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MULTIVALUED FUNCTIONS, FUZZY SUBSETS AND JOIN SPACES

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ABSTRACT

One has considered the Hypergroupoid $H_\Gamma = < H; o_\Gamma >$ associated with a multivalued function Γ from H to a set D, defined as follows:

$\forall\, x \in H,\ x\ o_\Gamma\ x = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$,

$\forall\, (y,z) \in H^2,\ y\ o_\Gamma\ z = y\ o_\Gamma\ y \cup z\ o_\Gamma\ z$,

and one has calculated the fuzzy grade $\partial(H_\Gamma)$ for several functions Γ defined on sets H, such that $|H| \in \{3, 4, 5, 6, 8, 9, 16\}$.

INTRODUCTION

The analysis of the connections between Hyperstructures and Fuzzy Sets dates since 1993 when Corsini defined and studied the join spaces H_μ obtained from the fuzzy set $< H, \mu >$, and a little later Zahedi and Ameri considered fuzzy hypergroups. These subjects were studied in the following years by several scientists in Romania, Iran, Greece, Italy, Canada.

In 1993 Corsini associated a hypergroupoid with every fuzzy subset, and he proved that this hypergroupoid is a join space [8].

In 2003 Corsini [14] associated a fuzzy set μ_H with every hypergroupoid $< H, o >$ and considered the sequence of the fuzzy subsets μ_H and of the join spaces H_μ constructed from a hypergroup. This sequence has been studied in depth for several classes of hypergroups by Corsini [14], Corsini–Cristea [16], [17], [18], Corsini–Leoreanu-Fotea [22], Corsini–Leoreanu–Iranmanesh [23], Cristea [25], [26], Stefanescu–Cristea [70], Leoreanu-Fotea V. – Leoreanu L. [53] .

In this paper one has considered the hypergroupoid $< H, o_\Gamma >$ associated with a multivalued function Γ from a set H to a set D, defined as follows

$\forall\, x \in H,\ x\ o_\Gamma\ x = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$,

$\forall$ (y,z) $\in H^2$, $y \, o_\Gamma \, z = y \, o_\Gamma \, y \cup z \, o_\Gamma \, z$

and one has calculated the fuzzy grade $\partial(H_\Gamma)$, for several functions Γ defined on sets H such that $|H| \in \{3, 4, 5, 6, 8, 9, 16\}$.

We can remark that we have ∂(H) = s+1, for all the examinated cases with the exception of $(1_3{}^6)$, $(2_3{}^6)$, $(3_3{}^6)$, $(1_2{}^9)$, if n = 2^sq , where m.c.d. (q,2) = 1.

We remember here some definitions, notations and results which will be the basis of what follows.

With every fuzzy subset (H; μ_A) of a set H , it is possible to associate a hypergroupoid <H; o_μ>, where the hyperoperation < o_μ> is defined by: $\forall$ (x,y) $\in H^2$,

(I) $$x \, o_\mu \, y = \{ z \mid \min\{ \mu_A(x), \mu_A(y)\} \leq \mu_A(z) \leq \max\{ \mu_A(x), \mu_A(y)\} \}$$

One proved [8] that <H; o_μ> is a join space.

With every hypergroupoid < H; o >, it is possible to associate a fuzzy subset, as follows:

Set $\forall (x,y) \in H^2$, $\forall u \in H$, $\mu_{x,y}(u) = 0 \Leftrightarrow u \notin x \, o \, y$

if $u \in x \, o \, y$, $\mu_{x,y}(u) = 1/|x \, o \, y|$,

set $\forall u \in H$, $A(u) = \sum_{(x,y)\in H^2} \mu_{x,y}(u)$, $Q(u) = \{ (x,y) \mid u \in x \, o \, y \}$, $q(u) = |Q(u)|$,

(II) $$\mu_H(u) = A(u) / q(u) ,$$ see [14].

So it is clear that, given a hypergroupoid < H; o >, a sequence of fuzzy subsets and of join spaces is determined $\mu_H = \mu_1 , \mu_2 , \ldots \mu_{m+1} , \ldots$, < H; o > = ${}_0H$, ${}_1H$, … ${}_mH$…, such that $\forall j \geq 1$, $\mu_j = \mu_{{}_{j-1}H}$, and ${}_jH$ is the join space associated, after (I), with μ_j .

We call "fuzzy grade of H", if it exists, the number ∂(H) (or f.g.(H)) = $\min\{ s \mid {}_mH \approx {}_{m+1}H \}$ and "strong fuzzy grade of H" , if it exists, the number s.f.g.(H) = $\min\{ s \mid {}_mH = {}_{m+1}H \}$, see [17].

In this paper one has determined

6 hypergroupoids of 3 elements such that ∂(H) = 0,
4 hypergroupoids of 3 elements such that ∂(H) = 1,
5 hypergroupoids of 4 elements such that ∂(H) = 0,
8 hypergroupoids of 4 elements such that ∂(H) = 1,
12 hypergroupoids of 4 elements such that ∂(H) = 2,
5 hypergroupoids of 4 elements such that ∂(H) = 3,
2 hypergroupoids of 5 elements such that ∂(H) = 1,
2 hypergroupoids of 6 elements such that ∂(H)= 1,
8 hypergroupoids of 6 elements such that ∂(H) = 2,
3 hypergroupoids of 6 elements such that ∂(H) = 3,
1 hypergroupoid of 8 elements such that ∂(H)= 4,
1 hypergroupoid of 9 elements such that ∂(H)= 2,
1 hypergroupoid of 16 elements such that ∂(H)= 5.

$ 1. Let Γ be a multivalued function from a set $H = \{u_1, u_2,..., u_n\}$ to a set D, i.e. $\Gamma : H \rightarrow P^*(D)$. Then we have the following

<u>THEOREM 1</u> If there exists $d \in D$, such that $\forall i, \Gamma(u_i) \ni d$, then $\partial(H_\Gamma) = 0$.

Indeed, we have $\forall i, x_i \, o_\Gamma \, x_i = \{u_j \mid \Gamma(u_j) \cap \Gamma(u_j) \neq \emptyset\} = H$, therefore $\forall(i, j), u_i \, o_\Gamma \, u_j = H$. Whence ${}_oH = T$, from which $\forall s, {}_sH = {}_oH$, so $\partial(H_\Gamma) = 0$.

<u>THEOREM 2</u> Let Γ be a multivalued function from a set H to a set D, that is $\Gamma : H \rightarrow P^*(D)$, and let $< o_\Gamma >$ be the hyperoperation defined in H :

$\forall x \in H, \qquad x o_\Gamma \, x = \{z \mid \Gamma(z) \cap \Gamma(x) \neq \emptyset\}$,

$\forall (y, z), \qquad y \, o_\Gamma \, z = y \, o_\Gamma \, y \cup z \, o_\Gamma \, z$.

Then the hypergroupoid $< H; o_\Gamma >$ is a commutative quasi-join space, that is $\forall (a, b, c, d) \in H^4$,

(j) $\qquad a / b \cap c / d \neq \emptyset \Rightarrow a \, o_\Gamma \, d \cap b \, o_\Gamma \, c \neq \emptyset$.

Let's suppose $a / b \cap c / d \ni v$, that is $a \in b \, o_\Gamma \, v$, $c \in d \, o_\Gamma \, v$. Then, since

$b \, o \, v = b \, o_\Gamma \, b \cup v \, o_\Gamma \, v, \qquad d \, o_\Gamma \, v = d \, o_\Gamma \, d \cup v \, o_\Gamma \, v$, and

$\forall (x, y) \in H^2, \qquad y \in x \, o_\Gamma \, x \Rightarrow x \in y \, o_\Gamma \, y$,

at least one of the following cases is verified

(I) $a \in b \, o_\Gamma \, b, \quad c \in d \, o_\Gamma \, d$, (II) $a \in b \, o_\Gamma \, b, \quad c \in v \, o_\Gamma \, v$

(III) $a \in v \, o_\Gamma \, v, \quad c \in d \, o_\Gamma \, d$, (IV) $a \in v \, o_\Gamma \, v, \quad c \in v \, o_\Gamma \, v$

(I) implies $b \in a \, o_\Gamma \, a$, whence $b \in a \, o_\Gamma \, d$, and we have also $b \in b \, o_\Gamma \, b \subseteq b \, o_\Gamma \, c$

(II) We find $b \in a \, o_\Gamma \, d \cap b \, o_\Gamma \, c$ as in (I).

(III) We obtain $c \in d \, o_\Gamma \, d \subseteq a \, o_\Gamma \, d$, and also $c \in c \, o_\Gamma \, c \subseteq b \, o_\Gamma \, c$.

(IV) implies $v \in a \, o_\Gamma \, a \subseteq a \, o_\Gamma \, d$ and also $v \in c \, o_\Gamma \, c \subseteq b \, o_\Gamma \, c$.

Therefore the implication (j) is always satisfied whence $< H; o_\Gamma >$ is a quasi-join space.

$ 2. Set H = $\{u_1, u_2, u_3\}$. Then there are functions $\Gamma : H \rightarrow P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 0, 1.

(1_0^3) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_2, d_3\}$. We have clearly

$_0H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\,u_3$	$u_2\,u_3$
u_3			$u_2\,u_3$

So $\mu_1(u_1) = 0.467$, $\mu_1(u_2) = \mu_1(u_3) = 0.417$.

It follows $_1H = {}_0H$.

By consequence $\partial(1_0^3) = 0$.

(2_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\,u_3$	$u_2\,u_3$
u_3			$u_2\,u_3$

One obtains $\mu_1(u_1) = 0.467$,

$\mu_1(u_2) = \mu_1(u_3) = 0.417$.

So $_1H = {}_0H$, then $\partial(2_0^3) = 0$.

(3_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_1\}$.

$_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		H	H
u_3			H

We have $_1H = {}_0H = T$,

$\partial(3_0^3) = 0$.

(4_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	$u_1\,u_2$	$u_1\,u_2$	H
u_2		$u_1\,u_2$	H
u_3			u_3

We obtain $\mu(1) = 0.417 = \mu(2)$,
$\mu(3) = 0.467$,
So $\partial(4_0^3) = 0$.

(5_0^3) Set $\Gamma(u_1) = \Gamma(u_2) = \{d_1\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	$u_1\ u_2$	$u_1\ u_2$	H
u_2		$u_1\ u_2$	H
u_3			u_3

As in (4_0^3), we obtain $\partial(5_0^3) = 0$.

(1_1^3) Let $|H| = 3 = |D|$. Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$.
So we have

$_0H$	u_1	u_2	u_3
u_1	u_1	$u_1\ u_2$	$u_1\ u_3$
u_2		u_2	$u_2\ u_3$
u_3			u_3

We have clearly

$\mu_1(u_1) = \mu_1(u_2) = \mu_1(u_3) = 0.6$.

Therefore we obtain $_1H = T$, whence $\partial(1_1^3) = 1$.

(2_1^3) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		$u_1\ u_2$	H
u_3			$u_1\ u_3$

We obtain : $\mu_1(u_1) = 0.37$,

$\mu_1(u_2) = \mu_1(u_3) = 0.354$.

By consequence,

$_1H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\ u_3$	$u_2\ u_3$
u_3			$u_2\ u_3$

So we have : $\mu_2(u_1) = 0.467$,

$\mu_2(u_2) = \mu_2(u_3) = 0.417$.

From this, we obtain $_2H = {}_1H$, whence $\partial(2_1^3) = 1$.

(3_1^3) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

${}_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		$u_1\, u_2$	H
u_3			$u_1\, u_3$

See (2_1^3).

So we obtain again

$\partial(3_1^3) = 1$.

(4_1^3) Set $H = \{u_1, u_2, u_3\}$, $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_1\}$.

So we have

${}_0H$	u_1	u_2	u_3
u_1	$u_1\, u_3$	H	H
u_2		$u_2\, u_3$	H
u_3			H

By consequence

$\mu_1(u_1) = 0.354 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.370$.

Therefore we obtain

${}_1H$	u_1	u_2	u_3
u_1	$u_1\, u_2$	$u_1\, u_2$	H
u_2		$u_1\, u_2$	H
u_3			u_3

Hence $\mu_2(u_1) = \mu_2(u_2) = 0.4167$, $\mu_2(u_3) = 0.467$. It follows ${}_2H = {}_1H$. Therefore $\partial(4_1^3) = 1$.

\$ 3. Set $H = \{u_1, u_2, u_3, u_4\}$. Then there are functions $\Gamma : H \rightarrow P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 0, 1, 2, 3.

(1_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_3, d_4\}$. Then we have

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\, u_3\, u_4$	$u_2\, u_3\, u_4$	$u_2\, u_3\, u_4$
u_3			$u_2\, u_3\, u_4$	$u_2\, u_3\, u_4$
u_4				$u_2\, u_3\, u_4$

We obtain $\mu_1(u_1) = 0.357$,

$\mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.300$.

By consequence ${}_1H = {}_0H$

and therefore $\partial(1_0^4) = 0$.

(2_0^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. Then

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

We have as in (1)

${}_0H = {}_1H$ so $\partial(2_0^4) = 0$.

(3_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. Also in this case

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

By consequence

${}_0H = {}_1H$ from which

$\partial(3_0^4) = 0$.

(4_0^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3, d_4\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		H	H	H
u_3			H	H
u_4				H

Clearly, ${}_1H = {}_0H = T$.

So $\partial(4_0^4) = 0$.

(1_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H
u_3			$u_2\ u_3$	$u_2\ u_3\ u_4$
u_4				u_4

Whence

$\mu_1(u_1) = \mu_1(u_3) = 0.333$

$\mu_1(u_2) = 0.344$, $\mu_1(u_4) = 0.405$.

from which we obtain ${}_1H$:

${}_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_3$	$u_1\ u_3$	$u_1\ u_3\ u_2$	H
u_2		$u_1\ u_3$	$u_1\ u_3\ u_2$	H
u_3			u_2	$u_2\ u_4$
u_4				u_4

Hence

$\mu_2(u_1) = \mu_2(u_3) = 0.36$,

$\mu_2(u_2) = 0.394$,

$\mu_2(u_4) = 0.429$.

By consequence ${}_2H = {}_1H$, then $\partial(1_1{}^4) = 1$.

($2_1{}^4$) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. So we have

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_4$

Whence

$\mu_1(u_1) = 0.405$

$\mu_1(u_2) = 0.344$, $\mu(u_3) = \mu(u_4) = 0.3$.

We obtain ${}_1H$:

${}_1H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	H	H
u_2		u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

So we have : $\mu_2(u_1) = 0.429$,

$\mu_2(u_2) = 0.394$, $\mu_2(u_3) = \mu_2(u_4) = 0.361$

whence one finds that

${}_2H = {}_1H$. It follows $\partial(2_1{}^4) = 1$.

($3_1{}^4$) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. So

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	$u_1\ u_3$	$u_1\ u_4$
u_2		u_2	$u_2\ u_3$	$u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

Then $\forall$ i, $\mu_1(u_i) = 0.571$.

By consequence ${}_1H = T$ and

therefore $\partial(3_1{}^4) = 1$.

(4_1^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_1\}$, $\Gamma(u_3) = \{d_2\}$, $\Gamma(u_4) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			$u_1\ u_3$	$u_1\ u_3\ u_4$
u_4				$u_1\ u_4$

So $\mu_1(u_1) = 0.328$,

$\mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.299$.

Hence

$_1H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\mu_2(u_1) = 0.357$

$\mu_2(u_2) = \mu_2(u_3) = \mu_2(u_4) = 0.3$

from which $_2H = {}_1H$. Therefore $\partial(4_1^4) = 1$.

(5_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

We have
$\mu_1(u_1) = \mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.333$.

So $_1H = T$, whence $\partial(5_1^4) = 1$.

(6_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

We have clearly

$\forall$ i, $\mu_1(u_2) = \mu_1(u_1)$.

Therefore $_1H = T$ and by consequence $\partial(6_1^4) = 1$.

(7_1^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_4\}$, $\Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

See (5_1^4) and (6_1^4).

We have $_1H = T$,

whence $\partial(7_1^{4)} = 1$.

(6_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\partial(6_0^4) = 0$.

(5_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\mu(1) = 0.357$,

$\mu(2) = \mu(3) = \mu(4) = 0.3$.

It follows $\partial(5_0^4) = 0$.

(1_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_2, d_3\}$, $\Gamma(u_4) = \{d_3, d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_4\ u_2\ u_3$

One obtains $\mu_1(u_1) = 0.256 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260$,

whence we obtain $_1H$:

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Therefore $_2H$ = T (the total hypergroup). Then $\partial(1_2^4) = 2$.

(2_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. Then

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	H	H
u_2		$u_1\ u_2\ u_3$	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_3\ u_4$

whence

$\mu_1(u_1) = 0.292 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.3$.

So, we have

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Then $\mu_1(u_1) = \mu_1(u_4) = \mu_1(u_2) = \mu_1(u_3)$,

whence $_2H$ = T, and by consequence

$\partial(2_2^4) = 2$.

(3_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_2, d_4\}$, $\Gamma(u_4) = \{d_3\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		H	H	H
u_3			$u_1\ u_2\ u_3$	H
u_4				$u_1\ u_2\ u_4$

We have $\mu_1(u_1) = \mu_1(u_2) = 0.260$, $\mu_1(u_3) = \mu_1(u_4) = 0.256$, whence we obtain

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Then $_2H = T$, from which $\partial(3_2^4) = 2$.

(4_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{ d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have:

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	$u_1\ u_2\ u_3$	H
u_2		$u_1\ u_2\ u_4$	H	$u_1\ u_2\ u_4$
u_3			$u_1\ u_3$	H
u_4				$u_2\ u_4$

Then $\mu_1(u_1) = 0.3 = \mu_1(u_2)$,

$\mu_1(u_3) = \mu_1(u_4) = 0.292$.

It follows

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore $_2H = T$, whence $\partial(4_2^4) = 2$.

(5_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_2\ u_3\ u_4$

So $\mu_1(u_1) = \mu_1(u_4) = 0.23$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260$.

By consequence, we obtain

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Therefore we have $_2H = T$, whence $\partial(5_2^4) = 2$.

(6_2^4) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_2, d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\,u_2$	$u_1\,u_2\,u_3$	H	H
u_2		$u_1\,u_2\,u_3$	H	H
u_3			$u_2\,u_3\,u_4$	$u_2\,u_3\,u_4$
u_4				$u_3\,u_4$

Hence $\mu_1(u_1) = 0.292 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.3$.

We obtain

${}_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\,u_4$	$u_1\,u_4$	H	H
u_4		$u_1\,u_4$	H	H
u_2			$u_2\,u_3$	$u_2\,u_3$
u_3				$u_2\,u_3$

whence $\forall$ i, $\mu_1(u_i) = \mu_1(u_1)$.

Then ${}_2H = T$. Therefore $\partial(6_2^4) = 2$.

(7_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$.

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\,u_2\,u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_2\,u_3\,u_4$

We have $\mu_1(u_1) = \mu_1(u_4) = 0.256$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260$.

So, we obtain ${}_1H$:

${}_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\,u_4$	$u_1\,u_4$	H	H
u_4		$u_1\,u_4$	H	H
u_2			$u_2\,u_3$	$u_2\,u_3$
u_3				$u_2\,u_3$

Then $\forall$ i, $\mu_2(u_i) = 0.389$, so ${}_2H = T$, and $\partial(7_2^4) = 2$.

(8_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

We obtain $\mu_1(u_1) = 0.389 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.476 = \mu_1(u_4)$.

By consequence,

${}_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore ${}_2H = T$ and $\partial(8_2^4) = 2$.

(9_2^4) Set $\Gamma(u_1) = \Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

See (8_2^4).

Therefore $\partial(9_2^4) = 2$.

(10_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We obtain

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	$u_1\ u_3\ u_4$	$u_1\ u_3\ u_4$
u_2		u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

whence $\mu_1(u_1) = \mu_1(u_2) = 0.476$,

$\mu(u_3) = \mu(u_4) = 0.389$.

So, we obtain

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Hence $_2H = T$, from which $\partial(10_2^4) = 2$.

(11_2^4) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_1\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_4$	H	H	$u_1\ u_3\ u_4$
u_2		$u_2\ u_3$	$u_2\ u_3\ u_4$	H
u_3			$u_2\ u_3\ u_4$	H
u_4				$u_1\ u_3\ u_4$

Hence $\mu_1(u_1) = \mu_1(u_3) = 0.291$,

$\mu_1(u_3) = \mu_1(u_4) = 0.3$.

One obtains $_1H$ as follows:

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

From $_1H$, one finds $\mu_2(u_i) = \mu_2(u_j)$, $\forall(i, j)$.

Therefore $_2H = T$, so $\partial(11_2^4) = 2$.

(12_2^4) Let $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_1\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_4$	H	H	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H
u_3			$u_2\ u_3$	H
u_4				$u_1\ u_4$

Then $\mu_1(u_1) = 0.3$, $\mu_1(u_2) = 0.3$,

$\mu_1(u_3) = 0.2917$, $\mu_1(u_4) = 0.2917$.

By consequence,

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore we have $_2H$ = T (the total hypergroup) whence $\partial(12_2^4)$= 2.

(1_3^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_2\}$, $\Gamma(u_4) = \{d_3\}$.

We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	$u_1\ u_2\ u_3$	H
u_2		H	H	H
u_3			$u_1\ u_2\ u_3$	H
u_4				$u_2\ u_4$

$\mu_1(u_1) = 0.272 = \mu_1(u_3)$,

$\mu_1(u_2) = 0.286$, $\mu_1(u_4) = 0.271$,

whence

$_1H$	u_2	u_1	u_3	u_4
u_2	u_2	$u_2\ u_1\ u_3$	$u_2\ u_1\ u_3$	H
u_1		$u_1\ u_3$	$u_1\ u_3$	$u_1\ u_3\ u_4$
u_3			$u_1\ u_3$	$u_1\ u_3\ u_4$
u_4				u_4

So we have $\mu_2(u_2) = 0.405 = \mu_2(u_4)$,

$\mu_2(u_1) = \mu_2(u_3) = 0.34$.

We obtain

$_2H$	u_1	u_3	u_2	u_4
u_1	$u_1\ u_3$	$u_1\ u_3$	H	H
u_3		$u_1\ u_3$	H	H
u_2			$u_2\ u_4$	$u_2\ u_4$
u_4				$u_2\ u_4$

We have clearly

$\mu_3(u_1) = \mu_3(u_3) = \mu_3(u_2) = \mu_3(u_4)$.

Therefore $_3H$ is the total hypergroup of order 4 and $\partial(1_3^4)$= 3.

(2_3^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_3, d_4\}$.

We obtain the following

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	H	H	H
u_2		H	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

We have $\mu_1(u_1) = 0.27083$,

$\mu_1(u_2) = 0.286$, $\mu_1(u_3) = \mu_1(u_4) = 0.272$.

Therefore the second hypergroupoid is

$_1H$	u_2	u_3	u_4	u_1
u_2	u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	H
u_3		$u_3\ u_4$	$u_3\ u_4$	$u_3\ u_4\ u_1$
u_4			$u_3\ u_4$	$u_3\ u_4\ u_1$
u_1				u_1

Hence $\mu_2(u_2) = \mu_2(u_1) = 0.405$,

$\mu_2(u_3) = \mu_2(u_4) = 0.369$.

By consequence we have again

$_2H$	u_2	u_1	u_3	u_4
u_2	$u_2\ u_1$	H	H	H
u_1		$u_2\ u_1$	H	H
u_3			$u_3\ u_4$	H
u_4				$u_3\ u_4$

From $_2H$ we obtain $_3H = T$. Then $\partial(2_3^4) = 3$.

(3_3^4) If $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	H	H	H
u_2		H	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

One finds the same sequence as in (2_3^4).

Therefore $\partial(3_3^4) = 3$

(4_3^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H	H
u_2		$u_1\ u_2\ u_3$	H	H
u_3			H	H
u_4				$u_3\ u_4$

So $\mu_1(u_1) = 0.272 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.286$, $\mu_1(u_4) = 0.271$.

Hence

$_1H$	u_3	u_1	u_2	u_4
u_3	u_3	$u_3\ u_1\ u_2$	$u_3\ u_1\ u_2$	H
u_1		$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_4$
u_2			$u_1\ u_2$	$u_1\ u_2\ u_4$
u_4				u_4

Then $\mu_2(u_3) = \mu_2(u_4) = 0.405$,

$\mu_2(u_1) = \mu_2(u_2) = 0.369$,

from which we obtain

$_2H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Hence $_3H = T$ and $\partial(4_3^4) = 3$.

\$ 4. Set $H = \{u_1, u_2, u_3, u_4, u_5\}$. Then there are functions $\Gamma : H \to P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 1, 2.

(1_1^5) Let $|H| = 5 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_5$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_5$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$
u_4				$u_3\ u_4$	$u_3\ u_4\ u_5$
u_5					u_5

So $\mu_1(u_1) = 0.29167 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.2936$, $\mu_1(u_5) = 0.370$.

We obtain

$_1H$	u_5	u_2	u_3	u_1	u_4
u_5	u_5	u_5 u_2 u_3	u_5 u_2 u_3	H	H
u_2		u_2 u_3	u_2 u_3	u_2 u_3 u_1 u_4	u_2 u_3 u_1 u_4
u_3			u_2 u_3	u_2 u_3 u_1 u_4	u_2 u_3 u_1 u_4
u_1				u_1 u_4	u_1 u_4
u_4					u_1 u_4

From this we have $\mu_2(u_5) = 0.348$,

$\mu_2(u_2) = \mu_2(u_3) = 0.3067$,

$\mu_2(u_1) = \mu_2(u_4) = 0.3$.

So $\mu_2(u_5) > \mu_2(u_2) = \mu(u_3) > \mu(u_1) = \mu(u_4)$.

It follows that $_2H = {}_1H$. Therefore $\partial(1_1^5) = 1$.

(2_2^5) Set $|H| = 5 = |D|$, $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$. So we have

$_0H$	u_1	u_2	u_3	u_4	u_5
u_1	u_1 u_2 u_3	u_1 u_2 u_3 u_4	H	u_1 u_2 u_3 u_4	u_1 u_2 u_3 u_5
u_2		u_1 u_2 u_3 u_4	H	u_1 u_2 u_3 u_4	H
u_3			H	H	H
u_4				u_2 u_3 u_4	u_2 u_3 u_4 u_5
u_5					u_3 u_5

Whence we obtain $\mu_1(u_1) = 0.228$,

$\mu_1(u_2) = 0.234$, $\mu_1(u_3) = 0.2447$,

$\mu_1(u_4) = \mu_1(u_1)$

$\mu_1(u_5) = 0.231$.

$_1H$	u_3	u_2	u_5	u_1	u_4
u_3	u_3	u_3 u_2	u_3 u_2 u_5	H	H
u_2		u_2	u_2 u_5	u_2 u_5 u_1 u_4	u_2 u_5 u_1 u_4
u_5			u_5	u_5 u_1 u_4	u_5 u_1 u_4
u_1				u_1 u_4	u_1 u_4
u_4					u_1 u_4

We have $\mu_2(u_3) = 0.3852$,

$\mu_2(u_2) = 0.3644$,

$\mu_2(u_5) = 0.3412$,

$\mu_2(u_1) = \mu_2(u_4) = 0.3208$.

So $_2H = {}_1H$ and by consequence $\partial(2_2^5) = 1$.

$ 5. Set $H = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then there are functions $\Gamma : H \to P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 1, 2, 3.

(1_1^6) Set $|H| = 6 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \Gamma(u_6) = \{ d_5, d_6\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	H	$u_1\ u_2\ u_4\ u_5\ u_6$	$u_1\ u_2\ u_4\ u_5\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	H	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_4\ u_5\ u_6$	$u_4\ u_5\ u_6$
u_6						$u_4\ u_5\ u_6$

So $\mu_1(u_1) = 0.231667$, $\mu_1(u_2) = 0.2284$, $\mu_1(u_3) = 0.22654$, $\mu_1(u_4) = 0.22656$, $\mu_1(u_5) = \mu_1(u_6) = 0.219$.

Hence we obtain

$_1H$	u_1	u_2	u_4	u_3	u_5	u_6
u_1	u_1	$u_1\ u_2$	$u_1\ u_2\ u_4$	$u_1\ u_2\ u_3\ u_4$	H	H
u_2		u_2	$u_2\ u_4$	$u_2\ u_4\ u_3$	$u_2\ u_4\ u_3\ u_5\ u_6$	$u_2\ u_4\ u_3\ u_5\ u_6$
u_4			u_4	$u_4\ u_3$	$u_4\ u_3\ u_5\ u_6$	$u_4\ u_3\ u_5\ u_6$
u_3				u_3	$u_3\ u_5\ u_6$	$u_3\ u_5\ u_6$
u_5					$u_5\ u_6$	$u_5\ u_6$
u_6						$u_5\ u_6$

Therefore $\mu_2(u_1) = 0.348$, $\mu_2(u_2) = 0.3315$, $\mu_2(u_4) = 0.317$, $\mu_2(u_3) = 0.303$, $\mu_2(u_5) = \mu_2(u_6) = 0.29$.

By consequence $_2H = {}_1H$, hence $\partial(1_1^6) = 1$.

(2_1^6) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_5, d_6\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	u_1	$u_1\ u_2\ u_3\ u_4$	H	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_3\ u_5\ u_6$	$u_1\ u_3\ u_5\ u_6$
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_3			$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_5					$u_3\ u_5\ u_6$	$u_3\ u_5\ u_6$
u_6						$u_3\ u_5\ u_6$

We obtain

$\mu_1(u_1) = 0.303$

$\mu_1(u_2) = 0.22469$

$\mu_1(u_3) = 0.24$

$\mu_1(u_4) = \mu_1(u_2) =$

$\mu_1(u_5) = \mu_1(u_6)$.

Setting $\{u_2, u_4, u_5, u_6\} = P$, we have

$_1H$	u_1	u_3	u_2	u_4	u_5	u_6
u_1	u_1	$u_1\ u_3$	H	H	H	H
u_3		u_3	H	H	H	H
u_2			P	P	P	P
u_4				P	P	P
u_5					P	P
u_6						P

One finds $_2H = {}_1H$. So $\partial(2_1^{\,6}) = 1$.

($1_2^{\,6}$) Set $|H| = 6 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. So we have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_4\ u_5$	$u_1\ u_2\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_6$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_6$
u_4				$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_4\ u_5$	$u_4\ u_5\ u_6$
u_6						u_6

$\mu_1(u_1) = 0.268$,

$\mu_1(u_2) = 0.267$,

$\mu_1(u_3) = 0.260$,

$\mu_1(u_4) = \mu_1(u_2)$,

$\mu_1(u_5) = \mu_1(u_1)$,

$\mu_1(u_6) = 0.348$.

Therefore we obtain

${}_1H$	u_6	u_1	u_5	u_2	u_4	u_3
u_6	u_6	$u_1\ u_5\ u_6$	$u_1\ u_5\ u_6$	$u_1\ u_5\ u_6\ u_2\ u_4$	$u_1\ u_5\ u_6\ u_2\ u_4$	H
u_1		$u_1\ u_5$	$u_1\ u_5$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4\ u_3$
u_5			$u_1\ u_5$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4\ u_3$
u_2				$u_2\ u_4$	$u_2\ u_4$	$u_2\ u_4\ u_3$
u_4					$u_2\ u_4$	$u_2\ u_4\ u_3$
u_3						u_3

So $\mu_2(u_6) = \mu_2(u_3) = 0.315$

$\mu_2(u_1) = \mu_2(u_5) = \mu_2(u_2) = \mu_2(u_4) = 0.279$.

Setting $\Gamma = \{u_1, u_5, u_2, u_4\}$, $Q = \{u_6, u_3\}$ we have

${}_2H$	u_1	u_5	u_2	u_4	u_6	u_3
u_1	P	P	P	P	H	H
u_5		P	P	P	H	H
u_2			P	P	H	H
u_4				P	H	H
u_6					Q	Q
u_3						Q

So we have

$\mu_1(u_1) = \mu_1(u_5) = \mu_1(u_2) = \mu_1(u_4) = 0.208$

$\mu_1(u_6) = \mu_1(u_3) = 0.233$

It follows ${}_3H = {}_2H$, by consequence $\partial(1_2^6) = 2$.

(2_2^6) Set $|D| = 6 = |H|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$, $\Gamma(u_6) = \{d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_4\ u_5\ u_6$	$u_1\ u_2\ u_5\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_5\ u_6$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_4\ u_5\ u_6$	$u_4\ u_5\ u_6$
u_6						$u_5\ u_6$

So we obtain

$\mu_1(u_1) = 0.2467 = \mu_1(u_6)$,

$\mu_1(u_2) = 0.243 = \mu_1(u_5)$,

$\mu_1(u_3) = \mu_1(u_4) = 0.2407$

Whence

$_1H$	u_1	u_6	u_2	u_5	u_3	u_4
u_1	$u_1\ u_6$	$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$	H	H
u_6		$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$	H	H
u_2			$u_2\ u_5$	$u_2\ u_5$	$u_2\ u_5$ $u_3\ u_4$	$u_2\ u_5$ $u_3\ u_4$
u_5				$u_2\ u_5$	$u_2\ u_5$ $u_3\ u_4$	$u_2\ u_5$ $u_3\ u_4$
u_3					$u_3\ u_4$	$u_3\ u_4$
u_4						$u_3\ u_4$

Hence
$\mu_2(u_1) = \mu_2(u_6) = \mu_2(u_3) = \mu_2(u_4) = 0.2667$,
$\mu_2(u_2) = \mu_2(u_5) = 0.2619$.

Therefore we set P = $\{u_1, u_6, u_3, u_4\}$. We obtain

$_2H$	u_1	u_6	u_3	u_4	u_2	u_5
u_1	P	P	P	P	H	H
u_6		P	P	P	H	H
u_3			P	P	H	H
u_4				P	H	H
u_2					$u_2\ u_5$	$u_2\ u_5$
u_5						$u_2\ u_5$

We have clearly $_3H = {_2H}$, whence $\partial(2_2^6) = 2$.

(3_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_4, d_5\}$, $\Gamma(u_4) = \{d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$, $\Gamma(u_6) = \{d_6\}$.

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2$ u_3	$u_1\ u_2$ $u_3\ u_4\ u_5$	$u_1\ u_2$ $u_3\ u_4\ u_5$	H	$u_1\ u_2$ $u_5\ u_6$
u_2		$u_1\ u_2$ u_3	$u_1\ u_2$ $u_3\ u_4\ u_5$	$u_1\ u_2$ $u_3\ u_4\ u_5$	H	$u_1\ u_2$ $u_3\ u_5\ u_6$
u_3			$u_2\ u_3$ $u_4\ u_5$	$u_2\ u_3$ $u_4\ u_5$	$u_2\ u_3$ $u_4\ u_5\ u_6$	$u_2\ u_3$ $u_4\ u_5\ u_6$
u_4				$u_3\ u_4$ u_5	$u_3\ u_4$ $u_5\ u_6$	$u_3\ u_4$ $u_5\ u_6$
u_5					$u_3\ u_4$ $u_5\ u_6$	$u_3\ u_4$ $u_5\ u_6$
u_6						$u_5\ u_6$

So we have
$\mu_1(u_1) = 0.233$,
$\mu_1(u_2) = 0.230$,
$\mu_1(u_3) = 0.228$,
$\mu_1(u_4) = 0.218$,
$\mu_1(u_5) = \mu_1(u_3) = 0.228$,
$\mu_1(u_6) = 0.231$.

We obtain

$_1H$	u_1	u_6	u_2	u_3	u_5	u_4
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_2$	$u_1\ u_6\ u_2\ u_3\ u_5$	$u_1\ u_6\ u_2\ u_3\ u_5$	H
u_6		u_6	$u_6\ u_2$	$u_6\ u_2\ u_3\ u_5$	$u_6\ u_2\ u_3\ u_5$	$u_6\ u_2\ u_3\ u_5\ u_4$
u_2			u_2	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5\ u_4$
u_3				$u_3\ u_5$	$u_3\ u_5$	$u_3\ u_5\ u_4$
u_5					$u_3\ u_5$	$u_3\ u_5\ u_4$
u_4						u_4

We have

$\mu_2(u_1) = 0.345 > \mu_2(u_6) = 0.326 > \mu_2(u_4) = 0.324 > \mu_2(u_2) = 0.306 > \mu_2(u_3) = \mu_2(u_5) = 0.296.$

Therefore we have

$_2H$	u_1	u_6	u_4	u_2	u_3	u_5
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_4$	$u_1\ u_6\ u_4\ u_2$	H	H
u_6		u_6	$u_6\ u_4$	$u_6\ u_4\ u_2$	$u_6\ u_4\ u_2\ u_3\ u_5$	$u_6\ u_4\ u_2\ u_3\ u_5$
u_4			u_4	$u_4\ u_2$	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$
u_2				u_2	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$
u_3					$u_3\ u_5$	$u_3\ u_5$
u_5						$u_3\ u_5$

We obtain now

$\mu_3(u_1) = 0.348 > \mu_3(u_6) = 0.33158 > \mu_3(u_4) = 0.317 > \mu_3(u_2) = 0.303 > \mu_3(u_3) = \mu_3(u_5) = 0.29.$

Therefore $_3H = {}_2H$ and it follows that $\partial(3_2^6) = 2$.

(4_2^6) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_2\}$, $\Gamma(u_5) = \{d_3\}$, $\Gamma(u_6) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_2		$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_5$	H
u_3			$u_2\, u_3\, u_5\, u_6$	H	$u_2\, u_3\, u_5\, u_6$	$u_2\, u_3\, u_5\, u_6$
u_4				$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_5					$u_2\, u_3\, u_5$	$u_2\, u_3\, u_5\, u_6$
u_6						$u_3\, u_6$

We find :

$\mu_1(u_1) = 0.210 = \mu_1(u_4)$

$\mu_1(u_2) = 0.221$

$\mu_1(u_3) = 0.216$

$\mu_1(u_5) = 0.208$

$\mu_1(u_6) = 0.219$.

Hence we obtain

${}_1H$	u_5	u_4	u_1	u_3	u_6	u_2
u_5	u_5	$u_5\, u_4\, u_1$	$u_5\, u_4\, u_1$	$u_5\, u_4\, u_1\, u_3$	$u_5\, u_4\, u_1\, u_3\, u_6$	H
u_4		$u_4\, u_1$	$u_4\, u_1$	$u_3\, u_1\, u_4$	$u_4\, u_1\, u_3\, u_6$	$u_4\, u_1\, u_3\, u_6\, u_2$
u_1			$u_4\, u_1$	$u_3\, u_1\, u_4$	$u_4\, u_1\, u_3\, u_6$	$u_4\, u_1\, u_3\, u_6\, u_2$
u_3				u_3	$u_3\, u_6$	$u_3\, u_6\, u_2$
u_6					u_6	$u_6\, u_2$
u_2						u_2

We have :

$\mu_2(u_5) = 0.324$

$\mu_2(u_2) = 0.345$

$\mu_2(u_6) = 0.326$

$\mu_2(u_3) = 0.3058$

$\mu_2(u_1) = \mu_2(u_4) = 0.296$.

Therefore $\mu_2(u_2) > \mu_2(u_6) > \mu_2(u_5) > \mu_2(u_3) > \mu_2(u_1) = \mu_2(u_4)$.

Therefore we have

${}_2H$	u_2	u_6	u_5	u_3	u_1	u_4
u_2	u_2	$u_2\, u_6$	$u_2\, u_6\, u_5$	$u_2\, u_6\, u_5\, u_3$	H	H
u_6		u_6	$u_6\, u_5$	$u_6\, u_5\, u_3$	$u_6\, u_5\, u_3\, u_1\, u_4$	$u_6\, u_5\, u_3\, u_1\, u_4$
u_5			u_5	$u_5\, u_3$	$u_5\, u_3\, u_1\, u_4$	$u_5\, u_3\, u_1\, u_4$
u_3				u_3	$u_3\, u_1\, u_4$	$u_3\, u_1\, u_4$
u_1					$u_1\, u_4$	$u_1\, u_4$
u_4						$u_1\, u_4$

We can see that ${}_3H = {}_2H$ and it follows that $\partial(4_2^6) = 2$.

(5_2^6) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$ $\Gamma(u_6) = \{d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	u_1	u_1 u_2 u_3	u_1 u_2 u_3 u_4	u_1 u_3 u_4 u_5	u_1 u_4 u_5 u_6	u_1 u_5 u_6
u_2		u_2 u_3	u_2 u_3 u_4	u_2 u_3 u_4 u_5	u_2 u_3 u_4 u_5 u_6	u_2 u_3 u_5 u_6
u_3			u_2 u_3 u_4	u_2 u_3 u_4 u_5	u_2 u_3 u_4 u_5 u_6	u_2 u_3 u_4 u_5 u_6
u_4				u_3 u_4 u_5	u_3 u_4 u_5 u_6	u_3 u_4 u_5 u_6
u_5					u_4 u_5 u_6	u_4 u_5 u_6
u_6						u_5 u_6

We obtain $\mu_1(u_1) = 0.348$, $\mu_1(u_2) = 0.268 = \mu_1(u_6)$, $\mu_1(u_3) = 0.2667 = \mu_1(u_5)$, $\mu_1(u_4) = 0.260$.

So, we have

${}_1H$	u_1	u_2	u_6	u_3	u_5	u_4
u_1	u_1	u_1 u_2 u_6	u_1 u_2 u_6	u_1 u_2 u_6 u_3 u_5	u_1 u_2 u_6 u_3 u_5	H
u_2		u_2 u_6	u_2 u_6	u_2 u_6 u_3 u_5	u_2 u_6 u_3 u_5	u_2 u_6 u_5 u_4 u_3
u_6			u_2 u_6	u_2 u_6 u_3 u_5	u_2 u_6 u_3 u_5	u_2 u_6 u_5 u_4 u_3
u_3				u_3 u_5	u_3 u_5	u_5 u_4 u_3
u_5					u_3 u_5	u_5 u_4 u_3
u_4						u_4

Now we obtain $\mu_2(u_1) = 0.324$ $\mu_2(u_4) = 0.315$, $\mu_2(u_5) = \mu_2(u_3) = \mu_2(u_2) = \mu_2(u_6) = 0.279$.

Setting $P = \{ u_2, u_6, u_3, u_5 \}$, we find ${}_2H$

${}_2H$	u_1	u_4	u_2	u_6	u_3	u_5
u_1	u_1 u_4	u_1 u_4	H	H	H	H
u_4		u_1 u_4	H	H	H	H
u_2			P	P	P	P
u_6				P	P	P
u_3					P	P
u_5						P

We have clearly ${}_3H = {}_2H$ whence $\partial(5_2^6) = 2$.

(6_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_4, d_5\}$, $\Gamma(u_4) = \{d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. So, denoting $\{u_i, u_{i+1},\ldots, u_{j-1}, u_j\}$ by u_i^j, we have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\,u_2$	u_1^3	u_1^5	H	u_1^5	$u_1\,u_2\,u_4\,u_6$
u_2		u_1^3	u_1^5	H	u_1^5	$u_1^4\,u_6$
u_3			u_2^5	u_2^6	u_2^5	u_2^6
u_4				u_3^6	u_3^6	u_3^6
u_5					u_3^5	u_3^6
u_6						$u_4\,u_6$

We have $\mu_1(u_1) = 0.233$,

$\mu_1(u_2) = 0.2302$,

$\mu_1(u_3) = \mu_1(u_4) = 0.228$,

$\mu_1(u_5) = 0.218$, $\mu_1(u_6) = 0.2308$.

Hence

${}_1H$	u_1	u_6	u_2	u_3	u_4	u_5
u_1	u_1	$u_1\,u_6$	$u_1\,u_6\,u_2$	$u_1\,u_6\,u_2\,u_3\,u_4$	$u_1\,u_6\,u_2\,u_3\,u_4$	H
u_6		u_6	$u_6\,u_2$	$u_6\,u_2\,u_3\,u_4$	$u_6\,u_2\,u_3\,u_4$	$u_6\,u_2\,u_3\,u_4\,u_5$
u_2			u_2	$u_2\,u_3\,u_4$	$u_2\,u_3\,u_4$	$u_2\,u_5\,u_3\,u_4$
u_3				$u_3\,u_4$	$u_3\,u_4$	$u_3\,u_4\,u_5$
u_4					$u_3\,u_4$	$u_3\,u_4\,u_5$
u_5						u_5

We have $\mu_2(u_1) = 0.345454$,

$\mu_2(u_6) = 0.326316$,

$\mu_2(u_2) = 0.305797$,

$\mu_2(u_3) = \mu_2(u_4) = 0.29615$,

$\mu_2(u_5) = 0.32424$.

From this, we have ${}_2H$ as follows

${}_2H$	u_1	u_6	u_5	u_2	u_3	u_4
u_1	u_1	$u_1\,u_6$	$u_1\,u_6\,u_5$	$u_1\,u_6\,u_5\,u_2$	H	H
u_6		u_6	$u_6\,u_5$	$u_6\,u_5\,u_2$	$u_6\,u_5\,u_2\,u_3\,u_4$	$u_6\,u_5\,u_2\,u_3\,u_4$
u_5			u_5	$u_5\,u_2$	$u_5\,u_2\,u_3\,u_4$	$u_5\,u_2\,u_3\,u_4$
u_2				u_2	$u_2\,u_3\,u_4$	$u_2\,u_3\,u_4$
u_3					$u_3\,u_4$	$u_3\,u_4$
u_4						$u_3\,u_4$

One can see that ${}_3H = {}_2H$, therefore $\partial(6_2^6) = 2$.

(7_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_3$	$u_1\ u_2$ $u_3\ u_4$	$u_1\ u_2\ u_3$ $u_4\ u_5$	H	$u_1\ u_3$ $u_4\ u_5$	$u_1\ u_3$ $u_4\ u_6$
u_2		$u_2\ u_3$ u_4	$u_1\ u_2$ $u_3\ u_4\ u_5$	$u_2\ u_3$ $u_4\ u_5\ u_6$	$u_2\ u_3$ $u_4\ u_5$	$u_2\ u_3$ $u_4\ u_6$
u_3			$u_1\ u_2$ $u_3\ u_4\ u_5$	H	$u_1\ u_2$ $u_3\ u_4\ u_5$	H
u_4				$u_2\ u_3$ $u_4\ u_5\ u_6$	$u_2\ u_3$ $u_4\ u_5\ u_6$	$u_2\ u_3$ $u_4\ u_5\ u_6$
u_5					$u_3\ u_4$ u_5	$u_3\ u_4$ $u_5\ u_6$
u_6						$u_4\ u_6$

We have $\mu_1(u_1) = 0.22$,
$\mu_1(u_2) = 0.20864$,
$\mu_1(u_3) = 0.22762$,
$\mu_1(u_4) = \mu_1(u_3)$
$\mu_1(u_5) = \mu_1(u_2) = 0.20864$,
$\mu_1(u_6) = \mu_1(u_1)= 0.22$.
Hence,

$\mu_1(u_3) = \mu_1(u_4) = 0.22762 > \mu_1(u_1) = \mu_1(u_6) = 0.22 > \mu_1(u_2) = \mu_1(u_5) = 0.20864$. We obtain

$_1H$	u_3	u_4	u_1	u_6	u_2	u_5
u_3	$u_3\ u_4$	$u_3\ u_4$	$u_3\ u_4$ $u_1\ u_6$	$u_3\ u_4$ $u_1\ u_6$	H	H
u_4		$u_3\ u_4$	$u_3\ u_4$ $u_1\ u_6$	$u_3\ u_4$ $u_1\ u_6$	H	H
u_1			$u_1\ u_6$	$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$
u_6				$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$
u_2					$u_2\ u_5$	$u_2\ u_5$
u_5						$u_2\ u_5$

We have
$\mu_2(u_3) = \mu_2(u_4) = 0.2667$,
$\mu_2(u_1) = 0.2619 = \mu_2(u_6)$,
$\mu_2(u_2) = \mu_2(u_5) = \mu_2(u_3) =$
$\mu_2(u_4) = 0.2667$.

Set $P = \{u_3, u_4, u_2, u_5\}$, $Q= \{u_1, u_6\}$. We obtain

$_2H$	u_3	u_4	u_2	u_5	u_1	u_6
u_3	P	P	P	P	H	H
u_4		P	P	P	H	H
u_2			P	P	H	H
u_5				P	H	H
u_1					Q	Q
u_6						Q

It follows that
$\mu_3(u_3) = \mu_3(u_4) = \mu_3(u_2) = \mu_3(u_5) =0.208$,
$\mu_3(u_1) = \mu_3(u_6) = 0.233$.

We have clearly $_3H = {_2H}$, so $\partial(7_2^6) = 2$.

(8_2^6) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$ $\Gamma(u_4) = \{d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_4\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_4\ u_6$
u_3			$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$
u_6						$u_4\ u_6$

We obtain $\mu_1(u_1) = 0.233$, $\mu_1(u_2) = 0.230247$, $\mu_1(u_3) = 0.228125$, $\mu_1(u_4) = \mu_1(u_3)$ $\mu_1(u_5) = 0.218518$, $\mu_1(u_6) = 0.230833$. From this, we have ${}_1H$.

${}_1H$	u_1	u_6	u_2	u_3	u_4	u_5
u_1	u_1	$u_1\ u_6$	$u_1\ u_2\ u_6$	$u_1\ u_6\ u_2\ u_3\ u_4$	$u_1\ u_6\ u_2\ u_3\ u_4$	H
u_6		u_6	$u_2\ u_6$	$u_2\ u_6\ u_3\ u_4$	$u_2\ u_6\ u_3\ u_4$	$u_2\ u_6\ u_3\ u_4\ u_5$
u_2			u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$
u_3				$u_3\ u_4$	$u_3\ u_4$	$u_3\ u_4\ u_5$
u_4					$u_3\ u_4$	$u_3\ u_4\ u_5$
u_5						u_5

From ${}_1H$ we obtain : $\mu_2(u_1) = 0.34545$, $\mu_2(u_6) = 0.3263$, $\mu_2(u_5) = 0.324242$, $\mu_2(u_2) = 0.305797$, $\mu_2(u_3) = \mu_2(u_4) = 0.296$.

Therefore we find ${}_2H$ as follows

${}_2H$	u_1	u_6	u_5	u_2	u_3	u_4
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_5$	$u_1\ u_6\ u_5\ u_2$	H	H
u_6		u_6	$u_6\ u_5$	$u_6\ u_5\ u_2$	$u_6\ u_5\ u_2\ u_3\ u_4$	$u_6\ u_5\ u_2\ u_3\ u_4$
u_5			u_5	$u_5\ u_2$	$u_5\ u_2\ u_3\ u_4$	$u_5\ u_2\ u_3\ u_4$
u_2				u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3					$u_3\ u_4$	$u_3\ u_4$
u_4						$u_3\ u_4$

From ${}_2H$ we obtain : $\mu_3(u_1) = 0.34848$, $\mu_3(u_6) = 0.331579$, $\mu_3(u_5) = 0.31739$ $\mu_3(u_2) = 0.302898$, $\mu_3(u_3) = \mu_3(u_4) = 0.29$

We have clearly ${}_3H = {}_2H$, by consequence $\partial(8_2^{\ 6}) = 2$.

(1_3^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$ $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\,u_2\,u_3$	$u_1\,u_2\,u_3\,u_4$	$u_1\,u_2\,u_3\,u_4\,u_5$	H	$u_1\,u_2\,u_3\,u_4\,u_5$	$u_1\,u_2\,u_3\,u_4\,u_6$
u_2		$u_1\,u_2\,u_3\,u_4$	$u_1\,u_2\,u_3\,u_4\,u_5$	H	$u_1\,u_2\,u_3\,u_4\,u_5$	$u_1\,u_2\,u_3\,u_4\,u_6$
u_3			$u_1\,u_2\,u_3\,u_4\,u_5$	H	$u_1\,u_2\,u_3\,u_4\,u_5$	H
u_4				$u_2\,u_3\,u_4\,u_5\,u_6$	$u_2\,u_3\,u_4\,u_5\,u_6$	$u_2\,u_3\,u_4\,u_5\,u_6$
u_5					$u_3\,u_4\,u_5$	$u_3\,u_4\,u_5\,u_6$
u_6						$u_4\,u_6$

We obtain $\mu_1(u_1) = 0.2006173$, $\mu_1(u_2) = 0.2005208$, $\mu_1(u_3) = 0.20714$, $\mu_1(u_4) = 0.211905$, $\mu_1(u_5) = 0.198765$, $\mu_1(u_6) = 0.206667$.

By consequence we have ${}_1H$.

${}_1H$	u_4	u_3	u_6	u_1	u_2	u_5
u_4	u_4	$u_4\,u_3$	$u_4\,u_3\,u_6$	$u_4\,u_3\,u_6\,u_1$	$u_4\,u_3\,u_6\,u_1\,u_2$	H
u_3		u_3	$u_3\,u_6$	$u_3\,u_6\,u_1$	$u_3\,u_6\,u_1\,u_2$	$u_3\,u_6\,u_1\,u_2\,u_5$
u_6			u_6	$u_6\,u_1$	$u_6\,u_1\,u_2$	$u_6\,u_1\,u_2\,u_5$
u_1				u_1	$u_1\,u_2$	$u_1\,u_2\,u_5$
u_2					u_2	$u_2\,u_5$
u_5						u_5

Hence we have

$\mu_2(u_4) = 0.354545 = \mu_2(u_5)$,

$\mu_2(u_3) = 0.34035 = \mu_2(u_2)$

$\mu_2(u_6) = 0.33188 = \mu_2(u_1)$

from which we obtain ${}_2H$.

${}_2H$	u_4	u_5	u_3	u_2	u_6	u_1
u_4	$u_4\,u_5$	$u_4\,u_5$	$u_4\,u_5\,u_3\,u_2$	$u_4\,u_5\,u_3\,u_2$	H	H
u_5		$u_4\,u_5$	$u_4\,u_5\,u_3\,u_2$	$u_4\,u_5\,u_3\,u_2$	H	H
u_3			$u_3\,u_2$	$u_3\,u_2$	$u_3\,u_2\,u_6\,u_1$	$u_3\,u_2\,u_6\,u_1$
u_2				$u_3\,u_2$	$u_3\,u_2\,u_6\,u_1$	$u_3\,u_2\,u_6\,u_1$
u_6					$u_6\,u_1$	$u_6\,u_1$
u_1						$u_6\,u_1$

From ${}_2H$ it follows

$\mu_3(u_4) = \mu_3(u_5) = \mu_3(u_6) = \mu_3(u_1)$ $= 0.26667$

$\mu_3(u_2) = \mu_3(u_3) = 0.26190$.

Set $P = \{u_4, u_5, u_6, u_1\}$, $Q = \{u_3, u_2\}$. Then we obtain ${}_3H$ as follows

${}_3H$	u_4	u_5	u_6	u_1	u_3	u_2
u_4	P	P	P	P	H	H
u_5		P	P	P	H	H
u_6			P	P	H	H
u_1				P	H	H
u_3					Q	Q
u_2						Q

From ${}_3H$, it follows that ${}_4H = {}_3H$ and we have finally $\partial(1_3^{\,6}) = 3$.

($2_3^{\,6}$) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\, u_2\, u_3$	$u_1\, u_2\, u_3\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_2		$u_1\, u_2\, u_3\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3, u_4, u_5$	$u_1, u_2\, u_3\, u_4\, u_6$
u_3			$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	H
u_4				$u_2\, u_3\, u_4\, u_5\, u_6$	$u_2\, u_3\, u_4\, u_5\, u_6$	$u_2\, u_3\, u_4, u_5\, u_6$
u_5					$u_3\, u_4\, u_5$	$u_3\, u_4\, u_5\, u_6$
u_6						$u_4\, u_6$

whence $\partial(1_3^{\,6}) = \partial(2_3^{\,6}) = 3$.

($3_3^{\,6}$) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$ $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\, u_2\, u_3$	$u_1\, u_2\, u_3\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_2		$u_1\, u_2\, u_3\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_3			$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	H
u_4				$u_2\, u_3\, u_4\, u_5\, u_6$	$u_2\, u_3\, u_4\, u_5\, u_6$	$u_2\, u_3\, u_4\, u_5\, u_6$
u_5					$u_3\, u_4\, u_5$	$u_3\, u_4\, u_5\, u_6$
u_6						$u_4\, u_6$

See ($1_3^{\,6}$).

We have $\partial(3_3^{\,6}) = \partial(1_3^{\,6}) = 3$.

\$ 6. (1_4^8) Set $H = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5, d_6, d_7\}$, $\Gamma(u_6) = \{d_7, d_8\}$, $\Gamma(u_7) = \{d_7\}$, $\Gamma(u_8) = \{d_8\}$.

So, denoting $\{u_i, u_{i+1},\ldots, u_{j-1}, u_j\}$ by u_i^j, we have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	u_1^3	u_1^4	u_1^5	u_1^5	u_1^7	$u_1^3\ u_5^8$	$u_1^3\ u_5^7$	$u_1^3\ u_6\ u_8$
u_2		u_1^4	u_1^5	u_1^5	u_1^7	H	u_1^7	$u_1^4\ u_6\ u_8$
u_3			u_1^5	u_1^5	u_1^7	H	u_1^7	$u_1^6\ u_8$
u_4				u_2^5	u_2^7	u_2^8	u_2^7	$u_2^6\ u_8$
u_5					u_3^7	u_3^8	u_3^7	u_3^8
u_6						u_5^8	u_5^8	u_5^8
u_7							u_5^7	u_5^8
u_8								$u_6\ u_8$

We have

$\mu_1(u_1) = 0.1756$,

$\mu_1(u_2) = 0.17470$**,**

$\mu_1(u_3) = 0.1754978$,

$\mu_1(u_4) = 0.1729$,

$\mu_1(u_5) = 0.1803$,

$\mu_1(u_6) = 0.1813$,

$\mu_1(u_7) = 0.175641 = \mu_1(u_1)$,

$\mu_1(u_8) = 0.19073$.

So $\mu_1(u_8) > \mu_1(u_6) > \mu_1(u_5) > \mu(u_7) = \mu(u_1) > \mu_1(u_3) > \mu_1(u_2) > \mu(u_4)$.

One obtains ${}_1H$ as follows

${}_1H$	u_4	u_2	u_3	u_1	u_7	u_5	u_6	u_8
u_4	u_4	$u_4\ u_2$	$u_4\ u_2\ u_3$	$u_4\ u_2\ u_3\ u_1\ u_7$	$u_4\ u_2\ u_3\ u_1\ u_7$	$u_4\ u_2\ u_3\ u_1\ u_7\ u_5$	$u_4\ u_2\ u_3\ u_1\ u_7\ u_5\ u_6$	H
u_2		u_2	$u_2\ u_3$	$u_2\ u_3\ u_1\ u_7$	$u_2\ u_3\ u_1\ u_7$	$u_2\ u_3\ u_1\ u_7\ u_5$	$u_2\ u_3\ u_1\ u_7\ u_5\ u_6$	$u_2\ u_3\ u_1\ u_7\ u_5\ u_6\ u_8$
u_3			u_3	$u_3\ u_1\ u_7$	$u_3\ u_1\ u_7$	$u_3\ u_1\ u_5\ u_7$	$u_3\ u_1\ u_7\ u_5\ u_6$	$u_3\ u_1\ u_7\ u_5\ u_6\ u_8$
u_1				$u_1\ u_7$	$u_1\ u_7$	$u_1\ u_7\ u_5$	$u_1\ u_7\ u_5\ u_6$	$u_1\ u_7\ u_5\ u_6\ u_8$
u_7					$u_1\ u_7$	$u_1\ u_7\ u_5$	$u_1\ u_7\ u_5\ u_6$	$u_1\ u_7\ u_5\ u_6\ u_8$
u_5						u_5	$u_5\ u_6$	$u_5\ u_6\ u_8$
u_6							u_6	$u_6\ u_8$
u_8								u_8

from which $\mu_2(u_4) = \mu_2(u_8) = 0.2890$, $\mu_2(u_2) = \mu_2(u_6) = 0.2724$,

$\mu_2(u_1) = \mu_2(u_7) = 0.247567$, $\mu_2(u_3) = \mu_2(u_5) = 0.2549$. So we obtain ${}_2H$.

$_2H$	u_4	u_8	u_2	u_6	u_3	u_5	u_1	u_7
u_4	$u_4\ u_8$	$u_4\ u_8$	$u_4\ u_8\ u_2\ u_6$	$u_4\ u_8\ u_2\ u_6$	$u_4\ u_8\ u_2\ u_6\ u_3\ u_5$	$u_4\ u_8\ u_2\ u_6\ u_3\ u_5$	H	H
u_8		$u_4\ u_8$	$u_4\ u_8\ u_2\ u_6$	$u_4\ u_8\ u_2\ u_6$	$u_4\ u_8\ u_2\ u_6\ u_3\ u_5$	$u_4\ u_8\ u_2\ u_6\ u_3\ u_5$	H	H
u_2			$u_2\ u_6$	$u_2\ u_6$	$u_2\ u_6\ u_3\ u_5$	$u_2\ u_6\ u_3\ u_5$	$u_2\ u_6\ u_3\ u_5\ u_1\ u_7$	$u_2\ u_6\ u_3\ u_5\ u_1\ u_7$
u_6				$u_2\ u_6$	$u_2\ u_6\ u_3\ u_5$	$u_2\ u_6\ u_3\ u_5$	$u_2\ u_6\ u_3\ u_5\ u_1\ u_7$	$u_2\ u_6\ u_3\ u_5\ u_1\ u_7$
u_3					$u_3\ u_5$	$u_3\ u_5$	$u_3\ u_5\ u_1\ u_7$	$u_3\ u_5\ u_1\ u_7$
u_5						$u_3\ u_5$	$u_3\ u_5\ u_1\ u_7$	$u_3\ u_5\ u_1\ u_7$
u_1							$u_1\ u_7$	$u_1\ u_7$
u_7								$u_1\ u_7$

Hence we have :

$\mu_3(u_4) = \mu_3(u_8) = \mu_3(u_1) = \mu_3(u_7) = 0.22619$, $\mu_3(u_2) = \mu_3(u_6) = \mu_3(u_3) = \mu_3(u_5) = 0.2197$.

Setting $P= \{u_4, u_8, u_1, u_7\}$, $Q = \{u_3, u_5, u_2, u_6\}$, we find $_3H$

$_3H$	u_4	u_8	u_1	u_7	u_2	u_6	u_3	u_5
u_4	P	P	P	P	H	H	T	H
u_8		P	P	P	H	H	H	H
u_1			P	P	H	H	H	H
u_7				P	H	H	H	H
u_2					Q	Q	Q	Q
u_6						Q	Q	Q
u_3							Q	Q
u_5								Q

From this, we obtain :
$\forall i,\ \mu_4(u_i) = 0.166$.

It follows $_4H = T$, whence $\partial(1_4^{\,8}) = 4$.

\$7. ($1_2^{\,9}$) Let $H=\{ u_i \mid 1 \le i \le 9\}$ and for $i \le 7$, set

$\Gamma(u_i) = \{ d_i, d_{i+1}, d_{i+2}\}$, $\Gamma(u_8) = \{d_8\}$, $\Gamma(u_9) = \{d_9\}$.

We obtain

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
u_1	u_1u_2 u_3	u_1 u_2 u_3 u_4	u_1 u_2 u_3 u_4 u_5	u_1 u_2 u_3 u_4 u_5 u_6	u_1 u_2 u_3 u_4 u_5 u_6 u_7	u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8	u_1 u_2 u_3 u_5 u_6 u_7 u_8 u_9	u_1 u_2 u_3 u_6 u_7 u_8	u_1 u_2 u_3 u_7 u_9
u_2		u_1 u_2 u_3 u_4	u_1 u_2 u_3 u_4 u_5	u_1 u_2 u_3 u_4 u_5 u_6	u_1 u_2 u_3 u_4 u_5 u_6 u_7	u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8	H	u_1 u_2 u_3 u_4 u_6 u_7 u_8	u_1 u_2 u_3 u_4 u_7 u_9
u_3			u_1 u_2 u_3 u_4 u_5	u_1 u_2 u_3 u_4 u_5 u_6	u_1u_2 u_3 u_4 u_5 u_6 u_7	u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8	H	u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8	u_1 u_2 u_3 u_4 u_5 u_7 u_9
u_4				u_2 u_3 u_4 u_5 u_6	u_2 u_3 u_4 u_5 u_6 u_7	u_2 u_3 u_4 u_5 u_6 u_7 u_8	u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9	u_2 u_3 u_4 u_5 u_6 u_7 u_8	u_2 u_3 u_4 u_5 u_6 u_7 u_9
u_5					u_3 u_4 u_5 u_6 u_7	u_3 u_4 u_5 u_6 u_7 u_8	u_3 u_4 u_5 u_6 u_7 u_8 u_9	u_3 u_4 u_5 u_6 u_7 u_8	u_3 u_4 u_5 u_6 u_7 u_9
u_6						u_4 u_5 u_6 u_7 u_8	u_4 u_5 u_6 u_7 u_8 u_9	u_4 u_5 u_6 u_7 u_8	u_4 u_5 u_6 u_7 u_8 u_9
u_7							u_5 u_6 u_7 u_8 u_9	u_5 u_6 u_7 u_8 u_9	u_5 u_6 u_7 u_8 u_9
u_8								u_6 u_7 u_8	u_6 u_7 u_8 u_9
u_9									u_7 u_9

From ${}_0H$ we have

$\mu_1(u_9) = 0.1729 > \mu_1(u_7) = 0.1648 > \mu_1(u_1) = 0.1616 > \mu_1(u_3) = 0.160 > \mu_1(u_6) = 0.1599$

$> \mu_1(u_8) = 0.1597 > \mu_1(u_2) = 0.159169 > \mu_1(u_5) = 0.157387 > \mu_1(u_4) = 0.159218.$

From these data, we obtain ${}_1H$ as follows

${}_1H$	u_9	u_7	u_1	u_3	u_6	u_8	u_4	u_2	u_5
u_9	u_9	u_9 u_7	u_9 u_7 u_1	u_9 u_7 u_1 u_3	u_9 u_7 u_1 u_3 u_6	u_9 u_7 u_1 u_3 u_6 u_8	u_9 u_7 u_1 u_3 u_6 u_8 u_4	u_9 u_7 u_1 u_3 u_6 u_8 u_2 u_4	H
u_7		u_7	u_7 u_1	u_7 u_1 u_3	u_7 u_1 u_3 u_6	u_7 u_1 u_3 u_6 u_8	u_7 u_1 u_3 u_6 u_8 u_4	u_7 u_1 u_3 u_6 u_8 u_2 u_4	u_7 u_1 u_3 u_6 u_8 u_2 u_5 u_4
u_1			u_1	u_1 u_3	u_1 u_3 u_6	u_1 u_3 u_6 u_8	u_1 u_3 u_6 u_8 u_4	u_1 u_3 u_6 u_8 u_2 u_4	u_1 u_3 u_6 u_8 u_2 u_5 u_4
u_3				u_3	u_3 u_6	u_3 u_6 u_8	u_3 u_6 u_8 u_4	u_3 u_6 u_8 u_2 u_4	u_3 u_6 u_8 u_2 u_5 u_4
u_6					u_6	u_6 u_8	u_6 u_8 u_4	u_6 u_8 u_2 u_4	u_6 u_8 u_2 u_5 u_4
u_8						u_8	u_8 u_4	u_8 u_2 u_4	u_8 u_2 u_5 u_4
u_4							u_4	u_2 u_4	u_2 u_5 u_4
u_2								u_2	u_5 u_2
u_5									u_5

From ${}_1H$ we obtain as follows, μ_2 and then ${}_2H$.

$\mu_2(u_9) = \mu_2(u_5) = 0.2740 > \mu_2(u_7) = \mu_2(u_2) = 0.261085 > \mu_2(u_1) = \mu_2(u_4) =$

$= 0.250716 > \mu_2(u_3) = \mu_2(u_8) = 0.24495 > \mu_2(u_6) = 0.243116.$

${}_2H$	u_9	u_5	u_7	u_2	u_1	u_4	u_3	u_8	u_6
u_9	u_5 u_9	u_5 u_9	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_3 u_8 u_1 u_4	u_5 u_9 u_7 u_3 u_8 u_2 u_1 u_4	H
u_5		u_5 u_9	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 $u_7 u_2$ u_1 u_4 u_3 u_8	u_5 u_9 u_7 u_2 u_1 u_4 $u_3 u_8$	H
u_7			u_7 u_2	u_7 u_2	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_6 u_1 u_4 u_3 u_8
u_2				u_7 u_2	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_6 u_1 u_4 u_3 u_8
u_1					u_1 u_4	u_1 u_4	u_1 u_4 u_3 u_8	u_1 u_4 u_3 u_8	u_1 u_4 u_6 u_3 u_8
u_4						u_1 u_4	u_1 u_4 u_3 u_8	u_1 u_4 u_3 u_8	u_1 u_4 u_6 u_3 u_8
u_3							u_3 u_8	u_3 u_8	u_3 u_8 u_6
u_8								u_3 u_8	u_3 u_8 u_6
u_6									u_6

From ${}_2H$ we obtain $\mu_3(u_9) = \mu_3(u_5) = 0.211805$, $\mu_3(u_7) = \mu_3(u_2) = 0.205433,$

$\mu_3(u_1) = \mu_3(u_4) = 0.20504,$ $\mu_3(u_3) = \mu_3(u_8) = 0.21155,$ $\mu_3(u_6) = 0.24407.$

Then we have ${}_3H$ as follows

${}_3H$	u_6	u_9	u_5	u_3	u_8	u_2	u_7	u_1	u_4
u_6	u_6	u_6 u_9 u_5	u_6 u_9 u_5	u_6 u_9 u_5 u_3 u_8	u_6 u_9 u_5 u_3 u_8	u_6 u_9 u_5 u_3 u_8 $u_2 u_7$	u_6 u_9 u_5 u_3 u_8 u_2 u_7	H	H
u_9		u_9 u_5	u_9 u_5	u_9 $u_5 u_3$ u_8	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 $u_3 u_8$ u_2 u_7 u_1 u_4	u_9 u_5 u_3 u_8 u_2 u_7 u_1 u_4
u_5			u_9 u_5	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7 u_1 u_4	u_9 u_5 $u_3 u_8$ u_2 u_7 u_1 u_4
u_3				u_3 u_8	u_3 u_8	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7 u_1 u_4	u_3 u_8 u_2 u_7 u_1 u_4
u_8					u_3 u_8	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7 u_1 u_4	u_3 u_8 u_2 u_7 u_1 u_4
u_2						u_2 u_7	u_2 u_7	u_2 u_7 u_1 u_4	u_2 u_7 u_1 u_4
u_7							u_2 u_7	u_2 u_7 u_1 u_4	u_2 u_7 u_1 u_4
u_1								u_1 u_4	u_1 u_4
u_4									u_1 u_4

It is possible to verify that the function $\varphi : {}_2H \to {}_3H$ defined as follows

$\varphi(u_3) = u_9$, $\varphi(u_8) = u_5$, $\varphi(u_1) = u_3$,

$\varphi(u_4) = u_8$, $\varphi(u_9) = u_1$, $\varphi(u_5) = u_4$, $\varphi(u_7) = u_7$,

$\varphi(u_2) = u_2$, $\varphi(u_6) = u_6$,

is a hypergroup isomorphism.

It follows that the fuzzy grade of $(1_2{}^9)$ is 2.

§ 8. $(1_5{}^{16})$ Set $H = \{u_i \mid 1 \le i \le 16\}$, $D = \{d_i \mid 1 \le i \le 16\}$, $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, and

$\forall\, i : i \le 13$, $\Gamma(u_i) = \{d_i, d_{i+1}, d_{i+2}\}$,

$\Gamma(u_{14}) = \{d_{15}, d_{16}\}$, $\Gamma(u_{15}) = \{d_{15}\}$, $\Gamma(u_{16}) = \{d_{16}\}$.

Since $\forall\, i$, we have $u_i \circ u_i = \{u_j \mid \Gamma(u_j) \cap \Gamma(u_i) \neq \varnothing\}$, it follows that we have

$u_1 \circ u_1 = \{ u_1, u_2, u_3\}$, $u_2 \circ u_2 = \{ u_1, u_2, u_3, u_4\}$,

$u_3 \circ u_3 = \{ u_1, u_2, u_3, u_4, u_5\}$,

$\forall\, i : 4 \le i \le 13$, $u_i \circ u_i = \{ u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$,

$u_{14} \circ u_{14} = \{ u_{13}, u_{14}, u_{15}, u_{16}\}$,

$u_{15} \circ u_{15} = \{ u_{13}, u_{14}, u_{15}\}$,

$u_{16} \circ u_{16} = \{ u_{14}, u_{16}\}$.

For ${}_0H$ we have the following table :

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
u_1	u_1^{3}	u_1^{4}	u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	$u_1^{3} u_5^{9}$	$u_1^{3} u_6^{10}$	$u_1^{3} u_7^{11}$	$u_1^{3} u_8^{12}$	$u_1^{3} u_9^{13}$	$u_1^{3} u_{10}^{14}$	$u_1^{3} u_{11}^{15}$	$u_1^{3} u_{13}^{16}$	$u_1^{3} u_{13}^{15}$	$u_1^{3} u_{14} u_{16}$
u_2		u_1^{4}	u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	u_1^{9}	$u_1^{4} u_6^{10}$	$u_1^{4} u_7^{11}$	$u_1^{4} u_8^{12}$	$u_1^{4}u_9^{13}$	$u_1^{4}u_{10}^{14}$	$u_1^{4} u_{11}^{15}$	$u_1^{4}u_{13}^{16}$	$u_1^{4}u_{13}^{15}$	$u_1^{4}u_{14} u_{16}$
u_3			u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	u_1^{9}	u_1^{10}	$u_1^{5} u_7^{11}$	$u_1^{5}u_8^{12}$	$u_1^{5}u_9^{13}$	$u_1^{5}u_{10}^{14}$	$u_1^{5}u_{11}^{15}$	$u_1^{5}u_{13}^{16}$	$u_1^{5}u_{13}^{15}$	$u_1^{5}u_{14} u_{16}$
u_4				u_2^{6}	u_2^{7}	u_2^{8}	u_2^{9}	u_2^{10}	u_2^{11}	$u_2^{6}u_8^{12}$	$u_2^{6}u_9^{13}$	$u_2^{6}u_{10}^{14}$	$u_2^{6}u_{11}^{15}$	$u_2^{6}u_{13}^{16}$	$u_2^{6}u_{13}^{15}$	$u_2^{6}u_{14} u_{16}$
u_5					u_3^{7}	u_3^{8}	u_3^{9}	u_3^{10}	u_3^{11}	u_3^{12}	$u_3^{7}u_9^{13}$	$u_3^{7}u_{10}^{14}$	$u_3^{7}u_{11}^{15}$	$u_3^{7}u_{13}^{16}$	$u_3^{7}u_{13}^{15}$	$u_3^{7}u_{14} u_{16}$
u_6						u_4^{8}	u_4^{9}	u_4^{10}	u_4^{11}	u_4^{12}	u_4^{13}	$u_4^{8}u_{10}^{14}$	$u_4^{8}u_{11}^{15}$	$u_4^{8}u_{13}^{16}$	$u_4^{8}u_{13}^{15}$	$u_4^{8}u_{14} u_{16}$
u_7							u_5^{9}	u_5^{10}	u_5^{11}	u_5^{12}	u_5^{13}	u_5^{14}	$u_5^{9}u_{11}^{15}$	$u_5^{9}u_{13}^{16}$	$u_5^{9}u_{13}^{15}$	$u_5^{9}u_{14} u_{16}$
u_8								u_6^{10}	u_6^{11}	u_6^{12}	u_6^{13}	u_6^{14}	u_6^{15}	$u_6^{10}u_{13}^{16}$	$u_6^{10}u_{13}^{15}$	$u_6^{10}u_{14} u_{16}$
u_9									u_7^{11}	u_7^{12}	u_7^{13}	u_7^{14}	u_7^{15}	$u_7^{11}u_{13}^{16}$	u_7^{15}	$u_7^{11}u_{14} u_{16}$
u_{10}										u_8^{12}	u_8^{13}	u_8^{14}	u_8^{15}	u_8^{16}	u_8^{15}	$u_8^{12}u_{14} u_{16}$
u_{11}											u_9^{13}	u_9^{14}	u_9^{15}	u_9^{16}	u_9^{15}	$u_9^{14}u_{16}$
u_{12}												u_{10}^{14}	u_{10}^{15}	u_{10}^{16}	u_{10}^{15}	$u_{10}^{14}u_{16}$
u_{13}													u_{11}^{15}	u_{11}^{16}	u_{11}^{15}	u_{11}^{16}
u_{14}														u_{13}^{16}	u_{13}^{16}	u_{13}^{16}
u_{15}															u_{13}^{15}	u_{13}^{16}
u_{16}																$u_{14} u_{16}$

From ${}_0H$ we obtain $\mu_1(u_{16}) = 0.15673$, $\mu_1(u_{15}) = 0.13992$, $\mu_1(u_{14}) = 0.141293$, $\mu_1(u_1) = 0.13867$, $\mu_1(u_2) = 0.134942$, $\mu_1(u_{13}) = 0.134215$, $\mu_1(u_3) = 0.132574$, $\mu_1(u_4) = 0.129700$, $\mu_1(u_5) = 0.128076$, $\mu_1(u_6) = 0.127554$, $\mu_1(u_7) = 0.126581$, $\mu_1(u_{12}) = 0.1283654$, $\mu_1(u_{11}) = 0.126441$, $\mu_1(u_{10}) = 0.126878$, $\mu_1(u_8) = 0.12671$, $\mu_1(u_9) = 0.126608$.

For ${}_1H$, set $v_1 = u_{16}$, $v_2 = u_{14}$, $v_3 = u_{15}$, $v_4 = u_1$, $v_5 = u_2$, $v_6 = u_{13}$, $v_7 = u_3$, $v_8 = u_4$, $v_9 = u_{12}$, $v_{10} = u_5$, $v_{11} = u_6$, $v_{12} = u_{10}$, $v_{13} = u_8$, $v_{14} = u_9$, $v_{15} = u_7$, $v_{16} = u_{11}$.

$\forall(i, j)$, such that $i \le j$ set $v_i^j = \{v_i, v_{i+1},...v_j\}$. So we have $v_i \, o_1 \, v_j = v_i^j$. For ${}_2H$ we have $v_1 \, o_2 \, v_1 = v_1 \, o_2 \, v_{16} = v_{16} \, o_2 \, v_{16} = \{v_1, v_{16}\}$, $v_2 \, o_2 \, v_2 = v_2 \, o_2 \, v_{15} = v_{15} \, o_2 \, v_{15} = \{v_2, v_{15}\}$. Generally, $v_i \, o_2 \, v_i = v_i \, o_2 \, v_{16-(i-1)} = v_{16-(i-1)} \, o_2 \, v_{16-(i-1)} = \{v_i, v_{16-(i-1)}\}$. For $i < j$, $v_i \, o_2 \, v_j = \bigcup_{i \le s \le j} v_s \, o_2 \, v_s$.

Set $P_1 = \{v_1, v_{16}, v_8, v_9\}$, $P_2 = \{v_2, v_{15}, v_7, v_{10}\}$, $P_3 = \{v_3, v_{14}, v_6, v_{11}\}$, $P_4 = \{v_4, v_{13}, v_5, v_{12}\}$.

Then for ${}_3H$, $\forall k$: $1 \le k \le 14$, we have $\forall(v_i, v_j) \in P_k \times P_k$, $v_i \, o_3 \, v_j = P_k$.

If $s < t$, $\forall(v_i, v_j) \in P_s \times P_t$, we have $v_i \, o_3 \, v_j = \bigcup_{s \le u \le t} P_u$. For ${}_4H$, setting

$Q_1 = P_1 \bigcup P_4$, $Q_2 = P_2 \bigcup P_3$, we have $\forall(v_i, v_j) \in Q_i \times Q_j$, $v_i \, o_4 \, v_j = Q_i \bigcup Q_j$.

By consequence, if $i \neq j$, $v_i \, o_4 \, v_j = H$ and $v_i \, o_4 \, v_i = Q_i$. Since $|Q_1| = |Q_2|$, we have $\forall \, v_i \in Q_1$, $\forall \, v_j \in Q_2$, $\mu_4(v_i) = \mu_4(v_j)$. It follows that ${}_5H = T$ (total hypergroup) and by consequence $\partial(1_5^{16}) = 5$.

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Term Functions and Fundamental Relation of Fuzzy Hyperalgebras

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Abstract

We introduce and study term functions over fuzzy hyperalgebras. We start from this idea that the set of nonzero fuzzy subsets of a fuzzy hyperalgebra can be organized naturally as a universal algebra, and constructing the term functions over this algebra. We present the form of generated subfuzzy hyperalgebra of a given fuzzy hyperalgebra as a generalization of universal algebras and multialgebras. Finally, we characterize the form of the fundamental relation of a fuzzy hyperalgebra.

Keywords: Hyperalgebra, Fuzzy hyperalgebra, Equivalence relation, Term function, Fundamental relation, Quotient set.

1 Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analysis their properties and applied them to groups, relational algebraic functions (see [15]). Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties ([7]). As it is well known, in 1965 Zadeh ([28]) introduced the notion of a set μ on a nonempty set X as a function from X to the unite real interval $\mathbb{I} = [0, 1]$ as a fuzzy set. In 1971, Rosenfeld ([25]) introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, many researchers are engaged in extending the concepts of abstract algebra to the framework of the fuzzy setting (for instance see [23]).

The study of fuzzy hyperstructure is an interesting research topic of fuzzy sets and applied to the theory of algebraic hyperstructure. As it is known a hyperoperation assigns to every pair of elements of H a nonempty subset of H, while a fuzzy hyperoperation assigns to every pair of elements of H a nonzero fuzzy set on H. Recently, Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups in [21]. This idea was followed by other researchers and extended to other branches of algebraic hyperstructures, for instance Leoreanu and Davvaz introduced and studied fuzzy hyperring notion in [13], Chowdhury in [5] studied fuzzy transposition hypergroups and Leoreanu studied fuzzy hypermodules in [15].

In this paper we follow the idea in [20] and introduced fuzzy hyperalgebras, as the largest class of fuzzy algebraic system. We introduce and study term functions over algebra of all nonzero fuzzy subsets of a fuzzy hyperalgebra, as an important tool to introduce fundamental relation on fuzzy hyperalgebra. Finally, we construct fundamental relation of fuzzy algebras and investigate its basic properties.

This paper is organized in four sections. In section 2 we gather the definitions and basic properties of hyperalgebras and fuzzy sets that we need to develop our paper. In section 3 we introduce term functions over the algebra of nonzero fuzzy subsets of a fuzzy hyperalgebra and we obtained some basic results on fuzzy hyperalgebras, in section 4 we will present the form of the fundamental relation of a fuzzy hyperalgebra.

2 Preliminaries

In this section we present some definitions and simple properties of hyperalgebras from [2] and [3], which will be used in the next sections. In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote for H^n the set of n-tuples over H (for more see [6] and [7]).

For a positive integer n a n-ary *hyperoperation* β on H is a function $\beta : H^n \rightarrow P^*(H)$. We say that n is the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \ldots, x_n) \in S^n$ implies that $\beta(x_1, \ldots, x_n) \subseteq S$. A *nullary hyperoperation* on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H.

A *hyperalgebraic system* or a *hyperalgebra* $\langle H, (\beta_i : i \in I)\rangle$ is the set H with together a collection $(\beta_i \mid i \in I)$ of hyperoperations on H.

A subset S of a hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ is a *subhyperalgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_{n_i}) \subseteq S$, whenever $(a_1, ..., a_{n_i}) \in S^{n_i}$. The *type* of $\mathbb{H}$ is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. In this paper we will assume that for every $i \in I$, the arity of β_i is n_i.

For $n > 0$ we extend an n-ary hyperoperation β on H to an n-ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1, ..., A_n \in P^*(H)$

$$\overline{\beta}(A_1, ..., A_n) = \bigcup\{\beta(a_1, ..., a_n) | a_i \in A_i (i = 1, ..., n)\}$$

It is easy to see that $\langle P^*(H), (\overline{\beta}_i : i \in I)\rangle$ is an algebra of the same type of $\mathbb{H}$.

Definition 2.1. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\overline{\mathbb{H}}$=$\langle \overline{H}, (\overline{\beta}_i : i \in I)\rangle$ be two similar hyperalgebras. A map h from $\mathbb{H}$ into $\overline{\mathbb{H}}$ is called a

(i) A *homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, ..., a_{n_i})) \subseteq \overline{\beta}_i(h(a_1), ..., h(a_{n_i}));$$

(ii) a *good homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that

$$h(\beta_i((a_1, ..., a_{n_i})) = \overline{\beta}_i(h(a_1), ..., h(a_{n_i})),$$

for more details about homomorphism of hyperalgebras see [12]. Let ρ be an equivalence relation on H. We can extend ρ on $P^*(H)$ in the following ways:

(i) Let $\{A, B\} \subseteq P^*(H)$. We write $A\overline{\rho}B$ iff

$$\forall a \in A, \exists b \in B, \text{ such that } a\rho b \quad \text{and} \quad \forall b \in B, \exists a \in A, \text{ such that } a\rho b.$$

(ii) we write $A\overline{\overline{\rho}}B$ iff $\forall a \in A, \forall b \in B$ we have $a\rho b$.

Definition 2.2. If $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a hyperalgebra and ρ be an equivalence relation on H. Then ρ is called *regular* (resp. *strongly regular*) if for every $i \in I$, and for all $a_1, ..., a_{n_i}, b_1, ..., b_{n_i} \in H$ the following implication holds:

$$a_1\rho b_1, ..., a_{n_i}\rho b_{n_i} \Rightarrow \beta_i(a_1, ..., a_{n_i})\overline{\rho}\beta_i(b_1, ..., b_{n_i})$$

$$(resp. \ \ a_1\rho b_1, ..., a_{n_i}\rho b_{n_i} \Rightarrow \beta_i(a_1, ..., a_{n_i})\overline{\overline{\rho}}\beta_i(b_1, ..., b_{n_i})).$$

Definition 2.3. Recall that for a nonempty set H, a fuzzy subset μ of H is a function

$$\mu : H \rightarrow [0, 1].$$

If μ_i is a collection of fuzzy subsets of H, then we define the fuzzy subset $\bigcap_{i\in I}\mu_i$ by:

$$(\bigcap_{i\in I}\mu_i)(x) = \bigwedge_{i\in I}\{\mu_i(x)\}, \quad \forall x \in H.$$

Definition 2.4. Let ρ be an equivalence relation on a hyperalgebra $\langle H, (\beta_i : i \in I)\rangle$ and μ and υ be two fuzzy subsets on H. We say that $\mu\rho\upsilon$ if the following two conditions hold:

(i) $\mu(a) > 0 \Rightarrow \exists b \in H : \upsilon(b) > 0$, and $a\rho b$

(ii) $\upsilon(x) > 0 \Rightarrow \exists y \in H : \mu(y) > 0$, and $x\rho y$.

3 Fuzzy Hyperalgebra and Term Functions

Definition 3.1. A *fuzzy n-ary hyperoperation* f^n on S is a map $f^n : S\times ...\times S \longrightarrow F^*(S)$, which associated a nonzero fuzzy subset $f^n(a_1, ..., a_n)$ with any n-tuple $(a_1, ..., a_n)$ of elements of S. The couple $\langle S, f^n\rangle$ is called a *fuzzy n-ary hypergroupoid*. A *fuzzy nullary hyperoperation* on S is just an element of $F^*(S)$; i.e. a nonzero fuzzy subset of S.

Definition 3.2. Let H be a nonempty set and for every $i \in I$, β_i be a fuzzy n_i-ary hyperoperation on H. Then $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ is called *fuzzy hyperalgebra*, where $(n_i : i \in I)$ is the type of this fuzzy hyperalgebra.

Definition 3.3. If $\mu_1, ..., \mu_{n_i}$ be n_i nonzero fuzzy subsets of a fuzzy hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$, we define for all $t \in H$

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge ... \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))$$

Finally, if $A_1, ..., A_{n_k}$ are nonempty subsets of H, for all $t \in H$

$$\beta_k(A_1, ..., A_{n_k})(t) = \bigvee_{(a_1,...,a_{n_k})\in H^{n_k}} (\beta_k(a_1, ..., a_{n_k})(t)).$$

If A is a nonempty subset of H, then we denote the characteristic function of A by χ_A. Note that, if $f : H_1 \longrightarrow H_2$ is a map and $a \in H_1$, then $f(\chi_a) = \chi_{f(a)}$.

Example 3.4.

(i) A *fuzzy hypergroupoid* is a fuzzy hyperalgebra of type (2), that is a set H together with a fuzzy hyperoperation $\circ$. A fuzzy hypergroupoid $\langle H, \circ\rangle$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$ is called *fuzzy hypersemigroup*[22]. In this case for any $\mu \in F^*(H)$, we define $(a \circ \mu)(r) = \bigvee_{t \in H}((a \circ t)(r) \wedge \mu(t))$ and $(\mu \circ a)(r) = \bigvee_{t \in H}(\mu(t) \wedge (t \circ a)(r))$ for all $r \in H$.

(ii) A *fuzzy hypergroup* is a fuzzy hypersemigroup such that for all $x \in H$ we have $x \circ H = H \circ x = \chi_H$ (fuzzy reproduction axiom)(for more details see [22]).

(iii) A *fuzzy hyperring* $\mathbb{R}$=$\langle R, \oplus, \odot \rangle$ ([13]) is a fuzzy hyperalgebra of type $(2,2)$, which in that the following axioms hold:

1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in R$;

2) $x \oplus R = R \oplus x = \chi_R$ for all $x \in R$;

3) $a \oplus b = b \oplus a$ for all $a, b \in R$;

4) $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in R$;

5) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ for all $a, b, c \in R$.

Example 3.5. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a hyperalgebra and μ be a nonzero fuzzy subset of H. Define the following fuzzy n-ary hyperoperations on H, for every $i \in I$ and for all $(a_1, ..., a_{n_i}) \in H^{n_i}$;

$$\beta_i^{\diamond}(a_1, ..., a_{n_i})(t) = \begin{cases} \mu(a_1) \bigwedge ... \bigwedge \mu(a_{n_i}) & t \in \beta(a_1, ..., a_{n_i}) \\ 0 & otherwise \end{cases}$$

and letting

$$\beta_i^{\circ}(a_1, ..., a_{n_i}) = \chi_{\{a_1, ..., a_{n_i}\}}.$$

Evidently $\mathbb{H}^{\diamond}$=$\langle H, (\beta_i^{\diamond} : i \in I) \rangle$, $\mathbb{H}^{\circ}$=$\langle H, (\beta_i^{\circ} : i \in I) \rangle$ are fuzzy hyperalgebras.

Theorem 3.6. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra, then for every $i \in I$ and every $a_1, ..., a_{n_i} \in H$ we have $\beta_i(\chi_{a_1}, ..., \chi_{a_{n_i}}) = \beta_i(a_1, ..., a_{n_i})$.

Definition 3.7. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra . A nonempty subset S of H is called a *subfuzzy hyperalgebra* if for $\forall i \in I, \forall a_1, ..., a_{n_i} \in S$, the following condition hold:

$$\beta_i(a_1, ..., a_{n_i})(x) > 0 \text{ then } x \in S.$$

We denote by $\mathcal{S}(\mathbb{U})$ the set of the subfuzzy hyperalgebras of $\mathbb{H}$.

Definition 3.8. Consider the fuzzy hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\phi \neq X \subseteq H$ be nonempty. Clearly, $\langle X\rangle = \bigcap\{B : B \in \mathcal{S}(\mathbb{H})|\ X \subseteq B\}$ with the fuzzy hyperoperations of $\mathbb{H}$ form a subfuzzy hyperalgebra of $\mathbb{H}$ called the *subfuzzy hyperalgebra of* $\mathbb{H}$ *generated* by the subset X . Evidently if X is a subfuzzy hyperalgebra for $\mathbb{H}$ then $\langle X\rangle = X$.

Theorem 3.9. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\phi \neq X \subseteq H$. We consider $X_0 = X$ and for any $k \in \mathbb{N}$,

$$X_{k+1} = X_k \cup \{a \in H \mid \exists i \in I, n_i \in \mathbb{N}, x_1, ..., x_{n_i} \in X_k; \beta_i(x_1, ..., x_{n_i})(a) > 0\}.$$

Then $\langle X\rangle = \bigcup_{k\in\mathbb{N}} X_k$.

Proof. Let $M = \bigcup_{k\in N} X_k$, and $\forall i \in I$, consider $t_1, ..., t_{n_i} \in M$ and $\beta_i(t_1, ..., t_{n_i})(x) > 0$. From $X_0 \subseteq X_1 \subseteq ... \subseteq X_k \subseteq ...$ it follows the existence of $m \in \mathbb{N}$ such that $t_1, ..., t_{n_i} \in X_m$, which implies, according to the definition of X_{m+1} that $x \in X_{m+1}$. Thus $x \in M$ and $M = \bigcup_{k\in\mathbb{N}} X_k$ is a subfuzzy hyperalgebra. From $X = X_0 \subseteq M$, by definition of the generated subfuzzy hyperalgebra, it results $\langle X\rangle \subseteq \langle M\rangle = M$. To prove the inverse inclusion we will show by induction on $k \in \mathbb{N}$ that $X_k \subseteq \langle X\rangle$ for any $k \in \mathbb{N}$, and we have $X_0 = X \subseteq \langle X\rangle$. We suppose that $X_k \subseteq \langle X\rangle$. From $\langle X\rangle \in \mathcal{S}(\mathbb{H})$ and the definition X_{k+1} we can deduce that $X_{k+1} \subseteq \langle X\rangle$. Hence $M \subseteq \langle X\rangle$. The two inclusion lead us to $M = \langle X\rangle$.□

Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra then, the set of the nonzero fuzzy subsets of H denoted by $F^*(H)$, can be organized as a universal algebra with the operations;

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge ... \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))$$

for every $i \in I$, $\mu_1, ..., \mu_{n_i} \in F^*(H)$ and $t \in H$. We denote this algebra by $\mathcal{F}^*(\mathbb{H})$.

In [13] Gratzer presents the algebra of the term functions of a universal algebra. If we consider an algebra $\mathbb{B}=\langle B, (\beta_i : i \in I)\rangle$ we call $n-$ary term functions on $\mathbb{B}$ $(n \in \mathbb{N})$ those and only those functions from B^n into B, which can be obtained by applying (i) and (ii) from bellow for finitely many times:

(i) the functions $e_i^n : B^n \to B$, $e_i^n(x_1, ..., x_n) = x_i$, $i = 1, ..., n$ are $n-$ary term functions on $\mathbb{B}$;

(ii) if $p_1, ..., p_{n_i}$ are $n-$ary term functions on $\mathbb{B}$, then $\beta_i(p_1, ..., p_{n_i}) : B^n \to B$,

$\beta_i(p_1, ..., p_{n_i})(x_1, ..., x_n) = \beta_i(p_1(x_1, ..., x_n), ..., p_{n_i}(x_1, ..., x_n))$ is also a $n-$ary term function on $\mathbb{B}$.

We can observe that (ii) organize the set of $n-$ary term functions over $\mathbb{B}$ $(P^{(n)}(\mathbb{B}))$ as a universal algebra, denoted by $\mathcal{B}^{(n)}(\mathbb{B})$.

If $\mathbb{H}$ is a fuzzy hyperalgebra then for any $n \in \mathbb{N}$, we can construct the algebra of $n-$ary term functions on $\mathcal{F}^*(\mathbb{H})$, denoted by $\mathcal{B}^{(n)}(\mathcal{F}^*(\mathbb{H})) = \langle P^{(n)}(\mathcal{F}^*(\mathbb{H})), (\beta_i : i \in I)\rangle$.

Theorem 3.10. A necessary and sufficient condition for $\mathcal{F}^*(\mathbb{B})$ to be a subalgebra of $\mathcal{F}^*(\mathbb{U})$ is that $\mathbb{B}$ is to be a subfuzzy hyperalgebra for $\mathbb{U}$.

Proof. Obvious.$\square$

The next result immediately follows from Theorem 3.10.

Corollary 3.11. (i) Let $\mathbb{H}=\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\mathbb{B}$ a subfuzzy hyperalgebra of $\mathbb{H}$, and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$,$(n \in \mathbb{N})$. If $\mu_1, ..., \mu_n \in F^*(B)$, then $p(\mu_1, ..., \mu_n) \in F^*(B)$.

(ii) Let $\mathbb{H}= \langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\mathbb{B}$ a subfuzzy hyperalgebra of $\mathbb{H}$, and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$,$(n \in \mathbb{N})$. If $x_1, ..., x_n \in B$, then $p(\chi_{x_1}, ..., \chi_{x_n}) \in F^*(B)$.$\square$

Theorem 3.12. Let $\mathbb{H}=\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and $\phi \neq X \subseteq H$. Then $a \in \langle X\rangle$ if and only if $\exists n \in \mathbb{N}, \exists p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$, and $\exists x_1, ..., x_n \in X$, such that

$p(\chi_{x_1}, ..., \chi_{x_n})(a) > 0$.

Proof. We denote

$$M = \{a \in H \mid \exists n \in \mathbb{N}, \exists p \in P^{(n)}(\mathcal{F}^*(\mathbb{H})), \quad \exists x_1, ..., x_n \in X : p(\chi_{x_1}, ...\chi_{x_n})(a) > 0\}.$$

For any $x \in X$ we have $e_1^1(\chi_x)(x) = \chi_x(x) = 1$, thus $x \in X$ and hence $X \subseteq M$. Also from Corollary 3.11 (ii), it follows that $p(\chi_{x_1}, ..., \chi_{x_n}) \in \mathcal{F}^*(\langle X \rangle)$, therefore $M \subseteq \langle X \rangle$. We will prove now that M is subfuzzy hyperalgebra of $\mathbb{H}$. For any $i \in I$, if $c_1, ..., c_{n_i} \in M$ and $\beta_i(c_1, ..., c_{n_i})(x) > 0$, we must show that $x \in M$. For $c_1, ..., c_{n_i} \in M$, it means that there exist $m_k \in \mathbb{N}, p_k \in P^{m_k}(\mathcal{F}^*(\mathbb{H})), x_1^k, ..., x_{m_k}^k \in X, k \in \{1, ..., n_i\}$, such that $p_k(\chi_{x_1^k}, ..., \chi_{x_{m_k}^k})(c_k) > 0$, $\forall k \in \{1, ..., n_i\}$. According to the Corollary 8.2 from [12], for any $n-$ary term function p over $\mathcal{F}^*(\mathbb{H})$ and for $m \geq n$ there exists an $m-$ary term function q over $\mathcal{F}^*(\mathbb{H})$, such that $p(\mu_1, ..., \mu_n) = q(\mu_1, ..., \mu_m)$, for all $\mu_1, ..., \mu_m \in F^*(H)$; this allows us to consider instead of $p_1, ..., p_{n_i}$ the term functions $q_1, ..., q_{n_i}$ all with the same arity $m = m_1 + ... + m_{n_i}$ and the elements $y_1, ..., y_m \in X$ (which are the elements $x_1^1, ..., x_{m_1}^1, ..., x_1^{n_i}, ..., x_{m_{n_i}}^{n_i}$), such that $q_k(\chi_{y_1}, ..., \chi_{y_m})(c_k) > 0$,$\forall k \in \{1, ..., n_i\}$. But we have
$\beta_i(q_1(\chi_{y_1}, ..., \chi_{y_m}), ..., q_{n_i}(\chi_{y_1}, ..., \chi_{y_m}))(x) =$

$$\bigvee_{(a_1,...,a_{n_i}) \in H^{n_i}} (q_1(\chi_{y_1}, ..., \chi_{y_m})(a_1) \wedge ... \wedge q_{n_i}(\chi_{y_1}, ..., \chi_{y_m})(a_{n_i}) \wedge \beta_i(a_1, ..., a_{n_i})(x)),$$

and for $(a_1, ..., a_{n_i}) = (c_1, ..., c_{n_i})$ we have $(\beta_i(q_1, ..., q_{n_i})(\chi_{y_1}, ..., \chi_{y_m}))(x) > 0$. On the other hands we have $\beta_i(q_1, ..., q_{n_i}) \in P^{(m)}(\mathcal{F}^*(\mathbb{H})), (m \in \mathbb{N})$, $y_1, ..., y_m \in X$ which implies that $x \in M$. Therefore, $M = \langle X \rangle$ and this complete the proof.□

Remark 3.13. If $\mathbb{H}$ has a fuzzy nullary hyperoperation then

$$< \phi >= \{a \in H \mid \exists \mu \in P^0(\mathcal{F}^*(\mathbb{H})), \text{ such that } \mu(a) > 0\}.$$

Recall that if $\mathbb{H}$=$\langle H, (\beta_i : i \in I) \rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I) \rangle$ are fuzzy hyperalgebras with the same type, then a map $h : H \to B$ is called a *good homomorphism* if for any $i \in I$ we have ;

$$h(\beta_i(a_1,...,a_{n_i})) = \beta_i(h(a_1),...,h(a_{n_i})), \forall a_1,...,a_{n_i} \in H.$$

An equivalence relation on H φ is said to be an *ideal* if for any $i \in I$ we have:

$$\beta_i(x_1,...,x_{n_i})(a) > 0 \quad \text{and} \quad x_k\varphi y_k(k \in \{1,...,n_i\}) \ \Rightarrow \ \exists b \in H : \beta_i(y_1,...,y_{n_i})(b) > 0 \quad \text{and} \quad a\varphi b.$$

For example the fuzzy regular relations on a fuzzy hypersemigroup are ideal equivalence. (for more details see [13, 21])

Definition 3.14. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra and φ an equivalence relation on H. Then H/φ can be described as a fuzzy hyperalgebra $\mathbb{H}/\varphi$ with the fuzzy hyperoperations:

$$\beta_i(\varphi(x_1),...,\varphi(x_{n_i}))(\varphi(x_{n_i+1})) = \bigvee_{x_k\varphi y_k} \beta_i(y_1,...,y_{n_i})(y_{n_i+1}).$$

Theorem 3.15. Let $h : H \to B$ be a good homomorphism of fuzzy hyperalgebras $\mathbb{H}$ and $\mathbb{B}$. Then the relation $\varphi = \{(x,y) \in H | h(x) = h(y)\}$ is an ideal relation on $\mathbb{H}$. Conversely, if φ is an ideal relation on $\mathbb{H}$, then $p = p_\varphi : H \to H/\varphi$ is homomorphism (which is not strong).

Proof. Straightforward.□

Remark 3.16. Let $\mathbb{H}$ and $\mathbb{B}$ be fuzzy hyperalgebras of the same type and h be a homomorphism from $\mathbb{H}$ into $\mathbb{B}$. We will construct the algebras $\mathcal{F}^*(\mathbb{H})$ and $\mathcal{F}^*(\mathbb{B})$. The homomorphism h induces a mapping $h' : \mathcal{F}^*(\mathbb{H}) \to \mathcal{F}^*(\mathbb{B})$ by $h'(\mu) = h(\mu)$, for any $\mu \in F^*(H)$.

Let us consider H a set and $F^*(H)$ the set of its nonzero fuzzy subsets. Let φ be an equivalence on H and let us consider the relation $\overline{\varphi}$ on $F^*(H)$ as follows:

$$\mu\overline{\varphi}\nu \Leftrightarrow \quad \forall a \in H : \mu(a) > 0 \quad \Rightarrow \quad \exists b \in H : \nu(b) > 0 \quad \text{and} \quad a\varphi b \text{ and}$$

$$\forall b \in H : \quad \nu(b) > 0 \quad \Rightarrow \quad \exists a \in H : \mu(a) > 0 \quad \text{and} \quad a\varphi b.$$

It is immediate that $\overline{\varphi}$ is an equivalence relation on $F^*(H)$. The next result immediately follows.

Theorem 3.17. An equivalence relation φ on a fuzzy hyperalgebra $\mathbb{H}$ is ideal if and only if $\overline{\varphi}$ is a congruence relation on $\mathcal{F}^*(\mathbb{H})$.

Proof. Let us suppose that φ is an ideal equivalence on $\mathbb{H}$ and let us consider $i \in I$ and $\mu_k, \nu_k \in F^*(H)$ nonzero and $\mu_k\overline{\varphi}\nu_k$, $k \in \{1, ..., n_i\}$. Then for any $a \in H$ such that $\beta_i(\mu_1, ..., \mu_{n_i})(a) > 0$, we have

$$\beta_i(\mu_1, ..., \mu_{n_i})(a) = \bigvee_{(x_1,...,x_{n_i}) \in H^{n_i}} \mu_1(x_1) \wedge ... \wedge \mu_{n_i}(x_{n_i}) \wedge \beta_i(x_1, ..., x_{n_i})(a).$$

Thus there exists $(x_1, ..., x_{n_i}) \in H^{n_i}$, such that $\mu_k(x_k) > 0$ for $k \in \{1, ..., n_i\}$ and $\beta_i(x_1, ..., x_{n_i})(a) > 0$. From the definition $\overline{\varphi}$ and hence there exists $(y_1, ..., y_{n_i}) \in H^{n_i}$, such that $\nu_k(y_k) > 0$ for $k \in \{1, ..., n_i\}$ and $x_k\varphi y_k$, and sice φ is an ideal and $\beta_i(x_1, ..., x_{n_i})(a) > 0$, there exists $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$ and $a\varphi b$. Analogously, it can be proved that for all $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$, there exists $a \in H$, such that $\beta_i(x_1, ..., x_{n_i})(a) > 0$ and $a\varphi b$. Hence, it is proved that $\overline{\varphi}$ is a congruence on $\mathcal{F}^*(\mathbb{H})$.

Conversely, let us take $i \in I$ and $a, x_k, y_k \in H$, $k \in \{1, ..., n_i\}$ such that $x_k\varphi y_k$ and $\beta_i(x_1, ..., x_{n_i})(a) > 0$. Obviously, $\chi_{x_k}\overline{\varphi}\chi_{y_k}$, $\forall k \in \{1, ..., n_i\}$, and because $\overline{\varphi}$ is a congruence on $\mathcal{F}^*(\mathbb{H})$ We can write $\beta_i(\chi_{x_1}, ..., \chi_{x_{n_i}})\overline{\varphi}\beta_i(\chi_{y_1}, ..., \chi_{y_{n_i}})$, hence $\beta_i(x_1, ..., x_{n_i})\overline{\varphi}\beta_i(y_1, ..., y_{n_i})$, which leads us to the existence $b \in H$, such that $\beta_i(y_1, ..., y_{n_i})(b) > 0$ and $a\varphi b$. This complete the proof.□

Corollary 3.18. (i) If $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ is a fuzzy hyperalgebra, φ is an ideal equivalence relation on $\mathbb{H}$ and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$ If for any nonzero, μ_k, ν_k, such that $\mu_k\overline{\varphi}\nu_k$

$k \in \{1, ..., n\}$, then $p(\mu_1, ..., \mu_n)\overline{\varphi}p(\nu_1, ..., \nu_n)$.

(ii) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra, φ an ideal equivalence relation on $\mathbb{H}$. If $x_k \varphi y_k$, $k \in \{1, ..., n\}$, $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$, $x_k, y_k \in H$. Then have $p(\chi_{x_1}, ..., \chi_{x_n})\overline{\varphi}p(\chi_{y_1}, ..., \chi_{y_n})$.

Let h be a homomorphism from $\mathbb{H}$ into $\mathbb{B}$ and take $\varphi = \{(x, y) \in H^2 \mid h(x) = h(y)\}$. Then we have $\overline{\varphi} = \{(\mu, \nu) \in (F^*(H))^2 \mid h'(\mu) = h'(\nu)\}$. Obviously, φ is an ideal of $\mathbb{H}$ if and only if $\overline{\varphi}$ is congruence on $\mathcal{F}^*(\mathbb{H})$.

Theorem 3.19. The map h is a homomorphism ofthe universal algebras $\mathcal{F}^*(\mathbb{H})$ and $\mathcal{F}^*(\mathbb{B})$ if and only if h is a good homomorphism between $\mathbb{H}$ and $\mathbb{B}$.

Proof. Straightforward.□

The next result immediately follows from Theorem 3.12.

Corollary 3.20. (i) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ be fuzzy hyperalgebras of the same type, $h : H \to B$ a homomorphism and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Then for all $\mu_1, ..., \mu_n \in F^*(H)$ we have $h'(p(\mu_1, ..., \mu_n)) = p(h'(\mu_1), ..., h'(\mu_n))$.

(ii) Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ be fuzzy hyperalgebras of the same type, $h : H \to B$ a homomorphism and $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Then for all $a_1, ..., a_n \in H$, we have $h'(p(\chi_{a_1}, ..., \chi_{a_n})) = p(h'(\chi_{a_1}), ..., h'(\chi_{a_n}))$.□

4 Fundamental Relation of Fuzzy Hyperalgebra

As it is known that if R is an strongly regular equivalence relation on a given hypergroup (resp. hypergroupoid, semihypergroup) H, then we can define a binary operation $\otimes$ on the quotient set H/R, the set of all equivalence classes of H with respect to R, such that $(H/R, \otimes)$ consists a group (resp. groupoid, semigroup). In fact the relation β^* is the smallest equivalences relation such that the quotient H/β^* is a group (resp. groupoid, semigroup) and it is called *fundamental relation* of H. The equivalence relation β^* was studied on hypergroups by many authors(for more details see [6]). As the fundamental relation plays an important role in the theory of algebraic hyperstructure it extended to other classes of algebraic hyperstructure, such as hyperrings, hypermodules, hypervectorspaces(for more details see [25], [26] and [27]). In [20] Pelea introduced and studied the fundamental relation of a multialgebra based on term functions. In the sequel we extend fundamental relation on fuzzy hyperalgebras and investigate its basic properties.

Let $\mathbb{B}=\langle B, (\beta_i : i \in I)\rangle$ be an universal algebra. If we add to the set of the operations of $\mathbb{B}$ the nullary operations corresponding to the elements of B, the $n-$ary term functions of this new algebra are called the *$n-$ary polynomial functions* of $\mathbb{B}$. The $n-$ary polynomial functions $P^n(\mathbb{B})$ of $\mathbb{B}$ form a universal algebra with the operations $(\beta_i : i \in I)$, denoted by $\mathcal{P}^{(n)}(\mathbb{B})$, $\mathcal{P}^{(n)}(\mathbb{B})=\langle P^n(\mathbb{B}), (\beta_i : i \in I)\rangle$.

Let $\mathbb{H}=\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra. For any $n \in \mathbb{N}$, we can construct the algebra $\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H}))$ of $n-$ary polynomial functions on $\mathcal{F}^*(\mathbb{H})$, ($\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H})) = \langle P^n(\mathcal{F}^*(\mathbb{H})), (\beta_i : i \in I)\rangle$) . Consider the subalgebra $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$ of $\mathcal{P}^{(n)}(\mathcal{F}^*(\mathbb{H}))$ obtained by adding to the operations $(\beta_i : i \in I)$ of $\mathcal{F}^*(\mathbb{H})$ only the nullary operations associated to the characteristic functions of the elements of H. Thus the elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$ $(n \in \mathbb{N})$ are those and only those functions from $(F^*(H))^n$ into $F^*(H)$ which can obtained by applying $(i), (ii), (iii)$ from bellow for finitely many times:

(i) the functions $C^n_{\chi_a} : (F^*(H))^n \to F^*(H)$, defined by setting $C^n_{\chi_a}(\mu_1, ..., \mu_n) = \chi_a$, for all $\mu_1, ..., \mu_n \in F^*(H)$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$, for every $a \in H$;

(ii) the functions $e^n_i : (F^*(H))^n \to F^*(H)$, $e^n_i(\mu_1, ..., \mu_n) = \mu_i$, for all $\mu_1, ..., \mu_n \in F^*(H)$, $i = 1, ..., n$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$;

(iii) if $p_1, ..., p_{n_i}$ are elements of $\mathcal{P}^{(n)}_H(\mathcal{F}^*(\mathbb{H}))$, and $i \in I$ then $\beta_i(p_1, ..., p_{n_i}) : (F^*(H))^n \to$

$F^*(H)$, defined by setting for all $\mu_1, ..., \mu_n \in F^*(H)$, $(\beta_i(p_1, ..., p_{n_i}))(\mu_1, ..., \mu_n) = \beta_i(p_1(\mu_1, ..., \mu_n), ..., p_{n_i}(\mu_1, ..., \mu_n))$ is also an element of $\mathcal{P}_H^{(n)}(\mathcal{F}^*(\mathbb{H}))$.

In the continue, we will use only polynomial functions from $\mathcal{P}_H^{(n)}(\mathcal{F}^*(\mathbb{H}))$. Thus we will drop the subscript with no danger of confusion.

Definition 4.1. Let α be the relation defined on H for $x, y \in H$ set $x\alpha y$ follows:

$$x\alpha y \iff p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0 \text{and } p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0, \text{ for some } p \in P^n(\mathcal{F}^*(\mathbb{H})), a_1, ..., a_n \in H.$$

It is clear that α is symmetric. Because for any $a \in H$, $e_1^1(\chi_a)(a) > 0$, the relation α is also reflexive. We take α^* to be the transitive closure of α. Then α^* is an equivalence relation on H.

Lemma 4.2. If $f \in P^1(\mathbb{F}^*(\mathbb{H}))$ and $a, b \in H$ satisfy $a\alpha^* b$ then $f(\chi_a)\overline{\overline{\alpha^*}}f(\chi_b)$.

Proof. By the definition of α^* : $a = y_1\alpha y_2\alpha ...\alpha y_m = b$ for some $m \in \mathbb{N}$ and $y_2, ..., y_{m-1} \in H$. Let $f(\chi_{y_i})(u_i) > 0$, $i = 1, ..., m$. Consider $1 \leq j < m$. Clearly $y_j\alpha y_{j+1}$ means that $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_j) > 0$ and $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_{j+1}) > 0$, for some $n_j \in \mathbb{N}, p_j \in P^{n_j}(\mathcal{F}^*(\mathbb{H}))$, $a_1, ..., a_n \in H$. Define the n_j-ary hyperoperation q_j on $F^*(H)$ by setting

$q_j(\chi_{x_1}, ..., \chi_{x_{n_j}}) = \bigvee\{f(\chi_t) : p_j(\chi_{x_1}, ..., \chi_{x_{n_j}})(t) > 0\}$ for all $x_1, ..., x_{n_j} \in H$. Clearly $q_j \in P^{n_j}(\mathcal{F}^*(\mathbb{H}))$ and for $x \in H$; $q_j(\chi_{a_1}, ..., \chi_{a_n})(x) = \bigvee_{p_j(\chi_{a_1}, ..., \chi_{a_n})(z) > 0} f(\chi_z)(x)$.

From $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_j) > 0$ and $p_j(\chi_{a_1}, ..., \chi_{a_n})(y_{j+1}) > 0$ we get

$$0 < f(\chi_{y_j})(u_j) \leq q_j(\chi_{a_1}, ..., \chi_{a_n})(u_j) \quad \text{and}$$

$$0 < f(\chi_{y_{j+1}})(u_{j+1}) \leq q_j(\chi_{a_1}, ..., \chi_{a_n})(u_{j+1})$$

proving $u_j\alpha u_{j+1}$. Thus $u_1\alpha^* u_m$. Since $f(\chi_a)(u_1) = f(\chi_{y_1})(u_1) > 0$ and $f(\chi_b)(u_m) = f(\chi_{y_m})(u_m) > 0$ were arbitrary, we obtain $f(\chi_a)\overline{\overline{\alpha^*}}f(\chi_b)$.□

Remark 4.3. For a given fuzzy hyperalgebra $\mathbb{H}$ and equivalence relation ρ on H, the set H/ρ can be considered as a hyperalgebra with the hyperoperations

$$\beta_i(\rho(a_1), ..., \rho(a_{n_i})) = \{\rho(z) \mid \beta_i(b_1, ..., b_{n_i})(z) > 0, b_k \in \rho(a_k), \forall k \in \{1, ..., n_i\}\} \quad (1)$$

for all $i \in I$.

Lemma 4.4. Let ρ be an equivalence relation on $\mathbb{H}$ such that $\mathbb{H}/\rho$ be an universal algebra . Then for any $n \in \mathbb{N}$, $p \in P^n(\mathcal{F}^*(\mathbb{H}))$ and $a_1, ..., a_n \in H$ the following gold:

$$p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0 \text{ and } p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0 \implies x\rho y.$$

Proof. We will prove this statement by induction over the steps of construction of an $n-$ary polynomial function(for $n \in \mathbb{N}$ arbitrary).

If $p = C^n_{\chi_a}$, from $C^n_{\chi_a}(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $C^n_{\chi_a}(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$ we deduce that $x = y = a$, thus $x\rho y$.

If $p = e^n_i$ with $i \in \{1, ..., n\}$, from $e^n_i(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $e^n_i(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$ we deduce that $x = y = a_i$, , and hence $x\rho y$.

We suppose that the statement holds for the $n-$ary polynomial functions $p_1, ..., p_{n_k}$ and we will prove it for the $n-$ary polynomial function $\beta_k(p_1, ..., p_{n_k})$. If

$0 < \beta_k(p_1, ..., p_{n_k})(\chi_{a_1}, ..., \chi_{a_n})(x) = \beta_k(p_1(\chi_{a_1}, ..., \chi_{a_n}), ..., p_{n_k}(\chi_{a_1}, ..., \chi_{a_n}))(x) =$

$$\bigvee_{(x_1,...,x_{n_k}) \in H^{n_k}} (p_1(\chi_{a_1}, ..., \chi_{a_n})(x_1) \wedge ... \wedge p_{n_k}(\chi_{a_1}, ..., \chi_{a_n})(x_{n_k}) \wedge \beta_k(x_1, ..., x_{n_k})(x))$$

and if we set y instead of x, above statement is true. Thus there exist

$x_1, ..., x_{n_k}, y_1, ..., y_{n_k} \in H$, such that $p_i(\chi_{a_1}, ..., \chi_{a_n})(x_i) > 0$ and $p_i(\chi_{a_1}, ..., \chi_{a_n})(y_i) > 0$, for $i \in \{1, ..., n_k\}$ and $\beta_k(x_1, ..., x_{n_k})(x) > 0$ and $\beta_k(y_1, ..., y_{n_k})(y) > 0$. Obviously, $x_i\rho y_i$ for all $i \in \{1, ..., n_k\}$ and according to (1) and by the hypothesis we obtain that $\rho(x) = \rho(y)$, i.e., $x\rho y$, as desired.□

The next result immediately follows from previous two lemmas.

Theorem 4.5. The relation α^* is the smallest equivalence relation on fuzzy hyperalgebra $\mathbb{H}$ such that $\mathbb{H}/\rho$ is an universal algebra.

We call $\mathbb{H}/\rho$, *fundamental universal algebra* of fuzzy hyperalgebra $\mathbb{H}$ such that $\mathbb{H}/\rho$.

Proof. At the first, we show that $\mathbb{H}/\rho$ is a universal algebra. For this we take any $x, y \in H$, such that $\alpha^*(x), \alpha^*(y) \in \beta_k(\alpha^*(a_1), ..., \alpha^*(a_{n_k}))$ for $k \in I$ and $a_1, ..., a_{n_k} \in H$. This means that there exist $x_1, ..., x_{n_k}, y_1, ..., y_{n_k} \in H$, such that $\beta_k(x_1, ..., x_{n_k})(x) > 0$ and $\beta_k(y_1, ..., y_{n_k})(y) > 0$ and $x_i \alpha^* a_i \alpha^* y_i$ for all $i \in \{1, ..., n_k\}$.

Applying Lemma 4.2 to the unary polynomial functions

$\beta_i(z, C^n_{\chi_{x_2}}, ..., C^n_{\chi_{x_{n_k}}})$, $\beta_i(C^n_{\chi_{y_1}}, z, C^n_{\chi_{x_3}}, ..., C^n_{\chi_{x_{n_k}}}), ..., , \beta_i(C^n_{\chi_{y_1}}, ..., C^n_{\chi_{y_{n_k-1}}}, z)$,

we obtain the following relations:

$$\beta_i(\chi_{x_1}, ..., \chi_{x_{n_k}}) \overline{\overline{\alpha^*}} \beta(\chi_{y_1}, \chi_{x_2}, ..., \chi_{x_{n_k}})$$

$$\beta_i(\chi_{y_1}, \chi_{x_2}, ..., \chi_{x_{n_k}}) \overline{\overline{\alpha^*}} \beta_i(\chi_{y_1}, \chi_{2_2}, \chi_{x_3} ..., \chi_{x_{n_k}})$$

$$\vdots$$

$$\beta_i(\chi_{y_1}, \chi_{y_2}, ..., \chi_{x_{n_k-1}}) \overline{\overline{\alpha^*}} \beta_i(\chi_{y_1}, \chi_{y_2}, ..., \chi_{y_{n_k}}),$$

which leads us to $x\alpha^* y$ (from definition α^*), i.e. $\alpha^*(x) = \alpha^*(y)$. Clearly, β_i in (1) is an operation on H/α^*, for any $i \in I$, and $\mathbb{H}/\alpha^*$ is a universal algebra. Now we prove that α^* is smallest. If ρ is an arbitrary equivalence relation on H such that H/ρ is a universal algebra, we can show that $\alpha^* \subseteq \rho$. If $x\alpha y$ then there exist $n \in \mathbb{N}$, $p \in P^n(\mathcal{F}^*(\mathbb{H}))$ and $a_1, ..., a_n \in H$ for which $p(\chi_{a_1}, ..., \chi_{a_n})(x) > 0$ and $p(\chi_{a_1}, ..., \chi_{a_n})(y) > 0$, and hence by Lemma 4.4 we have $x\rho y$, hence $\alpha \subseteq \rho$, which implies $\alpha^* \subseteq \rho$.□

Remark 4.6. For a given fuzzy hyperalgebra $\mathbb{H}$ and equivalence relation α^* on H. Let us define the operations of the universal algebra $\mathbb{H}/\alpha^*$ as follows :

$$\beta_i(\alpha^*(a_1), ..., \alpha^*(a_{n_i})) = \{\alpha^*(b) \mid \beta_i(a_1, ..., a_{n_i})(b) > 0\}.$$

Moreover, we can write

$$\beta_i(\alpha^*(a_1), ..., \alpha^*(a_{n_i})) = \alpha^*(b) \qquad \beta_i(a_1, ..., a_{n_i})(b) > 0.$$

Example 4.7. Let $\mathbb{H}$=$\langle H, \circ \rangle$ be a fuzzy hypersemigroup, i.e. a fuzzy hyperalgebra with one binary fuzzy hyperoperation $\circ$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$ (for more details see [21]). Let $\mathcal{F}^*(\mathbb{H})$=$\langle F^*(H), \circ\rangle$ be the universal algebra with one binary operation defined as follows:

$$(\mu \circ \nu)(r) = \bigvee_{x,y\in H} \mu(x) \wedge \nu(y) \wedge (x \circ y)(r) \qquad \forall \ \ \mu, \nu \in F^*(H), r \in H.$$

By distributivity of the lattice $([0,1], \vee, \wedge)$ and associativity of $\circ$ in H, we will prove that the operation $\circ$ in $\mathcal{F}^*(\mathbb{H})$ is associative. So for every $\mu, \nu, \eta \in F^*(H)$ and $r \in H$ we have

$$((\mu \circ \nu) \circ \eta)(r) = \bigvee_{x,y\in H} [(\mu \circ \nu)(x) \wedge \eta(y) \wedge (x \circ y)(r)] =$$

$$\bigvee_{x,y\in H} [(\bigvee_{p,q\in H} \mu(p) \wedge \nu(q) \wedge (p \circ q)(x)) \wedge \eta(y) \wedge (x \circ y)(r)] =$$

$$\bigvee_{p,q,y\in H} [\mu(p) \wedge \nu(q) \wedge \eta(y) \wedge (\bigvee_{x\in H} (p \circ q)(x) \wedge (x \circ y)(r))] =$$

$$\bigvee_{p,q,y\in H} [\mu(p) \wedge \nu(q) \wedge \eta(y) \wedge (\bigvee_{x\in H} (p \circ x)(r) \wedge (q \circ y)(x))] =$$

$$\bigvee_{p,x\in H} [\mu(p) \wedge (p \circ x)(r) \wedge (\bigvee_{q,y\in H} \nu(q) \wedge \eta(y) \wedge (q \circ y)(x))] =$$

$$\bigvee_{p,x\in H} [\mu(p) \wedge (p \circ x)(r) \wedge (\nu \circ \eta)(x)] = (\mu \circ (\nu \circ \eta))(r).$$

Consider now the universal algebra of polynomial functions of $\langle F^*(H), \circ\rangle$. The images of the elements of this algebra are the sums of nonzero fuzzy subsets of $\mathbb{H}$. Thus we can define α on H by:

$a\alpha b \Longleftrightarrow \exists x_1, ..., x_n \in H (n \in \mathbb{N})$: $(\chi_{x_1} \circ ... \circ \chi_{x_n})(a) > 0$ and $(\chi_{x_1} \circ ... \circ \chi_{x_n})(b) > 0$.

Consider the quotient set H/α^* with the hyperoperation

$$\alpha^*(a) \circ \alpha^*(b) = \{\alpha^*(c) \mid (a' \circ b')(c) > 0, \quad a'\alpha^* a, \quad b'\alpha^* b\}.$$

Really $\circ$ is an operation, because α^* is the fundamental relation on $\mathbb{H}$. Also

$\alpha^*(x) \circ \alpha^*(y) \circ \alpha^*(z)) = \alpha^*(x) \circ \alpha^*(k) = \alpha^*(l)$, where $(y \circ z)(k) > 0$ and $(x \circ k)(l) > 0$.

Therefore, $0 < (x \circ (y \circ z))(l) = ((x \circ y) \circ z)(l) = \bigvee_{p\in H} [(x \circ y)(p) \wedge (p \circ z)(l)]$. Thus there exists $p \in H$, such that $\alpha^*(l) = \alpha^*(p) \circ \alpha^*(z) = (\alpha^*(x) \circ \alpha^*(y)) \circ \alpha^*(z)$, that the operation $\circ$ in H/α^* is associative. Moreover, if $\mathbb{H}$=$\langle H, \circ\rangle$ be a fuzzy hypergroup, that is $x \circ H = H \circ x = \chi_H$, for every $x \in H$, since for every $\alpha^*(a), \alpha^*(b) \in H/\alpha^*$, there exist $\alpha^*(t), \alpha^*(s) \in H/\alpha^*$, such that, $\alpha^*(a) \circ \alpha^*(t) = \alpha^*(b)$ and $\alpha^*(s) \circ \alpha^*(a) = \alpha^*(b)$, it is concluded that $\mathbb{H}/\alpha^*$=$\langle H/\alpha^*, \circ\rangle$ is a group.

Example 4.8. Let $\mathbb{R}$=$\langle R, \oplus, \odot\rangle$ be a fuzzy hyperring. This means that $\langle R, \oplus\rangle$ is a commutative fuzzy hypergroup, $\langle R, \odot\rangle$ is a fuzzy hypersemigroup and for all $x, y, z \in R$ satisfies: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ (for more details see [13]). Let $\mathcal{F}^*(\mathbb{R})$=$\langle F^*(R), \oplus, \odot\rangle$ be the universal algebra with two binary operations defined as follows:

$$(\mu \oplus \nu)(r) = \bigvee_{x,y \in H} [\mu(x) \wedge \nu(y) \wedge (x \oplus y)(r)],$$

$$(\mu \odot \nu)(r) = \bigvee_{x,y \in H} [\mu(x) \wedge \nu(y) \wedge (x \odot y)(r)],$$

for all $\mu, \nu \in F^*(R)$, $r \in R$. Obviously, the operation $\oplus$ in $F^*(R)$ is commutative, and $\oplus$ and $\odot$ in $F^*(R)$ are associative. By distributivity of the lattice [0, 1] and distributivity $\odot$ with respect to $\oplus$ in R, we will prove that the operation $\odot$ in $F^*(R)$ is distributive with respect to the operation $\oplus$, too.

For every $\mu, \nu,\ eta \in F^*(R)$ and $r \in R$ we have:

$$(\mu \odot (\nu \oplus \eta))(r) = \bigvee_{x,y \in R} [\mu(x) \wedge (\nu \oplus \eta)(y) \wedge (x \odot y)(r)] =$$

$$\bigvee_{x,y \in R} [\mu(x) \wedge (\bigvee_{s,t \in R} \nu(s) \wedge \eta(t) \wedge (s \oplus t)(y)) \wedge (x \odot y)(r)] =$$

$$\bigvee_{x,y \in R} [\bigvee_{s,t \in R} (\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (s \oplus t)(y) \wedge (x \odot y)(r))] =$$

$$\bigvee_{x,s,t \in R} [\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (\bigvee_{y \in R} (x \odot y)(r) \wedge (s \oplus t)(y))] =$$

$$\bigvee_{x,s,t \in R} [\mu(x) \wedge \nu(s) \wedge \eta(t) \wedge (\bigvee_{p,q \in R} (x \odot s)(p) \wedge (x \odot t)(q) \wedge (p \oplus q)(r))] =$$

$\bigvee_{x,s,t\in R}[\bigvee_{p,q\in R}(\mu(x)\wedge\eta(t)\wedge(x\odot t)(q)\wedge\mu(x)\wedge\nu(s)\wedge(x\odot s)(p)\wedge(p\oplus q)(r))]=$
$\bigvee_{p,q\in R}[(\bigvee_{x,t\in R}\mu(x)\wedge\eta(t)\wedge(x\odot t)(q))\wedge(\bigvee_{x,s\in R}\mu(x)\wedge\nu(s)\wedge(x\odot s)(p))\wedge(p\oplus q)(r)]=$
$\bigvee_{p,q\in R}[(\mu\odot\eta)(q)\wedge(\mu\odot\nu)(p)\wedge(p\oplus q)(r)]=((\mu\odot\nu)\oplus(\mu\odot\eta))(r).$
And analogously, $(\mu\oplus\nu)\odot\eta=(\mu\odot\eta)\oplus(\nu\odot\eta)$. Now we can construct the universal algebra (with two binary operations) of the polynomial functions of $\mathcal{F}^*(\mathbb{R})$ for any $n\in\mathbb{N}$. The images of the elements of this algebra are the sums of products of nonzero fuzzy subsets of $\mathbb{R}$. Thus we can define α on $\mathbb{R}$ by;

$a\alpha b\Longleftrightarrow\exists x_{ij}\in R, i\in\{1,...,k_j\}, j\in\{1,...,l\}, k_j,l\in\mathbb{N}$:

$$(\oplus_{j=1}^{l}(\odot_{i=1}^{k_j}\chi_{x_{ij}}))(a)>0 \text{ and } (\oplus_{j=1}^{l}(\odot_{i=1}^{k_j}\chi_{x_{ij}}))(b)>0.$$

Consider the quotient set R/α^* withe two following hyperoperations :

$$\alpha^*(a)\oplus\alpha^*(b)=\{\alpha^*(c)\mid(a'\oplus b')(c)>0, a'\alpha^*a, b'\alpha^*b\}, \text{ and}$$

$$\alpha^*(a)\odot\alpha^*(b)=\{\alpha^*(c)\mid(a'\odot b')(c)>0, a'\alpha^*a, b'\alpha^*b\}$$

Actually $\oplus$ and $\odot$ are operations, because α^* is the fundamental relation on $\mathbb{R}$. By considering the previous example, obviously $\langle R/\alpha^*,\oplus\rangle$ is a commutative group. We verify the distributivity of $\odot$ with respect to $\oplus$ for the universal algebra $\mathbb{R}/\alpha^*=\langle R/\alpha^*,\oplus,\odot\rangle$. We have

$\alpha^*(a)\odot(\alpha^*(b)\oplus\alpha^*(c))=\alpha^*(a)\odot\alpha^*(d)=\alpha^*(e)$, where $(b\oplus c)(d)>0$ and $(a\odot d)(e)>0$
$0<(a\odot(b\oplus c))(e)=\bigvee_{p\in R}(a\odot p)(e)\wedge(b\oplus c)(p)$. Thus
$0<((a\odot b)\oplus(a\odot c))(e)=\bigvee_{x,y\in R}(a\odot b)(x)\wedge(a\odot c)(y)\wedge(x\oplus y)(e)$. Therefore, there exist $x,y\in R$ such that $\alpha^*(e)=\alpha^*(x)+\alpha^*(y)=(\alpha^*(a)+\alpha^*(b))\oplus(\alpha^*(a)\odot\alpha^*(c))$, and hence it was proved that $\alpha^*(a)\odot(\alpha^*(b)\oplus\alpha^*(c))=(\alpha^*(a)+\alpha^*(b))\oplus(\alpha^*(a)\odot\alpha^*(c))$. Analogously, we can prove that $(\alpha^*(b)\oplus\alpha^*(c))\odot\alpha^*(a))=(\alpha^*(b)\odot\alpha^*(a))\oplus(\alpha^*(c)\odot\alpha^*(a))$. Thus it concluded that $\mathbb{R}/\alpha^*=\langle R/\alpha^*,\oplus,\odot\rangle$ is a ring, as desired.□

Conclusion

We introduced and studied term functions over fuzzy hyperalgebras, as the largest class of fuzzy algebraic systems. We use the idea that the set of nonzero fuzzy subsets of a fuzzy hyperalgebra can be organized naturally as a universal algebra, and constructed the term functions over this algebra. We gave the form of generated subfuzzy hyperalgebra of a given fuzzy hyperalgebra as a generalization of universal algebras and multialgebras. Finally, we characterized the form of the fundamental relation of a fuzzy hyperalgebra, to construct the fundamental universal algebra corresponding to a given fuzzy hyperalgebra, and this result guarantee that that fundamental relation on any fuzzy algebraic hyperstructures, such as fuzzy hypergroups, fuzzy hyperrings, fuzzy hypermodules,... exists.

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Error Locating Codes Dealing with Repeated Low-Density Burst Errors

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Abstract. This paper presents a study of linear codes which are capable to detect and locate errors which are repeated low-density bursts of length b(fixed) with weight w or less. An illustration for such a kind of code has also been provided.

Keywords: Error locating codes, burst errors, burst errors of length of b(fixed), repeated low-density burst errors of length b(fixed).

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1 Introduction

Burst errors are the type of errors that occur quite frequently in several communication channels. Codes developed to detect and correct such errors have been studied extensively by many authors. Abramson [1959] developed codes which dealt with the correction of single and double adjacent errors, which was extended by Fire [1959] as a more general concept called 'burst errors'. A burst of length b is defined as follows:

Definition 1. A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non-zero.

The nature of burst errors differs from channel to channel depending upon the kind of channel. Chien and Tang [1965] proposed a modification in the definition of a burst and they defined a burst of length b, which shall be called as CT-burst of length b, as follows:

Definition 2. A CT-burst of length b is a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero.

Channels due to Alexander, Gryb and Nast [1960] fall in this category. This definition was further modified by Dass [1980] as follows:

Definition 3. A burst of length b(fixed) is an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions is among the first $n-b+1$ components.

This definition is useful for channels not producing errors near the end of a code word. In very busy communication channels errors repeat themselves. So is a situation when errors occur in the form of bursts. Dass, Garg and Zannetti [2008] studied this kind of repeated burst errors. They termed such errors as m-repeated burst errors of length b(fixed) which has been defined as follows:

Definition 4. An m-repeated bursts of length b(fixed) is an n-tuple whose only non-zero components are confined to m distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - mb + 1$ components.

In particular a 2-repeated bursts of length b(fixed) has been defined by Dass and Garg [2009(a)] as follows:

Definition 5. A 2-repeated bursts of length b(fixed) is an n-tuple whose only non-zero components are confined to 2 distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - 2b + 1$ components.

During the process of transmission some disturbances cause occurrence of burst errors in such a way that over a given length, some digits are received correctly while others get corrupted i.e. not all the digits inside a burst are in error. Such bursts are termed as low-density bursts [Wyner (1963)].

A low-density burst of length b(fixed) with weight w or less has been defined as follows:

Definition 6. A low-density burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero with at most w $(w \leq b)$ non-zero components within such b consecutive digits and the number of starting positions of the burst is among the first $n-b+1$ components.

Dass and Garg [2009(b)] studied codes which are capable to detect and/or correct m-repeated low-density bursts of length b(fixed) with weight w or less. They defined such codes as follows:

Definition 7. An m-repeated low-density burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to m distinct sets of b consecutive positions, the first component of each set is non-zero where each set can have at most w non-zero components $(w \leq b)$, and the number of its starting positions in an n-tuple is among the first $n-mb+1$ positions.

In particular, a 2-repeated low-density burst of length b(fixed) with weight w or less has been defined as follows:

Definition 8. A 2-repeated low-density burst of length b(fixed) with weight w or less is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive positions, the first component of each set is non-zero where each set can have at most w non-zero components $(w \leq b)$, and the number of its starting positions in an n-tuple is among the first $n-2b+1$ positions.

As an illustration, (21010000102000) is a 2-repeated low-density burst of length up to 6(fixed) with weight 3 or less over GF(3) whereas (001000011110) is a 2-repeated low-density burst of length at most 5(fixed) with weight 4 or less over GF(2).

In this paper we have presented a study of codes dealing with the location of such kind of errors occurring within a sub-block. The concept of error-locating codes, lying midway between error detection and error correction, was introduced by Wolf and Elspas [1963]. In this technique the block of received digits is to be regarded as subdivided into mutually exclusive sub-blocks and while decoding it is possible to detect the error and in addition the receiver is able to identify which particular sub-block contains error. Such codes are referred to as Error-Locating codes (EL-codes). Wolf and Elspas [1963] studied binary codes which are capable of detecting and locating a single sub-block containing random errors. A study of codes locating burst errors of length b(fixed) has been made by Dass and Kishanchand [1986]. Dass and Arora [2010] obtained bounds for codes capable of locating repeated burst errors of length b(fixed) occurring within a sub-block.

In this paper we have obtained bounds on the number of check digits required to locate 2-repeated low-density bursts of length b(fixed), and m-repeated low-density bursts of length b(fixed). An illustration of such a code has also been given. Development of such codes will economize in the number of parity-check digits required in comparison to the usual low-density burst error locating codes while considering such repeated bursts as single bursts.

The paper has been organized as follows. In section 2 the necessary condition for the detection and location of 2-repeated low-density burst of length b(fixed) with weight w or less has been derived. This is followed by a sufficient condition for the existence of such a code. An illustration of 2-repeated low-density burst of length b(fixed) with weight w or less over GF(2) has also been given. In section 3 a necessary condition for the detection and location of m-repeated low-density burst of length b(fixed) with weight w or less has been given followed by a sufficient condition for the existence of such a code.

In what follows we shall consider a linear code to be a subspace of n-tuples over $\mathrm{GF}(q)$. The block of n digits, consisting of r check digits and $k = n - r$ information digits, is considered to be divided into s mutually exclusive sub-blocks. Each sub-block contains $t = n/s$ digits.

2 2-Repeated Low-density Burst Error Locating Codes

In this section, we consider (n, k) linear codes over $\mathrm{GF}(q)$ that are capable of detecting and locating all 2-repeated low-density burst of length b(fixed) with weight w or less within a single sub-block.

It may be noted that an EL-code capable of detecting and locating a single sub-block containing an error which is in the form of a 2-repeated low-density bursts of length b(fixed) with weight w or less must satisfy the following conditions:

(a) The syndrome resulting from the occurrence of a 2-repeated low-density burst of length b(fixed) with weight w or less within any one sub-block must be distinct from the all zero syndrome.

(b) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b(fixed) with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated low-density burst of length b(fixed) with weight w or less *within* any other sub-block.

In this section we shall derive two results. The first result derives a lower bound on the number of check digits required for the existence of a linear code over GF(q) capable of detecting and locating a single sub-block containing errors that are 2-repeated low-density burst of length b(fixed) with weight w or less. In the second result, an upper bound on the number of check digits which ensures the existence of such a code has been derived.

As the code is divided into several blocks of length t each and we wish to detect a 2-repeated low-density burst of length b(fixed) with weight w or less, we may come across with a situation when the difference between $2b$ and t ($b+w$ and t) becomes narrow. We note that if $t-b+1 < b+w$ and if we consider any two 2-repeated low-density bursts x_1 and x_2 of length b(fixed) with weight w or less such that their non-zero components are confined to first $t-b+1$ positions with w components confining to some fixed w positions out of first b consecutive positions then their difference x_1 - x_2 is again a 2-repeated low-density burst of length b(fixed) with weight w or less. However if we do not restrict ourselves to first $t-b+1$ positions then we may not get a 2-repeated burst of length b(fixed) with weight w or less. This may be better understood with the help of the following examples:

Example 1. Let $t = 9$, $b = 4$, $w = 3$ and $q = 2$. So that $t - b + 1 = 6 < b + w (= 7)$.

Let $x_1 = (101101001)$ and $x_2 = (100101011)$.

Then x_1 and x_2 are 2-repeated low-density burst of length 4(fixed) with weight 3 or less whereas $x_1 - x_2 = (001000010)$ is not a 2-repeated burst of length 4(fixed).

Example 2. Let $t = 11$, $b = 5$, $w = 3$ and $q = 2$.

Let $x_1 = (10101010010)$ and $x_2 = (10101010001)$

Then x_1 and x_2 are 2-repeated low-density burst of length 5(fixed) with weight 3 or less whereas $x_1 - x_2 = (00000000011)$ which is not even a 2-repeated burst of length 4(fixed) what to talk of its weight.

So, accordingly we discuss the following cases:

Case 1: When $t - b + 1 \geq 2b$.

Let X be the collection of all those vectors in which all the non-zero components are confined to some fixed w positions out of two sets of b consecutive positions each i.e. l-th to $(l + b)$-th position and j-th to $(j + b)$-th position where $j > l + b$.

We observe that the syndromes of all the elements of X should be different; else for any x_1, x_2 belonging to X having the same syndrome would imply that the syndrome of $x_1 - x_2$ which is also an element of X and hence a 2-repeated low density burst of length b(fixed) with weight w or less within the same sub-block becomes zero; in violation of condition (a). Also, since the error locates a single sub-block containing errors that are 2-repeated low-density bursts of length b(fixed) of weight w or less, the syndromes produced by similar vectors in different sub-blocks must be distinct by condition (b).

Thus the syndromes of vectors which are 2-repeated low-density burst of length b(fixed) with weight w or less in fixed positions, whether in the same sub-block or in different sub-blocks, must be distinct. (It may be noted that the choice of different fixed components in different sub-blocks will also yield the same result).

As there are $(q^{2w}-1)$ distinct non-zero syndromes corresponding to the vectors in any one sub-block and there are s sub-blocks in all, so we must have atleast $(1+s(q^{2w}-1))$ distinct syndromes counting the all zero syndrome.

As maximum number of distinct syndromes available using r check bits is q^r, so there are q^r distinct syndromes in all, therefore we must have

$$q^r \geq \{1+s(q^{2w}-1)\} \tag{1}$$

where $t-b+1 \geq 2b$.

Case 2: When $b+w \leq t-b+1 < 2b$.

Let X be the collection of all those vectors in which all the non-zero components are confined to some w fixed positions out of first b components i.e first and b-th position and another set of w fixed positions out of $(b+1)$-th to $(t-b+1)$-th positions.

As discussed in case 1 the syndromes of all the elements of X is different.

In this case also, there are $(q^{2w}-1)$ distinct non-zero syndromes corresponding to the vectors in any one sub-block and there are s sub-blocks in all, so we must have atleast $(1+s(q^{2w}-1))$ distinct syndromes counting the all zero syndrome.

So, in this case also, we must have

$$q^r \geq \{1+s(q^{2w}-1)\} \tag{2}$$

where $b+w \leq t-b+1 < 2b$.

Case 3: When $t-b+1 < b+w$.

In this case consider X to be collection of all those vectors in which all the non-zero components are confined to some w fixed positions out of first b positions and $t-2b+1$ components from $(b+1)$-th to $(t-b+1)$-th positions. In this case there are $(q^{w+(t-2b+1)}-1)$ distinct non-zero syndromes corresponding to the vectors in any one sub-block. As and there are s sub-blocks in all, so we must have atleast $(1+s(q^{w+(t-2b+1)}-1))$ distinct syndromes counting the all zero syndrome.

Therefore in this case, we must have

$$q^r \geq \{1+s(q^{w+(t-2b+1)}-1)\} \tag{3}$$

where $t-b+1 < b+w$.

From (1), (2), and (3) we have

$$r \geq \begin{cases} \log_q\{1+s(q^{2w}-1)\} & \text{where } t-b+1 \geq 2b \\ & \text{and } b+w \leq t-b+1 < 2b \\ \log_q\{1+s(q^{w+(t-2b+1)}-1)\} & \text{where } t-b+1 < b+w. \end{cases}$$

Thus we have proved:

Theorem 1. *The number of parity check digits r in an (n,k) linear code subdivided into s sub-blocks of length t each, that locates a single corrupted* *sub-block containing errors that are 2-repeated low density burst of length b (fixed) with weight w or less is at least*

$$\begin{cases} \log_q\{1+s(q^{2w}-1)\} & \text{where } t-b+1 \geq 2b \\ & \text{and } b+w \leq t-b+1 < 2b\,. \\ \log_q\{1+s(q^{w+(t-2b+1)}-1)\} & \text{where } t-b+1 < b+w \end{cases}$$

Remark 1. For $w = b$, the weight consideration over the burst becomes redundant and the result coincides with Theorem 1[Dass and Arora [2010]], when the bursts considered are 2-repeated bursts of length b(fixed).

In the following result we derive another bound on the number of check digits required for the existence of such a code. The proof is based on the technique used to establish Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code [refer Sacks[1958], also Theorem 4.7 Peterson and Weldon[1972]]. This technique not only ensures the existence of such a code but also gives a method for the construction of such a code.

Theorem 2. *An (n, k) linear EL-code over* GF(q) *capable of detecting a 2-repeated low density burst of length b (fixed) with weight w or less ($w \leq b$) within a single sub-block and of locating that sub-block can always be constructed provided that*

$$q^{n-k} > [1+(q-1)]^{(b-1,w-1)}\{1+(q-1)(t-2b+1)[1+(q-1)]^{(b-1,w-1)}\} \cdot \left\{1+(s-1)\sum_{i=1}^{2}\binom{t-ib+i}{i}(q-1)^i\{[1+(q-1)]^{(b-1,w-1)}\}^i\right\} \quad (4)$$

where $[1+x]^{(m,r)}$ denotes the incomplete binomial expansion of $(1+x)^m$ up to the term x^r in ascending power of x, viz.

$$[1+x]^{(m,r)} = 1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r.$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n-k\times n)$ parity check matrix H by a synthesis procedure. For that we first construct a matrix H_1 from which the requisite parity check matrix H shall be obtained by reversing the order of the columns of each sub-block.

After adding $(s-1)t$ columns appropriately corresponding to the first $(s-1)$ sub-blocks, suppose that we have added the first $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ of the s-th sub-block also, out of which the first $b-1$ columns $h_1, h_2, \ldots, h_{b-1}$ may be chosen arbitrarily (non-zero). We now lay down the condition to add the j-th column h_j to H_1 as follows:

According to condition (a), for the detection of 2-repeated low-density burst of length b(fixed) with weight w or less in the s-th sub-block h_j should not be a linear combination of any $w-1$ or fewer columns among the immediately preceding $b-1$ columns $h_{j-b+1}, h_{j-b+2}, \ldots h_{j-1}$ together with any w or fewer columns from amongst some b consecutive columns from the first $j-b$ columns of the s-th sub-block.

i.e.

$$h_j \neq (\alpha_{j_1}h_{j_1}+\alpha_{j_2}h_{j_2}+\ldots+\alpha_{j_{w-1}}h_{j_{w-1}})\ +(\beta_{l_1}h_{l_1}+\beta_{l_2}h_{l_2}+\ldots+\beta_{l_w}h_{l_w}) \quad (5)$$

where $h_{j_1}, h_{j_2}, \ldots, h_{j_{w-1}}$ are any $w-1$ columns among $h_{j-b+1}, h_{j-b+2}, \ldots h_{j-1}$ and h_l's are any w columns from a set of b consecutive columns among the first $j-b$ columns of the s-th sub-block such that either all the coefficients β_{l_i}'s are zero or if the p-th coefficient β_{l_p} is the last non-zero coefficients then $b \leq p \leq j-b$;

$$\alpha_{j_i}\text{'s}, \beta_{l_i}\text{'s} \in \mathrm{GF}(q).$$

The number of ways in which the coefficients α_i's can be selected is $[1+(q-1)]^{(b-1,w-1)}$. To enumerate the coefficients β_i's is equivalent to enumerate the number of bursts of length b(fixed) with weight w or less in a vector of length $j-b$.

This number including the vector of all zeros [refer Theorem 1, Dass [1983]] is

$$1+(j-2b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}$$

So, the number of linear combinations on the right hand side of (5) is

$$[1+(q-1)]^{(b-1,w-1)}[1+(j-2b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}] \quad (6)$$

According to condition (b), for the location of 2-repeated low-density bursts of length b(fixed) with weight w or less, h_j should not be a linear combination of any $w-1$ or fewer columns among the immediately preceding the $b-1$ columns and any w columns from a set of b consecutive columns from the remaining $j-b$ columns of the s-th sub-block along with any w or less columns each from any of the two sets of b consecutive columns out of any one of the previously chosen $s-1$ sub-blocks, the coefficient of the last column of either both or one of the sets being non-zero.

The number of 2-repeated low-density bursts of length b(fixed) with weight w or less in a sub-block of length t [refer Dass and Garg [2009(b)]] is

$$\sum_{i=1}^{2}\binom{t-ib+i}{i}(q-1)^i\{[1+(q-1)]^{(b-1,w-1)}\}^i \quad (7)$$

Since there are $(s-1)$ previous sub-blocks, therefore number of such linear combinations becomes

$$(s-1)\sum_{i=1}^{2}\binom{t-ib+i}{i}(q-1)^i\{[1+(q-1)]^{(b-1,w-1)}\}^i \tag{8}$$

So, for the location of 2-repeated low-density burst of length b(fixed) with weight w or less the number of linear combinations to which h_j can not be equal to is the product of expr.(6) and expr.(8)

$$\text{i.e. expr.(6)} \times \text{ expr.(8)} \tag{9}$$

Thus the total number of linear combinations to which h_i can not be equal to is the sum of exp.(6) and exp.(9) At worst all these combinations might yield distinct sum.

Therefore h_i can be added to the s-th sub-block provided that

$$q^{n-k} > [1+(q-1)]^{(b-1,w-1)}\{1+(q-1)(j-2b+1)[1+(q-1)]^{(b-1,w-1)}\} \cdot \left\{1+(s-1)\sum_{i=1}^{2}\binom{t-ib+i}{i}(q-1)^i\{[1+(q-1)]^{(b-1,w-1)}\}^i\right\}$$

To obtain the length of the block as t we replace j by t in the above expression.

The required parity-check matrix H can be obtained from H_1 by reversing the order of the columns in each sub-block.

Remark 2. For $w = b$, the weight consideration over the burst becomes redundant and the inequality in Theorem 2 reduces to

$$\begin{aligned} q^{n-k} &> q^{b-1}\{1+(q-1)(t-2b+1)q^{b-1}\} \\ &\times \left\{1+(s-1)\sum_{i=1}^{2}\binom{t-ib+i}{i}(q-1)^i q^{i(b-1)}\right\} \end{aligned}$$

which coincides with the condition for the location of 2-repeated burst of length b(fixed) [refer Theorem 2, Dass and Arora [2010]].

We conclude this section with the following example:

Example 3. For an (27,15) linear code over GF(2) consider the following 12×27 matrix H which has been constructed by the synthesis procedure given in the proof of theorem 2 by taking $s = 3,\ t = 9,\ b = 3,\ w = 2.$

$$H = \begin{bmatrix}
0\,0\,0\,0\,0\,0\,0\,0\,1 & 1\,1\,1\,1\,1\,1\,0\,0\,0 & 1\,0\,1\,0\,0\,0\,1\,1\,1 \\
0\,0\,0\,0\,0\,0\,0\,1\,0 & 1\,1\,0\,1\,1\,1\,0\,0\,0 & 1\,0\,0\,0\,1\,0\,1\,1\,0 \\
0\,0\,0\,0\,0\,0\,1\,0\,0 & 1\,1\,1\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,1\,1\,1\,1\,0 \\
0\,0\,0\,0\,0\,1\,0\,0\,0 & 1\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,1\,0\,0\,0\,0 \\
0\,0\,0\,0\,1\,0\,0\,0\,0 & 1\,1\,1\,1\,0\,0\,0\,0\,0 & 1\,0\,1\,0\,0\,1\,1\,0\,0 \\
0\,0\,0\,1\,0\,0\,0\,0\,0 & 1\,1\,0\,1\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0\,1\,0\,0 \\
0\,0\,1\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0\,1\,0\,0 \\
0\,1\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,0\,1\,1\,0\,1\,1\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,1\,0\,0\,0\,0\,0 & 1\,1\,1\,1\,1\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,0\,1\,0\,0\,0\,0\,1 & 0\,0\,1\,1\,1\,1\,1\,1\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,0\,1\,0\,0\,1\,0 & 0\,1\,1\,1\,1\,0\,1\,1\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,1\,0\,1\,0\,0 & 0\,0\,1\,1\,0\,0\,1\,1\,1
\end{bmatrix}$$

The null space of this matrix can be used as a code to locate a sub-block of length $t = 9$ containing 2-repeated burst of length 3(fixed). From the error pattern syndrome Table 1 we observe that:

The syndromes of 2-repeated burst of length 3(fixed) within any sub-block are all non-zero showing thereby that the code detects all 2-repeated low-density bursts of length 3(fixed) with weight 2 or less occurring within a sub-block.

It has been verified through MS-Excel program that the syndromes of the 2-repeated bursts of length 3(fixed) with weight 2 or less in any sub-block is different from the syndrome of a 2-repeated burst of length 3(fixed) with weight 2 or less within any other sub-block.

Table 1

	Low density 2-repeated bursts of length 3(fixed)			Syndromes
		Sub-block - 1		
1	100100000	000000000	000000000	0000 0100 1000
2	100101000	000000000	000000000	0001 0100 1000
3	100110000	000000000	000000000	0000 1100 1000
4	101100000	000000000	000000000	0000 0110 1000
5	101101000	000000000	000000000	0001 0110 1000
6	101110000	000000000	000000000	0000 1110 1000
7	110100000	000000000	000000000	0000 0101 1000
8	110101000	000000000	000000000	0001 0101 1000
9	110110000	000000000	000000000	0000 1101 1000
10	100010000	000000000	000000000	0000 1000 1000
11	100010100	000000000	000000000	0010 1000 1000
12	100011000	000000000	000000000	0001 1000 1000
13	101010000	000000000	000000000	0000 1010 1000
14	101010100	000000000	000000000	0010 1010 1000
15	101011000	000000000	000000000	0001 1010 1000
16	110010000	000000000	000000000	0000 1001 1000
17	110010100	000000000	000000000	0010 1001 1000
18	110011000	000000000	000000000	0001 1001 1000
19	100001000	000000000	000000000	0001 0000 1000
20	100001010	000000000	000000000	0101 0000 1000
21	100001100	000000000	000000000	0011 0000 1000
22	101001000	000000000	000000000	0001 0010 1000
23	101001010	000000000	000000000	0101 0010 1000
24	101001100	000000000	000000000	0011 0010 1000
25	110001000	000000000	000000000	0001 0001 1000
26	110001010	000000000	000000000	0101 0001 1000
27	110001100	000000000	000000000	0011 0001 1000
28	100000100	000000000	000000000	0010 0000 1000
29	100000101	000000000	000000000	1010 0000 1000

	Low density 2-repeated bursts of length 3(fixed)			Syndromes
		Sub-block - 1		
30	100000110	000000000	000000000	0110 0000 1000
31	101000100	000000000	000000000	0010 0010 1000
32	101000101	000000000	000000000	1010 0010 1000
33	101000110	000000000	000000000	0110 0010 1000
34	110000100	000000000	000000000	0010 0001 1000
35	110000101	000000000	000000000	1010 0001 1000
36	110000110	000000000	000000000	0110 0001 1000
37	010010000	000000000	000000000	0000 1001 0000
38	010010100	000000000	000000000	0010 1001 0000
39	010011000	000000000	000000000	0001 1001 0000
40	010110000	000000000	000000000	0000 1101 0000
41	010110100	000000000	000000000	0010 1101 0000
42	010111000	000000000	000000000	0001 1101 0000
43	011010000	000000000	000000000	0000 1011 0000
44	011010100	000000000	000000000	0010 1011 0000
45	011011000	000000000	000000000	0001 1011 0000
46	010001000	000000000	000000000	0001 0001 0000
47	010001010	000000000	000000000	0101 0001 0000
48	010001100	000000000	000000000	0011 0001 0000
49	010101000	000000000	000000000	0001 0101 0000
50	010101010	000000000	000000000	0101 0101 0000
51	010101100	000000000	000000000	0011 0101 0000
52	011001000	000000000	000000000	0001 0011 0000
53	011001010	000000000	000000000	0101 0011 0000
54	011001100	000000000	000000000	0011 0011 0000
55	010000100	000000000	000000000	0010 0001 0000
56	010000101	000000000	000000000	1010 0001 0000
57	010000110	000000000	000000000	0110 0001 0000
58	010100100	000000000	000000000	0010 0101 0000
59	010100101	000000000	000000000	1010 0101 0000
60	010100110	000000000	000000000	0110 0101 0000
61	011000100	000000000	000000000	0010 0011 0000
62	011000101	000000000	000000000	1010 0011 0000
63	011000110	000000000	000000000	0110 0011 0000
64	001001000	000000000	000000000	0001 0010 0000
65	001001010	000000000	000000000	0101 0010 0000
66	001001100	000000000	000000000	0011 0010 0000

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 1		
67	001011000	000000000	000000000	0001 1010 0000
68	001011010	000000000	000000000	0101 1010 0000
69	001011100	000000000	000000000	0011 1010 0000
70	001101000	000000000	000000000	0001 0110 0000
71	001101010	000000000	000000000	0101 0110 0000
72	001101100	000000000	000000000	0011 0110 0000
73	001000100	000000000	000000000	0010 0010 0000
74	001000101	000000000	000000000	1010 0010 0000
75	001000110	000000000	000000000	0110 0010 0000
76	001010100	000000000	000000000	0010 1010 0000
77	001010101	000000000	000000000	1010 1010 0000
78	001010110	000000000	000000000	0110 1010 0000
79	001100100	000000000	000000000	0010 0110 0000
80	001100101	000000000	000000000	1010 0110 0000
81	001100110	000000000	000000000	0110 0110 0000
82	000100100	000000000	000000000	0010 0100 0000
83	000100101	000000000	000000000	1010 0100 0000
84	000100110	000000000	000000000	0110 0100 0000
85	000101100	000000000	000000000	0011 0100 0000
86	000101101	000000000	000000000	1011 0100 0000
87	000101110	000000000	000000000	0111 0100 0000
88	000110100	000000000	000000000	0010 1100 0000
89	000110101	000000000	000000000	1010 1100 0000
90	000110110	000000000	000000000	0110 1100 0000
91	100000000	000000000	000000000	0000 0000 1000
92	101000000	000000000	000000000	0000 0010 1000
93	110000000	000000000	000000000	0000 0001 1000
94	010000000	000000000	000000000	0000 0001 0000
95	010100000	000000000	000000000	0000 0101 0000
96	011000000	000000000	000000000	0000 0011 0000
97	001000000	000000000	000000000	0000 0010 0000
98	001010000	000000000	000000000	0000 1010 0000
99	001100000	000000000	000000000	0000 0110 0000
100	000100000	000000000	000000000	0000 0100 0000
101	000101000	000000000	000000000	0001 0100 0000
102	000110000	000000000	000000000	0000 1100 0000

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 1		
103	000010000	000000000	000000000	0000 1000 0000
104	000010100	000000000	000000000	0010 1000 0000
105	000011000	000000000	000000000	0001 1000 0000
106	000001000	000000000	000000000	0001 0000 0000
107	000001010	000000000	000000000	0101 0000 0000
108	000001100	000000000	000000000	0011 0000 0000
109	000000100	000000000	000000000	0010 0000 0000
110	000000101	000000000	000000000	1010 0000 0000
111	000000110	000000000	000000000	0110 0000 0000

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 2		
112	000000000	100100000	000000000	0011 0000 1100
113	000000000	100101000	000000000	1111 0000 1100
114	000000000	100110000	000000000	1111 0000 1111
115	000000000	101100000	000000000	1001 1010 0110
116	000000000	101101000	000000000	0101 1010 0110
117	000000000	101110000	000000000	0101 1010 0101
118	000000000	110100000	000000000	1101 1110 0010
119	000000000	110101000	000000000	0001 1110 0010
120	000000000	110110000	000000000	0001 1110 0001
121	000000000	100010000	000000000	0011 1100 0011
122	000000000	100010100	000000000	0011 1100 0010
123	000000000	100011000	000000000	1111 1100 0011
124	000000000	101010000	000000000	1001 0110 1001
125	000000000	101010100	000000000	1001 0110 1000
126	000000000	101011000	000000000	0101 0110 1001
127	000000000	110010000	000000000	1101 0010 1101
128	000000000	110010100	000000000	1101 0010 1100
129	000000000	110011000	000000000	0001 0010 1101
130	000000000	100001000	000000000	0011 1100 0000
131	000000000	100001010	000000000	0011 1100 0010
132	000000000	100001100	000000000	0011 1100 0001
133	000000000	101001000	000000000	1001 0110 1010
134	000000000	101001010	000000000	1001 0110 1000
135	000000000	101001100	000000000	1001 0110 1011
136	000000000	110001000	000000000	1101 0010 1110

	Low density 2-repeated bursts of length 3(fixed)			Syndromes
		Sub-block - 2		
137	000000000	110001010	000000000	1101 0010 1100
138	000000000	110001100	000000000	1101 0010 1111
139	000000000	100000100	000000000	1111 1100 0001
140	000000000	100000101	000000000	1111 1100 0101
141	000000000	100000110	000000000	1111 1100 0011
142	000000000	101000100	000000000	0101 0110 1011
143	000000000	101000101	000000000	0101 0110 1111
144	000000000	101000110	000000000	0101 0110 1001
145	000000000	110000100	000000000	0001 0010 1111
146	000000000	110000101	000000000	0001 0010 1011
147	000000000	110000110	000000000	0001 0010 1101
148	000000000	010010000	000000000	0010 1110 1101
149	000000000	010010100	000000000	0010 1110 1100
150	000000000	010011000	000000000	1110 1110 1101
151	000000000	010110000	000000000	1110 0010 0001
152	000000000	010110100	000000000	1110 0010 0000
153	000000000	010111000	000000000	0010 0010 0001
154	000000000	011010000	000000000	1000 0100 0111
155	000000000	011010100	000000000	1000 0100 0110
156	000000000	011011000	000000000	0100 0100 0111
157	000000000	010001000	000000000	0010 1110 1110
158	000000000	010001010	000000000	0010 1110 1100
159	000000000	010001100	000000000	0010 1110 1111
160	000000000	010101000	000000000	1110 0010 0010
161	000000000	010101010	000000000	1110 0010 0000
162	000000000	010101100	000000000	1110 0010 0011
163	000000000	011001000	000000000	1000 0100 0100
164	000000000	011001010	000000000	1000 0100 0110
165	000000000	011001100	000000000	1000 0100 0101
166	000000000	010000100	000000000	1110 1110 1111
167	000000000	010000101	000000000	1110 1110 1011
168	000000000	010000110	000000000	1110 1110 1101
169	000000000	010100100	000000000	0010 0010 0011
170	000000000	010100101	000000000	0010 0010 0111
171	000000000	010100110	000000000	0010 0010 0001
172	000000000	011000100	000000000	0100 0100 0101
173	000000000	011000101	000000000	0100 0100 0001

	Low density 2-repeated bursts of length 3(fixed)			Syndromes
		Sub-block - 2		
174	000000000	011000110	000000000	0100 0100 0111
175	000000000	001001000	000000000	0110 1010 1010
176	000000000	001001010	000000000	0110 1010 1000
177	000000000	001001100	000000000	0110 1010 1011
178	000000000	001011000	000000000	1010 1010 1001
179	000000000	001011010	000000000	1010 1010 1011
180	000000000	001011100	000000000	1010 1010 1000
181	000000000	001101000	000000000	1010 0110 0110
182	000000000	001101010	000000000	1010 0110 0100
183	000000000	001101100	000000000	1010 0110 0111
184	000000000	001000100	000000000	1010 1010 1011
185	000000000	001000101	000000000	1010 1010 1111
186	000000000	001000110	000000000	1010 1010 1001
187	000000000	001010100	000000000	0110 1010 1000
188	000000000	001010101	000000000	0110 1010 1100
189	000000000	001010110	000000000	0110 1010 1010
190	000000000	001100100	000000000	0110 0110 0111
191	000000000	001100101	000000000	0110 0110 0011
192	000000000	001100110	000000000	0110 0110 0101
193	000000000	000100100	000000000	1100 1100 1101
194	000000000	000100101	000000000	1100 1100 1001
195	000000000	000100110	000000000	1100 1100 1111
196	000000000	000101100	000000000	0000 1100 1101
197	000000000	000101101	000000000	0000 1100 1001
198	000000000	000101110	000000000	0000 1100 1111
199	000000000	000110100	000000000	0000 1100 1110
200	000000000	000110101	000000000	0000 1100 1010
201	000000000	000110110	000000000	0000 1100 1100
202	000000000	100000000	000000000	1111 1100 0000
203	000000000	101000000	000000000	0101 0110 1010
204	000000000	110000000	000000000	0001 0010 1110
205	000000000	010000000	000000000	1110 1110 1110
206	000000000	010100000	000000000	0010 0010 0010
207	000000000	011000000	000000000	0100 0100 0100
208	000000000	001000000	000000000	1010 1010 1010
209	000000000	001010000	000000000	0110 1010 1001
210	000000000	001100000	000000000	0110 0110 0110
211	000000000	000100000	000000000	1100 1100 1100

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 2		
212	000000000	000101000	000000000	0000 1100 1100
213	000000000	000110000	000000000	0000 1100 1111
214	000000000	000010000	000000000	1100 0000 0011
215	000000000	000010100	000000000	1100 0000 0010
216	000000000	000011000	000000000	0000 0000 0011
217	000000000	000001000	000000000	1100 0000 0000
218	000000000	000001010	000000000	1100 0000 0010
219	000000000	000001100	000000000	1100 0000 0001
220	000000000	000000100	000000000	0000 0000 0001
221	000000000	000000101	000000000	0000 0000 0101
222	000000000	000000110	000000000	0000 0000 0011

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 3		
223	000000000	000000000	100100000	1111 1110 0111
224	000000000	000000000	100101000	1101 0111 0011
225	000000000	000000000	100110000	1000 1110 1001
226	000000000	000000000	101100000	0111 0111 1000
227	000000000	000000000	101101000	0101 1110 1100
228	000000000	000000000	101110000	0000 0111 0110
229	000000000	000000000	110100000	1111 1110 1101
230	000000000	000000000	110101000	1101 0111 1001
231	000000000	000000000	110110000	1000 1110 0011
232	000000000	000000000	100010000	1000 1111 0110
233	000000000	000000000	100010100	0110 0000 0001
234	000000000	000000000	100011000	1010 0110 0010
235	000000000	000000000	101010000	0000 0110 1001
236	000000000	000000000	101010100	1110 1001 1110
237	000000000	000000000	101011000	0010 1111 1101
238	000000000	000000000	110010000	1000 1111 1100
239	000000000	000000000	110010100	0110 0000 1011
240	000000000	000000000	110011000	1010 0110 1000
241	000000000	000000000	100001000	1101 0110 1100
242	000000000	000000000	100001010	0011 0110 1011
243	000000000	000000000	100001100	0011 1001 1011
244	000000000	000000000	101001000	0101 1111 0011

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 3		
245	000000000	000000000	101001010	1011 1111 0100
246	000000000	000000000	101001100	1011 0000 0100
247	000000000	000000000	110001000	1101 0110 0110
248	000000000	000000000	110001010	0011 0110 0001
249	000000000	000000000	110001100	0011 1001 0001
250	000000000	000000000	100000100	0001 0000 1111
251	000000000	000000000	100000101	1001 0000 1110
252	000000000	000000000	100000110	1111 0000 1000
253	000000000	000000000	101000100	1001 1001 0000
254	000000000	000000000	101000101	0001 1001 0001
255	000000000	000000000	101000110	0111 1001 0111
256	000000000	000000000	110000100	0001 0000 0101
257	000000000	000000000	110000101	1001 0000 0100
258	000000000	000000000	110000110	1111 0000 0010
259	000000000	000000000	010010000	0111 0000 0100
260	000000000	000000000	010010100	1001 1111 0011
261	000000000	000000000	010011000	0101 1001 0000
262	000000000	000000000	010110000	0111 0001 1011
263	000000000	000000000	010110100	1001 1110 1100
264	000000000	000000000	010111000	0101 1000 1111
265	000000000	000000000	011010000	1111 1001 1011
266	000000000	000000000	011010100	0001 0110 1100
267	000000000	000000000	011011000	1101 0000 1111
268	000000000	000000000	010001000	0010 1001 1110
269	000000000	000000000	010001010	1100 1001 1001
270	000000000	000000000	010001100	1100 0110 1001
271	000000000	000000000	010101000	0010 1000 0001
272	000000000	000000000	010101010	1100 1000 0110
273	000000000	000000000	010101100	1100 0111 0110
274	000000000	000000000	011001000	1010 0000 0001
275	000000000	000000000	011001010	0100 0000 0110
276	000000000	000000000	011001100	0100 1111 0110
277	000000000	000000000	010000100	1110 1111 1101
278	000000000	000000000	010000101	0110 1111 1100
279	000000000	000000000	010000110	0000 1111 1010
280	000000000	000000000	010100100	1110 1110 0010
281	000000000	000000000	010100101	0110 1110 0011

Low density 2-repeated bursts of length 3(fixed)				Syndromes
		Sub-block - 3		
282	000000000	000000000	010100110	0000 1110 0101
283	000000000	000000000	011000100	0110 0110 0010
284	000000000	000000000	011000101	1110 0110 0011
285	000000000	000000000	011000110	1000 0110 0101
286	000000000	000000000	001001000	1010 0000 1011
287	000000000	000000000	001001010	0100 0000 1100
288	000000000	000000000	001001100	0100 1111 1100
289	000000000	000000000	001011000	1101 0000 0101
290	000000000	000000000	001011010	0011 0000 0010
291	000000000	000000000	001011100	0011 1111 0010
292	000000000	000000000	001101000	1010 0001 0100
293	000000000	000000000	001101010	0100 0001 0011
294	000000000	000000000	001101100	0100 1110 0011
295	000000000	000000000	001000100	0110 0110 1000
296	000000000	000000000	001000101	1110 0110 1001
297	000000000	000000000	001000110	1000 0110 1111
298	000000000	000000000	001010100	0001 0110 0110
299	000000000	000000000	001010101	1001 0110 0111
300	000000000	000000000	001010110	1111 0110 0001
301	000000000	000000000	001100100	0110 0111 0111
302	000000000	000000000	001100101	1110 0111 0110
303	000000000	000000000	001100110	1000 0111 0000
304	000000000	000000000	000100100	1110 1110 1000
305	000000000	000000000	000100101	0110 1110 1001
306	000000000	000000000	000100110	0000 1110 1111
307	000000000	000000000	000101100	1100 0111 1100
308	000000000	000000000	000101101	0100 0111 1101
309	000000000	000000000	000101110	0010 0111 1011
310	000000000	000000000	000110100	1001 1110 0110
311	000000000	000000000	000110101	0001 1110 0111
312	000000000	000000000	000110110	0111 1110 0001
313	000000000	000000000	100000000	1111 1111 1000
314	000000000	000000000	101000000	0111 0110 0111
315	000000000	000000000	110000000	1111 1111 0010
316	000000000	000000000	010000000	0000 0000 1010
317	000000000	000000000	010100000	0000 0001 0101
318	000000000	000000000	011000000	1000 1001 0101

	Low density 2-repeated bursts of length 3(fixed)			Syndromes
		Sub-block - 3		
319	000000000	000000000	001000000	1000 1001 1111
320	000000000	000000000	001010000	1111 1001 0001
321	000000000	000000000	001100000	1000 1000 0000
322	000000000	000000000	000100000	0000 0001 1111
323	000000000	000000000	000101000	0010 1000 1011
324	000000000	000000000	000110000	0111 0001 0001
325	000000000	000000000	000010000	0111 0000 1110
326	000000000	000000000	000010100	1001 1111 1001
327	000000000	000000000	000011000	0101 1001 1010
328	000000000	000000000	000001000	0010 1001 0100
329	000000000	000000000	000001010	1100 1001 0011
330	000000000	000000000	000001100	1100 0110 0011
331	000000000	000000000	000000100	1110 1111 0111
332	000000000	000000000	000000101	0110 1111 0110
333	000000000	000000000	000000110	0000 1111 0000

Remark 3. The space visible between vectors in the first column in Table 1 has been given to distinguish between different sub-blocks whereas the space given in the syndrome vector is for convenience.

Observation. Syndromes of some of the 2-repeated bursts of length 3(fixed) occurring within the second sub-block are same. For coding efficiency it is desired that the syndromes of the error patterns within any sub-block is identical whenever possible.

3 Location of m-Repeated Low-density burst of length b(fixed)

In this section a necessary and sufficient condition for the location of an m-repeated low-density burst of length b(fixed) with weight w or less has been given.

It may be noted that an EL-code capable of detecting and locating a single sub-block containing an error which is in the form of an m-repeated low-density burst of length b(fixed) with weight w or less $(w \leq b)$ must satisfy the following conditions:

(c) The syndrome resulting from the occurrence of an m-repeated low-density burst of length b(fixed) with weight w or less within any one sub-block must be distinct from the all zero syndrome.

(d) The syndrome resulting from the occurrence of any m-repeated low-density burst of length b(fixed) with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any m-repeated low-density burst of length b(fixed) with weight w or less *within* any other sub-block.

In this section we shall derive two results. The first result gives a lower bound on the number of check digits required for the existence of a linear code over GF(q) capable of detecting and locating a single sub-block containing errors that are m-repeated low-density bursts of length b(fixed) with weight w or less. In the second result, we derive an upper bound on the number of check digits which ensures the existence of such a code.

Theorem 3. *The number of parity check digits r in an (n, k) linear code subdivided into s sub-blocks of length t each, that locates a single corrupted sub-block containing errors that are 2-repeated low density bursts of length b (fixed) with weight w or less is at least*

$$\begin{cases} \log_q\{1 + s(q^{mw} - 1)\} & \textit{where } \ t - b + 1 \geq mb \\ & \textit{and } \ (m-1)b + w \leq t - b + 1 < mb \\ \log_q\{1 + s(q^{(m-1)w+(t-mb+1)} - 1)\} & \textit{where } \ t - b + 1 < (m-1)b + w. \end{cases} \quad (10)$$

The proof of this result is on the similar lines as that of proof of Theorem 1 so we omit the proof.

Remark 4. For $m = 2$ the result coincides with that of Theorem 1 when 2-repeated low-density bursts of length b(fixed) with weight w or less are considered.

Remark 5. For $m = 1$, the result obtained in (10) reduces to

$$\begin{cases} \log_q\{1+s(q^w-1)\} & \text{where } \ t-b+1 \geq b \\ & \text{and } \ w \leq t-b+1 < b \ . \\ \log_q\{1+s(q^{(t-b+1)}-1)\} & \text{where } \ t-b+1 < w \end{cases}$$

which is a case of detecting and locating a sub-block containing errors which are usual low-density bursts of length b(fixed) with weight w or less.

Remark 6. For $w = b$, the result obtained in (10) reduces to

$$r \geq \begin{cases} \log_q\{1+s(q^{mb}-1)\} & \text{where } \ t-b+1 \geq mb \\ \log_q\{1+s(q^{(t-b+1)}-1)\} & \text{where } \ t-b+1 < mb \end{cases}$$

which coincides with the result due to Dass and Arora [Theorem 3, 2010].

In the following result we derive another bound on the number of check digits required for the existence of such a code. As earlier the proof is based on the technique used to establish Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks, Theorem 4.7 Peterson and Weldon(1972)).

Theorem 4. *An* (n, k) *linear EL-code over* GF(q) *capable of detecting an* m*-repeated low density burst of length* b*(fixed) with weight* w *or less*

$(w \le b)$ within a single sub-block and of locating that sub-block can always be constructed provided that

$$\begin{aligned} q^{n-k} &> [1+(q-1)]^{(b-1,w-1)} \\ &\cdot \left\{ \sum_{i=0}^{m-1} \binom{t-(i+1)b+i}{i} (q-1)^i [1+(q-1)]^{(b-1,w-1)} \right\} \\ &\cdot \left\{ 1+(s-1)\sum_{i=1}^{m} \binom{t-ib+i}{i} (q-1)^i \{[1+(q-1)]^{(b-1,w-1)}\}^i \right\} \quad (11) \end{aligned}$$

where $[1+x]^{(m,r)}$ denotes the incomplete binomial expansion of $(1+x)^m$ up to the term x^r in ascending power of x, viz.

$$[1+x]^{(m,r)} = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{r}x^r.$$

As in theorem 3 we omit the proof because proof of this result is on the similar lines as that of proof of Theorem 2.

Remark 7. For $m = 2$ the result coincides with that of Theorem 2 when 2-repeated low-density bursts of length b(fixed) with weight w or less are considered.

Remark 8. For $m = 1$, the result obtained in (11) reduces to

$$q^{n-k} > [1+(q-1)]^{(b-1,w-1)}\{1+(s-1)(t-b+1)(q-1)[1+(q-1)]^{(b-1,w-1)}\}$$

which is a necessary condition for detecting and locating a sub-block containing errors which are usual low-density bursts of length b(fixed) with weight w or less.

Remark 9. For $w = b$, the result obtained in (11) reduces to

$$q^{n-k} > q^{b-1}\left\{\sum_{i=0}^{m-1}\binom{j-(i+1)b+i}{i}(q-1)^i q^{i(b-1)}\right\} \cdot \left\{1+(s-1)\sum_{i=1}^{m}\binom{j-(i+1)b+i}{i}(q-1)^i q^{i(b-1)}\right\}$$

which coincides with the result due to Dass and Arora [Theorem 4, 2010].

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Blockwise Repeated Burst Error Correcting Linear Codes

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Abstract

This paper presents a lower and an upper bound on the number of parity check digits required for a linear code that corrects a single sub-block containing errors which are in the form of 2-repeated bursts of length b or less. An illustration of such kind of codes has been provided. Further, the codes that correct m-repeated bursts of length b or less have also been studied.

Keywords: Error locating codes, error correction, burst errors, repeated burst errors
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I Introduction

Error detecting codes and Error correcting codes have been the traditional areas of study in the field of coding techniques on error control in digital data transmission. Wolf and Elspas [12] introduced a coding technique, *error-locating codes* (EL Codes), lying midway between error detection and error correction. In an error locating code, each block of received digits is regarded as being subdivided into mutually exclusive sub-blocks, and codes have been devised that permit the detection of errors occurring within a single sub-block, the sub-block containing errors being identified. In ordinary decision feedback systems using error detection the receiver tests each block of received digits for the presence of errors. If errors are detected, the receiver requests the retransmission of the corrupted block of digits alone and this process is repeated for each incoming block. One drawback of the conventional system is that long block lengths (which are desirable for increased coding efficiency) can result in a low data rate when the reception of large amount of data is called for. However, the use of EL codes can soften this conflict between short and long block lengths by providing an additional design parameter. The overall constraint block length can be long to provide efficient coding while the length of the sub-blocks can be relatively short in order to keep the data rate up.

Codes developed at the early stages were meant mainly to detect and correct random errors. However, it was observed later that in many channels the likelihood of the occurrence of errors is more in adjacent positions rather than their occurrence in a random manner. In this spirit, Abramson[1] developed codes correcting single and double adjacent errors. The concept of clustered errors, commonly called burst errors, was generalized further in the work due to Fire [7]. A burst, also known as an open loop burst, of length b may be defined as follows:

Definition 1. A burst of length b is a vector whose all non-zero components are among some b consecutive components, the first and the last of which is non-zero.

It was observed that in very busy communication channels, errors repeat themselves. Similar is a situation when errors occur in the form of a burst. The development of codes for such kind of repeated burst errors is useful for improving upon the efficiency of some communication channels. Not only do repeated bursts emerge as a natural generalization of bursts, but considering a recent study by Srinivas, Jain, Saurav and Sikdar [11], where the changes in the neuronal network properties during epileptiform activity *in vitro* in planar two-dimensional neuronal networks cultured on a multielectrode array using the *in vitro* model of stroke-induced epilepsy have been explored, we observe that the study of these codes is significant.

The study of codes that detect repeated open-loop bursts was initiated by Berardi, Dass and Verma [2] and for correction of such errors by Dass and Verma [6] . An m-repeated burst (open-loop) of length b is defined as follows:

Definition 2. An m-repeated burst of length b is a vector of length n whose only non-zero components are confined to m distinct sets of b consecutive components, the first and the last component of each set being non-zero.

For example, (001032000020310000313200) is a 3-repeated burst of length 4 over $GF(4)$.

In particular, a 2-repeated burst (open-loop) of length b is defined as:

Definition 3. A 2-repeated burst of length b is a vector of length n whose only non-zero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being non-zero.

Wolf and Elspas [12] obtained results in the form of bounds over the number of parity-check digits required for binary codes capable of detecting and locating a single sub-block containing random errors. A study of such error locating codes in which errors occur in the form of bursts was made by Dass [3]. Further, these results were extended to the codes correcting burst errors occurring within a sub-block (refer Dass and Tyagi [5]). In our earlier paper [4] the authors obtained bounds over the number of parity-check digits required for codes detecting 2-repeated and m-repeated bursts of length b or less occurring within a single sub-block, the sub-block containing errors being identified. In this paper we extend our study to the correction of repeated bursts occurring within a sub-block. The development of codes correcting repeated burst errors within a sub-block improves the efficiency of the communication channel as it reduces the number of parity check digits required. The results that follow have been described in terms of the following parameters: the block of n digits, consisting of r check digits, and $k = n - r$ information digits, is subdivided into s mutually exclusive sub-blocks, each sub-block containing $t = n/s$ digits.

II Bounds for codes correcting 2-repeated bursts

In this section, we obtain bounds on the number of parity check digits of a code capable of correcting 2-repeated bursts of length b or less occurring within a single sub-block.

We note that an (n, k) linear EL code over $GF(q)$ capable of detecting and locating a single sub-block containing 2-repeated burst of length b or less must satisfy the following two conditions:
(i) The syndrome resulting from the occurrence of any 2-repeated burst of length b or less within any one sub-block must be non-zero.
(ii) The syndrome resulting from the occurrence of any 2-repeated burst of length b or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated burst of length b or less within *any other* sub-block.

Further, an (n, k) linear code over $GF(q)$ capable of correcting an error requires the syndromes of any two vectors to be distinct irrespective of whether they belong to the same sub-block or different sub-blocks. So, in order to correct 2-repeated bursts of length b or less lying within a sub-block the following conditions need to be satisfied:
(iii) The syndrome resulting from the occurrence of any 2-repeated burst of length b or less within a single sub-block must be distinct from the syndrome resulting from any other 2-repeated burst of length b or less within the same sub-block.
(iv) The syndrome resulting from the occurrence of any 2-repeated burst of length b or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated burst of length b or less within *any other* sub-block.

Remark 1. We observe that condition (ii) is the same as condition (iv). Also, for computational purposes condition (i) is taken care of by condition (iii). From this we infer that correction of errors requires more strict conditions than location of errors. So we need to consider conditions (iii) and (iv) or equivalently conditions (ii) and (iii) for correction of the said type of errors.

We first obtain a lower bound over the number of parity check digits required for such a code.

Theorem 1. *The number of check digits r required for an (n, k) linear code over $GF(q)$, subdivided into s sub-blocks of length t each, that corrects 2-repeated bursts of length b or less lying within a single corrupted sub-block is atleast*

$$log_q\left\{1+s\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]\right\}. \quad (1)$$

Proof. Let V be an (n, k) linear code over $GF(q)$ that corrects 2-repeated burst of length b or less within a single corrupted sub-block. The maximum number of distinct syndromes available using r check digits is q^r. The proof proceeds by first counting the number of syndromes that are required to be distinct by the two conditions and then setting this number less than or equal to q^r.

Since the code is capable of correcting all errors which are 2-repeated bursts of length b or less within any single sub-block, any syndrome produced by a 2-repeated burst of length b or less in a given sub-block must be distinct from any such syndrome likewise resulting from another 2-repeated burst of length b or less in the same sub-block(refer to condition (iii)). Moreover, syndromes produced by 2-repeated bursts of length b or less in different sub-blocks must also be distinct by condition (iv).

Thus, the syndromes of vectors which are 2-repeated bursts, whether in the same sub-block or in different sub-blocks, must be distinct.
Since there are

$$q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1$$

2-repeated bursts of length b or less within one sub-block of length t, excluding the vector of all zeros(refer Dass and Verma (2008)) and there are s sub-blocks in all, we must have at least

$$1+s\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]$$

distinct syndromes, including the all zeros syndrome.
Therefore, we must have

$$q^r \geq 1+s\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]$$

i.e.

$$r \geq log_q\left\{1+s\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]\right\}.$$

□

Remark 2. By taking $s = 1$ the bound obtained in (1) reduces to

$$log_q\left(q^{2b-2}\left[q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right]\right)$$

which coincides with the result for correction of 2-repeated bursts obtained by Dass and Verma(2008).

In the following result, we derive another bound on the number of check digits required for the existence of such a code. The proof is based on the technique used to establish Varshamov-Gilbert-Sacks bound by constructing a parity check matrix for such a code (refer Sacks (1958) also Theorem 4.7, Peterson and Weldon (1972)). This technique not only ensures the existence of such a code but also gives a method for the construction of the code.

Theorem 2. *An (n, k) linear code over $GF(q)$ capable of correcting 2-repeated burst of length b or less occurring within a single sub-block of length t $(4b < t)$ can always be constructed using r check digits, where r is the smallest integer* *satisfying the inequality*

$$q^r > q^{2(b-1)}\Bigg\{q^{2(b-1)}\Bigg\{(q-1)^3\binom{t-4b+3}{3} + (q-1)^2\binom{t-4b+2}{2} + q(q-1)\binom{t-4b+1}{1} + q^2\Bigg\}$$
$$+\Bigg\{(s-1)\Big[(t-2b+1)(q-1)+1\Big]\times$$
$$\Big[q^{2(b-1)}\Big\{q + (q-1)^2\binom{t-2b+2}{2} + (q-1)\binom{t-2b+1}{1}\Big\} - 1\Big]\Bigg\}\Bigg\}. \quad (2)$$

Proof. We shall prove the result by constructing an appropriate $(n-k)\times n$ parity check matrix H for the desired code. Suppose that the columns of the first $s-1$ sub-blocks of H and the first $j-1$ columns $h_1, h_2, \cdots, h_{j-1}$ of the s^{th} sub-block have been appropriately added. We now lay down conditions to add the j^{th} column h_j to the s^{th} sub-block as follows:

Since the code is to correct 2-repeated bursts of length b or less within a single sub-block, therefore, by condition (iii), the syndrome of any 2-repeated burst in any sub-block must be different from the syndrome resulting from any other such burst within the same sub-block. Therefore the j^{th} column h_j can be added provided that h_j is not a linear combination of the immediately preceding $b-1$ or fewer columns $h_{j-b+1}, \cdots, h_{j-1}$ of the s^{th} sub-block together with any three distinct sets of b or fewer consecutive columns each from amongst the first $j-b$ columns $h_1, h_2, \cdots, h_{j-b}$. In other words,

$$h_j \neq (\alpha_1 h_{j-b+1} + \alpha_2 h_{j-b+2} + \cdots + \alpha_{b-1} h_{j-1}) +$$
$$\sum_{l=1}^{3}(\beta_{l_1} h_{l_1} + \beta_{l_2} h_{l_2} + \cdots + \beta_{l_b} h_{l_b}), \quad (3)$$

where $\alpha_i, \beta_{l_i} \in GF(q)$ and $l_b \leq j-b$.

The number of ways in which the coefficients α_i can be selected is clearly q^{b-1}. To enumerate the coefficients β_i is equivalent to enumerate the number of 3-repeated bursts of length b or less in a vector of length $j-b$ which is (refer Dass and Verma(2008))

$$q^{3(b-1)}\Bigg\{(q-1)^3\binom{j-4b+3}{3} + (q-1)^2\binom{j-4b+2}{2} + q(q-1)\binom{j-4b+1}{1} + q^2\Bigg\}.$$

Therefore, the total number of possible choices for α_i and β_i on the R.H.S of (3) is

$$q^{4(b-1)}\left\{(q-1)^3\binom{j-4b+3}{3}+(q-1)^2\binom{j-4b+2}{2}+q(q-1)\binom{j-4b+1}{1}+q^2\right\}. \quad (4)$$

Further, by condition (iv), h_j can be added to the s^{th} sub-block provided h_j is not a linear combination of the immediately preceding $b-1$ or fewer columns together with one set of b or fewer columns from amongst the first $j-b$ columns together with linear combination of any two sets of b or less consecutive columns within *any other* sub-block. i.e.

$$\begin{aligned} h_j \neq\ & (\alpha_1 h_{j-b+1} + \alpha_2 h_{j-b+2} + \cdots + \alpha_{b-1} h_{j-1}) + \\ & (\beta_1 h_i + \beta_2 h_{i+1} + \cdots + \beta_b h_{i+b-1}) + \\ & (\gamma_1 h_{i_1} + \gamma_2 h_{i_1+1} + \cdots + \gamma_b h_{i_1+b-1}) + \\ & (\delta_1 h_{i_2} + \delta_2 h_{i_2+1} + \cdots + \delta_b h_{i_2+b-1}) \end{aligned} \quad (5)$$

where $\alpha_p, \beta_p, \gamma_p, \delta_p \in GF(q), i+b-1 \leq j-b$ and not all γ_p and δ_p are zero. (The last two terms in the above sum correspond to any two sets of b or less consecutive columns within any one of the other sub-block.)

The number of ways in which the coefficients α_p can be selected is clearly q^{b-1}. To enumerate the coefficients β_p is equivalent to enumerate the number of bursts of length b or less in a vector of length $j-b$ which is $q^{b-1}[(j-2b+1)(q-1)+1]$ (refer Fire [7]). Therefore, the total number of possible choices for α_p and β_p on the R.H.S of (5) is

$$q^{2(b-1)}[(j-2b+1)(q-1)+1]. \quad (6)$$

Also, the number of linear combinations corresponding to the last two terms on the R.H.S. of (5) is the same as the number of 2-repeated bursts of length b or less within a sub-block of length t, excluding the vector of all zeros; which is (refer Dass and Verma (2008))

$$q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1.$$

Since there are $s-1$ previously chosen sub-blocks, the number of such linear combinations becomes

$$(s-1)\left[q^{2b-2}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]. \quad (7)$$

Thus, the number of linear combinations to which h_j can not be equal to is the product computed in expr. (6) and expr. (7). i.e.

$$expr.(6)\times expr.(7). \quad (8)$$

Thus, the total number of linear combinations that h_j can not be equal to is the sum of linear combinations in (4) and (8).
At worst, all these combinations might yield a distinct sum. Therefore, h_j can be added to the s^{th} sub- block of H provided that

$$q^r > q^{2(b-1)}\left\{q^{2(b-1)}\left\{(q-1)^3\binom{j-4b+3}{3}+(q-1)^2\binom{j-4b+2}{2}+q(q-1)\binom{j-4b+1}{1}+q^2\right\}\right.$$
$$+\left\{(s-1)\Big[(j-2b+1)(q-1)+1\Big]\times\right.$$
$$\left.\left.\left[q^{2(b-1)}\left\{q+(q-1)^2\binom{t-2b+2}{2}+(q-1)\binom{t-2b+1}{1}\right\}-1\right]\right\}\right\}.$$

For completing the s^{th} sub-block of length t, replacing j by t gives the result as stated in (2). □

Remark 3. By taking $s=1$ in (2) the bound reduces to

$$q^r > q^{4(b-1)}\left\{(q-1)^3\binom{t-4b+3}{3}+(q-1)^2\binom{t-4b+2}{2}+q(q-1)\binom{t-4b+1}{1}+q^2\right\}$$

which coincides with the condition for existence of a code correcting 2-repeated bursts of length b or less(refer Dass and Verma(2008)).

We conclude this section with an example.

Example 1 Consider a $(26,10)$ binary code with a 16×26 parity-check matrix

H given by

$$
H = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&1&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0
\end{bmatrix}
$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 2 by taking $b = 3$, $s = 2$, $t = 13$ over $GF(2)$ (MS Excel Program was used for the construction of the matrix). It can be seen from the Table 1 that the syndromes of all distinct 2-repeated bursts of length 3 or less whether in the same sub-block or in different sub-blocks are different, showing thereby that the code that is the null space of this matrix corrects all 2-repeated bursts of length 3 or less occurring within a sub-block.

Table 1
Error Patterns - Syndrome vectors

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1	1111110000000 0000000000000	1111110000000000	44	0111101000000 0000000000000	0111101000000000
2	1110111000000 0000000000000	1110111000000000	45	0111010100000 0000000000000	0111010100000000
3	1110011100000 0000000000000	1110011100000000	46	0111001010000 0000000000000	0111001010000000
4	1110001110000 0000000000000	1110001110000000	47	0111000101000 0000000000000	0111000101000000
5	1110000111000 0000000000000	1110000111000000	48	0111000010100 0000000000000	0111000010100000
6	1110000011100 0000000000000	1110000011100000	49	0111000001010 0000000000000	0111000001010000
7	1110000001110 0000000000000	1110000001110000	50	0111000000101 0000000000000	0111000000101000
8	1110000000111 0000000000000	1110000000111000	51	0111110000000 0000000000000	0111110000000000
9	1111010000000 0000000000000	1111010000000000	52	0111011000000 0000000000000	0111011000000000
10	1110101000000 0000000000000	1110101000000000	53	0111001100000 0000000000000	0111001100000000
11	1110010100000 0000000000000	1110010100000000	54	0111000110000 0000000000000	0111000110000000
12	1110001010000 0000000000000	1110001010000000	55	0111000011000 0000000000000	0111000011000000
13	1110000101000 0000000000000	1110000101000000	56	0111000001100 0000000000000	0111000001100000
14	1110000010100 0000000000000	1110000010100000	57	0111000000110 0000000000000	0111000000110000
15	1110000001010 0000000000000	1110000001010000	58	0111000000011 0000000000000	0111000000011000
16	1110000000101 0000000000000	1110000000101000	59	0111100000000 0000000000000	0111100000000000
17	1111100000000 0000000000000	1111100000000000	60	0111010000000 0000000000000	0111010000000000
18	1110110000000 0000000000000	1110110000000000	61	0111001000000 0000000000000	0111001000000000
19	1110011000000 0000000000000	1110011000000000	62	0111000100000 0000000000000	0111000100000000
20	1110001100000 0000000000000	1110001100000000	63	0111000010000 0000000000000	0111000010000000
21	1110000110000 0000000000000	1110000110000000	64	0111000001000 0000000000000	0111000001000000
22	1110000011000 0000000000000	1110000011000000	65	0111000000100 0000000000000	0111000000100000
23	1110000001100 0000000000000	1110000001100000	66	0111000000010 0000000000000	0111000000010000
24	1110000000110 0000000000000	1110000000110000	67	0111000000001 0000000000000	0111000000001000
25	1110000000011 0000000000000	1110000000011000	68	0111000000000 0000000000000	0111000000000000
26	1111000000000 0000000000000	1111000000000000	69	0011111100000 0000000000000	0011111100000000
27	1110100000000 0000000000000	1110100000000000	70	0011101110000 0000000000000	0011101110000000
28	1110010000000 0000000000000	1110010000000000	71	0011100111000 0000000000000	0011100111000000
29	1110001000000 0000000000000	1110001000000000	72	0011100011100 0000000000000	0011100011100000
30	1110000100000 0000000000000	1110000100000000	73	0011100001110 0000000000000	0011100001110000
31	1110000010000 0000000000000	1110000010000000	74	0011100000111 0000000000000	0011100000111000
32	1110000001000 0000000000000	1110000001000000	75	0011110100000 0000000000000	0011110100000000
33	1110000000100 0000000000000	1110000000100000	76	0011101010000 0000000000000	0011101010000000
34	1110000000010 0000000000000	1110000000010000	77	0011100101000 0000000000000	0011100101000000
35	1110000000001 0000000000000	1110000000001000	78	0011100010100 0000000000000	0011100010100000
36	1110000000000 0000000000000	1110000000000000	79	0011100001010 0000000000000	0011100001010000
37	0111111000000 0000000000000	0111111000000000	80	0011100000101 0000000000000	0011100000101000
38	0111011100000 0000000000000	0111011100000000	81	0011111000000 0000000000000	0011111000000000
39	0111001110000 0000000000000	0111001110000000	82	0011101100000 0000000000000	0011101100000000
40	0111000111000 0000000000000	0111000111000000	83	0011100110000 0000000000000	0011100110000000
41	0111000011100 0000000000000	0111000011100000	84	0011100011000 0000000000000	0011100011000000
42	0111000001110 0000000000000	0111000001110000	85	0011100001100 0000000000000	0011100001100000
43	0111000000111 0000000000000	0111000000111000	86	0011100000110 0000000000000	0011100000110000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
87	0011100000011 0000000000000	0011100000011000	134	0000111100000 0000000000000	0000111100000000
88	0011110000000 0000000000000	0011110000000000	135	0000111010000 0000000000000	0000111010000000
89	0011101000000 0000000000000	0011101000000000	136	0000111001000 0000000000000	0000111001000000
90	0011100100000 0000000000000	0011100100000000	137	0000111000100 0000000000000	0000111000100000
91	0011100010000 0000000000000	0011100010000000	138	0000111000010 0000000000000	0000111000010000
92	0011100001000 0000000000000	0011100001000000	139	0000111000001 0000000000000	0000111000001000
93	0011100000100 0000000000000	0011100000100000	140	0000111000000 0000000000000	0000111000000000
94	0011100000010 0000000000000	0011100000010000	141	0000011111100 0000000000000	0000011111100000
95	0011100000001 0000000000000	0011100000001000	142	0000011101110 0000000000000	0000011101110000
96	0011100000000 0000000000000	0011100000000000	143	0000011100111 0000000000000	0000011100111000
97	0001111110000 0000000000000	0001111110000000	144	0000011110100 0000000000000	0000011110100000
98	0001110111000 0000000000000	0001110111000000	145	0000011101010 0000000000000	0000011101010000
99	0001110011100 0000000000000	0001110011100000	146	0000011100101 0000000000000	0000011100101000
100	0001110001110 0000000000000	0001110001110000	147	0000011111000 0000000000000	0000011111000000
101	0001110000111 0000000000000	0001110000111000	148	0000011101100 0000000000000	0000011101100000
102	0001111010000 0000000000000	0001111010000000	149	0000011100110 0000000000000	0000011100110000
103	0001110101000 0000000000000	0001110101000000	150	0000011100011 0000000000000	0000011100011000
104	0001110010100 0000000000000	0001110010100000	151	0000011110000 0000000000000	0000011110000000
105	0001110001010 0000000000000	0001110001010000	152	0000011101000 0000000000000	0000011101000000
106	0001110000101 0000000000000	0001110000101000	153	0000011100100 0000000000000	0000011100100000
107	0001111100000 0000000000000	0001111100000000	154	0000011100010 0000000000000	0000011100010000
108	0001110110000 0000000000000	0001110110000000	155	0000011100001 0000000000000	0000011100001000
109	0001110011000 0000000000000	0001110011000000	156	0000011100000 0000000000000	0000011100000000
110	0001110001100 0000000000000	0001110001100000	157	0000001111110 0000000000000	0000001111110000
111	0001110000110 0000000000000	0001110000110000	158	0000001110111 0000000000000	0000001110111000
112	0001110000011 0000000000000	0001110000011000	159	0000001111010 0000000000000	0000001111010000
113	0001111000000 0000000000000	0001111000000000	160	0000001110101 0000000000000	0000001110101000
114	0001110100000 0000000000000	0001110100000000	161	0000001111100 0000000000000	0000001111100000
115	0001110010000 0000000000000	0001110010000000	162	0000001110110 0000000000000	0000001110110000
116	0001110001000 0000000000000	0001110001000000	163	0000001110011 0000000000000	0000001110011000
117	0001110000100 0000000000000	0001110000100000	164	0000001111000 0000000000000	0000001111000000
118	0001110000010 0000000000000	0001110000010000	165	0000001110100 0000000000000	0000001110100000
119	0001110000001 0000000000000	0001110000001000	166	0000001110010 0000000000000	0000001110010000
120	0001110000000 0000000000000	0001110000000000	167	0000001110001 0000000000000	0000001110001000
121	0000111111000 0000000000000	0000111111000000	168	0000001110000 0000000000000	0000001110000000
122	0000111011100 0000000000000	0000111011100000	169	0000000111111 0000000000000	0000000111111000
123	0000111001110 0000000000000	0000111001110000	170	0000000111101 0000000000000	0000000111101000
124	0000111000111 0000000000000	0000111000111000	171	0000000111110 0000000000000	0000000111110000
125	0000111101000 0000000000000	0000111101000000	172	0000000111011 0000000000000	0000000111011000
126	0000111010100 0000000000000	0000111010100000	173	0000000111100 0000000000000	0000000111100000
127	0000111001010 0000000000000	0000111001010000	174	0000000111010 0000000000000	0000000111010000
128	0000111000101 0000000000000	0000111000101000	175	0000000111001 0000000000000	0000000111001000
129	0000111110000 0000000000000	0000111110000000	176	0000000111000 0000000000000	0000000111000000
130	0000111011000 0000000000000	0000111011000000	177	0000000011111 0000000000000	0000000011111000
131	0000111001100 0000000000000	0000111001100000	178	0000000011101 0000000000000	0000000011101000
132	0000111000110 0000000000000	0000111000110000	179	0000000011110 0000000000000	0000000011110000
133	0000111000011 0000000000000	0000111000011000	180	0000000011100 0000000000000	0000000011100000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
181	0000000001111 0000000000000	0000000001111000	228	0101010100000 0000000000000	0101010100000000
182	0000000001110 0000000000000	0000000001110000	229	0101001010000 0000000000000	0101001010000000
183	0000000000111 0000000000000	0000000000111000	230	0101000101000 0000000000000	0101000101000000
184	1011110000000 0000000000000	1011110000000000	231	0101000010100 0000000000000	0101000010100000
185	1010111000000 0000000000000	1010111000000000	232	0101000001010 0000000000000	0101000001010000
186	1010011100000 0000000000000	1010011100000000	233	0101000000101 0000000000000	0101000000101000
187	1010001110000 0000000000000	1010001110000000	234	0101110000000 0000000000000	0101110000000000
188	1010000111000 0000000000000	1010000111000000	235	0101011000000 0000000000000	0101011000000000
189	1010000011100 0000000000000	1010000011100000	236	0101001100000 0000000000000	0101001100000000
190	1010000001110 0000000000000	1010000001110000	237	0101000110000 0000000000000	0101000110000000
191	1010000000111 0000000000000	1010000000111000	238	0101000011000 0000000000000	0101000011000000
192	1011010000000 0000000000000	1011010000000000	239	0101000001100 0000000000000	0101000001100000
193	1010101000000 0000000000000	1010101000000000	240	0101000000110 0000000000000	0101000000110000
194	1010010100000 0000000000000	1010010100000000	241	0101000000011 0000000000000	0101000000011000
195	1010001010000 0000000000000	1010001010000000	242	0101100000000 0000000000000	0101100000000000
196	1010000101000 0000000000000	1010000101000000	243	0101010000000 0000000000000	0101010000000000
197	1010000010100 0000000000000	1010000010100000	244	0101001000000 0000000000000	0101001000000000
198	1010000001010 0000000000000	1010000001010000	245	0101000100000 0000000000000	0101000100000000
199	1010000000101 0000000000000	1010000000101000	246	0101000010000 0000000000000	0101000010000000
200	1011100000000 0000000000000	1011100000000000	247	0101000001000 0000000000000	0101000001000000
201	1010110000000 0000000000000	1010110000000000	248	0101000000100 0000000000000	0101000000100000
202	1010011000000 0000000000000	1010011000000000	249	0101000000010 0000000000000	0101000000010000
203	1010001100000 0000000000000	1010001100000000	250	0101000000001 0000000000000	0101000000001000
204	1010000110000 0000000000000	1010000110000000	251	0101000000000 0000000000000	0101000000000000
205	1010000011000 0000000000000	1010000011000000	252	0010111100000 0000000000000	0010111100000000
206	1010000001100 0000000000000	1010000001100000	253	0010101110000 0000000000000	0010101110000000
207	1010000000110 0000000000000	1010000000110000	254	0010100111000 0000000000000	0010100111000000
208	1010000000011 0000000000000	1010000000011000	255	0010100011100 0000000000000	0010100011100000
209	1011000000000 0000000000000	1011000000000000	256	0010100001110 0000000000000	0010100001110000
210	1010100000000 0000000000000	1010100000000000	257	0010100000111 0000000000000	0010100000111000
211	1010010000000 0000000000000	1010010000000000	258	0010110100000 0000000000000	0010110100000000
212	1010001000000 0000000000000	1010001000000000	259	0010101010000 0000000000000	0010101010000000
213	1010000100000 0000000000000	1010000100000000	260	0010100101000 0000000000000	0010100101000000
214	1010000010000 0000000000000	1010000010000000	261	0010100010100 0000000000000	0010100010100000
215	1010000001000 0000000000000	1010000001000000	262	0010100001010 0000000000000	0010100001010000
216	1010000000100 0000000000000	1010000000100000	263	0010100000101 0000000000000	0010100000101000
217	1010000000010 0000000000000	1010000000010000	264	0010111000000 0000000000000	0010111000000000
218	1010000000001 0000000000000	1010000000001000	265	0010101100000 0000000000000	0010101100000000
219	1010000000000 0000000000000	1010000000000000	266	0010100110000 0000000000000	0010100110000000
220	0101111000000 0000000000000	0101111000000000	267	0010100011000 0000000000000	0010100011000000
221	0101011100000 0000000000000	0101011100000000	268	0010100001100 0000000000000	0010100001100000
222	0101001110000 0000000000000	0101001110000000	269	0010100000110 0000000000000	0010100000110000
223	0101000111000 0000000000000	0101000111000000	270	0010100000011 0000000000000	0010100000011000
224	0101000011100 0000000000000	0101000011100000	271	0010110000000 0000000000000	0010110000000000
225	0101000001110 0000000000000	0101000001110000	272	0010101000000 0000000000000	0010101000000000
226	0101000000111 0000000000000	0101000000111000	273	0010100100000 0000000000000	0010100100000000
227	0101101000000 0000000000000	0101101000000000	274	0010100010000 0000000000000	0010100010000000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
275	0010100001000 0000000000000	0010100001000000	322	0000101000001 0000000000000	0000101000001000
276	0010100000100 0000000000000	0010100000100000	323	0000101000000 0000000000000	0000101000000000
277	0010100000010 0000000000000	0010100000010000	324	0000010111100 0000000000000	0000010111100000
278	0010100000001 0000000000000	0010100000001000	325	0000010101110 0000000000000	0000010101110000
279	0010100000000 0000000000000	0010100000000000	326	0000010100111 0000000000000	0000010100111000
280	0001011110000 0000000000000	0001011110000000	327	0000010110100 0000000000000	0000010110100000
281	0001010111000 0000000000000	0001010111000000	328	0000010101010 0000000000000	0000010101010000
282	0001010011100 0000000000000	0001010011100000	329	0000010100101 0000000000000	0000010100101000
283	0001010001110 0000000000000	0001010001110000	330	0000010111000 0000000000000	0000010111000000
284	0001010000111 0000000000000	0001010000111000	331	0000010101100 0000000000000	0000010101100000
285	0001011010000 0000000000000	0001011010000000	332	0000010100110 0000000000000	0000010100110000
286	0001010101000 0000000000000	0001010101000000	333	0000010100011 0000000000000	0000010100011000
287	0001010010100 0000000000000	0001010010100000	334	0000010110000 0000000000000	0000010110000000
288	0001010001010 0000000000000	0001010001010000	335	0000010101000 0000000000000	0000010101000000
289	0001010000101 0000000000000	0001010000101000	336	0000010100100 0000000000000	0000010100100000
290	0001011100000 0000000000000	0001011100000000	337	0000010100010 0000000000000	0000010100010000
291	0001010110000 0000000000000	0001010110000000	338	0000010100001 0000000000000	0000010100001000
292	0001010011000 0000000000000	0001010011000000	339	0000010100000 0000000000000	0000010100000000
293	0001010001100 0000000000000	0001010001100000	340	0000001011110 0000000000000	0000001011110000
294	0001010000110 0000000000000	0001010000110000	341	0000001010111 0000000000000	0000001010111000
295	0001010000011 0000000000000	0001010000011000	342	0000001011010 0000000000000	0000001011010000
296	0001011000000 0000000000000	0001011000000000	343	0000001010101 0000000000000	0000001010101000
297	0001010100000 0000000000000	0001010100000000	344	0000001011100 0000000000000	0000001011100000
298	0001010010000 0000000000000	0001010010000000	345	0000001010110 0000000000000	0000001010110000
299	0001010001000 0000000000000	0001010001000000	346	0000001010011 0000000000000	0000001010011000
300	0001010000100 0000000000000	0001010000100000	347	0000001011000 0000000000000	0000001011000000
301	0001010000010 0000000000000	0001010000010000	348	0000001010100 0000000000000	0000001010100000
302	0001010000001 0000000000000	0001010000001000	349	0000001010010 0000000000000	0000001010010000
303	0001010000000 0000000000000	0001010000000000	350	0000001010001 0000000000000	0000001010001000
304	0000101111000 0000000000000	0000101111000000	351	0000001010000 0000000000000	0000001010000000
305	0000101011100 0000000000000	0000101011100000	352	0000000101111 0000000000000	0000000101111000
306	0000101001110 0000000000000	0000101001110000	353	0000000101101 0000000000000	0000000101101000
307	0000101000111 0000000000000	0000101000111000	354	0000000101110 0000000000000	0000000101110000
308	0000101101000 0000000000000	0000101101000000	355	0000000101011 0000000000000	0000000101011000
309	0000101010100 0000000000000	0000101010100000	356	0000000101100 0000000000000	0000000101100000
310	0000101001010 0000000000000	0000101001010000	357	0000000101010 0000000000000	0000000101010000
311	0000101000101 0000000000000	0000101000101000	358	0000000101001 0000000000000	0000000101001000
312	0000101110000 0000000000000	0000101110000000	359	0000000101000 0000000000000	0000000101000000
313	0000101011000 0000000000000	0000101011000000	360	0000000010111 0000000000000	0000000010111000
314	0000101001100 0000000000000	0000101001100000	361	0000000010101 0000000000000	0000000010101000
315	0000101000110 0000000000000	0000101000110000	362	0000000010110 0000000000000	0000000010110000
316	0000101000011 0000000000000	0000101000011000	363	0000000010100 0000000000000	0000000010100000
317	0000101100000 0000000000000	0000101100000000	364	0000000001011 0000000000000	0000000001011000
318	0000101010000 0000000000000	0000101010000000	365	0000000001010 0000000000000	0000000001010000
319	0000101001000 0000000000000	0000101001000000	366	0000000000101 0000000000000	0000000000101000
320	0000101000100 0000000000000	0000101000100000	367	1101110000000 0000000000000	1101110000000000
321	0000101000010 0000000000000	0000101000010000	368	1100111000000 0000000000000	1100111000000000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
369	1100011100000 0000000000000	1100011100000000	416	0110000000101 0000000000000	0110000000101000
370	1100001110000 0000000000000	1100001110000000	417	0110110000000 0000000000000	0110110000000000
371	1100000111000 0000000000000	1100000111000000	418	0110011000000 0000000000000	0110011000000000
372	1100000011100 0000000000000	1100000011100000	419	0110001100000 0000000000000	0110001100000000
373	1100000001110 0000000000000	1100000001110000	420	0110000110000 0000000000000	0110000110000000
374	1100000000111 0000000000000	1100000000111000	421	0110000011000 0000000000000	0110000011000000
375	1101010000000 0000000000000	1101010000000000	422	0110000001100 0000000000000	0110000001100000
376	1100101000000 0000000000000	1100101000000000	423	0110000000110 0000000000000	0110000000110000
377	1100010100000 0000000000000	1100010100000000	424	0110000000011 0000000000000	0110000000011000
378	1100001010000 0000000000000	1100001010000000	425	0110100000000 0000000000000	0110100000000000
379	1100000101000 0000000000000	1100000101000000	426	0110010000000 0000000000000	0110010000000000
380	1100000010100 0000000000000	1100000010100000	427	0110001000000 0000000000000	0110001000000000
381	1100000001010 0000000000000	1100000001010000	428	0110000100000 0000000000000	0110000100000000
382	1100000000101 0000000000000	1100000000101000	429	0110000010000 0000000000000	0110000010000000
383	1101100000000 0000000000000	1101100000000000	430	0110000001000 0000000000000	0110000001000000
384	1100110000000 0000000000000	1100110000000000	431	0110000000100 0000000000000	0110000000100000
385	1100011000000 0000000000000	1100011000000000	432	0110000000010 0000000000000	0110000000010000
386	1100001100000 0000000000000	1100001100000000	433	0110000000001 0000000000000	0110000000001000
387	1100000110000 0000000000000	1100000110000000	434	0110000000000 0000000000000	0110000000000000
388	1100000011000 0000000000000	1100000011000000	435	0011011100000 0000000000000	0011011100000000
389	1100000001100 0000000000000	1100000001100000	436	0011001110000 0000000000000	0011001110000000
390	1100000000110 0000000000000	1100000000110000	437	0011000111000 0000000000000	0011000111000000
391	1100000000011 0000000000000	1100000000011000	438	0011000011100 0000000000000	0011000011100000
392	1101000000000 0000000000000	1101000000000000	439	0011000001110 0000000000000	0011000001110000
393	1100100000000 0000000000000	1100100000000000	440	0011000000111 0000000000000	0011000000111000
394	1100010000000 0000000000000	1100010000000000	441	0011010100000 0000000000000	0011010100000000
395	1100001000000 0000000000000	1100001000000000	442	0011001010000 0000000000000	0011001010000000
396	1100000100000 0000000000000	1100000100000000	443	0011000101000 0000000000000	0011000101000000
397	1100000010000 0000000000000	1100000010000000	444	0011000010100 0000000000000	0011000010100000
398	1100000001000 0000000000000	1100000001000000	445	0011000001010 0000000000000	0011000001010000
399	1100000000100 0000000000000	1100000000100000	446	0011000000101 0000000000000	0011000000101000
400	1100000000010 0000000000000	1100000000010000	447	0011011000000 0000000000000	0011011000000000
401	1100000000001 0000000000000	1100000000001000	448	0011001100000 0000000000000	0011001100000000
402	1100000000000 0000000000000	1100000000000000	449	0011000110000 0000000000000	0011000110000000
403	0110111000000 0000000000000	0110111000000000	450	0011000011000 0000000000000	0011000011000000
404	0110011100000 0000000000000	0110011100000000	451	0011000001100 0000000000000	0011000001100000
405	0110001110000 0000000000000	0110001110000000	452	0011000000110 0000000000000	0011000000110000
406	0110000111000 0000000000000	0110000111000000	453	0011000000011 0000000000000	0011000000011000
407	0110000011100 0000000000000	0110000011100000	454	0011010000000 0000000000000	0011010000000000
408	0110000001110 0000000000000	0110000001110000	455	0011001000000 0000000000000	0011001000000000
409	0110000000111 0000000000000	0110000000111000	456	0011000100000 0000000000000	0011000100000000
410	0110101000000 0000000000000	0110101000000000	457	0011000010000 0000000000000	0011000010000000
411	0110010100000 0000000000000	0110010100000000	458	0011000001000 0000000000000	0011000001000000
412	0110001010000 0000000000000	0110001010000000	459	0011000000100 0000000000000	0011000000100000
413	0110000101000 0000000000000	0110000101000000	460	0011000000010 0000000000000	0011000000010000
414	0110000010100 0000000000000	0110000010100000	461	0011000000001 0000000000000	0011000000001000
415	0110000001010 0000000000000	0110000001010000	462	0011000000000 0000000000000	0011000000000000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
463	0001101110000 0000000000000	0001101110000000	510	0000011010100 0000000000000	0000011010100000
464	0001100111000 0000000000000	0001100111000000	511	0000011001010 0000000000000	0000011001010000
465	0001100011100 0000000000000	0001100011100000	512	0000011000101 0000000000000	0000011000101000
466	0001100001110 0000000000000	0001100001110000	513	0000011011000 0000000000000	0000011011000000
467	0001100000111 0000000000000	0001100000111000	514	0000011001100 0000000000000	0000011001100000
468	0001101010000 0000000000000	0001101010000000	515	0000011000110 0000000000000	0000011000110000
469	0001100101000 0000000000000	0001100101000000	516	0000011000011 0000000000000	0000011000011000
470	0001100010100 0000000000000	0001100010100000	517	0000011010000 0000000000000	0000011010000000
471	0001100001010 0000000000000	0001100001010000	518	0000011001000 0000000000000	0000011001000000
472	0001100000101 0000000000000	0001100000101000	519	0000011000100 0000000000000	0000011000100000
473	0001101100000 0000000000000	0001101100000000	520	0000011000010 0000000000000	0000011000010000
474	0001100110000 0000000000000	0001100110000000	521	0000011000001 0000000000000	0000011000001000
475	0001100011000 0000000000000	0001100011000000	522	0000011000000 0000000000000	0000011000000000
476	0001100001100 0000000000000	0001100001100000	523	0000001101110 0000000000000	0000001101110000
477	0001100000110 0000000000000	0001100000110000	524	0000001100111 0000000000000	0000001100111000
478	0001100000011 0000000000000	0001100000011000	525	0000001101010 0000000000000	0000001101010000
479	0001101000000 0000000000000	0001101000000000	526	0000001100101 0000000000000	0000001100101000
480	0001100100000 0000000000000	0001100100000000	527	0000001101100 0000000000000	0000001101100000
481	0001100010000 0000000000000	0001100010000000	528	0000001100110 0000000000000	0000001100110000
482	0001100001000 0000000000000	0001100001000000	529	0000001100011 0000000000000	0000001100011000
483	0001100000100 0000000000000	0001100000100000	530	0000001101000 0000000000000	0000001101000000
484	0001100000010 0000000000000	0001100000010000	531	0000001100100 0000000000000	0000001100100000
485	0001100000001 0000000000000	0001100000001000	532	0000001100010 0000000000000	0000001100010000
486	0001100000000 0000000000000	0001100000000000	533	0000001100001 0000000000000	0000001100001000
487	0000110111000 0000000000000	0000110111000000	534	0000001100000 0000000000000	0000001100000000
488	0000110011100 0000000000000	0000110011100000	535	0000000110111 0000000000000	0000000110111000
489	0000110001110 0000000000000	0000110001110000	536	0000000110101 0000000000000	0000000110101000
490	0000110000111 0000000000000	0000110000111000	537	0000000110110 0000000000000	0000000110110000
491	0000110101000 0000000000000	0000110101000000	538	0000000110011 0000000000000	0000000110011000
492	0000110010100 0000000000000	0000110010100000	539	0000000110100 0000000000000	0000000110100000
493	0000110001010 0000000000000	0000110001010000	540	0000000110010 0000000000000	0000000110010000
494	0000110000101 0000000000000	0000110000101000	541	0000000110001 0000000000000	0000000110001000
495	0000110110000 0000000000000	0000110110000000	542	0000000110000 0000000000000	0000000110000000
496	0000110011000 0000000000000	0000110011000000	543	0000000011011 0000000000000	0000000011011000
497	0000110001100 0000000000000	0000110001100000	544	0000000011001 0000000000000	0000000011001000
498	0000110000110 0000000000000	0000110000110000	545	0000000011010 0000000000000	0000000011010000
499	0000110000011 0000000000000	0000110000011000	546	0000000011000 0000000000000	0000000011000000
500	0000110100000 0000000000000	0000110100000000	547	0000000001101 0000000000000	0000000001101000
501	0000110010000 0000000000000	0000110010000000	548	0000000001100 0000000000000	0000000001100000
502	0000110001000 0000000000000	0000110001000000	549	0000000000110 0000000000000	0000000000110000
503	0000110000100 0000000000000	0000110000100000	550	0000000000011 0000000000000	0000000000011000
504	0000110000010 0000000000000	0000110000010000	551	1001110000000 0000000000000	1001110000000000
505	0000110000001 0000000000000	0000110000001000	552	1000111000000 0000000000000	1000111000000000
506	0000110000000 0000000000000	0000110000000000	553	1000011100000 0000000000000	1000011100000000
507	0000011011100 0000000000000	0000011011100000	554	1000001110000 0000000000000	1000001110000000
508	0000011001110 0000000000000	0000011001110000	555	1000000111000 0000000000000	1000000111000000
509	0000011000111 0000000000000	0000011000111000	556	1000000011100 0000000000000	1000000011100000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
557	1000000001110 0000000000000	1000000001110000	604	0100000110000 0000000000000	0100000110000000
558	1000000000111 0000000000000	1000000000111000	605	0100000011000 0000000000000	0100000011000000
559	1001010000000 0000000000000	1001010000000000	606	0100000001100 0000000000000	0100000001100000
560	1000101000000 0000000000000	1000101000000000	607	0100000000110 0000000000000	0100000000110000
561	1000010100000 0000000000000	1000010100000000	608	0100000000011 0000000000000	0100000000011000
562	1000001010000 0000000000000	1000001010000000	609	0100100000000 0000000000000	0100100000000000
563	1000000101000 0000000000000	1000000101000000	610	0100010000000 0000000000000	0100010000000000
564	1000000010100 0000000000000	1000000010100000	611	0100001000000 0000000000000	0100001000000000
565	1000000001010 0000000000000	1000000001010000	612	0100000100000 0000000000000	0100000100000000
566	1000000000101 0000000000000	1000000000101000	613	0100000010000 0000000000000	0100000010000000
567	1001100000000 0000000000000	1001100000000000	614	0100000001000 0000000000000	0100000001000000
568	1000110000000 0000000000000	1000110000000000	615	0100000000100 0000000000000	0100000000100000
569	1000011000000 0000000000000	1000011000000000	616	0100000000010 0000000000000	0100000000010000
570	1000001100000 0000000000000	1000001100000000	617	0100000000001 0000000000000	0100000000001000
571	1000000110000 0000000000000	1000000110000000	618	0100000000000 0000000000000	0100000000000000
572	1000000011000 0000000000000	1000000011000000	619	0010011100000 0000000000000	0010011100000000
573	1000000001100 0000000000000	1000000001100000	620	0010001110000 0000000000000	0010001110000000
574	1000000000110 0000000000000	1000000000110000	621	0010000111000 0000000000000	0010000111000000
575	1000000000011 0000000000000	1000000000011000	622	0010000011100 0000000000000	0010000011100000
576	1001000000000 0000000000000	1001000000000000	623	0010000001110 0000000000000	0010000001110000
577	1000100000000 0000000000000	1000100000000000	624	0010000000111 0000000000000	0010000000111000
578	1000010000000 0000000000000	1000010000000000	625	0010010100000 0000000000000	0010010100000000
579	1000001000000 0000000000000	1000001000000000	626	0010001010000 0000000000000	0010001010000000
580	1000000100000 0000000000000	1000000100000000	627	0010000101000 0000000000000	0010000101000000
581	1000000010000 0000000000000	1000000010000000	628	0010000010100 0000000000000	0010000010100000
582	1000000001000 0000000000000	1000000001000000	629	0010000001010 0000000000000	0010000001010000
583	1000000000100 0000000000000	1000000000100000	630	0010000000101 0000000000000	0010000000101000
584	1000000000010 0000000000000	1000000000010000	631	0010011000000 0000000000000	0010011000000000
585	1000000000001 0000000000000	1000000000001000	632	0010001100000 0000000000000	0010001100000000
586	1000000000000 0000000000000	1000000000000000	633	0010000110000 0000000000000	0010000110000000
587	0100111000000 0000000000000	0100111000000000	634	0010000011000 0000000000000	0010000011000000
588	0100011100000 0000000000000	0100011100000000	635	0010000001100 0000000000000	0010000001100000
589	0100001110000 0000000000000	0100001110000000	636	0010000000110 0000000000000	0010000000110000
590	0100000111000 0000000000000	0100000111000000	637	0010000000011 0000000000000	0010000000011000
591	0100000011100 0000000000000	0100000011100000	638	0010010000000 0000000000000	0010010000000000
592	0100000001110 0000000000000	0100000001110000	639	0010001000000 0000000000000	0010001000000000
593	0100000000111 0000000000000	0100000000111000	640	0010000100000 0000000000000	0010000100000000
594	0100101000000 0000000000000	0100101000000000	641	0010000010000 0000000000000	0010000010000000
595	0100010100000 0000000000000	0100010100000000	642	0010000001000 0000000000000	0010000001000000
596	0100001010000 0000000000000	0100001010000000	643	0010000000100 0000000000000	0010000000100000
597	0100000101000 0000000000000	0100000101000000	644	0010000000010 0000000000000	0010000000010000
598	0100000010100 0000000000000	0100000010100000	645	0010000000001 0000000000000	0010000000001000
599	0100000001010 0000000000000	0100000001010000	646	0010000000000 0000000000000	0010000000000000
600	0100000000101 0000000000000	0100000000101000	647	0001001110000 0000000000000	0001001110000000
601	0100110000000 0000000000000	0100110000000000	648	0001000111000 0000000000000	0001000111000000
602	0100011000000 0000000000000	0100011000000000	649	0001000011100 0000000000000	0001000011100000
603	0100001100000 0000000000000	0100001100000000	650	0001000001110 0000000000000	0001000001110000

Sub-block 1

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
651	0001000000111 0000000000000	0001000000111000	694	0000010010100 0000000000000	0000010010100000
652	0001001010000 0000000000000	0001001010000000	695	0000010001010 0000000000000	0000010001010000
653	0001000101000 0000000000000	0001000101000000	696	0000010000101 0000000000000	0000010000101000
654	0001000010100 0000000000000	0001000010100000	697	0000010011000 0000000000000	0000010011000000
655	0001000001010 0000000000000	0001000001010000	698	0000010001100 0000000000000	0000010001100000
656	0001000000101 0000000000000	0001000000101000	699	0000010000110 0000000000000	0000010000110000
657	0001001100000 0000000000000	0001001100000000	700	0000010000011 0000000000000	0000010000011000
658	0001000110000 0000000000000	0001000110000000	701	0000010010000 0000000000000	0000010010000000
659	0001000011000 0000000000000	0001000011000000	702	0000010001000 0000000000000	0000010001000000
660	0001000001100 0000000000000	0001000001100000	703	0000010000100 0000000000000	0000010000100000
661	0001000000110 0000000000000	0001000000110000	704	0000010000010 0000000000000	0000010000010000
662	0001000000011 0000000000000	0001000000011000	705	0000010000001 0000000000000	0000010000001000
663	0001001000000 0000000000000	0001001000000000	706	0000010000000 0000000000000	0000010000000000
664	0001000100000 0000000000000	0001000100000000	707	0000001001110 0000000000000	0000001001110000
665	0001000010000 0000000000000	0001000010000000	708	0000001000111 0000000000000	0000001000111000
666	0001000001000 0000000000000	0001000001000000	709	0000001001010 0000000000000	0000001001010000
667	0001000000100 0000000000000	0001000000100000	710	0000001000101 0000000000000	0000001000101000
668	0001000000010 0000000000000	0001000000010000	711	0000001001100 0000000000000	0000001001100000
669	0001000000001 0000000000000	0001000000001000	712	0000001000110 0000000000000	0000001000110000
670	0001000000000 0000000000000	0001000000000000	713	0000001000011 0000000000000	0000001000011000
671	0000100111000 0000000000000	0000100111000000	714	0000001001000 0000000000000	0000001001000000
672	0000100011100 0000000000000	0000100011100000	715	0000001000100 0000000000000	0000001000100000
673	0000100001110 0000000000000	0000100001110000	716	0000001000010 0000000000000	0000001000010000
674	0000100000111 0000000000000	0000100000111000	717	0000001000001 0000000000000	0000001000001000
675	0000100101000 0000000000000	0000100101000000	718	0000001000000 0000000000000	0000001000000000
676	0000100010100 0000000000000	0000100010100000	719	0000000100111 0000000000000	0000000100111000
677	0000100001010 0000000000000	0000100001010000	720	0000000100101 0000000000000	0000000100101000
678	0000100000101 0000000000000	0000100000101000	721	0000000100110 0000000000000	0000000100110000
679	0000100110000 0000000000000	0000100110000000	722	0000000100011 0000000000000	0000000100011000
680	0000100011000 0000000000000	0000100011000000	723	0000000100100 0000000000000	0000000100100000
681	0000100001100 0000000000000	0000100001100000	724	0000000100010 0000000000000	0000000100010000
682	0000100000110 0000000000000	0000100000110000	725	0000000100001 0000000000000	0000000100001000
683	0000100000011 0000000000000	0000100000011000	726	0000000100000 0000000000000	0000000100000000
684	0000100100000 0000000000000	0000100100000000	727	0000000010011 0000000000000	0000000010011000
685	0000100010000 0000000000000	0000100010000000	728	0000000010001 0000000000000	0000000010001000
686	0000100001000 0000000000000	0000100001000000	729	0000000010010 0000000000000	0000000010010000
687	0000100000100 0000000000000	0000100000100000	730	0000000010000 0000000000000	0000000010000000
688	0000100000010 0000000000000	0000100000010000	731	0000000001001 0000000000000	0000000001001000
689	0000100000001 0000000000000	0000100000001000	732	0000000001000 0000000000000	0000000001000000
690	0000100000000 0000000000000	0000100000000000	733	0000000000100 0000000000000	0000000000100000
691	0000010011100 0000000000000	0000010011100000	734	0000000000010 0000000000000	0000000000010000
692	0000010001110 0000000000000	0000010001110000	735	0000000000001 0000000000000	0000000000001000
693	0000010000111 0000000000000	0000010000111000			

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
736	0000000000000 1111110000000	0000111111111101	779	0000000000000 0111101000000	0000010010000100
737	0000000000000 1110111000000	0000111111101010	780	0000000000000 0111010100000	0000111111100101
738	0000000000000 1110011100000	0000111111110110	781	0000000000000 0111001010000	0000100101110100
739	0000000000000 1110001110000	0000111110110100	782	0000000000000 0111000101000	0001011010011100
740	0000000000000 1110000111000	0001110110110000	783	0000000000000 0111000010100	0000000000110111
741	0000000000000 1110000011100	0001001000100000	784	0000000000000 0111000001010	0011001010011010
742	0000000000000 1110000001110	0011101111001010	785	0000000000000 0111000000101	0001100000001101
743	0000000000000 1110000000111	0011101011001011	786	0000000000000 0111110000000	0000111111111001
744	0000000000000 1111010000000	0000101101101101	787	0000000000000 0111011000000	0000100100110110
745	0000000000000 1110101000000	0000011011001000	788	0000000000000 0111001100000	0000010010011000
746	0000000000000 1110010100000	0000110110101001	789	0000000000000 0111000110000	0000111110100111
747	0000000000000 1110001010000	0000101100111000	790	0000000000000 0111000011000	0001101101110000
748	0000000000000 1110000101000	0001010011010000	791	0000000000000 0111000001100	0001100100001100
749	0000000000000 1110000010100	0000001001111011	792	0000000000000 0111000000110	0010100111011101
750	0000000000000 1110000001010	0011000011010110	793	0000000000000 0111000000011	0011001110011011
751	0000000000000 1110000000101	0001101001000001	794	0000000000000 0111100000000	0000011011011011
752	0000000000000 1111100000000	0000011011011111	795	0000000000000 0111010000000	0000101101101001
753	0000000000000 1110110000000	0000110110110101	796	0000000000000 0111001000000	0000000000010100
754	0000000000000 1110011000000	0000101101111010	797	0000000000000 0111000100000	0000011011000111
755	0000000000000 1110001100000	0000011011010100	798	0000000000000 0111000010000	0000101100101011
756	0000000000000 1110000110000	0000110111101011	799	0000000000000 0111000001000	0001001000010000
757	0000000000000 1110000011000	0001100100111100	800	0000000000000 0111000000100	0000100101010111
758	0000000000000 1110000001100	0001101101000000	801	0000000000000 0111000000010	0010001011000001
759	0000000000000 1110000000110	0010101110010001	802	0000000000000 0111000000001	0001001100010001
760	0000000000000 1110000000011	0011000111010111	803	0000000000000 0111000000000	0000001001001011
761	0000000000000 1111000000000	0000001001001111	804	0000000000000 0011111100000	0000100100101000
762	0000000000000 1110100000000	0000010010010111	805	0000000000000 0011101110000	0000100101101010
763	0000000000000 1110010000000	0000100100100101	806	0000000000000 0011100111000	0001101101101110
764	0000000000000 1110001000000	0000001001011000	807	0000000000000 0011100011100	0001010011111110
765	0000000000000 1110000100000	0000010010001011	808	0000000000000 0011100001110	0011110100010100
766	0000000000000 1110000010000	0000100101100111	809	0000000000000 0011100000111	0011110000010101
767	0000000000000 1110000001000	0001000001011100	810	0000000000000 0011110100000	0000101101110111
768	0000000000000 1110000000100	0000101100011011	811	0000000000000 0011101010000	0000110111100110
769	0000000000000 1110000000010	0010000010001101	812	0000000000000 0011100101000	0001001000001110
770	0000000000000 1110000000001	0001000101011101	813	0000000000000 0011100010100	0000010010100101
771	0000000000000 1110000000000	0000000000000111	814	0000000000000 0011100001010	0011011000001000
772	0000000000000 0111111000000	0000110110100110	815	0000000000000 0011100000101	0001110010011111
773	0000000000000 0111011100000	0000110110111010	816	0000000000000 0011111000000	0000110110100100
774	0000000000000 0111001110000	0000110111111000	817	0000000000000 0011101100000	0000000000001010
775	0000000000000 0111000111000	0001111111111100	818	0000000000000 0011100110000	0000101100110101
776	0000000000000 0111000011100	0001000001101100	819	0000000000000 0011100011000	0001111111100010
777	0000000000000 0111000001110	0011100110000110	820	0000000000000 0011100001100	0001110110011110
778	0000000000000 0111000000111	0011100010000111	821	0000000000000 0011100000110	0010110101001111

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
822	0000000000000 0011100000011	0011011100001001	869	0000000000000 0000111100000	0000101101100001
823	0000000000000 0011110000000	0000111111111011	870	0000000000000 0000111010000	0000011010001101
824	0000000000000 0011101000000	0000010010000110	871	0000000000000 0000111001000	0001111110110110
825	0000000000000 0011100100000	0000001001010101	872	0000000000000 0000111000100	0000010011110001
826	0000000000000 0011100010000	0000111110111001	873	0000000000000 0000111000010	0010111101100111
827	0000000000000 0011100001000	0001011010000010	874	0000000000000 0000111000001	0001111010110111
828	0000000000000 0011100000100	0000110111000101	875	0000000000000 0000111000000	0000111111101101
829	0000000000000 0011100000010	0010011001010011	876	0000000000000 0000011111100	0001110111010110
830	0000000000000 0011100000001	0001011110000011	877	0000000000000 0000011101110	0011010000111100
831	0000000000000 0011100000000	0000011011011001	878	0000000000000 0000011100111	0011010100111101
832	0000000000000 0001111110000	0000000001001001	879	0000000000000 0000011110100	0000110110001101
833	0000000000000 0001110111000	0001001001001101	880	0000000000000 0000011101010	0011111100100000
834	0000000000000 0001110011100	0001110111011101	881	0000000000000 0000011100101	0001010110110111
835	0000000000000 0001110001110	0011010000110111	882	0000000000000 0000011111000	0001011011001010
836	0000000000000 0001110000111	0011010100110110	883	0000000000000 0000011101100	0001010010110110
837	0000000000000 0001111010000	0000010011000101	884	0000000000000 0000011100110	0010010001100111
838	0000000000000 0001110101000	0001101100101101	885	0000000000000 0000011100011	0011111000100001
839	0000000000000 0001110010100	0000110110000110	886	0000000000000 0000011110000	0000011010010001
840	0000000000000 0001110001010	0011111100101011	887	0000000000000 0000011101000	0001111110101010
841	0000000000000 0001110000101	0001010110111100	888	0000000000000 0000011100100	0000010011101101
842	0000000000000 0001111100000	0000100100101001	889	0000000000000 0000011100010	0010111101111011
843	0000000000000 0001110110000	0000001000010110	890	0000000000000 0000011100001	0001111010101011
844	0000000000000 0001110011000	0001011011000001	891	0000000000000 0000011100000	0000111111110001
845	0000000000000 0001110001100	0001010010111101	892	0000000000000 0000001111110	0011010001111110
846	0000000000000 0001110000110	0010010001101100	893	0000000000000 0000001110111	0011010101111111
847	0000000000000 0001110000011	0011111000101010	894	0000000000000 0000001111010	0011111101100010
848	0000000000000 0001111000000	0000110110100101	895	0000000000000 0000001110101	0001010111110101
849	0000000000000 0001110100000	0000101101110110	896	0000000000000 0000001111100	0001010011110100
850	0000000000000 0001110010000	0000011010011010	897	0000000000000 0000001110110	0010010000100101
851	0000000000000 0001110001000	0001111110100001	898	0000000000000 0000001110011	0011111001100011
852	0000000000000 0001110000100	0000010011100110	899	0000000000000 0000001111000	0001111111101000
853	0000000000000 0001110000010	0010111101110000	900	0000000000000 0000001110100	0000010010101111
854	0000000000000 0001110000001	0001111010100000	901	0000000000000 0000001110010	0010111100111001
855	0000000000000 0001110000000	0000111111111010	902	0000000000000 0000001110001	0001111011101001
856	0000000000000 0000111111000	0001001001011010	903	0000000000000 0000001110000	0000111110110011
857	0000000000000 0000111011100	0001110111001010	904	0000000000000 0000000111111	0010011101111011
858	0000000000000 0000111001110	0011010000100000	905	0000000000000 0000000111101	0000011111110001
859	0000000000000 0000111000111	0011010100100001	906	0000000000000 0000000111110	0011011000100001
860	0000000000000 0000111101000	0001101100111010	907	0000000000000 0000000111011	0010110001100111
861	0000000000000 0000111010100	0000110110010001	908	0000000000000 0000000111100	0001011010101011
862	0000000000000 0000111001010	0011111100111100	909	0000000000000 0000000111010	0011110100111101
863	0000000000000 0000111000101	0001010110101011	910	0000000000000 0000000111001	0000110011101101
864	0000000000000 0000111110000	0000001000000001	911	0000000000000 0000000111000	0001110110110111
865	0000000000000 0000111011000	0001011011010110	912	0000000000000 0000000011111	0010001111110111
866	0000000000000 0000111001100	0001010010101010	913	0000000000000 0000000011101	0000001101111101
867	0000000000000 0000111000110	0010010001111011	914	0000000000000 0000000011110	0011001010101101
868	0000000000000 0000111000011	0011111000111101	915	0000000000000 0000000011100	0001001000100111

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
916	0000000000000 0000000001111	0010101010010111	963	0000000000000 0101010100000	0000111111100100
917	0000000000000 0000000001110	0011101111001101	964	0000000000000 0101001010000	0000100101110101
918	0000000000000 0000000000111	0011101011001100	965	0000000000000 0101000101000	0001011010011101
919	0000000000000 1011110000000	0000111111111111	966	0000000000000 0101000010100	0000000000110110
920	0000000000000 1010111000000	0000111111101000	967	0000000000000 0101000001010	0011001010011011
921	0000000000000 1010011100000	0000111111110100	968	0000000000000 0101000000101	0001100000001100
922	0000000000000 1010001110000	0000111110110110	969	0000000000000 0101110000000	0000111111111000
923	0000000000000 1010000111000	0001110110110010	970	0000000000000 0101011000000	0000100100110111
924	0000000000000 1010000011100	0001001000100010	971	0000000000000 0101001100000	0000010010011001
925	0000000000000 1010000001110	0011101111001000	972	0000000000000 0101000110000	0000111110100110
926	0000000000000 1010000000111	0011101011001001	973	0000000000000 0101000011000	0001101101110001
927	0000000000000 1011010000000	0000101101101111	974	0000000000000 0101000001100	0001100100001101
928	0000000000000 1010101000000	0000011011001010	975	0000000000000 0101000000110	0010100111011100
929	0000000000000 1010010100000	0000110110101011	976	0000000000000 0101000000011	0011001110011010
930	0000000000000 1010001010000	0000101100111010	977	0000000000000 0101100000000	0000011011011010
931	0000000000000 1010000101000	0001010011010010	978	0000000000000 0101010000000	0000101101101000
932	0000000000000 1010000010100	0000001001111001	979	0000000000000 0101001000000	0000000000010101
933	0000000000000 1010000001010	0011000011010100	980	0000000000000 0101000100000	0000011011000110
934	0000000000000 1010000000101	0001101001000011	981	0000000000000 0101000010000	0000101100101010
935	0000000000000 1011100000000	0000011011011101	982	0000000000000 0101000001000	0001001000010001
936	0000000000000 1010110000000	0000110110110111	983	0000000000000 0101000000100	0000100101010110
937	0000000000000 1010011000000	0000101101111000	984	0000000000000 0101000000010	0010001011000000
938	0000000000000 1010001100000	0000011011010110	985	0000000000000 0101000000001	0001001100010000
939	0000000000000 1010000110000	0000110111101001	986	0000000000000 0101000000000	0000001001001010
940	0000000000000 1010000011000	0001100100111110	987	0000000000000 0010111100000	0000101101100000
941	0000000000000 1010000001100	0001101101000010	988	0000000000000 0010101110000	0000101100100010
942	0000000000000 1010000000110	0010101110010011	989	0000000000000 0010100111000	0001100100100110
943	0000000000000 1010000000011	0011000111010101	990	0000000000000 0010100011100	0001011010110110
944	0000000000000 1011000000000	0000001001001101	991	0000000000000 0010100001110	0011111101011100
945	0000000000000 1010100000000	0000010010010101	992	0000000000000 0010100000111	0011111001011101
946	0000000000000 1010010000000	0000100100100111	993	0000000000000 0010110100000	0000100100111111
947	0000000000000 1010001000000	0000001001011010	994	0000000000000 0010101010000	0000111110101110
948	0000000000000 1010000100000	0000010010001001	995	0000000000000 0010100101000	0001000001000110
949	0000000000000 1010000010000	0000100101100101	996	0000000000000 0010100010100	0000011011101101
950	0000000000000 1010000001000	0001000001011110	997	0000000000000 0010100001010	0011010001000000
951	0000000000000 1010000000100	0000101100011001	998	0000000000000 0010100000101	0001111011010111
952	0000000000000 1010000000010	0010000010001111	999	0000000000000 0010111000000	0000111111101100
953	0000000000000 1010000000001	0001000101011111	1000	0000000000000 0010101100000	0000001001000010
954	0000000000000 1010000000000	0000000000000101	1001	0000000000000 0010100110000	0000100101111101
955	0000000000000 0101111000000	0000110110100111	1002	0000000000000 0010100011000	0001110110101010
956	0000000000000 0101011100000	0000110110111011	1003	0000000000000 0010100001100	0001111111010110
957	0000000000000 0101001110000	0000110111111001	1004	0000000000000 0010100000110	0010111100000111
958	0000000000000 0101000111000	0001111111111101	1005	0000000000000 0010100000011	0011010101000001
959	0000000000000 0101000011100	0001000001101101	1006	0000000000000 0010110000000	0000110110110011
960	0000000000000 0101000001110	0011100110000111	1007	0000000000000 0010101000000	0000011011001110
961	0000000000000 0101000000111	0011100010000110	1008	0000000000000 0010100100000	0000000000011101
962	0000000000000 0101101000000	0000010010000101	1009	0000000000000 0010100010000	0000110111110001

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1010	0000000000000 0010100001000	0001010011001010	1057	0000000000000 0000101000001	0001011110010101
1011	0000000000000 0010100000100	0000111110001101	1058	0000000000000 0000101000000	0000011011001111
1012	0000000000000 0010100000010	0010010000011011	1059	0000000000000 0000010111100	0001111110001001
1013	0000000000000 0010100000001	0001010111001011	1060	0000000000000 0000010101110	0011011001100011
1014	0000000000000 0010100000000	0000010010010001	1061	0000000000000 0000010100111	0011011101100010
1015	0000000000000 0001011110000	0000010011011001	1062	0000000000000 0000010110100	0000111111010010
1016	0000000000000 0001010111000	0001011011011101	1063	0000000000000 0000010101010	0011110101111111
1017	0000000000000 0001010011100	0001100101001101	1064	0000000000000 0000010100101	0001011111101000
1018	0000000000000 0001010001110	0011000010100111	1065	0000000000000 0000010111000	0001010010010101
1019	0000000000000 0001010000111	0011000110100110	1066	0000000000000 0000010101100	0001011011101001
1020	0000000000000 0001011010000	0000000001010101	1067	0000000000000 0000010100110	0010011000111000
1021	0000000000000 0001010101000	0001111110111101	1068	0000000000000 0000010100011	0011110001111110
1022	0000000000000 0001010010100	0000100100010110	1069	0000000000000 0000010110000	0000010011001110
1023	0000000000000 0001010001010	0011101110111011	1070	0000000000000 0000010101000	0001110111110101
1024	0000000000000 0001010000101	0001000100101100	1071	0000000000000 0000010100100	0000011010110010
1025	0000000000000 0001011100000	0000110110111001	1072	0000000000000 0000010100010	0010110100100100
1026	0000000000000 0001010110000	0000011010000110	1073	0000000000000 0000010100001	0001110011110100
1027	0000000000000 0001010011000	0001001001010001	1074	0000000000000 0000010100000	0000110110101110
1028	0000000000000 0001010001100	0001000000101101	1075	0000000000000 0000001011110	0011000011110010
1029	0000000000000 0001010000110	0010000011111100	1076	0000000000000 0000001010111	0011000111110011
1030	0000000000000 0001010000011	0011101010111010	1077	0000000000000 0000001011010	0011101111101110
1031	0000000000000 0001011000000	0000100100110101	1078	0000000000000 0000001010101	0001000101111001
1032	0000000000000 0001010100000	0000111111100110	1079	0000000000000 0000001011100	0001000001111000
1033	0000000000000 0001010010000	0000001000001010	1080	0000000000000 0000001010110	0010000010101001
1034	0000000000000 0001010001000	0001101100110001	1081	0000000000000 0000001010011	0011101011101111
1035	0000000000000 0001010000100	0000000001110110	1082	0000000000000 0000001011000	0001101101100100
1036	0000000000000 0001010000010	0010101111100000	1083	0000000000000 0000001010100	0000000000100011
1037	0000000000000 0001010000001	0001101000110000	1084	0000000000000 0000001010010	0010101110110101
1038	0000000000000 0001010000000	0000101101101010	1085	0000000000000 0000001010001	0001101001100101
1039	0000000000000 0000101111000	0001101101111000	1086	0000000000000 0000001010000	0000101100111111
1040	0000000000000 0000101011100	0001010011101000	1087	0000000000000 0000000101111	0010111000011011
1041	0000000000000 0000101001110	0011110100000010	1088	0000000000000 0000000101101	0000111010010001
1042	0000000000000 0000101000111	0011110000000011	1089	0000000000000 0000000101110	0011111101000001
1043	0000000000000 0000101101000	0001001000011000	1090	0000000000000 0000000101011	0010010100000111
1044	0000000000000 0000101010100	0000010010110011	1091	0000000000000 0000000101100	0001111111001011
1045	0000000000000 0000101001010	0011011000011110	1092	0000000000000 0000000101010	0011010001011101
1046	0000000000000 0000101000101	0001110010001001	1093	0000000000000 0000000101001	0000010110001101
1047	0000000000000 0000101110000	0000101100100011	1094	0000000000000 0000000101000	0001010011010111
1048	0000000000000 0000101011000	0001111111110100	1095	0000000000000 0000000010111	0011001110101100
1049	0000000000000 0000101001100	0001110110001000	1096	0000000000000 0000000010101	0001001100100110
1050	0000000000000 0000101000110	0010110101011001	1097	0000000000000 0000000010110	0010001011110110
1051	0000000000000 0000101000011	0011011100011111	1098	0000000000000 0000000010100	0000001001111100
1052	0000000000000 0000101100000	0000001001000011	1099	0000000000000 0000000001011	0010000110001011
1053	0000000000000 0000101010000	0000111110101111	1100	0000000000000 0000000001010	0011000011010001
1054	0000000000000 0000101001000	0001011010010100	1101	0000000000000 0000000000101	0001101001000110
1055	0000000000000 0000101000100	0000110111010011	1102	0000000000000 1101110000000	0000111111111100
1056	0000000000000 0000101000010	0010011001000101	1103	0000000000000 1100111000000	0000111111101011

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1104	0000000000000 1100011100000	0000111111110111	1151	0000000000000 0110000000101	0001101001000101
1105	0000000000000 1100001110000	0000111110110101	1152	0000000000000 0110110000000	0000110110110001
1106	0000000000000 1100000111000	0001110110110001	1153	0000000000000 0110011000000	0000101101111110
1107	0000000000000 1100000011100	0001001000100001	1154	0000000000000 0110001100000	0000011011010000
1108	0000000000000 1100000001110	0011101111001011	1155	0000000000000 0110000110000	0000110111101111
1109	0000000000000 1100000000111	0011101011001010	1156	0000000000000 0110000011000	0001100100111000
1110	0000000000000 1101010000000	0000101101101100	1157	0000000000000 0110000001100	0001101101000100
1111	0000000000000 1100101000000	0000011011001001	1158	0000000000000 0110000000110	0010101110010101
1112	0000000000000 1100010100000	0000110110101000	1159	0000000000000 0110000000011	0011000111010011
1113	0000000000000 1100001010000	0000101100111001	1160	0000000000000 0110100000000	0000010010010011
1114	0000000000000 1100000101000	0001010011010001	1161	0000000000000 0110010000000	0000100100100001
1115	0000000000000 1100000010100	0000001001111010	1162	0000000000000 0110001000000	0000001001011100
1116	0000000000000 1100000001010	0011000011010111	1163	0000000000000 0110000100000	0000010010001111
1117	0000000000000 1100000000101	0001101001000000	1164	0000000000000 0110000010000	0000100101100011
1118	0000000000000 1101100000000	0000011011011110	1165	0000000000000 0110000001000	0001000001011000
1119	0000000000000 1100110000000	0000110110110100	1166	0000000000000 0110000000100	0000101100011111
1120	0000000000000 1100011000000	0000101101111011	1167	0000000000000 0110000000010	0010000010001001
1121	0000000000000 1100001100000	0000011011010101	1168	0000000000000 0110000000001	0001000101011001
1122	0000000000000 1100000110000	0000110111101010	1169	0000000000000 0110000000000	0000000000000011
1123	0000000000000 1100000011000	0001100100111101	1170	0000000000000 0011011100000	0000110110111000
1124	0000000000000 1100000001100	0001101101000001	1171	0000000000000 0011001110000	0000110111111010
1125	0000000000000 1100000000110	0010101110010000	1172	0000000000000 0011000111000	0001111111111110
1126	0000000000000 1100000000011	0011000111010110	1173	0000000000000 0011000011100	0001000001101110
1127	0000000000000 1101000000000	0000001001001110	1174	0000000000000 0011000001110	0011100110000100
1128	0000000000000 1100100000000	0000010010010110	1175	0000000000000 0011000000111	0011100010000101
1129	0000000000000 1100010000000	0000100100100100	1176	0000000000000 0011010100000	0000111111100111
1130	0000000000000 1100001000000	0000001001011001	1177	0000000000000 0011001010000	0000100101110110
1131	0000000000000 1100000100000	0000010010001010	1178	0000000000000 0011000101000	0001011010011110
1132	0000000000000 1100000010000	0000100101100110	1179	0000000000000 0011000010100	0000000000110101
1133	0000000000000 1100000001000	0001000001011101	1180	0000000000000 0011000001010	0011001010011000
1134	0000000000000 1100000000100	0000101100011010	1181	0000000000000 0011000000101	0001100000001111
1135	0000000000000 1100000000010	0010000010001100	1182	0000000000000 0011011000000	0000100100110100
1136	0000000000000 1100000000001	0001000101011100	1183	0000000000000 0011001100000	0000010010011010
1137	0000000000000 1100000000000	0000000000000110	1184	0000000000000 0011000110000	0000111110100101
1138	0000000000000 0110111000000	0000111111101110	1185	0000000000000 0011000011000	0001101101110010
1139	0000000000000 0110011100000	0000111111110010	1186	0000000000000 0011000001100	0001100100001110
1140	0000000000000 0110001110000	0000111110110000	1187	0000000000000 0011000000110	0010100111011111
1141	0000000000000 0110000111000	0001110110110100	1188	0000000000000 0011000000011	0011001110011001
1142	0000000000000 0110000011100	0001001000100100	1189	0000000000000 0011010000000	0000101101101011
1143	0000000000000 0110000001110	0011101111001110	1190	0000000000000 0011001000000	0000000000010110
1144	0000000000000 0110000000111	0011101011001111	1191	0000000000000 0011000100000	0000011011000101
1145	0000000000000 0110101000000	0000011011001100	1192	0000000000000 0011000010000	0000101100101001
1146	0000000000000 0110010100000	0000110110101101	1193	0000000000000 0011000001000	0001001000010010
1147	0000000000000 0110001010000	0000101100111100	1194	0000000000000 0011000000100	0000100101010101
1148	0000000000000 0110000101000	0001010011010100	1195	0000000000000 0011000000010	0010001011000011
1149	0000000000000 0110000010100	0000001001111111	1196	0000000000000 0011000000001	0001001100010011
1150	0000000000000 0110000001010	0011000011010010	1197	0000000000000 0011000000000	0000001001001001

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1198	0000000000000 0001101110000	0000100101101011	1245	0000000000000 0000011010100	0000100100000001
1199	0000000000000 0001100111000	0001101101101111	1246	0000000000000 0000011001010	0011101110101100
1200	0000000000000 0001100011100	0001010011111111	1247	0000000000000 0000011000101	0001000100111011
1201	0000000000000 0001100001110	0011110100010101	1248	0000000000000 0000011011000	0001001001000110
1202	0000000000000 0001100000111	0011110000010100	1249	0000000000000 0000011001100	0001000000111010
1203	0000000000000 0001101010000	0000110111100111	1250	0000000000000 0000011000110	0010000011101011
1204	0000000000000 0001100101000	0001001000001111	1251	0000000000000 0000011000011	0011101010101101
1205	0000000000000 0001100010100	0000010010100100	1252	0000000000000 0000011010000	0000001000011101
1206	0000000000000 0001100001010	0011011000001001	1253	0000000000000 0000011001000	0001101100100110
1207	0000000000000 0001100000101	0001110010011110	1254	0000000000000 0000011000100	0000000001100001
1208	0000000000000 0001101100000	0000000000001011	1255	0000000000000 0000011000010	0010101111110111
1209	0000000000000 0001100110000	0000101100110100	1256	0000000000000 0000011000001	0001101000100111
1210	0000000000000 0001100011000	0001111111100011	1257	0000000000000 0000011000000	0000101101111101
1211	0000000000000 0001100001100	0001110110011111	1258	0000000000000 0000001101110	0011110100011110
1212	0000000000000 0001100000110	0010110101001110	1259	0000000000000 0000001100111	0011110000011111
1213	0000000000000 0001100000011	0011011100001000	1260	0000000000000 0000001101010	0011011000000010
1214	0000000000000 0001101000000	0000010010000111	1261	0000000000000 0000001100101	0001110010010101
1215	0000000000000 0001100100000	0000001001010100	1262	0000000000000 0000001101100	0001110110010100
1216	0000000000000 0001100010000	0000111110111000	1263	0000000000000 0000001100110	0010110101000101
1217	0000000000000 0001100001000	0001011010000011	1264	0000000000000 0000001100011	0011011100000011
1218	0000000000000 0001100000100	0000110111000100	1265	0000000000000 0000001101000	0001011010001000
1219	0000000000000 0001100000010	0010011001010010	1266	0000000000000 0000001100100	0000110111001111
1220	0000000000000 0001100000001	0001011110000010	1267	0000000000000 0000001100010	0010011001011001
1221	0000000000000 0001100000000	0000011011011000	1268	0000000000000 0000001100001	0001011110001001
1222	0000000000000 0000110111000	0001000000000101	1269	0000000000000 0000001100000	0000011011010011
1223	0000000000000 0000110011100	0001111110010101	1270	0000000000000 0000000110111	0011011100100000
1224	0000000000000 0000110001110	0011011001111111	1271	0000000000000 0000000110101	0001011110101010
1225	0000000000000 0000110000111	0011011101111110	1272	0000000000000 0000000110110	0010011001111010
1226	0000000000000 0000110101000	0001100101100101	1273	0000000000000 0000000110011	0011110000111100
1227	0000000000000 0000110010100	0000111111001110	1274	0000000000000 0000000110100	0000011011110000
1228	0000000000000 0000110001010	0011110101100011	1275	0000000000000 0000000110010	0010110101100110
1229	0000000000000 0000110000101	0001011111110100	1276	0000000000000 0000000110001	0001110010110110
1230	0000000000000 0000110110000	0000000001011110	1277	0000000000000 0000000110000	0000110111101100
1231	0000000000000 0000110011000	0001010010001001	1278	0000000000000 0000000011011	0010100011101011
1232	0000000000000 0000110001100	0001011011110101	1279	0000000000000 0000000011001	0000100001100001
1233	0000000000000 0000110000110	0010011000100100	1280	0000000000000 0000000011010	0011100110110001
1234	0000000000000 0000110000011	0011110001100010	1281	0000000000000 0000000011000	0001100100111011
1235	0000000000000 0000110100000	0000100100111110	1282	0000000000000 0000000001101	0000101000011101
1236	0000000000000 0000110010000	0000010011010010	1283	0000000000000 0000000001100	0001101101000111
1237	0000000000000 0000110001000	0001110111101001	1284	0000000000000 0000000000110	0010101110010110
1238	0000000000000 0000110000100	0000011010101110	1285	0000000000000 0000000000011	0011000111010000
1239	0000000000000 0000110000010	0010110100111000	1286	0000000000000 1001110000000	0000111111111110
1240	0000000000000 0000110000001	0001110011101000	1287	0000000000000 1000111000000	0000111111101001
1241	0000000000000 0000110000000	0000110110110010	1288	0000000000000 1000011100000	0000111111110101
1242	0000000000000 0000011011100	0001100101011010	1289	0000000000000 1000001110000	0000111110110111
1243	0000000000000 0000011001110	0011000010110000	1290	0000000000000 1000000111000	0001110110110011
1244	0000000000000 0000011000111	0011000110110001	1291	0000000000000 1000000011100	0001001000100011

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1292	0000000000000 1000000001110	0011101111001001	1339	0000000000000 0100000110000	0000110111101110
1293	0000000000000 1000000000111	0011101011001000	1340	0000000000000 0100000011000	0001100100111001
1294	0000000000000 1001010000000	0000101101101110	1341	0000000000000 0100000001100	0001101101000101
1295	0000000000000 1000101000000	0000011011001011	1342	0000000000000 0100000000110	0010101110010100
1296	0000000000000 1000010100000	0000110110101010	1343	0000000000000 0100000000011	0011000111010010
1297	0000000000000 1000001010000	0000101100111011	1344	0000000000000 0100100000000	0000010010010010
1298	0000000000000 1000000101000	0001010011010011	1345	0000000000000 0100010000000	0000100100100000
1299	0000000000000 1000000010100	0000001001111000	1346	0000000000000 0100001000000	0000001001011101
1300	0000000000000 1000000001010	0011000011010101	1347	0000000000000 0100000100000	0000010010001110
1301	0000000000000 1000000000101	0001101001000010	1348	0000000000000 0100000010000	0000100101100010
1302	0000000000000 1001100000000	0000011011011100	1349	0000000000000 0100000001000	0001000001011001
1303	0000000000000 1000110000000	0000110110110110	1350	0000000000000 0100000000100	0000101100011110
1304	0000000000000 1000011000000	0000101101111001	1351	0000000000000 0100000000010	0010000010001000
1305	0000000000000 1000001100000	0000011011010111	1352	0000000000000 0100000000001	0001000101011000
1306	0000000000000 1000000110000	0000110111101000	1353	0000000000000 0100000000000	0000000000000010
1307	0000000000000 1000000011000	0001100100111111	1354	0000000000000 0010011100000	0000111111110000
1308	0000000000000 1000000001100	0001101101000011	1355	0000000000000 0010001110000	0000111110110010
1309	0000000000000 1000000000110	0010101110010010	1356	0000000000000 0010000111000	0001110110110110
1310	0000000000000 1000000000011	0011000111010100	1357	0000000000000 0010000011100	0001001000100110
1311	0000000000000 1001000000000	0000001001001100	1358	0000000000000 0010000001110	0011101111001100
1312	0000000000000 1000100000000	0000010010010100	1359	0000000000000 0010000000111	0011101011001101
1313	0000000000000 1000010000000	0000100100100110	1360	0000000000000 0010010100000	0000110110101111
1314	0000000000000 1000001000000	0000001001011011	1361	0000000000000 0010001010000	0000101100111110
1315	0000000000000 1000000100000	0000010010001000	1362	0000000000000 0010000101000	0001010011010110
1316	0000000000000 1000000010000	0000100101100100	1363	0000000000000 0010000010100	0000001001111101
1317	0000000000000 1000000001000	0001000001011111	1364	0000000000000 0010000001010	0011000011010000
1318	0000000000000 1000000000100	0000101100011000	1365	0000000000000 0010000000101	0001101001000111
1319	0000000000000 1000000000010	0010000010001110	1366	0000000000000 0010011000000	0000101101111100
1320	0000000000000 1000000000001	0001000101011110	1367	0000000000000 0010001100000	0000011011010010
1321	0000000000000 1000000000000	0000000000000100	1368	0000000000000 0010000110000	0000110111101101
1322	0000000000000 0100111000000	0000111111101111	1369	0000000000000 0010000011000	0001100100111010
1323	0000000000000 0100011100000	0000111111110011	1370	0000000000000 0010000001100	0001101101000110
1324	0000000000000 0100001110000	0000111110110001	1371	0000000000000 0010000000110	0010101110010111
1325	0000000000000 0100000111000	0001110110110101	1372	0000000000000 0010000000011	0011000111010001
1326	0000000000000 0100000011100	0001001000100101	1373	0000000000000 0010010000000	0000100100100011
1327	0000000000000 0100000001110	0011101111001111	1374	0000000000000 0010001000000	0000001001011110
1328	0000000000000 0100000000111	0011101011001110	1375	0000000000000 0010000100000	0000010010001101
1329	0000000000000 0100101000000	0000011011001101	1376	0000000000000 0010000010000	0000100101100001
1330	0000000000000 0100010100000	0000110110101100	1377	0000000000000 0010000001000	0001000001011010
1331	0000000000000 0100001010000	0000101100111101	1378	0000000000000 0010000000100	0000101100011101
1332	0000000000000 0100000101000	0001010011010101	1379	0000000000000 0010000000010	0010000010001011
1333	0000000000000 0100000010100	0000001001111110	1380	0000000000000 0010000000001	0001000101011011
1334	0000000000000 0100000001010	0011000011010011	1381	0000000000000 0010000000000	0000000000000001
1335	0000000000000 0100000000101	0001101001000100	1382	0000000000000 0001001110000	0000110111111011
1336	0000000000000 0100110000000	0000110110110000	1383	0000000000000 0001000111000	0001111111111111
1337	0000000000000 0100011000000	0000101101111111	1384	0000000000000 0001000011100	0001000001101111
1338	0000000000000 0100001100000	0000011011010001	1385	0000000000000 0001000001110	0011100110000101

Sub-block 2

S.No.	Error Vector	Syndrome	S. no.	Error vector	Syndrome
1386	0000000000000 0001000000111	0011100010000100	1429	0000000000000 0000010010100	0000101101011110
1387	0000000000000 0001001010000	0000100101110111	1430	0000000000000 0000010001010	0011100111110011
1388	0000000000000 0001000101000	0001011010011111	1431	0000000000000 0000010000101	0001001101100100
1389	0000000000000 0001000010100	0000000000110100	1432	0000000000000 0000010011000	0001000000011001
1390	0000000000000 0001000001010	0011001010011001	1433	0000000000000 0000010001100	0001001001100101
1391	0000000000000 0001000000101	0001100000001110	1434	0000000000000 0000010000110	0010001010110100
1392	0000000000000 0001001100000	0000010010011011	1435	0000000000000 0000010000011	0011100011110010
1393	0000000000000 0001000110000	0000111110100100	1436	0000000000000 0000010010000	0000000001000010
1394	0000000000000 0001000011000	0001101101110011	1437	0000000000000 0000010001000	0001100101111001
1395	0000000000000 0001000001100	0001100100001111	1438	0000000000000 0000010000100	0000001000111110
1396	0000000000000 0001000000110	0010100111011110	1439	0000000000000 0000010000010	0010100110101000
1397	0000000000000 0001000000011	0011001110011000	1440	0000000000000 0000010000001	0001100001111000
1398	0000000000000 0001001000000	0000000000010111	1441	0000000000000 0000010000000	0000100100100010
1399	0000000000000 0001000100000	0000011011000100	1442	0000000000000 0000001001110	0011100110010010
1400	0000000000000 0001000010000	0000101100101000	1443	0000000000000 0000001000111	0011100010010011
1401	0000000000000 0001000001000	0001001000010011	1444	0000000000000 0000001001010	0011001010001110
1402	0000000000000 0001000000100	0000100101010100	1445	0000000000000 0000001000101	0001100000011001
1403	0000000000000 0001000000010	0010001011000010	1446	0000000000000 0000001001100	0001100100011000
1404	0000000000000 0001000000001	0001001100010010	1447	0000000000000 0000001000110	0010100111001001
1405	0000000000000 0001000000000	0000001001001000	1448	0000000000000 0000001000011	0011001110001111
1406	0000000000000 0000100111000	0001100100100111	1449	0000000000000 0000001001000	0001001000000100
1407	0000000000000 0000100011100	0001011010110111	1450	0000000000000 0000001000100	0000100101000011
1408	0000000000000 0000100001110	0011111101011101	1451	0000000000000 0000001000010	0010001011010101
1409	0000000000000 0000100000111	0011111001011100	1452	0000000000000 0000001000001	0001001100000101
1410	0000000000000 0000100101000	0001000001000111	1453	0000000000000 0000001000000	0000001001011111
1411	0000000000000 0000100010100	0000011011101100	1454	0000000000000 0000000100111	0011111001000000
1412	0000000000000 0000100001010	0011010001000001	1455	0000000000000 0000000100101	0001111011001010
1413	0000000000000 0000100000101	0001111011010110	1456	0000000000000 0000000100110	0010111100011010
1414	0000000000000 0000100110000	0000100101111100	1457	0000000000000 0000000100011	0011010101011100
1415	0000000000000 0000100011000	0001110110101011	1458	0000000000000 0000000100100	0000111110010000
1416	0000000000000 0000100001100	0001111111010111	1459	0000000000000 0000000100010	0010010000000110
1417	0000000000000 0000100000110	0010111100000110	1460	0000000000000 0000000100001	0001010111010110
1418	0000000000000 0000100000011	0011010101000000	1461	0000000000000 0000000100000	0000010010001100
1419	0000000000000 0000100100000	0000000000011100	1462	0000000000000 0000000010011	0011100010110000
1420	0000000000000 0000100010000	0000110111110000	1463	0000000000000 0000000010001	0001100000111010
1421	0000000000000 0000100001000	0001010011001011	1464	0000000000000 0000000010010	0010100111101010
1422	0000000000000 0000100000100	0000111110001100	1465	0000000000000 0000000010000	0000100101100000
1423	0000000000000 0000100000010	0010010000011010	1466	0000000000000 0000000001001	0000000100000001
1424	0000000000000 0000100000001	0001010111001010	1467	0000000000000 0000000001000	0001000001011011
1425	0000000000000 0000100000000	0000010010010000	1468	0000000000000 0000000000100	0000101100011100
1426	0000000000000 0000010011100	0001101100000101	1469	0000000000000 0000000000010	0010000010001010
1427	0000000000000 0000010001110	0011001011101111	1470	0000000000000 0000000000001	0001000101011010
1428	0000000000000 0000010000111	0011001111101110			

III Bounds for codes correcting m-repeated bursts

In this section, we extend the results of previous section to the case of m-repeated bursts of length b or less occurring within a single sub-block.

Similar to the case of correction of 2-repeated burst occurring within a sub-block, an (n, k) linear code over $GF(q)$ capable of correcting any sub-block containing m-repeated burst of length b or less must satisfy the following two conditions:
(v) The syndrome resulting from the occurrence of any m-repeated burst of length b or less within a single sub-block must be distinct from the syndrome resulting from any other m-repeated burst within the same sub-block.
(vi) The syndrome resulting from the occurrence of any m-repeated burst of length b or less within a single sub-block must be distinct from the syndrome resulting likewise from any m-repeated burst of length b or less within any other sub-block.

We now present a lower bound on the number of parity check digits required for such a code.

Theorem 3. *The number of check digits r required for an (n, k) linear code over $GF(q)$, subdivided into s sub-blocks of length t each, that corrects m-repeated bursts of length b or less lying within a single corrupted sub-block is atleast*

$$log_q\left\{1 + s\left[q^{m(b-1)}\left(\binom{t-mb+m}{m}(q-1)^m + \sum_{l=0}^{m-1}\binom{t-mb+l}{l}(q-1)^l q^{m-1-l}\right) - 1\right]\right\}. \quad (9)$$

Proof. The proof of this result is on the similar lines as that of proof of Theorem 1 so we omit the proof.

□

Remark 4. By taking $s = 1$ the bound obtained in (9) reduces to

$$log_q\left\{q^{m(b-1)}\left(\binom{t-mb+m}{m}(q-1)^m + \sum_{l=0}^{m-1}\binom{t-mb+l}{l}(q-1)^l q^{m-1-l}\right)\right\}.$$

which coincides with the result for correction of m-repeated burst obtained by Dass and Verma(2008).

Remark 5. For $m = 2$, the bound obtained in (9) coincides with the bound obtained in (1) for the case of 2-repeated bursts.
In particular, for $m = 1$, the bound in (9) reduces to

$$1 + s\Big(q^{b-1}\big((t-b+1)(q-1)+1\big)-1\Big)$$

which reduces to the result for correction of burst of length b or less within a sub-block.

In the following result, we present another bound on the number of check digits required for the existence of the code considered in Theorem 3.

Theorem 4. *An (n, k) linear code over $GF(q)$ capable of correcting m-repeated burst of length b or less occurring within a single sub-block of length t $(2mb < t)$ can always be constructed using r check digits where r is the smallest integer satisfying the inequality*

$$\begin{aligned} q^r > & q^{m(b-1)}\Bigg\{q^{m(b-1)}\Bigg((q-1)^{2m-1}\binom{t-2mb+(2m-1)}{2m-1}+ \\ & \sum_{l=0}^{2m-2}(q-1)^l q^{2m-2-l}\binom{t-2mb+l}{l}\Bigg)+ \\ & \Bigg((s-1)\times\Bigg[(q-1)^{m-1}\binom{t-mb+(m-1)}{m-1}+ \\ & \sum_{l=0}^{m-2}(q-1)^l q^{m-2-l}\binom{t-mb+l}{l}\Bigg]\times \\ & \Bigg[q^{m(b-1)}\Bigg(\binom{t-mb+m}{m}(q-1)^m+ \\ & \sum_{l=0}^{m-1}\binom{t-mb+l}{l}(q-1)^l q^{m-1-l}\Bigg)-1\Bigg]\Bigg)\Bigg\}. \end{aligned} \tag{10}$$

Proof. As in Theorem 3, we omit the proof of this result since it can be derived on lines similar to that of Theorem 2.

□

Remark 6. By taking $s = 1$ in (10) the bound reduces to

$$q^r > q^{2m(b-1)}\left((q-1)^{2m-1}\binom{t-2mb+(2m-1)}{2m-1} + \sum_{l=0}^{2m-2}(q-1)^l q^{2m-2-l}\binom{t-2mb+l}{l}\right)$$

which coincides with the sufficient condition for existence of a code correcting m-repeated bursts(refer Dass and Verma(2008)).

Remark 7. For $m = 2$, the result obtained in Theorem 4 coincides with the result in Theorem 2, for the case of 2-repeated burst of length b or less.
For $m = 1$, the bound in (10) reduces to

$$q^{b-1}\left(q^{b-1}\Big[(q-1)(t-2b+1)+1\Big] + (s-1)\Big[q^{b-1}\Big((t-b+1)(q-1)+1\Big)-1\Big]\right)$$

which is the condition for existence of a code correcting bursts of length b or less within a sub-block.

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Gamma Modules

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Abstract

Let R be a Γ-ring. We introduce the notion of gamma modules over R and study important properties of such modules. In this regards we study submodules and homomorphism of gamma modules and give related basic results of gamma modules.

Keywords: Γ-ring, R_Γ-module, Submodule, Homomorphism.

1 Introduction

The notion of a Γ-ring was introduced by N. Nobusawa in [6]. Recently, W.E. Barnes [2], J. Luh [5], W.E. Coppage studied the structure of Γ-rings and obtained various generalization analogous of corresponding parts in ring theory. In this paper we extend the concepts of module from the category of rings to the category of R_Γ-modules over Γ-rings. Indeed we show that the notion of a gamma module is a generalization of a Γ-ring as well as a module over a ring, in fact we show that many, but not all, of the results in the theory of modules are also valid for R_Γ-modules. In Section 2, some definitions and results of $\Gamma - ring$ which will be used in the sequel are given. In Section 3, the notion of a Γ-module M over a $\Gamma - ring$ R is given and by many example it is shown that the class of Γ-modules is very wide, in fact it is shown that the notion of a Γ-module is a generalization of an ordinary module and a $\Gamma - ring$. In Section 3, we study the submodules of a given Γ-module. In particular, we that $L(M)$, the set of all submodules of a Γ-module M constitute a complete lattice. In Section 3, homomorphisms of Γ-modules are studied and the well known homomorphisms (isomorphisms) theorems of modules extended for Γ-modules. Also, the behavior of Γ-submodules under homomorphisms are investigated.

2 Preliminaries

Recall that for additive abelian groups R and Γ we say that R is a $\mathit{\Gamma} - ring$ if there exists a mapping

$$\cdot : R \times \Gamma \times R \longrightarrow R$$
$$(r, \gamma, r') \longmapsto \quad r\gamma r'$$

such that for every $a, b, c \in R$ and $\alpha, \beta \in \Gamma$, the following hold:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$;

$a(\alpha + \beta)c = a\alpha c + a\beta c$;

$a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

A subset A of a Γ-ring R is said to be a *right ideal* of R if A is an additive subgroup of R and $A\Gamma R \subseteq A$, where $A\Gamma R = \{a\alpha c |\ a \in A, \alpha \in \Gamma, r \in R\}$.

A *left ideal* of R is defined in a similar way. If A is both right and left ideal, we say that A is an *ideal* of R.

If R and S are Γ-rings. A pair (θ, φ) of maps from R into S such that

$i)$ $\theta(x+y) = \theta(x) + \theta(y)$;

$ii)$ φ is an isomorphism on Γ;

$iii)$ $\theta(x\gamma y) = \theta(x)\varphi(\gamma)\theta(y)$.

is called a *homomorphism* from R into S.

3 $\mathbf{R_\Gamma}$ -Modules

In this section we introduce and study the notion of modules over a fixed Γ-ring.

Definition 3.1. Let R be a Γ-ring. A (left) R_Γ-module is an additive abelian group M together with a mapping $. : R \times \Gamma \times M \longrightarrow M$ (the image of (r, γ, m) being denoted by $r\gamma m$), such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $r, r_1, r_2 \in R$ the following hold:

(M_1) $\quad r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$;

(M_2) $\quad (r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$;

(M_3) $\quad r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m$;

(M_4) $\quad r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m$.

A *right* $R_\Gamma - module$ is defined in analogous manner.

Definition 3.2. A (left) R_Γ-module M is *unitary* if there exist elements, say 1 in R and $\gamma_0 \in \Gamma$, such that, $1\gamma_0 m = m$ for every $m \in M$. We denote $1\gamma_0$ by 1_{γ_0}, so $1_{\gamma_0} m = m$ for all $m \in M$.

Remark 3.3. If M is a left R_Γ-module then it is easy to verify that $0\gamma m = r0m = r\gamma 0 = 0_M$. If R and S are Γ-rings then an $(R, S)_\Gamma$-bimodule M is both a left R_Γ-module and right S_Γ-module and simultaneously such that $(r\alpha m)\beta s = r\alpha(m\beta s) \quad \forall m \in M, \forall r \in R, \forall s \in S$ and $\alpha, \beta \in \Gamma$.

In the following by many examples we illustrate the notion of gamma modules and show that the class of gamma module is very wide.

Example 3.4. If R is a Γ-ring, then every abelian group M can be made into an R_Γ-module with trivial module structure by defining

$r\gamma m = 0 \quad \forall r \in R, \forall\gamma \in \Gamma, \forall m \in M.$

Example 3.5. Every Γ-ring R, is an R_Γ-module with $r\gamma(r, s \in R, \gamma \in \Gamma)$ being the Γ-ring structure in R, i.e. the mapping

$$. : R \times \Gamma \times R \longrightarrow R.$$

$$(r, \gamma, s) \longmapsto r.\gamma.s$$

Example 3.6. Let M be a module over a ring A. Define $\; . : A \times R \times M \longrightarrow M$, by $(a, s, m) = (as)m$, being the R-module structure of M. Then M is an A_A-module.

Example 3.7. Let M be an arbitrary abelian group and S be an arbitrary subring of $\mathbb{Z}$, the ring of integers. Then M is a $\mathbb{Z}_S$-module under the mapping

$$. : \mathbb{Z} \times S \times M \longrightarrow M$$

$$(n, n', x) \longmapsto nn'x$$

Example 3.8. If R is a Γ-ring and I is a left ideal of R .Then I is an R_Γ-module under the mapping $\; . : R \times \Gamma \times I \longrightarrow I$ such that $(r, \gamma, a) \longmapsto r\gamma a$.

Example 3.9. Let R be an arbitrary commutative Γ-ring with identity. A polynomial in one indeterminate with coefficients in R is to be an expression $P(X) = a_nX^n + +a_2X^2 + a_1X + a_0$ in which X is a symbol, not a variable and the set $R[x]$ of all polynomials is then an abelian group. Now $R[x]$ becomes to an R_Γ-module, under the mapping

$$. : R \times \Gamma \times R[x] \longrightarrow R[x]$$

$$(r, \gamma, f(x)) \longmapsto r.\gamma.f(x) = \textstyle\sum_{i=1}^{n}(r\gamma a_i)x^i.$$

Example 3.10. If R is a Γ-ring and M is an R_Γ-module. Set $M[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in M\}$. For $f(x) = \sum_{j=0}^{n} b_j x^j$ and $g(x) = \sum_{i=0}^{m} a_i x^i$, define the mapping

$$. : R[x] \times \Gamma \times M[x] \longrightarrow M[x]$$
$$(g(x), \gamma, f(x)) \longmapsto g(x)\gamma f(x) = \textstyle\sum_{k=1}^{m+n} (a_k.\gamma.b_k)x^k.$$

It is easy to verify that $M[x]$ is an $R[x]_\Gamma$-module.

Example 3.11. Let I be an ideal of a Γ-ring R. Then R/I is an R_Γ-module, where the mapping $. : R \times \Gamma \times R/I \longrightarrow R/I$ is defined by $(r, \gamma, r' + I) \longmapsto (r\gamma r') + I$.

Example 3.12. Let M be an R_Γ-module, $m \in M$. Letting $T(m) = \{t \in R \mid t\gamma m = 0 \ \forall \gamma \in \Gamma\}$. Then $T(m)$ is an R_Γ-module.

Proposition 3.12. Let R be a Γ-ring and $(M, +, .)$ be an R_Γ-module. Set $Sub(M) = \{X \mid X \subseteq M\}$, Then sub$(M)$ is an R_Γ-module.

proof. Define $\oplus : (A, B) \longmapsto A \oplus B$ by $A \oplus B = (A \backslash B) \cup (B \backslash A)$ for $A, B \in sub(M)$. Then $(Sub(M), \oplus)$ is an additive group with identity element $\emptyset$ and the inverse of each element A is itself. Consider the mapping:

$$\circ : R \times \Gamma \times Sub(M) \longrightarrow sub(M)$$
$$(r, \gamma, X) \longmapsto r \circ \gamma \circ X = r\gamma X,$$

where $r\gamma X = \{r\gamma x \mid x \in X\}$. Then we have

(i) $r \circ \gamma \circ (X_1 \oplus X_2) = r \cdot \gamma \cdot (X_1 \oplus X_2)$

$= r \cdot \gamma \cdot ((X_1 \backslash X_2) \cup (X_2 \backslash X_1)) = r \cdot \gamma \cdot (\{a \mid a \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}$

$= \{r \cdot \gamma \cdot a \mid a \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}$.

And

$r \circ \gamma \circ X_1 \oplus r \circ \gamma \circ X_2 = r \cdot \gamma \cdot X_1 \oplus r \cdot \gamma \cdot X_2$

$= (r \cdot \gamma \cdot X_1 \backslash r \cdot \gamma \cdot X_2) \cup (r \cdot \gamma \cdot X_2 \backslash r \cdot \gamma \cdot X_1)$

$= \{r \cdot \gamma \cdot x \mid x \in (X_1 \backslash X_2)\} \cup \{r \cdot \gamma \cdot x \mid x \in (X_2 \backslash X_1)\}.$

$= \{r \cdot \gamma \cdot x \mid x \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}.$

(ii) $(r_1 + r_2) \circ \gamma \circ X = (r_1 + r_2) \cdot \gamma \cdot X$

$= \{(r_1 + r_2) \cdot \gamma \cdot x \mid x \in X\} = \{r_1 \cdot \gamma \cdot x + r_2 \cdot \gamma \cdot x \mid x \in X\}$

$= r_1 \cdot \gamma \cdot X + r_2 \cdot \gamma \cdot X = r_1 \circ \gamma \circ X + r_2 \circ \gamma \circ X.$

(iii) $r \circ (\gamma_1 + \gamma_2) \circ X = r \cdot (\gamma_1 + \gamma_2) \cdot X$

$= \{r \cdot (\gamma_1 + \gamma_2) \cdot x \mid x \in X\} = \{r \cdot \gamma_1 \cdot x + r \cdot \gamma_2 \cdot x \mid x \in X\}$

$= r \cdot \gamma_1 \cdot X + r \cdot \gamma_2 \cdot X = r \circ \gamma_1 \circ X + r \circ \gamma_2 \circ X.$

(iv) $r_1 \circ \gamma_1 \circ (r_2 \circ \gamma_2 \circ X)$

$= r_1 \cdot \gamma_1 \cdot (r_2 \circ \gamma_2 \circ X)$

$= \{r_1.\gamma_1.(r_2 \circ \gamma_2 \circ x) | x \in X\}$

$= \{r_1.\gamma_1.(r_2.\gamma_2.x) \mid x \in X\} = \{(r_1.\gamma_1.r_2).\gamma_2.x \mid x \in X\} = (r_1.\gamma_1.r_2).\gamma_2.X.$

Corollary 3.13. If in Proposition 3.12, we define $\oplus$ by $A \oplus B = \{a + b | a \in A, b \in B\}$. Then $(Sub(M), \oplus, \circ)$ is an R_Γ-module.

Proposition 3.14. Let $(R, \circ)$ and $(S, \bullet)$ be Γ-rings. Let $(M, .)$ be a left R_Γ-module and right S_Γ-module. Then $A = \{\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M\}$ is a Γ-ring and A_Γ-module under the mappings

$$\star : A \times \Gamma \times A \longrightarrow A$$
$$(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \gamma, \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix}) \longmapsto \begin{pmatrix} r \circ \gamma \circ r_1 & r.\gamma.m_1 + m.\gamma.s_1 \\ 0 & s \bullet \gamma \bullet s_1 \end{pmatrix}.$$

□

Proof. Straightforward.

Example 3.15. Let $(R, \circ)$ be a Γ-ring . Then $R \oplus \mathbb{Z} = \{(r, s) \mid r \in R, s \in \mathbb{Z}\}$ is an left R_Γ-module, where $\oplus$ addition operation is defined $(r, n) \oplus (r', n') = (r +_R r', n +_\mathbb{Z} n')$ and the product $\cdot : R \times \Gamma \times (R \oplus \mathbb{Z}) \longrightarrow R \oplus \mathbb{Z}$ is defined $r' \cdot \gamma \cdot (r, n) \longrightarrow (r' \circ \gamma \circ r, n)$.

Example 3.16. Let R be the set of all digraphs (A digraph is a pair (V, E) consisting of a finite set V of vertices and a subset E of $V \times V$ of edges) and define addition on R by setting $(V_1, E_1) + (V_2, E_2) = (V_1 \cup V_2, E_1 \cup E_2)$. Obviously R is a commutative group since $(\emptyset, \emptyset)$ is the identity element and the inverse of every element is itself. For $\Gamma \subseteq R$ consider the mapping

$$\cdot : R \times \Gamma \times R \longrightarrow R$$
$$(V_1, E_1) \cdot (V_2, E_2) \cdot (V_3, E_3) = (V_1 \cup V_2 \cup V_3, E_1 \cup E_2 \cup E_3 \cup \{V_1 \times V_2 \times V_3\}),$$

under condition

$$\begin{aligned}(\emptyset, \emptyset) &= (\emptyset, \emptyset) \cdot (V_1, E_1) \cdot (V_2, E_2)(V_1, E_1) \cdot (\emptyset, \emptyset) \cdot (V_2, E_2) \\ &= (V_1, E_1) \cdot (\emptyset, \emptyset) \cdot (V_2, E_2) \\ &= (V_1, E_1) \cdot (V_2, E_2) \cdot (\emptyset, \emptyset).\end{aligned}$$

It is easy to verify that R is an R_Γ-module .

Example 3.17. Suppose that M is an abelian group. Set $R = M_{mn}$ and $\Gamma = M_{nm}$, so by definition of multiplication matrix subset $R^{(t)}_{mn} = \{(x_{ij}) \mid x_{tj} = 0 \ \forall \ j = 1, ...m\}$ is a right R_Γ-module. Also, $C^{(k)}_{mn} = \{x_{ij}) \mid x_{ik} = 0 \ \forall i = 1, ..., n\}$ is a left R_Γ-module.

Example 3.18. Let $(M, \bullet)$ be an R_Γ-module over Γ-ring $(R, .)$ and $S = \{(a, 0)|a \in R\}$. Then $R \times M = \{(a, m)|a \in R, m \in M\}$ is an S_Γ-module, where addition operation is defined by $(a, m) \oplus (b, m_1) = (a +_R b, m +_M m_1)$. Obviously, $(R \times M, \oplus)$ is an additive group. Now consider the mapping

$$\circ : S \times \Gamma \times (R \times M) \longrightarrow R \times M$$
$$((a, 0), \gamma, (b, m)) \longmapsto (a, 0) \circ \gamma \circ (b, m) = (a.\gamma.b, a \bullet \gamma \bullet m).$$

Then it is easy to verify that $R \times M$ is an S_Γ-module.

Example 3.19 Let R be a Γ-ring and $(M, .)$ be an R_Γ-module. Consider the mapping $\alpha : M \longrightarrow R$. Then M is an M_Γ-module, under the mapping

$$\circ : M \times \Gamma \times M \longrightarrow M$$

$$(m, \gamma, n) \longmapsto m \circ \gamma \circ n = (\alpha(m)).\gamma.n.$$

Example 3.20. Let $(R, \cdot)$ and $(S, \circ)$ be Γ- rings. Then

(i) The product $R \times S$ is a Γ- ring, under the mapping

$$((r_1, s_1), \gamma, (r_2, s_2)) \longmapsto (r_1 \cdot \gamma \cdot r_2, s_1 \circ \gamma \circ s_2).$$

(ii) For $A = \{\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \mid r \in R, s \in S\}$ there exists a mapping $R \times S \longrightarrow A$, such that $(r, s) \longrightarrow \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ and A is a Γ- ring. Moreover, A is an $(R \times S)_\Gamma$- module under the mapping

$$(R \times S) \times \Gamma \times A \longrightarrow A \ ((r_1, s_1), \gamma, \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix}) \longrightarrow \begin{pmatrix} r_1 \cdot \gamma \cdot r_2 & 0 \\ 0 & s_1 \circ \gamma \circ s_2 \end{pmatrix}.$$

Example 3.21. Let $(R, \cdot)$ be a Γ-ring. Then $R \times R$ is an R_Γ-module and $(R \times R)_\Gamma$- module. Consider addition operation $(a, b) + (c, d) = (a +_R c, b +_R d)$. Then $(R \times R, +)$ is an additive group. Now define the mapping $R \times \Gamma \times (R \times R) \longmapsto R \times R$ by $(r, \gamma, (a, b)) \longmapsto (r \cdot \gamma \cdot a, r \cdot \gamma \cdot b)$ and $(R \times R) \times \Gamma \times (R \times R) \longrightarrow R \times R$ by $((a, b), \gamma, (c, d)) \longmapsto (a \cdot \gamma \cdot c + b \cdot \gamma \cdot d, a \cdot \gamma \cdot d + b \cdot \gamma \cdot c)$. Then $R \times R$ is an $(R \times R)_\Gamma$- module.

4 Submodules of Gamma Modules

In this section we study submodules of gamma modules and investigate their properties. In the sequel R denotes a Γ-ring and all gamma modules are R_Γ-modules

Definition 4.1. Let $(M, +)$ be an R_Γ-module. A nonempty subset N of $(M, +)$ is said to be a (left) R_Γ-submodule of M if N is a subgroup of M and $R\Gamma N \subseteq N$, where

$R\Gamma N = \{r\gamma n | \gamma \in \Gamma, r \in R, n \in N\}$, that is for all $n, n' \in N$ and for all $\gamma \in \Gamma, r \in R$; $n - n' \in N$ and $r\gamma n \in N$. In this case we write $N \leq M$.

Remark 4.2. (i) Clearly $\{0\}$ and M are two trivial R_Γ-submodules of R_Γ-module M, which is called trivial R_Γ-submodules.

(ii) Consider R as R_Γ-module. Clearly, every ideal of Γ-ring R is submodule, of R as R_Γ-module.

Theorem 4.3. Let M be an R_Γ-module. If N is a subgroup of M, then the factor group M/N is an R_Γ-module under the mapping $. : R \times \Gamma \times M/N \longrightarrow M/N$ is defined $(r, \gamma, m + N) \longmapsto (r.\gamma.m) + N$.

Proof. Straight forward.

Theorem 4.4. Let N be an R_Γ-submodules of M. Then every R_Γ-submodule of M/N is of the form K/N, where K is an R_Γ-submodule of M containing N.

Proof. For all $x, y \in K, x + N, y + N \in K/N$; $(x + N) - (y + N) = (x - y) + N \in K/N$, we have $x - y \in K$, and $\forall r \in R \ \forall \gamma \in \Gamma, \forall x \in K$, we have

$$r\gamma(x + N) = r\gamma x + N \in K/N \Rightarrow r\gamma x \in K.$$

Then K is a R_Γ-submodule M . Conversely,it is easy to verify that $N \subseteq K \leq M$ then K/N is R_Γ-submodule of M/N. This complete the proof. $\Box$

Proposition 4.5. Let M be an R_Γ-module and I be an ideal of R. Let X be a nonempty subset of M. Then

$I\Gamma X = \{\sum_{i=1}^{n} a_i\gamma_i x_i \mid a_i \in I r_{\gamma i} \in \Gamma, x_i \in X, n \in \mathbb{N}\}$ is an R_Γ-submodule of M.

Proof. (i) For elements $x = \sum_{i=1}^{n} a_i\alpha_i x_i$ and $y = \sum_{j=1}^{m} x_{a'_j\beta_j y_j}$ of $I\Gamma X$, we have

$$x - y = \sum_{k=1}^{m+n} b_k\gamma_k z_k \in I\Gamma X.$$

Now we consider the following cases:

Case (1): If $1 \leq k \leq n$, then $b_k = a_k, \gamma_k = \alpha_k, z_k = x_k$.

$Case(2)$: If $n+1 \leq k \leq m+n$, then $b_k = -a'_{k-n}, \gamma_k = \beta_{k-n}, z_k = y_{k-n}$. Also (ii) $\forall r \in R, \forall \gamma \in \Gamma, \forall a = \sum_{i=1}^{n} a_i \gamma_i x_i \in I\Gamma X$, we have $r\gamma x = \sum_{i=1}^{n} r\gamma(a_i\gamma_i x_i) = \sum_{i=1}^{n}(r\gamma a_i)\gamma_i x_i$. Thus $I\Gamma X$ is an R_Γ-submodule of M. $\square$

Corollary 4.6. If M is an R_Γ-module and S is a submodule of M. Then $R\Gamma S$ is an R_Γ-submodule of M.

Let $N \leq M$. Define $N : M = \{r \in R | r\gamma m \quad \forall \gamma \in \Gamma \ \ \forall m \in M\}$.

It is easy to see that $N : M$ is an ideal of Γ ring R.

Theorem 4.7. Let M be an R_Γ-module and I be an ideal of R. If $I \subseteq (0 : M)$, then M is an $(R/I)_\Gamma$-module.

proof. Since R/I is Γ-ring, $define the mapping \bullet : (R/I) \times \Gamma \times M \longrightarrow M$ by

$(r+I, \gamma, m) \longmapsto r\gamma m.$. The mapping $\bullet$ is well-defined since $I \subseteq (0 : M)$. Now it is straight forward to see that M is an $(R/I)_\Gamma$-module. $\square$

Proposition 4.8. Let R be a Γ-ring, I be an ideal of R, and $(M, .)$ be a R_Γ-module. Then $M/(I\Gamma M)$ is an $(R/I)_\Gamma$- module.

Proof. First note that $M/(I\Gamma M)$ is an additive subgroup of M. Consider the mapping

$$\gamma \bullet (m + I\Gamma M) = r.\gamma.m + I\Gamma M$$

$)Now it is straight forward to see that$ M$is an$ (R/I)$_\Gamma$-module. $\square$

Proposition 4.9. Let M be an R_Γ-module and $N \leq M$, $m \in M$. Then

$(N : m) = \{a \in R \mid a\gamma m \in N \ \forall \gamma \in \Gamma\}$ is a left ideal of R.

Proof. Obvious.

Proposition 4.10. If N and K are R_Γ-submodules of a R_Γ-module M and if A, B are nonempty subsets of M then:

(i) $A \subseteq B$ implies that $(N : B) \subseteq (N : A)$;

(ii) $(N \cap K : A) = (N : A) \cap (K : A)$;

(iii) $(N : A) \cap (N : B) \subseteq (N : A + B)$, moreover the equality hold if $0_M \in A \cap B$.

proof. (i) Easy.

(ii) By definition, if $r \in R$, then $r \in (N \cap K : A) \Longleftrightarrow \forall a \in A r \in (N \cap K : a) \Longleftrightarrow \forall \gamma \in \Gamma;\ r\gamma a \in N \cap K \Longleftrightarrow r \in (N : A) \cap K : A)$.

(iii) If $r \in (N : A) \cap (N : B)$. Then $\forall \gamma \in \Gamma, \forall a \in A, \forall b \in B,\ r\gamma(a + b) \in N$ and $r \in (N : A + B)$.

Conversely, $0_M \in A + B \Longrightarrow A \cup B \subseteq A + B \Longrightarrow (N : A + B) \subseteq (N : A \cup B)$ by(i).

Again by using $A, B \subseteq A \cup B$ we have $(N : A \cup B) \subseteq (N : A) \cap (N : B)$. $\square$

Definition 4.11. Let M be an R_Γ-module and $\emptyset \neq X \subseteq M$. Then the generated R_Γ-submodule of M, denoted by $< X >$ is the smallest R_Γ-submodule of M containing X, i.e. $< X >= \cap\{N | N \leq M\}$, X is called the *generator* of $< X >$; and $< X >$ is finitely generated if $|X| < \infty$. If $X = \{x_1, ...x_n\}$ we write $< x_1, ..., x_n >$ instead $< \{x_1, ..., x_n\} >$. In particular, if $X = \{x\}$ then $< x >$ is called the *cyclic submodule* of M, generated by x.

Lemma 4.12. Suppose that M is an R_Γ-module. Then

(i) Let $\{M_i\}_{i \in I}$ be a family of R_Γ-submodules M. Then $\cap M_i$ is the largest R_Γ-submodule of M, such that contained in M_i, for all $i \in I$.

(ii) If X is a subset of M and $|X| < \infty$. Then

$< X >= \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j \gamma_j x_j | k, m \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$.

Proof. (i) It is easy to verify that $\cap_{i \in I} M_i \subseteq M_i$ is a R_Γ-submodule of M. Now suppose that $N \leq M$ and $\forall\ i \in I, N \subseteq M_i$, then $N \subseteq \cap M_i$.

(ii) Suppose that the right hand in (b) is equal to D. First, we show that D is an R_Γ-submodule containing X. $X \subseteq D$ and difference of two elements of D is belong to D and $\forall r \in R\ \forall \gamma \in \Gamma, \forall a \in D$ we have

$$r\gamma a = r\gamma(\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j \gamma_j x_j) = \sum_{i=1}^{m} n_i(r\gamma x_i) + \sum_{j=1}^{k}(r\gamma r_j)\gamma_j x_j \in D.$$

Also, every submodule of M containing X, clearly contains D. Thus D is the smallest

R_Γ-submodules of M, containing X. Therefore $< X >= D$. $\square$

For $N, K \leq M$, set $N + K = \{n + k | n \in N, K \in K\}$. Then it is easy to see that $M + N$ is an R_Γ-submodules of M, containing both N and K. Then the next result immediately follows.

Lemma 4.13. Suppose that M is an R_Γ-module and $N, K \leq M$. Then $N + K$ is the smallest submodule of M containing N and K.

Set $L(M) = \{N | N \leq M\}$. Define the binary operations $\vee$ and $\wedge$ on $L(M)$ by $N \vee K = N + K$ and$N \wedge K = N \cap K$. In fact $(L(M), \vee, \wedge)$ is a lattice. Then the next result immediately follows from lemmas 4.12. 4.13.

Theorem 4.13. $L(M)$ is a complete lattice.

5 Homomorphisms Gamma Modules

In this section we study the homomorphisms of gamma modules. In particular we investigate the behavior of submodules od gamma modules under homomorphisms.

Definition 5.1. Let M and N be arbitrary R_Γ-modules. A mapping $f : M \longrightarrow N$ is a *homomorphism* of R_Γ-modules (or an R_Γ-homomorphisms) if for all $x, y \in M$ and $\forall r \in R, \forall \gamma \in \Gamma$ we have

(i) $f(x + y) = f(x) + f(y)$;

(ii) $f(r\gamma x) = r\gamma f(x)$.

A homomorphism f is *monomorphism* if f is one-to-one and f is *epimorphism* if f is onto. f is called *isomorphism* if f is both monomorphism and epimorphism. We denote the set of all R_Γ-homomorphisms from M into N by $Hom_{R_\Gamma}(M, N)$ or shortly by

$Hom_{R_\Gamma}(M,N)$. In particular if $M=N$ we denote $Hom(M,M)$ by $End(M)$.

Remark 5.2. If $f: M \longrightarrow N$ is an R_Γ-homomorphism, then $Kerf = \{x \in M | f(x) = 0\}$, $Imf = \{y \in N | \exists x \in M; y = f(x)\}$ are R_Γ-submodules of M.

Example 5.3. For all R_Γ-modules A, B, the zero map $0: A \longrightarrow B$ is an R_Γ-homomorphism.

Example 5.4. Let R be a Γ-ring. Fix $r_0 \in \Gamma$ and consider the mapping $\phi: R[x] \longrightarrow R[x]$ by $f \longmapsto f\gamma_0 x$. Then ϕ is an R_Γ-module homomorphism, because $\forall r \in R,\ \forall \gamma \in \Gamma$ and $\forall f, g \in R[x]$:

$\phi(f+g) = (f+g)\gamma_0 x = f\gamma_0 x + g\gamma_0 x = \phi(f) + \phi(g)$ and

$\phi(r\gamma f) = r\gamma f \gamma_0 x = r\gamma\phi(f)$.

Example 5.5. If $N \leq M$, then the natural map $\pi: M \longrightarrow M/N$ with $\pi(x) = x + N$ is an R_Γ-module epimorphism with $ker\pi = N$.

Proposition 5.6. If M is unitary R_Γ-module and $End(M) = \{f: M \longrightarrow M | f \ \ is\ R_\Gamma - homomorphism\}$. Then M is an $End(M)_\Gamma$-module.

Proof. It is well known that $End(M)$ is an abelian group with usual addition of functions. Define the mapping

$$. : End(M) \times \Gamma \times M \longrightarrow M$$
$$(f, \gamma, m) \longmapsto f(1.\gamma.m) = 1\gamma f(m),$$

where 1 is the identity map. Now it is routine to verify that M is an $End(M)_\Gamma$-module.□

Lemma 5.7. Let $f: M \longrightarrow N$ be an R_Γ-homomorphism. If $M_1 \leq M$ and $N_1 \leq$. Then

$(i)\ \ Kerf \leq M\ ,\ Imf \leq N;$

$(ii)\ \ f(M_1) \leq Imf;$

$(iii)\ \ Kerf^{-1}(N_1) \leq M.$

Example 5.8. Consider $L(M)$ the lattice of R_Γ-submodules of M. We know that $(L(M),+)$ is a monoid with the sum of submodules. Then $L(M)$ is R_Γ-semimodule under the mapping

$. : R \times \Gamma \times T \longrightarrow T$, such that $(r, \gamma, N) \longmapsto r.\gamma.N = r\gamma N = \{r\gamma n | n \in N\}$.

Example 5.9. Let $\theta : R \longrightarrow S$ be a homomorphism of Γ-rings and M be an S_Γ-module. Then M is an R_Γ-module under the mapping $\bullet : R \times \Gamma \times M \longrightarrow M$ by $(r, \gamma, m) \longmapsto r \bullet \gamma \bullet m = \theta(r)$. Moreover if M is an S_Γ-module then M is a R_Γ-module for $R \subseteq S$.

Example 5.10. Let $(M, .)$ be an R_Γ-module and $A \subseteq M$. Letting $M^A = \{f | f : A \longrightarrow M \ \ \textit{is a map}\}$. Then M^A is an R_Γ-module under the mapping $\circ : R \times \Gamma \times M^A \longrightarrow M^A$ defined by $(r, \gamma, f) \longmapsto r \circ \gamma \circ f = r\gamma f(a)$, since M^A is an additive group with usual addition of maps.

Example 5.11. Let$(M, .)$ and $(N, \bullet)$ be R_Γ-modules. Then $Hom(M, N)$ is a R_Γ-module, under the mapping

$$\circ : R \times \Gamma \times Hom(M, N) \longrightarrow Hom(M, N)$$

$$(r, \gamma, \alpha) \longmapsto r \circ \gamma \circ \alpha,$$

$$\text{where } (r \bullet \gamma \bullet \alpha)(m) = r\gamma\alpha)(m).$$

Example 5.12. Let M be a left R_Γ-module and right S_Γ-module. If N be an R_Γ-module, then

(i) $Hom(M, N)$ is a left S_Γ-module. Indeed

$$\circ : S \times \Gamma \times Hom(M, N) \longrightarrow Hom(M, N)$$

$$(s, \gamma, \alpha) \longrightarrow s \circ \gamma \circ \alpha : M \longrightarrow N$$

$$m \longmapsto \alpha(m\gamma s)$$

(ii) $Hom(N, M)$ is right S_Γ-module under the mapping

$$\circ : Hom(N, M) \times \Gamma \times S \longrightarrow Hom(N, M)$$
$$(\alpha, \gamma, s) \longmapsto \alpha \circ \gamma \circ s : N \longrightarrow M$$
$$n \longmapsto \alpha(n).\gamma.s$$

Example 5.13. Let M be a left R_Γ-module and right S_Γ-module and $\alpha \in End(M)$ then α induces a right $S[t]_\Gamma$-module structure on M with the mapping

$$\circ : M \times \Gamma \times S[t] \longrightarrow M$$
$$(m, \gamma, \sum_{i=0}^{n} s_i t^i) \longmapsto m \circ \gamma \circ (\sum_{i=0}^{n} s_i t^i) = \sum_{i=0}^{n} (m\gamma s_i)\alpha^i$$

Proposition 5.14. Let M be a R_Γ-module and $S \subseteq M$. Then $S\Gamma M = \{\sum s_i \gamma_i a_i \mid s_i \in S, a_i \in M, \gamma_i \in \Gamma\}$ is an R_Γ-submodule of M.

Proof. Consider the mapping

$$\circ : R \times \Gamma \times (S\Gamma M) \longrightarrow S\Gamma M$$
$$(r, \gamma, \sum_{i=1}^{n} s_i \gamma_i a_i) \longmapsto \sum_{i=1}^{n} s_i \gamma_i (r\gamma a_i).$$

Now it is easy to check that $S\Gamma M$ is a R_Γ-submodule of M.

Example 5.16. Let $(R, .)$ be a Γ-ring. Let $\mathbb{Z}_2$, the cyclic group of order 2. For a nonempty subset A, set $Hom(R, \mathbb{B}^A) = \{f : R \longrightarrow \mathbb{B}^A\}$. Clearly $(Hom(R, \mathbb{B}^A), +)$ is an abelian group. Consider the mapping
$\circ : R \times \Gamma \times Hom(R, \mathbb{B}^A) \longrightarrow Hom(R, \mathbb{B}^A)$ that is defined

$$(r, \gamma, f) \longmapsto r \circ \gamma \circ f,$$

where $(r \circ \gamma \circ f)(s) : A \longrightarrow \mathbb{B}$ is defied by $(r \circ \gamma \circ f(s))(a) = f(s\gamma r)(a)$.

Now it is easy to check that $Hom(R, \mathbb{B}^A)$ is an Γ-ring.

Example 5.17. Let R and S be Γ-rings and $\varphi : R \longrightarrow S$ be a Γ-rings homomorphism. Then every S_Γ-module M can be made into an R_Γ-module by defining

$r\gamma x \ \ (r \in R, \gamma \in \Gamma, x \in M)$ to be $\varphi(r)\gamma x$. We says that the R_Γ-module structure M is given by pullback along φ.

Example 5.18. Let $\varphi : R \longrightarrow S$ be a homomorphism of Γ-rings then $(S, .)$ is an R_Γ-module. Indeed

$$\circ : R \times \Gamma \times S \longrightarrow S$$

$$(r, \gamma, s) \longmapsto r \circ \gamma \circ s = \varphi(r).\gamma.s$$

Example 5.19. Let $(M, +)$ be an R_Γ-module. Define the operation $\circ$ on M by $a \oplus b = b.a$. Then $(M, \oplus)$ is an R_Γ-module.

Proposition 5.20. Let R be a Γ-ring. If $f : M \longrightarrow N$ is an R_Γ-homomorphism and $C \leq kerf$, then there exists an unique R_Γ-homomorphism $\bar{f} : M/C \longrightarrow N$, such that for every $x \in M$; $Ker\bar{f} = Kerf/C$ *and* $Im\bar{f} = Imf$ *and* $\bar{f}(x + C) = f(x)$, also $\bar{f}$ is an R_Γ-isomorphism if and only if f is an R_Γ-epimorphism and $C = Kerf$. In particular $M/Kerf \cong Imf$.

Proof. Let $b \in x + C$ then $b = x + c$ for some $c \in C$, also $f(b) = f(x + c)$. We know f is R_Γ-homomorphism therefore $f(b) = f(x + c) = f(x) + f(c) = f(x) + 0 = f(x)$ (since $C \leq kerf$) then $\bar{f} : M/C \longrightarrow N$ is well defined function. Also $\forall\ x + C, y + C \in M/C$ and $\forall\ r \in R, \gamma \in \Gamma$ we have

(i) $\bar{f}((x+C)+(y+C)) = \bar{f}((x+y)+C) = f(x+y) = f(x)+f(y) = \bar{f}(x+C)+\bar{f}(y+C)$.

(ii) $\bar{f}(r\gamma(x + C)) = \bar{f}(r\gamma x + C) = f(r\gamma x) = r\gamma f(x) = r\gamma\bar{f}(x + C)$.

then $\bar{f}$ is a homomorphism of R_Γ-modules, also it is clear $Im\bar{f} = Imf$ and $\forall(x + C) \in ker\bar{f};\ x + C \in ker\bar{f} \Leftrightarrow \bar{f}(x + C) = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow x \in kerf$ then $ker\bar{f} = kerf/C$.

Then definition $\bar{f}$ depends only f, then $\bar{f}$ is unique. $\bar{f}$ is epimorphism if and only if f is epimorphism. $\bar{f}$ is monomorphism if and only if $ker\bar{f}$ be trivial R_Γ-submodule of M/C. In actually if and only if $Kerf = C$ then $M/Kerf \cong Imf$.□

Corollary 5.21. If R is a Γ-ring and M_1 is an R_Γ-submodule of M and N_1 is R_Γ-submodule of N, $f : M \longrightarrow N$ is a R_Γ-homomorphism such that $f(M_1) \subseteq N_1$ then f make a R_Γ-homomorphism $\bar{f} : M/M_1 \longrightarrow N/N_1$ with operation $m + M_1 \longmapsto f(m) + N_1$. $\bar{f}$ is R_Γ-isomorphism if and only if $Imf + N_1 = N,\ f^{-1}(N_1) \subseteq M_1$. In particular, if f is epimorphism such that $f(M_1) = N_1,\ kerf \subseteq M_1$ then f is a R_Γ-isomorphism.

proof. We consider the mapping $M \longrightarrow^f N \longrightarrow^\pi N/N_1$. In this case; $M_1 \subseteq f^{-1}(N_1) = ker\pi f\ (\forall m_1 \in M_1,\ f(m_1) \in N_1 \Rightarrow \pi f(m_1) = 0 \Rightarrow m_1 \in ker\pi f)$. Now we use Proposition 5.20 for map $\pi f : M \longrightarrow N/N_1$ with function $m \longmapsto f(m) + N_1$ and submodule M_1 of M.

Therefore, map $\bar{f} : M/M_1 \longrightarrow N/N_1$ that is defined $m + M \longmapsto f(m) + N_1$ is a R_Γ-homomorphism. It is isomorphism if and only if πf is epimorphism, $M_1 = ker\pi f$. But condition will satisfy if and only if $Imf + N_1 = N,\ f^{-1}(N_1) \subseteq M_1$. If f is epimorphism then $N = Imf = Imf + N_1$ and if $f(M_1) = N_1$ and $kerf \subseteq M_1$ then $f^{-1}(N_1) \subseteq M_1$ so $\bar{f}$ is isomorphism.$\square$

Proposition 5.22. Let B, C be R_Γ-submodules of M.

(i) There exists a R_Γ-isomorphism $B/(B \cap C) \cong (B + C)/C$.

(ii) If $C \subseteq B$, then B/C is an R_Γ-submodule of M/C and there is an R_Γ-isomorphism $(M/C)/(B/C) \cong M/B$.

Proof. (i) Combination $B \longrightarrow^j B + C \longrightarrow^\pi (B + C)/C$ is an R_Γ-homomorphism with kernel$= B \cap C$, because $ker\pi j = \{b \in B | \pi j(b) = 0_{(B+C)/C}\} = \{b \in B | \pi(b) = C\} = \{b \in B | b + C = C\} = \{b \in B | b \in C\} = B \cap C$ therefore, in order to Proposition 5.20., $B/(B \cap C) \cong Im(\pi j)(\star)$, every element of $(B + C)/C$ is to form $(b + c) + C$, thus $(b + c) + C = b + C = \pi j(b)$ then πj is epimorphism and $Im\pi j = (B + C)/C$ in attention $(\star)$, $B/(B \cap C) \cong (B + C)/C$.

(ii) We consider the identity map $i : M \longrightarrow M$, we have $i(C) \subseteq B$, then in order to apply Proposition 5.21. we have R_Γ-epimorphism $\bar{i} : M/C \longrightarrow M/B$ with $\bar{i}(m + C) = m + B$ by using (i). But we know $B = \bar{i}(m + C)$ if and only if $m \in B$ thus $ker\ \bar{i} = \{m + C \in M/C | m \in B\} = B/C$ then $ker\bar{i} = B/C \leq M/C$ and we have

$$M/B = Im\bar{i} \cong (M/C)/(B/C).\square$$

Let M be a R_Γ-module and $\{N_i | i \in \Omega\}$ be a family of R_Γ-submodule of M. Then $\cap_{i\in\Omega} N_i$ is a R_Γ-submodule of M which, indeed, is the largest R_Γ-submodule M contained in each of the N_i. In particular, if A is a subset of a left R_Γ-module$M$ then intersection of all submodules of M containing A is a R_Γ-submodule of M, called the submodule *generated* by A. If A generates all of the R_Γ-module, then A is a set of*generators* for M. A left R_Γ-module having a finite set of generators is *finitely generated*. An element m of the R_Γ-submodule generated by a subset A of a R_Γ-module M is a *linear combination* of the elements of A.

If M is a left R_Γ-module then the set $\sum_{i\in\Omega} N_i$ of all finite sums of elements of N_i is an R_Γ-submodule of M generated by $\cup_{i\in\Omega} N_i$. R_Γ-submodule generated by $X = \cup_{i\in\Omega} N_i$ is $D = \{\sum_{i=1}^{s} r_i\gamma_i a_i + \sum_{j=1}^{t} n_j b_j | a_i, b_j \in X, r_i \in R, n_j \in \mathbb{Z}, \gamma_i \in \Gamma\}$ if M is a unitary R_Γ-module then $D = R\Gamma X = \{\sum_{i=1}^{s} r_i\gamma_i a_i | r_i \in R, \gamma_i \in \Gamma, a_i \in X\}$.

Example 5.23. Let M, N be R_Γ-modules and $f, g : M \longrightarrow N$ be R_Γ-module homomorphisms. Then $K = \{m \in M \mid f(m) = g(m)\}$ is R_Γ-submodule of M.

Example 5.24. Let M be a R_Γ-module and let N, N' be R_Γ- submodules of M. Set $A = \{m \in M \mid m + n \in N'\ for\ some\ n \in N\}$ is an R_Γ-module of M containing N'.

Proposition 5.25. Let $(M, \cdot)$ be an R_Γ- module and M generated by A. Then there exists an R_Γ-homomorphism $R^{(A)} \longrightarrow M$, such that $f \longmapsto \sum_{a\in A, a\in supp(f)} f(a) \cdot \gamma \cdot a$.

Remark 5.26. Let R be a Γ- ring and let $\{(M_i, o_i) | i \in \Omega\}$ be a family of left R_Γ-modules. Then $\times_{i\in\Omega} M_i$, the Cartesian product of M_i's also has the structure of a left R_Γ-module under componentwise addition and mapping

$$\cdot : R \times \Gamma \times (\times M_i) \longrightarrow \times M_i$$
$$(r, \gamma, \{m_i\}) \longrightarrow r \cdot \gamma \cdot \{m_i\} = \{r o_i \gamma o_i m_i\}_\Omega.$$

We denote this left R_Γ-module by $\prod_{i\in\Omega} M_i$. Similarly, $\sum_{i\in\Omega} M_i = \{\{m_i\} \in \prod M_i | m_i = 0 \quad \text{for all but finitely many indices } \; i\}$ is a R_Γ-submodule of $\prod_{i\in\Omega} M_i$. For each h in Ω we have canonical R_Γ- homomorphisms $\pi_h : \prod M_i \longrightarrow M_h$ and $\lambda_h : M_h \longrightarrow \sum M_i$ is defined respectively by $\pi_h :< m_i >\longmapsto m_h$ and $\lambda(m_h) =< u_i >$, where

$$u_i = \begin{cases} 0 & i \neq h \\ m_h & i = h \end{cases}$$

The R_Γ-module $\prod M_i$ is called the (external)*direct product* of the R_Γ- modules M_i and the R_Γ- module $\sum M_i$ is called the (external) *direct sum* of M_i. It is easy to verify that if M is a left R_Γ-module and if $\{M_i | i \in \Omega\}$ is a family of left R_Γ-modules such that, for each $i \in \Omega$, we are given an R_Γ-homomorphism $\alpha_i : M \longrightarrow M_i$ then there exists a unique R_Γ- homomorphism $\alpha : M \longrightarrow \prod_{i\in\Omega} M_i$ such that $\alpha_i = \alpha\pi_i$ for each $i \in \Omega$. Similarly, if we are given an R_Γ-homomorphism $\beta_i : M_i \longrightarrow M$ for each $i \in \Omega$ then there exists an unique R_Γ- homomorphism $\beta : \sum_{i\in\Omega} M_i \longrightarrow M$ such that $\beta_i = \lambda_i\beta$ for each $i \in \Omega$.

Remark 5.27. Let M be a left R_Γ-module. Then M is a right R_Γ^{op}-module under the mapping

$$* : M \times \Gamma \times R^{op} \longrightarrow M$$
$$(m, \gamma, r) \longmapsto m * \gamma * r = r\gamma m.$$

Definition 5.28. A nonempty subset N of a left R_Γ-module M is *subtractive* if and only if $m + m' \in N$ and $m \in N$ imply that $m' \in N$ for all $m, m' \in M$. Similarly, N is *strong subtractive* if and only if $m + m' \in N$ implies that $m, m' \in N$ for all $m, m' \in M$.

Remark 5.29. (i) Clearly, every submodule of a left R_Γ-module is subtractive. Indeed, if N is a R_Γ-submodule of a R_Γ-module M and $m \in M, n \in N$ are elements satisfying $m + n \in N$ then $m = (m + n) + (-n) \in N$.

(ii) If $N, N' \subseteq N$ are R_Γ-submodules of an R_Γ-module M, such that N' is a subtractive R_Γ-submodule of N and N is a subtractive R_Γ-submodule of M then N' is a subtractive R_Γ-module of M.

Note. If $\{M_i | i \in \Omega\}$ is a family of (resp. strong) subtractive R_Γ-submodule of a left R_Γ-module M then $\cap_{i \in \Omega} M_i$ is again (resp. strong) subtractive. Thus every R_Γ -submodule of a left R_Γ-module M is contained in a smallest (resp. strong) subtractive R_Γ-submodule of M, called its (resp. strong) *subtractive closure* in M.

Proposition 5.30 Let R be a Γ-ring and let M be a left R_Γ -module. If N, N' and $N'' \leq M$ are submodules of M satisfying the conditions that N is subtractive and $N' \subseteq N$, then $N \cap (N' + N'') = N' + (N \cap N'')$.

Proof. Let $x \in N \cap (N' + N'')$. Then we can write $x = y + z$, where $y \in N'$ and $z \in N''$. by $N' \subseteq N$, we have $y \in N$ and so, $z \in N$, since N is subtractive. Thus $x \in N' + (N \cap N'')$, proving that $N \cap (N' + N'') \subseteq N' + (N \cap N'')$. The reverse containment is immediate.□

Proposition 5.31. If N is a subtractive R_Γ-submodule of a left R_Γ-module M and if A is a nonempty subset of M then $(N : A)$ is a subtractive left ideal of R.

Proof. Since the intersection of an arbitrary family of subtractive left ideals of R is again subtractive, it suffices to show that $(N : m)$ is subtractive for each element m. Let $a \in R$ and $b \in (N : M)$ (for $\gamma \in \Gamma$) satisfy the condition that $a + b \in (N : M)$. Then $a\gamma m + b\gamma m \in N$ and $b\gamma m \in N$ so $a\gamma m \in N$, since N is subtractive. Thus $a \in (N : M)$.□.

proposition 5.32. If I is an ideal of a Γ-ring R and M is a left R_Γ-module. Then

$N = \{m \in M \mid I\Gamma m = \{0\}\}$ is a subtractive R_Γ-submodule of M.

Proof. Clearly, N is an R_Γ-submodule of M. If $m, m' \in M$ satisfy the condition that m and $m + m'$ belong to N then for each $r \in I$ and for each $\gamma \in \Gamma$ we have $0 = r\gamma(m + m') = r\gamma m + r\gamma m'm' = r\gamma m'$, and hence $m' \in N$. Thus N is subtractive. $\square$

proposition 5.33. Let $(R, +, \cdot)$ be a Γ-ring and let M be an R_Γ-module and there exists bijection function $\delta : M \longrightarrow R$. Then M is a Γ-ring and M_Γ-module.

Proof. Define $\circ : M \times \Gamma \times M \longrightarrow M$ by $(x, \gamma, y) \longmapsto x \circ \gamma \circ y = \delta^{-1}(\delta(x) \cdot \gamma\delta(y))$. It is easy to verify that R is a Γ- ring. If M is a set together with a bijection function $\delta : X \longrightarrow R$ then the Γ-ring structure on R induces a Γ-ring structure $(M, \oplus, \odot)$ on X with the operations defined by $x \oplus y = \delta^{-1}(\delta(x) + \delta(y))$ and

$$x \odot \gamma \odot y = \delta^{-1}(\delta(x) \cdot \gamma \cdot \delta(y)).\square$$

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Homomorphism and quotient of fuzzy k-hyperideals

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Abstract

In [15], we introduced the notion of weak (resp. strong) fuzzy k-hyperideal. In this note we investigate the behavior of them under homomorphisms of semihyperrings. Also we define the quotient of fuzzy weak (resp. strong) k-hyperideals by a regular relation of semihyperring and obtain some results.

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1 Introduction

Following the introduction of fuzzy set by L. A. Zadeh in 1965 ([26]), the fuzzy set theory developed by Zadeh himself and can be found in mathematics and many applied areas. The concept of a fuzzy group was introduced by A. Rosenfeld in [24]. The notion of fuzzy ideals in a ring was introduced and studied by W. J. Liu [20]. T.K. Dutta and B. K. Biswas studied fuzzy ideals, fuzzy prime ideals of semirings in [14, 16] and they defined fuzzy ideals of semirings and fuzzy prime ideals of semirings and characterized fuzzy prime ideals of non-negative prime integers and determined all it's prime ideals. Recently, Y. B. Jun, J. Neggeres and H. S. Kim ([16]) extended the concept of a L-fuzzy (characteristic) ideal left(resp. right) ideal of a ring to a semiring. S. I. Baik and H. S. Kim introduced the notion of fuzzy k-ideals in semirings [6].

Also a hypergroup was introduced by F. Marty ([23]), today the literature on hypergroups and related structures counts 400 odd items [8, 9, 25]. Among the several contexts which they aries is hyperrings. First M. Krasner studied hyperrings, which is a triple $(R, +, .)$, where $(R, +)$ is a canonical hypergroup and $(R, .)$ is a semigroup, such that for all $a, b, c \in R$, $a(b + c) = ab + ac, (b + c)a = ba + ca$ ([18]). Zahedi and others introduced and studied the notion of fuzzy hyperalgebraic structures [3, 4, 5, 11, 12, 19, 27]. In [15] we introduced the notion of fuzzy weak (strong) k-hyperideal and then we obtained some related basic results. In this note we investigate the behavior of them under homomorphisms of semihyperrings. Also we define the quotient of fuzzy weak (strong) k-hyperideals by a regular relation of semihyperring and obtain some results.

2 Preliminaries

In this section we gather all definitions and simple properties we require of semihyperrings and fuzzy subsets and set the notions.

A map $\circ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or *join operation.*

A *hypergroupoid* is a set H with together a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in H$ is called a *semi-hypergroup* . A *hypergroup* is a semihypergroup such that $\forall x \in H$ we have $x \circ H = H = H \circ x$, which is called *reproduction* axiom.

Let H be a hypergroup and K a nonempty subset of H. Then K is a *subhypergroup* of H if itself is a hypergroup under hyperoperation restricted to K. Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H.

A set H together a hyperoperation $\circ$ is called a *polygroup* if the following conditions are satisfied:

(1) $(x \circ y) \circ z = x \circ (y \circ z) \;\; \forall x, y, z \in H$;

(2) $\exists e \in H$ as unique element such that $e \circ x = x = x \circ e \;\; \forall x \in H$;

(3) $\forall x \in H$ there exists an unique element, say $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$ (we denote x' by x^{-1}).

(4) $\forall x, y, z \in H, \;\; z \in x \circ y \Longrightarrow x \in z \circ y^{-1} \Longrightarrow y \in x^{-1} \circ z.$

A non-empty subset K of a polygroup $(H, \circ)$ is called a *subpolygroup* if $(K, \circ)$ is itself a polygroup. In this case we write $K <_P H$.

A commutative polygroup is called *canonical hypergroup.*

Definition 2.1. A hyperalgebra $(R, +, .)$ is called a semihyperring if and only if

(i) $(R,+)$ is a semihypergroup and $(R,.)$ is a semigroup;

(ii) $a.(a+b) = a.b + a.c$ and $(b+c).a = b.c + c.a$ $\forall a,b,c \in R$.

A semihyperring is called with zero, if there exists an element, say $0 \in R$ such that $0.x = 0 = x.0$ and $0 + x = x = x + 0$ $\forall x \in R$.

Also a semihyperring $(R,+,.)$ is called a *hyperring* provided $(R,+)$ is a canonical hypergroup.

A hyperring $(R,+,.)$ is called

(i) *commutative* if and only if $a.b = b.a$ $\forall a,b \in R$;

(ii)*with identity*, if there exists an element, say $1 \in R$ such that $1.x = x.1 = x$ $\forall x \in R$.

Let $(R,+,.)$ be a hyperring, a nonempty subset S of R is called a subhyperring of R if $(S,+,.)$ is itself a hyperring.

Definition 2.2. A subhyperring I of a hyperring R is a *(resp. left) right hyperideal* of R provided that (resp. $x.r \in I$) $r.x \in I$ $\forall r \in R$, $\forall x \in I$. I is called a *hyperideal* if I is both left and right hyperideal.

We use $I = [0,1]$, the real unit interval as a chain with the usual ordering, in which $\bigwedge$ stands for infimum (inf) (or intersection) and $\bigvee$ stands for supremum (sup) (or union), for the degree of membership.

A *fuzzy subset* of a given set X is a mapping $\mu : X \longrightarrow I$. We denote the set of all fuzzy subsets of X by $FS(X)$. For $\mu \in FS(X)$, the level subset of μ is defined by

$$\mu_t = \{x \in X|\ \mu(x) \geq t\} \quad \forall t \in I.$$

For a fuzzy subset μ of X we denote by $Im(\mu)$ the image of μ. Let $\{\mu_i \mid$

$i \in I\}$ be a family of fuzzy subsets, *intersection* of μ_i's is defined by

$$(\bigcap_{i\in I}\mu_i)(x) = \bigwedge_{i\in I}\mu_i(x).$$

Definition 2.3. Let $(G, .)$ be a group and $\mu \in FS(G)$. Then μ is said to be a *fuzzy subgroup* of G if $\forall x, y \in G$ we have :

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$;

(ii) $\mu(x^{-1}) \geq \mu(x)$.

Definition 2.4. If $f : X \longrightarrow Y$ be a function and $\mu \in FS(X)$, then we say μ is $f-$invariant if and only if

$$f(a) = f(b) \Longrightarrow \mu(a) = \mu(b).$$

In the sequel by R we mean a semihyperring.

Definition 2.5.[1] A nonempty subset I of R is called

(i) a *left (resp. right) hyperideal* of R if and only if

(1) $(I, +)$ is a semihypergroup of $(R, +)$;

(2) $rx \in I$ (resp. $xr \in I$), for all $r \in R$ and for all $x \in I$.

(ii) a *hyperideal* of R if it is both a left and a right hyperideal of R. By $I <_h R$, we mean hyperideal of R.

(iii) a left hyperideal I of R is called *weak left k-hyperideal* of R if for $a \in I$ and $x \in R$ we have

$$a + x \subseteq I \;\text{ or }\; x + a \subseteq I \quad\Longrightarrow\quad x \in I.$$

A left hyperideal I of R is called *strong left k-hyperideal* of R if for $a \in I$ and $x \in R$ we have

$$a + x \approx I \;\text{ or }\; x + a \approx I \quad\Longrightarrow\quad x \in I,$$

where by $A \approx B$, we mean $A \cap B \neq \emptyset$, for all nonempty subsets A and B of R.

A right(resp. strong) weak k-hyperideal is defined dually. A two sided (resp. strong) weak k-hyperileal or simply a (resp. strong) weak k-hyperideal is both left and right (resp. strong) weak k-hyperideal. We denote $I <_{w.k.h} R$ (resp. $I <_{s.k.h} R$) for weak (resp. strong) k-hyperideal of R.

Clearly, every (strong) weak $k-$ hyperideal is a hyperideal, but the converse is not true.

Example. Consider $\mathbb{Z}$, the set of integer numbers. Define new hyperoperations $\oplus$ and $\circ$ on $\mathbb{Z}$ as follow

$$m \oplus n = \{m, n\} \qquad \text{and} \qquad m \circ n = mn \quad \forall m, n \in \mathbb{Z}.$$

Clearly $(\mathbb{Z}, \oplus, \circ)$ is a semihyperring. Now it is easy to verify that $I =< 2 >= \{2k \mid k \in \mathbb{Z}\}$, is a hyperideal of $\mathbb{Z}$, but it isn't strong $k-$hyperideal, since $3 \oplus 2 = \{3, 2\} \approx I$ and $2 \in I$ but $3 \notin I$.

Definition 2.6 .[7] Let R and S be semihyperrings. A mapping $f : R \longrightarrow S$ is said to be

(i) *homomorphism* if and only if

$f(x + y) \subseteq f(x) + f(y)$ and

$f(x.y) = f(x).f(y) \ \ \forall x, y \in R.$

(ii) *good homomorphism* if and only if

$f(x + y) = f(x) + f(y)$ and

$f(x.y) = f(x).f(y) \ \ \forall x, y \in R.$

Definition 2.7 .[15] A fuzzy subset μ of a semihyperring R is called a *fuzzy* *left hyperideal* of R if and only if

(i) $\bigwedge_{z\in x+y} \mu(z) \geq \mu(x) \bigwedge \mu(y) \quad \forall x, y \in R;$

(ii) $\mu(xy) \geq \mu(y) \quad \forall x, y \in R.$

A fuzzy right hyperideal is defined dually. A fuzzy left and right hyperideal is called a fuzzy hyperideal. We denote $\mu <_{f.h} R$ for fuzzy hyperideal of R.

Definition 2.8.[15] A fuzzy hyperideal μ of R is called

(i) a *weak fuzzy k-hyperideal* of R if and only if

$$\mu(x) \geq [(\bigwedge_{u\in x+y} \mu(u)) \bigvee (\bigwedge_{v\in y+x} \mu(v))] \bigwedge \mu(y) \qquad \forall x, y \in R.$$

(ii) a *strong fuzzy k-hyperideal* of R if and only if

$$\mu(x) \geq (\mu(z) \vee \mu(z^{'})) \wedge \mu(y) \qquad \forall z \in x + y, \forall z^{'} \in y + x.$$

Note that if $(R, +)$ is a commutative semihyperring, then the above conditions reduce to the following conditions:

$$\mu(x) \geq (\bigwedge_{u\in x+y} \mu(u)) \bigwedge \mu(y) \qquad \forall x, y \in R.$$

and

$$\mu(x) \geq \mu(z) \wedge \mu(y) \qquad \forall z \in x + y.$$

We denote by $\mu <_{w.f.k.h} R$ (resp. $\mu <_{s.f.k.h} R$), for a weak fuzzy $k-$hyperideal (resp. strong fuzzy $k-$hyperideal) of R.

Proposition 2.9.[15] Let R be a semihyperring and $\mu \in FS(R)$. Then

(i) μ is a fuzzy hyperideal of R if and only if every nonempty level subset, μ_t is a hyperideal of R.

(ii) μ is a weak fuzzy k-hyperideal of R if and only if every nonempty level subset, μ_t is a weak k-hyperideal of R.

(iii) μ is a strong fuzzy k-hyperideal of R if and only if every nonempty level subset, μ_t is a strong k-hyperideal of R.

Lemma 2.10. Let R be a semihyperring with zero and μ be a fuzzy hyperideal of R. Then $\mu(x) \leq \mu(0) \quad$ for all $x \in R$.

3 Homomorphisms of Fuzzy k-Hyperideals

In this section we investigate the behavior of fuzzy weak (strong) k-hyperideals under homomorphisms of semihyperrings.

proposition 3.1. Let $f : R \longrightarrow R^{'}$ be a homomorphism of semihyperrings. If $\nu <_{s.f.k.h} R'$, then $f^{-1}(\nu) <_{s.f.k.h} R$.

proof. We know that $f^{-1}(\nu)(x) = \nu(f(x))$. Let $x,\ y \in R$ and $z \in x + y$, then we have $f(z) \in f(x + y) \subseteq f(x) + f(y)$, and since $\nu <_{f.h} R'$, it concluded that $\nu(f(z)) \geq \nu(f(x)) \wedge \nu(f(y))$.

Also

$$\nu(f(xy)) = \nu(f(x)f(y)) \geq \nu(f(x)) \vee \nu(f(y)).$$

Therefore $f^{-1}(\nu) <_{f.h} R$.

Now let $z \in x + y$ and $z' \in y + x$, thus $f(z) \in f(x) + f(y)$ and $f(z') \in f(y) + f(x)$, then $\nu <_{s.f.k.h} R'$ implies that

$$\nu(f(x)) \geq [\nu(f(z)) \vee \nu(f(z'))] \wedge \nu(f(y))$$

as required.

proposition 3.2. Let $f : R \longrightarrow R'$ be a good homomorphism of semihyperrings. If $\nu <_{w.f.k.h} R'$, then $f^{-1}(\nu) <_{w.f.k.h} R$.

proof. We know that $f^{-1}(\nu)(x) = \nu(f(x))$. First we prove that $f^{-1}(\nu)$ is a fuzzy hyperideal of R. Let $x,\ y \in R$ and $z \in x + y$, we should prove that

$$\nu(f(z)) \geq \nu(f(x)) \wedge \nu(f(y)) \qquad (1)$$

(1) is valid because ν is a fuzzy hyperideal and f is a good homomorphism, then for $z \in x + y$ we have $f(z) \in f(x + y) = f(x) + f(y)$.

Also similar previous proposition

$$\nu(f(xy)) \geq \nu(f(x)) \vee \nu(f(y)).$$

Therefore $f^{-1}(\nu) <_{f.h} R$.

Now we prove that $f^{-1}(\nu) <_{w.f.k.h} R$, that is

$$f^{-1}(\nu)(x) \geq \{(\bigwedge_{t \in x+y} f^{-1}(\nu)(t)) \bigvee (\bigwedge_{t' \in y+x} f^{-1}(\nu)(t'))\} \bigwedge f^{-1}(\nu)(y) \qquad (2).$$

Note that since f is a good homomorphism, then $t \in x + y$ if and only if $f(t) \in f(x) + f(y)$, and also $\nu <_{w.f.k.h} R'$, we have

$$\nu(f(x)) \geq \{(\bigwedge_{f(t) \in f(x)+f(y)} \nu(f(t))) \bigvee (\bigwedge_{f(t') \in f(y)+f(x)} \nu(f(t')))\} \bigwedge \nu(f(y)).$$

The last relation implies (2), and this complete the proof.

proposition 3.3. Let $f : R \longrightarrow R'$ be a good epimorphism of semihyperrings. If $\mu <_{w.f.k.h} R$ (resp. $\mu <_{s.f.k.h} R$) and μ be $f-$invariant, then $f(\mu) <_{w.f.k.h} R'$ (resp. $f(\mu) <_{s.f.k.h} R$).

proof. First we show that $f(\mu) <_{f.h} R'$.

Let $a,\ b \in R^{'}$ and $c \in a+b$, we should prove that

$$f(\mu(c)) \geq f(\mu(a)) \wedge f(\mu(b)).$$

We have

$$\begin{aligned} f(\mu(c)) &= \bigvee_{z \in f^{-1}(c)} \mu(z), \\ f(\mu(a)) &= \bigvee_{x \in f^{-1}(a)} \mu(x), \\ f(\mu(b)) &= \bigvee_{y \in f^{-1}(b)} \mu(y). \end{aligned}$$

Since μ is $f-$invariant, then

$$\exists z_0 \in f^{-1}(c),\ f(\mu(c)) = \mu(z_0),$$

$$\exists x_0 \in f^{-1}(a),\ f(\mu(a)) = \mu(x_0),$$

$$\exists y_0 \in f^{-1}(b),\ f(\mu(b)) = \mu(y_0),$$

therefore

$$\begin{aligned} f(z_0) = c,\ f(x_0) = a,\ f(y_0) = b &\Longrightarrow f(z_0) \in f(x_0) + f(y_0) \\ &\Longrightarrow z_0 \in x_0 + y_0 \qquad (\text{ f is a good homomorphism }) \\ &\Longrightarrow \mu(z_0) \geq \mu(x_0) \wedge \mu(y_0) \qquad (\mu <_{f.h} R) \\ &\Longrightarrow f(\mu(c)) \geq f(\mu(a)) \wedge f(\mu(b)). \end{aligned}$$

For proving the second condition of a fuzzy hyperideal, we should prove that

$$f(\mu)(r^{'}x^{'}) \geq f(\mu)(x^{'}) \vee f(\mu)(r^{'}) \quad \forall\ r^{'}, x^{'} \in R^{'}$$

Since f is onto, then $r' = f(r)$ and $x' = f(x)$ for some r and x in R, Thus

$$\begin{aligned} f(\mu)(r^{'}x^{'}) &= \bigvee_{rx\in f^{-1}(r^{'}x^{'})} \mu(rx) \\ &= \mu(r_0x_0) \quad \exists r_0 \in f^{-1}(r'), x_0 \in f^{-1}(x') \quad (\mu \text{ is } f\text{-invariant}) \\ &\geq \mu(x_0) \vee \mu(r_0) \qquad (\ \mu <_{f.h} R) \\ &= f(\mu)(x^{'}) \vee f(\mu)(r^{'}) \quad (\mu \text{ is } f\text{-invariant}). \end{aligned}$$

Therefore

$$f(\mu)(r'x') \geq f(\mu)(r') \vee f(\mu)(x').$$

Now we prove that $f(\mu) <_{w.f.k.h} R^{'}$. Let $a,\ b \in R$, we show that

$$f(\mu)(a) \geq [(\bigwedge_{t\in a+b} f(\mu)(t)) \bigvee (\bigwedge_{t^{'}\in b+a} f(\mu)(t^{'}))] \bigwedge f(\mu)(b) \qquad (1)$$

Since f is onto and μ is $f-$invariant, then

$$f(\mu)(a) = \mu(x_0),\ f(\mu)(t) = \mu(z_0),\ f(\mu)(t^{'}) = \mu(z_0^{'}),\ f(\mu)(b) = \mu(y_0),$$

where

$$x_0 \in f^{-1}(a),\ y_0 \in f^{-1}(b),\ z_0 \in f^{-1}(t),\ z^{'} \in f^{-1}(t^{'}).$$

Hence (1) reduced to the form

$$\mu(x_0) \geq [(\bigwedge_{t\in a+b} \mu(z_0)) \bigvee (\bigwedge_{t^{'}\in b+a} \mu(z_0^{'}))] \bigwedge \mu(y_0) \qquad (2)$$

On the other hand from above discussion and since f is a good homomorphism $t \in a + b$ if and only if $f(z_0) \in f(x_0) + f(y_0)$ if and only if $z_0 \in x_0 + y_0$. Similarly, $t' \in b + a$ if and only if $z_0' \in y_0 + x_0$.

Therefore by (2), it is enough that we prove that

$$\mu(x_0) \geq [(\bigwedge_{z_0 \in x_0 + y_0} \mu(z_0)) \bigvee (\bigwedge_{z_0' \in y_0 + x_0} \mu(z_0'))] \bigwedge \mu(y_0),$$

but clearly the last statement is true, since $\mu <_{w.f.k.h} R$. This complete the proof.

In this part we define the quotient of fuzzy weak (strong) k-hyperideals by a regular relation of semihyperring

Let R be a semihyperring and θ be an equivalence relation on R. Naturally we can extend θ to $\overline{\theta}$ to the subsets of R as follow:

Let A, B be nonempty subsets of R. Define

$$A\overline{\theta}B \Longleftrightarrow \quad \forall a \in A \ \ \exists b \in B : \quad a\theta b, \ \ \forall b \in B \ \ \exists \ \ a \in A : \quad b\theta a.$$

An equivalence relation θ on R is said to be *regular* if for all $a, \ b, \ x \in R$ we have

(i) $a\theta b \Longrightarrow (a+x)\overline{\theta}(b+x)$ and $(x+a)\overline{\theta}(x+b)$,

(ii) $a\theta b \Longrightarrow (ax)\theta(bx)$ and $(xa)\theta(bx)$.

By $R : \theta$ we mean the set of all equivalence classes with respect to θ, that is

$$R : \theta = \{r_\theta | r \in R\}.$$

Remark 3.4. We know that if R is a semihyperring and θ is a regular equivalence relation on R , then $R : \theta$ by hyperoperations $\oplus$ and $\odot$ is defined as follow

$$x_\theta \oplus y_\theta = \{x_\theta | z \in x + y\},$$

$$x_\theta \odot y_\theta = (xy)_\theta.$$

is a semihyperring. For $\mu \in FS(R)$, define $(\mu : \theta)(x_\theta) = \bigvee_{y \in x_\theta} \mu(y)$. Also we know that the mapping $\varphi : R \longrightarrow R : \theta$ defined by $\varphi(a) = a_\theta$ is a good epimorphism. Now if $\mu <_{w.f.k.h} R$ and μ be $\varphi-$invariant then by proposition 3.3 it concludes that $\varphi(\mu) = \mu : \theta <_{w.f.k.h} R : \theta$.

Proposition 3.5. If $\mu <_{w.f.k.h} R$ and R has zero, then $\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$ is a weak $k-$hyperideal of R.

Proof. First we prove that $\mu_* <_h R$. For $x,\ y \in \mu_*$ and $z \in x + y$, then $\mu(z) \geq \mu(x) \wedge \mu(y) = \mu(0)$, hence by Lemma 2.10 $\mu(z) = \mu(0)$, therefore $z \in \mu_*$.

Let $r \in R$ and $x \in \mu_*$, then we have

$$\begin{aligned}
\mu(rx) &\geq \mu(r) \vee \mu(x) \\
&= \mu(r) \vee \mu(0) \qquad (\ x \in \mu_*\ ) \\
&= \mu(0) \qquad (\text{ by Lemma 2.10 }) \\
&\Longrightarrow \mu(rx) = \mu(0) \qquad (\text{ by Lemma 2.10 }) \\
&\Longrightarrow rx \in \mu_*.
\end{aligned}$$

Now suppose $r + x \subseteq \mu_*$ or $x + r \subseteq \mu_*$ and $x \in \mu_*$, we show that $r \in \mu_*$.

From $\mu <_{w.f.k.h} R$ then we have :

$$\mu(r) \geq [(\bigwedge_{z \in r+x} \mu(z)) \bigvee (\bigwedge_{z' \in x+r} \mu(z'))] \bigwedge \mu(x).$$

Since $\mu(x) = \mu(0)$ and $\bigwedge_{z \in r+x} \mu(z) = \mu(0)$ and $\bigwedge_{z' \in x+r} \mu(z') = \mu(0)$, then $\mu(r) \geq \mu(0)$, and then by Lemma 2.10, $\mu(r) = \mu(0)$. Therefore $\mu_* <_{w.k.h} R$.

Proposition 3.6. If $\mu <_{s.f.k.h} R$, then $\mu^* = \{x \in R \mid \mu(x) > 0\}$ is a strong k-hyperideal of R.

Proof. Let $x,\ y \in \mu^*$ and $z \in x+y$, then by hypothesis yields

$$\mu(z) \geq \mu(x) \wedge \mu(y) > 0,$$

thus $z \in \mu^*$.

If $r \in R$ and $x \in \mu^*$, then we have

$$\mu(rx) \geq \mu(r) \vee \mu(x) \geq \mu(x) > 0,$$

therefore $rx \in \mu^*$. Similarly $xr \in \mu^*$. Thus $\mu^* <_h R$.

Now if $r + x \approx \mu^*$ or $x + r \approx \mu^*$ and $x \in \mu^*$.

By hypothesis we have

$$\mu(r) \geq (\mu(z) \vee \mu(z')) \wedge \mu(x) > 0 \qquad \forall z \in r+x \approx \mu^*,\ \forall z' \in x+r \approx \mu^*,$$

that is $r \in \mu^*$, and hence $\mu^* <_{s.k.h} R$.

Proposition 3.7. Let R be a semihyperring with zero and $x, y \in R$:

(i) If $\mu <_{w.f.k.h} R$ and $\mu(t) = \mu(0) = \mu(t')$ for all $t \in x+y$ and $t' \in y+x$, then $\mu(x) = \mu(y)$.

(ii) If $\mu <_{s.f.k.h} R$ and $\mu(u) = \mu(0) = \mu(v)$ for some $u \in x+y$ and $v \in y+x$, then $\mu(x) = \mu(y)$.

Proof. (i) Since $\mu <_{w.f.k.h} R$ and $\mu(t) = \mu(0) = \mu(t')$ for all $t \in x+y$ and $t' \in y+x$, then $\bigwedge\limits_{t \in x+y} \mu(t) = \mu(0) = \bigwedge\limits_{t' \in y+x} \mu(t')$, thus

$$\begin{aligned} \mu(x) &\geq [(\bigwedge_{t\in x+y} \mu(t)) \bigvee (\bigwedge_{t'\in y+x} \mu(t'))] \bigwedge \mu(y) \\ &= \mu(0) \wedge \mu(y) \\ &= \mu(y) \qquad\qquad \text{(by Lemma 2.10)} \\ &\Longrightarrow \mu(x) \geq \mu(y). \end{aligned}$$

Similarly we conclude that $\mu(y) \geq \mu(x)$. Therefore $\mu(x) = \mu(y)$.

(ii) Suppose $u \in x + y$ and $v \in y + x$ such that $\mu(u) = \mu(0) = \mu(v)$, since $\mu <_{s.f.k.h} R$, then

$$\begin{aligned}
\mu(y) \geq (\mu(u) \vee \mu(v)) \wedge \mu(x) &= \mu(0) \wedge \mu(x) \quad (\text{ by hypothesis }) \\
&= \mu(x) \qquad\qquad (\text{by Lemma 2.10}) \\
&\Longrightarrow \mu(y) \geq \mu(x).
\end{aligned}$$

Similarly we obtain $\mu(x) \geq \mu(y)$. Therefore $\mu(x) = \mu(y)$.

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