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Rough sets applied in sublattices and ideals of lattices

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Abstract

The purpose of this paper is the study of rough hyperlattice. In this regards we introduce rough sublattice and rough ideals of lattices. We will proceed by obtaining lower and upper approximations in these lattices.

Keywords: rough set, lower approximation, upper approximation, rough sublattice, rough ideal

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1 Introduction

Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory (for more details see[3, 13]). Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. A lattice is a set on which two operations are defined, called *join* and *meet* and denoted by $\vee$

and $\wedge$, which satisfy the *idempotent*, *commutative* and *associative* laws, as well as the *absorption* laws:

$a \vee (b \wedge a) = a,$

$a \wedge (b \vee a) = a.$

Lattices are better behaved than partially ordered sets lacking upper or lower bounds.

The concept of rough set was originally proposed by Pawlak [21, 22] as a formal tool for modeling and processing incomplete information in information systems. Since then the subject has been investigated in many papers (see [20, 23, 24]). The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. Some authors, for example, Bonikowaski [5], Iwinski [15], and Pomykala and Pomykala [24] studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski [15]. In this paper we concentrates on the relationship between rough sets and lattice theory. We introduce the notion of rough sublattices (resp. ideals) of lattices, and investigate some properties of lower and upper approximations in lattices.

2 Preliminaries

Suppose that U is a non-empty set. A *partition* or *classification* of U is a family P of non-empty subsets of U such that each element of U is contained in exactly one element of P. Recall that an *equivalence relation* on a set U is a reflexive, symmetric, and transitive binary relation on U. Each partition P induces an equivalence relation θ on U by setting:

$x\theta y \Leftrightarrow$ x and y are in the same class of P.

Conversely, each equivalence relation θ on U induces a partition P of U whose classes have the form $[x]_\theta = \{y \in U \mid x\theta y\}$.

Given a non-empty universe U, by $P(U)$ we will denote the power set on U. If θ is an equivalence relation on U then for every $x \in U$, $[x]_\theta$ denotes the equivalence class of θ determined by x. For any $X \subseteq U$, we write X^c to denote the complementation of X in U, that is the set $U \setminus X$.

Definition 2.1. [8] A pair(U, θ); where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an *approximation space*.

Definition 2.2. [8] For an approximation space (U, θ), by a *rough approximation* in (U, θ) we mean a mapping $\mathfrak{A} : P(U) \to P(U) \times P(U)$ defined by for every $X \in P(U)$, $\mathfrak{A}(X) = (\underline{\mathfrak{A}}(X), \overline{\mathfrak{A}}(X))$ where $\underline{\mathfrak{A}}(X) = \{x \in X \mid [x]_\theta \subseteq X\}$, $\overline{\mathfrak{A}}(x) = \{x \in X \mid [x]_\theta \cap X \neq \emptyset\}$. $\underline{\mathfrak{A}}(X)$ is called a *lower rough approximation* of X in (U, θ), where as $\overline{\mathfrak{A}}(X)$ is called *upper rough approximation* of X in (U, θ).

Definition 2.3. [8] Given an approximation space (U, θ) a pair $(A, B) \in P(U) \times P(U)$ is called a *rough set* in (U, θ) iff $(A, B) = \mathfrak{A}(X)$ for some $X \in P(U)$.

For the sake of illustration, let (U, θ) is an approximation space, where:
$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, and an equivalence relation θ with the following equivalence classes:
$E_1 = \{x_1, x_4, x_8\}$,
$E_2 = \{x_2, x_5, x_7\}$,
$E_3 = \{x_3\}$,
$E_4 = \{x_6\}$,
Let $X = \{x_3, x_5\}$, then $\underline{\mathfrak{A}}(X) = \{x_3\}$ and $\overline{\mathfrak{A}}(X) = \{x_2, x_3, x_5, x_7\}$ and so $(\{x_3\}, \{x_2, x_3, x_5, x_7\}) = \mathfrak{A}(X)$ is a rough set.
The reader will find in [18,21-25] a deep study of rough set theory.

Definition 2.4. [7] A subset $X$of U is called $definable$ if $\underline{\mathfrak{A}}(X) = \overline{\mathfrak{A}}(X)$. If $X \subseteq U$ given by a predicate P and $x \in U$, then:
1. $x \in \underline{\mathfrak{A}}(X)$ means that x certainly has property P,
2. $x \in \overline{\mathfrak{A}}(X)$ means that x possibly has property P,
3. $x \in U \setminus \overline{\mathfrak{A}}(X)$ means that x definitely does not have property P.

When $\mathfrak{A}(A) \sqsubseteq \mathfrak{A}(B)$, we say that $\mathfrak{A}(A)$ is a *rough subset* of $\mathfrak{A}(B)$. Thus in the case of rough sets $\mathfrak{A}(A)$ and $\mathfrak{A}(B)$, $\mathfrak{A}(A) \sqsubseteq \mathfrak{A}(B)$ if and only if $\underline{\mathfrak{A}}(A) \subseteq \underline{\mathfrak{A}}(B)$ and $\overline{\mathfrak{A}}(A) \subseteq \overline{\mathfrak{A}}(B)$. This property of rough inclusion has all the properties of set inclusion. The rough complement of $\mathfrak{A}(A)$ denoted by $\mathfrak{A}^c(A)$ is defined by: $\mathfrak{A}^c(A) = (U \setminus \overline{\mathfrak{A}}(A), U \setminus \underline{\mathfrak{A}}(A))$. Also, we can define $\mathfrak{A}(A) \setminus \mathfrak{A}(B)$ as follows:
$\mathfrak{A}(A) \setminus \mathfrak{A}(B) = \mathfrak{A}(A) \sqcap \mathfrak{A}^c(B) = (\underline{\mathfrak{A}}(A) \setminus \overline{\mathfrak{A}}(B), \overline{\mathfrak{A}}(A) \setminus \underline{\mathfrak{A}}(B))$.

Let L be a lattice and $S \subseteq L$, If S is a lattice, then S is called a *sublattice* of L. A sublattice I is called an *ideal* of L, if $a \in L$ and $x \in I$ imply $a \wedge x \in L$

(see[2]).

Let ρ be an equivalence relation on L and x, y, $z \in L$.

(1) ρ is called a *congruence relation* if $x\rho y$ implies $(x \vee z)\rho(y \vee z)$ and $(x \wedge z)\rho(y \wedge z)$.

(2) ρ is called a *complete congruence relation* if $[x]_\rho \vee [y]_\rho = [x \vee y]_\rho$, and $[x]_\rho \wedge [y]_\rho = [x \wedge y]_\rho$.

If ρ is a congruence relation on L, then it is easy to verify that $[x]_\rho \vee [y]_\rho \subseteq [x \vee y]_\rho$, $[x]_\rho \wedge [y]_\rho \subseteq [x \wedge y]_\rho$.

3 Rough ideals of lattices

Throughout this paper L denotes a lattice. Let ρ be an equivalence relation on L and X be a non-empty subset of L. When $U = L$ and θ is the above equivalence relation, then we use the pair (L, ρ) instead of the approximation space (U, θ). Also, in this case we use the symbols $\underline{\mathfrak{A}}_\rho(X)$ and $\overline{\mathfrak{A}}_\rho(X)$ instead of $\underline{\mathfrak{A}}(X)$ and $\overline{\mathfrak{A}}(X)$.

Proposition 3.1. For every approximation space (L, ρ), where ρ is an equivalence relation, and every subsets A, $B \subseteq L$, we have:

(1) $\underline{\mathfrak{A}}_\rho(A) \subseteq A \subseteq \overline{\mathfrak{A}}_\rho(A)$;
(2) $\underline{\mathfrak{A}}_\rho(\emptyset) = \emptyset = \overline{\mathfrak{A}}_\rho(\emptyset)$;
(3) $\underline{\mathfrak{A}}_\rho(L) = L = \overline{\mathfrak{A}}_\rho(L)$;
(4) If $A \subseteq B$, then $\underline{\mathfrak{A}}_\rho(A) \subseteq \underline{\mathfrak{A}}_\rho(B)$, and $\overline{\mathfrak{A}}_\rho(A) \subseteq \overline{\mathfrak{A}}_\rho(B)$;
(5) $\underline{\mathfrak{A}}_\rho(\underline{\mathfrak{A}}_\rho(A)) = \underline{\mathfrak{A}}_\rho(A)$;
(6) $\overline{\mathfrak{A}}_\rho(\overline{\mathfrak{A}}_\rho(A)) = \overline{\mathfrak{A}}_\rho(A)$;
(7) $\overline{\mathfrak{A}}_\rho(\underline{\mathfrak{A}}_\rho(A)) = \underline{\mathfrak{A}}_\rho(A)$;
(8) $\underline{\mathfrak{A}}_\rho(\overline{\mathfrak{A}}_\rho(A)) = \overline{\mathfrak{A}}_\rho(A)$;
(9) $\underline{\mathfrak{A}}_\rho(A) = (\overline{\mathfrak{A}}_\rho(A^c))^c$;
(10)$\overline{\mathfrak{A}}_\rho(A) = (\underline{\mathfrak{A}}_\rho(A^c))^c$;
(11)$\underline{\mathfrak{A}}_\rho(A \cap B) = \underline{\mathfrak{A}}_\rho(A) \cap \underline{\mathfrak{A}}_\rho(B)$;
(12)$\overline{\mathfrak{A}}_\rho(A \cap B) \subseteq \overline{\mathfrak{A}}_\rho(A) \cap \overline{\mathfrak{A}}_\rho(B)$;
(13)$\underline{\mathfrak{A}}_\rho(A \cup B) \supseteq \underline{\mathfrak{A}}_\rho(A) \cup \underline{\mathfrak{A}}_\rho(B)$;
(14)$\overline{\mathfrak{A}}_\rho(A \cup B) = \overline{\mathfrak{A}}_\rho(A) \cup \overline{\mathfrak{A}}_\rho(B)$;
(15)$\underline{\mathfrak{A}}_\rho([x]_\rho) = \overline{\mathfrak{A}}_\rho([x]_\rho)$ for all $x \in L$;

Proof. (15) $\underline{\mathfrak{A}}_\rho([x]_\rho) = \{y \in L \mid [y]_\rho \subseteq [x]_\rho\} = [x]_\rho$, and $\overline{\mathfrak{A}}_\rho([x]_\rho) = \{y \in L \mid [y]_\rho \cap [x]_\rho \neq \emptyset\} = [x]_\rho$. Hence $\underline{\mathfrak{A}}_\rho([x]_\rho) = \overline{\mathfrak{A}}_\rho([x]_\rho)$.

The other parts of the proof is similar to the [17, Theorem 2.1] and [7, Proposition 4.1]. □

The following example shows that the converse of (12) and (13) in Proposition 3.1 are not true.

Example 3.2. Let $L = \{1, 2, ..., 8\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be an equivalence relation on L with the following equivalence classes:
$[1]_\rho = \{1, 4, 8\}$,
$[2]_\rho = \{2, 5, 7\}$,
$[3]_\rho = \{3\}$,
$[6]_\rho = \{6\}$,
and $A = \{3, 5, 7\}$, $B = \{2, 6\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{6\}$,
$\underline{\mathfrak{A}}_\rho(A \cup B) = \{2, 3, 5, 6, 7\}$,
$\overline{\mathfrak{A}}_\rho(A) = \{2, 3, 5, 7\}$,
$\overline{\mathfrak{A}}_\rho(B) = \{2, 5, 6, 7\}$,
$\overline{\mathfrak{A}}_\rho(A \cap B) = \emptyset$,
and so $\overline{\mathfrak{A}}_\rho(A) \cap \overline{\mathfrak{A}}_\rho(B) \nsubseteq \overline{\mathfrak{A}}_\rho(A \cap B)$, $\underline{\mathfrak{A}}_\rho(A \cup B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \cup \underline{\mathfrak{A}}_\rho(B)$.

Corollary 3.3. For every approximation space (L, ρ),
(i) For every $A \subseteq L$, $\underline{\mathfrak{A}}_\rho(A)$ and $\overline{\mathfrak{A}}_\rho(A)$ are definable sets,
(ii) For every $x \in L$, $[x]_\rho$ is definable set.

Proof. It is immediately by Proposition 3.1 (parts (5), (6), (7), (8) and (15)). □

If A and B are non-empty subsets of L, let $A \wedge B$ and $A \vee B$ denotes the following sets:

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\},\ A \vee B = \{a \vee b \mid a \in A, b \in B\}.$$

Proposition 3.4. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B) = \overline{\mathfrak{A}}_\rho(A \wedge B)$.

Proof. Suppose z be any element of $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$, then $z = a \wedge b$ for some $a \in \overline{\mathfrak{A}}_\rho(A)$, $b \in \overline{\mathfrak{A}}_\rho(B)$, hence $[a]_\rho \cap A \neq \emptyset$ and $[b]_\rho \cap B \neq \emptyset$ and so there exist $x \in [a]_\rho \cap A$ and $y \in [b]_\rho \cap B$. Therefore $x \wedge y \in A \wedge B$

and $x \wedge y \in [a]_\rho \wedge [b]_\rho = [a \wedge b]_\rho$ hence $[a \wedge b]_\rho \cap (A \wedge B) \neq \emptyset$ and so $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B) \subseteq \overline{\mathfrak{A}}_\rho(A \wedge B)$.

Conversely, let $x \in \overline{\mathfrak{A}}_\rho(A \wedge B)$ then $[x]_\rho \cap (A \wedge B) \neq \emptyset$ hence there exists $y \in [x]_\rho$ and $y \in A \wedge B$ and so $y = a \wedge b$ for some $a \in A$ and $b \in B$. Now we have $x \in [y]_\rho = [a \wedge b]_\rho = [a]_\rho \wedge [b]_\rho$. Then there exist $x' \in [a]_\rho$ and $y' \in [b]_\rho$ such that $x = x' \wedge y'$. Since $a \in [x']_\rho \cap A$ and $b \in [y']_\rho \cap B$, hence $x' \in \overline{\mathfrak{A}}_\rho(A)$ and $y' \in \overline{\mathfrak{A}}_\rho(B)$, which yields that $x = x' \wedge y' \in \overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$ and so $\overline{\mathfrak{A}}_\rho(A \wedge B) \subseteq \overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$. □

Proposition 3.5. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\overline{\mathfrak{A}}_\rho(A) \vee \overline{\mathfrak{A}}_\rho(B) = \overline{\mathfrak{A}}_\rho(A \vee B)$.

Proof. The proof is similar to the proof of Proposition 3.4, by considering the suitable modification by using the definition of $A \vee B$. □

Proposition 3.6. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B) \subseteq \underline{\mathfrak{A}}_\rho(A \wedge B)$.

Proof. Suppose x be any element of $\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B)$ then $x = a \wedge b$ for some $a \in \underline{\mathfrak{A}}_\rho(A)$ and $b \in \underline{\mathfrak{A}}_\rho(B)$. Hence $[a]_\rho \subseteq A$ and $[b]_\rho \subseteq B$. Since $[a \wedge b]_\rho = [a]_\rho \wedge [b]_\rho \subseteq A \wedge B$, we get $a \wedge b \in \underline{\mathfrak{A}}_\rho(A \wedge B)$ and so $x \in \underline{\mathfrak{A}}_\rho(A \wedge B)$. □

The following example shows that the converse of Proposition 3.6 is not true.

Example 3.7. Let $L = \{0, 1, 2, ..., 11\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be a complete congruence relation on L with the following equivalence classes:
$[0]_\rho = \{0, 1, 2\}$,
$[3]_\rho = \{3, 4, 5\}$,
$[6]_\rho = \{6, 7, 8\}$,
$[9]_\rho = \{9, 10, 11\}$,
and $A = \{1, 3, 4, 5\}$, $B = \{0, 1, 2, 6, 8\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3, 4, 5\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$,
$A \wedge B = \{0, 1, 2, 3, 4, 5\}$
$\underline{\mathfrak{A}}_\rho(A \wedge B) = \{0, 1, 2, 3, 4, 5\}$,
$\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$
and so $\underline{\mathfrak{A}}_\rho(A \wedge B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B)$.

Proposition 3.8. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B) \subseteq \underline{\mathfrak{A}}_\rho(A \vee B)$.

Proof. The proof is similar to the proof of Proposition 3.6, by considering the suitable modification by using the definition of $A \vee B$. □

The following example shows that $\underline{\mathfrak{A}}_\rho(A \vee B) \subseteq \underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B)$ does not hold in general.

Example 3.9. Let $L = \{0, 1, 2, ..., 8\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be a complete congruence relation on L with the following equivalence classes:
$[0]_\rho = \{0, 1, 2\}$,
$[3]_\rho = \{3, 4\}$,
$[5]_\rho = \{5, 6, 7, 8\}$,
and $A = \{3, 4, 5, 7\}$, $B = \{0, 1, 2, 3, 6, 8\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3, 4\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$,
$A \vee B = \{3, 4, 5, 6, 7, 8\}$,
$\underline{\mathfrak{A}}_\rho(A \vee B) = \{3, 4, 5, 6, 7, 8\}$,
$\underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B) = \{3, 4\}$,
and so $\underline{\mathfrak{A}}_\rho(A \vee B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B)$

Lemma 3.10. Let ρ_1 and ρ_2 be two complete congruence relations on L such that $\rho_1 \subseteq \rho_2$ and let A be a non-empty subset of L, then:
(i) $\underline{\mathfrak{A}}_{\rho_2}(A) \subseteq \underline{\mathfrak{A}}_{\rho_1}(A)$,
(ii) $\overline{\mathfrak{A}}_{\rho_1}(A) \subseteq \overline{\mathfrak{A}}_{\rho_2}(A)$.

Proof. It is straightforward. □

The following Corollary follows from Lemma 3.10.

Corollary 3.11. Let ρ_1 and ρ_2 be two complete congruence relations on L and A a non-empty subset of L, then:
(i) $\underline{\mathfrak{A}}_{\rho_1}(A) \cap \underline{\mathfrak{A}}_{\rho_2}(A) \subseteq \underline{\mathfrak{A}}_{(\rho_1 \cap \rho_2)}(A)$,
(ii) $\overline{\mathfrak{A}}_{(\rho_1 \cap \rho_2)}(A) \subseteq \overline{\mathfrak{A}}_{\rho_1}(A) \cap \overline{\mathfrak{A}}_{\rho_2}(A)$.

Proposition 3.12. Let ρ be a congruence relation on L, and J be an ideal of L, then $\overline{\mathfrak{A}}_\rho(J)$ is an ideal of L.

Proof. Suppose $a, b \in \overline{\mathfrak{A}}_\rho(J)$ and $r \in L$, then $[a]_\rho \cap J \neq \emptyset$ and $[b]_\rho \cap J \neq \emptyset$. So there exist $x \in [a]_\rho \cap J$ and $y \in [a]_\rho \cap J$. Since J is an ideal of L, we have $x \vee y \in J$ and $x \vee y \in [a]_\rho \vee [b]_\rho \subseteq [a \vee b]_\rho$. Hence $[a \vee b]_\rho \cap J \neq \emptyset$ which implies $a \vee b \in \overline{\mathfrak{A}}_\rho(J)$. Also, we have $r \wedge x \in J$ and $r \wedge x \in [r]_\rho \wedge [a]_\rho \subseteq [r \wedge a]_\rho$. So $[r \wedge a]_\rho \cap J \neq \emptyset$ which implies $r \wedge a \in \overline{\mathfrak{A}}_\rho(J)$. Therefore $\overline{\mathfrak{A}}_\rho(J)$ is an ideal of L. □

Similarly, if ρ is a congruence relation on L and J is a sublattice of L, then $\overline{\mathfrak{A}}_\rho(J)$ is a sublattice of L.

Proposition 3.13. Let ρ be a complete congruence relation on L, and J be an ideal of L, then $\underline{\mathfrak{A}}_\rho(J)$ is an ideal of L.

Proof. Suppose a, $b \in \underline{\mathfrak{A}}_\rho(J)$ and $r \in L$, then $[a]_\rho \subseteq J$ and $[b]_\rho \subseteq J$. So $[a \vee b]_\rho = [a]_\rho \vee [b]_\rho \subseteq J$, and $[r \wedge a]_\rho = [a]_\rho \wedge [b]_\rho \subseteq J$. Hence $a \vee b \in \underline{\mathfrak{A}}_\rho(J)$ and $r \wedge a \in \underline{\mathfrak{A}}_\rho(J)$. □

Similarly, if ρ is a complete congruence relation on L and J is a sublattice of L, then $\underline{\mathfrak{A}}_\rho(J)$ is a sublattice of L.

Definition 3.14. Let ρ be a congruence relation on L and $\mathfrak{A}_\rho(A) = (\underline{\mathfrak{A}}_\rho(A), \overline{\mathfrak{A}}_\rho(A))$ a *rough set* in the approximation space (L, ρ). If $\underline{\mathfrak{A}}_\rho(A)$ and $\overline{\mathfrak{A}}_\rho(A)$ are ideals (resp. sublattice) of L, then we call $\mathfrak{A}_\rho(A)$ a *rough ideal* (resp. *sublattice*). Note that a rough sublattice also is called a *rough lattice*.

Corollary 3.15. (i) Let ρ, be a congruence relation on L, and I an ideal of L then $\mathfrak{A}_\rho(I)$ is a rough ideals.

(ii) Let ρ be a complete congruence relation on L and J a sublattice of L, then $\mathfrak{A}_\rho(J)$ is a rough lattice.

Proof. It is obtained by 3.12 and 3.13. □

Let L and L' be two lattices, a map $f : L \to L'$ is said to be *homomor−phism* or (*lattice homomorphism*) if for all a, $b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$, and $f(a \vee b) = f(a) \vee f(b)$.

Now, let L and L' be two lattices and $f : L \to L'$ a homomorphism. It is well known, $\theta = \{(a, b) \in L \times L \mid f(a) = f(b)\} \subseteq L \times L$ is a congruence relation on L. Because if $a\theta b$ then $f(a) = f(b)$ and for all $z \in L$, we have $f(a \wedge z) = f(a) \wedge f(z) = f(b) \wedge f(z) = f(b \wedge z)$. Therefor $(a \wedge z)\ \theta\ (b \wedge z)$, and similarly $(a \vee z)\ \theta\ (b \vee z)$.

Theorem 3.16. Let L and L' be two lattices and $f : L \to L'$ a homomorphism. If A is a non-empty subset of L, then $f(\overline{\mathfrak{A}}_\theta(A)) = f(A)$.

Proof. Since $A \subseteq \overline{\mathfrak{A}}_\theta(A)$ it follows that $f(A) \subseteq f(\overline{\mathfrak{A}}_\theta(A))$.

Conversely, let $y \in f(\overline{\mathfrak{A}}_\theta(A))$. Then there exists an element $x \in \overline{\mathfrak{A}}_\theta(A)$, such that $f(x) = y$, so we have $[x]_\theta \cap A \neq \emptyset$. Thus there exists an element $a \in [x]_\theta \cap A$. Then $a \in [x]_\theta$, hence $x\theta a$, and so $f(x) = f(a) \in f(A)$, therefore $f(\overline{\mathfrak{A}}_\theta(A)) \subseteq f(A)$. □

Let $f : L \to L'$ be a homomorphism and A a subset of L, Since $\underline{\mathfrak{A}}_\theta(A) \subseteq A$ it follows that $f(\underline{\mathfrak{A}}_\theta(A)) \subseteq f(A)$. But the following example shows that, in general, $f(\underline{\mathfrak{A}}_\theta(A)) \neq f(A)$.

Example 3.17. Let $(L, \wedge, \vee)$ and $(L', \wedge, \vee)$ be two lattices where $L = \{1, 2, 3, 4\}$; and $L' = \{5, 6, 7\}$; and for all s, t in L or L', $s \wedge t$= $min\{s, t\}$ and $s \vee t = max\{s, t\}$. The map $f : L \to L'$ given by

$$f(4) = f(3) = 7,\ f(2) = 6,\ f(1) = 5,$$

is a homomorphism. We have $\theta = \{3, 4\}$. Suppose $A = \{1, 2\}$, then $f(A) = \{5, 6\}$, $\underline{\mathfrak{A}}_\theta(A) = \emptyset$ and $f(\underline{\mathfrak{A}}_\theta(A)) = \emptyset$, and so $f(\underline{\mathfrak{A}}_\theta(A)) \neq f(A)$.

The lower and upper approximations can be presented in an equivalent form as follows:

Let L be a lattice, ρ a congruence relation on L, and A a non-empty subset of L. Then we define $\overline{\vee}$ and $\overline{\wedge}$ on $L/\rho = \{[x]_\rho \mid x \in L\}$, by

$$[x]_\rho \overline{\vee} [y]_\rho = [x \vee y]_\rho,\ [x]_\rho \overline{\wedge} [y]_\rho = [x \wedge y]_\rho.$$

This relation is well-defined, since if $[x_1]_\rho = [x_2]_\rho$ and $[y_1]_\rho = [y_2]_\rho$, then $x_1 \rho x_2$ and $y_1 \rho y_2$. Since ρ is a congruence relation we have $(x_1 \vee y_1)\rho(x_2 \vee y_1)$ and $(x_2 \vee y_1)\rho(x_2 \vee y_2)$. Then $(x_1 \vee y_1)\rho(x_2 \vee y_2)$, so $[x_1 \vee y_1]_\rho = [x_2 \vee y_2]_\rho$. Therefore $[x_1]_\rho \overline{\vee} [y_1]_\rho = [x_2]_\rho \overline{\vee} [y_2]_\rho$.

It is easy to see that $(L/\rho, \overline{\vee}, \overline{\wedge})$, is a lattice. Also if $A \neq \emptyset$, and $A \subseteq L$ put $\underline{\underline{\mathfrak{A}}}_\rho(A) = \{[x]_\rho \in L/\rho \mid [x]_\rho \subseteq A\}$ and $\overline{\overline{\mathfrak{A}}}_\rho(A) = \{[x]_\rho \in L/\rho \mid [x]_\rho \cap A \neq \emptyset\}$.

Proposition 3.18. Let ρ be a congruence relation on L and J be an ideal of L, then $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ.

Proof. Assume that $[a]_\rho$, $[b]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \in L/\rho$. Then $[a]_\rho \cap J \neq \emptyset$ and $[b]_\rho \cap J \neq \emptyset$, so there exist $x \in [a]_\rho \cap J$ and $y \in [b]_\rho \cap J$. Since J is an ideal of L, we have $x \vee y \in J$ and $r \wedge x \in J$. Also, we have $x \vee y \in [a]_\rho \vee [b]_\rho \subseteq [a \vee b]_\rho$, and $r \wedge x \in [r]_\rho \wedge [a]_\rho \subseteq [r \wedge a]_\rho$. Therefore $[a \vee b]_\rho \cap J \neq \emptyset$ and $[r \wedge a]_\rho \cap J \neq \emptyset$, which imply $[a]_\rho \vee [b]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \wedge [a]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$. Therefore $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ. $\square$

Proposition 3.19. Let ρ be a complete congruence relation on L and J be an ideal of L, then $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ.

Proof. Assume that $[a]_\rho$, $[b]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \in L/\rho$. Then $[a]_\rho \subseteq J$ and $[b]_\rho \subseteq J$. Since J is an ideal of L, we have $a \vee b \in J$ and $r \wedge a \in J$ Therefore $[a]_\rho \vee [b]_\rho = [a \vee b]_\rho \subseteq J \vee J = J$, and $[r]_\rho \wedge [a]_\rho = [r \wedge a]_\rho \subseteq J$, which imply $[a]_\rho \vee [b]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \wedge [a]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$. Therefore $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ. $\square$

Proposition 3.20. (i) Let ρ be a congruence relation on L and J a sublattice of L, then $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is a sublattice of L/ρ.

(ii) Let ρ be a complete congruence relation on L and J a sublattice of L, then $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is a sublattice of L/ρ.

Proof. Similar to the proof of propositions 3.13, 3.18 and 3.19. $\square$

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Application of point method in risk evaluation for railway transport

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Abstract

The paper is dealing with risk assessment affecting the hazardous substances shipping by rail; there are identified and assessed risks during the work process. The point method is applied to evaluate how serious risks are. In conclusion, there are suggested particular measures to reduce or eliminate the risks. The main priority of the system should consist in providing a safe workplace, or minimizing and eliminating undesirable factors.

Keywords: transport, accident, emergency, hazardous substance, railway, risks assessment

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1 Introduction

Safety belongs to basic prerequisites in the transport process; therefore, the emergence of rail accidents as well as emergencies cannot be passed over in the transport process particularly in cases of shipping hazardous substances. Every responsible person involved in transporting hazardous substances is obliged to comply with the relevant rules and regulations so that risks could be prevented as much as possible.

There are a number of methods able to anticipate and mitigate the impacts of accidents. All of these methods follow their purpose and are limited by restrictions. This paper is presenting the point method application. The risk assessment is a highly complex process considering various criteria. Having identified threatening sources of risks and factors, assessment and subsequent managing risk can follow.

2 Current situation

Occurrence and consequences of emergencies and accidents is a worldwide problem. An accident is such an activity of transport participants occurring in case of conflict with legal standards and regulations.

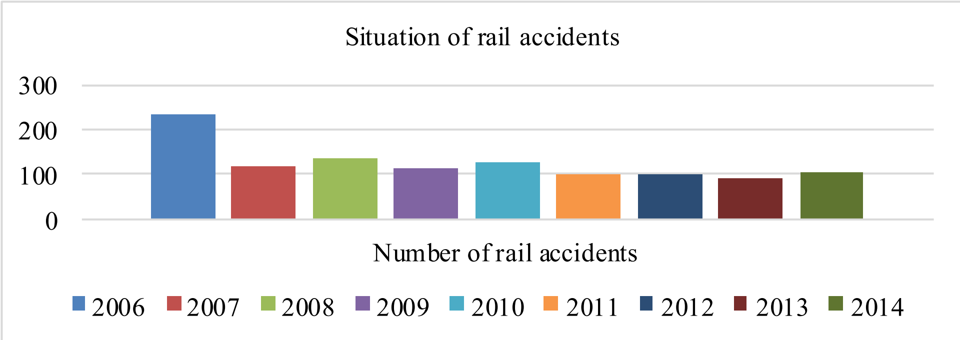
There is an incorrect movement of means of transport, interaction with one another or collision with other traffic participants with consequences resulting in damage, destruction or deterioration of means, vehicles, communications and further damage. This fact is accompanied by damage to health or fatalities caused to participants of accidents. [1]

Thorough cooperation of stakeholders as well as institutions can support significantly the smooth railway operation. Therefore, it becomes necessary to evaluate the situation and take measures while considering both the complex and partial situation solution processes.

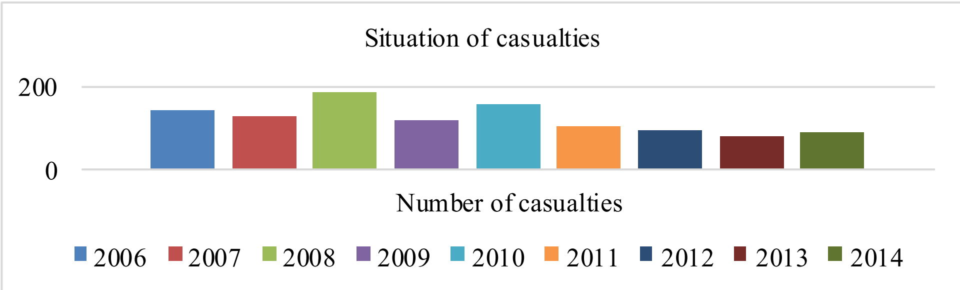
The available statistical data characterized the situations as follows: in the Czech Republic, a total of 1,100 accidents with 1,083 fatalities happened on the railways within the period 2006-2014.

Year	Number of accidents	Number of casualties
2006	233	141
2007	115	126
2008	133	183
2009	113	118
2010	125	155
2011	99	103
2012	97	92
2013	91	76
2014	104	89

Table 1 Number of rail accidents in the Czech Republic and number of relevant casualties [2]



Graph 1 Number of rail accidents in the Czech Republic [2]



Graf 2 Number of relevant casualties in the Czech Republic [2]

Although the shipment by rail seems comparatively safe, it is not entirely without risk. The accidents occurrence is affected by aspects such as human factor, technical condition of the train, technical condition of railway superstructure natural conditions as well as the transported goods. The risk and affects are much higher in case of shipping hazardous substances.

2.1 Risk assessment

Nowadays, there are high requirements for performance and work effort of employees; they dominate the threat resulting in working environment safety. Employers often do not realize that safe workplace can improve the quality of the entire work process.

Considering all the factors affecting the safe working environment is the basis for risks assessment in the work process.

Risk analysis is a method for identifying and assessing factors, which may threaten individual activities and objectives of the organization. We can use it for the risks identification, to which the enterprise is exposed to in terms of external and internal perspectives. It is based on identification of risks factors, developing scenarios, assessing the likelihood and consequences, and, finally, financial costs, in case that the emergency occurs. It is the basis for risk management and prevention of crisis situations in the enterprise. [2]

The point method, extended risk definition, was selected to assess risk in our case. The point method is classified as one of most frequently used methods for risks assessment. The level of risk is expressed by combining the value of the likelihood of risks, possible consequence and the effect of the occupational safety and health (OSH); having assessed, it is assigned to the relevant group of final risk. This method is focused on the protecting human life.

R (risk) = P (probability) x D (consequence) x V (effect of OSH level), [3]

P – probability establishes the option estimation that the undesirable event occurs. It is expressed by assigning specific numbers 1 - 5 (Table 2),

D – consequence expresses the seriousness of the consequence of the emergency occurrence; it is defined by five stages with assigned values from 1 to 5 (Table 3),

V – OSH level impact: this parameter comprises consideration of management level, the time of action period of threats, staff qualification, work ethic, the level of prevention, condition and age of technical equipment, maintenance level, the effect of work environment, workplace detachability, etc. (Table 4).

Point value	Verbal expression
1	Improbable
2	Random
3	Probable
4	Highly probable
5	Permanent

Table 2 Probability estimation [4]

Point value	Verbal expression
1	Negligible effect on probability and injury consequences
2	Little effect on probability and injury consequences
3	Considerable effect
4	Significant, big effect
5	More significant effects

Table 3 Consequence estimation [4]

Point value	Verbal expression
1	Damage to health and work activity
2	Injury followed by sick leave
3	More serious injury resulting in hospitalization
4	Severe occupational injury with permanent consequences
5	Fatal occupational injury

Table 4 OSH impact estimation [4]

Risk – final indicator, which is the product of all three parameters of the risk value. The lowest value can reach 1 and the highest 125. According to point range, the risk is classified into five categories. (Table 4).

Risk	Risk category	Point range	Safety assessment	Safety measures requirement
Negligible	I	1-4	Acceptable safety	Taking measures not required
Moderate	II	5-10	Acceptable risk at increased attention	System is classified as safe; improvement can be achieved, redress can be planned
Critical	III	11-50	Risk cannot be accepted without taking protective measures	Safety measures should be taken
Undesirable	IV	51-100	Inadequate safety, high possibility of injuries	Immediate corrective measures or short-term measures have to be taken
Unacceptable	V	100-125	Dangerous system, permanent threat of injury	Immediate cessation of activity, exclusion from operation

Table 5 Final risk range [4]

2 Point method application while transporting hazardous substances by rail

The carriage of hazardous substances by rail accounts for a significant share of total rail freight. Emergencies as well as accidents occur at shipping process resulting from hazardous substances characteristics. Number and scope of rail accidents is affected by many factors, which can be called causes resulting in consequences of various extents.

Each hazardous substance has its characteristics, according to which the material should be packed, loaded and stowed, shipped via adequate route and unloaded. The employees are frequently a significant element at giving rise to an accident: it is caused by activities, either intentional non-compliance with regulations and rules or by ignorance. These accidents affect the smooth flow of work process and shipping hazardous substances and threaten the very persons involved as well as people around. They may also affect significantly the property of residents within the accident as well as the environment (soil and water contamination, air toxic pollution). Therefore, all the time it is necessary to inspect and train the staff being focused on preventing accidents. The following table highlights the possible threats, which might arise during the rail-transport work process.

Number	Responsible action	Profession	Possible threat due to non-compliance with regulations
1.	Goods loading	loader	- goods loading, which may react together, omitting the tank failure (rupture), failure to comply with test date, improper packaging (certified package of I, II, III group) and goods labelling, damage to goods, incorrectly completed waybill, inappropriate use of wagon for the particular goods
2.	Tank labelling	shipper	- assigning wrong UN code, excessive number of pieces, overload, assigning improper parameters to a particular category of hazardous substances, different data in waybill in terms of labelling tanks/wagons or particular content
3.	Tank cleaning	tank cleaner	- failure to comply with the rules on safety equipment, sparking at a soiled tank - threat to health state of an employee (fatality)

4.	Train dispatch	conductor and chief guard	- faulty inspection of the train formation and brakes, improper connecting and disconnecting rail vehicles
5.	Maintenance of train-set and tanks	train maintenance worker	- tank leaks, unclosed dome lid, cracks, bulging, violent damage, improper securing the bottom valve, missing protective caps, blind fastening screws (leakage of hazardous substance, fire, explosion)
6.	Track maintenance	track engineer	- neglected maintenance of tracks, sliding rails, not removing snow, icing, vegetation, outdated track (derailment)
7.	Security devices inspection	Railway transport worker-specialist (shunter, train dispatcher, switchman, switch supervisor, signalman, announcer, level-crossing operator)	- improper position of sliding rail/derailer, forgotten stop (derailment), faulty signalling (collision with a car, person)
8.	Shipping process	engine driver	- demanding route (steep descent, sharp bends), collisions with objects, cars, people, gases leakage into the environment
9.	Loading inspection	security advisor	- improper purchase of vehicles, faulty testing of means of transport, inadequately trained staff

Table 6 Application of point method for expressing threat and risk identification

Number	Risk value P x D x V	Risk category
1.	2 x 2 x 3 12	III
2.	3 x 3 x 2 18	III
3.	2 x 4 x 5 40	III
4.	3 x 3 x 2 18	III
5.	4 x 5 x 4 80	IV
6.	2 x 3 x 2 12	III
7.	2 x 3 x 4 24	III
8.	2 x 3 x 3 12	III
9.	2 x 2 x 1 4	I

Table 7 Results of point method

4 Proposal of measures to reduce risks

Risks presented in table 6, to a greater or lesser extent, affect the occurrence of accidents and emergencies. Knowledge of possible threats can result in taking measures, which might encourage risk reduction or elimination.

Rail transport brings risks of different levels. Some risks are determined by illegal action of a third party (terroristic attack, criminality); therefore, these threats cannot be controlled properly.

List of threats resulting from the assessment of risks in terms of transporting hazardous substances by rail:

- rigorous assessment of the particular goods characteristics and safe loading,
- modernization and inspection of used wagons and security devices,
- improvement and checking used packaging/containers,
- inspection of proper filling and pumping tanks,
- thorough inspection of labelling and marking wagons,
- data checking in a waybill and wagon labelling/marking,
- observing number of loaded units, not overloading wagons,
- applying adequate protective equipment and compliance with regulations at tank cleaning,

- regular and complex inspection of the technical condition of the train, tanks and brakes,
- observing time-period checks,
- rigorous track inspection,
- tracks modernization,
- regular removing obstacles from the railway track (vegetation, snow, icing),
- weather forecasting and thorough evaluation of transport options,
- assessing and selecting route that is appropriate for shipping,
- goods inspection while transported,
- timely reporting in case of a terrorist attack or other unlawful entry of a third party,
- assessment and investigation of accidents and their causes so that recurrence of accidents due to same causes could be avoided,
- proper planning of work process,
- responsible performing work by employees,
- creating friendly work environment by superiors,

regular training: acquainting employees with risks, which might affect their work, work process knowledgeability, knowledge and compliance with relevant legislation, compliance with OSH, knowledge to provide first aid help.

5 Conclusions

Nowadays, risk identification belongs to a significant and inseparable prevention component leading to higher quality and safer working environment. The point method application does not have to provide objective assessment, and final risk determination does not result in accurate values. However, its benefit consists in identification of risks, which threaten the smooth transport by rail. Risks assessment results are highly significant for taking suggested measures encouraging occupational health.

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Fundamental hoop-algebras

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Abstract

In this paper, we investigate some results on hoop algebras and hyper hoop-algebras. We construct a hoop and a hyper hoop on any countable set. Then using the notion of the fundamental relation we define the fundamental hoop and we show that any hoop is a fundamental hoop and then we construct a fundamental hoop on any non-empty countable set.

Keywords: hoop algebras, hyper hoop algebras, (strong) regular relation,fundamental relations.

2000 AMS subject classifications: 20N20, 14L17, 97H50, 03G25,06F35.

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1 Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids were originally introduced by Bosbach in [7] under the name of complementary semigroups. It was proved that a hoop is a meet-semilattice. Hoop-algbras then investigated by Büchi and Owens in an unpublished manuscript [8] of 1975, and they have been studied by Blok and Ferreirim[2],[3], and Aglianò et.al.[1]. The study of hoops is motivated by researchers both in universal algebra and algebraic logic.In recent years, hoop theory was enriched with deep structure theorems.

Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic(see Theorem 3.8 of [1]) introduced by Hájek in [13]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops and MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

Hypersructure theory was introduced in 1934[15], by Marty. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography[9]. Many researchers have worked on this area. The authors applied hyper structure theory on hyper hoop and introduced and studied hyper hoop algebras in [17]and[16].

In this paper, we investigate some new results on hoop-algebras and hyper hoop-algebras. We construct a hoop and a hyper hoop on any countable set. Then using the notion of the fundamental relation we define the fundamental hoop.

2 Preliminaries

First, we recall following basic notions of the hypergroup theory from[10]:
Let A be a non-empty set. A hypergroupoid is a pair $(A, \odot)$, where $\odot : A \times A \longrightarrow P(A) - \{\emptyset\}$ is a binary hyperoperation on A. If associativity low holds, then $(A, \odot)$ is called a semihypergroup, and it is said to be commutative if $\odot$ is commutative. An element $1 \in A$ is called a unit, if $a \in 1 \odot a \cap a \odot 1$, for all $a \in A$ and is called a scaler unit, if $1 \odot a = a \odot 1 = \{a\}$, for all $a \in A$. Note that if $B, C \subseteq A$, then we consider $B \odot C$ by $B \odot C = \bigcup_{b\in B, c\in C} (b \odot c)$. (See [10])

Definition 2.1. [3] A *hoop-algebra* or briefly *hoop* is an algebra $(A, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that, (HP1): $(A, \odot, 1)$ is a commutative monoid and for all $x, y, z \in A$, (HP2): $x \rightarrow x = 1$, (HP3): $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and (HP4): $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$. On hoop A we define "$x \leq y$" if and only if $x \rightarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A.

Definition 2.2. [17] A *hyper hoop-algebra* or briefly, a *hyper hoop* is a non-empty set A endowed with two binary hyperoperations $\odot, \rightarrow$ and a constant 1 such that, for all $x, y, z \in A$ satisfying the following conditions,
(HHA1) $(A, \odot, 1)$ is a commutative semihypergroup with 1 as the unit,
(HHA2) $1 \in x \rightarrow x$,
(HHA3) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$,
(HHA4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,
(HHA5) $1 \in x \rightarrow 1$,

(HHA6) if $1 \in x \to y$ and $1 \in y \to x$ then $x = y$,
(HHA7) if $1 \in x \to y$ and $1 \in y \to z$ then $1 \in x \to z$.

In the sequel we will refer to the hyper hoop $(A, \odot, \to, 1)$ by its universe A. On hyper hoop A, we define $x \leq y$ if and only if $1 \in x \to y$. If A is a hyper hoop, it is easy to see that $\leq$ is a partial order relation on A. Moreover, for all $B, C \subseteq A$ we define $B \ll C$ iff there exist $b \in B$ and $c \in C$ such that $b \leq c$ and define $B \leq C$ iff for any $b \in B$ there exists $c \in C$ such that $b \leq c$. A hyper hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.

Proposition 2.3. In any hyper hoop $(A, \odot, \to, 1)$, if $x \odot y$ and $x \to y$ are singletons, for any $x, y \in A$, then $(A, \odot, \to, 1)$ is a hoop. Then hyper hoops are a generalization of hoops and every hoop is a trivial hyper hoop.

Proposition 2.4. [17] Let A be a *hyper hoop*. Then for all $x, y, z \in A$ and $B, C, D \subseteq A$, the following hold,
(HHA8) $x \odot y \ll z \Leftrightarrow x \leq y \to z$,
(HHA9) $B \odot C \ll D \Leftrightarrow B \ll C \to D$,
(HHA10) $z \to y \leq (y \to x) \to (z \to x)$,
(HHA11) $z \to y \ll (x \to z) \to (x \to y)$,
(HHA12) $1 \odot 1 = \{1\}$.

Notations: Let **R** be an equivalence relation on hyper hoop A and $B, C \subseteq A$. Then $B\mathbf{R}C$, $B\overline{\mathbf{R}}C$ and $B\overline{\overline{\mathbf{R}}}C$ denoted as follows,
(i) $B\mathbf{R}C$ if there exist $b \in B$ and $c \in C$ such that $b\mathbf{R}c$,
(ii) $B\overline{\mathbf{R}}C$ if for all $b \in B$ there exists $c \in C$ such that $b\mathbf{R}c$ and for all $c \in C$ there exists $b \in B$ such that $b\mathbf{R}c$,
(iii) $B\overline{\overline{\mathbf{R}}}C$ if for all $b \in B$ and $c \in C$, we have $b\mathbf{R}c$.

Remark 2.5. It is clear that $B\overline{\mathbf{R}}C$ and $C\overline{\mathbf{R}}D$ imply that $B\overline{\mathbf{R}}D$, for all $B, C, D \subseteq A$.

Definition 2.6. [17] Let **R** be an equivalence relation on hyper hoop A. Then **R** is called a *regular relation* on A if and only if for all $x, y, z \in A$,
(i) if $x\mathbf{R}y$, then $x \odot z\overline{\mathbf{R}}y \odot z$,
(ii) if $x\mathbf{R}y$, then $x \to z\overline{\mathbf{R}}y \to z$ and $z \to x\overline{\mathbf{R}}z \to y$,
(iii) if $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, then $x\mathbf{R}y$.

Definition 2.7. [17] Let **R** be an equivalence relation on hyper hoop A. Then **R** is called a strong regular relation on A if and only if, for all $x, y, z \in A$,
(i) if $x\mathbf{R}y$, then $x \odot z\overline{\overline{\mathbf{R}}}y \odot z$,
(ii) if $x\mathbf{R}y$, then $x \to z\overline{\overline{\mathbf{R}}}y \to z$ and $z \to x\overline{\overline{\mathbf{R}}}z \to y$,

Theorem 2.8. *[17] Let* $\mathbf{R}$ *be a regular relation on hyper hoop* A *and* $\frac{A}{\mathbf{R}}$ *be the set of all equivalence classes respect to* $\mathbf{R}$*, that is* $\frac{A}{\mathbf{R}} = \{[x] | x \in A\}$*. Then* $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ *is a hyper hoop, which is called the quotient hyper hoop of* A *respect to* $\mathbf{R}$*, where for all* $[x], [y] \in \frac{A}{\mathbf{R}}$*,*

$$[x] \otimes [y] = \{[t] | t \in x \odot y\} \quad \text{and} \quad [x] \hookrightarrow [y] = \{[z] | z \in x \to y\}$$

Theorem 2.9. *[17] Let* $\mathbf{R}$ *be a strong regular relation on hyper hoop* A*. Then* $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ *is a hoop which is called the quotient hoop of* A *respect to* $\mathbf{R}$*.*

Theorem 2.10. *[4] Let* X *and* Y *be two sets such that* $|X| = |Y|$*. If* $(Y, \leq, 0)$ *is a well-ordered set, then there exists a binary order relation on* X *and* $x_0 \in X$*, such that* $(X, \leq, x_0)$ *is a well-ordered set.*

Lemma 2.11. *[14] Let* X *be an infinite set. Then for any set* $\{a, b\}$*, we have* $|X \times \{a, b\}| = |X|$*.*

3 Constructing of hoops

In this section, we show that we can construct a hoop on any non-empty countable set.

Lemma 3.1. *Let* A *and* B *be two sets such that* $|A| = |B|$*. If* A *is a hoop, then we can construct a hoop on* B *by using of* A*.*

Proof. *Since* $|A| = |B|$*, there exists a bijection* $\varphi : A \to B$*. For any* $b_1, b_2 \in B$*. We define the binary operations* $\odot_B$ *and* $\to_B$ *on* B *by,*

$$b_1 \odot_B b_2 = \varphi(a_1 \odot_A a_2) \quad \text{and} \quad b_1 \to_B b_2 = \varphi(a_1 \to_A a_2)$$

where $b_1 = \varphi(a_1)$*,* $b_2 = \varphi(a_2)$ *and* $a_1, a_2 \in A$*. It is easy to show that* $\odot_B$ *and* $\to_B$ *are well-defined. Moreover, for any* $b \in B$ *we define* 1_B *as* $1_B = \varphi(1_A)$*. Now, by some modification we can show that* $(B, \odot_B, \to_B, 1_B)$ *is a hoop.*□

Lemma 3.2. *For any* $k \in \mathbb{N}$*, we can construct a hoop on* $\mathbb{W}_k = \{0, 1, 2, 3, ..., k - 1\}$*.*

Proof. *Let* $k \in \mathbb{N}$*. We define the operations "*$\odot$*" and "*$\to$*", on* $\mathbb{W}_k$ *as follows, for all* $a, b \in \mathbb{W}_k$*,*

$$a \odot b = \begin{cases} 0 & \text{if } a + b \leq k - 1, \\ a + b - k + 1 & \text{otherwise} \end{cases}$$

$$a \to b = \begin{cases} k - 1 & \text{if } a \leq b, \\ k - 1 - a + b & \text{otherwise} \end{cases}$$

Now, we show that $(\mathbb{W}_k, \odot, \rightarrow, k-1)$ is a hoop,
***(HP1)**: Since, $+$ is commutative, hence $\odot$ is commutative. Now, we show that $\odot$ is associative on $\mathbb{W}_k$. For all $a, b, c \in \mathbb{W}_k$,*
***Case 1**: If $a+b \leq k-1$ and $b+c \leq k-1$, then $(a \odot b) \odot c = (0) \odot c = 0$ and $a \odot (b \odot c) = a \odot 0 = 0$ and so $(a \odot b) \odot c = a \odot (b \odot c)$.*
***Case 2**: If $a+b > k-1$ and $b+c \leq k-1$, since $a+b+c \leq 2(k-1)$ and so $a+b+c-k+1 \leq k-1$, we get $(a \odot b) \odot c = (a+b-k+1) \odot c = 0$. On the other hand, $a \odot (b \odot c) = a \odot 0 = 0$ and then $(a \odot b) \odot c = a \odot (b \odot c)$.*
***Case 3**: If $a+b > k-1$ and $b+c > k-1$, then $(a \odot b) \odot c = (a+b-k+1) \odot c$ and $a \odot (b \odot c) = a \odot (b+c-k+1)$. If $a+b+c \leq 2k$ then $(a \odot b) \odot c = a \odot (b \odot c) = 0$ and if $a+b+c > 2k$ then $(a \odot b) \odot c = a \odot (b \odot c) = a+b+c-2k+2$.*
***Case 4**: Let $a+b \leq k-1$ and $b+c > k-1$. This case is similar to the Case 2. Now, we have $0 \odot k-1 = 0$ and if $0 \neq a \in \mathbb{W}_k$, we have $a+(k-1) > k-1$ and so $a \odot (k-1) = a+k-1-k+1 = a$. Then $(k-1)$ is the identity of $(\mathbb{W}_k, \odot)$ and so $(\mathbb{W}_k, \odot, k-1)$ is a commutative monoid.*
***(HP2)**: It is clear that, for all $a \in \mathbb{W}_k$, $a \rightarrow a = k-1$.*
***(HP3)**: Let $a, b, c \in \mathbb{W}_k$. We show that $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 1**: If $a+b \leq k-1$ and $a \leq b \leq c$, then $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$ and $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 2**: If $a+b \leq k-1$ and $a \leq c < b$, $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$ and since $k-1-b+c \geq a$, $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1-b+c) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 3**: If $a+b \leq k-1$ and $b \leq a \leq c$, then $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$ and $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 4**: If $a+b \leq k-1$ and $b \leq c < a$, then $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$ and $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 5**: If $a+b \leq k-1$ and $c \leq b \leq a$, then $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$. On the other hand since $a+b \leq k-1$, we get $a+b-c \leq k-1$, $a \leq (k-1-b+c)$ and $a \rightarrow (k-1-b+c) = k-1$. Then $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1-b+c) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 6**: If $a+b \leq k-1$ and $c \leq a < b$, then $(a \odot b) \rightarrow c = 0 \rightarrow c = k-1$. On the other hand since $a+b \leq k-1$, we get $a+b-c \leq k-1$, $a \leq (k-1-b+c)$ and $a \rightarrow (k-1-b+c) = k-1$. Then $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1-b+c) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 7**: Let $a+b > k-1$ and $a \leq b \leq c$. Since $a \leq b \leq c$, we get $a+b-c \leq a \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \rightarrow c = (a+b-k+1) \rightarrow c = k-1$. On the other hand, $a \rightarrow (b \rightarrow c) = a \rightarrow (k-1) = k-1$. Hence, $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*
***Case 8**: Let $a+b > k-1$ and $a \leq c < b$. Since $a \leq c < b$ we get $a+b-c \leq b \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \rightarrow c = (a+b-k+1) \rightarrow c = k-1$. On the other hand, since $k-1-b+c \geq c \geq a$, we get $a \rightarrow (b \rightarrow c) = a \rightarrow$*

$(k-1-b+c) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.
***Case 9**: Let $a+b > k-1$ and $b \leq a \leq c$. Since $b \leq a \leq c$, we get $a+b-c \leq a \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \to c = (a+b-k+1) \to c = k-1$. On the other hand since $k-1-b+c \geq c \geq a$, we get $a \to (b \to c) = a \to (k-1-b+c) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.*
***Case 10**: Let $a+b > k-1$ and $b \leq c < a$. Since $b \leq c < a$, we get $a+b-c \leq a \leq k-1$ and so $a+b-k+1 \leq c$. Then $(a \odot b) \to c = (a+b-k+1) \to c = k-1$. On the other hand $a \to (b \to c) = a \to (k-1) = k-1$. Hence, $(a \odot b) \to c = a \to (b \to c)$.*
***Case 11**: If $a+b > k-1$ and $c \leq b \leq a$, then $(a \odot b) \to c = (a+b-k+1) \to c$ and $a \to (b \to c) = a \to (k-1-b+c)$. Hence, if $a+b-c \leq k-1$, then $(a \odot b) \to c = a \to (b \to c) = k-1$ and if $a+b-c > k-1$, then $(a \odot b) \to c = a \to (b \to c) = 2k-2-a-b+c$.*
***Case 12**: If $a+b > k-1$ and $c \leq a < b$, then $(a \odot b) \to c = (a+b-k+1) \to c$ and $a \to (b \to c) = a \to (k-1-b+c)$. Hence, if $a+b-c \leq k-1$, then $(a \odot b) \to c = a \to (b \to c) = k-1$ and if $a+b-c > k-1$, then $(a \odot b) \to c = a \to (b \to c) = 2k-2-a-b+c$*
***(HP4)**: Now, we show that $(a \to b) \odot a = (b \to a) \odot b$, for all $a, b \in \mathbb{W}_k$.*
***Case 1**: If $a \leq b$, then $(a \to b) \odot a = (k-1) \odot a = a$ and $(b \to a) \odot b = (k-1-b+a) \odot b = k-1-b+a+b-k+1 = a$. Hence, $(a \to b) \odot a = (b \to a) \odot b$.*
***Case 2**: If $a > b$, then $(a \to b) \odot a = (k-1-a+b) \odot a = k-1-a+b+a-k+1 = b$ and $(b \to a) \odot b = (k-1) \odot b = b$. Hence, $(a \to b) \odot a = (b \to a) \odot b$.*
Therefore, $(\mathbb{W}_k, \odot, \to, k-1)$ is a hoop.□

Theorem 3.3. *Let A be a finite set. Then there exist binary operations $\odot$ and $\to$ and constant 1 on A, such that $(A, \odot, \to, 1)$, is a hoop.*

Proof. *Let A be a finite set. Then, there exists $k \in \mathbb{N}$ such that $|A| = |\mathbb{W}_k|$. Now, by Lemma 3.2, $(\mathbb{W}_k, \odot, \to, 1)$ is a hoop and so by Lemma 3.1, there exist binary operations $\odot$ and $\to$, and constant 1 on A , such that $(A, \odot, \to, 1)$ is a hoop.*□

Lemma 3.4. *Let $1 < n \in \mathbb{Q}$. Then there exist binary operations $\odot$ and $\to$ on $E = \mathbb{Q} \cap [1, n]$, such that $(E, \odot, \to, n)$ is a hoop.*

Proof. *For any $1 < n \in E$, we define the binary operations $\odot$ and $\to$ on E as follows, for all $a, b \in E$,*

$$a \odot b = \begin{cases} 1 & \text{if } ab \leq n, \\ \frac{ab}{n} & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} n & \text{if } a \leq b, \\ \frac{nb}{a} & \text{otherwise} \end{cases}$$

Clearly, $\odot$ and $\to$ are well-defined on E. Now, we show that $(E, \odot, \to, n)$ is a hoop.

(HP1): *For all $a \in E$, if $a \neq 1$, since $an > n$ we have $a \odot n = n \odot a = \frac{an}{n} = a$ and if $a = 1$, we have $a \odot n = 1 \odot n = 1 = a$. Then n is the identity element of $(E, \odot)$. Now, we show that $\odot$ is associative on E. Let $a, b, c \in E$,*
Case 1: *If $ab \leq n$ and $bc \leq n$, then $(a \odot b) \odot c = 1 \odot c = 1$. On the other hand $a \odot (b \odot c) = a \odot (1) = 1$. Then $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case 2: *If $ab \leq n$ and $bc > n$, then $(a \odot b) \odot c = 1 \odot c = 1$. On the other hand $b \odot c = \frac{bc}{n}$ and then $a \odot (b \odot c) = a \odot (\frac{bc}{n})$. Since $\frac{abc}{n} = \frac{ab}{n} c \leq c \leq n$, we get $a \odot (b \odot c) = 1$ and so $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case3: *If $ab > n$ and $bc > n$, then $(a \odot b) \odot c = (\frac{ab}{n}) \odot c$. On the other hand $a \odot (b \odot c) = a \odot (\frac{bc}{n})$. If $\frac{abc}{n} \leq n$, then $(a \odot b) \odot c = a \odot (b \odot c) = 1$ and if $\frac{abc}{n} > n$, then $(a \odot b) \odot c = a \odot (b \odot c) = \frac{abc}{n^2}$. Hence, $(a \odot b) \odot c = a \odot (b \odot c)$.*
Case 4: *Let $ab > n$ and $bc \leq n$. This case is similar to the Case 2.*
It is clear that, for all $a, b \in E$, $a \odot b = b \odot a$. Hence, $(E, \odot, n)$ is a commutative monoid.
(HP2): *It is clear that, for all $a \in E$, we have $a \to a = n$.*
(HP3): *For all $a, b, c \in E$, we have the following cases,*
Case 1: *If $b \leq c$ and $ab \leq n$, then $a \to (b \to c) = a \to n = n$ and $(a \odot b) \to c = 1 \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 2: *If $b \leq c$ and $ab > n$, then $a \to (b \to c) = a \to n = n$ and since $\frac{a}{n} < 1$, we get $\frac{ab}{n} < b \leq c$ and so $(a \odot b) \to c = \frac{ab}{n} \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 3: *If $b > c$ and $ab \leq n$, since $ab \leq n \leq nc$ and so $a \leq \frac{nc}{b}$, then $a \to (b \to c) = a \to \frac{nc}{b} = n$. On the other hand, $(a \odot b) \to c = 1 \to c = n$. Then $a \to (b \to c) = (a \odot b) \to c$.*
Case 4: *If $b > c$ and $ab > n$, then $a \to (b \to c) = a \to \frac{nc}{b}$ and $(a \odot b) \to c = \frac{ab}{n} \to c$. We have, $a \leq \frac{nc}{b}$ if and only if $\frac{ab}{n} \leq c$, and so $a \to (b \to c) = (a \odot b) \to c$.*
HP4: *For all $a, b \in E$, we have the following cases,*
Case 1: *If $a \leq b$, then $a \odot (a \to b) = a \odot n = \frac{an}{n} = a$ and $b \odot (b \to a) = b \odot \frac{na}{b} = \frac{bna}{bn} = a$ and so $a \odot (a \to b) = b \odot (b \to a)$.*
Case 2: *If $a > b$, then $a \odot (a \to b) = a \odot \frac{nb}{a} = \frac{anb}{an} = b$ and $b \odot (b \to a) = b \odot n = \frac{bn}{n} = b$ and so $a \odot (a \to b) = b \odot (b \to a)$.*
Therefore, $(E, \odot, \to, n)$ is a hoop.□

Theorem 3.5. *Let A be an infinite countable set. Then there exist binary operations $\odot$ and $\to$ and constant 1 on A, such that $(A, \odot, \to, 1)$ is a hoop.*

Proof. *Let A be an infinite countable set and $E = Q \cap [1, n]$. Then by Lemma 3.4, $(E, \odot, \to, 1)$ is an infinite countable hoop and $|A| = |E|$. Hence, by Lemma 3.1, there exist binary operations $\odot$ and $\to$ and constant 1, such that $(A, \odot, \to, 1)$ is a hoop.*□

Corollary 3.6. For any non-empty countable set A, we can construct a hoop on A.

Proof. Let A be a non-empty countable set. Then, A is a finite set, or an infinite countable set . Then by the Theorems 3.3 and 3.5, the proof is clear.□

4 Constructing of some hyper hoops

In this section first we show that the Cartesian product of hoops is a hyper hoop and then we construct a hyper hoop by any non-empty countable set.

Theorem 4.1. *Let $(A, \odot_A, \rightarrow_A, 1_A)$ and $(B, \odot_B, \rightarrow_B, 1_B)$ be two hoops. Then there exist hyperoperations $\odot$, $\rightarrow$ and constant 1 on $A \times B$ such that $(A \times B, \odot, \rightarrow, 1)$ is a hyper hoop.*

Proof. *For any $(a_1, b_1), (a_2, b_2) \in A \times B$, we define the binary hyperoperations $\odot$, $\rightarrow$ on $A \times B$ by,*

$$(a_1, b_1) \odot (a_2, b_2) = \{(a_1 \odot_A a_2, b_1), (a_1 \odot_A a_2, b_2)\},$$
$$(a_1, b_1) \rightarrow (a_2, b_2) = \begin{cases} \{(a_1 \rightarrow_A a_2, b_2), (a_1 \rightarrow_A a_2, 1_B)\} & \text{if } b_1 = b_2, \\ \{(a_1 \rightarrow_A a_2, b_2)\} & \text{otherwise} \end{cases}$$

and constant $1 = (1_A, 1_B)$. It is easy to show that the hyperoperations are well-defined. Now, we show that $(A \times B, \odot, \rightarrow, 1)$ is a hyper hoop.
***(HHA1)**: Since $\odot_A$, is associative and commutative, we get $\odot$ is associative and commutative. Moreover, for all $(a, b) \in A \times B$, we have $(a, b) \odot (1_A, 1_B) = \{(a \odot_A 1_A, b), (a \odot_A 1_A, 1_B)\} \ni (a, b)$. Then $(A \times B, \odot, \rightarrow, 1)$ is a commutative semihypergroup with 1 as the unit, where $1 = (1_A, 1_B)$.*
***(HHA2)**: For all $(a, b) \in A \times B$, we have*

$$(a, b) \rightarrow (a, b) = \{(a \rightarrow_A a, b), (a \rightarrow_A a, 1_B)\} =$$

$$\{(a \rightarrow_A a, b), (1_A, 1_B)\} \ni (1_A, 1_B) = 1$$

***(HHA3)**: For all $(a_1, b_1), (a_2, b_2) \in A \times B$, we have the following cases,*
***Case 1**: If $b_1 \neq b_2$, then,*

$$\begin{aligned} ((a_1, b_1) \rightarrow (a_2, b_2)) \odot (a_1, b_1) &= \{(a_1 \rightarrow a_2, b_2)\} \odot (a_1, b_1) \\ &= \{((a_1 \rightarrow a_2) \odot_A a_1, b_1), ((a_1 \rightarrow a_2) \odot_A a_1, \\ &\quad b_2)\} \\ &= \{((a_2 \rightarrow a_1) \odot_A a_2, b_1), ((a_2 \rightarrow a_1) \odot_A a_2, \\ &\quad b_2)\} \\ &= ((a_2, b_2) \rightarrow (a_1, b_1)) \odot (a_2, b_2) \end{aligned}$$

***Case 2**: If $b_1 = b_2$, then,*

$$\begin{aligned}((a_1,b_1)\to(a_2,b_2))\odot(a_1,b_1) &= \{(a_1\to a_2,b_2),(a_1\to a_2,1_B)\}\odot(a_1,b_1)\\ &= \{((a_1\to a_2)\odot_A a_1,b_1),((a_1\to a_2)\odot_A a_1,\\ &\quad b_2),((a_1\to a_2)\odot_A a_1,1_B)\}\\ &= \{((a_2\to a_1)\odot_A a_2,b_1),((a_2\to a_1)\odot_A a_2,\\ &\quad b_2),((a_2\to a_1)\odot_A a_2,1_B)\}\\ &= ((a_2,b_2)\to(a_1,b_1))\odot(a_2,b_2)\end{aligned}$$

***(<u>HHA4</u>)**: For all $(a_1,b_1),(a_2,b_2),(a_3,b_3)\in A\times B$, we have the following cases,*
***Case 1**: If $b_1 = b_2 = b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3),((a_2\to_A a_3),\\ &\quad 1_B)\}\\ &= \{(a_1\to_A(a_2\to_A a_3),1_B),(a_1\to_A(a_2\to_A\\ &\quad a_3),b_3)\}\\ &= \{((a_1\odot_A a_2)\to_A a_3,1_B),((a_1\odot_A a_2)\to_A\\ &\quad a_3),b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

***Case 2**: If $b_1 \neq b_2 = b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3),((a_2\to_A a_3),\\ &\quad 1_B)\}\\ &= \{(a_1\to_A(a_2\to_A a_3),1_B),(a_1\to_A(a_2\to_A\\ &\quad a_3),b_3)\}\\ &= \{(a_1\odot_A a_2)\to_A(a_3,1_B),((a_1\odot_A a_2)\to_A\\ &\quad a_3),b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

***Case 3**: If $b_1 = b_2 \neq b_3$,*

$$\begin{aligned}(a_1,b_1)\to((a_2,b_2)\to(a_3,b_3)) &= (a_1,b_1)\to\{((a_2\to_A a_3),b_3)\}\\ &= \{a_1\to_A(a_2\to_A a_3),b_3)\}\\ &= \{((a_1\odot_A a_2)\to_A a_3,b_3)\}\\ &= ((a_1,b_1)\odot(a_2,b_2))\to(a_3,b_3)\end{aligned}$$

Case 4: *If* $b_1 \neq b_2 \neq b_3$,

$$\begin{aligned}(a_1,b_1) \to ((a_2,b_2) \to (a_3,b_3)) &= (a_1,b_1) \to \{((a_2 \to_A a_3), b_3)\} \\ &= \{(a_1 \to_A (a_2 \to_A a_3), b_3)\} \\ &= \{((a_1 \odot_A a_2) \to_A a_3, b_3)\} \\ &= ((a_1,b_1) \odot (a_2,b_2)) \to (a_3,b_3)\end{aligned}$$

(HHA5): *For all* $(a,b) \in A \times B$, *we have the following cases,*
Case 1: *If* $b = 1_B$, *then* $(a,b) \to (1_A,1_B) = \{(a \to 1_A, 1_B), (a \to 1_A, b \to 1_B)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
Case 2: *If* $b \neq 1_B$, *then* $(a,b) \to (1_A,1_B) = \{(a \to 1_A, 1_B)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
(HHA6): *For all* $(a_1,b_1),(a_2,b_2) \in A \times B$, *if* $(1_A,1_B) \in (a_1,b_1) \to (a_2,b_2)$ *and* $(1_A,1_B) \in (a_2,b_2) \to (a_1,b_1)$, *then we have the following cases,*
Case 1: *If* $b_1 \neq b_2$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_1, b_1)\}$. *Hence,* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_1$ *and* $1_B = b_1 = b_2$. *Since* A *is a hoop, we get* $a_1 = a_2$ *and so* $(a_1,b_1) = (a_2,b_2)$
Case 2: *If* $b_1 = b_2$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2), (a_1 \to_A a_2, 1_B)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_1, b_1), (a_2 \to_A a_1, 1_B)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_1$. *Since* A *is a hoop, we get* $a_1 = a_2$ *and by assumption, we have* $b_1 = b_2$. *So* $(a_1,b_1) = (a_2,b_2)$.
(HHA7): *For all* $(a_1,b1),(a_2,b_2),(a_3,b_3) \in A \times B$, *let* $(1_A,1_B) \in (a_1,b_1) \to (a_2,b_2)$ *and* $(1_A,1_B) \in (a_2,b_2) \to (a_3,b_3)$. *Then we consider the following cases:*
Case 1: *If* $b_1 = b_2 = b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, 1_B), (a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, 1_B), (a_2 \to_A a_3, b_3)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3), (a_1 \to_A a_3, 1_B)\} = \{(1_A,b_3),(1_A,1_B)\} \ni (1_A,1_B)$.
Case 2: *If* $b_1 \neq b_2 = b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, 1_B), (a_2 \to_A a_3, b_3)\}$. *Hence* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$ *and* $b_2 = b_3 = 1_B$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.
Case 3: *Let* $b_1 = b_2 \neq b_3$. *Then proof is similar to the Case 2.*
Case 4: *If* $b_1 \neq b_2 \neq b_3$, *then* $(1_A,1_B) \in \{(a_1 \to_A a_2, b_2)\}$ *and* $(1_A,1_B) \in \{(a_2 \to_A a_3, b_3)\}$. *Hence,* $1_A = a_1 \to_A a_2$ *and* $1_A = a_2 \to a_3$ *and* $b_2 = b_3 = 1_B$. *Since* A *is a hoop, we get* $1_A = a_1 \to_A a_3$. *Hence,* $(a_1,b_1) \to (a_3,b_3) = \{(a_1 \to_A a_3, b_3)\} = \{(1_A,1_B)\} \ni (1_A,1_B)$.

Therefore, $(A \times B, \odot, \to, 1)$ *is a hyper hoop, where* $1 = (1_A,1_B)$.□

Lemma 4.2. *Let* A *and* B *be two sets such that* $|A| = |B|$. *If* $(A, \odot_A, \to_A, 1_A)$ *is a hyper hoop, then there exist hyperoperations* $\odot_B$, $\to_B$ *and constant* 1_B *on* B, *such that* $(B, \odot_B, \to_B, 1_B)$ *is a hyper hoop and* $(A, \odot_A, \to_A, 1_A) \cong (B, \odot_B, \to_B, 1_B)$.

Proof. *Since $|A| = |B|$, then there exists a bijection $\varphi : A \to B$. For any $b_1, b_2 \in B$, there exist $a_1, a_2 \in A$ such that $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$. Then we define the hyperoperations $\odot_B, \to_B$ on B by, $b_1 \odot_B b_2 = \{\varphi(a)|a \in a_1 \odot a_2\}$, and $b_1 \to_B b_2 = \{\varphi(a)|a \in a_1 \to a_2\}$. It is easy to show that $\odot_B, \to_B$ are well-defined and $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop, where $1_B = \varphi(1_A)$. Now, we define the map $\theta : (A, \odot_A, \to_A, 1_A) \to (B, \odot_B, \to_B, 1_B)$ by $\theta(x) = \varphi(x)$. Since φ is a bijection then θ is a bijection and it is easy to see that θ is a homomorphism and so it is an isomorphism.*□

Corollary 4.3. For any non-empty countable set A and any hoop B, we can construct a hyper hoop on $A \times B$.

Proof. By Corollary 3.6, we can construct a hoop on A and by Theorem 4.1, we can construct a hyper hoop on $A \times B$.□

Corollary 4.4. Let A be an infinite countable set. We can construct a hyper hoop on A.

Proof. Let A be an infinite countable set. Then by Corollary 3.6, we can construct a hoop on A. Now, By Theorem 3.3, for arbitrary elements x, y not belonging to A, we can define operations $\odot$ and $\to$ on the set $\{x, y\}$, such that $(\{x, y\}, \odot, \to)$ is a hoop. Then by Theorem 4.1, we can construct a hyper hoop on $A \times \{x, y\}$. Then by Lemma 2.11 and 4.2, there exists a hyper hoop on A.□

5 Fundametal hoops

In this section we apply the β^* relation to the hyper hoops and obtain some results. Then we show that any hoop is a fundamental hoop.

Let $(A, \odot, \to, 1)$ be a hyper hoop and $U(A)$ denote the set of all finite combinations of elements of A with respect to $\odot$ and $\to$. Then, for all $a, b \in A$, we define $a\beta b$ if and only if $\{a, b\} \subseteq u$, where $u \in U(A)$, and $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ with $z_1 = a, z_{m+1} = b$ such that $\{z_i, z_{i+1}\} \subseteq u_i \subseteq U(A)$, for $i = 1, ..., m$ (In fact β^* is the transitive closure of the relation β).

Theorem 5.1. *Let A be a hyper hoop. Then β^* is a strong regular relation on A.*

Proof. *Let $a\beta^* b$, for $a, b \in A$. Then there exist $x_1, ..., x_{n+1} \in A$ with $x_1 = a, x_{n+1} = b$ and $u_i \in U(A)$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, for $1 \leq i \leq n$. Let $z_i \in x_i \to c$, for all $1 \leq i \leq n+1, c \in A$. Then we have,*

$$\{z_i, z_{i+1}\} \subseteq (x_i \to c) \cup (x_{i+1} \to c) \subseteq u_i \to c \subseteq U(A), \textit{for all } 1 \leq i \leq n.$$

Hence, $z_1 \beta^* z_{n+1}$, where $z_1 \in a \to c$ and $z_{n+1} \in b \to c$ and so $a \to c\overline{\overline{\beta^*}}b \to c$. Similarly, we can show that $c \to a\overline{\overline{\beta^*}}c \to b$. Now, by the same way we can prove *that $a\beta^*b$ implies $a\odot c\overline{\overline{\beta^*}}b\odot c$, for all $c\in A$. Hence, β^* is a strong regular relation on A.*$\square$

Corollary 5.2. Let A be a hyper hoop. Then $(\frac{A}{\beta^*},\otimes,\hookrightarrow)$ is a hoop, where $\otimes$ and $\hookrightarrow$ are defined by Theorem 2.8.

Proof. By Theorem 2.9 the proof is clear.$\square$

Theorem 5.3. *Let A be a hyper hoop. Then the relation β^* is the smallest equivalence relations γ defined on A such that the quotient $\frac{A}{\gamma}$ is a hoop with operations*

$$\gamma(x)\otimes\gamma(y)=\gamma(t):t\in x\odot y \quad \text{and} \quad \gamma(x)\hookrightarrow\gamma(y)=\gamma(z):z\in x\to y$$

where $\gamma(x)$ is equivalence class of x with respect to the relation γ.

Proof. *By Corollary 5.2, $\frac{A}{\beta^*}$ is a hoop. Now, let γ be an equivalence relation on A such that $\frac{A}{\gamma}$ is a hoop. Let $x\beta y$, for $x,y\in A$ and $\pi:A\to\frac{A}{\gamma}$ be the natural projection such that $\pi(x)=\gamma(x)$. It is clear that π is a homomorphism of hyper hoops. Then there exists $u\in U(A)$ such that $\{x,y\}\subseteq u$. Since π is a homomorphism of hyper hoops, we get $|\pi(u)|=|\gamma(u)|=1$. Since $\{\pi(x),\pi(y)\}\subseteq\pi(u)$ and $|\pi(u)|=1$, we get $\pi(x)=\pi(y)$ and so $\gamma(x)=\gamma(y)$ i.e. $x\gamma y$. Hence, $\beta\subseteq\gamma$. Now, let $a\beta^*b$, for $a,b\in A$. Then there exist $x_1,...,x_{n+1}\in A$, such that $a=x_1\beta x_2,...,\beta x_{n+1}=b$. Since $\beta\subseteq\gamma$, we get $a=x_1\gamma x_2,...,\gamma x_{n+1}=b$. Then since γ is a transitive relation on A, we get $a\gamma b$ and so $\beta^*\subseteq\gamma$.*$\square$

Corollary 5.4. The relation β^* is the smallest strong regular relation on hyper hoop A.

Proof. The proof is straightforward.$\square$

Lemma 5.5. *If A_1 and A_2 are two hyper hoops, then the Cartesian product $A_1\times A_2$ is a hyper hoop with the unit $(1_{A_1},1_{A_2})$ by the following hyperoperations, for $(x_1,y_1),(x_2,y_2)\in A_1\times A_2$,*

$$(x_1,y_1)\odot(x_2,y_2)=\{(a,b)|a\in x_1\odot x_2,b\in y_1\odot y_2\},$$
$$(x_1,y_1)\to(x_2,y_2)=\{(a',b')|a'\in x_1\to x_2,b'\in y_1\to y_2\}$$

Proof. *The proof is straightforward.*$\square$

Lemma 5.6. *Let A_1 and A_2 be two hyper hoops. Then, for $a,c\in A_1$ and $b,d\in A_2$, we have $(a,b)\beta^*_{A_1\times A_2}(c,d)$ if and only if $a\beta^*_{A_1}c$ and $b\beta^*_{A_2}d$.*

Proof. *We know that $u\in U(A_1\times A_2)$ if and only if there exist $u_1\in U(A_1)$ and $u_2\in U(A_2)$ such that $u=u_1\times u_2$. Then $(a,b)\beta^*_{A_1\times A_2}(c,d)$ if and only if there exist $u_1\in U(A_1)$ and $u_2\in U(A_2)$ such that $\{(a,b),(c,d)\}\subseteq u_1\times u_2$ if and only if $\{a,c\}\subseteq u_1$ and $\{b,d\}\subseteq u_2$ if and only if $a\beta^*_{A_1}c$ and $b\beta^*_{A_2}d$.*$\square$

Theorem 5.7. *Let A_1 and A_2 be two hyper hoops. Then $\frac{A_1\times A_2}{\beta^*_{A_1\times A_2}} \cong \frac{A_1}{\beta^*_{A_1}} \times \frac{A_2}{\beta^*_{A_2}}$.*

Proof. *Let $\varphi : \frac{A_1\times A_2}{\beta^*} \to \frac{A_1}{\beta^*_{A_1}} \times \frac{A_2}{\beta^*_{A_2}}$ be defined by $\varphi(\beta^*(x,y)) = (\beta^*_{A_1}(x), \beta^*_{A_2}(y))$, where $\beta^* = \beta^*_{A_1\times A_2}$ By Lemma 5.5, $\frac{A_1\times A_2}{\beta^*}$ is well-define. It is clear that φ is onto. By Lemma 5.6, we have $\beta^*(x_1,y_1) = \beta^*(x_2,y_2)$ if and only if $\beta^*_{A_1}(x_1) = \beta^*_{A_2}(x_2)$ and $\beta^*_{A_2}(y_1) = \beta^*_{A_2}(y_2)$, for any $(x_1,y_1),(x_2,y_2) \in A_1 \times A_2$. So, φ is well defined and one to one. Also, by considering the hyperoperations $\otimes$ and $\hookrightarrow$ defined in Theorem 2.8, we have,*

$$
\begin{aligned}
\varphi(\beta^*(x_1,y_1) \hookrightarrow \beta^*(x_2,y_2)) &= \varphi(\{\beta^*(a,b)|a \in x_1 \to x_2, b \in y_1 \to y_2\}) \\
&= \{\varphi(\beta^*(a,b))|a \in x_1 \to x_2, b \in y_1 \to y_2\} \\
&= \{(\beta^*_{A_1}(a), \beta^*_{A_2}(b))|a \in x_1 \to x_2, b \in y_1 \to y_2\} \\
&= (\beta^*_{A_1}(x_1) \hookrightarrow \beta^*_{A_1}(x_2), \beta^*_{A_2}(y_1) \hookrightarrow \beta^*_{A_2}(y_2)) \\
&= (\beta^*_{A_1}(x_1), \beta^*_{A_2}(y_1)) \hookrightarrow (\beta^*_{A_1}(x_2), \beta^*_{A_2}(y_2)) \\
&= \varphi(\beta^*(x_1,y_1)) \hookrightarrow \varphi(\beta^*(x_2,y_2))
\end{aligned}
$$

Similarly, we can show that $\varphi(\beta^*(x_1,y_1)\otimes\beta^*(x_2,y_2)) = \varphi(\beta^*(x_1,y_1))\otimes\varphi(\beta^*(x_2, y_2))$. Moreover, it is clear that $\varphi(\beta^*(1_{A_1}, 1_{A_2})) = (\beta^*(1_{A_1}), \beta^*(1_{A_2}))$. Hence, φ is an isomorphism.□

Corollary 5.8. Let $A_1, A_2, ..., A_n$ be hyper hoops. Then,

$$\frac{A_1\times A_2\times....\times A_n}{\beta^*_{A_1\times A_2\times....\times A_n}} \cong \frac{A_1}{\beta^*_1} \times \frac{A_2}{\beta^*_2} \times .... \times \frac{A_n}{\beta^*_n}$$

Proof. The proof is straightforward.□

Theorem 5.9. *Let A and B be two sets such that $|A| = |B|$. If $(A, \odot_A, \to_A, 1_A)$ is a hyper hoop, then there exist hyperoperations $\odot_B$ and $\to_B$ and constant 1_B on B such that $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop and $\frac{(A,\odot_A,\to_A,1_A)}{\beta^*_A} \cong \frac{(B,\odot_B,\to_B,1_b)}{\beta^*_B}$.*

Proof. *Since $|A| = |B|$, then by Lemma 4.2, there exist binary hyperoperations $\odot_B$ and $\to_B$, such that $(B, \odot_B, \to_B, 1_B)$ is a hyper hoop. Moreover, there exists an isomorphism $f : (A, \odot_A, \to_A, 1_A) \to (B, \odot_B, \to_B, 1_B)$, such that $f(1_A) = 1_B$. Now, we define $\varphi : \frac{(A,\odot_A,\to_A,1_A)}{\beta^*_A} \to \frac{(B,\odot_B,\to_B,1_B)}{\beta^*_B}$ by $\varphi(\beta^*_A(x)) = \beta^*_B(f(x))$. Since f is an isomorphism, φ is onto. Let y_1 , $y_2 \in B$. Then there exist $a_1, a_2 \in A$ such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then $\beta^*_A(a_1) = \beta^*_A(a_2)$ iff $a_1\beta^*_A a_2$ iff there exists $u \in U(A)$ such that $\{a_1, a_2\} \subseteq u$ iff there existes $f(u) \in U(B) : \{f(a_1), f(a_2)\} \subseteq f(u)$ iff $\beta^*_B(b_1) = \beta^*_B(b_2)$ iff $\beta^*_B(f(a_1)) = \beta^*_B(f(a_2))$. Then φ is well-defined and one to one. Also, by consid-*

ering the hyperoperations $\otimes$ and $\hookrightarrow$ defined in Theorem 2.8, we have,

$$\begin{array}{rcl}\varphi(\beta^*_A(a_1)\otimes\beta^*_A(a_2)) & = & \varphi_{t\in a_1\odot a_2}(\beta^*_A(t))=\beta^*_{t\in a_1\odot a_2}(f(t))\\ & = & \beta^*_{t'\in f(a_1\odot a_2)}(t')=\beta^*_{t'\in f(a_1)\odot f(a_2)}(t')=\beta^*_B(f(a_1))\otimes\beta^*_B\\ & & (f(a_2))\\ & = & \varphi(\beta^*_A(a_1))\otimes\varphi(\beta^*_A(a_2))\end{array}$$

By the same way, we can show that

$$\varphi(\beta^*_A(a_1)\hookrightarrow\beta^*_A(a_2))=\varphi(\beta^*_A(a_1))\hookrightarrow\varphi(\beta^*_A(a_2))$$

Since f is an isomorphism, we get $\varphi(\beta^*_A(1_A))=\beta^*_B(f(1_A))=\beta^*_B(1_B)$. Hence, φ is an isomorphism.$\square$

Definition 5.10. Let A be a hoop algebra. Then A is called a *fundamental hoop*, if there exists a nontrivial hyper hoop B, such that $\frac{B}{\beta^*_B}\cong A$

Theorem 5.11. *Every hoop is a fundamental hoop.*

Proof. *Let A be a hoop. Then by Theorem 4.1, for any hoop B, $A\times B$ is a hyper hoop. By considering the hyperoperations $\odot$ and $\rightarrow$ defined in Theorem 4.1, we get that any finite combination $u\in U(A\times B)$ is the form of $u=\{(a,x_i)|a\in A, x_i\in B\}$. Hence, for any $(a_1,b_1),(a_2,b_2)\in A\times B$,*

$$\begin{array}{c}(a_1,b_1)\beta^*(a_2,b_2)\Leftrightarrow\exists u\in U(A\times B)\text{ such that}\\ \{(a_1,b_1),(a_2,b_2)\}\subseteq u\Leftrightarrow a_1=a_2\end{array}$$

Hence, for any $(a,b)\in A\times B$, $\beta^*(a,b)=\{(a,x)|x\in B\}$.
Now, we define the map $\psi:\frac{A\times B}{\beta^*}\rightarrow A$ by, $\psi(\beta^*(a,b))=a$. It is clear that,

$$\beta^*(a_1,b_1)=\beta^*(a_2,b_2)\Leftrightarrow a_1=a_2\Leftrightarrow\psi(\beta^*(a_1,b_1))=\psi(\beta^*(a_2,b_2)).$$

Then, ψ is well defined and one to one. In the following, we show that ψ is a homomorphism. For this we have,

$$\begin{array}{rcl}\psi(\beta^*(a_1,b_1)\otimes\beta^*(a_2,b_2)) & = & \psi(\beta^*(u,v)):(u,v)\in(a_1,b_1)\odot(a_2,b_2)\\ & = & \psi(\beta^*(u,v)):(u,v)\in\{((a_1\odot a_2),b_1),((a_1\odot\\ & & a_2),b_2)\}\\ & = & \{u|u\in a_1\odot a_2\}=a_1\odot a_2\\ & = & \psi(\beta^*(a_1,b_1))\odot\psi(\beta^*(a_2,b_2))\end{array}$$

and similarly, for the operation $\hookrightarrow$, we have the following cases,
***Case 1**: If $b_1 \neq b_2$, then,*

$$\begin{aligned}
\psi(\beta^*(a_1,b_1) \hookrightarrow \beta^*(a_2,b_2)) &= \psi(\beta^*(u,v)) : (u,v) \in (a_1,b_1) \to (a_2,b_2) \\
&= \psi(\beta^*(u,v)) : (u,v) \in \{((a_1 \to a_2), b_2)\} \\
&= \{u | u \in a_1 \to a_2\} = a_1 \to a_2 \\
&= \psi(\beta^*(a_1,b_1)) \to \psi(\beta^*(a_2,b_2))
\end{aligned}$$

***Case 2**:If $b_1 = b_2$, then,*

$$\begin{aligned}
\psi(\beta^*(a_1,b_1) \hookrightarrow \beta^*(a_2,b_2)) &= \psi(\beta^*(u,v)) : (u,v) \in (a_1,b_1) \to (a_2,b_2) \\
&= \psi(\beta^*(u,v)) : (u,v) \in \{((a_1 \to a_2), b_2), ((a_1 \to \\
&\quad a_2), 1_B)\} \\
&= \{u | u \in a_1 \to a_2\} = a_1 \to a_2 \\
&= \psi(\beta^*(a_1,b_1)) \to \psi(\beta^*(a_2,b_2))
\end{aligned}$$

Clearly, $\psi(\beta^*(1_A, 1_B) = 1_A$ and ψ is onto. Therefore, ψ is an isomorphism i.e. $\frac{A \times B}{\beta^*} \cong A$ and so A is fundamental.□

Corollary 5.12. For any non-empty countable set A, we can construct a fundamental hoop on A.

Proof. By Corollary 3.6 and Theorem 5.11 the proof is clear.□

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Application of mathematical software in solving the problems of electricity

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Abstract

At the present time great emphasis is put on making accessible new knowledge to students through information and communication technologies in effort to facilitate and introduce objects, phenomena and reality. Information and communication technologies complement and develop traditional methods such as direct observation, manipulation with objects, experiment. It is justified mainly at teaching natural sciences. The possibilities of solving physical problem with the use of software tools are presented in the paper.

Keywords: information and communication technologies, electrical circuit, Kirchhoffov´s lows, MS Excel, Matlab.

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1 Introduction

Information and communication technologies currently present a set of modern means that are used for preparation, processing and distribution of data and information, but also process control with the aim of achieving more effective results and searching for optimal problem solutions at various fields and areas of human activities [1], [2]. Information and communication technologies significantly influence even university education. Information and communication technologies provide incomparably bigger information basis as it was several years ago. This gradually changes the style of teaching and makes teachers implement new technologies not only in direct pedagogical activity, but also at its preparation. Implementation of information and communication technologies into education enables new forms of university studies. We can stimulate the interest of students in studies of natural science subjects as mathematics, physics, chemistry, create conditions for educational individualization and improve conditions to raise the quality of education by a suitable combination of traditional and modern teaching methods [3].

In teaching physics there exist possibilities for effective and suitable integration of information and communication technologies into schooling system. One of them is physical problem solution with the support of computer. This paper concretely presents the solution of physical problem from the part Physics – Power and Magnetism by the use of mathematical software MS Excel a Matlab.

2 Physical analysis of the problem

Problem: Figure out the currents in individual circuit branches in Fig. 1, if source voltage and resistance are: U_{01}=10 V, U_{02}=20 V, U_{03}=15 V, U_{04}=10 V, R_1=10 Ω, R_2=15 Ω, R_3=30 Ω, R_4=20 Ω, R_5=10 Ω, R_6=15 Ω, R_7=10 Ω.

Solution: Kirchhoff´s rules are used to figure out the currents in the circuit [4]. The first of Kirchhoff´s rules describes the law of electric charge preservation: *The sum of all the currents flowing into the junction point must equal the sum of all the currents leaving the point,* i.e. $\sum_{k=1}^{n} I_k = 0$. The second of Kirchhoff´s rules forms the law of electric energy preservation for electric circuits: *Algebraic sum of electromotive voltages in any closed part of the electrical network is equal to the sum of ohmic voltages at individual branches of this closed part,* i.e. $\sum_{i=1}^{m} U_{ei} = \sum_{k=1}^{n} R_k I_k$.

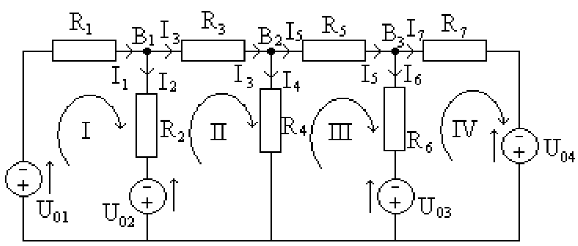

Fig. 1. Circuit

Based on the first and the second of Kirchhoff´s rules (Fig. 1) for the currents and electromotive voltages, it is valid

$$\begin{aligned}
B_1 &: \quad I_1 - I_2 - I_3 = 0 \\
B_2 &: \quad I_3 - I_4 - I_5 = 0 \\
B_3 &: \quad I_5 - I_6 - I_7 = 0 \\
I &: \quad R_1 I_1 + R_2 I_2 = U_{01} - U_{02} \\
II &: \quad -R_2 I_2 + R_3 I_3 + R_4 I_4 = U_{02} \\
III &: \quad -R_4 I_4 + R_5 I_5 + R_6 I_6 = -U_{03} \\
IV &: \quad -R_6 I_6 + R_7 I_7 = U_{03} - U_{04}
\end{aligned}$$

The set of 7 equations on 7 unknown quantities I_1, I_2, ..., I_7 was obtained. Numeric values are inducted for the known quantities and we have

$$\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
I_3 - I_4 - I_5 &= 0 \\
I_5 - I_6 - I_7 &= 0 \\
10 I_1 + 15 I_2 &= -10 \\
-15 I_2 + 30 I_3 + 20 I_4 &= 20 \\
-20 I_4 + 10 I_5 + 15 I_6 &= -15 \\
-15 I_6 + 10 I_7 &= 5
\end{aligned}$$

3 Analytic solution of the problem

Based on analysis of the problem and use of electrical laws the system of 7 equations in 7 variables was obtained, where analytic solution is not simple. In general it is possible to solve the system of *n* equations in *n* variables in three ways:

1. solving the system of linear equations by means of Cramer´s Rule,
2. solving the system of linear equations by means of inversion matrix,
3. solving the system of linear equations by Gauss elimination method.

Gauss elimination method appears to be a suitable method of solving the system of *n* equations in *n* variables, if $n > 3$ [5]. By means of equivalent line adjustment the matrix of the system of equations, which is augmented by the second column (so called augmented matrix of the system) to a triangle shape, is modified. We write to such an augmented matrix an appropriate system, which is equivalent with the original system, i.e. it has the same family of solutions. Frobenius norm and its consequences can be used to solve such a modified system.

The system of heterogeneous equations can be solved only if the rank of a matrix is equal to the rank of an augmented matrix of the system.

Consequence 1: If $h(\mathrm{A}) = h(\mathrm{A}') = n$ *(n is the number of unknowns), then the system has only one solution.*

Consequence 2: If $h(\mathrm{A}) = h(\mathrm{A}') < n$ *(n is the number of unknowns), then the system has infinite number of solutions and* $n - h$ *unknowns can be arbitrarily selected.*

Consequence 3: If $h(\mathrm{A}) \neq h(\mathrm{A}')$ *, then the system has no solution.*

We get the values of unknowns by gradual substitution into previous equations.

The system of equations is written into the form of an augmented matrix and we get by means of equivalent line adjustment

$$\left(\begin{array}{ccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 10 & 15 & 0 & 0 & 0 & 0 & 0 & -10 \\ 0 & -15 & 30 & 20 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & -20 & 10 & 15 & 0 & -15 \\ 0 & 0 & 0 & 0 & 0 & -15 & 10 & 5 \end{array}\right) \approx$$

$$\approx \left(\begin{array}{ccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 20 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 10 & 15 & 0 & -15 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15 & 10 & 5 \\ 10 & 15 & 0 & 0 & 0 & 0 & 0 & -10 \end{array}\right) \begin{array}{c} \\ \\ \\ \\ \\ \\ -10R_1 \end{array} \approx \ldots$$

$$\approx \left(\begin{array}{ccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 20 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 10 & 15 & 0 & -15 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1590 & -1060 & -530 \\ 0 & 0 & 0 & 0 & 0 & -1590 & -960 & 420 \end{array}\right) \begin{array}{c} \\ \\ \\ \\ \\ \\ +R_6 \end{array} \approx$$

$$\approx \left(\begin{array}{ccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -15 & 30 & 20 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 10 & 15 & 0 & -15 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1590 & -1060 & -530 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2020 & -110 \end{array}\right)$$

The rank of a matrix is equal to the rank of an augmented matrix, i.e. $h(\mathrm{A}) = h(\mathrm{A}') = n$ (n is the number of unknowns), then the system has one solution that is determined from an appropriate system:

$$-2020I_7 = -110 \quad \Rightarrow \quad I_7 = \frac{11}{202} = 0{,}05446$$

$$1590I_6 - 1060I_7 = -530 \quad \Rightarrow \quad I_6 = \frac{-530 + 1060I_7}{1590}$$

$$I_6 = \frac{-530 + 1060\frac{11}{202}}{1590} = -\frac{30}{101} = -0{,}29703$$

$$I_5 - I_6 - I_7 = 0 \quad \Rightarrow \quad I_5 = I_6 + I_7$$

$$I_5 = -\frac{30}{101} + \frac{11}{202} = -\frac{49}{202} = -0{,}24257$$

$$-20I_4 + 10I_5 + 15I_6 = -15 \quad \Rightarrow \quad I_4 = \frac{-15 - 10I_5 - 15I_6}{-20}$$

$$I_4 = \frac{-15 - 10\left(-\frac{49}{202}\right) - 15\left(-\frac{30}{101}\right)}{-20} = \frac{41}{101} = 0{,}40594$$

$$I_3 - I_4 - I_5 = 0 \quad \Rightarrow \quad I_3 = I_4 + I_5$$

$$I_3 = \frac{41}{101} + \left(-\frac{49}{202}\right) = \frac{33}{202} = 0{,}16337$$

$$-15I_2 + 30I_3 + 20I_4 = 20 \quad \Rightarrow \quad I_2 = \frac{20 - 30I_3 - 20I_4}{-15}$$

$$I_2 = \frac{20 - 30\left(\frac{33}{202}\right) - 20\left(\frac{41}{101}\right)}{-15} = -\frac{47}{101} = -0{,}46534$$

$$I_1 - I_2 - I_3 = 0 \quad \Rightarrow \quad I_1 = I_2 + I_3$$

$$I_1 = \left(-\frac{47}{101}\right) + \frac{33}{202} = -\frac{61}{202} = -0{,}30198$$

We get the following current values in the circuit:

$$\begin{aligned} I_1 &= -0{,}30198 \text{ A} \\ I_2 &= -0{,}46534 \text{ A} \\ I_3 &= \ \ 0{,}16337 \text{ A} \\ I_4 &= \ \ 0{,}40594 \text{ A} \\ I_5 &= -0{,}24257 \text{ A} \\ I_6 &= -0{,}29703 \text{ A} \\ I_7 &= \ \ 0{,}05446 \text{ A} \end{aligned}$$

It results from the negative current values that currents in the circuit are in the opposite direction as we selected.

Analytic solution of the system of 7 equations in 7 variables by Gauss elimination method requires not only knowledge of linear algebra (matrix algebra), but also good mathematical skills and time. Numerical solution of the system of equations by means of various mathematical software tools such as MS Excel, Mathematica or MATLAB is much more easier.

4 Using software tools at the problem solution

4.1 The problem solution by means of MS Excel

In current computing technique it is possible to use standard programs for the matrix inversion up to relatively big number of equations (hundreds of variables). One of the possibilities is the solution in MS Excel [6]. We write the system of equations in matrix form

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{A}.x = \mathbf{b}$$

where **A** is the matrix of coefficients, ***x*** is the vector of unknowns and **b** is the vector of the second members. We get by multiplying $\mathbf{A}^{-1}$ from the left $\mathbf{A}^{-1} \cdot \mathbf{A} \cdot x = x = \mathbf{A}^{-1} \cdot \mathbf{b}$. If we calculate the inversion matrix, x_k unknowns can be obtained by multiplication of the matrix and vector, which is procedure that is optimized very well and is the part of standard libraries of subprograms. To calculate the inversion matrix MINVERSE functions from the offer of MS Excel More Functions is used. To calculate the roots of the system of equations ($\mathbf{A}^{-1}\mathbf{b}$) MMULT function is used. The result of the solution can be found in Fig. 2.

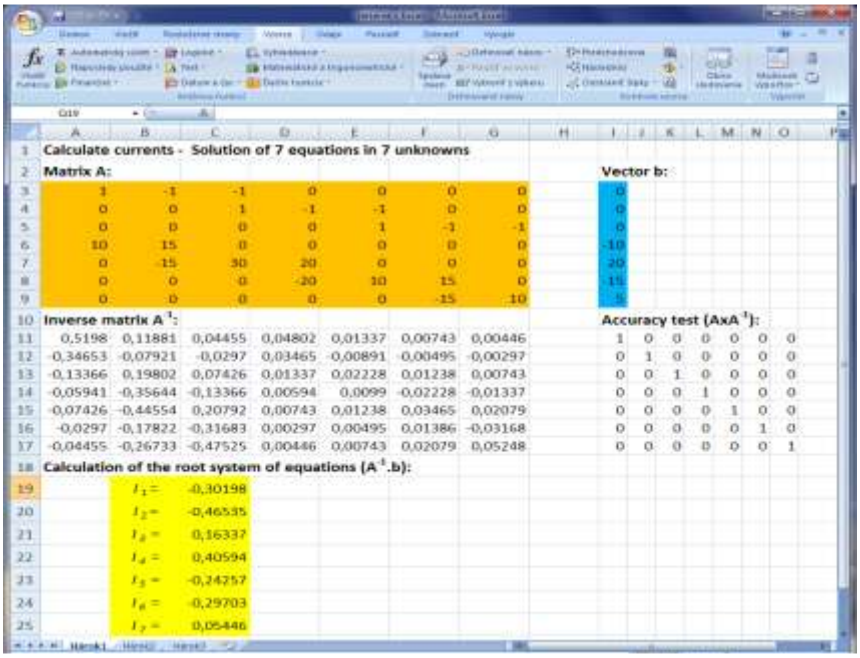

Fig. 2. Numerical solution of the set of equations in MS Excel

Another possibility to solve the system of equations is to use the MS Excel Solutionist [7], [8]. From the task and solution of the problem in the Solutionist we get (Fig. 3)

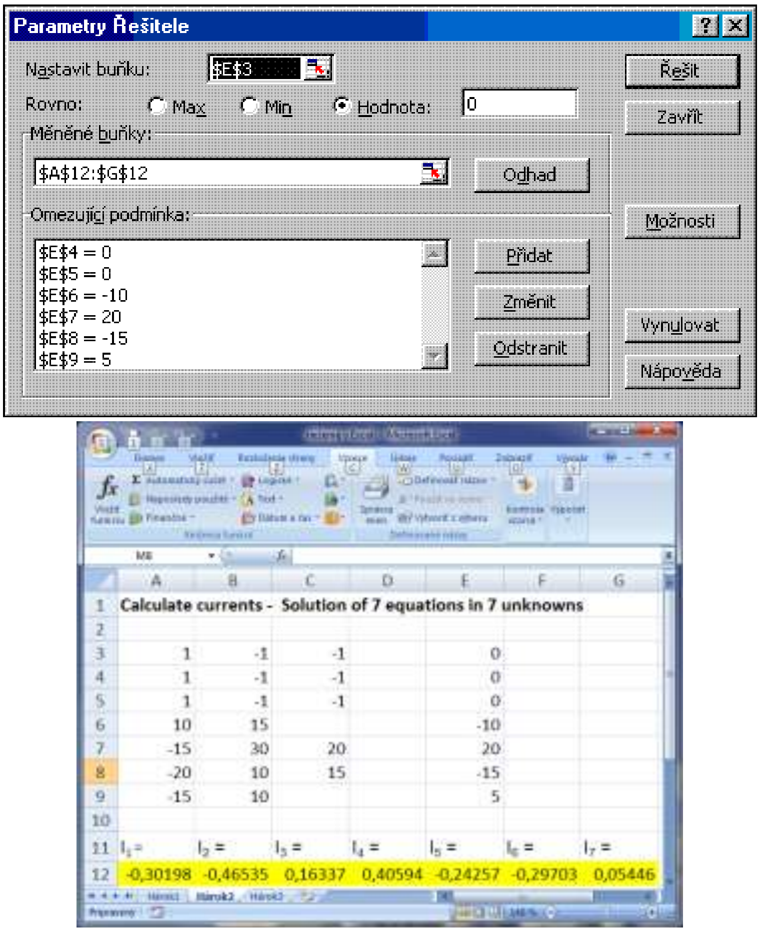

Fig. 3. Numerical solution of the system of problems in the MS Excel Solutionist

From both solutions in MS Excel we get the following current values in the circuit

$$I_1 = -0{,}30198\ \text{A},\ \ I_2 = -0{,}46535\ \text{A},\ \ I_3 = \ \ 0{,}16337\ \text{A},\ \ I_4 = \ \ 0{,}40594\ \text{A},$$
$$I_5 = -0{,}24257\ \text{A},\ \ I_6 = -0{,}29703\ \text{A},\ \ I_7 = \ \ 0{,}05446\ \text{A}$$

It results from the negative values of the current that currents have reverse directions as it was selected.

4.2 Problem solution by means of Matlab

MATLAB presents highly functional language for technical calculations. It integrates the calculations, visualization and programming into simply usable environment where the problems and solutions are expressed in natural form [9],

[10]. The field is the basic data type of this interactive system. This property together with number of built-in functions enables relatively easy solution of many technical problems, mainly those that lead to the vector or matrix formulations, in much shorter time as solution in classic program languages. To calculate the currents I_1, I_2, ..., I_7 the method of node voltage is used. This method comes from the fact that $(u-1)$ equations is written by means of Kirchhoff´s first law applied to suitably selected nodes. In these equations the equations of Kirchhoff´s second law written for appropriate loops are implicitly included. That is why voltages on tree´s branches are selected as unknowns at the method of node voltage. To determine node voltages it is necessary to solve (u-1) equations. After calculation of node voltages the currents of the circuit are determined.

We write for B_1, B_2 and B_3 nodes according to Kirchhoff´s first law

$$B_1: \qquad I_1 - I_2 - I_3 = 0$$
$$B_2: \qquad I_3 - I_4 - I_5 = 0$$
$$B_3: \qquad I_5 - I_6 - I_7 = 0$$

It is possible to express above mentioned currents by means of known node voltages with regard to the selected reference node

$$B_1: \qquad \frac{U_{01}-U_{B1}}{R_1} - \frac{U_{B1}-U_{02}}{R_2} - \frac{U_{B1}-U_{B2}}{R_3} = 0$$
$$B_2: \qquad \frac{U_{B1}-U_{B2}}{R_3} - \frac{U_{B2}}{R_4} - \frac{U_{B2}-U_{B3}}{R_5} = 0$$
$$B_3: \qquad \frac{U_{B2}-U_{B3}}{R_5} - \frac{U_{B3}-U_{03}}{R_6} - \frac{U_{B3}-U_{04}}{R_7} = 0$$

We write the equations in the matrix form

$$\begin{bmatrix} 1/R_1 + 1/R_2 + 1/R_3 & -1/R_3 & 0 \\ 1/R_3 & -(1/R_3 + 1/R_4 + 1/R_5) & 1/R_5 \\ 0 & -1/R_5 & 1/R_5 + 1/R_6 + 1/R_7 \end{bmatrix} \begin{bmatrix} U_{B1} \\ U_{B2} \\ U_{B3} \end{bmatrix} =$$
$$= \begin{bmatrix} U_{01}/R_1 + U_{02}/R_2 \\ 0 \\ U_{03}/R_6 + U_{04}/R_7 \end{bmatrix}$$

To solve such written equations the matrix solution in Matlab is used. We form the m-file *prudy.m* (Fig. 4)

```
U01=10; U02=20; U03=15; U04=10;
R1=10; R2=15; R3=30; R4=20; R5=10; R6=15; R7=10;

A=[(1/R1+1/R2+1/R3), -1/R3             , 0    ;
    1/R3           , -(1/R3+1/R4+1/R5) , 1/R5 ;
    0              ,-1/R5              , (1/R5+1/R6+1/R7)];

I=[(U01/R1+U02/R2);0;U03/R6+U04/R7];

U=A\I;

I1=(U01-U(1))/R1;
I2=(U(1)-U02)/R2;
I3=(U(1)-U(2))/R3;
I4=U(2)/R4;
I5=(U(2)-U(3))/R5;
I6=(U(3)-U03)/R6;
I7=(U(3)-U04)/R7;

fprintf('\n');
fprintf('I1 = %8.5f A\n', I1);
fprintf('I2 = %8.5f A\n', I2);
fprintf('I3 = %8.5f A\n', I3);
fprintf('I4 = %8.5f A\n', I4);
fprintf('I5 = %8.5f A\n', I5);
fprintf('I6 = %8.5f A\n', I6);
fprintf('I7 = %8.5f A\n', I7);
```

Fig. 4. M-file *prudy.m* for calculation of matrices and currents

After solving the system of equations we get the values of node voltages, which are converted to the currents in branches of the circuit. The result of solution is launching the script of *prudy.m* and print of results.

```
>> prudy
I1 = -0.30198 A
I2 = -0.46535 A
I3 =  0.16337 A
I4 =  0.40594 A
I5 = -0.24257 A
I6 = -0.29703 A
I7 =  0.05446 A
```

Another possibility of the problem solution in Matlab is use of Symbolic Math Toolbox, which provides functions for solution and graphic description of mathematical functions. Tool panel provides libraries of functions in common mathematical areas such as mathematical analysis, linear algebra, algebraic and common differential equations and so on. Symbolic Math Toolbox uses MuPAD language as a part of its calculus core. The language has a extensive set of functions, which are optimized to create and operate symbolic arithmetical expressions. To solve the system of equations *linsolve* ([*eqs*], [*vars*]) function was used, where *eqs* is a list or a set of linear equations or arithmetical expressions, *vars* is a list or a set of unknowns to solve for: typically identifiers or indexed identifiers. The solution of the system can be found in Fig. 5, where $x = I_1$, $y = I_2$, $z = I_3$, $k = I_4$, $l = I_5$, $m = I_6$, $n = I_7$:

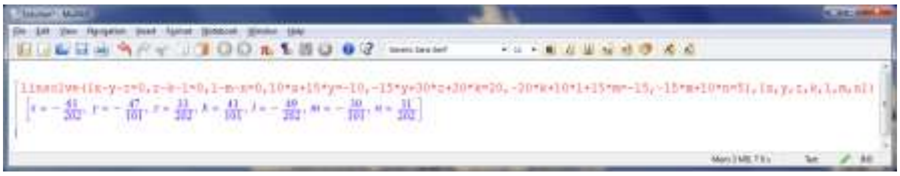

Fig. 5. Numeric solution of the system of equations in MuPad

The same values are obtained from the problem solution in Matlab as in the case of the problem solution in MS Excel.

5 Conclusions

It accrues from the solution results that solution of the system of equations of the physical problem in an analytic way as well as by using mathematical software tools leads to certain numeric values. Analytic solution of the system of *n* equations in *n* variables requires certain mathematical knowledge and skills to solve matrices. Use of modern software tools to solve the system of equations facilitates the problem solution. On the other side it requires certain computing skills. The physical problem being solved points out importance and necessity of using modern information and communication technology means and their utilization in educational process that makes “learning” for pupils and students more interesting and attractive.

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Solvable groups derived from fuzzy hypergroups

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Abstract

In this paper we introduce the smallest equivalence relation ξ^* on a finite fuzzy hypergroup S such that the quotient group S/ξ^*, the set of all equivalence classes, is a solvable group. The characterization of solvable groups via strongly regular relation is investigated and several results on the topic are presented.

Key words: Fuzzy hypergroups, strongly regular relation, solvable groups, fundamental relation.

2000 AMS subject classifications: 8A72; 20N20, 20F18, 20F19.

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1 Introduction

In mathematics, more specifically in the field of group theory, a solvable group or soluble group is a group that can be constructed from Abelian groups using extensions. Equivalently, a solvable group is a group whose derived series terminates in the trivial subgroup. All Abelian groups are trivially solvable a subnormal series being given by just the group itself and the trivial group. But non-Abelian groups may or may not be solvable. A small example of a solvable, non-nilpotent group is the symmetric group S3. In fact, as the smallest simple non-Abelian group is A5, (the alternating group of degree 5) it follows that every group with order less than 60 is solvable. The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures. This can be considered into three groups. A *first* group of papers studies *crisp* hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [3, 4] and then continued by other researchers. A *second* group of papers concerns the *fuzzy hyperalgebras*. This is a direct extension of the concept of fuzzy algebras. This was initiated by Zahedi in [12]. A *third* group was introduced by Corsini and Tofan in [5]. The basic idea in this group of papers is the following: a multioperation assigns to every pair of elements of S a non-empty subset of S, while a fuzzy multioperation assigns to every pair of elements of S a nonzero fuzzy set on S. This idea was continuated by Sen, Ameri and Chowdhury in [10] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relation α^* on fuzzy hypersemigroups is studied in [1].Also in [8], the smallest strongly regular equivalence relation γ^* on a fuzzy hypersemigroup S such that S/γ^* is a commutative semigroup is studied. In this paper, we introduce and study the fundamental relation ξ^* of a finite fuzzy hypergroup S such that S/ξ^* is a solvable group. Finally, we introduce the concept of ξ-part of a fuzzy hypergroup and we determines necessary and sufficient conditions such that the relation ξ to be transitive.

2 Preliminary

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0, 1]$. We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S, \quad (\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a, b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergropoid.*

Definition 2.1. *A fuzzy hypergropoid $(S, \circ)$ is called a fuzzy hypersemigroup if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S*

$$(a \circ \mu)(r) = \begin{cases} \bigvee\limits_{t \in S}((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

for all $r \in S$.

Definition 2.2. *Let μ, ν be two fuzzy subsets of a fuzzy hypergropoid $(S, \circ)$. Then we define $\mu \circ \nu$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.*

Definition 2.3. *A fuzzy hypersemigroup $(S, \circ)$ is called fuzzy hypergroup if $x \circ S = S \circ x = \chi_S$, for all $x \in S$, where χ_S is characteristic function of S.*

Example 2.1. *Consider a fuzzy hyperoperation $\circ$ on a non-empty set S by $a \circ b = \chi_{\{a,b\}}$, for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy hypersemigroup and fuzzy hypergroup as well.*

Theorem 2.1. *Let $(S, \circ)$ be a fuzzy hypersemigroup. Then $\chi_a \circ \chi_b = a \circ b$, for all $a, b \in S$.*

Definition 2.4. *Let ρ be an equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, we define two relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ on $F^*(S)$ as follows: for $\mu, \nu \in F^*(S)$; $\mu\overline{\rho}\nu$ if $\mu(a) > 0$ then there exists $b \in S$ such that $\nu(b) > 0$ and $a\rho b$, also if $\nu(x) > 0$ then there exists $y \in S$, such that $\mu(y) > 0$ and $x\rho y$. $\mu\overline{\overline{\rho}}\nu$ if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, $x\rho y$.*

Definition 2.5. *An equivalence relation ρ on a fuzzy hypersemigroup $(S, \circ)$ is said to be (strongly) fuzzy regular if $a\rho b, a'\rho b'$ implies $a \circ a' \; \overline{\rho} \; b \circ b' (a \circ a' \; \overline{\overline{\rho}} \; b \circ b')$.*

If ρ is a equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, then we consider the following hyperoperation on the quotient set S/ρ as follows:
for every $a\rho, b\rho \in S/\rho$

$$a\rho \oplus b\rho = \{c\rho : (a' \circ b')(c) > 0, a\rho a', b\rho b'\}$$

Theorem 2.2. *[2] Let $(S, \circ)$ be a fuzzy hypersemigroup and ρ be an equivalence relation on S. Then*
(i) the relation ρ is fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a hypersemigroup.
(ii) the relation ρ is strongly fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a semigroup.

3 New strongly regular relation ξ_n^*

Now in this paper we introduce and analyze a new strongly regular relation ξ_n^* on a fuzzy hypergroup S such that the quotient group S/ξ_n^* is solvable.

Definition 3.1. *Let (S, o) be a fuzzy hypergroup. We define*
1) $L_0(S) = S$
2) $L_{k+1}(S) = \{t \in S \mid (xy)(r) > 0, (tyx)(r) > 0, \quad in \quad which \quad x, y \in L_k(S), \quad for \quad some \quad r \in S\}$.
for all $k \geq 0$. Suppose that $n \in \mathbb{N}$ and $\xi_n = \cup_{m \geq 1} \xi_{m,n}$, where $\xi_{1,n}$ is the diagonal relation and for every integer $m > 1, \xi_{m,n}$ is the relation defined as follows:
$a\xi_{m,n}b \Longleftrightarrow \exists x_1, ..., x_m \in H(m \in \mathbb{N}), \exists \sigma \in \mathbb{S}_m : \sigma(i) = i, \quad if \quad z_i \notin L_n(H) : (x_1 o ... o x_m)(a) > 0$ *and* $(x_{\sigma_1} o ... o x_{\sigma_m})(b) > 0$.

It is clear that ξ_n is symmetric. Define for any $a \in S$, $a(a) = (\chi_a)(a) = 1$, thus ξ_n is reflexive. We take ξ_n^* to be transitive closure of ξ_n. Then it is an equivalence relation on H.

Corolary 3.1. *For every $n \in \mathbb{N}$, we have $\alpha^* \subseteq \xi_n^* \subseteq \gamma^*$.*

Theorem 3.1. *For every $n \in \mathbb{N}$, the relation ξ_n^* is a strongly regular relation.*

Proof. Suppose $n \in \mathbb{N}$. Clearly, $\xi_{m,n}$ is an equivalence relation. First we show that for each $x, y, z \in S$

$$x\xi_n y \Rightarrow xz\overline{\overline{\xi_n}}yz, \quad zx\overline{\overline{\xi_n}}zy \qquad (*).$$

If $x\xi_n y$, then there exists $m \in \mathbb{N}$ such that $x\xi_{m,n}y$, and so there exist $(z_1, \ldots, z_m) \in S^m$ and $\sigma \in S_m$ such that if $z_i \notin L_n(S)$ then $\prod_{i=1}^{m} z_i(x) > 0, \prod_{i=1}^{m} z_{\sigma(i)}(y) > 0$. Let $z \in S$, for any r, s such that $(xz)(r) > 0$ and $(yz)(s) > 0$. We have $((\prod_{i=1}^m z_i)z)(r) = \bigvee_p \{(\prod_{i=1}^{m} z_i)(p) \wedge (pz)(r)\}$. Let $p = x$, then $((\prod_{i=1}^{m} z_i)(z))r > 0, \sigma(i) = i, \quad if \quad z_i \notin L_n(S)$, $((\prod_{i=1}^{m} z_\sigma(i))(z))(s) = \bigvee_q \{(\prod_{i=1}^{m} z_\sigma(i))(q) \wedge (qz)(s)\}$. Let $q = y$, then $((\prod_{i=1}^{m} z_\sigma(i))(z))(s) > 0$, and $\sigma(i) = i, \quad if \quad z_i \notin L_n(S)$. Now suppose that $z_{m+1} = z$ and we define

$\sigma' \in S^m+1$: $\sigma'(i) = \begin{cases} \sigma(i), & \forall i \in \{1,2,\ldots,m\} \\ m+1, & i = m+1. \end{cases}$ Thus for all $r, s \in S$; $(\prod_{i=1}^{m} z_i)(r) > 0$, $(\prod_{i=1}^{m} z'_\sigma)(s) > 0$; $\sigma'(i) = i$ if $z_i \notin L_n(S)$. Therefore $xz\overline{\overline{\xi_n}}yz$. Now if $x\xi_n^* y$, then there exists $k \in \mathbb{N}$ and $u_0 = x, u_1, \ldots, u_k = y \in S$ such that $u_0 = x\xi_n u_1 \xi_n u_2 \xi_n \ldots \xi_n u_m = y$, by the above result we have $u_0 z = xz\overline{\overline{\xi_n}}u_1 z\overline{\overline{\xi_n}}u_2 z\overline{\overline{\xi_n}} \ldots \overline{\overline{\xi_n}}u_k z = yz$ and so $xz\overline{\overline{\xi_n}}yz$. Similarly we can show that $zx\overline{\overline{\xi_n}}zy$. Therefore ξ_n^* is a strongly regular relation on S. □

Proposition 3.1. *For every $n \in \mathbb{N}$, we have $\xi_{n+1}^* \subseteq \xi_n^*$.*

Proof. Let $x\xi_{n+1}y$ so $\exists(z_1,\ldots,z_m) \in S^m; \exists\delta \in S_m : \delta(i) = i$ if $z_i \notin L_{n+1}(S)$, such that $(\prod_{i=1}^{m} z_i)(x) > 0$, $(\prod_{i=1}^{m} z_{\delta(i)})(y) > 0$. Now let $\delta_1 = \delta$, since $L_{n+1}(S) \subseteq L_n(S)$ so $x\xi_n y$.□

The next result immediately follows from previous theorem.

Corolary 3.2. *If S is a commutative hypergroup, then $\beta^* = \xi_n^*$.*

A group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 1$ in which, $G^{(0)} = G$, $G^{(1)} = G'$, commutator subgroup of G, and inductively $G^{(i)} = (G^{(i-1)})'$.

Theorem 3.2. *If S is a fuzzy hypergroup and φ is a strongly regular relation on S, then*

$$L_{k+1}(S/\varphi)) = \langle \bar{t} \mid t \in L_k(S) \rangle$$

for $k \in \mathbb{N}$.

Proof. Suppose that $G = S/\varphi$ and $\overline{x} = \varphi(x)$ for all $x \in S$. We prove the theorem by induction on k. For $k = 0$ we have $L_1(G) = \langle \bar{t} \mid t \in L_0(S) \rangle$. Now suppose that $\overline{a} = \bar{t}$ where $t \in L_{k+1}(S)$ so there exist $r_1 \in S$; $(xy)(r_1) > 0$, $(tyx)(r_1) > 0$ in which $x, y \in L_k(S)$. Then $\overline{xy} = \overline{z_1}$; $(xy)(z_1) > 0$ and so $\overline{xy} = \overline{r_1}$. Also $\overline{tyx} = \overline{z_2}$; $(tyx)(z_2) > 0$ and $\overline{tyx} = \overline{r_1} = \overline{xy}$ which implies that $\bar{t} = [\overline{x}, \overline{y}]$. By hypotheses of induction we conclude that $\bar{t} \in L_{k+1}(G)$. Hence $\overline{a} = [\bar{t}, \bar{s}] \in L_{k+2}(G)$. Conversely, let $\overline{a} \in L_{k+2}(G)$. Then $\overline{a} = [\overline{x}, \overline{y}]$, where $\overline{x}, \overline{y} \in L_{k+1}(G)$, so by hypotheses of induction we have $\overline{x} = \overline{u}$ and $\overline{y} = \overline{v}$, where $u, v \in L_k(S)$. Let $c \in S$; $(uv)(c) > 0$ we show that there exists $t \in S$ such that $(tvu)(c) > 0$. Since $S \circ u = \chi_S$ and $c \in S$ then there exists $r \in S$ such that $(ru)(c) > 0$ and so by $r \in S = S \circ v$ there exist $t \in S$; $(tv)(r) > 0$. Therefore $(tvu)(c) = \bigvee_n((tv)(n) \wedge (nu)(c)) \geq (tv)(r) \wedge (ru)(c) > 0$. Thus $(uv)(c) > 0$, $(tvu)(c) > 0$ which implies that $t \in L_{k+1}(S)$. Now since

$\overline{uv} = \overline{c} = \overline{tvu}$, then $\overline{t} = [\overline{u}, \overline{v}] = [\overline{x}, \overline{y}] = \overline{a}$ and $t \in L_{k+1}(S)$. Therefore, $\overline{a} = \overline{t} \in \langle \overline{t}; t \in L_{k+1}(S) \rangle$.□

Theorem 3.3. *S/ξ_n^* is a solvable group of class at most $n+1$.*

Proof. Using Theorem 3.2, $L_k(S/\xi_n^*)$is an Abelian group and $L_{k+1}(S/\xi_n^*) = \{e\}$. □

4 On solvable groups derived from finite fuzzy hypergroups

In this section we introduce the smallest strongly relation ξ^* on a finite fuzzy hypergroup S such that H/ξ^* is a solvable group.

Definition 4.1. *Let S be a finite fuzzy hypergroup. Then we define the relation ξ^* on S by*

$$\xi^* = \bigcap_{n \geq 1} \xi_n^*.$$

Theorem 4.1. *The relation ξ^* is a strongly regular relation on a finite fuzzy hypergroup S such that S/ξ^* is a solvable group.*

Proof. Since $\xi^* = \bigcap_{n \geq 1} \xi_n^*$, it is easy to see that ξ^* is a strongly regular relation on S. By using Proposition 3.1, we conclude that there exists $k \in \mathbb{N}$ such that $\xi_{k+1}^* = \xi_k^*$. Thus $\xi* = \xi_k^*$ for some $k \in \mathbb{N}$. □

Theorem 4.2. *The relation ξ^* is the smallest strongly regular relation on a finite fuzzy hypergroup S such that S/ξ^* is a solvable group.*

Proof. Suppose ρ is a strongly regular relation on S such that $K = S/\rho$ is a solvable group of class c. Suppose that $x\xi y$. Then $x\xi_n y$, for some $n \in \mathbb{N}$ and so there exists $m \in \mathbb{N}$ such that

$x\xi_{mn}y \iff \exists (z_1, ..z_m) \in S^m, \exists \delta \in S_m : \delta(i) = i$ if $z_i \notin L_n(S)$ such that $(\prod_{i=1}^m z_i)(x) > 0$, $(\prod_{i=1}^m z_{\delta(i)})(y) > 0$,

$$L_{c+1}(S/\rho) = \langle \rho(t); t \in L_c(S) \rangle = \{\rho(e)\},$$

and so $\rho(z_i) = \rho(e)$, for every $z_i \in L_c(S)$. Therefore $\rho(x) = \rho(y)$, which implies that $x\rho y$.□

5 Transitivity of ξ^*

In this section we introduce the concept of ξ-part of a fuzzy hypergroup and we determine necessary and sufficient condition such that the relation ξ to be transitive.

Definition 5.1. *Let X be a non-empty subset of S. Then we say that X is a ξ-part of S if the following condition holds: for every $k \in \mathbb{N}$ and $(z_1, ..., z_m) \in H^m$ and for every $\sigma \in S_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$, and there exists $x \in X$ such that $(\prod_{i=1}^{m} z_i)(x) > 0$, then for all $y \in S \backslash X$, $(\prod_{i=1}^{m} z_{\sigma(i)})(y) = 0$.*

Theorem 5.1. *Let X be a non-empty subset of a fuzzy hypergroup S. Then the following conditions are equivalent:*
1) X is a ξ-part of S,
2) $x \in X$, $x\xi y \Longrightarrow y \in X$,
3) $x \in X$, $x\xi^* y \Longrightarrow y \in X$.

Proof. (1) $\Longrightarrow$ (2) if $(x, y) \in S^2$ is a pair such that $x \in X$ and $x\xi y$, then there exist $(z_1, ..., z_i) \in S^k$; $(\prod_{i=1}^{m} z_i)(x) > 0$, $(\prod_{i=1}^{m} z_{\sigma(i)})(y) > 0$ and $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$. Since X is a ξ-part of S, we have $y \in X$.
(2) $\Longrightarrow$ (3) Suppose that $(x, y) \in S^2$ is a part such that $x \in X$ and $x\xi^* y$. Then there is $(z_1, ..., z_i) \in S^k$ such that $x = z_0 \xi z_1 \xi ... \xi z_k = y$. Now by using (2) k-times we obtain $y \in X$.
(3) $\Longrightarrow$ (1) For every $k \in \mathbb{N}$ and $(z_1, ..., z_i) \in S^k$ and for every $\sigma \in S_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$, then there exists $x \in X$; $(\prod_{i=1}^{m} z_i)(x) > 0$ and there exist $y \in S \backslash X$; $(\prod_{i=1} z_{\sigma(i)})(y) > 0$, then $x\xi_n y$ and so $x\xi y$. Therefore by (3) we have $y \in X$ which is a contradiction.□

Theorem 5.2. *The following conditions are equivalent:*
1) for every $a \in H$, $\xi(a)$ is a ξ-part of S,
2) ξ is transitive.

Proof. (1) $\Longrightarrow$ (2) Suppose that $x\xi^* y$. Then there is $(z_1, ..., z_i) \in S^k$ such that $x = z_0 \xi z_1 \xi ... \xi z_k = y$, since $\xi(z_i)$ for all $0 \leq i \leq k$, is a ξ-part, we have $z_i \in \xi(z_{i-1})$, for all $1 \leq i \leq k$. Thus $y \in \xi(x)$, which means that $x\xi y$.
(2) $\Longrightarrow$ (1) Suppose that $x \in S$, $z \in \xi(x)$ and $z\xi y$. By transitivity of ξ, we have $y \in \xi(x)$. Now according to the last theorem, $\xi(x)$ is a ξ-part of S.□

Definition 5.2. *The intersection of all ξ-parts which contain A is called ξ-closure of A in S and it will be denoted by $K(A)$.*

In what follows, we determine the set $W(A)$, where A is a non-empty subset of S. We set
1) $W_1(A) = A$ and
2) $W_{n+1}(A) = \{x \in S \mid \exists(z_1, ..., z_i) \in S^k : \quad (\prod_{i=1}^{m} z_{(i)})(x) > 0, \exists\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{n\geq 1} L_n(S)$ and there exists $a \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0\}$.
We denote $W(A) = \bigcup_{n\geq 1} W_n(A)$.

Theorem 5.3. *For any non-empty subset of S, the following statements hold:*
1) $W(A) = K(A)$,
2) $K(A) = \cup_{a\in A} K(a)$.

Proof. 1) It is enough to prove:
a) $W(A)$ i a ξ-part,
b) if $A \subseteq B$ and B is a ξ-part, then $W(A) \subseteq B$.
In order to prove (a), suppose that $a \in W(A)$ such that $(\prod_{i=1} z_i)(a) > 0$ and $\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{n\geq 1} L_n(S)$. Therefore, there exists $n \in \mathbb{N}$ such that $(\prod_{i=1}^{m} z_i)(a) > 0$ $a \in W_n(A)$. Now if there exists $t \in S$ such that $(\prod_{i=1} z_{\sigma(i)})(t) > 0$ we obtain $t \in W_{n+1}(A)$. Therefore, $t \in W(A)$ which is a contradiction. Thus $(\prod_{i=1}^{m} z_{\sigma(i)})(t) = 0$ and so $W(A)$ is a ξ-part. Now we prove (b) by induction on n. We have $W_1(A) = A \subseteq B$. Suppose that $W_n(A) \subseteq B$. We prove that $W_{n+1}(A) \subseteq B$. If $z \in W_{n+1}(A)$, then there exists $k \in \mathbb{N}$; $(z_1, ..., z_k) \in S^k$; $(\prod_{i=1}^{m} z_i)(z) > 0$ and there exists $\sigma \in S_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t\geq 1} L_t(S)$ and there exists $t \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma_i})(t) > 0$, since $W_n(A) \subseteq B$ we have $t \in B$ and $(\prod_{i=1}^{m} z_{\sigma_i})(t) > 0$. Now since B is ξ-part , $(\prod_{i=1}^{m} z_i)(z) > 0$ then $z \in B$.

2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part 1), we have $K(A) = \cup_{n\geq 1} W_n(A)$ and $W_1(A) = A = \cup_{a\in A}\{a\}$. It is enough to prove that $W_n(A) = \cup_{a\in A} W_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n. Suppose it is true for n. We prove that $W_{n+1}(A) = \cup_{a\in A} W_{n+1}(a)$. If $z \in W_{n+1}(A)$, then there exists $k \in \mathbb{N}$, $(z_1, ..., z_k) \in S^k$; $(\prod_{i=1}^{m} z_i)z > 0$ and there exists $\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t\geq 1} L_t(S)$ and there exist $a \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0$. By the hypotheses of induction there exists $a \in W_n(A) = \cup_{b\in A} W_n(b)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a') > 0$ for some $a' \in W_n(b)$ in which $b \in A$. Therefore, $z \in W_{n+1}(b)$, and so $W_{n+1}(A) \subseteq \cup_{b\in A} W_{n+1}(b)$. Hence $K(A) = \cup_{a\in A} K(a)$.□

Theorem 5.4. *The following relation is equivalence relation on H.*

$$xWy \Longleftrightarrow x \in W(y),$$

for every $(x, y) \in S^2$, where $W(y) = W(\{y\})$.

Proof. It is easy to see that W is reflexive and transitive. We prove that W is symmetric. To this, we check that:
1) for all $n \geq 2$ and $x \in S$, $W_n(W_2(x)) = W_{n+1}(x)$,
2) $x \in W_n(y)$ if and only if $y \in W_n(x)$.
We prove (1) by induction on n.

$W_2(W_2(x)) = \{z \mid \exists q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1} a_i)(z) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i, if\ z_i \notin \cup_{s\geq 1} L_s(S)$ *and* $\exists y \in W_2(x)$; $(\prod_{i=1}^{m} a_{\sigma(i)})(y) > 0\} = W_3(x)$. Now we proceed by induction on n. Suppose $W_n(W_2(x)) = W_{n+1}(x)$ then
$W_{n+1}(W_2(x)) = \{z \mid \exists q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1}^{m} a_i)(z) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i$, *if* $z_i \notin \cup_{s\geq 1} L_s(S)$ *and* $\exists t \in W_n(W_2(x)); (\prod_{i=1}^{m} a_{\sigma(i)})(t) > 0\} = W_{n+2}(x)$. Now we prove (2) by induction on n, too. It is clear that $x \in W_2(y)$ if and only if $y \in W_2(x)$. Suppose $x \in W_n(y)$ if and only if $y \in W_n(x)$. Let $x \in W_{n+1}(y)$, then there exists $q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1}^{m} a_i)(x) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i$, *if* $a_i \notin$

$\cup_{s\geq 1}L_s(S)$ *and* $\exists t \in W_n(y); (\prod_{i=1}^{m} a_{\sigma(i)})t > 0$. Now, $(\prod_{i=1}^{m} a_i)(x) > 0, x \in W_1(x)$ and $(\prod_{i=1}^{m} a_{\sigma(i)})(t) > 0$ implies that $t \in W_2(x)$. Since $t \in W_n(y)$, then by hypotheses of induction $y \in W_n(t)$ and we see that $t \in W_2(x)$, therefore $y \in W_n(W_2(x)) = W_{n+1}(x)$. □

Remark 5.1. *If S is a fuzzy hypergroup, then S/ξ^* is a group. We define $\omega_S = \phi^{-1}(1_{S/\xi^*})$, in which $\phi : S \to S/\xi^*$ is the canonical projection.*

Lemma 5.1. *If S is a fuzzy hypergroup and M is a non-empty subset of S, then*

(i) $\phi^{-1}(\phi(M)) = \{x \in S : (\omega_S M)(x) > 0\} = \{x \in S : (M\omega_S)(x) > 0\}$

(ii) If M is a ξ part of S, then $\phi^{-1}(\phi(M)) = M$.

Proof. *(i)* Let $x \in S$ and $(t, y) \in \omega_S \times M$ such that $(ty)(x) > 0$, so $\phi(x) = \phi(t) \oplus \phi(y) = 1_{S/\xi^*} \oplus \phi(y) = \phi(y)$, therefore $x \in \phi^{-1}(\phi(y)) \subset \phi^{-1}(\phi(M))$. Conversely, for every $x \in \phi^{-1}(\phi(M))$, there exists $b \in M$ such that $\phi(x) = \phi(b)$. By reproducibility, $a \in S$ exists such that $(ab)(x) > 0$, so $\phi(b) = \phi(x) = \phi(a) \oplus \phi(b)$. This implies $\phi(a) = 1_{S/\xi^*}$ and $a \in \phi^{-1}(1_{S/\xi^*}) = \omega_S$. Therefore $(\omega_S M)(x) > 0$.

In the same way, we can prove that $\phi^{-1}(\phi(M)) = \{x \in S : (M\omega_S)(x) > 0\}$.

(ii) We know $M \subseteq \phi^{-1}(\phi(M))$. If $x \in \phi^{-1}(\phi(M))$, then there exists $b \in M$ such that $\phi(x) = \phi(b)$. Therefore $x \in \xi^*(x) = \xi^*(b)$. Since M is a ξ part of S and $b \in M$, by Lemma 5.1, we conclude $\xi^*(b) \subseteq M$ and $x \in M$. □

Definition 5.3. *Let $(S, \cdot)$ be a fuzzy hypergroup. $K \subseteq S$ is called a fuzzy subhypergroup of S if*

i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in S$

ii) $a \cdot K = \chi_K$, for all $a \in K$.

Theorem 5.5. *ω_S is a fuzzy subhypergroup of S, which is also a ξ-part of S.*

Proof. Clearly, $\omega_S \subseteq S$ and so $(a\cdot b)\cdot c = a\cdot(b\cdot c)$, for all $a, b, c \in \omega_S$. Now we show that $\omega_S y = \chi_{\omega_S}$ for all $y \in \omega_S$. Let $x, y \in \omega_S$, then there exists $u \in S$ such that $(uy)(x) > 0$. Therefore, $\overline{uy} = \overline{x}$, which implies that $\overline{u} = 1$. Thus $u \in \omega_S$. Consequently, $\omega_S y = \chi_{\omega_S}$. Hence, ω_S is a fuzzy subhypergroup of S. Now we prove that $K(y) = \phi^{-1}(\phi(\{y\})) = \{x \in S : (\omega_S y)(x) > 0\} = \omega_S$.

$$\begin{aligned} z \in \phi^{-1}(\phi(\{y\})) &\iff \varphi(z) = \varphi(y) \\ &\iff \xi^*(z) = \xi^*(y) \\ &\iff z\xi^* y \\ &\iff z \in \xi^*(z) = \omega(\{y\}) = K(y). \end{aligned}$$

Also since $y \in \omega_S$, then $\{x \in S : (\omega_S y)(x) > 0\} = \{x \in S : (\chi_{\omega_S})(x) > 0\} = \omega_S$. Therefore $K(y) = \omega_S$ and so ω_S is ξ part. $\Box$

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Neutrosophic filters in BE-algebras

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Abstract

In this paper, we introduce the notion of (implicative) neutrosophic filters in BE-algebras. The relation between implicative neutrosophic filters and neutrosophic filters is investigated and we show that in self distributive BE-algebras these notions are equivalent.

Keywords: BE-algebra, neutrosophic set, (implicative) neutrosophic filter.

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1 Introduction

Neutrosophic set theory was introduced by Smarandache in 1998 ([10]). Neutrosophic sets are a new mathematical tool for dealing with uncertainties which are free from many difficulties that have troubled the usual theoretical approaches. Research works on neutrosophic set theory for many applications such as information fussion, probability theory, control theory, decision making, measurement theory, etc. Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures ([3, 4, 5]). Since then many researchers worked in this area and lots of literatures had been produced about the theory of neutrosophic set. In the neutrosophic set one can have elements which have paraconsistent information (sum of components > 1), others incomplete information (sum of components < 1), others consistent information (in the case when the sum of components =1) and others interval-valued components (with no restriction on their superior or inferior sums).

H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra ([6]). B.L. Meng give a procedure which generated a filter by a subset in a transitive BE-algebra ([7]). A. Walendziak introduced the notion of a normal filter in BE-algebras and showed that there is a bijection between congruence relations and filters in commutative BE-algebras ([11]). A. Borumand Saeid and et al. defined some types of filters in BE-algebras and showed the relationship between them ([1]). A. Rezaei and et al. discussed on the relationship between BE-algebras and Hilbert algebras ([9]). Recently, A. Rezaei and et al. introduced the notion of hesitant fuzzy (implicative) filters and get some results on BE-algebras ([8]).

In this paper, we introduce the notion of (implicative) neutrosophic filters and study it in details. In fact, we show that in self distributive BE-algebras concepts of implicative neutrosophic filter and neutrosophic filter are equivalent.

2 Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel:

Definition 2.1. [6] By a BE-algebra we shall mean an algebra $\mathfrak{X} = (X; *, 1)$ of type $(2, 0)$ satisfying the following axioms:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

From now on, $\mathfrak{X}$ is a BE-algebra, unless otherwise is stated. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. A BE-algebra $\mathfrak{X}$ is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$. A BE-algebra $\mathfrak{X}$ is said to be commutative if satisfies:

$$(x * y) * y = (y * x) * x, \text{ for all } x, y \in X.$$

Proposition 2.1. [11] If $\mathfrak{X}$ is a commutative BE-algebra, then for all $x, y \in X$,

$$x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

We note that "$\leq$" is reflexive by (BE1). If $\mathfrak{X}$ is self distributive then relation "$\leq$" is a transitive ordered set on X, because if $x \leq y$ and $y \leq z$, then

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Hence $x \leq z$. If $\mathfrak{X}$ is commutative then by Proposition 2.1, relation "$\leq$" is antisymmetric. Hence if $\mathfrak{X}$ is a commutative self distributive BE-algebra, then relation "$\leq$" is a partial ordered set on $\mathfrak{X}$.

Proposition 2.2. [6] In a BE-algebra $\mathfrak{X}$, the following hold:

(i) $x * (y * x) = 1$,

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

A subset F of X is called a filter of $\mathfrak{X}$ if it satisfies: (F1) $1 \in F$, (F2) $x \in F$ and $x * y \in F$ imply $y \in F$. Define

$$A(x, y) = \{z \in X : x * (y * z) = 1\},$$

which is called an upper set of x and y. It is easy to see that $1, x, y \in A(x, y)$, for any $x, y \in X$. Every upper set $A(x, y)$ need not be a filter of $\mathfrak{X}$ in general.

Definition 2.2. [1] A non-empty subset F of X is called an implicative filter if satisfies the following conditions:

(IF1) $1 \in F$,

(IF2) $x * (y * z) \in F$ and $x * y \in F$ imply that $x * z \in F$, for all $x, y, z \in X$.

If we replace x of the condition (IF2) by the element 1, then it can be easily observed that every implicative filter is a filter. However, every filter is not an implicative filter as shown in the following example.

Example 2.1. Let $X = \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then $F = \{1, a\}$ is a filter of X, but it is not an implicative filter, since $1 * (a * b) = 1 * a = a \in F$ and $1 * a = a \in F$ but $1 * b = b \notin F$.

Definition 2.3. [10] Let X be a set. A neutrosophic subset A of X is a triple (T_A, I_A, F_A) where $T_A : X \to [0, 1]$ is the membership function, $I_A : X \to [0, 1]$ is the indeterminacy function and $F_A : X \to [0, 1]$ is the nonmembership function. Here for each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are all standard real numbers in $[0, 1]$.

We note that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, for all $x \in X$. The set of neutrosophic subset of X is denoted by NS(X).

Definition 2.4. [10] Let A and B be two neutrosophic sets on X. Define $A \leq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$, for all $x \in X$.

Definition 2.5. Let $\mathfrak{X}_1 = (X_1; *, 1)$ and $\mathfrak{X}_2 = (X_2; \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \to X_2$ is called a homomorphism if, for all $x_1, x_2 \in X_1$ $f(x_1 * x_2) = f(x_1) \circ f(x_2)$. It is clear that if $f : X_1 \to X_2$ is a homomorphism, then $f(1) = 1'$.

3 Neutrosophic Filters

Definition 3.1. A neutrosophic set A of $\mathfrak{X}$ is called a *neutrosophic filter if satisfies the following conditions:*

(NF1) $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$,

(NF2) $\min\{T_A(x * y), T_A(x)\} \leq T_A(y)$, $\min\{I_A(x * y), I_A(x)\} \geq I_A(y)$ and $\min\{F_A(x * y), F_A(x)\} \geq F_A(y)$, *for all* $x, y \in X$.

The set of neutrosophic filter of $\mathfrak{X}$ is denoted by NF($\mathfrak{X}$).

Example 3.1. In Example 2.1, put $T_A(1) = 0.9$, $T_A(a) = T_A(b) = 0.5$, $I_A(1) = 0.2$, $I_A(a) = I_A(b) = 0.35$ and $F_A(1) = 0.1$, $F_A(a) = F_A(b) = 0$. Then $A = (T_A, I_A, F_A)$ is a neutrosophic filter.

Proposition 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

(i) *if* $x \leq y$, *then* $T_A(x) \leq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \geq F_A(y)$,

(ii) $T_A(x) \leq T_A(y * x)$, $I_A(x) \geq I_A(y * x)$ and $F_A(x) \geq F_A(y * x)$,

(iii) $\min\{T_A(x), T_A(y)\} \leq T_A(x * y)$, $\min\{I_A(x), I_A(y)\} \geq I_A(x * y)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(x * y)$,

(iv) $T_A(x) \leq T_A((x*y)*y)$, $I_A(x) \geq I_A((x*y)*y)$ and $F_A(x) \geq F_A((x*y)*y)$,

(v) $\min\{T_A(x), T_A(y)\} \leq T_A((x * (y * z)) * z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A((x * (y * z)) * z)$ and
$\min\{F_A(x), F_A(y)\} \geq F_A((x * (y * z)) * z)$,

(vi) *if* $\min\{T_A(y), T_A((x * y) * z)\} \leq T_A(z * x)$, *then* T_A *is order reversing and* I_A, F_A *are order* (i.e. if $x \leq y$, then $T_A(y) \leq T_A(x)$, $I_A(y) \geq I_A(x)$ and $F_A(y) \geq F_A(x)$)

(vii) *if* $z \in A(x, y)$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A(z)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(z)$

(viii) *if* $\prod_{i=1}^{n} a_i * x = 1$, *then* $\bigwedge_{i=1}^{n} T_A(a_i) \leq T_A(x)$, $\bigwedge_{i=1}^{n} I_A(a_i) \geq I_A(x)$ and $\bigwedge_{i=1}^{n} F_A(a_i) \geq F_A(x)$ *where* $\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\ldots (a_1 * x) \ldots))$.

Proof. (i). Let $x \leq y$. Then $x * y = 1$ and so

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A(x * y)\} \leq T_A(y),$$

$$I_A(x) = \min\{I_A(x), I_A(1)\} = \min\{I_A(x), I_A(x * y)\} \geq I_A(y),$$

$$F_A(x) = \min\{F_A(x), F_A(1)\} = \min\{F_A(x), F_A(x * y)\} \geq F_A(y).$$

(ii). Since $x \leq y * x$, by using (i) the proof is clear.

(iii). By using (ii) we have

$$\min\{T_A(x), T_A(y)\} \leq T_A(y) \leq T_A(x * y),$$

$$\min\{I_A(x), I_A(y)\} \geq I_A(y) \geq I_A(x * y),$$

$$\min\{F_A(x), F_A(y)\} \geq F_A(y) \geq F_A(x * y).$$

(iv). It follows from Definition 3.1,

$$\begin{aligned} T_A(x) &= \min\{T_A(x), T_A(1)\} \\ &= \min\{T_A(x), T_A((x * y) * (x * y))\} \\ &= \min\{T_A(x), T_A(x * ((x * y) * y))\} \\ &\leq T_A((x * y) * y). \end{aligned}$$

Also, we have

$$\begin{aligned} I_A(x) &= \min\{I_A(x), I_A(1)\} \\ &= \min\{I_A(x), I_A((x * y) * (x * y))\} \\ &= \min\{I_A(x), I_A(x * ((x * y) * y))\} \\ &\geq I_A((x * y) * y) \end{aligned}$$

and

$$\begin{aligned} F_A(x) &= \min\{F_A(x), F_A(1)\} \\ &= \min\{F_A(x), F_A((x * y) * (x * y))\} \\ &= \min\{F_A(x), F_A(x * ((x * y) * y))\} \\ &\geq F_A((x * y) * y). \end{aligned}$$

(v). From (iv) we have

$$\begin{aligned} \min\{T_A(x), T_A(y)\} &\leq \min\{T_A(x), T_A((y * (x * z)) * (x * z))\} \\ &= \min\{T_A(x), T_A((x * (y * z)) * (x * z))\} \\ &= \min\{T_A(x), T_A(x * (x * (y * z)) * z))\} \\ &\leq T_A((x * (y * z)) * z)), \end{aligned}$$

$$\begin{aligned} \min\{I_A(x), I_A(y)\} &\geq \min\{I_A(x), I_A((y * (x * z)) * (x * z))\} \\ &= \min\{I_A(x), I_A((x * (y * z)) * (x * z))\} \\ &= \min\{I_A(x), I_A(x * (x * (y * z)) * z))\} \\ &\geq I_A((x * (y * z)) * z)) \end{aligned}$$

and

$$\begin{aligned}\min\{F_A(x), F_A(y)\} &\geq \min\{F_A(x), F_A((y*(x*z))*(x*z))\} \\ &= \min\{F_A(x), F_A((x*(y*z))*(x*z))\} \\ &= \min\{F_A(x), F_A(x*(x*(y*z))*z))\} \\ &\geq F_A((x*(y*z))*z)).\end{aligned}$$

(vi). Let $x \leq y$, that is, $x * y = 1$.

$T_A(y) = \min\{T_A(y), T_A(1*1)\} = \min\{T_A(y), T_A((x*y)*1)\} \leq T_A(1*x) = T_A(x)$,

$I_A(y) = \min\{I_A(y), I_A(1*1)\} = \min\{I_A(y), I_A((x*y)*1)\} \geq I_A(1*x) = I_A(x)$,

$F_A(y) = \min\{F_A(y), F_A(1*1)\} = \min\{F_A(y), F_A((x*y)*1)\} \geq F_A(1*x) = F_A(x)$.

(vii). Let $z \in A(x, y)$. Then $x * (y * z) = 1$. Hence

$$\begin{aligned}\min\{T_A(x), T_A(y)\} &= \min\{T_A(x), T_A(y), T_A(1)\} \\ &= \min\{T_A(x), T_A(y), T_A(x*(y*z))\} \\ &\leq \min\{T_A(y), T_A(y*z)\} \\ &\leq T_A(z).\end{aligned}$$

Also, we have

$$\begin{aligned}\min\{I_A(x), I_A(y)\} &= \min\{I_A(x), I_A(y), I_A(1)\} \\ &= \min\{I_A(x), I_A(y), I_A(x*(y*z))\} \\ &\geq \min\{I_A(y), I_A(y*z)\} \\ &\geq I_A(z),\end{aligned}$$

and

$$\begin{aligned}\min\{F_A(x), F_A(y)\} &= \min\{F_A(x), F_A(y), F_A(1)\} \\ &= \min\{F_A(x), F_A(y), F_A(x*(y*z))\} \\ &\geq \min\{F_A(y), F_A(y*z)\} \\ &\geq F_A(z).\end{aligned}$$

(viii). The proof is by induction on n. By (vii) it is true for $n = 1, 2$. Assume that it satisfies for $n = k$, that is,

$\prod_{i=1}^{k} a_i * x = 1 \Rightarrow \bigwedge_{i=1}^{k} T_A(a_i) \leq T_A(x), \bigwedge_{i=1}^{k} I_A(a_i) \geq I_A(x)$ and $\bigwedge_{i=1}^{k} F_A(a_i) \geq F_A(x)$ for all $a_1, \ldots, a_k, x \in X$.

Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$, for all $a_1, \ldots, a_k, a_{k+1}, x \in X$. Then $\bigwedge_{i=2}^{k+1} T_A(a_i) \leq T_A(a_1 * x)$, $\bigwedge_{i=2}^{k+1} I_A(a_i) \geq I_A(a_1 * x)$, and $\bigwedge_{i=2}^{k+1} F_A(a_i) \geq F_A(a_1 * x)$.

Since A is a neutrosophic filter of $\mathfrak{X}$, we have

$$\bigwedge_{i=1}^{k+1} T_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} T_A(a_i)), T_A(a_1)\} \leq \min\{T_A(a_1 * x), T_A(a_1)\} \leq T_A(x),$$

$$\bigwedge_{i=1}^{k+1} I_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} I_A(a_i)), I_A(a_1)\} \geq \min\{I_A(a_1 * x), I_A(a_1)\} \geq I_A(x)$$

and

$$\bigwedge_{i=1}^{k+1} F_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} F_A(a_i)), F_A(a_1)\} \geq \min\{F_A(a_1 * x), F_A(a_1)\} \geq F_A(x).$$

□

Theorem 3.1. *If $\{A_i\}_{i \in I}$ is a family of neutrosophic filters in $\mathfrak{X}$, then $\bigcap_{i \in I} A_i$ is too.*

Theorem 3.2. *Let $A \in \mathrm{NF}(\mathfrak{X})$. Then the sets*

(i) $X_{T_A} = \{x \in X : T_A(x) = T_A(1)\}$,

(ii) $X_{I_A} = \{x \in X : I_A(x) = I_A(1)\}$,

(iii) $X_{F_A} = \{x \in X : F_A(x) = F_A(1)\}$,

are filters of $\mathfrak{X}$.

Proof. (i). Obviously, $1 \in X_{h_A}$. Let $x, x * y \in X_{T_A}$. Then $T_A(x) = T_A(x * y) = T_A(1)$. Now, by (NF1) and (NF2), we have

$$T_A(1) = \min\{T_A(x), T_A(x * y)\} \leq T_A(y) \leq T_A(1).$$

Hence $T_A(y) = T_A(1)$. Therefore, $y \in X_{T_A}$.

The proofs of (ii) and (iii) are similar to (i).□

Definition 3.2. A neutrosophic set A of $\mathfrak{X}$ is called an implicative neutrosophic filter of $\mathfrak{X}$ if satisfies the following conditions:

(INF1) $T_A(1) \geq T_A(x)$,

(INF2) $T_A(x * z) \geq \min\{T_A(x * (y * z)), T_A(x * y)\}$,
$I_A(x * z) \leq \min\{I_A(x * (y * z)), I_A(x * y)\}$ and
$F_A(x * z) \leq \min\{F_A(x * (y * z)), F_A(x * y)\}$, for all $x, y, z \in X$.

The set of implicative neutrosophic filter of $\mathfrak{X}$ is denoted by INF($\mathfrak{X}$). If we replace x of the condition (INF2) by the element 1, then it can be easily observed that every implicative neutrosophic filter is a neutrosophic filter. However, every neutrosophic filter is not an implicative neutrosophic filter as shown in the following example.

Example 3.2. Let $X = \{1, a, b, c, d\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $\mathfrak{X} = (X; *, 1)$ is a BE-algebra. Define a neutrosophic set A on $\mathfrak{X}$ as follows:

$$T_A(x) = \begin{cases} 0.85 & \text{if } x = 1, a \\ 0.12 & otherwise \end{cases}$$

and $I_A(x) = F_A(x) = 0.5$, for all $x \in X$.

Then clearly $A = (T_A, I_A, F_A)$ is a neutrosophic filter of $\mathfrak{X}$, but it is not an implicative neutrosophic filter of $\mathfrak{X}$, since

$$T_A(b * c) \ngeq \min\{T_A(b * (d * c)), T_A(b * d)\}.$$

Theorem 3.3. *Let $\mathfrak{X}$ be a self distributive BE-algebra. Then every neutrosophic filter is an implicative neutrosophic filter.*

Proof. Let $A \in \mathrm{NF}(\mathfrak{X})$ and $x \in X$. Obvious that $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$. By self distributivity and (NF2), we have

$$\min\{T_A(x*(y*z)), T_A(x*y)\} = \min\{T_A((x*y)*(x*z)), T_A(x*y)\} \leq T_A(x*z),$$

$$\min\{I_A(x*(y*z)), I_A(x*y)\} = \min\{I_A((x*y)*(x*z)), I_A(x*y)\} \geq I_A(x*z)$$

and

$$\min\{F_A(x*(y*z)), F_A(x*y)\} = \min\{F_A((x*y)*(x*z)), F_A(x*y)\} \geq F_A(x*z).$$

Therefore $A \in \mathrm{INF}(\mathfrak{X})$.□

Let $t \in [0, 1]$. For a neutrosophic filter A of $\mathfrak{X}$, t-level subset which denoted by $U(A;t)$ is defined as follows:

$$U(A;t) := \{x \in A : t \leq T_A(x), I_A(x) \leq t \text{ and } F_A(x) \leq t\}$$

and strong t-level subset which denoted by $U(A;t)_>$ as

$$U(A;t)_> := \{x \in A : t < T_A(x), I_A(x) < t \text{ and } F_A(x) < t\}.$$

Theorem 3.4. *Let $A \in \mathrm{NS}(\mathfrak{X})$. The following are equivalent:*

(i) $A \in \mathrm{NF}(\mathfrak{X})$,

(ii) $(\forall t \in [0, 1])$ $U(A;t) \neq \emptyset$ *imply* $U(A;t)$ *is a filter of* $\mathfrak{X}$.

Proof. (i)⇒(ii). Let $x, y \in X$ be such that $x, x * y \in U(A;t)$, for any $t \in [0, 1]$. Then $t \leq T_A(x)$ and $t \leq T_A(x*y)$. Hence $t \leq \min\{T_A(x), T_A(x*y)\} \leq T_A(y)$. Also, $I_A(x) \leq t$ and $I_A(x * y) \leq t$ and so $t \geq \min\{I_A(x), I_A(x * y)\} \geq I_A(y)$. By a similar argument we have $t \geq \min\{F_A(x), F_A(x * y)\} \geq F_A(y)$. Therefore, $y \in U(A;t)$.

(ii)⇒(i). Let $U(A;t)$ be a filter of $\mathfrak{X}$, for any $t \in [0, 1]$ with $U(A;t) \neq \emptyset$. Put $T_A(x) = I_A(x) = F_A(x) = t$, for any $x \in X$. Then $x \in U(A;t)$. Since $U(A;t)$ is a filter of $\mathfrak{X}$, we have $1 \in U(A;t)$ and so $T_A(x) = t \leq T_A(1)$. Now, for any $x, y \in X$, let $T_A(x * y) = I_A(x * y) = F_A(x * y) = t_{x*y}$ and
$T_A(x) = I_A(x) = F_A(x) = t_x$. Put $t = \min\{t_{x*y}, t_x\}$. Then $x, x * y \in U(A;t)$, so $y \in U(A;t)$. Hence $t \leq T_A(y)$, $t \geq I_A(y)$, $t \geq F_A(y)$ and so

$$\min\{T_A(x * y), T_A(x)\} = \min\{t_{x*y}, t_x\} = t \leq T_A(y),$$

$$\min\{I_A(x * y), I_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq I_A(y),$$

and

$$\min\{F_A(x * y), F_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq F_A(y).$$

Therefore, $A \in \mathrm{NF}(\mathfrak{X})$.□

Theorem 3.5. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then we have*

$$(\forall a, b \in X)\ (\forall t \in [0,1])\ (a, b \in U(A;t) \Rightarrow A(a,b) \subseteq U(A;t)).$$

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{X})$. Let $a, b \in X$ be such that $a, b \in U(A;t)$. Then $t \leq T_A(a)$ and $t \leq T_A(b)$. Let $c \in A(a,b)$. Hence $a * (b * c) = 1$. Now, by Proposition 3.1(v) and (BE3), we have

$$t \leq \min\{T_A(a), T_A(b)\} \leq T_A((a * (b * c) * c)) = T_A(1 * c) = T_A(c),$$

$$t \geq \min\{I_A(a), I_A(b)\} \geq I_A((a * (b * c) * c)) = I_A(1 * c) = I_A(c)$$

and

$$t \geq \min\{F_A(a), F_A(b)\} \geq F_A((a * (b * c) * c)) = F_A(1 * c) = F_A(c).$$

Then $c \in U(A;t)$. Therefore, $A(a,b) \subseteq U(A;t)$.□

Corolary 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

$$(\forall t \in [0,1])\ (U(A;t) \neq \emptyset \Rightarrow U(A;t) = \bigcup_{a,b \in U(A;t)} A(a,b)).$$

Proof. It is sufficient prove that $U(A;t) \subseteq \bigcup_{a,b \in U(A;t)} A(a,b)$. For this, assume that $x \in U(A;t)$. Since $x * (1 * x) = 1$, we have $x \in A(x,1)$. Hence

$$U(A;t) \subseteq A(x,1) \subseteq \bigcup_{x \in U(A;t)} A(x,1) \subseteq \bigcup_{x,y \in U(A;t)} A(x,y).$$

□

Theorem 3.6. *Let* $\mathfrak{X}$ *be a self distributive BE-algebra and* $A \in \mathrm{NF}(\mathfrak{X})$. *Then the following conditions are equivalent:*

(i) $A \in \mathrm{INF}(\mathfrak{X})$,

(ii) $T_A(y * (y * x)) \leq T_A(y * x)$, $I_A(y * (y * x)) \geq I_A(y * x)$ *and* $F_A(y * (y * x)) \geq F_A(y * x)$,

(iii) $\min\{T_A((z * (y * (y * x))), T_A(z)\} \leq T_A(y * x)$, $\min\{I_A((z * (y * (y * x))), I_A(z)\} \geq I_A(y * x)$ *and* $\min\{F_A((z * (y * (y * x))), F_A(z)\} \geq F_A(y * x)$.

Proof. (i)⇒(ii). Let $A \in \mathrm{NF}(\mathfrak{X})$. By (INF1) and (BE1) we have

$$\begin{aligned} T_A(y*(y*x)) &= \min\{T_A(y*(y*x)), T_A(1)\} \\ &= \min\{T_A(y*(y*x)), T_A(y*y)\} \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} I_A(y*(y*x)) &= \min\{I_A(y*(y*x)), I_A(1)\} \\ &= \min\{I_A(y*(y*x)), I_A(y*y)\} \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} F_A(y*(y*x)) &= \min\{F_A(y*(y*x)), F_A(1)\} \\ &= \min\{F_A(y*(y*x)), F_A(y*y)\} \\ &\geq F_A(y*x). \end{aligned}$$

(ii)⇒(iii). Let A be a neutrosophic filter of $\mathfrak{X}$ satisfying the condition (ii). By using (NF2) and (ii) we have

$$\begin{aligned} \min\{T_A(z*(y*(y*x))), T_A(z)\} &\leq T_A(y*(y*x)) \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} \min\{I_A(z*(y*(y*x))), I_A(z)\} &\geq I_A(y*(y*x)) \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(z*(y*(y*x))), F_A(z)\} &\geq F_A(y*(y*x)) \\ &\geq F_A(y*x). \end{aligned}$$

(iii)⇒(i). Since

$$x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)),$$

we have $T_A(x*(y*z)) \leq T_A((x*y)*(x*(x*z)))$,
$I_A(x*(y*z)) \geq I_A((x*y)*(x*(x*z)))$ and
$F_A(x*(y*z)) \geq F_A((x*y)*(x*(x*z)))$, by Proposition 3.1(i). Thus

$$\begin{aligned} \min\{T_A(x*(y*z)), T_A(x*y)\} &\leq \min\{T_A((x*y)*(x*(x*z))), T_A(x*y)\} \\ &\leq T_A(x*z). \end{aligned}$$

$$\begin{aligned}\min\{I_A(x*(y*z)), I_A(x*y)\} &\geq \min\{I_A((x*y)*(x*(x*z))), I_A(x*y)\} \\ &\geq I_A(x*z)\end{aligned}$$

and

$$\begin{aligned}\min\{F_A(x*(y*z)), F_A(x*y)\} &\geq \min\{F_A((x*y)*(x*(x*z))), F_A(x* \\ y)\} &\geq F_A(x*z).\end{aligned}$$

Therefore, $A \in \mathrm{INF}(\mathfrak{X})$. Let $f : X \to Y$ be a homomorphism of BE-algebras

and $A \in \mathrm{NS}(\mathfrak{X})$.
Define tree maps $T_{A^f} : X \to [0,1]$ such that $T_{A^f}(x) = T_A(f(x))$,
$I_{A^f} : X \to [0,1]$ such that $I_{A^f}(x) = I_A(f(x))$ and $F_{A^f} : X \to [0,1]$ such that $F_{A^f}(x) = F_A(f(x))$, for all $x \in X$. Then T_{A^f}, I_{A^f} and F_{A^f} are well-define and $A^f = (T_{A^f}, I_{A^f}, F_{A^f}) \in \mathrm{NS}(\mathfrak{X})$.□

Theorem 3.7. *Let $f : X \to Y$ be an onto homomorphism of BE-algebras and $A \in \mathrm{NS}(\mathfrak{Y})$. Then $A \in \mathrm{NF}(\mathfrak{Y})$ (resp. $A \in \mathrm{INF}(\mathfrak{Y})$) if and only if $A^f \in \mathrm{NF}(\mathfrak{X})$ (resp. $A^f \in \mathrm{INF}(\mathfrak{X})$).*

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{Y})$. For any $x \in X$, we have

$$T_{A^f}(x) = T_A(f(x)) \leq T_A(1_Y) = T_A(f(1_X)) = T_{A^f}(1_X),$$

$$I_{A^f}(x) = I_A(f(x)) \geq I_A(1_Y) = I_A(f(1_X)) = I_{A^f}(1_X)$$

and

$$F_{A^f}(x) = F_A(f(x)) \geq F_A(1_Y) = F_A(f(1_X)) = F_{A^f}(1_X).$$

Hence (NF1) is valid. Now, let $x, y \in X$. By (NF1) we have

$$\begin{aligned}\min\{T_{A^f}(x*y), T_{A^f}(x)\} &= \min\{T_A(f(x*y)), T_A(f(x))\} \\ &= \min\{T_A(f(x)*f(y)), T_A(f(x))\} \\ &\leq T_A(f(y)) \\ &= T_{A^f}(y)\end{aligned}$$

Also,

$$\begin{aligned}\min\{I_{A^f}(x*y), I_{A^f}(x)\} &= \min\{I_A(f(x*y)), I_A(f(x))\} \\ &= \min\{I_A(f(x)*f(y)), I_A(f(x))\} \\ &\geq I_A(f(y)) \\ &= I_{A^f}(y).\end{aligned}$$

By a similar argument we have $\min\{F_{A^f}(x*y), F_{A^f}(x)\} \geq F_{A^f}(y)$. Therefore, $A^f \in \mathrm{NF}(\mathfrak{X})$.

Conversely, Assume that $A^f \in \mathrm{NF}(\mathfrak{X})$. Let $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Then

$$T_A(y) = T_A(f(x)) = T_{A^f}(x) \leq T_{A^f}(1_X) = T_A(f(1_X)) = T_A(1_Y),$$

$$I_A(y) = I_A(f(x)) = I_{A^f}(x) \geq I_{A^f}(1_X) = I_A(f(1_X)) = I_A(1_Y)$$

and

$$F_A(y) = F_A(f(x)) = F_{A^f}(x) \geq F_{A^f}(1_X) = F_A(f(1_X)) = F_A(1_Y),$$

Now, let $x, y \in Y$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we have

$$\begin{aligned}\min\{T_A(x*y), T_A(x)\} &= \min\{T_A(f(a)*f(b)), T_A(f(a))\}\\ &= \min\{T_A(f(a*b)), T_A(f(a))\}\\ &= \min\{T_{A^f}(a*b), T_{A^f}(a)\}\\ &\leq T_{A^f}(b)\\ &= T_A(f(b))\\ &= T_A(y).\end{aligned}$$

Also, we have

$$\begin{aligned}\min\{I_A(x*y), I_A(x)\} &= \min\{I_A(f(a)*f(b)), I_A(f(a))\}\\ &= \min\{I_A(f(a*b)), I_A(f(a))\}\\ &= \min\{I_{A^f}(a*b), I_{A^f}(a)\}\\ &\geq I_{A^f}(b)\\ &= I_A(f(b))\\ &= I_A(y).\end{aligned}$$

By a similar argument we have $\min\{F_A(x*y), F_A(x)\} \geq F_A(y)$.
Therefore, $A \in \mathrm{NF}(\mathfrak{Y})$.□

4 Conclusion

F. Smarandache as an extension of intuitionistic fuzzy logic introduced the concept of neutrosophic logic and then several researchers have studied of some neutrosophic algebraic structures. In this paper, we applied the theory of neutrosophic sets to BE-algebras and introduced the notions of (implicative) neutrosophic filters and many related properties are investigated.

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