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History and new possible research directions of hyperstructures

Piergiulio Corsini

Abstract We present a summary of the origins and current developments of the theory of algebraic hyperstructures. We also sketch some possible lines of research .

Keywords. Hypergroupoid, hypergroups, fuzzy sets.

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1 The origins of the theory of hyperstructures

Hyperstructure theory was born in 1934, when Marty at the 8th Congress of Scandinavian Mathematiciens, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and rational fractions. In the following years, around the 40's, several others worked on this subject: in France, the same Marty, Krasner, Kuntzman, Croisot, in USA Dresher and Ore, Prenowitz, Eaton, Griffith, Wall (who introduced a generalization of hypergroups, where the hyperproduct is a multiset, i.e. a set in which every element has a certain multiplicity); in Japan Utumi, in Spain San Juan, in Russia Vikhrov, in Uzbekistan Dietzman, in Italy Zappa.
In the 50's and 60's they worked on hyperstructures, in Romania Benado, in Czech Republic Drbohlav, in France Koskas, Sureau, In Greece Mittas, Stratigopoulos, in Italy Orsatti, Boccioni, in USA Prenowitz, Graetzer , Pickett, McAlister, in Japan Nakano, in Yugoslavia Dacic.

But it is above all since 70' that a more luxuriant flourishing of hyperstructures has been and is seen in Europe, Asia, America, Australia.

2. The most important definitions

Definition 1 Let H be a nonempty set and $P^*(H)$ the family of the nonempty subsets of H. A *multivalued operation* (said also *hyperoperation*) < o > on H is a function which associates with every pair $(x, y) \in H^2$ a non empty subset of H denoted x o y.
An *algebraic hyperstructure* or simply a *hyperstructure* is a non empty set H, endowed with one or more hyperoperations.
A nonempty set H endowed with an hyperoperation < o > is called *hypergroupoid* and is denoted <H ; o >. If $\{A, B\} \subseteq P^*(H)$, A o B denotes the set $\cup_{a\in A,\ b\in B}$ a o b.

Definition 2 A hypergroupoid <H ; o> is called *semi-hypergroup* if

(I) $\forall (x, y, z) \in H^3$, (x o y) o z = x o (y o z).

A hypergroupoid <H ; o> is called *quasi-hypergroup* if

(II) $\forall (a, b) \in H^2$, $\exists (x, y) \in H^2$ such that $a \in$ b o x , $a \in$ y o b.

Definition 3 A hyperoperation is said *weak associative* if

(III) $\forall (x, y, z) \in H^3$, (x o y) o z $\cap$ x o (y o z) $\neq \varnothing$ (see [141]).

Definition 4 A hypergroupoid <H ; ○> is called *hypergroup* if satisfies both (I) and (II).

Definition 5 A hyperoperation < ○> is said *commutative* if

(IV) $\forall (a, b) \in H^2$, a o b = b o a.

Definition 6 A hyperoperation is said *weak commutative* if

(V) $\forall (x, y) \in H^2, x \text{ o } y \cap y \text{ o } x \neq \varnothing$.

Definition 7 A *H_v – group* is a quasi-hypergroup <H ; o > such that the hyperoperation < o > is weak associative.

Let <H ; ∘> be a commutative hypergroup. We denote with a / b the set $\{x \mid a \in x \text{ o } b\}$.

Definition 8 A hypergroupoid <H ; o> is called *join space* if it is a commutative hypergroup such that the following implication is satisfied

(VI) $\forall (a, b, c, d) \in H^4, a / b \cap c / d \neq \varnothing \Rightarrow a \text{ o } d \cap b \text{ o } c \neq \varnothing$.

3. The recent history of the theory

Currently one works successfully in hyperstructures, in several continents, I shall remember only some names of hyperstructure scientists since 1970.

Around the 70's and 80's, hyperstructures where cultivated especially:

- in Greece by Mittas and his school (Canonical Hypergroups and their applications, Vougiouklis and his school (H_v – groups);
- in Italy: by Corsini (Homomorphisms, Join Spaces, Quasicanonical Hypergroups, Complete Hypergroups, 1-Hypergroups, Cyclic Hypergroups etc.) and his school, Tallini G. (Hypergroups associated with Projective Planes), Rota, Procesi Ciampi (Hyperrings);
- in USA: by Prenowitz and Jantoshak (Join Spaces and Geometries, Homomorphisms), Roth (Character of hypergroups, Canonical hypergroups), Comer (Polygroups, Representations of hypergroups);
- in France by Krasner and Sureau, (Structure of Hypergroups), Koskas, (Semihypergroups associated with Groupoids). Deza;

- in Canada Rosenberg (Hypergroups associated with graphs, binary relations).

Around the 90's and more recently, many papers appeared, made in Europe, Asia, Asia , America and Australia.

Europe:

- In Greece
  - at Thessaloniki (Aristotle Univ.), Mittas, Konstantinidou, Serafimidis Kehagias, Ioulidis, Yatras, Synefaki,
  - at Alexandroupolis (Democritus Univ. of Thrace), T. Vougiouklis, Dramalidis, S.Vougiouklis,
  - at Patras (Patras Univ.), Stratigopoulos,
  - at Orestiada (Democritus Univ. of Thrace), Spartalis,
  - at Athens , Ch. Massouros, G. Massouros, G. Pinotsis;
- in Romania
  - at Iasi (Cuza Univ.), V. Leoreanu, Cristea, Tofan, Gontineac., Volf, L. Leoreanu,
  - at Cluj Napoca Babes Bolyai Univ.), Purdea, Pelea, Calugareanu,
  - at Constanta (Ovidius Univ.) Stefanescu;
- in Czech Republic
  - at Praha (Karlos Univ.) Kepka, Jezek, Drbohlav, (Agricultural Univ.) Nemec,
  - at Brno (Brno Univ. of Technology) J. Chvalina, (Military Academy of Brno) Hoskova, (Technical Univ. of Brno) L. Chvalinova, (Masaryk Univ.) Novotny, (University of Defence) Rackova, at Olomouc, (Palacky Univ.) Hort,
  - at Vyskov (Military Univ. of Ground Forces) Moucka;
- in Montenegro
  - at Podgorica (Univ. of Podgorica) Dasic, Rasovic;
- in Slovakia

- at Bratislava, (Comenius University), Kolibiar, (Slovak Techn. Univ.) Jasem,
- at Kosice, (Matematickz ustav SAV), Jakubik, (Safarik Univ.), Lihova, Repasky, Csontoova;

- in Italy
  - at Udine (Udine Univ.) Corsini,
  - at Messina (Messina Univ.) De Salvo, Migliorato, Lo Faro, Gentile,
  - at Rome (Universita' La Sapienza) G. Tallini, M. Scafati-Tallini, Rota, Procesi Ciampi, Peroni,
  - at Pescara (G. d'Annunzio Univ.) A. Maturo, S. Doria, B. Ferri,
  - at Teramo (Univ. di Teramo) Eugeni,
  - at L'Aquila (Univ. dell'Aquila) Innamorati, L. Berardi,
  - at Brescia (Universita' Cattolica del Sacro Cuore), Marchi,
  - at Lecce (Universita' di Lecce), Letizia, Lenzi,
  - at Palermo, (Univ. di Palermo), Falcone,
  - at Milano, (Politecnico), Mercanti, Cerritelli, Gelsomini;
- in France
  - at Clermont-Ferrand (Universite' des Math. Pures et Appl.) Sureau, M. Gutan, C. Gutan,
  - at Lyon, (Universite' Lyon 1), Bayon, Lygeros;
- in Spain
  - at Malaga, (Malaga Univ.) Martinez, Gutierrez, De Guzman, Cordero;
- in Finland
  - at Oulu, (Univ. of Oulu), Nieminen, Niemenmaa.

America

- In USA
  - at Charleston (The Citadel) Comer,
  - at New York (Brooklyn College, CUNY), Jantosciak,
  - at Cleveland, Ohio, (John Carroll Univ.), Olson, Ward,

- in Canada
    - at Montreal, (Universite' de Montreal), Rosenberg, Foldes,

Asia

- In Thailand
    - at Bangkok, (Chulalongkorn Univ), Kemprasit, Punkla , Chaopraknoi, Triphop, Tumsoun,
    - at Samutprakarn, (Hauchievw Chalermprakiet Univ.), Juntakharajorn,
    - at Phitsanulok, (Naresuan Univ.), C. Namnak.
- in Iran
    - at Babolsar (Mazandaran Univ.) Ameri, Razieh Mahjoob, Moghani, Hedayati, Alimohammadi,
    - at Yazd (Yazd Univ.) Davvaz, Koushky,
    - at Kerman, (Shahid Bahonar Univ.) Zahedi, Molaei, Torkzadeh, Khorashadi Zadeh, Hosseini, Mousavi, (Islamic Azad Univ.) Borumand Saeid
    - at Kashan (Univ. of Kashan) Ashrafi, Ali Hossein Zadeh,
    - at Tehran (Tehran Univ.) Darafsheh, Morteza Yavary, (Tarbiat Modarres Univ.) Iranmanesh, Iradmusa, Madanshekaf, (Iran Univ. of Sci. and Technology) Ghorbany, Alaeyan, (Shahid Beheshti Univ.) Mehdi Ebrahimi, Karimi, Mahmoudi,
    - at Zahedan (Sistan and Baluchestan Univ.) Borzooei, Hasankhani, Rezaei,
    - at Zanjan (Institute for Advanced Studies in Basic Sciences) Barghi
    - at Sari-Branch, (Islamic Azad Univ.), Roohi.
- in Korea
    - at Chinju (Gyeongsang National Univ.) Young Bae Jun, E. H. Roh,
    - at Taejon, (Chungnam National Univ.) Sang Cho Chung,

- (Taejon Univ.) Byung-Mun Choi,
    - at Chungju (Chungju National Univ.) K.H. Kim.
- in India
    - at Kolkata, (Uni. of Calcutta), M.K. Sen, Dasgupta,Chowdhury,
    - at Tiruchendur, Tamilnadu (Adinatar College of Arts and Sciences), Asokkumar,Velrajan,
- in China
    - at Chongqing, (Chongqing three Gorges Univ.) Yuming Feng,
    - at Xi'an, (Northwest Univ.), Xiao Long Xin,
    - at Enshi, Hubei Province (Hubei Institute for Nationalities), Janming Zhan, Xueling Ma;
- in Japon
    - at Tokyo, (Hitotsubashi Univ. Kunitachi), Machida,
    - at Tagajo, Miyagi, (Tohoku Gakuin Univ.), Shoji Kyuno;
- in Jordan
    - at Karak, (Mu'tah Univ.) M.I. Al Ali;
- in Israel
    - at Ramat Gan, (Bar - Ilan Univ.), Feigelstock.

4. Join Spaces, Fuzzy Sets, Rough Sets

The *Join Spaces* were introduced by Prenowitz in the 40's and were utilized by him and later by him together with Jantosciak, to construct again several kinds of Geometry. Join spaces had already many other applications, as to Graphs, (Nieminen, Rosenberg, Bandelt, Mulder, Corsini), to Median Algebras (Bandelt-Hedlikova) to Hypergraphs (Corsini, Leoreanu), to Binary Relations (Chvalina, Rosenberg, Corsini, Corsini, Leoreanu, De Salvo-Lo Faro).

Fuzzy Sets were introduced in the 60's by an Iranian scientist who lives in USA, Zadeh [144]. He and others, in the following decades, found surprising applications to almost every field of science and knowledge: from engineering to sociology, from agronomy to linguistic, from biology to computer science, from medicine to economy. From psychology to statistics and so on. They are now cultivated in all the world. Let's remember what is a Fuzzy Set. We know that a subset A of an universe H, can be represented as a function, the characteristic functions χ_A from H to the set {0,1}. The notion of fuzzy subset generalizes that one of characteristic function. One considers instead of the functions χ_A, functions μ_A from H to the closed real interval [0,1]. These functions, called "membership functions" express the degree of belonging of an element $x \in H$ to a subset A of H. To consider in a problem, a fuzzy subset instead of an usual (Cantor) subset, corresponds usually to think according to a multivalued logic instead of a bivalent logic. The reply to many questions in the science, and in the life, often is not possible in a dichotomic form, but it has a great variety of nuances. A Cantor subset A of the universe H, can be represented as the class of objects which satisfy a certain property p , so an element x does not belong to A if it does not satisfy p . But in the reality an object can satisfy p in a certain measure. Whence the advisability to size the satisfaction of p by a real number $\mu_A(x) \in [0,1]$.

Rough Sets, which have been proved to be a particular case of fuzzy sets (see [8]) are they also an important instrument for studying in depth some subjects of applied science. The first idea of rough set appears in the book by Shaefer [134] as pair of "inner and outer reductions" (see pages 117-119), in the context of Probability and Scientific Inference, but they became a well known subject of research in pure and applied mathematics, since Pawlak [121] considered them again and proved their utility in some topics of Artificial Intelligence as Decision Making, Data Analysis, Learning Machines, Switching Circuits.

Connections between fuzzy sets and algebraic hyperstructures were considered for the first time by Rosenfeld [130]. Many others worked in the same direction, that is studied algebraic structures endowed also with a fuzzy structure (see [1], [2], [3], [54],…, [59]).

Hyperstructures endowed with a fuzzy structure were considered by Ameri and Zahedi, Tofan, Davvaz, Borzooei, Hasankhani, Bolurian and others.

Definition 9 A *fuzzy subset* of a set H is a pair (H; μ_A) where μ_A is a function $\mu_A : H \to [0,1]$, A is the set $\{ x \in H \mid \mu_A(x) = 1 \}$.

Corsini proved in 1993, [17], that to every fuzzy subset of a set H one can associate a join space, where the hyperoperation is defined as follows:

(I) $\forall (x, y) \in H^2$, $x \bullet y = \{z \mid \min\{\mu(x), \mu(y)\} \leq \mu(z) \leq \max\{\mu(x), \mu(y)\}\}$.

Moreover in 2003 Corsini proved [29] that to every hypergroupoid <H, o> one can associate a fuzzy subset. See (II) :

(II) $\forall u \in H$, let $Q(u) = \{(x, y) \in H^2 \mid u \in x \text{ o } y \}$, $q(u) = |Q(u)|$, $A(u) = \Sigma_{\{x,y\} \subseteq Q(u)} (1 / |x \text{ o } y|)$, $\mu_A'(u) = A(u)/q(u)$.

If we have a hypergroupoid <H, ○"> and <○"> is a weak hyperoperation, we can associate with H the following fuzzy subset
(II") Set $m_{x,y}(u)$ the multiplicity of the element u in the hyperproduct $x \circ y$. Set
$\mu_{x,y}(u) = m_{x,y}(u) / \Sigma (m_{x,y}(v) \mid v \in H, m_{x,y}(v) > 0)$
$Q(u) = \{ (a,b) \in H^2 \mid m_{a,b}(u) > 0 \}$, $q(u) = |Q(u)|$
$A(u) = \Sigma_{\{x,y\} \subset Q(u)} \mu_{x,y}(u)$, $\mu'(u) = A(u)/q(u)$.

Weak hyperstructures were introduced by Vougiouklis (1981) and were studied by many people especially by Vougiouklis and Spartalis.

So every fuzzy subset (and every hypergroupoid) determines a sequence of join spaces and of fuzzy subsets

Connections between hyperstructures and fuzzy sets have been considered by many people. In particular (I) and (II) opened research lines studied in deep by several scientists. In this context several papers have been made in Italy, Romania, Greece, Iran, for example by Corsini, Leoreanu, Cristea, Serafimidis, Kehagias, Konstantinidou, Rosenberg. Hyperstructures endowed also with a fuzzy structure have been considered especially in Iran by Ameri, Zahedi, Davvaz and many others.

From (I) and (II) follows clearly that every hypergroupod (o fuzzy subset) determines a sequence of fuzzy subsets and hypergroupoids or of hypergroupoids and fuzzy subset) which is obtained applying consecutevely (II) e (I) (oppure (I) e (II))

The *fuzzy grade* (*minimum lengh* of such sequences) has been calculated for several classes of hyperstructures: Corsini-Cristea for i.p,s, hypergroups (a particular case of *canonical hypergroups*), and 1-hypergroups (hypergroups such that if ω is the heart of the hypergroup, $|\omega|= 1$). Corsini - Leoreanu for hypergroups associated with hypergraphs, Leoreanu for hypergroups associated with *rough sets*. Corsini and Cristea for *complete hypergroups*.

Let H be a set, R an equivalence relation in H and $\forall x \in H$, we denote the equivalence class of x by R(x).

It is known that with every binary relation R defined in a set H, a partial hypergroupoid corresponds defined

$\forall\ (x,y) \in H^2$, $x \cdot_R x = \{u \mid xRu\}$, $x \cdot_R y = x \cdot_R x \cup y \cdot_R y$

This structure that under certain conditions is a hypergroup was introduced by Rosenberg in 1996, see [102] and afterwards studied also by Corsini in Multiple Logic and Applications (1997) and by Corsini - Leoreanu in Algebra Universalis n. 43 (2000).
Hypergroupoids associated with multivalued functions, have been analyzed by Corsini and Razieh Majoob in 2010, see [40].

5. New lines of research

1) A research line could be to calculate the Fuzzy Grade of the hypergroupoid associated with a Binary relation
2) In Bull. Greek Math Society, Corsini has associated in different ways hypergropoids with an ordered set. It could be interesting to study the sequences of join spaces determined by these hypergroupoids.
3) Another research line could be to study the sequence of Join Spaces determined by a hypergroupoid endowed with a weak hyperoperation.
4) It would be interesting also to consider the sequence of fuzzy sets and join spaces determined by a *Chinese Hypergroupoid* (see [24])
5) In [25], [26] one has associated a hypergroupoid with a factor space, that is , given a function f from an universe U to a set of states X(f), and and a fuzzy subset of U, μ_f called the extension of f, one has considered the hyperoperation in U, $< \circ^f_{\mu f} >$. defined:
$x \circ^f_{\mu f} y = \{ w \mid \mu_f(w) \in [\sup. \{ \mu_f(z) \mid f(z) = f(x) \}, \sup. \{ \mu_f(v) \mid f(v) = f(y) \}] \}$
One proposes to determine the fuzzy grade of the hypergroupoid $<U; \circ^f_{\mu f}>$.
6) Set A a non empty set and $\mathcal{F}$ the set of functions f: $A \to P^*(A)$ such that $\cup_{x \in A} f(x) = A$. One considers the following hyperoperations in $\mathcal{F}(A)$, see [31]

(i) $f \circ_1 g = \{ h \in \mathcal{F} \mid \forall x \in A, h(x) \subset f(g(x)) \}$,

(ii) $f \circ_2 g = \{ h \in \mathcal{F} \mid \forall x \in A, h(x) \subset f(x) \cup g(x) \}$,

(iii) Let's suppose now $<A ; ò>$ to be a hypergroupoid. Then set for every (f.g) $\in \mathcal{F} \mathcal{X} \mathcal{F}$

$$f \circ_3 g = \{ h \in \mathcal{F} \mid \forall x \in A, h(x) \subset f(x) \,ò\, g(x) \},$$

Problems: Let's suppose $|A| = 3$
i* Determine the fuzzy grade of the hypergroupoid (i)
ii* Determine the fuzzy grade of the hypergroupoid (ii)
iii * Determine the fuzzy grade of the hypergroupoid (iii) , for some hypergroupoid $<A ; ò>$

(7) It is known that every hypergraph <Γ ; $\{A_i\}$ > determines a sequence of quasi-hypergroups $Q_0, Q_1, \ldots\ldots, Q_m$ (see [18], Th. 6). Set, for every k, m_k the membership function associated with Q_k and $J(Q_k)$ the corresponding join space. Set σ^* the sequence < Q_0 , μ_0 , $J(Q_o)$, μ_1 , $J(Q_1)$,….., μ_m , $J(Q_m)$ > , and set σ the sequence of membership functions and join spaces determined by the hypergroupoid Q_0 after (I) and (II)

It would be interesting to compare the two sequences σ and σ^* .

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Bar and Theta Hyperoperations

Thomas Vougiouklis
Democritus University of Thrace, School of Education
681 00 Alexandroupolis, Greece
tvougiou@eled.duth.gr

Abstract

In questionnaires the replacement of the scale of Likert by a bar was suggested in 2008 by Vougiouklis & Vougiouklis. The use of the bar was rapidly accepted in social sciences. The bar is closely related with fuzzy theory and has several advantages during both the filling-in questionnaires and mainly in the research processing. In this paper we relate hyperstructure theory with questionnaires and we study the obtained hyperstructures which are used as an organising device of the problem.

Key words: Hyperstructures, H_v-structures, hopes, ∂-hopes.

Subject Classification: 20N20, 16Y99.

1. Introduction

Hyperstructures are called the algebraic structures equipped with at least one hyperoperation i.e. a multivalued operation. We have abbreviated the 'hyperoperation' by *'hope'* [24]. Therefore, if in a set H at least one hope $\cdot:H\times H\to P(H)-\{\varnothing\}$ is defined, then $(H,\cdot)$ is called a hypergroupoid. The *H_v-structures* introduced in 1990 [15], is the largest class of hyperstructures. The H_v-structures satisfy the *weak axioms* where the non-empty intersection replaces the equality. In $(H,\cdot)$ we abbreviate by

WASS the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing,\ \forall x,y,z\in H$ and by

COW the *weak commutativity*: $xy\cap yx\neq\varnothing,\ \forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called *H_v-semigroup* if it is *WASS*, and it is called *H_v-group* if it is reproductive H_v-semigroup, i.e. $xH=Hx=H,\ \forall x\in H$. The hyperstructure $(R,+,\cdot)$ is called *H_v-ring* if both hopes (+) and (·) are *WASS*, the reproduction axiom is valid for (+) and (·) is *weak distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

The main tool to study all hyperstructures are the fundamental relations β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp., [17]. An element is called *single* if its fundamental class is singleton.

A way to find the fundamental classes is given by analogous theorems to the following [17],[18],[19],[20],[21],[5]:

Theorem. Let $(H,\cdot)$ be an H_v-group and denote $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\}\subset \boldsymbol{u}$ where $\boldsymbol{u}\in\boldsymbol{U}$. Then β^* is the transitive closure of β.

Analogous theorems for the relations γ^* in H_v-rings and ε^* in H_v-modules and H_v-vector spaces, are also proved. These relations were introduced and first studied by T.Vougiouklis, see [17]. One can see generalizations of the classical hyperstructure theory in several papers and books as [3],[4],[6],[17].

Fundamental relations are used for general definitions [17],[20]. Thus, in the general definition of the H_v-field, the γ^* is used: An H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field. This definition includes all the well known definitions of hyperfields [15], [17], as special cases.

Motivations. The motivation for H_v-structures is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group by any subgroup is a hypergroup. Now, the quotient of a group by any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [15].

Specifying this motivation we remark: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R,\cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups on the same set. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists $f\in Aut(H,*)$ such that $xy\subset f(x*y)$, $\forall x,y\in H$. Then, we write $\cdot\leq *$ and say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

Theorem (The Little Theorem), [17],[18]. In all H_v-structures and for all hopes, which are defined on them, greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on H_v-structures so, to a correspondence between hyperstructures and posets. The determination of all H_v-groups and H_v-rings is hard. In this direction there are many results by R. Bayon and N. Lygeros [1].

Definitions 1 [19],[20]. Let $(H,\cdot)$ be hypergroupoid. We *remove* $h\in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$, therefore, h does not appear in the hyperstructure.

Most of H_v-structures are used in Representation (abbreviate by *rep*) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [16],[17]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role.

The rep problem by H_v-matrices is the following:

H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type $m\times n$ and $n\times r$, respectively, is a set of $m\times r$ H_v-matrices, defined in a usual manner:

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{\, \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus\boldsymbol{\Sigma} a_{ik}\cdot b_{kj} \,\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition [17].

Definition 2. Let $(H,\cdot)$ be H_v-group, $(R,+,\cdot)$ H_v-ring, $\mathbf{M}_R=\{(a_{ij}) \mid a_{ij}\in R\}$, then any map

$$\mathbf{T}\!: H\to\mathbf{M}_R\!: h\to\mathbf{T}(h) \quad \text{with} \quad \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2)\neq\varnothing, \quad \forall h_1,h_2\in H,$$

is called *H_v-matrix rep*. If $\mathbf{T}(h_1h_2)\subset\mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is an *inclusion rep*, if $\mathbf{T}(h_1h_2)=\mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is a *good rep*.

Hyperoperations on any type of matrices can be defined:

Definition 3 [13],[8]. Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$, with $1\le s\le m$, $1\le t\le n$.

Then *helix-projection* is a map $\underline{st}$: $\mathbf{M}_{m\times n}\to\mathbf{M}_{s\times t}$: $A\to A\underline{st} = (\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij} = \{\, a_{i+\kappa s,j+\lambda t} \mid 1\le i\le s,\ 1\le j\le t \ \text{and}\ \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n \,\}$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called *helix-addition*, by

$$\oplus:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ and $s=\min(n,u)$. We define the *helix-multiplication*, by

$$\otimes:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{m\times v}): (A,B)\to A\otimes B = A\underline{ms}\cdot B\underline{sv} = (\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

2. Basic definitions

One can see basic definitions, results, applications and generalizations on hyperstructure theory, not only for H_v-structures, in the books [3],[4],[6],[17] and the survey papers [2],[5],[7],[14],[20],[21]. Here we present some definitions related to our problem.

In a hypergroupoid $(H,\cdot)$ the powers of $h\in H$ are $h^1=\{h\},\dots, h^n = h\circ h\circ\dots\circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts with all possible patterns of parentheses put on them. An H_v-semigroup $(H,\cdot)$ is called *cyclic of period s*, if there exists a g (*generator*) and a number s, the minimum, such that $H= h^1\cup \dots \cup h^s$. The cyclicity for the infinite period is defined in [14]. If there is an h and a number s, the minimum, such that $H=h^s$, then $(H,\cdot)$ is called *single-power cyclic of period s*.

In 1989 Corsini & Vougiouklis introduced a method to obtain stricter algebraic structures from given ones through hyperstructure theory. This method was introduced before of the H_v-structures, but in fact the H_v-structures appeared in the procedure.

Definition. *The uniting elements method* is the following: Let G be a structure and d be a property, which is not valid, and it is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. Then quotient of G/d by the fundamental relation β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid.

An application of the uniting elements is if more than one property desired. The reason for this is some of the properties lead straighter to the classes: commutativity and the reproductivity are easily applicable. One can do this because the following is valid:

Theorem [17],[21]. Let $(G,\cdot)$ be groupoid, and $F=\{f_1,\dots,f_m,f_{m+1},\dots,f_{m+n}\}$ a system of equations on G consisting of two subsystems $F_m=\{f_1,\dots,f_m\}$, $F_n=\{f_{m+1},\dots, f_{m+n}\}$. Let σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then we have $(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$.

Definition 4 [17],[21]**.** Let $(F,+,\cdot)$ be an H_v-field, $(\boldsymbol{V},+)$ be a COW H_v-group and there exists an external hope

$$\cdot: F\times \boldsymbol{V} \to P(\boldsymbol{V}): (a,x) \to ax$$

such that, for all a,b in F and x,y in $\boldsymbol{V}$ we have

$$a(x+y) \cap (ax+ay) \neq \varnothing, \quad (a+b)x \cap (ax+bx) \neq \varnothing, \quad (ab)x \cap a(bx) \neq \varnothing,$$

then $\boldsymbol{V}$ is called an *H_v-vector space* over F.

In the case of an H_v-ring instead of H_v-field then the *H_v-modulo* is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence relation such that the quotient $\boldsymbol{V}/\varepsilon^*$ is a vector space over the fundamental field F/γ^*.

The general definition of an H_v-Lie algebra over a field F was given in as follows:

Definition 5. Let $(\boldsymbol{L},+)$ be an H_v-vector space over the field $(F,+,\cdot)$, $\varphi: F\to F/\gamma^*$ be the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi': \boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Consider the bracket (commutator) hope:

$$[\ ,\ ] : \boldsymbol{L}\times\boldsymbol{L} \to P(L): (x,y) \to [x,y]$$

then $\boldsymbol{L}$ is an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+ \lambda_2x_2,y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \varnothing$$

$$[x, \lambda_1y_1+ \lambda_2y] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \varnothing, \ \forall\ x,x_1,x_2,y,y_1,y_2 \in\boldsymbol{L}, \quad \lambda_1,\lambda_2 \in F$$

(L2) $[x,x] \cap \omega_L \neq \varnothing$, $\forall x \in \boldsymbol{L}$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]]) \cap \omega_L \neq \varnothing$, $\forall x,y \in \boldsymbol{L}$.

We remark that this is a very general definition therefore one can use special cases in order to face several problems in applied sciences [12],. Moreover, from this definition we can see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3).

3. ∂-hopes

In [22] an extremely large class of hopes introduced called theta:

Definition 6. Let H be a set equipped with n operations (or hopes) $\otimes_1,\ldots,\otimes_n$ and a map (or multivalued map) $f:H\to H$ (or $f:H\to P(H)-\{\varnothing\}$, resp.), then n hopes $\partial_1,\ldots,\partial_n$ on H are defined, called *theta-hopes*, (*∂-hopes*), by putting

$$x\partial_i y = \{f(x)\otimes_i y,\ x\otimes_i f(y)\ \},\quad \forall x,y\in H \ \text{ and }\ i\in\{1,2,\ldots,n\}$$

or, in the case where $\otimes_i$ is hope or f is multivalued map, we have

$$x\partial_i y = (\ f(x)\otimes_i y)\cup(x\otimes_i f(y)\ ),\ \forall x,y\in H \text{ and } i\in\{1,2,\ldots,n\}$$

If $\otimes_i$ is associative, then ∂_i is WASS. Remark that one can use several maps f instead of only one, in a similar way.

In a groupoid $(G,\cdot)$, or a hypergroupoid, with a ∂-hope, one can study several properties like the following ones:

Reproductivity. For the reproductivity we must have

$$x\partial G= \cup_{g\in G}\{f(x)\cdot g,\ x\cdot f(g)\}= G \quad\text{and}\quad G\partial x= \cup_{g\in G}\{f(g)\cdot x,\ g\cdot f(x)\}= G.$$

If $(\cdot)$ is reproductive, then (∂) is reproductive: $\cup_{g\in G}\{f(x)\cdot g\} = G$.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre, then (∂) is commutative. If $(\cdot)$ is *COW* then (∂) is *COW*.

Unit elements. u is right unit if $x\partial u=\{f(x)\cdot u,x\cdot f(u)\}\ni x$. So $f(u)=e$, if e is a unit in $(G,\cdot)$. The elements of the kernel of f, are the units of (G,∂).

Inverse elements. Let $(G,\cdot)$ be a monoid with unit e and u be a unit in (G,∂), then $f(u)=e$. For given x, the x' is an inverse with respect to u, if $x\partial x'=\{f(x)\cdot x',x\cdot f(x')\}\ni u$ and $x'\partial x=\{f(x')\cdot x,x'\cdot f(x)\}\ni u$. So, $x'=(f(x))^{-1}u$ and

$x'=u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Proposition 7. Let $(G,\cdot)$ be a group then, for all maps $f: G\to G$, the (G,∂) is an H_v-group.

One can define ∂-hopes on rings and more complicated structures (or hyper-structures or H_v-structures), where more than one ∂-hopes can be defined.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x)$, $t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Example. Let $(G,\cdot)$ be a group and $f(x)=a$ the *constant map* on G, then $x\partial y=\{ay,xa\}$, $\forall x,y\in G$. The $(G,\partial)/\beta^*$ is singleton, indeed, we have $a^{-1}\partial(a^{-1}\cdot x)=\{x,e\}$ $\forall x\in G$, so $x\beta e$, $\forall x\in G$, thus $\beta^*(x)=\beta^*(e)$. For $f(x)=e$ we obtain $x\partial y=\{x,y\}$, the smallest incidence hope.

Propositions 8. Let $g\in G$ is a generator of the group $(G,\cdot)$. Then,

(a) for every f, g is a generator in (G,∂), with period at most n.

(b) suppose that there exists an element w such that $f(w)=g$, then the element w is a generator in (G,∂), with period at most n.

Definitions 9. Let $(G,\cdot)$ be a groupoid and $f_i:G\to G$, $i\in I$, be a set of maps. Take the map $f_\cup:G\to \boldsymbol{P}(G)$ such that $f_\cup(x)=\{f_i(x) \mid i\in I\}$, i.e. *the union* of $f_i(x)$. We call *union ∂-hopes*, if we consider the map $f_\cup(x)$. Special case: the union of f with the identity, i.e. $\underline{f}= f\cup(id)$, so $\underline{f}(x)=\{x,f(x)\}$, $\forall x\in G$, which is called *b∂-hope*. We denote the b∂-hope by $(\underline{\partial})$, so

$$x\underline{\partial}y = \{ xy, f(x)\cdot y, x\cdot f(y) \},\ \forall x,y\in G.$$

Remark that $\underline{\partial}$ contains the operation $(\cdot)$ so it is a b-hope. If $f: G\to P(G)-\{\varnothing\}$, then the b$\partial$-hope is defined by using the map $\underline{f}(x)=\{x\}\cup f(x)$, $\forall x\in G$.

A construction between ∂ and $\underline{\partial}$ is the one which obtained by using special maps.

Definition 10. Let $(G,\cdot)$ be a groupoid and $f:G\to G$ be a map, we call *basic set* of the map f the set $B = B_f = \{ x\in G: f(x)=x \}$. Then, if $B\neq\varnothing$, we have

$$x\partial y = xy,\ \forall x,y\in B,$$

$$x\partial y = \{ xy, x\cdot f(y) \},\ \forall x\in B,\ y\in G-B,$$

$$x\partial y = \{ f(x)\cdot y, xy \},\ \forall x\in G-B,\ y\in B,$$

$$x\partial y = \{ f(x)\cdot y, x\cdot f(y) \}, \ \forall x,y\in G\text{-}B.$$

For $(G,\cdot)$ groups, we obtain the following:

- If B is a subgroup of $(G,\cdot)$, then (B,∂) is a sub-H_v-group of (G,∂).
- If $e\in B$, then e is a unit of (B,∂) because it belongs into the kernel of f.
- Inverses: If u is a unit of (B,∂), then $x\in G$, has an inverse in (G,∂) if $f(x)u = uf(x)$. Therefore an element $x\in B$ has an inverse iff $xu=ux$. If $e\in B$ then the element x^{-1} is an inverse of x in (G,) as well.

Proposition 11. Let $g\in G$ is a generator of the group $(G,\cdot)$. If $g\in B$ then g is a generator in (G,∂), $\forall f$, with period at most n.

There is connection of ∂-hopes with other hyperstructures:

Example. *Merging and ∂.* If $(H,\cdot)$ is a groupoid and $\underline{h}\in H$ merges with the $h\in H$, then h does not appeared and we have for the merge $(H,\circ)$,

$$\underline{h}\circ x = \{\underline{h}\cdot x, h\cdot x\}, \ x\circ\underline{h} = \{x\cdot\underline{h}, x\cdot h\}, \ \underline{h}\circ\underline{h} = \{\underline{h}\cdot\underline{h}, \underline{h}\cdot h, h\cdot\underline{h}, h\cdot h\}$$

and in rest cases $(\circ)$ coincides with $(\cdot)$, so we have merge $(H\text{-}\{h\},\circ)$.

Similarly, if $(H,\cdot)$ is a hypergroupoid then we have

$$\underline{h}\circ x= (\underline{h}\cdot x)\cup(h\cdot x), \ x\circ\underline{h}= (x\cdot\underline{h})\cup(x\cdot h), \ \underline{h}\circ\underline{h}= (\underline{h}\cdot\underline{h})\cup(\underline{h}\cdot h)\cup(h\cdot\underline{h})\cup(h\cdot h)$$

In order to see a connection of merge with the ∂-hope, consider the map f such that $f(\underline{h})=h$ and $f(x)=x$ in the rest cases. Then in $(H\text{-}\{h\},\partial)$ we have, $\forall x,y\in H\text{-}\{h\}$

$$\underline{h}\partial x = \{h\cdot x, \underline{h}\cdot x\}, \quad x\partial\underline{h} = \{x\cdot\underline{h}, x\cdot h\} \quad \text{and} \quad \underline{h}\partial\underline{h} = \{h\cdot\underline{h}, \underline{h}\cdot h\}$$

and in the rest cases (∂) coincides with $(\cdot)$. Therefore, $\partial \leq\circ$, or

$$\underline{h}\partial\underline{h} = \{h\cdot\underline{h}, \underline{h}\cdot h\} \ \subset \ \{\underline{h}\cdot\underline{h}, \underline{h}\cdot h, h\cdot\underline{h}, h\cdot h\}= \underline{h}\circ\underline{h}$$

and in the remaining cases we have $\partial\equiv\circ$.

Example. *P-hopes* [14]. Let $(G,\cdot)$ be a commutative semigroup and $P\subset G$. Consider the multivalued map f such that $f(x) = P\cdot x$, $\forall x\in G$.

Then we have $x\partial y = x\cdot y\cdot P$, $\forall x,y\in G$.

So the ∂-hope coincides with the well known class of P-hopes [20].

One can define theta-hopes on rings and other more complicate structures, where more than one ∂-hopes can be defined. Moreover, one can replace structures by hyper ones or by H_v-structures, as well [23],[24].

Definition 12. Let $(R,+,\cdot)$ be a ring and $f:R\to R$, $g:R\to R$ be two maps. We define two hyperoperations (∂_+) and $(\partial\cdot)$, called both *theta-operations*, on R as follows

$$x\partial_+ y = \{f(x)+y,\ x+f(y)\ \} \quad \text{and} \quad x\partial\cdot y = \{g(x)\cdot y,\ x\cdot g(y)\ \},\ \forall x,y\in G.$$

A hyperstructure $(R,+,\cdot)$, where $(+)$, $(\cdot)$ be hyperoperations which satisfy all H_v-ring axioms, except the weak distributivity, will be called *H_v-near-ring.*

Proposition 13. Let $(R,+,\cdot)$ ring and $f:R\to R$, $g:R\to R$ maps. The hyperstructure $(R,\partial_+,\partial\cdot)$, called *theta*, is an H_v-near-ring. Moreover $(+)$ is commutative.

Proof. First, one can see that all properties of an H_v-ring, except the distributivity, are valid. For the distributivity we have, $\forall x,y,z\in R$, $\quad x\partial\cdot(y\partial_+z) \cap (x\partial\cdot y)\partial_+(x\partial\cdot z) = \varnothing$. ■

In order more properties to be valid, the ∂ can be replaced by $\underline{\partial}$.

Proposition 14. Let $(R,+,\cdot)$ ring and $f:R\to R$, $g:R\to R$ maps, then $(R,\underline{\partial}_+,\partial\cdot)$, is an H_v-ring.

Proof. The only one axiom we have to see is the distributivity. So, $\forall x,y,z\in R$,

$$x\partial\cdot(y\underline{\partial}_+z) = \{g(x)(y+z),\ g(x)(f(y)+z),\ g(x)(y+f(z)),\ xg(y+z),\ xg(f(y)+z),\ xg(y+f(z))\}$$

and $\quad (x\partial\cdot y)\underline{\partial}_+(x\partial\cdot z) =$

$$\{g(x)(y+z),\ f(g(x)y)+g(x)z,\ g(x)y+f(g(x)z),\ g(x)y+xg(z),\ f(g(x)y)+xg(z),$$
$$g(x)y+f(xg(z)),\ xg(y)+g(x)z,\ f(xg(y))+g(x)z,\ xg(y)+f(g(x)z),\ x(g(y)+g(z)),$$
$$f(xg(y))+xg(z),\ xg(y)+f(xg(z))\}.$$

So $\quad x\partial\cdot(y\underline{\partial}_+z) \cap (x\partial\cdot y)\underline{\partial}_+(x\partial\cdot z) =\{\ g(x)(y+z)\} \neq \varnothing$.

Therefore, $(R,\underline{\partial}_+,\partial\cdot)$ is an H_v-ring. ■

Remark. If $(R,+,\cdot)$ ring and $f:R\to R$, $g:R\to R$ maps, then $(R,\partial_+,\underline{\partial}\cdot)$ is still an H_v-near-ring.

Theorems 15. (a) Consider the group of integers $(\boldsymbol{Z},+)$ and let $n\neq 0$ be a natural number. Take the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in\boldsymbol{Z}$-$\{0\}$. Then $(\boldsymbol{Z},\partial)/\beta^* \cong (\boldsymbol{Z}_n,+)$.

(b) Consider the ring of integers $(\mathbf{Z},+,\cdot)$ and let $n\neq 0$ be a natural. Consider the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in\boldsymbol{Z}$-$\{0\}$. Then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-near-ring, with

$$(\boldsymbol{Z},\partial_+,\partial\cdot)/\gamma^* \cong \boldsymbol{Z}_n.$$

Special case of the above is for n=p, prime, then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-field.

Proposition 16. Let $(\boldsymbol{V},+,\cdot)$ be an algebra over the field $(F,+,\cdot)$ and $f:\boldsymbol{V}\to\boldsymbol{V}$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\boldsymbol{V},+,\partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have more strong properties.

Definition 17. Let $\boldsymbol{L}$ be a Lie algebra, defined on an algebra $(\boldsymbol{V},+,\cdot)$ over the field $(F,+,\cdot)$ with Lie bracket $[x,y]=xy-yx$. Consider a map $f:\boldsymbol{L}\to\boldsymbol{L}$, then the ∂-hope is defined by

$$x\partial y = \{f(x)y-f(y)x,\ f(x)y-yf(x),\ xf(y)-f(y)x,\ xf(y)-yf(x)\}$$

Proposition 18. Let $(\boldsymbol{V},+,\cdot)$ be an algebra over the field $(F,+,\cdot)$ and $f:\boldsymbol{V}\to\boldsymbol{V}$ be a linear map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\boldsymbol{V},+,\partial)$ is an H_v-algebra over F, with respect to Lie bracket, where the weak anti-commutatinity and the inclusion linearity is valid.

If $(G,\cdot)$ is a semigroup then, for every f, the b∂-operation $(\underline{\partial})$ is *WASS.*

4. HypreStructures in questionnaires

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including the social ones. These applications range from biomathematics and hadronic physics to automata theory, to mention but a few. This theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too.

In several papers, such as [2],[4],[11],[12] one can find numerous applications; similarly, in the books [4], [6] a wide variety of applications is also presented.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis. The suggestion is the following [10]:

Definition 19. *"In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ———————————— 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question".

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, see [9], [25].

Now we state our main problem for this paper by using this bar and we can describe in mathematical model using theta-hopes.

Problem 20. In the research processing suppose that we want to use Likert scale dividing the continuum [01] both by, first, into equal steps (segments) and, second, into equal-area spaces according to Gauss distribution [9], [25]. If we consider both types of divisions into n segments, then the continuum [01] is divided into 2n-1 segments, if n is odd number and into 2(n-1) segments, if n is even number. We can number the segments and we can consider as an organized devise the group ($\boldsymbol{Z}_k,\oplus$) where k=2n-1 or 2(n-1). Then we can obtain several hyperstructures using ∂-hopes as the following way: We can have two partitions of the final segments, into n classes either using the division into equal steps or the Gauss distribution by putting in the same class all segments that belong (a) to the equal step or (b) to equal-area spaces according to Gauss distribution. Then we can consider two kinds of maps (i) a multi-map where every element corresponds to the hole class or (ii) a map where every element corresponds to one special fixed element of the same class. Using these maps we define the ∂-hopes and we obtain the corresponding H_v-structure.

An example for the case (i) is the following:

Example 21. Suppose that we take the case of the Likert scale with 5 equal steps: [0-1.24-2.48-3.72-4.96-6.2] and the Gauss 5 equal areas: [0-2.4-2.9-3.3-3.8-6.2] we have 9 segments as follows

[0 –1.24 – 2.4 – 2.48 – 2.9 – 3.3 – 3.72 – 3.8 – 4.96 – 6.2]

Therefore, if we consider the set $\boldsymbol{Z}_9$ and if we name the segments by 1, 2,…, 8, 0, then if we consider the equal steps partition: {1}, {2,3}, {4,5,6}, {7,8}, {0} we take, according to the above construction the multi-map f such that f(1)={1}, f(2)={2,3}, f(3)={2,3}, f(4)={4,5,6}, f(5)={4,5,6}, f(6)={4,5,6}, f(7)={7,8}, f(8)={7,8}, f(0)={0}, then we obtain the following table:

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**0**
1	2	3,4	3,4	5,6,7	5,6,7	5,6,7	0,8	0,8	1
2	3,4	4,5	4,5,6	6,7,8	6,7,8	6,7,8,9	0,1	0,1,2	2,3
3	3,4	4,5,6	5,6	0,6,7,8	0,6,7,8	0,6,7,8	0,1,2	0,1,2	2,3
4	5,6,7	6,7,8	0,6,7,8	0,1,8	0,1,2,8	0,1,2,3,8	2,3,4	2,3,4,5	4,5,6
5	5,6,7	6,7,8	0,6,7,8	0,1,2,8	0,1,2	0,1,2,3	2,3,4	3,4,5	4,5,6
6	5,6,7	6,7,8,9	0,6,7,8	0,1,2,3,8	0,1,2,3	1,2,3	2,3,4,5	3,4,5	4,5,6
7	0,8	0,1	0,1,2	2,3,4	2,3,4	2,3,4,5	5,6	5,6,7	7,8
8	0,8	0,1,2	0,1,2	2,3,4,5	3,4,5	3,4,5	5,6,7	6,7	7,8
0	1	2,3	2,3	4,5,6	4,5,6	4,5,6	7,8	7,8	0

5. Hyperstructures in several scales obtained from the bar

Now we represent a mathematic model on obtained from the Problem 20:

Construction 22. Consider a group $(\mathbf{G},\cdot)$ and suppose take a partition $\mathbf{G}_i, i\in I$ of the $\mathbf{G}$. Select and fix an element g_i of each partition class $\mathbf{G}_i$, and consider the map

$$f: \mathbf{G}\to\mathbf{G} \quad \text{such that} \quad f(x)= g_i\,, \quad \forall x\in\mathbf{G}_i,$$

then $(\mathbf{G},\partial)$ is an H_v-group. Moreover the fundamental group $(G/R,\cdot)/\beta^*$ is (up to isomorphism) a subgroup of the corresponding fundamental group $(G,\partial)/\beta^*$.

Proof. First, we remark that the ∂-H_v-group $(\mathbf{G},\partial)$ is an H_v-group because this is true for all given maps. Now, let us call R the given partition. For all $x\in\mathbf{G}_i$ and $y\in\mathbf{G}_j$ we have $x\partial y=\{g_iy,xg_j\}$, thus we remark that the elements g_iy and xg_j belong to the same R class. Therefore, the β^*-classes with respect to ∂, are subsets of the β^*-classes with respect to the R-classes. The fundamental group $(G/R,\cdot)/\beta^*$ is (up to isomorphism) a subgroup of the corresponding fundamental group $(G,\partial)/\beta^*$. ■

Theorem 23. In the above construction, if one of the selected elements is the unit element e of the group $(\mathbf{G},\cdot)$, otherwise, if there exist an element $z\in\mathbf{G}$ such that $f(z)=e$, then we have

$$(G/R,\cdot)/\beta^* = (G,\partial)/\beta^*.$$

Proof. Since there exist $z\in\mathbf{G}$ such that $f(z)=e$, then for all $x\in\mathbf{G}_i$, we have $f(x)=g_i$, consequently, $f(e)=e$. Moreover, for all $x\in\mathbf{G}_i$, we have

$$x\partial e = \{g_i\cdot e,\, x\cdot e\}= \{g_i,\, x\},$$

thus, x belongs to the fundamental class to g_i with respect to ∂-hope. So $\mathbf{G}_i \subset\beta^*(g_i)$ and from the above theorem we obtain that

$$(G/R,\cdot)/\beta^* = (G,\partial)/\beta^*. \blacksquare$$

In hypergroups does not necessarily exist any unit element and if there exists a unit this is not necessarily unique. Moreover the ∂-hopes do not have always the unit element of the group as unit for the corresponding ∂-hope. This is so because

$$e\partial e = \{f(e)e, ef(e)\} = \{f(e)\}.$$

However for the above hyperstructure we have the following:

Proposition 24. Suppose $(\mathbf{G},\cdot)$ be a group and $\mathbf{G}_i, i\in I$ be a partition of $\mathbf{G}$. For any class we fix a $g_i\in\mathbf{G}_i$, and take the map $f: \mathbf{G}\to\mathbf{G}: f(x)=g_i, \forall x\in\mathbf{G}_i$. If for the unit element e, in $(\mathbf{G},\cdot)$, we have $f(e)=e$, i.e. e is any fixed element, then e is also a unit element of the H_v-group $(\mathbf{G},\partial)$. Moreover $(f(x))^{-1}$ is an inverse element in the ∂-H_v-group $(\mathbf{G},\partial)$, of x.

Proof. For all $x\in\mathbf{G}$ we have

$$x\partial e = \{f(x)e, xe\} = \{f(x), x\}\ni x.$$

Thus, e is a unit element in $(\mathbf{G},\partial)$.

Moreover, $\forall x\in G$, denoting $(f(x))^{-1}$ the inverse of f(x) in $(\mathbf{G},\cdot)$, we have

$$x\partial(f(x))^{-1} = \{f(x)\cdot(f(x))^{-1}, x\cdot f((f(x))^{-1})\} = \{e, x\cdot f((f(x))^{-1})\}\ni e.$$

Therefore the element $(f(x))^{-1}$ is an inverse of x with respect to ∂.

This theorem states that the inverse g_i^{-1} in $(\mathbf{G},\cdot)$, of every fixed element g_i, is also an inverse in $(\mathbf{G},\partial)$ of all elements which belong to their partition class $\mathbf{G}_i$. Finally, remark that some of the elements of G may have more than one inverse in $(\mathbf{G},\partial)$. $\blacksquare$

Now we conclude with an example of the above Construction 22 on our main Problem 20:

Example 11. In the case of the Likert scale with 6 equal steps: [0-1-2.1-3.1-4.1-5.1-6.2] and the Gauss 6 equal areas: [0-2.23-2.73-3.1-3.47-3.97-6.2] we have 10 segments as follows

[0 – 1 – 2.1 – 2.23 – 2.73 – 3.1 – 3.47 – 3.97 – 4.1 – 5.1 – 6.2]

Therefore, if we consider the set $\boldsymbol{Z}_{10}$ and if we name the segments by 1, 2,…, 9, 0, then if we consider the Gauss partition: {1,2,3}, {4}, {5}, {6}, {7}, {8,9,0} we take, according to the above Theorem, the map f such that f(1)={1}, f(2)={1}, f(3)={1}, f(4)={4}, f(5)={5}, f(6)={6}, f(7)={7}, f(8)={0}, f(9)={0}, f(0)={0}, then we obtain the following table:

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**0**
1	2	2,3	2,4	5	6	7	8	1,9	0,1	1
2	2,3	3	3,4	5,6	6,7	7,8	8,9	2,9	0,2	1,9
3	2,4	3,4	4	5,7	6,8	7,9	0,8	3,9	0,3	1,3
4	5	5,6	5,7	8	9	0	1	2,4	3,4	4
5	6	6,7	6,8	9	0	1	2	3,5	4,5	5
6	7	7,8	7,9	0	1	2	3	4,6	5,6	6
7	8	8,9	0,8	1	2	3	4	5,7	6,7	7
8	1,9	2,9	3,9	2,4	3,5	4,6	5,7	8	8,9	0,8
9	0,1	0,2	0,3	3,4	4,5	5,6	6,7	8,9	9	0,9
0	1	1,9	1,3	4	5	6	7	0,8	0,9	0

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On geometrical hyperstructures of finite order

Achilles Dramalidis

School of Sciences of Education,

Democritus University of Thrace,

681 00 Alexandroupolis, Greece

adramali@psed.duth.gr

Abstract

It is known that a concrete representation of a finite *k*-dimensional Projective Geometry can be given by means of marks of a Galois Field GF [p^n], denoted by PG(k, p^n).
In this geometry, we define hyperoperations, which create hyperstructures of finite order and we present results, propositions and examples on this topic. Additionally, we connect these hyperstructures to Join Spaces.

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1. Introduction

The algebraic hyperstructures, which constitute a generalization of the ordinary algebraic structures, were introduced by Marty in 1934 [5]. Since then, many researchers worked on hyperstructures. The results of this work, as well as, applications of the hyperstructures theory can be found in the books [2] and [3]. Vougiouklis in 1991 introduced a larger class than the known hyperstructures, so called H_v-structures [8] and all about them can be found in his book [9].
Let us give some basic definitions, appearing in [3], [9]:

Let H be a set, $P'(H)$ the family of nonempty subsets of H and $(\cdot)$ a hyperoperation in H, that is

$$\cdot : H\times H \to P'(H)$$

If $(x,y)\in H\times H$, its image under $(\cdot)$ is denoted by $x\cdot y$ or xy. If A , $B \subseteq H$ then $A\cdot B$ or AB is given by $AB = \cup \{xy / x\in A , y\in B\}$.
xA is used for $\{x\}A$ and Ax for $A\{x\}$. Generally, the singleton $\{x\}$ is identified with its member x.
The hyperoperation $(\cdot)$ is called *associative* in H if

$$(xy)z = x(yz) \text{ for all } x,y,z\in H$$

The hyperoperation $(\cdot)$ is called *commutative* in H if

$$xy = yx \text{ for all } x,y\in H$$

A hypergroupoid $(H,\cdot)$ that satisfies reproducibility, $xH = Hx = H$ for all $x\in H$, and associativity, is called *hypergroup*.
A *join operation* $(\cdot)$ [6] in a set J is a mapping of $J\times J$ into the family of subsets of J. A *join space* is defined as a system $(J,\cdot)$, where $(\cdot)$ is a join operation in the arbitrary set J, which satisfies the postulates:

$$\text{i) } a\cdot b\neq\varnothing \quad \text{ii) } a\cdot b = b\cdot a \quad \text{iii) } (a\cdot b)\cdot c=a\cdot(b\cdot c)$$

$$\text{iv) } a/b \cap c/d\neq\varnothing \Rightarrow a\cdot d \cap b\cdot c\neq\varnothing \quad \text{v) } a/b = \{x\in J / a\in b\cdot x\}\neq\varnothing .$$

The H_v-structures are hyperstructures satisfying the weak axioms, where the non-empty intersection replaces the equality.

Let $H\neq\varnothing$ be a set equipped with the hyperoperations $(+)$, $(\cdot)$, then the *weak associativity* in $(\cdot)$ is given by the relation

$$(x\cdot y)\cdot z \cap x\cdot(y\cdot z) \neq \varnothing, \ \forall x,y,z\in H.$$

The $(\cdot)$, is called *weak commutative* if

$$x\cdot y \cap y\cdot x \neq \varnothing, \ \forall x,y\in H.$$

The hyperstructure $(H,\cdot)$ is called *H_v-semigroup* if $(\cdot)$ is weak associative and it is called *H_v-quasigroup* if the reproduction axiom is valid, i.e. $x\cdot H=H\cdot x=H$, $\forall x\in H$.

The hyperstructure $(H,\cdot)$ is called *H_v-group* if it is an H_v-quasigroup and an H_v-semigroup. It is called *H_v-commutative group* if it is an H_v-group and the weak commutativity is valid.

The *weak distributivity* of $(\cdot)$ with respect to $(+)$ is given for all $x,y,z \in H$, by

$$x\cdot(y+z) \cap (x\cdot y+x\cdot z) \neq \emptyset \;,\; (x+y)\cdot z \cap (x\cdot z+y\cdot z) \neq \emptyset.$$

Using these axioms, the *H_v-ring*, which is the largest class of algebraic systems that satisfy ring-like axioms, is defined to be the triple $(H,+,\cdot)$, where in both $(+)$ and $(\cdot)$ the weak associativity is valid, the weak distributivity is also valid and $(+)$ is *reproductive*, i.e $x+H = H+x = H$, $\forall x \in H$.

An H_v-ring $(R,+,\cdot)$ is called *dual H_v-ring* if the hyperstructure $(R,\cdot,+)$ is also an H_v-ring [4].

Let $(H,\cdot)$ be a hypergroup or an H_v-group. The β^* relation is defined as the smallest equivalence relation, one can say also congruence, such that, the quotient H/β^* is a group.

The β^* is called *fundamental equivalence relation*.

2. Representation of the geometry of a *k*-dimensional space by means of Galois Fields

Veblen and Bussey [7] have defined a finite projective geometry, which is said to be a geometry of a k-dimensional space, in the following way.

It consists of a set of elements, called points for suggestiveness, which are subjected to the following five conditions or postulates:

I. The set contains a finite number of points. It contains one or more subsets called lines, each of which contains at least three points.

II. If A and B are distinct points, there is one and only one line that contains both A and B.

III. If A, B, C are noncollinear points and if a line l contains a point D of the line AB and a point E of the line BC but does not contain A or B or C, then the line l contains a point F of the line CA.

IV. If m is an integer less than k, not all the points considered are in the same m-space.

V. If (IV) is satisfied, there exists in the set of points considered no (k+1)-space.

Furthermore, a point is called *0*-space, a line is called *1*-space and a plane is called *2*-space.

By means of marks of a Galois field, we shall now give a concrete representation of a finite k-dimensional projective geometry.

We denote a point of the geometry by the ordered set of coordinates $(\mu_0, \mu_1, \mu_2,\ldots, \mu_k)$, where $\mu_0, \mu_1, \mu_2,\ldots, \mu_k$ are marks of the GF[p^n], at least one of which is different from zero. It is understood that the foregoing symbol $(\mu_0, \mu_1, \mu_2,\ldots, \mu_k)$ denotes the same point as the symbol $(\mu\mu_0, \mu\mu_1, \mu\mu_2,\ldots, \mu\mu_k)$, where μ is one of the p^n-1 nonzero marks of the field.

The ordered set of marks $\mu_0, \mu_1, \mu_2,\ldots, \mu_k$ may be chosen in $(p^n)^{\kappa+1}$ ways, but since the symbol (0, 0, 0,….,0) is excepted, then it may be chosen in $(p^n)^{\kappa+1}-1$ ways. So, there exists $(p^n)^{\kappa+1}-1$ points. In this totality, each point is represented in p^n-1 ways (there are p^n-1 nonzero marks in the field) and thus, it follows that the number of points defined is

$$\frac{(p^n)^{k+1}-1}{p^n-1} = 1 + p^n + \ldots.. + p^{kn}$$

This representation of the finite κ-dimensional projective geometry by means of the marks of the GF[p^n] constitute the projective geometry PG(κ, p^n) [1].

Now, for the line containing the two distinct points $(\mu_0, \mu_1, \mu_2,\ldots, \mu_k)$ and $(\nu_0, \nu_1, \nu_2,\ldots, \nu_k)$ we consider the set of points :

$$(\mu\mu_0 + \nu\nu_0 , \mu\mu_1 + \nu\nu_1 , \mu\mu_2 + \nu\nu_2 , \ldots\ldots, \mu\mu_k + \nu\nu_k)$$

where μ and ν run independently over the marks of the GF[p^n], subjected to the condition that μ and ν shall not be simultaneously zero.

Then the number of possible combinations of μ and ν is $(p^n)^2 - 1$ and for each of these the corresponding symbol denotes a point, since not all the k+1 coordinates are zero. But the same point is represented p^n-1 times, due to the factor of proportionality involved in the definition of points. Therefore, a line so defined contains

$$\frac{(p^n)^2-1}{p^n-1} = p^n + 1 \text{ points.}$$

It is obvious that any two points on the line may be used in this way to define the same line.
The five postulates given above for the *k*-dimensional space are satisfied by the concrete elements thus introduced [1].

3. On a hypergroup of finite order

Let us denote by V the set of the elements of the PG(κ, p^n) and for x,y∈V let us denote by l_{xy} the line which is defined by the points x and y. By l_{x-} is denoted the line which is defined by the point x and any other point of V.
We define the hyperoperation (·) on V, as follows :

Definition 1. For every x,y∈V , $\cdot : V\times V \to P'(V)$, such that

$$x\cdot y = \begin{cases} x & if \;\; x = y \\ l_{xy} & if \;\; x \neq y \end{cases}$$

Obviously, the above hyperoperation is a commutative one, since

$$x\cdot y = l_{xy} = l_{yx} = y\cdot x \quad \text{for every } x,y\in V \text{ and } x \neq y.$$

One can compare the above defined hyperoperation with the join operation [6], when Euclidean Geometry is converted into Join Spaces by defining *ab* with $a \neq b$, to be the open segment, whose endpoints are *a* and b. Moreover, *aa* is defined to be *a*.

Proposition 2. For every noncollinear x,y,z∈V, $|x\cdot(y\cdot z)| = |(x\cdot y)\cdot z| = p^{2n} + p^n + 1$.

Proof. All the lines defined in V are having one point in common, at most. First, let us calculate the points of the set $x\cdot(y\cdot z)$. For $y \neq z$, the set $y\cdot z = l_{yz}$ consists of p^n+1 points, including y and z. On the other hand, the point x ($x \neq y,z$), with each of the p^n+1 points of the line l_{yz}, creates p^n+1 lines of the type $l_{x\text{-}}$ which are having the point x in common. This means that the p^n+1 lines of the type $l_{x\text{-}}$ are having no other point in common. So, each line $l_{x\text{-}}$ is having p^n different points from the others. Then it follows that the set $x\cdot(y\cdot z)$ consists of $(p^n+1)\cdot p^n+1= p^{2n}+p^n+1$ different points.

Similarly, it arises that $|(x\cdot y)\cdot z| = p^{2n}+p^n+1$.■

Proposition 3. The hyperstructure $(V, \cdot)$ is a hypergroup.
Proof. Easily follows, that for every $x\in V$

$$x\cdot V = \bigcup_{v\in V}(x\cdot v)=\bigcup_{v\in V}(v\cdot x) = V\cdot x = V$$

Now, for every $x, y, z\in V$

if $x = y = z$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = x$
if $x = y \neq z$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xz}$
if $x = z \neq y$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if $y = z \neq x$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if $x \neq y \neq z$ and x, y, z collinear, then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if x, y, z noncollinear then

for the line l_{yz} containing the two distinct points $y(y_0, y_1,..,y_k)$ and $z(z_0, z_1,\ldots, z_k)$ we take the set of points :

$$(\mu y_0 + \nu z_0 , \mu y_1 + \nu z_1 ,\ldots\ldots, \mu y_k + \nu z_k)$$

where μ and ν run independently over the marks of the $GF[p^n]$ subjected to the condition that μ and ν shall not be simultaneously zero.

Let $x(x_0, x_1,...,x_k)$. Then for the set $x\cdot(y\cdot z)$ we take the set of points :

$$\big(\rho x_0 + \lambda(\mu y_0 + \nu z_0) , \rho x_1 + \lambda(\mu y_1 + \nu z_1) ,\ldots\ldots, \rho x_k + \lambda(\mu y_k + \nu z_k)\big) \quad (1)$$

where ρ and λ run independently over the marks of the $GF[p^n]$ subjected to the condition that ρ and λ shall not be simultaneously zero.

Let $w \in x\cdot(y\cdot z)$, then the coordinates of the point w is of the form of (1).

For some $i \in 0,1,\ldots,k$ we have

$$\rho x_i+\lambda(\mu y_i+\nu z_i) = \rho x_i+\lambda\mu y_i+\lambda\nu z_i = \rho x_i+\lambda\mu y_i+\nu' z_i, \text{ where } \nu' \in GF[p^n]$$

If $\rho=0$ (2) then $\rho=\lambda 0$ for every $\lambda \in GF[p^n]$ and then

$$\rho x_i + \lambda\mu y_i + \nu' z_i = \lambda 0 x_i + \lambda\mu y_i + \nu' z_i = \lambda(0x_i + \mu y_i) + \nu' z_i$$

If $\rho \neq 0$ (3) then $\rho=\lambda\mu'$ for every $\lambda, \mu' \in GF[p^n]-\{0\}$ and then

$$\rho x_i + \lambda\mu y_i + \nu' z_i = \lambda\mu' x_i + \lambda\mu y_i + \nu' z_i = \lambda(\mu' x_i + \mu y_i) + \nu' z_i$$

The coordinates of the points of the set $(x\cdot y)\cdot z$ are of the form

$$\kappa(\tau x_i + \tau' y_i) + \kappa' z_i$$

where $\tau, \tau', \kappa, \kappa'$ run independently over the marks of the $GF[p^n]$ subjected to the condition that τ, τ' and κ, κ' shall not be simultaneously zero.

Due to the conditions (2) and (3) we get that

$w \in x\cdot(y\cdot z) \Rightarrow w \in (x\cdot y)\cdot z$ which means that $x\cdot(y\cdot z) \subset (x\cdot y)\cdot z$.

In a similar way, it can be proven that for $w' \in (x\cdot y)\cdot z \Rightarrow w' \in x\cdot(y\cdot z)$, which means that $(x\cdot y)\cdot z \subset x\cdot(y\cdot z)$. So,

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z \quad \text{for every } x,y,z \in V. \blacksquare$$

Remark 4. For the hypergroup $(V,\cdot)$, since $\{x,y\} \subset x\cdot y$ $\forall x,y \in V$, the V/β^* is a singleton.

Proposition 5. The hypergroup $(V, \cdot)$ is a Join Space.

Proof. Since the hyperoperation $(\cdot)$ is commutative, the hyperstructure $(V, \cdot)$ is a commutative hypergroup.

Moreover, let $a/b \cap c/d \neq \varnothing$, $a,b,c,d \in V$. Then, there exists $w \in V$ such that $w \in a/b$ which implies that $a \in l_{bw}$ and $w \in c/d$ which implies that $c \in l_{dw}$. Since the lines of V are having one point in common at most, the lines l_{bw} and l_{dw} intersect at w.

Let the ordered set of coordinates of the points w, a, d be $(w_0, w_1,...,w_k)$, $(a_0, a_1,...,a_k)$, $(d_0, d_1,...,d_k)$ respectively. Then, the coordinates of the point b will be of the form $(\lambda a_0 + \mu w_0, \lambda a_1 + \mu w_1, \ldots\ldots, \lambda a_k + \mu w_k)$, where $\lambda,\mu \in GF[p^n]$ and the coordinates of the point c will be of the form $(\kappa d_0 + \rho w_0, \kappa d_1 + \rho w_1, \ldots \ldots\ldots, \kappa a_k + \rho w_k)$, where $\kappa,\rho \in GF[p^n]$. Since the points w, a, d do not belong to the line l_{bc} , the marks λ,μ,κ,ρ of the $GF[p^n]$ are not zero.

Now, l_{bc} consists of the points of the form

$$(\nu\lambda a_0+\nu\mu w_0+ \tau\kappa d_0+\tau\rho w_0 , \nu\lambda a_1+\nu\mu w_1+ \tau\kappa d_1+\tau\rho w_1 ,\ldots\ldots, \nu\lambda a_k+\nu\mu w_k+ \tau\kappa d_k+\tau\rho w_k),$$

where ν and τ run independently over the marks of the $GF[p^n]$ and they are not simultaneously zero.

It is known that for the nonzero marks μ and ρ, there exist nonzero marks ν and τ such that: $\nu\mu+\tau\rho = 0$. Then, we get

$\nu\mu w_0 + \tau\rho w_0 = (\nu\mu+\tau\rho)w_0 = 0$, $\nu\mu w_1 + \tau\rho w_1 = (\nu\mu+\tau\rho)w_1 = 0$,…….., $\nu\mu w_k + \tau\rho w_k = (\nu\mu+\tau\rho)w_k = 0$.

In that case, the point $(\nu\lambda a_0+\tau\kappa d_0 , \nu\lambda a_1+\tau\kappa d_1 ,\ldots\ldots., \nu\lambda a_k+\tau\kappa d_k)$ of the line l_{bc} is additionally a point of the line l_{ad}. So, the lines l_{bc}, l_{ad} intersect and then:

$$a\cdot d \cap b\cdot c \neq \varnothing \text{ for all } a,b,c,d \in V . \blacksquare$$

4. On a H_v-group of finite order

Now, we define a new hyperoperation ($\circ$) on V as follows :

Definition 6. For every $x,y \in V$, $\circ : V\times V \to P'(V)$, such that

$$x\circ y = \begin{cases} x & if \;\; x = y \\ l_{xy} - \{x\} & if \;\; x \neq y \end{cases}$$

Every line of the set V contains p^n+1 points. The hyperoperation $(\circ)$ is weak commutative, since the lines $l_{xy}-\{x\}$ and $l_{yx}-\{y\}$ are having $p^n+1-2 = p^n-1$ points in common, so

$$(x\circ y) \cap (y\circ x) \neq \varnothing \quad \text{for every } x,y\in V .$$

Proposition 7. For every noncolliner $x,y,z\in V$,

$$|x\circ(y\circ z)| = p^{2n} > |(x\circ y)\circ z| = p^{2n} - p^n + 1$$

Proof. Since the line $y\circ z$ does not contain the point y, we get that $|y\circ z| = p^n$. The point x creates p^n points with each of the p^n points of the line $y\circ z$ (the point x is not participating according to the hyperoperation $(\circ)$). So, $|x\circ(y\circ z)| = p^n\cdot p^n = p^{2n}$.
On the other hand, the line $x\circ y$ consists of p^n points. Each of these points creates p^n points each time together with the point z, but since the point z appears p^n times, we get

$$|(x\circ y)\circ z| = p^n\cdot p^n - p^n+1 = p^{2n} - p^n + 1.$$

Since p is prime and $n\in \mathbb{N}$, easily follows that $|x\circ(y\circ z)| > |(x\circ y)\circ z|$.■

Proposition 8. The hyperstructure $(V, \circ)$ is an H_v-group.
Proof. Indeed, for every $x\in V$

$$x\circ V = \bigcup_{v\in V}(x\circ v) = (x\circ x) \cup \left(\bigcup_{v\in V-\{x\}}(x\circ v)\right) = x\cup\left(\bigcup_{v\in V-\{x\}}(l_{xv}-\{x\})\right) = V$$

since every line l_{xv} always contains the point $v\in V$.
On the other hand, for every $x\in V$

$$V\circ x = \bigcup_{v\in V}(v\circ x) = (x\circ x) \cup \left(\bigcup_{v\in V-\{x\}}(v\circ x)\right) = x\cup\left(\bigcup_{v\in V-\{x\}}(l_{vx}-\{v\})\right) = V$$

Indeed, having the fact that every line of V contains at least 3 points, for every line $l_{vx}-\{v\}=v\circ x$ there exists at least one point $v'\in l_{vx}-\{v\}$, that $v\in l_{v'x}-\{v'\} = v'\circ x$. So,

$$x\circ V = V\circ x = V \quad \text{for every } x\in V.$$

The hyperoperation ($\circ$) is weak associative, since for every $x,y,z\in V$

$$x\circ(y\circ z) \supset x\circ z \ni z \quad \text{and} \quad (x\circ y)\circ z \supset y\circ z \ni z$$

But, we shall go further on, proving that the inclusion on the right parenthesis is valid, i.e $(x\circ y)\circ z \subset x\circ(y\circ z)$.

From the proposition 7 we get that $|x\circ(y\circ z)| = p^{2n}$, since the p^n+1 points of the line l_{xy} are not contained into the set $x\circ(y\circ z)$.
Similarly, the set $(x\circ y)\circ z$ does not contain the points of the line l_{xy}, since

$$(x\circ y)\circ z = (l_{xy} - \{x\})\circ z = (x_1\circ z) \cup (x_2\circ z) \cup \ldots\ldots \cup (x_{p^n-1} \circ z) \cup (y\circ z),$$

where $x_1, x_2, \ldots\ldots, x_{p^n-1} \in l_{xy}$.

Also, the set $(x\circ y)\circ z$ does not contain the points of the line l_{xz}, since $x\circ y = l_{xy} - \{x\}$.
As the lines l_{xy} and l_{xz} intersect at the point x, they don't have any other points in common. That means that the $p^{2n}-p^n+1$ points of the set $(x\circ y)\circ z$ (proposition 7), are also points of the set $x\circ(y\circ z)$.
So, we proved that $(x\circ y)\circ z \subset x\circ(y\circ z)$, which, generally, means that

$$(x\circ y)\circ z \cap x\circ(y\circ z) \neq \varnothing \quad \text{for every } x,y,z\in V. \blacksquare$$

Remark 9. For the H_v-group $(V,\circ)$, since $y\in x\circ y$ $\forall x,y\in V$, the V/β^* is a singleton.

Since, $y\in x\circ y$ for every $x,y\in V$, we get the following proposition:

Proposition 10. Every element of the H_v-commutative group $(V,\circ)$ is simultaneously a right zero and a left unit element.

Example 11. By means of the marks of a Galois Field, we shall now give a concrete representation of a finite *2*-dimensional projective geometry.
We denote a point of the geometry by the ordered set of coordinates (μ_0, μ_1, μ_2). The μ_0, μ_1, μ_2 are marks of the $GF[2^2]$ defined by means of the function x^2+x+1. At least one of μ_0, μ_1, μ_2 is different from

zero. Let us denote the marks of the GF[2^2] by 0, 1, a, b, then we have the following tables :

+	**0**	**1**	**a**	**b**
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	**0**	**1**	**a**	**b**
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

The ordered set of marks μ_0, μ_1, μ_2 may be chosen by $(2^2)^{2+1} = 64$ ways, but since the symbol (0, 0, 0) is excepted, it may be chosen by 64-1=63 ways. So, there exist 63 points. In this totality, each point is represented by 3 ways (3 sets of symbols, since there are 3 nonzero marks in the field). Then the number of points defined is $63 \div 3 = 21$.
This representation of the finite *2*-dimensional projective geometry by means of the marks of the GF[2^2], constitute the projective geometry PG(2, 2^2).
The 21 points of PG(2, 2^2) = V will be denoted by letters in accordance with the following scheme :

A(001) B(010) C(011) D(01a) E(01b) F(100) G(101)

H(10a) I(10b) J(110) K(111) L(11a) M(11b) N(1a0)

O(1a1) P(1aa) Q(1ab) R(1b0) S(1b1) T(1ba) U(1bb)

Now, for the line containing the two distinct points A(001) and B(010) we take the set of points :

$$(\mu 0 + \nu 0 , \mu 0 + \nu 1 , \mu 1 + \nu 0)$$

where μ and ν run independently over the marks of the GF[2^2] subjected to the condition that μ and ν shall not be simultaneously zero.
Then the number of possible combinations of the μ and ν is $(2^2)^2 - 1 = 15$. For each of these combinations, the corresponding symbol

denotes a point. But the same point is represented by 3 of these combinations of μ and ν, due to the factor of proportionality involved in the definition of points. So, we get the following scheme:

A : **(001)** , (00a) , (00b)
B : **(010)** , (0a0) , (0b0)
C : **(011)** , (0aa) , (0bb)
D : **(01a)** , (0b1) , (0ab)
E : **(01b)** , (0ba) , (0a1)

Therefore, a line so defined, contains the $15 \div 3 = 5$ points A,B,C,D,E. It is obvious that any two points on the line may be used in this way to define the same line.

The 21 lines are those given in the following scheme and the letters in a given column denoting a line:

A	A	A	A	A	B	B	B	B	C	C	C	C	D	D	D	D	E	E	E	E
B	F	J	N	R	F	G	H	I	F	G	H	I	F	G	H	I	F	G	H	I
C	G	K	O	S	J	K	L	M	K	J	M	L	L	M	J	K	M	L	K	J
D	H	L	P	T	N	O	P	Q	P	Q	N	O	Q	P	O	N	O	N	Q	P
E	I	M	Q	U	R	S	T	U	U	T	S	R	S	R	U	T	T	U	R	S

Let us take the noncolliner points A, B, F, then

$$A\circ(B\circ F) = A\circ\{F, J, N, R\} =$$
$$=\{F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U\}.$$
$$(A\circ B)\circ F = \{B, C, D, E\}\circ F = \{F, J, N, R, K, P, U, L, Q, S, M, O, T\}.$$

Then, it follows that $(A\circ B)\circ F \subset A\circ(B\circ F)$.

Also, for the hyperoperation $(\cdot)$ we proved that $|x\cdot(y\cdot z)| = |(x\cdot y)\cdot z| = p^{2n} + p^{n} + 1$ and since the set $V = PG(2, 2^2)$ consists of 21 points, it follows that

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z = V \quad \text{for every } x,y,z\in V.$$

5. On a dual H_v-ring of finite order

Working on dual H_v-rings $(H,+,\cdot)$, one needs to prove not only the weak distributivity of $(\cdot)$ with respect to $(+)$ but also the weak distributivity of $(+)$ with respect to $(\cdot)$.

Since the set V is now equipped with the hyperoperations $(\cdot)$ and $(\circ)$ mentioned above, the next propositions 12 to 19 serve the above purpose.
Similarly, as in proposition 2, we can prove that:

Proposition 12. For every noncollinear $x,y,z\in V$

$$|(x\cdot y)\cdot(x\cdot z)| = |(x\cdot z)\cdot(y\cdot z)| = p^{2n} + p^n + 1$$

Following a similar procedure, as in proposition 3 and according to the propositions 2 and 12, we get the next proposition:

Proposition 13. For every $x,y,z\in V$

$$x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \quad \text{and} \quad (x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z)$$

Proposition 14. For every noncolliner $x,y,z\in V$,

$$|x\cdot(y\circ z)| = p^{2n} + 1 \quad \text{and} \quad |(x\cdot y)\circ(x\cdot z)| = p^{2n} + p^n + 1$$

Proof. First we consider the set $x\cdot(y\circ z)$. Since the point y does not belong to the line $y\circ z$, we get that $|y\circ z| = p^n$. Also, for every $w\in y\circ z$ we get that $|x\cdot w| = p^n + 1$. Since the point x appears p^n times in the set $x\cdot(y\circ z)$, we have

$$|x\cdot(y\circ z)| = (p^n + 1)\, p^n - p^n + 1 = p^{2n} + 1.$$

Consider now the set $(x\cdot y)\circ(x\cdot z)$. Each of the lines $x\cdot y$ and $x\cdot z$ contains $p^n + 1$ points, having in common only the point x, since the points x,y,z are noncollinear. Then, we get the following:

i) $x\circ x = x$, by definition.

ii) Due to the hyperoperation $(\circ)$, the point $x\in x\cdot y$ together with the rest p^n points of the line $x\cdot z$ create the p^n points of the line $x\cdot z$.

iii) Due to the hyperoperation $(\circ)$, the point $y\in x\cdot y$ together with the $p^n + 1$ points of the line $x\cdot z$, create $(p^n + 1) - 2 = p^n - 1$ points each time. Indeed, the point y does not participate (by definition) and the point $w\in x\cdot z$ (which appears due to the hyperoperation $y\circ w$), already exists due to the hyperoperation $x\circ w$ of the case (ii).

iv) Also, the point $y \in (x \cdot y) \circ (x \cdot z)$. Indeed, since there exists $w' \in x \cdot y$ such that $y \in w' \circ x$ (where $x \in x \cdot z$).

So, from the above 4 cases we get that:

$$|(x \cdot y) \circ (x \cdot z)| = 1 + p^n + (p^n - 1)(p^n + 1) + 1 = p^{2n} + p^n + 1. \blacksquare$$

In a similar way, we get the following proposition:

Proposition 15. For every noncolliner $x,y,z \in V$

$$|(x \circ y) \cdot z| = p^{2n} + 1 \quad \text{and} \quad |(x \cdot z) \circ (y \cdot z)| = p^{2n} + p^n + 1$$

Proposition 16. For every noncolliner $x,y,z \in V$,

$$|x \circ (y \cdot z)| = |(x \circ y) \cdot (x \circ z)| = p^{2n} + p^n$$

Proof. First, consider the set $x \circ (y \cdot z)$. The line $y \cdot z$ consists of $p^n + 1$ points. The point x (due to the hyperoperation ($\circ$)) together with the points of the line $y \cdot z$ creates each time $(p^n + 1)\, p^n = p^{2n} + p^n$ points, since x is not participating.

Consider now, the set $(x \circ y) \cdot (x \circ z)$. Each of the lines $x \circ y$ and $x \circ z$ contains p^n different points, since the point x is not participating. Then, we get the following:

i) The point $y \in x \circ y$ (due to the hyperoperation ($\cdot$)), together with every $w \in x \circ z$ creates $p^n + 1$ points each time, but since the point y appears p^n times we get that the number of points in this case is $(p^n + 1)\, p^n - p^n + 1 = p^{2n} + 1$.

ii) The point $w' \in (x \circ y) - \{y\}$ (due to the hyperoperation ($\cdot$)), together with the point $w \in x \circ z$ creates each time the point w', since the point w already exists from case (i). The number of those w''s is p^n-1.

Then, the set $(x \circ y) \cdot (x \circ z)$ consists of $(p^{2n} + 1) + (p^n - 1) = p^{2n} + p^n$ points. As we mentioned, x is the only point which is not participating. $\blacksquare$

Similarly, we get the following two propositions:

Proposition 17. For every noncolliner $x,y,z \in V$

$$|(x\cdot y)\circ z| = p^{2n} \quad \text{and} \quad |(x\circ z)\cdot(y\circ z)| = p^{2n} + p^n + 1$$

Proposition 18. For every noncollinear $x,y,z \in V$

$$|(x\circ y)\circ(x\circ z)| = p^{2n} \quad \text{and} \quad |(x\circ z)\circ(y\circ z)| = p^{2n} + p^n + 1$$

Following a similar procedure as above and according to the proposition 18 we get the next proposition:

Proposition 19. For every $x,y,z \in V$

$$x\circ(y\circ z) = (x\circ y)\circ(x\circ z) \quad \text{and} \quad (x\circ y)\circ z \subset (x\circ z)\circ(y\circ z)$$

Proposition 20. The hyperstructure $(V, ▫, ▪)$, where $▫, ▪ \in \{\cdot, \circ\}$, is a dual H_v-ring.

Proof. There are four hyperstructures: $(V,\cdot,\cdot)$, $(V,\circ,\circ)$, $(V,\cdot,\circ)$, $(V,\circ,\cdot)$. The hyperoperations $(\cdot)$, $(\circ)$ (by propositions 3 and 8 respectively) are satisfying the reproduction axiom.
The hyperoperation $(\cdot)$ (by proposition 3) is associative and the hyperoperation $(\circ)$ (by proposition 8) is weak associative.
Now, for the distributivity or the weak distributivity of (▪) with respect to (▫) we have the following cases:
By proposition 13:

$$x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \quad \text{and} \quad (x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z) \text{ for every } x,y,z \in V.$$

By proposition 19:

$$x\circ(y\circ z) = (x\circ y)\circ(x\circ z) \quad \text{and} \quad (x\circ y)\circ z \cap (x\circ z)\circ(y\circ z) \neq \varnothing \quad \forall\, x,y,z \in V.$$

Following a similar procedure as for the distributivity of the above hyperoperations and taking into account the propositions 16, 17, 18, 19 we get that:

$$x\circ(y\cdot z) = (x\circ y)\cdot(x\circ z) \quad \text{for every } x,y,z \in V.$$

On the right side, $(x\cdot y)\circ z \subset (x\circ z)\cdot(y\circ z)$ is valid, which means that

$$(x\cdot y)\circ z \cap (x\circ z)\cdot(y\circ z) \neq \varnothing \quad \text{for every } x,y,z \in V.$$

Also, $x\cdot(y\circ z) \subset (x\cdot y)\circ(x\cdot z)$, which means that

$$x\cdot(y\circ z) \cap (x\cdot y)\circ(x\cdot z) \neq \varnothing \quad \text{for every } x,y,z\in V.$$

Finally, on the right hand side $(x\circ y)\cdot z \subset (x\cdot z)\circ(y\cdot z)$, which means that

$$(x\circ y)\cdot z \cap (x\cdot z)\circ(y\cdot z) \neq \varnothing \quad \text{for every } x,y,z\in V.$$

So, the hyperstructure $(V,\square,\blacksquare)$, where $\square,\blacksquare \in \{\cdot,\circ\}$, is a dual H_v-ring.■

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Combination of survival probabilities of the components in a system. An application to long-term financial valuation[1]

Salvador Cruz Rambaud[2]

Abstract. The Net Present Value (NPV) is a well-known method to value an investment project. Nevertheless, this methodology exhibits a serious problem when the used discounting function decreases very rapidly, especially in (very) long-term projects, because the future cash-flows are not significant in the expression of the NPV. For this reason, this paper introduces a methodology to correct the discounting function used for valuing. To do this, a new operation between discounting functions is defined by reducing the (cumulative) instantaneous discount rate corresponding of the valuing discounting function with another appropriate discounting function. The result is a new discounting function which can be more adequate to value this class of investment projects.

Keywords. Combination, discounting function, (cumulative) instantaneous discount rate, Net Present Value, investment project.

1. Introduction

[1] This paper has been partially supported by the project "Valoración de proyectos gubernamentales a largo plazo: obtención de la tasa social de descuento", reference: P09-SEJ-05404, Proyectos de Excelencia de la Junta de Andalucía and Fondos FEDER.

[2] Departamento de Dirección y Gestión de Empresas. Universidad de Almería (Spain), e-mail: scruz@ual.es.

It is well-known that traditionally the exponential discounting has been used in the valuation of investment projects, as discounting function. But the main problem that exhibits this type of discount is the geometrical diminishing of its corresponding factors. In effect, the expression $(1+i)^{-t}$ decreases exponentially whereby future cash-flows, being very important, are not significant in the expression of the Net Present Value (NPV). This is the reason whereby our aim is to considerer a diminishing discount rate which would imply, at least, a decay of the corresponding (cumulative) instantaneous discount rate.

On the other hand, in a previous work, Cruz and Muñoz (2005 and 2007) introduced a new point of view of determining the social rate of discount and, more concretely, the discount function to be applied in the valuation of (very) long-term environmental and governmental projects. To do this, they started from the hazard rate of the system to which the project we are trying to value is addressed. In this way, if we are trying to value the construction of a public good (for example, a highway), the hazard rate corresponding to this construction along his useful life will supply us its survival probability (defined as the complement to the unit of the corresponding distribution function) which we will identify with the discounting function to be used in the valuation.

Thus, the instantaneous hazard rate of an investment is identified with the instantaneous discount rate corresponding to the discounting function necessary to value the project. As a consequence, the discounting function will be the survival probability of the system. More widely, the following table establishes the correspondence between several concepts from Finance (see, for example, Gil, 1993) and from Reliability Theory (see, for example, Barlow and Proschan, 1996).

Reliability Theory		**Finance**	
Survival probability (distribution tail)	$S(t) = 1 - F(t)$	Discounting function	$A(t)$
Instantaneous hazard rate	$h(t) = -\dfrac{\mathrm{d}S(t)/\mathrm{d}t}{S(t)}$	Instantaneous discount rate	$\delta(t) = -\dfrac{\mathrm{d}A(t)/\mathrm{d}t}{A(t)}$

Density function	$f(t) = -\frac{\mathrm{d}S(t)}{\mathrm{d}t}$	Cumulative inst. discount rate	$\nu(t) = -\frac{\mathrm{d}A(t)}{\mathrm{d}t}$
Conditional probability	$\frac{1-F(t+s)}{1-F(t)}$	Discounting factor	$\frac{A(t+s)}{A(t)}$

Table 1. Correspondence of concepts from Reliability Theory and Finance.

Starting from this methodology, we can obtain a (cumulative) instantaneous discount rate with two important advantages. The first one is that this magnitude is variable and the second one is that it can be diminishing. In this way, we agree with Harvey's (1986) position who proposes the hyperbolic or hyperbola-like discounting function and later (1994) defends variable discount rates.

Harvey (1994) examines "the reasonableness for public policy analysis of non-constant discounting method that, unlike constant discounting, can accord considerable importance to outcomes in the distant future". In his work, he proposes a method with positive discount rates that decrease and converge to zero as time converges to infinity.

The organization of this paper is as follows. In Section 2, we introduce a new algebraic operation between the survival probabilities of two components in a system. Taking into account Table 1, this is the same as define an algebraic operation between two discounting functions. Section 2 introduces a novel classification of discounting functions in singular and regular ones. Later, Section 4 presents a noteworthy application of Section 2 for the valuation of (very) long-term investment projects, avoiding the problems exhibited by a rapidly decreasing discounting function. Finally, Section 5 summarizes and concludes.

2. Combination of survival probabilities of the components in a system. Combination of discounting functions

Let us consider a structure composed by two independent components whose i-th component (i = 1, 2) has probability

$p_i(t)=1-F_i(t)$ of still being operative at time t. If $h(\mathbf{p})$ is the structure reliability function, where $\mathbf{p}=(p_1,p_2)$, and $S(t)=1-F(t)$ is the probability of the structure survival past time t, then since

$$S(t)=h(\mathbf{p}(t)),$$

we obtain:

$$-\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\partial h}{\partial p_1}\left(-\frac{\mathrm{d}p_1}{\mathrm{d}t}\right)+\frac{\partial h}{\partial p_2}\left(-\frac{\mathrm{d}p_2}{\mathrm{d}t}\right). \tag{1}$$

It is well-known that $-\frac{\mathrm{d}S}{\mathrm{d}t}:=f(t)$ is the *density function* of the variable T describing the useful life of the system, and $-\frac{\mathrm{d}p_1}{\mathrm{d}t}:=f_1(t)$ and $-\frac{\mathrm{d}p_2}{\mathrm{d}t}:=f_2(t)$ are the *density* functions of variables T_1 and T_2 describing the useful life of components 1 and 2, respectively.

A noteworthy case is that in which the density function of the structure survival is the density function of component 1 but reduced by the effect of the survival probability of component 2, that is to say:

$$f(t)=f_1(t)\cdot p_2(t)=-\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}p_2(t). \tag{2}$$

In this case,

$$S(t):=(p_1\otimes p_2)(t)=1-\int_0^t-\frac{\mathrm{d}S}{\mathrm{d}x}\mathrm{d}x=$$
$$=1-\int_0^t-\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}p_2(x)\mathrm{d}x=$$
$$1-\int_0^t p_2(x)[-\mathrm{d}p_1(x)]=1-\int_0^t f_1(x)p_2(x)\mathrm{d}x. \tag{3}$$

that is, $-\mathrm{d}p_1(t)$, as an approximation of $-\Delta p_1 = -p_1(t+h) + p_1(t)$, is reduced by the survival probability of component 2. This new function $p_1 \otimes p_2$ will be called the *combination of survival probabilities* p_1 and p_2.

Taking into account Table 1, we can export this concept to Finance. Thus, the *combination* $A_1 \otimes A_2$ *of two discounting functions* A_1 and A_2:

$$A(t) := (A_1 \otimes A_2)(t) = 1 - \int_0^t -\frac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}x =$$

$$= 1 - \int_0^t -\frac{\mathrm{d}A_1(x)}{\mathrm{d}x} A_2(x)\mathrm{d}x =$$

$$1 - \int_0^t A_2(x)\left[-\mathrm{d}A_1(x)\right] = 1 - \int_0^t \nu_1(x) A_2(x)\mathrm{d}x \tag{4}$$

can be interpreted as a methodology to reduce the time perception of an objective discounting function (A_1) by the effect of another discounting function (A_2). This is because the more aged the individuals, the less time perception. In effect, a year of future time is not the same for a person *r* years old than a person *s* years old, being $r > s$. In this case, the time perception is greater for the second one.

In what follows and taking into account the aim of this paper, we will only refer to the combination of two discounting functions.

Example 1 Let us consider the combination of two simple discounting functions of parameters *d* and *d′*:

$$A(t) = 1 - d \cdot t\left(1 - d'\frac{t}{2}\right).$$

The following proposition supplies a preliminary basic inequality.

Proposition 1 $A_1(t) < A(t) < 1 - \left[1 - A_1(t)\right]A_2(t)$.

Proof. In effect, for the first inequality, as $0 < A_2(x) < 1$, it is verified that:

$$\int_0^t v_1(x)A_2(x)\mathrm{d}x < \int_0^t v_1(x)\mathrm{d}x$$

and so

$$A(t) = 1 - \int_0^t v_1(x)A_2(x)\mathrm{d}x > 1 - \int_0^t v_1(x)\mathrm{d}x = A_1(t).$$

For the second inequality, as function A_2 is strictly decreasing, take into account that:

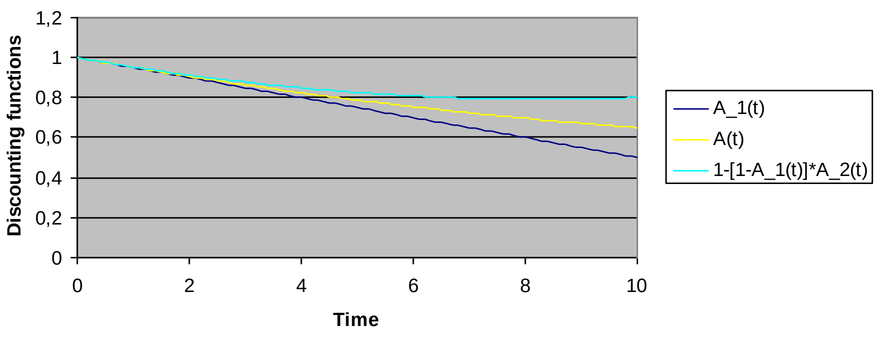
$$\int_0^t v_1(x)A_2(x)\mathrm{d}x > A_2(t)\int_0^t v_1(x)\mathrm{d}x = [1 - A_1(t)]A_2(t).$$

Thus,

$$A(t) = 1 - \int_0^t v_1(x)A_2(x)\mathrm{d}x < 1 - [1 - A_1(t)]A_2(t). \quad \square$$

A graphic representation of the discounting function obtained in Example 1 for $d = 0.05$ and $d' = 0.06$, and a confirmation of the result deduced in Proposition 1, can be seen in Figure 1.

Figure 1. Confirming Proposition 1

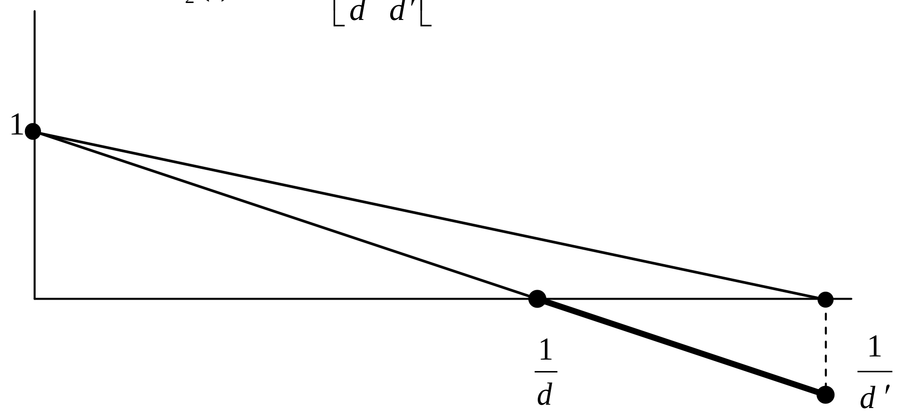


With respect to the temporal domain, $D(t)$, of the new discounting function, two cases can occur ($D_1(t)$ and $D_2(t)$ are the time discounting domains of the discounting functions A_1 and A_2, respectively):

- If $D_2(t) \subseteq D_1(t)$, then $D(t) = D_2(t)$.
- If $D_1(t) \subset D_2(t)$, then $D_1(t) \subseteq D(t) \subseteq D_2(t)$, because, if $D_1(t) = [0, t_1]$, there can exist a non-empty interval $[t_1, t_2[\subseteq D_2(t) - D_1(t)$ where $A_1 < 0$ and $\frac{\mathrm{d}A_1}{\mathrm{d}t} < 0$. In this case, $D(t) = D_1(t) \cup [t_1, t_2[$. Observe that eventually $D(t)$ can coincide with $D_2(t)$.

In Example 1, $D_1(t) = \left[0, \frac{1}{d}\right[$ and $D_2(t) = \left[0, \frac{1}{d'}\right[$. Consequently, two cases can occur:

- If $d \leq d'$, $\frac{1}{d'} \leq \frac{1}{d}$ and so $D_2(t) \subseteq D_1(t)$. Thus, $D(t) = D_2(t) = \left[0, \frac{1}{d'}\right[$.
- If $d > d'$, $\frac{1}{d} < \frac{1}{d'}$ and so $D_1(t) \subset D_2(t)$. As A_1 is decreasing and $A_2(t) > 0$ in $\left[\frac{1}{d}, \frac{1}{d'}\right[$:



then there exists a t_2 such that $D(t) = D_1(t) \cup \left[\frac{1}{d}, t_2\right[$. To calculate t_2, we have to solve the equation:

$$dd'\frac{t^2}{2} - dt + 1 = 0,$$

which only has a solution if and only if $d' < \frac{d}{2}$. In this case the obtained solution is $t = \frac{d \pm \sqrt{d^2 - 2dd'}}{dd'} = \frac{1 \pm \sqrt{1 - 2\frac{d'}{d}}}{d'}$, from where $t_2 = \frac{1 - \sqrt{1 - 2\frac{d'}{d}}}{d'}$, which obviously lesser than $\frac{1}{d'}$. On the other hand, writing the solution as $t_2 = \frac{d - \sqrt{(d-d')^2 - d'^2}}{dd'}$, we can show that $t_2 > \frac{d - (d - d')}{dd'} = \frac{1}{d}$.

Definition 1 Let A_1 and A_2 be two discounting functions. The *ordinary product* of both functions, denoted by $A_1 \cdot A_2$, is defined in the following way:

$$(A_1 \cdot A_2)(t) = A_1(t) \cdot A_2(t).$$

Observe that this algebraic operation reflects the "multiplicative" superposition of the effects due to both discounting functions over a certain temporal interval.

Definition 2 Let A_1 and A_2 be two discounting functions. The *reduced sum* of both functions, denoted by $A_1 \oplus A_2$, is defined in the following way:

$$(A_1 \oplus A_2)(t) = A_1(t) + A_2(t) - 1 .$$

Once defined these algebraic operations, we can enunciate the following

Proposition 2 $(A_1 \otimes A_2) \oplus (A_1 \otimes A_2) = A_1 \cdot A_2$.

Proof. In effect, by calculating the integral $\int_0^t \nu_1(x) A_2(x) \mathrm{d}x$ by parts, we have:

$$\int_0^t \nu_1(x) A_2(x) \mathrm{d}x = -A_1(t) \cdot A_2(t) + 1 - \int_0^t A_1(x) \nu_2(x) \mathrm{d}x ,$$

from where we can easily deduce the required equality. □

The following theorem relates the convexity of discounting functions A and A_1.

Theorem 1 If A_1 is convex, then $A = A_1 \otimes A_2$ is also convex, independently of the convexity or concavity of A_2.

Proof. From Equation (4), $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t)$. Differentiating again with respect to *x*:

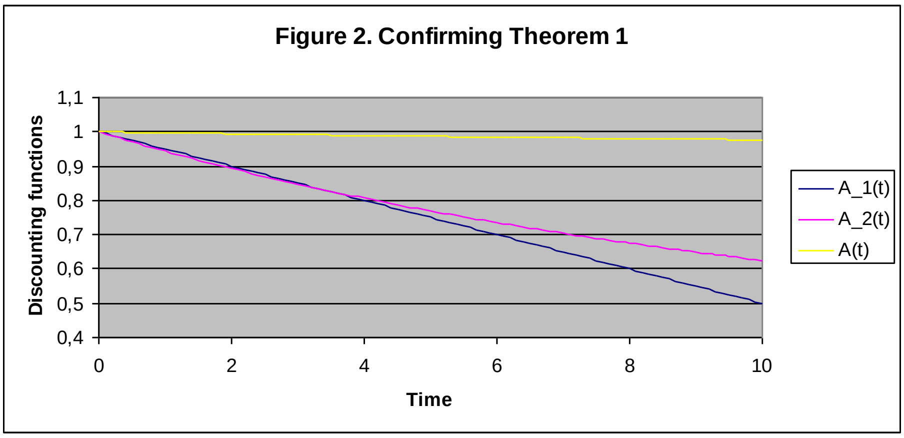
$$\frac{\mathrm{d}^2 S(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 A_1(t)}{\mathrm{d}t^2} A_2(t) + \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} .$$

As p_1 is convex, $\frac{\mathrm{d}^2 A_1(t)}{\mathrm{d}t^2} > 0$. Moreover, as $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}A_2(t)}{\mathrm{d}t}$ are negative, and obviously $A_2(t) > 0$, then *S* is convex. □

Example 2 Let us consider the combination of the simple discounting function of parameters d and the hyperbolic discounting of parameter i:

$$A(t) = 1 - d \cdot \ln(1 + i \cdot t).$$

A graphic representation of the discounting function obtained in Example 2 for $d = 0.05$ and $i = 0.06$, and a confirmation of the result deduced in Theorem 1, can be seen in Figure 2.



Obviously, the operation $\otimes$ does not verify the commutative property, but we can easily show the following proposition. Observe that there exists an "exchange by quotient" between the cumulative instantaneous discount rates of the two combinations of discounting functions towards the instantaneous discount rates corresponding to components 1 and 2.

Proposition 2 The following equality holds:

$$\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}(A_2 \otimes A_1)(t)} = \frac{\delta_1(t)}{\delta_2(t)}.$$

Proof. It is obvious taking into account that $\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t)$, $\frac{\mathrm{d}(A_2 \otimes A_1)(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} A_1(t)$, $\delta_1(t) = -\frac{\mathrm{d}A_1(t)/\mathrm{d}t}{A_1(t)}$ and $\delta_2(t) = -\frac{\mathrm{d}A_2(t)/\mathrm{d}t}{A_2(t)}$. □

We can check the result obtained in Proposition 2 with the following example.

Example 3 Combination of two exponential discounting functions of parameters i and i' ($i < i'$):
$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \ln(1+i)\left[(1+i)(1+i')\right]^{-t}$ and $\frac{\mathrm{d}A'(t)}{\mathrm{d}t} = \ln(1+i')\left[(1+i)(1+i')\right]^{-t}$.

Therefore, we can obviously check that:

$$\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}(A_2 \otimes A_1)(t)} = \frac{\ln(1+i)}{\ln(1+i')} = \frac{\delta_1(t)}{\delta_2(t)}.$$

Proposition 3 The operation $\otimes$ does not verify the commutative property except for equal elements. Moreover, it is cancellative on the left and on the right.

Proof. In effect, if $(A_1 \otimes A_2)(t) = (A_2 \otimes A_1)(t)$, then $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t) = \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} A_1(t)$ and, consequently, $\delta_1(t) = -\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} \cdot \frac{1}{A_1(t)} = -\frac{\mathrm{d}A_2(t)}{\mathrm{d}t} \cdot \frac{1}{A_2(t)} = \delta_2(t)$. Thus $A_1(t) = A_2(t)$. On the other hand, if $(A_1 \otimes A_2)(t) = (A_1 \otimes A_3)(t)$, by definition, $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t) = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_3(t)$ and then $A_2(t) = A_3(t)$. Finally, if

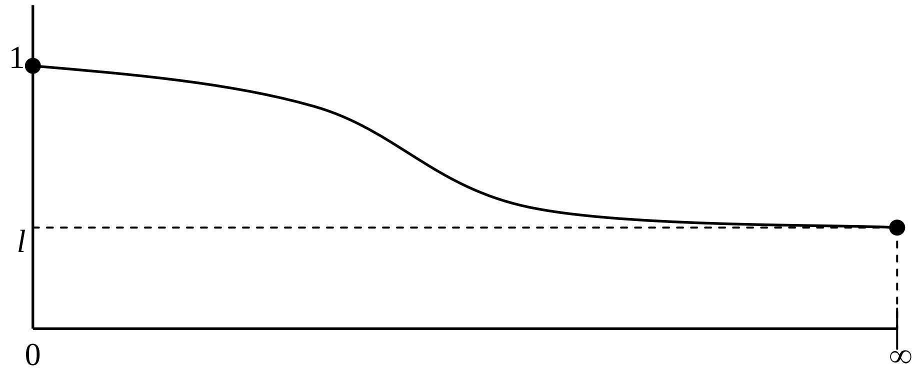
$(p_1 \otimes p_3)(t) = (p_2 \otimes p_3)(t)$, by definition, $\frac{dA_1(t)}{dt} A_3(t) = \frac{dA_2(t)}{dt} A_3(t)$ and then $A_1(t) = A_2(t)$. □

3. Singular and regular discounting functions

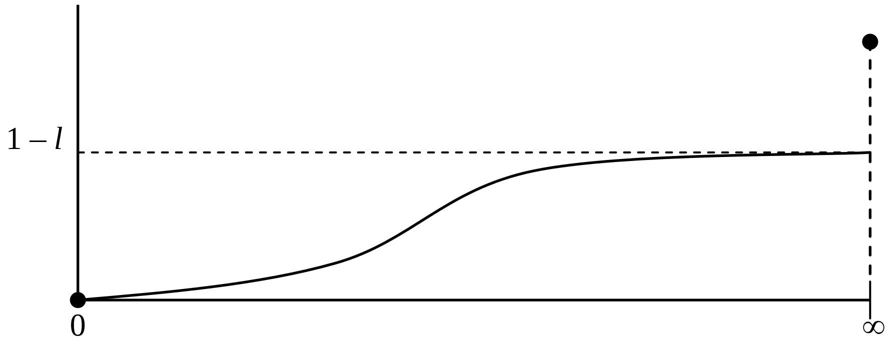
Definition 3 (Maravall, 1970) A discounting function $A(t)$ is said to be *singular* if $\lim_{t\to\infty} A(t) \neq 0$ or there exists a real number t_0 such that $A(t_0) \neq 0$. Otherwise, $A(t)$ is said to be *regular*.

Example 4 The discounting function is $A(t) = \frac{1 + i \cdot t}{1 + j \cdot t}$, where $i < j$, is singular because, obviously, $\lim_{t\to\infty} A(t) = \frac{i}{j}$. Obviously, hyperbolic discounting is regular.

A singular discounting function is a peculiar discounting function which has a horizontal asymptote at $y = l$, where l can be interpreted as the mass of probability at infinity of the corresponding distribution function. Representing this function in the extended real numbers:



and so its corresponding distribution function:



Definition 3 provides a *classification of discounting functions*:

1. *Singular discounting functions with bounded domain*: $D(t) = [0, t_0[$ and $A(t_0) \neq 0$. More specifically, $0 < A(t_0) < 1$
2. *Regular discounting functions with bounded domain*: $D(t) = [0, t_0[$ and $A(t_0) = 0$.
3. *Singular discounting functions*: $D(t) = [0, +\infty[$ and $\lim_{t \to \infty} A(t) \neq 0$. More specifically, $0 < \lim_{t \to \infty} A(t) < 1$.
4. *Regular discounting functions*: $D(t) = [0, +\infty[$ and $\lim_{t \to \infty} A(t) = 0$.

It is possible to provide some results on all possible combinations of different class of discounting functions. For instance, it can be shown that the combination of two singular discounting functions with bounded domain is also singular with bounded domain (see Example 1) and that the combination of a singular and a regular discounting function with bounded domain is singular with bounded domain. Finally, next examples show the result of combining of some well-known discounting function.

Example 5 Combination of a hyperbolic discounting function of parameter *i* and a simple discounting of parameter *d*:

$$A(t)=1-\frac{1}{i}\left(1-\frac{1}{1+it}\right)-\frac{d}{i^2}\left[1-\ln(1+it)-\frac{1}{1+it}\right].$$

Example 6 Combination of two hyperbolic discounting functions of parameters *i* and *j*:

$$A(t)=1+\frac{j}{(i-j)^2}\ln\frac{(1+it)^i}{(1+jt)^j}-\frac{i}{i-j}\left(1-\frac{1}{1+it}\right) \text{ (singular).}$$

Example 7 Combination of a simple discounting function of parameter *d* and an exponential discounting function of parameter *k*:

$$A(t)=1-\frac{d}{k}\left(1-e^{-kt}\right) \text{ (singular).}$$

Example 8 Combination of an exponential discounting function of parameter *k* and a simple discounting function of parameter *d*:

$$A(t)=(1-dt)e^{-kt}+\frac{d}{k}\left(1-e^{-kt}\right) \text{ (singular).}$$

4. The combination of discounting functions in the valuation of governmental projects

Consider the case in which a government must decide if a (very) long-term investment project is feasible. It is well-known that, to valuate this project, the most important discounting function to be used in the net present value (*NPV*) formula is the exponential one $A_1(t)=(1+i)^{-t}$, being *i* the technical interest rate:

$$NPV=-A+\sum_{k=1}^{n}CF_k\cdot A_1(k),$$

where:

- NPV is the net present value of the project;
- A is the initial payment of the project;
- n is the useful life of the project;
- CF_k is the *k*-th cash-flow corresponding to the project.

Assume that the survival probability of the system or the perception time of the population is described by the discounting function $A_2(t)$. In this case, it could be convenient to reinforce the first discounting function with the aim of preserve the future cash-flows. Thus, the formula to be employed would be:

$$NPV = -A + \sum_{k=1}^{n} CF_k \cdot (A_1 \otimes A_2)(k),$$

leading to smaller discount rates, more appropriate to value the aforementioned governmental projects.

5. Conclusion

In (very) long-term project appraisal (for example, governmental and environmental projects), the exponential discounting function has been traditionally used to update the future cash-flows at the present moment. Despite its generalized use, exponential discounting presents an obvious problem: the geometric diminishing of the actualization factors "almost annihilates" the most distant cash-flows. Therefore, it is necessary to increase the discounting function with the aim of reaching a higher presence of the further cash-flows.

To do this, there are several procedures. The methodology used in this paper is based on the idea of a diminishing perception of future time. Indeed, empirical researches show that for most people the larger the age of the person, the shorter the time periods. Thus, the "perceived" discounted amounts must be lesser and this fact must be reflected in the mathematical expression of the "true" discounting function. In this work, the reduction in the discounted values can be reached with another discounting function through the so-called combination of discounting functions.

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THE TRANSPOSITION AXIOM IN HYPERCOMPOSITIONAL STRUCTURES

Ch. G. Massouros [a,b] and G. G. Massouros [a,b]

[a] *Technological Institute of Chalkis, GR34400, Evia, Greece*
[b] *54, Klious st., GR15561, Cholargos-Athens, Greece*

ABSTRACT. The hypergroup (as defined by F. Marty), being a very general algebraic structure, was subsequently quickly enriched with additional axioms. One of these is the transposition axiom, the utilization of which led to the creation of join spaces (join hypergroups) and of transposition hypergroups. These hypergroups have numerous applications in geometry, formal languages, the theory of automata and graph theory.

This paper deals with transposition hypergroups. It also introduces the transposition axiom to weaker structures, which result from the hypergroup by the removal of certain axioms, thus defining the transposition hypergroupoid, the transposition semi-hypergroup and the transposition quasi-hypergroup. Finally, it presents hypercompositional structures with internal or external compositions and hypercompositions, in which the transposition axiom is valid. Such structures emerged during the study of formal languages and the theory of automata through the use of hypercompositional algebra.

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1. THE TRANSPOSITION AXIOM IN HYPERGROUPS

Hypercompositional structures are algebraic structures equipped with multivalued compositions, which are called hyperoperations or hypercompositions. A hypercomposition in a non-void set H is a function from the Cartesian product $H \times H$ to the powerset $P(H)$ of H. Hypercompositional structures came into being through the notion of the *hypergroup*. The hypergroup was introduced by F. Marty in 1934, during the 8$^{\text{th}}$ congress of the Scandinavian

Mathematicians [18]. F. Marty used hypergroups in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups. A hypergroup, which is a generalization of the group, satisfies the following axioms:

i. $(ab)c = a(bc)$ for all $a,b,c \in H$ (associativity),

ii. $aH = Ha = H$ for all $a \in H$ (reproduction).

Note that, if «·» is a hypercomposition in a set H and A, B are subsets of H, then $A \cdot B$ signifies the union $\bigcup_{(a,b)\in A\times B} a \cdot b$. In both cases, aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a. In [18], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|\,b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H \mid a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, W. Prenowitz [48] denoted division in commutative hypergroups by a / b. Later on, J. Jantosciak used the notation a / b for right division and $b \backslash a$ for left division [14]. Notations $a : b$ and $a..b$ have also been used correspondingly for the above two types of division [21].

In [14] and then in [15], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition ab with the hypercomposition ba. One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds:

> *Given a theorem, the dual statement resulting from interchanging the order of hypercomposition "·" (and, necessarily, interchanging of the left and the right divisions), is also a theorem.*

This principle is used throughout this paper. The following properties are direct consequences of axioms (i) and (ii) and the principle of duality is used in their proofs [see also 20, 21]:

Property 1.1. *$ab \neq \varnothing$ is valid for all the elements a,b of a hypergroup H.*

P r o o f. Suppose that $ab = \varnothing$ for some $a,b \in H$. Per reproduction, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd.

Property 1.2. *$a/b \neq \varnothing$ and $a \backslash b \neq \varnothing$ for all the elements a,b of a hypergroup H.*

P r o o f. Per reproduction, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \varnothing$. Dually, $a \backslash b \neq \varnothing$.

Property 1.3. *In a hypergroup H, the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.*

P r o o f. Suppose that $x/a \neq \varnothing$ for all $x,a \in H$. Thus, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$ for all $x \in H$, and so $H \subseteq Ha$. Next, since $Ha \subseteq H$ for all $a \in H$, it follows that $H = Ha$. Per duality, $H = aH$. Conversely now, per Property 1.2, the reproduction axiom implies that $a/b \neq \varnothing$ and $a \backslash b \neq \varnothing$ for all a,b in H.

Property 1.4. *In a hypergroup H equalities (i) $H = H/a = a/H$ and (ii) $H = a \backslash H = H \backslash a$ are valid for all a in H.*

P r o o f. (i) Per Property 1.1, the result of hypercomposition in H is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$. (ii) follows by duality.

The hypergroup (as defined by F. Marty), being a very general algebraic structure, was enriched with additional axioms, some less and some more powerful. These axioms led to the creation of more specific types of hypergroups.

One of these axioms is the transposition axiom. It was introduced by W. Prenowitz, who used it in commutative hypergroups. W. Prenowitz called the resulting hypergroup *join space* [48]. Thus, join space (or join hypergroup) is defined as a commutative hypergroups H, in which
$a/b \cap c/d \neq \varnothing$ implies $ad \cap bc \neq \varnothing$ for all $a,b,c,d \in H$ (transposition axiom) is true. This type of hypergroup has been widely utilized in the study of

Geometry via the use of hypercompositional algebra tools which function without any need of Cartesian or other coordinate-type systems [48, 49]. Later, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a,b,c,d \in H .$$

He named this particular hypergroup *transposition hypergroup* and studied its properties in [14].

The transposition axiom also emerged in the hypercompositional structures which surfaced during the study of formal languages through the use of hypercompositional algebra tools [see, for example, 6, 7, 27, 32, 33, 35, 36, 42, 44; see also 7, 12, 13 for other occurrences of the join space]. The manner in which these structures emerged will be discussed in paragraph 3. In the present paragraph we will only deal with the mathematical description of join space classes which resulted from the theory of formal languages and automata. The basic concept which generated these types of join spaces is the incorporation of a special neutral element e into a transposition hypergroup. This neutral element e possesses the property $ex = xe \subseteq \{e,x\}$ for every element x of the hypergroup and was named *strong*. Thus, the fortification of transposition hypergroups by an identity element came into being.

Therefore a *fortified transposition hypergroup* is a transposition hypergroup H for which the following axioms are valid:

i. $ee = e$,
ii. $x \in ex = xe$ for all $x \in H$,
iii. for every $x \in H - \{e\}$ there exists a unique $y \in H - \{e\}$, such that $e \in xy$ and, furthermore, y satisfies $e \in yx$.

If the commutativity is valid in H, then H is called a *fortified join hypergroup*.

Theorem 1.1. *In a fortified transposition hypergroup H, the identity is strong.*

P r o o f. It must be proven that $ex \subseteq \{e,x\}$ for all x in H. This is true for $x = e$. Let $x \neq e$. Suppose that $y \in ex$. Then, $x \in e \backslash y$. However, $x \in e / x^{-1}$, since $e \in xx^{-1}$. Thus, $e \backslash y \approx e / x^{-1}$ and transposition yields $e = ee \approx yx^{-1}$. Hence, $y \in \{x,e\}$.

Theorem 1.2. *In a fortified transposition hypergroup H, the strong identity is unique.*

P r o o f. Suppose that u is an identity distinct from e. It then follows that there exists z distinct from u, such that $u \in ez$. But, $ez \subseteq \{e,z\}$, so $u \in \{e,z\}$, which is a contradiction.

It is worth noting that a transposition hypergroup H becomes a quasicanonical hypergroup, if it incorporates a scalar identity, i.e. an identity e with the property $ex = xe = x$ for all x in H. Moreover, a join hypergroup is a canonical hypergroup, if it contains a scalar identity [14, 20, 23].

A hypergroup H with a strong identity e has a natural partition. Let

$$A = \{x \in H \mid ex = xe = \{e,x\}\} \text{ and } C = \{x \in H - \{e\} \mid ex = xe = e\}.$$

Then, $H = A \cup C$ and $A \cap C = \varnothing$. A member of A is an *attractive element* and a member of C is a *canonical element*. See [39] for the origin of terminology.

Fortified join hypergroups and fortified transposition hypergroups have been studied in a series of papers [see, for example, 15, 22, 33, 37, 39, 43], in which several very interesting properties of these types of hypergroups were revealed. The following was proven, among others [15]:

Structure Theorem. *A transposition hypergroup H containing a strong identity e is isomorphic to the expansion of a quasicanonical hypergroup $C \cup \{e\}$ by the transposition hypergroup A of all attractive elements through the idempotent e.*

Moreover, from the theory of automata resulted the *transposition polysymmetrical hypergroup* [24, 42, 45], i.e. a transposition hypergroup H, having an identity (or neutral) element e, such that $ee = e$, $x \in ex = xe$ for all $x \in H$ and also, for every $x \in H - \{e\}$ there exists at least one element $x' \in H - \{e\}$, (called symmetric or two-sided inverse of x), such that $e \in xx'$ and $e \in x'x$. The set of the symmetric elements of x is denoted by $S(x)$ and is called the *symmetric set* of x. A commutative transposition polysymmetrical hypergroup is called a *join polysymmetrical hypergroup*.

Theorem 1.3. *If a polysymmetrical transposition hypergroup contains a strong identity e, then this identity is unique.*

Analytical examples of the above hypergoup types are presented in [28]. A thorough study of transposition hypergroups with idempotent identity is presented [30]

2. THE TRANSPOSITION AXIOM IN HYPERGROUPOIDS

In the previous paragraph it was mentioned that the hypergroup was enriched with further axioms, a fact which led to the creation of specific hypergroup families. However, mathematical research also followed the reverse course. Certain axioms were removed from the hypergroup and the resulting weaker structures were studied. Thus, the pair $(H,\cdot)$, where H is a non-empty set and "$\cdot$" a hypercomposition, was named *partial hypegroupoid*, while it was called *hypegroupoid* if $ab\neq\varnothing$ for all $a,b\in H$. A hypergroupoid in which the associativity is valid, was called *semi-hypergroup*, while it was called *quasi-hypergroup*, if only the reproductivity is valid. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c\cap a(bc)\neq\varnothing$ for all $a,b,c\in H$, were named *H_V-groups* [55]. Certain properties of these structures, which are analogous to those of hypergroups, are presented herein.

Property 2.1. *If the weak associativity is valid in a hypergroupoid, then this hypergroupoid is not partial.*

P r o o f. Suppose that $ab=\varnothing$ for some $a,b\in H$. Then, $(ab)c=\varnothing$ for any $c\in H$. Therefore, $(ab)c\cap a(bc)=\varnothing$, which is absurd. Hence, ab is non-void.

The following is a direct consequence of the above property:

Property 2.2. *The result of the hypercomposition in an H_V-group H is always a non-empty set.*

Property 2.3. *A hypergroupoid H is a quasi-hypergroup, if the results of induced hypercompositions in it are non-void.*

P r o o f. Suppose that $x/a\neq\varnothing$ is valid for all $x,a\in H$. Then, there exists $y\in H$, such that $x\in ya$. Therefore, $x\in Ha$ for all $x\in H$ and so $H\subseteq Ha$. But $Ha\subseteq H$ is also valid for all $a\in H$. Hence, $H=Ha$. By duality, $aH=H$. Thus, H is a quasi-hypergroup.

Property 2.4. *$a/b\neq\varnothing$ and $b\backslash a\neq\varnothing$ is valid for all the elements a,b of a quasi-hypergroup H.*

P r o o f. Per equality $H=Hb$, there exists $y\in H$, such that $a\in yb$ for every $a\in H$. Thus, $y\in a/b$ and, therefore, $a/b\neq\varnothing$. $b\backslash a\neq\varnothing$, per the principle of duality.

Property 2.5. *In a quasi-hypergroup* H *, the equalities* $H = a / H = H \backslash a$ *are valid for all* a *in* H *.*

P r o o f. Let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a / y$. Hence, $H \subseteq a / H$. Moreover, $a / H \subseteq H$. Therefore, $H = a / H$. The other equality follows by duality.

Property 2.6. *In any non-partial hypergroupoid H, the equalities* $H = H / a = a \backslash H$ *are valid for all* a *in* H *.*

P r o o f. Since the result of the hypercomposition in a non-partial hypergroupoid is always a non-empty set, there exists $y \in H$ such that $y \in xa$ for every $x \in H$. This implies that $x \in y / a$. Hence, $H \subseteq H / a$. Moreover, $H / a \subseteq H$. Therefore, $H = H / a$. The other equality follows by duality.

The following is a direct consequence of Properties 2.5 and 2.6 above:

Property 2.7. *In any* H_V*-group H, the equalities (i)* $H = H / a = a / H$ *and (ii)* $H = a \backslash H = H \backslash a$ *are valid for all* a *in* H *.*

Extensive work has been done on the construction of hypergroupoids, on their enumeration and on the study of their structure (see, for example, [3, 4, 5, 6, 9, 10, 11, 29, 50, 51, 52, 54]). As mentioned above, this direction pertained to researching hypercompositional structures resulting from the weakening of the structure of the hypergoup. The opposite direction pertained to researching hypercompositional structures resulting from the reinforcement of the structure of the hypergoup. These two directions are combined in [31], via the introduction of the transposition axiom into the H_V-group, thus leading to the following definition:

Definition 2.1. *An* H_V*-group (H,·) is called* ***transposition*** $\boldsymbol{H_V}$***-group****, if it satisfies the transposition axiom:*

$$b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for all } a,b,c,d \in H .$$

A transposition H_V*-group (H,·) is called* ***join*** $\boldsymbol{H_V}$***-group,*** *if H is a commutative* H_V*-group, while it is called* ***weak join*** $\boldsymbol{H_V}$***-group,*** *if H is an* H_V*-commutative group.*

The fortified transposition H_V-group was also defined in [31], in a manner analogous to the definition of the fortified transposition hypergroup, as follows:

Definition 2.2. *A transposition* H_V*-group (H,·) is called* ***fortified****, if H contains an element* e *, which satisfies the axioms:*

i. $ee = e$,

ii. $x \in ex = xe$ *for all* $x \in H$,

iii. *for every* $x \in H-\{e\}$ *there exists a unique* $y \in H-\{e\}$, *such that* $e \in xy$ *and, furthermore, y satisfies* $e \in yx$.

If "·" is commutative, then H is called a ***fortified join H$_V$-group***.

Properties of the structure above, as well as relevant examples are presented in [31]. The elements of the fortified transposition H$_V$-group are partitioned into canonical and attractive, exactly as in hypergroups.

Proposition 2.1. *Let H be a fortified transposition H$_V$-group and suppose that* x,y *are attractive elements with* $y \neq x^{-1}$. *Then,* $x,y \in xy$ *and* $x,y \in yx$.

P r o o f. Since x is an attractive element, $ex = xe = \{e,x\}$ is valid. Therefore, $e/x = x \backslash e = \{e,x^{-1}\}$. Moreover, $y/y = \{z \mid y \in zy\}$. Hence, $e \in y/y$. Thus, $y/y \cap x \backslash e \neq \varnothing$ which, per the transposition axiom, results into $ey \cap yx \neq \varnothing$ or, equivalently, $\{e,y\} \cap yx \neq \varnothing$. Since $y \neq x^{-1}$, it follows that $y \in yx$. Similarly, $x \in yx$ and, per duality, $x,y \in xy$.

Corollary 2.1. *A fortified transposition H$_V$-group containing exclusively attractive elements is weakly commutative.*

As can be observed, the transposition axiom is not dependent on the two hypergroup axioms (asssosiativity and reproduction) and their consequences. Therefore, the transposition axiom can be introduced even into a partial hypergroupoid. Thus, the notions of the *transposition hypergroupoid*, of the *transposition quasi-hypergroup* and of the *transposition semi-hypergroup* emerge. If the commutativity is also valid in the above, the notions of the *join hypergroupoid*, of the *join quasi-hypergroup* and of the *join semi-hypergroup* emerge as well. The following proposition is analogous to the one used in [31] for the construction of transposition H$_V$-groups. The proof of this proposition, as well as of Proposition 2.3 below, is quite straightforward, albeit long, since all the possible cases must be verified.

Proposition 2.2. *Let H be a hypergroupoid (either partial or non- partial) or a quasi-hypergroup. Also, let an arbitrary subset* I_{ab} *of H be associated to each pair of elements* $(a,b) \in H^2$. *If* $\bigcap_{a,b \in H} I_{ab} \neq \varnothing$, *then H endowed with the hypercomposition*: $a*b = ab \cup I_{ab}$, $a,b \in H$ *is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or* *a join quasi-hypergroup, if the commutativity is valid in* H *and* $I_{ab} = I_{ba}$ *for all* $a,b \in H$.

Corollary 2.1. *If H is a hypergroupoid (either partial or non- partial) or a quasi-hypergroup and w is an arbitrary element of H, then H endowed with the hypercomposition*

$$x * y = xy \cup \{x, y, w\}$$

is a transposition hypergroupoid or a transposition quasi-hypergroup respectively, while it is a join hypergroupoid or a join quasi-hypergroup, if the commutatity is valid in H.

Proposition 2.3. *Let* H *be a set with more that two elements and let* w *be an arbitrary element in* H. *Two hypercompositions are defined in* H *as follows:*

$a \circ_l b = \{a, w\}$ *for all* $a,b \in H$ *and* $a \circ_r b = \{b, w\}$ *for all* $a,b \in H$.

Then, $(H, \circ_l)$ *and* $(H, \circ_r)$ *are transposition semi-hypergroups.*

3. The Transposition Axiom in Hypercompositional Structures with Internal Compositions

M. Krasner was the first to expand hypercompositional structures via the creation of structures containing composition and hypercompositions. Thus, in 1956, he replaced the additive group of a field with a special hypergroup, thereby introducing the *hyperfield*. He then used the hyperfield as the proper algebraic tool, in order to define a certain approximation of complete valued fields by sequences of such fields [16, 17]. Later, he introduced a more general structure, which relates to hyperfields in the same way rings relate to fields. He called this structure *hyperring*. Additional hypercompositional structures, similar to the above, introduced by various researchers, soon followed. Examples of those are the ***superring*** and the ***superfield***, in which both the addition and the multiplication are hypercompositions [47]. Additionally, the study of formal languages introduced structures in which the hypercompositional component is a join hypergroup.

Indeed, let A be an alphabet, let A^* denote the set of the words defined over A and let λ be the empty word. Then, set A^* is a semigroup with regard to the concatenation of the worlds. This semigroup has λ as its neutral element, since $\lambda a = a\lambda = a$ for all a in A^*. In addition, the expression $a+b$, where a and b are words over A, is used in formal languages theory to denote «either a or

b ». Based on the fact that $a+b$ is in essence a biset, hypercomposition $a+b=\{a,b\}$ appears in the word set A^*. It has been proven that A^* is a join hypergroup [32, 33] with regard to this hypercomposition. This hypergroup was named *B(iset)-hypergroup*. However, since A^* is a semigroup with regard to world concatenation and since it has been proven that world concatenation is distributive with regard to the hypercomposition, a new hypercompositional structure thus emerged. This structure was named hyperringoid.

Definition 3.1. *A **hyperringoid** is a non-empty set Y equipped with an operation "·" and a hyperoperation "+", such that:*

i) (Y,+) is a hypergroup,

ii) (Y, ·) is a semigroup,

iii) the operation "·" is distributive on both sides of the hyperoperation "+ ".

If (Y,+) is a join hypergroup, (Y,+,·) is called *join hyperringoid*. The join hyperringoid that results from a B-hypergroup is called *B-hyperringoid* and the special B-hyperringoid that appears in the theory of formal languages is the *linguistic hyperringoid*. Join hyperringoids are studied in [38, 40, 41].

Another notion in the theory of formal languages is the null word, the introduction of which resulted from the theory of automata. The null word is symbolized with 0 and is bilaterally absorbing with regard to word concatenation. Therefore, the extension of the composition and of the hypercomposition onto $A^*\cup\{0\}$ results into the following:

$$0a=a0=0, \quad 0+a=a+0=\{0,a\} \quad \text{for all } a\in A^*.$$

With these extensions, structure $(A^*\cup\{0\},+,\cdot)$ continues to be a hyperringoid, which, however now also has an absorbing element. The additive structure of these hyperringoid comprises a fortified join hypergroup. Thus, a new hypercompositional structure appeared:

Definition 3.2. *If the additive part of a hyperringoid is a fortified join hypergroup whose zero element is bilaterally absorbing with respect to the multiplication, then, this hyperringoid is named **join hyperring**. A **join hyperdomain** is a join hyperring which has no divisors of zero. A **proper join hyperring** is a join hyperring which is not a Krasner hyperring. A join hyperring K is called **join hyperfield** if $K^*=K-\{0\}$ is a multiplicative group.*

Join hyperrings are studied in [25, 41].

Moreover, hypercompositional structures having external operations and hyperoperations on hypergroups appeared [see, for example, 1, 2, 19, 56]. The notions of the set of operators and hyperoperators from a hyperringoid Y over an arbitrary non-void set M were introduced in [33, 34], in order to describe the action of the state transition function in the theory of Automata. Y is a set of *operators* over M, if there exists an external operation from $M \times Y$ to M, such that $(s\kappa)\lambda = s(\kappa\lambda)$ for all $s \in \mathrm{M}$ and $\kappa, \lambda \in Y$ and, moreover, $s1 = s$ for all $s \in \mathrm{M}$, when Y is a unitary hyperringoid. If there exists an external hyperoperation from $M \times Y$ to $P(M)$ which satisfies the above axiom, with the variation that $s \in s1$ when Y is a unitary hyperringoid, then Y is a set of *hyperoperators* over M. If M is a hypergroup and Y is a hyperringoid of operators over M, such that, for each $\kappa, \lambda \in Y$ and $s, t \in M$, the axioms: (i) $(s+t)\lambda = s\lambda + t\lambda$, (ii) $s(\kappa + \lambda) \subseteq s\kappa + s\lambda$ hold, then M is called *right hypermoduloid* over Y. If Y is a set of hyperoperators, then M is called *right supermoduloid*. If the second of the above axioms holds as an equality, then the hypermoduloid is called *strongly distributive*. There is a similar definition of the *left hypermoduloid* and the *left supermoduloid* over Y, in which the elements of Y operate from the left side. When M is both right and left hypermoduloid (resp. supermoduloid) over Y, it is called *Y-hypermoduloid* (resp. *Y-supermoduloid*) [33, 34]. If M is a canonical hypergroup, the set of operators Y is a hyperring and, if $s1 = s$, $s0 = 0$ for all $s \in M$, then M is named *right hypermodule,* while it is named *right supermodule* if Y is a set of hyperoperators [26]. A study of external operations and hyperoperations on hypergroups is carried out in [26].

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Ch. G. Massouros,
e mail: masouros@teihal.gr
URL: http://www.teihal.gr/gen/profesors/massouros/index.htm

G. G. Massouros,
e mail: germasouros@gmail.com
URL: https://sites.google.com/site/gerasimosgmassouros/

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An Optimization Framework for "Build-or-Buy" Strategy for component Selection in a Fault Tolerant Modular Software System under Recovery Block Scheme

P.C.Jha* Ritu Arora** U.Dinesh Kumar***

*Department of Operational Research , University of Delhi,India
**Maharaja Agrasen Institute of Technology, GGSIP University, Delhi,India.
***Indian Institute of Management, Bangalore, India

*jhapc@yahoo.com ** arora_ritu21@yahoo.co.in ***dineshk@iimb.ernet.in

Abstract

This paper discusses a framework that helps developers to decide whether to buy or build components of software architecture. Two optimization models have been proposed. First model is Bi-criteria optimization model based on decision variables in order to maximize the software reliability with simultaneous minimization of the overall cost of the system. The second optimization model deals with the issue of compatibility.

Keywords : Modular software, software reliability, software cost, fault tolerance, software components, recovery block scheme

1. Introduction

Science and technology demand high quality software for making improvement and breakthroughs. Today, computer hardware and software permeates our modern society. The newest cameras, VCRs, and automobiles cannot be controlled and operated without computers. When the requirement for and dependencies on computer increases, the possibility of crises from computer failures also increases. Software systems are developed as per the requirements given by the users. While developing the software, quality and reliability of the software are two key factors. Reliability of a software system is defined as the probability that software operates without failure in a specified environment, during a specified exposure period. Introduction of redundancy in the parts of the hardware and/or software components is one of the most followed ways to improve the reliability of the system under development. A careful use of redundancy may allow the system to tolerate faults. Despite that we still cannot guarantee error free software. A way of handling unknown and unpredictable software failures is through fault tolerance. One way to reduce the risks of software design faults and thus enhance software dependability is to use software fault tolerance techniques. Software fault tolerance techniques are employed during the procurement, or development, of the software. They enable a system to tolerate software faults remaining in the system after its development. When a fault occurs, these techniques provide mechanisms to the software system to prevent system failure from occurring. There are two structural methodologies for Fault Tolerant System i.e. Recovery Block Scheme and N-Version Scheme. In this paper, we will discuss optimization model for recovery block. Non functional aspects play a significant role in determining software quality. Given the fact that lack of proper handling of non functional aspects (Cysneiros et al, [5]) of a software application has led to a series of software failures, nonfunctional attributes such as reliability security and performance should be considered during the component selection phase of software development. This paper discusses a framework that helps developers to decide whether buying or building components of software architecture on the base of cost and non functional factors. While developing software, components can be both bought as COTS (Commercial Off-The Shelf) products, and probably adapted to work in the software system, or they can be developed in-house. This decision is known as "build-or-buy decision". This decision affects the overall cost and reliability of the system. Most of today's software systems include one or more COTS products. COTS are pieces of software that can be reused by software projects to build new systems. Benefits of COTS based development include significant reduction in the development cost, time and improvement in the dependability requirement. No changes are normally made to their source codes. COTS components are used without any code modification and inspection. The components, which are not available in the market or cannot be purchased economically, can be developed within the organization and are known as in-house built components. Kapur et al [8] discussed issues related to reliability of systems through weighted maximization of system quality subject to budgetary constraint.

This paper discusses the issues related with reliability of the software systems and cost produced by integrating COTS or in-house build components. Large software system has modular structure to perform a set of functions. Each function is performed by different modules having different alternatives for each module. In case a COTS component is selected then different versions are available for each alternative and only one version will be selected for each alternative of a module. If a component is in-house build component, then the alternative of a module is selected. A schematic representation of the software system is given in Figure 1. We are selecting the components for modules to maximize the system reliability by simultaneously minimizing the cost. The frequency with which the functions are used is not same for all of them and not all the modules are called during the execution of a function, the software has in its menu. Software whose failure can have bad effects afterwards can be made fault tolerant through redundancy at module level (Belli and Jadrzejowicz, [1]). We assume that functionally equivalent and independently developed alternatives (i.e In-house or COTS) for each module are available with an estimated reliability and cost. The first optimization model (optimization model-I) of this paper maximizes the system reliability with simultaneously minimizing the cost. The model contains four problems (P1), (P2), (P3) and (P4). Problem (P1) is not in normalized form, therefore, it has been normalized and transformed into problem (P3) and (P4). The second optimization model (optimization model-II) considers the issue of compatibility between different alternatives of modules as it is observed that some COTS components cannot integrate with all the alternatives of another module. The models discussed are illustrated with numerical example.

2. Notations

R : System quality measure

f_l : Frequency of use, of function l

s_l : Set of modules required for function l

R_i : Reliability of module i

L : Number of functions, the software is required to perform

n : Number of modules in the software

m_i : Number of alternatives available for module i

V_{ij} : Number of versions available for alternative j of module i

N_{ij}^{tot} : Total number of tests performed on the in- house developed instance (i.e. alternative j of module i)

N_{ij}^{suc} : Number of successful (i .e failure free) test performed on the in-house developed instance (i.e. alternative j of module i)

t_1 : Probability that next alternative is not invoked upon failure of the current alternative

t_2 : Probability that the correct result is judged wrong.

t_3 : Probability that an incorrect result is accepted as correct.

X_{ij} : Event that output of alternative j of module i is rejected.

Y_{ij} : Event that correct result of alternative j of module i is accepted.

s_{ij} : Reliability of alternative j of module i

r_{ijk} : Reliability of version k of alternative j for module i

C_{ijk} : Cost of version k of alternative j for module i

r_{ijk} : Reliability of version k of alternative j for module i

d_{ijk} : Delivery time of version k of alternative j for module i

c_{ij} : Unitary development cost for alternative j of module i

t_{ij} : Estimated development time for alternative j of module i

τ_{ij} : Average time required to perform a test case for alternative j of module i

π_{ij} : Probability that a single execution of software fails on a test case chosen from a certain input distribution

y_t : $\begin{cases} 0, \text{ if } t^{th} \text{ constraint is active} \\ 1, \text{ if } t^{th} \text{ constraint is inactive} \end{cases}$

y_{ij} : $\begin{cases} 1 \text{ if the } j\text{th alternative of } i\text{th module is in-house developed.} \\ 0 \text{ otherwise} \end{cases}$

x_{ijk} : $\begin{cases} 1, \text{ if the } k^{th} \textit{version} \text{ of } j^{th} \text{ COTS alternative of the } i^{th} \text{ module is chosen} \\ 0, \text{ otherwise} \end{cases}$

z_{ij} : Binary variable taking value 0 or 1

$\begin{cases} 1, & \text{if alternative } j \text{ is present in module } i \\ 0, & \text{otherwise} \end{cases}$

3. Optimization Models

The first optimization model is developed for the following situations which also hold good for the second model, but with additional assumptions related to compatibility among alternatives of a module.

The following assumptions are common for the optimization models are:

1. Software system consists of a finite number of modules.
2. Software system is required to perform a known number of functions. The program written for a function can call a series of modules $(\le n)$. A failure occurs if a module fails to carry out an intended operation.
3. Codes written for integration of modules don't contain any bug.
4. Several alternatives are available for each module. Fault tolerant architecture is desired in the modules (it has to be within the specified budget). Independently developed alternatives (primarily COTS/ In-House components) are attached in the modules and work similar to the recovery block scheme discussed in (Berman et al., [2] and Kumar, [9]).
5. The cost of an alternative is the development cost, if developed in house; otherwise it is the buying price for the COTS product.
6. Different In- house alternatives with respect to unitary development cost, estimated development time, average time and testability of a module are available.
7. Cost, reliability and development time of an in-house component can be specified by using basic parameters of the development process, e.g., a component cost may depend on a measure of developer skills, or the component reliability depends on the amount of testing.
8. Different versions with respect to cost, reliability and delivery time of a module are available.
9. Other than available cost-reliability versions of an alternative, we assume the existence of virtual versions, which has a negligible reliability of 0.001, zero cost and zero delivery time. These components are denoted by index one in the third subscript of x_{ijk}, C_{ijk} and r_{ijk}. for example r_{ij1} denotes the reliability of first version of alternatives j for module i.

3.1 Model Formulation

Let S be a software architecture made of n modules having m_i alternatives available for each module and each COTS alternatives has different versions.

3.1.1 Build versus Buy Decision

For each module i, if an alternative is bought (i.e. some $x_{ijk} = 1$) then there is no in-house development (i.e. $y_{ij} = 0$) and vice versa.

$$y_{ij} + \sum_{k=1}^{V_{ij}} x_{ijk} = 1;\ \ i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m_i$$

3.1.2 Redundancy Constraint

The equation stated below guarantees that redundancy is allowed for the components.

$$y_{ij} + \sum_{k=2}^{V_{ij}} x_{ijk} = z_{ij};\ \ i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m_i$$

$$x_{ij1} + z_{ij} = 1;\ \ i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m_i$$

$$\sum_{j=1}^{m_i} z_{ij} \geq 1;\ \ i = 1, 2, ...n$$

3.1.3 Probability of Failure Free In-house Developed Components

The possibility of reducing the probability that the j^{th} alternative of i^{th} module fails by means of a certain amount of test cases (represented by the variable N_{ij}^{tot}). Cortellessa et al [4] define the probability of failure on demand of an in-house developed j^{th} alternative of i^{th} module, under the assumption that the on-field users' operational profile is the same as the one adopted for testing (Bertolino and Strigini, [3]). *Basing on the testability definition, we can assume that the number N_{ij}^{suc} of successful (i.e. failure free) tests performed on j^{th} alternative of same module.*

$$N_{ij}^{suc} = \left(1 - \pi_{ij}\right) N_{ij}^{tot}\ \ ; i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m_i$$

Let A be the event " N_{ij}^{suc} failure – free test cases have been performed " and B be the event " the alternative is failure free during a single run ".If ρ_{ij} is the probability that the in- house developed alternative is failure free during a single run given that N_{ij}^{suc} test cases have been successfully performed, from the Bayes Theorem we get

$$\rho_{ij} = P(B/A) = \frac{P(A/B)P(B)}{P(A/B)P(B) + P(A/\overline{B})P(\overline{B})}$$

The following equalities come straightforwardly:

$$P(A/B) = 1;\ \ P(B) = 1 - \pi_{ij}\ ;\ P(A/\overline{B}) = (1 - \pi_{ij})^{N_{ij}^{suc}}\ ;\ \ P(\overline{B}) = \pi_{ij}$$

therefore, we have

$$\rho_{ij} = \frac{1-\pi_{ij}}{\left(1-\pi_{ij}\right)+\pi_{ij}\left(1-\pi_{ij}\right)^{N_{ij}^{suc}}}; \quad i=1,2,....,n \text{ and } j=1,2,....,m_i$$

3.1.4 Reliability equation of both in-house and COTS components

The reliability (s_{ij}) of j^{th} alternative of i^{th} module of the software.

$$s_{ij} = \rho_{ij} y_{ij} + r_{ij};\ i=1,2,....,n \text{ and } j=1,2,....,m_i$$

where $r_{ij} = \sum_{k=1}^{V_{ij}} r_{ijk} x_{ijk};\ i=1,2,....,n$ and $j=1,2,....,m_i$

3.1.5 Delivery time constraint

The maximum threshold T has been given on the delivery time of the whole system. In case of a COTS components the delivery time is simply given by d_{ijk}, whereas for an in- house developed alternative the delivery time shall be expressed as $(t_{ij} + \tau_{ij} N_{ij}^{tot})$.

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij}\left(t_{ij} + \tau_{ij} N_{ij}^{tot}\right) + \sum_{k=1}^{V_{ij}} d_{ijk} x_{ijk} \right) \le T$$

3.2 Objective Function

3.2.1 Reliability objective function

Reliability objective function maximizes the system quality (in terms of reliability) through a weighted function of module reliabilities. Reliability of modules that are invoked more frequently during use is given higher weights. Analytic Hierarchy Process (AHP) can be effectively used to calculate these weights.

$$\text{Maximize } R(X) = \sum_{l=1}^{L} f_l \prod_{i \in s_l} R_i$$

where R_i is the reliability of module i of the system under Recovery Block stated as follows.

$$R_i = \sum_{j=1}^{m_i} z_{ij}\left[\prod_{k=1}^{j-1} P\left(X_{ik}\right)^{z_{ij}}\right] P\left(Y_{ij}\right)^{z_{ij}};\ i=1,2,.........n$$

$$P\left(X_{ij}\right) = \left(1-t_1\right)\left[\left(1-s_{ij}\right)\left(1-t_3\right) + s_{ij} t_2\right]$$

$$P\left(Y_{ij}\right) = s_{ij}\left(1-t_2\right)$$

3.2.2 Cost objective function

Cost objective function minimizes the overall cost of the system. The sum of the cost of all the modules is selected from "build – or - buy" strategy. The in-house development cost of the alternative j of module i can be expressed as $c_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)$

$$\text{Minimize C(X)}= \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)y_{ij}+\sum_{k=1}^{V_{ij}}C_{ijk}x_{ijk}\right)$$

3.3 Optimization Model I

In the optimization model it is assumed that the alternatives of a module are in a Recovery Block. In recovery block more than one alternative of a program exist. For COTS based software multiple alternatives of a module can be purchased from different vendors. Each module works under a recovery block. Upon invocation of a module the first alternative is executed and the result is submitted for acceptance test. If it is rejected, the second alternative is executed with the original inputs. The same process continues through attached alternative until a result is accepted or the whole recovery block (module) fails. Fault tolerance in a recovery block is achieved by increasing the number of redundancies.

Problem (P1)

$$\text{Maximize } R(X)=\sum_{l=1}^{L} f_l \prod_{i\in s_l} R_i \qquad (1)$$

$$\text{Minimize C(X)}= \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)y_{ij}+\sum_{k=1}^{V_{ij}}C_{ijk}x_{ijk}\right) \qquad (2)$$

Subject to $\quad X \in S = \{x_{ijk}$ and y_{ij} are binary variable/

$$R_i = \sum_{j=1}^{m_i} z_{ij}\left[\prod_{k=1}^{j-1} P\left(X_{ik}\right)^{z_{ij}}\right] P\left(Y_{ij}\right)^{z_{ij}};\ i=1,2,\dots\dots\dots n \qquad (3)$$

$$P\left(X_{ij}\right) = \left(1-t_1\right)\left[\left(1-s_{ij}\right)\left(1-t_3\right)+s_{ij}t_2\right]$$

$$P\left(Y_{ij}\right) = s_{ij}\left(1-t_2\right)$$

$$N_{ij}^{suc} = \left(1-\pi_{ij}\right)N_{ij}^{tot} \quad ,i=1,2,\dots,n \text{ and } j=1,2,\dots,m_i \qquad (4)$$

$$\rho_{ij} = \frac{1-\pi_{ij}}{\left(1-\pi_{ij}\right)+\pi_{ij}\left(1-\pi_{ij}\right)^{N_{ij}^{suc}}}; \quad i=1,2,....,n \text{ and } j=1,2,....,m_i \tag{5}$$

$$s_{ij} = \rho_{ij} y_{ij} + r_{ij}; \; i=1,2,....,n \text{ and } j=1,2,....,m_i \tag{6}$$

$$y_{ij} + \sum_{k=1}^{V_{ij}} x_{ijk} = 1; \; i=1,2,....,n \text{ and } j=1,2,....,m_i \tag{7}$$

$$y_{ij} + \sum_{k=2}^{V_{ij}} x_{ijk} = z_{ij}; \quad i=1,2,....,n \text{ and } j=1,2,....,m_i \tag{8}$$

$$x_{ij1} + z_{ij} = 1; \; i=1,2,....,n \text{ and } j=1,2,....,m_i \tag{9}$$

$$\sum_{j=1}^{m_i} z_{ij} \geq 1 \; ; \; i=1,2,....,n \tag{10}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)+\sum_{k=1}^{V_{ij}} d_{ijk}x_{ijk}\right) \leq T \tag{11} \quad \}$$

Where X is a vector of elements : x_{ijk} and y_{ij}; $i=1,2,.....n$; $j=1,2,....,m_i$; k=1,2,....V_{ij}

3.3.1 Normalization

The problem (P1) is Bi- criteria optimization problem in which on one hand system reliability is maximized and other hand cost of selected components to form / assemble the system is minimized. The reliability which is unit free is measured between zero and one whereas cost has its unit. Two objectives can be converted to single objective programming problem either if both objectives are of same unit or if both objectives can be made unit free. To make cost function unit free, the following transformation is used.

$$c = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=1}^{V_{ij}} C_{ijk}, \qquad \bar{c} = \sum_{i=1}^{n}\sum_{j=1}^{m_i} c_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)$$

Now $\overline{C_{ijk}} = \dfrac{C_{ijk}}{c+\bar{c}}$, $\overline{c_{ij}} = \dfrac{c_{ij}\left(t_{ij}+\tau_{ij}N_{ij}^{tot}\right)}{c+\bar{c}}$ and $\overline{C}_{ijk} + \bar{c}_{ij} = 1$

The resulting problem then can be rewritten as follows.

Problem (P2) *Maximize* $\mathrm{F}_1(X) = \sum_{l=1}^{L} f_l \prod_{i\in s_l} R_i$

$$\textit{Minimize} \quad F_2(X) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(\overline{c_{ij}}\, y_{ij} + \sum_{k=1}^{V_{ij}} \overline{C_{ijk}}\, x_{ijk}\right)$$

Subject to $\quad X \in S$

The problem (P2) can further be written as vector optimization problem as.

Problem (P3) *Vector Max* $F(X)$

Subject to $X \in S$

where $F(X) = (F_1(X), F_2(X))^T$

3.3.2 Finding Properly Efficient Solution

Definition 1 (Steuer, [10]): A feasible solution $X^* \in S$ is said to be an efficient solution for the below problem if there exists no $X \in S$ such that $F(X) \geq F(X^*)$ and $F(X) \neq F(X^*)$

Definition 2 (Steuer, [10]): An efficient solution $X^* \in S$ is said to be an properly efficient solution for the problem (P2) if there exist $\alpha > 0$ such that for each r

$(F_r(X) - F_r(X^*))/(F_j(X^*) - F_j(X)) < \alpha$ for some j with $F_j(X) \leq F_j(X^*)$ and $F_r(X) > F_r(X^*)$ for $X \in S$.

Using Geoffrion's scalarization the problem (P2) reduces to

Problem (P4)

Maxize $\quad Z= \lambda_1 F_1 - \lambda_2 F_2$

Subject to $\quad X \in S$

$\lambda_1 + \lambda_2 = 1 \qquad \lambda_1, \lambda_2 \geq 0$

Lemma(Geoffrion,[6]): The optimal solution of the problem (P4) for fixed λ_1 and λ_2 is a properly efficient solution for the problem (P3) as well as (P1).

3.4 Optimization Model II

Optimization model II is an extension of optimization model I. As explained in the introduction, it is observed that some alternatives of a module may not be compatible with alternatives of another module (Jung and Choi, [7]). The next optimization model II addresses this problem. It is done, incorporating additional constraints in the optimization models. This constraint can be represented as $x_{gsq} \leq x_{hu_t c}$, which means that if alternative s for module g is chosen, then alternative $u_t, t=1,........z$ have to be chosen for module h. We also assume that if two alternatives are compatible, then their versions are also compatible.

$$x_{gsq}-x_{hu_tc} \leq My_t,\ q=2,.......,V_{gs},\ c=2,......,V_{hu_t},\ s=1,.....,m_g \qquad (12)$$

$$\sum y_t = z\left(V_{hu_t}-2\right) \qquad (13)$$

Constraint (12) and (13) make use of binary variable y_t to choose one pair of alternatives from among different alternative pairs of modules. Problem (P3) can be transformed to another optimization problem using compatibility constraints and if more than one alternative compatible component is to be chosen for redundancy, constraint (13) can be relaxed as follows.

$$\sum y_t \leq z\left(V_{hu_t}-2\right) \qquad (14)$$

4. Illustrative Examples

Consider a software system having two modules with more than one alternative for each module. The data sets for COTS and in-house developed components are given in Table-1 and table II, respectively. Let $L=3,\ s_1=\{1,2,3\},\ s_2=\{1,3\},\ s_3=\{2\},\ f_1=0.5,\ f_2=0.3$ and $f_3=0.2$. It is also assumed that $t_1=.01, t_2=.05$ and $t_3=.01$

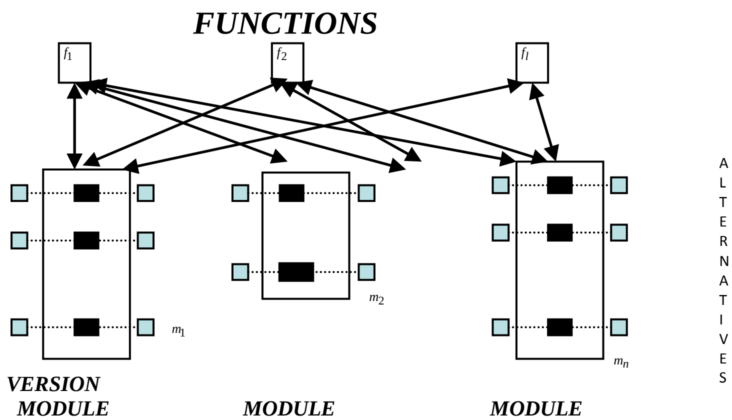

Figure 1 Structure of the software

Table 1: Data set for COTS components

	Alternatives	Versions								
		1			2			3		
		Cost	Reliability	Delivery Time	Cost	Reliability	Delivery Time	Cost	Reliability	Delivery Time
1	1	0	0.001	0	14	0.90	3	11	0.88	4
	2	0	0.001	0	12.5	0.86	4	18	0.92	2
	3	0	0.001	0	17	0.90	2	15	0.88	3
2	1	0	0.001	0	13	0.87	4	17.5	0.86	2
	2	0	0.001	0	11	0.91	5	12	0.89	4
	3	0	0.001	0	18	0.89	2	15	0.86	3
	4	0	0.001	0	13	0.86	4	14	0.88	3
3	1	0	0.001	0	16	0.85	3	18	0.90	2
	2	0	0.001	0	16	0.89	3	17	0.87	2

Table 2Data set for In-House conponents

Module i	Alternatives	t_{ij}	τ_{ij}	c_{ij}	π_{ij}
1	1	8	0.005	5	0.002
	2	6	0.005	4	0.002
	3	7	0.005	4	0.002
2	1	9	0.005	5	0.002
	2	5	0.005	2	0.002
	3	6	0.005	4	0.002
	4	5	0.005	3	0.002
3	1	6	0.005	4	0.002
	2	5	0.005	3	0.002

4.1 Optimization Model – I

Table 3 presents the solution for optimization model I. The problem is solved using software package LINGO (Thiriez, [11]). The solution to the model gives the optimal component selection for the software system along with the corresponding cost and reliability of the overall system. The sensitivity analysis to the delivery time constraint has been performed. It is clearly seen from the table that when the delivery time was 10 units, then only COTS components were selected. When the delivery time increases along with the COTS components, in house build components were also selected. When the delivery time was 12 units, only one in-house component was developed with the minimum cost 79 units attained at reliability level 0.85.Our system cost decreases while the corresponding reliability increases because the components developed in-house decreases the cost initially but later if the level of reliability has to be kept at 0.90 then by increasing delivery time by 5 and 9 units respectively, more in-house build components were selected which in turn increases the cost and reliability of the overall system. Redundancy is also there in all the four cases.

Table 3: Solution of Optimization Model I

Case No.	Delivery Time	COTS	IN-House	System Reliability	Overall system Cost	Joint Objective Value
1	10	$x_{111} = x_{123} = x_{132} = 1$ $x_{211} = x_{221} = x_{232} = x_{242} = 1$ $x_{311} = x_{322} = 1$	Nil	0.84	82	0.66
2	12	$x_{111} = x_{123} = x_{132} = 1$ $x_{211} = x_{232} = x_{241} = 1$ $x_{311} = x_{322} = 1$	$y_{22} = 1$	0.85	79	0.68
3	17	$x_{111} = x_{123} = x_{132} = 1$ $x_{211} = x_{221} = x_{232} = 1$ $x_{311} = 1$	$y_{24} = y_{32} = 1$	0.93	86	0.74
4	21	$x_{111} = x_{132} = 1$ $x_{211} = x_{221} = x_{232} = 1$ $x_{311} = 1$	$y_{12} = y_{24} = y_{32} = 1$	0.94	92	0.75

4.2 Optimization Model-II

To illustrate optimization model for compatibility, we use previous results.

Case 1. Delivery Time is assumed to be 10 units.

We assume third alternative of second module is compatible with second and third alternatives of first module.

$x_{111} = x_{123} = x_{133} = 1$

$x_{211} = x_{221} = x_{232} = x_{242} = 1$

$x_{311} = x_{322} = 1$

It is observed that due to the compatibility condition, third alternative of first module is chosen as it is compatible with third alternative of second module. The system reliability for the above solution is 0.84 and cost is 81 units.

Case 2. Delivery Time is assumed to be 12 units.

We assume second alternative of third module is compatible with second and third alternatives of first module.

$y_{22} = 1$;

$x_{111} = x_{123} = x_{133} = 1$

$x_{211} = x_{232} = x_{241} = 1$

$x_{311} = x_{322} = 1$

It is observed that due to the compatibility condition, third alternative of first module is chosen as it is compatible with second alternative of third module. The system reliability for the above solution is 0.85 and cost is 77 units.

Case 3. Delivery Time is assumed to be 10 units.

We assume third alternative of second module is compatible with second and third alternatives of first module.

$y_{24} = y_{32} = 1$

$x_{111} = x_{123} = x_{133} = 1$

$x_{211} = x_{221} = x_{232} = 1$

$x_{311} = 1$

It is observed that due to the compatibility condition, third alternative of first module is chosen as it is compatible with third alternative of second module. The system reliability for the above solution is 0.94 and cost is 84 units.

5. Conclusions

We have presented optimization models that supports the decision whether to buy software components or to build them in-house upon designing structure. A fault tolerant software structure for Recovery block scheme is discussed. A numerical example is presented to support these models. When delivery time is small then all the COTS components were selected and redundancy is allowed. But as the delivery time increases along with the COTS components in-house components were also selected and different impacts on cost and reliability were considered. Redundancy was also there in all the cases.

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