{S(\phi t, \phi t, \phi t) + S(\phi t, \phi t, \psi t) + S(\phi t, \phi t, \mu t)} \right] + k_2 S(\phi t, \phi t, \phi t),
 \end{aligned}$$

which implies $S(\mu t, \mu t, \phi t) = 0$ proving $\mu t = \phi t$.

Therefore we have $\phi t = \psi t = \mu t = a(\text{say})$,

proving that t is a coincident point of μ, ψ and ϕ .

Since the pairs (ϕ, μ) and (ϕ, ψ) are weakly compatible, we have $\phi\mu t = \mu\phi t$ and $\phi\psi t = \psi\phi t$ which implies $\phi a = \psi a = \mu a$.

Now we have,

$$\begin{aligned} S(\phi a, \phi a, a) &= S(\mu a, \mu a, \psi t) \\ &\leq k_1 \left[\frac{S(\phi a, \phi a, \mu a) \cdot S(\phi t, \phi t, \psi t)}{S(\phi a, \phi a, \phi t) + S(\phi a, \phi a, \mu t) + S(\phi t, \phi t, \mu a)} \right] \\ &\quad + k_2 S(\phi a, \phi a, \phi t) \\ &\leq k_1 \left[\frac{S(\phi a, \phi a, \phi a) \cdot S(a, a, a)}{S(\phi a, \phi a, a) + S(\phi a, \phi a, a) + S(a, a, \phi a)} \right] + k_2 S(\phi a, \phi a, a) \\ &\leq k_2 S(\phi a, \phi a, a), \end{aligned}$$

leading to a contradiction, giving that $S(\phi a, \phi a, a) = 0$ implies $\phi a = a$.

Hence $\phi a = \psi a = \mu a = a$, showing that a is a common fixed point of μ, ψ and ϕ .

We now prove that the common fixed point is unique.

Suppose $a' (\neq a)$ is another common fixed point of μ, ψ and ϕ .

That is $a' = \phi a' = \psi a' = \mu a'$.

We have

$$\begin{aligned} S(a, a, a') &= S(\mu a, \mu a, \psi a') \\ &\leq k_1 \left[\frac{S(\phi a, \phi a, \mu a) \cdot S(\phi a', \phi a', \psi a')}{S(\phi a, \phi a, \phi a') + S(\phi a, \phi a, \psi a') + S(\phi a', \phi a', \mu a)} \right] \\ &\quad + k_2 S(\phi a, \phi a, \phi a') \\ &\leq k_1 \left[\frac{S(a, a, a) \cdot S(a', a', a')}{S(a, a, a') + S(a, a, a') + S(a, a, a')} \right] + k_2 S(a, a, a') \\ &\leq k_2 S(a, a, a'), \end{aligned}$$

which is a contradiction since $k_2 < 1$.

Therefore $S(a, a, a') = 0$ implies $a = a'$, Proving the uniqueness. $\square$

Corolary 3.1. Let P be a subset of a S -metric space (Y, S) . Suppose that ϕ, μ are two selfmaps of P satisfy

(i) $\mu(Y) \subseteq \phi(Y)$ and $(\phi(P), S)$ is complete.

(ii)

$$\begin{aligned} &S(\mu y_1, \mu y_1, \mu y_2) \\ &\leq k_1 \left\{ \frac{S(\phi y_1, \phi y_1, \mu y_1) \cdot S(\phi y_2, \phi y_2, \mu y_2)}{S(\phi y_1, \phi y_1, \phi y_2) + S(\phi y_1, \phi y_1, \mu y_2) + S(\phi y_2, \phi y_2, \mu y_1)} \right\} \\ &\quad + k_2 S(\phi y_1, \phi y_1, \phi y_2) \end{aligned}$$

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for every $y_1, y_2 \in P$ and $k_1, k_2 \in [0, 1)$ with $2k_1 + k_2 < 1$.

(iii) The pair (μ, ϕ) is weakly compatible.

Then μ and ϕ have a unique common fixed point.

Proof. On taking $\psi = \mu$ in Theorem 3.1, the corollary follows. □

4 Conclusion

In this paper, a common fixed theorem for three weakly compatible selfmaps of a S-metric space is established with the aid of an associated sequence of three selfmaps. Moreover, we deduce a common fixed point theorem for two selfmaps. As S-metric space is a robust generalization of metric space, our theorem generalizes the theorem in literature.

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Some Results in Hausdorff Neutrosophic Metric Spaces on Hutchinson-Barnsley Operator

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Abstract

The main purpose of this paper is to prove the Neutrosophic contraction properties of the Hutchinson-Barnsley operator on the Neutrosophic hyperspace with respect to the Hausdorff Neutrosophic metrics. Also we discuss about the relationships between the Hausdorff Neutrosophic metrics on the Neutrosophic hyperspaces. Our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces to the Neutrosophic metric spaces.

Keywords: Contraction, Hutchinson-Barnsley Operator, Metric Space, Hausdorff Neutrosophic Metric Spaces, Hyperspace.

2010 AMS subject classification: 03E72, 54E35, 54A40, 46S40[‡]

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Received on May 16th, 2022. Accepted on June 28th, 2022. Published on June 30th, 2022. doi: 10.23755/rm.v43i0.782. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

The Fractal Analysis was introduced by Mandelbrot in 1975 [8] and popularized by various mathematicians [6], [3], [4]. Sets with non-integral Hausdorff dimension, which exceeds its topological dimension, are called Fractals by Mandelbrot [8]. Hutchinson [6] and Barnsley [3] initiated and developed the Hutchinson-Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Iterated Function System (IFS) of contractions. That is, Hutchinson[6] introduced an operator on hyperspace called as Hutchinson-Barnsley operator (HB operator) to define a fractal set as a unique fixed point by using the Banach Contraction Theorem in the metric spaces. Recently in [5], [15], HB operator properties were analyzed in fuzzy metric spaces. Here we introduce the concepts and properties of HB operator in the intuitionistic fuzzy metric spaces.

Atanassov [2] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [10] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space. In 1998, Smarandache [12,13] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Basset et al. [1] Explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making. In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Later Jeyaraman et al., [7, 11] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [14] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we prove the neutrosophic contraction properties of the HB operator on the neutrosophic hyperspace with respect to the Hausdorff neutrosophic metrics. Also we discuss about the relationships between the Hausdorff neutrosophic metrics on the neutrosophic hyperspaces. Here our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces.

2. Preliminaries

Definition 2.1. [3] Let (Σ, d) be a metric space and $\mathcal{K}_0(\Sigma)$ be the collection of all non-empty compact subsets of Σ .

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Define $d(\zeta, Q) := \inf_{y \in Q} d(\zeta, y)$ and $d(P, Q) := \sup_{x \in P} d(x, Q)$ for all $\zeta \in \Sigma$ and $P, Q \in \mathcal{K}_0(\Sigma)$. The Hausdorff metric or Hausdorff distance (H_d) is a function $(H_d) : \mathcal{K}_0(\Sigma) \times \mathcal{K}_0(\Sigma) \rightarrow \mathbb{R}$ defined by $H_d(P, Q) = \max\{d(P, Q), d(Q, P)\}$.

Then H_d is a metric on the hyperspace of compact sets $\mathcal{K}_0(\Sigma)$ and hence $(\mathcal{K}_0(\Sigma), H_d)$ is called a Hausdorff metric space.

Theorem 2.2. [3] If (Σ, d) is a complete metric space, then $(\mathcal{K}_0(\Sigma), H_d)$ is also a complete metric space.

Definition 2.3. [3] Let (Σ, d) be a metric space and $f_n : \Sigma \rightarrow \Sigma$, $n = 1, 2, \dots, N_0$ ($N_0 \in \mathbb{N}$) be N_0 - contraction mappings with the corresponding contractivity ratios k_n , $n = 1, 2, \dots, N_0$. The system $\{\Sigma; f_n, n = 1, 2, \dots, N_0\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System with the ratio $k = \max_{n=1}^{N_0} k_n$. Then the Hutchinson Barnsley Operator (HBO) of the IFS is a function $F : \mathcal{K}_0(\Sigma) \rightarrow \mathcal{K}_0(\Sigma)$ defined by $F(Q) = \bigcup_{n=1}^{N_0} f_n(Q)$, for all $Q \in \mathcal{K}_0(\Sigma)$.

Theorem 2.4. [3] Let (Σ, d) be a metric space. Let $\{\Sigma; f_n, n = 1, 2, \dots, N_0; N_0 \in \mathbb{N}\}$ be an IFS. Then, the HBO (F) is a contraction mapping on $(\mathcal{K}_0(\Sigma), H_d)$.

Theorem 2.5. [3] Let (Σ, d) be a complete metric space and $\{\Sigma; f_n, n = 1, 2, 3 \dots, N_0; N_0 \in \mathbb{N}\}$ be an IFS. Then, there exists only one compact invariant set $P_\infty \in \mathcal{K}_0(\Sigma)$ of the HBO (F) or equivalently, F has a unique fixed point namely $P_\infty \in \mathcal{K}_0(\Sigma)$.

Definition 2.6. [3] The fixed point $P_\infty \in \mathcal{K}_0(\Sigma)$ of the HBO F described in the Theorem (2.5) is called the Attractor (Fractal) of the IFS. Sometimes $P_\infty \in \mathcal{K}_0(\Sigma)$ is called as Fractal generated by the IFS and so called as IFS Fractal.

Definition 2.7. A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm, if $*$ satisfies the following conditions:

- (a) $*$ is commutative and associative;
- (b) $*$ is continuous
- (c) $a * 1 = a$ for all $a \in [0,1]$;
- (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.8. A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm, if $\diamond$ satisfies the following conditions:

- (a) $\diamond$ is commutative and associative;
- (b) $\diamond$ is continuous
- (c) $a \diamond 0 = a$ for all $a \in [0,1]$;
- (d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.9. A 6-tuple $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ is said to be an Neutrosophic Metric Space (shortly NMS), if Σ is an arbitrary set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ and Y are neutrosophic on $\Sigma \times \Sigma$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma, \lambda, \mu \in R^+$.

- (i) $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1; 0 \leq \Theta(\zeta, \eta, \lambda) \leq 1; 0 \leq Y(\zeta, \eta, \lambda) \leq 1$;
- (ii) $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + Y(\zeta, \eta, \lambda) \leq 3$;
- (iii) $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;
- (iv) $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$ for $\lambda > 0$
- (v) $\Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu) \leq \Xi(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
- (vi) $\Xi(\zeta, \eta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous ;
- (vii) $\lim_{\lambda \rightarrow \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;
- (viii) $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
- (ix) $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$ for $\lambda > 0$;
- (x) $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu) \geq \Theta(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
- (xi) $\Theta(\zeta, \eta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
- (xii) $\lim_{\lambda \rightarrow \infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
- (xiii) $Y(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
- (xiv) $Y(\zeta, \eta, \lambda) = Y(\eta, \zeta, \lambda)$ for $\lambda > 0$;
- (xv) $Y(\zeta, \eta, \lambda) \diamond Y(\eta, \delta, \mu) \geq Y(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
- (xvi) $Y(\zeta, \eta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is neutrosophic continuous;
- (xvii) $\lim_{\lambda \rightarrow \infty} Y(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

Then, (Ξ, Θ, Y) is called an NMS on Σ . The functions Ξ, Θ and Y denote degree of closedness, naturalness and non-closedness between ζ and η with respect to λ respectively.

Example 2.10. Let (Σ, d) be a metric space. Let Ξ_d, Θ_d and Y_d be the functions defined on $\Sigma^2 \times (0, \infty)$ by $\Xi_d(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$, $\Theta_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$ and $Y_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi_d, \Theta_d, Y_d, *, \diamond)$ is a

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NMS which is called standard NMS, and $(\mathcal{E}_d, \theta_d, \gamma_d)$ is called as standard NM induced by the metric d .

Definition 2.11. Let $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ be a NMS. The open ball $B(\zeta, r, \lambda)$ with $\zeta \in \Sigma$ and radius r , $0 < r < 1$, with respect to $\lambda > 0$ is defined as $B(\zeta, r, \lambda) = \{\eta \in \Sigma : \mathcal{E}(\zeta, \eta, \lambda) > 1 - r, \theta(\zeta, \eta, \lambda) < r \text{ and } \gamma(\zeta, \eta, \lambda) < r\}$. Define

$$\tau_{(\mathcal{E}, \theta, \gamma)} = \left\{ P \subset \Sigma : \text{for each } \zeta \in P, \exists \lambda > 0 \text{ and } r \in (0, 1) \text{ such that } B(\zeta, r, \lambda) \subset P \right\}.$$

Then $\tau_{(\mathcal{E}, \theta, \gamma)}$ is a topology on Σ induced by a NFM $(\mathcal{E}, \theta, \gamma)$.

The topologies induced by the metric and the corresponding standard NM are the same.

Proposition 2.12. The metric space (Σ, d) is complete if and only if the standard NMS $(\Sigma, \mathcal{E}_d, \theta_d, \gamma_d, *, \diamond)$ is complete.

Definition 2.13. A neutrosophic fuzzy B-contraction (neutrosophic fuzzy Sehgal contraction) on an NMS $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ is a self-mapping f on Σ for which

$$\mathcal{E}(f(\zeta), f(\eta), k\lambda) \geq \mathcal{E}(\zeta, \eta, \lambda), \quad \theta(f(\zeta), f(\eta), k\lambda) \leq \theta(\zeta, \eta, \lambda) \text{ and}$$

$$\gamma(f(\zeta), f(\eta), k\lambda) \leq \gamma(\zeta, \eta, \lambda) \text{ for all } \zeta, \eta \in \Sigma \text{ and } \lambda > 0,$$

where k is a fixed constant in $(0, 1)$.

3. Hausdorff Neutrosophic Metric Spaces

Definition 3.1. Let $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ be a NMS and $\tau_{(\mathcal{E}, \theta, \gamma)}$ be the topology induced by the NM $(\mathcal{E}, \theta, \gamma)$. We shall denote by $\mathcal{K}_0(X)$, the set of all non-empty compact subsets of $(\Sigma, \tau_{(\mathcal{E}, \theta, \gamma)})$.

$$\text{Define } \mathcal{E}(\zeta, Q, \lambda) := \sup_{\eta \in Q} \mathcal{E}(\zeta, \eta, \lambda), \quad \mathcal{E}(P, Q, \lambda) := \inf_{\zeta \in P} \mathcal{E}(\zeta, Q, \lambda),$$

$$\theta(\zeta, Q, \lambda) := \inf_{\eta \in Q} \theta(\zeta, \eta, \lambda), \quad \theta(P, Q, \lambda) := \sup_{\zeta \in P} \theta(\zeta, Q, \lambda) \text{ and}$$

$$\gamma(\zeta, Q, \lambda) := \inf_{\eta \in Q} \gamma(\zeta, \eta, \lambda), \quad \gamma(P, Q, \lambda) := \sup_{\zeta \in P} \gamma(\zeta, Q, \lambda),$$

for all $\zeta \in \Sigma$ and $P, Q \in \mathcal{K}_0(X)$.

Then, we define the Hausdorff NM $(H_{\mathcal{E}}, H_{\theta}, H_{\gamma}, *, \diamond)$ as

$$H_{\mathcal{E}}(P, Q, \lambda) = \min\{\mathcal{E}(P, Q, \lambda), \mathcal{E}(Q, P, \lambda)\},$$

$$H_\theta(P, Q, \lambda) = \max\{\theta(P, Q, \lambda), \theta(Q, P, \lambda)\} \text{ and } H_Y(P, Q, \lambda) = \max\{Y(P, Q, \lambda), Y(Q, P, \lambda)\}.$$

Here (H_Ξ, H_θ, H_Y) is a NM on the hyperspace of compact sets, $\mathcal{K}_0(\Sigma)$ and hence $(\mathcal{K}_0(\Sigma), H_\Xi, H_\theta, H_Y, *, \diamond)$ is called a Hausdorff NMS.

Proposition 3.2. Let (Σ, d) be a metric space. Then the Hausdorff NM $(H_{\Xi_d}, H_{\theta_d}, H_{Y_d})$ of the standard NM (Ξ_d, θ_d, Y_d) coincides with the standard NM $(H_{\Xi_d}, H_{\theta_d}, H_{Y_d})$ of the Hausdorff metric (H_d) on $\mathcal{K}_0(\Sigma)$.

$$\text{ie., } H_{\Xi_d}(P, Q, \lambda) = \Xi_{H_d}(P, Q, \lambda), H_{\theta_d}(P, Q, \lambda) = \theta_{H_d}(P, Q, \lambda) \text{ and}$$

$$H_{Y_d}(P, Q, \lambda) = Y_{H_d}(P, Q, \lambda), \text{ for all } P, Q \in \mathcal{K}_0(\Sigma) \text{ and } \lambda > 0.$$

Proof: Fix $\lambda > 0$ and $P, Q \in \mathcal{K}_0(\Sigma)$. We recall that

$$\sup_{\beta \in Q} \Xi_d(\alpha, \beta, \lambda) = \frac{\lambda}{\lambda + \inf_{\beta \in Q} d(\alpha, \beta)}, \inf_{\beta \in Q} \theta_d(\alpha, \beta, \lambda) = \frac{1}{1 + \frac{\lambda}{\inf_{\beta \in Q} d(\alpha, \beta)}} \text{ and}$$

$$\inf_{\beta \in Q} Y_d(\alpha, \beta, \lambda) = \frac{1}{\frac{\lambda}{\inf_{\beta \in Q} d(\alpha, \beta)}}, \text{ for all } \alpha \in P. \text{ It follows that}$$

$$\Xi_d(\alpha, Q, \lambda) = \frac{\lambda}{\lambda + d(\alpha, Q)}, \theta_d(\alpha, Q, \lambda) = \frac{1}{1 + \frac{\lambda}{d(\alpha, Q)}} \text{ and } Y_d(\alpha, Q, \lambda) = \frac{1}{\frac{\lambda}{d(\alpha, Q)}} \text{ for all } \alpha \in P.$$

$$\text{Then } \inf_{\alpha \in P} \Xi_d(\alpha, Q, \lambda) = \frac{\lambda}{\lambda + \sup_{\alpha \in P} d(\alpha, Q)}, \sup_{\alpha \in P} \theta_d(\alpha, Q, \lambda) = \frac{1}{1 + \frac{\lambda}{\sup_{\alpha \in P} d(\alpha, Q)}} \text{ and}$$

$$\sup_{\alpha \in P} Y_d(\alpha, Q, \lambda) = \frac{1}{\sup_{\alpha \in P} \frac{\lambda}{d(\alpha, Q)}}.$$

It follows that

$$\Xi_d(P, Q, \lambda) = \frac{\lambda}{\lambda + d(P, Q)}, \theta_d(P, Q, \lambda) = \frac{1}{1 + \frac{\lambda}{d(P, Q)}} = \frac{d(P, Q)}{\lambda + d(P, Q)} \text{ and}$$

$$Y_d(P, Q, \lambda) = \frac{1}{\frac{\lambda}{d(P, Q)}} = \frac{d(P, Q)}{\lambda}.$$

Similarly, we obtain

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$$\Xi_d(Q, P, \lambda) = \frac{\lambda}{\lambda + d(Q, P)}, \quad \theta_d(Q, P, \lambda) = \frac{d(Q, P)}{\lambda + d(Q, P)} \text{ and } \gamma_d(Q, P, \lambda) = \frac{d(Q, P)}{\lambda}.$$

Therefore, $H_{\Xi_d}(P, Q, \lambda) = \Xi_{H_d}(P, Q, \lambda)$, $H_{\theta_d}(P, Q, \lambda) = \theta_{H_d}(P, Q, \lambda)$ and

$H_{\gamma_d}(P, Q, \lambda) = \gamma_{H_d}(P, Q, \lambda)$. The proof is complete.

Using the Proposition 3.2., we can easily compute distances with respect to the Hausdorff NM $(H_{\Xi_d}, H_{\theta_d}, H_{\gamma_d})$ of the standard NM $(\Xi_d, \theta_d, \gamma_d)$ by computing distances with respect to the Hausdorff metric (H_d) implied by the metric d . Here, we illustrate this situation with two examples.

Example 3.3. Let $(\mathbb{R}, d)$ be the Euclidean metric space and $P = [\alpha_1, \alpha_2]$ and $Q = [\beta_1, \beta_2]$ be two compact intervals of $\mathbb{R}$. Then $d(P, Q) = |\alpha_1 - \beta_1|$ and $d(Q, P) = |\alpha_2 - \beta_2|$ and hence $H_d(P, Q) = \max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}$; So, by Proposition (3.2), We have

$$\Xi_d(P, Q, \lambda) = \frac{\lambda}{\lambda + \max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}}, \quad \theta_d(P, Q, \lambda) = \frac{\max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}}{\lambda + \max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}} \text{ and}$$

$$\gamma_d(P, Q, \lambda) = \frac{\max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}}{\lambda}, \text{ for all } \lambda > 0.$$

Example 3.4. Let (Σ, d) be the discrete metric space such that $|\Sigma| \geq 2$. Let P and Q be two non-empty finite subsets of Σ , with $P \neq Q$. Then $d(P, Q) = 1 = d(Q, P)$ and hence $H_d(P, Q) = 1$; so by Proposition 3.2., we have

$$H_{\Xi_d}(P, Q, \lambda) = \frac{\lambda}{\lambda + 1}, H_{\theta_d} = \frac{1}{1 + \lambda} \text{ and } H_{\gamma_d} = \frac{1}{\lambda}, \text{ for all } \lambda > 0.$$

Definition 3.5. Let $(\Sigma, \Xi, \theta, \gamma, *, \diamond)$ be an NMS and $\tau_{(\Xi, \theta, \gamma)}$ be the topology induced by (Ξ, θ, γ) . We observe that $(\mathcal{K}_0(\mathcal{K}_0(\Sigma)), H_{H_{\Xi}}, \mathcal{H}_{H_{\theta}}, \mathcal{H}_{H_{\gamma}}, *, \diamond)$ is also an NMS, where $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ is the hyperspace of all non-empty compact subsets of $(\mathcal{K}_0(\Sigma), H_{\Xi}, H_{\theta}, H_{\gamma}, *, \diamond)$ and $(\mathcal{H}_{H_{\Xi}}, \mathcal{H}_{H_{\theta}}, \mathcal{H}_{H_{\gamma}})$ is the Hausdorff NM on $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ implied by the Hausdorff NM $(\Xi_d, \theta_d, \gamma_d)$ on $\mathcal{K}_0(\Sigma)$. That is, for all $P \in \mathcal{K}_0(\Sigma)$ and $\mathfrak{P}, \mathfrak{Q} \in \mathcal{K}_0(\mathcal{K}_0(\Sigma))$,

$$H_{H_{\Xi}}(\mathfrak{P}, \mathfrak{Q}) = \min\{H_{\Xi}\mathfrak{P}, \mathfrak{Q}, H_{\Xi}(\mathfrak{Q}, \mathfrak{P})\}, H_{H_{\theta}}(\mathfrak{P}, \mathfrak{Q}) = \max\{H_{\theta}(\mathfrak{P}, \mathfrak{Q}), H_{\theta}(\mathfrak{Q}, \mathfrak{P})\}$$

$$\text{and } H_{H_{\gamma}}(\mathfrak{P}, \mathfrak{Q}) = \max\{H_{\gamma}(\mathfrak{P}, \mathfrak{Q}), H_{\gamma}(\mathfrak{Q}, \mathfrak{P})\} \text{ where}$$

$$\begin{aligned} H_{\Xi}(\mathfrak{P}, \mathfrak{Q}) &:= \inf_{p \in \mathfrak{P}} H_{\Xi}(P, \mathfrak{Q}), H_{\Xi}(P, \mathfrak{Q}) := \sup_{Q \in \mathfrak{Q}} H_{\Xi}(P, Q), \\ H_{\theta}(\mathfrak{P}, \mathfrak{Q}) &:= \sup_{p \in \mathfrak{P}} H_{\theta}(P, \mathfrak{Q}), H_{\theta}(P, \mathfrak{Q}) := \inf_{Q \in \mathfrak{Q}} H_{\theta}(P, Q) \text{ and} \\ H_{\gamma}(\mathfrak{P}, \mathfrak{Q}) &:= \sup_{p \in \mathfrak{P}} H_{\gamma}(P, \mathfrak{Q}), H_{\gamma}(P, \mathfrak{Q}) := \inf_{Q \in \mathfrak{Q}} H_{\gamma}(P, Q). \end{aligned}$$

Proposition 3.6. Let (Σ, d) be a metric space and let $(\mathcal{K}_0(\Sigma), H_d)$ and $(\mathcal{K}_0(\mathcal{K}_0(\Sigma)), \mathcal{H}_{H_d})$ be the corresponding Hausdorff metric spaces. Then, the Hausdorff NM $(\mathcal{H}_{\Xi_{H_d}}, \mathcal{H}_{\theta_{H_d}}, \mathcal{H}_{\gamma_{H_d}})$ of the standard NM $(\Xi_{H_d}, \theta_{H_d}, \gamma_{H_d})$ coincides with the standard NM $(\Xi_{\mathcal{H}_{H_d}}, \theta_{\mathcal{H}_{H_d}}, \gamma_{\mathcal{H}_{H_d}})$ of the Hausdorff metric $(\mathcal{H}_{H_d})$ on $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$, ie. $\mathcal{H}_{\Xi_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda) = \Xi_{\mathcal{H}_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda)$, $\mathcal{H}_{\theta_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda) = \theta_{\mathcal{H}_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda)$ and $\mathcal{H}_{\gamma_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda) = \gamma_{\mathcal{H}_{H_d}}(\mathfrak{P}, \mathfrak{Q}, \lambda)$ for all $\mathfrak{P}, \mathfrak{Q} \in \mathcal{K}_0(\mathcal{K}_0(\Sigma))$ and $\lambda > 0$.

Proof: Proposition 3.2. completes the proof.

4. Neutrosophic Hutchinson-Barnsley Operator

In this section, we define the Neutrosophic Iterated Function System (NIFS) and Neutrosophic HB Operator on the NMS.

Definition 4.1. Let $(\Sigma, \Xi, \theta, \gamma, *, \diamond)$ be an NMS and $f_n: \Sigma \rightarrow \Sigma, n = 1, 2, 3 \dots N_0 (N_0 \in \mathbb{N})$ be N_0 - neutrosophic B-contractions. Then the system $\{\Sigma; f_n, n = 1, 2, 3 \dots N_0\}$ is called a NIFS of neutrosophic B-contractions in the NMS $(\Sigma, \Xi, \theta, \gamma, *, \diamond)$.

Definition 4.2. Let $(\Sigma, \Xi, \theta, \gamma, *, \diamond)$ be a NMS. Let $\{\Sigma; f_n, n = 1, 2, 3 \dots N_0\}$ be an NIFS of neutrosophic B-contractions. Then the Neutrosophic Hutchinson-Barnsley Operator (NHBO) of the NIFS is a function $F: \mathcal{K}_0(\Sigma) \rightarrow \mathcal{K}_0(\Sigma)$ defined by $F(Q) = \cup_{n=1}^{N_0} f_n(Q)$, for all $Q \in \mathcal{K}_0(\Sigma)$.

Definition 4.3. Let $(\Sigma, \Xi, \theta, \gamma, *, \diamond)$ be a complete NMS. Let $f_n: \Sigma \rightarrow \Sigma, n = 1, 2, 3 \dots N_0 (N_0 \in \mathbb{N})$ be a NIFS of neutrosophic B-contractions and F be the NHBO of the NIFS. We say that the set $P_{\infty} \in \mathcal{K}_0(\Sigma)$ is Neutrosophic Attractor (Neutrosophic Fractal) of the given NIFS, if P_{∞} is a unique fixed point of the NHBO F. Such $P_{\infty} \in \mathcal{K}_0(\Sigma)$ is also called as Fractal generated by the NIFS and so called NIFS Fractal of neutrosophic B-contractions.

Properties of NHBO

Now, we prove the interesting results about the neutrosophic B-contraction properties of operators with respect to the Hausdorff neutrosophic metric on $\mathcal{K}_0(\Sigma)$.

Theorem 4.4. Let (Σ, d) be a metric space. Let $f: \Sigma \rightarrow \Sigma$ be a contraction function on (Σ, d) , with a contractivity ratio k . Then

$$H_{\varepsilon_d}(f(P), f(Q), \lambda) \geq H_{\varepsilon_d}(P, Q, \lambda), \quad H_{\theta_d}(f(P), f(Q), \lambda) \leq H_{\theta_d}(P, Q, \lambda) \text{ and} \\ H_{\gamma_d}(f(P), f(Q), \lambda) \leq H_{\gamma_d}(P, Q, \lambda), \text{ for all } P, Q \in \mathcal{K}_0(\Sigma) \text{ and } \lambda > 0.$$

Proof: Fix $\lambda > 0$ and let $P, Q \in \mathcal{K}_0(\Sigma)$. Since f is contraction on (Σ, d) with the contractivity ratio $k \in (0, 1)$ and by Theorem 2.4. for the case $\theta = 1$, we have

$$H_d(f(P), f(Q)) \leq kH_d(P, Q). \text{ Since } \lambda > 0 \text{ and } k \in (0, 1),$$

$$\frac{k\lambda}{k\lambda + H_d(f(P), f(Q))} \geq \frac{k\lambda}{k\lambda + kH_d(P, Q)} = \frac{\lambda}{\lambda + H_d(P, Q)}, \quad \frac{H_d(f(P), f(Q))}{k\lambda + H_d(f(P), f(Q))} \leq \frac{kH_d(P, Q)}{k\lambda + kH_d(P, Q)} = \frac{H_d(P, Q)}{\lambda + H_d(P, Q)}$$

and $\frac{H_d(f(P), f(Q))}{k\lambda} \leq \frac{kH_d(P, Q)}{k\lambda} = \frac{H_d(P, Q)}{\lambda}$.

By using the above inequalities and the Proposition 3.2., we have

$$H_{\varepsilon_d}(f(P), f(Q), k\lambda) = \varepsilon_{H_d}(f(P), f(Q), k\lambda) \\ = \frac{k\lambda}{k\lambda + H_d(f(P), f(Q))} \geq \frac{\lambda}{\lambda + H_d(P, Q)} \\ = \varepsilon_{H_d}(P, Q, \lambda) = H_{\varepsilon_d}(P, Q, \lambda),$$

$$H_{\theta_d}(f(P), f(Q), k\lambda) = \theta_{H_d}(f(P), f(Q), k\lambda) \\ = \frac{kH_d(f(P), f(Q))}{k\lambda + H_d(f(P), f(Q))} \leq \frac{H_d(P, Q)}{\lambda + H_d(P, Q)} \\ = \theta_{H_d}(P, Q, \lambda) = H_{\theta_d}(P, Q, \lambda) \text{ and}$$

Similarly, $H_{\gamma_d}(f(P), f(Q), k\lambda) = \gamma_{H_d}(f(P), f(Q), k\lambda)$

$$= \frac{kH_d(f(P), f(Q))}{k\lambda + H_d(f(P), f(Q))} \leq \frac{H_d(P, Q)}{\lambda + H_d(P, Q)} \\ = \gamma_{H_d}(P, Q, \lambda) = H_{\gamma_d}(P, Q, \lambda).$$

The above theorem 4.4. shows that f is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $(H_{\varepsilon_d}, H_{\theta_d}, H_{\gamma_d})$ implied by the standard metric $(\varepsilon_d, \theta_d, \gamma_d)$, if f is contraction on a metric space (Σ, d) . The following theorem is somewhat generalization of the Theorem 4.4.

Theorem 4.5. Let $(\Sigma, \Xi, \Theta, Y, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_{\Xi}, H_{\Theta}, H_Y, *, \diamond)$ be the corresponding Hausdorff NMS. Suppose $f: \Sigma \rightarrow \Sigma$ be a neutrosophic B-Contraction on $(\Sigma, \Xi, \Theta, Y, *, \diamond)$. Then for $k \in (0, 1)$,
 $H_{\Xi}(f(P), f(Q), k\lambda) \geq H_{\Xi}(P, Q, \lambda)$, $H_{\Theta}(f(P), f(Q), k\lambda) \leq H_{\Theta}(P, Q, \lambda)$ and
 $H_Y(f(P), f(Q), k\lambda) \leq H_Y(P, Q, \lambda)$ for all $P, Q \in \mathcal{K}_0(\Sigma)$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q \in \mathcal{K}_0(\Sigma)$. For given $k \in (0, 1)$, we get
 $\Xi(f(\zeta), f(\eta), k\lambda) \geq \Xi(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $\Xi(f(\zeta), f(\eta), k\lambda) \geq \Xi(\zeta, \eta, \lambda)$,
for all $\zeta \in P$ and $\eta \in Q$,
 $\sup_{\eta \in Q} \Xi(f(\zeta), f(\eta), k\lambda) \geq \sup_{\eta \in Q} \Xi(\zeta, \eta, \lambda)$, for all $\zeta \in P$,
 $\Xi(f(\zeta), f(Q), k\lambda) \geq \Xi(\zeta, Q, \lambda)$, for all $\zeta \in P$,
 $\inf_{\zeta \in P} \Xi(f(\zeta), f(Q), k\lambda) \geq \inf_{\zeta \in P} \Xi(\zeta, Q, \lambda)$, $\Xi(f(P), f(Q), k\lambda) \geq \Xi(P, Q, \lambda)$.
Similarly $\Xi(f(Q), f(\zeta), k\lambda) \geq \Xi(Q, P, \lambda)$.
Hence $H_{\Xi}(f(P), f(Q), k\lambda) \geq H_{\Xi}(P, Q, \lambda)$
Now, $\Theta(f(\zeta), f(\eta), k\lambda) \leq \Theta(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $\Theta(f(\zeta), f(\eta), k\lambda) \leq \Theta(\zeta, \eta, \lambda)$,
for all $\zeta \in P$ and $\eta \in Q$,
 $\inf_{\eta \in Q} \Theta(f(\zeta), f(\eta), k\lambda) \leq \inf_{\eta \in Q} \Theta(\zeta, \eta, \lambda)$, for all $\zeta \in P$,
 $\Theta(f(\zeta), f(Q), k\lambda) \leq \Theta(\zeta, Q, \lambda)$, for all $\zeta \in P$,
 $\sup_{\zeta \in P} \Theta(f(\zeta), f(Q), k\lambda) \leq \sup_{\zeta \in P} \Theta(\zeta, Q, \lambda)$.
 $\Theta(f(P), f(Q), k\lambda) \leq \Theta(P, Q, \lambda)$.
Similarly $\Theta(f(Q), f(\zeta), k\lambda) \leq \Theta(Q, P, \lambda)$.
Hence $H_{\Theta}(f(P), f(Q), k\lambda) \leq H_{\Theta}(P, Q, \lambda)$ and
 $Y(f(\zeta), f(\eta), k\lambda) \leq Y(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $Y(f(\zeta), f(\eta), k\lambda) \leq Y(\zeta, \eta, \lambda)$,
for all $\zeta \in P$ and $\eta \in Q$
 $\inf_{\eta \in Q} Y(f(\zeta), f(\eta), k\lambda) \leq \inf_{\eta \in Q} Y(\zeta, \eta, \lambda)$, for all $\zeta \in P$
 $Y(f(\zeta), f(Q), k\lambda) \leq Y(\zeta, Q, \lambda)$, for all $\zeta \in P$,
 $\sup_{\zeta \in P} Y(f(\zeta), f(Q), k\lambda) \leq \sup_{\zeta \in P} Y(\zeta, Q, \lambda)$.
 $Y(f(P), f(Q), k\lambda) \leq Y(P, Q, \lambda)$.
Similarly $Y(f(Q), f(\zeta), k\lambda) \leq Y(Q, P, \lambda)$.
Hence $H_Y(f(P), f(Q), k\lambda) \leq H_Y(P, Q, \lambda)$. This completes the proof.

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The above Theorem 4.5. shows that f is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $H_{\mathcal{E}}, H_{\theta}, H_Y$, if f is neutrosophic B-contraction on neutrosophic metric space $(\Sigma, \mathcal{E}, \theta, Y, *, \diamond)$.

Lemma 4.6. Let $(\Sigma, \mathcal{E}, \theta, Y, *, \diamond)$ be a NMS. If $Q, R \subset \Sigma$ such that $Q \subset R$, then $\mathcal{E}(\zeta, Q, \lambda) \leq \mathcal{E}(\zeta, R, \lambda)$, $\theta(\zeta, Q, \lambda) \geq \theta(\zeta, R, \lambda)$ and $Y(\zeta, Q, \lambda) \geq Y(\zeta, R, \lambda)$ for all $\zeta \in \Sigma$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $\zeta \in \Sigma$ and $Q, R \subset \Sigma$ such that $Q \subset R$. Then,

$$\begin{aligned}\mathcal{E}(\zeta, Q, \lambda) &= \sup_{q \in Q} \mathcal{E}(\zeta, q, \lambda) \leq \sup_{q \in R} \mathcal{E}(\zeta, q, \lambda) = \mathcal{E}(\zeta, R, \lambda), \\ \theta(\zeta, Q, \lambda) &= \inf_{q \in Q} \theta(\zeta, q, \lambda) \geq \inf_{q \in R} \theta(\zeta, q, \lambda) = \theta(\zeta, R, \lambda) \text{ and} \\ Y(\zeta, Q, \lambda) &= \inf_{q \in Q} Y(\zeta, q, \lambda) \geq \inf_{q \in R} Y(\zeta, q, \lambda) = Y(\zeta, R, \lambda).\end{aligned}$$

Lemma 4.7. Let $(\Sigma, \mathcal{E}, \theta, Y, *, \diamond)$ be a NMS. If $Q, R \subset \Sigma$ such that $Q \subset R$, then $\mathcal{E}(P, Q, \lambda) \leq \mathcal{E}(P, R, \lambda)$, $\theta(P, Q, \lambda) \geq \theta(P, R, \lambda)$ and $Y(P, Q, \lambda) \geq Y(P, R, \lambda)$ for all $P \subset \Sigma$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q, R \subset \Sigma$ such that $Q \subset R$. By the lemma 4.6., we have

$$\begin{aligned}\mathcal{E}(P, Q, \lambda) &= \inf_{p \in P} \mathcal{E}(p, Q, \lambda), \mathcal{E}(P, Q, \lambda) \leq \mathcal{E}(p, Q, \lambda), \text{ for all } p \in P \\ \mathcal{E}(P, Q, \lambda) &\leq \mathcal{E}(p, R, \lambda), \text{ for all } p \in P, \mathcal{E}(P, Q, \lambda) \leq \inf_{p \in P} \mathcal{E}(p, R, \lambda), \\ \mathcal{E}(P, Q, \lambda) &\leq \mathcal{E}(P, R, \lambda). \\ \text{Similarly, by the lemma 4.6.} \\ \theta(P, Q, \lambda) &= \sup_{p \in P} \theta(p, Q, \lambda), \theta(P, Q, \lambda) \geq \theta(p, Q, \lambda) \text{ for all } p \in P, \\ \theta(P, Q, \lambda) &\geq \theta(p, R, \lambda) \text{ for all } p \in P, \theta(P, Q, \lambda) \geq \sup_{p \in P} \theta(p, R, \lambda), \\ \theta(P, Q, \lambda) &\geq \theta(P, R, \lambda) \text{ and} \\ Y(P, Q, \lambda) &= \sup_{p \in P} Y(p, Q, \lambda), Y(P, Q, \lambda) \geq Y(p, Q, \lambda) \text{ for all } p \in P, \\ Y(P, Q, \lambda) &\geq Y(p, R, \lambda) \text{ for all } p \in P, Y(P, Q, \lambda) \geq \sup_{p \in P} Y(p, R, \lambda) \\ Y(P, Q, \lambda) &\geq Y(P, R, \lambda).\end{aligned}$$

Lemma 4.8. Let $(\Sigma, \mathcal{E}, \theta, Y, *, \diamond)$ be a NMS. If $P, Q, R \subset \Sigma$, then

$\mathcal{E}(P \cup Q, R, \lambda) = \min\{\mathcal{E}(P, R, \lambda), \mathcal{E}(Q, R, \lambda)\}$, $\theta(P \cup Q, R, \lambda) = \max\{\theta(P, R, \lambda), \theta(Q, R, \lambda)\}$ and $Y(P \cup Q, R, \lambda) = \max\{Y(P, R, \lambda), Y(Q, R, \lambda)\}$, for all $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q, R \subset \Sigma$. Then

$$\begin{aligned}\mathcal{E}(P \cup Q, R, \lambda) &= \inf_{\zeta \in P \cup Q} \mathcal{E}(\zeta, R, \lambda) = \min\left\{\inf_{p \in P} \mathcal{E}(p, R, \lambda), \inf_{q \in Q} \mathcal{E}(q, R, \lambda)\right\} \\ &= \min\{\mathcal{E}(P, R, \lambda), \mathcal{E}(Q, R, \lambda)\}, \\ \theta(P \cup Q, R, \lambda) &= \sup_{\zeta \in P \cup Q} \theta(\zeta, R, \lambda) = \max\left\{\sup_{p \in P} \theta(p, R, \lambda), \sup_{q \in Q} \theta(q, R, \lambda)\right\} \\ &= \max\{\theta(P, R, \lambda), \theta(Q, R, \lambda)\} \text{ and} \\ Y(P \cup Q, R, \lambda) &= \sup_{\zeta \in P \cup Q} Y(\zeta, R, \lambda) = \max\left\{\sup_{p \in P} Y(p, R, \lambda), \sup_{q \in Q} Y(q, R, \lambda)\right\} \\ &= \max\{Y(P, R, \lambda), Y(Q, R, \lambda)\}\end{aligned}$$

Lemma 4.9. Let $(\Sigma, \mathcal{E}, \theta, Y, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_{\mathcal{E}}, H_{\theta}, H_Y, *, \diamond)$ be the corresponding Hausdorff NMS. If $P, Q, R, S \in \mathcal{K}_0(\Sigma)$ then

$$\begin{aligned}H_{\mathcal{E}}(P \cup Q, R \cup S, \lambda) &\geq \min\{H_{\mathcal{E}}(P, R, \lambda), H_{\mathcal{E}}(Q, S, \lambda)\}, \\ H_{\theta}(P \cup Q, R \cup S, \lambda) &\leq \max\{H_{\theta}(P, R, \lambda), H_{\theta}(Q, S, \lambda)\} \text{ and} \\ H_Y(P \cup Q, R \cup S, \lambda) &\leq \max\{H_Y(P, R, \lambda), H_Y(Q, S, \lambda)\}, \text{ for all } \lambda > 0.\end{aligned}$$

Proof: Fix $\lambda > 0$. Let $P, Q, R, S \in \mathcal{K}_0(\Sigma)$.

By using Lemma 4.7. and Lemma 4.8., we get

$$\begin{aligned}\mathcal{E}(P \cup Q, R \cup S, \lambda) &= \min\{\mathcal{E}(P, R \cup S, \lambda), \mathcal{E}(Q, R \cup S, \lambda)\} \\ &\geq \min\{\mathcal{E}(P, R, \lambda), \mathcal{E}(Q, S, \lambda)\} \\ &\geq \min\{H_{\mathcal{E}}(P, R, \lambda), H_{\mathcal{E}}(Q, S, \lambda)\}.\end{aligned}$$

Similarly, $\mathcal{E}(R \cup S, P \cup Q, \lambda) \geq \min\{H_{\mathcal{E}}(P, R, \lambda), H_{\mathcal{E}}(Q, S, \lambda)\}$.

Hence, $H_{\mathcal{E}}(P \cup Q, R \cup S, \lambda) \geq \min\{H_{\mathcal{E}}(P, R, \lambda), H_{\mathcal{E}}(Q, S, \lambda)\}$.

$$\begin{aligned}\theta(P \cup Q, R \cup S, \lambda) &= \max\{\theta(P, R \cup S, \lambda), \theta(Q, R \cup S, \lambda)\} \\ &\leq \max\{\theta(P, R, \lambda), \theta(Q, S, \lambda)\} \\ &\leq \max\{H_{\theta}(P, R, \lambda), H_{\theta}(Q, S, \lambda)\}.\end{aligned}$$

Similarly, $\theta(R \cup S, P \cup Q, \lambda) \leq \max\{H_{\theta}(P, R, \lambda), H_{\theta}(Q, S, \lambda)\}$.

Hence, $H_{\theta}(P \cup Q, R \cup S, \lambda) \leq \max\{H_{\theta}(P, R, \lambda), H_{\theta}(Q, S, \lambda)\}$ and

$$\begin{aligned}Y(P \cup Q, R \cup S, \lambda) &= \max\{Y(P, R \cup S, \lambda), Y(Q, R \cup S, \lambda)\} \\ &\leq \max\{Y(P, R, \lambda), Y(Q, S, \lambda)\} \\ &\leq \max\{H_Y(P, R, \lambda), H_Y(Q, S, \lambda)\}.\end{aligned}$$

Similarly, $Y(R \cup S, P \cup Q, \lambda) \leq \max\{H_Y(P, R, \lambda), H_Y(Q, S, \lambda)\}$.

Hence, $H_Y(P \cup Q, R \cup S, \lambda) \leq \max\{H_Y(P, R, \lambda), H_Y(Q, S, \lambda)\}$. This completes the proof.

The following theorem is a generalized version of the Theorem 4.5.

Theorem 4.10. Let $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_{\mathcal{E}}, H_{\theta}, H_{\gamma}, *, \diamond)$ be the corresponding Hausdorff NMS. Suppose $f_n: \Sigma \rightarrow \Sigma, n = 1, 2, \dots, N_0; N_0 \in \mathbb{N}$, is a neutrosophic B-Contraction on $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$. Then the neutrosophic HBO is also a neutrosophic B-Contraction on $(\mathcal{K}_0(\Sigma), H_{\mathcal{E}}, H_{\theta}, H_{\gamma}, *, \diamond)$.

Proof: Fix $\lambda > 0$. Let $P, Q \in \mathcal{K}_0(\Sigma)$. By using the Lemma 4.9. and Theorem 4.5. for a given $k \in (0, 1)$, we get

$$\begin{aligned}
 H_{\mathcal{E}}(F(P), F(Q), k\lambda) &= H_{\mathcal{E}}\left(\bigcup_{n=1}^{N_0} f_n(P), \bigcup_{n=1}^{N_0} f_n(Q), k\lambda\right) \\
 &\geq \min_{n=1}^{N_0} H_{\mathcal{E}}(f_n(P), f_n(Q), k\lambda) \geq H_{\mathcal{E}}(P, Q, \lambda), \\
 H_{\theta}(F(P), F(Q), k\lambda) &= H_{\theta}\left(\bigcup_{n=1}^{N_0} f_n(P), \bigcup_{n=1}^{N_0} f_n(Q), k\lambda\right) \\
 &\leq \max_{n=1}^{N_0} H_{\theta}(f_n(P), f_n(Q), k\lambda) \leq H_{\theta}(P, Q, \lambda) \text{ and} \\
 H_{\gamma}(F(P), F(Q), k\lambda) &= H_{\gamma}\left(\bigcup_{n=1}^{N_0} f_n(P), \bigcup_{n=1}^{N_0} f_n(Q), k\lambda\right) \\
 &\leq \max_{n=1}^{N_0} H_{\gamma}(f_n(P), f_n(Q), k\lambda) \leq H_{\gamma}(P, Q, \lambda). \text{ This completes the proof.}
 \end{aligned}$$

From the above Theorem 4.10., we conclude that the operator F is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $(H_{\mathcal{E}}, H_{\theta}, H_{\gamma})$, if f_n is neutrosophic B-contraction on an neutrosophic metric space $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ for each $n \in \{1, 2, \dots, N_0\}$.

5. Hausdorff Neutrosophic Metrics On $\mathcal{K}_0(\Sigma)$ and $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$

Now, we investigate the relationships between the hyperspaces $\mathcal{K}_0(\Sigma)$ and $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ and the Hausdorff neutrosophic metrics $H_{\mathcal{E}}$ and $\mathcal{H}_{H_{\mathcal{E}}}$.

Theorem 5.1. Let $(\Sigma, \mathcal{E}, \theta, \gamma, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_{\mathcal{E}}, H_{\theta}, H_{\gamma}, *, \diamond)$ and $(\mathcal{K}_0(\mathcal{K}_0(\Sigma)), \mathcal{H}_{H_{\mathcal{E}}}, \mathcal{H}_{H_{\theta}}, \mathcal{H}_{H_{\gamma}}, *, \diamond)$ be the corresponding Hausdorff Neutrosophic hyper spaces. Let $\mathfrak{P}, \mathfrak{Q} \in \mathcal{K}_0(\mathcal{K}_0(\Sigma))$ be such that $\{p \in P : P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\} \in \mathcal{K}_0(\Sigma)$. Then $H_{\mathcal{E}}(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q : Q \in \mathfrak{Q}\}, \lambda) \geq \mathcal{H}_{H_{\mathcal{E}}}(\mathfrak{P}, \mathfrak{Q}, \lambda)$,

$$H_{\theta}(\{p \in P : P \in \mathfrak{P}\}, \{q \in : Q \in \mathfrak{Q}\}, \lambda) \leq \mathcal{H}_{H_{\theta}}(\mathfrak{P}, \mathfrak{Q}, \lambda) \text{ and}$$

$$H_{\gamma}(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q : Q \in \mathfrak{Q}\}, \lambda) \leq \mathcal{H}_{H_{\gamma}}(\mathfrak{P}, \mathfrak{Q}, \lambda) \text{ for all } \lambda > 0.$$

Proof: Fix $\lambda > 0$. Firstly, we note that

$$\begin{aligned} \mathcal{E}(Q, \{p \in P : P \in \mathfrak{P}\}, \lambda) &= \inf_{q \in Q} \mathcal{E}(q, \{p \in P : P \in \mathfrak{P}\}, \lambda) \\ &= \inf_{q \in Q} \sup_{\{p \in P : P \in \mathfrak{P}\}} \mathcal{E}(q, p, \lambda) \\ &= \inf_{q \in Q} \sup_{P \in \mathfrak{P}} \sup_{p \in P} \mathcal{E}(q, p, \lambda) \geq \sup_{P \in \mathfrak{P}} \inf_{q \in Q} \sup_{p \in P} \mathcal{E}(q, p, \lambda) \\ &= \sup_{P \in \mathfrak{P}} \mathcal{E}(Q, P, \lambda). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{E}(\{q \in Q, Q \in \mathfrak{Q}\}, \{p \in P : P \in \mathfrak{P}\}, \lambda) &= \inf_{\{q \in Q : Q \in \mathfrak{Q}\}} \mathcal{E}(q, \{p \in P : P \in \mathfrak{P}\}, \lambda) \\ &= \inf_{Q \in \mathfrak{Q}} \inf_{q \in Q} \mathcal{E}(q, \{p \in P : P \in \mathfrak{P}\}, \lambda) = \inf_{Q \in \mathfrak{Q}} \mathcal{E}(Q, \{p \in P : P \in \mathfrak{P}\}, \lambda) \\ &\geq \inf_{Q \in \mathfrak{Q}} \sup_{P \in \mathfrak{P}} \mathcal{E}(Q, P, \lambda). \end{aligned}$$

Similarly, $\mathcal{E}(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\}, \lambda) \geq \inf_{P \in \mathfrak{P}} \sup_{Q \in \mathfrak{Q}} \mathcal{E}(P, Q, \lambda)$.

Hence, $H_{\mathcal{E}}(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\}, \lambda)$

$$\begin{aligned} &= \min \left\{ \mathcal{E}(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\}, \lambda), \right. \\ &\quad \left. \mathcal{E}(\{q \in Q, Q \in \mathfrak{Q}\}, \{p \in P : P \in \mathfrak{P}\}, \lambda) \right\} \\ &\geq \min \left\{ \inf_{P \in \mathfrak{P}} \sup_{Q \in \mathfrak{Q}} \mathcal{E}(P, Q, \lambda), \inf_{Q \in \mathfrak{Q}} \sup_{P \in \mathfrak{P}} \mathcal{E}(Q, P, \lambda) \right\} \\ &\geq \min \left\{ \inf_{P \in \mathfrak{P}} \sup_{Q \in \mathfrak{Q}} H_{\mathcal{E}}(P, Q, \lambda), \inf_{Q \in \mathfrak{Q}} \sup_{P \in \mathfrak{P}} H_{\mathcal{E}}(Q, P, \lambda) \right\} \\ &= \min \{H_{\mathcal{E}}(\mathfrak{P}, \mathfrak{Q}, \lambda), H_{\mathcal{E}}(\mathfrak{Q}, \mathfrak{P}, \lambda)\} \\ &= \mathcal{H}_{H_{\mathcal{E}}}(\mathfrak{P}, \mathfrak{Q}, \lambda). \end{aligned}$$

Secondly, we note that

$$\theta(Q, \{p \in P : P \in \mathfrak{P}\}, \lambda) = \sup_{q \in Q} \theta(q, \{p \in P : P \in \mathfrak{P}\}, \lambda)$$

$$\begin{aligned}
 &= \sup_{q \in Q} \inf_{\{p \in P: P \in \mathfrak{P}\}} \theta(q, p, \lambda) = \sup_{q \in Q} \inf_{P \in \mathfrak{P}} \inf_{p \in P} \theta(q, p, \lambda) \\
 &\leq \inf_{P \in \mathfrak{P}} \sup_{q \in Q} \inf_{p \in P} \theta(q, p, \lambda) = \inf_{P \in \mathfrak{P}} \theta(Q, P, \lambda).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \theta(\{q \in Q: Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) &= \sup_{\{q \in Q: Q \in \mathfrak{Q}\}} \theta(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) \\
 &= \sup_{Q \in \mathfrak{Q}} \sup_{q \in Q} \theta(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{Q \in \mathfrak{Q}} \theta(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda) \\
 &\leq \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} \theta(Q, P, \lambda).
 \end{aligned}$$

Similarly, $\theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda) \leq \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} \theta(P, Q, \lambda)$.

Hence, $H_\theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda)$

$$\begin{aligned}
 &= \max \left\{ \theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda), \right. \\
 &\quad \left. \theta(\{q \in Q: Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) \right\} \\
 &\leq \max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} \theta(P, Q, \lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} \theta(Q, P, \lambda) \right\} \\
 &\leq \max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} H_\theta(P, Q, \lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} H_\theta(Q, P, \lambda) \right\} \\
 &= \max \{H_\theta(\mathfrak{P}, \mathfrak{Q}, \lambda), H_\theta(\mathfrak{Q}, \mathfrak{P}, \lambda)\} \\
 &= \mathcal{H}_{H_\theta}(\mathfrak{P}, \mathfrak{Q}, \lambda) \text{ and}
 \end{aligned}$$

Lastly, we note that

$$\begin{aligned}
 Y(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda) &= \sup_{q \in Q} Y(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) \\
 &= \sup_{q \in Q} \inf_{\{p \in P: P \in \mathfrak{P}\}} Y(q, p, \lambda) = \sup_{q \in Q} \inf_{P \in \mathfrak{P}} \inf_{p \in P} Y(q, p, \lambda) \\
 &\leq \inf_{P \in \mathfrak{P}} \sup_{q \in Q} \inf_{p \in P} Y(q, p, \lambda) = \inf_{P \in \mathfrak{P}} Y(Q, P, \lambda)
 \end{aligned}$$

It follows that

$$\begin{aligned}
 Y(\{q \in Q: Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) &= \sup_{\{q \in Q: Q \in \mathfrak{Q}\}} Y(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) \\
 &= \sup_{Q \in \mathfrak{Q}} \sup_{q \in Q} Y(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{Q \in \mathfrak{Q}} Y(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda) \\
 &\leq \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} Y(Q, P, \lambda).
 \end{aligned}$$

Similarly, $Y(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda) \leq \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} Y(P, Q, \lambda)$.

Hence, $H_Y(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda)$

$$\begin{aligned}
 &= \max \left\{ Y(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda), \right. \\
 &\quad \left. Y(\{\{q \in Q: Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) \right\} \\
 &\leq \max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} Y(P, Q, \lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} Y(Q, P, \lambda) \right\} \\
 &\leq \max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} H_Y(P, Q, \lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} H_Y(Q, P, \lambda) \right\} \\
 &= \max \{H_Y(\mathfrak{P}, \mathfrak{Q}, \lambda), H_Y(\mathfrak{Q}, \mathfrak{P}, \lambda)\} = \mathcal{H}_{H_Y}(\mathfrak{P}, \mathfrak{Q}, \lambda).
 \end{aligned}$$

The proof is complete.

6. Conclusions

In this paper, we proved the neutrosophic contraction properties of the Hutchinson-Barnsley operator on the neutrosophic hyperspace with respect to the Hausdorff neutrosophic metrics. Also we discussed about the relationships between the Hausdorff neutrosophic metrics on the neutrosophic hyperspaces. This paper will lead our direction to develop the Hutchinson-Barnsley Theory in the sense of neutrosophic B-contractions in order to define a fractal set in the neutrosophic metric spaces as a unique fixed point of the Neutrosophic HBO.

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Connections between ideals of semisimple EMV-algebras and set-theoretic filters

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Abstract

In this paper, we mainly study connections between ideals of the semisimple EMV-algebra M and filters on some nonempty set Ω . We show that there is a bijection between the set of all closed ideals of M and the set of all filters on Ω . We get that this correspondence also holds between the set of all closed prime ideals of M and the set of all weak ultrafilters on Ω . We prove that the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω are homeomorphic.

Keywords: Semisimple EMV-algebra; Ideal; Filter; Closure operation; Closed ideal

2020 AMS subject classifications: 06D99 ¹

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¹Received on January 12, 2022. Accepted on May 12, 2022. Published on September 25, 2022.
doi: 10.23755/rm.v43i0.786. ISSN: 1592-7415. eISSN: 2282-8214. © The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

An MV-algebra is an algebra $(M; \oplus, *, 0)$ of type $(2, 1, 0, 0)$ which has the top element 1. The study of MV-algebras is very in-depth and comprehensive, which has important applications in other areas of mathematical research. There are close connections between ideals of a semisimple MV-algebra and filters on some associated nonempty set. Moreover, there exists a bijection between the set of all closed ideals of a semisimple MV-algebra and the set of all filters on some nonempty set. For more details about it, we recommend the monographs Cignoli et al. [2013], Lele et al. [2021].

An EMV-algebra is an algebra $(M; \vee, \wedge, \oplus, 0)$ of type $(2, 2, 2, 0)$, which is a new class of algebraic structures. EMV-algebras cannot guarantee the existence of the top element 1, which are the generalizations of MV-algebras. MV-algebras are termwise equivalent to EMV-algebras with the top element, Dvurečenskij and Zahiri [2019].

We shall mainly study connections between ideals of a semisimple EMV-algebra M and filters on Ω , where $M \subseteq [0, 1]^\Omega$ and $[0, 1]^\Omega$ is an EMV-clan of fuzzy functions on some nonempty set Ω . This paper is organized as follows. In Section 2, we give some basic notions and theorems on EMV-algebras, which will be used in the paper. In Section 3, we start by introducing the limits of $f \in M$ along a filter F on Ω . We study the connections between ideals of M and filters on Ω . In Section 4, we define a closure operation on M . We exhibit a one-to-one correspondence between the set of all closed ideals of M and the set of all filters on Ω . We show that there is a homeomorphism between the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω . In addition, there is an example of an ideal that is a non-closed ideal, and some properties of closed ideals are listed.

2 Preliminaries

In this section, we introduce some basic notions and theorems on an EMV-algebra, which will be used in the following sections.

A filter F on a nonempty set Ω is a collection of subsets of Ω satisfying (i) the intersection of two elements in F again belongs to it and (ii) for all $S \in F$, $S \subseteq T \subseteq \Omega$ implies that $T \in F$. By (ii), we have $\Omega \in F$ for any filter F on Ω . A filter F is called proper if $\emptyset \notin F$. It is obvious that if F_1 and F_2 are filters on Ω , $F_1 \cap F_2$ is also a filter of Ω . In fact, for all $S_1, S_2 \in F_1 \cap F_2$, we get $S_1 \cap S_2 \in F_1 \cap F_2$. Moreover, for any $S \in F_1 \cap F_2$ and $S \subseteq T \subseteq \Omega$, which implies $T \in F_1$ and $T \in F_2$. So $T \in F_1 \cap F_2$. We have shown that $F_1 \cap F_2$ is a filter on Ω .

Definition 2.1. ([Cignoli et al., 2013, Definition 1.1.1]) An MV-algebra is an algebra $(M; \oplus, *, 0, 1)$ of type $(2, 1, 0, 0)$ such that $(M; \oplus, 0)$ is a commutative monoid, and for all $x, y \in M$ satisfying the following axioms:

$$(MV1) \ x^{**} = x;$$

$$(MV2) \ x \oplus 0^* = 0^*;$$

$$(MV3) \ (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x.$$

For all $x, y \in [0, 1]$, the real interval $[0, 1]$ with the operations $x \oplus y = \min\{x + y, 1\}$ and $x^* = 1 - x$ is an MV-algebra. Let $(M; +, 0)$ be a monoid. An element $a \in M$ is called idempotent if it satisfies the equation $a + a = a$. We denote the set of all idempotent elements of M by $\mathcal{I}(M)$. We recommend Cignoli et al. [2013] for MV-algebras.

EMV-algebras as the generalizations of MV-algebras have many important properties. We recommend Dvurečenskij and Zahiri [2019] for EMV-algebras.

Definition 2.2. ([Dvurečenskij and Zahiri, 2019, Definition 3.1]) An EMV-algebra is an algebra $(M; \vee, \wedge, \oplus, 0)$ with type $(2, 2, 2, 0)$ satisfying the followings:

(EMV1) $(M; \vee, \wedge, 0)$ is a distributive lattice with the least element 0;

(EMV2) $(M; \oplus, 0)$ is a commutative ordered monoid with the neutral element 0;

(EMV3) for all $a, b \in \mathcal{I}(M)$ with $a \leq b$ and for each $x \in [a, b]$, the element $\lambda_{a,b}(x) = \min\{y \in [a, b] \mid x \oplus y = b\}$ exists in M , and $([a, b]; \oplus, \lambda_{a,b}, a, b)$ is an MV-algebra;

(EMV4) for any $x \in M$, there is $a \in \mathcal{I}(M)$ such that $x \leq a$.

EMV-algebras cannot guarantee the existence of the top element 1. An ideal I of an EMV-algebra M is a nonempty subset satisfying (i) for all $x, y \in I$, $x \oplus y \in I$ and (ii) for each $y \in I$ and $x \in M$, $x \leq y$ can deduce $x \in I$. Let $\text{Ideal}(M)$ to denote the set of all ideals of M . An ideal I of M is proper if $I \neq M$. A proper ideal I is called prime if for any $x, y \in M$, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$. We use $\mathcal{P}(M)$ to denote the set of all prime ideals of M . An ideal I of M is maximal if for all $x \in M \setminus I$, we have $\langle I \cup \{x\} \rangle = M$, where $\langle I \cup \{x\} \rangle = \{z \in M \mid z \leq a \oplus n.x \text{ for some } a \in I \text{ and some } n \in \mathbb{N}\}$. The set of all maximal ideals of M is denoted by $\text{Max}I(M)$. It is well known that any maximal ideal of M must be prime ([Dvurečenskij and Zahiri, 2019]). An EMV-algebra M is semisimple if and only if $\text{Rad}(M) = \{0\}$, where $\text{Rad}(M) \triangleq \cap \{I \mid I \in \text{Max}I(M)\}$. The set $\text{Rad}(M)$ is called the radical of M .

For two EMV-algebras $(M_1; \vee, \wedge, \oplus, 0)$ and $(M_2; \vee, \wedge, \oplus, 0)$, a mapping $\Phi : M_1 \longrightarrow M_2$ is called an EMV-homomorphism if Φ preserves the operations $\vee, \wedge, \oplus$ and 0, and for each $b \in \mathcal{I}(M_1)$ and for each $x \in [0, b]$, we have $\Phi(\lambda_b(x)) = \lambda_{\Phi(b)}(\Phi(x))$. Every MV-homomorphism is also an EMV-homomorphism, but the converse is not necessarily true ([Dvurečenskij and Zahiri, 2019]). A mapping $s : M \longrightarrow [0, 1]$ is said a state-morphism on M if s is an EMV-homomorphism from

the EMV-algebra M into the EMV-algebra of the real interval $([0, 1]; \vee, \wedge, \oplus, 0)$ with top element, such that there exists an element $x \in M$ with $s(x) = 1$. The set $\text{Ker}(s) = \{x \in M \mid s(x) = 0\}$ is called the kernel of the state-morphism s ([Dvurečenskij and Zahiri, 2019]).

Theorem 2.1. ([Dvurečenskij and Zahiri, 2019, Theorem 4.2 (ii)]) *Let M be an EMV-algebra and s be a state-morphism on M . Then $\text{Ker}(s)$ is a maximal ideal of M . In addition, there is a unique maximal ideal I of M such that $s = s_I$, where $s_I : x \mapsto x/I$ for all $x \in M$.*

Definition 2.3. ([Dvurečenskij and Zahiri, 2019, Definition 4.9]) *Let Ω be a nonempty set. A system $T \subseteq [0, 1]^\Omega$ is called an EMV-clan if it satisfies the following conditions:*

- (1) $0 \in T$ such that $0(w) = 0$ for all $w \in \Omega$;
- (2) if $a \in T$ is a 0-1-valued function, then $a - f \in T$ for each $f \in T$ with $f(w) \leq a(w)$ for all $w \in \Omega$, and if $f, g \in T$ with $f(w), g(w) \leq a(w)$ for all $w \in \Omega$, then $f \oplus g \in T$, where $(f \oplus g)(w) = \min\{f(w) + g(w), a(w)\}$ for all $w \in \Omega$;
- (3) for each $f \in T$, there exists a 0-1-valued function $a \in T$ such that $f(w) \leq a(w)$ for all $w \in \Omega$;
- (4) for given $w \in \Omega$, there exists $f \in T$ such that $f(w) = 1$.

From Dvurečenskij and Zahiri [2019, Proposition 4.10], we see that any EMV-clan can be organized into an EMV-algebra. That is, every EMV-clan on some $\Omega \neq \emptyset$ is an EMV-algebra, see Dvurečenskij and Zahiri [2019].

3 Ideals of semisimple EMV-algebras and filters on associated nonempty sets

Let M be a semisimple EMV-algebra. By Dvurečenskij and Zahiri [2019, Theorem 4.11], there is an EMV-clan $[0, 1]^\Omega$ on some $\Omega \neq \emptyset$ such that M is an EMV-subalgebra of $[0, 1]^\Omega$. In this section, for a semisimple EMV-algebra $M \subseteq [0, 1]^\Omega$, we shall define the notion of limits along a filter. The connections between ideals of M and filters on Ω are studied. For each $f \in M$ and for all $\varepsilon > 0$, we denote $D(f, \varepsilon) = \{x \in \Omega \mid f(x) < \varepsilon\}$.

Definition 3.1. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $M \subseteq [0, 1]^\Omega$. For any $f \in M$ and $t \in [0, 1]$, we call that f converges to t along F if for every $\varepsilon > 0$, there is $S \in F$ such that $|f(S) - t| < \varepsilon$.*

Proposition 3.1. *Let M be a semisimple EMV-algebra and F be a proper filter on Ω such that $M \subseteq [0, 1]^\Omega$. Then for each $f \in M$, there has at most one limit along F .*

Proof. The proof is similar to Lele et al. [2021, Proposition 2.2]. $\square$

For any $f \in M$, the limit of f along a proper filter F on Ω does not necessarily exist. But it would be unique if it exists by Proposition 3.1. We denote it by $\lim_F f$.

Let I be an ideal of M and F be a filter on Ω . We define

$$\mathbf{F}_I = \{S \subseteq \Omega \mid D(f, \varepsilon) \subseteq S \text{ for some } f \in I \text{ and } \varepsilon > 0\}$$

and

$$\mathbf{I}_F = \{f \in M \mid f \text{ converges to } 0 \text{ along } F\} = \{f \in M \mid D(f, \varepsilon) \in F \text{ for all } \varepsilon > 0\}.$$

Proposition 3.2. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $M \subseteq [0, 1]^\Omega$. For all $f, g \in M$:*

- (1) *If $\lim_F f$ and $\lim_F g$ exist, then $\lim_F (f \oplus g)$ exists and $\lim_F (f \oplus g) = \lim_F f \oplus \lim_F g$.*
- (2) *If $\lim_F f$ exists, then $\lim_F \lambda_a(f)$ exists and $\lim_F \lambda_a(f) = \lambda_a(\lim_F f)$, where a is an idempotent element of M such that $f \in [0, a]$.*

Proof. (1) Suppose that $f, g \in M$, $\lim_F f$ and $\lim_F g$ exist. There exists an idempotent element $a \in \mathcal{I}(M)$ such that $f, g \in [0, a]$. Also, we have $\lim_F f, \lim_F g \leq a(x)$ for all $x \in \Omega$. In the MV-algebra $([0, a]; \oplus, \lambda_a, 0, a)$, $\lim_F f$ and $\lim_F g$ also exist. By Lele et al. [2021, Lemma 2.4], we have $\lim_F (f \oplus g)$ exists and $\lim_F (f \oplus g) = \lim_F f \oplus \lim_F g$.

(2) Recall that $\lambda_a(f) = \min\{z \in [0, a] \mid z \oplus f = a\}$, where $a \in \mathcal{I}(M)$ with $f \in [0, a]$. Since $([0, a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra, the result follows from Lele et al. [2021, Lemma 2.4]. $\square$

Recall that an ultrafilter U on Ω is a filter which is maximal, in other words, any filter that contains it is equal to it. An ultrafilter U on Ω is equally a collection of subsets of Ω satisfying (i) U is proper, (ii) the intersection of two subsets in the collection belongs to it and (iii) for any subset V , $V \in U$ if and only if $\Omega \setminus V \notin U$, see Garner [2020, Definition 2]. From (iii), we see that $\Omega \in U$ for any ultrafilter U on Ω . We shall show that the limits along an ultrafilter exist.

Proposition 3.3. *Let M be a semisimple EMV-algebra and U be an ultrafilter on Ω such that $M \subseteq [0, 1]^\Omega$. Then, for any $f \in M$, there has a unique limit along U .*

Proof. Suppose that there is no $t \in [0, 1]$ such that $\lim_U f = t$. That is, for any $t \in [0, 1]$, there exists $\varepsilon_0 > 0$ such that $f^{-1}(O_t) \notin U$, where $O_t = (t - \varepsilon_0, t + \varepsilon_0)$. In fact, if for all $\varepsilon > 0$, there exists $t_0 \in [0, 1]$ such that $f^{-1}(O_{t_0}) \in U$, where $O_{t_0} = (t_0 - \varepsilon, t_0 + \varepsilon)$. It follows that $\lim_U f = t_0$, which is a contradiction. Since $[0, 1]$ is compact, for each open covering $\{O_t \mid t \in [0, 1]\}$ of $[0, 1]$, where $O_t = (t - \varepsilon, t + \varepsilon)$, there exists a finite subset $\{O_{t_1}, O_{t_2}, \dots, O_{t_n}\}$ such that $[0, 1] = \bigcup_{i=1}^n O_{t_i}$. Since U is an ultrafilter on Ω , we have $\bigcup_{i=1}^n f^{-1}(O_{t_i}) =$

$f^{-1}(\bigcup_{i=1}^n O_{t_i}) = f^{-1}([0, 1]) = \Omega \in U$. By Garner [2020, Definition 2], there is $j \in \{1, 2, \dots, n\}$ such that $f^{-1}(O_{t_j}) \in U$, which is a contradiction. Hence, f has at least one limit along U .

By Proposition 3.1, the uniqueness of the limit is clear. $\square$

Theorem 3.1. *Let M be a semisimple EMV-algebra and U be an ultrafilter on Ω such that $M \subseteq [0, 1]^\Omega$. Consider the mapping $\Phi_U : M \rightarrow [0, 1]$ given by $\Phi_U(f) = \lim_U f$, where $f \in M$. Then Φ_U is an EMV-homomorphism with $\text{Ker}(\Phi_U) = \mathbf{I}_U$.*

Proof. Let $\Phi_U : M \rightarrow [0, 1]$ be a mapping defined by $\Phi_U(f) = \lim_U f$, where $f \in M$. By Proposition 3.3, the limit of f along U is unique. So Φ_U is well-defined. For all $f, g \in M$, there is $a \in \mathcal{I}(M)$ such that $f, g \in [0, a]$ and $([0, a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra. Now we consider the restriction of Φ_U on $[0, a]$. From Lele et al. [2021, Proposition 2.6] we see that $\Phi_U|_{[0, a]}$ is an MV-homomorphism. Clearly, $\Phi_U(0) = 0$. Also, we have $\Phi_U(f \oplus g) = \Phi_U(f) \oplus \Phi_U(g)$, $\Phi_U(f \vee g) = \Phi_U(f) \vee \Phi_U(g)$ and $\Phi_U(f \wedge g) = \Phi_U(f) \wedge \Phi_U(g)$. That is, Φ_U is an EMV-homomorphism. In addition, $\text{Ker}(\Phi_U) = \{f \in M \mid \lim_U f = 0\} = \mathbf{I}_U$. $\square$

Theorem 3.2. *Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. We have the followings:*

- (1) *For each ideal I of M , $\mathbf{F}_I$ is a filter on Ω . Moreover, if I is proper, then $\mathbf{F}_I$ is proper.*
- (2) *For each filter F on Ω , $\mathbf{I}_F$ is an ideal of M . Moreover, if F is proper, then $\mathbf{I}_F$ is proper.*

Proof. (1) Let I be an ideal of M .

(i) For all $\varepsilon > 0$ and $f \in I$, we have $D(f, \varepsilon) = \{x \in \Omega \mid f(x) < \varepsilon\} \subseteq \Omega$. Then $\Omega \in \mathbf{F}_I$.

(ii) Let $S_1 \subseteq S_2 \subseteq \Omega$ and $S_1 \in \mathbf{F}_I$. There exist $f \in I$ and $\varepsilon > 0$ such that $D(f, \varepsilon) \subseteq S_1 \subseteq S_2$. This implies that $S_2 \in \mathbf{F}_I$.

(iii) Suppose that $S_1, S_2 \in \mathbf{F}_I$. There exist $f, g \in I$ and $\varepsilon, \delta > 0$ such that $D(f, \varepsilon) \subseteq S_1$ and $D(g, \delta) \subseteq S_2$. It follows that $D(f, \varepsilon) \cap D(g, \delta) \subseteq S_1 \cap S_2$. In addition, since $D(f \oplus g, \min(\varepsilon, \delta)) \subseteq D(f, \varepsilon) \cap D(g, \delta)$ and $f \oplus g \in I$, we have $D(f, \varepsilon) \cap D(g, \delta) \in \mathbf{F}_I$. By (ii), it now follows that $S_1 \cap S_2 \in \mathbf{F}_I$. So $\mathbf{F}_I$ is a filter on Ω .

Let I be a proper ideal. Suppose that $\mathbf{F}_I$ is not proper. Then $\emptyset \in \mathbf{F}_I$. So there exist $f \in I$ and $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in \Omega$. We choose $N \geq 1$ such that $f(x) \geq \varepsilon \geq \frac{1}{N}$. Then $Nf \in I$ and $Nf(x) \geq 1$. It implies that $1 \in I$ and $I = M$, which is a contradiction. Therefore, $\mathbf{F}_I$ is proper.

(2) Let F be a filter on Ω .

(i) Since $0 \in \mathbf{I}_F$, we have $\mathbf{I}_F \neq \emptyset$.

(ii) For all $f, g \in \mathbf{I}_F$, by Proposition 3.2, we have $\lim_F(f \oplus g) = \lim_F f \oplus \lim_F g = 0$. So $f \oplus g \in \mathbf{I}_F$.

(iii) Suppose that $f \in M$, $g \in \mathbf{I}_F$ and $f \leq g$. We have $\lim_F f \leq \lim_F g = 0$. Then $f \in \mathbf{I}_F$. Therefore, $\mathbf{I}_F$ is an ideal of M .

Let F be a proper filter. If $\mathbf{I}_F$ is not proper, then $\mathbf{I}_F = M$. For all $f \in \mathbf{I}_F = M$, for all $\varepsilon > 0$, we have $D(f, \varepsilon) \in F$. There exists $a \in \mathcal{I}(M)$ such that $f \leq a$ and $a \in M = \mathbf{I}_F$. So for any $x \in \Omega$, there is $g(x) > 0$ such that $a(x) \geq g(x)$, where $g \in [0, a]$. It follows that $\emptyset = D(a, g(x)) \in F$, which is a contradiction. Hence, $\mathbf{I}_F$ is proper. $\square$

Proposition 3.4. *Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. Then we have the followings:*

- (1) *For each ideal I of M , $I \subseteq \mathbf{I}_{F_I}$.*
- (2) *For each filter F on Ω , $\mathbf{F}_{\mathbf{I}_F} \subseteq F$.*
- (3) *For each filter F on Ω , $\mathbf{F}_{\mathbf{I}_F} = F$ if $\{0, 1\}^\Omega \subseteq M$.*

Proof. The proof is similar to Lele et al. [2021, Proposition 2.8]. $\square$

Proposition 3.5. *Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. We have the followings:*

- (1) *If $\{0, 1\}^\Omega \subseteq M$, then for each maximal ideal K of M , $\mathbf{F}_K$ is an ultrafilter on Ω .*
- (2) *$\mathbf{I}_U$ is a maximal ideal of M if U is an ultrafilter on Ω .*
- (3) *If $\{0, 1\}^\Omega \subseteq M$, the converse of (2) is true.*

Proof. (1) Let K be a maximal ideal of M and $S \subseteq \Omega$. Suppose $S \notin \mathbf{F}_K$. We will show that $\Omega \setminus S \in \mathbf{F}_K$.

We define $f \in M$ by

$$f(x) = \begin{cases} 0 & x \in S, \\ 1 & x \notin S. \end{cases}$$

Then we have $D(f, 0.5) = S \notin \mathbf{F}_K$. It follows that $f \notin K$. Let $b \in \mathcal{I}(M)$ such that $f \in [0, b]$. It follows from $f \notin K$ that $f \notin K_b$, where $K_b = K \cap [0, b]$. Since K is a maximal ideal of M , by Dvurečenskij and Zahiri [2019, Proposition 3.22], K_b is a maximal ideal of the MV-algebra $([0, b]; \oplus, \lambda_b, 0, b)$. By the maximality of K_b , there exists $n \geq 1$ such that $\lambda_b(nf) \in K_b$. Then $\lambda_b(nf) \in K$. Notice that $nf = f$, which follows that $\lambda_b(f) = \lambda_b(nf) \in K$. In addition, we also have $\Omega \setminus S = \Omega \setminus D(f, 0.5) = D(\lambda_b(f), 0.5) \in \mathbf{F}_K$. Hence, by Freiwald [2014, Chapter IX, Theorem 3.5], $\mathbf{F}_K$ is an ultrafilter on Ω .

(2) Let U be an ultrafilter on Ω . From Theorem 3.1, there is an EMV-homomorphism $\Phi_U : M \rightarrow [0, 1]$ defined by $\Phi_U(f) = \lim_U f$. Since $M \subseteq [0, 1]^\Omega$ is semisimple, for given $w \in \Omega$, there is $f \in M$ such that $f(w) = 1$. So for

$\{w\} \subseteq \Omega \in U$ and all $\varepsilon > 0$, we have $f(\{w\}) \subseteq (1 - \varepsilon, 1 + \varepsilon)$, which implies that there exists $f \in M$ such that $\Phi_U(f) = \lim_U f = 1$. Hence, Φ_U is a state-morphism on M . By Theorem 2.1, $\text{Ker}(\Phi_U) = \mathbf{I}_U$ is a maximal ideal of M .

(3) If $\mathbf{I}_U$ be a maximal ideal of M . Then $\mathbf{F}_{\mathbf{I}_U}$ is an ultrafilter on Ω by (1). By Proposition 3.4 (3), $U = \mathbf{F}_{\mathbf{I}_U}$ is an ultrafilter. $\square$

Proposition 3.6. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $\{0, 1\}^\Omega \subseteq M \subseteq [0, 1]^\Omega$. Then for any $f \in M$, F is an ultrafilter if and only if f has a unique limit along F .*

Proof. $\Rightarrow$: If F is an ultrafilter. By Proposition 3.3 we see that f has a unique limit along F .

$\Leftarrow$: Suppose that f has a unique limit along F , where $f \in M$. Consider the mapping $\Phi_F : M \rightarrow [0, 1]$ defined by $\Phi_F(f) = \lim_F f$. We have that Φ_F is well-defined. By the proof of Proposition 3.5, Φ_F is a state-morphism on M . So $\text{Ker}(\Phi_F) = \mathbf{I}_F$ is a maximal ideal of M by Theorem 2.1. Therefore, F is an ultrafilter on Ω by Proposition 3.5 (3). $\square$

4 Closed ideals of semisimple EMV-algebras

In this section, we introduce the notions of closure operations and c -closed ideals on EMV-algebras. We get a bijection between the set of all closed ideals of M and the set of all filters on Ω . We exhibit a homeomorphism between the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω .

Definition 4.1. *A closure operation on an EMV-algebra M is a mapping $c : \text{Ideal}(M) \rightarrow \text{Ideal}(M)$ satisfying the following conditions: for all $I, J \in \text{Ideal}(M)$,*

- (C1) $I \subseteq I^c$;
- (C2) if $I \subseteq J$, then $I^c \subseteq J^c$;
- (C3) $I^{cc} = I^c$; where $I^c = c(I)$.

Proposition 4.1. *Let M be a semisimple EMV-algebra and $M \subseteq [0, 1]^\Omega$. For each ideal I of M , we denote $I^c = \mathbf{I}_{F_I}$. Then c is a closure operation on M .*

Proof. The proof is similar to Lele et al. [2021, Proposition 3.1]. $\square$

An ideal I of M is called c -closed if $I^c = I$. We frequently prefer to call an ideal is closed instead of c -closed. The set of all closed ideals of M is denoted by $\mathcal{C}(M)$. In the subsequent sections, we shall mainly study closed ideals of M , where the closure operation is given by Proposition 4.1. Now we show that any maximal ideal must be contained in $\mathcal{C}(M)$.

Proposition 4.2. *Let M be a semisimple EMV-algebra and $M \subseteq [0, 1]^\Omega$. Every maximal ideal of M is a closed ideal.*

Proof. Let I be a maximal ideal of M . $\mathbf{I}_{F_I}$ is a proper ideal by Theorem 3.2. By Proposition 3.4 (1), we have $I \subseteq \mathbf{I}_{F_I}$. Suppose $I \subsetneq \mathbf{I}_{F_I}$. For any $f \in \mathbf{I}_{F_I} \setminus I$, by the maximality of I , we have $M = \langle I \cup \{f\} \rangle \subseteq \mathbf{I}_{F_I}$, which is a contradiction. So $I = \mathbf{I}_{F_I}$. We have shown that I is closed. $\square$

Theorem 4.1. *Let M be a semisimple EMV-algebra such that $\{0, 1\}^\Omega \subseteq M \subseteq [0, 1]^\Omega$. Then there is a bijection between the set of all closed ideals of M and the set of all filters on Ω .*

Proof. Let $F(\Omega)$ to denote the set of all filters on Ω . Define two mappings:

$\Theta : \mathcal{C}(M) \longrightarrow F(\Omega)$ by $\Theta(I) = \mathbf{F}_I$ and $\Upsilon : F(\Omega) \longrightarrow \mathcal{C}(M)$ by $\Upsilon(F) = \mathbf{I}_F$. By Theorem 3.2 and Proposition 3.4(3), Θ and Υ are well-defined. For any $I \in \mathcal{C}(M)$ and $F \in F(\Omega)$, we get $\Theta\Upsilon(F) = \Theta(\mathbf{I}_F) = \mathbf{F}_{\mathbf{I}_F} = F$ and $\Upsilon\Theta(I) = \Upsilon(\mathbf{F}_I) = \mathbf{I}_{\mathbf{F}_I} = I$. So $\Theta\Upsilon$ and $\Upsilon\Theta$ are identical mappings. Hence, Θ is a bijection. $\square$

Remark 4.1. *From Theorem 4.1, we get a one-to-one correspondence between the set of all closed ideals of M and the set of all filters on Ω . We shall study the restriction of this correspondence. We define $\mathcal{C}_M(M) = \{I \in \mathcal{C}(M) \mid I \in \text{Max}I(M)\}$ and $F_U(\Omega) = \{F \mid F \text{ is an ultrafilter on } \Omega\}$. Suppose that $\{0, 1\}^\Omega \subseteq M \subseteq [0, 1]^\Omega$. It is easy to verify that there is also a bijection between $\mathcal{C}_M(M)$ and $F_U(\Omega)$.*

In fact, define two mappings $\Psi : F_U(\Omega) \longrightarrow \mathcal{C}_M(M)$ given by $\Psi(U) = \mathbf{I}_U$ and $\Psi' : \mathcal{C}_M(M) \longrightarrow F_U(\Omega)$ given by $\Psi'(I) = \mathbf{F}_I$. From Proposition 3.4 (3) and Proposition 3.5 we see that Ψ and Ψ' are well-defined. Similar to Theorem 4.1, we can prove that Ψ is a bijection.

Next, we will study a special class of filters on Ω , which corresponds to closed prime ideals of M . A filter F on Ω is called a weak ultrafilter if $\mathbf{I}_F$ is a prime ideal of M . We denote the set of all weak ultrafilters on Ω by $W(\Omega)$.

Proposition 4.3. *Let M be a semisimple EMV-algebra and $M \subseteq [0, 1]^\Omega$. Every ultrafilter on Ω is a weak ultrafilter.*

Proof. Let F be an ultrafilter on Ω . Then $\mathbf{I}_F$ is a maximal ideal of M by Proposition 3.5 (2). So $\mathbf{I}_F$ is prime ([Dvurečenskij and Zahiri, 2019]). Hence, F is a weak ultrafilter. $\square$

Proposition 4.4. *Let M be a semisimple EMV-algebra and $M \subseteq [0, 1]^\Omega$. If I is a prime ideal of M , $\mathbf{F}_I$ is a weak ultrafilter on Ω .*

Proof. Let I be a prime ideal of M . Then $\mathbf{F}_I$ is proper. It follows that $\mathbf{I}_{\mathbf{F}_I}$ is a proper ideal by Theorem 3.2. Suppose that $f \wedge g \in \mathbf{I}_{\mathbf{F}_I}$ for $f, g \in M$. We get $D(f \wedge g, \varepsilon) \in \mathbf{F}_I$ for all $\varepsilon > 0$. Since $D(f, \varepsilon), D(g, \varepsilon) \subseteq D(f \wedge g, \varepsilon) \in \mathbf{F}_I$, we have that at least one of $D(f, \varepsilon)$ and $D(g, \varepsilon)$ is nonempty. That is, $f \in \mathbf{I}_{\mathbf{F}_I}$ or $g \in \mathbf{I}_{\mathbf{F}_I}$. In fact, suppose that $D(f, \varepsilon)$ and $D(g, \varepsilon)$ are empty sets. It follows that $\emptyset = D(f \wedge g, \varepsilon) \in \mathbf{F}_I$, which is a contradiction. We have shown that $\mathbf{F}_I$ is a weak ultrafilter on Ω . $\square$

Theorem 4.2. *Let M be a semisimple EMV-algebra such that $\{0, 1\}^\Omega \subseteq M \subseteq [0, 1]^\Omega$. Then there is a bijection between the set of all closed prime ideals of M and the set of all weak ultrafilters on Ω .*

Proof. Let $\mathcal{P}_c(M)$ to denote the set of all closed prime ideals of M . Define two mappings:

$\Phi : \mathcal{P}_c(M) \longrightarrow W(\Omega)$ defined by $\Phi(I) = \mathbf{F}_I$ and $\Gamma : W(\Omega) \longrightarrow \mathcal{P}_c(M)$ defined by $\Gamma(F) = \mathbf{I}_F$.

The mappings Φ and Γ are well-defined by Proposition 4.4, Proposition 3.4 (3) and the definition of weak ultrafilters.

For any $I \in \mathcal{P}_c(M)$ and $F \in W(\Omega)$, we have $\Gamma\Phi(I) = \Gamma(\mathbf{F}_I) = \mathbf{I}_{\mathbf{F}_I} = I$ and $\Phi\Gamma(F) = \Phi(\mathbf{I}_F) = \mathbf{F}_{\mathbf{I}_F} = F$. So $\Phi\Gamma$ and $\Gamma\Phi$ are identical mappings. Hence, Φ is a bijection. $\square$

Lemma 4.1. *Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. Then there is a topology on the space $W(\Omega)$ which has $\mathcal{B}_w \triangleq \{\mathcal{U}_w(f) \mid f \in M\}$ as a basis, where $\mathcal{U}_w(f) = \{F \in W(\Omega) \mid f \notin \mathbf{I}_F\}$ for $f \in M$.*

Proof. For any $F \in W(\Omega)$, there is $f \in M \setminus \mathbf{I}_F$ such that $F \in \mathcal{U}_w(f) \in \mathcal{B}_w$ since $\mathbf{I}_F$ is prime.

Furthermore, for all $f, g \in M$, suppose that $F \in \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. Then $f \notin \mathbf{I}_F$ and $g \notin \mathbf{I}_F$. We have $f \wedge g \notin \mathbf{I}_F$ since $\mathbf{I}_F$ is a prime ideal of M , which follows that $\mathcal{U}_w(f) \cap \mathcal{U}_w(g) \subseteq \mathcal{U}_w(f \wedge g)$. For any $F \in \mathcal{U}_w(f \wedge g)$, we have $f \wedge g \notin \mathbf{I}_F$. It implies that $D(f \wedge g, \varepsilon_0) \notin F$ for some $\varepsilon_0 > 0$. It follows from $D(f, \varepsilon_0), D(g, \varepsilon_0) \subseteq D(f \wedge g, \varepsilon_0) \notin F$ and $F \in W(\Omega)$ that $f \notin \mathbf{I}_F$ and $g \notin \mathbf{I}_F$. Then $\mathcal{U}_w(f \wedge g) \subseteq \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. So $\mathcal{U}_w(f \wedge g) = \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. That is, for any $F \in \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$, there is $\mathcal{U}_w(f \wedge g) \in \mathcal{B}_w$ such that $F \in \mathcal{U}_w(f \wedge g) \subseteq \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$.

We have shown that the sets $\mathcal{U}_w(f)$ form a basis of the topology on $W(\Omega)$. $\square$

From Lemma 4.1, we get a space $W(\Omega)$ whose topology is the topology generated by $\mathcal{B}_w$. The open sets on $W(\Omega)$ are sets $\bigcup_{\mathcal{U}_w(f) \in \mathcal{B}_w'} \mathcal{U}_w(f)$, where $\mathcal{B}_w' \subseteq \mathcal{B}_w$ and $f \in M$. When we refer to the topological space $W(\Omega)$, it will be with reference to the topology $\{\bigcup_{\mathcal{U}_w(f) \in \mathcal{B}_w'} \mathcal{U}_w(f) \mid \mathcal{B}_w' \subseteq \mathcal{B}_w\}$ ([Munkres, 2000]).

Lemma 4.2. *Let M be a semisimple EMV-algebra and $M \subseteq [0, 1]^\Omega$. The sets $\mathcal{U}_c(f), f \in M$ form a basis of the topology on $\mathcal{P}_c(M)$, where $\mathcal{U}_c(f) = \{I \in \mathcal{P}_c(M) \mid f \notin I\}$ for $f \in M$.*

Proof. We denote $\mathcal{B}_c = \{\mathcal{U}_c(f) \mid f \in M\}$.

For any $I \in \mathcal{P}_c(M)$, there is $f \in M \setminus I$ such that $I \in \mathcal{U}_c(f) \in \mathcal{B}_c$ since I is proper.

It is obvious that $\mathcal{U}_c(f) \cap \mathcal{U}_c(g) \subseteq \mathcal{U}_c(f \wedge g)$. Suppose that $I \in \mathcal{U}_c(f \wedge g)$. Then $f \wedge g \notin I = \mathbf{I}_{F_I}$, where $f, g \in M$. Similar to Lemma 4.1, we have $f \notin \mathbf{I}_{F_I} = I$ and $g \notin \mathbf{I}_{F_I} = I$. It implies that $\mathcal{U}_c(f \wedge g) \subseteq \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$. So $\mathcal{U}_c(f \wedge g) = \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$. That is, for any $I \in \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$, there is $\mathcal{U}_c(f \wedge g) \in \mathcal{B}_c$ such that $I \in \mathcal{U}_c(f \wedge g) \subseteq \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$.

Hence, we have shown that $\mathcal{B}_c$ as the basis of the topology on $\mathcal{P}_c(M)$. $\square$

By Lemma 4.2 and Munkres [2000], the topology on $\mathcal{P}_c(M)$ is the topology generated by $\mathcal{B}_c$ where the open sets are sets $\bigcup_{\mathcal{U}_c(f) \in \mathcal{B}_c'} \mathcal{U}_c(f)$, where $\mathcal{B}_c' \subseteq \mathcal{B}_c$ and $f \in M$.

Theorem 4.3. *Let M be a semisimple EMV-algebra such that $\{0, 1\}^\Omega \subseteq M \subseteq [0, 1]^\Omega$. Then the two topological spaces $\mathcal{P}_c(M)$ and $W(\Omega)$ are homeomorphic.*

Proof. Consider the two well-defined bijections Φ and Γ defined by Theorem 4.2.

(1) Φ is continuous. Without loss of generality, we shall prove that the preimage of any $\mathcal{U}_w(f)$ in $W(\Omega)$ is open in $\mathcal{P}_c(M)$. We have $\Phi^{-1}(\mathcal{U}_w(f)) = \Gamma(\mathcal{U}_w(f)) = \{\mathbf{I}_F \mid f \notin \mathbf{I}_F\}$. For any $\mathbf{I}_F \in \Gamma(\mathcal{U}_w(f))$, where $F \in W(\Omega)$ and $f \notin \mathbf{I}_F$, by Proposition 3.4 (3), we have $\mathbf{I}_F \in \mathcal{P}_c(M)$. Then $\mathbf{I}_F \in \mathcal{U}_c(f)$. So $\Gamma(\mathcal{U}_w(f)) \subseteq \mathcal{U}_c(f)$. Moreover, for any $I \in \mathcal{U}_c(f)$, then $I \in \mathcal{P}_c(M)$ and $f \notin I$. We have $\mathbf{F}_I \in W(\Omega)$ and $f \notin I = \mathbf{I}_{F_I}$. It implies that $I \in \Gamma(\mathcal{U}_w(f))$. So $\mathcal{U}_c(f) \subseteq \Gamma(\mathcal{U}_w(f))$. Hence, $\Phi^{-1}(\mathcal{U}_w(f)) = \Gamma(\mathcal{U}_w(f)) = \mathcal{U}_c(f)$ is an open set in $\mathcal{P}_c(M)$.

(2) Γ is continuous. We shall prove $\Gamma^{-1}(\mathcal{U}_c(f)) = \mathcal{U}_w(f)$. We have $\Gamma^{-1}(\mathcal{U}_c(f)) = \Phi(\mathcal{U}_c(f)) = \{\mathbf{F}_I \mid f \notin I\}$. For any $F \in \mathcal{U}_w(f)$, we get $F \in W(\Omega)$ and $f \notin \mathbf{I}_F$. By Proposition 3.4 (3), we see that $\mathbf{I}_F \in \mathcal{P}_c(M)$ and $F = \mathbf{F}_{\mathbf{I}_F} \in \Phi(\mathcal{U}_c(f))$. So $\mathcal{U}_w(f) \subseteq \Phi(\mathcal{U}_c(f))$. For each $\mathbf{F}_I \in \Phi(\mathcal{U}_c(f))$, where $I \in \mathcal{P}_c(M)$ and $f \notin I = \mathbf{I}_{F_I}$. It follows that $\mathbf{F}_I \in \mathcal{U}_w(f)$. So $\Phi(\mathcal{U}_c(f)) \subseteq \mathcal{U}_w(f)$. Thus $\Gamma^{-1}(\mathcal{U}_c(f)) = \Phi(\mathcal{U}_c(f)) = \mathcal{U}_w(f)$ is an open set in $W(\Omega)$.

We have shown that Φ is a homeomorphism between $\mathcal{P}_c(M)$ and $W(\Omega)$. $\square$

Example 4.1. *There exist non-closed ideals.*

Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. Suppose that I is an ideal of M . It is obvious that $\mathbf{I}_{F_I} = \{f \in M \mid \forall \varepsilon > 0, \exists \delta > 0 \text{ and } g \in I \text{ such that } g^{-1}([0, \delta)) \subseteq f^{-1}([0, \varepsilon))\}$. In fact, for each $f \in \mathbf{I}_{F_I}$, we have $D(f, \varepsilon) \in \mathbf{F}_I$

for all $\varepsilon > 0$. So there exist $g \in I$ and $\delta > 0$ such that $D(g, \delta) \subseteq D(f, \varepsilon)$. It follows that $g^{-1}([0, \delta)) \subseteq f^{-1}([0, \varepsilon))$.

Let $M = [0, 1]^{\mathbb{Z}^+}$, where all operations given by Definition 2.3 and Dvurečenskij and Zahiri [2019, Proposition 4.10]. Let $I = \{f \in M \mid \text{for all but finitely many } n \in \mathbb{Z}^+ \text{ such that } f(n) = 0\}$. It follows from $(f \oplus g)(n) = \min\{f(n) + g(n), a(n)\}$ and simple exercises that I is an ideal of M , where $f, g \in I$ and $a \in M$ is a 0-1-valued function such that $f(n), g(n) \leq a(n)$ for all $n \in \mathbb{Z}^+$.

Consider f given by $f(n) = \frac{n+1}{n^2+1}$ ($n \in \mathbb{Z}^+$). Clearly, $f \in M \setminus I$. It is easy to see that $f(n) \rightarrow 0$ when $n \rightarrow \infty$. That is, for all $\varepsilon > 0$, there is $N \in \mathbb{Z}^+$ such that $f(n) < \varepsilon$ when $n > N$. Now we consider $g \in M$ defined by

$$g(n) = \begin{cases} \frac{1}{n} & 1 \leq n \leq N, \\ 0 & n > N. \end{cases}$$

Then $g \in I$ and $D(g, \delta) \subseteq D(f, \varepsilon)$ for $\delta = \min\{\frac{1}{N+1}, \varepsilon\}$. It implies that $g^{-1}([0, \delta)) \subseteq f^{-1}([0, \varepsilon))$. So $f \in \mathbf{I}_{\mathbf{F}_I}$. We have shown that $\mathbf{I}$ is a non-closed ideal.

Definition 4.2. Let M be an EMV-algebra and I be an ideal of M . Then I is called radical if $I = \text{Rad}(M)$, where $\text{Rad}(M)$ is the radical of M .

Proposition 4.5. Let M be a semisimple EMV-algebra such that $M \subseteq [0, 1]^\Omega$. The following conditions are satisfied:

- (1) The intersection of closed ideals of M is also a closed ideal.
- (2) An ideal I of M is closed if I is radical.

Proof. (1) Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a family of closed ideals of M . For each $\beta \in \Lambda$, it follows from $\bigcap_{\alpha \in \Lambda} I_\alpha \subseteq I_\beta$ that $(\bigcap_{\alpha \in \Lambda} I_\alpha)^c \subseteq I_\beta^c = I_\beta$. Then $(\bigcap_{\alpha \in \Lambda} I_\alpha)^c \subseteq \bigcap_{\beta \in \Lambda} I_\beta = \bigcap_{\alpha \in \Lambda} I_\alpha$. Since $\bigcap_{\alpha \in \Lambda} I_\alpha \subseteq (\bigcap_{\alpha \in \Lambda} I_\alpha)^c$, we have $(\bigcap_{\alpha \in \Lambda} I_\alpha)^c = \bigcap_{\alpha \in \Lambda} I_\alpha$. So $\bigcap_{\alpha \in \Lambda} I_\alpha \in \mathcal{C}(M)$.

(2) Suppose that I is radical. It implies that $I = \bigcap \{K \mid K \in \text{Max}I(M)\}$. So by Proposition 4.2 and (1), I is closed. $\square$

5 Conclusion

For a semisimple EMV-algebra M such that $M \subseteq [0, 1]^\Omega$, we introduce the notion of limits along a filter on Ω , which is unique if it exists. For all ultrafilters U on Ω and for all $f \in M$, we give an EMV-homomorphism Φ_U with kernel equal to $\mathbf{I}_U$, which is defined by $\Phi_U(f) = \lim_U f$. We study connections between ideals of M and filters on Ω . We define closure operations and closed ideals on EMV-algebras. We show that there is a bijection between the set of all closed ideals of M and the set of all filters on Ω . We show that there is a homeomorphism

between the topological space $\mathcal{P}_c(M)$ and the topological space $W(\Omega)$. We give an example of a non-closed ideal and some properties of closed ideals.

Assume that F is a filter of the proper EMV-algebra M and I is an ideal of M . We can show that $\mathbf{I}_F = \{\lambda_a(x) \mid x \in F, a \in \mathcal{I}(M), x \leq a\}$ is an ideal of M . If F is a maximal filter of M , $\mathbf{I}_F$ is a maximal ideal of M can be proved. We can also get that $\mathbf{F}_I = \{\lambda_a(x) \mid x \in I, a \in \mathcal{I}(M) \setminus I, x < a\}$ is a filter of M under the assumption that $\forall a \in \mathcal{I}(M), a \notin I \implies (\forall b \in \mathcal{I}(M), a < b) \lambda_b(a) \in I$.

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Existence and Uniqueness of solution of Volterra Integrodifferential Equation of Fractional Order via S-Iteration

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Abstract

In this paper, we study the existence, uniqueness and other properties of solutions of Volterra integrodifferential equation of fractional order involving the Caputo fractional derivative. The tool employed in the analysis is based on application of S -iteration method. Since the study of qualitative properties in general required differential and integral inequalities, but here S -iteration method itself has equally important contribution to study various properties such as dependence on initial data, closeness of solutions and dependence on parameters and functions involved therein. An example in support of the all-established results is given.

Keywords: Existence and uniqueness; Normal S -iterative method; Fractional derivative; Continuous dependence; Closeness; Parameters.

2020 AMS subject classifications: 35A01, 35A02, 34A12, 26A33, 35B30, 45D05, 47H10. ¹

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¹Received on May 28th, 2022. Accepted on December 25th, 2022. Published on December 30th, 2022. doi: 10.23755/rm.v43i0.791. ISSN: 1592-7415. eISSN: 2282-8214. © The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

We consider the following Volterra integrodifferential equation of fractional order involving the Caputo fractional derivative of the type:

$$(D_{*a}^\alpha)y(t) = \mathcal{F}\left(t, y(t), \int_a^t h(s, y(s))ds\right), \quad (1)$$

for $t \in I = [a, b]$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(a) = c_j, \quad j = 0, 1, 2, \dots, n - 1, \quad (2)$$

where $\mathcal{F} : I \times X \times X \rightarrow X$, $h : I \times X \rightarrow X$ are continuous functions and c_j ($j = 0, 1, 2, \dots, n - 1$) are given elements in X .

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1, 3, 4, 5, 6, 8, 9, 18, 19, 20, 21, 22, 23, 24, 31, 32]). The most of iterations devoted for both analytical and numerical approaches. The S -iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1)-(2) and its variants have been studied by several researchers under variety of hypotheses by using different techniques, [2, 7, 10, 11, 12, 13, 14, 15, 16, 26, 27, 29, 30] and some of references cited therein. In recently, Soltuz and Grosan [33] have studied the special version of equation (1) for different qualitative properties of solutions. Authors are motivated by the work of Sahu [31] and influenced by [5,33].

The main objective of this paper is to use normal S -iteration method to establish the existence and uniqueness of solution of the initial value problem (1)-(2) and other qualitative properties of solutions.

2 Preliminaries

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I = [a, b]$ denotes an interval of the real line $\mathbb{R}$. For the fractional order α , $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, we define $B = C^r(I, X)$, (where $r = n$ for $\alpha \in \mathbb{N}$ and $r = n - 1$ for $\alpha \notin \mathbb{N}$), as a Banach space of all r times continuously differentiable functions from I into X , endowed with the norm

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$$\|y\|_B = \sup\{\|y(t)\| : y \in B\}, \quad t \in I.$$

Definition 2.1 (28). *The Riemann Liouville fractional integral (left-sided) of a function $h \in C^1[a, b]$ of order $\alpha \in \mathbf{R}_+ = (0, \infty)$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds, \quad t \in I$$

where Γ is the Euler gamma function.

Definition 2.2 (28). *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. Then the expression*

$$D_a^\alpha h(t) = \frac{d^n}{dt^n} [I_a^{n-\alpha} h(t)], \quad t \in [a, b]$$

is called the (left-sided) Riemann Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3 (25). *Let $h \in C^n[a, b]$ and $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. Then the expression*

$$(D_{*a}^\alpha)h(t) = I_a^{n-\alpha} h^{(n)}(t), \quad t \in [a, b]$$

is called the (left-sided) Caputo derivative of h of order α .

Lemma 2.1 (17). *If the function $f = (f_1, \dots, f_n) \in C^1[a, b]$, then the initial value problems*

$$(D_{*a}^{\alpha_i})y(t) = f_i(t, y_1, \dots, y_n), \quad y_i^{(k)}(0) = c_k^i, \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, m_i$$

where $m_i < \alpha_i \leq m_i + 1$ is equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I_a^{\alpha_i} f_i(t, y_1, \dots, y_n), \quad 1 \leq i \leq n.$$

As a consequence of the Lemma 2.1, it is easy to observe that if $y \in B$ and $\mathcal{F} \in C^1[a, b]$, then $y(t)$ satisfies the integral equation

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, y(s), \int_a^s h(\sigma, y(\sigma)) d\sigma\right) ds, \quad (3)$$

which is equivalent to (1)-(2).

We need the following pair of known results:

Theorem 2.1. ([31], p.194) Let C be a nonempty closed convex subset of a Banach space X and $T : C \rightarrow C$ a contraction operator with contractivity factor $m \in [0, 1)$ and fixed point x^* . Let α_k and β_k be two real sequences in $[0, 1]$ such that $\alpha \leq \alpha_k \leq 1$ and $\beta \leq \beta_k < 1$ for all $k \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_k, v_k and w_k in C as follows:

S-iteration process:
$$\begin{cases} u_{k+1} = (1 - \alpha_k)Tu_k + \alpha_kTy_k, \\ y_k = (1 - \beta_k)u_k + \beta_kTu_k, k \in \mathbb{N}. \end{cases}$$

Picard iteration:
$$v_{k+1} = Tv_k, k \in \mathbb{N}.$$

Mann iteration process:
$$w_{k+1} = (1 - \beta_k)w_k + \beta_kTw_k, k \in \mathbb{N}.$$

 Then we have the following:

- (a) $\|u_{k+1} - x^*\| \leq m^k \left[1 - (1 - m)\alpha\beta \right]^k \|u_1 - x^*\|$, for all $k \in \mathbb{N}$.
- (b) $\|v_{k+1} - x^*\| \leq m^k \|v_1 - x^*\|$, for all $k \in \mathbb{N}$.
- (c) $\|w_{k+1} - x^*\| \leq \left[1 - (1 - m)\beta \right]^k \|w_1 - x^*\|$, for all $k \in \mathbb{N}$.

Moreover, the S -iteration process is faster than the Picard and Mann iteration processes.

Definition 2.4. ([31], p.194) In particular, for $\alpha_k = 1, k \in \mathbb{N} \cup \{0\}$ in the S -iteration process, then it reduces to as follows:

$$\begin{cases} u_0 \in C, \\ u_{k+1} = Ty_k, \\ y_k = (1 - \xi_k)u_k + \xi_kTu_k, k \in \mathbb{N} \cup \{0\}. \end{cases} \quad (4)$$

This is called normal S -iteration method.

Note: For our convenience, we replaced β_k in the S -iteration process by ξ_k .

Lemma 2.2. ([33], p.4) Let $\{\beta_k\}_{k=0}^\infty$ be a nonnegative sequence for which one assumes there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ one has satisfied the inequality

$$\beta_{k+1} \leq (1 - \mu_k)\beta_k + \mu_k\gamma_k, \quad (5)$$

where $\mu_k \in (0, 1)$, for all $k \in \mathbb{N} \cup \{0\}$, $\sum_{k=0}^\infty \mu_k = \infty$ and $\gamma_k \geq 0, \forall k \in \mathbb{N} \cup \{0\}$.

Then the following inequality holds

$$0 \leq \limsup_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \gamma_k. \quad (6)$$

3 Existence and Uniqueness of Solutions via S –iteration

Now, we are able to state and prove the following main theorem which deals with the existence and uniqueness of solutions of the problem (1)-(2).

Theorem 3.1. *Assume that there exist functions $p, q \in C(I, \mathbb{R}_+)$ such that*

$$\|\mathcal{F}(t, u_1, u_2) - \mathcal{F}(t, v_1, v_2)\| \leq p(t) [\|u_1 - v_1\| + \|u_2 - v_2\|] \quad (7)$$

and

$$\|h(t, u_1) - h(t, v_1)\| \leq q(t)\|u_1 - v_1\|,$$

for $t \in I$. If $\Theta = I_a^\alpha p(t)(1 + (b - a)Q) < 1$ (where $Q = \sup_{a \leq t \leq b} q(t)$), then the iterative sequence $\{y_k\}_{k=0}^\infty$ generated by normal S –iteration method (4) with the real control sequence $\{\xi_k\}_{k=0}^\infty$ in $[0, 1]$ satisfying $\sum_{k=0}^\infty \xi_k = \infty$, converges to a unique point $y \in B$, which is the required solution of the equations (1)-(2) with the following estimate:

$$\|y_{k+1} - y\|_B \leq \frac{\Theta^{k+1}}{e^{(1-\Theta) \sum_{i=0}^k \xi_i}} \|y_0 - y\|_B. \quad (8)$$

Proof. Let $y(t) \in B$ and define the operator

$$\begin{aligned} (Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!} (t - a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} \mathcal{F}\left(s, y(s), \int_a^s h(\sigma, y(\sigma)) d\sigma\right) ds, \quad t \in I. \end{aligned} \quad (9)$$

Let $\{y_k\}_{k=0}^\infty$ be iterative sequence generated by normal S –iteration method (4) for the operator given in (9).

We will show that $y_k \rightarrow y$ as $k \rightarrow \infty$.

From (4), (9) and assumption, we obtain

$$\begin{aligned} &\|y_{k+1}(t) - y(t)\| \\ &= \|(Ty_k)(t) - (Ty)(t)\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{c_j}{j!} (t - a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) ds \right\| \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, y(s), \int_a^s h(\sigma, y(\sigma)) d\sigma\right) ds \Big\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) \right. \\
& \quad \left. - \mathcal{F}\left(s, y(s), \int_a^s h(\sigma, y(\sigma)) d\sigma\right) \right\| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k(s) - y(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - y(\sigma)\| d\sigma \right] ds.
\end{aligned} \tag{10}$$

Now, we estimate

$$\begin{aligned}
\|z_k(t) - y(t)\| &= \left[(1 - \xi_k) \|y_k(t) - y(t)\| + \xi_k \|(Ty_k)(t) - (Ty)(t)\| \right] \\
&\leq (1 - \xi_k) \|y_k(t) - y(t)\| + \xi_k \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\quad \times \left[\|y_k(s) - y(s)\| + \int_a^s q(\sigma) \|y_k(\sigma) - y(\sigma)\| d\sigma \right] ds. \tag{11}
\end{aligned}$$

Now, by taking supremum in the inequalities (10) and (11), we obtain

$$\begin{aligned}
\|y_{k+1} - y\|_B &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k - y\|_B + \int_a^s q(\sigma) \|z_k - y\|_B d\sigma \right] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k - y\|_B + (b-a)Q \|z_k - y\|_B \right] ds \\
&\leq I_a^\alpha p(t) \left(1 + (b-a)Q \right) \|z_k - y\|_B \\
&= \Theta \|z_k - y\|_B
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\|z_k - y\|_B &\leq \left[(1 - \xi_k) \|y_k - y\|_B + \xi_k \Theta \|y_k - y\|_B \right] \\
&= \left[1 - \xi_k (1 - \Theta) \right] \|y_k - y\|_B,
\end{aligned} \tag{13}$$

respectively.

Therefore, using (13) in (12), we have

$$\|y_{k+1} - y\|_B \leq \Theta \left[1 - \xi_k (1 - \Theta) \right] \|y_k - y\|_B. \tag{14}$$

Thus, by induction, we get

$$\|y_{k+1} - y\|_B \leq \Theta^{k+1} \prod_{j=0}^k \left[1 - \xi_j (1 - \Theta) \right] \|y_0 - y\|_B. \tag{15}$$

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Since $\xi_k \in [0, 1]$ for all $k \in \mathbb{N} \cup \{0\}$, the definition of Θ and $\xi_k \leq 1$ yields,

$$\Rightarrow \xi_k \Theta < \xi_k$$

$$\Rightarrow \xi_k (1 - \Theta) < 1, \forall k \in \mathbb{N} \cup \{0\}. \quad (16)$$

From the classical analysis, we know that

$$1 - x \leq e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \quad x \in [0, 1].$$

Hence by utilizing this fact with (16) in (15), we obtain

$$\begin{aligned} \|y_{k+1} - y\|_B &\leq \Theta^{k+1} e^{-(1-\Theta) \sum_{j=0}^k \xi_j} \|y_0 - y\|_B \\ &= \frac{\Theta^{k+1}}{e^{(1-\Theta) \sum_{i=0}^k \xi_i}} \|y_0 - y\|_B. \end{aligned} \quad (17)$$

Since $\sum_{k=0}^{\infty} \xi_k = \infty$,

$$e^{-(1-\Theta) \sum_{j=0}^k \xi_j} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (18)$$

Hence, using this, the inequality (17) implies $\lim_{k \rightarrow \infty} \|y_{k+1} - y\|_B = 0$ and therefore, we have $y_k \rightarrow y$ as $k \rightarrow \infty$. $\square$

Remark: It is an interesting to note that the inequality (17) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the problem (1)-(2) for $t \in I$.

4 Continuous dependence via S -iteration

In this section, we shall deal with continuous dependence of solution of the problem (1) on the initial data, functions involved therein and also on parameters.

4.1 Dependence on initial data

Suppose $y(t)$ and $\bar{y}(t)$ are solutions of (1) with initial data

$$y^{(j)}(a) = c_j, \quad j = 0, 1, 2, \dots, n-1, \quad (19)$$

and

$$\bar{y}^{(j)}(a) = d_j, \quad j = 0, 1, 2, \dots, n-1, \quad (20)$$

respectively, where c_j, d_j are elements of the space X .

Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equation (1) with the initial conditions (20):

$$\begin{aligned} (\bar{T}\bar{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, \bar{y}(s), \int_a^s h(\sigma, \bar{y}(\sigma)) d\sigma\right) ds, \quad t \in I. \end{aligned} \quad (21)$$

We shall deal with the continuous dependence of solutions of equations (1) on initial data.

Theorem 4.1. *Suppose the function $\mathcal{F}$ in equation (1) satisfies the condition (7). Consider the sequences $\{y_k\}_{k=0}^\infty$ and $\{\bar{y}_k\}_{k=0}^\infty$ generated normal S -iterative method associated with operators T in (9) and $\bar{T}$ in (21), respectively with the real sequence $\{\xi_k\}_{k=0}^\infty$ in $[0, 1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. If the sequence $\{\bar{y}_k\}_{k=0}^\infty$ converges to $\bar{y}$, then we have*

$$\|y - \bar{y}\|_B \leq \frac{3M}{(1 - \Theta)}, \quad (22)$$

where

$$M = \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} (b-a)^j.$$

Proof. From iteration (4) and equations (9); (21) and assumptions, we obtain

$$\begin{aligned} &\|y_{k+1}(t) - \bar{y}_{k+1}(t)\| \\ &= \|(Tz_k)(t) - (\bar{T}\bar{z}_k)(t)\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) ds \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) ds \right\| \\ &\leq \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} (b-a)^j \end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) \right. \\
& \quad \left. - \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) \right\| ds \\
& \leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k(s) - \bar{z}_k(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - \bar{z}_k(\sigma)\| d\sigma \right] ds.
\end{aligned} \tag{23}$$

Recalling the equations (12) and (13), the above inequality becomes

$$\|y_{k+1} - \bar{y}_{k+1}\|_B \leq M + \Theta \|z_k - \bar{z}_k\|_B, \tag{24}$$

and similarly, it is seen that

$$\|z_k - \bar{z}_k\|_B \leq \xi_k M + \left[1 - \xi_k(1 - \Theta)\right] \|y_k - \bar{y}_k\|_B. \tag{25}$$

Therefore, using (25) in (24) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned}
\|y_{k+1} - \bar{y}_{k+1}\|_B & \leq M + \|z_k - \bar{z}_k\|_B \\
& \leq M + \xi_k M + \left[1 - \xi_k(1 - \Theta)\right] \|y_k - \bar{y}_k\|_B \\
& \leq 2\xi_k M + \xi_k M + \left[1 - \xi_k(1 - \Theta)\right] \|y_k - \bar{y}_k\|_B \\
& \leq \left[1 - \xi_k(1 - \Theta)\right] \|y_k - \bar{y}_k\|_B + \xi_k(1 - \Theta) \frac{3M}{(1 - \Theta)}.
\end{aligned} \tag{26}$$

We denote

$$\begin{aligned}
\beta_k & = \|y_k - \bar{y}_k\|_B \geq 0, \\
\mu_k & = \xi_k(1 - \Theta) \in (0, 1), \\
\gamma_k & = \frac{3M}{(1 - \Theta)} \geq 0.
\end{aligned}$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (26) satisfies all the conditions of Lemma 2.2 and hence, we have

$$0 \leq \limsup_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \gamma_k$$

$$\begin{aligned} \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \limsup_{k \rightarrow \infty} \frac{3M}{(1 - \Theta)} \\ \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \frac{3M}{(1 - \Theta)}. \end{aligned} \quad (27)$$

Using the assumptions, $\lim_{k \rightarrow \infty} y_k = y$, $\lim_{k \rightarrow \infty} \bar{y}_k = \bar{y}$, we get from (27) that

$$\|y - \bar{y}\|_B \leq \frac{3M}{(1 - \Theta)}, \quad (28)$$

which shows that the dependency of solutions of the equations (1)-(2) and (1) with the initial conditions (20) on given initial data. $\square$

4.2 Closeness of solution via S -iteration

Consider the problem (1)-(2) and the corresponding problem

$$(D_{*a}^\alpha) \bar{y}(t) = \bar{\mathcal{F}}\left(t, \bar{y}(t), \int_a^t h(s, \bar{y}(s)) ds\right), \quad (29)$$

for $t \in I = [a, b]$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, with the given initial conditions

$$\bar{y}^{(j)}(a) = d_j, \quad j = 0, 1, 2, \dots, n - 1, \quad (30)$$

where $\bar{\mathcal{F}}$ is defined as $\mathcal{F}$ and d_j ($j = 0, 1, 2, \dots, n - 1$) are given elements in X .

Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equations (29)-(30)

$$\begin{aligned} (\bar{T}\bar{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!} (t - a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} \bar{\mathcal{F}}\left(s, \bar{y}(s), \int_a^s h(\sigma, \bar{y}(\sigma)) d\sigma\right) ds, \quad t \in I. \end{aligned} \quad (31)$$

The next theorem deals with the closeness of solutions of the problems (1)-(2) and (29)-(30).

Theorem 4.2. Consider the sequences $\{y_k\}_{k=0}^\infty$ and $\{\bar{y}_k\}_{k=0}^\infty$ generated normal S -iterative method associated with operators T in (9) and $\bar{T}$ in (31), respectively with the real sequence $\{\xi_k\}_{k=0}^\infty$ in $[0, 1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Assume that

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(i) all conditions of Theorem 3.1 hold, and $y(t)$ and $\bar{y}(t)$ are solutions of (1)-(2) and (29)-(30) respectively,

(ii) there exist non negative constant ϵ such that

$$\|\mathcal{F}(t, u_1, u_2) - \bar{\mathcal{F}}(t, u_1, u_2)\| \leq \epsilon, \quad \forall t \in I. \quad (32)$$

If the sequence $\{\bar{y}_k\}_{k=0}^{\infty}$ converges to $\bar{y}$, then we have

$$\|y - \bar{y}\|_B \leq \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)}. \quad (33)$$

Proof. From iteration (4) and equations (9); (31) and hypotheses, we obtain

$$\begin{aligned} & \|y_{k+1}(t) - \bar{y}_{k+1}(t)\| \\ &= \|(Tz_k)(t) - (\bar{T}\bar{z}_k)(t)\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) ds \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \bar{\mathcal{F}}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) ds \right\| \\ &\leq \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} (b-a)^j \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) \right. \\ &\quad \left. - \bar{\mathcal{F}}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) \right\| ds \\ &\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma\right) \right. \\ &\quad \left. - \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) \right\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) \right. \\ &\quad \left. - \bar{\mathcal{F}}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma\right) \right\| ds \\ &\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \epsilon ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k(s) - \bar{z}_k(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - \bar{z}_k(\sigma)\| d\sigma \right] ds \\
 & \leq M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \\
 & + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \left[\|z_k(s) - \bar{z}_k(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - \bar{z}_k(\sigma)\| d\sigma \right] ds.
 \end{aligned} \tag{34}$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$\|y_{k+1} - \bar{y}_{k+1}\|_B \leq M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} + \Theta \|z_k - \bar{z}_k\|_B, \tag{35}$$

and similarly, it is seen that

$$\|z_k - \bar{z}_k\|_B \leq \xi_k \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \left[1 - \xi_k (1 - \Theta) \right] \|y_k - \bar{y}_k\|_B. \tag{36}$$

Therefore, using (36) in (35) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality becomes

$$\begin{aligned}
 & \|y_{k+1} - \bar{y}_{k+1}\|_B \\
 & \leq \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \|z_k - \bar{z}_k\|_B \\
 & \leq \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \xi_k \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \left[1 - \xi_k (1 - \Theta) \right] \|y_k - \bar{y}_k\|_B \\
 & \leq 2\xi_k \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \xi_k \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right] + \left[1 - \xi_k (1 - \Theta) \right] \|y_k - \bar{y}_k\|_B \\
 & \leq \left[1 - \xi_k (1 - \Theta) \right] \|y_k - \bar{y}_k\|_B + \xi_k (1 - \Theta) \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)}.
 \end{aligned} \tag{37}$$

We denote

$$\begin{aligned}
 \beta_k &= \|y_k - \bar{y}_k\|_B \geq 0, \\
 \mu_k &= \xi_k (1 - \Theta) \in (0, 1), \\
 \gamma_k &= \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)} \geq 0.
 \end{aligned}$$

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The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (37) satisfies all the conditions of Lemma 2.2 and hence, we have

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \gamma_k \\ \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \limsup_{k \rightarrow \infty} \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)} \\ \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)}. \end{aligned} \quad (38)$$

Using the assumptions, $\lim_{k \rightarrow \infty} y_k = y$, $\lim_{k \rightarrow \infty} \bar{y}_k = \bar{y}$, we get from (38) that

$$\|y - \bar{y}\|_B \leq \frac{3 \left[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \right]}{(1 - \Theta)}, \quad (39)$$

which shows that the dependency of solutions of IVP (1)-(2) on the function involved on the right hand side of the given equation. $\square$

Remark: The inequality (39) relates the solutions of the problems (1)-(2) and (29)-(30) in the sense that, if $\mathcal{F}$ and $\bar{\mathcal{F}}$ are close as $\epsilon \rightarrow 0$, then not only the solutions of the problems (1)-(2) and (29)-(30) are close to each other (i.e. $\|y - \bar{y}\|_B \rightarrow 0$), but also depends continuously on the functions involved therein and initial data.

4.3 Dependence on Parameters

We next consider the following problems

$$(D_{*a}^\alpha)y(t) = \mathcal{F}\left(t, y(t), \int_a^t h(s, y(s))ds, \mu_1\right), \quad (40)$$

for $t \in I = [a, b]$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(a) = c_j, \quad j = 0, 1, 2, \dots, n - 1, \quad (41)$$

and

$$(D_{*a}^\alpha)\bar{y}(t) = \mathcal{F}\left(t, \bar{y}(t), \int_a^t h(s, \bar{y}(s))ds, \mu_2\right), \quad (42)$$

for $t \in I = [a, b]$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, with the given initial conditions

$$\bar{y}^{(j)}(a) = d_j, \quad j = 0, 1, 2, \dots, n - 1, \quad (43)$$

where $\mathcal{F} : I \times X \times X \times \mathbb{R} \rightarrow X$ is continuous function, c_j, d_j ($j = 0, 1, 2, \dots, n - 1$) are given elements in X and constants μ_1, μ_2 are real parameters.

Let $y(t), \bar{y}(t) \in B$ and following steps from the proof of Theorem 3.1, define the operators for the equations (40) and (42), respectively

$$\begin{aligned} (Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, y(s), \int_a^s h(\sigma, y(\sigma)) d\sigma, \mu_1\right) ds, \quad t \in I; \end{aligned} \quad (44)$$

and

$$\begin{aligned} (\bar{T}\bar{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, \bar{y}(s), \int_a^s h(\sigma, \bar{y}(\sigma)) d\sigma, \mu_2\right) ds, \quad t \in I. \end{aligned} \quad (45)$$

The following theorem states the continuous dependency of solutions on parameters.

Theorem 4.3. *Consider the sequences $\{y_k\}_{k=0}^\infty$ and $\{\bar{y}_k\}_{k=0}^\infty$ generated normal S -iterative method associated with operators T in (44) and $\bar{T}$ in (45), respectively with the real sequence $\{\xi_k\}_{k=0}^\infty$ in $[0, 1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Assume that*

(i) $y(t)$ and $\bar{y}(t)$ are solutions of (40)-(41) and (42)-(43) respectively,

(ii) there exist functions $\bar{p}, r \in C(I, \mathbb{R}_+)$ such that

$$\|\mathcal{F}(t, u_1, u_2, \mu_1) - \mathcal{F}(t, v_1, v_2, \mu_1)\| \leq \bar{p}(t) [\|u_1 - v_1\| + \|u_2 - v_2\|],$$

and

$$\|\mathcal{F}(t, u_1, u_2, \mu_1) - \mathcal{F}(t, u_1, u_2, \mu_2)\| \leq r(t) |\mu_1 - \mu_2|.$$

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If the sequence $\{\bar{y}_k\}_{k=0}^{\infty}$ converges to $\bar{y}$, then we have

$$\|y - \bar{y}\|_B \leq \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})}, \quad (46)$$

where $\bar{\Theta} = I_a^\alpha \bar{p}(t) (1 + (b - a)Q) < 1$, $t \in I$.

Proof. From iteration (4) and equations (44); (45) and hypotheses, we obtain

$$\begin{aligned} & \|y_{k+1}(t) - \bar{y}_{k+1}(t)\| \\ &= \|(Tz_k)(t) - (\bar{T}\bar{z}_k)(t)\| \\ &= \left\| \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma, \mu_1\right) ds \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma, \mu_2\right) ds \right\| \\ &\leq \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} (b-a)^j \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma, \mu_1\right) \right. \\ &\quad \left. - \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma, \mu_2\right) \right\| ds \\ &\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, z_k(s), \int_a^s h(\sigma, z_k(\sigma)) d\sigma, \mu_1\right) \right. \\ &\quad \left. - \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma, \mu_1\right) \right\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma, \mu_1\right) \right. \\ &\quad \left. - \mathcal{F}\left(s, \bar{z}_k(s), \int_a^s h(\sigma, \bar{z}_k(\sigma)) d\sigma, \mu_2\right) \right\| ds \\ &\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} r(s) |\mu_1 - \mu_2| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \bar{p}(s) \left[\|z_k(s) - \bar{z}_k(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - \bar{z}_k(\sigma)\| d\sigma \right] ds \\ &\leq M + |\mu_1 - \mu_2| I_a^\alpha r(t) \end{aligned}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \bar{p}(s) \left[\|z_k(s) - \bar{z}_k(s)\| + \int_a^s q(\sigma) \|z_k(\sigma) - \bar{z}_k(\sigma)\| d\sigma \right] ds. \quad (47)$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$\|y_{k+1} - \bar{y}_{k+1}\|_B \leq M + |\mu_1 - \mu_2| I_a^\alpha r(t) + \bar{\Theta} \|z_k - \bar{z}_k\|_B, \quad (48)$$

and similarly, it is seen that

$$\|z_k - \bar{z}_k\|_B \leq \xi_k \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] + \left[1 - \xi_k (1 - \bar{\Theta}) \right] \|y_k - \bar{y}_k\|_B. \quad (49)$$

Therefore, using (49) in (48) and using hypothesis $\bar{\Theta} < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned} & \|y_{k+1} - \bar{y}_{k+1}\|_B \\ & \leq \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] + \|z_k - \bar{z}_k\|_B \\ & \leq \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] + \xi_k \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] \\ & \quad + \left[1 - \xi_k (1 - \bar{\Theta}) \right] \|y_k - \bar{y}_k\|_B \\ & \leq 2\xi_k \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] + \xi_k \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right] \\ & \quad + \left[1 - \xi_k (1 - \bar{\Theta}) \right] \|y_k - \bar{y}_k\|_B \\ & \leq \left[1 - \xi_k (1 - \bar{\Theta}) \right] \|y_k - \bar{y}_k\|_B + \xi_k (1 - \bar{\Theta}) \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})}. \quad (50) \end{aligned}$$

We denote

$$\begin{aligned} \beta_k &= \|y_k - \bar{y}_k\|_B \geq 0, \\ \mu_k &= \xi_k (1 - \bar{\Theta}) \in (0, 1), \\ \gamma_k &= \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})} \geq 0. \end{aligned}$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (50) satisfies all the conditions of Lemma 2.2 and hence we have

$$0 \leq \limsup_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \gamma_k$$

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$$\begin{aligned} \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \limsup_{k \rightarrow \infty} \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})} \\ \Rightarrow 0 &\leq \limsup_{k \rightarrow \infty} \|y_k - \bar{y}_k\|_B \leq \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})}. \end{aligned} \quad (51)$$

Using the assumption $\lim_{k \rightarrow \infty} y_k = y$, $\lim_{k \rightarrow \infty} \bar{y}_k = \bar{y}$, we get from (51) that

$$\|y - \bar{y}\|_B \leq \frac{3 \left[M + |\mu_1 - \mu_2| I_a^\alpha r(t) \right]}{(1 - \bar{\Theta})}, \quad (52)$$

which shows the dependence of solutions of the problem (1)-(2) is on parameters μ_1 and μ_2 . $\square$

Remark: The result dealing with the property of a solution called “dependence of solutions on parameters”. Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

5 Example

We consider the following problem:

$$(D_*^\alpha)y(t) = \frac{3t}{5} \left[\frac{t - \sin(y(t))}{2} + \frac{1}{9} \int_0^t \frac{e^{-s}}{(2+s)^2} y(s) ds \right], \quad (53)$$

for $t \in [0, 1]$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(0) = c_j, \quad j = 0, 1, 2, \dots, n - 1. \quad (54)$$

Comparing this equation with the equation (1), we get $\mathcal{F} \in C(I \times \mathbb{R}^2, \mathbb{R})$, with

$$\mathcal{F}\left(t, y(t), \int_0^t h(s, y(s)) ds\right) = \frac{3t}{5} \left[\frac{t - \sin(y(t))}{2} + \frac{1}{9} \int_0^t \frac{e^{-s}}{(2+s)^2} y(s) ds \right]$$

and

$$h(t, y(t)) = \frac{3t}{45} \frac{e^{-t}}{(2+t)^2} y(t).$$

Now, one can easily show that

$$\begin{aligned} & \left| \mathcal{F}(t, y(t), z(t)) - \mathcal{F}(t, \bar{y}(t), \bar{z}(t)) \right| \\ & \leq \frac{3t}{5} \left[\frac{1}{2} \left| \sin(y(t)) - \sin(\bar{y}(t)) \right| + \frac{1}{9} \left| z(t) - \bar{z}(t) \right| \right] \\ & \leq \frac{3t}{10} \left[\left| y - \bar{y} \right| + \left| z - \bar{z} \right| \right], \end{aligned} \quad (55)$$

and

$$\left| h(t, y(t)) - h(t, z(t)) \right| \leq \frac{3t}{45} \frac{e^{-t}}{(2+t)^2} \left| y - z \right|, \quad (56)$$

where $p(t) = \frac{3t}{10}$, and $q(t) = \frac{3t}{45} \frac{e^{-t}}{(2+t)^2}$. Therefore, we have

$$Q = \sup_{t \in [0,1]} \{q(t)\} = \frac{3}{180} = \frac{1}{60}.$$

Thus, we the estimate

$$\begin{aligned} \Theta &= I_a^\alpha p(t) (1 + (b-a)Q) \\ &= I_a^\alpha \frac{3t}{10} \left(1 + \frac{1}{60} \right) \\ &= \frac{3}{10} \left(1 + \frac{1}{60} \right) (I_a^\alpha)(t) \\ &= \frac{61}{200} (I_a^\alpha)(t) \\ &= \frac{61}{200} \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\leq \frac{1}{\Gamma(\alpha+2)}, \quad (t \leq 1). \end{aligned} \quad (57)$$

Therefore, the condition $\Theta < 1$ is satisfied only if $\frac{1}{\Gamma(\alpha+2)} < 1$.

We define the operator $T : B \rightarrow B$ by

$$\begin{aligned} (Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!} t^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{3s}{5} \left[\frac{s - \sin(y(s))}{2} + \frac{1}{9} \int_0^s \frac{e^{-\sigma}}{(2+\sigma)^2} y(\sigma) d\sigma \right] ds, \end{aligned} \quad (58)$$

for $t \in I$. Since all conditions of Theorem 3.1 are satisfied and so by its conclusion, the sequence $\{y_n\}$ associated with the normal S -iterative method (4) for the operator T in (58) converges to a unique solution $y \in B$.

6 Conclusions

Firstly, we proved the main result, which address the existence and uniqueness of the solution to the IVP (1)-(2) by the method of normal S -iteration. Next, we discussed various properties of solutions like continuous dependence on the initial data, closeness of solutions, and dependence on parameters and functions involve therein. Finally, we provided an appropriate example to support all of the findings.

Acknowledgement: The authors are very grateful to the referees for their comments and remarks.

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Multi-block two repeated fixed burst error correcting linear codes

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Abstract

During the digital transmission of information, errors are bound to occur. The errors may be random or burst errors. In this paper, we have obtained necessary and sufficient conditions for the existence of linear codes over $GF(q)$ that are capable of correcting 2-repeated burst errors of length b_1 (fixed) and 2-repeated burst errors of length b_2 (fixed) within sub-blocks of unequal lengths. We have also constructed a $(14, 22)$ parity-check matrix for such a code.

Keywords: Linear codes; parity-check matrix; 2-repeated burst of length b (fixed); blockwise correction; syndromes.

2020 AMS subject classifications: 94B20; 94B65. ¹

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¹Received on June 03, 2022. Revised on October 04, 2022. Accepted on October 19, 2022.

Published on October 21, 2022. DOI: 10.23755/rm.v43i0.792. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Today, we live in a world flooded with digital information and communication channels. A good communication system is one that conveys information from one point to another point without any distortion. When information is transmitted through digital channels, noise gets added which corrupts the information sent. So, we need to develop codes that are capable of correcting such distorted information so that the information received by the receiver is accurate.

It was seen that when data is transmitted, errors occur in adjacent positions rather than random positions. This led to the development of codes that could correct single errors and double adjacent errors by Abramson [1]. In these types of errors, digits in adjacent or consecutive positions are in error. Further, when the data is transmitted, lots of bits are transmitted in a fraction of seconds while the noise is for a longer period, so the probability of occurrence of errors in the adjacent positions is more than the probability of occurrence of random errors. So it becomes very important to develop the codes that can correct burst errors. A *burst of length b* , also known as Open loop burst error, has been defined by Fire [8] as a vector whose only non-zero components are among some b consecutive components, the first and the last of which are non-zero.

Another category of burst errors was introduced by Chien and Tang [3]. They defined *CT-burst of length b* as a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero. This definition was further modified by Dass [4] and he defined a *burst of length b (fixed)* as an n -tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions is among the first $n - b + 1$ components. For example, (001011000) is a burst of length 4 but can also be regarded as a burst of length upto 7(fixed) over $GF(2)$.

As more and more information is getting transmitted, the communication channels become extremely busy. This leads to the repetition of errors where errors may be random or burst errors. This led to the definition of repeated bursts of length b or less [2]. A *2-repeated burst of length b or less* is a vector whose non-zero components are in the form of two non-overlapping sets of b or less consecutive digits. For example, (001011100) is a 2-repeated burst of length 3 or less over $GF(2)$. In this paper, we consider repeated bursts of length b (fixed) [6].

A *2-repeated burst of length b (fixed)* is defined as an n -tuple whose only non-zero components are confined to 2 distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - 2b + 1$ components [6]. We denote a 2-repeated burst of length b (fixed) by $2 - RB_f(b)$. For example, (10010000) is a 2-repeated burst of length atmost 3(fixed) and (001000001000) is a 2-repeated burst of length upto 4(fixed) over $GF(2)$.

The development of codes dealing with repeated burst errors improves the efficiency of the communication channels as the requirement of the parity-check digits is lesser in this case as compared to when we consider such errors as simple burst errors.

In this paper, we are dealing with (n, k) linear codes over $GF(q)$, where q is some power of a prime. The code of length n consists of two subblocks of length n_1 and n_2 such that $n = n_1 + n_2$. Also, the code of length n consists of $k = n - r$ information digits, where r denotes the number of parity check digits. The distance between two codewords is taken to be their Hamming distance [9]. The codes under consideration are able to correct all burst errors of length b_1 (fixed) in a block of length n_1 and all burst errors of length b_2 (fixed) in a block of length n_2 . For an (n, k) linear code that corrects $2 - RB_f(b_1)$ in a sub-block of length n_1 and also corrects $2 - RB_f(b_2)$ in a sub-block of length n_2 , following conditions must be satisfied:

1. The syndromes of all $2 - RB_f(b_1)$ occurring in the first sub-block (which is of length n_1) must be non-zero and distinct.
2. The syndromes of all $2 - RB_f(b_2)$ occurring in the second sub-block (which is of length n_2) must be non-zero and distinct.
3. The syndromes of all $2 - RB_f(b_1)$ occurring in the first sub-block must be different from the syndromes of all $2 - RB_f(b_2)$ occurring in the second sub-block.

2 Necessary Condition

In this section, we obtain a necessary condition for existence of an (n, k) linear code over $GF(q)$ that corrects 2-repeated bursts of length b_1 (fixed) ($2 - RB_f(b_1)$) in a block of length n_1 and also corrects 2-repeated bursts of length b_2 (fixed) ($2 - RB_f(b_2)$) in a block of length n_2 , where $n = n_1 + n_2$. For a code that corrects any type of error patterns, the error patterns must lie in distinct cosets of the standard array or in other words, must have distinct syndromes. So, we get the following theorem.

Theorem 2.1. *For an (n, k) code over $GF(q)$ that corrects $2 - RB_f(b_1)$ in a block of length n_1 and also corrects $2 - RB_f(b_2)$ in a block of length n_2 , where $(n = n_1 + n_2)$*

$$\log_q \left\{ 1 + \sum_{i=1}^2 (q-1)^i \left\{ \binom{n_1 - ib_1 + i}{i} q^{i(b_1-1)} + \binom{n_2 - ib_2 + i}{i} q^{i(b_2-1)} \right\} \right\} \quad (1)$$

number of parity-check digits are necessary.

Proof. For $n = n_1 + n_2$, consider an (n, k) linear code that corrects $2 - RB_f(b_1)$ in a block of length n_1 and $2 - RB_f(b_2)$ in a block of length n_2 . For a code to be able to correct such errors, the syndromes of these errors should be distinct. Also, the total number of syndromes is q^{n-k} . So, q^{n-k} must be greater than the number of syndromes corresponding to such errors.

Now, the number of syndromes corresponding to $2 - RB_f(b_1)$ in the first block of length n_1 is the same as the number of such errors in a vector of length n_1 , which is [5]

$$\sum_{i=1}^2 (q-1)^i \binom{n_1 - ib_1 + i}{i} q^{i(b_1-1)}. \quad (2)$$

Further, the number of syndromes corresponding to the error $2 - RB_f(b_2)$ in the second block of length n_2 is the same as the number of such errors in a vector of length n_2 , which is [5]

$$\sum_{i=1}^2 (q-1)^i \binom{n_2 - ib_2 + i}{i} q^{i(b_2-1)}. \quad (3)$$

Now, the total number of syndromes, including the all zero syndrome is q^{n-k} . So, for the desired code, we must have

$$q^{n-k} \geq 1 + \text{expression (2)} + \text{expression (3)}.$$

Taking log, we get the desired result.

Remark: For $n_1 = n_2$ and $b_1 = b_2$ we obtain that

$$1 + 2 \left\{ \sum_{i=1}^2 (q-1)^i \binom{n_2 - ib_2 + i}{i} q^{i(b_2-1)} \right\} \quad (4)$$

number of parity-check digits are necessary for existence of a code that corrects $2 - RB_f(b_1)$ in a block of length n_1 and also corrects $2 - RB_f(b_2)$ in a block of length n_2 , which coincides with the necessary number of parity-check digits required for the location and correction of such errors in a code of length n with 2 subblocks of length $n_1 (= n_2)$ each [7].

3 Sufficient Condition

We now obtain an upper bound on the number of parity-check digits required for existence of the code considered in Theorem 2.1. The proof uses the technique of

Varshmov-Gilbert-Sacks bound [11, 10, 12]. This technique not only helps us to construct a parity-check matrix for the required code but also ensures the existence of such a code.

Theorem 3.1. *For existence of an (n, k) code over $GF(q)$ that corrects $2 - RB_f(b_1)$ in a block of length n_1 and also corrects $2 - RB_f(b_2)$ in a block of length n_2 , where $(n = n_1 + n_2)$,*

$$\begin{aligned} & \text{Max} \left[q^{b_2-1} \sum_{i=0}^3 \binom{n_2 - (i+1)b_2 + i}{i} (q-1)^i q^{i(b_2-1)}, \right. \\ & q^{b_1-1} \sum_{i=0}^3 \binom{n_1 - (i+1)b_1 + i}{i} (q-1)^i q^{i(b_1-1)} \\ & + \{q^{b_1-1} \{1 + (n_1 - 2b_1 + 1)(q-1)q^{b_1-1}\} \\ & \left. \times \sum_{i=1}^2 \binom{n_2 - ib_2 + i}{i} (q-1)^i q^{i(b_2-1)} \} \right] \end{aligned}$$

number of parity-check digits are sufficient.

Proof. We construct a parity-check matrix of the desired code. In the process of construction, we first add columns of the block of length n_2 (second block) followed by the columns of the block of length n_1 (first block), to obtain a preliminary matrix H' . After completing the process of addition of columns, we reverse the order of the columns of H' to obtain the desired parity-check matrix H .

Suppose first $j-1, j \leq n_2$, columns have been appropriately added to the second block. Since, the code is capable of correcting $2 - RB_f(b_2)$ in a block of length n_2 , so j th column is chosen based on the following condition:

For the correction of $2 - RB_f(b_2)$ in second block of length n_2 , we should have the syndromes of all $2 - RB_f(b_2)$ in this block to be distinct. This ensures that the sum or difference of two bursts is not a code word. So, an $n - k$ tuple can be added as the j th column, h_j , provided that it is not a linear combination of immediately preceding $b_2 - 1$ columns together with any three distinct sets of b_2 (fixed) consecutive columns out of the first $j - b_2$ columns of the second block i.e.

$$\begin{aligned} h_j & \neq (\alpha_1 h_{j-b_2+1} + \alpha_2 h_{j-b_2+2} + \cdots + \alpha_{b_2-1} h_{j-1}) + \\ & + (\beta_1 h_{r_1+1} + \beta_2 h_{r_1+2} + \cdots + \beta_{b_2} h_{r_1+b_2}) \\ & + (\gamma_1 h_{r_2+1} + \gamma_2 h_{r_2+2} + \cdots + \gamma_{b_2} h_{r_2+b_2}) \\ & + (\delta_1 h_{r_3+1} + \delta_2 h_{r_3+2} + \cdots + \delta_{b_2} h_{r_3+b_2}), \end{aligned} \quad (5)$$

where either all β_r 's, γ_r 's, δ_r 's are zero or if the last non-zero coefficient amongst these is say θ_k , then $b_2 \leq k \leq j - b_2$.

Number of ways α 's may be chosen is q^{b_2-1} . To count β_r 's, γ_r 's, δ_r 's is equivalent to enumerating the number of $3 - RB_f(b_2)$ in a vector of length $j - b_2$ which, including the zero vector, is [5]

$$\sum_{i=0}^3 \binom{j - (i+1)b_2 + i}{i} (q-1)^i q^{i(b_2-1)}. \quad (6)$$

So, the total number of linear combinations on R.H.S. of (5) is

$$q^{b_2-1} \sum_{i=0}^3 \binom{j - (i+1)b_2 + i}{i} (q-1)^i q^{i(b_2-1)}.$$

Proceeding like this we can add the last column h_{n_2} provided it is not equal to

$$q^{b_2-1} \sum_{i=0}^3 \binom{n_2 - (i+1)b_2 + i}{i} (q-1)^i q^{i(b_2-1)} \quad (7)$$

linear combinations.

After adding the last column, n_2 of second block we proceed further to add the $(n_2 + 1)$ th, $(n_2 + 2)$ th, $\dots$ columns to H' such that the code is able to correct $2 - RB_f(b_1)$ in a block of length n_1 and also corrects $2 - RB_f(b_2)$ in a block of length n_2 .

For that, we should ensure that in the code so constructed:

1. The syndromes of all $2 - RB_f(b_1)$ in a block of length n_1 are not equal.
2. The syndromes of any two 2-repeated bursts(fixed), one of which is a $2 - RB_f(b_1)$ in a block of length n_1 and the other is a $2 - RB_f(b_2)$ in a block of length n_2 (columns corresponding to which have already been chosen), are different.

So as the first requirement, k th ($k > n_2$) column can be chosen provided that it is not a linear combination of immediately preceding $b_1 - 1$ columns together with any three distinct sets of b_1 (fixed) consecutive columns out of the first $k - n_2 - b_1$ columns of the already added second block. i.e.

$$\begin{aligned} h_k \neq & (\zeta_1 h_{k-b_1+1} + \zeta_2 h_{k-b_1+2} + \dots + \zeta_{b_1-1} h_{k-1}) + \\ & + (\lambda_1 h_{l_1+1} + \lambda_2 h_{l_1+2} + \dots + \lambda_{b_1} h_{l_1+b_1}) \\ & + (\mu_1 h_{l_2+1} + \mu_2 h_{l_2+2} + \dots + \mu_{b_1} h_{l_2+b_1}) \\ & + (\nu_1 h_{l_3+1} + \nu_2 h_{l_3+2} + \dots + \nu_{b_1} h_{l_3+b_1}), \end{aligned} \quad (8)$$

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where either all λ_l 's, μ_l 's, ν_l 's are zero or if the last non-zero coefficient amongst these is say τ_p , then $b_1 \leq p \leq k - b_1$, and ζ_l 's, λ_l 's, μ_l 's, ν_l 's $\in GF(q)$.

Number of ways ζ 's may be chosen is q^{b_1-1} . Choosing λ_l 's, μ_l 's, ν_l 's is equivalent to enumerating the number of $3 - RB_f(b_1)$ in a vector of length $k - n_2 - b_1$. So this number, including the vector of all zeros, is [5]

$$\sum_{i=0}^3 \binom{k - n_2 - (i+1)b_1 + i}{i} (q-1)^i q^{i(b_1-1)}. \quad (9)$$

Now, according to the second condition, k th column should not be a linear combination of immediately preceding $b_1 - 1$ columns $h_{k-b_1+1}, h_{k-b_1+2}, \dots, h_{k-1}$, together with b_1 consecutive columns from previous $k - b_1$ columns and two sets of distinct b_2 columns from the second block. So,

$$\begin{aligned} h_k \neq & (\phi_1 h_{k-b_1+1} + \phi_2 h_{k-b_1+2} + \dots + \phi_{b_1-1} h_{k-1}) + \\ & + (\chi_1 h_{l_1+1} + \chi_2 h_{l_1+2} + \dots + \chi_{b_1} h_{l_1+b_1}) \\ & + (\psi_1 h_{r_1+1} + \psi_2 h_{r_1+2} + \dots + \psi_{b_2} h_{r_1+b_2}) \\ & + (\omega_1 h_{r_2+1} + \omega_2 h_{r_2+2} + \dots + \omega_{b_2} h_{r_2+b_2}), \end{aligned} \quad (10)$$

where h_l 's are columns corresponding to the first block, h_r 's are columns corresponding to the second block. Also, here either all χ_l 's are zero or if χ_p is the last non-zero coefficient amongst these, then $b_1 \leq p \leq k - b_1$. Also not all of ψ_r 's and ω_r 's are zero and which so ever is the last non-zero coefficient amongst these, say ρ_p then $b_2 \leq p \leq n_2$.

Number of ways in which ϕ 's may be chosen is q^{b_1-1} . Number of ways in which χ_r 's, can be chosen is equivalent to the number of bursts of length b_1 (fixed) in a vector of length $k - n_2 - b_1$. This number, including the zero vector is [4]

$$1 + (k - n_2 - 2b_1 + 1)(q-1)q^{b_1-1}. \quad (11)$$

So, first two expressions of (10) represent

$$q^{b_1-1} \{1 + (k - n_2 - 2b_1 + 1)(q-1)q^{b_1-1}\} \quad (12)$$

linear combinations.

Further, number of linear combinations corresponding to ψ_r 's and ω_r 's is the same as the number of $2 - RB_f(b_2)$ in a vector of length n_2 . So, this number is (not including the zero vector) [5]

$$\sum_{i=1}^2 \binom{n_2 - ib_2 + i}{i} (q-1)^i q^{i(b_2-1)}. \quad (13)$$

So, the total number of linear combinations to which h_k can not be equal is

$$\begin{aligned} & q^{b_1-1} \sum_{i=0}^3 \binom{k - n_2 - (i+1)b_1 + i}{i} (q-1)^i q^{i(b_1-1)} \\ & + \{q^{b_1-1} \{1 + (k - n_2 - 2b_1 + 1)(q-1)q^{b_1-1}\} \\ & \times \sum_{i=1}^2 \binom{n_2 - ib_2 + i}{i} (q-1)^i q^{i(b_2-1)}\}. \end{aligned}$$

Proceeding like this, we can add the n th column provided h_n is not equal to

$$\begin{aligned} & q^{b_1-1} \sum_{i=0}^3 \binom{n_1 - (i+1)b_1 + i}{i} (q-1)^i q^{i(b_1-1)} \\ & + \{q^{b_1-1} \{1 + (n_1 - 2b_1 + 1)(q-1)q^{b_1-1}\} \\ & \times \sum_{i=1}^2 \binom{n_2 - ib_2 + i}{i} (q-1)^i q^{i(b_2-1)}\} \end{aligned} \quad (14)$$

number of linear combinations.

Complete matrix H' can be constructed if both the blocks are added. So, for the existence of the matrix we require $q^{n-k} > \text{Max}[(7), (14)]$. Further, the parity-check matrix H is then obtained by reversing the order of the columns of H' .

4 Illustration: Parity-check Matrix

In this section, we have constructed a $(14, 22)$ parity-check matrix H , for a $(22, 8)$ linear code over $GF(2)$ that can correct $2 - RB_f(2)$ in the first sub-block of length 9 and $2 - RB_f(3)$ in the second sub-block of length 13. The construction is based on the synthesis process discussed in Theorem 3.1. The null space of this matrix is the required code.

Multi-block two repeated fixed burst error . . .

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix has been constructed using Python. The following also has been verified using Python.

- (a) The syndromes of all $2 - RB_f(2)$ within the first sub-block and $2 - RB_f(3)$ within the second sub-block are all different from zero.
- (b) The syndromes of all $2 - RB_f(2)$ occurring in the first sub-block are all different.
- (c) The syndromes of all $2 - RB_f(3)$ occurring in the second sub-block are all distinct.
- (d) The syndromes of all $2 - RB_f(2)$ occurring in the first sub-block are different from syndromes of all $2 - RB_f(3)$ occurring within the second sub-block.

The error vectors and syndromes are listed in Table 1.

Table 1
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
10100000000000000000	00010011101110	01000110000000000000	00100001010101
10010000000000000000	00111110010111	01000011000000000000	00100101110000
10001000000000000000	00111010110000	01000001100000000000	10100111100010
10000100000000000000	00110111011100	01111000000000000000	00000011010010
10000010000000000000	00110001101010	01101100000000000000	00001010011001
10000001000000000000	00110011111001	01100110000000000000	00000001000011
11100000000000000000	00110100001101	01100011000000000000	00000101100110
11010000000000000000	00011001110100	01100001100000000000	10000111110100
11001000000000000000	00011101010011	00101000000000000000	00101001011110
11000100000000000000	00010000111111	00100100000000000000	00100100110010
11000010000000000000	00010110001001	00100010000000000000	00100010000100
11000001000000000000	00010100011010	00100001000000000000	00100000010111
10110000000000000000	00011110000001	00111000000000000000	00100100110001
10011000000000000000	00110111011111	00110100000000000000	00101001011101
10001100000000000000	00111110010100	00110010000000000000	00101111101011
10000110000000000000	00110101001110	00110001000000000000	00101101111000
10000011000000000000	00110001101011	00101100000000000000	00101101111010
10000001100000000000	10110011111001	00100110000000000000	00100110100000
11110000000000000000	00111001100010	00100011000000000000	00100010000101
11011000000000000000	00010000111100	00100001100000000000	10100000010111
11001100000000000000	00011001110111	00111100000000000000	00100000010101
11000110000000000000	00010010101101	00110110000000000000	00101011001111
11000011000000000000	00010110001000	00110011000000000000	00101111101010
11000001100000000000	10010100011010	00110001100000000000	10101101111000
01010000000000000000	00101010001100	00010100000000000000	00001001001011
01001000000000000000	00101110101011	00010010000000000000	00001111111101
01000100000000000000	00100011000111	00010001000000000000	00001101101110
01000010000000000000	00100101110001	00011100000000000000	00000000000011
01000001000000000000	00100111100010	00011010000000000000	00000110110101
01110000000000000000	00001010011010	00011001000000000000	00000100100110
01101000000000000000	00001110111101	00010110000000000000	00001011011001
01100100000000000000	00000011010001	00010011000000000000	00001111111100
01100010000000000000	00000101100111	00010001100000000000	10001101101110
01100001000000000000	00000111110100	00011110000000000000	00000010010001
01011000000000000000	00100011000100	00011011000000000000	00000110110100
01001100000000000000	00101010001111	00011001100000000000	10000100100110

Multi-block two repeated fixed burst error . . .

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
000010100000000000000000	00001011011010	0000000000000000000010000	000010000000000
000010010000000000000000	00001001001001	000000000000000000001000	000100000000000
000011100000000000000000	00001111111110	000000000000000000000100	001000000000000
000011010000000000000000	00001101101101	000000000101000000000000	10000000001010
000010110000000000000000	00001011011011	000000000010100000000000	00000000010100
000010011000000000000000	10001001001001	000000000001010000000000	00000000101000
000011110000000000000000	00001111111111	000000000000101000000000	00000001010000
000011011000000000000000	10001101101101	000000000000010100000000	00000010100000
000001010000000000000000	00000100100101	000000000000001010000000	00000101000000
000001110000000000000000	00000110110111	000000000000000101000000	00001010000000
000001011000000000000000	10000100100101	000000000000000010100000	00010100000000
000001111000000000000000	10000110110111	000000000000000001010000	00101000000000
100000000000000000000000	00110011111000	00000000000000000001010	01010000000000
010000000000000000000000	00100111100011	00000000000000000000101	10100000000000
001000000000000000000000	00100000010110	000000000110000000000000	10000000000110
000100000000000000000000	00001101101111	000000000011000000000000	00000000001100
000010000000000000000000	00001001001000	000000000001100000000000	00000000011000
000001000000000000000000	00000100100100	000000000000110000000000	00000000110000
000000100000000000000000	00000010010010	000000000000011000000000	00000001100000
000000010000000000000000	00000000000001	000000000000001100000000	00000011000000
110000000000000000000000	00010100011011	000000000000000110000000	00000110000000
011000000000000000000000	00000111110101	000000000000000011000000	00001100000000
001100000000000000000000	00101101111001	000000000000000001100000	00011000000000
000110000000000000000000	00000100100111	00000000000000000001100	00110000000000
000011000000000000000000	00001101101100	00000000000000000000110	01100000000000
000001100000000000000000	00000110110110	000000000111000000000000	10000000001110
000000110000000000000000	00000010010011	000000000011100000000000	00000000011100
000000011000000000000000	10000000000001	000000000001110000000000	00000000111000
000000001000000000000000	10000000000010	000000000000111000000000	00000001110000
000000000100000000000000	00000000000100	000000000000011100000000	00000011100000
000000000010000000000000	00000000001000	000000000000001110000000	00000111000000
000000000001000000000000	000000000010000	000000000000000111000000	00011100000000
000000000000100000000000	00000001000000	0000000000000000111000	00111000000000
000000000000010000000000	00000010000000	000000000000000001110	01110000000000
000000000000001000000000	000000100000000	0000000000000000000111	11100000000000

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000001001000000000	10000000010010	0000000001010000100000	10000100001010
0000000001000100000000	100000000100010	0000000001010000010000	10001000001010
0000000001000010000000	10000001000010	0000000001010000001000	10010000001010
0000000001000001000000	10000010000010	0000000001010000000100	10100000001010
0000000001000000100000	10000100000010	0000000001011010000000	10000001011010
0000000001000000010000	10001000000010	0000000001010101000000	10000010101010
0000000001000000001000	10010000000010	0000000001010010100000	10000101001010
0000000001000000000100	10100000000010	0000000001010001010000	10001010001010
0000000001001010000000	10000001010010	0000000001010000101000	10010100001010
0000000001000101000000	10000010100010	0000000001010000010100	10101000001010
0000000001000010100000	10000101000010	0000000001010000001010	11010000001010
0000000001000001010000	10001010000010	0000000001010000000101	00100000001010
0000000001000000101000	10010100000010	0000000001011100000000	10000000111010
0000000001000000010100	10101000000010	0000000001010110000000	10000001101010
0000000001000000001010	11010000000010	0000000001010011000000	10000011001010
0000000001000000000101	00100000000010	0000000001010001100000	10000110001010
0000000001001100000000	10000000110010	0000000001010000110000	10001100001010
0000000001000110000000	10000001100010	0000000001010000011000	10011000001010
0000000001000011000000	10000011000010	0000000001010000001100	10110000001010
0000000001000001100000	10000110000010	0000000001010000000110	11100000001010
0000000001000000110000	10001100000010	0000000001011110000000	10000001111010
0000000001000000011000	10011000000010	0000000001010111000000	10000011101010
0000000001000000001100	10110000000010	0000000001010011100000	10000111001010
0000000001000000000110	11100000000010	0000000001010001110000	10001110001010
0000000001001110000000	10000001110010	0000000001010000111000	10111000001010
0000000001000111000000	10000011100010	0000000001010000011100	10111000001010
0000000001000011100000	10000111000010	0000000001010000001110	11110000001010
0000000001000001110000	10001110000010	0000000001010000000111	01100000001010
0000000001000000111000	10011100000010	0000000001101000000000	10000000010110
0000000001000000011100	10111000000010	0000000001100100000000	10000000100110
0000000001000000001110	11110000000010	0000000001100010000000	10000001000110
0000000001000000000111	01100000000010	0000000001100001000000	10000010000110
0000000001011000000000	10000000011010	0000000001100000100000	10000100000110
0000000001010100000000	10000000101010	0000000001100000010000	10001000000110
0000000001010010000000	10000001001010	0000000001100000001000	10010000000110
0000000001010001000000	10000010001010	0000000001100000000100	10100000000110
0000000001010000100000	10000010001010	0000000001100000000010	10100000000110

Multi-block two repeated fixed burst error . . .

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000001101010000000	10000001010110	0000000001110000101000	10010100001110
0000000001100101000000	10000010100110	0000000001110000010100	10101000001110
0000000001100010100000	10000101000110	0000000001110000001010	11010000001110
0000000001100001010000	10001010000110	0000000001110000000101	00100000001110
0000000001100000101000	10010100000110	0000000001111100000000	10000000111110
0000000001100000010100	10101000000110	0000000001110110000000	10000001101110
0000000001100000001010	11010000000110	0000000001110011000000	10000011001110
0000000001100000000101	00100000000110	0000000001110001100000	10000110001110
0000000001101100000000	10000000110110	0000000001110000110000	10001100001110
0000000001100110000000	10000001100110	0000000001110000011000	10011000001110
0000000001100011000000	10000011000110	0000000001110000001100	10110000001110
0000000001100001100000	10000110000110	0000000001110000000110	11100000001110
0000000001100000110000	10001100000110	0000000001111110000000	10000001111110
0000000001100000011000	10011000000110	0000000001110111000000	10000011101110
0000000001100000001100	10110000000110	0000000001110011100000	10000111001110
0000000001100000000110	11100000000110	0000000001110001110000	10001110001110
0000000001101110000000	10000001110110	0000000001110000111000	10011100001110
0000000001100111000000	10000011100110	0000000001110000011100	10111000001110
0000000001100011100000	10000111000110	0000000001110000001110	11110000001110
0000000001100001110000	10001110000110	0000000001110000000111	01100000001110
0000000001100000111000	10011100000110	0000000000100100000000	00000000100100
0000000001100000011100	10111000000110	0000000000100010000000	00000001000100
0000000001100000001110	11110000000110	0000000000100001000000	00000010000100
0000000001100000000111	01100000000110	0000000000100000100000	00000100000100
0000000001111000000000	10000000011110	0000000000100000010000	00001000000100
0000000001110100000000	10000000101110	0000000000100000001000	00010000000100
0000000001110010000000	10000001001110	0000000000100000000100	00100000000100
0000000001110001000000	10000010001110	0000000000100101000000	00000010100100
0000000001110000100000	10000100001110	0000000000100010100000	00000101000100
0000000001110000010000	10001000001110	0000000000100001010000	00001010000100
0000000001110000001000	10010000001110	0000000000100000101000	00010100000100
0000000001110000000100	10100000001110	0000000000100000010100	00101000000100
0000000001110100000000	10000001011110	0000000000100000001010	01010000000100
0000000001110101000000	10000010101110	0000000000100000000101	10100000000100
0000000001110010100000	10000101001110	0000000000100110000000	00000001100100
0000000001110001010000	10001010001110	0000000000100011000000	00000011000100

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000000100001100000	000001100000100	0000000000101000111000	00011100010100
0000000000100000110000	000011000000100	0000000000101000011100	00111000010100
0000000000100000011000	000110000000100	0000000000101000001110	01110000010100
0000000000100000001100	001100000000100	0000000000101000000111	11100000010100
0000000000100000000110	011000000000100	0000000000110100000000	00000000101100
0000000000100111000000	00000011100100	0000000000110010000000	00000001001100
0000000000100011100000	00000111000100	0000000000110001000000	00000010001100
0000000000100001110000	00001110000100	0000000000110000100000	00000100001100
0000000000100000111000	00011100000100	0000000000110000010000	00001000001100
0000000000100000011100	00111000000100	0000000000110000001000	00010000001100
0000000000100000001110	01110000000100	0000000000110000000100	00100000001100
0000000000100000000111	11100000000100	0000000000110101000000	00000010101100
0000000000101100000000	00000000110100	0000000000110010100000	00000101001100
0000000000101010000000	00000001010100	0000000000110001010000	00001010001100
0000000000101001000000	00000010010100	0000000000110000101000	00010100001100
0000000000101000100000	00000100010100	0000000000110000010100	00101000001100
0000000000101000010000	00001000010100	0000000000110000001010	01010000001100
0000000000101000001000	00010000010100	0000000000110000000101	10100000001100
0000000000101000000100	00100000010100	0000000000110110000000	00000001101100
0000000000101101000000	00000010110100	0000000000110011000000	00000011001100
0000000000101010100000	00000101010100	0000000000110001100000	00000110001100
0000000000101001010000	00001010010100	0000000000110000110000	00001100001100
0000000000101000101000	00010100010100	0000000000110000011000	00011000001100
0000000000101000010100	00101000010100	0000000000110000001100	00110000001100
0000000000101000001010	01010000010100	0000000000110000000110	01100000001100
0000000000101000000101	10100000010100	0000000000110111000000	00000011101100
0000000000101110000000	00000001110100	0000000000110011100000	00000111001100
0000000000101011000000	00000011010100	0000000000110001110000	00001110001100
0000000000101001100000	00000110010100	0000000000110000111000	00011100001100
0000000000101000110000	00001100010100	0000000000110000011100	00111000001100
0000000000101000011000	00011000010100	0000000000110000001110	01110000001100
0000000000101000001100	00110000010100	0000000000110000000111	11100000001100
0000000000101000000110	01100000010100	0000000000111100000000	00000000111100
0000000000101111000000	00000011110100	0000000000111010000000	00000001011100
0000000000101011100000	00000111010100	0000000000111001000000	00000010011100
0000000000101001110000	00001110010100	0000000000111000100000	00000010001100

Multi-block two repeated fixed burst error . . .

Table 1 contd. . . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000000111000010000	00001000011100	0000000000010011000000	00000011001000
0000000000111000001000	00010000011100	0000000000010001100000	00000110001000
0000000000111000000100	00100000011100	0000000000010000110000	00001100001000
0000000000111101000000	00000010111100	0000000000010000011000	00011000001000
0000000000111010100000	00000101011100	0000000000010000001100	00110000001000
0000000000111001010000	00001010011100	0000000000010000000110	01100000001000
0000000000111000101000	00010100011100	0000000000010011100000	00000111001000
0000000000111000010100	00101000011100	0000000000010001110000	00001110001000
0000000000111000001010	01010000011100	0000000000010000111000	00011100001000
0000000000111000000101	10100000011100	0000000000010000011100	00111000001000
0000000000111110000000	00000001111100	0000000000010000001110	01110000001000
0000000000111011000000	00000011011100	0000000000010000000111	11100000001000
0000000000111001100000	00000110011100	0000000000010110000000	00000001101000
0000000000111000110000	00001100011100	0000000000010101000000	00000010101000
0000000000111000011000	00011000011100	0000000000010100100000	00000100101000
0000000000111000001100	00110000011100	0000000000010100010000	00001000101000
0000000000111111000000	00000011111100	0000000000010100001000	00010000101000
0000000000111011100000	00000111011100	0000000000010110100000	00000101101000
0000000000111001110000	00001110011100	0000000000010101010000	00001010101000
0000000000111000111000	00011100011100	0000000000010100101000	00010100101000
0000000000111000011100	00111000011100	0000000000010100010100	00101000101000
0000000000111000001110	01110000011100	0000000000010100001010	01010000101000
0000000000111000000111	11100000011100	0000000000010100000101	10100000101000
0000000000010010000000	00000001001000	0000000000010111000000	00000011101000
0000000000010001000000	00000010001000	0000000000010101100000	00000110101000
0000000000010000100000	00000100001000	0000000000010100110000	00001100101000
0000000000010000010000	00001000001000	0000000000010100011000	00011000101000
0000000000010000001000	00010000001000	0000000000010100001100	00110000101000
0000000000010000000100	00100000001000	0000000000010100000110	01100000101000
0000000000010010100000	00000101001000	0000000000010111100000	00000111101000
0000000000010001010000	00001010001000	0000000000010101110000	00001110101000
0000000000010000101000	00010100001000	0000000000010100111000	00011100101000
0000000000010000010100	00101000001000	0000000000010100011100	00111000101000
0000000000010000001010	01010000001000	0000000000010100001110	01110000101000
0000000000010000000101	10100000001000	0000000000010100000111	11100000101000

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000000011010000000	00000001011000	0000000000001111100000	00000011111000
0000000000011001000000	00000010011000	0000000000001110110000	00000110111000
0000000000011000100000	00000100011000	0000000000001110011000	00001100111000
0000000000011000010000	00001000011000	0000000000001110001100	00011000111000
0000000000011000001000	00010000011000	0000000000001110000110	00110000111000
0000000000011000000100	00100000011000	0000000000001110000011	01100000111000
0000000000011010100000	00000101011000	0000000000001111110000	00000111111000
0000000000011001010000	00001010011000	0000000000001110111000	00001110111000
0000000000011000101000	00010100011000	0000000000001110011100	00011100111000
0000000000011000010100	00101000011000	0000000000001110001110	00111000111000
0000000000011000001010	01010000011000	0000000000001110000111	01110000111000
0000000000011000000101	10100000011000	0000000000001110000011	11100000111000
0000000000011011000000	00000011011000	0000000000001001000000	00000010010000
0000000000011001100000	00000110011000	0000000000001000100000	00000100010000
0000000000011000110000	00001100011000	0000000000001000010000	00001000010000
0000000000011000011000	00011000011000	0000000000001000001000	00010000010000
0000000000011000001100	00110000011000	0000000000001000000100	00100000010000
0000000000011000000110	01100000011000	0000000000001001010000	00001010010000
0000000000011011100000	00000111011000	0000000000001000101000	00010100010000
0000000000011001110000	00001110011000	0000000000001000010100	00101000010000
0000000000011000111000	00011100011000	0000000000001000001010	01010000010000
0000000000011000011100	00111000011000	0000000000001000000101	10100000010000
0000000000011000001110	01110000011000	0000000000001001100000	00000110010000
0000000000011000000111	11100000011000	0000000000001000110000	00001100010000
0000000000011110000000	00000001111000	0000000000001000011000	00011000010000
0000000000011101000000	00000010111000	0000000000001000001100	00110000010000
0000000000011100100000	00000100111000	0000000000001000000110	01100000010000
0000000000011100010000	00001000111000	0000000000001001110000	00001110010000
0000000000011100001000	00010000111000	0000000000001000111000	00011100010000
0000000000011100000100	00100000111000	0000000000001000011100	00111000010000
0000000000011101010000	00000101111000	0000000000001000001110	01110000010000
0000000000011101010000	00001010111000	0000000000001000000111	11100000010000
0000000000011100101000	00010100111000	0000000000001011000000	00000011010000
0000000000011100010100	00101000111000	0000000000001010100000	00000101010000
0000000000011100001010	01010000111000	0000000000001010010000	00001001010000
0000000000011100000101	10100000111000	0000000000001010001000	00010001010000

Multi-block two repeated fixed burst error . . .

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000000001010000100	00100001010000	0000000000001111000000	00000011110000
0000000000001011010000	00001011010000	0000000000001110100000	00000101110000
0000000000001010101000	00010101010000	0000000000001110010000	00001001110000
0000000000001010010100	00101001010000	0000000000001110001000	00010001110000
0000000000001010001010	01010001010000	0000000000001110000100	00100001110000
0000000000001010000101	10100001010000	0000000000001111010000	00001011110000
0000000000001011100000	00000111010000	0000000000001110101000	00010101110000
0000000000001010110000	00001101010000	0000000000001110010100	00101001110000
0000000000001010011000	00011001010000	0000000000001110001010	01010001110000
0000000000001010001100	00110001010000	0000000000001110000101	10100001110000
0000000000001010000110	01100001010000	0000000000001111100000	00000111110000
0000000000001011110000	00001111010000	0000000000001110110000	00001101110000
0000000000001010111000	00011101010000	0000000000001110011000	00011001110000
0000000000001010011100	00111001010000	0000000000001110001100	00110001110000
0000000000001010001110	01110001010000	0000000000001110000110	01100001110000
0000000000001010000111	11100001010000	0000000000001111110000	00001111110000
0000000000001101000000	00000010110000	0000000000001110111000	00011101110000
0000000000001100100000	00000100110000	0000000000001110011100	00111001110000
0000000000001100010000	00001000110000	0000000000001110001110	01110001110000
0000000000001100001000	00010000110000	0000000000001110000111	11100001110000
0000000000001100000100	00100000110000	0000000000001001000000	00000100100000
0000000000001101010000	00001010110000	0000000000001000100000	00001000100000
0000000000001100101000	00010100110000	0000000000001000010000	00010000100000
0000000000001100010100	00101000110000	0000000000001000001000	00100000100000
0000000000001100001010	01010000110000	0000000000001001010000	00010100100000
0000000000001100000101	10100000110000	0000000000001000101000	00101000100000
0000000000001101100000	00000110110000	0000000000001000010100	01010000100000
0000000000001100110000	00001100110000	0000000000001000001010	10100000100000
0000000000001100011000	00011000110000	0000000000001001100000	00001100100000
0000000000001100001100	00110000110000	0000000000001000110000	00011000100000
0000000000001100000110	01100000110000	0000000000001000011000	00110000100000
0000000000001101110000	00001110110000	0000000000001000001110	01100000100000
0000000000001100111000	00011100110000	0000000000001001110000	00011100100000
0000000000001100011100	00111000110000	0000000000001000111000	00111000100000
0000000000001100001110	01110000110000	0000000000001000011110	01110000100000
0000000000001100000111	11100000110000	0000000000001000001111	11100000100000

Table 1 contd. . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
000000000000101100000	00000110100000	0000000000000111101000	00010111100000
000000000000101010000	00001010100000	0000000000000111010100	00101011100000
000000000000101001000	00010010100000	0000000000000111001010	01010011100000
000000000000101000100	00100010100000	0000000000000111000101	10100011100000
000000000000101101000	00010110100000	0000000000000111110000	00001111100000
000000000000101010100	00101010100000	0000000000000111011000	00011011100000
000000000000101001010	01010010100000	0000000000000111001100	00110011100000
000000000000101000101	10100010100000	0000000000000111000110	01100011100000
000000000000101110000	00001110100000	0000000000000111111000	00011111100000
000000000000101011000	00011010100000	0000000000000111011100	00111011100000
000000000000101001100	00110010100000	0000000000000111001110	01110011100000
000000000000101000110	01100010100000	0000000000000111000111	11100011100000
000000000000101111000	00011110100000	000000000000010010000	00001001000000
000000000000101011100	00111010100000	000000000000010001000	00010001000000
000000000000101001110	01110010100000	000000000000010000100	00100001000000
000000000000101000111	11100010100000	000000000000010010100	00101001000000
000000000000110100000	00000101100000	000000000000010001010	01010001000000
000000000000110010000	00001001100000	000000000000010000101	10100001000000
000000000000110001000	00010001100000	000000000000010011000	00011001000000
000000000000110000100	00100001100000	000000000000010001100	00110001000000
000000000000110101000	00010101100000	000000000000010000110	01100001000000
00000000000011010100	00101001100000	000000000000010011100	00111001000000
000000000000110001010	01010001100000	000000000000010001110	01110001000000
000000000000110000101	10100001100000	000000000000010000111	11100001000000
000000000000110110000	00001101100000	000000000000010110000	00001101000000
000000000000110011000	00011001100000	000000000000010101000	00010101000000
000000000000110001100	00110001100000	000000000000010100100	00100101000000
000000000000110000110	01100001100000	000000000000010110100	00101101000000
000000000000110111000	00011101100000	000000000000010101010	01010101000000
000000000000110011100	00111001100000	000000000000010100101	10100101000000
000000000000110001110	01110001100000	000000000000010111000	00011101000000
000000000000110000111	11100001100000	000000000000010101100	00110101000000
000000000000111100000	00000111100000	000000000000010100110	01100101000000
000000000000111010000	00001011100000	000000000000010111100	00111101000000
000000000000111001000	00010011100000	000000000000010101110	01110101000000
000000000000111000100	00100011100000	000000000000010100111	11100101000000

Multi-block two repeated fixed burst error . . .

Table 1 contd. . . .
Error Patterns - Syndrome vectors

Error Vector	Syndrome	Error vector	Syndrome
0000000000000011010000	00001011000000	0000000000000001011100	00111010000000
0000000000000011001000	00010011000000	0000000000000001010110	01101010000000
0000000000000011000100	00100011000000	0000000000000001011110	01111010000000
0000000000000011010100	00101011000000	0000000000000001010111	11101010000000
0000000000000011001010	01010011000000	0000000000000001101000	00010110000000
0000000000000011000101	10100011000000	0000000000000001100100	00100110000000
0000000000000011011000	00011011000000	0000000000000001101010	01010110000000
0000000000000011001100	00110011000000	0000000000000001100101	10100110000000
0000000000000011000110	01100011000000	0000000000000001101100	00110110000000
0000000000000011011100	00111011000000	0000000000000001100110	01100110000000
0000000000000011001110	01110011000000	0000000000000001101110	01110110000000
0000000000000011000111	11100011000000	0000000000000001100111	11100110000000
0000000000000011110000	00001111000000	0000000000000001111000	00011110000000
0000000000000011101000	00010111000000	0000000000000001110100	00101110000000
0000000000000011100100	00100111000000	0000000000000001111010	01011110000000
0000000000000011101010	00101111000000	0000000000000001110101	10101110000000
0000000000000011101010	01010111000000	0000000000000001111100	00111110000000
0000000000000011100101	10100111000000	0000000000000001110110	01101110000000
0000000000000011111000	00011111000000	0000000000000001111110	01111110000000
0000000000000011101100	00110111000000	0000000000000001110111	11101110000000
0000000000000011100110	01100111000000	0000000000000001001000	00100100000000
0000000000000011111100	00111111000000	0000000000000001001010	10100100000000
0000000000000011101110	01110111000000	0000000000000001001110	01100100000000
0000000000000011100111	11100111000000	0000000000000001001111	11100100000000
0000000000000010010000	00010010000000	0000000000000001011000	00110100000000
0000000000000010001000	00100010000000	0000000000000001011010	10110100000000
0000000000000010010100	01010010000000	0000000000000001011110	01110100000000
0000000000000010001010	10100010000000	0000000000000001011111	11110100000000
0000000000000010011000	00110010000000	0000000000000001101000	00101100000000
0000000000000010001100	01100010000000	0000000000000001101010	10101100000000
0000000000000010011100	01110010000000	0000000000000001101110	01101100000000
0000000000000010001111	11100010000000	0000000000000001101111	11101100000000
0000000000000010110000	00011010000000	0000000000000001111000	00111100000000
0000000000000010101000	00101010000000	0000000000000001111010	10111100000000
0000000000000010110100	01011010000000	0000000000000001111110	01111100000000
0000000000000010101010	10101010000000	0000000000000001111111	11111100000000

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Generalized Continuous Integral Wavelet Transform and Its Properties

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Abstract

In this paper, the generalized admissibility condition has been obtained and corresponding generalized continuous integral wavelet transforms have been defined. Two new theorems on generalized continuous integral wavelet transforms are established.

Keywords: $L^2(\mathbb{R})$ -space, basic wavelet, generalized admissibility condition, generalized continuous wavelet transforms.

2020 AMS subject classifications: 40A30, 42C15. ¹

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¹Received on June 3, 2022. Accepted on December 16, 2022. Published on December 30, 2022.
doi: 10.23755/rm.v43i0.793. ISSN: 1592-7415. eISSN: 2282-8214. ©Manoj Kumar. This paper is published under the CC-BY licence agreement.

1 Introduction

Wavelet Analysis has wide applications in Science, Engineering and Technology. Its applications have very important role in data analysis, signals and image processions. Wavelet transforms and wavelet expansions are two mean features of this analysis.

A function $\varphi \in L^2(\mathbb{R})$ is called a basic wavelet if it make certain the admissibility condition,

$$C_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$. This admissibility condition is extended as

$$C_{\varphi_1, \varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1(\xi)} \hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty.$$

We notice that if $\varphi_1 = \varphi_2 = \varphi$ then C_{φ_1, φ_2} reduces to C_φ .

In this paper, the admissibility condition C_{φ_1, φ_2} has generalized and new more general admissibility condition $C_{c\varphi_1, d\varphi_2}$ has been obtained as following

$$C_{c\varphi_1, d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)} \hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi < \infty,$$

where $1 \leq c < 2$, $1 \leq d < 2$.

It's worth noting that

- (1) if $c = d = 1$ then $C_{c\varphi_1, d\varphi_2}$ reduces to C_{φ_1, φ_2} ;
- (2) if $c = d = 1$, $\varphi_1 = \varphi_2 = \varphi$ then $C_{c\varphi_1, d\varphi_2}$ reduces to C_φ .

To conduct a more in-depth investigation in this area, the generalized integral wavelet transform is defined and inverse integral wavelet transform is obtained with the help of generalized admissibility condition. The aim of this paper is also to study the property of generalized continuous integral wavelet transform and a function K of $L^p(\mathbb{R})$, $1 < p < \infty$.

2 Definitions and Preliminaries

Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ and the Fourier transforms of φ_1 and φ_2 be

$$\hat{\varphi}_1(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \varphi_1(x) dx \text{ and } \hat{\varphi}_2(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \varphi_2(x) dx$$

respectively. Suppose that φ_1 and φ_2 satisfy the condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1(\xi)\hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty, \quad (\text{Daubechies [1992]}).$$

Define

$$\varphi_i^{a,b}(x) = |a|^{-\frac{1}{2}} \varphi_i \left(\frac{x-b}{a} \right), \quad i = 1, 2.$$

$\varphi_i^{a,b}$, $i = 1, 2$, are obtained by first dilating by $a \neq 0$ and then translating by b . The normalization has been taken in such a way that $\|\varphi_i^{(a,b)}\|_{L^2(\mathbb{R})} = \|\varphi_i\|_{L^2(\mathbb{R})}$. We assume that $\|\varphi_i\|_2 = 1$.

The continuous integral wavelet transform of $f \in L^2(\mathbb{R})$ by φ_1 is defined by

$$\begin{aligned} (W_{\varphi_1} f)(a, b) &= |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \overline{\varphi_1 \left(\frac{x-b}{a} \right)} dx \\ &= \int_{-\infty}^{\infty} f(x) \overline{\varphi_1^{a,b}(x)} dx. \\ &= \langle f, \varphi_1^{a,b} \rangle. \end{aligned}$$

The admissibility condition is generalized as; if

$$C_{\varphi_1, \varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1(\xi)} \hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty, \quad (1)$$

then for any $f \in L^2(\mathbb{R})$, we have the L^2 - inversion formula

$$f(x) = C_{\varphi_1, \varphi_2}^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{\varphi_1} f)(a, b) \varphi_2^{(a,b)}(x) \frac{dad b}{a^2}. \quad (2)$$

Define the functions ${}^c\varphi_1, {}^d\varphi_2 \in L^2(\mathbb{R})$ as follows

$${}^c\varphi_1(x) = {}^c_1\varphi_{b,a}(x) = \left(\frac{|a|}{c} \right)^{-\frac{1}{2}} \varphi_1 \left(\frac{cx-b}{a} \right) \quad (3)$$

and

$${}^d\varphi_2(x) = {}^d_2\varphi_{b,a}(x) = \left(\frac{|a|}{d} \right)^{-\frac{1}{2}} \varphi_2 \left(\frac{dx-b}{a} \right), \quad \text{where } 1 \leq c, d < 2. \quad (4)$$

We observe that

$$\|{}^c\varphi_1\|_{L^2(\mathbb{R})} = \|\varphi_1\|_{L^2(\mathbb{R})} = 1$$

and

$$\|{}^d\varphi_2\|_{L^2(\mathbb{R})} = \|\varphi_2\|_{L^2(\mathbb{R})} = 1.$$

The Fourier transform of ${}^c\varphi_1$ is given by

$$\begin{aligned} {}^c\hat{\varphi}_1(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} {}^c\varphi_{b,a}(x) dx \\ &= \int_{-\infty}^{\infty} e^{-i\xi x} \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \varphi_1\left(\frac{cx-b}{a}\right) dx, \quad \frac{cx-b}{a} = u \\ &= \int_{-\infty}^{\infty} e^{-i\xi\left(\frac{b+au}{c}\right)} \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \varphi_1(u) \frac{a}{c} du \\ &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{-i\left(\frac{b}{c}\right)\xi} \int_{-\infty}^{\infty} e^{-i\left(\frac{a\xi}{c}\right)u} \varphi_1(u) du \\ &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{-i\left(\frac{b}{c}\right)\xi} \hat{\varphi}_1\left(\frac{a\xi}{c}\right). \end{aligned} \quad (5)$$

In the same way, the Fourier transform of ${}^d\varphi_2$ is given by

$${}^d\hat{\varphi}_2(\xi) = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} e^{-i\left(\frac{b}{d}\right)\xi} \hat{\varphi}_2\left(\frac{a\xi}{d}\right). \quad (6)$$

In this paper, the admissibility condition C_{φ_1, φ_2} has generalized and new more possible general admissibility condition $C_{c\varphi_1, d\varphi_2}$ has been obtained as following

$$C_{c\varphi_1, d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1\left(\frac{\xi}{cd}\right) \hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi < \infty,$$

where $1 \leq c < 2$, $1 \leq d < 2$.

It is important to note that

- (1) if $c = d = 1$ then $C_{c\varphi_1, d\varphi_2}$ reduces to C_{φ_1, φ_2} ,
- (2) if $c = d = 1$, $\varphi_1 = \varphi_2 = \varphi$ then $C_{c\varphi_1, d\varphi_2}$ reduces to

$$C_{\varphi} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty \quad (7)$$

which is the admissibility condition for a function φ to be a wavelet. If $\varphi \in L^1(\mathbb{R})$, then $\hat{\varphi}$ is continuous and the condition (7) is satisfied only if

$$\hat{\varphi}(0) = 0 \text{ or } \int_{-\infty}^{\infty} \varphi(x) dx = 0.$$

There are following two advantages of selecting two different functions φ_1 and φ_2 :

Generalized Continuous Integral Wavelet Transform and Its Properties

1. φ_1 & φ_2 may have different aspects;
2. $C_{\varphi_1, \varphi_2} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1(\xi)\hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty$ even φ_1 & φ_2 both may not satisfy admissibility conditions.

Similarly

- (a) ${}^c\varphi_1$ & ${}^d\varphi_2$ may have different aspects;
- (b) $C_{{}^c\varphi_1, {}^d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1(\frac{\xi}{cd})\hat{\varphi}_2(\frac{\xi}{d})|}{\sqrt{cd}|\xi|} d\xi < \infty$ even ${}^c\varphi_1$ & ${}^d\varphi_2$ both may not satisfy admissibility conditions.

The continuous integral wavelet transform related to the basic wavelet ${}^c\varphi_1$ is given by

$$\begin{aligned}
 (W_{{}^c\varphi_1} f)(b, a) &= \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \overline{{}^c\varphi_1\left(\frac{cx-b}{a}\right)} dx \\
 &= \langle f, {}^c\varphi_1 \rangle \\
 &= \frac{1}{2\pi} \langle \hat{f}, {}^c\hat{\varphi}_1 \rangle \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) {}^c\hat{\varphi}_{b,a}(\xi) d\xi \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{{}^c\hat{\varphi}_{b,a}(\xi)} d\xi \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{i(\frac{b}{c})\xi} \overline{{}^c\hat{\varphi}_1\left(\frac{a\xi}{c}\right)} d\xi, \quad (\text{by (5)}) \\
 &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{b}{c})\xi} \hat{f}(\xi) \overline{{}^c\hat{\varphi}_1\left(\frac{a\xi}{c}\right)} d\xi, \\
 \text{i.e. } (W_{{}^c\varphi_1} f)(b, a) &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} \mathfrak{F}^{-1}[\hat{f}(\xi) \overline{{}^c\hat{\varphi}_1\left(\frac{a\xi}{c}\right)}] \left(\frac{b}{c}\right).
 \end{aligned}$$

Thus the Fourier transform of $W_{{}^c\varphi_1}$ is given by

$$\mathfrak{F}[(W_{{}^c\varphi_1} f)(b, a)] = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} \hat{f}(\xi) \overline{{}^c\hat{\varphi}_1\left(\frac{a\xi}{c}\right)}. \quad (8)$$

Similarly, the integral wavelet transform related to the basic wavelet ${}^d\varphi_2$ is given

by

$$\begin{aligned} (W_{d\varphi_2}g)(b, a) &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{b}{d})\xi} \hat{g}(\xi) \hat{\varphi}_2\left(\frac{a\xi}{d}\right) d\xi \\ &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} \mathfrak{F}^{-1}[\hat{g}(\xi) \hat{\varphi}_2\left(\frac{a\xi}{d}\right)]\left(\frac{b}{d}\right). \end{aligned}$$

Therefore

$$\mathfrak{F}[(W_{d\varphi_2}g)(b, a)] = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} \overline{\hat{g}(\xi) \hat{\varphi}_2\left(\frac{a\xi}{d}\right)}. \quad (9)$$

Throughout this paper, we choose ${}^c\varphi_1$ and ${}^d\varphi_2$ such that

- (i) ${}^c\varphi_1, {}^d\varphi_2 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$;
- (ii) $x {}^c\varphi_1, x {}^d\varphi_2 \in L^1(\mathbb{R})$;
- (iii) ${}^d\varphi_2 \in L_2^s(\mathbb{R})$, $s > 0$,

where $L_2^s(\mathbb{R})$ is the Liouville space with the norm

$$\|\varphi\|_{L_2^s(\mathbb{R})} = \|(1 + |\xi|)^{\frac{s}{2}} \hat{\varphi}(\xi)\|_{L^2(\mathbb{R})}, \quad (\text{Nikol'skii [1995]}).$$

In this paper, two new theorems has been established in the following forms:

Theorem 2.1. *Let ${}^c\varphi_1$ and ${}^d\varphi_2$ be the basic wavelets which define the integral wavelet transforms $W_{c\varphi_1}$ and $W_{d\varphi_2}$ respectively. Then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c\varphi_1}f)(b, a) \overline{(W_{d\varphi_2}g)(b, a)} \frac{dad b}{a^2} = C_{c\varphi_1, d\varphi_2} \langle f, g \rangle$$

for all $f, g \in L^2(\mathbb{R})$, where

$$C_{c\varphi_1, d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1\left(\frac{\xi}{cd}\right) \hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi.$$

Furthermore, for any $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ at which f is continuous,

$$f(x) = \frac{1}{C_{c\varphi_1, d\varphi_2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c\varphi_1}f)(b, a) {}^d\varphi_2\left(\frac{x}{b, a}\right) \frac{dad b}{a^2}.$$

Proof.

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c\varphi_1} f)(b, a) \overline{(W_{d\varphi_2} g)(b, a)} \frac{dad b}{a^2} \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dad b}{a^2} (W_{c\varphi_1} f)(b, a) \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\frac{b}{d})\xi} \overline{\hat{g}(\xi)} \hat{\varphi}_2\left(\frac{a\xi}{d}\right) d\xi \\
 &= \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} \frac{1}{2\pi} \overline{\hat{g}(\xi)} \hat{\varphi}_2\left(\frac{a\xi}{d}\right) d\xi \int_{-\infty}^{\infty} (W_{c\varphi_1} f)(b, a) e^{-i(\frac{b}{d})\xi} db \\
 &= \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} \frac{1}{2\pi} \overline{\hat{g}(\xi)} \hat{\varphi}_2\left(\frac{a\xi}{d}\right) \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} \overline{\hat{f}(\xi)} \hat{\varphi}_1\left(\frac{a\xi}{cd}\right) d\xi \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi \int_{-\infty}^{\infty} \frac{\overline{\hat{\varphi}_1\left(\frac{a\xi}{cd}\right)} \hat{\varphi}_2\left(\frac{a\xi}{d}\right)}{\sqrt{cd}|a|} da, \quad a\xi = u \\
 &= \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle \int_{-\infty}^{\infty} \frac{\overline{\hat{\varphi}_1\left(\frac{u}{cd}\right)} \hat{\varphi}_2\left(\frac{u}{d}\right)}{\sqrt{cd}|u|} du \\
 &= \langle f, g \rangle \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)} \hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi \\
 &= C_{c\varphi_1, d\varphi_2} \langle f, g \rangle,
 \end{aligned}$$

where

$$\begin{aligned}
 C_{c\varphi_1, d\varphi_2} &= \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)} \hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi, \quad \frac{\xi}{d} = v \\
 &= \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{v}{c}\right)} \hat{\varphi}_2(v)|}{\sqrt{cd}|v|} d v \\
 &= \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{c}\right)} \hat{\varphi}_2(\xi)|}{\sqrt{cd}|\xi|} d\xi.
 \end{aligned} \tag{10}$$

This is the generalized admissibility condition for the functions ${}^c\varphi_1$ and ${}^d\varphi_2$ to be the basic wavelets.

For $c = d = 1$ and ${}^c\varphi_1 = {}^d\varphi_2 = \varphi$, the condition (10) reduces to the condition

$$C_{\varphi} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi$$

which is the admissibility condition for the single function φ to be a wavelet, (Chui [1992]).

Furthermore,

$$\begin{aligned}
 C_{c_{\varphi_1}, d_{\varphi_2}} \langle f, g \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) \overline{(W_{d_{\varphi_2}} g)(b, a)} \frac{dad b}{a^2} \\
 \Rightarrow \langle f, g \rangle &= \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) \overline{\left(\left(\frac{|a|}{d} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(x) \varphi \left(\frac{dx-b}{a} \right) dx \right)} \frac{dad b}{a^2} \\
 &= \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) \left(\int_{-\infty}^{\infty} \overline{g(x)} {}^d_2 \varphi_{b,a}(x) dx \right) \frac{dad b}{a^2} \\
 &= \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) {}^d_2 \varphi_{b,a}(x) \overline{g(x)} \frac{dad b}{a^2} dx \\
 &= \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) {}^d_2 \varphi_{b,a}(x) \frac{dad b}{a^2}, g \rangle, \forall f, g \in L^2(\mathbb{R}).
 \end{aligned}$$

Therefore

$$f(x) = \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{c_{\varphi_1}} f)(b, a) {}^d_2 \varphi_{b,a}(x) \frac{dad b}{a^2}.$$

The Theorem 2.1 has thus been proven. $\square$

Theorem 2.2. If K is a function in $L^p(\mathbb{R})$, $1 < p < \infty$ such that

$$\hat{K}(\xi) = \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{|a| \geq |\xi|} {}^d \hat{\varphi}_2(a) \overline{{}^c \hat{\varphi}_1(a)} \frac{da}{|a|}. \quad (11)$$

Then it satisfies the following properties:

- (i) K is continuous and bounded.
- (ii) $|K(x)| \leq \frac{C}{1+\alpha x^2}$, where $0 < C < \infty$, $1 \leq \alpha < 2$.
- (iii) $\|K\|_q = O(1)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof.

- (i) Define the function K which Fourier transform is given by

$$\hat{K}(\xi) = \frac{1}{C_{c_{\varphi_1}, d_{\varphi_2}}} \int_{|a| \geq |\xi|} {}^d \hat{\varphi}_2(a) \overline{{}^c \hat{\varphi}_1(a)} \frac{da}{|a|}.$$

Therefore

$$\begin{aligned}
 |\hat{K}(\xi)| &\leq \frac{1}{|C_{c_{\varphi_1}, d_{\varphi_2}}|} \int_{|a| \geq |\xi|} |{}^d \hat{\varphi}_2(a)| |{}^c \hat{\varphi}_1(a)| \frac{da}{|a|} \\
 &< \infty
 \end{aligned}$$

i.e.,

$$|\hat{K}(\xi)| < C.$$

Also by Cauchy-Schwartz's inequality, we have

$$\begin{aligned} |\hat{K}(\xi)| &\leq \frac{1}{C_{c\varphi_1, d\varphi_2}} \left(\int_{-\infty}^{\infty} |^d \hat{\varphi}_2(a)|^2 da \right)^{\frac{1}{2}} \left(\int_{|a| \geq |\xi|} \frac{|^c \hat{\varphi}_1(a)|^2}{|a|^2} da \right)^{\frac{1}{2}} \\ &= \frac{1}{C_{c\varphi_1, d\varphi_2}} \left(\int_{|a| \geq |\xi|} \left(\frac{|^c \hat{\varphi}_1(a)|}{|a|^{1+s}} \right)^2 da \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |a|^{2s} |^d \hat{\varphi}_2(a)|^2 da \right)^{\frac{1}{2}} \\ &\leq \frac{1}{C_{c\varphi_1, d\varphi_2}} \frac{1}{|\xi|^{1+s}} \left(\int_{|a| \geq |\xi|} |^c \hat{\varphi}_1(a)|^2 da \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} (1 + |a|^2)^s |^d \hat{\varphi}_2(a)|^2 da \right)^{\frac{1}{2}} \\ &\leq C |\xi|^{-1-s} \| ^c \hat{\varphi}_1 \|_{L^2(\mathbb{R})} \| ^d \hat{\varphi}_2 \|_{L^2_s(\mathbb{R})} \\ &\leq C (1 + |\xi|)^{-1-s}. \end{aligned}$$

Thus the function K is continuous and bounded.

(ii) We have

$$\begin{aligned} x^2 |K(x)| &\leq \int_{-\infty}^{\infty} |\hat{K}''(\xi)| d\xi \\ \Rightarrow \alpha x^2 |K(x)| &\leq \int_{-\infty}^{\infty} |\alpha \hat{K}''(\xi)| d\xi, \quad 1 < \alpha < 2. \end{aligned} \quad (12)$$

Since the function K is bounded, therefore, to prove our result (ii), it is sufficient to prove that the right hand side of above inequality (12) is bounded.

The Fourier transform of $^c \varphi_1$ is given by

$$^c \hat{\varphi}_1(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} ^c \varphi_1(x) dx.$$

Differentiating both sides w.r.t. ξ and applying Leibnitz's rule for differentiation under the sine of integral, we have

$$\begin{aligned} \frac{d}{d\xi} ^c \hat{\varphi}_1(\xi) &= ^c \hat{\varphi}_1'(\xi) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi} e^{-i\xi x} ^c \varphi_1(x) dx \\ &= \int_{-\infty}^{\infty} e^{-i\xi x} (-ix) ^c \varphi_1(x) dx \\ &= -i \int_{-\infty}^{\infty} e^{-i\xi x} (x ^c \varphi_1(x)) dx. \end{aligned}$$

Since $x \text{ } ^c\varphi_1 \in L^1(\mathbb{R})$ therefore ${}^c\hat{\varphi}_1$ is differentiable. Similarly ${}^d\hat{\varphi}_2$ is also differentiable.

Since

$$\hat{K}(\xi) = \frac{1}{C_{c\varphi_1, d\varphi_2}} \int_{|a| \geq |\xi|} {}^d\hat{\varphi}_2(a) \overline{{}^c\hat{\varphi}_1(a)} \frac{da}{|a|}.$$

Applying the above process again, we have

$$\hat{K}'(\xi) = \frac{1}{C_{c\varphi_1, d\varphi_2}} \left(\frac{{}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1(-\xi)}}{(-\xi)} - \frac{{}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1(\xi)}}{\xi} \right).$$

Again differentiating w. r. t. ξ , we have

$$\begin{aligned} \hat{K}''(\xi) &= \frac{1}{C_{c\varphi_1, d\varphi_2}} \left[\frac{1}{(-\xi)^2} \left\{ (-\xi) \left(- {}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1'(-\xi)} - {}^d\hat{\varphi}_2'(-\xi) \overline{{}^c\hat{\varphi}_1(-\xi)} \right) \right. \right. \\ &\quad \left. \left. + {}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1(-\xi)} \right\} - \frac{1}{\xi^2} \left\{ \xi \left({}^d\hat{\varphi}_2'(\xi) \overline{{}^c\hat{\varphi}_1(\xi)} + {}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1'(\xi)} \right) \right. \right. \\ &\quad \left. \left. - {}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1(\xi)} \right\} \right] \\ &= \frac{1}{C_{c\varphi_1, d\varphi_2}} \left[\left(\frac{{}^c\hat{\varphi}_1(-\xi) {}^d\hat{\varphi}_2'(-\xi) + {}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1'(-\xi)}}{\xi} \right) + \frac{\overline{{}^c\hat{\varphi}_1(-\xi)} {}^d\hat{\varphi}_2(-\xi)}{\xi^2} \right. \\ &\quad \left. - \left(\frac{\overline{{}^c\hat{\varphi}_1(\xi)} {}^d\hat{\varphi}_2'(\xi) + {}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1'(\xi)}}{\xi} \right) + \frac{\overline{{}^c\hat{\varphi}_1(\xi)} {}^d\hat{\varphi}_2(\xi)}{\xi^2} \right] \\ &= \frac{1}{C_{c\varphi_1, d\varphi_2}} \left[\frac{1}{\xi^2} \left({}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1(\xi)} + {}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1(-\xi)} \right) \right. \\ &\quad \left. - \frac{1}{\xi} \left\{ \left({}^d\hat{\varphi}_2'(\xi) \overline{{}^c\hat{\varphi}_1(\xi)} + {}^d\hat{\varphi}_2(\xi) \overline{{}^c\hat{\varphi}_1'(\xi)} \right) \right. \right. \\ &\quad \left. \left. - \left({}^d\hat{\varphi}_2'(-\xi) \overline{{}^c\hat{\varphi}_1(-\xi)} + {}^d\hat{\varphi}_2(-\xi) \overline{{}^c\hat{\varphi}_1'(-\xi)} \right) \right\} \right]. \end{aligned}$$

Since ${}^c\hat{\varphi}_1(0) = {}^d\hat{\varphi}_2(0) = 0$ and ${}^c\hat{\varphi}_1', {}^d\hat{\varphi}_2'$ are continuous and bounded, using Taylor's formula at the origin, we have

$$|{}^c\hat{\varphi}_1(\xi)| \leq C|\xi|$$

and

$$|{}^d\hat{\varphi}_2(\xi)| \leq C|\xi|.$$

Using the above inequalities and the boundedness of the functions ${}^c\hat{\varphi}_1', {}^d\hat{\varphi}_2'$, we have

$$\int_{|\xi| < 1} |\alpha \hat{K}''(\xi)| d\xi \leq C. \quad (13)$$

Generalized Continuous Integral Wavelet Transform and Its Properties

Again, since ${}^c\hat{\varphi}_1$ and ${}^d\hat{\varphi}_2$ are in the space $L^2(\mathbb{R})$ therefore using the Cauchy-Schwartz inequality, we have

$$\int_{|\xi|>1} |\alpha \hat{K}''(\xi)| d\xi \leq C. \quad (14)$$

By using the inequalities (12), (13) and (14), the proof of (ii) of the Theorem 2.2 has been completed.

(iii)

$$\begin{aligned} \|K\|_q^q &= \int_{-\infty}^{\infty} |K(x)|^q dx \leq \int_{-\infty}^{\infty} \frac{C^q}{(1 + \alpha x^2)^q} dx, \quad 1 < \alpha < 2 \\ &\leq 2C^q \int_0^{\infty} \frac{dx}{(1 + \alpha x^2)^q}, \quad 1 < q < \infty \\ &\leq 2C^q \int_0^{\infty} \frac{dx}{(1 + x^2)^q} \\ &= 2C^q \left(\int_0^1 \frac{dx}{(1 + x^2)^q} + \int_1^{\infty} \frac{dx}{(1 + x^2)^q} \right) \\ &\leq 2C^q \left(\int_0^1 \frac{dx}{(1 + x^2)^q} + \int_1^{\infty} \frac{dx}{x^{2q}} \right). \end{aligned}$$

Since $\int_0^1 \frac{dx}{(1+x^2)^q}$ and $\int_1^{\infty} \frac{dx}{x^{2q}}$ are convergent at 0 and ∞ respectively. Therefore

$$\|K\|_q^q = O(1).$$

Consequently

$$\|K\|_q = O(1).$$

In this way, the proof of the Theorem 2.2 has been completed. $\square$

3 Conclusions

The generalized integral wavelet transform is defined and inverse integral wavelet transform is obtained with the help of generalized admissibility condition in the form of Theorem 2.1. Also, the property of generalized continuous integral wavelet transform and a function K of $L^p(\mathbb{R})$, $1 < p < \infty$, have been studied in Theorem 2.2.

Acknowledgements

This paper and the research behind it would not have been possible without the exceptional support of my supervisor, Prof. Shyam Lal. I am also grateful for the insightful comments offered by the anonymous peer reviewers at this paper.

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Theta open sets in N -topology

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Abstract

This article elaborates about a novel approach on defining θ -open sets and its properties in N -topological space. We establish various continuous maps and discuss the relationship with θ -continuous maps using N -topology. We procure important results in θ -irresolute maps by θ open sets. Finally, we develop the idea of θ -connectedness properties and some new type of separation axioms in this space.

Keywords: N -topology, $N\tau\theta$ -open, $N^\#\theta$ -irresolute, $N\tau\theta$ -connected, $N\tau\theta$ -hausdorff, $N\tau\theta$ -normal, $N\tau\theta$ -regular.

2010 AMS subject classification: 54A05, 54A10, 54C08, 54D05.[‡]

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[‡] Received on June 15, 2022. Accepted on October 23, 2022. Published on December 30, 2022.

doi:10.23755/rm.v43i0.797. ISSN: 1592-7415. eISSN: 2282-8214. ©Loyala Foresith Spencer et al. This paper is published under the CC-BY licence agreement.

1. Introduction

J.C. Kelly [1] introduced bitopological space and its essential role to study the shapes of objects. In topological spaces, The notion of α -open sets was initiated by Njastad [2] and θ -open sets, θ -closed sets and θ -closure are established by Velicko [3]. Lellis Thivagar et al. [4] originated the structure of N -topology and its continuous properties, weak forms [5] which are all helps to form more topologies on a non-empty set.

Lellis Thivagar et al. [6] introduced $N\tau\theta$ -open sets and $N\tau\theta$ -interior concepts in N -topology using $N\tau\theta$ -closed sets. In this article, we explored θ -open sets in N -topological space in a different way and introduced some other properties. We defined its continuous maps and investigated the relationship with other continuous maps. Also, we derive important results in θ -irresolute maps by θ -open sets. Further, introduced the concept of θ -connectedness and examined their properties. Finally, some sorts of separation axioms in N -topological space were discussed.

2. Preliminaries

In this part, we recollect some definitions and results which are beneficial in the sequent.

Definition 2.1 [1] A bitopological space (Z, P, Q) is a set Z with two topologies, P and Q on it.

Definition 2.2 [2] If Z is a non empty set and $\tau_1, \tau_2, \tau_3, \dots, \tau_N$ are N arbitrary topologies defined on Z and the collection

$N\tau = \{P \subseteq Z : P = (\bigcup_{i=1}^n A_i) \cup (\bigcap_{i=1}^n B_i), A_i, B_i \in \tau_i\}$ persuades the axioms:

- (i) $Z, \emptyset \in N\tau$
- (ii) $\bigcup_{i=1}^{\infty} P_i \in N\tau$ for all $\{P_i\}_{i=1}^{\infty} \in N\tau$
- (iii) $\bigcap_{i=1}^n P_i \in N\tau$ for all $\{P_i\}_{i=1}^n \in N\tau$.

Then $(Z, N\tau)$ is claimed as N topological space and $N\tau$ is called N topology.

Definition 2.3 [3] In $(Z, N\tau)$, $U \subseteq Z$ is said to be

- (i) $N\tau$ semi-open set if $U \subseteq N\tau cl(N\tau int(U))$
- (ii) $N\tau$ pre-open set if $U \subseteq N\tau int(N\tau cl(U))$
- (iii) $N\tau\alpha$ -open set if $U \subseteq N\tau int(N\tau cl(N\tau int(U)))$
- (iv) $N\tau\beta$ -open set if $U \subseteq N\tau cl(N\tau int(N\tau cl(U)))$

Proposition 2.4 [2] Let the two N topological spaces are $(Z, N\tau)$ and $(Y, N\sigma)$. A mapping $\eta : (Z, N\tau) \rightarrow (Y, N\sigma)$ is N^* continuous on Z if $\eta^{-1}(V) \in N\tau O(Z)$ for every $V \in N\sigma O(Y)$.

Definition 2.5 [3] Let $(Z, N\tau)$ and $(Y, N\sigma)$ are two N topological spaces. A mapping $\eta : (Z, N\tau) \rightarrow (Y, N\sigma)$ is $N^*\alpha$ continuous (resp. N^* semi continuous, N^* pre continuous, $N^*\beta$ continuous) on Z if the inverse image of every $N\sigma$ open set in Y is a $N\tau\alpha$ open set (resp. $N\tau$ semi open, $N\tau$ pre open, $N\tau\beta$ open) in Z .

Definition 2.6 [6] Suppose $(X, N\tau)$ be a N topological space and $P \subseteq X$. An element $s \in X$ is said to be $N\tau$ - θ cluster point of P if $P \cap N\tau cl(G) \neq \emptyset$ for every $N\tau$ open set G containings. The set of all $N\tau$ - θ cluster points of P is called $N\tau$ - θ closure of P and is denoted by $N\tau-cl_\theta(P)$. A subset P of X is said to be $N\tau\theta$ closed in X if $N\tau cl_\theta(P) = P$. The complement of $N\tau\theta$ closed set is $N\tau\theta$ open.

Definition 2.7 [6] Let $P \subseteq (X, N\tau)$ and an element $x \in P$ is $N\tau\theta$ -interior point of P if $N\tau \theta cl(G) \subseteq P$ for some $N\tau$ open set G containing x . The set of all $N\tau\theta$ interior points of P is called $N\tau\theta$ -interior of P and is denoted by $N\tau-int_\theta(P)$.

3. $N\tau\theta$ -open sets and its properties

In this segment, we strive to define $N\tau\theta$ -open sets in a different way and study its common properties which are θ -neighborhood, θ -limit point, θ -derived set, θ -frontier, θ -exterior in N topological space. We introduce $N\tau$ -regular open and its connection with $N\tau\theta$ -open sets.

Definition 3.1 A set $S \subseteq (P, N\tau)$ is called $N\tau\theta$ -open if for every $p \in S$, $\exists$ a $N\tau O(G)$ such that $p \in G \subseteq N\tau cl(G) \subseteq S$.

Example 3.2 Let $P = \{1,2,3,4\}$, For $N = 3$, $\tau_1 = \{\emptyset, \{1,2\}, \{1,2,3\}, P\}$, $\tau_2 = \{\emptyset, \{3\}, \{3,4\}, P\}$, $\tau_3 = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}, P\}$ and $3\tau = \{\emptyset, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3\}, P\}$. Then $3\tau\theta O(P) = \{\emptyset, \{1,2\}, \{3,4\}, P\}$.

Remark 3.3 The collection of all $N\tau\theta$ -open sets produces a N topology and is signified as $N\tau_\theta$.

Definition 3.4 In $(P, N\tau)$, M is said to be $N\tau$ - θ neighborhood of u if $\exists L \in N\tau_\theta$ such that $u \in L \subseteq M$ where $u \in P$ and $M \subseteq P$.

Definition 3.5 Let $S \subseteq (P, N\tau)$ and $p \in P$ is called $N\tau\theta$ limit point of S , if for each $N\tau\theta$ open set $\mathcal{H}$ containing p , $\mathcal{H} \cap \{S - \{p\}\} \neq \emptyset$.

Definition 3.6 The collection of all $N\tau\theta$ limit points of S is called $N\tau\theta$ -derived set of S and it is signified as $N\tau \theta_D(S)$.

Theorem 3.7 If $P, Q \subseteq (Z, N\tau)$ then

- (i) $N\tau_D(P) \subset N\tau \theta_D(P)$ where $N\tau_D(P)$ is the $N\tau$ derived set of P

- (ii) If $P \subset Q$ then $N\tau \theta_D(P) \subset N\tau \theta_D(Q)$
- (iii) $N\tau \theta_D(P) \cup N\tau \theta_D(Q) = N\tau \theta_D(P \cup Q)$
- (iv) $N\tau \theta_D(P \cap Q) \subset N\tau \theta_D(P) \cap N\tau \theta_D(Q)$
- (v) $N\tau \theta_D(N\tau \theta_D(P)) - P \subset N\tau \theta_D(P)$
- (vi) $N\tau \theta_D(P \cup N\tau \theta_D(P)) \subset P \cup N\tau \theta_D(P)$.

Proof:

- (i) It is enough to perceive that every $N\tau\theta$ -open set is $N\tau$ open.
- (ii), (iii), (iv) are obvious.
- (v) If $x \in N\tau \theta_D(N\tau \theta_D(P)) - P$ and $V \in N\tau\theta O(Z)$ and $x \in V$ then $V \cap (N\tau \theta_D(P) - \{x\}) \neq \emptyset$. If $y \in V \cap N\tau \theta_D(P) - \{x\}$ then $y \in V$ and $y \in N\tau \theta_D(P)$. $V \cap (P - \{y\}) \neq \emptyset$. Now if $z \in V \cap (P - \{y\})$ then $z \neq x$ for $z \in P$ and $x \notin P$. $V \cap (P - \{x\}) \neq \emptyset$. Hence $x \in N\tau \theta_D(P)$.
- (vi) Let $x \in N\tau \theta_D(P \cup N\tau \theta_D(P))$. If $x \in P$ then the result is true. Now, If $x \in N\tau \theta_D(P \cup N\tau \theta_D(P)) - P$ then for $N\tau$ - θ open set V containing x then $V \cap (P \cup N\tau \theta_D(P) - \{x\}) \neq \emptyset$. So $V \cap (P - \{x\}) \neq \emptyset$ or $V \cap (N\tau \theta_D(P) - \{x\}) \neq \emptyset$. Now it arrives by (v) that $V \cap (P - \{x\}) \neq \emptyset$. Hence $x \in N\tau \theta_D(P)$ and $N\tau \theta_D(P \cup N\tau \theta_D(P)) \subset P \cup N\tau \theta_D(P)$.

Theorem 3.8 If $S \subseteq (P, N\tau)$ then $S \cup N\tau \theta_D(S) \subset N\tau\text{-}cl_\theta(S)$.

Proof: Since $N\tau \theta_D(S) \subset N\tau\text{-}cl_\theta(S)$, $S \cup N\tau \theta_D(S) \subset N\tau\text{-}cl_\theta(S)$.

Definition 3.9 Let $(P, N\tau)$ be a N topological space. The θ -border of a set $S \subset P$, denoted by $N\tau \theta_b(S)$ is $S - (N\tau\text{-}int_\theta(S))$

Theorem 3.10 If $S \subseteq (P, N\tau)$ then the following statements hold:

- (i) $N\tau_b(S) \subset N\tau \theta_b(S)$ where $N\tau_b(S)$ denotes the $N\tau$ -border of S
- (ii) $S = N\tau\text{-}int_\theta(S) \cup N\tau \theta_b(S)$
- (iii) $N\tau\text{-}int_\theta(S) \cap N\tau \theta_b(S) = \emptyset$
- (iv) S is a $N\tau$ θ -open set if and only if $N\tau \theta_b(S) = \emptyset$
- (v) $N\tau\text{-}int_\theta(N\tau \theta_b(S)) = \emptyset$
- (vi) $N\tau \theta_b(N\tau \theta_b(S)) = N\tau \theta_b(S)$
- (vii) $N\tau \theta_b(S) = S \cap N\tau\text{-}cl_\theta(P - S)$.

Proof: Obvious

Definition 3.11 Let $(P, N\tau)$ be a N topological space. The θ -frontier of a set $S \subset P$, signified by $N\tau \theta_{Fr}(S)$ is defined as $N\tau \theta_{Fr}(S) = N\tau\text{-}cl_\theta(S) - N\tau\text{-}int_\theta(S)$.

Example 3.12 Let $P = \{1,2,3,4\}$, For $N = 3$, $\tau_1 = \{\emptyset, \{1,2\}, \{1,2,3\}, P\}$, $\tau_2 = \{\emptyset, \{3\}, \{3,4\}, P\}$, $\tau_3 = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}, P\}$ and $3\tau = \{\emptyset, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3\}, P\}$. Let $S = \{1,2,3\}$ and $3\tau\theta\text{-}cl(S) = P$, $3\tau\text{-}int_\theta(S) = \{1,2\}$. Then $3\tau \theta_{Fr}(S) = \{3,4\}$.

Proposition 3.13 If $S \subseteq (P, N\tau)$ then the following statements hold:

- (i) $N\tau_{Fr}(S) \subset N\tau \theta_{Fr}(S)$ where $N\tau_{Fr}(S)$ denotes the $N\tau$ -frontier of S .
- (ii) $N\tau-cl_{\theta}(S) = N\tau-int_{\theta}(S) \cup N\tau \theta_{Fr}(S)$
- (iii) $N\tau-int_{\theta}(S) \cap N\tau \theta_{Fr}(S) = \emptyset$
- (iv) $N\tau\theta_b(S) \subset N\tau \theta_{Fr}(S)$
- (v) $N\tau \theta_{Fr}(S) = N\tau-cl_{\theta}(S) \cap N\tau-cl_{\theta}(P - S)$
- (vi) $N\tau \theta_{Fr}(S) = N\tau \theta_{Fr}(P - S)$
- (vii) $N\tau int_{\theta}(S) = S - (N\tau \theta_{Fr}(S))$.
- (viii) $N\tau \theta_{Fr}(S)$ is $N\tau$ -closed.

Definition 3.14 The θ -exterior of a set $S \subseteq (P, N\tau)$, signified by $N\tau \theta_{Ext}(S)$ is determined as $N\tau \theta_{Ext}(S) = N\tau int_{\theta}(P - S)$.

Example 3.15 Let $P = \{1,2,3,4\}$ and $S = \{1,2,3\}$. For $N = 3$, $\tau_1 = \{\emptyset, \{1,2\}, \{1,2,3\}, P\}$, $\tau_2 = \{\emptyset, \{3\}, \{3,4\}, P\}$, $\tau_3 = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}, P\}$ and $3\tau = \{\emptyset, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3\}, P\}$. Then $3\tau \theta_{Ext}(S) = \emptyset$.

Proposition 3.16 If $P, Q \subseteq (Z, N\tau)$ then the following statements hold:

- (i) $N\tau \theta_{Ext}(P) \subset N\tau_{Ext}(P)$ where $N\tau_{Ext}(P)$ denotes the $N\tau$ -exterior of P .
- (ii) $N\tau \theta_{Ext}(P)$ is $N\tau$ -open
- (iii) $N\tau \theta_{Ext}(P) = N\tau int_{\theta}(Z - P) = Z - (N\tau-cl_{\theta}(P))$
- (iv) $N\tau \theta_{Ext}(N\tau \theta_{Ext}(P)) = N\tau int_{\theta}(N\tau-cl_{\theta}(P))$
- (v) If $P \subset Q$, then $N\tau \theta_{Ext}(P) \supset N\tau \theta_{Ext}(Q)$
- (vi) $N\tau \theta_{Ext}(P \cup Q) = N\tau \theta_{Ext}(P) \cup N\tau \theta_{Ext}(Q)$
- (vii) $N\tau \theta_{Ext}(P \cap Q) = N\tau \theta_{Ext}(P) \cap N\tau \theta_{Ext}(Q)$
- (viii) $N\tau \theta_{Ext}(Z) = \emptyset$
- (ix) $N\tau \theta_{Ext}(\emptyset) = Z$
- (x) $N\tau \theta_{Ext}(Z - N\tau \theta_{Ext}(P)) \subset N\tau \theta_{Ext}(P)$
- (xi) $N\tau int_{\theta}(P) \subset N\tau \theta_{Ext}(N\tau \theta_{Ext}(P))$
- (xii) $Z = N\tau int_{\theta}(P) \cup N\tau \theta_{Ext}(P) \cup N\tau \theta_{Fr}(P)$.

Definition 3.17 A set E is $N\tau$ -regular open if $E = N\tau-int(N\tau-cl(E))$.

Theorem 3.18 If $\mathcal{H} \subset (P, N\tau)$ is $N\tau\theta$ -open and $x \in \mathcal{H}$ then $\exists$ a $N\tau$ -regular open set $\mathcal{G} \ni x \in \mathcal{G} \subset N\tau-cl(\mathcal{G}) \subset \mathcal{H}$.

Proof: Since H is $N\tau\theta$ open, there exist an $N\tau$ open set $W \ni x \in W \subset N\tau-cl(W) \subset H$. But $N\tau-int(N\tau-cl(W)) = G$ is $N\tau$ -regular open and it follows that $x \in \mathcal{G} \subset N\tau-cl(\mathcal{G}) \subset \mathcal{H}$ due to the fact that $N\tau-cl(W) \subset \mathcal{H}$.

Corollary 3.19 The set $\mathcal{H}$ is $N\tau\theta$ -open $\Leftrightarrow$ for each $y \in \mathcal{H}$, there exist a $N\tau$ -regular open $\mathcal{G}$ such that $y \in \mathcal{G} \subset N\tau-cl(\mathcal{G}) \subset \mathcal{H}$.

Definition 3.20 A space $(P, N\tau)$ is $N\tau$ almost-regular if $\forall p \in P$ and $N\tau$ -regular closed set $\mathcal{G}$ not containing p , there exists disjoint $N\tau$ open sets L and $M \ni x \in L$ and $\mathcal{G} \subset M$.

4. $N^*\theta$ -continuous maps

In this segment, we establish various continuous maps and discuss the relationship with θ -continuous maps in N topological space. Moreover, explored Urysohn space, θ -irresolute and its properties using N -topology. Throughout this segment P, Q and R represents the N -topological spaces $(P, N\tau)$, $(Q, N\sigma)$ and $(R, N\rho)$ respectively.

Definition 4.1 A mapping $\eta: P \rightarrow Q$ is N^* almost-continuous at $u \in P \Leftrightarrow$ for all $N\sigma$ -open set $\mathcal{H} \subset Q$ containing $\eta(u)$, $N\tau-cl(\eta^{-1}(\mathcal{H}))$ is a $N\tau$ -neighborhood of u . If this convinced for every $u \in P$, then η is said to be N^* almost continuous.

Definition 4.2 A mapping $\eta: P \rightarrow Q$ is $N^*\theta$ -continuous if $\eta^{-1}(\mathcal{H}) \in N\tau_\theta \forall \mathcal{H} \in N\sigma$.

Definition 4.3 A mapping $\eta: P \rightarrow Q$ is N^* weakly continuous at $u \in P$ if for every $N\sigma$ open set $\mathcal{H}, \eta(u) \in \mathcal{H} \subset Q \ni$ an $N\tau$ open set $\mathcal{G}, u \in \mathcal{G} \subset P \ni \eta(\mathcal{G}) \subseteq N\sigma-cl(\mathcal{H})$. If this convinced for every $u \in P$ then η is N^* weakly continuous.

Definition 4.4 A N topological space $(P, N\tau)$ is $N\tau$ -urysohn if for every $u \neq v \in P, \ni$ an $N\tau$ open sets $\mathcal{G}$ and $\mathcal{H}, u \in \mathcal{G}$ and $v \in \mathcal{H} \ni N\tau-cl(\mathcal{G}) \cap N\tau-cl(\mathcal{H}) = \emptyset$.

Proposition 4.5 If η, γ be N^* weakly-continuous from P to a $N\sigma$ urysohn space Q then $A = \{u \in P: \eta(u) = \gamma(u)\}$ is a $N\tau$ closed set.

Corollary 4.6 Let η, γ be N^* weakly-continuous from P to a $N\sigma$ -urysohn space Q and if η, γ coincide on a $N\tau$ dense subset of P then $\eta = \gamma$ everywhere.

Corollary 4.7 Let η, γ be $N^*\theta$ -continuous from P to a $N\sigma$ -urysohn space Q and if η, γ agree on a $N\tau\theta$ dense subset of P then $\eta = \gamma$ everywhere.

Corollary 4.8 Let η, γ be N^* strongly θ -continuous from P to a $N\sigma$ Hausdorff space Q . If η, γ agree on a $N\tau\theta$ dense subset of P then $\eta = \gamma$ all over.

Definition 4.9 A mapping $\eta: P \rightarrow Q$ is N^* faintly continuous if $\eta^{-1}(U)$ is $N\tau$ -open for every $N\sigma$ - θ open set $U \in Q$

Definition 4.10 A mapping $\eta: P \rightarrow Q$ is N^* quasi θ -continuous if $\eta^{-1}(U)$ is $N\tau\theta$ -open for every $N\sigma\theta$ -open set $U \in Q$.

Proposition 4.11 If a mapping $\eta: P \rightarrow Q$ is N^* weakly continuous then η is N^* faintly continuous.

Proof: If $u \in P$ and G is a $N\sigma\theta$ open $\ni \eta(u) \in G \subseteq Q$ then $\exists$ an $N\sigma$ open set $M \ni \eta(u) \in M \subset N\sigma-cl(M) \subset G$. Since, η is N^* weakly-continuous then $\exists$ an $N\tau$ open set K containing $u \ni \eta(K) \subset N\sigma-cl(M) \subset G$. Consequently η is N^* faintly-continuous.

Proposition 4.12 Every N^* weakly θ -continuous is N^* quasi θ -continuous.

Theorem 4.13 Every N^* weakly-continuous need not be $N^*\theta$ -continuous.

Proof: This can be proved by an simple example.

Let $P = \{l, m, n, o\}$. For $N = 3$, $\tau_1 = \{\emptyset, \{l, m, n\}, P\}$, $\tau_2 = \{\emptyset, \{n\}, P\}$ $\tau_3 = \{\emptyset, \{n, o\}, P\}$ and $3\tau = \{\emptyset, \{l, m, n\}, \{n\}, \{n, o\}, P\}$. Let $Q = \{r, s, t, u\}$. For $N = 3$, $\sigma_1 = \{\emptyset, \{r, s\}, \{u\}, \{r, s, u\}, Q\}$, $\sigma_2 = \{\emptyset, \{s, u\}, \{s, t, u\}, Q\}$, $\sigma_3 = \{\emptyset, \{s\}, \{r, s\}, Q\}$ and $3\sigma = \{\emptyset, \{r, s\}, \{s\}, \{u\}, \{s, u\}, \{r, s, u\}, \{s, t, u\}, Q\}$. Define $\gamma: P \rightarrow Q$ by $\gamma(l) = r$, $\gamma(m) = s$, $\gamma(n) = t$, $\gamma(o) = u$. Here γ is 3^* weakly continuous but not a $3^*\theta$ continuous.

Lemma 4.14 If $\eta: P \rightarrow Q$ is N^* faintly continuous and $\gamma: P \rightarrow Q$ is N^* quasi θ continuous (res. N^* strongly θ continuous) then $\gamma \circ \eta: P \rightarrow R$ is N^* faintly continuous (res., N^* continuous).

Lemma 4.15 If $\eta: P \rightarrow Q$ is N^* continuous and $\gamma: Q \rightarrow R$ N^* faintly-continuous then $\gamma \circ \eta: P \rightarrow R$ is N^* faintly continuous.

Lemma 4.16 If $\eta: P \rightarrow Q$ is a N^* quasi θ -continuous and $\gamma: Q \rightarrow R$ is N^* quasi θ continuous then $\gamma \circ \eta: P \rightarrow R$ is N^* quasi θ continuous. (res. N^* strongly θ -continuous).

Proposition 4.17 If $\eta: (P, N\tau) \rightarrow (Q, N\sigma)$ is a mapping then the following are similar:

- (1) η is $N^*\theta$ -continuous
- (2) The inverse image of each $N\sigma$ closed set in Q is a $N\tau$ - θ closed set in P
- (3) $N\tau-cl_\theta[\eta^{-1}(\mathcal{H})] \subseteq \eta^{-1}[N\sigma-cl(\mathcal{H})]$ for every $\mathcal{H} \subseteq Q$
- (4) $\eta[N\tau-cl_\theta(\mathcal{G})] \subseteq N\sigma-cl[\eta(\mathcal{G})]$, for every $\mathcal{G} \subseteq P$
- (5) $\eta^{-1}(N\sigma-int(\mathcal{G})) \subseteq N\tau-int_\theta(\eta^{-1}(\mathcal{G}))$ for every $\mathcal{G} \subseteq Q$.

Proof: (1) $\Rightarrow$ (2) Let $\mathcal{H} \subseteq Q$ be an $N\sigma$ closed. Since η is $N^*\theta$ continuous $\eta^{-1}(Q - \mathcal{H}) = P - \eta^{-1}(\mathcal{H})$ is $N\tau$ θ open. Hence $\eta^{-1}(\mathcal{H})$ is $N\tau$ - θ closed in P .

(2) $\Rightarrow$ (3) Since $N\sigma-cl(\mathcal{H})$ is $N\sigma$ closed $\forall \mathcal{H} \subseteq Q$ then $\eta^{-1}(N\sigma-cl(\mathcal{H}))$ is $N\tau\theta$ -closed. Hence $\eta^{-1}(N\sigma-cl(\mathcal{H})) = N\tau-cl_\theta[\eta^{-1}(N\sigma-cl(\mathcal{H}))] \supseteq N\tau-cl_\theta[\eta^{-1}(\mathcal{H})]$.

(3) $\Rightarrow$ (4) Let $\mathcal{G} \subseteq P$ and $\eta(\mathcal{G}) = V$. Then $N\tau-cl_\theta(\eta^{-1}(V)) \subseteq \eta^{-1}(N\sigma-cl(V))$. Thus $N\tau-cl_\theta(\mathcal{G}) \subseteq N\tau-cl_\theta[\eta^{-1}(\eta(\mathcal{G}))] \subseteq \eta^{-1}(N\sigma-cl(\eta(\mathcal{G})))$. Hence, it follows that $\eta(N\tau-cl_\theta(\mathcal{G})) \subseteq N\sigma-cl(\eta(\mathcal{G}))$.

(4) $\Rightarrow$ (5) Let $\mathcal{G} \subseteq Q$. By (4) $\eta[N\tau-cl_\theta(P - \eta^{-1}(\mathcal{G}))] \subseteq N\sigma-cl_\theta(\eta(P - \eta^{-1}(\mathcal{G})))$ and $\eta[P - (N\tau-int_\theta(\eta^{-1}(\mathcal{G})))] \subseteq N\sigma-cl(Q - \mathcal{G}) = Q - (N\sigma-int(\mathcal{G}))$. Therefore $P - [N\tau-int_\theta(\eta^{-1}(\mathcal{G}))] \subseteq \eta^{-1}(Q - (N\sigma-int(\mathcal{G})))$ and hence $\eta^{-1}(N\sigma-int(\mathcal{G})) \subseteq N\tau-int_\theta(\eta^{-1}(\mathcal{G}))$ for every $\mathcal{G} \subseteq Q$.

(5) $\Rightarrow$ (1) If $\mathcal{G} \subseteq Q$ be an $N\sigma$ open set and $\eta^{-1}(N\sigma-int(\mathcal{G})) \subseteq N\tau-int_\theta(\eta^{-1}(\mathcal{G}))$. Then $\eta^{-1}(\mathcal{G}) \subseteq N\tau-int_\theta(\eta^{-1}(\mathcal{G}))$. But $N\tau-int_\theta(\eta^{-1}(\mathcal{G})) \subseteq \eta^{-1}(\mathcal{G})$. Hence $N\tau-int_\theta(\eta^{-1}(\mathcal{G})) = \eta^{-1}(\mathcal{G})$. Therefore $\eta^{-1}(\mathcal{G})$ is $N\tau\theta$ -open in P .

Proposition 4.18 Let $S \subseteq Q \subseteq P$, Q is $N\tau\theta$ -open in P and S is $N\sigma\theta$ -open in Q . Then S is $N\tau\theta$ -open in P .

Proposition 4.19 If a mapping $\eta: (P, N\tau) \rightarrow (Q, N\sigma)$ and $\{U_v: v \in I\}$ is a cover of $P \ni U_v \in N\tau_\theta$ for every $v \in I$. Suppose $\eta|_{U_v}: U_v \rightarrow Q$ is $N^*\theta$ -continuous for every $v \in I$ and then η is $N^*\theta$ -continuous.

Proof: Let $\mathcal{H} \subseteq Q$ be $N\sigma$ open set. Now $(\eta|_{U_v})^{-1}(\mathcal{H})$ is $N\tau\theta$ -open in U_v for every $v \in I$. We know that U_v is $N\tau\theta$ -open in P for each $v \in I$. By proposition 4.18, $(\eta|_{U_v})^{-1}(\mathcal{H})$ is $N\tau\theta$ -open in P for every $v \in I$. But $\eta^{-1}(\mathcal{H}) = \cup \{(\eta|_{U_v})^{-1}(\mathcal{H}): v \in I\}$ then $\eta^{-1}(\mathcal{H}) \in N\tau_\theta$ since $N\tau_\theta$ is N -topology on P . This implies η is $N^*\theta$ -continuous.

Proposition 4.20 If $(P, N\tau)$ and $(Q, N\sigma)$ are N topological spaces and $\eta: P \rightarrow Q$ is a mapping then the following are identical.

- (a) $\eta: P \rightarrow (Q, N\sigma)$ is N^* faintly continuous.
- (b) $\eta: P \rightarrow (Q, N\sigma_\theta)$ is N^* continuous.
- (c) $\eta^{-1}(U)$ is $N\tau$ open for every $N\sigma\theta$ -open set $U \in Q$.
- (d) $\eta^{-1}(U)$ is $N\tau$ closed for every $N\sigma\theta$ -closed set $U \in Q$.

Proof: The implication follows from definitions.

Theorem 4.21 If a mapping $\eta: P \rightarrow Q$ is N^* weakly continuous then η is N^* faintly continuous.

Proposition 4.22 A N^* faintly continuous map need not be N^* weakly continuous.

Proof: The following example proves this proposition. Let $P = \{0, 1\}$, For $N = 2$, $\tau_1 = \{\emptyset, P\}$, $\tau_2 = \{\emptyset, P, \{1\}\}$ and $2\tau = \{\{\emptyset, P, \{1\}\}$ and let $Q = \{l, m, n\}$ with $\sigma_1 = \{\emptyset, Q, \{l\}\}$, $\sigma_2 = \{\emptyset, Q, \{m\}, \{l, m\}\}$ and $2\sigma = \{\emptyset, Q, \{l\}, \{m\}, \{l, m\}\}$. Now if $\eta: P \rightarrow Q$ is defined as $\eta(0) = l$ and $\eta(1) = m$. Then η is not 2^* weakly continuous at $l = 0$, but η is 2^* -faintly continuous since the only $2\sigma\theta$ -open set in Q is Q itself.

Corollary 4.23 If $(Q, N\sigma)$ is $N\sigma$ almost-regular and $\eta: Q \rightarrow (Q, N\sigma)$ then the following are equivalent:

- (a) η is N^* faintly continuous

- (b) η is N^* weakly continuous
- (c) η is $N^*\theta$ -continuous
- (d) η is N^* almost-continuous.

Proposition 4.24 If $\eta: P \rightarrow Q$ is N^* faintly continuous and $K \subset P$, then $\eta|_K: K \rightarrow Q$ is N^* faintly -continuous.

Proposition 4.25 If $\eta: P \rightarrow Q$ is $N^*\theta$ continuous and $\gamma: Q \rightarrow R$ is $N^*\theta$ continuous then $\gamma\eta$ is $N^*\theta$ -continuous.

Definition 4.26 A mapping $\eta: (P, N\tau) \rightarrow (Q, N\sigma)$ is N^* strongly continuous (N^* strongly θ -continuous) at $u \in P$ if for every $N\sigma$ open set $\mathcal{H}, \eta(u) \in H \subset Q$, there exist an $N\tau$ -open set $\mathcal{G}, u \in \mathcal{G} \subset P$ such that $\eta(N\tau-cl(\mathcal{G})) \subseteq \mathcal{H}$. If this convinced at every $u \in P$, then η is N^* strongly-continuous (N^* strongly θ -continuous)

Proposition 4.27 Let $\eta: P \rightarrow Q$ be N^* strongly θ continuous and if $\gamma: Q \rightarrow R$ is N^* -continuous then $\gamma\eta$ is N^* strongly θ -continuous.

Proposition 4.28 Let $\eta: P \rightarrow Q$ be $N^*\theta$ -continuous and if $\gamma: Q \rightarrow R$ be N^* strongly θ -continuous then $\gamma\eta$ is $N^*\theta$ -continuous.

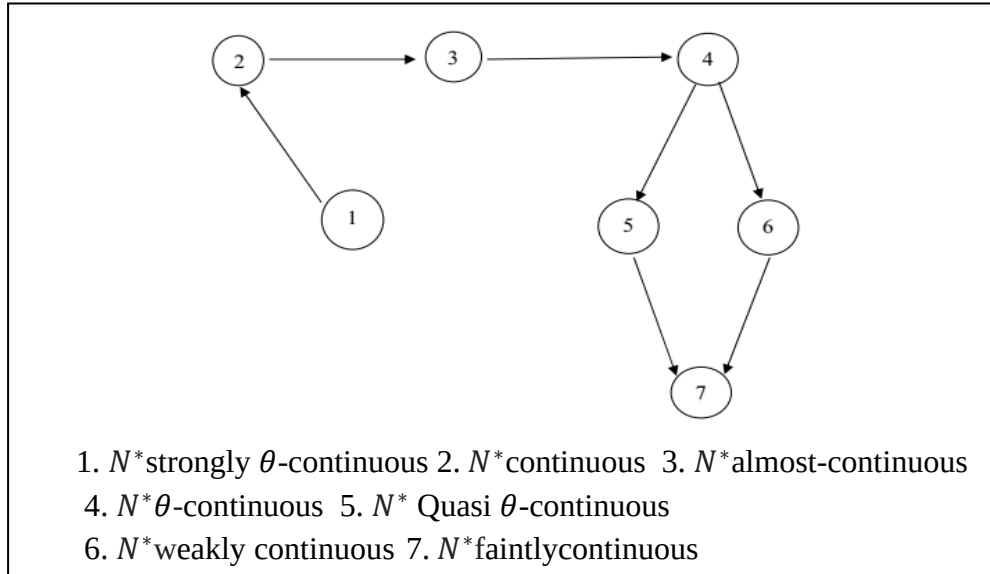


Figure 1. The relationship between $N^*\theta$ -continuous maps and other continuous maps.

Definition 4.29 A mapping $\eta: (P, N\tau) \rightarrow (Q, N\sigma)$ is $N^\#$ - θ irresolute if $\eta^{-1}(\mathcal{G}) \in N\tau_\theta$ for every $\mathcal{G} \in N\tau_\theta$.

Proposition 4.30 If $\eta: P \rightarrow Q$ is a mapping then the following are identical:

- (i) η is $N^\#$ - θ irresolute

- (ii) $\eta^{-1}(U)$ is $N\tau$ - θ closed for every $N\sigma$ - θ closed set $U \in Q$
- (iii) $N\tau\text{-}cl_\theta[\eta^{-1}(\mathcal{H})] \subseteq \eta^{-1}[N\sigma\text{-}cl_\theta(\mathcal{H})]$ for every $\mathcal{H} \subseteq Q$
- (iv) $\eta[N\tau\text{-}cl_\theta(\mathcal{G})] \subseteq N\sigma\text{-}cl_\theta[\eta(\mathcal{G})]$ for every $\mathcal{G} \subseteq P$.
- (v) $\eta^{-1}[N\sigma\text{-}int_\theta(M)] \subseteq N\tau\text{-}int_\theta[\eta^{-1}(Q)]$ for every $M \subseteq Q$.

Proposition 4.31 A mapping $\eta: P \rightarrow Q$ is $N^\# \theta$ -irresolute $\Leftrightarrow$ for every point u in P and every $N\sigma$ - θ open M in Q with $\eta(u) \in M$, there is a $N\tau$ - θ open set L in P $\ni u \in L$, $\eta(L) \subseteq M$.

Proof: Necessity: Let $u \in P$ and $M \in N\sigma_\theta$ $\ni f(u) \in M$. Let $L = \eta^{-1}(M)$. Since η is $N^\# \theta$ -irresolute, L is $N\tau$ - θ open in P . Also $u \in \eta^{-1}(M) = L$ as $\eta(u) \in M$. Hence $\eta(L) = \eta(\eta^{-1}(M)) \subseteq M$. **Sufficiency:** Let $M \in N\sigma_\theta$ and let $L = \eta^{-1}(M)$. we prove that L is $N\tau$ - θ open in P . Now, let $x \in L$. It implicit $\eta(x) \in M$. By conjecture, $\exists L_x \in N\tau_\theta$ $\ni x \in L_x$ and $\eta(L_x) \subseteq M$. Then $L_x \subseteq \eta^{-1}[\eta(L_x)] \subseteq \eta^{-1}(M) = L$. Thus $L = \cup \{L_x : x \in L\}$. These results L is $N\tau$ - θ open in P and η is $N^\# \theta$ -irresolute.

Proposition 4.32 A mapping $\eta: (P, N\tau) \rightarrow (Q, N\sigma)$ is $N^\# \theta$ -irresolute $\Leftrightarrow$ for every $u \in P$, the inverse image of every $N\sigma$ - θ -neighborhood of $\eta(u)$, is a $N\tau$ - θ -neighborhood of u .

Proof: Necessity Let $u \in P$ and L be a $N\sigma$ - θ -neighborhood of $\eta(u)$. If there exist $M \in N\sigma_\theta$ such that $\eta(u) \in M \subseteq L$. This implies that $u \in \eta^{-1}(M) \subseteq \eta^{-1}(L)$. Since η is $N^\# \theta$ -irresolute, so $\eta^{-1}(M) \in N\tau_\theta$. Hence $\eta^{-1}(L)$ is $N\tau$ - θ -neighborhood of u . **Sufficiency:** let $L \in N\sigma_\theta$ and $M = \eta^{-1}(L)$. Let $x \in M$ then $\eta(x) \in L$. But then L being $N\sigma$ - θ -open set is a $N\sigma$ - θ -neighborhood of $\eta(x)$. So by hypothesis $M = \eta^{-1}(L)$ is a $N\tau$ - θ -neighborhood of x . By definition, $\exists M_x \in N\tau_\theta$ $\ni x \in M_x \subseteq M$. Hence $M = \cup \{M_x : x \in M\}$ and M is a $N\tau$ - θ -open in P . So η is $N^\# \theta$ -irresolute.

5. $N\tau$ - θ -connected and separation axioms

In this chapter, we establish the ideas of $N\tau$ - θ -connected, $N\tau$ - θ hausdorff, $N\tau$ - θ regular, $N\tau$ - θ normal spaces. Throughout this section $(P, N\tau)$ represents a N topological space on P .

Definition 5.1 $(P, N\tau)$ is $N\tau$ - θ -disconnected if $P = L \cup M$ where $L \cap M = \emptyset$ and $L \neq \emptyset, M \neq \emptyset$. Or else, P is $N\tau$ - θ -connected.

Proposition 5.2 In $(P, N\tau)$, the subsequent are identical:

- (i) P is $N\tau$ - θ -connected.
- (ii) $\emptyset, P \subseteq P$ are the only $N\tau$ - θ -open and $N\tau$ - θ -closed sets.
- (iii) $P \neq L \cup M, L \neq \emptyset, M \neq \emptyset ; L \cap M = \emptyset$ where L & M are $N\tau$ - θ -open sets.
- (iv) There is no $N^* \theta$ -continuous function $\eta: P \rightarrow E$ is surjective.

Proposition 5.3 $(P, N\tau)$ is $N\tau$ connected $\Leftrightarrow$ it is $N\tau\theta$ -connected.

Proposition 5.4 $(P, N\tau)$ is $N\tau\theta$ -connected $\Leftrightarrow$ for every $S \subseteq P$, $S \neq \emptyset$ such that $N\tau \theta_{Fr}(S) \neq \emptyset$.

Proof: Necessity: assuming P is $N\tau \theta$ -disconnected. Then there exist $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$ such that $P = A \cup B$. Here both A and B are $N\tau\theta$ -open and $N\tau\theta$ -closed in P . Therefore $A' = A^0 = \bar{A}$ but $N\tau \theta_{Fr}(A) = N\tau-cl_\theta(A) - N\tau int_\theta(A)$. Hence $N\tau \theta_{Fr}(A) = \emptyset$ and this is contrary to the conjecture. Consequently P must be $N\tau\theta$ -connected.
Sufficiency: let P be $N\tau\theta$ -connected and suppose $\exists S \subset P, S \neq \emptyset \ni N\tau \theta_{Fr}(S) = \emptyset$. Now $N\tau-cl_\theta(S) = N\tau int_\theta(S) \cup N\tau \theta_{Fr}(S) = S \cup N\tau \theta_{Fr}(N\tau-cl_\theta(S))$. Hence $S = N\tau int_\theta(S) = N\tau-cl_\theta(S)$ showing that S is both $N\tau\theta$ -open and $N\tau\theta$ -closed in P and therefore P is a $N\tau\theta$ -disconnected which is a denial. Hence every proper subset of P must have a non empty $N\tau\theta$ -frontier.

Definition 5.5 A N topological space P is

- (i) $N\tau$ - θ hausdorff if every two disjoint elements p, q in P then $\exists$ distinct $N\tau$ - θ open sets $G, H \ni p \in G$ and $q \in H$
- (ii) $N\tau$ - θ regular if for every $N\tau$ closed set C and every point $x \notin C$, $\exists$ disjoint $N\tau$ - θ open sets $G, H \ni x \in G$ and $C \subseteq H$
- (iii) $N\tau$ - θ normal if for any two distinct $N\tau$ closed sets $C, D \subseteq P$, $\exists$ distinct $N\tau$ - θ open sets $G, H \ni C \subseteq G$ and $D \subseteq H$.

Remark 5.6 Every $N\tau$ - θ hausdorff (resp. $N\tau$ - θ regular, $N\tau$ - θ normal) space is $N\tau$ -hausdorff (res. $N\tau$ -regular, $N\tau$ -normal).

Proposition 5.7 In $(P, N\tau)$ the subsequent are identical:

- (i) P is $N\tau$ - θ hausdorff
- (ii) let $a \in P$ for $b \neq a$, $\exists$ a $N\tau\theta$ -open set G , $a \in G \ni b \notin N\tau-cl_\theta(G)$
- (iii) for every $a \in P$, $D = \bigcap \{N\tau \theta-cl(G) : G \text{ is } N\tau\theta\text{-open and } a \in G\} = \{a\}$.

Proof:

- (i) $\Rightarrow$ (ii) For any different elements $a, b \in P$, $\exists$ disjoint $N\tau\theta$ -open sets $G, H \ni a \in G$ and $b \in H$. Hence $b \notin N\tau-cl_\theta(G)$
- (ii) $\Rightarrow$ (iii) Suppose $x \in D$, for each $b \neq a \ni$ a $N\tau\theta$ -open set G , $x \in G \ni b \notin N\tau-cl_\theta(G)$. Thus $b \notin D$ since b is arbitrary $D = \{a\}$.
- (iii) $\Rightarrow$ (i) If $a, b \in X$ then $\ni a \neq b$. suppose $\exists$ a $N\tau\theta$ -open set G , $x \in G \ni b \notin N\tau-cl_\theta(G)$. By remark 5.6, $\exists$ a $N\tau \theta$ -open set H , $b \in H \ni G \cap H = \emptyset$. Therefore P is $N\tau$ - θ hausdorff.

Proposition 5.8 In $(P, N\tau)$, the subsequent are identical:

- (i) P is $N\tau\theta$ - regular
- (ii) for every $z \in P$ and an $N\tau$ open set G , $z \in G$ then $\exists$ a $N\tau\theta$ -open set $H \ni$:
 $z \in H \subseteq N\tau\text{-}cl_\theta(H) \subseteq G$
- (iii) for every $z \in P$ and $N\tau$ closed set C with $z \notin C$ then $\exists$ a $N\tau\theta$ -open set V , $z \in V \ni$:
 $D \cap N\tau\text{-}cl_\theta(H) = \emptyset$.

Proof:

(i) $\Rightarrow$ (ii) Suppose $z \in P$ and an $N\tau$ open set G , $z \in G$ then $\exists$ disjoint $N\tau\theta$ -open sets $H, W \ni z \in H$ and $P - G \subseteq W$. Since $N\tau\text{-}cl_\theta(H) \cap (P - G) \subseteq N\tau\text{-}cl_\theta(H) \cap W = \emptyset$ then $N\tau\text{-}cl_\theta(H) \subseteq G$. Therefore $z \in H \subseteq N\tau\text{-}cl_\theta(H) \subseteq G$.

(ii) $\Rightarrow$ (iii) Suppose $z \in P$, and C be an $N\tau$ closed set with $z \notin C$ then $\exists$ $N\tau\theta$ -open set V , $z \in V \ni V \subseteq N\tau\text{-}cl_\theta(V) \subseteq P - C$. Hence $N\tau\text{-}cl_\theta(V) \cap C = \emptyset$.

(iii) $\Rightarrow$ (i) Suppose $z \in P$, and C be an $N\tau$ closed set with $z \notin C$, then $\exists$ $N\tau\theta$ -open set V , $z \in V \ni N\tau\text{-}cl_\theta(V) \cap C = \emptyset$. Since $P - N\tau\text{-}cl_\theta(V)$ is a $N\tau\theta$ -open set and $C \subseteq P - N\tau\text{-}cl_\theta(V)$. Further, $V \cap P - N\tau\text{-}cl_\theta(V) = \emptyset$. Hence P is $N\tau\theta$ -regular.

Proposition 5.9 In $(P, N\tau)$, the subsequent are identical:

- (i) P is $N\tau - \theta$ normal
- (ii) if for every $N\tau$ closed set G and an $N\tau$ open set $H \supset G$, $\exists$ a $N\tau\theta$ -open set K ,
 $G \subseteq K \ni N\tau\text{-}cl_\theta(K) \subseteq H$
- (iii) for every two disjoint $N\tau$ closed sets G and L then $\exists$ a $N\tau\theta$ -open set K , $G \subseteq K \ni$:
 $N\tau\text{-}cl_\theta(K) \cap L = \emptyset$.

Proof:

(i) $\Rightarrow$ (ii) Suppose G is $N\tau$ closed set and H is a $N\tau$ open set, $H \supset G$. Now G and $P - H$ are disjoint $N\tau$ closed sets in P . So Then there exist a disjoint $N\tau\theta$ -open sets $K, W \ni G \subseteq K$ and $P - H \subseteq W$. We know that $N\tau\text{-}cl_\theta(K) \subseteq N\tau\text{-}cl_\theta(P - W) = P - W$. Consequently, $N\tau\text{-}cl_\theta(K) \subseteq P - W \subseteq H$.

(ii) $\Rightarrow$ (iii) Suppose $G, L \subseteq P$ are two distinct $N\tau$ closed sets. Now $G \subseteq P - L$ and $P - L$ is $N\tau$ open. As presumption, there exist $N\tau\theta$ -open sets K , $K \supseteq G \ni N\tau\text{-}cl_\theta(K) \subseteq P - L$ and hence $N\tau\text{-}cl_\theta(K) \cap L = \emptyset$.

(iii) $\Rightarrow$ (i) Suppose $G, L \subseteq P$ are two distinct $N\tau$ closed sets then there exist a $N\tau\theta$ -open sets K containing $G \ni N\tau\text{-}cl_\theta(K) \cap L = \emptyset$ and $L \subseteq P - (N\tau\text{-}cl_\theta(K))$. Since K and $P - (N\tau\text{-}cl_\theta(K))$ are distinct $N\tau\theta$ -open sets then P is $N\tau$ normal.

Definition 5.10 A N topological space P is $N\tau T_1$ -space (res., $N\tau T_1^\theta$ space) if $\forall p, q \in P$ with $p \neq q$, $\exists$ an $N\tau$ open set G, H (res., $N\tau$ - θ open) $\ni p \in G, q \notin G$ and $q \in H, p \notin H$.

Proposition 5.11 Let $(P, N\tau)$ be a $N\tau T_1^\theta$ space. The following holds:

- (i) if P is $N\tau$ - θ regular then P is $N\tau$ - θ hausdorff
- (ii) if P is $N\tau$ - θ normal then P is $N\tau$ - θ regular.

Proof:

(i) Let P is $N\tau$ - θ regular. When P is a $N\tau T_1$ -space, $\forall a, b \in P$ with $a \neq b$ then there exist a distinct $N\tau$ open sets G and $H \ni a \in G, b \notin G$ and $b \in H, a \notin H$ which implies $a \notin P - G$ and $b \notin P - H$. Since P is $N\tau$ - θ regular, $\exists$ distinct $N\tau\theta$ -open sets $A, B \ni a \in A$ and $P - G \subseteq B$. Since $b \in P - G$ then $b \in B$. Thus P is $N\tau$ - θ hausdorff.

(ii) Proof is similar.

6. Conclusion

In this article, we explored θ -open sets in N -topological space through a different approach. Moreover, we investigated the relationship of θ -continuous maps, θ -connectedness properties and some new type of separation axioms in this space. Thus, this concept can be used in defining several new topological spaces in addition to several types of mappings.

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Fair Fuzzy Matching in Middle Fuzzy Graph

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Abstract

A fuzzy matching is a set of edges in which an edge does not incident on a vertex with same membership value. If every vertex of fuzzy graph is M -Plunged then the fuzzy matching is called as fair fuzzy matching. In this paper, we introduce the new concept of fair fuzzy matching in Middle fuzzy graph. We discussed some properties based on these concepts in an Absolute Fuzzy Labeling Graph.

Keywords: Fuzzy Middle Graph; fuzzy fair matching; condensation.

2010 AMS subject classification: 03E72, 05C72, 05C78‡

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‡ Received on September 15, 2022. Accepted on December 18, 2022. Published on December 30, 2022. DOI:10.23755/rm.v43i0.820. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

The concept of fuzzy set was introduced by Lotfi A. Zadeh in 1965[13]. He developed a mathematical theory to deal with uncertainty. The advantage of replacing the classical sets by Zadeh's fuzzy sets is that it gives greater accuracy. Rosenfeld [8] introduced the notion of fuzzy graphs in the year 1975. Fuzzy matching concept in fuzzy graph is introduced by S.Yahya Mohamed and S. Suganthi [11] and Complete matching domination in fuzzy labelling graph[12].

Middle graph of a fuzzy graph is an unary operation. It is used to extend the graph together with its fair fuzzy matching. Here we consider an absolute fuzzy graph with even number of vertices because fair fuzzy matching exists only for even number of vertices.

2. Preliminaries

Definition 2.1

Let γ be a non-empty set. A **fuzzy graph** $G = (\alpha, \beta)$ is a combination of two functions, $\alpha: \gamma \rightarrow [0,1]$ and $\beta: \gamma \times \gamma \rightarrow [0,1]$ where for all u, v belongs to γ we have $\beta(u, v) \leq \min\{\alpha(u), \alpha(v)\}$.

Definition 2.2

Two edges of a fuzzy labeling graph $G = (\alpha, \beta)$ is said to be **neighbors** to each other if they incident on a vertex with same membership value.

Definition 2.3

A fuzzy subset M of $\beta(v_i, v_{i+1})$, $1 \leq i \leq n$ is called a **fuzzy matching** M in fuzzy labeling graph $G = (\alpha, \beta)$ if its elements are links (neither self loop nor parallel edges) and no two are neighboring in G . The two ends of an edge in M are said to be **fuzzy tied** under M .

Example 2.4

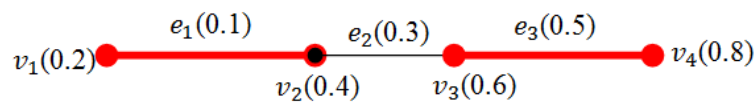


Figure 2.1

In figure 2.1, $\beta = \{e_1(0.1), e_2(0.3), e_3(0.5)\}$, and $f_M = \{e_1(0.1), e_3(0.5)\}$.

Definition 2.5

The vertex v is said to be **M -fuzzy plunged** or **fuzzy plunged by M** if it belongs to the circumstance of the elements of matching M .

Example 2.6

Consider the following fuzzy graph given in figure 2.2

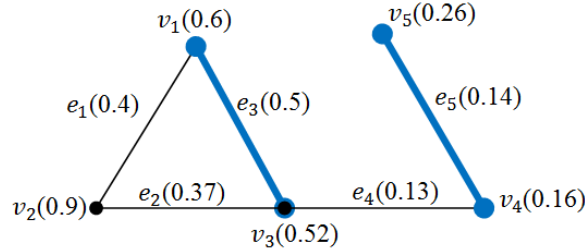


Figure 2.2

In fig 2.2 $M = \{e_3(0.5), e_5(0.14)\}$ is one of the matching in G . Then the vertices $\{v_1(0.6), v_3(0.52), v_4(0.16) \text{ and } v_5(0.26)\}$ are fuzzy plunged by M .

Definition 2.7

Every vertex of the fuzzy labeling graph is fuzzy M plunged then the fuzzy matching M is said to be **fuzzy fair matching**. It is denoted by F_{fm} .

Example 2.8

Consider the following fuzzy graph given in figure 2.3

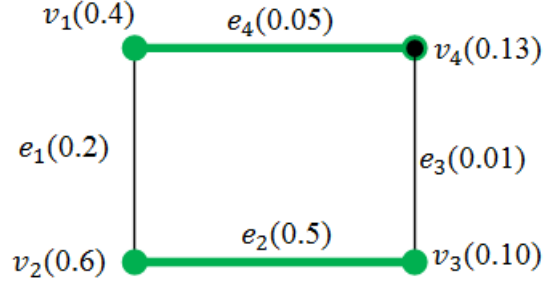


Figure 2.3

In fig 2.3, $M = \{e_2(0.5), e_4(0.05)\}$ is one of the fuzzy matching in fuzzy labeling graph G in which all vertices are fuzzy plunged by M . Hence, M is a fuzzy fair matching.

Definition 2.9

Let M be a fuzzy matching in a fuzzy labeling graph $G = (\alpha, \beta)$. Then M –fuzzy interchanging path (MFIP) is a path whose edges appear alternatively in $(\beta - M)$ and M .

Definition 2.10

An Absolute fuzzy labeling graph $G = (\alpha, \beta)$ is a fuzzy graph G in which $\beta(v_i, v_j) > 0$ for all $v_i, v_j \in V$ and $\beta(v_i, v_j) = \min\{\alpha(v_i), \alpha(v_j)\}$ for all $v_i, v_j \in V$. It is denoted by *AFG*.

Example 2.11

Consider the following fuzzy graph given in figure 2.4

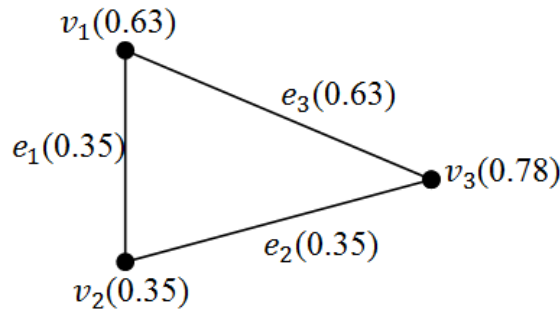


Figure 2.4

3. Main Results

Definition 3.1

Let $G = (\alpha, \beta)$ be a fuzzy graph and M be a fair fuzzy matching in $G = (\alpha, \beta)$. Then the condensation of M is defined as each fair fuzzy matching convert into a single vertex.

Definition 3.2

The minimum number of fair matching to cover all the edges of a fuzzy graph is called the condensation number.

Definition 3.3

Let $G = (\alpha, \beta)$ be a fuzzy graph and M be a fair fuzzy matching in $G = (\alpha, \beta)$. The Middle fuzzy graph $M(G)$ contains

- (i) The vertex set as the union of all vertices in G and the condensation of fair matching. (ie.,) $V(G) \cup C(M)$.
- (ii) The edge set as there exists an edge between v_i and v_j with $\beta(v_i, v_j) > 0$ for all $v_i, v_j \in G$ and also each $C(M)$ has an edge $\beta(v_i, v_j) > 0$ for all $v_i, v_j \in G$, $\beta(v_i, v_j) = 0$ for all $v_i, v_j \in C(M)$.

Theorem 3.4

Let $G = (\alpha, \beta)$ be a fuzzy absolute labelling graph with fuzzy fair matching M . Then the number of edges in $M(G)$ is $3n(n-1)/2$ where n is even.

Proof:

Consider $G = (\alpha, \beta)$ be a fuzzy absolute labelling graph with fuzzy fair matching M .

To find the number of edges in Middle fuzzy graph of G .

The number of edges in $M(G)$

$$\begin{aligned} &= \beta(v_i, v_j) > 0 \text{ for all } v_i, v_j \in G \\ &\quad + \text{number lines joining } C(M) \text{ to all vertices in } G \\ &= n(n-1)/2 + |C(M)| |V(G)| \\ &= \frac{n(n-1)}{2} + (n-1)n \\ &= n(n-1)[1/2 + 1] \end{aligned}$$

Hence the number of edges in $M(G)$ is $3n(n-1)/2$.

Theorem 3.5

Every middle graph of an absolute fuzzy graph with n vertices has $(n-1)$ condensation number. Here n is even.

Proof:

Let $G = (\alpha, \beta)$ be an absolute fuzzy graph and M be the fuzzy fair matching in G .

Now, we can construct the fair matching to cover all edges of G .

The E -count of each fair matching is $n/2$ and the number of edges in an absolute fuzzy graph is $n(n-1)/2$.

For $n = 4$, three distinct fair matching need to cover all edges of G and E -count of each fair matching is 2. Hence $C(M)$ contains 3 points.

For $n = 6$, E -count of each fuzzy fair matching is 3 and five distinct fair matching cover all edges of G . Then these fair matching are converted into five points. Hence the condensation number is 5.

Similarly for $n = 8$, seven fair matching need to cover all edges of G . Then these fair matching are converted into seven points. Hence the condensation number is 7.

In general, an absolute fuzzy labelling graph with n vertices has $(n-1)$ condensation number.

Theorem 3.6

Let $G = (\alpha, \beta)$ be an absolute fuzzy graph with even number of vertices and M be a fair matching in G . Then middle graph of G does not contain fair fuzzy matching.

Proof:

Let $G = (\alpha, \beta)$ be an absolute fuzzy graph with even number of vertices and M be a fair matching in G .

An absolute fuzzy graph with n vertices has $(n-1)$ condensation point. Now we construct the middle graph of G . Also by the definition of middle graph these $(n-1)$ points adjacent to all vertices in G .

Hence $M(G)$ contains the vertex set as union of $V(G)$ and $C(M)$.

Therefore, $M(G)$ contains (even + odd) number of vertices. Always $M(G)$ contains odd number of vertices. Also $M(G)$ is not an absolute graph.

Then we can find a fuzzy matching which does not cover all vertices of $M(G)$ because $C(M)$ adjacent to all vertices in G . Also there exist $\beta(v_i, v_j) > 0 \forall v_i \in C(M)$ and $v_j \in G$. So that the required matching is not fair. Hence $M(G)$ does not have fair fuzzy matching.

Definition 3.7

The number of edges incident to v is called **grade of v** .

Theorem 3.8

Let $G = (\alpha, \beta)$ be an absolute fuzzy graph with even number of vertices and M be a fair matching in G . Then maximum grade is $2(n - 1)$ and minimum grade is n .

Proof:

Let $G = (\alpha, \beta)$ be an absolute fuzzy graph with even number of vertices and M be a fair matching in G .

By the definition, The middle fuzzy graph $M(G)$ contains

- (i) The vertex set as the union of all vertices in G and the condensation of fair matching (ie.,) $V(G) \cup C(M)$.
- (ii) The edge set as there exists an edge between v_i and v_j with $\beta(v_i, v_j) > 0$ for all $v_i, v_j \in G$ and also each $C(M)$ has an edge $\beta(v_i, v_j) > 0$ for all $v_i, v_j \in G$, $\beta(v_i, v_j) = 0$ for all $v_i, v_j \in C(M)$. But $\beta(v_i, v_j) = 0 \forall v_i, v_j \in C(M)$.

Hence the maximum grade of $M(G)$ is $2(n - 1)$ also the minimum grade of $M(G)$ is n .

4 Conclusions

In this paper, we introduced the new concept of fair fuzzy matching in an Middle fuzzy graph. Also we defined condensation of fair fuzzy matching and condensation point. In Future, we will find total fuzzy graph, central fuzzy graph using fuzzy matching and fair fuzzy matching.

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An adaptive Neural Network approach to predict the Capital Adequacy Ratio

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Abstract

Financial institutions, policy makers and regulatory authorities need to implement stress tests in order to test both resilience and the consequences of adverse shocks. The European Central Bank and the European Banking Authority regularly conduct these tests, whose importance is more and more evident after the financial crisis of 2007-2008. The stress tests' non-linear features of variables and scenarios triggered the need of general and robust strategies to perform this task. In this paper we want to introduce an adaptive Neural Network approach to predict the Capital Adequacy Ratio (CAR), which is one of the main ratios monitored to retrieve useful information along many stress test procedures. The Neural Network approach is based on a comparison between feed-forward and recurrent networks, and is run after a meaningful pre-processing operations definition. Results show that our approach is able to successfully predict CAR by using both Neural Networks and recurrent networks.

Keywords: Capital Adequacy Ratio; Stress Tests, Neural Network Approach.¹

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¹Received on July 20, 2022. Accepted on September 20, 2022. Published on September 25, 2022. doi: 10.23755/rm.v43i0.841. ISSN: 1592-7415. eISSN: 2282-8214. ©Di Tollo et al. This paper is published under the CC-BY licence agreement.

1 Introduction

Banks' bankruptcy may have catastrophic effects over the overall economy, since the contagion effect it may trigger could lead to a generalised overall crisis [47]. To this extent the activity of banking supervision, and the role and the authority of bank regulation, play a big role in preventing (or reducing the effect of) the banks' bankruptcy [11, 44]. Although aimed to different targets, many of these supervision exercises are designed to maintain a sufficient banks' level of capital adequacy to allocate specific reserves aimed to face expected losses and to protect themselves against excessive credit expansion. Authorities' regulations impose constraints over these reserves, even though banks often impose themselves reserves higher than the ones imposed by the regulations. These constraints are defined by a minimum capital adequacy ratio that measures the level of capital as a function of the risk borne by the bank [27].

In this framework, *systemic risk* represents the risk of breakdown of the entire financial system: this can be triggered by the misbehavior of a single component of the overall financial system, that triggers negative impacts on the overall system. This scenario can be observed on the timeline of years 2007-2008, starting from some *cracks* in the subprime mortgage markets leading to a worldwide financial crisis: this confirmed the idea that the more complex and non-linear a system, the higher the probability of the system to fail [23, 28]. In order to prevent these failures (and/or to quantify financial (un)stability), financial institutions resort to *stress tests*, which are non linear tools used to assess the magnitude of an exogenous shock and to determine a collapse threshold. These tools operate by investigating both the local stress level (i.e. a single bank) and how the shock is globally spread. Back in the 1990s, stress tests were intended for testing the *resilience* and the stability of a financial institution to reach a certain credibility. Later, stress tests, have been used to check the stability and vulnerability of single financial institutions and the overall banking systems [16]. Based on the evaluation aim and on the implications of the findings, stress tests can be classified into two major groups: microprudential stress tests, which are forward-looking supervisory instruments for determining the liquidity adequacy of individual banks in relation to their portfolio risks [4]; macroprudential stress tests, which consists of two different types of approaches: the bottom-up and the top-down approach [16]. In the bottom-up approach, the effect is measured using data on individual portfolios. On the other hand, in the top-down approach, the impact is estimated by using aggregated data. Many computational methods have been introduced to perform stress test and to predict bank failures: Discriminant Analysis [37], Logit and Probit analysis [8], Neural Networks [53], just to name a few. There are also contributions that compared different methods: for instance, [2] investigates different methods such as Logistic Regression (LR), Linear Discriminant

Analysis (LDA), Random Forests (RF), Support Vector Machines (SVM), Neural Networks (NN) and Random Forests of Conditional Inference Trees (CRF); [18] compared Generalized Linear Models and Generalized Additive Models, and concluded that Generalized Linear Mixed Models have a better ability to predict troubled businesses.

During a stress-test procedure many variables are taken into account to monitor the financial institutions, and there exist several studies aimed to assess the relative *importance* of these variables, in order to select what variable to monitor to get the most accurate information as possible. Many contributions focus in determining the relationships between Capital Ratios and bank failures [1], and to understand whether these ratios are useful to assess the regulatory capital adequacy [42]. In this context, a careful investigation of Capital Ratios is crucial for both the regulatory authority and the bank itself, since it has been shown that the familiar banking characteristics for identifying a distress-prone bank identified fragile banks effectively during the global crisis without new information and are likely to continue to work well in the future [40]. Amongst many variables (e.g., profitability, liquidity, solvency, productivity, asset quality, see [54]), Capital Adequacy Ratio (CAR) has a prominent place, and has been used in the predictor set by many works [39, 31, 14, 38, 33, 5, 48]. Recently, some contributions proposed to predict it as an indicator of financial health [49]. In our contribution we want to expand this framework by implementing an adaptive Neural Network approach to predict CAR from a well established set of indicators, and to provide banks and regulators with useful information about their stress-test activity.

Our contribution is organised as follows: Section 2 reports the main literature on the topic; Sections 3 and 4 outline the set of data and the pre-processing operation performed on it; Section 5 introduces the methods used in this paper; Section 6 comments the main results and Section 7 concludes the paper.

2 Literature review

CAR represents a particularly relevant topic for assessing the risks to which banks are exposed [6]. In fact, for the construction of the CAR index, credit risk, market risk, interest rate risk and exchange rate risk are considered. In this sense, the regulatory authorities define the CAR as a significant indicator of safety and stability as it considers capital as a useful element to absorb losses [34]. Currently, the Capital Adequacy Ratios (CARs) defined by the minimum ratio of capital to risk weighted assets are 8% under Basel II and 10.5% under Basel III [3, 12]. Based on this, CAR represents a factor of analysis by regulators to determine capital adequacy for banks and to perform stress tests [29]. In order to aggregate the information coming from the literature, we performed an analysis on bibliographic

data using the software *VOSviewer* (Figure 1) to create a keyword co-occurrence map in order to analyze the main CAR literature in our field of analysis.

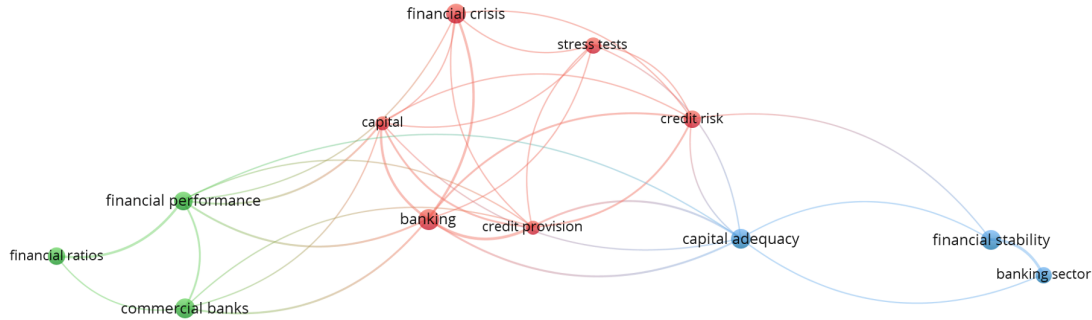


Figure 1: VOSviewer: Capital Adequacy Ratio

From the analysis of the data, it emerges that several authors analyze the required minimum levels of the CAR by evaluating macroeconomic indicators [46], [7], financial indicators [50], multi credit rating indicators [45]. Furthermore, many authors carry out stress tests on CAR to verify the effects of economic crises [58], stability [25] and resilience [15] of banks, along with macro stress test for resilience assessment [20]. Recent studies are moving towards identifying the most important variables for future projections of the CAR. In particular, [50] carry out a study on South Korean national banks using Random Forest Boruta algorithms, Random Forest Recursive Feature Elimination, and Bayesian Regularization Neural Networks. Other contributions use CAR to benchmark the performances of banks in stress tests [3, 12, 24, 27, 29, 32, 59].

The goal of our contribution is to assess whether we can use stress-testing to effectively benchmark the performance of a bank in a precise scenario, and to this extent we need to choose a metrics that can precisely fill that role. CAR is apt to measures the financial soundness of banks in absorbing a reasonable amount of loss, and on the basis of the central role that the CAR assumes in the assessments of banks and on the basis of the guidelines of the literature on the analysis of the minimum levels of the CAR, our work aims to accurately predict the CAR by using quantitative methods. In this framework the quantitative research about stress-testing has been twofold: on one side, to predict the banks' bankruptcy; on the other side, to assess the different variables features and capability to explain the default. According to [54], Neural Networks are widely used in contributions related to the first side, while its application about the other side are still limited. We can start our discussion by pointing out that along with stress testing, a key topic is the prediction of various risks, that was based on traditional probability

and statistical theories [9], but that could lead to non-linear formulations or to taking into account just a few variables, hence triggering the we need of complex and non linear models, also due to the needs of a more interconnected world, not only in financial terms. For this reason researchers and risk managers avail themselves of the usage of Artificial Neural Networks and Deep Learning to stress testing activities and predict high volatility periods. Since more hidden layers are in a Neural Network means a more complex modelling interaction effect, in finance forecasts, large collections of data often require dynamic data relationships that are difficult or impossible to specify under a complete model [9]. On the other hand, deep learning models can identify and manipulate dynamic non-linear data connections that are invisible to any current financial economic theory and may deliver more reliable predictive outcomes than traditional approaches [30]. Although Artificial Neural Networks and Deep Learning have several applications in the financial field such as credit scoring, predictions and forecasts in financial crisis and bankruptcy, we want to focus on how Artificial Neural Networks and Deep Learning Methods are related to stress testing. Financial stability is essential to the economic growth of countries and individuals. Regulatory agencies and foreign organisations carried out stress testing activities to determine the stability of the financial system even earlier than 2007, but failed to anticipate the unprecedented economic implications of the crisis. For this reason ever more stress testing exercises were created and used from the authorities with a glance to the consequences of an interconnected financial system in the macroeconomic environment. For example, the European Banking Authority (namely, EBA) approach uses simplified assumptions that cover only particular risks to individual bank balance sheets depending on the macro-economic scenario. One of the major drawbacks of the European Banking Authority approach is the static financial statement expectation, which allows assets and liabilities to stay stable over the horizon considered without any appreciation of management decisions or new loans. Macroeconomic feedback impacts, such as the influence of large insolvent firms on the global economy, are not generally welcomed assumptions in these systems. This kind of test aims the planning binaries of an after crisis recovery behaviour. However, [52] show that the main problem with respect to the European Banking Authority approach is that this mechanism does not provide an early alarm to avoid being completely disarmed in front of a shock. In [52] it is also provided a solution to this weakness of the model. They propose a Neural intelligence for which financial or macroeconomic disturbances extend to the bank's balance sheets while simultaneously building a large Neural Network with macro and financial factors. The model is capable of gathering more knowledge concealed in a large data set and allows for complex non-linear interactions that materialize under adverse macroeconomic conditions and financial strain. This methodology examines the financial system independently, without relying on the

forecasts of the single banks. As a result of the cited paper, comparing the static stress test models with dynamic ones, prove that the deep learning framework can become a useful tool and can improve the early warning mechanism's signaling ability to anticipate future financial issues and failures of individual banks. The authors, finally, compare the performance of the Deep Learning technique with the classic stress test models, such as the constant balance sheet approach and the dynamic balance sheet approach to satellite modelling. They reveal that the prediction error of the CAR dropped significantly under the Deep Learning Method due to its improved performance in simulating the one-year gains and losses of financial institutions. For this reason, Deep Learning Architecture may become a useful tool for macro prudential stress testing and can improve the early warning mechanism's ability to anticipate future financial crises and failures of individual banks.

Now that we have a measure by which we can benchmark banks, we need to find a way by which we can predict the CAR of banks based on certain factors which is what we need in order to stress test banks. The more factors we can incorporate in our predictions the better since it will reflect better a real-world situation and make our stress testing much more realistic.

3 Data set

Our data set consists of worldwide banks' financial indicators; along with stress financial indicators, we have considered also macro-economic indicators, in order to identify the propagation of systemic shocks that propagate into the financial institutions. We have retrieved quarterly observations that covers a period of 12 years (2007 to 2019).

Data was collected from different sources: stress financial indicators have been collected from the Federal Deposit Insurance Corporation² website³; macro-economics indicators were collected from the Federal Reserve Economic Data⁴ website⁵. The sample period covers twelve years: we have collected quarterly data referring to 672 banks and financial institutions between 2007 and 2019, hence we dispose of 34944 observations. The sample includes missing and noisy

²*FDIC is an independent agency created by the Congress to maintain stability and public confidence in the nation's (USA) financial system. The FDIC insures deposits; examines and supervises financial institutions for safety, soundness, and consumer protection; makes large and complex financial institutions resolvable; and manages receiverships*

³*referred to as <https://www.fdic.gov/FDIC> in what follows*

⁴*referred to as <https://fred.stlouisfed.org/FRED> in what follows*

⁵*Researchers at the St. Louis Fed contribute to monetary policy discussions by advising on a range of topics, especially in preparation for Federal Open Market Committee meetings (from the <https://www.fdic.gov/FDIC> website, accessed on 2021, January 29th).*

Table 1: Variables used in the experimental phase and the category they belong to: the label *FIN* denotes financial indicators and *MAC* denotes macro-economic indicators.

Name	Description	<i>FIN / MAC</i>
net loan	Net loans and leases exposure	<i>FIN</i>
loss allow	Loss allowance to loans	<i>FIN</i>
dep	Total deposits	<i>FIN</i>
yield ea	Yield on earning assets	<i>FIN</i>
fundc ea	cost of funding earning assets	<i>FIN</i>
inc aa	Noninterest income to average assets	<i>FIN</i>
CAR	Total risk-based capital ratio	<i>FIN</i>
tot asst	Average total assets	<i>FIN</i>
tot eq	Average total equity	<i>FIN</i>
tot loan	Average total loans	<i>FIN</i>
risk dens	Risk weight density	<i>FIN</i>
GDP growth	Gross Domestic Product growth	<i>MAC</i>
export growth	US real exports of goods and services growth	<i>MAC</i>
debt GDP	US public debt to GDP	<i>MAC</i>
govex GDP	US government expenditure to GDP	<i>MAC</i>
inflat	Implicit price deflator as a measure of US inflation	<i>MAC</i>
HPI growth	House Price Index growth	<i>MAC</i>
unemp	Unemployment rate (age 15-64)	<i>MAC</i>
Yield 10Y	10-year US sovereign bonds yields	<i>MAC</i>
SP500 ret	SP 500 quarterly returns	<i>MAC</i>

values. Please notice that we have chosen a sample period that does not contain sub-periods denoting the emergence of a crisis, since we want to develop a methodology for *ordinary* periods, in which systemic shocks are more difficult to detect. Collected data show a number of correct entries which is smaller than its theoretical value: this is due to missing and noisy data, and could lead to misbehavior of the Neural Network approach, hence we had to devise pre-processing operations, that are outlined in what follows.

4 Data pre-processing

Data analysis is a key point in all experimental settings, and it is always performed in order to understand its features, to detect anomalies (if any), and to represent data without losing useful information. Based on the observations outlined by [13, 21], we apply the following data pre-processing operations.

Table 2: Variables used in the experimental phase: overall main statistics before pre-processing operations.

Name	Mean	STD	Kurt.	Skewn.	Min	Max
net loan	684062.10	2845178	128.86	10.17	0	75190000
loss allow	10811.40	92866.23	1170.43	29.11	0	5752000
dep	848100.51	4672760	498.48	18.76	68	2180000
yield ea	4.61	1.21	20.01	2.38	0.07	26.96
fundc ea	1.24	0.88	2.67	1.32	0	16.59
inc aa	1.47	18.11	722.73	26.40	-15.95	601.27
CAR	23.68	15.42	113.18	6.08	0.75	725.80
tot asst	1114221	5587407	302.58	14.46	2816	2110361
tot eq	125720.10	566772.30	129.17	10.33	539.75	14389800
tot loan	683913.22	2852402	129.57	10.19	0	71201027
risk dens	60.09	14.28	0.63	0.06	8.43	192.24
GDP growth	1.73	2.35	4.05	-1.53	-8.45	5.51
exp. growth	3.63	8.17	5.14	-1.19	-28.65	25.84
debt GDP	93.95	11.27	1.38	-1.49	61.65	105.18
govex GDP	0.34	0.01	-1.09	0.37	0.31	0.37
inflat	100.43	4.51	-0.94	0.12	91.70	111.25
HPI growth	201.26	20.39	0.22	0.92	176.86	264.31
unemp	7.22	1.89	-1.40	-0.07	3.78	10.05
Yield 10Y	2.64	0.72	-0.02	0.70	1.56	4.84
SP500 ret	0.02	0.06	6.76	-1.47	-0.27	0.17

4.1 Removal and replacement

When collecting data, one may incur in missing and incorrect values. Previous contributions related to Neural Network approaches [21] suggested to remove indicators containing more than 30% of missing and wrong values. Our set of data does not contain such indicators, so we are using the whole set of variables in our experimental phase. Anyhow, many indicators show missing and wrong values, so we replace missing values (due to computational errors) with the upper limit of the normalization (see what follows), and wrong values with the indicator's average over time.

4.2 Normalization

Normalization is a general procedure performed in order to feed the Neural Network with data belonging to the same range: many contributions stress the importance of performing meaningful normalization, and many formulas are sug-

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gested [35]. In our case we used the logarithmic transformation that has been already introduced by [21], defined as follows:

$$\bar{x}_i = \log_u (|\min(0, x_{min})| + x_i + 1), \quad (1)$$

where x_i represents the value before normalisation of input x for firm i , and $\bar{x}_i$ represents its normalised value. Please notice that we have defined

$$u$$

such that

$$u = x_{\max} + 1$$

this has been imposed in order to have $\bar{x}_i \in [0, 1]$.

Table 3: Main statistics of overall financial and macro-economic indicators after pre-processing operations.

Name	Mean	STD	Kurtosis	Skewness	Min	Max
net loan	0.67	0.08	1.41	0.42	0	1
loss allow	0.48	0.10	1.29	0.25	0	1
dep	0.64	0.07	1.38	0.58	0.22	1
yield ea	0.53	0.06	5.19	-0.07	0.02	1
fundc ea	0.30	0.14	-0.28	0.53	0	1
inc aa	0.41	0.06	6.68	0.05	0	1
CAR	0.50	0.07	1.23	0.86	0.08	1
tot asst	0.65	0.07	1.38	0.66	0.42	1
tot eq	0.63	0.08	1.36	0.71	0.38	1
tot loan	0.67	0.08	1.43	0.43	0	1
risk dens	0.79	0.05	1.48	-0.74	0.44	1
GDP growth	1.06	0.31	2.58	-1.52	0	1
export growth	0.99	0.2	10.45	-3.09	0	1
debt GDP	0.97	0.02	2.11	-1.71	0.89	1
govex GDP	0.93	0.03	-1.09	0.36	0.86	1
inflat	0.98	0.01	-1.17	0.03	0.96	1
HPI growth	0.96	0.01	-0.83	0.01	0.92	1
unemp	0.86	0.10	-1.27	-0.28	0.65	1
Yield 10Y	0.79	0.11	-0.93	0.10	0.55	1
SP500 ret	1.12	0.57	-1.08	-0.09	0	1

4.3 Correlation analysis

We have performed a correlation analysis in order to understand whether some kind of correlation arise amongst variables defined in Section 3 and to avoid feeding the network with highly-correlated indicators. We have tested Pearson's, Kendall's, and Spearman's correlation, leading to similar trends in the obtained correlations. In what follows we will refer to Spearman's ranked based correlation. We have decided to remove from the predictor set the indicators showing a correlation with a given portion (i.e., j) of other indicators greater than a given threshold (i.e., h). In order to determine the value of j and h we have defined parameter-tuning procedure via REVAC (see [43]). The values found have been $j = \frac{1}{3}$ and $h = 0.70$. On the basis of these values, we have decided to remove indicators showing a correlation with 30% of the other indicators greater than 0.7. These indicators are: net_{loan} , $loss_{allow}$, dep , tot_{asst} , tot_{eq} , tot_{loan} . They will not be considered in what follows: 6 indicators have been removed from the predictors set, corresponding to 24 quarterly indicators that will not be used to feed the Neural Networks' nodes.

As for the Neural Network experiments, in what follows we are outlining results of the experiments run by using data before the pre-processing operations (referred to as *full model*) and data after the pre-processing operations (referred to as *reduced model*).

5 Experimental analysis

In this section we are introducing the methods used to perform our experimental analysis: the Neural Network approach will be detailed in Section 5.1, along with the main components needed to define its use, i.e., the network topologies (Section 5.1.1) and the partitioning of the set of data to enforce generalisation (Section 5.1.2). Then, we are introducing the methods we are comparing our approach with (Linear Regression in Section 5.2, and Generalised Linear Models in Section 5.3), along with the metrics used for our comparisons in Section 5.4.

5.1 Neural networks

In this section we are introducing our Neural Network approach: Artificial Neural Networks [22] can be referred to as algorithms that mimic the behavior of the human brain to perform complex tasks, and they are used to grasp non-functional relationships over the data. They are composed of elementary units (*neurons*) which are connected to each other via weighted and oriented links (*synapses*). Neurons may have different functions: the *input* neurons receive data

from external sources; the *output* neurons show the computed output values; the *hidden* neurons are used to perform computations. During the *learning* phase, the weights associated to the synapses iteratively change over time, accordingly to a specific algorithm: several algorithms have been proposed for this *learning* phase: *Back-Propagation* [56], *Quasi-Newton methods* [55], *Levenberg-Marquardt algorithm* [26], just to name a few. The *learning* phase may be organised following three different paradigms: *supervised* learning [36]; *unsupervised* learning [10] and *reinforcement* learning [57]. In what follows, we are training networks introduced in Section 5.1.1 by using *back-propagation* algorithm, in order to minimize the network test set's root mean square error (RMSE) defined as

$$\sqrt{\frac{1}{n} \sum_{i=1}^n (e_i - a_i)^2}, \quad (2)$$

where n is the test set size, e_i is expected output value corresponding to pattern i , and a_i is the actual network output corresponding to input pattern i .

5.1.1 Network topologies

For our experiments we are using two different Neural topologies: a *feed-forward*⁶ architecture with 80 inputs nodes, referred to as *standard* network (see Figure 2), and a variant in which inputs neurons corresponding to the same indicator are grouped by 4 before feeding the first feed-forward layer⁷ (see Figure 3).

Please notice that, as for the cardinality of hidden neurons, many rules of thumb exist, suggesting different formulas to compute the number of hidden layers and the number of hidden neurons [19]. We have decided not to use any of these rules, resorting to an adaptive method to determine the optimal hidden neurons structure: this procedure has been proposed by [17], and it aims to minimize the network's error (in our scenario, the Eq. 2) calculated for each of the data set at hand. This adaptive procedure starts with a single hidden neuron and iteratively add one neuron until no improvement on the Eq. 2 is found over the last user-defined K iterations, and is outlined in Algorithm 1.

⁶A feed-forward network features neurons grouped into layers $(1, 2, \dots, l_{max})$: each neuron belonging to layer i ($i < l_{max}$) is associated to synapses that connect itself to all neurons belonging to layer $i + 1$.

⁷These four values correspond to past observations spreading over one year, since for each indicator i corresponding to time t the input pattern contains the value of the indicators collected at time t , together with the 3 previously quarterly collected values.

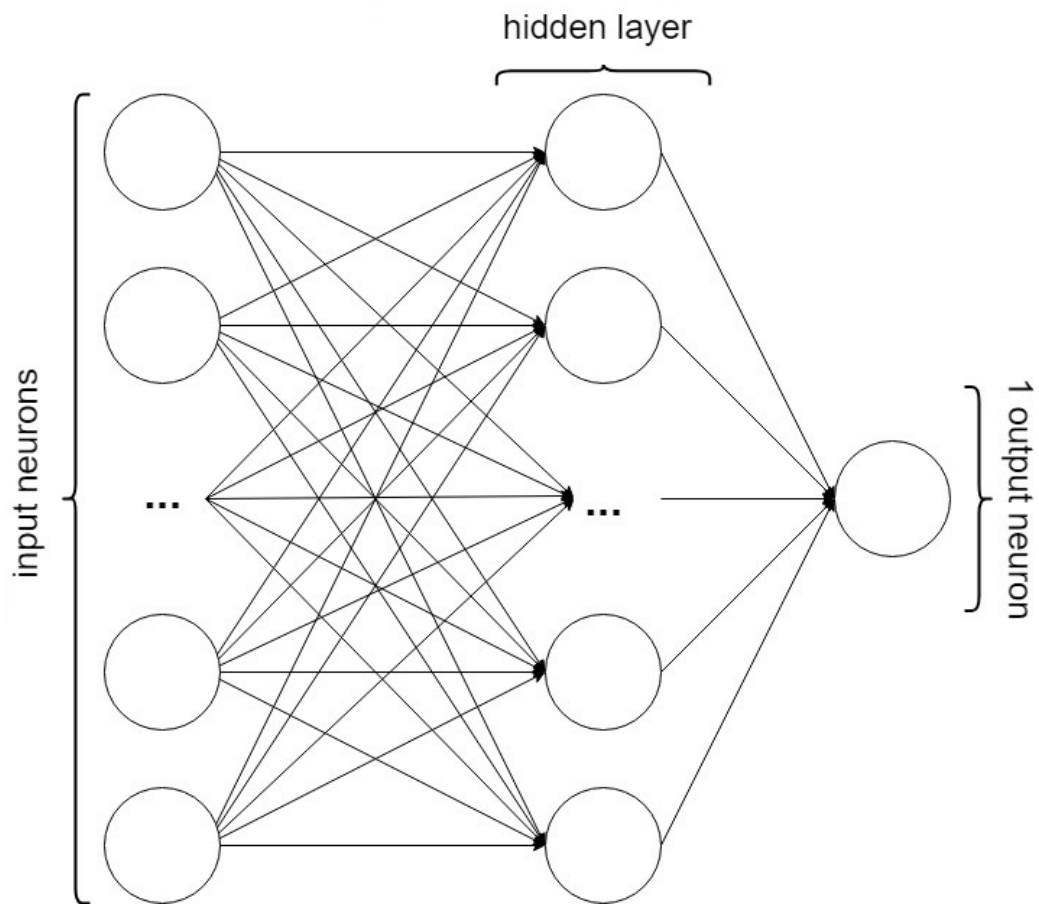


Figure 2: *Standard network.*

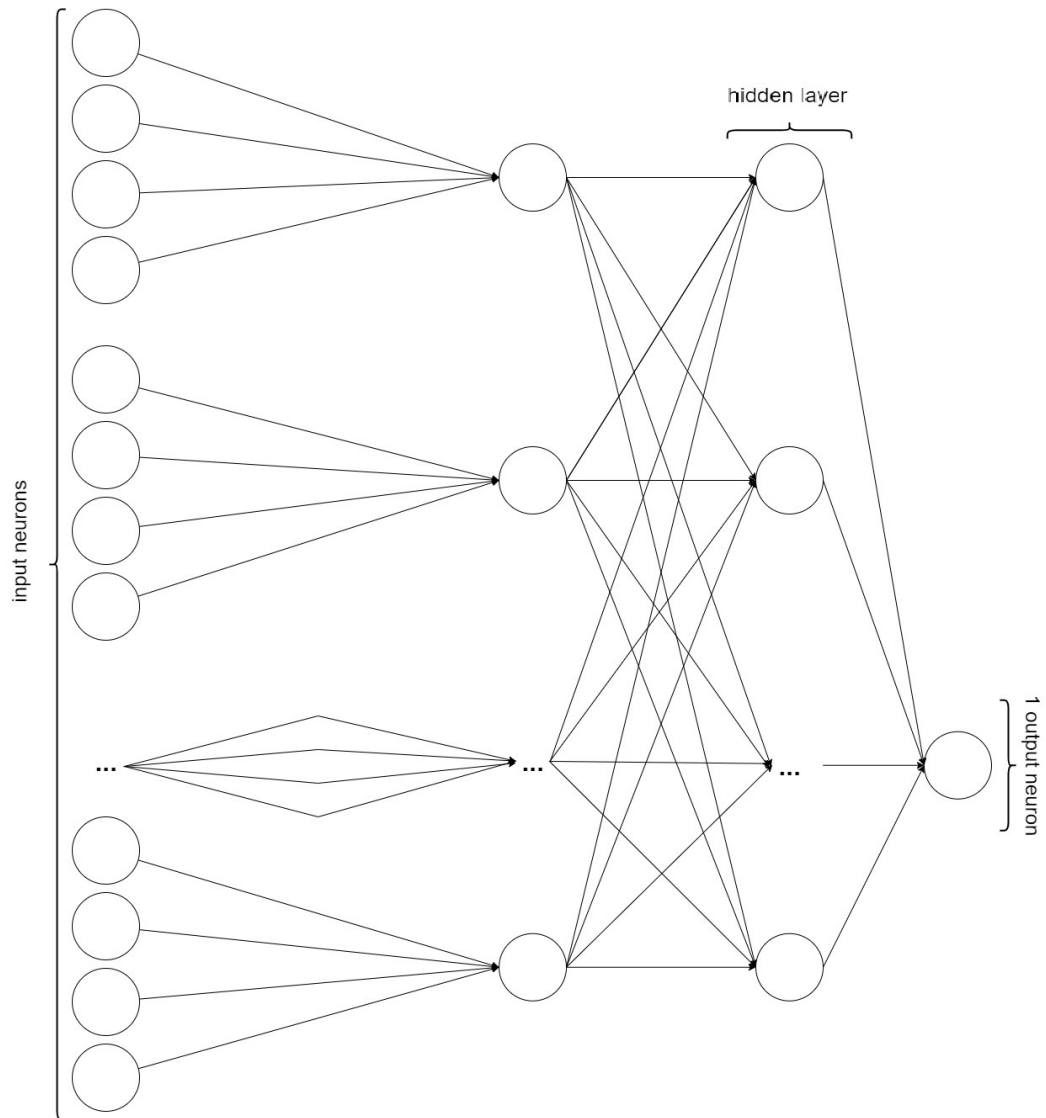


Figure 3: *Ad-hoc* network: input neurons are grouped by four, indicating the observations over a year of the same variable.

Algorithm 1: Adaptive hidden neurons computation for Neural Networks

Initialization: observational data set

The optimal network topology w.r.t. the error defined in Eq. 2. b in $1, \dots, \#$ sub-sampling runs $X_b \leftarrow$ Dataset at b^{th} sub-sample $X_{train_b} \leftarrow$ Training set at b^{th} sub-sample run $X_{test_b} \leftarrow$ Test set at b^{th} sub-sample run i in $1, \dots, \#$ of hidden layer $k \leftarrow 0$ $j \leftarrow 1$ $k \leq K$ **train** net Net_{ij} on X_{train_b} **compute** $RMSE_{ij}$ error with Eq. 2 on the X_{test_b} $Overall_{ij} \leq$ **all** $RMSE_{ij}$ $BestNet \leftarrow Net_{ij}$ $Neurons \leftarrow (i, j)$ $Overall_{ij} > RMSE_{Neurons}$ $k++$ $k \leftarrow 0$ $j++$ $BestNet_b \leftarrow BestNet$ $Neurons_b \leftarrow Neurons$ **return** $BestNet$, $RSME=Eq. 2$, $Neurons$

5.1.2 Training and test set

In our experiments we are exploiting the *supervised* learning, meaning that during the *learning* phase, for each input pattern, we are also providing the desired output value, that in our scenario corresponds to CAR: all other indicators considered in Table 3 will define the input pattern for each financial institution. In order to grasp the time dynamics, for each input indicator i we are providing to the network the value of the indicators collected at time t , together with the values collected at time $(t - 1)$, $(t - 2)$, and $(t - 3)$. During the Neural Network learning we have to identify two disjoint sets of observations out of the overall 34944 observations: the *training set*, that will be used to determine the synapses' weights, and the *test set*, that will be used to determine the network performance and to stop the learning. According to [19], we have decided to split the overall data by randomly allocating the 70% of its observations to the *training set*, and the remaining 30% to the *test set*. This random allocation has been repeated 50 times, each time determining a different *train-test* partition. We have then run our Neural network approaches on all obtained partitions, and in what follows we are reporting, for each Neural approach, the average and standard deviation statistics over the 50 partitions.

5.2 Linear Regression

Linear Regression is used to model the relationship between two or multiple parameters by fitting a linear equation on the observed data. Usually, this is done using the *least-square* regression that minimizes the sum of squares of the vertical deviation from each data point on the line. The algorithm aims to reduce this sum by selecting the most appropriate constants in the equation representative of the regression line.

A Linear Regression line has an equation of the form $Y = a + bX$ where X is the explanatory variable and Y is the dependent variable. The slope of the line is b , and a is the intercept (the value of y when $x = 0$) [60].

5.3 Generalised Linear Models (GLMs)

The basic Linear Regression predicts a certain value as a linear combination of a specific set of observed values, meaning that a change in one or multiple predictors affects the response variable. However, for complex data, Linear Regression is not very effective, and in these cases one may resort to Generalized Linear Models [41], that allow response variables to have arbitrary distributions (rather than normal distributions), and define an arbitrary function of the response variable (the link function) to vary linearly with the predictors, rather than assuming that the response itself must vary linearly. A generalized Linear Model (GLM) consists of three elements: Linear predictor; Link function; Probability distribution or exponential family. The linear predictor is the linear combination of parameter b and explanatory variable x . The link function is what links the linear predicted and the probability distribution: there are many link functions and usually, they are used depending on the features of data we are trying to predict and in which range are we expecting it to be.

5.4 Evaluation of the model

Once we build a model through Linear Regressions (or GLMs), we need to measure the correctness of this model. This can be done via different statistical measures, that are used to benchmark the performance of predictive models.

5.4.1 Root mean squared error

The root mean squared errors represents the standard deviation of the prediction errors. by that, we mean that it tells us how concentrated the data is around the line that we predict. To calculate it we can take the square root of the Mean squared errors.

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_i)^2}.$$

where n is the number of predictions, Y_i the observed values, and $\hat{Y}_i$ being the actual prediction of that variable.

5.4.2 R-Squared

The R-Squared represents how well the data fits on the regression line. More

generally, it is used to analyze how the difference in one variable can be explained by other variables. In the case of regression, we can reason in percentages and say that the closer the measure is to 1 the closer the points are to the regression line up to 1 where 100% of the points are on the line. Generally, this would mean that the higher R-squared is the better the results we have but this can be false in some edge cases. It is calculated by squaring the correlation coefficient calculated with this formula

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}$$

where SS_{res} is the sum of squares of residuals, and SS_{tot} is the residual sum of squares.

5.4.3 F-statistic

The *F-statistic* in a regression is a value that represents how well you improved the regression line compared to a regression line with all the coefficients = 0. if your model significantly improved the model fit then you will get a better *F-statistic*. But before taking into account the *F-value* one must first look at the *P-value* that is calculated at the same time as the *F-statistic*. With the *F-statistic* calculation comes the *P-value*. Usually the *P-value* is looked at before taking the *F-statistic* into account. If the *P-value* is lower than the alpha level, then we can reject the null hypothesis and we can consider the *F-value*, otherwise the *F-values* is worthless.

6 Results and discussion

In this section we report the principal results to build a model that is performs well on our benchmarks.

All Neural approaches have been implemented in Python, exploiting the library Tensor-Flow. Experiments have been run on a on a cluster with AMD Opteron 2216 dual core CPUs running at 2.4 GHz with 2x1 MB L2 cache and 4 GB of RAM under Cluster Rocks distribution built on top of CentOS 5.3 Linux.

Table 4 reports the RMSE of the experiments run with both the *Standard* and *Ad-hoc* networks. We have performed 50 runs of the adaptive procedure devised in Algorithm 2 and reported the minimum, maximum, mean, median, and standard deviation of the RMSE distribution, for each possible instantiation of the pair $[network, model]$.

	Standard Reduced model	Standard Full model	Ad-hoc Reduced model	Ad-hoc Full model
Min	0.070	0.071	0.040	0.040
Max	0.074	0.076	0.045	0.051
Mean	0.072	0.073	0.042	0.045
Std	0.001	0.001	0.001	0.003
Median	0.072	0.073	0.042	0.046

Table 4: RMSE of the experiments run with the *Standard* and *Ad-hoc* networks. For each columns. For each column, statistics over 50 runs of the proposed adaptive procedure are reported.

As a first remark, we can see that the pre-processing operation have a valid role in improving the networks' performances, in both *Standard* and *Ad-hoc* networks. Then, we can see that the *Ad-hoc* networks' error is lower than the *Standard* one. This confirms the results found by [17] and [21], in which authors exploit the fact that the *Ad-hoc* network is able to grasp the temporal dependence of inputs.

Then, we are presenting the results obtained with Linear Regression and with GLMs (both developed using the Sci-kit learn library on Python[51]), and then, we are comparing them with the Neural Network approach devised in section 5.1. As a first experiment, we have implemented a regression approach over the whole dataset, and the results are shown in table 5: in this table we report the R^2 relative to experiments performed with Linear Regression and GLMs, along with two variants: the Adjusted R-Squared (that takes into account the number of predictors) and the Predicted R-Squared (that takes into account overfitting). Please notice that for GLM we have used the *pseudo R-squared* (for sake of definition) and that we also report the *P-value* of the *F-test* (i.e., the probability of obtaining an *F-statistic* value that is greater than the model's *F-value*, under the null hypothesis that the regression model is not significant: low positive values identifies good fit). We have performed experiments by using as predictors both the total set of variables identified after the pre-processing phase (identified by the entry *Whole set of data* in the table), and a limited set of predictors composed of all predictors that are significant for the regression according to their *P-value* (identified by the entry *All observations, limited set of predictors*). We see that the reduction of predictors does not improve the goodness of the fit according to the different metrics used, so in what follows we are using the whole set of predictors. In this direction, we remark that Generalised linear models lead to a better *R – squared*, but this comes at the cost of a higher overfitting, as witnessed by the lower value of the *PredictedR – squared*. Linear Regression instead, leads to a worse (but still acceptable) *R – squared*, but its difference with the *PredictedR – squared*

is lower, showing a better robustness of the approach. Please notice that all regressions are significant according to the P -value of the F -test. We recall that these experiments have been performed on the whole set of data. In what follows we will describe experiments performed on different partitions of *training/test* sets, in order to compare these approaches with our Neural Network approach.

Table 5: Linear Regression and GLMs over the whole set of data: measures of the goodness of the fit.

Data Used	Model	statistic	values
All observations, limited set of predictors	Linear Regression	Predicted R-squared	0.77
		R-squared	0.78
		Adj R-squared	0.78
Whole set of data	Linear Regression	Predicted R-squared	0.77
		R-squared	0.78
		Adj R-squared	0.78
		P-value of the F-test	0.0
Whole set of data	GLM	Predicted R-squared	0.71
		Pseudo R-Squared	0.91
		P-value of the F-test	0.0

Please notice that the previous results have been obtained on the whole set of data. In order to test the generalization capability of our approach, we have split the whole set of data in 50 different *training/testing* partitions (i.e., the same partitioning generated in Section 5.1.2), built our model to fit the training set, and assessed the goodness of the fit on the test set. This has been done for all regression detailed in Table 5, along with the Neural approaches devised in Table 4, and the goodness of fit (assessed by the $R - squared$) has been reported in Table 6, as computed on the error distributions displayed in Table 4.

By looking the results, we see that the best results are offered by the *Ad-hoc* Neural Networks, but also the Standard network performs fairly well. This is due to the generalisation skill of the network, able to prevent the overfitting we have found on the aforementioned regression approaches. This confirms the goodness of our adaptive procedure, that has been proposed by [17] in a different context, but that can be tailored to the different application scenarios.

Table 6: Experiments with Linear Regression, Generalised Linear Models, and Neural Networks (Standard and Ad-hoc). Statistics of the R-Squared computed on the test sets of 50 different *training/testing* partitions.

Model	statistic	values
Linear Regression	min	0.76
	max	0.80
	mean	0.77
	std	0.01
GLM	min	0.69
	max	0.72
	mean	0.70
	std	0.01
Standard NN	min	0.71
	max	0.90
	mean	0.81
	std	0.15
Ad-hoc NN	min	0.64
	max	0.92
	mean	0.84
	std	0.13

7 Concluding remarks

The bankruptcy of banks may lead to a huge catastrophic effect over the overall economy, since the contagion effect it may trigger could lead to a generalised overall crisis. To this extent, the activity of banking supervision, and the role and the authority of bank regulation, play a big role, since they may prevent (or reduce the effect of) the banks' bankruptcy. Although aimed to different targets, many of these supervision exercise are designed to maintain a sufficient level of capital adequacy: in a way, the banks have to allocate specific reserves to face expected losses and to protect themselves from excessive credit expansion. Bank regulations impose constraints over these reserves, even though banks operate themselves preventively against unexpected crises, and their reserves are often higher than the ones imposed by the regulations. The CAR is one of the main indicators monitored by the banks themselves and by the supervising authorities in order to assess the bank health, and in our contribution we have devised an adaptive Neural Network approach to predict the CAR, and compared the obtained results with standard approaches such as *Linear Regression* and *Generalised Linear Models*.

Results show that Neural Networks may be successfully used to predict the CAR, and that their outcomes compare favourably with standard methods when used jointly with meaningful pre-processing operations. In future research, modern recurrent neural networks (Long Short-Term Memory or Gate Recurrent Units) and 1D-convolutional Neural Networks could be used to exploit the time dependency of the data.

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Group Decision Making in Conditions of Uncertainty using Fermat's Weak Fuzzy Graphs and Beal's Weak Fuzzy Graphs

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Abstract

Decision making is a process of solving problems for choosing the best alternative. The best way to illustrate the alternatives and relation between them is a graph. Developing a fuzzy graph is the convenient way of illustration if there is uncertainty in alternatives or in their relation. In group decision making problems, according to a group of experts, the relation between alternatives involves measure of preference and non preference. Intuitionistic fuzzy graph has limitations to model such problems. In n- Pythagorean fuzzy graphs the hesitancy degree and other decision tools are restricted to second degree. To overcome the flaws of intuitionistic fuzzy graphs and n- Pythagorean fuzzy graphs, we introduced Fermat's Fuzzy Graphs in 2022. In this paper the decision tools are generalized for Fermat's Fuzzy Graphs. A practical example of selection of investment scheme is illustrated. Finally, Beal's Fuzzy graphs is developed as generalization of Fermat's Fuzzy Graphs.

Keywords: Fuzzy Graph; Weak Fuzzy Graph; Fermat's Fuzzy Graph; Fermat's Weak Fuzzy Graphs; Beal's Fuzzy Graph; Beal's Weak Fuzzy Graphs.

2010 AMS subject classification: 03E72,05C72.

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‡Received on September 15, 2022. Accepted on December 15, 2022. Published on December 30, 2022.
DOI:10.23755/rm.v43i0.854. ISSN: 1592-7415. eISSN: 2282-8214. © Nishad et al. This paper is published under the CC-BY licence agreement.

1. Introduction

L A Zadeh in 1965[1] introduced fuzzy sets to describe the vagueness phenomena in real world problems. In 1975[2], Azriel Rosenfeld introduced fuzzy graphs. A. Prasanna and T M Nishad introduced weak fuzzy graphs in 2021 [3]. In 2009, Hongmei and Lianhua defined Interval Valued Fuzzy Graph (IVFG) [4] and in 2013 Talebi and Rashmanlou studied properties of isomorphism and complement of an IVFG[5]. To overcome the flaws of Intuitionistic Fuzzy Graph in simulation, Muhammed Akram, Amna Habib, etc. discussed specific types of Pythagorean Fuzzy Graphs and applications to decision making in 2018[6]. Fermat's Weak Fuzzy graph and hesitancy degree in general scale are discussed by B.M Harif and T.M Nishad in 2022[7]. The American Banker and amateur mathematician Mr. Daniel Andrew Beal formulated the Beal's conjecture in 1993 [8] as a generalization of Fermat's Conjecture. The contents of this article are as follows.

In section 2 some fundamental concepts of Fuzzy Graphs and Fermat's Fuzzy Graphs are reviewed. Section 3 illustrates the mathematical model of a group decision making problem using Fermat's weak Fuzzy Graph. Section 4 describes the generalized decision tools for Fermat's Fuzzy Graphs. A practical example of selection of investment scheme is illustrated in section 5. In section 6, the fundamental concepts of Beal's Fuzzy Graph and some theorems are developed. The whole article is concluded in section 7.

2. Some Fundamental concepts of Fuzzy Graphs

A mapping $m: A \rightarrow [0,1]$ from a non empty set A is a fuzzy subset of A . A fuzzy relation r on the fuzzy subset m , is a fuzzy subset of $A \times A$. A is assumed as finite non empty set.

Definition 2.1: Suppose A is the underlying set. A fuzzy graph is a pair of functions $G: (m, r)$ where fuzzy subset $m: A \rightarrow [0,1]$, the fuzzy relation r on m is denoted by $r: A \times A \rightarrow [0,1]$, such that for all $u, v \in A$, we have $r(u, v) \leq m(u) \wedge m(v)$ where $\wedge$ stands minimum.

$G^*: (m^*, r^*)$ denotes the underlying crisp graph of a fuzzy graph $G: (m, r)$ where $m^* = \{u \in A / m(u) > 0\}$ and $r^* = \{(u, v) \in A \times A / r(u, v) > 0\}$. The nodes u and v are known as neighbours if $r(u, v) > 0$.

Definition 2.2.: A fuzzy graph $G: (m, r)$ is a strong fuzzy graph if $r(u, v) = m(u) \wedge m(v), \forall (u, v) \in r^*$.

Definition 2.3: A fuzzy graph $G: (m, r)$ is a weak fuzzy graph if $r(u, v) < m(u) \wedge m(v)$ for all $(u, v) \in r^*$.

Definition 2.4: A Fermat's Fuzzy Set (FFS) on a universal set A is a set of 3 tuples of the form $F = \{(u, I_F(u), O_F(u))\}$ where $I_F(u)$ and $O_F(u)$ represents the membership and non membership degrees of $u \in A$ and $I_F(u), O_F(u)$ satisfy $0 \leq I_F^n(u) + O_F^n(u) \leq 1$ for all $u \in A, n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

Definition 2.5: A Fermat's fuzzy relation (FFR) R on $A \times A$ is a set of 3 tuples of the form $R = \{(uv, I_R(uv), O_R(uv))\}$ where $I_R(uv)$, and $O_R(uv)$ represents the membership degree and non membership degree of uv in R and $I_R(uv), O_R(uv)$ satisfy $0 \leq I_R^n(uv) + O_R^n(uv) \leq 1$ for all $uv \in A \times A$. FFR need not be symmetric. Hence $I_R(uv)$ need not be equal to $I_R(vu)$.

Definition 2.6: A Fermat's fuzzy graph (FFG(n)) on a non empty set A is a pair $G : (\sigma, \mu)$ with σ as FFS on A and μ as FFR on A such that $I_\mu(uv) \leq I_\sigma(u) \wedge I_\sigma(v)$, $O_\mu(uv) \geq O_\sigma(u) \vee O_\sigma(v)$ and $0 \leq I_\mu^n(uv) + O_\mu^n(uv) \leq 1$ for all $u, v \in A, n \in \mathbb{N} = \{1, 2, 3, \dots\}$.where

$I_\mu : A \times A \rightarrow [0, 1]$ and $O_\mu : A \times A \rightarrow [0, 1]$ represents the membership and non membership functions of μ respectively.

Definition 2.7: A Fermat's fuzzy preference relation (FFPR) on the set of nodes $N = \{x_1, x_2, \dots, x_n\}$ is represented by a matrix $M = (m_{ij})_{n \times n}$, where $m_{ij} = (x_i x_j, I(x_i x_j), O(x_i x_j))$ for all $i, j = 1, 2, 3, \dots, n$. Let $m_{ij} = (I_{ij}, O_{ij})$ where I_{ij} indicates the degree to which the node x_i is preferred to node x_j and O_{ij} denotes the degree to which the node x_i is not preferred to the node x_j and $\pi_{ij} = \sqrt[n]{1 - I_{ij}^n - O_{ij}^n}$ is interpreted as hesitancy degree, with the conditions, $I_{ij}, O_{ij} \in [0, 1]$, $0 \leq I_{ij}^n + O_{ij}^n \leq 1$, $I_{ij} = O_{ji}$, $I_{ii} = O_{ii} = 0.5$ for all $i, j = 1, 2, 3, \dots, n$.

Definition 2.8: A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be Fermat's Strong fuzzy Graph FSFG(n) with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v)$, $O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$

Definition 2.9: A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be Fermat's Weak Fuzzy Graph FWFG(n) with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) < I_\sigma(u) \wedge I_\sigma(v)$, $O_\mu(uv) > O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$

Definition 2.10 : A Fermat's fuzzy graph $G : (\sigma, \mu)$ is said to be complete FFG with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v)$, $O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$

$O_{\sigma}(v)$ for all $u, v \in \sigma^*$.

3. Modeling of Group Decision Making Problem

Example 3.1: Mr. X from India wish to invest money in any of the following 5 schemes that helps him better financial security in future.

1. Public Provident Fund $\square$ S1 $\square$
2. National Saving Certificate $\square$ S2 $\square$
3. Atal Pension Yojana $\square$ S3 $\square$
4. National Pension Scheme $\square$ S4 $\square$
5. Sovereign Gold Bonds $\square$ S5 $\square$

He consulted with 4 experts and they advised the merits and demerits of each particular scheme comparing with other. The aggregate of information FFPR is prepared as relation matrices. How can he select the best Scheme?

Modeling: Suppose the 5 schemes are S1, S2, S3, S4 and S5. Consider the discrete set of alternatives $A = \{S1, S2, S3, S4, S5\}$. Since the alternatives are present, assign the membership degree as 1 and non membership degree 0 to each alternatives. Consider the set of experts as $\{E1, E2, E3, E4\}$. Since each experts gives the acceptance and rejection reasons comparing every pair of alternatives, the aggregate of information FFPR can be represented as relation matrices. This data represents a FFG(n). If in the given FFPRs, all the membership values are in (0,1) and non membership values are greater than 0 then the given FFG(n) will be FWFG(n).

4. Decision tools for Fermat's Fuzzy Graph

In decision making, the Optimal Score having maximum rank is considered as best choice. The scores to rank the alternatives can be calculated using score function $S(p_i)$. Here p_i is the collective Fermat's Fuzzy Element which can be obtained using Fermat's Fuzzy Weighted Averaging Operator FFWA. The weight of each expert can be obtained using deviations of each experts and the deviations can be calculated from difference matrices. The entries in difference matrix is calculated using Fermats Fuzzy Hamming distance between Fermat's Fuzzy Elements.

4.1 Fuzzy Averaging operator FFA

$$FFA(p_{i1}^{(k)}, p_{i2}^{(k)}, \dots, p_{im}^{(k)}) = \left(\sqrt[n]{1 - \left(\prod_{j=1}^m (1 - I_{\mu_{ij}}^n) \right)^{1/m}}, \left(\prod_{j=1}^m (O_{\mu_{ij}}) \right)^{1/m} \right), i = 1, 2, 3, \dots, m.$$

Fermat's Fuzzy Element (FFE) $p_i^{(k)}$ indicates preference of each expert E_k over each pair of alternatives.

It is determined using Fermat's Fuzzy Averaging operator FFA

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$p_i^{(k)} = FFA(p_{i1}^{(k)}, p_{i2}^{(k)}, \dots, p_{im}^{(k)}), i=1, 2, 3, \dots, m$. FFE is used in calculation of FFWA.

4.2 Fermat's Fuzzy Hamming distance between FFEs.

From the given FFPR,

$$d(p_{ij}^{(l)}, p_{ij}^{(k)}) = \frac{1}{2} \left(\left| I_{\mu_{p_{ij}^{(l)}}}^n - I_{\mu_{p_{ij}^{(k)}}}^n \right| + \left| O_{\mu_{p_{ij}^{(l)}}}^n - O_{\mu_{p_{ij}^{(k)}}}^n \right| + \left| \pi_{p_{ij}^{(l)}}^n - \pi_{p_{ij}^{(k)}}^n \right| \right), \text{ where}$$

$$\pi_{p_{ij}^{(l)}} = \sqrt[n]{1 - I_{\mu_{p_{ij}^{(l)}}}^n - O_{\mu_{p_{ij}^{(l)}}}^n}, \pi_{p_{ij}^{(k)}} = \sqrt[n]{1 - I_{\mu_{p_{ij}^{(k)}}}^n - O_{\mu_{p_{ij}^{(k)}}}^n} \text{ are hesitancy degree.}$$

4.3 Difference matrix

$$D_{lk} = (d_{ij}^{(lk)})_{m \times m} = d(p_{ij}^{(l)}, p_{ij}^{(k)})_{m \times m}$$

4.4 Average Values of Difference Matrix

The equation to determine average values of difference matrix $\overline{d}_{lk} = \frac{1}{m^2} \sum_{i=1}^m \sum_{j=1}^m d_{ij}^{(lk)}$

4.5 Deviation of expert E_r from remaining experts

The equation to determine deviation of expert E_r from remaining experts $d_r = \sum_{k=1, k \neq r}^s \overline{d}_{rk}$

4.6 Weight of experts w_r .

The equation to determine weight of experts $w_r = \frac{(d_r)^{-1}}{\sum_{r=1}^s (d_r)^{-1}}, r=1, 2, \dots, s$.

4.7 Fermat's Fuzzy Weighed Averaging operator FFWA

Fermat's Fuzzy Weighed Averaging operator FFWA to compute collective Fermat's Fuzzy Element p_i over other alternatives is

$$p_i = FFWA(p_i^{(1)}, p_i^{(2)}, \dots, p_i^{(s)}) = \left(\sqrt[n]{1 - \prod_{k=1}^s (1 - I_{\mu_{p_i^{(k)}}}^n)^{w_k}}, \prod_{k=1}^s (O_{\mu_{p_i^{(k)}}}^n)^{w_k} \right) = (I_{p_i}, O_{p_i})$$

4.8 Score Function

Score function $S(p_i)$ to rank the alternatives $u_i, i=1, 2, \dots, m$,

$$S(p_i) = (I_{p_i}^n - O_{p_i}^n)(I_{\sigma}(u_i) - O_{\sigma}(u_i)), u_i, i=1, 2, \dots, m.$$

5. Illustration of a Practical Example

In example 3.1, Suppose the aggregate of information FFPR is given as following relation matrices.

Data 1. The information from E_1 in the form of Relation Matrix.

	C1	C2	C3	C4	C5
C1	(0.5,0.5)	(0.6,0.8)	(0.7,0.9)	(0.6,0.9)	(0.3,0.9)
C2	(0.8,0.6)	(0.5,0.5)	(0.5,0.8)	(0.6,0.7)	(0.7,0.9)
C3	(0.9,0.7)	(0.8,0.5)	(0.5,0.5)	(0.4,0.9)	(0.8,0.4)
C4	(0.9,0.6)	(0.7,0.6)	(0.9,0.4)	(0.5,0.5)	(0.8,0.5)
C5	(0.9,0.3)	(0.9,0.7)	(0.4,0.8)	(0.5,0.8)	(0.5,0.5)

Data 2. The information from E_2 in the form of Relation Matrix.

	C1	C2	C3	C4	C5
C1	(0.5,0.5)	(0.6,0.7)	(0.7,0.8)	(0.8,0.8)	(0.3,0.8)
C2	(0.7,0.6)	(0.5,0.5)	(0.5,0.8)	(0.6,0.7)	(0.7,0.8)
C3	(0.8,0.7)	(0.8,0.5)	(0.5,0.5)	(0.4,0.9)	(0.8,0.5)
C4	(0.8,0.8)	(0.7,0.6)	(0.9,0.4)	(0.5,0.5)	(0.8,0.5)
C5	(0.8,0.3)	(0.8,0.7)	(0.5,0.8)	(0.5,0.8)	(0.5,0.5)

Data 3. The information from E_3 in the form of Relation Matrix.

	C1	C2	C3	C4	C5
C1	(0.5,0.5)	(0.5,0.8)	(0.6,0.9)	(0.7,0.9)	(0.4,0.9)
C2	(0.8,0.5)	(0.5,0.5)	(0.5,0.8)	(0.6,0.7)	(0.5,0.9)
C3	(0.9,0.6)	(0.8,0.5)	(0.5,0.5)	(0.4,0.9)	(0.6,0.4)
C4	(0.9,0.7)	(0.7,0.6)	(0.9,0.4)	(0.5,0.5)	(0.8,0.5)
C5	(0.9,0.4)	(0.9,0.5)	(0.4,0.6)	(0.5,0.8)	(0.5,0.5)

Data 4. The information from E_4 in the form of Relation Matrix.

	C1	C2	C3	C4	C5
C1	(0.5,0.5)	(0.6,0.8)	(0.7,0.9)	(0.7,0.9)	(0.3,0.9)
C2	(0.8,0.6)	(0.5,0.5)	(0.5,0.8)	(0.7,0.7)	(0.7,0.9)
C3	(0.9,0.7)	(0.8,0.5)	(0.5,0.5)	(0.4,0.8)	(0.8,0.4)
C4	(0.9,0.7)	(0.7,0.7)	(0.8,0.4)	(0.5,0.5)	(0.8,0.5)
C5	(0.9,0.3)	(0.9,0.7)	(0.4,0.8)	(0.5,0.8)	(0.5,0.5)

The above data represents a FFG(n). Among the relations, $0.8+0.8=0.9+0.7=0.7+0.9=1.6$ is the maximum sum among measures of acceptance and corresponding rejection. Since there is more than one pair with the same sum, we break the tie by comparing the sum of powers and selecting the pair that brings maximum sum. Here $0.8^2+0.8^2=1.28 <$

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$0.9^2 + 0.7^2 = 1.3$. So we consider sum of higher powers of 0.9 and 0.7 till we get a sum ≤ 1 . Note that $0.9^3 + 0.7^3 = 1.072 > 1$. But $0.9^4 + 0.7^4 = 0.8962 < 1$. So the FFG(n) is FFG(4). Since alternatives are present, the membership value 1 and non membership value 0 have to be assigned to each alternatives. In the given FFRs, all the membership and non membership values are in (0,1). Hence the given FFG(4) is FWFG(4).

Fermat's Fuzzy Elements are

$$P_1^{(1)} = (0.5854, 0.7816), P_2^{(1)} = (0.6624, 0.6853), P_3^{(1)} = (0.7732, 0.5753),$$

$$P_4^{(1)} = (0.8196, 0.5144), P_5^{(1)} = (0.7786, 0.5827)$$

$$P_1^{(2)} = (0.6544, 0.7090), P_2^{(2)} = (0.6231, 0.6694), P_3^{(2)} = (0.7301, 0.6015),$$

$$P_4^{(2)} = (0.7896, 0.5448), P_5^{(2)} = (0.6855, 0.5827)$$

$$P_1^{(3)} = (0.5725, 0.7816), P_2^{(3)} = (0.6304, 0.6608), P_3^{(3)} = (0.7435, 0.5578),$$

$$P_4^{(3)} = (0.8196, 0.5305), P_5^{(3)} = (0.7786, 0.5448)$$

$$P_1^{(4)} = (0.6129, 0.7816), P_2^{(4)} = (0.6803, 0.6853), P_3^{(4)} = (0.7732, 0.5619),$$

$$P_4^{(4)} = (0.7896, 0.5471), P_5^{(4)} = (0.7786, 0.5827)$$

From the difference matrices and the average values of difference matrices we get the deviations $d_1 = 0.196528$, $d_2 = 0.341216$, $d_3 = 0.261024$ and $d_4 = 0.242128$. Then the weights of experts are $w_1 = 0.31842$, $w_2 = 0.18340$, $w_3 = 0.23974$ and $w_4 = 0.25845$.

Now the collective Fermat's Fuzzy Elements are $p_1 = (I_{p1}, O_{p1}) = (0.60466, 0.76775)$,

$p_2 = (I_{p2}, O_{p2}) = (0.65365, 0.67642)$, $p_3 = (I_{p3}, O_{p3}) = (0.75922, 0.57224)$, $p_4 = (I_{p4}, O_{p4}) = (0.80715, 0.53211)$ and $p_5 = (I_{p5}, O_{p5}) = (0.76520, 0.57338)$.

The corresponding score function gives the following scores

$$S(p_1) = -0.21377, S(p_2) = -0.02680, S(p_3) = 0.22503, S(p_4) = 0.34427 \text{ and } S(p_5) = 0.23476$$

Since $S(p_4)$ is the maximum score, the best Choice is S_4 , the National Pension Scheme.

6. Beal's Fuzzy Graph BFG(m,n)

If the membership value of acceptance (or rejection) is given a limit (say α) then the membership value (say β) of rejection (or acceptance) is assumed to be governed by the in equation $\beta^n \leq 1 - \alpha^m$ for some $m, n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Therefore the generalization of FFG(n) has importance.

Definition 6.1: A Beal's Fuzzy Set (BFS) on a universal set A is a set of 3 tuples of the form $F=\{(u, I_F(u), O_F(u))\}$ where $I_F(u)$ and $O_F(u)$ represents the membership and non membership degrees of $u \in A$ and $I_F(u), O_F(u)$ satisfy $0 \leq I_F^m(u) + O_F^n(u) \leq 1$ for all $u \in A, m, n \in \mathbb{N}=\{1,2,3,.. \}$.

Definition 6.2: A Beal's fuzzy relation (BFR) R on $A \times A$ is a set of 3 tuples of the form $R = \{ (uv, I_R(uv), O_R(uv)) \}$ where $I_R(uv)$, and $O_R(uv)$ represents the membership degree and non membership degree of uv in R and $I_R(uv), O_R(uv)$ satisfy $0 \leq I_R^m(uv) + O_R^n(uv) \leq 1$ for all $uv \in A \times A$. BFR need not be symmetric. Hence $I_R(uv)$ need not be equal to $I_R(vu)$.

Definition 6.3: A Beal's fuzzy graph BFG(m,n) on a non empty set A is a pair $G : (\sigma, \mu)$ with σ as BFS on A and μ as BFR on A such that $I_\mu(uv) \leq I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) \geq O_\sigma(u) \vee O_\sigma(v)$ and $0 \leq I_\mu^m(uv) + O_\mu^n(uv) \leq 1$ for all $u, v \in A, m, n \in \mathbb{N}=\{1,2,3,.. \}$.where

$I_\mu : A \times A \rightarrow [0,1]$ and $O_\mu : A \times A \rightarrow [0,1]$ represents the membership and non membership functions of μ respectively.

Definition 6.4: A Beal's fuzzy graph $G : (\sigma, \mu)$ is said to be Beal's Strong fuzzy Graph BSFG(m,n) with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$.

Definition 6.5: A Beal's fuzzy graph $G : (\sigma, \mu)$ is said to be Beal's Weak Fuzzy Graph BWFG(m,n) with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) < I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) > O_\sigma(u) \vee O_\sigma(v)$ for all $uv \in \mu^*$

Definition 6.6 : A Beal's fuzzy graph $G : (\sigma, \mu)$ is said to be complete BFG with underlying crisp graph $G^* : (\sigma^*, \mu^*)$ if $I_\mu(uv) = I_\sigma(u) \wedge I_\sigma(v), O_\mu(uv) = O_\sigma(u) \vee O_\sigma(v)$ for all $u, v \in \sigma^*$.

Theorem 6.1: When $m = n$, BFG (m,n) $\Rightarrow$ FFG(n) and BFG (1,1) $\Rightarrow$ FFG(1) which is an intuitionistic fuzzy graph.

Proof. Directly follows from the definitions.

i.e, Beal's Fuzzy graph is generalization of Fermat's Fuzzy Graph and Fermat's Fuzzy Graph is generalization of Intuitionistic Fuzzy Graph.

Theorem 6.2: BWFG (m-1,n-1) $\Rightarrow$ BWFG(m,n) but the converse is not true.

Proof. Let $G : (\sigma, \mu)$ be a BWFG (n-1) with σ as BFS on A and μ as BFR on A .

Since $I_\mu(uv) < 1, I_\mu^{m-1}(uv) < 1 \Rightarrow I_\mu^m(uv) < I_\mu^{m-1}(uv) < 1$, for all $m \in \mathbb{N} \rightarrow (1)$

Similarly since $O_\mu(uv) < 1, O_\mu^{n-1}(uv) < 1 \Rightarrow O_\mu^n(uv) < O_\mu^{n-1}(uv)$

Therefore $I_{\mu}^{m-1}(uv) + O_{\mu}^{n-1}(uv) \leq 1 \Rightarrow I_{\mu}^m(uv) + O_{\mu}^n(uv) \leq 1$.

Hence $\text{BWFG}(m-1, n-1) \Rightarrow \text{BWFG}(m, n)$.

It is obvious from equation (1) that the converse is not true.

7 Conclusion

In this article some fundamental concepts of Fuzzy Graphs and Fermat's Fuzzy Graphs are reviewed. The decision tools are generalized for Fermat's Fuzzy Graphs. Application of Fermat's Weak Fuzzy Graph in modeling group decision making problem is illustrated with a practical example. The fundamental concepts of Beal's Fuzzy Graph are developed. The applications of FFG(n) and BFG(m, n) in various fields of science, social science and engineering are under research.

Acknowledgement

The authors wish to thank all the reviewers of this article.

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A Wavelet Collocation Method For Some Fractional Models

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Abstract

This article presents an effective numerical approach based on the operational matrix of fractional order integration of Haar wavelets for dealing with the fractional models of the mixing and the Newton law of cooling problems. A general procedure of obtaining the fractional integration operational matrix of Haar wavelets which converts the fractional models into a system of algebraic equations is derived so that the computation is very simple and it is much effective than the conventional numerical methods. The reliability and the applicability of the current numerical technique for fractional models are examined by comparing the achieved results with the precise solutions.

Keywords: Caputo fractional derivative; Haar wavelets; Operational matrix.

2020 AMS subject classifications: 26A33, 34A08, 65T60. ¹

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¹Received on September 15, 2022. Accepted on December 15, 2022. Published on December 30, 2022. DOI: 10.23755/rm.v43i0.859. ISSN: 1592-7415. eISSN: 2282-8214. c The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Consideration of derivatives and integrals of any order makes fractional calculus as an extension of integer order calculus. In recent years, several fractional models have drawn much attention in diverse disciplines of science and technology, such as, economics, medicine, earthquake, colored noise, signal processing, electromagnetism, electrochemistry, fluid-dynamic traffic model, physical chemistry, finance, social science, electric circuit, mechanical system and so on. Besides modeling, the solution techniques and their reliabilities are most important to catch critical points at which a sudden divergence, convergence or bifurcation starts. Therefore, high accuracy solutions are always needed. So, in the last few years, many researchers have proposed several efficient numerical methods for the numerical solutions of fractional models such as, Laplace transform [Podlubny [1999]], Fractional Differential Transform Method (FDTM) [Arikoglu and Ozkol [2009]], Variational Iteration Method (VIM) [Das [2009]], Power Series Method [Odibat and Shawagfeh [2007]], Adomian decomposition method (ADM) [El-Wakil et al. [2006]], Fractional Difference Method (FDM) [Meerschaert and Tadjeran [2006]] and so on.

Wavelet theory has drawn a lot of attention due to widespread applications, particularly in the field of image processing, signal processing, time-frequency analysis, edge extrapolation and fast algorithms. A special kind of function known as a wavelet has a number of convenient features including compact support, orthogonality, exact representation of polynomials to a certain degree and ability to represent functions at different levels of resolution. Therefore, for solving fractional models, numerous operational matrices based on different wavelets are employed, such as Haar wavelets [Chen et al. [2012], Mohammadi et al. [2019], Shah et al. [2017]], Bernoulli wavelets [Rahimkhani and Moeti [2018]], Legendre wavelets [ur Rehman and Khan [2011]], Euler wavelets [Aruldoss et al. [2021]], Chebyshev wavelets [Farooq et al. [2019]], Muntz-Legendre polynomial [Rahimkhani et al. [2018]], Chelyshkov polynomials [Talaie and Asgari [2018]], laguerre polynomials [Ji and Hou [2020]], Bernstein polynomials [Kadkhoda [2020]], Lagrange polynomials [Sabermahani et al. [2018]], spectral method [Talaie and Asgari [2018]], wavelet based numerical inversion Laplace transform [Aruldoss and Balaji [2022]] and so on.

Since the Haar wavelets are constructed from piecewise constant functions, they are the mathematically simplest wavelets among all wavelet families. The advantages of Haar wavelet technique are its ease of use, less computation costs and more precise outcomes.

The main objective of the current work is to get numerical solutions for fractional models of mixing problems and Newton law of cooling problems, using a numerical scheme based on Haar wavelets.

This paper is organised as follows. Some essential terminology and mathematical background information for fractional calculus are discussed in Section 2. Functional approximation using Haar wavelets and the operational matrix for fractional integration of Haar wavelets are covered in section 3. In Section 4, we demonstrate the applicability and the high accuracy of the present numerical algorithm for the fractional models of mixing and Newton law of cooling problems. Finally, the conclusion is made in Section 5.

2 Preliminaries

Here, we cover fractional order integral and differential operators along with some fundamental fractional calculus concepts.

Definition 2.1 (Diethelm [2004], Miller and Ross [1993], Oldham and Spanier [1974], Podlubny [1999]). *The Riemann-Liouville integral of fractional order $\lambda \geq 0$, of the function $h(x) \in L^1([0, \infty))$, is given by*

$$I^\lambda h(x) = \begin{cases} \frac{1}{\Gamma(\lambda)} \int_0^x (x-v)^{\lambda-1} h(v) dv, & \lambda > 0. \\ h(x), & \lambda = 0. \end{cases}$$

Let $h(x), g(x) \in L^1([0, \infty))$, $a, b \in \mathbb{R}$, $\nu > -1$, $\lambda \geq 0$. Then the following properties are attained.

- (i) $I^\lambda (ah(x) + bg(x)) = aI^\lambda h(x) + bI^\lambda g(x)$.
- (ii) $I^\lambda x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\lambda+1)} x^{\lambda+\nu}$.

Definition 2.2 (Diethelm [2004], Miller and Ross [1993], Oldham and Spanier [1974], Podlubny [1999]). *The Caputo fractional derivative of fractional order $\lambda \geq 0$, of the function $h(x) \in L^1([0, \infty))$, is given by*

$$D^\lambda h(x) = I^{m-\lambda} (D^m h(x)) = \begin{cases} \frac{1}{\Gamma(m-\lambda)} \int_0^x \frac{h^{(m)}(v)}{(x-v)^{\lambda-m+1}} dv, & m-1 < \lambda < m, m \in \mathbb{N}. \\ \frac{d^m}{dx^m} h(x), & \lambda = m \in \mathbb{N}. \end{cases}$$

Let $h(x), g(x) \in L^1([0, \infty))$, $a, b \in \mathbb{R}$, $\lambda \geq 0$.

Then the following properties are attained.

- (i) $D^\lambda k = 0$, where k is a constant.
- (ii) $D^\lambda (ah(x) + bg(x)) = a(D^\lambda h(x)) + b(D^\lambda g(x))$.
- (iii) $D^\lambda (I^\lambda h(x)) = h(x)$.
- (iv) $I^\lambda (D^\lambda h(x)) = h(x) - \sum_{k=0}^{m-1} h^{(k)}(0^+) \frac{x^k}{k!}$, $m-1 < \lambda \leq m$,

where $m \in \mathbb{N}$, $x > 0$ and $h^{(k)}(0^+) := \lim_{x \rightarrow 0^+} D^{(k)}h(x)$, $k = 0, 1, 2, \dots, m-1$.

3 Haar wavelets approximations of functions

The Haar wavelet and its fractional integration operational matrix are presented here.

3.1 Haar wavelets

Alfred Haar, a Hungarian mathematician, developed the Haar functions in 1910. A square wave with magnitude of 1 in some interval and zero in all other intervals is referred to as the orthogonal set of Haar functions. The first curve is $h_1(x) = 1$ on the entire interval $0 \leq x \leq T$, $T \in \mathbb{R}^+$. The mother wavelet or the fundamental square wave which also spans the entire interval $[0, T]$ is represented by the second curve. By performing the two operations of translation and dilation on $h_1(x)$, all other subsequent curves are produced. The scaling function $h_1(x)$ for the family of Haar wavelets is defined as

$$h_1(x) = \begin{cases} \frac{1}{\sqrt{2^{J+1}}}, & \text{for } x \in [0, T), \\ 0, & \text{otherwise.} \end{cases}$$

The haar wavelet family for $x \in [0, T]$ is defined as

$$h_i(x) = \sqrt{\frac{m}{2^{J+1}}} \begin{cases} 1, & \text{for } x \in [\frac{kT}{m}, \frac{(k+0.5)T}{m}), \\ -1, & \text{for } x \in [\frac{(k+0.5)T}{m}, \frac{(k+1)T}{m}), \\ 0, & \text{otherwise.} \end{cases}$$

where $m = 2^l$, $l = 1$ to J , $J \in N$, indicates the level of resolution and the translation parameter. The maximum level of resolution is J . The index i is calculated using $i = m + k + 1$. the minimal value of i is 2 and the maximal value of i is $N = 2^{J+1}$.

3.2 Square integrable functions approximation

With regard to Haar wavelets, every function in $f(t)$ from $L^2([0, T])$ can be expressed as

$$f(t) = \sum_{i=1}^{\infty} c_i h_i(t), \quad (1)$$

where the following inner product resolves the coefficients c_i .

$$\langle f(t), h_i(t) \rangle = \int_0^T f(t) h_i(t) dt.$$

If the series (1) is truncated, $f(t)$ can be roughly calculated as

$$f(t) \approx \sum_{i=1}^N c_i h_i(t) = CH(t), \quad (2)$$

where $N = 2^{J+1}$, $C = [c_1, c_2, \dots, c_N]_{1 \times N}$ is the coefficient vector, $H(t) = [h_1(t), h_2(t), \dots, h_N(t)]'$.

Discretizing (2) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1, 2, 3, \dots, N$, we attain

$$F \simeq CH_{N \times N}, \quad (3)$$

where $F = [f(t_1), f(t_2), \dots, f(t_N)]_{1 \times N}$ and $H_{N \times N} = [H(t_1), H(t_2), \dots, H(t_N)]$ is a Haar wavelet coefficient matrix of order N . Especially, for $J = 2$ and $T = 1$, the Haar wavelet coefficient matrix becomes

$$H_{8 \times 8} = \begin{pmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.3536 & 0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & -0.3536 & -0.3536 \\ 0.5000 & 0.5000 & -0.5000 & -0.5000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 & 0.5000 & -0.5000 & -0.5000 \\ 0.7071 & -0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 & -0.7071 \end{pmatrix}$$

3.3 Haar wavelets operational matrix of fractional integration

According to [Kilicman and Al Zhour [2007]], the N Block pulse functions(BPFs) on $[0, T)$ are given by,

$$b_m(t) = \begin{cases} 1, & [\frac{(m-1)T}{N}, \frac{mT}{N}), \\ 0, & otherwise, \end{cases}$$

where $m = 1, 2, 3, \dots, N, N \in \mathbb{N}$.

For $t \in [0, T)$,

$$b_n(t)b_m(t) = \begin{cases} b_m(t), & m = n. \\ 0, & m \neq n. \end{cases}$$

and

$$\int_0^T b_n(t)b_m(t)dt = \begin{cases} \frac{T}{N}, & m = n. \\ 0, & m \neq n. \end{cases}$$

Every function $f(t) \in L^2(0, T)$ can be written in terms of BPFs as

$$f(t) \simeq \sum_{m=1}^N f_m b_m(t) = f' B_N(t),$$

where $f = [f_1, f_2, \dots, f_N]'$, $f_m = \frac{N}{T} \int_{\frac{(m-1)T}{N}}^{\frac{mT}{N}} f(t)b_m(t)dt$ and $B_N(t) = [b_1(t), b_2(t), \dots, b_N(t)]'$.

The relationship between the Haar wavelets and BPfs is given by

$$H(t) = H_{N \times N} B_N(t). \quad (4)$$

According to [Kilicman and Al Zhour [2007]], the integration I^λ , with fractional order $\lambda \geq 0$ of $B_N(t)$ is approximated as

$$I^\lambda (B_N(t)) \simeq E_{N \times N}^\lambda B_N(t), \quad (5)$$

where $E_{N \times N}^\lambda$ is the Block pulse operational matrix of $I^\lambda (B_N(t))$ given by

$$E_{N \times N}^\lambda = \left(\frac{T}{N} \right)^\lambda \frac{1}{\Gamma(\lambda + 2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{N-2} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{N-3} \\ 0 & 0 & 0 & 1 & \dots & \xi_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \xi_1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

with $\xi_r = (r+1)^{\lambda+1} - 2r^{\lambda+1} + (r-1)^{\lambda+1}$.

The integration I^λ with fractional order $\lambda \geq 0$ of $H(t)$ is approximated as,

$$I^\lambda(H(t)) \simeq P_{N \times N}^\lambda H(t), \quad (6)$$

where $P_{N \times N}^\lambda$ is referred to as the Haar wavelets operational matrix of N for fractional integration with order $\lambda \geq 0$.

Using (4) and (5), we arrive

$$I^\lambda(H(t)) = I^\lambda(H_{N \times N} B_N(t)) = H_{N \times N} I^\lambda(B_N(t)) \simeq H_{N \times N} E_{N \times N}^\lambda B_N(t). \quad (7)$$

Thus, combining (6) and (7), we attain

$$P_{N \times N}^\lambda H(t) \simeq I^\lambda(H(t)) \simeq H_{N \times N} E_{N \times N}^\lambda B_N(t) = H_{N \times N} E_{N \times N}^\lambda (H_{N \times N})^{-1} H(t)$$

and so

$$P_{N \times N}^\lambda \simeq H_{N \times N} E_{N \times N}^\lambda (H_{N \times N})^{-1}. \quad (8)$$

Especially, for $J = 2$, $T = 1$ and $\lambda = 0.5$, the Haar wavelets operational matrix for fractional integration becomes,

$$P_{8 \times 8}^{0.5} = \begin{pmatrix} 0.7523 & -0.2203 & -0.1102 & -0.0580 & -0.0551 & -0.0290 & -0.0224 & -0.0189 \\ 0.2203 & 0.3116 & -0.1102 & 0.1623 & -0.0551 & -0.0290 & 0.0878 & 0.0391 \\ 0.0580 & 0.1623 & 0.2203 & -0.0350 & -0.0779 & 0.1148 & -0.0275 & -0.0045 \\ 0.1102 & -0.1102 & 0 & 0.2203 & 0 & 0 & -0.0779 & 0.1148 \\ 0.0189 & 0.0391 & 0.1148 & -0.0045 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0224 & 0.0878 & -0.0779 & -0.0275 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0290 & -0.0290 & 0 & 0.1148 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0551 & -0.0551 & 0 & -0.0779 & 0 & 0 & 0 & 0.1558 \end{pmatrix}$$

4 Illustrative Examples

The following fractional models provide an insight into the applicability and the effectiveness of the proposed numerical approach based on Haar wavelets.

Example 4.1 (Kreyszig [2007]). *Mixing Problem: A tank holds 1000 litres of water with 200 kilograms of salt dissolved in it. The tank receives 50 litres of brine every minute, each containing $(1+\cos t)$ kilograms of dissolved salt. The mixture runs out evenly since it is constantly being stirred. Find the tank salt content at any moment t by solving for $y(t)$.*

Solution:

First step. Modeling. The fractional order time rate of change $\frac{d^\lambda y(t)}{dt^\lambda}$ of the amount of salt $y(t)$ in the tank at any time t equals the inflow of salt minus the outflow of

the salt. The salt inflow rate is $50(1 + \cos t)$ kg/min. Since the tank always contains 1000 litres of brine, the salt outflow rate is $0.05y(t)$.

$$\text{Thus } \frac{d^\lambda y(t)}{dt^\lambda} = 50(1 + \cos t) - 0.05y(t), \quad 0 < \lambda \leq 1, \quad 0 \leq t < T, \quad (9)$$

with the initial condition $y(0) = 200$. When $\lambda = 1$, the exact solution of (9) is

$$y(t) = 1000 + 2.494\cos(t) + 49.88\sin(t) - 802.5e^{-0.05t}.$$

Second step. Numerical solution.

$$\text{Suppose } \frac{d^\lambda y(t)}{dt^\lambda} \approx CH(t). \quad (10)$$

$$\text{Then, } y(t) = I^\lambda (y^\lambda(t)) \approx CP^\lambda H(t) + y(0). \quad (11)$$

Using (10), (11) in (9), we have

$$CH(t) + 0.05 (CP^\lambda H(t) + 200) - 50(1 + \cos t) = 0. \quad (12)$$

We can attain the coefficient vector C by solving the equation(12) at the collocation points. Using the coefficient vector C in (11), we get the numerical solutions of (9) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	245.9747	223.5968	211.6746	206.5021	205.6014	7.1053e-03
0.1875	262.5291	242.6841	227.3894	218.5203	216.7208	1.5601e-02
0.3125	270.0022	254.4603	239.8020	229.8287	227.6261	2.3793e-02
0.4375	274.5230	263.3066	250.4973	240.5658	238.2246	3.1552e-02
0.5625	277.1876	270.1939	259.8753	250.7210	248.4280	3.8758e-02
0.6875	278.4453	275.5373	268.1011	260.2539	258.1535	4.5299e-02
0.8125	278.5408	279.5608	275.2559	269.1170	267.3255	5.1073e-02
0.9375	277.6367	282.4078	281.3853	277.2637	275.8764	5.5989e-02

Table 1: The achieved absolute errors and numerical solutions of Example 4.1 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.2 (Kreyszig [2007]). *Mixing Problem: A tank holds 500 litres of water with 80 kilograms of salt dissolved in it. 20 kilograms of salt are dissolved in 20 litres of water, and this inflow occurs once every minute. A uniform mixture at a rate of 20 litres/min exists. Calculate the tank's salt content $y(t)$ at time t .*

Solution:

First step. Modeling. The fractional order time rate of change $\frac{d^\lambda y(t)}{dt^\lambda}$ of the amount

of salt $y(t)$ in the tank at any time t equals the inflow of salt minus the outflow of the salt. The salt inflow rate is 20kg/min . Since the tank always contains 500 litres of brine, the salt outflow rate is $0.04y(t)$.

$$\text{Thus, } \frac{d^\lambda y(t)}{dt^\lambda} = 20 - 0.04y(t), \quad 0 < \lambda \leq 1, \quad 0 \leq t < T, \quad (13)$$

with the initial condition $y(0) = 80$. When $\lambda = 1$, the exact solution of (13) is

$$y(t) = 500 - 420e^{-0.04t}.$$

Second step. Numerical solution.

$$\text{Suppose } \frac{d^\lambda y(t)}{dt^\lambda} \approx CH(t). \quad (14)$$

$$\text{Then, } y(t) = I^\lambda (y^\lambda(t)) \approx CP^\lambda H(t) + y(0). \quad (15)$$

Using (14), (15) in (13), we have

$$CH(t) + 0.04 (CP^\lambda H(t) + 80) - 20 = 0. \quad (16)$$

We can attain the coefficient vector C by solving the equation(16) at the collocation points. Using the coefficient vector C in (15), we get the numerical solutions of (13) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	88.6354	84.4211	82.1845	81.2159	81.0474	1.3070e-03
0.1875	91.8360	88.0451	85.1477	83.4755	83.1369	1.2962e-03
0.3125	93.4299	90.3737	87.5417	85.6353	85.2160	1.2854e-03
0.4375	94.5790	92.2397	89.6813	87.7390	87.2848	1.2747e-03
0.5625	95.4957	93.8406	91.6570	89.8011	89.3432	1.2641e-03
0.6875	96.2661	95.2620	93.5132	91.8294	91.3914	1.2535e-03
0.8125	96.9350	96.5516	95.2761	93.8289	93.4293	1.2430e-03
0.9375	97.5286	97.7393	96.9628	95.8028	95.4571	1.2326e-03

Table 2: The achieved absolute errors and numerical solutions of Example 4.2 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.3 (Kreyszig [2007]). *Newton's law of cooling: A room that is 22°C in temperature has a thermometer in it that reads 5°C . Find the thermometer reading at any time t if the temperature is 12°C one minute later.*

Physical details:

According to the experiments, the fractional order time rate of change $\frac{dV^\lambda}{dt^\lambda}$ of the

temperature V of a body is directly related to the difference between V and the temperature V_A of the surrounding medium.

$$\text{Thus } \frac{d^\lambda V}{dt^\lambda} = k(V - V_A), \quad (17)$$

where $0 < \lambda \leq 1$, $0 \leq t < T$, k is the constant of proportionality, which is assumed to be -0.5306282 when $0 < \lambda \leq 1$, with the initial condition $V(0) = 5$. When $\lambda = 1$, the exact solution of (17) is

$$V(t) = 22 - 17e^{-0.5306282t}.$$

$$\text{Suppose } \frac{d^\lambda V}{dt^\lambda} \approx CH(t). \quad (18)$$

$$\text{Then, } V(t) = I^\lambda \left(\frac{d^\lambda V}{dt^\lambda} \right) \approx CP^\lambda H(t) + V(0). \quad (19)$$

Using (18), (19) in (17), we have

$$CH(t) - 0.5306282 (CP^\lambda H(t) + 5 - 22) = 0. \quad (20)$$

We can attain the coefficient vector C by solving the equation(20) at the collocation points. Using the coefficient vector C in (19), we get the numerical solutions of (17) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	8.7029	7.1024	6.1026	5.6305	5.5457	8.8512e-03
0.1875	9.7989	8.6288	7.5086	6.7620	6.6021	7.9085e-03
0.3125	10.2629	9.4660	8.5301	7.7714	7.5906	7.0503e-03
0.4375	10.5811	10.0819	9.3698	8.6915	8.5157	6.2697e-03
0.5625	10.8246	10.5743	10.0892	9.5369	9.3814	5.5603e-03
0.6875	11.0226	10.9861	10.7199	10.3170	10.1915	4.9161e-03
0.8125	11.1896	11.3405	11.2814	11.0390	10.9496	4.3317e-03
0.9375	11.3343	11.6517	11.7867	11.7087	11.6590	3.8020e-03

Table 3: The achieved absolute errors and numerical solutions of Example 4.3 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.4 (Kreyszig [2007]). *Newton's law of cooling: A metal bar that is 20°C in temperature is submerged in boiling water. Find the temperature of the bar at any time t if the temperature of the bar is 51.5°C , after one minute of heating.*

Physical details:

According to the experiments, the fractional order time rate of change $\frac{dV^\lambda}{dt^\lambda}$ of the temperature V of a body is directly related to the difference between V and the temperature V_A of the surrounding medium.

$$\text{Thus } \frac{d^\lambda V}{dt^\lambda} = k(V - V_A), \quad (21)$$

where $0 < \lambda \leq 1, 0 \leq t < T$, k is the constant of proportionality, which is assumed to be -0.500463 when $0 < \lambda \leq 1$, with the initial condition $V(0) = 20$. When $\lambda = 1$, the exact solution of (21) is

$$V(t) = 100 - 80e^{-0.500463t}.$$

$$\text{Suppose } \frac{d^\lambda V}{dt^\lambda} \approx CH(t). \quad (22)$$

$$\text{Then, } V(t) = I^\lambda \left(\frac{d^\lambda V}{dt^\lambda} \right) \approx CP^\lambda H(t) + V(0). \quad (23)$$

Using (22), (23) in (21), we have

$$CH(t) - 0.500463 (CP^\lambda H(t) + 20 - 100) = 0. \quad (24)$$

We can attain the coefficient vector C by solving the equation(24) at the collocation points. Using the coefficient vector C in (23), we get the numerical solutions of (21) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	36.6409	29.3975	24.9118	22.8043	22.4264	3.7166e-02
0.1875	41.6282	36.2681	31.1983	27.8475	27.1321	3.3425e-02
0.3125	43.7581	40.0722	35.7956	32.3651	31.5523	3.0000e-02
0.4375	45.2215	42.8837	39.5932	36.5011	35.7043	2.6869e-02
0.5625	46.3438	45.1396	42.8612	40.3155	39.6046	2.4006e-02
0.6875	47.2575	47.0320	45.7381	43.8492	43.2682	2.1392e-02
0.8125	48.0296	48.6651	48.3085	47.1323	46.7095	1.9007e-02
0.9375	48.6989	50.1026	50.6297	50.1890	46.7095	1.6833e-02

Table 4: The achieved absolute errors and numerical solutions of Example 4.4 using the proposed strategy for $J = 2$ and $T = 1$.

5 Conclusions

In this article, the Haar wavelet-based numerical scheme was successfully employed to optimize the fractional models of mixing and Newton's law of cooling problems. The functions and the components of the fractional models are

discretized by utilizing the operational matrix of fractional integration of Haar wavelets. Accordingly, the present numerical algorithm is found to be appropriate for fractional models that can be expressed as differential equations of integer or fractional orders subject to initial or boundary conditions. Error analysis demonstrates that the numerical solutions quickly approach the correct solution. The numerical results demonstrate that this current numerical approach is efficient and straightforward for fractional models.

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A multidisciplinary approach to decompose decision-making process in the context of intertemporal choices

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Abstract

This paper takes place in the context of intertemporal choices through a multidisciplinary approach involving behavioral finance, mathematical finance, temperament theory and multicriteria models of decision support techniques. The motivation at the basis of the present study is the need to investigate the discrepancy between the normative models assumed by classical finance and the actual actions of the investors. Therefore, a methodology that allows to understand the extent to which decision-making process depends on the cognitive and emotional characteristics of the individual is presented. To achieve the aim, preferences are discussed in terms of the decision-maker's limited rationality through the relationship between decision inconsistency, impatience, and subjective perception of time. The applicability of our methodology addresses the strategic personalization. In fact, the irrational attitudes associated with investors' seemingly superficial choices have led to the need for customer profiling operations in the European regulatory framework, introduced in 2014/65/EU Directive.

Keywords: anomalies; impatience; intertemporal choices; perception of time; strategic personalization.

2010 AMS subject classification: 91B08.*

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* Received on October 19, 2022. Accepted on November 2, 2022. Published on December 30, 2022. doi: 10.23755/rm.v43i0.880. ©Martino and Ventre. This paper is published under the CC-BY license agreement.

1 Introduction

Intertemporal choices are the type of decision in which consequences are distributed over time. Most daily decisions fall under this definition: buying a car, deciding on a university plan, and undertaking an investment (Lu and Yang, 2015). The uncertainty that characterizes the decision-making context of intertemporal choices makes evaluating and selecting the alternative extremely difficult, both because of the amount of information required to analyze the available prospects and because of the emotional intensity of the decision maker involved. The Discounted Utility Model defined by Samuelson (1952) is the essential mathematical reference for intertemporal preferences. However, when compared with empirical evidence, the preferences expected by the model do not correspond to the actual actions of the decision-maker. Hyperbolic formulations of the discount function are more in line with the description of the individual's behavior than the exponential formulation assumed by the Discounted Utility Model. The application of cognitive psychology to decision-making behavior has led to the realization that decision-making is subject to a condition of bounded rationality (Simon, 1957) and systematic distortions of the cognitive process (Kahnemnan, 2017). Furthermore, the countless behavioral differences clarify the impossibility of studying cognitive processes without considering a categorization of the client (Pompian, 2008, 2012).

To formalize a method able to quantify the mechanisms responsible for hyperbolic discounting and to project the analysis of the decision-making process into strategic personalization, a multidisciplinary approach is used. The disciplines used are: mathematical finance to mathematically describe the decision-maker's behavior, personality theory to identify behavioral classes, behavioral finance to justify the shift from choice to decision style, multicriteria models to get at a decomposition of the decision making process with respect to the factors considered in the other disciplines used.

To investigate the influence of the individual's cognitive and behavioral attitudes on the evaluation and selection of alternatives, factors affecting choice are quantified by the degree of impatience and subjective time bias. These values are then used to define the weights of the Analytic Hierarchical Process (Saaty, 1980, 2008), allowing a decomposition of the decision-making process with respect to the classes of temperaments (Keirse, 1998) and the anomalies of the Discounted Utility model (Loewenstein and Thaler, 1989, 1992).

The applicability of the present work refers to the Markets in Financial Instruments Directive, 2014/65/EU (MiFID 2), in which the importance of individual client characteristics is emphasized. It requires the analysis of risk appetite or risk aversion and the importance of communication, knowledge, and understanding relating to portfolio management.

The paper develops as follows: in the second section, decision inconsistency is discussed and contextualized with respect to behavioral theories; in the third section, the methodology is presented; discussion and conclusion follow.

2 Psychological mechanisms of hyperbolic discounting

The Discounted Utility Model assumes that the utility of an intertemporal prospectus $(x_1, t_1; \dots; x_n, t_n)$, is calculated as $\sum_{i=1}^n U(x_i)f(t_i)$, where $f(t_i)$ is the discount function and $U(x_i)$ is the cardinal utility of the outcome x_i . The performance of the discount function is crucial to the selection of alternatives and depends on how the individual perceives the indeterminacy of the future. Initially, the discount function was assumed to have an exponential trend. Empirical evidence (Green and Myerson, 1996; Dasgupta and Maskin, 2005; Lu and Yang, 2015) motivated the formulation of discount functions with a hyperbolic trend that better represents actual investor behavior (Green and Myerson, 1997). The fundamental elements of the Discounted Utility Model are discount rates and impatience. The first is defined as $\rho(t) = -\frac{f'(t)}{f(t)}$ and indicates the rate at which the discount function decreases over standard period (Read, 2004); the latter is impatience which indicates the amount of money one is willing to lose to receive a unit of money immediately (Cruz Rambaud and Muñoz Torrecillas, 2016). The exponential model assumes a constant value for the discount rate and impatience, while in hyperbolic models both quantities decrease over time (Prelec, 2004).

To clarify the consequences of exponential discounting, let be SS and LL two results, one smaller and sooner (SS) and the other larger and later (LL). A rational decision-maker can indifferently choose one of the two available options, the choice depends on how the uncertainty of the future is perceived. The choice is temporal consistent if the preference remains constant over time, as shown in Figure 1 for (a) and (b). Hyperbolic discounting means that preferences are not constant over time generating the mechanism of temporal inconsistency, as shown in Figure 1 for (c).

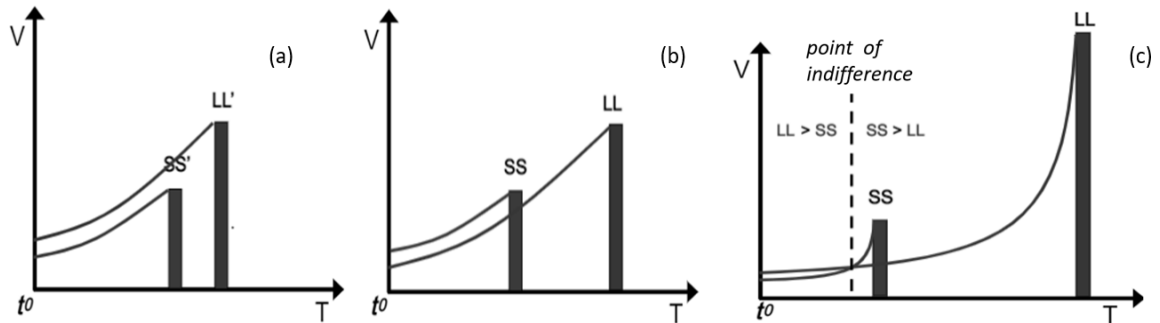


Figure 1: Images (a) and (b) describe temporally consistent preferences; image (c) shows temporally inconsistent choices.

Ventre et al. (2022 (a)) investigated the phenomenon of temporal inconsistency by describing the anomalies of the Discounted Utility Model (Loewenstein and Thaler, 1989) in terms of decreasing impatience. They made it possible to understand that the gap between empirical and normative preferences lies in the emotional drives that condition the decision-maker. Considering that the uncertainty of the future generates a haste

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mechanism that makes individuals more impatient in the first period, inconsistency can be seen as incoherence in the decision-making process when assessing prospects that are close or not to the present (Ventre et al., 2022 (a)). Table 1 shows some results of Ventre et al. (2022, (a)).

The description in Table 1 of the bias-related anomalies defines a connection between the behavioral attitudes of individuals and their decision-making process. The direct extension of this result turns to personalized behavioral finance. Indeed, psychology has made it clear that individuals are prone to different biases to varying degrees according to their behavioral characteristics (Kahnemnan and Tversky, 1979; Pompian, 2008, 2012; Pompian and Longo, 2005). Over the years, various theories that have tried to classify individuals in behavioral groups (Keirsey and Bates, 1984; Myers, 1980; Jung, 1923). The consequence is that, since different behavior defines different decision-making styles, the way of dealing with inter-temporal choices is different. For this reason, Pompian (2012) defined classes of behavioral investors by cross-referencing behavioral finance studies and Keirsey (Keirsey and Bates, 1984) temperaments.

The relationship between temperaments and BIT is shown in Table 2. A further important result in this context is described by McKenna et al. (2003), who define different communication modes for each BIT.

In addition to a behavioral classification, the Temporal Approach Theory (Zimbardo and Boyd, 2008), shown in Table 3, suggests that how individuals relate to time also influences decision-making.

Although the objective nature of chronological time is obviously recognized, the term subjective time denotes everyone's perception of time passing. From a neurological point of view, it is still unclear how and where the subjective perception of time develops in mind.

With respect to decision-making, Wittman et. al (2010) proved that the perception of time can be deviated from the experimental context, as it depends on the characteristics of stimuli (Fraisse, 1984) and how these interact with the emotional sphere of the decision-maker (Noulhiane et al., 2007 ; Droit-Volet and Gil, 2009). Therefore, the evaluation of intertemporal prospects is strongly influenced by subjective time (Agostino et al., 2021; Zauberman et al., 2009) as it affects emotional factors and the perception of the uncertainty of the future. By exploiting the mathematical properties of the discount function, it is possible to observe that the gap between empirical evidence and the normative model is directly related to a different perception of the time instants considered in the schedules to be evaluated. Let's observe that a hyperbolic discount function is characterized by a greater speed in the initial period, related to the property of decreasing in impatience (Cruz Rambaud and Muñoz Torrecillas, 2016; Prelec, 2004).

Anomaly	Description	Results by Ventre et al. (2022, (a))
<i>Delay effect</i> (Loewenstein and Thaler, 1989)	$(x, s) \sim (y, t) \Rightarrow$ $(x, s + h) < (y, t + h)$ $s < t, 0 < x < y$	The rate of variation of the discount function $f(t)$ is decreasing
<i>Magnitude effect</i> (Benzion et al., 1989, Noor, 2011)	$(x, s) \sim (y, t) \Rightarrow$ $(X, s) < (Y, t)$ $s < t, x < y, x < X < Y :$ $\frac{U(y)}{U(x)} = \frac{U(Y)}{U(X)}$	$(x, s) \sim (y, t) \Rightarrow (X, s) < (Y, t)$ $\Leftrightarrow U(y) - U(x) < U(Y) - U(X)$
<i>Sign Effect</i> (Loewenstein and Thaler, 1989; Cruz Rambaud and Sánchez Pérez, 2018)	$(x, s) \sim (y, t) \Rightarrow$ $(-x, s) > (-y, t)$ $s < t, 0 < x < y$	If the utility of a result is calculated as $V(x)$, where $\varepsilon > 1$ (according to the Prospect Theory (Kahneman and Tversky, 1979)) and U is concave ($U'' < 0$), then $(x, s) \sim (y, t) \Rightarrow (-x, s) > (-y, t)$

Table 1: Some results by Ventre et al. (2022, (a)).

Keirsey's Temperament	BIT
<i>Artisan</i> . It is the temperament which acts. Spontaneous and lovers of freedom	<i>Accumulator</i> . Interested in accumulating wealth
<i>Guardian</i> . Cautious, reliable, and disciplined, concrete and organized with a strong sense of responsibility and duty	<i>Conservative</i> . Protector of their heritage
<i>Idealist</i> . Interested in personal growth and development, skilled to inspire and clarify	<i>Follower</i> . Not interested in financial matters
<i>Rational</i> . The most theoretical temperament and attracted by problem solving.	<i>Independent</i> . The investment is procedure to be understood and controlled

Table 2: Keirsey's Temperament and BIT.

Temporal approach	Meaning
Positive past	Individuals oriented towards good memories of the past
Negative past	Individuals addressing the past with a negative attitude
Hedonistic present	Individuals living in the present and enjoying the moment
Fatalistic present	Individuals who live life as if everything were already written
Future	Individuals aiming at their own projects and actions
Transcendent future	Individuals more focused on what comes after life

Table 3: Psychological theory of the Temporal Approach.

Figure 2 shows in which relation $f(t)$ and $\tilde{f}(t)$ are, where $f(t)$ is an exponential discount function and $\tilde{f}(t)$ is an hyperbolic discount function.

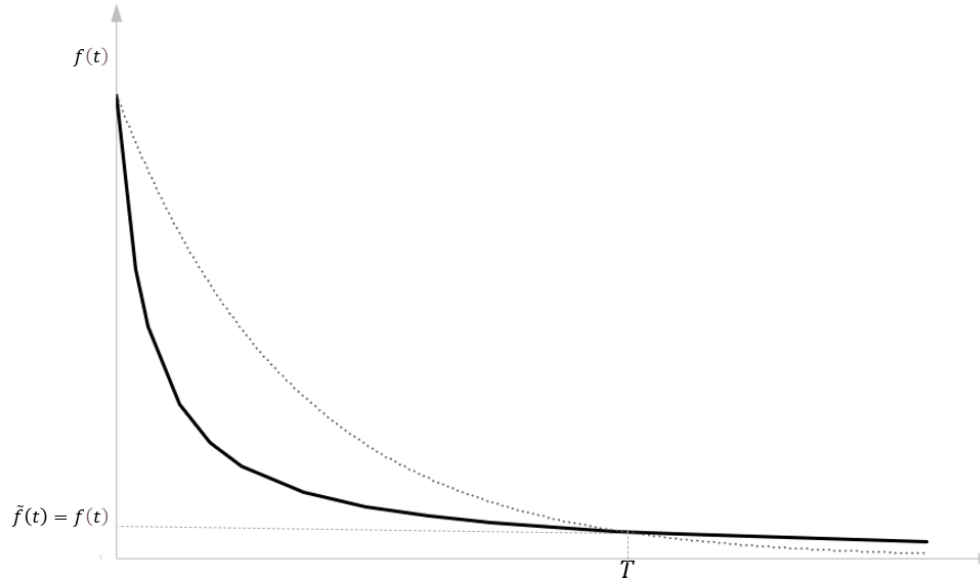


Figure 2: Relationship between a hyperbolic discount function (in black) and an exponential discount function (in grey).

So $\forall t \in [0, T]$, $\tilde{f}(t) < f(t)$ and from the decreasing monotony property of the discount function, there exists $\tilde{t} < t$ such that $f(t) = \tilde{f}(\tilde{t})$. Similarly, $\forall t \in [T, +\infty]$ $\tilde{f}(t) > f(t)$, i.e., exists a point $\tilde{t} > t$ such that $f(t) = \tilde{f}(\tilde{t})$. What has been observed is equivalent to stating that the subjective perception of time is decisive for hyperbolic discounting as it reduces the perceived intervals up to the T point and dilates them after the T point. From this perspective, it is possible to define a measure of subjective time-related inconsistency. If $y(t)$ is the best exponential function that approximates $\tilde{f}(t)$ a measure for inconsistency based on time perception bias is $d = \max_{t \in [0, T]} \{\tilde{f}(t) - y(t)\}$. A study investigating the relationship linking decreased impatience, discount rate

and subjective perception of time was conducted by Ventre et al. (2022, (b)) and Ventre and Martino (2022). Specifically, through a direct comparison of consistent and inconsistent preferences, the influence of time perception in intertemporal choices is formalized and empirically proven.

3 Decomposition of the decision-making process using AHP

The Analytical Hierarchical Process (Saaty, 1980, 2008) is a multi-criteria method to support the decisional process. The structure is based on directed graphs (Knuth, 1973) and its implementation allows the decomposition of a problem into its most basic components. Its characteristics are encapsulated in its name: the technique is analytical because it decomposes the main problem; it is a process because it develops with precise actions; it is hierarchical because the structure consists of several levels.

The first level represents the general objective, which in our case, is to understand how much the combination of temperament and inconsistency weighs on the evaluation of the available intertemporal perspectives. The structure is constituted as shown in Figure 3 and it represents an extension of the structure presented by Ventre et. al (2022, (c)).

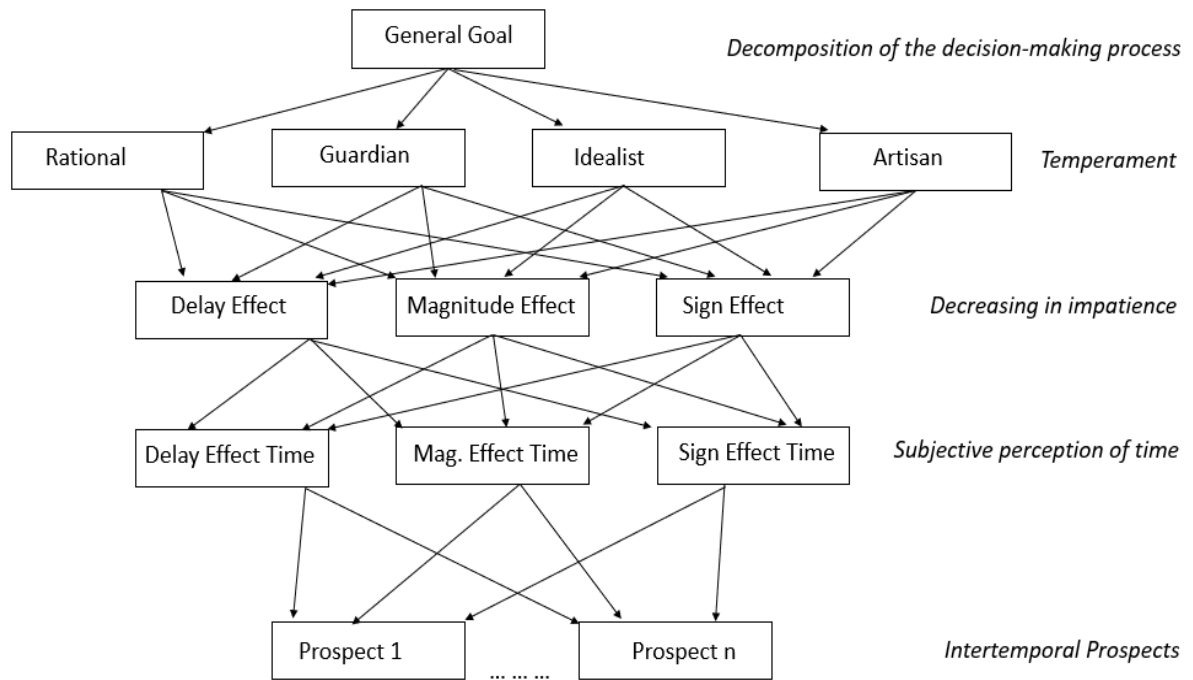


Figure 3: AHP structure.

To construct the pairwise comparisons matrix to obtain the local weights, and thus the global weights, we can exploit the information obtained in Section 2.

The matrix of the first level can be constructed by considering the "The Keirsey Temperament Sorter II" (Keirsey, 1988). The questionnaire determines a vector

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indicating for each score how much temperament is represented for the individual. The pairwise comparison matrix is defined in Table 4.

General Goal	Rational	Guardian	Idealist	Artisan
Rational	1	$\frac{\text{score of } G}{\text{score of } R}$	$\frac{\text{score of } I}{\text{score of } R}$	$\frac{\text{score of } A}{\text{score of } R}$
Guardian	$\frac{\text{score of } R}{\text{score of } G}$	1	$\frac{\text{score of } I}{\text{score of } G}$	$\frac{\text{score of } A}{\text{score of } G}$
Idealist	$\frac{\text{score of } R}{\text{score of } I}$	$\frac{\text{score of } G}{\text{score of } I}$	1	$\frac{\text{score of } A}{\text{score of } I}$
Artisan	$\frac{\text{score of } R}{\text{score of } A}$	$\frac{\text{score of } G}{\text{score of } A}$	$\frac{\text{score of } I}{\text{score of } A}$	1

Table 4: Pairwise comparison matrix between Temperament and General Objective.

For the level of impatience decrease, it is possible to calculate the hyperbolic factor for each anomaly with respect to each temperament. The hyperbolic factor, for the indifference pairs $(x, s) \sim (y, t)$ $(x, s + \sigma) \sim (y, t + \tau)$ $s < t, x < y, \tau > 0$ is defined as (Rohde, 2010):

$$H(s, t, y, \tau) = \frac{\tau - \sigma}{t\sigma - s\tau}$$

The questionnaire for constructing the hyperbolic factor and the relative pairwise comparison matrix are in Table 5 and Table 6, respectively.

As discussed in Section 2, the hyperbolic trend defined by the decrease in impatience is related to the subjective perception of time. By constructing a discount function using the interpolation technique for each anomaly, it is possible to obtain the degree of inconsistency as the maximum distance between the hyperbolic discount function and its best exponential approximation. The questionnaire for constructing the hyperbolic function and the relative pairwise comparison matrix are given in Table 7 and Table 8 respectively. The questionnaire proposed in Table 7 is based on the application conducted by Ventre et al. (2022, (b)) and Ventre and Martino (2022).

Instead, the last pairwise comparison matrix is defined by a panel of experts who determine how much the prospectus could contribute to that type of anomaly. In fact, since the decision-making context of intertemporal choices is so complex that the selection of an alternative can be modelled as a multi-criteria group decision problem (Wu et al., 2022).

Anomaly	Variable	Question
<i>Delay Effect</i> t fixed, s fixed, τ fixed	x	You must receive a sum of y_{DELAY} in t months, not before the set date but, alternatively, there is the possibility to immediately collect a certain result by reducing the total to be collected. To accept the offer, how much at least do you want to receive today?
y_{DELAY} fixed	σ	You must receive a sum of y_{DELAY} in t+ τ months, not before the set date but you are given the opportunity to anticipate the application and collect a result of x instead of y_{DELAY} . To accept the offer, how long do you want to receive the x-digit?
<i>Magnitude Effect</i> t fixed, s fixed, τ fixed	x	You must receive a sum of $y_{MAG.}$ in t months, not before the set date but, alternatively, there is the possibility to immediately collect a certain result by reducing the total to be collected. To accept the offer, how much at least do you want to receive today?
$y_{MAG.}$ fixed $y_{DELAY} >$ $y_{MAG.}$	σ	You must receive a sum of $y_{MAG.}$ in t+ τ months, not before the set date but you are given the opportunity to anticipate the application and collect a result of x instead of $y_{MAG.}$ To accept the offer, how long do you want to receive the x-digit?
<i>Sign Effect</i> t fixed, s fixed, τ fixed	x	You must pay a sum of y_{SIGN} in t months, not before the set date but you are given the option to pay a lower result as long as the payment is paid on the same day. What is a figure that you think you could offer to accept your offer?
y_{SIGN} fixed y_{SIGN} $= -y_{DELAY}$	σ	You must pay a sum of y_{SIGN} in t+ τ months, not before the set date but you are given the opportunity to pay x instead of y_{SIGN} . To accept the offer, how soon would you commit to making payment?

Table 5: Questions to obtain the pairwise comparison matrices between temperaments and anomalies.

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Temperament i	Delay Effect	Magnitude Effect	Sign Effect
Delay Effect	1	H_{DELAY}/H_{MAG}	H_{DELAY}/H_{SIGN}
Magnitude Effect	H_{MAG}/H_{DELAY}	1	H_{MAG}/H_{SIGN}
Sign Effect	H_{SIGN}/H_{DELAY}	H_{SIGN}/H_{MAG}	1

Table 6: Pairwise comparison matrices between temperaments i and anomalies.

Anomaly	Question	Function
<i>Delay Effect</i> $U_D(x(0))$ fixed $t_0 = 0$	You have to receive $U(x(t_i))$ euros today, how much do you want to receive in t_{i+1} days to consider the offer equivalent?	$f(t) = \begin{cases} f(0) = 1 \\ f(t_{i+1}) = \frac{f(t_i) * U(x(t_i))}{U(x(t_{i+1}))} \end{cases}$
<i>Magnitude Effect</i> $U_M(x(0))$ fixed $U_M(x(0)) < U_D(x(0))$ $t_0 = 0$	You have to receive $U(x(t_i))$ euros today, how much do you want to receive in t_{i+1} days to consider the offer equivalent?	
<i>Sign Effect</i> $U_S(x(0))$ fixed $U_S(x(0)) = -U_D(x(0))$ $t_0 = 0$	You must pay $U(x(t_i))$ euros today but you don't have the affordability. As far as, how much are you willing to pay in t_{i+1} days to consider the offer equivalent?	

Table 7: Questions to obtain the discount function for each anomaly.

Anomaly i	Delay Effect	Magnitude Effect	Sign Effect
Delay Effect	1	d_{DELAY}/d_{MAG}	d_{DELAY}/d_{SIGN}
Magnitude Effect	d_{MAG}/d_{DELAY}	1	d_{MAG}/d_{SIGN}
Sign Effect	d_{SIGN}/d_{DELAY}	d_{SIGN}/d_{MAG}	1

Table 8: Pairwise comparison matrices between temperaments i and subjective perception of time.

4 Discussion

The present work has discussed a multidisciplinary approach to decompose the decisional process in the context of intertemporal choices. The purpose of the paper is to present a method for investigating the behavior of individuals in a decision-making context of uncertainty. The applicability refers to the definition of classes of decision-makers based on decision-making style and behavioral characteristics. The creation of investor profiles were introduced following the major crises of the past decade to protect individuals. Uncertainty analysis is fundamental to the study of asset pricing and entrepreneurship (Gifford, 2003). Therefore, the present work turned to a classification based on the dynamics responsible for the anomalies in the Discounted Utility Model.

The quantification of cognitive factors has been obtained through the concept of decreasing impatience and subjective perception of time. The proposed method has been justified in Section 2 showing that the behavior between the discount and hyperbolic function differs due to the lack of elements in the normative model that do not consider the emotionality and cognitive limitations of the decision maker. In fact, the decrease in impatience can be interpreted as the inconsistency in the decision-making process between the short and the long run, the time perception bias can be associated with the projection mechanism that individuals implement to evaluate an event that has not yet happened (Loewenstein, 1996; Loewenstein et al., 2003; Loewenstein et al., 1998; Wheeler et al., 1997). Developments in cognitive psychology presented in Section 2 motivated the projection of the results discussed within personality theory.

This made it possible to construct the AHP structure to decompose the decision-making process into its main components and weigh the influence of behavioral attitudes and cognitive abilities on the evaluation of alternatives.

The originality of our approach lies in incorporating the following motivations into a single methodology: temporal inconsistency is related to the emotional drives of the decision-making context; behavioral differences among Keirsey temperaments indicate the impossibility of determining strategies that give the same degree of satisfaction to

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all individuals; psychological mechanisms and propensities for cognitive biases for each individual must be taken into account to investigate decision making; the lack of emotional factors in the normative model results in a different perception of time and a decreasing degree of impatience; to improve the choices of individuals, it is necessary to project considerations in the context of personalized behavioral finance in which strategic diversification must propose different strategies, different ways of representing and communicating intertemporal prospects.

5 Conclusions

The applicability of the present model is related to the investor profile creation operations required by the European Directive MiFID 2 (Markets in Financial Instruments Directive, 2014/65/UE). Since the decision maker suffers from systematic cognitive distortions, limited rationality, and emotional impulses that distort decision-making, financial advisors must assess the suitability of the client and the appropriateness of proposed alternatives.

The broad definition of intertemporal choices and the multidisciplinary approach used allow for numerous lines of future research. In the field of behavioral finance, understanding how subjective perception of time affects decision-making can contribute to understanding how individuals interface with investment and debt dynamics (Hoang and Hoxha, 2016). Because the perception of time is linked to the emotional sphere (McLoughlin, 2019; Droit-Volet and Meck, 2007), the relationship between investor emotions and financial assets can be further explored (Schmeling, 2009). Finally, the description of risk as a feeling (Loewenstein et al., 2001) determines a connection between time and prospect theory by improving the description of risk propensity or aversion (Wang et al., 2011; Broihanne et al., 2014; Sun and Li, 2010).

Regarding neuroscience, because time has neurological origins, our multidisciplinary approach can help clarify the difference between impulsivity and impatience, which are instead confused in the context of intertemporal choices (Berlinet al., 2004; McLeish, 2007). However, because impulsivity is described as a lack of self-control (Monterosso and Ainslie, 1999), it can be seen as a behavioral attitude. Instead, impatience is a feeling versus expectation.

In conclusion, the present paper has proposed a multidisciplinary method to calibrate strategies according to behavioral profile and quantification of inconsistency. In the perspective of strategic personalization, the applicability also extends into the social sphere considering that hyperbolic preferences are associated with procrastination mechanisms, depressive attitudes (Petry, 2001; Prelec, 2004), and gambling addiction (Calluso et al., 2020). Indeed, as argued by Thaler and Sunstein (2019) it is possible to guide individuals toward better alternatives by making use of the same cognitive biases.

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The Extension Of Generalized Intuitionistic Topological Spaces

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Abstract

In this paper, irresolute functions in generalized intuitionistic topological spaces were introduced. Regarding these functions, we attempted to unveil the notions of some minimal and maximal irresolute functions. In addition, the generalized intuitionistic topological spaces were extended by using their open sets which are finer than of it and their basic characterizations were investigated. Some continuous functions in the extension of generalized intuitionistic topological spaces are also been discussed in this paper.

Keywords: $mn-\mu_I$ -ops, $mx-\mu_I$ -ops, $P\mu_I$ -ops, $S\mu_I$ -ops, $mn-\mu_I$ -cts, $mx-\mu_I$ -cts, $mn-\mu_I$ -irresolute, $mx-\mu_I$ -irresolute.

2020 AMS subject classifications: 54A05, 54C08, 54C10. ¹

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¹Received on November 1st, 2022. Accepted on December 29th, 2022. Published on December 30th, 2022. doi: 10.23755/rm.v43i0.949. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

The concept of an intuitionistic set which is a generalization of an ordinary set and the specialization of an intuitionistic fuzzy set was given by Coker[2]. After that time, intuitionistic topological spaces were introduced [3]. A.Csaszar[1] introduced many closed sets in generalized topological spaces based on their basics. In 2019 [9], some new generalized closed sets in ideal nano topological spaces were developed. In 2022 [6], we have introduced a new type of topology called generalized intuitionistic topological spaces with the help of intuitionistic closed sets. After that time we introduced and studied μ_I -maps in generalized intuitionistic topological spaces. In addition we have introduced and defined a new structure of minimal and maximal μ_I -open sets in generalized intuitionistic topological spaces. In 2011 [10], the subject like minimal and maximal continuous, minimal and maximal irresolute, T-min space etc. were investigated on basic topological spaces.

In 2022 [7], the characterizations of $nI\alpha$ -closed sets are proved. In that paper authors has been used Kuratowski's closure operator. Taking it as an inspiration we introduce μ_I -irresolute functions in generalized intuitionistic topological spaces throughout this paper. Also, some minimal and maximal μ_I -irresolute functions were introduced and studied in detail.

The aim of this paper is, to introduce the $\mu_I(A)$ -topology which is finer than μ_I -topology by using the formula $U \cup (V \cap A)$, where U and V are μ_I -open. In addition, some important and interesting results were discussed by using μ_I -continuous maps on the extension of μ_I -topology. Also, some counterexamples are given to support this work.

2 Preliminaries

Definition 2.1 (6). A μ_I topology on a non-empty set X is a family of intuitionistic subsets of X satisfying the following axioms:

1. $\emptyset \in \mu_I$
2. Arbitrary union of elements of μ_I belongs to μ_I .

For a GITS (X, μ_I) , the elements of μ_I are called μ_I -open sets(briefly μ_I -ops) and the complement of μ_I -open sets are called μ_I -closed sets(briefly μ_I -cds).

Note:[6] $C_{\mu_I}(\emptyset) \neq \emptyset$, $C_{\mu_I}(X) = X$, $I_{\mu_I}(\emptyset) = \emptyset$ and $I_{\mu_I}(X) \neq X$.

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Definition 2.2 (6). Let (X, μ_I) be a GITS.

1. A proper non-null μ_I -ops G of (X, μ_I) is said to be a mn- μ_I -ops if any μ_I -ops which is contained in G is $\emptyset$ or G .
2. A proper non-null μ_I -ops $G (\neq M_{\mu_I})$ of (X, μ_I) is said to be a mx- μ_I -ops set if any μ_I -ops which contains G is M_{μ_I} or G .

Definition 2.3 (6). Let (X, μ_I) and (Y, σ_I) be the topological spaces. A map $f: (X, \mu_I) \rightarrow (Y, \sigma_I)$ is called,

1. mn- μ_I -cts if $f^{-1}(G)$ is a μ_I -ops in (X, μ_I) for every mn- μ_I -ops G in (Y, σ_I) .
2. mx- μ_I -cts if $f^{-1}(G)$ is a μ_I -ops in (X, μ_I) for every mx- μ_I -ops set G in (Y, σ_I) .

Results: [6]

1. Every μ_I -cts map is mn- μ_I -cts.
2. Every μ_I -cts map is mx- μ_I -cts.
3. Mn- μ_I -cts and mx- μ_I -cts maps are independent of each other.
4. If $f: (X, \mu_I) \rightarrow (Y, \sigma_I)$ is μ_I -cts and $g: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ is mn- μ_I -cts then $g \circ f: (X, \mu_I) \rightarrow (Z, \rho_I)$ is mn- μ_I -cts.
5. $f: (X, \mu_I) \rightarrow (Y, \sigma_I)$ is μ_I -cts and $g: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ is mx- μ_I -cts then $g \circ f: (X, \mu_I) \rightarrow (Z, \rho_I)$ is mx- μ_I -ops.

Definition 2.4 (4). Let X be a μ_I -topological spaces. A subset A of X is said to be μ_I -dense if $C_{\mu_I}(A) = X$. Clearly, X is the only μ_I -closed set dense in (X, μ_I) .

Theorem 2.1. Let (X, μ_I) be a GITS with closed under intersection property. Then $C_{\mu_I}(A \cup B) = C_{\mu_I}(A) \cup C_{\mu_I}(B)$.

Proof: Since $A \subset A \cup B$ and $B \subset A \cup B$, $C_{\mu_I}(A) \subset C_{\mu_I}(A \cup B)$ and $C_{\mu_I}(B) \subset C_{\mu_I}(A \cup B)$. Now we have to prove the second part, Since $A \subseteq C_{\mu_I}(A)$ and $B \subseteq C_{\mu_I}(B)$, $A \cup B \subseteq C_{\mu_I}(A) \cup C_{\mu_I}(B)$ which is μ_I -closed. Then $C_{\mu_I}(A \cup B) \subseteq C_{\mu_I}(A) \cup C_{\mu_I}(B)$. Hence the theorem.

3 μ_I -irresolute in GITS

Definition 3.1. A mapping $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ is said to be a

1. semi μ_I -irresolute function(briefly $S\mu_I$ -irresolute) if the inverse image of semi μ_I -open sets(briefly $S\mu_I$ -ops) in (Y, σ_I) is $S\mu_I$ -op in (X, μ_I) .
2. pre μ_I -irresolute function(briefly $P\mu_I$ -irresolute) if the inverse image of pre μ_I -open sets(briefly $P\mu_I$ -ops) in (Y, σ_I) is $P\mu_I$ -op in (X, μ_I) .
3. $\alpha\mu_I$ -irresolute function if the inverse image of $\alpha\mu_I$ -ops in (Y, σ_I) is $\alpha\mu_I$ -open in (X, μ_I) .
4. $\beta\mu_I$ -irresolute function if the inverse image of $\beta\mu_I$ -ops in (Y, σ_I) is $\beta\mu_I$ -open in (X, μ_I) .

Theorem 3.1. Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a semi μ_I -irresolute function if and only if the inverse image of semi μ_I -cds in (Y, σ_I) is semi μ_I -closed in (X, μ_I) .

Proof:

Necessary part: Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a semi μ_I -irresolute function and A be a semi μ_I -cds in (Y, σ_I) . Since f is $S\mu_I$ -irresolute, $\mathbb{k}^{-1}(Y - A) = X - \mathbb{k}^{-1}(A)$ is $S\mu_I$ -open in (X, μ_I) . Hence $\mathbb{k}^{-1}(A)$ is $S\mu_I$ -closed in (X, μ_I) .

Sufficient part: Assume that $\mathbb{k}^{-1}(A)$ is $S\mu_I$ -closed in (X, μ_I) for each $S\mu_I$ -closed set in (Y, σ_I) . Let V be a $S\mu_I$ -ops in (Y, σ_I) which yields that $Y - V$ is $S\mu_I$ -cds in (Y, σ_I) . Then we get $\mathbb{k}^{-1}(Y - V) = X - \mathbb{k}^{-1}(V)$ is $S\mu_I$ -closed in (X, μ_I) this implies $\mathbb{k}^{-1}(V)$ is $S\mu_I$ -open in (X, μ_I) . Hence $\mathbb{k}$ is $S\mu_I$ -irresolute.

Theorem 3.2. If $\mathbb{k}$ is $S\mu_I$ -irresolute then $\mathbb{k}$ is $S\mu_I$ -cts.

Proof: Suppose $\mathbb{k}$ is $S\mu_I$ -irresolute. Let A be any $S\mu_I$ -ops in (Y, σ_I) . Since every μ_I -ops is $S\mu_I$ -open and since A is $S\mu_I$ -open, $\mathbb{k}^{-1}(A)$ is $S\mu_I$ -open in (X, μ_I) . Hence $\mathbb{k}$ is $S\mu_I$ -cts.

Remark 3.1. Since every $S\mu_I$ -ops need not be μ_I -open, we cannot deduce the reversal concept of the above statement.

Theorem 3.3. Let (X, μ_I) , (Y, σ_I) and (Z, ρ_I) be three μ_I -topological spaces. For any $S\mu_I$ -irresolute map $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and any $S\mu_I$ -cts $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ the composition $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is $S\mu_I$ -cts.

Proof: Let A be any μ_I -ops in (Z, ρ_I) . Since $\mathbb{h}$ is $S\mu_I$ -cts, $\mathbb{h}^{-1}(A)$ is $S\mu_I$ -open in (Y, σ_I) . By using $\mathbb{k}$ is semi μ_I -irresolute, we get $\mathbb{k}^{-1}[\mathbb{h}^{-1}(A)]$ is $S\mu_I$ -open in (X, μ_I) .

But $\mathbb{k}^{-1}[\mathbb{h}^{-1}(A)] = (\mathbb{h} \circ \mathbb{k})^{-1}(A)$. Therefore, inverse image of μ_I -ops in (Z, ρ_I) is $S\mu_I$ -open in (X, μ_I) . Hence $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is $S\mu_I$ -cts.

Theorem 3.4. If $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ are both $S\mu_I$ -irresolute then $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is also $S\mu_I$ -irresolute.

Proof: Let A be any $S\mu_I$ -ops in (Z, ρ_I) . Since $\mathbb{k}$ and $\mathbb{h}$ are $S\mu_I$ -irresolute, $\mathbb{h}^{-1}(A)$ is $S\mu_I$ -open in (Y, σ_I) and $\mathbb{k}^{-1}[\mathbb{h}^{-1}(A)]$ is $S\mu_I$ -open in (X, μ_I) . Hence $(\mathbb{h} \circ \mathbb{k})^{-1}(A) = \mathbb{k}^{-1}[\mathbb{h}^{-1}(A)]$ is $S\mu_I$ -open and so $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is $S\mu_I$ -irresolute.

Theorem 3.5. Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a $P\mu_I$ -irresolute (resp. $\alpha\mu_I$ -irresolute and $\beta\mu_I$ -irresolute) function if and only if the inverse image of $P\mu_I$ -closed (resp. $\alpha\mu_I$ -closed and $\beta\mu_I$ -closed) sets in (Y, σ_I) is $P\mu_I$ -closed (resp. $\alpha\mu_I$ -closed and $\beta\mu_I$ -closed) in (X, μ_I) .

Proof: We can prove this theorem as we have done in the theorem 3.2.

Theorem 3.6. If f is $P\mu_I$ -irresolute (resp. $\alpha\mu_I$ -irresolute and $\beta\mu_I$ -irresolute) then f is $P\mu_I$ -continuous (resp. $\alpha\mu_I$ -cts and $\beta\mu_I$ -cts).

Proof: We can prove this theorem as we have done in the theorem 3.3.

Remark 3.2. Since every $P\mu_I$ -open (resp. $\alpha\mu_I$ -open and $\beta\mu_I$ -open) set need not be μ_I -open, we cannot deduce the reversal concept of the above statement.

Theorem 3.7. Let (X, μ_I) , (Y, σ_I) and (Z, ρ_I) be three μ_I -topological spaces. For any $P\mu_I$ -irresolute (resp. $\alpha\mu_I$ -irresolute and $\beta\mu_I$ -irresolute) map $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and any $P\mu_I$ -cts (resp. $\alpha\mu_I$ -cts and $\beta\mu_I$ -cts) $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ the composition $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is $P\mu_I$ -cts (resp. $\alpha\mu_I$ -cts and $\beta\mu_I$ -cts).

Proof: We can prove this theorem as we have done in the theorem 3.5.

Theorem 3.8. If $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ are both $P\mu_I$ -irresolute (resp. $\alpha\mu_I$ -irresolute and $\beta\mu_I$ -irresolute) then $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is also $P\mu_I$ -irresolute (resp. $\alpha\mu_I$ -irresolute and $\beta\mu_I$ -irresolute).

Proof: We can prove this theorem as we have done in the theorem 3.6

4 Minimal and Maximal μ_I -irresolute

Definition 4.1. Let (X, μ_I) and (Y, σ_I) be the topological spaces. A map $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ is called,

1. mn - μ_I -irresolute if the inverse image of every mn - μ_I -ops in (Y, σ_I) is mn - μ_I -open in (X, μ_I) .

2. $mx-\mu_I$ -irresolute if the inverse image of every $mx-\mu_I$ -ops in (Y, σ_I) is $mx-\mu_I$ -open in (X, μ_I) .

Example 4.1. Let $X = \{a, b, c, d\}$ and $Y = \{t, u, v, w\}$ with $\mu_I = \{\emptyset, < X, \emptyset, \{b\} >, < X, \emptyset, \{d\} >, < X, \{a, d\}, \emptyset >, < X, \{a\}, \emptyset >, < X, \emptyset, \emptyset >, < X, \emptyset, \{c, d\} >, < X, \emptyset, \{c\} >, < X, \{d\}, \emptyset >, < X, \{d\}, \{b\} >\}$ and $\sigma_I = \{\emptyset, < X, \emptyset, \{v\} >, < X, \emptyset, \{w\} >, < X, \emptyset, \{u, v\} >, < X, \emptyset, \emptyset >, < X, \{v\}, \emptyset >, < X, \{v\}, \{w\} >\}$. Define $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ by $\mathbb{k}(a) = t, \mathbb{k}(b) = w, \mathbb{k}(c) = u$ and $\mathbb{k}(d) = v$. Hence $\mathbb{k}$ is a $mn-\mu_I$ -irresolute map.

Theorem 4.1. Every $mn-\mu_I$ -irresolute map is $mn-\mu_I$ -cts.

Proof: Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a $mn-\mu_I$ -irresolute map. Let G be any $mn-\mu_I$ -ops in (Y, σ_I) . Since $\mathbb{k}$ is $mn-\mu_I$ -irresolute, $\mathbb{k}^{-1}(A)$ is a $mn-\mu_I$ -ops in (X, μ_I) . That is $\mathbb{k}^{-1}(A)$ is a μ_I -ops in (X, μ_I) Hence $\mathbb{k}$ is $mn-\mu_I$ -cts.

Remark 4.1. The reversal statement of the above theorem is not necessarily true. In example 4.3, $\mathbb{k}$ is $mn-\mu_I$ -cts but not $mn-\mu_I$ -irresolute. Since $\mathbb{k}^{-1}(jX, w, \emptyset_i) = jX, b, \emptyset_i$ which is not minimal μ_I -open in (X, μ_I) .

Theorem 4.2. Every $mx-\mu_I$ -irresolute map is $mx-\mu_I$ -cts.

Proof: We can prove this theorem as we have done in the theorem 4.4.

Remark 4.2. The reversal statement of the above theorem is not necessarily true. In example 4.2, $\mathbb{k}$ is $mx-\mu_I$ -cts but not $mx-\mu_I$ -irresolute. Since $\mathbb{k}^{-1}(jX, v, w_i) = jX, d, b_i$ which is not $mx-\mu_I$ -open in (X, μ_I) .

Remark 4.3. In example 4.2, $\mathbb{k}$ is a $mn-\mu_I$ -irresolute map but not $mx-\mu_I$ -irresolute. In example 4.3, $\mathbb{k}$ is a $mx-\mu_I$ -irresolute map but not $mn-\mu_I$ -irresolute. That is $mn-\mu_I$ -irresolute maps and $mx-\mu_I$ -irresolute maps are independent of each other.

Remark 4.4. Since $mn-\mu_I$ -ops and $mx-\mu_I$ -ops are independent of each other,

1. $mn-\mu_I$ -irresolute and $mx-\mu_I$ -cts are independent of each other.
2. $mx-\mu_I$ -irresolute and $mn-\mu_I$ -cts are independent of each other.

Theorem 4.3. Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a $mn-\mu_I$ -irresolute map if and only if the inverse image of each $mx-\mu_I$ -closed in (Y, σ_I) is a $mx-\mu_I$ -closed in (X, μ_I) .

Proof: We can prove this theorem by using the result, if G is a $mn-\mu_I$ -ops if and only if G^c is a $mx-\mu_I$ -closed set.

Theorem 4.4. If $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ are $mn-\mu_I$ -irresolute then $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is a $mn-\mu_I$ -irresolute map.

Proof: Let G be any $mn-\mu_I$ -ops in (Z, ρ_I) . Since $\mathbb{h}$ is $mn-\mu_I$ -irresolute, $\mathbb{h}^{-1}(G)$ is a $mn-\mu_I$ -ops in (Y, σ_I) . Also since $\mathbb{k}$ is $mn-\mu_I$ -irresolute, $\mathbb{k}^{-1}[\mathbb{h}^{-1}(G)] = (\mathbb{h} \circ \mathbb{k})^{-1}(G)$ is a $mn-\mu_I$ -ops in (X, μ_I) . Hence $\mathbb{h} \circ \mathbb{k}$ is $mn-\mu_I$ -irresolute.

Theorem 4.5. Let $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ be a $mx\text{-}\mu_I$ -irresolute map if and only if the inverse image of each $mn\text{-}\mu_I$ -closed in (Y, σ_I) is a $mn\text{-}\mu_I$ -closed in (X, μ_I) .

Proof: We can prove this theorem by using the result, if G is a $mx\text{-}\mu_I$ -ops if and only if G^c is a $mn\text{-}\mu_I$ -cds.

Theorem 4.6. If $\mathbb{k}: (X, \mu_I) \rightarrow (Y, \sigma_I)$ and $\mathbb{h}: (Y, \sigma_I) \rightarrow (Z, \rho_I)$ are $mx\text{-}\mu_I$ -irresolute then $\mathbb{h} \circ \mathbb{k}: (X, \mu_I) \rightarrow (Z, \rho_I)$ is a $mx\text{-}\mu_I$ -irresolute map.

Proof: Similar to that of theorem 4.11.

5 The Simple Extension of μ_I -topology over a μ_I -set

In (X, μ_I) a subset A of X , we denote by $\mu_I(A)$ the simple extension of μ_I over A , that is the collection of sets $U \cup (V \cap A)$, where $U, V \in \mu_I$. Note that $\mu_I(A)$ is finer than μ_I .

Theorem 5.1. If A is μ_I -dense subset of the space (X, μ_I) , then A is also μ_I -dense in $(X, \mu_I(A))$.

Proof: Since $\mu_I(A)$ is finer than μ_I , $\mu_I \subset \mu_I(A)$. This gives $C_{\mu_I(A)}(A) \subset C_{\mu_I}(A)$. To prove $C_{\mu_I}(A) \subset C_{\mu_I(A)}(A)$. Let $x \in C_{\mu_I}(A)$ and let G be a μ_I -ops of x in $\mu_I(A)$. Then $x \in G = H \cup (J \cap A)$ where $H, J \in \mu_I$. If $x \in H$ then $H \cap A \neq \emptyset$ and $G \cap A \neq \emptyset$. If $x \in J \cap A$ then $J \cap A \neq \emptyset$ and $G \cap A \neq \emptyset$. Hence $x \in C_{\mu_I(A)}(A)$. Therefore $C_{\mu_I(A)}(A) = C_{\mu_I}(A)$.

Theorem 5.2. Let (X, μ_I) be a μ_I -topological space with closed under intersection property. Let A be a μ_I -dense subset of the space (X, μ_I) . Then for every μ_I -open subset G of the space $(X, \mu_I(A))$ we have $C_{\mu_I}(G) = C_{\mu_I(A)}(G)$ and for every μ_I -closed subset F of the space $(X, \mu_I(A))$ we have $I_{\mu_I}(F) = I_{\mu_I(A)}(F)$.

Proof: Let $V \in \mu_I$. Since $\mu_I(A)$ is finer than μ_I , $C_{\mu_I(A)}(V) \subset C_{\mu_I}(V)$. Now to prove, $C_{\mu_I}(V) \subset C_{\mu_I(A)}(V)$. Let $x \in C_{\mu_I}(V)$ and let G be a μ_I -open neighborhood of x in $(X, \mu_I(A))$. Then $x \in G = H \cup (J \cap A)$ where $H, J \in \mu_I$. If $x \in H$ then $H \cap V \neq \emptyset$. Again if $x \in J \cap A \subset J$ then $J \cap V \neq \emptyset$ and hence $J \cap V \cap A \neq \emptyset$, since $J \cap V \in \mu_I$ and since A is μ_I -dense. Thus also in this case $G \cap V \neq \emptyset$ and hence $x \in C_{\mu_I(A)}(V)$. This implies $C_{\mu_I}(V) \subset C_{\mu_I(A)}(V)$. Henceforth $C_{\mu_I}(V) = C_{\mu_I(A)}(V)$ for each $V \in \mu_I$. Let $G \in \mu_I(A)$ then $G = H \cup (J \cap A)$ where $H, J \in \mu_I$. Clearly $C_{\mu_I}(H) = C_{\mu_I(A)}(H)$. Since $J \in \mu_I(A)$ and since A is a μ_I -dense subset of $(X, \mu_I(A))$, $C_{\mu_I(A)}(J \cap A) = C_{\mu_I(A)}(J) = C_{\mu_I}(J) = C_{\mu_I}(J \cap A)$. Thus $C_{\mu_I(A)}(G) = C_{\mu_I}(H) \cup C_{\mu_I}(J \cap A) = C_{\mu_I}(H \cup (J \cap A)) = C_{\mu_I}(G)$. Proceeding like this we can prove $I_{\mu_I}(F) = I_{\mu_I(A)}(F)$.

Corollary 5.1. Let (X, μ_I) be a GITS with closed under intersection property. If A is a μ_I -dense subset of the space (X, μ_I) . Then for every $V \in \mu_I(A)$ we have $I_{\mu_I}(C_{\mu_I}(V)) = I_{\mu_I(A)}(C_{\mu_I(A)}(V))$. Hence the set V is a regular μ_I -open subset of

(X, μ_I) if and only if it is regular μ_I -open in $(X, \mu_I(A))$.

Proof: From the previous theorem we have $I_{\mu_I}(C_{\mu_I}(V)) = I_{\mu_I}(C_{\mu_I(A)}(V)) = I_{\mu_I(A)}(C_{\mu_I(A)}(V))$.

6 The characterization of extension on μ_I -topology

Remark 6.1. If $\mathbb{k}: (X, \mu_I(A)) \rightarrow (Y, \sigma_I)$ is μ_I -cts. Then the restriction of $\mathbb{k}$ on (X, μ_I) [Shortly, $\mathbb{k}|(X, \mu_I)$] need not be μ_I -cts.

Example 6.1. Let $X = \{a, b, c\}$ and $Y = \{u, v, w\}$ with $\mu_I = \{\emptyset, \langle X, \emptyset, \{a\} \rangle, \langle X, \emptyset, \{b\} \rangle, \langle X, \emptyset, \emptyset \rangle, \langle X, \emptyset, \{a, b\} \rangle, \langle X, \{a, b\}, \emptyset \rangle\}$, $\mu_I(A) = \{\emptyset, \langle X, \emptyset, \{a\} \rangle, \langle X, \emptyset, \{b\} \rangle, \langle X, \emptyset, \emptyset \rangle, \langle X, \emptyset, \{a, b\} \rangle, \langle X, \{a, b\}, \emptyset \rangle, \langle X, \{b\}, \emptyset \rangle\}$ and $\sigma_I = \{\emptyset, \langle X, \emptyset, \{u\} \rangle, \langle X, \emptyset, \{v\} \rangle, \langle X, \emptyset, \emptyset \rangle, \langle X, \{v\}, \emptyset \rangle\}$. Define $\mathbb{k}: (X, \mu_I(A)) \rightarrow (Y, \sigma_I)$ by $\mathbb{k}(a) = u$, $\mathbb{k}(b) = v$ and $\mathbb{k}(c) = w$. Hence $\mathbb{k}$ is $\mu_I(A)$ -cts. But $\mathbb{k}|(X, \mu_I(A))$ is not μ_I -cts, since $\mathbb{k}^{-1}(\langle X, \{v\}, \emptyset \rangle) = \langle X, \{b\}, \emptyset \rangle \notin \mu_I$.

Remark 6.2. Since $\mu_I(A)$ is finer than μ_I , some elements of $\mu_I(A)$ does not belongs to μ_I and the elements of $\mu_I(A)$ which is not in μ_I need not be mn- μ_I -open in (X, μ_I) . For, $U \subset U \cup (V \cap A) \notin \mu_I$ and $U \in \mu_I(A)$, $U \cup (V \cap A)$ should not be mn- μ_I -open in $(X, \mu_I(A))$. By the previous example, we may conclude that every mx- μ_I -ops in $(X, \mu_I(A))$ need not be μ_I -open in (X, μ_I) .

Remark 6.3. A function $\mathbb{k}$ is mn- $\mu_I(A)$ -cts in $(X, \mu_I(A))$ then $\mathbb{k}|(X, \mu_I)$ is mn- μ_I -cts. In example 6.2, A function f is mx- $\mu_I(A)$ -cts in $(X, \mu_I(A))$ then $f|(X, \mu_I)$ need not be mx- μ_I -cts.

7 Conclusions

In example 4.2, k is a mn- μ_I -irresolute map but not mx- μ_I -irresolute and in example 4.3, k is a mx- μ_I -irresolute map but not mn- μ_I -irresolute. This examples evinces mn- μ_I -irresolute maps and mx- μ_I -irresolute maps are independent of each other. Remark 6.1 propounded the restriction of the function K on (X, μ_I) need not be a μ_I -continuous function. In remark 6.3, we discussed the connections between minimal μ_I -open sets in (X, μ_I) and in $(X, \mu_I(A))$. We hope that we improved some results concerning $\mu_I(A)$ -topological spaces. We will extend our research in kernel and contra continuous of μ_I -topological spaces.

Acknowledgements

My completion of this paper could not have been accomplished without the support of my guide and I cannot express enough thanks to my guide for the continued support and encouragement

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