RATIO mathematica

Journal of Foundations and Applications of Mathematics / Rivista di Fondamenti e Applicazioni della Matematica

23/12

ISSN: 1592-7415

Number 23 – 2012

RATIO MATHEMATICA

Journal of Foundations and Applications of Mathematics

Via Lucania, 1 – 64026 ROSETO DEGLI ABRUZZI (Italy)

Autorizzazione n. 9/90 del 10-07-1990 del Tribunale di Pescara

ISSN 1592-7415
Ratio Mathematica
[Testo stampato]

ISSN 2282-8214
Ratio Mathematica
[Online]

RATIO MATHEMATICA 23 (2012), 3–20 ISSN: 1592-7415

General ω-hyperstructures and certain applications of those

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Abstract

The aim of this paper is to investigate general hyperstructures construction of which is based on ideas of A. D. Nezhad and R. S. Hashemi. Concept of general hyperstructures considered by the above mentioned authors is generalized on the case of hyperstructures with hyperoperations of countable arity. Specifications of treated concepts to examples from various fields of the mathematical sturctures theory are also included.

Key words: Action of a hyperstructure on a set, general n-hyperstructure, transformation hypergroup, Fredholm integral operator, ordinary and partial differential operator

MSC2010: Primary: 20N20; Secondary: 37L99, 68Q70

1 Preliminaries

A resent book [7] contains a wealth of applications of hyperstructure theory developed since 1934, see [20]. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatoric, codes, artificial intelligence, and probability. In section 2 and 3 we give some basic definition and then, we consider three types of actions. In section 4 we present some applications. In section 5 there are described some applications of formerly investigated hyperstructures and corrected certain mistake from [14]. There is a connection between automata and dynamical systems realized by infinite automata without outputs and discrete dynamical systems.They are also shortly called as action of semigroups or groups on a given phase (or state) set. In connection with non-deterministic automata or with multifunctions (relations) on algebraic structures and topological spaces seems to be natural to investigate actions of multistructures on sets of various objects. Various generalizations of the above mentioned classical concepts are possible. Some motivating factors are coming from the general system theory [11, 23]; one illustrating example below is just based on the concept of a general time system. In [8, 9] there are investigated various binary relation and hyperstructures.

Notice, the binary relation on a binary hyperstructure, e.g. on a semihypergroup, are quite natural created by inner translations.

We use [1, 3, 7, 10, 21, 22, 26] for terminology and notations which are not define here. We suppose that the reader is familiar with some well-known notation such as presentation in hyperstructure theory. The following facts are some definitions and propositions in the theory of hyperstructure which we need for formulation of our results and in the proofs of the main results. For x from an ordered set H we denote by $[x)_\leq = \{y \in H \mid x \leq y\}$ the *upper end* generated by x.

The following lemma is called Ends Lemma.

Lemma 1.1. [3, 6, 24, 25] *Let $(H, \circ, \leq)$ be an ordered semigroup. Let $a \star b = [a \circ b)_\leq$ for any $a, b \in H$. The following conditions are equivalent:*

1) For any pair $a, b \in H$ there exists a pair $c, d \in H$ such that $b \circ c \leq a$, $c \circ d \leq a$.

2) A hypergroupoid $(H, \star)$ associated with $(H, \circ, \leq)$ satisfies the associativity law and the reproduction axioms, i.e., $(H, \star)$ is a hypergroup.

Dually we can define the Beginnings Lemma:

Lemma 1.2. [6] *Let $(H, \circ, \leq)$ be an ordered semigroup. Let $a \star b = (a \circ b]_\leq$ for any $a, b \in H$. The following conditions are equivalent:*

1) For any pair $a, b \in H$ there exists a pair $c, d \in H$ such that $b \circ c \geq a$, $c \circ d \geq a$.

2) A hypergroupoid $(H, \star)$ associated with $(H, \circ, \leq)$ satisfies the associativity law and the reproduction axioms, i.e., $(H, \star)$ is a hypergroup.

Quasi-order hypergroups have been introduced and studied by J. Chvalina. The following definition can be found e.g. in [7, 12, 24].

Definition 1.1. *A hypergroup* $(H, \star)$ *such that the following condition are satisfied:*

1) $a \in a^2 = a^3$ *for any* $a \in H$,

2) $a \star b = a^2 \cup b^2$ *for any pair* $a, b \in H$ *is called* a quasi-order hypergroup*. If moreover the unique square root condition:*

3) $a, b \in H$, $a^2 = b^2$ *implies* $a = b$

is satisfied then $(H, \star)$ *is called* an order hypergroup*.*

Definition 1.2. [19] *A hypergroup* $(G, \star)$ *is called* a transposition hypergroup *if it satisfies the transposition axiom: For all* $a, b, c, d \in G$ *the relation* $b \setminus a \cap c/d \neq \emptyset$ *implies* $a \star d \cap b \star c \neq \emptyset$*.*
The sets $b \setminus a = \{x \in G | a \in b \star x\}, c/d = \{x \in G | c \in x \star d\}$ *are called* left and right extensions*, respectively.*

Definition 1.3. [15, 16, 17] *Let* X *be a set,* $(G, \bullet)$ *be a (semi)hypergroup and* $\pi\colon X \times G \to X$ *a mapping such that*

$$\pi(\pi(x,t), s) \in \pi(x, t \bullet s), \ \text{where } \pi(x, t \bullet s) = \{\pi(x,u); u \in t \bullet s)\} \qquad (1)$$

for each $x \in X$, $s, t \in G$*. Then* (X, G, π) *is called a* discrete transformation (semi)hypergroup *or an* action of the (semi)hypergroup G *on the phase set* X*. The mapping* π *is usually said to be simply* an action*.*

Remark 1.1. [16] *The condition* (1) *used above is called* Generalized Mixed Associativity Condition*, shortly GMAC.*

2 General hyperstuctures and ω-hyperstructures

Throughout this paper, the symbol X, Y will denote two non-empty sets, where $P^*(X \cup Y)$ denotes the set of all non-empty subsets of $X \cup Y$.

A general hyperstructure is formed by two non-empty sets X, Y together with a hyperoperation

$$* : X \times Y \longrightarrow P^*(X \cup Y), \quad (x, y) \mapsto x * y \subseteq (X \cup Y) \smallsetminus \emptyset.$$

Remark 2.1. *A general hyperoperation* $*: X \times Y \longrightarrow P^*(X \cup Y)$ *yields a map of powersets determined by this hyperoperation. Thus the map* $\otimes : P^*(X) \times P^*(Y) \longrightarrow P^*(X \cup Y)$ *is defined by* $A \otimes B = \bigcup\limits_{a \in A, b \in B} a * b$.

Conversely an general hyperoperation on $P^*(X) \times P^*(Y)$ yields a general hyperoperation on $X \times Y$, defined by $x * y = \{x\} \otimes \{y\}$.

In the above definition if $A \subseteq X,\ B \subseteq Y,\ x \in X,\ y \in Y$, then we define,

$$A * y = A * \{y\} = \bigcup_{a \in A} a * y, \quad x * B = \{x\} * B = \bigcup_{b \in B} x * b,$$

$$A \otimes B = \bigcup_{a \in A, b \in B} a * b.$$

Remark 2.2. *If* $X = Y = H$*, then we obtain the classical hyperstructure theory. The concept of general hyperstructure with a hyperoperation which is a mapping*

$$*: X \times Y \longrightarrow P^*(X \cup Y)$$

mentioned above (used by A. D. Nezhad and R. S. Hashemi) allows straightforward generalization onto case of "hyperoperation of an arbitrary finite arity" *or* direct generalization to hyperoperation of countable arity.

Let us define the general ω-hyperstructure:

Definition 2.1. *Let* ω *be the smallest infinite countable ordinal. As usually, we consider the smallest infinite ordinal* ω *as the set of all smaller ordinals, i.e. as the domain of all finite ordinals (non-negative integers). Let* $\{X_k; k \in \omega\}$ *be a system of non-empty sets. By an* general ω-hyperstructure *we mean the pair* $(\{X_k; k \in \omega\}, *_\omega)$*, where* $*_\omega \colon \prod\limits_{k \in \omega} X_k \to \mathcal{P}^*\Big(\bigcup\limits_{k \in \omega} X_k\Big)$ *is a mapping assigning to any sequence* $\{x_k\}_{k \in \omega} \in \prod\limits_{k \in \omega} X_k$ *a non-empty subset* $*_\omega(\{x_k\}_{k \in \omega}) \subset \bigcup\limits_{k \in \omega} X_k$.

Similarly as above, with this hyperoperation there is associated a mapping of power sets

$$\otimes_\omega \colon \prod_{k \in \omega} \mathcal{P}^*(X_k) \to \mathcal{P}^*\Big(\bigcup_{k \in \omega} X_k\Big)$$

defined by

$$\otimes_\omega\big(\{A_k\}_{k \in \omega}\big) = \bigcup \Big\{ *_\omega\big(\{x_k\}_{k \in \omega}\big); \big(\{x_k\}_{k \in \omega}\big) \in \prod_{k \in \omega} A_k \Big\}.$$

Let us formulate the special case:

Definition 2.2. [13] *Let $n \in \omega$ be an arbitrary positive integer, $n \geq 1$. Let $\{X_k; k = 1, \ldots, n\}$ be a system of non-empty sets. By a* general n-hyperstructure *we mean the pair*

$$(\{X_k; k = 1, \ldots, n\}, *_n),$$

where $*_n\colon \prod_{k=1}^{n} X_k \to \mathcal{P}^*\Big(\bigcup_{k=1}^{n} X_k\Big)$ is a mapping assigning to any n-tuple $(x_1, \ldots, x_n) \in \prod_{k=1}^{n} X_k$ a non-empty subset $*_n(x_1, \ldots, x_n) \subset \bigcup_{k=1}^{n} X_k$.

Similarly as above, with this hyperoperation there is associated a mapping of power sets

$$\otimes_n\colon \prod_{k=1}^{n} \mathcal{P}^*(X_k) \to \mathcal{P}^*\Big(\bigcup_{k=1}^{n} X_k\Big)$$

defined by

$$\otimes_n(A_1, \ldots, A_n) = \bigcup\Big\{*_n(x_1, \ldots, x_n); (x_1, \ldots, x_n) \in \prod_{k=1}^{n} A_k\Big\}.$$

This construction is based on an idea of Nezhad and Hashemi for $n = 2$. Hyperstructures with n-ary hyperoperations are investigated among others in [2, 27].

Definition 2.3. *Let $\mathbb{G}_1(\omega) = \big(\{X_k; k \in \omega\}, *_\omega\big)$, $\mathbb{G}_2(\omega) = \big(\{Y_k; k \in \omega\}, \bullet_\omega\big)$, be a pair of general ω-hyperstructures. By a* good homomorphism *$H\colon \mathbb{G}_1(\omega) \to \mathbb{G}_2(\omega)$ we mean any system of mappings $H = \{h_k\colon X_k \to Y_k\}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \prod X_k & \xrightarrow{*_\omega} & \mathcal{P}^*\big(\bigcup_{k\in\omega} X_k\big) \\ {\scriptstyle \prod_{k\in\omega} h_k}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi^\sharp} \\ \prod Y_k & \xrightarrow{\bullet_\omega} & \mathcal{P}^*\big(\bigcup_{k\in\omega} Y_k\big) \end{array} \tag{D1}$$

Here $\prod_{k\in\omega} h_k(\{x_k\}_{k\in\omega}) = \{h_k(x_k)\}_{k\in\omega}$ for any sequence $\{x_k\}_{k\in\omega}$ and $\varphi^\sharp\colon \mathcal{P}^*\big(\bigcup_{k\in\omega} X_k\big) \to \mathcal{P}^*\big(\bigcup_{k\in\omega} Y_k\big)$ is the lifting of a mapping $\varphi\colon \bigcup_{k\in\omega} X_k \to \bigcup_{k\in\omega} Y_k$ defined by the mathematical induction. For $x \in X_0$ we put $\varphi(x) = h_0(x)$. Suppose $\varphi\colon \bigcup_{j=0}^{k} X_j \to \bigcup_{j=0}^{k} Y_j$ is well-defined. Then for any $x \in X_{k+1} \setminus \bigcup_{j=0}^{k} X_j$

we put $\varphi(x) = h_{k+1}(x)$. Then using mathematical induction the mapping $\varphi\colon \bigcup\limits_{k\in\omega} X_k \to \bigcup\limits_{k\in\omega} Y_k$ is well-defined.

If all mappings $h_k \in H$ are bijections (or isomorphism if all H_k, Y_k are endowed with some structures) we call the H the isomorphism of ω- hyperstructures $G_1(\omega)$, $G_2(\omega)$.

3 General ω-hyperstructures created by ordered sets and by differential operators

As a certain generalization of the general n-hyperstructure from [13], Example 3.2, we will construct the following structure:

Example 3.1. *Consider a countable system of pairwise disjoint ordered sets $(X_k, \leq_k)$, $k \in \omega$ and for $x \in X_k$ let us denote $[x)_k = \{y \in X_k; x \leq_k y\}$, i.e. $[x)_k$ is the principal end generated by the element x within the ordered set $(X_k, \leq_k)$. Further, put*

$$*_\omega\big(\{x_k\}_{k\in\omega}\big) = \bigcup_{k\in\omega} [x_k)_k$$

for any sequence $*_\omega\big(\{x_k\}_{k\in\omega}\big) \in \prod\limits_{k\omega} X_k$. Then $*_\omega\big(\{x_k\}_{k\in\omega}\big) \subseteq \bigcup\limits_{k\in\omega} X_k$, thus

$$\mathbb{G}(\omega) = \big(\{X_k; k \in \omega\}, *_\omega\big)$$

is a general ω-hyperstructure in the sense of the above definition.
If $\mathbb{H}(\omega) = \big(\{Y_k; k \in \omega\}, \bullet_\omega\big)$ is a general ω-hyperstructure such that $(Y_k, \preceq_k)$, $k \in \omega$ are pairwise disjoint ordered sets and

$$\bullet_\omega(\{y_k\}_{k\in\omega}) = \bigcup_{k\in\omega} [y_k)_k \subseteq \bigcup_{k\in\omega} Y_k$$

for any sequence $\{y_k\}_{k\in\omega} \in \prod\limits_{k\in\omega} Y_k$ we consider a system $h_k\colon (X_k, \leq_k) \to (Y_k, \preceq_k)$, $k \in \omega$, of strongly isotone mappings, i.e. for any $x \in X_k$ there holds $h_k\big([x_k)_k\big) = \big[h_k(x_k)\big)_k$. Then denoting $H = \{h_k\colon X_k \to Y_k; k \in \omega\}$ we obtain that H is a good homomorphism of the general ω-hyperstructure; $\mathbb{G}(\omega)$ into the general ω-hyperstructure $\mathbb{H}(\omega)$.

Indeed, consider an arbitrary sequence $\{x_k\}_{k\in\omega} \in \prod\limits_{k\in\omega} X_k$. As above denote by $\varphi\colon \mathcal{P}^*\big(\bigcup\limits_{k\in\omega} X_k\big) \to \mathcal{P}^*\big(\bigcup\limits_{k\in\omega} Y_k\big)$ the lifting of the mapping $\varphi\colon \bigcup\limits_{k\in\omega} X_k \to \bigcup\limits_{k\in\omega} Y_k$

induced by the system $\{h_k\colon X_k \to Y_k; k \in \omega\}$*—here in such a way that* $\varphi|X_k = h_k$*. Then for any sequence* $\{x_k\}_{k\in\omega} \in \prod_{k\in\omega} X_k$ *we have*

$$\begin{aligned}\varphi\big(*_\omega(\{x_k\}_{k\in\omega})\big) &= \varphi\Big(\bigcup_{k\in\omega}[x_k)_k\Big) = \bigcup_{k\in\omega}\varphi\big([x_k)_k\big) = \bigcup_{k\in\omega} h_k\big([x_k)_k\big)\\ &= \bigcup_{k\in\omega}\big[h_k(x_k)_k\big) = \bullet_\omega\big(\{h_k(x_k)\}_{k\in\omega}\big)\\ &= \bullet_\omega\Big(\prod_{k\in\omega} h_k\big(\{x_k\}_{k\in\omega}\big)\Big),\end{aligned}$$

i.e. $\varphi \circ *_\omega = \bullet_\omega \circ \prod_{k\in\omega} h_k$*, thus the diagram*

$$\begin{array}{ccc} \prod X_k & \xrightarrow{*_\omega} & \mathcal{P}^*\Big(\bigcup_{k\in\omega} X_k\Big) \\ {\scriptstyle\prod_{k\in\omega} h_k}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^\sharp} \\ \prod Y_k & \xrightarrow{\bullet_\omega} & \mathcal{P}^*\Big(\bigcup_{k\in\omega} Y_k\Big) \end{array} \tag{D2}$$

is commutative.

From the above example there follows immediately the following assertion.

Proposition 3.1. *Let* $(X_k, \leq_k)$*,* $(Y_k, \preceq_k)$*,* $k \in \omega$*, be two countable collections of pairwise disjoint ordered sets and* $\mathbb{G}(\omega)$*,* $\mathbb{H}(\omega)$ *be the corresponding* ω*-general hyperstructures. Suppose* $(X_k, \leq_k) \cong (Y_k, \preceq_k)$ *for each* $k \in \omega$ *and* $h_k\colon (X_k, \leq_k) \to (Y_k, \preceq_k)$ *are corresponding order-isomorphisms. Then we have* $\mathbb{G}(\omega) \cong \mathbb{H}(\omega)$*.*

Example 3.2. *Let* $J \subset \mathbb{R}$ *be an open interval,* $\mathbb{C}^n(J)$ *be the ring (with respect to usual addition and multiplication of functions) of all real functions* $f\colon J \to \mathbb{R}$ *with continuous derivatives up to the order* $n \geq 0$ *including. Denote*

$$L(p_0, p_1, \dots, p_{n-1})\colon \mathbb{C}^n(J) \to \mathbb{C}^n(J)$$

the linear differential operator defined by

$$L(p_0, p_1, \dots, p_{n-1})(y) = \frac{\mathrm{d}^n y(x)}{\mathrm{d}x^n} + \sum_{s=0}^{n-1} p_s(x)\frac{\mathrm{d}^s y(x)}{\mathrm{d}x^s}$$

where $y \in \mathbb{C}^n(J)$ *and* $p_s \in \mathbb{C}^n(J)$*,* $s = 0, 1, \dots, n-1$*. In accordance with* [4, 5] *we put*

$$\mathbb{LA}_n(J) = \{L(p_0, \dots, p_{n-1}); p_k \in \mathbb{C}^n(J)\}.$$

Instead of $L(p_{1,0}, p_{1,1}, \dots, p_{1,n-1})$ *we write* $L(\vec{p}_1)$. *We put* $L(\vec{p}_1) \leq L(\vec{p}_2)$ *whenever*
$L(\vec{p}_j) = L(p_{j,0}, \dots, p_{j,n-1})$, $j = 1, 2$, $p_{1,s}(x) \leq p_{2,s}(x)$, $s = 0, 1, \dots, n-1$, $x \in J$ *and* $p_{1,0}(x) \equiv p_{2,0}(x)$. *Defining*

$$*_n\big(L(\vec{p}_1), L(\vec{p}_2), \dots, L(\vec{p}_n)\big) = \bigcup_{k=1}^{n} \{L(\vec{p}) \in \mathbb{LA}_k(J); L(\vec{p}_k) \leq L(\vec{p})\}$$

for any n-tuple $\big(L(\vec{p}_1), L(\vec{p}_2), \dots, L(\vec{p}_n)\big) \in \prod_{k=1}^{n} \mathbb{LA}_k(J)$ *we obtain that* $\mathbb{L}(n) = \big(\{\mathbb{LA}_k(J); k = 1, 2, \dots, n\}, *_n\big)$ *is a general n-hyperstructure.*

Of course, $\mathbb{LA}_1(J)$ *is the set of all first-order linear differential operators of the form* $L(p_0)(y) = y'(x) + p_0(x)y$, *where* $p_0 \in \mathbb{C}(J)$ *and* $y \in \mathbb{C}^1(J)$. *Evidently* $\mathbb{LA}_j(J) \cap \mathbb{LA}_k(J) = \emptyset$ *whenever* $j \neq k$.

It is to be noted that if $k, m \in \{1, 2, \dots, n\}$ are fixed different integers then setting $X = \mathbb{LA}_k(J)$, $Y = \mathbb{LA}_m(J)$ we obtain from the above construction an example of a general hyperstructure in sense of Nezhad and Hashemi. If, moreover $X = Y = \mathbb{LA}_n(J)$ then the resulting general hyperstructure is an order hypergroup of linear differential n-order operators in the sense of [3], chap. IV, or [4, 5].

Theorem 3.1. *Let* ω *be the smallest infinite countable ordinal,* $J \subseteq \mathbb{R}$ *be an open interval. If*

$$\mathbb{L}(J; \omega) = \Big(\prod_{k\in\omega} \mathbb{LA}_k(J), *_\omega, \mathcal{P}^*\big(\bigcup_{k\in\omega} \mathbb{LA}_k(J)\big)\Big)$$

is the general ω*-hyperstructure of ordinary linear differential operators and*

$$\mathbb{S}(J; \omega) = \Big(\prod_{k\in\omega} \mathbb{VA}_k(J), \bullet_\omega, \mathcal{P}^*\big(\bigcup_{k\in\omega} \mathbb{VA}_k(J)\big)\Big)$$

is the general ω*-hyperstructure of solution spaces of linear ordinary homogeneous differential equations associated with* $\mathbb{L}(J; \omega)$. *Then we have*

$$\mathbb{L}(J; \omega) \cong \mathbb{S}(J; \omega),$$

i.e. in the commutative diagram

$$\begin{array}{ccc}
\prod_{k\in\omega} \mathbb{LA}_k(J) & \xrightarrow{*_\omega} & \mathcal{P}^*\Big(\bigcup_{k\in\omega} \mathbb{LA}_k(J)\Big) \\
{\scriptstyle \prod_{k\in\omega} \Phi_k}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi^\sharp} \\
\prod_{k\in\omega} \mathbb{VA}_k(J) & \xrightarrow{\bullet_\omega} & \mathcal{P}^*\Big(\bigcup_{k\in\omega} \mathbb{VA}_k(J)\Big)
\end{array} \tag{D3}$$

arrows $\prod_{k\in\omega} \Phi_k$, $\varphi^\sharp$ *are bijections.*

Proof. By [4, 5] we have $\Phi_k\colon \mathbb{LA}_k(J) \to \mathbb{VA}_k(J)$ is a group-isomorphism for any $k \in \omega$ thus $\prod\limits_{k\in\omega} \Phi_k\colon \prod\limits_{k\in\omega} \mathbb{LA}_k(J) \to \prod\limits_{k\in\omega} \mathbb{VA}_k(J)$ is a bijection. Since $\{\mathbb{LA}_k(J); k \in \omega\}, \{\mathbb{VA}_k(J); k \in \omega\}$ are pairwise disjoint families we have that the mapping $\varphi\colon \bigcup\limits_{k\in\omega} \mathbb{LA}_k(J) \to \bigcup\limits_{k\in\omega} \mathbb{VA}_k(J)$ such that $\varphi|\mathbb{LA}_k(J) = \Phi_k$, $k \in \omega$ is a well-defined bijection hence the bijection $\mathcal{P}^*\Big(\bigcup\limits_{k\in\omega} \mathbb{LA}_k(J)\Big) \to \mathcal{P}^*\Big(\bigcup\limits_{k\in\omega} \mathbb{VA}_k(J)\Big)$ is also well-defined.

Now, for an arbitrary sequence $\{L_n\}_{n\in\omega} \in \prod\limits_{k\in\omega} \mathbb{LA}_k(J)$ we obtain that

$$\bullet_\omega\Big(\big(\prod_{k\in\omega}\Phi_k\big)\{L_n\}_{n\in\omega}\Big) = \bullet_\omega\big\{\Phi_n(L_n)\big\}_{n\in\omega} = \bullet_\omega\{V_n\}_{n\in\omega} =$$
$$= \varphi\Big(*_\omega\big\{\Phi_n^{-1}(L_n)\big\}_{n\in\omega}\Big) = \varphi^\sharp\big(*_\omega\{L_n\}_{n\in\omega}\big),$$

since the hyperoperation "$\bullet_\omega$" is associated with the hyperoperation "$*_\omega$" . Therefore the diagram D3 in the Theorem 3.1 is commutative. $\square$

Let $\big\{(S_k, \cdot, \leq_k); k \in \omega\big\}$ be a system of quasi-ordered semigroups. Define a mapping $\odot_\omega\colon \prod\limits_{k\in\omega} S_k \to \mathcal{P}^*\Big(\bigcup\limits_{k\in\omega} S_k\Big)$ by the rule

$$\odot_\omega(x_1, \ldots, x_n) = \bigcup_{k\in\omega} [x_k^2)_{\leq_k}$$

for any sequence $\{x_k\}_{k\in omega} \in \prod\limits_{k\in\omega} S_k$. Then the general ω-hyperstructure is called the *general ω-hyperstructure determined by the Ends Lemma* or shortly *EL-determined general ω-hyperstructure.*

Corollary of Theorem 3.1

Let ω be the smallest infinite countable ordinal, $\omega \neq 0$, $J \subseteq \mathbb{R}$ be an open interval. Let $\mathbb{L}^{EL}(J; n) = \Big(\prod\limits_{k\in\omega} \mathbb{LA}_k(J), *_n, \mathcal{P}^*\big(\bigcup\limits_{k\in\omega} \mathbb{LA}_k(J)\big)\Big)$ be the EL-determined general ω-hyperstructure of all linear ordinary differential operators of all orders $k \in \omega$.

Let $\mathbb{S}^{EL}(J; n) = \Big(\prod\limits_{k\in\omega} \mathbb{VA}_k(J), *_n, \mathcal{P}^*\big(\bigcup\limits_{k\in\omega} \mathbb{VA}_k(J)\big)\Big)$ be the EL-determined general ω-hyperstructure of solutions of homogeneous linear ordinary differential equations $Ly = 0$, $L \in \bigcup\limits_{k\in\omega} \mathbb{LA}_k(J)$. Then $\mathbb{L}^{EL}(J; n) \cong \mathbb{S}^{EL}(J; n)$.

In the above construction we can use a finite sequence of positive integers $\{m_1, m_2, \ldots, m_n\}$ and then define the ω-hyperoperation $\odot_\omega\big(\{x_k\}_{k\in\omega}\big) = \bigcup\limits_{k\in\omega} [x_k^{m_k})_{\leq_k}$ for any ω sequence $\{x_k\}_{k\in\omega} \in \prod\limits_{k\in\omega} S_k$.

4 General R-hyperstuctures (or L-hyperstuctures)

In the paper [13], due to ideas of Nezhad and Hashemi, there are considered General R-hyperstuctures (or L-hyperstuctures) in the following sense. A general Right hyperstructure (or Left-hyperstructure) consist of two non-empty sets X, Y together with a hyperoperation:

$$\begin{array}{rcl} *_R : X \times Y \longrightarrow P^*(X) & \text{or} & *_L \quad : X \times Y \longrightarrow P^*(Y) \\ (x,y) \mapsto x *_R y \subseteq X, & & (x,y) \mapsto x *_L y \subseteq Y. \end{array}$$

To be more precise a *general Right hyperstructure (or Left hyperstructure)* is the quadruple $(X, Y, \mathcal{P}^*(X), *_R)$ or $(X, Y, \mathcal{P}^*(X), *_L)$, shortly *general R-hyperstructure* or *general L-hyperstructure.*

The set of points $Y_R x = \{x * y : y \in Y\}$ that can be reached from a given point $x \in X$ by the R-hyperoperation of two non-empty sets X, Y, is called the R-hyperorbit of x.

If $Y_R x = X$ for all $x \in X$. Then the set Y is said to be R-hypertransitive on Y. If $Y_R x = \{x\}$. Then x is called a R-hyperfixed point to the R-hyperoperation. The set $\{y \in Y : x *_R y = \{x\}\}$, is called R-hyperisotopy set at x.

Example 4.1. *Let $X \neq \emptyset$ be an arbitrary set, $f\colon X \to X$ be a mapping, i.e. the pair (X, f) is a monounary algebra. Put $Y = \mathbb{N}$ (the set of all positive integers) and define $*_R^f\colon X \times Y \to \mathcal{P}^*(X)$ by the rule $x *_R^f n = \{f^k(x); k \in \mathbb{N}, n \leq k\}$. Then the quadruple $(X, Y, \mathcal{P}^*(X), *_R^f)$ is a general Right hyperstructure, i.e. R-hyperstructure. (Here, f^k is the k-th iteration of f).*

Example 4.2. *Let T be a linearly ordered set (i.e. a chain) with the least element. Then T is called a* time scale *or* time axis. *Suppose $A \neq \emptyset \neq B$ are arbitrary sets and S is a binary relation between sets of mappings (impulses) A^T, B^T, i.e. $S \subset A^T \times B^T$. Then the triad (A^T, B^T, S) is called a* general time system *with input space A^T, the output space B^T and with input-output relation (or the transition relation) S—cf.[23]. Now, denote $X = A^T$, $Y = B^T$ and define $*_L^S\colon X \times Y \to \mathcal{P}^*(Y)$ by $x *_L^S y = S(x) = \{u \in Y; x\, S\, u\}$ for any pair of time-impulses $x\colon T \to A, y\colon T \to B$. Then we obtain the quadruple $(X, Y, \mathcal{P}^*(X), *_L^S)$ which is a general Left hyperstructure, i.e. a general L-hyperstructure.*

In the above mentioned classical monography [23] as a general system is considered a relation $S \subseteq \prod_{i \in I} V_i$ on non-void abstract sets. However—in detail—the index set I is decomposed into two subsets I_x, I_y and by *input-output* system is ment a relation $S \subseteq X \times Y$, where $X = \prod\limits_{i \in I_x} V_i$ is called the input object and $Y = \prod\limits_{i \in I_y} V_i$ is termed as the output object. (cf. [23], Definition 1.2).

In the connection with general ω-hyperstructures we introduce input-output systems with inner evaluated input and output objects. Denote $K = \{1, 2, \dots, n\}$. By an IEO-*general system*, i.e. an *Input-Output general system with Inner Evaluated Objects* we mean a qudruple $(\prod\limits_{k \in K} X_k, \prod\limits_{k \in K} Y_k, \mathrm{R}, F_{\mathrm{R}})$, where X_k, Y_k are non-empty sets, $\mathrm{R} \subseteq \prod\limits_{k \in K} X_k \times \prod\limits_{k \in K} Y_k$ is a domain full binary relation, i.e. $Dom\, \mathrm{R} = \prod\limits_{k \in K} X_k = \{x \in \prod\limits_{k \in K} X_k; \exists y \in \prod\limits_{k \in K} : x \,\mathrm{R}\, y\}$, $*_x \colon \prod\limits_{k \in K} X_k \to \mathcal{P}(\bigcup\limits_{k \in K} X_k)$, $*_y \colon \prod\limits_{k \in K} Y_k \to \mathcal{P}(\bigcup\limits_{k \in K} Y_k)$ are mappings (i.e. n-ary multioperations)—thus $(\prod\limits_{k \in K} X_k, *_x)$,$(\prod\limits_{k \in K} Y_k, *_y)$ are general n-ary hyperstructures and $F_{\mathrm{R}} \colon \mathcal{P}(\bigcup\limits_{k \in K} X_k) \to \mathcal{P}(\bigcup\limits_{k \in K} Y_k)$ is a mapping defined in this way:

For a non-empty set $M \subset \bigcup\limits_{k \in K} X_k$, i.e. $M \in \mathcal{P}(\bigcup\limits_{k \in K} X_k)$ we denote $M_X^- = \big\{[x_1, \dots, (x_1, \dots, x_n)_n] \in \prod\limits_{k \in K} X_k; *_X(x_1, \dots, (x_1, \dots, x_n)_n) \cap M \neq \emptyset\big\}$. Then we put $F_{\mathrm{R}}(M) = *_y\big(\mathrm{R}(M_X^-\big) \subset \bigcup\limits_{k \in K} Y_k$. If $M_X^- = \emptyset$ we define $F_{\mathrm{R}}(M) = \emptyset$. (This definition includes the case $F_{\mathrm{R}}(\emptyset) = \emptyset$). Hence the diagram

$$
\begin{array}{ccc}
\prod\limits_{k \in K} X_k & \xrightarrow{\;*_X\;} & \mathcal{P}\big(\bigcup\limits_{k \in K} X_k\big) \\
{\scriptstyle \mathrm{R}}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{\mathrm{R}}} \\
\prod\limits_{k \in K} Y_k & \xrightarrow{\;*_Y\;} & \mathcal{P}\big(\bigcup\limits_{k \in k} Y_k\big)
\end{array}
\tag{D4}
$$

is commutative.

Now, in the case when $X_k = A$, $Y_k = B$ for all $k \in K$, then denoting the index set K by T (the time set-continuous, which means a linearly ordered continuum with the minimal element, or discrete $\mathbb{N}_0$) we obtain $\prod\limits_{k \in K} X_k = A^T$, $\prod\limits_{k \in K} Y_k = B^T$ and $\bigcup\limits_{k \in K} X_k = A$ $\bigcup\limits_{k \in K} Y_k = B$. Thus the above IEO-general system turns to the above mentioned general time system (here with an inner evaluation) $(A^T, B^T, R, *_A, *_B)$, where $F_R \circ *_A = *_B \circ R$.

By a homomorphism of an IEO-general system $(\prod_{k\in K} X_k, \prod_{k\in K} Y_k, \mathrm{R}, *_X, *_Y)$ into another IEO-general system $(\prod_{k\in K} U_k, \prod_{k\in K} V_k, \mathrm{S}, *_U, *_V)$ we mean a quadruple of mappings $H = [\Phi_{XU}, \Phi_{YV}, \Phi_{PXU}, \Phi_{PYV}]$, $\Phi_{XU}\colon \prod_K X_k \to \prod_K U_k$; $\Phi_{YV}\colon \prod_K Y_k \to \prod_K V_k$; $\Phi_{PXU}\colon \mathcal{P}(\bigcup_K X_k \to \mathcal{P}(\bigcup_K U_k)$; $\Phi_{PYV}\colon \mathcal{P}(\bigcup_K Y_k) \to \mathcal{P}(\bigcup_K V_k)$ such that the diagram

$$
\begin{tikzcd}
\prod\limits_{k\in K} U_k \arrow[rrr, "*_U"] \arrow[ddd, "S"'] & & & \mathcal{P}\Big(\bigcup\limits_{k\in K} U_k\Big) \arrow[ddd, "F_S"] \\
& \prod\limits_{k\in K} X_k \arrow[ul, "\Phi_{XU}"'] \arrow[r, "*_X"] \arrow[d, "R"'] & \mathcal{P}\Big(\bigcup\limits_{k\in K} X_k\Big) \arrow[ur, "\Phi_{PXU}"] \arrow[d, "F_R"] & \\
& \prod\limits_{k\in K} Y_k \arrow[dl, "\Phi_{YU}"] \arrow[r, "*_Y"] & \mathcal{P}\Big(\bigcup\limits_{k\in K} Y_k\Big) \arrow[dr, "\Phi_{PYV}"'] & \\
\prod\limits_{k\in K} V_k \arrow[rrr, "*_V"] & & & \mathcal{P}\Big(\bigcup\limits_{k\in K} V_k\Big)
\end{tikzcd}
\tag{D5}
$$

is commutative. In concrete cases of modelling special systems, the used objects and their mappings take a concrete interpretation.

4.1 L-hyperoperation (or R-hyperoperation) of a hyperstucture on a non-empty set

In this paragraph, we recall two definitions of a L-hyperoperation (or R-hyperoperation) of a hyperstucture on a non-empty set. Let us make our point clear with some examples.

Definition 4.1. [13] *Let $(G, \star)$ be a hyperstructure and X be a non-empty set.*

A generalized L-hyperaction *of G on X is a L-hyperoperation $\psi : G \times X \longrightarrow \mathcal{P}^*(X)$ such that the following axioms are satisfied:*

1) *For all $g, h \in G$ and $x \in X$, $\psi(g \star h, x) \subseteq \psi(g, \psi(h, x))$,*

2) *For all $g \in G$, $\psi(g, X) = X$.*

For any $g \in G$ and $A \subseteq X$, $\psi(g, A) = \bigcup\limits_{x \in A} \psi(g, x)$, also for any $x \in X$ and $B \subseteq G$, $\psi(B, x) = \bigcup\limits_{b \in B} \psi(b, x)$. If in the axiom 1) of definition the equality holds, the generalized R-hyperaction is called strong.

As application of the above concepts we mention the classical *interval binary hyperoperation on a linearly ordered group.* See [18]. In detail if $(G, \cdot, \leq)$ is a linearly ordered group then we define a binary hyperoperation $*\colon G \times G \to \mathcal{P}^*(G)$ by

$$\begin{aligned} a * b &= \big[\min\{a, b\}\big)_{\leq} \cap \big(\max\{a, b\}\big]_{\leq} = \big[\min\{a, b\}, \max\{a, b\}\big]_{\leq} \\ &= \big\{x \in G; \min\{a, b\} \leq x \leq \max\{a, b\}\big\} \end{aligned}$$

(which is a closed interval) where $\min\{a, b\}$, $\max\{a, b\}$ is the least element, the greatest element of the set $\{a, b\}$, respectively. It is easy to verify that the obtained hypergroupoid $(G, *)$ is an extensive commutative hypergroup. This hypergroup we obtain even in the case if we restrict ourselves onto the set G_+ of all positive elements of the linearly ordered group $(G, \cdot, \leq)$, (cf. the proof of Proposition 4.1).

Proposition 4.1. *Let $(G, \cdot, \leq)$ be a linearly ordered group, G_+ be its subset of all positive elements (i.e. the positive cone) endowed with the interval binary hyperoperation "$*_L$". Define a mapping*

$$\psi_G\colon G_+ \times G \to \mathcal{P}^*(G)$$

by

$$\psi_G(a, b) = (a + b]_{\leq} = \{x \in G; x \leq a + b\}$$

for all pairs $(a, b) \in G_+ \times G$. Then the quadruple $\big(G_+, G, \mathcal{P}^*(G), \psi_G\big)$ is the generalized L-hyperoperation of the commutative extensive hypergroup $(G, *_L)$ on the group $(G, +, \leq)$.

Proof. For the proof see [13]. $\square$

5 Homomorphism of transformation semihypergroups

Definition 5.1. *Let (X, G, ψ), (Y, H, η) be two generalized transformation semihypergroups (GTS). A pair of mappings $\Phi = [\mu, \varphi]$ such that $\mu\colon G \to H$ is a homomorphism of semihypergroups and $\varphi\colon X \to Y$ is a mapping, is said to be a homomorphism of GTS (X, G, ψ) into GTS (Y, H, η) if for any pair $[g, x] \in G \times X$ the equality*

$$\eta\big(\mu(g), \varphi(x)\big) = \varphi\big(\psi(g, x)\big)$$

is satisfied, i.e. the diagram, where $\varphi^*\colon \mathcal{P}^*(X) \to \mathcal{P}^*(Y)$ is the corresponding liftation of the mapping $\varphi\colon X \to Y$,

$$\begin{array}{ccc} G \times X & \xrightarrow{\psi} & \mathcal{P}^*(X) \\ {\scriptstyle \mu\times\varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi^*} \\ H \times Y & \xrightarrow{\omega} & \mathcal{P}^*(Y) \end{array} \tag{D6}$$

commutes.

Example 5.1. *Let X, Y be equivalent non-empty sets and $f\colon X \to X$, $h\colon Y \to Y$ be mappings such that mono-unary algebras $(X, f) \cong (Y, h)$. Denote $G = \{f^n; n \in \mathbb{N}_0\}$, $H = \{h^n, n \in \mathbb{N}_0\}$ and define binary hyperoperations*

$$\star\colon G \times G \to \mathcal{P}^*(G), \qquad \bullet\colon H \times H \to \mathcal{P}^*(H),$$

by

$$f^n \star f^m = \{f^k; k \in \mathbb{N}_0, m+n \le k\} \quad \text{and} \quad h^n \bullet h^m = \{h^k; k \in \mathbb{N}_0, m+n \le k\}.$$

Define mappings $\psi\colon G \times X \to \mathcal{P}^*(X)$, $\eta\colon H \times Y \to \mathcal{P}^*(Y)$, by the same rule $\psi(f^n, x) = \big\{f^k(x); k \in \{0, n, n+1, n+2, \dots\}\big\}$, $\omega(h^n, y) = \big\{h^k(y); k \in \{0, n, n+1, n+2, \dots\}\big\}$. Suppose $\xi\colon (X, f) \to (Y, h)$ is an isomorphism and $\varphi\colon (X, f) \to (Y, h)$ a homomorphism of the mono-unary algebra (X, f) onto the mono-unary algebra (Y, h). Denote $\Phi = [\mu, \varphi]$ the pair of mappings such that $\mu(f^n) = \xi \circ f^n \circ \xi^{-1}$. Then Φ is a II-homomorphism of the generalized transformation semihypergroup (X, G, ψ) into the GTS (Y, H, ω).

Indeed, for an arbitrary pair $[f^n, x] \in G \times X$ we have

$$\begin{aligned} \varphi\big(\psi(f^n, x)\big) &= \varphi\{x, f^n(x), f^{n+1}(x), \dots\} = \big\{\varphi(x), \varphi\big(f^n(x)\big), \varphi\big(f^{n+1}(x)\big), \dots\big\} \\ &= \big\{\varphi(x), \varphi\big(h^n(x)\big), \varphi\big(h^{n+1}(x)\big), \dots\big\} = \omega\big(h^n, \varphi(x)\big) \\ &= \omega\big(\xi \circ f^n \circ \xi^{-1}, \varphi(x)\big) = \omega\big(\mu(f^n), \varphi(x)\big) = \omega\big(\mu \times \varphi)[f^n, x]\big). \end{aligned}$$

The following example of generalized transformation hypergroup is based on consideration published in [4, 5].

Example 5.2. *Let $J \subset \mathbb{R}$ be an open interval and denote $C^\infty(J)$ the ring of all infinitely differentiable functions on J. Let us consider the set $\mathbb{LA}_n(J)$, $n \in \mathbb{N}$, of linear differential operators of the n-th order in the form*

$$L(p_0, \ldots, p_{n-1}) = \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} p_k(x) \frac{d^k}{dx^k}.$$

Where $p_k \in C^\infty(J)$, $k = 0, 1, \ldots, n-1$; $L(p_0, \ldots, p_{n-1})\colon C^\infty(J) \longrightarrow C^\infty(J)$, thus

$$L(p_0, \ldots, p_{n-1})(f) = f^{(n)}(x) + p_{n-1}(x) f^{(n-1)}(x) + \cdots + p_0(x) f(x),\ f \in C^\infty(J).$$

Let δ_{ij} stand for the Kronecker symbol δ. For any but fixed $m \in \{0, 1, \ldots, n-1\}$ we denote by

$$\mathbb{LA}_n(J)_m = \big\{L(p_0, \ldots, p_{n-1}) | p_k \in C^\infty(J), p_m > 0\big\}.$$

Shortly we put $\mathbf{p} = (p_0(x), \ldots, p_{n-1}(x)), x \in J$ and on the set $\mathbb{LA}_n(J)_m$ we define a binary operation "$\circ_m$" and a binary relation $\leq_m$ in this way:

$$L(\mathbf{p}) \circ_m L(\mathbf{q}) = L(\mathbf{u})$$

where $u_k(x) = p_m(x) q_k(x) + (1 - \delta_{km}) p_k(x), x \in J, 0 \leq k \leq n-1$, and

$$L(\mathbf{p}) \leq_m L(\mathbf{q})$$

whenever $p_k(x) \leq q_k(x), k \neq m, k \in \{0, 1, \ldots, n-1\}, p_m(x) = q_m(x), x \in J$.

It is easy to verify that $(\mathbb{LA}_n(J)_m, \circ_m, \leq_m)$ is an ordered noncommutative group with the neutral element $D(\mathbf{w})$, where $D(\mathbf{w}) = (w_0, \ldots, w_{n-1})$, $w_k(x) = \delta_{km}$. An inverse to any $D(\mathbf{q})$ is $D^{-1}(\mathbf{q}) = \Big(\frac{-q_0}{q_m}, \ldots, \frac{1}{q_m}, \ldots, \frac{-q_{n-1}}{q_m}\Big)$.

Let $(\mathbb{Z}, +, \leq)$ be an additive group of all integers with an usual ordering "$\leq$". Then by Lemma 1.1 the structure $(\mathbb{Z}, \star)$, where

$$\star\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathcal{P}^*(\mathbb{Z})$$

was defined by $k \star l = [k + l)_\leq$ is a hypergroup.

For fixed $D(\mathbf{q}) \in \mathbb{LA}_n(J)_m$ we define an action $\psi_q : \mathbb{Z} \times \mathbb{LA}_n(J)_m \longrightarrow \mathcal{P}^*(\mathbb{LA}_n(J)_m)$ as follows,

$$\psi_q(k, L(\mathbf{p})) = \{L^t(\mathbf{q}) \circ_m L(\mathbf{p}) | t \leq k\}.$$

So $(\mathbb{LA}_n(J)_m, \mathbb{Z}, \varphi_q)$ is a generalized transformation hypergroup.

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RATIO MATHEMATICA 23 (2012), 21–38 ISSN: 1592-7415

(Intuitionistic) Fuzzy Grade of a Hypergroupoid: A Survey of Some Recent Researches

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Abstract

This paper aims to present a short survey on two numerical functions determined by a hypergroupoid, called the fuzzy grade and the intuitionistic fuzzy grade of a hypergroupoid. It starts with the main construction of the sequences of join spaces and (intuitionistic) fuzzy sets associated with a hypergroupoid. After some computations of the above grades, we discuss some similarities and differences between the two grades for the complete hypergroups and for the i.p.s. hypergroups. We conclude with some open problems.

Key words: (Intuitionistic) Fuzzy Grade, Join Space, I.p.s. Hypergroup, Complete Hypergroup.

MSC2012: 20N20; 03E72.

1 Introduction

The study of hyperstructures connected with fuzzy sets represents a growing and new line of research spanning over the last twenty years. Till now one can distinguish three principal approaches of this theme: the study of new crisp hyperoperations obtained by means of fuzzy sets; the study of fuzzy subhyperstructures (i.e. fuzzy sets those level sets are crisp hyperstructures); the study of structures endowed with fuzzy hyperoperations and called fuzzy hyperstructures.

We concentrate here on the first line of reserach. Its origin is due to Corsini [12] in 1993, when he defined a join space from a nonempty set endowed with a fuzzy set. Another definition of a join space induced by a fuzzy set was given in 1997 by Ameri and Zahedi [1]. The Corsini's direction was then investigated by Corsini and Leoreanu [16]. Ten years later, the same Corsini [13] considered the inverse problem: to define a fuzzy set from a hypergroupoid. Iterating both constructions he obtained a sequence of join spaces and fuzzy sets that stops when there exist two consecutive non-isomorphic join spaces. The length of the sequence, i.e. the number of the non-isomorphic join spaces in it, was called by Corsini and Cristea [14, 15] the fuzzy grade of a hypergroupoid. At the beginning the sequence was studied for some particular hypergroups: the complete hypergroups and the i.p.s. hypergroups. Recently this grade has been investigated with a general and innovative method by Cristea [22, 23], Ştefănescu and Cristea [30], Angheluţă and Cristea [2, 3] making use of an ordered n-tuple determined by an equivalence relation. We will see the details in the next section.

One of the principal concerns of several researchers is that to obtain new hyperstructures using modern tools for investigation of uncertain problems, like intuitionistic fuzzy sets, rough sets, soft sets, interval-valued fuzzy sets. Following Corsini's idea, Cristea and Davvaz [24] associated with any hypergroupoid a sequence of join spaces and intuitionistic fuzzy sets with the length called the intuitionistic fuzzy grade. Although the fuzzy grade and the intuitionistic fuzzy grade have similar definitions, the corresponding associated sequences of join spaces have different properties. In order to highlight the differences between them, Davvaz, Hassani-Sadrabadi and Cristea [25, 26, 27] have investigated this problem for the complete hypergroups and i.p.s. hypergroups of order less than 8. In this survey our main goal is to realize a comparison between the two sequences.

The second section of the paper gives a survey of the construction of the sequences of join spaces associated with a hypergroupoid or with a non-empty set endowed with an (intuitionistic) fuzzy set. Section 3 is dedicated to some computations of the two grades: the fuzzy and the intuitionistic fuzzy grade of a hypergroupoid. In Section 4 we focus our comparison between the two grades on the cases of the complete hypergroups and of the i.p.s. hypergroups of order less than 8. In the last section we conclude with the statement of some open problems.

2 Sequences of the join spaces associated with a hypergroupoid

We briefly recall the construction of the sequences of join spaces and fuzzy sets/intuitionistic fuzzy sets associated with a hypergroupoid. For a comprehensive overview of hypergroups theory the reader is refereed to [11, 17, 31] and for fuzzy and intuitionistic fuzzy sets theory see [32, 5, 6, 7, 8]. We will use the terminology and the notations from [13, 23, 30].

Let X be a nonempty set. A *fuzzy set* A in X is characterized by a *membership function* $\mu_A : X \longrightarrow [0,1]$, where for any $x \in X$, the value $\mu_A(x)$ represents the *grade of membership* of x in A. More generally, any function $\mu : X \longrightarrow [0,1]$ is called a *fuzzy subset* of X. We denote by $FS(X)$ the set of all fuzzy subsets of X.

An *Atanassov's intuitionistic fuzzy set* A in X is a triplet of the form $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$, where, for any $x \in X$, the *degree of membership* of x (namely $\mu_A(x)$) and the *degree of non-membership* of x (namely $\lambda_A(x)$) verify the relation $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$; for simplicity, we denote it by $A = (\mu, \lambda)$. We denote by $IFS(X)$ the set of all intuitionistic fuzzy subsets of X.

For any nonempty set H endowed with a fuzzy subset $\alpha \in FS(H)$, Corsini[12] defined a join space $(H, \circ_\alpha)$, where the hyperproduct is defined as it follows:

$$x \circ_\alpha y = \{z \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(z) \leq \alpha(x) \vee \alpha(y)\} \tag{1}$$

Conversely, from any hypergroupoid $(H, \circ)$ Corsini[13] defined a fuzzy subset $\widetilde{\mu} \in FS(H)$ by the following formula:

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{q(u)} \tag{2}$$

where $Q(u) = \{(a,b) \in H^2 \mid u \in a \circ b\}, \quad q(u) = |Q(u)|$.

If $Q(u) = \emptyset$, we set $\widetilde{\mu}(u) = 0$.

In 2010 Cristea and Davvaz[24] defined in a similar way an intuitionistic fuzzy subset associated with a hypergroupoid $(H, \circ)$:

$$\overline{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \qquad \overline{\lambda}(u) = \frac{\displaystyle\sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}}{n^2} \tag{3}$$

Iterating equations (1) and (2) one obtains a sequence $({}^iH = (H, \circ_i), \widetilde{\mu}_{i-1})_{i\geq 1}$ of join spaces and fuzzy sets associated with H. The sequence stops when there exist two consecutive isomorphic join spaces. The length of this sequence, that is the number of the non-isomorphic join spaces in the sequence, is called the *fuzzy grade* of the hypergroupoid H. More exactly we have the following definition.

Definition 2.1. [14] A hypergroupoid H has the *fuzzy grade* $m \in \mathbb{N}^*$, and we write $f.g.(H) = m$ if, for any $1 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic and, for any $s > m$, ${}^sH$ is isomorphic with ${}^mH$.
We say that the hypergroupoid H has the *strong fuzzy grade* m, and we write $s.f.g.(H) = m$, if $f.g.(H) = m$ and, for all $s, s > m$, ${}^sH = {}^mH$.

Similarly, iterating equations (1) and (3) one obtains two sequences $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ and $({}^iH = (H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ of join spaces and intuitionistic fuzzy sets associated with H. The sequences stop when there exist two consecutive isomorphic join spaces and their lengths are called the *lower/upper intuitionistic fuzzy grade* of the hypergroupoid H. Let's see better their definitions.

Definition 2.2. [24] A set H endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *lower intuitionistic fuzzy grade* m and we write $l.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $(H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i})$ and $(H, \circ_{\overline{\mu}_{i+1} \wedge \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}_sH$ is isomorphic with ${}_{m-1}H$.

Definition 2.3. [24] A set H endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *upper intuitionistic fuzzy grade* m and we write $u.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $(H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i})$ and $(H, \circ_{\overline{\mu}_{i+1} \vee \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}^sH$ is isomorphic with ${}^{m-1}H$.

The construction of the above sequences can start in two different ways:

1. from a hypergroupoid $(H, \circ)$;

2. from a nonempty set H endowed with a fuzzy set or an intuitionistic fuzzy set.

It is easy to prove that in both cases one obtains the same fuzzy grade. But what can we say about the other two sequences of join spaces and intuitionistic fuzzy sets?

If we start the construction from a hypergroupoid $(H, \circ)$, we obtain only one sequence of join spaces, because the first join spaces ${}_0H = (H, \circ_{\overline{\mu}\wedge\overline{\lambda}})$ and ${}^0H = (H, \circ_{\overline{\mu}\vee\overline{\lambda}})$ are always isomorphic. Indeed, since, for any $x \in H$, the sum $\overline{\mu}(x) + \overline{\lambda}(x)$ is always constant, it follows that $\overline{\mu}(x) = \overline{\mu}(y)$ if and only if $\overline{\lambda}(x) = \overline{\lambda}(y)$.
Therefore in this case $l.i.f.g.(H) = u.i.f.g.(H)$ and it is simply called the *intuitionistic fuzzy grade* of H (shortly $i.f.g.(H)$).

On the other side, if we start the construction from a nonempty set H endowed with an intuitionistic fuzzy set, one may obtain two different sequences with different lengths. We only illustrate this case by the following example.

Example 2.1. [24] On $H = \{a, b, c, d\}$ we define the Atanassov's intuitionistic fuzzy set:

$$\begin{array}{llll} \overline{\mu}(a) = 0.25 & \overline{\mu}(b) = 0.25 & \overline{\mu}(c) = 0.30 & \overline{\mu}(d) = 0.10 \\ \overline{\lambda}(a) = 0.40 & \overline{\lambda}(b) = 0.40 & \overline{\lambda}(c) = 0.50 & \overline{\lambda}(d) = 0.90. \end{array}$$

To start with, we construct the first sequence of join spaces and we determine its $l.i.f.g.(H)$.

For the associated join space

$\circ_{\overline{\mu}\wedge\overline{\lambda}}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
c	$\{a,b,c\}$	$\{a,b,c\}$	c	H
d	$\{a,b,d\}$	$\{a,b,d\}$	H	d

we calculate the Atanassov's intuitionistic fuzzy set associated with H as in (3) and we obtain the following values:

$$\begin{array}{llll} \overline{\mu}_1(a) = 31/96 & \overline{\mu}_1(b) = 31/96 & \overline{\mu}_1(c) = 17/96 & \overline{\mu}_1(d) = 17/96 \\ \overline{\lambda}_1(a) = 12/96 & \overline{\lambda}_1(b) = 12/96 & \overline{\lambda}_1(c) = 26/96 & \overline{\lambda}_1(d) = 26/96, \end{array}$$

therefore $\overline{\mu}_1 \wedge \overline{\lambda}_1(a) = \overline{\mu}_1 \wedge \overline{\lambda}_1(b) < \overline{\mu}_1 \wedge \overline{\lambda}_1(c) = \overline{\mu}_1 \wedge \overline{\lambda}_1(d)$ and thereby it results the following join space

$\circ_{\overline{\mu}_1\wedge\overline{\lambda}_1}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	H	H
b	$\{a,b\}$	$\{a,b\}$	H	H
c	H	H	$\{c,d\}$	$\{c,d\}$
d	H	H	$\{c,d\}$	$\{c,d\}$

For any $x \in H$, we compute $\overline{\mu}_2(x) = 4/16$ and $\overline{\lambda}_2(x) = 2/16$; thus, for any $x, y \in H$, $x \circ_{\overline{\mu}_2\wedge\overline{\lambda}_2} y = H$. So the first sequence of join spaces associated with H has 3 elements, that is $l.i.f.g.(H) = 3$.

Now, in order to determine the $u.i.f.g.(H)$, we start with the join space

$\circ_{\overline{\mu}\vee\overline{\lambda}}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	H
b	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	H
c	$\{a,b,c\}$	$\{a,b,c\}$	c	$\{c,d\}$
d	H	H	$\{c,d\}$	d

Note that $\langle {}_0H, \circ_{\overline{\mu}\wedge\overline{\lambda}}\rangle$ and $\langle {}^0H, \circ_{\overline{\mu}\vee\overline{\lambda}}\rangle$ are not isomorphic. Since

$$\begin{array}{llll} \overline{\mu}_1(a) = 13/48 & \overline{\mu}_1(b) = 13/48 & \overline{\mu}_1(c) = 13/48 & \overline{\mu}_1(d) = 9/48 \\ \overline{\lambda}_1(a) = 9/48 & \overline{\lambda}_1(b) = 9/48 & \overline{\lambda}_1(c) = 9/48 & \overline{\lambda}(d) = 13/48 \end{array}$$

it follows that $\overline{\mu}_1 \vee \overline{\lambda}_1(x) = 13/48$, whenever $x \in H$, which means that, for any $x, y \in H$, $x \circ_{\overline{\mu}_1\vee\overline{\lambda}_1} y = H$. Therefore the second sequence of join spaces associated with H has 2 elements, that is $u.i.f.g.(H) = 2$.

Sometimes the computations of the values of the membership functions $\widetilde{\mu}_i, \overline{\mu}_i, \overline{\lambda}_i$, $i \geq 0$ could take much time, thus it is useful to determine the (intuitionistic) fuzzy grade of a hypergroupoid without calculating all these values. In order to realize this we introduce some notations.

With any join space ${}^iH$ in the sequence of join spaces and fuzzy sets corresponding to H, one may associate an *ordered chain* $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an *ordered r-tuple* $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

1. $\forall j \geq 1$: $x, y \in {}^iC_j \Longleftrightarrow \widetilde{\mu}_{i-1}(x) = \widetilde{\mu}_{i-1}(y)$,
2. for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\widetilde{\mu}_{i-1}(x) < \widetilde{\mu}_{i-1}(z)$
3. ${}^ik_j = |{}^iC_j|$, for all j.

We have similar notations for the sequence $({}_iH = (H, \circ_{\overline{\mu}_i\wedge\overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$.

With any join space ${}_iH$ one may associate an *ordered chain* $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an *ordered r-tuple* $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

1. $\forall j \geq 1$: $x, y \in {}^iC_j \Longleftrightarrow \overline{\mu}_i \wedge \overline{\lambda}_i(x) = \overline{\mu}_i \wedge \overline{\lambda}_i(y)$,

2. for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\overline{\mu}_i \wedge \overline{\lambda}_i(x) < \overline{\mu}_i \wedge \overline{\lambda}_i(z)$

3. ${}^ik_j = |{}^iC_j|$, for all j.

One of the crucial questions we have to ask is: **When two consecutive join spaces in these sequences are isomorphic?**

A first answer was given by Corsini and Leoreanu [16] in 1995.

Theorem 2.1. [16] *Let ${}^iH$ and ${}^{i+1}H$ be two consecutive join spaces associated with H determined by the membership functions $\widetilde{\mu}_{i-1}$ and $\widetilde{\mu}_i$, where ${}^iH = \bigcup_{l=1}^{r_1} C_l$, ${}^{i+1}H = \bigcup_{l=1}^{r_2} C'_l$ and $(k_1, k_2, ..., k_{r_1})$ is the r_1-tuple associated with ${}^iH$, $(k'_1, k'_2, ..., k'_{r_2})$ is the r_2-tuple associated with ${}^{i+1}H$.*

The join spaces ${}^iH$ and ${}^{i+1}H$ are isomorphic if and only if $r_1 = r_2$ and $(k_1, \ldots, k_{r_1}) = (k'_1, \ldots, k'_{r_1})$ or $(k_1, \ldots, k_{r_1}) = (k'_{r_1}, \ldots, k'_1)$.

A similar theorem can be formulated also for the join spaces in the sequence $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i \geq 0}$.

In order to use this result we need to determine both r-tuples associated with ${}^iH$ and ${}^{i+1}H$. But Ştefănescu and Cristea [30] have given a sufficient condition such that two consecutive join spaces are not isomorphic, that is the sequence doesn't stop, using only one r-tuple.

Theorem 2.2. [30] *Let $(k_1, k_2, ..., k_r)$ be the r-tuple associated with the join space ${}^iH$. If $(k_1, \ldots, k_r) = (k_r, k_{r-1}, \ldots, k_1)$, then the join spaces ${}^iH$ and ${}^{i+1}H$ are not isomorphic.*

A second crucial problem arises: **does there exist a hypergroupoid H such that $s.f.g.(H)/i.f.g.(H) = n$, whenever n is a natural number?** The anser is yes, and an example is given in the following fundamental result.

Theorem 2.3. [30, 24] *Let $H = \{x_1, x_2, \ldots, x_s\}$, where $s = 2^n$, $n \in \mathbb{N}^* \setminus \{1, 2\}$, be the commutative hypergroupoid defined by the hyperproduct*

$$\begin{aligned} & x_i \circ x_i = x_i, 1 \leq i \leq n, \\ & x_i \circ x_j = \{x_i, x_{i+1}, \ldots, x_j\}, 1 \leq i < j \leq n. \end{aligned}$$

Then $s.f.g.(H) = i.f.g.(H) = n$. Moreover, H is a join space.

3 Computation of the fuzzy grade/intuitionistic fuzzy grade of a hypergroupoid

Let $(H, \circ)$ be a hypergroupoid. We define the following equivalence on H

$$xR_{\widetilde{\mu}}y \Longleftrightarrow \widetilde{\mu}(x) = \widetilde{\mu}(y).$$

We determine $f.g.(H)$ when $|H/R_{\widetilde{\mu}}| \in \{2, 3\}$.

Theorem 3.1. [2] *Let $(H, \circ)$ be a finite hypergroupoid.*

1. *If $|H/R_{\widetilde{\mu}}| = 2$, that is (k_1, k_2) is the pair associated with H, then $s.f.g.(H) = 2$ whenever $k_1 = k_2$, and $f.g.(H) = 1$ otherwise.*
2. *If $|H/R_{\widetilde{\mu}}| = 3$, that is (k_1, k_2, k_3) is the triple associated with H, and*
   (a) *if $k_1 = k_2 = k_3$, then $f.g.(H) = 2$*
   (b) *if $k_1 = k_3 \neq k_2$, then $f.g.(H) = 2$ whenever $k_2 \neq 2k_3$ and $s.f.g.(H) = 3$, otherwise*
   (c) *if $k_1 < k_2 = k_3$, then $f.g.(H) = 1$*
   (d) *if $k_1 = k_2 < k_3$, then $f.g.(H) = 1$ whenever $P = 2k_3^3 - 8k_1^3 - k_1^2k_3 + 5k_1k_3^2 > 0$ and $f.g.(H) = 3$, otherwise*
   (e) *if $k_1 \neq k_2 \neq k_3$, there is no precise order between $\widetilde{\mu}_1(x), \widetilde{\mu}_1(y), \widetilde{\mu}_1(z)$.*

A similar result we obtain for the quotient $H/R_{\overline{\mu}\wedge\overline{\lambda}}$.

Theorem 3.2. [3] *Let $(H, \circ)$ be a finite hypergroupoid.*

1. *If $|H/R_{\overline{\mu}\wedge\overline{\lambda}}| = 2$, that is (k_1, k_2) is the pair associated with H, then $i.f.g.(H) = 2$ whenever $k_1 = k_2$, and $i.f.g.(H) = 1$, otherwise.*
2. *If $|H/R_{\overline{\mu}\wedge\overline{\lambda}}| = 3$, that is (k_1, k_2, k_3) is the triple associated with H, and*
   (a) *if $k_1 = k_2 = k_3$, then $i.f.g.(H) = 2$*
   (b) *if $k_1 = k_3 \neq k_2$, then $i.f.g.(H) = 2$ whenever $k_2 \neq 2k_3$ and $i.f.g.(H) = 3$, otherwise*
   (c) *if $k_1 < k_2 = k_3$, then $i.f.g.(H) = 1$ whenever $2k_2^2 > 3k_1k_2 + 3k_1^2$ and $i.f.g.(H) = 3$, otherwise*
   (d) *if $k_1 = k_2 < k_3$, then $i.f.g.(H) = 1$ whenever $k_3 \neq 2k_1$ and $i.f.g.(H) = 2$, otherwise*
   (e) *if $k_1 \neq k_2 \neq k_3$, then there is no precise order between $\overline{\mu}_1(x) \wedge \overline{\lambda}_1(x), \overline{\mu}_1(y) \wedge \overline{\lambda}_1(y), \overline{\mu}_1(z) \wedge \overline{\lambda}_1(z)$.*

Angheluţă and Cristea [3] have studied the intuitionistic fuzzy grade of a hypergroupoid in the case of some particular ternaries. We integrate here this theorem with similar results for the fuzzy grade. In the following the ternary (k_1, k_2, k_3) associated with H is intended respected with the equivalence $R_{\widetilde{\mu}}$ (when we talk about the fuzzy grade) and $R_{\overline{\mu}\wedge\overline{\lambda}}$, respectively, (when we talk about the intuitionistic fuzzy grade).

Proposition 3.1. *Let H be a hypergroupoid of cardinality $2k$, $k \geq 3$, with the ternary associated with H of the type $(k, k-1, 1)$. Then $s.f.g.(H) = i.f.g.(H) = 1$.*

Generalizing, we obtain the same result for the ternary $(pk, p(k-1), p)$, with $p, k \in \mathbb{N} \setminus \{0, 1\}$.

Proof. We prove only that $s.f.g.(H) = 1$. For the proof of the fact that $i.f.g.(H) = 1$, see [3].

Let H be a hypergroupoid such that $|H| = 2k$, $k \geq 3$, and $(k, k-1, 1)$ be the ternary associated with H determined by the equivalence $R_{\widetilde{\mu}}$. Thus H can be written as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = k, |C_2| = k-1, |C_3| = 1$. Using equation (2), for any $x \in C_1, y \in C_2, z \in C_3$, one obtains

$$\widetilde{\mu}_1(x) = \frac{4k^2 - k - 1}{3k^2(2k-1)}$$

$$\widetilde{\mu}_1(y) = \frac{8k^3 + 2k^2 - 12k + 4}{2k(2k-1)(3k^2-1)}$$

$$\widetilde{\mu}_1(z) = \frac{4k-2}{k(4k-1)}$$

After some computations that we omit here for lack of space, it results the following relation

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z),$$

which means that $(k, k-1, 1)$ is the ternary associated with the join space ${}^1H$ and thereby $s.f.g.(H) = 1$. □

Proposition 3.2. *Let $(2k, k, 1)$, with $k \geq 2$, be the ternary associated with a hypergroupoid H. Then*

1. *$s.f.g.(H) = 1$.*

2. *$i.f.g.(H) = 3$ and the sequence of join spaces is cyclic, that is ${}_{3l}H \simeq {}_3H$, ${}_{3l+1}H \simeq {}_1H$, ${}_{3l+2}H \simeq {}_2H$, for any $l \in \mathbb{N}^*$.*

Proof. Considering $(2k, k, 1)$, with $k \geq 2$, be the ternary associated with a hypergroupoid H respected with the equivalence $R_{\widetilde{\mu}}$, we decompose H as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = 2k, |C_2| = k, |C_3| = 1$. Using again equation (2), for any $x \in C_1, y \in C_2, z \in C_3$, one obtains

$$\widetilde{\mu}_1(x) = \frac{15k + 11}{6(2k+1)(3k+1)}$$

$$\widetilde{\mu}_1(y) = \frac{21k^2 + 58k + 25}{3(k+1)(3k+1)(5k+6)}$$

$$\widetilde{\mu}_1(z) = \frac{13k^2 + 10k + 1}{(k+1)(3k+1)(6k+1)}$$

For any $k \geq 2$, it follows that

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z),$$

that is $s.f.g.(H) = 1$.

For the second part of the theorem see [3]. $\square$

Proposition 3.3. *Let $(h, k, 1)$, with $2 \leq h \leq k$, be the ternary associated with a hypergroupoid H.*

1. *Then $s.f.g.(H) = 1$.*
2. *If $h = 2$ and*
   (a) *$k = 2$, then $i.f.g.(H) = 3$;*
   (b) *$k = 3$, then $i.f.g.(H) = 2$;*
   (c) *$k \geq 4$, then $i.f.g.(H) = 3$ and the sequence of join spaces associated with H is cyclic.*
3. *If $h > 2$, then $i.f.g.(H) = 1$, whenever $k \geq h$.*

Proof. Since $(h, k, 1)$, with $2 \leq h \leq k$, is the ternary associated with a hypergroupoid H respected with the equivalence $R_{\widetilde{\mu}}$, we write H as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = h, |C_2| = k, |C_3| = 1$. Calculating the values of the membership function $\widetilde{\mu}_1$ by using the formula (2), we obtain, for any $x \in C_1, y \in C_2, z \in C_3$, that

$$\widetilde{\mu}_1(x) = \frac{h^2+3k^2+4hk+3h+5k}{(h+k)(h+k+1)(h+2k+2)}$$

$$\widetilde{\mu}_1(y) = \frac{3h^2k^2+7h^2k+4hk^3+9hk^2+7hk+k^4+4k^3+3k^2+2h^2}{(h+k)(h+k+1)(k+1)(2hk+2h+k^2+2k)}$$

$$\widetilde{\mu}_1(z) = \frac{5hk+3h+3k^2+4k+1}{(h+k+1)(k+1)(2h+2k+1)}$$

Similarly as in the previous two lemmas, we prove that

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z)$$

and thus $s.f.g.(H) = 1$.

For the second part of the theorem see [3].

$\square$

4 (Intuitionistic) fuzzy grade of special types of hypergroups

In this section we deal with two classes of hypergroups: the complete hypergroups and the hypergroups with partial scalar identities, shortly called i.p.s. hypergroups. We determine the fuzzy and intuitionistic fuzzy grades of all these hypergroups of order less than 8. This part of the paper is based on the articles of Corsini and Cristea [14, 15], Cristea [21], Cristea and Davvaz [24], Angheluţă and Cristea [2, 3], Davvaz, Hassani-Sadrabadi and Cristea [25, 26, 27].

4.1 The complete hypergroups

We start with the characterization of the complete hypergroups.

A hypergroup $(H, \circ)$ is a *complete hypergroup* if it can be written as the union of its subsets $H = \cup_{g \in G} A_g$, where

1. $(G, \cdot)$ is a group;
2. for any $(g_1, g_2) \in G^2$, $g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$;
3. if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.

Let $G = \{g_1, \ldots, g_m\}$ be a finite group. Then with $H = \bigcup_{i=1}^{m} A_{g_i}$ we may associate an m-tuple $[k_1, \ldots, k_m]$, where $k_i = |A_{g_i}|$.
Using these notations and based on the fact that, for any $u \in H$, $\exists! g_u \in G : u \in A_{g_u}$, Corsini [13] has determined the formula for the membership function $\widetilde{\mu}$ associated with a complete hypergroup as the following one:

$$\widetilde{\mu}(u) = \frac{1}{|A_{g_u}|}.$$

Similarly, the formula of the membership functions $\overline{\mu}, \overline{\lambda}$ for a complete hypergroup is simpler that those in the general case, as Cristea and Davvaz proved [24].
Defining on H the following equivalence

$$u \sim v \Longleftrightarrow \exists g \in G : u, v \in A_g,$$

one obtains that:

$$\bar{\mu}(u) = \frac{|Q(u)|}{|A_{g_u}|} \cdot \frac{1}{n^2}, \quad \bar{\lambda}(u) = \left(\sum_{v \notin \hat{u}} \frac{|Q(v)|}{|A_{g_v}|} \right) \cdot \frac{1}{n^2}.$$

We notice that, if G_1 and G_2 are non isomorphic groups of the same cardinality and H_1, H_2 are the complete hypergroups obtained by them, then $f.g.(H_1) = f.g.(H_2)$, but their $i.f.g.$ may be different.

Cristea [21] determined the fuzzy grade of all complete hypergroups of order less than 7. Later on, Davvaz together with Hassani-Sadrabadi and Cristea [27] determined their intuitionistic fuzzy grade. We recall here these results.

Theorem 4.1. *Let H be a complete hypergroup of order $n \leq 6$.*

1. *There are two non isomorphic complete hypergroups of order three that have $s.f.g.(H) = i.f.g.(H) = 1$.*

2. *Among the five non isomorphic complete hypergroups of order four, three of them have $s.f.g.(H) = i.f.g.(H) = 1$ and two of them have $s.f.g.(H) = i.f.g.(H) = 2$.*

3. *There are 12 non isomorphic complete hypergroups of order 5. All of them have $s.f.g.(H) = 1$. Nine of them have $i.f.g.(H) = 1$ and the other three have $i.f.g.(H) = 3$.*

4. *There are* 21 *non isomorphic complete hypergroups of order* 6*:*
   - 17 *with* $s.f.g.(H) = 1$ *and* 4 *with* $s.f.g.(H) = 2$.
   - 16 *with* $i.f.g.(H) = 1$, 3 *with* $i.f.g.(H) = 2$ *and* 2 *with* $i.f.g.(H) = 3$.

Angheluţă and Cristea [2] have studied the fuzzy grade of some particular complete hypergroups. It remains an open problem, and not a very easy one, to determine their intuitionistic fuzzy grade.

Theorem 4.2. *Let* H *be a complete hypergroup of type*

1. $[\underbrace{p,p,...,p}_{k\ \ times}, kp]$, *where* $n = |H| = 2kp$. *Then* $s.f.g.(H) = 2$.

2. $[\underbrace{p,p,\ldots,p}_{s\ \ times}, \underbrace{k,k,\ldots,k}_{t\ \ times}, ps]$, $2 \leq p < k < ps$, $n = |H| = 2ps + kt$.
   - *for* $n = 4ps$, $s.f.g.(H) = 3$,
   - *for* $kt \neq 2ps$, $f.g.(H) = 2$.

3. $[\underbrace{k,k,\ldots,k}_{l\ \ times}, \underbrace{p,p,\ldots,p}_{s\ \ times}, \underbrace{s,s,\ldots,s}_{p\ \ times}, \underbrace{l,l,\ldots,l}_{k\ \ times}]$, $2 \leq k < p < s < l$, $n = |H| = 2(ps + kl)$.
   - *if* $kl = ps$, *then* $s.f.g.(H) = 3$,
   - *otherwise* $f.g.(H) = 2$.

4. $[\underbrace{1,\ldots,1}_{k\ \ times}, \underbrace{2,\ldots,2}_{\frac{k}{2}\ \ times}, \underbrace{3,\ldots,3}_{\frac{k}{3}\ \ times}, \ldots, \underbrace{m-1,\ldots,m-1}_{\frac{k}{m-1}\ \ times}, k]$,
   $k = m.c.m.(1, 2, \ldots, m-1)$, $n = |H| = mk = 2^{s-1}k$.
   Then $s.f.g.(H) = s$.

4.2 The i.p.s. hypergroups

An *i.p.s. hypergroup* (i.e. a hypergroup with partial scalar identities) is a particular type of canonical hypergroup. We recall that a *canonical hypergroup* is a commutative hypergroup $(H, \circ)$ with a scalar identity, where every element $a \in H$ has a unique inverse a^{-1} such that it satisfies the following properties

1. if $y \in a \circ x$, then $x \in a^{-1} \circ y$;

2. $x \in x \circ a \Longrightarrow x = x \circ a$

The canonical hypergroups were introduced for the first time by Krasner [28] as the additive structures of the hyperfields and then Mittas [29] studied them independently from the others operations. Later on Corsini determined all i.p.s. hypergroups of order less than 9, proving that they are strongly canonical.

Corsini and Cristea [14, 15] calculated the fuzzy grade for all i.p.s. hypergroups of order less than 8 and their intuitionistic fuzzy grade have been determined by Davvaz, Hassan-sadrabadi and Cristea [25, 26]. Here we summarize these results.

Theorem 4.3. *1. There exists one i.p.s. hypergroup H of order 3 and $f.g.(H) = i.f.g.(H) = 1$.*

2. There exist 3 i.p.s. hypergroups of order 4 with $f.g.(H_1) = i.f.g.(H_1) = 1$, $f.g.(H_2) = 1, i.f.g.(H_2) = 2$, $f.g.(H_3) = i.f.g.(H_3) = 2$.

3. There exist 8 i.p.s. hypergroups of order 5: one has $f.g.(H) = 2$, all the others have $f.g.(H) = 1$; four of them have $i.f.g.(H) = 1$, three have $i.f.g.(H) = 2$ and one is with $i.f.g.(H) = 3$.

4. There exist 19 i.p.s. hypergroups of order 6: 14 of them have $f.g.(H) = 1$, 4 have $f.g.(H) = 2$ and one has $f.g.(H) = 3$;
ten of them have $i.f.g.(H) = 1$, eight of them have $i.f.g.(H) = 2$ and only for one of them we find $i.f.g.(H) = 3$.

5. There exist 36 i.p.s. hypergroups of order 6: 27 of them with $f.g.(H) = 1$, 8 with $f.g.(H) = 2$, 1 with $f.g.(H) = 3$;
10 of them with $i.f.g.(H) = 1$, 10 with $i.f.g.(H) = 2$, 9 with $i.f.g.(H) = 3$, 5 with $i.f.g.(H) = 4$, 2 with $i.f.g.(H) = 5$.
Moreover, for 18 of them we find that the associated sequences of join spaces and intuitionistic fuzzy sets are cyclic.

5 Future work

Based on the definition of the intuitionistic fuzzy set associated with a hypergroupoid [24], Ashgari-Larimi and Cristea [4] have defined the Atanassov's intuitionistic fuzzy index of a hypergroupoid. It would be interesting to obtain some relations between this index and the intuitionistic fuzzy grade.

We think that, finding necessary and sufficient conditions in order that the sequences of join spaces $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ and $({}^iH = (H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ associated with a nonempty set H endowed with an intuitionistic fuzzy set (like in Section 2) coincide, could be another line of research on this argument.

We also intend to find in our future work some possible connection between hypergroups and automata. For example, what can we say about its (intuitionistic) fuzzy grade? Does there exist an automaton with (intuitionistic) fuzzy grade of its state hypergroup equal to a given natural number?

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RATIO MATHEMATICA 23 (2012), 39–50 ISSN: 1592-7415

H_v-semigroups as noise pollution models in urban areas

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Abstract

Poor urban planning may give rise to noise pollution, since side-by-side industrial and residential buildings can result in noise pollution in the residential area. In this paper we represent the noise pollution with an H_v-semigroup. More specific, we introduce the concept of right reproductive H_v-semigroup which seems to be a useful tool to study the noise pollution problem in urban areas.

Key words: H_v-structures, H_v-group, H_v-semigroup

MSC2010: 20N20.

1 Introduction

The H_v-structures, introduced in the Fourth AHA Congress [13], where the known axioms were replaced by weaker ones. More precisely in axioms like associativity, commutativity and so on, the equality was replaced by the non-empty intersection. In $(H, \cdot)$ we abbreviate by
WASS, the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$ and by
COW, the *weak commutativity*: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$

Recall some basic definitions [14]:

Definition 1.1 *Let H be a non-empty set and $\cdot : H \times H \to \mathbf{P}(H) - \{\varnothing\}$ be a hyperoperation defined on H. Also, we have abbreviated the "hyperoperation" by "hope"[16]. The $(H, \cdot)$ is called H_v-semigroup if it is WASS and it is called*

H_v-group if it is H_v-semigroup and the reproduction axiom $x \cdot H = H \cdot x = H, \forall x \in H$, is valid. The hyperstructure $(R, +, \cdot)$ is called H_v-ring if both hopes $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing, (x+y)z \cap (xz+yz) \neq \varnothing, \forall x, y, z \in R.$$

An H_v-ring $(R, +, \cdot)$ is called dual H_v-ring [7] if the hyperstructure $(R, \cdot, +)$ is H_v-ring, too.

Let $(H, \cdot)$ be a hypergroupoid. An element $e \in H$ is called *right unit* element if $a \in a \cdot e, \forall a \in H$ and is called *left unit* element if $a \in e \cdot a, \forall a \in H$. The element $e \in H$ is called *unit* element if $a \in a \cdot e \cap e \cdot a, \forall a \in H$. An element $e \in H$ is called *right scalar* unit element if $a = a \cdot e, \forall a \in H$ and is called *left scalar* unit element if $a = e \cdot a, \forall a \in H$. It is called *scalar unit* element if $a = a \cdot e = e \cdot a, \forall a \in H$. Let $(H, \cdot)$ be a hypergroupoid endowed with at least one unit element. An element $a' \in H$ is called an inverse element of the element $a \in H$, if there exists a unit element $e \in H$ such that $e \in a' \cdot a \cap a \cdot a'$. The element $a \in H$ is called *right simplifiable* element (resp. *left*) if $\forall x, y \in H$ the following is valid:

$$x \cdot a = y \cdot a \Rightarrow x = y \text{ (resp. } a \cdot x = a \cdot y \Rightarrow x = y).$$

Moreover, let us define here: If $x \in x \cdot y$ (resp. $x \in y \cdot x$) $\forall y \in H$, then, x is called *left absorbing-like* element (resp. *right absorbing-like element*).The n^{th} power of an element h, denoted h^s, is defined to be the union of all expressions of n times of h, in which the parentheses are put in all possible ways. An H_v-group $(H, \cdot)$ is called *cyclic* with finite period with respect to $h \in H$, if there exists a positive integer s, such that $H = h^1 \cup h^2 \cup ... \cup h^s$. The minimum such an s is called period of the generator h. If all generators have the same period, then H is cyclic with period. If there exists $h \in H$ and s positive integer, the minimum one, such that $H = h^s$ then H is called *single-power cyclic* and h is a generator with *single-power period s.* The cyclicity in the infinite case is defined similarly. Thus, for example, the H_v-group $(H, \cdot)$ is called *single-power cyclic with infinite period* with generator h if every element of H belongs to a power of h and there exists $s_0 \geq 1$, such that $\forall s \geq s_0$ we have: $h^1 \cup h^2 \cup ... \cup h^{s-1} \subset h^s$.
An element $a \in H$ is called *idempotent* element if $a^2 = a$.
The main tool to study H_v-groups, is the fundamental relation β^*. The relation β^* is defined in H_v-groups, as the smallest equivalence relation on H,

such that the quotient would be group. It is called the fundamental group and β^* is called the *fundamental equivalence* relation on H. The relation β is defined on an H_v-group in the same way as in a hypergroup.
The basic Theorem is the following [14]: Let $(H, \cdot)$ be an H_v-group and denote by $\mathbf{U}$ the set of all finite products of elements of H. We define the relation β on H by setting xβy iff $\{x, y\} \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then β^* is the transitive closure of β.
An element $s \in H$ is called *single* if $\beta^*(s) = \{s\}$. The set of all single elements is denoted by S_H and if $S_H \neq \varnothing$ then one can find easily the fundamental classes.
A way to find the fundamental classes is given in [6],[13],[14].

For more definitions, results and applications onH_v-structures, see also books [4],[5],[14].

2 The noise problem

Noise pollution is displeasing human, animal or machine-created sound that disrupts the activity or balance of human or animal life. The source of most outdoor noise worldwide is not only transportation systems (including motor vehicle noise, aircraft noise and rail noise), but, noise caused by people as well (audio entertainment systems, electric megaphones and loud people) [9]. The fact that noise pollution is also a cause of annoyance, is that, a 2005 study by Spanish researchers found that in urban areas households are willing to pay approximately 4 Euros per decibel per year for noise reduction [2]. Poor urban planning may give rise to noise pollution, since side-by-side industrial and residential buildings can result in noise pollution in the residential area.

We set the following problem: *The noise pollution that comes from a certain block of flats in urban areas, obviously annoys not only the block of flats itself but possibly neighboring blocks of flats or buildings, as well.*

If every city is considered as a set Ω with elements the blocks of flats or the buildings, then, the above situation could be described with an algebraic hyperstructure and its properties. In this paper, we present the *right reproductive H_v-semigroup*, as a tool to study the noise pollution problem in urban areas. One can find hyperstructures on related problems in several survey papers like [10], [11],[12] and papers with a wide variety of applications [1], [3], [15].

Definition 2.1 *A hypergroupoid (**H**,*), such that, the weak associativity holds and* $\forall x \in \mathbf{H}$*, **H***x=**H** , is called **right reproductive*** $\mathbf{H_v}$***-semigroup**. A hypergroupoid **H**,*), such that, the weak associativity holds and* $\forall x \in \mathbf{H}$*, x***H**=**H** , is called **left reproductive*** $\mathbf{H_v}$***-semigroup***

Now we give the following definition:

Definition 2.2 *Let* $\mathbf{\Omega} \neq \varnothing$ *and* $f : \mathbf{\Omega} \to \mathbf{P}(\mathbf{\Omega})$ *be a map, then we define a hyperoperation* r_L*:* $\mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{P}(\mathbf{\Omega})$*, on* $\mathbf{\Omega}$ *as follows:* $\forall x, y \in \mathbf{\Omega}$*, we set*

$$xr_Ly = f(x) \cup \{x\}$$

We call the above hyperoperation (r_L) *noise hyperoperation. Remark that the noise hyperoperation, always contains the element* $x \in \mathbf{\Omega}$*. That means that the element* $x \in \mathbf{\Omega}$ *could be considered as the representative of the elements of the set* xr_Ly*. So, we symbolize:*

$$xr_Ly = f(x) \cup \{x\} = [x]$$

If, $\forall x \in \mathbf{\Omega}, x \in f(x)$ *then the hyperoperation is simplified as*

$$xr_Ly = f(x) = [x]$$

Therefore, the noise hyperoperation (r_L) *depends only on the left element. That means that if one composes an element x, on the left, with any other element y, on the right, then the result is always the same set [x].*

Example 2.1 *Consider a set* $\mathbf{\Omega} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ *and a map* $f : \mathbf{\Omega} \to \mathbf{P}(\mathbf{\Omega})$ *such that:*

$$f(x_1) = \{x_2\}, f(x_2) = \{x_2, x_3\}, f(x_3) = \{x_2\}, f(x_4) = \{x_4\}$$

$$f(x_5) = \{x_5, x_6, x_7\}, f(x_6) = \{x_6, x_7\}, f(x_7) = \{x_5\},$$

$$f(x_8) = \{x_8, x_9\}, f(x_9) = \varnothing.$$

Then, as in the defined above noise hyperoperation:

$$[x_1] = f(x_1) \cup \{x_1\} = \{x_1, x_2\}, [x_2] = f(x_2) = \{x_2, x_3\},$$

$$[x_3] = f(x_3) \cup \{x_3\} = \{x_2, x_3\}, [x_4] = f(x_4) = \{x_4\},$$

$$[x_5] = f(x_5) = \{x_5, x_6, x_7\}, [x_6] = f(x_6) = \{x_6, x_7\}$$

$$[x_7] = f(x_7) \cup \{x_7\} = \{x_5, x_7\}, [x_8] = f(x_8) = \{x_8, x_9\},$$

$$[x_9] = f(x_9) \cup \{x_9\} = \{x_9\}.$$

Then, the "multiplication" table of (r_L) *is given by:*

$\mathbf{r_L}$	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_8}$	$\mathbf{x_9}$
$\mathbf{x_1}$	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2	x_1, x_2
$\mathbf{x_2}$	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_2	x_2, x_3	x_2, x_3
$\mathbf{x_3}$	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_3	x_2, x_2	x_2, x_3	x_2, x_3
$\mathbf{x_4}$	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
$\mathbf{x_5}$	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7	x_5, x_6, x_7
$\mathbf{x_6}$	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7	x_6, x_7
$\mathbf{x_7}$	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7	x_5, x_7
$\mathbf{x_8}$	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9	x_8, x_9
$\mathbf{x_9}$	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9

Example 2.2 *Let $\mathbf{X} \neq \varnothing$ and $m : \mathbf{X} \to [0,1]$ be a fuzzy subset of $\mathbf{X}$. We define the hyperoperation $(\circ)$ on $\boldsymbol{X}$ as follows:*
$\forall x, y \in \mathbf{X}$, *$\circ : \mathbf{X} \times \mathbf{X} \to \mathbf{P}(\mathbf{X})$, such that*

$$x \circ y = \{z \in \mathbf{X} / m(z) = m(x)\}$$

Then, consider the map $f(x) = \{z \in \mathbf{X} / m(z) = m(x)\}$. Since $x \in f(x), \forall x \in \mathbf{X}$, as above we have the hyperoperation (r_L) on $\boldsymbol{X}$ as follows : $\forall x, y \in \mathbf{X}$, $r_L : \mathbf{X} \times \mathbf{X} \to \mathbf{P}(\mathbf{X})$, such that $x r_L y = f(x)$.

3 The right reproductive H_v-semigroup

Some properties of $(\mathbf{r_L})$

1. $x r_L \mathbf{\Omega} = [x], \forall x \in \mathbf{\Omega}$
2. $[x] r_L y = [x] r_L [y] \supset [x], \forall y \in \mathbf{\Omega}$
3. $x^2 = x r_L x = [x], \forall x \in \mathbf{\Omega}$

Proposition 3.1 *The hypergroupoid $(\mathbf{\Omega}, r_L)$ is an H_v-semigroup.*

Proof. We have to prove that the weak associativity holds. Indeed, $\forall x, y, z \in \mathbf{\Omega}$

$$x r_L (y r_L z) = \bigcup_{v \in y r_L z} (x r_L v) = \bigcup_{v \in [y]} (x r_L v) = [x]$$

$$(x r_L y) r_L z = \bigcup_{w \in x r_L y} (w r_L z) = \bigcup_{w \in [x]} (w r_L z) = \bigcup_{w \in [x]} [w] \supset [x]$$

Therefore $(x r_L y) r_L z \supset x r_L (y r_L z)$, so

$(x r_L y) r_L z \cap x r_L (y r_L z) \neq \varnothing, \forall x, y, z \in \mathbf{\Omega}$. $\square$

Proposition 3.2 $\forall x \in \Omega, \Omega r_L x = \Omega$ *and* $x r_L \Omega = [x]$.

Proof.

$$\forall x \in \Omega, \Omega r_L x = \bigcup_{\omega \in \Omega} (\omega r_L x) = \bigcup_{\omega \in \Omega} [\omega] = \Omega.$$

On the other hand,

$$\forall x \in \mathbf{\Omega}, x r_L \mathbf{\Omega} = \bigcup_{\omega \in \Omega} (x r_L \omega) = [x]. \square$$

By propositions (3.1) and (3.2), we get that:

Proposition 3.3 *The hypergoupoid* $(\mathbf{\Omega}, r_L)$ *is a right reproductive* H_v*-semigroup.*

Remark that the right reproductive H_v-semigroup $(\mathbf{\Omega}, r_L)$ is an H_v-group if, $\forall x \in \mathbf{\Omega}$, we have $x r_L \mathbf{\Omega} = \mathbf{\Omega}$.

Proposition 3.4 *The strong associativity of* (r_L) *is valid iff we have*

$$\bigcup_{w \in [x]} (w r_L z) = [x], \forall x, z \in \mathbf{\Omega}$$

Proof. Let $(x, y, z) \in \mathbf{\Omega^3}$, such that $(x r_L y) r_L z = x r_L (y r_L z)$, then

$$(x r_L y) r_L z = x r_L (y r_L z) \Rightarrow [x] r_L z = x r_L [y] \Rightarrow [x] r_L z = [x] \Rightarrow \bigcup_{w \in [x]} (w r_L z) = [x].$$

Now,let $(x, y, z) \in \mathbf{\Omega^3}$, such that

$$\bigcup_{w \in [x]} (w r_L z) = [x]$$

then,

$$(x r_L y) r_L z = [x] r_L z = \bigcup_{w \in [x]} (w r_L z) = [x]$$

$$x r_L (y r_L z) = x r_L [y] = [x]. \square$$

For the hyperoperation (r_L), we shall check conditions such that the strong or the weak commutativity is valid.

Proposition 3.5 . *If* $y \in [x]$ *and* $x \in [y], \forall x, y \in \mathbf{\Omega}$, *then the weak commutativity of* (r_L) *is valid. The strong commutativity of* (r_L) *is valid, iff* $[x] = [y], \forall x, y \in \mathbf{\Omega}$.

Proof. Let $y \in [x]$ and $x \in [y], \forall x, y \in \mathbf{\Omega}$, then

$$y \in [x] \text{ and } x \in [y] \Rightarrow x, y \in [x] \text{ and } x, y \in [y] \Rightarrow [x] \cap [y] \neq \varnothing \Rightarrow$$

$$\Rightarrow (x r_L y) \cap (y r_L x) \neq \varnothing.$$

The proof for the strong commutativity is straightforward.□

Proposition 3.6 *Let* $(\mathbf{\Omega}, +, r_L)$ *be an* H_v*-ring. If* $x r_L \mathbf{\Omega} = \mathbf{\Omega}, \forall x \in \mathbf{\Omega}$ *then the hyperstructure* $(\mathbf{\Omega}, +, r_L)$ *is a dual* H_v*-ring, i.e. both* $(\mathbf{\Omega}, +, r_L)$ *and* $(\mathbf{\Omega}, r_L, +)$ *are* H_v*-rings.*

Proof. From the remark of proposition 3.3, we have that the $(\mathbf{\Omega}, r_L)$ is an H_v-group. For the weak distributivity of $(+)$ with respect to (r_L) we have: $\forall x, y, z \in \mathbf{\Omega}$

$$x + (y r_L z) \supset x + y$$

and

$$(x+y) r_L (x+z) = \bigcup_{s \in x+y, t \in x+z} (s r_L t) \supset \bigcup_{s \in x+y} s = x + y$$

So,

$$[x + (y r_L z)] \cap [(x+y) r_L (x+z)] \neq \varnothing, \forall x, y, z \in \mathbf{\Omega}$$

Similarly, the weak distributivity of $(+)$ with respect to (r_L) from the right side. □

4 Special elements

Since $x \in x r_L y, \forall x, y \in \mathbf{\Omega}$, the next proposition is obvious:

Proposition 4.1 *(a) All the elements of* $\mathbf{\Omega}$ *are right unit elements with respect to* (r_L)*.*
(b) All the elements of $\mathbf{\Omega}$ *are left absorbing-like elements with respect to* (r_L)*.*

Proposition 4.2 *The left scalar elements of the* H_v*-semigroup* $(\mathbf{\Omega}, r_L)$*, are left absorbing elements.*

Proof. Let $u \in \mathbf{\Omega}$ be a left scalar unit element, then $u r_L x = x, \forall x \in \mathbf{\Omega}$. But since $u \in u r_L x, \forall x \in \mathbf{\Omega}$, we get that $u r_L x = u, \forall x \in \mathbf{\Omega}$.□

Proposition 4.3 *The right scalar unit elements of the* H_v*-semigroup* $(\mathbf{\Omega}, r_L)$*, are idempotent elements.*

Proof. Let $\alpha \in \mathbf{\Omega}$ be a right scalar unit element, then $xr_L\alpha = x, \forall x \in \mathbf{\Omega}$. So, $\alpha r_L \alpha = \alpha \Rightarrow \alpha^2 = \alpha$.$\square$

Proposition 4.4 *If there exists $x \in \mathbf{\Omega}$ such that f(x)=x or [x]=x, then x is left absorbing element and every element of $(\mathbf{\Omega}, r_L)$ is right scalar unit of x.*

Proof. Suppose there exist $x \in \mathbf{\Omega}$ such that [x]=x, then $\forall y \in \mathbf{\Omega}$:

$$xr_Ly = [x] = x.$$

That means that x is left absorbing element and every element of $(\mathbf{\Omega}, r_L)$ is right scalar unit of x.$\square$

Since all the elements of the H_v-semigroup $(\mathbf{\Omega}, r_L)$ are right unit elements, let us denote [7] by $I^l_{r_L}(x, y)$ the set of the left inverses of the element $x \in \mathbf{\Omega}$, associated with the right unit $y \in \mathbf{\Omega}$, with respect to the hyperoperation (r_L). The set of the right inverses of the element $x \in \mathbf{\Omega}$, associated with the right unit $y \in \mathbf{\Omega}$, with respect to the hyperoperation (r_L), is denoted by $I^r_{r_L}(x, y)$.

Proposition 4.5 $y \in I^l_{r_L}(x, y)$

Proof. Let $x' \in \mathbf{\Omega}$ such that $x' \in I^l_{r_L}(x, y) \Rightarrow y \in x'r_Lx$. But $\forall x \in \mathbf{\Omega}$ the relation $y \in yr_Lx$ is valid. That means that $y \in I^l_{r_L}(x, y)$.$\square$

Proposition 4.6 $I^r_{r_L}(x, y) = \mathbf{\Omega}$ *if and only if* $y \in [x]$.

Proof. Let $y \in \mathbf{\Omega}$ be right unit element and $x \in \mathbf{\Omega}$,then

$$y \in [x] \Leftrightarrow y \in xr_Lx', \forall x' \in \mathbf{\Omega} \Leftrightarrow x' \in I^r_{r_L}(x, y), \forall x'\mathbf{\Omega} \Leftrightarrow I^r_{r_L}(x, y) = \mathbf{\Omega}.\square$$

Since $x \in [x], \forall x \in \mathbf{\Omega}$,the following is obvious.

Corollary 4.1 $I^r_{r_L}(x, x) = \mathbf{\Omega}$

Remark 4.1 *Notice that, according to the example 2.1, the elements x_4 and x_9 are idempotent elements, since $x_4^2 = x_4$ and $x_9^2 = x_9$. They are, also, left absorbing elements, since $x_4r_Lx = x_4$ and $x_9r_Lx = x_9, \forall x \in \mathbf{\Omega}$. Also, taking for example, the element x_2 of $\mathbf{\Omega}$, notice that $I^l_{r_L}(x, x_2) = \{x_1, x_2, x_3\}$, $\forall x \in \mathbf{\Omega}$. Even more, since $x_2 \in [x_1]$ we get that $I^r_{r_L}(x_1, x_2) = \mathbf{\Omega}$ and $I^r_{r_L}(x_1, x_1) = \mathbf{\Omega}$.*

5 Applications

As we mentioned above, the noise pollution in urban areas coming from a spot, annoys a certain area in which the noisy spot belongs to. That was the motivation which led to the mathematical expression $xr_Ly = [x], \ \ \forall x, y \in \mathbf{\Omega}$. That means that if a city is considered as a set $\mathbf{\Omega}$ with elements its buildings (or spots which could produce noise pollution), then every building (or a spot) x, which is a source of noise pollution, together with any other building (or a spot) y of the city, will affect anyhow the noise pollution area [x], where $x \in [x]$ and maybe y. It is clear, that the source of the noise pollution x, could not be seen as the center of a cyclic disk, but as any spot of a certain area which is affected by x.

We shall try to explain some of the properties of the noise hyperoperation (r_L) developed above, in terms of noise pollution problems in urban areas. The property $x \in xr_Ly, \forall y \in \mathbf{\Omega}$ means that the building x, as a source of noise pollution, first of all, annoys the residents of the building x.
The property $r_L[y] = [x]$ means that the source of noise pollution x together with any region [y] is not only independent on the spots of the region [y] but the noise pollution region remains [x], as well.
The property $[x]r_Ly = [x]r_L[y] \supset [x]$ means that the noise pollution region that results when either the noise pollution region [x] operates with the spot y or with the region [y], is the same and anyhow this noise pollution region is bigger than [x].
The property $xr_Ly = xr_Lz$ means that [x] remains the noise pollution region when x as a source of noise pollution affects any other spot of the city $\mathbf{\Omega}$. Continuously, the relation $xr_L\mathbf{\Omega} \neq \mathbf{\Omega}$ means that, the noise pollution region coming from spot x, can't affect the whole city $\mathbf{\Omega}$. The weak associativity which is expressed by the inclusion on the left parenthesis, i.e. $(xr_Ly)r_Lz \supset xr_L(yr_Lz)$ actually means that, the noise pollution region coming from the noise pollution region [x] together with any spot, is not only bigger than that one which comes from the noise pollution spot x together with any other region but includes it, as well.
An absorbing element, as in the relation $\alpha r_L x = \alpha$, could be considered as a spot surrounded by a wall or a forest, which doesnt annoy any other spot of the city $\mathbf{\Omega}$.

Since the weak associativity is valid, the concept of transitive closure can be applied here, in order to obtain the fundamental β^* classes. The actual meaning of this situation is that the city $\mathbf{\Omega}$ can be divided, using the noise hyperoperation, in a partition, where every fundamental class does not annoy any other blocks of flats from other fundamental classes. The next example gives an idea:

Example 5.1 *According to the example 2.1, consider now that* $\mathbf{\Omega}$ *is a city where* $\mathbf{\Omega} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$*. From "multiplication" table of* (r_L)*,we obtain that* $\beta^*(x_1) = \{x_1, x_2, x_3\}$*,*

$$\beta^*(x_4) = \{x_4\}, \beta^*(x_5) = \{x_5, x_6, x_7\}, \beta^*(x_8) = \{x_8, x_9\}$$

. So,the fundamental semi-group $\mathbf{\Omega}/\beta^*$ *is:*

$$\mathbf{\Omega}/\beta^* = \big\{\{x_1, x_2, x_3\}, \{x_4\}, \{x_5, x_6, x_7\}, \{x_8, x_9\}\big\}$$

and the "multiplication" table is:

$\circ$	$\underline{\mathbf{x_1}}$	$\underline{\mathbf{x_4}}$	$\underline{\mathbf{x_5}}$	$\underline{\mathbf{x_8}}$
$\underline{\mathbf{x_1}}$	$\underline{x_1}$	$\underline{x_1}$	$\underline{x_1}$	$\underline{x_1}$
$\underline{\mathbf{x_4}}$	$\underline{x_4}$	$\underline{x_4}$	$\underline{x_4}$	$\underline{x_4}$
$\underline{\mathbf{x_5}}$	$\underline{x_5}$	$\underline{x_5}$	$\underline{x_5}$	$\underline{x_5}$
$\underline{\mathbf{x_8}}$	$\underline{x_8}$	$\underline{x_8}$	$\underline{x_8}$	$\underline{x_8}$

In other words and beyond the mathematical content of the present example, the city $\mathbf{\Omega}$ was divided into four regions, where every region (fundamental class) does not annoy any other spot belonging to the rest regions. So, one could consider that among the four regions there exists a green park, full of trees, which absorbs the possible noise pollution caused by any of the four regions. Since $\beta^*(x_4) = \{x_4\}$, the element $x_4 \in \mathbf{\Omega}$ (spot or building of the city) is a single element and that means that it doesn't annoy any other spot of the city $\mathbf{\Omega}$, so it can be considered as the remotest spot of the city.

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RATIO MATHEMATICA 23 (2012), 65–80 ISSN: 1592-7415

EL–hyperstructures: an overview

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Abstract

This paper gives a current overview of theoretical background of a special class of hyperstructures constructed from quasi / partially ordered (semi) groups using a construction known as the "Ends lemma". The paper is a collection of both older and new results presented at AHA 2011.

Key words: group, hyperstructure, partial ordering, quasi ordering, semigroup.

MSC2010: 20N20.

1 Introduction

This paper is a written version of a lecture given at the *11th International Conference on Algebraic Hyperstructures and Applications* in October 2011. Together with results new at that time I presented some older results published in journals or proceedings of rather local impact thus unknown to the international hyperstructure community. The new results presented at the conference were meanwhile published as [20]. Therefore, this article instead of presenting new unpublished results gives an overview of what has so far been achieved in the area of *EL*–hyperstructures. For proofs of respective theorems as well as for examples explaining their meaning cf either [20] or works indicated throughout the paper. Where not stated otherwise all definitions of hyperstructure concepts or properties of hyperstructures are used in the sense of the standard book [3].

2 The "Ends lemma"

The *EL*–hyperstructures are hyperstructures constructed from quasi / partially ordered (semi)groups using the "Ends lemma", which has the form of the following theorems from [4]. The notation $[a)_\leq$ used below stands for the set $\{x \in H; a \leq x\}$, where the properties of H are specified in the respective theorems (typically $(H, \cdot, \leq)$ is a quasi-ordered / partially ordered set / semigroup / group).

Lemma 2.1 *([4], Theorem 1.3, p. 146) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}'(S)$ defined by*

$$a * b = [a \cdot b)_\leq \tag{1}$$

is associative. The semi-hypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

In accordance with other papers regarding this topic, the hyperstructure $(S, *)$ constructed in this way will further on be called the *associated* hyperstructure to the single-valued structure $(S, \cdot)$ or *an "Ends lemma"–based hyperstructure*, or an *EL–hyperstructure* for short. Instead of S the carrier set will from Lemma 2.3 onward be denoted by H.

Lemma 2.2 *([4], Theorem 1.4, p. 147) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

1^0 *For any pair $(a, b) \in S^2$ there exists a pair $(c, c') \in S^2$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$*

2^0 *The associated semi-hypergroup $(S, *)$ is a hypergroup.*

Remark 2.1 *If $(S, \cdot, \leq)$ is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S, \cdot, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.*

Remark 2.2 *The wording of the above lemmas is the exact translation of theorems from [4]. The respective proofs, however, do not change in any way, if we regard* quasi-ordered *structures instead of* partially ordered *ones as the anti-symmetry of the relation $\leq$ is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi–ordered structures.*

The "Ends lemma" was later extended (cf e.g. [23]). Notice that if $(H, \cdot)$ is commutative, then $(H, *)$ is a join space. Also notice that unlike in the original "Ends lemma" the underlying single-valued structure in the following theorem is a group (not a semigroup).

Lemma 2.3 *([23], Theorem 4) Let $(H, \cdot, \leq)$ be a quasi-ordered group and $(H, *)$ be the associated hypergroupoid. Then $(H, *)$ is the transposition hypergroup.*

Initially, the typical use of the "Ends lemma" was creating hyperstructures and proving or deriving their properties at random without any (or with a very limited) theoretical background. This model is used in e.g. [6, 8, 13, 21, 23]. In order to overcome this inconvenience, theoretical background of the "Ends lemma" is being developed.

3 Extending the lemma, identities and inverses

After the "Ends lemma" extension, i.e. Lemma 2.3, was proved, there arose a question of whether one can go any further to stronger hyperstructures such as canonical hypergroups, strongly canonical hypergroups, etc. A positive answer to this question would mean that numerous ring-like analogies of *EL*–hyperstructures could be studied extensively. Unfortunately, the answer – obtained in [18] – turned out to be negative.

Theorem 3.1 *Let $(H, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(H, *)$ be its associated transposition hypergroup. Then:*

1. *$(H, *)$ does not have a scalar identity.*
2. *Regardless of commutativity, $(H, *)$ cannot be a canonical hypergoup.*

Naturally, if we regard the definition of a canonical hypergroup, 2 immediately follows from 1.

Once it was established that looking for scalar identities in *EL*–hyperstructures based on groups is of no point, at least the issue of identities was explored. In [18] the following simple results concerning identities were obtained.

Theorem 3.2 *Let $(H, *)$ be the semi-hypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$ with the identity u.*

1. *An element* $e \in H$ *is an identity of* $(H,*)$ *if and only if* $e \leq u$.

2. *If* $(H,\cdot)$ *is a group, then the identity of* $(H,\cdot)$ *is an identity of* $(H,*)$.

Again, 2 is naturally an immediate corollary of 1 yet we set it aside due to uniqueness of the single-valued group identity.

Lemma 3.1 *Let* $(H,*)$ *be the join space associated to a quasi-ordered commutative group* $(H,\cdot,\leq)$. *If an element* $e \in H$ *is an identity of* $(H,*)$, *then* $e \leq e^{-1}$.

Since the concept of an *inverse* in a hyperstructure is defined using the concept of an identity, the issue of inverses was touched upon in the same paper and the following result was obtained for the set $i(a)$ of inverses of an arbitrary element $a \in H$ in $(H,*)$.

Theorem 3.3 *Let* $(H,*)$ *be the transposition hypergroup associated to a quasi-ordered group* $(H,\cdot,\leq)$. *Then for an arbitrary* $a \in H$ *there holds*

$$i(a) = \{a' \in H; a' \leq a^{-1}\} = (a^{-1}]_{\leq},$$

where a^{-1} *is the inverse of* a *in* $(H,\cdot)$.

Corollary 3.1 *Let* $(H,*)$ *be the transposition hypergroup associated to a quasi-ordered group* $(H,\cdot,\leq)$. *Then* $(H,*)$ *is regular.*

4 Ring-like hyperstructures

Since it turned out that once we start with groups the "Ends lemma" cannot effectively be used to construct canonical hypergroups, the scope for use of this idea in the area of ring-like hyperstructures narrowed. Recall that there is a great variety of definitions of ring-like hyperstructures, yet many of them including the most often used one – that of *Krasner hyperring* – is built on canonical hypergroups. However, the idea of limits of the "Ends lemma" in the area of hyperstructures with two (hyper)operations is still worth exploring. In [19] (published before the author was able to get [12]) three possible extensions are suggested and explored:

1. Let $(H,+)$ and $(H,\cdot)$ be two single-valued structures. We can define a hyperoperation using one of the operations $+$ or $\cdot$ by e.g. $a*b = [a+b)_{\leq}$ – thus we get an *EL*–hyperstructure $(H,*)$. The hyperstructure will then be a triplet $(H,*,\cdot)$ where $*$ is a hyperoperation based on the single-valued operation $+$.

2. Let $(H,+)$ and $(H,\cdot)$ be two single-valued structures. We can define two hyperoperations, each based on one single-valued operation, i.e. for an arbitrary pair $(a,b) \in H^2$ we can define $a * b = [a+b)_\leq$ and $a \circ b = [a \cdot b)_\leq$. Thus we get a triplet $(H,*,\circ)$, where $*$ and $\circ$ are hyperoperations.

3. However, we can also start with a single single-valued structure $(H,\cdot)$ and using it define a hyperoperation $*$ by $a * b = [a \cdot b)_\leq$. The hyperstructure will then be a triplet $(H,*,\cdot)$ where $*$ is a hyperoperation based on the single-valued operation $\cdot$.

If we have a triplet $(H,+,\cdot)$, where symbols $+$ and $\cdot$ may stand for both single-valued and multivalued operation, then for an arbitrary triplet $(a,b,c) \in H^3$ we may either ask that $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$ or we may ask that inclusions holds instead of equalities.[1] Each of the three ways to create ring-like hyperstructures was explored with respect to both of these types of distributivity and the following results were obtained in [19]. Notice the variety of conditions imposed on the respective structures (group / semigroup, quasi-ordered / partially ordered, single-valued / multivalued).

Definition 4.1 [cf [28], p. 21, included as plain text] $(R,+,.)$ *is a hyperring in the general sense if* $(R,+)$ *is a hypergroup*[2], $(\cdot)$ *is associative hyperoperation and the distributive law*[3] $x(y+z) \subseteq xy + xz$, $(x+y)z \subseteq xz + yz$ *is satisfied for every* x,y,z *of* R. *Additive hyperring is the one of which only* $(+)$ *is a hyperoperation, multiplicative hyperring is the one of which only* $(\cdot)$ *is a hyperoperation.* [...] $(R,+,\cdot)$ *will be called semihyperring if* $(+),(\cdot)$ *are associative hyperoperations, where* $(\cdot)$ *is distributive with respect to* $(+)$. *The rest of definitions are analogous. If the equality in the distributive law is valid, then the hyperring is called strong or good.*

Theorem 4.1 *Let* $(H,+,\cdot)$ *be a ring such that* $(H,+)$ *is a group,* $(H,\cdot)$ *a semigroup and* $\leq$ *quasi-ordering on* H *such that* $(H,+,\leq)$ *is a quasi-ordered group and* $(H,\cdot,\leq)$ *is a quasi-ordered semigroup. Further, for an arbitrary pair of elements* $(a,b) \in H^2$ *define* $a * b = [a+b)_\leq$ *and* $a \circ b = [a\cdot b)_\leq$. *Then* $(H,*,\circ)$ *is a hyperring in the general sense.*

[1] The former distributivity is sometimes called *good* distributivity while the latter is often called *weak* distributivity.

[2] Vougiouklis uses the term *hypergroup of Marty*.

[3] Vougiouklis uses the sign $\subset$ in the sense of $\subseteq$.

Theorem 4.2 *Let $(H,+)$ be a semigroup, $(H,\cdot)$ a group and $\leq$ quasi-ordering on H such that $(H,+,\leq)$ is a quasi-ordered semigroup and $(H,\cdot,\leq)$ is a quasi-ordered group. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a * b = [a+b)_\leq$ and $a \circ b = [a \cdot b)_\leq$. Finally, let $\cdot$ distribute over $+$ from both left and right. Then $(H,*,\circ)$ is a good semihyperring in the sense of Definition 4.1.*

Theorem 4.3 *Let $(H,+,\cdot)$ be a ring such that $(H,+)$ is a group with neutral element 0, $(H\setminus\{0\},\cdot)$ a group and $\leq$ quasi-ordering on H such that $(H,+,\leq)$ and $(H,\cdot,\leq)$ are quasi-ordered groups. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a*b = [a+b)_\leq$ and $a \circ b = [a \cdot b)_\leq$. Then $(H,*,\circ)$ is a good hyperring in the general sense.*

Theorem 4.4 *Let $(H,+)$ be a group and $(H,\cdot,\leq)$ a quasi-ordered semigroup and for an arbitrary pair of elements $(a,b) \in H^2$ define $a \circ b = [a \cdot b)_\leq$. Further, let $(H,+,\cdot)$ be such that the operation $\cdot$ distributes over the operation $+$ from both left and right. Then $(H,+,\circ)$ is a good multiplicative hyperring.*

Definition 4.2 [cf [14], Definition 2.1 and Remark] *A hyperalgebra $(R,+,\cdot)$ is called a semihyperring if and only if*

(i) *$(R,+)$ is a semihypergroup;*

(ii) *$(R,\cdot)$ is a semigroup;*

(iii) *$\forall a,b,c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$*

If we replace (iii) by

$$\forall a,b,c \in R, a \cdot (b+c) \subseteq a \cdot b + a \cdot c \text{ and } (b+c) \cdot a \subseteq b \cdot a + c \cdot a$$

we say that R is a weak distributive semihyperring. A semihyperring is called with zero element, if there exists a unique element $0 \in R$ such that $0 + x = x = x + 0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in R$. [...] A semihyperring is called a hyperring provided $(R,+)$ is a canonical hypergroup.

Theorem 4.5 *Let $(H,\cdot,\leq)$ be a quasi-ordered semigroup such that $\cdot$ is a commutative idempotent operation. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a \circ b = [a \cdot b)_\leq$. Then $(H,\circ,\cdot)$ is a weak distributive semihyperring.*

Unfortunately, from Theorem 3.1 there follows that Krasner hyperrings cannot be constructed using the "Ends lemma" if the underlying single-valued structure $(H,+)$ is a group. However, there are weaker structures such as e.g. *hyperringoids*, which are defined as semihyperrings in the sense of Definition 4.2 where $(R, +)$ is a join space[4], for the construction of which the "Ends lemma" might still be used.

The assumptions of the following theorem seem rather complicated. The reason is simple: the requirement "$(H, +)$ is a group" results in trivialities. Notice that condition 1 is the condition used in Lemma 2.1 – the one which secures that $(H, *)$ is a hypergroup.

Theorem 4.6 *Let $(H, +)$ be a commutative semigroup, $(H, \cdot)$ a group and $\leq$ quasi-ordering on H such that*

1. *to every pair of elements $(a, b) \in H^2$ such that $a \leq b$ there exists a pair of elements $(c, c') \in H^2$ such that $b + c \leq a$, $c' + b \leq a$,*
2. *$(H, +, \leq)$ is a quasi-ordered semigroup and*
3. *$(H, \cdot, \leq)$ is a quasi-ordered group.*

Moreover, for an arbitrary pair of elements $(a, b) \in H^2$ define $a * b = [a + b)_{\leq}$. Finally, suppose that $\cdot$ distributes over $+$ from both left and right. Then

if $(H, *)$ satisfies the transposition axiom, then $(H, *, \cdot)$ is a hyperringoid.

Corollary 4.1 *If in Theorem 4.6 we suppose that $(H, +, \leq)$ is a quasi-ordered semigroup without any further assumptions, then $(H, *, \cdot)$ is a semi-hyperring in the sense of Definition 4.2.*

Theorem 4.7 *Let $(H, +, \leq)$ be a non-trivial quasi-ordered group with neutral element 0 such that $\leq$ is not the identity relation, $(H \setminus \{0\}, \cdot)$ a group. Moreover, for an arbitrary pair of elements $(a, b) \in H^2$ define $a * b = [a + b)_{\leq}$. Finally, suppose that $\cdot$ distributes over $+$ from both left and right. Then $(H, *, \cdot)$ is a weak distributive hyperringoid.*

5 The issue of a subhyperstructure

In order to proceed to the study of properties of *EL*–hyperstructures, one must clarify the concept of a subhyperstructure of an *EL*–hyperstructure

[4] This definition is used in [3], Chapter 6. On contrary Massouros brothers in [17] call this hyperstructure a *join* hyperringoid, while they call hyperringoid a hyperstructure such that $(R, +)$ is a hypergroup only. Further on in Theorem 4.6 the definition used in [3] is regarded.

since there are two possible approaches to it. If we regard a quasi-ordered semigroup $(H, \cdot, \leq)$ and define a hyperoperation $*$ on H by

$$a * b = [a \cdot b)_{\leq} = \{x \in H; a \cdot b \leq x\} \tag{2}$$

for an arbitrary pair of elements $(a, b) \in H^2$, we may in a subsemigroup $(G, \cdot)$ of the semigroup $(H, \cdot)$ set either

$$a *_G b = [a \cdot b)_{\leq_G} = \{x \in G; a \cdot b \leq x\} \tag{3}$$

or

$$a *_H b = [a \cdot b)_{\leq_H} = \{x \in H; a \cdot b \leq x\} \tag{4}$$

and thus create either $(G, *_G)$ or $(G, *_H)$. None of these concepts is obviously the only possible and "correct" one since the definition of both can be justified. In a way, the idea of $*_H$ conforms to the idea of the "Ends lemma" better. Thus in [22], which discussed the issue of subhyperstructures of *EL*–hyperstructures, it was this concept that was favoured. The concept of the *upper set* was introduced.[5]

Definition 5.1 *Let $(H, \cdot, \leq)$ be a partially ordered semigroup and let G be a nonempty subset of H.*

1. *If for an arbitrary element $g \in G$ there holds $[g)_{\leq} \subseteq G$, we call G an upper end of H.*

2. *If there exists an element $g \in G$ such that there exists an element $x \in H \setminus G$ such that $g \leq x$ (i.e. $x \in [g)_{\leq}$)[6], we say that G is not an upper end of H because of the element x.*

Among other results concerning hyperoperation $*_H$ (4) the following was proved.

Theorem 5.1 *Let $(H, *)$ be the semihypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$. Suppose that G is an upper end of H. If $(G, \cdot)$ is a subgroup of $(H, \cdot)$, then $(G, *)$ is a subhypergroup of $(H, *)$.*

[5] The concept itself is naturally not a new invention; the definition was only tailored for use in the "Ends lemma" context.

[6] We could – probably more properly since $x \notin G$ – write $g < x$ and $x \in [g)_{\leq} \setminus \{g\}$ yet in the definition we keep the $\leq$ notation of the Ends lemma for consistency reasons.

Proposition 5.1 *Let $(H, *)$ be the semihypergroup associated to a partially ordered semigroup $(H, \cdot, \leq)$ and $G \subseteq H$ nonempty. If $(G, *)$ is a subhypergroup of $(H, *)$, then $(G, \cdot)$ is a subsemigroup of $(H, \cdot)$ and G is an upper end of H such that for any pair $(a, b) \in G^2$ there exists a pair $(c, c') \in G^2$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$.*

Theorem 5.2 *Let $(H, *)$ be the semihypergroup associated to a partially ordered semigroup $(H, \cdot, \leq)$. Further, let $G \subseteq H$ be non-empty and such that $(G, \cdot)$ is a subgroupoid of $(H, \cdot)$, and let the relation $\leq_G$ be a restriction of $\leq$ on G, i.e. $\leq_G = \leq \cap (G \times G)$.*[7] *Finally – if it exists – denote u the identity of $(H, \cdot)$ and define a new hyperoperation $*_G : G \times G \to P^*(G)$ for arbitrary elements $a, b \in G$ by (3), i.e. by*

$$a *_G b = [a \cdot b)_{\leq_G} = \{x \in G; a \cdot b \leq_G x\}.$$

Then

1. *$(G, \cdot)$ is a semigroup if and only if $(G, *_G)$ is a semihypergroup.*
2. *$(G, \cdot)$ is a monoid if and only if $(G, *_G)$ is a semihypergroup and $u \in G$.*
3. *If $(G, \cdot)$ is a group, then $(G, *)$ is a transposition hypergroup.*
4. *If $(G, *)$ is a hypergroup, then $(G, \cdot)$ is a semigroup such that for any pair $(a, b) \in G^2$ there exists a pair $(c, c') \in G^2$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$.*

6 Properties of *EL*–hyperstructures and their subhyperstructures

Results in this section were presented at AHA 2011 and later included in [20].

Theorem 6.1 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$ and $(G, \cdot)$ its subgroup such that G is an upper end of H. Then $(G, *)$, where $*$ is defined for an arbitrary pair $(a, b) \in H^2$ as $a * b = [a \cdot b)_{\leq_H}$, is invertible and closed in H.*

[7] The exact quote from [22] at this place reads "for arbitrary elements $a, b \in G$ let $a \leq b \Rightarrow a \leq_G b$".

Theorem 6.2 *Let $(H,*)$ be the hypergroup associated to a quasi-ordered group $(H,\cdot,\leq)$ and $(G,*)$ its arbitrary subhypergroup associated to a subgroup $(G,\cdot)$ of $(H,\cdot)$, where G is an upper end of H (i.e. as defined in Theorem 5.1 using hyperoperation $*_H$). Denote u the identity of $(H,\cdot)$. Then*

1. *G is ultraclosed if and only if for any $h\in H$ such that $h\leq u$ it follows that $h\in G$.*
2. *If $G\neq H$ and if $(H,\cdot,\leq)$ has the smallest element, then $(G,*)$ is not ultraclosed.*
3. *If $(H,\cdot)$ or $(H,*)$ is commutative, then G is a complete part of H if and only if for every $h\in H$ such that $h\leq u$ there is $h\in G$.*

Theorem 6.3 *Let $(H,*)$ be the associated hypergroup of a quasi-ordered group $(H,\cdot,\leq)$ and $(G,*)$ its arbitrary subsemihypergroup associated to a subsemigroup $(G,\cdot)$ of $(H,\cdot)$.*[8] *If for arbitrary $x\in H$ and $g\in G$ there holds $x\cdot g\cdot x^{-1}\in G$, then $(G,*)$ is normal.*

Corollary 6.1 *Let $(H,*)$ be the hypergroup associated to a quasi-ordered group $(H,\cdot,\leq)$ and $(G,\cdot)$ its normal subgroup such that G is an upper end of H. Then $(G,*)$, where $*$ is defined as $a*b=[a\cdot b)_{\leq_H}$, is reflexive.*

Theorem 6.4 *Let $(H,*)$ be the hypergroup associated to a quasi-ordered group $(H,\cdot,\leq)$. Then $(H,*)$ is reversible.*

As far as *regularity* of *EL*–hyperstructures is concerned, cf Theorem 3.3 and its corollary. In the following theorem notice that by a subhypergroup we mean a subhyperstructure defined by hyperoperation $*_H$ (4). This is important to consider since the definition of inner irreducibility relies on subhyperstructures as a commutative hypergroup $(H,*)$ is called *inner irreducible* if for any pair of its subhypergroups G_1,G_2 such that $G_1*G_2=H$ there holds $G_1\cap G_2\neq\emptyset$.

Theorem 6.5 *Let $(H,*)$ be the associated hypergroup of a partially ordered commutative group $(H,\cdot,\leq)$.*

1. *If for every $x\in H$ such that x,x^{-1} are incomparable with respect to $\leq$ there is either $[x)_\leq\cap[x^{-1})_\leq\neq\emptyset$ or $(x]_\leq\cap(x^{-1}]_\leq\neq\emptyset$, then $(H,*)$ is inner irreducible.*

[8] In this context the fact whether we define a subhyperstructure by means of $*_H$ (4) or $*_G$ (3) is not important.

2. *If $(H, \leq)$ is a linear ordered set or if $(H, \leq)$ has the smallest or the greatest element, then $(H, *)$ is inner irreducible.*

Naturally, 2 is a corollary of 1 since in linear ordered sets all elements are comparable.

7 The issue of origins of a hypergroup

The "Ends lemma" describes a way to construct semihypergroups from quasi-ordered semigroups and hypergroups from semigroups with a special property. We know that groups are such structures that this property holds trivially. This means that we know that using the "Ends lemma" we may create a hypergroup from a group.

Thus one can ask: *if have a hypergroup (itself or as a subhypergroup of a larger structure) created in the "Ends lemma" fashion, is there a way to determine whether its underlying single-valued structure is a group or a semigroup?* Answering this question is not academic only as proofs of some of the above theorems have to answer this question in a rather complicated way. This issue is also connected to the issue of the converse of the "Ends lemma", which was already necessary to complete some proofs of theorems on subhyperstructures. The proof of the following theorem can be found in [22].

Theorem 7.1 *Let $(H, \cdot)$ be a non-trivial groupoid and $\leq$ a binary partial ordering on H such that for an arbitrary pair of elements $(a, b) \in H^2$, $a \leq b$, and for arbitrary $c \in H$ there holds $c \cdot a \leq c \cdot b$ and $a \cdot c \leq b \cdot c$. Further define a hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ for an arbitrary pair of elements $(a, b) \in H^2$ by $a * b = [a \cdot b)_\leq = \{x \in H; a \cdot b \leq x\}$. Then if the hyperoperation $*$ is associative, then the single-valued operation $\cdot$ is associative too. Furthermore, if there exists an element $e \in H$ such that for every $a \in H$ there holds $a * e = e * a = [a)_\leq$, then this element e is the identity of the semigroup $(H, \cdot)$.*

Notice that if the relation $\leq$ is not antisymmetric, the above theorem is not true. This is caused by the fact that only for antisymmetric relations $\leq$ there holds that $[a)_\leq = [b)_\leq$ implies that $a = b$. Indeed, suppose a simple two element set $M = \{a, b\}$ where the relation $\leq$ is defined as $a \leq a, a \leq b, b \leq a, b \leq b$. This reflexive and transitive relation $\leq$ is obviously not antisymmetric and there holds $[a)_\leq = [b)_\leq$ yet $a \neq b$.

A simple way of distinguishing between semigroups and groups is the study of idempotent elements, i.e. elements which (if we ignore the identity) exist in semigroups only. In [20] a few basic results concerning idempotent elements are included.

Theorem 7.2 *Let $(H, *)$ be the semihypergroup associated to a quasi–ordered semigroup $(H, \cdot, \leq)$. For an arbitrary element $a \in H$ there holds*

$a * a = \{a\} \Leftrightarrow$ *a is an idempotent and simultaneously a maximal element of $(H, \cdot, \leq)$.*

Corollary 7.1 *Let $(H, *)$ be the associated hypergroup of such a quasi-ordered semigroup $(H, \cdot, \leq)$ that at least two distinct elements $a, b \in H$ are in relation $\leq$ (i.e. $\leq$ is not trivial). If there exists an element $a \in H$ such that $a * a = \{a\}$, then $(H, \cdot)$ is not a group.*

Corollary 7.2 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$ and $(G, *)$ a subhypergroup of $(H, *)$.*[9]

1. *Denote u the identity of $(H, \cdot)$. If u is the maximal element of $(G, \leq)$ and at least two distinct elements $a, b \in G$ are in relation $\leq$, then $(G, \cdot)$ is a subsemigroup of $(H, \cdot)$ yet not a subgroup of $(H, \cdot)$.*

2. *If for two distinct elements $a, b \in H$ there holds $a * a = \{a\}$, $b * b = \{b\}$, then H does not have the greatest element. Also – obviously – $(H, \cdot)$ is not a group.*

8 "Ends lemma" in a broader context

Naturally, the "Ends lemma" is not a revolutionary stand alone concept. The study of relation of hyperstructures and ordered sets or binary relations is included in [3] as chapter 3. This part of the "canonical" book on hyperstructures was inspired by works of Chvalina (especially [5]) and Rosenberg (especially [24]). Results concerning the relation of hyperstructures and ordered sets have also been included in [12], another "canonical" book on hyperstructure theory. Among older works concerning relation of ordered sets and hyperstructures there is e.g. [2], which studies the relation in general giving a number of possible ways to create hyperstructures from ordered sets, and [29], in which a concept in a way similar to the "Ends lemma" may be found. Recent works related to the concept discussed in this article include e.g. works of Cristea and Ştefănescu which deal with n–ary relations on hypergroups (e.g. [9, 10]) or study of fundamental relations on hypergroupoids associated with binary relations (such as [11]), or works of e.g. Spartalis or Massouros (such as [16, 25, 26]).

[9] The statement is valid for subhypergroups based on both $*_H(4)$ or $*_G(3)$.

9 Open issues

There are many loose ends that wait to be tied. Most importantly, the full potential of the property included in Lemma 2.1 must be explored. This is closely connected to the problem of reversing the "Ends lemma", which has been partly answered by Theorem 7.1, and to the problem of telling the origins of the hypergroup for which the idea of idempotent elements is only a first and insufficient attempt. Clarifying this issue would also help in the study of ring-like *EL*–hyperstructures and in the study of such properties of *EL*–hyperstructures that rely on the concept of a subhypergroup.

Naturally, it might be very useful to set the issue of *EL*–hyperstructures in a broader perspective of hyperstructures constructed from binary operations of single-valued (semi)groups.

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RATIO MATHEMATICA 23 (2012), 51–64 ISSN:1592-7415

Directed Graphs representing isomorphism classes of C-Hypergroupoids

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Abstract

We investigate the relation of directed graphs and hyperstructures by virtue of the *graph hyperoperation*. A new class of graphs arises in this way representing isomorphism classes of **C**-hypergroupoids and we present the 17 such graphs that correspond to the 73 **C**-hypergroupoids associated with binary relations on three element sets. As it is shown they constitute an upper semilattice with respect to graph inclusion.

Key words: Hyperoperations, Hypergroupoids, Directed Graphs.

MCS2010: 20N20, 68R10, 97K30.

1 Introduction

The correlation between hyperstructures and binary relations has been intensively investigated in the last 20 years by several researchers ([10], [11], [12], [14], [18], [6], [7], [8]) while of particular importance are the hypergroupoids that derive from binary relations - known as ***C**-hypergroupoids* - which were introduced by Corsini in [9] (see also [19], [20], [22], [21]).

The purpose of the present paper is to further expand the ongoing research on hypergroupoids by employing concepts from Graph Theory. While the foundations of graph theory can be traced back to L. Euler and his "Königsberg bridge problem" [1] (see also [13, 2]), its growth in recent years has been explosive covering a large number of disciplines ranging from mathematical foundations of computer science [3, 4] to physical chemistry [5] and natural language processing [17]. In Section 2, basic concepts and results on directed graphs and hypergroupoids are presented. In Section 3 we define the *graph hyperoperation* which is actually Corsini's hyperoperation applied on graphs. A particular class of directed graphs arises in this way, representing isomorphism classes of **C**-hypergroupoids, namely *Corsini's graphs*. As it is evident from their construction, Corsini's graphs constitute a useful apparatus in order to represent and arrange the hypergroupoid classes they represent which thus results in a hierarchy inside the class of ***C**-hypergroupoids*. We identify and present the 17 Corsini's graphs with 3 nodes and we find that they constitute an upper semilattice with graph inclusion as the partial order.

2 Preliminaries on Hyperstructures and Graph Theory

A partial hypergroupoid is a pair $(H, *)$, where H is a non-empty set, and $*$ is a hyperoperation i.e.

$$* : H \times H \to \mathcal{P}(H), \quad (x, y) \mapsto x * y.$$

If $A, B \in \mathcal{P}(H)$-$\{\emptyset\}$, then

$$A * B = \bigcup_{a \in A, b \in B} a * b.$$

We denote by $a * B$ (respectively, $A * b$) the hyperproduct $A * B$ in the case that the set A (respectively, the set B) is the singleton $\{a\}$ (respectively, $\{b\}$). Moreover, $(H, *)$ is called hypergroupoid if $x * y \neq \emptyset$, for all $x, y \in H$ and it is called a degenerative (respectively, total) hypergroupoid in the case that for all $x, y \in H$, $x * y = \emptyset$ (respectively, $x * y = H$).

Given a binary relation $R \subseteq H \times H$ the *Corsini's hyperoperation* (cf. [9])

$$*_R : H \times H \to \mathcal{P}(H)$$

is defined in the following way:

$$(x, y) \mapsto x *_R y = \{z \in H \mid (x, z) \in R \text{ and } (z, y) \in R\}.$$

The hyperstructure $(H, *_R)$ is called *Corsini's partial hypergroupoid associated with the binary relation* R or simply *partial* ***C****-hypergroupoid* and is denoted H_R (cf. [19], [20]). In the case that $x *_R y \neq \emptyset$, for all $x, y \in H$, then $(H, *_R)$ is called **C**-hypergroupoid. It can be easily seen that a partial **C**-hypergroupoid H_R is a **C**-hypergroupoid if and only if it holds $R \circ R = H \times H$, where $\circ$ is the usual relation composition. Let $R \subseteq H \times H$ be a binary relation on the set $H = \{x_1, x_2, \ldots, x_n\}$ then the $n \times n$ matrix

$$M_R = [m_{i,j}]_{n\times n}, \text{ with } m_{i,j} = 1 \text{ if } (x_i, x_j) \in R \text{ and } m_{i,j} = 0, \text{ else}$$

is called the boolean matrix of R.

Formally a *concrete directed graph* G is a pair (V_G, E_G) where:

- V_G is a finite set, the elements of which we call vertices and
- $E_G \subseteq V_G \times V_G$ is a set of ordered pairs of V_G the elements of which we call edges.

A vertex is simply drawn as a node and an edge as an arrow connecting two vertices the head and the tail of the edge. A graph $G' = (V_{G'}, E_{G'})$ is a *subgraph* of the graph $G = (V_G, E_G)$ if it holds $V'_G \subseteq V_G$ and $E'_G \subseteq E_G$. In the other direction, a *supergraph* of a graph G is a graph that has G as a subgraph. We say that the graph H is included in the graph G ($H \leq G$) if G has a subgraph that is equal or isomorphic to H. The relation $\leq$, which is called *graph inclusion*, is a graph invariant.

Given a graph $G = (V_G, E_G)$ and $v \in V_G$ the number of edges that "leave" the vertex v is called the *out degree* of v and the number of edges that "enter" the vertex is called the *in degree* of v. Moreover we denote by $degreeout(G)$ the set of all the out degrees of G's vertices and similarly for $degreein(G)$. The *order* of a graph is the number of its vertices, i.e. $|V_G|$, and the size of a graph is the number of its edges, i.e. $|E_G|$. A loop is an edge whose head and tail is the same vertex. An edge is multiple if there is another edge with the same head and the same tail. A graph is called simple if it has no multiple edges. A vertex is called isolated if there is no edge connected to it.

Two graphs G and H are said to be isomorphic if there exists an isomorphism f between the vertices of the two graphs that respects the edges, i.e. it holds $(x, y) \in E_G$ if and only if $(f(x), f(y)) \in E_H$. Since the specific sets V_G, E_G chosen to define a concrete directed graph G are actually irrelevant we don't distinguish between two isomorphic graphs. Hence the following definition of an abstract graph. The equivalence class of a concrete directed graph with respect to isomorphism is called an *abstract directed graph* or simply *graph*. A graph property is called *invariant* if it is invariant under graph isomorphisms. Examples of graph invariants are

- order, size and diameter (the longest of the shortest path lengths between pairs of vertices)
- *vertex (edge) connectivity*, the smallest number of vertices (edges) whose removal disconnects the graph
- *vertex (edge) chromatic number*, the minimum number of colors needed to color all vertices (edges) so that adjacent vertices (edges) have a different color
- *vertex (edge) covering number*, the minimal number of vertices (edges) needed to cover all edges (vertices)

In what follows we consider simple graphs without isolated vertices.

3 The Graph Hyperoperation

Given a concrete directed graph $G = (V_G, E_G)$, we introduce the graph hyperoperation $\circ_G$ defined on the nodes of the graph G:

$$\circ_G : V_G \times V_G \to \mathcal{P}(V_G), \quad (x, y) \mapsto x \circ_G y = \{z \in V_G \mid (x, z), (z, y) \in E_G\}.$$

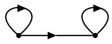
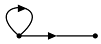
It can be easily seen that it holds $\circ_G = *_{E_G}$, where $*_{E_G}$ is the Corsini's hyperoperation derived by the binary relation E_G. Hence $\circ_G$ structures the set V_G into a (partial) **C**-hypergroupoid called the *Corsini's (partial) hypergroupoid associated with* G. If the hyperoperation $\circ_G$ structures V_G into a **C**-hypergroupoid then the same holds for every graph G' isomorphic with G, hence this property is a graph invariant. An (abstract) graph is called *Corsini's graph* if the hyperoperation $\circ_G$ structures V_G into a **C**-hypergroupoid.

Remark 3.1 *A hyperoperation for undirected graphs has been presented in [15] by virtue of spanning trees.*

Proposition 3.1 *The Graph hyperoperation associated with* $G = (V_G, E_G)$ *structures* V_G *into a* **C***-hypergroupoid if and only if, there exists a path of length 2 between every pair of nodes of* G.

Proof. The Corsini's hyperoperation associated with G is a hypergroupoid if and only if $x \circ_G y \neq \emptyset$ for all $x, y \in V_G$ if and only if for every pair of nodes $x, y \in V_G$ there exists a node $z \in V_G$ such that $(x, z), (z, y) \in E_G$, if and only if there exists a path of length 2 between x and y. $\square$

It is easy to prove that there are two Corsini's graphs with two nodes, depicted in the following figure.

We note that the second graph is actually a subgraph of the first.

Remark 3.2 *The adjacency matrix A_G of the graph G is identical with the boolean matrix M_{E_G} of the relation E_G.*

From this remark we obtain an alternative proof of Proposition 3.1. Indeed as it is stated in [9] a hyperoperation derived from the binary relation $R \subseteq H \times H$ structures H into a hypergroupoid if and only if the Boolean matrix M_R of the relation has the property $M_R^2 = \mathbf{1}$, where $\mathbf{1}$ is the matrix that has the unit at every entry. Since in the case of a hyperoperation associated with a graph G this matrix is equal with the adjacency matrix A_G of G it follows that $A_G^2 = \mathbf{1}$. Hence there exists a path of length 2 between any given pair of vertices.

Proposition 3.2 *The number of Corsini's graphs with order n is always smaller than the number of different Corsini's hypergroupoids derived from all binary relations $R \subseteq H \times H$ with card$|H| = n$.*

Proof. Indeed, although there exist isomorphic graphs with different adjacency matrices, different binary relations $R \subseteq H \times H$ always correspond to different Boolean matrices. Hence there exist different Corsini's hypergroupoids corresponding to two distinct Boolean matrices that represent the same graph up to isomorphism. $\square$

As an example for the above proposition we recall that there are three different Corsini's hypergroupoids deriving from binary relations with 2 elements (cf. [9]) but only two Corsini's graphs as we noted before.

Proposition 3.3 *If G is a Corsini's graph then every graph G' with $G \leq G'$ is also Corsini's.*

Proof. This is obtained by applying Proposition 3.1 since if there exists a path of length 2 between two nodes in a graph G then there exists such a path in every supergraph of G. $\square$

Now let $H = \{1, 2, 3\}$, $R \subseteq H \times H$ and M_R the 3×3 boolean matrix of R. We say that M_R has the form (p_1, p_2, p_3) if

$$|\{j \in H \mid (k, j) \in R\}| = p_k \text{ for all } k \in H$$

or equivalently if, for all $k \in H$, the sum of the elements of the kth line of M_R is p_k. We call the matrix M_R *good* if the Corsini hyperoperation $*_R$ structures $(H, *_R)$ into a **C**-hypergroupoid. As it is shown in [9] there are 30 good matrices with form (p_1, p_2, p_3) such that $p_1+p_2+p_3 = 6$. More precisely there are 12 matrices such that $p_i = 2$, where $i \in H$, and 18 matrices such that $p_i = 1$, $p_j = 2$, $p_k = 3$, where $i, j, k \in H$.

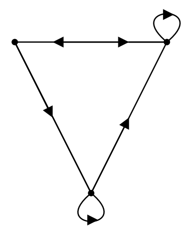
Proposition 3.4 *The 30 good boolean matrices with form (p_1, p_2, p_3), where $p_1 + p_2 + p_3 = 6$, correspond to 7 different non-isomorphic Corsini's graphs.*

Proof. First we examine the 12 good matrices with form $(2, 2, 2)$. The matrices

$$\begin{pmatrix}1&1&0\\1&0&1\\1&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\1&1&0\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\1&0&1\\0&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&0\\0&1&1\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\1&1&0\\0&1&1\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\0&1&1\\1&0&1\end{pmatrix},$$

which are respectively the matrices $(1), (3), (5), (8), (10)$ and (12) of [9], all represent Corsini's graphs isomorphic with the following graph.

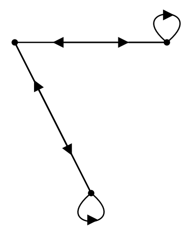


(G_1)

The matrices

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix},$$

which are respectively the matrices $(2), (6)$ and (9) of [9], correspond to graphs isomorphic with the graph below.

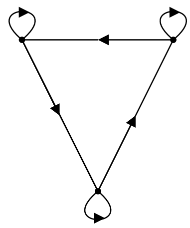


(G_2)

The matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

which are respectively the matrices (4) and (7) of [9], both represent Corsini's graphs isomorphic with the next graph.

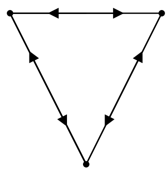


(G_3)

The matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

is the matrix (11) of [9] and represents the following Corsini's graph.



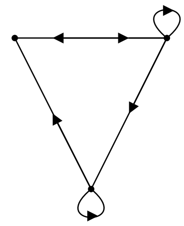
(G_4)

Now we examine the rest 18 good matrices with the desired form. The matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which are respectively the matrices $(13), (16), (20), (23), (27)$ and (30) of [9], represent Corsini's graphs isomorphic with the following graph.



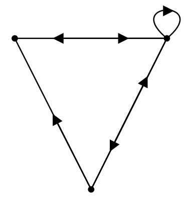
(G_5)

The matrices

$$\begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix},\quad \begin{pmatrix}0&1&1\\0&0&1\\1&1&1\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&1&1\\1&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&1\\1&1&1\\0&1&0\end{pmatrix},\quad \begin{pmatrix}1&1&1\\1&0&0\\1&1&0\end{pmatrix},\quad \begin{pmatrix}1&1&1\\1&0&1\\1&0&0\end{pmatrix},$$

which are respectively the matrices $(14), (18), (19), (24), (25)$ and (29) of [9], represent Corsini's graphs isomorphic with the following graph.



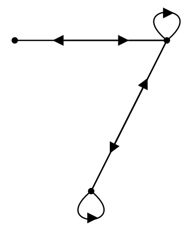
(G_6)

The matrices

$$\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix},\quad \begin{pmatrix}1&0&1\\0&0&1\\1&1&1\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&1&1\\0&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&0\\1&1&1\\0&1&0\end{pmatrix},\quad \begin{pmatrix}1&1&1\\1&0&0\\1&0&1\end{pmatrix},\quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix},$$

which are respectively the matrices $(15), (17), (19), (22), (25)$ and (28) of [9], represent Corsini's graphs isomorphic with the next graph.



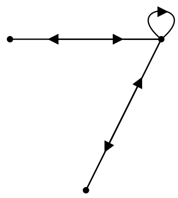
(G_7)

□

Remark 3.3 *A boolean matrix with form* (p_1, p_2, p_3) *corresponds to a graph* G *with a* $degreeout(G) = \{p_1, p_2, p_3\}$. *Hence graphs* G_1-G_4 *have* $degreeout(G) = \{2, 2, 2\}$ *and graphs* G_5-G_7 *have* $degreeout(G) = \{1, 2, 3\}$.

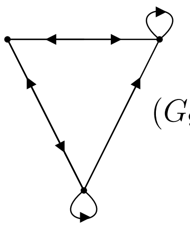
The number of Corsini's graphs with order n is equal with the number of boolean $n \times n$ matrices forming non-isomorphic hypergroupoids. Although Massouros and Tsitouras [16] have calculated this number - for $n = 3$ there are 17 such matrices - no actual representation of these 17 hypergroupoid classes has been demonstrated in [16]. Such a representation is of much greater importance than merely computing their number since it will allow us to compare and correlate these isomorphism classes. In what follows we present the remaining Corsini's graphs and moreover we discover that they constitute an upper semilattice with respect to graph inclusion. This hierarchy actually determines a hierarchy inside the set of 73 **C**-hypergroupoids (cf. [22]) deriving from binary relations on 3 elements.

Since in the previous proposition we found 7 Corsini's graphs with 3 vertices it follows that there are 10 more. From these 1 has $degreeout(G) = \{1, 1, 3\}$

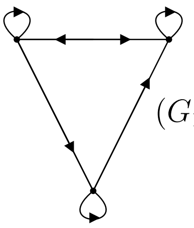


(G_8)

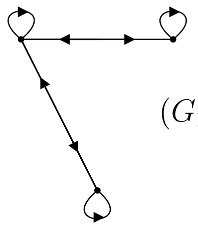
5 have $degreeout(G) = \{2, 2, 3\}$



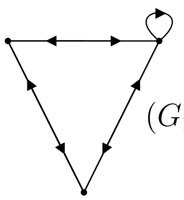
(G_9)



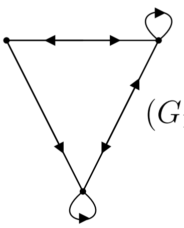
(G_{10})



(G_{11})

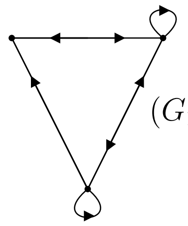


(G_{12})



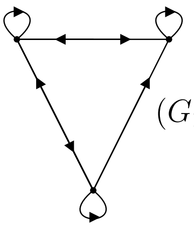
(G_{13})

1 has $degreeout(G) = \{1, 3, 3\}$

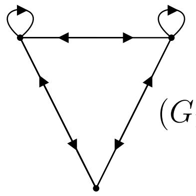


(G_{14})

2 have $degreeout(G) = \{2, 3, 3\}$

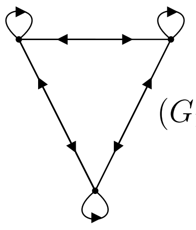


(G_{15})



(G_{16})

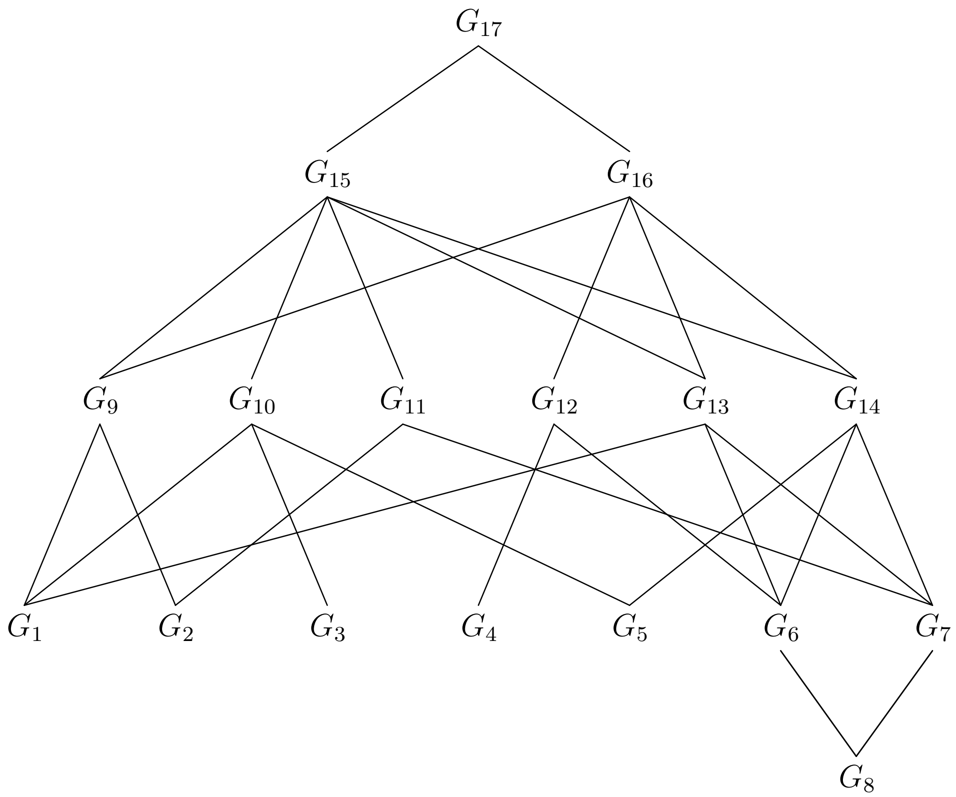
and 1 (the complete graph with 3 nodes) has $degreeout(G) = \{3, 3, 3\}$.



(G_{17})

As it is implied by Proposition 3.3, the graphs G_1-G_{17} form a partially ordered set with respect to graph inclusion. More precisely they form the following upper semilattice as it can be verified by merely inspecting the drawings of graphs G_1-G_{17}.

Directed Graphs representing isomorphism classes of **C**-Hypergroupoids



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RATIO MATHEMATICA 23 (2012), 81–86 ISSN: 1592-7415

Closed, Reflexive, Invertible, and Normal Subhypergroups of Special Hypergroups

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Abstract

In [5] J. Jantosciak introduced several special types of subhypergroups (invertible, closed, normal, reflexive) of a general hypergroup and studied their relationship. In this article, the full description of such subhypergroups in hypergroups induced by quasiordered groups is given. Further, it is shown that there are no such non-trivial subhypergroups in quasiorder hypergroups.

Key words: Hypergroup, transposition hypergroup, join space, closed, reflexive, invertible, and normal subhypergroup, quasiordered group, quasiorder hypergroup.

MSC 2010: 20N20.

1 Introduction

We will sum up basic concepts from hypergroup theory and results which will be needed in the following text.

- Let $H \neq \emptyset$. A mapping $*\colon H \times H \to \mathscr{P}^*$ is called *binary hyperoperation* on H. The pair $(H, *)$ is *hypergroupoid.*
- A hypergroupoid $(H, *)$ is *extensive* if $\{a, b\} \subseteq a * b$ for any $a, b \in H$.
- A hypergroupoid $(H, *)$ is called *hypergroup* if the hyperoperation $*$ is associative, i.e

$$(x * y) * z = x * (y * z) \qquad \text{for any } x, y, z \in H,$$

and *the reproduction axiom*

$$a * H = H = H * a \qquad \text{for any } a \in H$$

is satisfied.

- Let $(H, *)$ be a hypergroup and $S \subseteq H$. Assume that $a * b \subseteq S$ for any $a, b \in S$. Thus $(S, *)$ is an associative hypergroupoid (so called *semihypergroup*). If, moreover, it satisfies the reproduction axiom, that is, it is a hypergroup, we say that $(S, *)$ is *subhypergroup* of $(H, *)$.
- A hypergroup $(H, *)$ is called *transposition hypergroup* if *the transposition axiom* is satisfied, that is, for any quadruple of elements $a, b, c, d \in H$ the implication:

  $$\text{If } b\backslash a \approx c/d, \text{ then } a * d \approx b * c$$

  holds, where

  $$b\backslash a = \{x \in H : a \in b * x\},$$
  $$c/d = \{x \in H : c \in x * d\}$$

  are *left* and *right extensions*, respectively.
  (For two set A, B, the symbol $A \approx B$ means that A and B are incident, i.e. $A \cap B \neq \emptyset$.)
- A commutative transposition hypergroup $(H, *)$ is called *join space.*

The following concepts play the key role in the formulation of the main results.

Definition 1.1 *A subhypergroup $(S, *)$ of a hypergroup $(H, *)$ is called*

- closed *if $a/b \subseteq S$ and $b\backslash a \subseteq S$ for any $a, b \in S$,*
- invertible *if $a/b \approx S$ implies $b/a \approx S$ and $b\backslash a \approx S$ implies $a\backslash b \approx S$ for any $a, b \in H$,*
- reflexive *if $a\backslash S = S/a$ for any $a \in H$,*
- normal *if $a * S = S * a$ for any $a \in H$.*

J. Jantosciak (see [5]) proved that

- invertible subhypergroup of any hypergroup is closed,
- closed and normal subhypergroup of a transposition hypergroup is reflexive,
- invertible and normal subhypergroup of a transposition hypergroup is closed and reflexive.

2 Hypergroups Induced by Quasiordered Groups

First the relationship among closed, normal, invertible, and reflexive subhypergroups of hypergroups induced by quasiordering will be tackled.

Definition 2.1 An ordered (quasiordered) group *is a triple* $(H, \cdot, \leq)$*, where* $(H, \cdot)$ *is a group and "*$\leq$*" is an ordering (quasiordering) on* H *having* the substitution property *on* $(H, \cdot)$*, that is for any quadruple* $a, b, c, d \in H$ *such that* $a \leq b$*,* $c \leq d$*, there is* $a \cdot c \leq b \cdot d$*.*

A hyperoperation is naturally induced on each (quasi)ordered group $(H, \cdot, \leq)$. For $x \in H$, let us denote $[x)_{\leq}$ *the principal end* generated by x, that is $[x)_{\leq} = \{y \in H : x \leq y\}$. Analogously *the principal beginning* $(x]_{\leq}$ is defined. For $x, y \in H$, let us denote $x * y = [x \cdot y)_{\leq}$. Then $(H, *)$ is a hypergroupoid associated with $(H, \cdot, \leq)$. The following result can be proven (see [3, 6, 7]):

Theorem 2.1 *Let* $(H, \cdot, \leq)$ *be a quasiordered group and* $(H, *)$ *the hypergroupoid associated with it. Then* $(H, *)$ *is the transposition hypergroup.*

Let $(G, \cdot, \leq)$ be *a quasiordered group*, $(G, *)$, where $a * b = [a \cdot b)_{\leq}$, be the induced (transposition) hypergroup. Let $(S, *)$ be its subhypergroup, that is, $[x \cdot y)_{\leq} \subset S$ holds for $x, y \in S$, namely $x \cdot y \in S$.

For the right and left extensions we have:

$$\begin{aligned} a/b &= \{x \in G : a \in x * b\} = \{x \in G : x \cdot b \leq a\} = \\ &= \{x \in G : x \leq a \cdot b^{-1}\}, \\ b \backslash a &= \{x \in G : a \in b * x\} = \{x \in G : b \cdot x \leq a\} = \\ &= \{x \in G : x \leq b^{-1} \cdot a\}. \end{aligned}$$

Theorem 2.2 *A subhypergroup* $(S, *)$ *is closed iff* $(S, \cdot)$ *is a subgroup of the group* $(G, \cdot)$ *and* $(a]_{\leq} \cup [a)_{\leq} \subset S$ *for any* $a \in S$*.*

Proof. *Necessity:*

For $a \in S$ there is $a/a = (e]_{\leq}$, hence $e \in S$ (e is the neutral element).

If $a \in S$, then $e/a = (a^{-1}]_{\leq}$, hence $a^{-1} \in S$.

If $a \in S$, then $a/e = (a]_{\leq}$, hence $(a]_{\leq} \subset S$. Because $(S, *)$ is a subhypergroup we also get $a * e = [a)_{\leq} \subset S$.

Sufficiency: Let $a, b \in S$. Then $a \cdot b^{-1} \in S$, so $(a \cdot b^{-1}]_{\leq} = a/b \subset S$. Analogously for $b \backslash a$. $\square$

Theorem 2.3 *A subhypergroup $(S,*)$ is closed iff it is invertible.*

Proof. Each invertible subhypergroup is closed.

Now, let us assume that $(S,*)$ is closed. If $a/b \approx S$, there exists $x \in S$ such that $x \leq a \cdot b^{-1}$. Due to the previous theorem $a \cdot b^{-1} \in S$ and also $(a \cdot b^{-1})^{-1} = b \cdot a^{-1} \in S$, therefore $b/a \subset S$. Especially, $b/a \approx S$. Analogously the statement for $b\backslash a$ can be proven. □

Theorem 2.4 *A subhypergroup $(S,*)$ is reflexive iff the following property holds: If for $x, y \in G$ the element $x \cdot y$ is covered by an element of S, then $y \cdot x$ is also covered by an element of S.*

Proof. The following equalities hold:

$$a\backslash S = \bigcup_{b\in S} a\backslash b = \bigcup_{b\in S} (a^{-1} \cdot b]_{\leq}\,,$$
$$S/a = \bigcup_{b\in S} b/a = \bigcup_{b\in S} (b \cdot a^{-1}]_{\leq}\,.$$

Hence, $x \in a\backslash S$ iff $b \in S$ exists such that $x \leq a^{-1} \cdot b$, that is, $a \cdot x \leq b$. Analogously $x \in S/a$ iff $c \in S$ exists such that $x \cdot a \leq c$. This implies the statement. □

If $(S,*)$ is a closed subhypergroup, according to the previous result, the condition "to be comparable with an element of S" is equivalent with the condition "to be in S". Therefore, we get:

Corollary 2.1 *A closed subhypergroup $(S,*)$ is reflexive iff the following property holds: If for $x, y \in G$ the element $x \cdot y \in S$, then also $y \cdot x \in S$.*

Theorem 2.5 *A subhypergroup $(S,*)$ is normal iff the following property holds: If $x \cdot y$, where $x, y \in G$, covers an element of S, then $y \cdot x$ also covers an element of S.*

Proof. The following equalities hold:

$$a * S = \bigcup_{b\in S} a * b = \bigcup_{b\in S} [a \cdot b)_{\leq}\,,$$
$$S * a = \bigcup_{b\in S} b * a = \bigcup_{b\in S} [b \cdot a)_{\leq}\,.$$

Hence, $x \in a * S$ iff $b \in S$ exists such that $a \cdot b \leq x$, that is, $b \leq a^{-1} \cdot x$. Analogously $x \in S * a$ iff $c \in S$ exists such that $c \leq x \cdot a^{-1}$. This implies the statement. □

In case $(S, *)$ is a closed subhypergroup, analogously to reflexive subhypergroups we have:

Corollary 2.2 *A closed subhypergroup $(S, *)$ is normal iff the following property holds: If for $x, y \in G$ the element $x \cdot y \in S$, than also $y \cdot x \in S$.*

Joining the previous two results we get:

Corollary 2.3 *Let $(S, *)$ be a closed subhypergroup. Then $(S, *)$ is normal iff it is reflexive.*

3 Quasiorder Hypergroups

Now the relationship among closed, normal, invertible, and reflexive subhypergroups of quasiorder hypergroups will be tackled.

If R is a quasiordering on H we denote $R(a) = \{x \in H : a\,R\,x\}$. Further, for $A \subseteq H$ we set $R(A) = \bigcup_{a \in A} R(a)$.

Theorem 3.1 *Let (H, R) be a quasiordered set. For any pair $a, b \in H$ we set*

$$a *_R b = R(a) \cup R(b) = R(\{a, b\}).$$

Then $(H, *_R)$ is commutative extensive hypergroup.

For the proof see [3, p. 150, Th. 2.1].

Definition 3.1 *A* hypergroup $(H, *)$ *is said* quasiorder *if the following conditions are satisfied:*

- $a \in a^3 \subseteq a^2$,
- $a * b = a^2 \cup b^2$

for any $a, b \in H$.

The following theorem characterizes all quasiorder hypergroups.

Theorem 3.2 *A hypergroup $(H, *)$ is quasiorder iff a quasiordering R on H exists such that for any $a, b \in H$ there is*

$$a * b = R(a) \cup R(b) = R(\{a, b\}).$$

For the proof see [1, p. 96].

Theorem 3.3 *Let $(G, *)$ be a quasiorder hypergroup. Then any subhypergroup of $(G, *)$ is reflexive and normal. Moreover, $(G, *)$ contains no proper closed or invertible subhypergroup.*

Proof. Suppose that $(G, *)$ is a quasiorder hypergroup and R is a quasiordering from the previous theorem.

The hyperoperation $*$ is commutative, hence any subhypergroup $(S, *)$ is reflexive and normal.

Further,

$$a/b = \{x \in G : a \in b * x\} = \{x \in G : a \in R(b) \cup R(x)\}.$$

Especially $a/a = G$. If $(S, *)$ is closed, then necessarily $S = G$. Therefore, no proper closed subhypergroup exists.

Analogously, any invertible subhypergroup is closed, therefore, no proper invertible subhypergroup exists. □

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Ratio Mathematica, 23, 2012

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