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RATIO MATHEMATICA

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Editors

Franco Eugeni and Antonio Maturo

Scientific Committee

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LES HYPERTREILLIS TRES FINS

Cornelia Gutan *

Résumé. Dans ce travail on caractérise les semi-hypergroupes idempotents commutatifs très fins. Comme application on détermine la structure des hypertreillis très fins.

Abstract. In this paper, the very thin commutative idempotent semigroups are characterized. As application, the structure of all very thin hyperlattices is determined.

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1. Introduction

L'étude des hypertreillis a été commencée par M. Konstantinidou et J. Mittas dans [15] (voir aussi [6] et [14]). Des résultats remarquables dans ce domaine ont aussi été obtenus par R. Procesi-Ciampi et R. Rota ([17]) qui ont donné un théorème de représentation de type Stone pour une classe particulière d'hypertreillis, la classe des hyperalgèbres de Boole, comme réponse à un problème de S. Comer (voir [2], Problème 15).

Définition 1.1. Un *hypertreillis* est un ensemble non vide L , muni d'une hyperopération $\dot{\vee}$ et d'une opération $\wedge$ tels que, pour tous x, y, z de L , les conditions suivantes sont satisfaites :

- | | |
|--|---|
| i) $x \in x \dot{\vee} x$ | i') $x = x \wedge x$ |
| ii) $x \dot{\vee} y = y \dot{\vee} x$ | ii') $x \wedge y = y \wedge x$ |
| iii) $(x \dot{\vee} y) \dot{\vee} z = x \dot{\vee} (y \dot{\vee} z)$ | iii') $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ |
| iv) $x \in x \dot{\vee} (x \wedge y)$ | iv') $x \in x \wedge (x \dot{\vee} y)$ |
| v) $x \in x \dot{\vee} y$ si et seulement si $x \wedge y = y$. | |

Si $(L, \dot{\vee})$ satisfait les conditions i)-iii), donc si $(L, \dot{\vee})$ est un semi-hypergroupe commutatif très fin ayant la propriété $x \in x \dot{\vee} x$, pour tout $x \in L$, on dit que $(L, \dot{\vee})$ est un *semi-hypertreillis*.

L'exemple le plus connu d'hypertreillis est le suivant.

Exemple 1.2.

Soit $(L, \vee, \wedge)$ un treillis modulaire. On définit sur L une hyperopération par

$$x \dot{\vee} y = \{ z \in L \mid x \vee z = y \vee z = x \vee y \}.$$

Alors $(L, \dot{\vee}, \wedge)$ est un hypertreillis (voir [1], [4], [12] ou [16]). En particulier, pour le treillis $(\mathcal{P}(X), \cup, \cap)$ de toutes les parties d'un ensemble quelconque X on a

$A \dot{\vee} B = \{C \in \mathcal{P}(X) \mid A \Delta B \subseteq C \subseteq A \cup B\}$ et $A \wedge B = A \cap B$, pour tous A et B de $\mathcal{P}(X)$. L'hypertreillis $(\mathcal{P}(X), \dot{\vee}, \wedge)$ est même une hyperalgèbre de Boole (voir [17]).

On ne connaît pas beaucoup d'autres exemples d'hypertreillis. Ici on obtient de nouveaux exemples : les hypertreillis très fins.

Les hyperstructures très fines sont des structures algébriques multivalentes mais qui s'écartent le moins possible des structures traditionnelles dont les opérations sont univalentes. Ainsi, par exemple, un hypergroupe est un ensemble non vide H muni d'une opération qui fait correspondre, à chaque couple d'éléments x et y , un produit xy qui est une partie non vide quelconque de H au lieu d'être formé simplement d'un unique élément. Un hypergroupe très fin est alors un hypergroupe où tous les produits xy sont des singletons sauf un seul $ab = A$ que l'on qualifie d'*exceptionnel*. Cette notion a été introduite par T. Vougiouklis dans l'étude des représentations des hypergroupes par des hypermatrices et des permutations généralisées. Les hyperstructures très fines sont de plus intéressantes car elles ont des applications directes dans diverses branches de mathématiques comme la théorie de groupes, combinatoire, probabilités et statistique, cryptographie, informatique théorique ...

L'étude du lien entre un hypergroupe très fin et la famille de ses groupoïdes sous-jacents lorsque l'hypergrou-

poïde vérifie des conditions supplémentaires comme la (faible) associativité ou la reproductibilité a été faite dans [9] et [11]. Les hypergroupes et les H_V -groupes ont été caractérisés dans [7]. De même, les semi-hypergroupes commutatifs très fins ont été caractérisés dans [10]. On mentionne aussi que des résultats sur les hyperstructures très fines ont été obtenu dans [8], [13] et [18].

Dans ce travail on caractérise les hypertreillis très fins.

Remarquons que si $(L, \vee, \wedge)$ est un treillis modulaire alors l'hypertreillis $(L, \dot{\vee}, \wedge)$ construit dans l'exemple 1.2 est très fin si et seulement si $\text{Card } L = 2$ (cela découle du fait que $x \dot{\vee} x = \{z \in L \mid z \leq x\}$, pour tout $x \in L$). On déduit que $(\mathcal{P}(X), \dot{\vee}, \wedge)$ est un hypertreillis très fin si et seulement si $\text{Card } X = 1$.

2. Les semihypertreillis très fins

Dans la suite, pour tout semi-groupe commutatif $(S, \cdot)$ on désigne par $K(S)$ le noyau de S (s'il existe). Notons que si S est un semi-groupe commutatif idempotent alors $K(S)$ existe si et seulement si S a un élément zéro z et alors $K(S) = \{z\}$. De même, lorsque I est un idéal du semi-groupe S et $u \in S$ on note par λ_u la translation intérieure $\lambda_u : I \rightarrow I$, $\lambda_u(x) = u \cdot x$, pour tout $x \in I$. On désigne par $E(f)$ le sous-ensemble de S défini par $E(f) = \{s \in S \mid \lambda_s = f\}$, où $f : I \rightarrow I$ est une application.

On a le résultat suivant, établi dans [9].

Théorème 2.1.

Soient H un semi-hypergroupe commutatif très fin dont (a, a) est le couple exceptionnel. On considère

$S = H \setminus \{a\}$, $B = (aa) \cap S$ et $\varphi : S \rightarrow H$ l'application définie par $\varphi(s) = as$, pour tout $s \in S$. Soient $T = \varphi^{-1}(a)$, $I = S \setminus T$ et $\psi = \varphi|_I : I \rightarrow I$ la restriction de la fonction φ à I . Alors :

- i) *S est un sous-semi-groupe de H .*
- ii) *si T est non vide alors T est un sous-semi-groupe de S .*
- iii) *si I est non vide alors I est un idéal de S .*

De plus, lorsque $a \in aa$ il existe $\alpha \in S$ tel que $\psi = \lambda_\alpha$ et :

- I. *si $T = S$ alors $B = K(S)$.*
- II. *si $T = \emptyset$ alors $\emptyset \neq B \subset E(\lambda_\alpha)$, où α est un élément idempotent de S .*
- III. *si $\emptyset \subsetneq T \subsetneq S$ alors seulement les trois situations suivantes sont possibles :*
 - i) *$B = K(T) \subset E(\lambda_{\alpha^2})$, où $\alpha \in B$.*
 - ii) *$B = K(T) \cup \{\beta\}$, où β est un élément idempotent de I tel que $\beta = \beta \cdot T$, $\alpha \in K(T)$ et $K(T) \subset E(\lambda_\alpha)$.*
 - iii) *$\emptyset \neq B \subset E(\lambda_\alpha) \cap I$ et $B = B \cdot t$, pour tout $t \in T$, où α est un élément de I tel que $\lambda_{\alpha^2} = \lambda_\alpha$. ■*

On utilise le théorème précédent pour obtenir la structure des semi-hypertreillis très fins.

Soit $(S, \vee)$ un semi-groupe commutatif idempotent. Alors $(S, \vee)$ est un semi-treillis par rapport à une relation d'ordre

définie par $x \leq y$ si et seulement si $x \vee y = y$, o x et y de S . On considère aussi $H = S \cup \{a\}$, où a est un élément qui n'appartient pas à S .

On utilise ces données pour réaliser les constructions suivantes.

Construction 1.

Soit α un élément quelconque dans S et T un sous-ensemble de S tel que $T \leq \alpha$. Si T est non vide alors T est un sous-semi-groupe de S . De même, si $I = S \setminus T$ est non vide alors I est un idéal de S . On définit sur H une hyperopération par :

$$\begin{aligned} a \dot{\vee} a &= \{a, \alpha\}, \\ a \dot{\vee} T &= T \dot{\vee} a = a, \\ a \dot{\vee} x &= x \dot{\vee} a = \alpha \vee x, \text{ pour tout } x \in I \\ x \dot{\vee} y &= x \vee y, \text{ pour tous } x, y \text{ dans } S. \end{aligned}$$

On désigne par $H_1(S, a, \alpha)$ l'hyperstructure ainsi obtenue (elle correspond au cas I du Théorème 2.1 si α est le plus grand élément de S ; sinon elle résulte du cas III. i).

Construction 2.

Soit $\alpha \in S$. En analysant les cas II et respectivement III. iii) on déduit qu'il faut et il suffit de choisir T un sous-ensemble de S de sorte que $T < \alpha$ et $x \vee y \in T$ si et seulement si x, y sont tous les deux dans T . Cela découle du fait que si T est non vide alors T est un sous-semi-groupe de S et $I = S \setminus T$ est un idéal de S (bien évidemment $I \neq \emptyset$ car $\alpha \in I$). On considère sur H une hyperopération définie par :

$$\begin{aligned}
a \dot{\vee} a &= \{a, \alpha\}, \\
a \dot{\vee} T &= T \dot{\vee} a = a, \\
a \dot{\vee} x &= x \dot{\vee} a = \alpha \vee x, \text{ pour tout } x \in I, \\
x \dot{\vee} y &= x \vee y, \text{ pour tous } x, y \text{ dans } S.
\end{aligned}$$

On désigne par $H_2(S, a, \alpha, T)$ cette hyperstructure (elle correspond au cas II du Théorème 2.1 si $T = \emptyset$; sinon, elle correspond au cas III. iii).

En particulier, lorsque α est tel que si $x < \alpha$ et $y < \alpha$ alors on a aussi $x \vee y < \alpha$, on peut prendre $T < \alpha$.

Il reste à analyser le cas III. ii).

Construction 3.

Soient α et β deux éléments du semi-groupe S et T un sous-ensemble dans S tel que $T \leq \alpha$. Alors T est un sous-semigroupe de S .

Le cas III. ii) est possible si et seulement si $\beta = \beta \vee t$, pour tout $t \in T$, et $\beta \vee u = \alpha \vee u$, pour tout $u \in I = S \setminus T$.

Compte tenu de la manière dont T est défini, la condition $\beta = \beta \vee t$, pour tout $t \in T$, est équivalente à $\alpha < \beta$.

De même, la condition $\beta \vee u = \alpha \vee u$, pour tout $u \in I$, est équivalente à l'égalité

$$\{s \in S \mid \alpha \leq s\} = \{\alpha\} \cup \{s \in S \mid \beta \leq s\}.$$

En effet, supposons $\beta \vee u = \alpha \vee u$, pour tout $u \in I$ (donc pour tout u tel que $u \not\leq \alpha$).

Evidemment $\{s \in S \mid \alpha \leq s\} \supset \{\alpha\} \cup \{s \in S \mid \beta \leq s\}$. On démontre maintenant l'inclusion réciproque.

Soit $s \in S, \alpha < s$. Donc $s \not\leq \alpha$. Alors $\beta \vee s = \alpha \vee s = s$, d'où $\beta \leq s$.

Réciproquement, supposons

$$\{s \in S \mid \alpha \leq s\} = \{\alpha\} \cup \{s \in S \mid \beta \leq s\}.$$

On considère $u \in I$ (c'est-à-dire $u \not\leq \alpha$). Comme $\alpha < \beta \leq \beta \vee u$ on a $\alpha < \beta \vee u$. On obtient $\alpha \vee u \leq \beta \vee u$. D'autre part $\alpha < \alpha \vee u$ (si $\alpha = \alpha \vee u$ alors $u \leq \alpha$, qui est une contradiction) donc $\beta \leq \alpha \vee u$ et, par conséquent, $\beta \vee u \leq \alpha \vee u$.

On voit ainsi que pour le cas III.ii) il faut choisir α et β dans S de sorte que $\alpha < \beta$ et que l'intervalle $[\alpha, \beta]$ ne contienne aucun autre élément à l'exception de α et β , donc $[\alpha, \beta] = \{\alpha, \beta\}$. On considère sur H une hyperopération définie par :

$$\begin{aligned} a \dot{\vee} a &= \{a, \alpha, \beta\}, \\ a \dot{\vee} T &= T \dot{\vee} a = a, \\ a \dot{\vee} x &= x \dot{\vee} a = \beta \vee x, \text{ pour tout } x \in I, \\ x \dot{\vee} y &= x \vee y, \text{ pour tous } x, y \text{ dans } S. \end{aligned}$$

On note par $H(S, a, \alpha, \beta)$ l'hyperstructure obtenue de cette manière.

En utilisant les constructions 1, 2 et 3 on obtient le théorème-résumé de la "typologie" suivant.

Théorème 2.2.

Soit H un semi-hypertreillis très fins. Alors il existe un semi-groupe commutatif idempotent $(S, \vee)$ et un élément $a \notin S$ tels que $H = S \cup \{a\}$ et H est de l'un des types suivants :

- $H = (H_1(S, a, \alpha), \dot{\vee})$, où $\alpha \in S$ et T est un sous-ensemble de S tel que $T \leq \alpha$;

- $H = (H_2(S, a, \alpha, T), \dot{\vee})$, où α est un élément de S et $T < \alpha$ est un sous-ensemble de S de sorte que $x \vee y \in T$ si et seulement si $x \in T$ et $y \in T$;
- $H = (H(S, a, \alpha, \beta), \dot{\vee})$, où α, β sont des éléments de S de sorte que $\alpha < \beta$, $[\alpha, \beta] = \{\alpha, \beta\}$ et T est un sous-ensemble de S tel que $T \leq \alpha$. ■

3. Les hypertreillis très fins

Dans ce paragraphe on détermine la structure des hypertreillis très fins.

Lemme 3.1.

Soit L un ensemble non vide sur lequel on définit une hyperopération $\dot{\vee}$ telle que $(L, \dot{\vee})$ soit un semi-hypertreillis très fin dont (a, a) est le couple exceptionnel. Sur L on considère aussi une opération $\wedge$ qui fait de L un semi-treillis. Alors $(L, \dot{\vee}, \wedge)$ est un hypertreillis très fin si et seulement si les conditions suivantes sont satisfaites :

- (1) pour tout x, y de $L \setminus \{a\}$, $x \wedge y = x$ si et seulement si $x \dot{\vee} y = y$;
- (2) $\{x \in L \setminus \{a\} \mid a \wedge x = x\} = \{x \in L \setminus \{a\} \mid a \dot{\vee} x = a\}$;
- (3) $\{x \in L \setminus \{a\} \mid a \wedge x = a\} = \{x \in L \setminus \{a\} \mid a \dot{\vee} x = x\}$.

Démonstration :

Supposons que $(L, \dot{\vee}, \wedge)$ est un hypertreillis très fin dont le couple exceptionnel soit (a, a) . On utilise la condition v) (Définition 1.1) pour démontrer que les conditions (1), (2) et (3) sont satisfaites.

Si dans la condition v) on prend x et y dans $L \setminus \{a\}$

alors on a $x \wedge y = x$ si et seulement si $x \dot{\vee} y = y$. Donc on obtient la condition (1).

Pour démontrer (2) on applique v) pour le couple (a, x) , où $x \in L \setminus \{a\}$. Alors $a \in a \dot{\vee} x$, c'est-à-dire $a = a \dot{\vee} x$, équivaut à $x = x \wedge a$.

De la même manière, pour $x \in L \setminus \{a\}$ et $y = a$, de v) on déduit (3).

Réciproquement, supposons que les conditions (1), (2) et (3) sont vérifiées. Pour montrer que $(L, \dot{\vee}, \wedge)$ est un hypertreillis très fin il reste à vérifier les conditions iv) et iv') de la Définition 1.1.

Les conditions iv) et iv') sont évidemment satisfaites si $x = y = a$. Il faut encore vérifier les trois autres situations possibles.

Soient $x = a$ et $y \in L \setminus \{a\}$. Il faut démontrer que $a \in a \dot{\vee} (a \wedge y)$, respectivement $a \in a \wedge (a \dot{\vee} y)$. Si $a \wedge y = a$ alors évidemment $a \in a \dot{\vee} (a \wedge y)$. Supposons $a \wedge y = z \in L \setminus \{a\}$. Alors $a \wedge z = z$ et puisque $z \in L \setminus \{a\}$, d'après (2), il s'ensuit que $a \dot{\vee} z = a$. Donc $a \dot{\vee} (a \wedge y) = a$. Si $a \dot{\vee} y = a$, la condition iv') est vérifiée. Supposons $a \dot{\vee} y = z$, où $z \in L \setminus \{a\}$. Alors $a \dot{\vee} z = z$ et, de (3), on déduit $a \wedge z = a$. Donc $a \wedge (a \dot{\vee} y) = a$.

Soient $x \in L \setminus \{a\}$ et $y = a$. Alors lorsque $x \wedge a = a$, d'après (3) on a $a \dot{\vee} x = x$, c'est-à-dire $x \dot{\vee} (x \wedge a) = x$. De même, lorsque $x \wedge a = z \in L \setminus \{a\}$ il s'ensuit que $z \wedge x = z$ et, de (1), on a alors $z \dot{\vee} x = x$. Donc $x = x \dot{\vee} (x \wedge a)$. En ce qui concerne iv') si $x \dot{\vee} a = a$ alors, d'après (2), $x \wedge a = x$, donc $x \wedge (x \dot{\vee} a) = x$. Sinon, supposons que $x \dot{\vee} a = z$, où $z \in L \setminus \{a\}$.

Alors $x \dot{\vee} z = z$ et puisque la condition (1) est satisfaite, on a $x \wedge z = x$, donc $x \wedge (x \dot{\vee} a) = x$.

Il reste à analyser le cas x et y dans $L \setminus \{a\}$. Evidemment, si $x \wedge y = a$, alors $x \wedge a = a$ et, en utilisant (3) on déduit $a \dot{\vee} x = x$. Donc $x = x \dot{\vee} (x \wedge y)$. Si $x \wedge y = z \in L \setminus \{a\}$ alors $x \wedge z = z$ et conformément à (1), $x \dot{\vee} z = x$, donc $x = x \dot{\vee} (x \wedge y)$. Pour démontrer iv') remarquons que si x et y sont dans $L \setminus \{a\}$ alors, du Théorème 2.1, $x \dot{\vee} y = z \in L \setminus \{a\}$. Donc $x \dot{\vee} z = z$ et de (1), $x \wedge z = x$. On déduit $x \wedge (x \dot{\vee} y) = x$. ■

On considère

$$T = \{x \in L \setminus \{a\} \mid a \dot{\vee} x = a\} \text{ et } \tilde{T} = \{x \in L \setminus \{a\} \mid a \dot{\vee} x = x\}.$$

Alors les conditions (2) respectivement (3) sont équivalentes à

$$T = \{x \in L \setminus \{a\} \mid a \wedge x = x\} \text{ et } \tilde{T} = \{x \in L \setminus \{a\} \mid a \wedge x = a\}.$$

Si $(L, \dot{\vee}, \wedge)$ est un hypertreillis très fin alors $(L, \dot{\vee})$ est un semi-hypertreillis très fin. Donc il existe un sous-semi-groupe S de L tel que $L = S \cup \{a\}$, où $a \notin S$. De plus, sur S on a une relation d'ordre $\leq$ telle que $\sup_{\leq}(x, y)$ existe pour tous x, y de S et si $\vee = \dot{\vee} \mid_{S \times S}$ alors $x \vee y = \sup_{\leq}(x, y)$.

D'autre part, $(L, \wedge)$ est un semi-treillis, donc on a sur L une relation d'ordre $\preceq$ de sorte que pour tous x, y de L il existe $\inf_{\preceq}(x, y)$ et $x \wedge y = \inf_{\preceq}(x, y)$.

Du Lemme 3.1 on déduit maintenant que $(L, \dot{\vee}, \wedge)$ est un hypertreillis très fin si et seulement si

(*) les relations $\leq$ et $\preceq$ coïncident sur $L \setminus \{a\}$ et $T \prec a \prec \tilde{T}$.

On remarque aussi que si x, y sont deux éléments de

$L \setminus \{a\}$ tels que $x \wedge y = a$ alors $a \prec x, a \prec y$, donc x, y sont dans $\tilde{T}$ et il n'existe pas $t \in \tilde{T}$ tel que $t \leq x, t \leq y$.

On étudie maintenant de quelle manière il faut définir une opération sur $(H_1(S, a, \alpha), \dot{\vee})$, respectivement sur $(H_2(S, a, \alpha, T), \dot{\vee})$ et sur $(H(S, a, \alpha, \beta), \dot{\vee})$ pour que l'hyperstructure ainsi obtenues soient des hypertreillis très fins.

L'ordre $\leq$ induit sur $S \cup \{a\}$ un ordre $\preceq$ par $T \prec a \prec \tilde{T}$.

Il faut alors que $\inf_{\preceq}(x, y) = x \Delta y$ existe pour tout couple (x, y) d'éléments de $S \cup \{a\}$.

Par rapport à T , et à $\tilde{T}$, il y a trois cas possibles, où $T \subset S, \tilde{T} \subset I$.

- I. Pour $(H_1(S, a, \alpha), \dot{\vee})$ on a $T \leq \alpha$ et $\alpha < \tilde{T}$.
- II. De même, pour $(H_2(S, a, \alpha, T), \dot{\vee})$ on a $T < \alpha$ et $\alpha \leq \tilde{T}$.
- III. Si on considère $(H(S, a, \alpha, \beta), \dot{\vee})$ alors $T \leq \alpha$ et $\beta \leq \tilde{T}$.

Dans les deux derniers cas $\tilde{T}$ a un plus petit élément, donc $x \Delta y = x \wedge y \in S$, pour tous x, y de S . Par conséquent, on obtient le théorème suivant, qui donne la structure des hypertreillis très fins correspondants à $(H_2(S, a, \alpha, T), \dot{\vee})$, respectivement à $(H(S, a, \alpha, \beta), \dot{\vee})$.

Théorème 3.2.

Soient $(S, \vee, \wedge)$ un treillis et a un élément qui n'appartient pas à S . Alors $(H_2(S, a, \alpha, T), \dot{\vee}, \Delta)$ respectivement $(H(S, a, \alpha, \beta), \dot{\vee}, \Delta)$ est un hypertreillis très fin si et seulement si l'opération Δ est définie sur $H = S \cup \{a\}$ par :

$$\begin{aligned}
a \Delta a &= a \Delta \tilde{T} = \tilde{T} \Delta a = a, \\
a \Delta x &= x \Delta a = \alpha \wedge x, \text{ pour tout } x \in S \setminus \tilde{T}, \\
x \Delta y &= x \wedge y, \text{ pour tous } x, y \text{ dans } S.
\end{aligned}$$

Démonstration :

Supposons que $(H_2(S, a, \alpha, T), \dot{\vee}, \Delta)$ est un hypertreillis très fin. Alors $T \prec a \prec \alpha \leq \tilde{T}$.

Si $v \in T, v \wedge a = v = \alpha \wedge v$. Soit $v \in S \setminus (T \cup \tilde{T})$. On a $\alpha \wedge v \prec v$ et $\alpha \wedge v \prec \alpha$. De $\alpha \wedge v \prec \alpha$ il s'ensuit $\alpha \wedge v \in T$, donc $\alpha \wedge v \prec a$. Par conséquent $\alpha \wedge v$ est un minorant de l'ensemble $\{a, v\}$. Si u est un autre minorant de l'ensemble $\{a, v\}$ alors $u \prec v$ et $u \prec a$. On déduit que $u \in T$, donc $u \prec \alpha$. Par conséquent $u \prec \alpha \wedge v$.

Le résultat concernant $(H(S, a, \alpha, \beta), \dot{\vee}, \Delta)$ découle d'un raisonnement analogue. ■

On donne maintenant la construction de l'hypertreillis très fin qui correspond à $(H_1(S, a, \alpha), \dot{\vee})$. D'après les conditions (*) il s'ensuit qu'il faut considérer $(S, \leq)$ un treillis. Donc pour tout couple d'éléments de S il faut qu'il existe $\sup_{\leq}(x, y)$ et $\inf_{\leq}(x, y)$. On pose

$$\sup_{\leq}(x, y) = x \vee y \text{ et } \inf_{\leq}(x, y) = x \wedge y.$$

On considère $\alpha \in S$ et $T, \tilde{T}$ des sous-ensembles de S tels que $T \leq \alpha, \alpha < \tilde{T}$. La relation d'ordre $\leq$ considérée sur S induit sur $S \cup \{a\}$ un ordre $\preceq$ défini par $\preceq = \leq \cup (T \times \{a\}) \cup (\{a\} \times \tilde{T}) \cup \{(a, a)\}$. De plus, $(S \cup \{a\}, \preceq)$ doit

être un semi-treillis, i.e. pour tout couple (x, y) de $S \cup \{a\}$ il existe $\inf_{\leq}(x, y) = x \Delta y$. Alors

$$\begin{aligned} a \Delta a &= a, \\ a \Delta t &= t, \text{ pour tout } t \in T, \\ a \Delta \tilde{t} &= \tilde{t}, \text{ pour tout } \tilde{t} \in \tilde{T}. \end{aligned}$$

Avec un raisonnement similaire aux deux cas précédents on déduit $a \Delta x = \alpha \wedge x$, pour tout $x \in S \setminus (T \cup \tilde{T})$.

Supposons qu'il existe x et y dans S tels que $x \Delta y = a$. Alors $a \prec x, a \prec y$, donc $x, y \in \tilde{T}$ et, par conséquent, $\alpha < x$ et $\alpha < y$. De plus, si $u \in S$ est un minorant de l'ensemble $\{x, y\}$ alors $u \prec a$, donc $u \leq \alpha$. On déduit que $x \wedge y = \alpha$.

Ainsi $x \Delta y = a$, pour tous x, y dans S tels que $x \wedge y = \alpha$, et $x \Delta y = x \wedge y$, pour tous x, y dans S tels que $x \wedge y \neq \alpha$.

Compte tenu de ces remarques on a le résultat suivant.

Théorème 3.3.

Soit $(S, \vee, \wedge)$ un treillis et a un élément qui n'appartient pas à S . Alors $(H_1(S, a, \alpha), \dot{\vee}, \Delta)$ est un hypertreillis très fin si et seulement si l'opération Δ est définie sur $H = S \cup \{a\}$ par :

$$\begin{aligned} a \Delta a &= a = x \Delta y, \text{ pour tous } x, y \text{ dans } S \text{ tels que } x \wedge y = \alpha; \\ x \Delta y &= x \wedge y, \text{ pour tous } x, y \text{ dans } S \text{ tels que } x \wedge y \neq \alpha; \\ a \Delta x &= x \Delta a = x, \text{ pour tout } x \in T \cup \tilde{T}; \\ a \Delta x &= x \Delta a = \alpha \wedge x, \text{ pour tout } x \in S \setminus (T \cup \tilde{T}). \end{aligned} \quad \blacksquare$$

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**PROPERTIES OF HYPERPRODUCTS
AND
THE RELATION β IN QUASIHYPERGROUPTS**

Marin GUTAN *

Abstract : Some properties of the complete parts in hypergroupoids are established. Applying these properties to the case of quasihypergroups H for which H/β^* is a quasigroup, a necessary and sufficient condition for the transitivity of the relation β is proved. Consequently, several classes of quasihypergroups in which β is transitive are obtained (for instance, in any finite quasihypergroup with identity β is a transitive relation). Then, in the case of quasihypergroups having underlying groups, the relation β is completely determined.

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1. INTRODUCTION

Let H be a hypergroupoid, $n \geq 2$ be an integer and $x_1, \dots, x_n$ be elements of H . If γ is a grouping of the indices $1, 2, \dots, n$ in this order, then the product of the elements

x_i respecting γ is denoted $\prod_{\substack{i=1 \\ (\gamma)}}^n x_i$. Denote by $\Gamma(n)$ all the

groupings of the indices $1, 2, \dots, n$ in this order. Using these notations consider :

$$P_1(H) = \{\{x\} \mid x \in H\},$$

$$P_n(H) = \left\{ \prod_{\substack{i=1 \\ (\gamma)}}^n x_i \mid x_1, \dots, x_n \in H, \gamma \in \Gamma(n) \right\},$$

$$P_*(H) = \bigcup_{n=1}^{\infty} P_n(H).$$

By means of hyperproducts of $P(H)$ we define on H a chain of relations $(\beta_n)_{n \geq 1}$ as follows : $x \beta_n y$ if and only if there exists $Q \in P_n(H)$ such that $x, y \in Q$.

It is evident that $\beta_1 = Id(H)$ and β_n are symmetrical.

Consider the relation $\beta = \bigcup_{n=1}^{\infty} \beta_n$ which is reflexive and symmetrical. Its transitive closure $\beta^* = \beta \cup \beta \circ \beta \cup \dots$ is an equivalence relation on H and H/β^* is a groupoid. Hence using the relation β^* we can define functors between categories of hypergroupoids and categories of groupoids which permit to reduce some problems on hyperstructures to easier others on univalent structures.

The relation β^* in hypergroups has been studied by many authors like M. Koskas ([9]), P. Corsini ([1], [2]),

Y. Sureau ([11]), D. Freni ([4], [5], [6]), M. De Salvo ([3]), R. Migliorato ([10]).

There is an interesting problem concerning the relations β and β^* :

PROBLEM 1. When $\beta^* = \beta$? (Find the classes of hypergroupoids for which the corresponding relation β is transitive.)

A first important answer to this problem was obtained in 1991 by D. Freni ([4]). He proved that in hypergroups the relations β and β^* coincide.

The semihypergroups for which the relation β is transitive are characterised in [7].

In connection with Problem 1 we mention the following problem proposed by T. Vougiouklis ([12]).

PROBLEM 2. Do the relations β and β^* coincide in weakly associative quasihypergroups?

In this paper we extend some properties of the complete parts from semihypergroups to hypergroupoids. Using these properties we treat Problem 1 in the particular case of quasihypergroups.

2. COMPLETE PARTS AND THE RELATION β IN HYPERGROUPOIDS

The notion of complete part in hypergroups has been introduced and studied by M. Koskas in [9]. Then P. Corsini

([1], [2]), Y. Sureau ([11]), D. Freni ([4]), M. De Salvo ([3]), R. . Migliorato ([10]) have established connections between the complete parts and the heart of a hypergroup.

In the following some properties of the complete parts in (semi) hypergroups (see [1] and [7]) are extended to hypergroupoids. Using complete parts, a characterization of hypergroupoids in which the relation β is transitive, analogues with that for semihypergroups obtained in [7], is given.

Let H be a hypergroupoid. A subset A of H is a *complete part* if for every $Q \in P(H)$ such that $Q \cap A \neq \emptyset$ we have $Q \subset A$.

Remark that $\emptyset$ and H are complete parts of H . Also, the intersection of any family of complete parts of H is a complete part in H .

For a subset X of H denote by $\mathcal{C}(X)$ the intersection of all complete parts of H containing X . It is easy to verify that $\mathcal{C}(X)$ is the smallest complete part of H containing X (called the *complete closure* of X).

The following properties hold :

- (1) $X \subset \mathcal{C}(X)$.
- (2) If $X \subset X'$ then $\mathcal{C}(X) \subset \mathcal{C}(X')$.
- (3) $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$.
- (4) $\mathcal{C}(X) = \bigcup_{x \in X} \mathcal{C}(x)$, where $\mathcal{C}(x) = \mathcal{C}(\{x\})$.

As for an associative hyperoperation, we can associate to any subset X of H an ascending chain of subsets $(\mathcal{C}_n(X))_{n \in \mathbb{N}}$ defined by the following two relations :

$$\text{i)} \quad \mathcal{C}_0(X) = X$$

$$\text{ii)} \quad \mathcal{C}_{n+1}(X) = \cup \{Q \in P(H) \mid Q \cap \mathcal{C}_n(X) \neq \emptyset\}.$$

Using the chain $(\mathcal{C}_n(X))_{n \in \mathbb{N}}$ we can obtain the complete closure of X , as it is shown in the next result.

2.1. PROPOSITION. *Let X be a subset in a hypergroupoid H . Then the following properties hold :*

$$(5) \quad \mathcal{C}_n(X) = \bigcup_{x \in X} \mathcal{C}_n(x), \text{ where } \mathcal{C}_n(x) = \mathcal{C}_n(\{x\}).$$

$$(6) \quad \mathcal{C}_n(\mathcal{C}_m(X)) = \mathcal{C}_{n+m}(X).$$

$$(7) \quad \mathcal{C}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(X).$$

PROOF:

(5) As $\mathcal{C}_n(x) \subset \mathcal{C}_n(X)$ whenever $x \in X$, it follows that $\left(\bigcup_{x \in X} \mathcal{C}_n(x)\right) \subset \mathcal{C}_n(X)$. We prove the converse inclusion by induction on n . If $n = 0$ then the equality (5) holds. Assume that $\mathcal{C}_{n-1}(X) \subset \bigcup_{x \in X} \mathcal{C}_{n-1}(x)$, for $n \in \mathbb{N}^*$, and consider $y \in \mathcal{C}_n(X)$. This means that there exists $Q \in P(H)$ such that $y \in Q$ and $Q \cap \mathcal{C}_{n-1}(X) \neq \emptyset$. Then, by hypothesis, $Q \cap \mathcal{C}_{n-1}(x) \neq \emptyset$, for some $x \in X$. Therefore $Q \subset \mathcal{C}_n(x)$, that is $y \in \left(\bigcup_{x \in X} \mathcal{C}_n(x)\right)$, which proves that the converse inclusion holds.

(6) We proceed once again by induction on $n \in \mathbb{N}$.

For $n = 0$ the relation (6) is valid because $\mathcal{C}_0(\mathcal{C}_m(X)) = \mathcal{C}_m(X)$.

Suppose now that $\mathcal{C}_{n-1}(\mathcal{C}_m(X)) = \mathcal{C}_{m+n-1}(X)$, where $n \in \mathbb{N}^*$. Then

$$\mathcal{C}_n(\mathcal{C}_m(X)) = \cup\{Q \in P(H) \mid Q \cap \mathcal{C}_{n-1}(\mathcal{C}_m(X)) \neq \emptyset\} = \cup\{Q \in P(H) \mid Q \cap \mathcal{C}_{m+n-1}(X) \neq \emptyset\} = \mathcal{C}_{n+m}(X).$$

(7) In order to prove that $\mathcal{C}(X) \subset \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(X)$ it is sufficient to establish that $A = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(X)$ is a complete part of H .

Let $Q \in P(H)$ such that $Q \cap A \neq \emptyset$. Then $Q \cap \mathcal{C}_n(X) \neq \emptyset$, for some $n \in \mathbb{N}$. Consequently $Q \subset \mathcal{C}_{n+1}(X) \subset A$ and thus $\mathcal{C}(X) \subset \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(X)$. On the other hand, from i) and ii), by induction on n , we get that $\mathcal{C}_n(X) \subset \mathcal{C}(X)$, for every $n \in \mathbb{N}$. Hence $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n(X) \subset \mathcal{C}(X)$. ■

Note that we also have the properties below :

(8) If $Q \in P(H)$ then $\mathcal{C}(Q) = \mathcal{C}(x)$, for any $x \in Q$.

(9) $\mathcal{C}_n(x)y \subset \mathcal{C}_n(xy) \supset x\mathcal{C}_n(y)$, for any x and y in H .

(10) $\mathcal{C}(x)y \subset \mathcal{C}(xy) \supset x\mathcal{C}(y)$, for any x and y in H .

The connexion between the relation β^* and the complete parts of a hypergroupoid is given by the following result.

2.2. PROPOSITION. *Let x and y be two elements of a hypergroupoid H . Then $x\beta^*y$ if and only if $\mathcal{C}(x) = \mathcal{C}(y)$.*

PROOF :

In order to prove the required equivalence suppose first that $x\beta^*y$ and show that $\mathcal{C}(x) = \mathcal{C}(y)$. It suffices to establish that for the couples (x, y) of elements of H satisfying $x\beta y$. If $x\beta y$ then there exists $Q \in P(H)$ which contains both x and y . Therefore, according to (8), $\mathcal{C}(x) = \mathcal{C}(Q) = \mathcal{C}(y)$. Consequently, if $x\beta^*y$ then $\mathcal{C}(x) = \mathcal{C}(y)$.

Assume now that $\mathcal{C}(x) = \mathcal{C}(y)$. Then $x \in \mathcal{C}_n(y)$, for some $n \in \mathbb{N}$. We prove by induction on n that $x\beta^*y$. For $n = 0$ this is true because $x = y$. Assume this is also true for any integer $k < n$ and prove that the corresponding assertion for n is true, too. As $x \in \mathcal{C}_n(y)$ there exists $Q \in P(H)$ such that $x \in Q$ and $Q \cap \mathcal{C}_{n-1}(y) \neq \emptyset$. Let $z \in Q \cap \mathcal{C}_{n-1}(y)$. From $x \in Q$ we get $\{x\}\overset{=}{\beta^*}Q$, whence $x\beta^*z$. On the other hand, as $z \in \mathcal{C}_{n-1}(y)$, by inductive hypothesis, $z\beta^*y$. Hence $x\beta^*y$. ■

If R is a relation on the hypergroupoid H we define on $\mathcal{P}^*(H) = \{X \subset H \mid X \neq \emptyset\}$ two others relations R and $\overset{=}{R}$ by :

$$A\overline{R}B \text{ iff } \begin{cases} \forall a \in A, \exists b \in B \text{ such that } aRb \\ \forall b \in B, \exists a \in A \text{ such that } aRb \end{cases}$$

$$A\overset{=}{R}B \text{ iff } (\forall a \in A, \forall b \in B \text{ we have } aRb).$$

The relation $\overset{=}{\beta}$ which intervene in the previous proof is obtained in this manner.

Using the previous results we obtain the following characterization of the transitivity of the relation β in hypergroupoids.

2.3. THEOREM. *The relation β is transitive in a hypergroupoid H if and only if*

$$(*) \quad \mathcal{C}(x) = \mathcal{C}_1(x), \text{ for any } x \in H.$$

PROOF :

Suppose β is transitive. In order to prove that $\mathcal{C}(x) = \mathcal{C}_1(x)$, for any $x \in H$, it suffices to establish that $\mathcal{C}_1(x)$ is a complete part of H . Let $Q \in P(H)$ such that $Q \cap \mathcal{C}_1(x) \neq \emptyset$ and let $y \in Q \cap \mathcal{C}_1(x)$. We have to show that $Q \subset \mathcal{C}_1(x)$.

It is obvious that $x\beta y$. As $y\beta z$, for $z \in Q$, we obtain that $x\beta z$. Consequently there exists Q' in $P(H)$ containing both x and z , whence $z \in \mathcal{C}_1(x)$.

Conversely, suppose that $(*)$ holds. Consider x, y, z elements of H such that $x\beta y$ and $y\beta z$. Then $x\beta^* z$ and thus $\mathcal{C}(x) = \mathcal{C}(z)$. It follows that $z \in \mathcal{C}_1(x)$. Hence, there exists Q in $P(H)$ which contains both x and z , that is $x\beta z$. ■

Several examples of semihypergroups for which the relation β is not transitive are presented in [7]. However there is no known example of quasihypergroup for which the associated relation β is not transitive.

Conjecture. If H is a quasihypergroup then β is transitive on H .

If we deal with quasihypergroups H such that H/β^* is a quasigroup we can give a necessary and sufficient condition for the transitivity of β , more simple than the previous condition $(*)$.

2.4. THEOREM. *Let H be a quasihypergroup such that H/β^* is a quasigroup. Then the relation β is transitive in H if and only if*

$$(**) \quad \text{there exists } x \text{ in } H \text{ for which } \mathcal{C}(x) = C_1(x).$$

PROOF :

The result to prove is a direct consequence of the following.

2.5. LEMMA. *Let H be a quasihypergroup such that H/β^* is a quasigroup. Then the following assertions are valid :*

- a) $\mathcal{C}(xy) = \mathcal{C}(x)y$, for every x and y in H .
- b) If there exist x in H and $n \in \mathbb{N}$ such that $\mathcal{C}(x) = C_n(x)$ then $\mathcal{C}(y) = C_n(y)$, for every $y \in H$.

PROOF :

a) According to (10), $\mathcal{C}(x)y \subset \mathcal{C}(xy)$. In order to prove the converse inclusion let $t \in \mathcal{C}(xy)$. As $t \in H = Hy$ we get that $t \in uy$, for some $u \in H$. Therefore $\beta^*(t) = \beta^*(x)\beta^*(y) = \beta^*(u)\beta^*(y)$, whence $\beta^*(t) = \beta^*(u)$. Thus $t \in \mathcal{C}(u)y = \mathcal{C}(x)y$, that is $\mathcal{C}(xy) \subset \mathcal{C}(x)y$.

b) Let $y \in H$. Then there exists $u \in H$ such that $y \in xu$. Hence $\mathcal{C}(y) = \mathcal{C}(xu) = \mathcal{C}(x)u = C_n(x)u \subset C_n(xu) = C_n(y)$.

It follows that $\mathcal{C}(y) = C_n(y)$, for every $y \in H$. ■

2.6. COROLLARY. *Let H be a finite quasihypergroup. Then the relation β is transitive in H if and only if H verifies the condition $(**)$.*

2.7. REMARK. If H is a infinite quasihypergroup then H/β^* is not necessarily a quasigroup. This follows from the example below.

2.8. EXAMPLE. Let G be a groupoid and $(A_x)_{x \in G}$ be a family of nonempty disjoint sets. On $H = \bigcup_{x \in G} A_x$ we define a hyperoperation by $a \cdot b = A_{xy}$, where $a \in A_x, b \in A_y, x \in G$ and $y \in G$. Then $(H, \cdot)$ is a hypergroupoid for which $\beta^* = \beta = \beta_2$ and $H/\beta^* \simeq G$.

If we take $G = (\mathbb{N}, *)$, where $x * y = |x - y|$, by the above construction we obtain a quasihypergroup for which $H/\beta^* \simeq (\mathbb{N}, *)$ is not a quasigroup.

3. CLASSES OF QUASIHYPERGROUUPS IN WHICH THE RELATION β IS TRANSITIVE

In this section, we present some of the most important classes of quasihypergroups in which we can prove that the relation β is transitive. We use Theorems 2.3 and 2.4, established in the previous section.

3.1. THE CASE OF HYPERGROUPS

3.1.1. PROPOSITION. *Let H be a hypergroup.*

- i) *If $a \in H$ and $Q \in P_n(H)$, where $n \in \mathbb{N}^*$, then there exists $Q' \in P_n(H)$ such that $Q \subset Q'a$.*
- ii) *If Q and Q' are two elements of $P(H)$ such that*

$Q \cap Q' \neq \emptyset$ and $a \in Q'$ then $(Q \cup \{a\}) \subset Q''$, for some $Q'' \in P(H)$.

iii) $\mathcal{C}(x) = \mathcal{C}_1(x)$, for any $x \in H$.

PROOF :

i) Consider $x_1, \dots, x_n$ in H such that $Q = x_1 \dots x_n$. As $Ha = H$ there exists $x'_n \in H$ such that $x_n \in x'_n a$. Then, taking $Q' = x_1 \dots x_{n-1} x'_n$ we get $Q \subset Q' a$.

ii) Let $b \in Q \cap Q'$. Then, because of the reproductibility, then exists $c \in H$ such that $a \in bc$. According to i) we have $Q \subset Q_1 a$, for some $Q_1 \in P(H)$. Therefore $Q \subset Q_1 a \subset Q_1 bc \subset Q_1 Q' c$ and $a \in bc \subset Qc \subset Q_1 ac \subset Q_1 Q' c$. Hence for $Q'' = Q_1 Q' c$ we obtain that $(Q \cup \{a\}) \subset Q''$ and $Q'' \in P(H)$.

iii) It suffices to show that $\mathcal{C}_1(x)$ is a complete part of H , for every $x \in H$. Let $Q \in P(H)$ such that $Q \cap \mathcal{C}_1(x) \neq \emptyset$. According to the definition of $\mathcal{C}_1(x)$ there exists $Q' \in P(H)$ such that $x \in Q'$ and $Q \cap Q' \neq \emptyset$. From ii), we get $(Q \cup \{x\}) \subset Q''$ for some $Q'' \in P(H)$, where $Q \subset Q'' \subset \mathcal{C}_1(x)$. ■

Using this above proposition and Theorem 2.3 we have :

3.1.2. PROPOSITION. *The relation β is transitive in any hypergroup.* ■

3.1.3. REMARK. Another proof for this previous result have been given by D. Freni in [4] using the notion of heart of a hypergroup.

We present now an interesting consequence of Proposition 3.1.1 for finite hypergroups.

3.1.4. PROPOSITION. *Let H be a finite hypergroup and Q, Q' be in $P(H)$ such that $Q \cap Q' \neq \emptyset$. Then there exists $Q'' \in P(H)$ such that $Q \cup Q' \subset Q''$.* ■

3.2. THE CASE OF QUASIHYPERGROUPOIDS WITH IDENTITY

Recall that an element e of a hypergroupoid H is an identity if $x \in (ex \cap xe)$, for every $x \in H$.

We get the following result.

3.2.1. LEMMA. *Let H be a quasihypergroup in which there exists at least one identity e . Then $C(e) = C_1(e)$.*

PROOF :

It suffices to show that $C_1(e)$ is a complete part of H . Let $Q \in P(H)$ such that $Q \cap C_1(e) \neq \emptyset$. Hence there exists $Q' \in P(H)$ such that $e \in Q'$ and $Q \cap Q' \neq \emptyset$. Consider $x \in Q \cap Q'$. Then $e \in x'x$, for some $x' \in H$. Taking $Q'' = (x'Q')Q \in P(H)$ we have $e \in x'x \subset (x'e)x \subset (x'Q')Q = Q''$ and $Q \subset eQ \subset (x'x)Q \subset (x'Q')Q = Q''$. Hence $Q \subset C_1(e)$. ■

Using the previous Lemma and Theorem 2.4 we get :

3.2.2. THEOREM. *Let H be a quasihypergroup such that H/β^* is a quasigroup. If H contains at least one identity then the relation β is transitive in H .* ■

Here are now two remarkable consequences of Theorem 3.2.2.

3.2.3. PROPOSITION. *Let H be a weakly associative quasihypergroup (i.e., according to [10], a H_V -group). If H has at least one identity then the relation β is transitive in H .*

PROOF :

It follows from Theorem 3.2.2 and because H/β^* is a group whenever H is a H_V -group. ■

We mention that another proof of Proposition 3.2.3 has been given by D.Freni in Corollary 2.4 of [5].

3.3. THE CASE OF QUASIHYPERGROUPOIDS CONTAINING SINGLE ELEMENTS

An element x of a hypergroupoid H is *single* if $\mathcal{C}(x) = \{x\}$ (see [12]). The following result holds.

3.3.1. THEOREM. *Let H be a quasihypergroup such that H/β^* is a quasigroup. If there exists $x \in H$ such that $\mathcal{C}(x) \in P(H)$ then $\beta = \beta^*$.*

PROOF :

If $\mathcal{C}(x) = Q \in P(H)$ then $\mathcal{C}_1(x) = Q = \mathcal{C}(x)$ and, according to Theorem 2.4, $\beta = \beta^*$. ■

An immediate consequence of Theorem 3.3.1 is :

3.3.2. PROPOSITION. *Let H be a quasihypergroup such that H/β^* is a quasigroup. If H contains a single element then β is transitive.* ■

A similar result for H_V -groups has been obtained by T. Vougiouklis (see [12], Theorem 1.3.3).

3.4. THE CASE OF QUASIHYPERGROUPTS HAVING UNDERLYING GROUPTS

Let $(H, \cdot)$ be a quasihypergroup which has an underlying group $(H, \cdot)$, that is $x \cdot y \subset xy$, for every x and y in H . Then $(H, \cdot)$ is a weakly associative quasihypergroup, also called a H_b -group (see T. Vougiouklis, [12]).

In the following we show how to determine the relation β^* for this kind of quasihypergroups. Notice that according to Theorem 3.2.2 we have $\beta^* = \beta$.

Consider the set $A = \cup\{(x \cdot y)^{-1}(xy) \mid x, y \in H\}$ and let N be the normal closure of A (i.e. N is the least normal divisor of the group $(H, \cdot)$ containing the set A).

Consider the canonical projections $q : H \rightarrow H/N$, respectively $\pi : H \rightarrow H/\beta^*$.

Then $q(xy) = \{t \cdot N \mid t \in xy\}$ and $q(x) \cdot q(y) = (x \cdot y) \cdot N$, whence $q(xy) = q(x) \cdot q(y)$, for every x and y in H . Therefore the heart ω_H of H is included in N . On the other hand, as $1 = x^{-1} \cdot x \subset x^{-1}x$, for every $x \in H$, it follows that $\pi(x^{-1}) = \pi(x)^{-1}$. Using this equality we have that $x^{-1} \cdot y \in \omega_H$ and also $a^{-1} \cdot x \cdot a \in \omega_H$, whenever x and y are in ω_H and $a \in H$. Hence ω_H is a normal subgroup of H . More than that, as $\pi((xy)^{-1}(xy)) = \pi(1)$, it follows that $A \subset \omega_H$. Therefore $N \subset \omega_H$. Hence $\omega_H = N$. It results that H/β^* coincides with H/N . We get that $x\beta^*y$ if and only if $x\beta y$, that is, if and only if $xN = yN$.

A direct consequence of these remarks are the following results.

3.4.1. THEOREM. *Let $(H, \cdot)$ be a quasihypergroup having an underlying group $(H, \cdot)$. Then the heart ω_H of $(H, \cdot)$ is the normal closure of the set $\cup\{(x \cdot y)^{-1}(xy) \mid x, y \in H\}$ and the relations β and β^* coincide in $(H, \cdot)$. ■*

3.4.2. PROPOSITION. *Let H be a H_V -group having only one proper hyperproduct. Then the relations β and β^* coincide in H .*

PROOF :

According to [8], H is either a hypergroup or a H_b -group. Hence $\beta = \beta^*$. ■

We mention that in [12] some particular case of Proposition 3.4.1 are studied (see Examples 1.2.3, 1.2.4 and 1.2.5).

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FUZZY SUB- F-POLYGROUPS

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چکیده: در این مقاله مفاهیم زیر F -پلی گروه فازی و روابط هم‌ارزی فازی یک F -پلی گروه تعریف شده‌اند و قضایایی اثبات گردیده‌اند.

Abstrace: The concepts of scalars, fuzzy sub-F-polygroups and fuzzy congruence relations of an F-polygroup are defined and some theorems are proved.

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1. PRELIMINARIES

A fuzzy subset [3] of a non-empty set A is a function from A to $[0, 1]$. Throughout this paper I is the unit interval $[0, 1] \subseteq \mathbb{R}$ and I^A is the set of all fuzzy subsets of A . If $\mu \in I^A$, then by $\text{supp}(\mu)$ we mean the set $\{x \in A \mid \mu(x) \neq 0\}$. Let $\mu, \eta \in I^A$. Then $\mu \leq \eta$ iff $\mu(x) \leq \eta(x)$, For all $x \in A$.

Definition 1.1. Let μ, η and $\mu_\alpha \in I^A$ where α is in the index set Λ . We define the fuzzy subsets $\mu \cap \eta, \mu \cup \eta, \bigcap_{\alpha \in \Lambda} \mu_\alpha$ and $\bigcup_{\alpha \in \Lambda} \mu_\alpha$ as follows:

- (i) $(\mu \cap \eta)(x) = \min\{\mu(x), \eta(x)\},$
- (ii) $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\},$
- (iii) $(\bigcap_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf_{\alpha \in \Lambda} \mu_\alpha(x),$
- (iv) $(\bigcup_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup_{\alpha \in \Lambda} \mu_\alpha(x),$ for all $x \in A$.

Definition 1.2. Let $a \in A, t \in I$. They by a fuzzy point a_t of A we mean the fuzzy subset of A given as:

$$a_t(x) = \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.3 [5] Let $A \neq \emptyset$ and $I_*^A = I^A \setminus \{o\}$, where o is the function which is identically 0. Then

(i) by an F-hyperoperation $*$ on A we mean a function from $A \times A$ to I_*^A ; in other words for any $a, b \in A, a * b$ is a non-empty fuzzy subset of A .

(ii) if $\mu, \eta \in I_*^A$, then $\mu * \eta \in I_*^A$ is defined by:

$$\mu * \eta = \bigcup_{x \in \text{supp}(\mu), y \in \text{supp}(\eta)} x * y.$$

Notation 1.4. Let $\mu \in I_*^A, B, C \in P^*(A)$ and $a \in A$. Then

- (i) $a * \mu$ and $\mu * a$ denote $\chi_{\{a\}} * \mu$ and $\mu * \chi_{\{a\}}$ respectively,
- (ii) $a * B, B * a, \mu * B, B * \mu$ and $B * C$ denote $\chi_{\{a\}} * \chi_B, \chi_B * \chi_{\{a\}}, \mu * \chi_B, \chi_B * \mu$ and $\chi_B * \chi_C$ respectively.

Definition 1.5 [5]. Let $\mathcal{F}$ be a non-empty set and “ $*$ ” an F -hyperoperation on $\mathcal{F}$. Then $(\mathcal{F}, *)$ is called an F -polygroup iff

- (i) $(x * y) * z = x * (y * z), \forall x, y, z \in \mathcal{F}$,
- (ii) there exists an element $e_F \in \mathcal{F}$ such that $x \in \text{supp}(x * e_F \cap e_F * x), \forall x \in \mathcal{F}$ (In this case we say e_F is an F -identity element of $\mathcal{F}$.),
- (iii) for each $x \in \mathcal{F}$, there exists a unique element $x' \in \mathcal{F}$ such that $e_F \in \text{supp}(x * x' \cap x' * x)$, (x' is called the F -inverse of x and is denoted by x_F^{-1} .),
- (iv) $z \in \text{supp}(x * y) \Rightarrow x \in \text{supp}(z * y^{-1}) \Rightarrow y \in \text{supp}(x^{-1} * z), \forall x, y, z \in \mathcal{F}$. (This property is called the F -reversibility of $\mathcal{F}$ with respect to $*$.)

When there is no ambiguity, for simplicity of notation we use e and x^{-1} instead of e_F and x_F^{-1} respectively.

If $\mathcal{F}$ is an F -polygroup and $x * y = y * x, \forall x, y \in \mathcal{F}$, then $\mathcal{F}$ is said to be Abelian.

Henceforth $\mathcal{F}$ will denote an F -polygroup with hyperoperation “ $*$ ” and e will denote the F -identity of $\mathcal{F}$.

Definition 1.6 [5]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then H is called an F -subpolygroup iff

- (i) if $x \in H$, then $x^{-1} \in H$;
- (ii) $\text{supp}(x * y) \subseteq H, \forall x, y \in H$.

In this case we write: $H <_{F-P} \mathcal{F}$.

Note that condition (ii) of the above Definition is equivalent to $x * y \leq \chi_H, \forall x, y \in H$.

Lemma 1.7 [5]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then $H <_{F-P} \mathcal{F}$ if and only if $\text{supp}(x * y^{-1}) \subseteq H, \forall x, y \in H$.

Definition 1.8 [5]. Let $H <_{F-P} \mathcal{F}$. Then

- (i) H is said to be weak normal in $\mathcal{F}(H \triangleleft_{F-P}^w \mathcal{F})$ iff $x * H * x^{-1} \leq \chi_H, \forall x \in \mathcal{F}$,
- (ii) H is said to be normal in $\mathcal{F}(H \triangleleft_{F-P} \mathcal{F})$ iff $x * H * x^{-1} = \chi_H, \forall x \in \mathcal{F}$.

Notation: Let $H \triangleleft_{F-P} \mathcal{F}$. Then $\mathcal{F}/H = \{x * H : x \in \mathcal{F}\}$.

Theorem 1.9 [6]. Let $H \triangleleft_{F-P} \mathcal{F}$. Define the F -hyperoperation " $\square$ "

on $\mathcal{F}/H$ as follows:

$$\begin{aligned} \square : \mathcal{F}/H \times \mathcal{F}/H &\longrightarrow I_{*}^{\mathcal{F}/H} \\ (x * H, y * H) &\longmapsto x * H \square y * H \end{aligned}$$

where

$$(x * H \square y * H)(z * H) = (x * y * H)(z), \quad \forall z * H \in \mathcal{F}/H.$$

Then $(\mathcal{F}/H, \square)$ is an F -polygroup called the quotient F -polygroup.

Definition 1.10. A fuzzy binary relation R on a set X (i.e. $R \in I^{X \times X}$) is said to be a fuzzy similarity relation if it satisfies for all $x, y, z \in X$:

- (S1) reflexivity: $R(x, x) = 1$;
- (S2) symmetry: $R(x, y) = R(y, x)$;
- (S3) transitivity: $\min\{R(x, y), R(y, z)\} \leq R(x, z)$.

2. MAIN RESULTS

Definition 2.1. Let $\mu \in I^{\mathcal{F}}$. Then μ is a fuzzy sub- F -polygroup of $\mathcal{F}$ iff

- (i) $\mu(z) \geq \min\{\mu(x), \mu(y)\}, \forall z \in \text{supp}(x * y), \forall x, y \in \mathcal{F}$.
- (ii) $\mu(x^{-1}) \geq \mu(x), \forall x \in \mathcal{F}$.

Note that this definition is a generalization of Definition 4.1 of [4]. Clearly (ii) implies that $\mu(x^{-1}) = \mu(x), \forall x \in \mathcal{F}$ and (i) implies that $\mu(e) \geq \mu(x), \forall x \in \mathcal{F}$.

Definition 2.2. Let ξ be a binary fuzzy relation between two F -polygroups $\mathcal{F}, \mathcal{F}'$ (i.e. $\xi \in I^{\mathcal{F} \times \mathcal{F}'}$). Then ξ is called an FP-relation iff:

- (i) $(e, e') \in \text{supp}(\xi)$, where e and e' are the identity elements of $\mathcal{F}$ and $\mathcal{F}'$ respectively;

- (ii) $\xi(a, b) \leq \xi(a^{-1}, b^{-1}) \forall a, b \in \mathcal{F}$,
 (iii) $\min\{\xi(a, b), \xi(c, d)\} \leq \xi(x, y), \forall x \in \text{supp}(a * c), y \in \text{supp}(b * d)$, for all $(a, b), (c, d) \in \mathcal{F} \times \mathcal{F}'$.

Clearly (ii) implies that $\xi(a, b) = \xi(a^{-1}, b^{-1}), \forall a, b \in \mathcal{F}$.

Theorem. 2.3. If ξ is an FP-relation between $\mathcal{F}, \mathcal{F}'$ and if $K <_{F-P} \mathcal{F}'$, then the subset H of $\mathcal{F}$ which is defined as follows:

$$H = \{x \in \mathcal{F} : (x, y) \in \text{supp}(\xi), \text{ for some } y \in K\}$$

is an F-subpolygroup of $\mathcal{F}$.

Proof. Since $(e, e') \in \text{supp}(\xi)$, so $e \in H$. Now let $x \in H$. Then there is $y \in K$ such that $\xi(x, y) > 0$. Since $\xi(x, y) \leq \xi(x^{-1}, y^{-1})$ and $y^{-1} \in K$, we get $x^{-1} \in H$. At present let $x_1, x_2 \in H$ and $t \in \text{supp}(x_1 * x_2)$. Thus $(x_1, y_1) \& (x_2, y_2) \in \text{supp}(\xi)$, for some $y_1, y_2 \in K$. Thus

$$0 < \min\{\xi(x_1, y_1), \xi(x_2, y_2)\} \leq \xi(t, w)$$

where $w \in \text{supp}(y_1 * y_2) \subseteq K$. Hence $t \in H$.

Theorem 2.4. If ξ is a reflexive FP-relation on an F-polygroup $\mathcal{F}$, then ξ is a fuzzy similarity relation on $\mathcal{F}$.

Proof. First we prove the condition S_2 of Definition 1.10. Let $(a, b) \in \mathcal{F} \times \mathcal{F}$. Then

$$\begin{aligned} & \xi(a, b) \\ &= \min\{\xi(a, b), \xi(b^{-1}, b^{-1})\}, \text{ by reflexivity of } \xi \\ &\leq \xi(t, e), \quad \forall t \in \text{supp}(a * b^{-1}), \text{ since } e \in \text{supp}(b * b^{-1}) \\ &\leq \xi(t^{-1}, e), \quad \forall t \in \text{supp}(a * b^{-1}) \\ &= \min\{\xi(t^{-1}, e), \xi(a, a)\}, \text{ by reflexivity of } \xi \\ &\leq \xi(b, a), \text{ since } b \in \text{supp}(t^{-1} * a). \end{aligned}$$

Similarly $\xi(b, a) \leq \xi(a, b)$. Hence ξ is symmetric. Now we prove the condition S_3 of Definition 1.10. For any $a, b, d \in \mathcal{F}$ we

have:

$$\begin{aligned}\min\{\xi(a, b), \xi(b, d)\} &\leq \xi(b, d) \\ &= \min\{\xi(b, d), \xi(b^{-1}, b^{-1})\} \\ &\leq \xi(e, w), \quad \forall w \in \text{supp}(d * b^{-1}).\end{aligned}$$

Since, $\min\{\xi(a, b), \xi(b, d)\} \leq \xi(a, b)$. Therefore

$$\begin{aligned}\min\{\xi(a, b), \xi(b, d)\} &\leq \min\{\xi(e, w), \xi(a, b)\} \\ &\leq \xi(a, d), \text{ since } d \in \text{supp}(w * b).\end{aligned}$$

Thus ξ is transitive.

Definition 2.5. An FP-relation on an F-polygroup $\mathcal{F}$ which is also a fuzzy similarity relation is called a fuzzy congruence relation on $\mathcal{F}$.

Definition 2.6. Let ξ be a fuzzy congruence relation on $\mathcal{F}$, then the fuzzy subset $\xi < e >$ of $\mathcal{F}$ is defined by $\xi < e > (x) = \xi(e, x)$, for all $x \in \mathcal{F}$.

Theorem 2.7. If ξ is a fuzzy congruence relation on an F-polygroup $\mathcal{F}$, then the fuzzy subset $\xi < e >$ is a fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. Let $x, y \in \mathcal{F}$. Then $\min\{\xi < e > (x), \xi < e > (y)\} = \min\{\xi(e, x), \xi(e, y)\} \leq \xi(e, z)$, $\forall z \in \text{supp}(x * y)$.

Also we have $\xi < e > (x) = \xi(e, x) \leq \xi(e, x^{-1}) = \xi < e > (x^{-1})$. Therefore $\xi < e >$ is a fuzzy sub-F-polygroup of $\mathcal{F}$.

Definition 2.8. Let μ be a fuzzy sub-F-polygroup of $\mathcal{F}$. Then μ is said to be normal iff for all $x, y \in \mathcal{F}$

$$\mu(z) = \mu(z'), \quad \forall z \in \text{supp}(x * y), \quad \forall z' \in \text{supp}(y * x).$$

Note that the above definition generalizes Definition 2.3 of [4].

Remark 2.9. It is obvious that if μ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$, then $\mu(z) = \mu(z'), \forall z, z' \in \text{supp}(x * y), \forall x, y \in \mathcal{F}$.

Theorem 2.10. Let μ be a fuzzy sub-F-polygroup of $\mathcal{F}$. Then the following condition are equivalent:

- (i) μ is normal.
- (ii) For all $x, y \in \mathcal{F}, \mu(z) = \mu(y), \forall z \in \text{supp}(x * y * x^{-1})$
- (iii) For all $x, y \in \mathcal{F}, \mu(z) \geq \mu(y), \forall z \in \text{supp}(x * y * x^{-1})$
- (iv) For all $x, y \in \mathcal{F}, \mu(z) = \mu(y), \forall z \in \text{supp}(x^{-1} * y^{-1} * x * y)$.

Proof. The proof is similar to the proof of Theorem 2.5 of [4].

Corollary 2.11. If ξ is a fuzzy congruence relation on $\mathcal{F}$, then $\xi < e > \in I^{\mathcal{F}}$ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. By Theorem 2.7, $\xi < e >$ is a fuzzy sub-F-polygroup of $\mathcal{F}$. Now let $x, y \in \mathcal{F}$ and $z \in \text{supp}(x * y * x^{-1})$. Then $z \in \text{supp}(t * x^{-1})$, for some $t \in \text{supp}(x * y)$. Hence we have:

$$\begin{aligned}
 \xi < e > (z) &= \xi(e, z) \\
 &\geq \min\{\xi(x, t), \xi(x^{-1}, x^{-1})\}, \\
 &= \xi(x, t), \text{ since } \xi \text{ is reflexive} \\
 &\geq \min\{\xi(x, x), \xi(e, y)\}, \text{ since } t \in \text{supp}(x * y) \\
 &= \xi(e, y), \text{ since } \xi \text{ is reflexive} \\
 &= \xi < e > (y).
 \end{aligned}$$

Therefore by Theorem 2.10 (iii), $\xi < e >$ is normal.

Corollary 2.12. $H \triangleleft_{F-P}^w \mathcal{F}$ if and only if χ_H is a normal fuzzy sub-F-polygroup of $\mathcal{F}$.

Proof. The proof follows from Definition 2.1, Theorem 2.10 (iii).

Corollary 2.13. Let $H \triangleleft_{F-P} \mathcal{F}$. Then $x * H * x^{-1}$ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$, for all $x \in \mathcal{F}$.

Proof. It is obvious.

Remark 2.14. It is well-known that in the fuzzy group theory, every fuzzy subgroup of an Abelian group is normal. But the following example shows that this is not true in the case of F-polygroups. At first we have the following theorem:

Theorem 2.15 [6]. Let (A, o) be a polygroup [1]. Then $(A, *)$ is an F-polygroup where $x * y = \chi_{xoy}$, $\forall x, y \in A$.

Example 2.16. If $(H = \{e, a, b, c\}, .)$ is Klein's four-group, then (H, o) is a polygroup (see [2]) where the hyperoperation "o" is defined as follows:

$$\begin{aligned}xoy &= \{x, y, x.y\}, \text{ if } x \neq y^{-1}, \quad x, y \neq e, \\xox^{-1} &= x^{-1}ox = H, \text{ if } x \neq e, \\xoe &= eox = \{x\} \text{ for every } x \in H.\end{aligned}$$

Now let "*" be the F-hyperoperation induced by "o" (i.e. $x * y = \chi_{xoy}, \forall x, y \in H$). Then by Theorem 2.15 $(H, *)$ is an F-polygroup. Let $\beta, \gamma \in [0, 1]$ such that $\beta < \gamma$. Define a fuzzy subset μ of H as follows:

$$\mu(a) = \mu(b) = \mu(c) = \beta, \mu(e) = \gamma.$$

Then it is easy to see that μ is a fuzzy sub-F-polygroup of $(H, *)$. But since $e \in \text{supp}(a * a)$, $a \in \text{supp}(a * a)$ and $\mu(e) \neq \mu(a)$, we get μ is not normal.

Theorem 2.17. Let $\mathcal{F}$ be an Abelian F-polygroup and μ a normal fuzzy sub-F-polygroup of $\mathcal{F}$ such that $\mu(e) = 1$. Define $\xi \in I^{\mathcal{F} \times \mathcal{F}}$ as follows:

$$\xi(x, y) = \mu(z), \text{ for some arbitrary element } z \in \text{supp}(x * y^{-1})$$

Then ξ is a fuzzy congruence relation and $\mu = \xi < e >$.

Proof. By Remark 2.9, ξ is well-defined. Clearly $\xi(e, e) > 0$. Now for all $(x, y) \in \mathcal{F} \times \mathcal{F}$ we have:

$$\xi(x, y) = \mu(w), \quad w \in \text{supp}(x * y^{-1})$$

$$\begin{aligned}
&= \mu(w^{-1}), w^{-1} \in \text{supp}(y * x^{-1}) \\
&= \text{supp}(x^{-1} * y), \text{ by commutativity of } * \\
&= \xi(x^{-1}, y^{-1}).
\end{aligned}$$

Now we show that

$$\begin{aligned}
\min\{\xi(a_1, b_1), \xi(a_2, b_2)\} &\leq \xi(x, y), \forall x \in \text{supp}(a_1 * a_2) \\
&\quad, \forall y \in \text{supp}(b_1 * b_2).
\end{aligned}$$

Let $x \in \text{supp}(a_1 * a_2)$, $y \in \text{supp}(b_1 * b_2)$ and $t \in \text{supp}(x * y^{-1})$ be arbitrary. Then we have: $t \in \text{supp}(x * y^{-1}) \subseteq \text{supp}(a_1 * b_1^{-1} * a_2 * b_2^{-1})$. Thus $t \in \text{supp}(s * w)$, for some $s \in \text{supp}(a_1 * b_1^{-1})$ and $w \in \text{supp}(a_2 * b_2^{-1})$. Therefore we get

$$\begin{aligned}
\xi(x, y) &= \mu(t), \text{ by definition of } \xi \\
&\geq \min\{\mu(s), \mu(w)\}, \text{ since } \mu \text{ is a fuzzy sub-F-polygroup} \\
&= \min\{\xi(a_1, b_1), \xi(a_2, b_2)\}.
\end{aligned}$$

Consequently ξ is an FP-relation. Since $\xi(x, x) = \mu(e) = 1$, then ξ is reflexive. Hence by Theorem 2.4 and Definition 2.5, ξ is a fuzzy congruence relation.

Since $\mu(x) = \mu(x^{-1})$, $\forall x \in \mathcal{F}$, we get that $\mu = \xi < e >$.

Corollary 2.18. Let μ be a fuzzy subset of an Abelian F-polygroup, $\mathcal{F}$. Then μ is a normal fuzzy sub-F-polygroup of $\mathcal{F}$ and $\mu(e) = 1$ if and only if there exists a congruence relation ξ on $\mathcal{F}$ such that $\mu = \xi < e >$.

Proof. The proof follows from Theorem 2.17, and Corollary 2.11.

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M-POLYSYMMETRICAL HYPERRINGS

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ABSTRACT - In this paper a special case of canonical polysymmetrical hyperrings, the M-Polysymmetrical hyperrings, are studied and investigated in detail. Certain methods of their construction from rings are presented and a new class of hyperstructures, the M-Polysymmetrical superrings, is introduced.

1. INTRODUCTION

Mittas, the first one of the co-authors, in his paper [6] (in the French Academy of Sciences) has introduced two new classes of hypercompositional structures along with some of their fundamental properties. These classes are special cases of canonical polysymmetrical hypergroups and hyperrings (MITTAS [9], [10]). Yatras, the second co-author of this paper, has studied the first of the above classes. He has named the hypercompositional structures that appear in it, *M-Polysymmetrical hypergroups* and he has already presented the first results in [11]. Going on with his study, he has also presented results on the subhypergroups of the M-polysymmetrical hypergroup (YATRAS [12]) and on the homomorphisms of the M-polysymmetrical hypergroup (YATRAS [13]). Here we give the initial results that we have reached during our study of the hypercompositional structures of the second of the above classes. Likewise the M-Polysymmetrical hypergroups, we have given them the name *M-Polysymmetrical hyperrings*.

We give the relevant definitions:

Definition 1.1

A set H is called a *M-polysymmetrical hypergroup* (M-P-H) if it is endowed with a hyperoperation $x+y$ that satisfies the following axioms:

1. $(x+y)+z = x+(y+z)$ for every $x, y, z \in H$
2. $x+y = y+x$ for every $x, y \in H$
3. $(\exists 0 \in H) (\forall x \in H) [x \in 0+x]$
4. $(\forall x \in H) (\exists x' \in H) [x+x' = 0]$

(x' is an **opposite** or **symmetrical** of x , with regard to the considered 0, and the set of the opposites $S(x) = \{x' \in H : x+x' = 0\}$ is the **symmetrical** set of x).

5. For every $x, y, z \in H$, $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$ we have: $z \in x+y \Rightarrow z' \in x'+y'$.

We remind that in such a hypergroup, when x runs in H the sets $C(x) = 0+x$ form a partition of H , which is notified by $\text{mod}(0)$, or simply (0) and for which we have $x \equiv y (0) \Leftrightarrow 0+x = 0+y \Leftrightarrow C(x) = C(y)$. Also, for every $x \in H$, $x' \in S(x)$ we have $S(x) = C(x')$ and the set of classes, $H/(0) = G(H)$ is an abelian group (MITTAS [6], YATRAS [11]).

Definition 1.2

A set A , endowed with a hyperoperation $x+y$ (addition) and an operation xy (multiplication), is called *M-polysymmetrical hyperring* (M-P.HR) (and with abuse of definition, hyperring, since it is not such in the sense of the papers KRASNER [2], MASSOYROS [3], MITTAS [5], [7]) if it satisfies the axioms:

- I. The hyperstructure $(A, +)$ is a M-P.H.
- II. The structure $(A, \cdot)$ is a semigroup.
- III. The multiplication is bilaterally distributive over addition i.e.
 $(x+y)z = xz+yz, z(x+y) = zx+zy$ for every $x, y, z \in A$.

As an example of those structures we give the one that appears in MITTAS [6], which is also the one that motivated their definition.

Example 1.1

Let $(K, +, \cdot)$ be a commutative algebraically closed field with characteristic p . Also let $n \neq 1$ be a coprime to p number and E_n be the multiplicative group of the n -th roots of the unity of K . We define the following hyperoperation in K :

$$x \dot{+} y = \{z \in K : x^n + y^n = z^n\}$$

The $x \dot{+} y$ is a class modulo E_n in K and it holds that:

$$x \dot{+} y = (xE_n) \dot{+} (yE_n)$$

It can easily be proved that K , endowed with the above hyperoperation is a M-P.H. This M-P.H combined with the multiplication in K gives the hyperstructure $(K, \dot{+}, \cdot)$ which is a M-P.HR and especially a M-polysymmetrical hyperfield as it will appear in the following.

We proceed now to a general study of the M-P.HR.

2. GENERALITIES

We can easily notice that the neutral element 0 of the additive hypergroup, is a *bilaterally absorbing* element of the multiplicative semigroup, that is:

$$a0 = 0a = 0 \quad \text{for every } a \in A$$

Remarks 2.1

- a) Obviously we have $A \neq \emptyset$.
- b) Every ring is a M-P.HR. A M-P.HR, which is not a ring, is called **proper**.
- c) If there exists a scalar element $s \in A$, then M-P.H $(A, +)$ is an abelian group (YATRAS [11]) and thus A is a ring.
- d) If $W = \{x \in A : 0+x = x\}$, then W is a multiplicatively permissible subset (DUBREIL-JACOTIN [1]) of A . Since $x \in W \Rightarrow 0+x = x$ for every $a \in A$ we have:

$$a(0+x) = a0+ax = 0+ax = ax$$

thus $ax \in W$, i.e. $AW \subseteq W$. In the same way $WA \subseteq W$.

In the following, we will accept that all the terms which have been defined in the theory of the rings and which are dependent only on the multiplicative structure e.g. divisors, (units in the sense that every element which has a reverse with regard to the multiplicative unity element), divisors of zero, associate elements, unitary rings, commutative rings, integral rings etc. will have the same meaning for the M-P.HR, as it is for the hyperrings MITTAS [5], [7]. The only exception is the term «domain» (integral), that will be substituted with the term «hyperdomain». As a result in the case under consideration it will be named **M-polysymmetrical hyperdomain** (M-P.HD), that is, an integral (i.e. a commutative for which the cancellation law for multiplication holds) and unitary M-P.HR. Another special form of M-P.HR is the **M-polysymmetrical hyperfield** (M-P.HF), that is, a M-P.HR whose set $A^* = A - \{0\}$ is a group with regard to the multiplication. We also denote that, obviously, all the properties of the rings which are dependent only on the multiplication and which are being proved without the intervention of the addition, also hold in the theory of M-P.HR (as well as in the theory of the hyperrings MASSOUROS - MITTAS [4], MITTAS [7]).

Remark 2.2

In the ring or (canonical) hyperring theory, satisfying the cancellation law, is equivalent to not having divisors of zero. However, the same does not hold for the M-P.HR, where obviously satisfying the cancellation law does not imply that there exist divisors of zero ($x \neq 0$ and $xy = 0 \Rightarrow xy = x0 \Rightarrow y = 0$, thus x is a non-divisor of zero element), while for the converse we generally have that if $x \in A^*$ a non-divisor of zero (e.g. from the left) then $xy = xz \Rightarrow xy + xz' = xz + xz' \Rightarrow x(y+z') = x(z+z') = 0 \Rightarrow 0 \in y+z' \Rightarrow y+z' = 0 \Rightarrow y+z+z' = 0+z \Rightarrow 0+y = 0+z \Rightarrow z=y \text{ mod}(0)$.

And so the proposition:

Proposition 2.1

If A is a M-P.HR and $x \in A^$ is a non-divisor of zero (from one side for instance) then*

$$xy = xz \Rightarrow z=y \text{ mod}(0)$$

Examples 2.1

1. The set resulting from the set C of the complex numbers after the subtraction of the non zero real numbers and the pure imaginary numbers i.e. $A = C - (R^* \cup iR^*)$, endowed with the hyperaddition

$$x \dot{+} y = \begin{cases} \{z \in A : \operatorname{Re} z = \operatorname{Re} x + \operatorname{Re} y, \operatorname{Im} z \in R^*\}, & \text{if } \operatorname{Re} x + \operatorname{Re} y \neq 0 \\ 0 & , \text{ if } \operatorname{Re} x + \operatorname{Re} y = 0 \end{cases}$$

and with multiplication

$$x \circ y = \text{Re}x \cdot \text{Re}y + i \cdot \text{Im}x \cdot \text{Im}y$$

is a M-P.HR. This can be proved with the verification of the axioms. We remark that, if $z \in C$, $\text{Re}z = a$, $a \in R^*$ then the class $C(z) \bmod(0)$ is

$$C(z) = X_a = \{a+iy, y \in R^*\}$$

(i.e. it is a line parallel to the imaginary axis), while for the symmetrical of z it is $S(z) = \{-a+iy, y \in R^*\}$. Obviously for $a, x, y \in A$, $a \neq 0$, $a \circ x = a \circ y \Rightarrow x = y$, i.e. the cancellation law is valid and as a result the M-P.HR $(A, +, \circ)$ has no divisors of zero. The number $e = 1+i$ is the unitary element. Also, assuming that the commutative law of the multiplication is valid, then the M-P.HR $(A, +, \circ)$ is a M-P.HD and moreover a commutative one. Obviously, for every $z = x+iy \in A^*$ the $w = \frac{1}{x} + \frac{i}{y}$ is the multiplicative inverse of z

$$(z \circ w = 1+i).$$

2. It is obvious that, if we consider the Im instead of Re and $\text{Re}z \in R^*$ instead of $\text{Im}z \in R^*$ in the above definition of the hyperoperation, then the deriving hyperstructure is also a commutative M-P.HF.

3. We easily conclude that the previous examples can give more M-P.HRs which become M-P.HDs if instead of C , we consider its subset E of the Gaussian integers and instead of R , the set Z of integers. In other words, we consider the set $B = E - (Z^* \cup i Z^*)$, enriched with similar hyperoperations and operation to the ones that we have previously used in A .

4. If we equip the set G of an abelian group $(G, +)$ with the multiplication $xy=0$ for every $x, y \in G$, then it becomes a ring, the *zero ring*. Likewise, in the case of the hyperstructures if we equip a M-P.H $(H, +)$ with the zero multiplication as above, then it becomes a M-P.HR, the *zero M-P.HR*. Obviously in this M-P.HR every $x \in H^*$ is a zero divisor [see proposition 2.14].

In the following we will see others M-P.HRs which are neither M-P.HDs, and thus nor M-P.HFs. Based on the above information regarding the M-P.HF we have the following propositions which are presented without proof, since their proof depends only on the multiplication.

Proposition 2.2

Every commutative M-P.HF is a M-P.HD.

Proposition 2.3

Every integral M-P.HR having a finite number of elements is a M-P.HF.

Corollary 2.1

Every M-P.HD having a finite number of elements is a M-P.HF.

Regarding the M-P.HR we initially give the following lemma and theorem from MITTAS [6].

Lemma 2.1

If $x, y \in A$, then:

- i) $xC(y) = C(x)y = C(xy) = C(x_1y_1)$
- ii) $C(x)C(y) = C(xy) = C(x_1y_1)$

for every $x_1 \in C(x), y_1 \in C(y)$.

Theorem 2.1

The set $A/(0) = \{C(x), x \in A\}$ of equivalent classes $\text{mod}(0)$, is a ring with respect to the addition $C(x)+C(y) (=x+y)$ and the multiplication $C(x)C(y) (=C(xy))$, which is called *ring of reduction* or simply *ring* of M-P.HR $(A, +, \cdot)$, symbolized $R(A)$.

This theorem, combined with the study that has been presented in MITTAS [6], YATRAS [11] concerning the relation of the M-P.H to the abelian groups, shows that, generally, the M-P.HRs are closely related to the rings.

Obviously, from the relation $C(x)C(y) = C(x_1y_1)$, for every $x, y \in A, x_1 \in C(x), y_1 \in C(y)$ derives that the equivalence $\text{mod}(0)$ in A is compatible to the multiplication and consequently a *normal* equivalence relation in M-P.HR $(A, +, \cdot)$. According to the relevant theory (MITTAS [7]) the quotient set $R(A) = A/(0)$ is a hyperring, which, as it is mentioned in the above theorem, is a ring in the case of the M-P.HR.

More specifically, we have the relevant to the above theorem proposition:

Proposition 2.4

If the M-P.HR $(A, +, \cdot)$ is M-P.HF, then the set $A/(0)$ of the equivalent classes $\text{mod}(0)$ is a field called *field of reduction* or simply *field* of M-P.HF $(A, +, \cdot)$ symbolized $F(A)$.

Taking into consideration the previous theorem the proof becomes easy. Obviously the unity element of $F(A)$ is the class $C(1)$. Every class $C(x)$ has its reverse $C(x^{-1})$ (1 symbolizes the multiplication unity of A where there is no danger of confusion).

Next we present the following propositions showing the relation of the various M-P.HRs $(A, +, \cdot)$ to the corresponding resulting rings of reduction $A/(0)$ without their proofs for reasons of simplicity and keeping this text short.

Proposition 2.5

If A is a commutative M-P.HR, then $A/(0)$ is also a commutative ring.

Conversely, we have the proposition:

Proposition 2.6

If the ring of reduction $A/(0)$ of M-P.HR $(A, +, \cdot)$ is commutative then:

$$xy \equiv yx \pmod{(0)} \text{ for every } x, y \in A$$

and conversely.

Proposition 2.7

If the cancellation law for multiplication holds in A then it holds in $A/(0)$ as well.

Proposition 2.8

If A is a unitary M-P.HR, then $A/(0)$ is a unitary ring.

Proposition 2.9

If the M-P.HR A has zero divisors, then the ring $A/(0)$ has also zero divisors and reversely. Thus if one of them has no zero divisors, neither has the other.

Therefore we have the following proposition based on the above ones:

Proposition 2.10

If A an integral M-P.HR, then $A/(0)$ is an integral ring, and thus, if A is a M-P.HD, then $A/(0)$ is an integral domain.

Based on lemma 2.1 we are led to the following proposition which is significant, especially for the construction of the M-P.HRs from the rings. Such constructions are presented in the following.

Proposition 2.11

If A is a M-P.HR for which the cancellation law for multiplication holds (especially, if A is an integral, or a M-P.HD or even a M-P.HF) then the

classes $\text{mod}(0)$ have the same cardinality i.e. $|C(x)| = |C(y)|$ for every $x, y \in A^*$.

Proof

Obviously $xy \neq 0$. According to lemma 2.1 we have

$$C(x)y = xC(y)$$

From the above relation derives that for every $x_1 \in C(x)$ there is $y_1 \in C(y)$. So $x_1y = xy_1$, and because of the cancellation law, there exists only one y_1 like this. So, for every choice of x, y from their classes, a mapping is defined:

$$\varphi_{xy} : C(x) \longrightarrow C(y)$$

which is bijective as it can easily be concluded. And so the proposition.

Thus the classes $\text{mod}(0)$ in example 1.1 have the same finite cardinality n , while in the 1st and 2nd of examples 2.1, they have the cardinality of the continuum. Also in the 3rd of examples 2.1 those classes have the cardinality of the countable. Moreover in the 4th (of examples 2.1), where the condition of the above proposition does not hold, there exist zero M-P.HRs with classes having the same cardinality and also others with a different one.

Remark 2.3

Generally if A is an arbitrary M-P.HR and if for two elements $x, y \in A^*$ (for every $x_1, x_2 \in C(x)$, $y_1, y_2 \in C(y)$) we have:

$$xy_1 = xy_2 \Rightarrow y_1 = y_2 \text{ and } x_1y = x_2y \Rightarrow x_1 = x_2,$$

then $|C(x)| = |C(y)|$.

Next we give the following properties of M-P.HRs omitting the proofs:

Properties 2.1

First we introduce the definition of a multiplication of an integer n with an element $x \in H$ (where H is a M-PH) as follows:

$$n \cdot x = \begin{cases} x + x + \dots + x & n \text{ times for } n > 0, n \neq 1 \\ 0 & \text{for } n = 0 \\ x' + x' + \dots + x' & n \text{ times for } n < 0, n \neq -1 \end{cases}$$

for $n=1$ we define $1 \cdot x = 0 + x$

for $n=-1$ we define $(-1) \cdot x = 0 + x'$, $x' \in S(x)$ arbitrary, (YATRAS [11]).

So we have

1st. i) For every $x, y, z, t \in A$

$$(x+y)(z+t) = xz + yz + xt + yt$$

ii) For every $m, n \in \mathbb{Z}$ and $x \in A$

$$(m \cdot x)(n \cdot x) = (mn)x^2$$

And continuing we have:

2nd. $x'y \in S(xy)$, $xy' \in S(xy)$ for every $x, y \in A$ and thus $x'y \equiv xy' \pmod{0}$.

3rd. For every $x, y \in A$ we have:

$$S(xy) = xS(y) = S(x)y$$

4th. For every $x \in A$ and $n \in \mathbb{N}^*$ we have:

$$S^n(x) = \begin{cases} C^n(x) & \text{if } n \text{ even} \\ C^{n-1}(x)S(x) & \text{if } n \text{ odd} \end{cases}$$

5th. For every $x \in A$ and $n \in \mathbb{N}^*$ is valid that:

$$C^n(x) = C(x^n)$$

6th. For every $x, y, z \in A$ we have:

$$C(x)(y+z) = C(x)C(y) + C(x)C(z)$$

and

$$(y+z) \cdot C(x) = C(y)C(x) + C(z)C(x)$$

7th. For every $x, y, z \in A$ we have:

$$xC(y+z) = xC(y) + xC(z)$$

and

$$C(y+z)x = C(y)x + C(z)x$$

Remarks 2.4

- If the M-P.HR $(A, +, \cdot)$ is unitary and 1 is its unitary element, then 1 is unique (and obviously $1 \neq 0$ for $A \neq \{0\}$).
- If the M-P.HR $(A, +, \cdot)$ is unitary, then the set U of its *units* (i.e. of its elements which are inverse with regard to 1), is obviously a group and $U \subseteq A^*$, as in the case of rings.
- If the M-P.HR $(A, +, \cdot)$ is commutative, then the binomial formula for $(x+y)^n$ holds, i.e.:

$$(x+y)^n = x^n + nx^{n-1}y + \dots + \binom{n}{k}x^{n-k}y^k + \dots + y^n \text{ for every } x, y \in A$$

The binomial formula generally holds for every M-P.HR for x, y commutative elements. In addition since $z \in x+y \Rightarrow x+y = 0+z$, we have:

$$(x+y)^n = (0+z)^n = 0+z^n$$

which also is obviously valid in the case of the non commutative M-P.HR.

d) If $(A, +, \cdot)$ M-P.HF then:

- $xA^* = A^*x = A^*$ for every $x \in A^*$
- $A^* = U$ and conversely (if A unitary M-P.HR and $A^* = U$, then A M-P.HF).
- for $a \neq 0$ each one of the equations $ax=b$, $xa=b$ has one and only one solution in A and conversely (that is, if A a unitary M-P.HR, $A \neq \{0\}$ and the equations $ax=b$, $xa=b$ have one and only one solution in A each, then A is M-P.HF).

- e) If A unitary M-P.HR and $1 \in W$, then A is a ring. (Indeed, since $AW \subseteq W$, $WA \subseteq W$ [remark 2.1,d] and $1 \in W$, then $A=W$ and since $0+x=x$ for every $x \in W$, the structure $(A,+)$ is a group and thus $(A,+,\cdot)$ is a ring).

Let $R(A) = A/(0)$ be the ring of reduction of A and let $(\tilde{G}, \dot{+})$ be a group of choice of the additive hypergroup $(A,+)$ of M-P.HR $(A,+,\cdot)$ (YATRAS [11]). We define in $\tilde{G}$ the following operation starting from the group of reduction G of $(A,+)$ and through the already known bijective mapping (YATRAS [11])

$$f : G \longrightarrow \tilde{G} \text{ with } f(C) = x_c \in \tilde{G}$$

(where as it is known the x_c is arbitrarily chosen in every class mod(0) of $(A,+)$ element),

$$x \circ y = f[C(x)C(y)] = f[C(xy)]$$

for every $x,y \in A$ and thus for the mapping f we also have

$$f[C(x)C(y)] = f[C(x)] \circ f[C(y)]$$

for every $x,y \in A$.

This operation, in combination with the addition

$$x \dot{+} y = f[C(x)+C(y)]$$

in $\tilde{G}$, endows $\tilde{G}$ with the structure of a ring.

Indeed:

- a) $(x \circ y) \circ z = [f(C(x)C(y))] \circ f(C(z)) = [f(C(xy))] \circ f(C(z)) = f(C(xy)C(z)) = f(C(xyz)) = f(C(x)C(yz)) = f(C(x)) \circ [f(C(yz))] = f(C(x)) \circ [f(C(y)C(z))] = x \circ (y \circ z)$
for every $x,y,z \in \tilde{G}$.

- b) $x \circ (y \dot{+} z) = f(C(x)) \circ f(C(y)+C(z)) = f(C(x)) \circ f(C(y+z)) = f(C(x)C(y+z)) = f(C(x)C(y)+C(x)C(z)) = f(C(xy)+C(xz)) = f(C(xy)) \dot{+} f(C(xz)) = x \circ y \dot{+} x \circ z$
for every $x,y,z \in \tilde{G}$ and from the property 2.1, 6th.

In the same way we have $(y \dot{+} z) \circ x = y \circ x \dot{+} z \circ x$ for every $x,y,z \in \tilde{G}$

So, as a result we have for the M-P.HR the following proposition which is analogous to the proposition of the M-P.H.

Proposition 2.12

In every M-P.HR $(A, +, \cdot)$ there exists a subset $\tilde{R}$ of A which has the structure of a ring, isomorphic to the ring of reduction $A/(0)$.

We call the ring $(\tilde{R}, +, \cdot)$, **ring of choice** of $(A, +, \cdot)$.

Example 2.2

In the M-P.HF $(C, +, \cdot)$ of example 1.1 (where G is the set of complex numbers) the set $F(C) = C/(0)$ with elements $C(x) = 0+x = x\epsilon_n$, $x \in C$, equipped with the operations

$$C(x) + C(y) = (x\epsilon_n) + (y\epsilon_n)$$

$$C(x)C(y) = (x\epsilon_n)(y\epsilon_n) = xy\epsilon_n$$

for every $x, y \in C$ and $C(x), C(y) \in C/(0)$ is obviously its field of reduction. Also if:

i) we consider as a group of choice $\tilde{R}$ of the additive hypergroup $(C, +)$ the one which derives as follows: we take as a distinct element of every class $C(x)$, the element $x_c \in x\epsilon_n$ which has a principal value of argument the minimum of the principal values of the arguments of the numbers $z \in x\epsilon_n$.

ii) Through the bijective mapping

$$f : x\epsilon_n = C(x) \longrightarrow x_c$$

we define the operation

$$x \circ y = f[C(x)C(y)] = f[C(xy)]$$

for every $x, y \in C$.

iii) we consider the addition of the group $\tilde{R}$

$$x \dot{+} y = f[C(x) + C(y)]$$

for every $x, y \in C$.

Then $(\tilde{R}, +, \cdot)$ becomes a field (field of choice) which is isomorphic to the field of reduction $F(C)$.

Remarks 2.5

- Obviously we will have more rings of choice, for more choices of elements from classes $\text{mod}(0)$ of A as distinct. All of these are isomorphic to each other since they are also isomorphic to the ring of choice.
- Every ring of choice $\tilde{R}$ is obviously a proper subset of A when A is a proper M-P.HR.

- c) More rings with the same operations as the ones mentioned can result from one M-P.HR if we consider the mapping $g : R(A) \longrightarrow A$ bijective in A , instead of the mapping f , without necessarily having $g(C) \in C$ for $C \in R(A)$.

Now we will deal with the construction of the M-P.HRs from one ring. As it has been indicated in MITTAS [6] this is a more complex problem than the corresponding one of the construction of the M-P.Hs with a given abelian group as a group of reduction, as it derives from proposition 2.3 of YATRAS [11], and from the consequence 3 of theorem 1 of §1 of MITTAS [6]. Indeed, we start from a set E with the structure of a multiplicative semigroup which has an absorbing element and which also has as a subset a ring $(A, +, \cdot)$, such that:

- i) Its multiplication being the restriction of the corresponding of the semigroup $(E, \cdot)$ in A .
 ii) The zero (0) of A being the absorbing element of the semigroup $(E, \cdot)$.

Apart from these conditions we assume that:

- iii) There is a partition R of E with the property:

$$x C_R(y) = C_R(x) y = C_R(xy) \text{ for every } x, y \in E$$

- iv) There is also a bijective mapping f of the quotient set E/R on A such that:

$$f^{-1}(x) = C_R(x) \text{ for every } x \in A.$$

[where $C_R(x)$ is the class of $E \bmod (R)$ that contains the element x].

- v) $C_R(0) = \{0\}$

We also consider the hyperoperation:

$$x \oplus y = f^{-1}[f(C_R(x)) + f(C_R(y))] \text{ for every } x, y \in E$$

defined in E through the additive group $(A, +)$ of the ring A , and the operation:

$$x \circ y = xy \text{ for every } x, y \in E$$

As it is already known (YATRAS [11]) the hyperoperation $x \oplus y$ makes E a M-P.H whose group of reduction $E/(0)$ coincides with E/R . Also the operation $x \circ y$ is associative in E since $(E, \cdot)$ is a semigroup. Furthermore for every $x, y, z \in E$ we have:

$$\begin{aligned} x \circ (y \oplus z) &= x(y \oplus z) = x(y_1 \oplus z_1) = x f^{-1}(y_1 + z_1) = x C_R(y_1 + z_1) = \\ &= x C_R(w_1) = C_R(x w_1) = C_R(x) w_1 = C_R(x_1) w_1 = C_R(x_1 w_1) = \\ &= f^{-1}(x_1 w_1) = f^{-1}[x_1(y_1 + z_1)] = f^{-1}(x_1 y_1 + x_1 z_1) = C_R(x_1 y_1 + x_1 z_1) = \\ &= x_1 y_1 \oplus x_1 z_1 = C_R(x_1 y_1) \oplus C_R(x_1 z_1) = x_1 C_R(y_1) \oplus x_1 C_R(z_1) = \\ &= x_1 C_R(y) \oplus x_1 C_R(z) = C_R(x_1) y \oplus C_R(x_1) z = C_R(x) y \oplus C_R(x) z = \\ &= C_R(xy) \oplus C_R(xz) = xy \oplus yz = x \circ y \oplus y \circ z \end{aligned}$$

and in the same manner $(y \oplus z) \circ x = y \circ x \oplus z \circ x$.

In other words the multiplication is bilaterally distributive over the hyperoperation.

Consequently the hyperstructure $(E, \oplus, \circ)$ is a M-P.HR.

Thus we have the proposition:

Proposition 2.13

Let E be a set with the structure of a multiplicative semigroup that has an absorbing element, having as a subset, a ring $(A, +, \cdot)$, whose multiplication is the restriction of the corresponding one of the semigroup $(E, \cdot)$ in A and the zero (0) of A is the absorbing element of the semigroup $(E, \cdot)$. Then if

i) *there is a partition R of E having the property*

$$xC_R(y) = C_R(x)y = C_R(xy) \text{ for every } x, y \in E$$

ii) *there is a bijective mapping of the quotient set E/R on A such that for every $x \in A$*

$$f^{-1}(x) = C_R(x)$$

[where $C_R(x)$ is the class of $E \bmod(R)$ that contains the element x]

and

$$\text{iii) } C_R(0) = \{0\}$$

the hyperoperation $x \oplus y = f^{-1}[f(C_R(x)) + f(C_R(y))]$ defined on E through the group $(A, +)$ of the ring and the operation $x \circ y = xy$ (that is the operation of the semigroup and the ring) make E a M-P.HR whose ring of reduction $E/(0)$ coincides with E/R .

More specifically if E is an arbitrary set and A its subset with the structure of a zero ring (i.e. a ring every element of which is its zero divisor, that is for every $x, y \in A$ and $x \neq 0, y \neq 0$ $xy=0$ is valid) and R is a partition of E and f is a bijective mapping of the quotient set E/R on A such that for every $x \in A$, $f^{-1}(x) = C_R(x)$ and $C_R(0) = \{0\}$, then the previous hyperoperation $x \oplus y$, and the operation $x \circ y = f(C_R(x)) \cdot f(C_R(y)) = x_1 y_1$ for every $x, y \in E$ with $x_1, y_1 \in A$ make E a M-P.HR.

Indeed

$$\begin{aligned} x \circ (y \oplus z) &= x \circ f^{-1}(y_1 + z_1) = x \circ f^{-1}(w_1) = x \circ C_R(w_1) = \\ &= x_1 w_1 = x_1 (y_1 + z_1) = x_1 y_1 + x_1 z_1 = 0 = C_R(0) = C_R(x_1 y_1 + x_1 z_1) = \\ &= f^{-1}(x_1 y_1 + x_1 z_1) = x_1 y_1 \oplus x_1 z_1 = x \circ y \oplus x \circ z \end{aligned}$$

and in the same way $(y \oplus z) \circ x = y \circ x \oplus z \circ x$ for every $x, y, z \in E$.

Additionally for every $x, y, x^*, y^* \in E$ we have:

$$\begin{aligned} x \oplus y &= x \oplus C_R(y) = C_R(x) \oplus y = C_R(x) \oplus C_R(y) \\ x \circ y &= x \circ C_R(y) = C_R(y) \circ x = C_R(x) \circ C_R(y) \end{aligned}$$

$$x \oplus y = x * \oplus y^*, x \circ y = x * \circ y^* \quad \text{with } x \equiv x^* \pmod{R} \text{ and } y \equiv y^* \pmod{R}.$$

Consequently we have:

Proposition 2.14

Every zero ring can give a M-P.HR, which is called zero M-P.HR.

Remark 2.6

From the above it derives that we can construct all the M-P.HRs $(A, +, \cdot)$ whose ring of reduction is isomorphic to a given ring $(\Delta, +, \cdot)$. We consider arbitrary semigroups $(A, *)$ with absorbing element $0 \in A$ and surjective mappings $\varphi : A \longrightarrow \Delta$ such that, if we set

$$\varphi^{-1}(\varphi(x)) = C_R(x) \quad \text{for every } x, y \in A$$

such that

$$\text{i) } \varphi^{-1}(\varphi(0)) = C_R(0) = \{0\} \quad (\text{singleton, symbolized again } 0)$$

$$\text{ii) } \varphi(x * y) = \varphi(x)\varphi(y)$$

and

$$\text{iii) } x * C_R(y) = C_R(x) * y = C_R(x * y)$$

Starting from every such mapping we consider the hyperoperation $x \dot{+} y = \varphi^{-1}(\varphi(x) + \varphi(y))$ and the multiplication $x * y$. Under the above hyperoperation and multiplication the structure $(A, \dot{+}, *)$ is a M-P.HR with ring of reduction isomorphic to $(\Delta, +, \cdot)$. As it is already known (MITTAS [6], YATRAS [11]) the structure $(A, \dot{+})$ is a M-P.H and furthermore it holds:

$$\begin{aligned} x * (y \dot{+} z) &= x * \varphi^{-1}(\varphi(y) + \varphi(z)) = x * \varphi^{-1}(\varphi(w)) = \\ &= x * C_R(w) = C_R(x * w) = \varphi^{-1}(\varphi(x * w)) = \\ &= \varphi^{-1}(\varphi(x) \cdot \varphi(w)) = \varphi^{-1}[\varphi(x) \cdot (\varphi(y) + \varphi(z))] = \varphi^{-1}(\varphi(x)\varphi(y) + \varphi(x)\varphi(z)) = \\ &= \varphi^{-1}(\varphi(x * y) + \varphi(x * z)) = x * y \dot{+} x * z. \end{aligned}$$

Similarly for $(y \dot{+} z) * x$.

Hence the structure $(A, \dot{+}, *)$ is a M-P.HR with the ring $(\Delta, +, \cdot)$ as a ring of reduction.

Examples 2.3

1. Having as a starting point the example 1 of YATRAS [11] we consider the set of the rational numbers Q and the partition $O \cup X$ where $O = \{0\}$ and $X = Q^*$. Then the set $H = \{0\} \cup Q^* = Q$ equipped with the hyperoperation $x \dot{+} y$ of the

previously mentioned example, that is $0+0=0$, $0+x = x+0 = Q^*$, $x+y = y+x = 0$ for every $x,y \in Q^*$, and multiplication the usual multiplication of the rational numbers is a M-P.HF.

The set $F(Q) = \{C(0), C(x)\} = \{O, X\}$ is the field of reduction, with addition:

$$O+O=O, \quad O+X=X+O=X, \quad X+X=O$$

and multiplication

$$OX= XO=O, \quad XX=X$$

$F(Q)$ is obviously isomorphic to the field of integers mod 2. The set $\tilde{R} = \{0, q\}$ with q any arbitrary element of Q^* , is the field of choice that is defined through the mapping:

$$f: H \longrightarrow \tilde{R} \quad \text{with } f(0) = 0 \text{ and } f(x) = q, \quad x \in Q^*$$

Generally we can consider X to be an arbitrary multiplicative group and as a result we have the construction of all the M-P.HFs that arise from a multiplicative group whose field of reduction is isomorphic to the field of integers mod 2.

2. Having as a starting point example 2 of YATRAS [11] we take again the set of rational numbers Q and we define in it the partition $O \cup X \cup Y$, where $O = \{0\}$, $X = Q_+^*$ and $Y = Q_-^*$. Then the set $H = \{0\} \cup Q_+^* \cup Q_-^* = Q$ equipped with the hyperoperation $x+y$ of the above mentioned example and operation the usual multiplication of the rational numbers becomes a M-P.HF. The field of reduction is the set $F(Q) = \{C(0), C(x), C(y)\} = \{O, X, Y\}$ with addition

$$O+O=O, \quad O+X=X+O=X, \quad X+Y=Y+X=O, \quad X+X=Y, \quad Y+Y=X$$

and multiplication

$$OX= XO=O, \quad OY= YO=O, \quad XX=X, \quad YY=X, \quad XY=YX=Y$$

which is obviously isomorphic to the field of integers mod 3.

The set $\tilde{R} = \{0, p, q\}$, with $p \in Q_+^*$ and $q \in Q_-^*$ arbitrarily chosen, is the field of choice that is defined through the mapping.

$$f: H \longrightarrow \tilde{R} \quad \text{with } f(0)=0, \quad f(x)=p, \quad x \in Q_+^* \quad \text{and } f(y)=q, \quad y \in Q_-^*$$

We reach an analogous M-P.HF if we consider the set R of the real numbers and the partition $\{0\} \cup R_+^* \cup R_-^*$. More generally we can consider an arbitrary multiplicative group and its partition $X \cup Y$. In this way we have the construction of all the M-P.HFs which derive from a multiplicative group and whose field of reduction is isomorphic to the field of integers mod 3.

3. Based on the example of YATRAS [12] we consider the M-P.H $(H, \dot{+})$

with $H = \bigcup_{i \in \mathbb{Z}} X^i$ which appears in it and which is defined through the parti-

tion R of H into classes $X^i, i \in \mathbb{Z}$ (the set of integers) with $X^0 = \{x_a^0\} = \{0\}$ and with hyperoperation

$$x_a^i \dot{+} x_b^j = X^{i+j}$$

Next the set of classes $H/R = H/(0) = \{X^i, i \in \mathbb{Z}\}$ becomes a semigroup isomorphic to $\mathbb{Z}$ based on the isomorphism $\mathbb{Z} \ni i \longrightarrow X^i \in H/(0)$ with multiplication

$$X^i X^j = X^{ij}$$

Consequently the structure $(H/(0), +, \cdot)$ becomes a ring isomorphic to $(\mathbb{Z}, +, \cdot)$. Now, in order to make the M-P.H $(H, \dot{+})$ a M-P.HR we define an associate multiplication xy in H taking into consideration that since the ring $(H/(0), +, \cdot)$ is an integral domain, it does not have divisors of zero with regard to this multiplication. If we further assume that it also verifies the cancellation law according to proposition 2.11, we have that the classes X^i ($i \neq 0$) have the same cardinality. Moreover applying lemma 2.1 in class $X^1 = C(1)$ which contains the unity element $1 \in \mathbb{Z}$, we have that $x C(1) = C(1) \cdot 1 = C(1) C(1) = C(1)$, that is $x X^1 = X^1$ for every $x \in X^1$ and consequently we have that the class X^1 is a group.

Furthermore denoting the elements of H as x_a^i with $x_a^i \in X^i$ and $a \in I$, where I is a set of indexes, which as it easily results from the consideration of $x_a^i \in X^1$, has the structure of a group we have for the multiplication of H the following:

$$x_a^i x_b^j = x_{ab}^{ij}$$

It derives that $1 = x_1^1 \in X^1$ is the unitary element of H under this multiplication (because $1 \cdot x_a^i = x_1^1 x_a^i = x_a^i$). The hypercompositional structure $(H, \dot{+}, \cdot)$ defined in the above mentioned way, is an integral M-P.HR and more precisely a M-P.HD. We can easily see that what is mentioned in remark 2.6 is satisfied through the surjective mapping

$$\varphi : H \longrightarrow H/R \text{ with } \varphi(x_a^i) = X^i.$$

As a partial case we can consider the classes X^i with cardinality 2, that is $X^i = \{x_1^i, x_2^i\}$ with the set of indexes a group isomorphic to the group $\{\bar{1}, \bar{2}\}$ of the classes $(\neq \bar{0})$ of the integers mod 3.

4. Obviously the above construction of M-P.HR can also be applied if instead of $\mathbb{Z}$ we consider any integral ring. Thus we construct M-P.HRs that are also M-P.HFs, if we consider as ring Δ the field of classes of integers mod(p) with $p \in \mathbb{N}$ (the set of naturals) prime number. E.g. for $p=5$, $\Delta = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, so $H = X^0 \cup X^1 \cup X^2 \cup X^3 \cup X^4$ and e.g. $X^i = \{x_1^i, x_2^i\}$ or more generally $X^i = \{x_1^i, \dots, x_{q-1}^i\}$ where $I = \{1, \dots, q-1\}$ isomorphic to the multiplicative group of the classes of integers mod(q) with $q \in \mathbb{N}$ prime number.

5. Finally we give one more example in which the M-P.HR that results according to the above way, does not have classes mod(0) with the same cardinality. Indeed let the ring $\Delta = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ of classes mod 4 and $H = X^0 \cup \dots \cup X^3$. Here in the ring $H/(0)$ which is isomorphic to Δ , we obviously have $X^2 X^2 = X^0$ and thus for every $x_a^2 x_b^2 \in X^2$ we will have $x_a^2 x_b^2 = 0$. Trying to construct M-P.HRs with classes that have different cardinalities, we consider (due to simplicity): $X^1 = \{x_1^1, x_2^1\} = \{1, a\}$, $X^2 = \{x_1^2\} = \{b\}$, $X^3 = \{x_1^3, x_2^3\} = \{c, d\}$ (while $X^0 = \{x_a^0\} = \{0\}$) and the set of indexes $I = \{1, 2\}$ for the multiplication to be a group isomorphic to the group $\{\bar{1}, \bar{2}\}$ mod 3. The equality of certain products of elements is necessary for the validity of lemma 2.1, e.g. in order to be

$$x_a^1 X^2 = X^1 x_1^2$$

it must be $x_a^1 X^2 = x_1^1 x_1^2 = x_2^1 x_1^2 = x_1^2 = x_2^2$

and $X^1 x_1^2 = \{x_1^1, x_2^1\} x_1^2 = \{x_1^2, x_2^2\}$

since it is also $X^1 X^2 = X^2$

So we get the following table, after having checked all the cases of products in H :

.	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	1	b	d	c
b	0	b	b	0	b	b
c	0	c	d	b	1	a
d	0	d	c	b	a	1

and the corresponding one for the addition, with the use of the same symbolism for the elements is:

+	0	1	a	b	c	d
0	0	{1,a}	{1,a}	b	{c,d}	{c,d}
1	{1,a}	b	b	{c,d}	0	0
a	{1,a}	b	b	{c,d}	0	0
b	b	{c,d}	{c,d}	0	{1,a}	{1,a}
c	{c,d}	0	0	{1,a}	b	b
d	{c,d}	0	0	{1,a}	b	b

As it can be seen from the above tables the cancellation law for the multiplication does not hold in this M-P.HR.

Remark 2.7

According to the above examples 4 and 5, we can generalize the method and proceed to the construction of all the M-P.HRs with ring of reduction isomorphic to the ring of classes of integers mod(p), with the same cardinality when p is not prime. However, in this case, since the ring of reduction does have zero divisors, the M-P.HRs which have this one as ring of reduction do not satisfy the cancellation law. This fact proves that the fundamental proposition 2.11 provides only a sufficient condition for the same cardinality of the classes mod(0) of the M-P.HR.

Lastly, if we consider the multiplication as a hyperoperation in a M-P.HRs we are led to a new polysymmetrical hyperstructure analogous to the superrings (MITTAS [8]). So, based on the introductory example of the hyperpolynomials (i.e. polynomials with coefficients from a hyperring), which appears in MITTAS [8] we can consider in an analogous way the **M-hyperpolynomials**, which due to the polysymmetrical of their addition, give a polysymmetrical M-hyperstructure.

So we have the definition:

Definition 2.2

We name **M-polysymmetrical superring** (M-P.SR) a set S equipped with two hyperoperations $x+y$ - addition - and xy - multiplication - which verify the following conditions:

- I. S is a M-P.H under the addition (called **additive** hypergroup of the M-P.SR).
- II. S is a semihypergroup under the multiplication (called **multiplicative** semihypergroup of the M-P.SR) the 0 of which is a bilateral absorbing element.

$$\begin{array}{lll} \text{i.e.:} & \text{i)} & xy \subseteq S \\ & \text{ii)} & (xy)z = x(yz) \\ & \text{iii)} & x0 = 0x = 0 \end{array} \quad \text{for every } x,y,z \in S$$

- III. The multiplication is bilaterally distributive over addition i.e.
 $(x+y)z = xz+yz$, $z(x+y) = zx+zy$ for every $x,y,z \in S$

Remark 2.8

As in the case of rings and of M-P.HRs we can define various kinds of M-P.SRs such as commutative M-P.SRs, unitary M-P.SRs etc. Another special type of M-P.SRs is the **M-polysymmetrical superfield** (M-P.SF), that is a M-P.SR whose set $S^* = S - \{0\}$ is a multiplicative hypergroup.

It is obvious that by starting with the M-P.SRs we are led to various interesting hyperstructures, as it happens for instance with the consideration of the multiplicative semihypergroup S , as a M-P.H under the multiplication.

Regarding the M-P.SRs we set the following proposition which is analogous to 2.13 and the theorem that follows, while their special study will be the subject of a different paper.

Proposition 2.15

Let E be a set and A its subset with the structure of a ring without divisors of zero. Also, let 0 to be its neutral element and let for every $x \in A$, $-x$ to be its opposite. Then if:

- i) *there exists a partition R of E and a bijective mapping of the quotient set E/R on A such that $x \in A$, $f^{-1}(x) = C_R(x)$ for every $x \in A$ and*
- ii) $C_R(0) = \{0\}$

then the hyperoperations $x \oplus y = f^{-1}[f(C_R(x)) + f(C_R(y))]$ and $x \circ y = f^{-1}[f(C_R(x)) \cdot f(C_R(y))]$ defined in E through the ring A make E a M-P.SR.

Proof

Firstly the hyperoperation $x \oplus y$ makes E a M-P.H (YATRAS [11]). Next we notice that for every $x, y \in E$ the hyperproduct $x \circ y$ is a class mod(R) of E . We also have that

$$x \circ y = x \circ z \Leftrightarrow y \equiv z \text{ mod}(R)$$

Indeed, if we set $f(C_R(x)) = x_1$ etc. we have

$$x \circ y = x \circ z \Leftrightarrow f^{-1}(x_1 y_1) = f^{-1}(x_1 z_1) \Leftrightarrow x_1 y_1 = x_1 z_1 \Leftrightarrow y_1 = z_1 \Leftrightarrow C_R(y) = C_R(z) \Leftrightarrow y \equiv z \text{ mod}(R)$$

Also, we obviously have:

$$x \circ y = x^* \circ y^* \text{ for every } x^* \in C_R(x), y^* \in C_R(y)$$

Proceeding with the proof we see that the associative law for the multiplication is valid. Indeed if we set $x \circ y = C_R(w)$ we have:

$$f[(x \circ y) \circ z] = f(C_R(w) \circ z) = \{f(w^* \circ z), w^* \in C_R(w)\} = f(w \circ z) = \\ f(C_R(w)) \cdot f(C_R(z)) = w_1 z_1 = (x_1 y_1) z_1 \\ [\text{since } f(x \circ y) = f(C_R(w)) = w_1 = x_1 y_1 = f(C_R(x)) \cdot f(C_R(y))].$$

But in A $(x_1 y_1) z_1 = x_1 (y_1 z_1)$. Thus $f[(x \circ y) \circ z] = f[x \circ (y \circ z)]$ and consequently the associate law holds.

Also

$$x \circ 0 = f^{-1}[f(C_R(x)) \cdot f(C_R(0))] = f^{-1}(x_1 \cdot 0) = f^{-1}(0) = C_R(0) = 0$$

In the same way $0 \circ x = 0$.

Yet, if we set $y \oplus z = C_R(w)$ we have:

$$f[x \circ (y \oplus z)] = f[x \circ C_R(w)] = \{f(x \circ w^*), w^* \in C_R(w)\} = f(x \circ w) = \\ f(C_R(x)) \cdot f(C_R(w)) = x_1 w_1 = x_1 (y_1 + z_1)$$

$$[\text{since } f(y \oplus z) = f(C_R(w)) = w_1 = y_1 + z_1 = f(C_R(y)) + f(C_R(z))].$$

However, in the set A we have $x_1(y_1 + z_1) = x_1 y_1 + x_1 z_1$ and so

$$x_1 y_1 + x_1 z_1 = f(C_R(x)) f(C_R(y)) + f(C_R(x)) f(C_R(z)) = \\ = f(x \circ y) + f(x \circ z) = f(x \circ y \oplus x \circ z).$$

Thus

$$f[x \circ (y \oplus z)] = f[x \circ y \oplus x \circ z] \text{ and since } f \text{ is a bijective mapping it holds} \\ x \circ (y \oplus z) = x \circ y \oplus x \circ z$$

In the same way we have $(y \oplus z) \circ x = y \circ x \oplus z \circ x$.

Consequently $(E, \oplus, \circ)$ is a M-P.SR.

Lastly, we can have the following theorem, which derives in an analogous way to the corresponding one 3.1 of YATRAS [11]:

Theorem 2.2

Let $(A, +, \cdot)$ be a ring without divisors of zero. If we correspond bijectively to A a family $\tilde{A}$ of disjoint sets, from which the corresponding to the neutral element of A is a singleton, and if we make this family, (through the properly defined operations $X \oplus Y, X \circ Y$) a ring isomorphic to A , then the set

$$S = \bigcup_{X \in \tilde{A}} X \text{ with hyperoperations } x + y = X \oplus Y \text{ and } x * y = X \circ Y \text{ for every}$$

$x, y \in S, (x, y) \in X \times Y, X, Y \in \tilde{A}$ is a M-P.SR.

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OVER THE CONSTRUCTION OF AN HYPERSTRUCTURE OF QUOTIENTS FOR A MULTIPLICATIVE HYPERRING

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Summary - In this paper we construct a weak hyperfield of quotients for a class of multiplicative hyperrings.

First of all we want to recall some algebraic definitions that will be used through the paper.

An hyperring $(A, \oplus, \cdot)$ is a set A with an hyperoperation $\oplus$ and a product $\cdot$ such that the following properties hold:

- i) $\forall a, b, c \in A : a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
- ii) $\forall a, b \in A : a \oplus b = b \oplus a$,
- iii) $\exists 0 \in A / \forall a \in A : 0 \oplus a = a \oplus 0 = a$,
- iv) $\forall a \in A \exists ! a' \in A : a \oplus a' = 0, (a' = -a)$,
- v) $\forall a, b, c \in A / a \in b \oplus c \Rightarrow c \in a - b (c \in a \oplus b')$,
- vi) $\forall a, b, c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- vii) $\forall a, b, c \in A : (a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$,
- viii) $\forall a, b, c \in A : a \cdot (b \oplus c) = a \cdot b \oplus a \cdot c$,
- ix) $\forall a \in A : a \cdot 0 = 0 \cdot a = 0$,

We recall that $(A, \oplus)$ satisfying i), ii), iii), iv) and v) is called canonical hypergroup.

Let us observe ([3]) that axiom v) is equivalent to v)' and also to v)'':

$$v)' \quad \forall a, b \in A : -(a \oplus b) = -a - b,$$

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$$v) \forall a, b, c, d \in A : (a \oplus b) \cap (c \oplus d) \neq \emptyset \Rightarrow (c - a) \cap (b - d) \neq \emptyset.$$

An hyperring A is called hyperfield if $(A^*, \cdot)$ is a group, where $A^* = A \setminus \{0\}$.

A multiplicative hyperring $(A, +, \bullet)$ is an abelian group $(A, +)$ together with an hyperproduct satisfying the following properties:

- i) $\forall a, b, c \in A : a \bullet (b \bullet c) = (a \bullet b) \bullet c$;
- ii) $\forall a, b, c \in A : (a + b) \bullet c \subseteq a \bullet c + b \bullet c$;
- iii) $\forall a, b, c \in A : a \bullet (b + c) \subseteq a \bullet b + a \bullet c$;
- iv) $\forall a, b \in A : (-a) \bullet b = a \bullet (-b) = -(a \bullet b)$.

If a multiplicative hyperring satisfies, instead of properties ii) and iii), the following ones:

- ii)' $\forall a, b, c \in A : (a + b) \bullet c = a \bullet c + b \bullet c$
- iii)' $\forall a, b, c \in A : a \bullet (b + c) = a \bullet b + a \bullet c$

then $(A, +, \bullet)$ is called strongly distributive. Moreover $(A, +, \bullet)$ is strongly left (right) distributive if ii)' (iii)') holds.

In [2] J. Mittas studied the possibility of immersion for an hyperring in an hyperfield and constructed the hyperfield of quotients for a particular hyperring. In this paper we want to solve the analogous problem for a multiplicative hyperring.

Let $(A, +, \bullet)$ be a multiplicative, strongly distributive, commutative hyperring; from now on we will write ab instead of $a \bullet b$ and, for any two sets X and Y , $X \approx Y$ if and only if $X \cap Y \neq \emptyset$. We suppose that the hyperring A satisfies the following properties: i) if $ab \approx 0c$, $a \neq 0 \Rightarrow b = 0$; ii) $\forall X, Y, Z, W \in P^*(A) = P(A) \setminus \{\emptyset\} / XY \approx ZW \Rightarrow (\forall x \in X \text{ and } \forall w \in W \exists y \in Y \text{ and } z \in Z / xy \approx zw)$.

For this structure we can prove the following lemmas:

I.- For any $a, b \in A$ if $0 \in ab$ and $a \neq 0$ then $b = 0$.

Proof. - Since A is strongly distributive $0 \in 0c \forall c \in A$, thus from i) $b = 0$ follows.

II.- For any $X, Y \in P^*(A)$ and for any $d \in A \setminus \{0\}$, $dX \approx dY \Leftrightarrow X \approx Y$.

Proof. - Since $dX = \bigcup dx$, $x \in X$, and $dY = \bigcup dy$, $y \in Y$, thus, from the hypothesis, there exists $z \in dX \cap dY$, that is $z \in dx$ and $z \in dy$, for some $x \in X$

and $y \in Y$; this implies, from the strong distributivity, $0 \in d(x-y) = dx - dy$ and, since $d \neq 0$, $x=y$, thus $X \approx Y$. The inverse implication is obvious.

III.- For any $X, Y \in P^*(A)$ and for any $d_i \in A \setminus \{0\}$, $(X+Y)d_1 \bullet \dots \bullet d_k = (Xd_1 \bullet \dots \bullet d_k) + (Yd_1 \bullet \dots \bullet d_k)$.

Proof. We can prove the lemma for $k=1$; the general case will follow as a consequence of the associative property. Now $(X+Y)d = \{z / z \in (x+y)d, x \in X, y \in Y\} = \{z / z \in xd + yd, x \in X, y \in Y\} = Xd + Yd$.

Denoting by A^* the set $A \setminus \{0\}$ and by H the set $A \times A^* = \{(a,b) / a \in A, b \in A^*\}$, we define in H the following relation $(a,b) \rho (c,d) \Leftrightarrow ad \approx bc$; for this relation the following properties hold:

IV.- $(a,b) \rho (0,d) \Leftrightarrow a=0$.

Proof. - Since $(a,b) \rho (0,d) \Leftrightarrow ad \approx 0b$ then, from i) and $d \neq 0$, $a=0$ follows. Vice-versa, since $0 \in 0d \cap 0b$, then $(0,b) \rho (0,d)$, $\forall b, d \in A^*$.

V.- For any $a, b, c, d, f \in A$, if $ad \approx bc$, then $adf \approx bcf$.

Proof. - If $z \in ad \cap bc$ then $zf \subseteq adf$ and $zf \subseteq bcf$; thus $adf \approx bcf$.

VI.- ρ is an equivalence relation.

Proof. - Reflexivity and symmetry are for ρ immediate consequence of commutativity in A . As for transitivity let $(a,b) \rho (c,d)$ and $(c,d) \rho (e,f)$; then $ad \approx bc$ and $cf \approx de$ that is there exist $z, w \in A$ such that $z \in ad \cap bc$, $w \in cf \cap de$, thus $zw \subseteq adcf \cap bcde$. So $afdc \approx bedc$ and, since $d \neq 0$, from proposition II we obtain $afc \approx bec$; now, if $c \neq 0$, again from proposition II $af \approx be$, while if $c=0$ then, from proposition IV, $a=e=0$. In both cases it results $(a,b) \rho (e,f)$ as requested.

Let now $K = H/\rho$, we want to define in K two commutative hyperoperations $\oplus$ and $\otimes$ in such a way that the following conditions hold:

- 1) $(K, \oplus)$ is a canonical hypergroup;
- 2) $\forall x, y, z \in K : x \otimes (y \otimes z) = (x \otimes y) \otimes z$;
- 3) $\forall x, y, z \in K : x \otimes (y \oplus z) \subseteq (x \otimes y) \oplus (x \otimes z)$;
- 4) $\forall x, y \in K : -(x \otimes y) = (-x) \otimes y = x \otimes (-y)$;
- 5) $\exists 1 \in K / \forall x \in K : x \in x \otimes 1$;
- 6) $\forall x \in K \setminus \{0\} \exists y \in K \setminus \{0\} / 1 \in x \otimes y$.

For any $[(a,b)], [(c,d)] \in K$ let us define $[(a,b)] \oplus [(c,d)] = [(ad+bc, bd)] = \{(s,t) / s \in ad+bc, t \in bd\}$ and $[(a,b)] \otimes [(c,d)] = [(ac, bd)] = \{(s,t) / s \in ac, t \in bd\}$; first

In order to prove the existence of a zero element in $(K, \oplus)$ we first recall that, as a consequence of proposition IV, $[(0, y)] = \{(0, d) / d \in A^*\}$, moreover $[(a, b)] \oplus [(0, y)] = \{[(s, t)] / s \in ay + b0, t \in by\} = [(a, b)]$; in fact, since $(by)a \subseteq \subseteq ayb + b0b$, we have $bya \approx (ay + b0)b$ and, from ii), $\forall s \in ay + b0 \forall t \in by$ it results $sb \approx ta$. This proves that $[(0, y)]$ is a zero for $\oplus$; let us now prove that it is unique. To do this let $[(x, y)] \in K$ such that $[(a, b)] \oplus [(x, y)] = [(a, b)]$, that is $\forall (s, t) \in H$ such that $s \in ay + bx, t \in by$ then $sb \approx ta$. From this it follows that $ayb + bxb \approx bya$, then there exists $z \in (ayb + bxb) \cap bya, z = z' + z'', z' \in ayb$ and $z'' \in bxb, z \in bya$; that is $z'' \in bya - bya$ or $(bya - bya) \cap bxb \neq \emptyset$. Thus $((a - a)y)b \approx \approx (bx)b$ and, from proposition II, $0y \approx bx$ with $b \neq 0$; because of i) $x = 0$ and the uniqueness is proved.

We want now to verify the existence and uniqueness, for any $[(a, b)] \in K$, of an element $[(z, w)] \in K$ such that $[(a, b)] \oplus [(z, w)] = [(0, y)]$. First of all we observe that $[(a, b)] \oplus [(-a, b)] = \{[(s, t)] / s \in ab - ab, t \in bb\}$ and this set trivially contains $[(0, t)]$. As for the uniqueness let $[(z, w)] \in K$ such that $[(0, y)] \in [(a, b)] \oplus \oplus [(z, w)] = \{[(s, t)] / s \in aw + bz, t \in bw\}$; then $0 \in aw + bz$ or $0 = u + v, u \in aw, v \in bz$ that is $v = -u - (aw) = (-a)w$ which implies $(-a)w \approx bz$ and this means $[(z, w)] = [(-a, b)]$.

Finally we must prove condition v) in the definition of canonical hypergroup which is equivalent to the following condition: $-([(a, b)] \oplus [(c, d)]) = (-[(a, b)]) \oplus \oplus (-[(c, d)]) = [(-a, b)] \oplus [(-c, d)]$.

To prove such equality, we first must prove that $-([(a, b)] \oplus [(c, d)]) \subseteq [(-a, b)] \oplus \oplus [(-c, d)]$; to do this let $[(s, t)]$ such that $s \in ad + bc, t \in bd$. Thus $-s \in -(ad + bc) = -ad - bc$ that is $-s = (-z) + (-w), z \in ad, w \in bc$; from this, since $-(ad) = (-a)d$ and $-(bc) = b(-c)$, $[(s, t)] \in [(-a, b)] \oplus [(-c, d)]$ follows. Similarly the inverse inclusion can be proved.

From all that has been proved we obtain the requested result. Moreover we can prove the following proposition:

X.- In $(K, \oplus, \otimes)$ the following hold:

- $\alpha) \forall x, y, z \in K : x \otimes (y \otimes z) = (x \otimes y) \otimes z;$
- $\beta) \exists 1 \in K / \forall x \in K : x \in x \otimes 1;$
- $\gamma) \forall x, y, z \in K : x \otimes (y \oplus z) \subseteq (x \otimes y) \oplus (x \otimes z);$
- $\delta) \forall x, y \in K : -(x \otimes y) = (-x) \otimes y = x \otimes (-y);$
- $\varepsilon) \forall x \in K, x \text{ different from zero}, \exists y \in K, y \text{ different from zero, such that } 1 \in x \otimes y.$

Proof. - $\forall [(a, b)], [(c, d)], [(e, f)] \in K$ we have $([(a, b)] \otimes [(c, d)]) \otimes [(e, f)] = \{[(s, t)] / s \in ac, t \in bd\} \otimes [(e, f)] = \{[(x, y)] / x \in se, y \in tf\} = \{[(x, y)] / x \in (ac)e, y \in (bd)f\};$

of all we observe that, because of i), t is always different from zero since b and d are different from zero.

Let us now prove that $\oplus$ and $\otimes$ are well defined by proving the following two propositions.

VII.- If $(a', b') \in [(a, b)]$ and $(c', d') \in [(c, d)]$ then $[(ad+bc, bd)] = [(a'd'+b'c', b'd')]$.

Proof. - In order to prove the requested result we must prove that for any $[(s', t')] \in [(a'd'+b'c', b'd')]$ there exists $[(s, t)] \in [(ad+bc, bd)]$ such that $(s', t') \rho(s, t)$ and vice-versa. If $(a', b') \in [(a, b)]$ and $(c', d') \in [(c, d)]$ then $a'b \approx b'a$ and $c'd \approx d'c$ from which we can have $(a'd'+b'c')bd \approx b'd'(ad+bc)$; in fact, as for proposition V, we have $a'bdd' \approx b'add'$ and $c'dbb' \approx d'cbb'$ from which the set $a'bdd'+c'dbb'$ intersects the set $b'add'+d'cbb'$, by proposition III we have, as requested, $(a'd'+b'c')bd \approx b'd'(ad+bc)$. Now let $(s', t') \in H$ such that $s' \in a'd'+b'c'$ and $t' \in b'd'$; from $(a'd'+b'c')bd \approx b'd'(ad+bc)$ and ii) there must exist $(s, t) \in H$ with $s \in ad+bc$ and $t \in bd$ with the condition $s't \approx t's$. Thus $(s', t') \rho(s, t)$. Similarly it can be proved that for any $[(s, t)] \in [(ad+bc, bd)]$ there exists $[(s', t')] \in [(a'd'+b'c', b'd')]$ such that $(s', t') \rho(s, t)$; thus $\oplus$ is well defined.

VIII.- If $(a', b') \in [(a, b)]$ and $(c', d') \in [(c, d)]$, then $[(ac, bd)] = [(a'c', b'd')]$.

Proof. - As in proposition VII we prove that for any $[(s', t')] \in [(a'c', b'd')]$ there exists $[(s, t)] \in [(ac, bd)]$ such that $(s', t') \rho(s, t)$ and vice-versa. If $(a', b') \in [(a, b)]$ and $(c', d') \in [(c, d)]$ then $a'b \approx b'a$ and $c'd \approx d'c$ that is $a'bc'd \approx b'ad'c$ or $a'c'bd \approx b'd'ac$; from ii) the requested result follows.

At this point we can study the properties of hyperstructure $(K, \oplus, \otimes)$.

IX.- $(K, \oplus)$ is a canonical hypergroup.

Proof. As for associativity we have $([(a, b)] \oplus [(c, d)]) \oplus [(e, f)] = \{[(s, t)] / s \in ad+bc, t \in bd\} \oplus [(e, f)] = \{[(z, w)] / [(z, w)] \in [(s, t)] \oplus [(e, f)]\} = \{[(z, w)] / z \in sf+te, w \in tf, s \in ad+bc, t \in bd\} = \{[(z, w)] / z \in (ad+bc)f+(bd)e, w \in (bd)f\}$, while $[(a, b)] \oplus ([[(c, d)] \oplus [(e, f)]] = [(a, b)] \oplus \{[(u, v)] / u \in cf+de, v \in df\} = \{[(p, q)] / [(p, q)] \in [(a, b)] \oplus [(u, v)]\} = \{[(p, q)] / p \in av+bu, q \in bv, u \in cf+de, v \in df\} = \{[(p, q)] / p \in a(df)+b(cf+de), q \in b(df)\}$ and, from associativity and distributivity in A , the two sets are equal. Commutativity of $\oplus$ follows from commutativity of $+$ and $\bullet$ in A as can be easily proved.

moreover $[(a,b)] \otimes [(c,d)] \oplus [(e,f)] = [(a,b)] \otimes \{[(u,v)] / u \in ce, v \in df\} = \{[(x,y)] / x \in au, y \in bv\} = \{[(x,y)] / x \in a(ce), y \in b(df)\}.$

Thus associativity for $\otimes$ follows from the analogous property in $(A, \bullet)$. Similarly commutativity in $(K, \otimes)$ follows from commutativity in $(A, \bullet)$.

As for the existence of a unity let us consider the element $[(x,x)] \in K$, $x \neq 0$; obviously $[(x,x)] = [(y,y)] \quad \forall y \in A^*$. For this element and for any other element $[(a,b)] \in K$ it results: $[(a,b)] \otimes [(x,x)] = \{[(s,t)] / s \in ax, t \in bx\}$. At this point we observe that, since $\{a\}bx = \{b\}ax$, from ii) we have that, $\forall s \in bx$, $t \in ax$ such that $as \approx bt$ and this implies that $[(a,b)] \in [(a,b)] \otimes [(x,x)]$.

Let us now prove property γ ; $\forall [(a,b)], [(c,d)], [(e,f)] \in K$ it results $[(a,b)] \otimes [(c,d)] \oplus [(e,f)] = [(a,b)] \otimes \{[(s,t)] / s \in cf+de, t \in df\} = \{[(x,y)] / x \in as, y \in bt\} = \{[(x,y)] / x \in a(cf+de), y \in b(df)\} = \{[(x,y)] / x \in acf+ade, y \in bdf\}$ and $[(a,b)] \otimes [(c,d)] \oplus [(a,b)] \otimes [(e,f)] = \{[(s,t)] / s \in ac, t \in bd\} \oplus \{[(u,v)] / u \in ae, v \in bf\} = \{[(z,w)] / z \in sv+tu, w \in tv\} = \{[(z,w)] / z \in acbf+bdae, w \in bdbf\} = \{[(z,w)] / z \in b(acf+ade), w \in b(bdf)\}$. Since $[(x,y)] \otimes [(b,b)] \ni [(x,y)]$, the previous set contains $\{[(x,y)] / x \in acf+ade, y \in bdf\}$, that is:

$$[(a,b)] \otimes [(c,d)] \oplus [(e,f)] \subseteq [(a,b)] \otimes [(c,d)] \oplus [(a,b)] \otimes [(e,f)].$$

Moreover, in order to prove δ), we have $-([(a,b)] \otimes [(c,d)]) = -\{[(s,t)] / s \in ac, t \in bd\} = \{[(s,t)] / -s \in (-a)c, t \in bd\} = \{[(s,t)] / -s \in (-a)c, t \in bd\} = \{[(s,t)] / -s \in (-a)c, t \in bd\} = -([(a,b)] \otimes [(c,d)]) = [(a,b)] \otimes [-(c,d)]$.

Finally, $\forall [(a,b)] \in K$, $a \neq 0$, it results $[(a,b)] \otimes [(b,a)] = \{[(s,t)] / s \in ab, t \in ba\} \ni [(x,x)]$, $x \neq 0$, and this ends the proof.

We want now to prove, under particular hypothesis, that there exists a substructure of K which is weakly isomorphic to A . To do this let us remember that an element $1 \in A^*$ is called weak unity if and only if $\forall x \in A$ it results $1 \bullet x \approx x$ ([4]). Then, if $(A, +, \bullet)$ is a multiplicative, strongly distributive, commutative hyperring such that: i) if $a \bullet b \approx 0 \bullet c$, $a \neq 0 \Rightarrow b = 0$; ii) $\forall X, Y, Z, W \in P^*(A) / XY \approx ZW \Rightarrow (\forall x \in X \text{ and } \forall w \in W \exists y \in Y \text{ and } z \in Z / x \bullet y \approx z \bullet w)$; iii) $\exists 1 \in A / \forall x \in A$ it results $1 \bullet x \approx x$, it is possible to prove the following two results:

XI.-The map $\varphi : A \longrightarrow K$ defined as $\varphi(a) = [(a, 1)]$, $\forall a \in A$, is a weak monomorphism.

Proof. - Because of proposition II φ is injective; in fact, if $[(a, 1)] = [(b, 1)]$, then $a \bullet 1 \approx b \bullet 1$ and, from proposition II, this implies $a = b$. Moreover $\varphi(a+b) = [(a+b, 1)]$ while $\varphi(a) \oplus \varphi(b) = [(a, 1)] \oplus [(b, 1)] = \{[(s,t)] / s \in a \bullet 1 + b \bullet 1, t \in 1 \bullet 1\}$ and, since $a+b \in a \bullet 1 + b \bullet 1$ and $1 \in 1 \bullet 1$, then $\varphi(a+b) \in \varphi(a) \oplus \varphi(b)$.

Finally it results $\varphi(a \bullet b) = \{ \varphi(x) / x \in a \bullet b \} = \{ [(x, 1)] / x \in a \bullet b \} \subseteq \varphi(a) \otimes \varphi(b) = [(a, 1)] \otimes [(b, 1)] = \{ [(s, t)] / s \in a \bullet b, t \in 1 \bullet 1 \}$.

XII.- Each element of K belongs to a product $x \otimes y$ where $x \in \text{Im} \varphi$ and y is such that there exists $y' \in \text{Im} \varphi : y \otimes y'$ contains $[(z, z)]$.

Proof. - For $[(a, b)] \in K$ it results $[(a, b)] \in [(a, 1)] \otimes [(1, b)] = \{ [(s, t)] / s \in a \bullet 1, t \in 1 \bullet b \}$.

As a consequence of what has been proved we will call $(K, \oplus, \otimes)$ the weak hyperfield of quotients for $(A, +, \bullet)$.

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UNE NOTE SUR LES GROUPES PERMUTABLES

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RÉSUMÉ On montre, dans cette Note, que la catégorie $\mathcal{P}$ des groupes permutables est balancée.

INTRODUCTION.

D'abord, on rappelle quelques définitions et résultats importants concernant les groupes totalement permutables et les groupes permutables.

Définition 1.1. Soit G un groupe et soit n un entier plus grand que 1. On dit qu'un n -uple $(x_1, \dots, x_n)$ d'éléments de G est permutable s'il existe une permutation nontriviale $\sigma \in S_n$, ainsi que:

$$x_1 \cdot \dots \cdot x_n = x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)}$$

On dit qu'un sous-ensemble $\{x_1, \dots, x_n\}$ d'éléments de G est permutable s'il existe deux permutations différentes s et t en S_n ainsi que:

$$x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)} = x_{\tau(1)} \cdot \dots \cdot x_{\tau(n)}$$

Définition 1.2. Un groupe G s'appelle *groupe totalement n -permutable* (ou $n > 1$) si chaque n -uple $(x_1, \dots, x_n)$ d'éléments de G est permutable.

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On note par P_n la classe des groupes totalement n -permutables et on note par $P = \bigcup_{n>1} P_n$ la classe des groupes totalement permutables.

Définition 1.3. Un groupe G s'appelle *groupe n -permutable* (ou $n>1$) si chaque sous ensemble $(x_1, \dots, x_n)$ d'éléments de G est permutable.

On note par Q_n la classe des groupes n -permutables et on note par $Q = \bigcup_{n>1} Q_n$ la classe des groupes permutables.

Théorème 1.4. ([2]) Un groupe fini G , ayant un nombre fini de générateurs est totalement permutable, si et seulement si, G est fini-par-abélien, c'est-à-dire s'il existe un sous groupe normal N , qui est abélien et a un index fini en G .

Théorème 1.5. ([3]) Un groupe G est totalement permutable, si et seulement si, G est fini par-abélien par-fini, c'est-à-dire s'il existe un sous-groupe normal N , d'index fini, ainsi que N' est fini. (N' est le sous-groupe dérivé de N .)

Un rôle important dans l'étude les groupes totalement permutables et des groupes permutables est joué par les FC-groupes.

Définition 1.6. Un élément $x \in G$ s'appelle *FC-élément* si x a un nombre fini des conjugués en G , c-à-dire $|G:C_G(x)| < \infty$.

Théorème 1.7. (Baer) ([6], Lemme 4.31) Pour chaque groupe G , les FC-éléments forment un sous-groupe caractéristique, qui s'appelle *FC-centre*.

Définition 1.8. Un groupe G s'appelle *FC-groupe* si G coïncide avec son FC-centre.

Théorème 1.9. ([3], Theorem 4.1) La classe des FC-groupes totalement permutables et la classe des groupes avec le sous--groupe dérivé fini coïncident.

Théorème 1.10. Un groupe G est permutable si G est fini par-abélien par-fini, c-à-dire s'il existe un sous-groupe normal N , d'index fini, ainsi que N' est fini.

Corollaire 1.11. La classe des groupes totalement permutables et la classe des groupes permutables coïncident.

Proposition 1.12. ([1], Proposition 2.10) Pour chaque $n \geq 3$, il existe un groupe G_n , d'ordre $(2^{j_n} - 1)j_n$ qui est n -permutable, mais il n'est pas totalement n -permutable, où $j_n = [\log_2(n!)]$.

LA CATÉGORIE $\mathcal{P}$.

On introduira une catégorie, notée par $\mathcal{P}$, qu'on appelle la catégorie des groupes permutables, pour laquelle:

- les objets de $\mathcal{P}$ sont tous les groupes permutables, c-à-dire $\text{Ob}\mathcal{P} = \mathcal{P} = \mathcal{Q}$;
- les morphismes de $\mathcal{P}$ sont les morphismes de groupes;
- le composé de morphismes est le composé de morphismes de groupes.

Remarques

1. La catégorie $\mathcal{P}$ est une sous-catégorie pleine de la catégorie des groupes Gr .
2. Dans la catégorie $\mathcal{P}$ on peut distinguer une chaîne des sous-catégories $\{Q_n\}_{n \geq 1}$ où par Q_n on a noté la catégorie des groupes n -permutables.

Q_n est une sous-catégorie pleine de $\mathcal{P}$ et pour tous $m, n \in \mathbb{N}^*$, $1 < m < n$, Q_m est une sous-catégorie pleine de Q_n .

Définition 2.1. Une catégorie s'appelle *balancée* si chacun de ses monomorphismes est un morphisme injectif et chacun de ses épimorphismes est un morphisme surjectif.

Quoique sur la catégorie Gr on sache que c'est une catégorie balancée ([4], p.27), on ne peut pas dire la même chose sur toutes ses sous-catégories pleines. Il y a des sous-catégories pleines en Gr qui sont balancées, par exemple, la catégorie des groupes abéliens Ab ([4], p.26), mais il existe encore des sous-catégories pleines de Gr qui ne sont pas balancées, par exemple, la catégorie des groupes divisibles Div ([4], p.32).

Par la suite, on montrera que $\mathcal{P}$ est balancée.

Proposition 2.2. Si l'inclusion canonique $i:G_1 \rightarrow G_2$ est un épimorphisme en Gr, alors $G_1 = G_2$.

Démonstration. Soit $*$ un élément fixé, ainsi que $*$ n'appartient pas à l'ensemble facteur $(G_2 / G_1)_d = \{G_1 * g | g \in G_2\}$. Soit K le groupe des permutations de l'ensemble $(G_2 / G_1)_d \cup \{*\}$.

On définit les monomorphismes $t, s:G_1 \rightarrow K$ de cette manière:

$$\begin{aligned} t(g) &= t_g \text{ avec } t_g(\hat{g}) = \hat{g}g' \text{ et } t_g(*) = * \text{ et} \\ s(g) &= s_g = \sigma t_g \sigma, \text{ où } \sigma \in K \text{ est définie par } \sigma(\hat{1}) = *, \\ \sigma(*) &= \hat{1} \text{ et } \sigma(\hat{g}) = \hat{g}, \text{ pour tout } \hat{g} \neq \hat{1} \end{aligned}$$

On vérifie immédiatement que t et s sont des morphismes de groupes. De plus, pour tout $g \in G_1$, on a $t(g) = s(g)$.

Puisque i est un épimorphisme on obtient $t = s$.

Pour tout $g \in G_2$ on a: $\hat{g} = t_g(\hat{1}) = s_g(\hat{1}) = (\sigma t_g \sigma)(\hat{1}) = \sigma t_g(*) = \sigma(*) = \hat{1}$ d'où $G_1 = G_2$.

La démonstration reste encore valable si on change la catégorie Gr avec chaque sous-catégorie Gr d'elle. La seule différence est qu'il faut satisfaire la condition que le groupe K soit un objet de la catégorie C.

Par conséquent, on obtient le lemme suivant:

Lemme 2.3. Si l'inclusion canonique de $i:G_1 \rightarrow G_2$ est un épimorphisme en P et si K , le groupe des permutations de l'ensemble $(G_2 / G_1)_d \cup \{*\}$ où $*$ $\notin (G_2 / G_1)_d$ est un groupe permutable, alors $G_1 = G_2$.

Théorème 2.4. La catégorie $\mathcal{P}$ est balancée.

Démonstration. La démonstration du fait que chaque monomorphisme de $\mathcal{P}$ est un morphisme injectif est similaire à celle pour Gr.

Soit $u:G_1 \rightarrow G_2$ un épimorphisme en $\mathcal{P}$ ($G_1, G_2 \in \mathcal{P}$). En décomposant canoniquement $u = i u_1$, où $u_1:G_1 \rightarrow \text{Im } u$ et $i:\text{Im } u \rightarrow G_2$, on obtient que i est un épimorphisme en $\mathcal{P}$.

En appliquant pour G_2 le Théorème 1.10 il résulte l'existence du sous-groupe N , normal en G_2 , d'index fini, ainsi que N' soit fini. Par conséquent, $N \cdot \text{Im } u \leq G_2$ et $|G_2:(N \cdot \text{Im } u)| < \infty$.

En décomposant maintenant $i = i_2 i_1$, où $i_1:\text{Im } u \rightarrow N \cdot \text{Im } u$ et $i_2:N \cdot \text{Im } u \rightarrow G_2$ il résulte que i_2 est un épimorphisme en $\mathcal{P}$.

Soit K le groupe des permutations de l'ensemble $(G_2 / N \cdot \text{Im } u)_d \cup \{*\}$ où $* \notin (G_2 / N \cdot \text{Im } u)_d$.

On supposera qu'il existe N , sous-groupe normal en G_2 , $M \neq G_2$, ainsi que $\text{Im } u$ soit un sous-groupe en M . On considérera du groupe G_2 au groupe facteur G_2 / M la projection canonique p et le morphisme nul 0 . Parce que $p = 0$ en $\text{Im } u$, il résulte $p = 0$ en G_2 , en vertu du fait que i est un épimorphisme. On obtient $M = G_2$, contradiction.

Donc, le plus petit sous-groupe normal de G qui contient $\text{Im } u$ est G_2 . Par conséquent, chaque $n \in N$ peut être écrit $n = n_1 \cdot u(x) \cdot n_1^{-1}$ où $n_1 \in N$, $x \in G_1$.

Si $N \subset \text{Im } u$, alors on obtient $G_2 = \text{Im } u$ (Lemme 2.3).

Si $N \not\subset \text{Im } u$, alors il existe $n \in N - \text{Im } u$ et pour ce n , $n_1 \in N - \text{Im } u$.

On obtient $[n_1, n^{-1}] = u(x) \cdot n^{-1}$ et parce que N' est fini il résulte que $\{u(x) \cdot n^{-1}\}_{n \in N}$ est un ensemble fini, donc $|G_2:\text{Im } u| < \infty$.

Conformément au Lemme 2.3, on obtient $G_2 = \text{Im } u$.

Donc, u est un morphisme surjectif en $\mathcal{P}$.

Proposition 2.5. Pour tout u , un monomorphisme en $\mathcal{P}$, il existe $n_0 \in \mathbb{N}^*$, $n_0 \geq 2$, ainsi que u est un monomorphisme en Q_m , pour tout $m \geq n_0$. Pour tout u , un épimorphisme en $\mathcal{P}$, il existe $n_0 \in \mathbb{N}^*$, $n_0 \geq 2$, ainsi que u est épimorphisme en Q_m pour tout $m \geq n_0$.

Démonstration. On applique la définition d'un monomorphisme dans une catégorie

$n_0 = \max(K(G_1), K(G_2))$ où $K(G) = \min\{n \in \mathbb{N}^*, n \geq 2, G \in Q_n\}$

Fait analogue pour les épimorphismes.

Proposition 2.6. La catégorie $\mathcal{P}$ a des noyaux et des conoyaux de double flèche.

Démonstration. En Gr , pour $u, v: X \rightarrow Y$ il existe $(K, k) = \text{Ker}(u, v)$ où K est un sous-groupe de X . Donc, si $x \in \tilde{P}$, alors $K \in P$. Fait analogue, en $\text{Gr}(p; Y/N) = \text{Coker}(u, v)$ où N est le sous-groupe normal engendré de $u(x)v(x)^{-1}$, $x \in X$ et si $Y \in P$, alors $Y/N \in P$.

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HYPERSTRUCTURES AS TOOL TO COMPARE URBAN PROJECTS¹

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ABSTRACT - The objective of this work is the application of theories of fuzzy sets and hyperstructures to the problem of selection among alternative urban projects, so to identify those ones pursue a predetermined set of heterogeneous objectives in the best way.

The recourse to fuzzy sets depends on the consciousness that all the planning stages are in condition of uncertainty. So it is necessary to adopt techniques that permit to study the uncertainty about the influence that a decision could have on the validity of the project. The hyperstructures, instead, in this case represent a way to define some rules to choose.

1. The Evaluation in the prospect of a Sustainable Urban Growth

The complex problem of redevelopment planning has consolidated a major consciousness of urgency to adopt suitable evaluation procedures for the projects, so to render "*equitable*" and "*transparent*" the decision process. Besides, we think that it is possible to combine quality and feasibility of architecture and environment through the comparison among different projects.

The quality of places, the respect and exploitation of natural landscapes, the protection of the consolidated urban systems, the preservation of cultural resources are now the hearth of every experience of town planning.

The public interventions in restoration and redevelopment plans have to aim at improving the arrangement of the territory through projects, which tend towards "*integrated maintenance*" of natural and cultural resources. In

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fact we think that an urban development will be achieved not only through economical and social factors, but also through cultural elements, recovering the historical, environmental, architectural values of a place.

For this reason with "*sustainable urban development*" we mean a balanced urban growth, as the result of an urban policy that aims at improving the quality of urban scenery to obtain positive economic effects.

Then the integrated maintenance of physical resources put a set of heterogeneous objectives. In order to achieve these objectives it is necessary to arrange suitable procedures of evaluation about the effects produced by an urban project, in such a way as to analyse all the aspects involved in the urban planning.

In the last National Conference of Architects (Florence, March 1997) the social role of the Architect has been specified: he has to operate on territory, exploiting and watching over the available resources and examining the entire question about the urban transformations.

Public works in urban questions can't be seen only in an economical point of view. Really, we have to examine various impacts of an urban project (environmental, cultural, social, technical, psychological...) to determine the feasibility and to optimise the pursuit of a predetermined set of various objectives.

In other words, the aim of the evaluation will be to define the measure in which an urban project increases the general standard of welfare for the involved community, considering the complexity of the consequences of a redevelopment project on the contest of intervention.

For this reason we think that an accurate evaluation of projects concerning the exploitation of existent resources could be made considering opportunely also the symbolic esthetical-cultural characters of the interested area, getting over the traditional valuing procedures which consider only the monetary aspects.

To introduce also the qualitative aspects into a valuing procedure it is necessary to refer to "*multidimensional techniques*".

In presence of various objectives these procedures represent a rational answer to problems of decision making among a series of alternatives, improving the traditional economical analysis of project whether to estimate the not monetary benefits or to consider the multiplicity of social groups involved in an urban planning.

In this viewpoint we think that the "*global feasibility*" of an urban project could be valued examining some general objectives, within which defining a set of criteria to achieve by carrying out the project.

2. The mathematical model

The problem about the examination of a set of alternative solutions is put to verify the feasibility of each project and to single out the best proposals. It is a question of valuing all the projects according to some parameters.

In order to deal with this problem of evaluation we consider the following mathematical model:

- a) We determine a set $F = \{F_1, F_2, \dots, F_m\}$, called *set of feasibilities*, whose elements are the objectives that we think any projects must attain.
- b) We select a set $C = \{C_1, C_2, \dots, C_n\}$, called *set of criteria*, whose elements represent the ways to measure the degree in which each project realises each objective.
- c) We consider a function $W: \Omega = F \times C \rightarrow [0,1]$, called *weight function*, that to any $\omega_{ij} = (F_i, C_j) \in \Omega$ associates a number w_{ij} that indicate the weight of ω_{ij} in the model. We suppose that the sum of all the w_{ij} is 1. We denote with W also the matrix, called *weight matrix* with elements the w_{ij} . The elements φ_i of the marginal column of W are the weights of the feasibilities and the elements γ_j of the marginal row are the weights of the criteria. In practice it is convenient to consider m functions $K_i: C \rightarrow [0,1]$, called *conditional weight functions*, such that, for any i , the sum of the $K_i(C_j)$, $j=1, \dots, n$, is 1. The matrix K with elements $k_{ij} = K_i(C_j)$ is called *conditional weight matrix*. If $\Phi = \text{diag}(\varphi_1, \varphi_2, \dots, \varphi_m)$ is the diagonal matrix with elements the feasibility weights, we have $W = \Phi K$.
- d) For any project P we assign a function $E^P: \Omega = F \times C \rightarrow [0,1]$, called *efficaciousness function of P*, such that to any $\omega_{ij} = (F_i, C_j) \in \Omega$ associates a number e_{ij}^P that indicates the measure in which P satisfies the pair (F_i, C_j) .
- e) The matrix R^P with generic element $r_{ij}^P = k_{ij} e_{ij}^P$ is called *relative utility matrix of P* and the matrix U^P with generic element $u_{ij}^P = w_{ij} e_{ij}^P = r_{ij}^P \varphi_i$ is the *utility matrix or representative matrix of P*.

In this paper we consider as feasibilities for an urban project the following aspects:

environmental: evaluation of project about the effects on natural and built ambient;

aesthetic-cultural: evaluation of the project about the effects on the historical, artistic and archaeological interests;

economical: evaluation of the project to verify the increase of general welfare in the considered area;

financial: evaluation of the project to control the measure of costs-return ratio;

technical: evaluation of the project about the effects on constraints of construction and regulation;

social: evaluation of the project about the effects on the persons directly and indirectly involved by the redevelopment and the improvement on functional characters of the site;

procedural: evaluation of project about the administrative-procedural aspects, in particular with regard to the conditions of agreements and the modality of entrusting of works.

Besides, for an urban project some of the selected criteria are:

- reduction of the urban decay, through the urbanistic and functional rejoining of vacant sites or urban areas without identity;
- compatibility of project with the morphology and with the geologic conditions of the site;
- offering of new possibility to enjoy the landscape;
- preservation of artistic-historical-architectural values, through the protection of typological, structural and formal values that are peculiar of buildings which have created the historic ancient environment;
- consistent use of colours and materials as regard to the conformation of the urban environment;
- compatibility of the project with the rules of the urbanistic instrumentation;
- improvement in the equipment of social services;
- compatibility of the project with the local and state rules so to simplify the administrative procedures.

Any criterion C_j may be valid only for some feasibilities. In this case we put $w_{ij}=0$ for any F_i that not consider the criterion C_j .

The elements of all the matrices considered are numbers belonging to the interval $[0,1]$. So any matrix can be seen as fuzzy set with universe set Ω . The matrix K can be seen also as a *fuzzy partition*, if we consider its columns as fuzzy sets with universe the set of feasibilities.

For each $H \subseteq \Omega$, $H \neq \emptyset$, we denote with U_H^P the part of U^P constituted by the elements u_{ij}^P such that $(F_i, C_j) \in H$. U_H^P is also a fuzzy set with the universe set H .

We have a particular case of the treated theory if we consider the matrices of incidence as matrices associated to projects. That is, for each pair (F_i, C_j) and for each project P , we put $u_{ij}^P=1$ if the pair (F_i, C_j) is satisfied by the project P , and $u_{ij}^P=0$ if the pair is not satisfied. Then the matrix U^P of the project P can be interpreted as a subset of Ω , that is the set of the $(F_i, C_j) \in \Omega$ such that $u_{ij}^P=1$ and $U_H^P = U^P \cap H$.

Denote with $\wp$ the family of projects and with $\mathfrak{F}$ the family of fuzzy sets on Ω . Let U be the function that associates $U^P \in \mathfrak{F}$ to each $P \in \wp$.

Generally, for each $H \subseteq \Omega$, $H \neq \emptyset$, let $\mathfrak{F}_H$ be the family of fuzzy sets on H and let U_H be the function that associates to each $P \in \wp$ the $U_H^P \in \mathfrak{F}_H$.

For each pair (M, N) of elements of $\mathfrak{F}_H$ we put

$$M \leq_H N \Leftrightarrow m_{ij} \leq n_{ij}, \forall (F_i, C_j) \in H.$$

The pair $(\mathfrak{I}_H, \leq_H)$ is a lattice. If M and N are any two elements of $\mathfrak{I}_H$ we denote with $M \wedge_H N$ the infimum of N and M , and with $M \vee_H N$ their supremum.

For each pair (P, Q) of projects, we put:

$$P \leq_H Q \Leftrightarrow U_H(P) \leq_H U_H(Q), \quad (2.1)$$

$$P \approx_H Q \Leftrightarrow (P \leq_H Q \text{ and } Q \leq_H P) \quad (2.2)$$

The $\leq_H$ is a preorder relation on $\wp$, called *H-preorder* on $\wp$. If $H=\Omega$ it is called *global preorder*, on the contrary it is called *partial preorder*. The $\approx_H$ is an equivalence relation on $\wp$.

Given n projects $P_1, P_2, \dots, P_n$, we call *lower bound* of them *relatively to H* or *H-lower bound* every project P such that $P \leq_H P_i, \forall i \in \{1, 2, \dots, n\}$ and we denote the set of H-lower bounds with $LBH(P_1, P_2, \dots, P_n)$. An element P of this set is said to be *maximal* if for each $Q \in LBH(P_1, P_2, \dots, P_n)$, $P \leq_H Q \Rightarrow P \approx_H Q$. We denote the set of maximal lower bounds relatively to H with $\delta_H(P_1, P_2, \dots, P_n)$.

We call *upper bound* of $P_1, P_2, \dots, P_n$ *relatively to H* or *H-upper bound* every project P such that $P_i \leq_H P, \forall i \in \{1, 2, \dots, n\}$ and we denote the set of H-upper bounds with $UBH(P_1, P_2, \dots, P_n)$. An element P of this set is said to be *minimal* if for each $Q \in UBH(P_1, P_2, \dots, P_n)$, $Q \leq_H P \Rightarrow P \approx_H Q$. Let $\sigma_H(P_1, P_2, \dots, P_n)$ be the set of minimal upper bounds of $P_1, P_2, \dots, P_n$ relatively to H .

It is easy to prove the following

$$(P \in \delta_H(P_1, P_2, \dots, P_n), P \approx_H Q) \Rightarrow Q \in \delta_H(P_1, P_2, \dots, P_n), \quad (2.3)$$

$$(P \in \sigma_H(P_1, P_2, \dots, P_n), P \approx_H Q) \Rightarrow Q \in \sigma_H(P_1, P_2, \dots, P_n). \quad (2.4)$$

Evidently, if $\emptyset \subset H \subset I \subset \Omega$ then the $\leq_H$ and $\approx_H$ are, respectively, extensions of the $\leq_I$ and $\approx_I$. So, for each n -tuple of projects $(P_1, P_2, \dots, P_n)$, we have

$$LBH(P_1, P_2, \dots, P_n) \supseteq LBI(P_1, P_2, \dots, P_n) \text{ and } UBH(P_1, P_2, \dots, P_n) \supseteq UBI(P_1, P_2, \dots, P_n).$$

3. Hyperstructures associated to projects

Now we show how it is possible to define some particularly very significant hyperstructures of projects for geometrical and practical interpretation of the implications of the preorder relations considered.

At this aim, we recall some definitions about the hyperstructures. We assume the terminology given in (Corsini, [1]), (Vougiouklis, [8]), (Migliorato, [6]) and (Maturo, [5]).

Definition 3.1 A *hypergroupoid* (S, α) , is a non empty set S with a function $\alpha: S \times S \rightarrow P^*(S) = P(S) - \{\emptyset\}$, called *hyperoperation*. The image of the pair (x, y) is noted αxy and is called *hyperproduct* by x and y .

For any pair (H, K) of subsets of S different from $\emptyset$, we denote by $H\alpha K$ the union of all the sets $x\alpha y$ with $x \in H$ and $y \in K$. The hyperproducts $a\alpha K$ and $H\alpha a$, $a \in S$, are considered equal, respectively, to $\{a\}\alpha K$ and $H\alpha\{a\}$.

$\forall n \in \mathbb{N}$, $\forall (x_1, x_2, \dots, x_n) \in S^n$, the set $\mathfrak{I}_\alpha(x_1, x_2, \dots, x_n)$ of all the hyperproducts generated by $(x_1, x_2, \dots, x_n)$ is given, by induction, as follows: $\mathfrak{I}_\alpha(x_1) = \{x_1\}$ and, for $n > 1$, $\mathfrak{I}_\alpha(x_1, x_2, \dots, x_n)$ is the set of all the hyperproducts $K = F\alpha G$, with $F = \{x_1, x_2, \dots, x_h\}$, $G = \{x_{h+1}, x_2, \dots, x_n\}$, $h \in \{1, 2, \dots, n-1\}$.

Any element of $\mathfrak{I}_\alpha(x_1, x_2, \dots, x_n)$ is called *block of order n* or *n-block* of S generated by $(x_1, x_2, \dots, x_n)$.

Definition 3.2 A hypergroupoid (S, α) is said to be

(H1) *semihypergroup* if, $\forall x, y, z \in S$, $x\alpha(y\alpha z) = (x\alpha y)\alpha z$;

(H2) *quasihypergroup* if, $\forall x \in S$, $x\alpha S = S\alpha x = S$;

(H3) *commutative* if, $\forall x, y \in S$, $x\alpha y = y\alpha x$;

(H4) *hypergroup* if it is a semihypergroup and a quasihypergroup.

Definition 3.3 A hypergroupoid (S, α) is said to be

(W1) *weak commutative* if, $\forall x, y \in S$, $x\alpha y \cap y\alpha x \neq \emptyset$;

(W2) *weak associative* if, $\forall x, y, z \in S$, $x\alpha(y\alpha z) \cap (x\alpha y)\alpha z \neq \emptyset$;

(W3) *feebly associative* if, $\forall n \in \mathbb{N}$ and $\forall (x_1, x_2, \dots, x_n) \in S^n$, the intersection of all the blocks belonging to $\mathfrak{I}_\alpha(x_1, x_2, \dots, x_n)$ is different from $\emptyset$.

Definition 3.4 Let (S, α) be a hypergroupoid and let $\emptyset \neq T \subseteq S$. (T, α) is called a *subhypergroupoid* of S if, $\forall x, y \in T$, $x\alpha y \subseteq T$. A subhypergroupoid T of S is called *subhypergroup* if (T, α) is a hypergroup. If $a\alpha b \cap T \neq \emptyset, \forall a, b \in T$, the pair (T, β) , with $a\beta b = a\alpha b \cap T, \forall a, b \in T$, is called *substructure generated by T*.

Finally any hypergroupoid (T, γ) such that, $\forall a, b \in T$, $a\gamma b \subseteq a\alpha b \cap T$, is called *partial substructure* of (S, α) with domain T .

Now, we assume S equal to the set $\wp$ of projects. For any $H \subseteq \Omega$, $H \neq \emptyset$, we consider the following functions:

$$\delta_H: (P, Q) \in \wp^2 \rightarrow \delta_H(P, Q), \quad (3.1)$$

$$\sigma_H: (P, Q) \in \wp^2 \rightarrow \sigma_H(P, Q). \quad (3.2)$$

A project I_H is said to be an *H-inferior project* if

$$\forall P \in \wp, U_H(I_H) \leq_H U_H(P). \quad (3.3)$$

A project I is called *inferior project* if it is an *H-inferior project* for any $H \subseteq \Omega$, $H \neq \emptyset$.

A project S_H is said to be an *H-superior project* if

$$\forall P \in \wp, U_H(P) \leq_H U_H(S_H). \quad (3.4)$$

A project S is called *superior project* if it is an H -superior project for any $H \subseteq \Omega$, $H \neq \emptyset$.

Eventually by adding two fictitious projects, we can suppose that there exist, in $\wp$, an H -superior project and an H -inferior project. In this hypothesis δ_H and σ_H are hyperoperations on $\wp$ and so we have the following

Theorem 3.5 If an H -superior project S_H exists, then the pair $(\wp, \sigma_H)$ is a commutative hypergruppoid. Similarly, if an H -inferior project I_H exists, then the pair $(\wp, \delta_H)$ is a commutative hypergruppoid.

Proof Let S_H be an H -superior project. Then, for any $P, Q \in \wp$, S_H is an upper bound of (P, Q) and so $\sigma_H(P, Q) \neq \emptyset$. Since $\sigma_H(P, Q) = \sigma_H(Q, P)$, then $(\wp, \sigma_H)$ is a commutative hypergruppoid. Analogously the existence of an H -inferior project I_H implies that $(\wp, \delta_H)$ is a commutative hypergruppoid. $\square$

Theorem 3.6 If $U_H(\wp)$ is a sublattice of $(\mathfrak{I}_H, \leq_H)$, then the pairs $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative semihypergroups.

Proof Suppose $U_H(\wp)$ is a sublattice of $(\mathfrak{I}_H, \leq_H)$. Since $U_H(\wp)$ is finite then there exist the maximum S_H and the minimum I_H with respect $\leq_H$, that are, respectively, an H -superior project and a H -inferior project. By theorem 3.5 it follows that $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative hypergroupoids.

By hypothesis that $U_H(\wp)$ is a sublattice of $(\mathfrak{I}_H, \leq_H)$, $\forall P, Q \in \wp$, we have,

$$P\delta_H Q = \{R \in \wp : U_H(R) = U_H(P) \wedge_H U_H(Q)\}, \quad (3.5)$$

$$P\sigma_H Q = \{R \in \wp : U_H(R) = U_H(P) \vee_H U_H(Q)\}. \quad (3.6)$$

Since $\wedge_H$ and $\vee_H$ are associative, theorem follows. $\square$

Generally, $U_H(\wp)$ is not a sublattice of $(\mathfrak{I}_H, \leq_H)$ and so it is not assured the existence of the H -superior and H -inferior projects. Therefore, if such projects exist, the associative properties of δ_H and σ_H are not valid. In this case, we have the following

Theorem 3.7 If an H -superior project S_H exists, then the pair $(\wp, \sigma_H)$ is a commutative and weak associative hypergruppoid. Similarly, if an H -inferior project I_H exists, then the pair $(\wp, \delta_H)$ is a commutative and weak associative hypergruppoid.

Proof Let S_H be an H -superior project. By theorem 3.5 we have that the pair $(\wp, \sigma_H)$ is a commutative hypergruppoid. Therefore, for any $P, Q, R \in \wp$, S_H is an upper bound of (P, Q, R) and so $\sigma_H(P, Q, R) \neq \emptyset$. Let Y be any element of $\sigma_H(P, Q, R)$. Then $Y \in \text{UBH}(P, Q)$ and so there exists a $Z \in \sigma_H(P, Q)$ such that

$Z \leq_H Y$ and $Y \in \text{UBH}(Z, R)$. Let T be an element of $\sigma_H(Z, R)$ such that $T \leq_H Y$. T is an upper bound of (P, Q, R) . Since $Y \in \sigma_H(P, Q, R)$ we have $T \approx_H Y$ and so $Y \in \sigma_H(Z, R)$. Then, $\forall P, Q, R \in \wp$, it results $(P\sigma_H Q)\sigma_H R \supseteq \sigma_H(P, Q, R)$. Analogously, we can prove that $P\sigma_H(Q\sigma_H R) \supseteq \sigma_H(P, Q, R)$. So $\forall P, Q, R \in \wp$, we have

$$(P\sigma_H Q)\sigma_H R \cap P\sigma_H(Q\sigma_H R) \supseteq \sigma_H(P, Q, R) \neq \emptyset, \quad (3.7)$$

and the weak associative property of σ_H is proved.

In a similar way we prove that, $\forall P, Q, R \in \wp$,

$$(P\delta_H Q)\delta_H R \cap P\delta_H(Q\delta_H R) \supseteq \delta_H(P, Q, R) \neq \emptyset \quad (3.8)$$

and so we conclude that also δ_H is a weak associative hyperoperation. $\square$

Practically, in the hypotheses of the previous theorem, given two projects P_1 and P_2 , it is possible to single out at least a project Q such that, with respect H , is not superior to P_1 and P_2 and is not inferior to any project not superior to P_1 and P_2 .

Besides, we can single out at least a project R , such that, with respect H , is not inferior to P_1 and P_2 , and is not superior to any project not inferior to P_1 and P_2 .

The hyperproduct δ_H gives the set of all the projects that are *H-maximal lower bounds* of P_1 and P_2 . They are the projects with the better qualities, with respect H , in the set of the projects not superior to P_1 and P_2 . They can be chosen from an economic point of view.

The hyperproduct σ_H gives the set of all the projects that are *H-minimal upper bounds* of P_1 and P_2 . Those ones present all the better qualitative characteristics, with respect H , as regard to the pair (P_1, P_2) . They are a *compromise* between the solution prefigured by both projects P_1 and P_2 and every proposal P_3 that is not inferior to P_1 and P_2 .

In decision making problems, often the economical-financial factor is so important that the decider is induced to choose the *H-minimal upper bounds* projects. Those ones make it possible to achieve all the pre-established objectives in a satisfying measure, at a lower price than the amount implied by another project P_3 that achieves the same objectives.

If the U^P are *incidence matrices*, than $U_H(\wp)$ is a family of subsets of H , the operations $\wedge_H$ and $\vee_H$ in this family are the set operations of intersection and union and the order relation induced by these ones is reduced to the inclusion among the subsets of H belonging to $U_H(\wp)$.

If I and S are, respectively, an inferior and a superior project, then

- the matrix $U(I)$ specifies a set of pairs that must be necessary satisfied, that is $U(P) \supseteq U(I)$, $\forall P \in \wp$;
- the matrix $U(S)$ represents the largest set of pairs that can be satisfied, that is $U(P) \subseteq U(S)$, $\forall P \in \wp$.

Two particular fictitious projects that can be usefully used, respectively, as inferior and superior projects are the following:

- a) the *null project* O with efficaciousness matrix E^O having all the elements equal to 0 and so with U^O equal to null matrix;
- b) the *fantastic project* F such that E^F has all the elements equal to 1 and so with $U^F = W$.

In many situations it is necessary to consider only projects such that the elements of the efficaciousness matrices are not inferior to fixed values, called *threshold values*. In this case we have to consider as inferior project, a project T , real or fictitious, called *threshold project*, with efficaciousness matrix E^T such that any e_{ij}^T represents the lowest acceptable grade of satisfaction of the pair (F_i, C_j) by any project.

Now we give some properties of hypergruppoids $(\wp, \delta_H)$ and $(\wp, \sigma_H)$. Precisely, given n projects $P_1, P_2, \dots, P_n$, we study the relations among the blocks generated by these elements and the sets $\delta_H(P_1, P_2, \dots, P_n)$ of their maximal lower bounds and $\sigma_H(P_1, P_2, \dots, P_n)$ of their minimal upper bounds.

Theorem 3.8 For each $n > 1$, for each $k \in \{1, 2, \dots, n\}$ and for each n -tuple of projects $(P_1, P_2, \dots, P_n)$, it results

$$\delta_H(P_1, P_2, \dots, P_n) \subseteq \delta_H(P_1, P_2, \dots, P_k) \delta_H(P_{k+1}, P_{k+2}, \dots, P_n), \quad (3.5)$$

$$\sigma_H(P_1, P_2, \dots, P_n) \subseteq \sigma_H(P_1, P_2, \dots, P_k) \sigma_H(P_{k+1}, P_{k+2}, \dots, P_n). \quad (3.6)$$

Proof We prove the (3.6). In a similar way the (3.5) is verified. It results evidently

$$\sigma_H(P_1, P_2, \dots, P_n) \subseteq \text{UBH}(P_1, P_2, \dots, P_k) \cap \text{UBH}(P_{k+1}, P_{k+2}, \dots, P_n).$$

Now, suppose $P \in \sigma_H(P_1, P_2, \dots, P_n)$. We have

$$\exists Q \in \sigma_H(P_1, P_2, \dots, P_k), \exists R \in \sigma_H(P_{k+1}, P_{k+2}, \dots, P_n) : P \in \text{UBH}(Q, R).$$

If a project S belongs to $\text{UBH}(Q, R)$ and $P \geq_H S$ then we have that $S \in \text{UBH}(P_1, P_2, \dots, P_k) \cap \text{UBH}(P_{k+1}, P_{k+2}, \dots, P_n)$ and so $S \in \text{UBH}(P_1, P_2, \dots, P_n)$.

Since $P \in \sigma_H(P_1, P_2, \dots, P_n)$, $P \geq_H S$, it follows that $S \in \sigma_H(P_1, P_2, \dots, P_n)$ and so $S \approx_H P$. Then $P \in Q \sigma_H R \subseteq \sigma_H(P_1, P_2, \dots, P_k) \sigma_H(P_{k+1}, P_{k+2}, \dots, P_n)$. This implies (4.6). $\square$

For each $n \geq 1$ and for each n -tuple of projects $(P_1, P_2, \dots, P_n)$, we denote with $\mathfrak{I}_\delta(P_1, P_2, \dots, P_n)$ the family of blocks associated to the n -tuple as regards the hyperoperation δ_H and with $\mathfrak{I}_\sigma(P_1, P_2, \dots, P_n)$ that one as regards the hyperoperation σ_H . We have the following

Theorem 3.9 $\forall n > 1$, $\forall n$ -tuple of projects $(P_1, P_2, \dots, P_n)$, $\forall K_\delta \in \mathfrak{I}_\delta(P_1, P_2, \dots, P_n)$ and $\forall K_\sigma \in \mathfrak{I}_\sigma(P_1, P_2, \dots, P_n)$, it results

$$\delta_H(P_1, P_2, \dots, P_n) \subseteq K_\delta, \quad (3.7)$$

$$\sigma_H(P_1, P_2, \dots, P_n) \subseteq K_\sigma. \quad (3.8)$$

Proof We prove (4.8). In a similar way (4.7) is verified. The theorem is evident for $n=2$ because $\mathfrak{I}_\sigma(P_1, P_2)$ has the only element $\sigma_H(P_1, P_2)$. We suppose that the theorem is true for $n \leq k$, for a generic $k \geq 2$ and we verify that it is true also for $n=k+1$.

In fact, suppose $K_\sigma \in \mathfrak{I}_\sigma(P_1, P_2, \dots, P_{k+1})$. Then $\exists h: 1 \leq h \leq k$ such that $K_\sigma = K_1 \sigma_H K_2$, with $K_1 \in \mathfrak{I}_\sigma(P_1, P_2, \dots, P_h)$ and $K_2 \in \mathfrak{I}_\sigma(P_{h+1}, P_{h+2}, \dots, P_{k+1})$. For hypothesis of induction

$$\sigma_H(P_1, P_2, \dots, P_h) \subseteq K_1, \quad \sigma_H(P_{h+1}, P_{h+2}, \dots, P_{k+1}) \subseteq K_2.$$

For the previous theorem we have

$$\sigma_H(P_1, P_2, \dots, P_{k+1}) \subseteq \sigma_H(P_1, P_2, \dots, P_h) \sigma_H \sigma_H(P_{h+1}, P_{h+2}, \dots, P_{k+1})$$

and so

$$\sigma_H(P_1, P_2, \dots, P_{k+1}) \subseteq K_\sigma = K_1 \sigma_H K_2. \quad \square$$

Theorem 3.10 The pairs $(\wp, \delta_H)$ and $(\wp, \sigma_H)$ are commutative and feebly associative hypergruppoids.

Proof By previous theorem we can observe that, $\forall n \in \mathbb{N}$ and $\forall P_1, P_2, \dots, P_n \in \wp$, $\delta_H(P_1, P_2, \dots, P_n)$ is included in the intersection of all blocks K_δ belonging to $\mathfrak{I}_\delta(P_1, P_2, \dots, P_n)$ and, similarly, the intersection of all blocks K_σ belonging to $\mathfrak{I}_\sigma(P_1, P_2, \dots, P_n)$ includes $\sigma_H(P_1, P_2, \dots, P_n)$. $\square$

By two examples we verify that, generally, the associative property doesn't hold and we can have two possibilities:

- the intersection of all blocks K_σ belonging to $\mathfrak{I}_\sigma(P_1, P_2, \dots, P_n)$ is equal to $\sigma_H(P_1, P_2, \dots, P_n)$;
- $\sigma_H(P_1, P_2, \dots, P_n)$ is strictly contained in the intersection of all K_σ belonging to $\mathfrak{I}_\sigma(P_1, P_2, \dots, P_n)$.

Example 3.11 Let $\wp = \{P_0, P_1, \dots, P_9\}$ be the set of projects. For each pair A, B of subset of $\wp$ we put $A \leq_H B$ if for each $a \in A$ $e \in B$ we have $a \leq_H b$. We impose the following conditions

$$\{P_0\} \leq_H \{P_1, P_2, P_3\}, \quad \{P_1\} \leq_H \{P_4, P_5\}, \quad \{P_2\} \leq_H \{P_4, P_5, P_6\},$$

$$\{P_3\} \leq_H \{P_5, P_6\}, \quad \{P_4, P_5\} \leq_H \{P_7\}, \quad \{P_5, P_6\} \leq_H \{P_8\}, \quad \{P_7, P_8\} \leq_H \{P_9\}.$$

These conditions determine an antisymmetric relation. The reflexive and transitive closure of this relation is an order relation and for each $A \subseteq \wp$, $\delta_H(A)$ and $\sigma_H(A)$ are not empty.

In particular we have

$$\sigma_H(P_1, P_2, P_3) = \{P_5\}, \quad P_1 \sigma_H (P_2 \sigma_H P_3) = \{P_5, P_8\}, \quad (P_1 \sigma_H P_2) \sigma_H P_3 = \{P_5, P_7\}.$$

The two blocks generated by (P_1, P_2, P_3) are different, so the associative property doesn't hold. Their intersection is equal to $\sigma_H(P_1, P_2, P_3)$.

Example 3.12 Let $\wp = \{P_0, P_1, \dots, P_7\}$ be the set of projects. We impose the following conditions

$$\{P_0\} \leq_H \{P_1, P_2, P_3\}, \{P_1\} \leq_H \{P_4, P_5\}, \{P_2\} \leq_H \{P_4, P_5, P_6\}, \\ \{P_3\} \leq_H \{P_5, P_6\}, \{P_4, P_5, P_6\} \leq_H \{P_7\}$$

These conditions determine an antisymmetric relation. The reflexive and transitive closure of this relation is an order relation and for each $A \subseteq \wp$, $\delta_H(A)$ and $\sigma_H(A)$ are not empty.

In particular we have

$$\sigma_H(P_1, P_2, P_3) = \{P_5\}, P_1 \sigma_H(P_2 \sigma_H P_3) = \{P_5, P_7\}, (P_1 \sigma_H P_2) \sigma_H P_3 = \{P_5, P_7\}.$$

The two blocks generated by (P_1, P_2, P_3) are equal. Their intersection is different from $\sigma_H(P_1, P_2, P_3)$.

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WEAK HYPERSTRUCTURES ON SMALL SETS¹

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Abstract The class of the weak hyperstructures is larger than the known ones originated from the hypergroup in the sense of Marty. The weak hyperstructures are also called H_v -structures. In this paper we deal with some classes of H_v -groups defined on sets with three elements.

H_v -STRUCTURES ON SMALL SETS

A hyperstructure is called H_v -structure if it satisfies the structure-like axioms where the related axioms are replaced by the weak ones, i.e. the equality is replaced by the non-empty intersection. For example the **weak associativity** in a H_v -group $(H, \bullet)$ is

$$(xy)z \cap x(yz) \neq \emptyset \quad \text{for all } x, y, z \in H$$

and the **weak commutativity** is

$$xy \cap yx \neq \emptyset \quad \text{for all } x, y \in H.$$

Similarly, in a H_v -ring $(H, +, \bullet)$ the left weak distributivity is

$$x(y+z) \cap (xy+xz) \neq \emptyset \quad \text{for all } x, y, z \in H.$$

The main tool to study the H_v -structures remains as in all hyperstructures, the fundamental relation. That is, the smallest equivalence relation such that the quotient would be the corresponding structure. So the quotient of a H_v -

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group by the fundamental relation β^* is a group, the smallest one. Similarly, the quotient of a H_V -ring by the fundamental relation β^* is a ring [2] and so on. According the above one can define the H_V -semigroup, the H_V -quasigroup and so on.

The H_V -structures have been, introduced in [6] and properties as well as basic facts on the topic can be found in the book [4] and for the fundamental relation of hyperstructures in [1].

In the paper [5] the set of all H_V -groups with two elements is given. Moreover the set of H_V -groups with three elements is calculated under the assumption that they contain a scalar unit element. In this paper we determine some more general cases in a set with three elements.

Let $(H, \bullet)$, $(H, *)$ be two H_V -groups, defined on the same set H . The hyperoperation $(\bullet)$ is called *less or equal* than $(*)$, if and only if there exists a $f \in \text{Aut}(H, *)$ such that

$$xy \subseteq f(x*y) \quad \text{for all } x, y \text{ in } H.$$

Then $(*)$ is called greater than $(\bullet)$ and we say that $(H, *)$ contains $(H, \bullet)$. A H_V -group is called *minimal* if contains no other H_V -group defined in the same set. A basic consequence of the above definition is the following.

THEOREM 1

Greater hyperoperation from the one of a given H_V -group defines a H_V -group. The weak commutativity is also valid for every greater hyperoperation.

In our study we need to recall the following properties.

PROPERTIES 2

Let $(\bullet)$ is a hyperoperation in the set H , then

- (i) If $(\bullet)$ is weak commutative then

$$a(ba) \cap (ab)a \neq \emptyset \quad \text{for all } a, b \in H.$$

- (ii) If $(\bullet)$ is commutative then for all $a, b \in H$,

$$a(ab) \cap (aa)b \neq \emptyset \quad \text{implies} \quad b(aa) \cap (ba)a \neq \emptyset.$$

- (iii) If e is a unit of the hypergroupoid $(H, \bullet)$ we have

$$(xy)z \cap x(yz) \neq \emptyset \quad \text{for all } x, y, z \in H \text{ such that } e \in \{x, y, z\}.$$

Finally we recall the following theorem from [5], [4].

THEOREM 3

Let $H = \{e, a, b\}$ be a set with three elements. Suppose we want to define in H a hyperoperation $(\bullet)$ for which the element e is a scalar unit. Denote by a four (aa, ab, ba, bb) the products needed to be defined. Then, the set of all H_v -groups defined in H has a set of minimal the following ones, given with their isomorphic:

- $$\begin{aligned} M_1 & (e, b, b, \{e, a\}) \cong (\{e, b\}, a, a, e) \\ M_2 & (e, \{e, b\}, \{e, b\}, \{a, b\}) \cong (\{a, b\}, \{e, a\}, \{e, a\}, e) \\ M_3 & (e, H, H, a) \cong (b, H, H, e) \\ M_4 & (e, \{a, b\}, \{a, b\}, e) \\ M_5 & (e, H, H, b) \cong (a, H, H, e) \\ M_6 & (a, \{e, b\}, \{e, b\}, a) \cong (b, \{e, a\}, \{e, a\}, b) \\ M_7 & (a, H, H, b) \\ M_8 & (b, e, e, a) \\ M_9 & (\{e, b\}, \{e, a\}, b, H) \cong (H, a, \{e, b\}, \{e, a\}) \\ M_{10} & (\{e, b\}, b, \{e, a\}, H) \cong (H, \{e, b\}, a, \{e, a\}) \end{aligned}$$

In the above theorem we observe that if

$$ee = \{e\} \text{ or } \{e, a\} \text{ or } \{e, b\} \text{ or } H$$

then we need the above fours because the reproductivity must be valid. In other words in these cases the minimal H_v -groups remain the same. Therefore we proved the following.

THEOREM 4

Suppose we want to define in $H = \{e, a, b\}$ a hyperoperation $(\bullet)$ for which

$$\begin{aligned} ex = xe = x & \text{ for all } x \neq e \text{ and} \\ ee = \{e\} & \text{ or } \{e, a\} \text{ or } \{e, b\} \text{ or } H \end{aligned}$$

such that $(H, \bullet)$ is H_v -group. Then, we must have greater or equal four (aa, ab, ba, bb) than the ones $M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}$ given in Theorem 3.

THEOREM 5

Suppose we want to define in $H = \{e, a, b\}$ a hyperoperation $(\bullet)$ for which the element e is a two-sided unit and $x^2 = H$, for $x \in \{a, b\}$, such that $(H, \bullet)$ is H_v -group. Denote by the ordered 7-tuples

$$(ee, ea, ae, eb, be, ab, ba)$$

the products needed to be defined. Then, the set of all H_v -groups defined in H has a set of minimal the following ones, given with their isomorphic:

$$\begin{aligned}\Sigma_1 & (e, a, a, b, b, e, e) \\ \Sigma_2 & (e, a, a, b, b, a, a) \cong (e, a, a, b, b, b, b) \\ \Sigma_3 & (e, a, a, b, b, a, b) \\ \Sigma_4 & (e, a, a, b, b, b, a) \\ \Sigma_5 & (e, a, a, b, b, e, \{a, b\}) \cong (e, a, a, b, b, \{a, b\}, e) \\ \Sigma_6 & (e, a, a, b, \{b, e\}, e, a) \cong (e, a, \{a, e\}, b, b, b, e) \\ \Sigma_7 & (e, \{a, e\}, a, b, b, e, b) \cong (e, a, a, \{b, e\}, b, a, e)\end{aligned}$$

Proof

First observe that the reproduction axiom is always valid. To find these 7-tuples we worked in the following procedure. Since we have only to check the weak associativity we start with the smallest possible cases in the 7-tuples and we checked the weak associativity.

In case that the weak associativity is not valid we enlarged the products up to the sets in order the weak associativity to be valid. In this case there is the possibility the obtained hyperoperation is not minimal. Therefore we checked smaller hyperoperations up to the sets the weak associativity is not valid. However, in our special case the last step is not needed since the reproductivity is always valid.

Special case : in the case that e is a scalar unit then the set of minimal is given above $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$.

THEOREM 6

Suppose we want to define in $H = \{e, a, b\}$ a hyperoperation $(\bullet)$ for which e is unit and a is scalar, such that $(H, \bullet)$ is H_v -group. Then, we have two cases

$$(i) \quad aa = b \quad \text{and} \quad (ii) \quad aa = e$$

In the case (i) the only one minimal H_v -group is the group defined in H . In the case (ii) the four (ee, eb, be, bb) is the one needed to be defined. Then the set of minimal is the following:

$$\begin{aligned}F_1 & (e, H, H, b) \\ F_2 & (e, b, b, \{e, a\}) \\ F_3 & (e, \{a, b\}, \{a, b\}, \{e, b\}) \\ F_4 & (e, \{e, b\}, \{e, b\}, \{a, b\}).\end{aligned}$$

Proof

(i) We observe that

$$a(ab) = ae = a \quad \text{and} \quad (aa)b = bb$$

therefore $a \in bb$. So, in this case we obtain the group as the minimal.

(ii) We follow the method given in Theorem 5 where we have always to check the reproduction axiom as well.

We conclude our study with the results for two general cases. In the proofs we followed the method described above however one can see that these cases are more laborious. The first one theorem can be considered as the Boolean type H_v -groups defined in set with three elements.

THEOREM 7

Suppose we want to define in $H = \{e, a, b\}$ a H_v -group $(H, \bullet)$ for which e is unit and the relation $x^2 = x, \forall x \in H$, is valid. Denote by the ordered 6-tuples

$$(ea, ae, eb, be, ab, ba)$$

the products needed to be defined. Then the set of all H_v -groups defined in H has a set of minimals the following ones given with their isomorphic.

- $S_1 \quad (H, H, \{e, b\}, \{e, b\}, a, a) \cong (\{e, a\}, \{e, a\}, H, H, b, b)$
- $S_2 \quad (\{e, a\}, H, \{e, b\}, H, a, b)$
- $S_3 \quad (H, H, \{e, b\}, H, a, e) \cong (\{e, a\}, H, H, H, e, b)$
- $S_4 \quad (\{a, b\}, H, \{e, b\}, b, a, \{e, a\}) \cong (\{e, a\}, a, \{a, b\}, H, \{e, b\}, b)$
- $S_5 \quad (a, H, \{e, b\}, \{a, b\}, a, \{e, b\}) \cong (\{e, a\}, \{a, b\}, b, H, \{e, a\}, b)$
- $S_6 \quad (\{e, a\}, H, \{e, b\}, \{e, b\}, a, \{a, b\}) \cong (\{e, a\}, \{e, a\}, \{e, b\}, H, \{a, b\}, b)$
- $S_7 \quad (a, H, \{e, b\}, b, a, H) \cong (\{e, a\}, a, b, H, H, b)$
- $S_8 \quad (H, \{e, a\}, H, \{e, b\}, b, a)$
- $S_9 \quad (H, \{e, a\}, H, H, b, e) \cong (H, H, H, \{e, b\}, e, a)$
- $S_{10} \quad (\{a, b\}, \{e, a\}, H, b, b, \{e, a\}) \cong (H, a, \{a, b\}, \{e, b\}, \{e, b\}, a)$
- $S_{11} \quad (a, \{e, a\}, H, \{a, b\}, b, \{e, b\}) \cong (H, \{a, b\}, b, \{e, b\}, \{e, a\}, a)$
- $S_{12} \quad (\{e, a\}, \{e, a\}, H, \{e, b\}, b, \{a, b\}) \cong (H, \{e, a\}, \{e, b\}, \{e, b\}, \{a, b\}, a)$
- $S_{13} \quad (a, \{e, a\}, H, b, b, H) \cong (H, a, b, \{e, b\}, H, a)$
- $S_{14} \quad (H, H, H, H, e, e)$
- $S_{15} \quad (\{a, b\}, H, H, b, e, \{e, a\}) \cong (H, a, \{a, b\}, H, \{e, b\}, e)$
- $S_{16} \quad (a, H, H, \{a, b\}, e, \{e, b\}) \cong (H, \{a, b\}, b, H, \{e, a\}, a)$
- $S_{17} \quad (\{e, a\}, H, H, \{e, b\}, e, \{a, b\}) \cong (H, \{e, a\}, \{e, b\}, H, \{a, b\}, e)$
- $S_{18} \quad (a, H, H, b, e, H) \cong (H, a, b, H, H, e)$
- $S_{19} \quad (\{a, b\}, \{a, b\}, b, b, \{e, a\}, \{e, a\}) \cong (a, a, \{a, b\}, \{a, b\}, \{e, b\}, \{e, b\})$
- $S_{20} \quad (a, \{a, b\}, b, \{a, b\}, \{e, a\}, \{e, b\})$

$$\begin{aligned}
 S_{21} & (\{e,a\},\{a,b\},b,\{e,b\},\{e,a\},\{a,b\}) \cong (a,\{e,a\},\{e,b\},\{a,b\},\{a,b\},\{e,b\}) \\
 S_{22} & (a,\{a,b\},b,b,\{e,a\},H) \cong (a,a,b,\{a,b\},H,\{e,b\}) \\
 S_{23} & (\{a,b\},a,\{a,b\},b,\{e,b\},\{e,a\}) \\
 S_{24} & (\{e,a\},a,\{a,b\},\{e,b\},\{e,b\},\{a,b\}) \cong (\{a,b\},\{e,a\},\{e,b\},b,\{a,b\},\{e,a\}) \\
 S_{25} & (a,a,\{a,b\},b,\{e,b\},H) \cong (\{a,b\},a,b,b,H,\{e,a\}) \\
 S_{26} & (\{e,a\},\{e,a\},\{e,b\},\{e,b\},\{a,b\},\{a,b\}) \\
 S_{27} & (a,\{e,a\},\{e,b\},b,\{a,b\},H) \cong (\{e,a\},a,b,\{e,b\},H,\{a,b\}) \\
 S_{28} & (a,a,b,b,H,H)
 \end{aligned}$$

THEOREM 8

Suppose in $H = \{e,a,b\}$ we want to define a H_v -group $(H, \bullet)$ for which e is unit and xy is a singleton for all $x,y \in \{a,b\}$. Denote by the ordered 9-tuples

$$(ee, ea, eb, ae, aa, ab, be, ba, bb)$$

the products needed to be defined. Then the set of all H_v -groups defined in H has a set of minimals the following ones given with their isomorphic.

$$\begin{aligned}
 I_1 & (e,H,\{e,b\},H,a,a,\{e,b\},a,a) \cong (e,\{e,a\},H,\{e,a\},b,b,H,b,b) \\
 I_2 & (e,H,\{e,b\},H,a,a,\{e,b\},a,b) \cong (e,\{e,a\},H,\{e,a\},a,b,H,b,b) \\
 I_3 & (e,\{e,a\},\{e,b\},H,a,a,H,b,b) \\
 I_4 & (e,H,H,\{e,a\},a,b,\{e,b\},a,b) \\
 I_5 & (e,\{e,a\},\{e,b\},\{e,a\},a,b,\{e,b\},b,a) \cong (e,\{e,a\},\{e,b\},\{e,a\},b,a,\{e,b\},a,b) \\
 I_6 & (e,\{a,b\},\{e,b\},\{a,b\},e,a,\{e,b\},a,b) \cong (e,\{e,a\},\{a,b\},\{e,a\},a,b,\{a,b\},b,e) \\
 I_7 & (e,a,H,a,e,b,H,b,b) \cong (e,H,b,H,a,a,b,a,e) \\
 I_8 & (e,H,H,H,a,e,\{e,b\},a,b) \cong (e,H,H,\{e,a\},a,b,H,e,b) \\
 I_9 & (e,\{e,a\},H,H,a,e,H,b,b) \cong (e,H,\{e,b\},H,a,a,H,e,b) \\
 I_{10} & (e,H,H,\{a,b\},e,e,\{e,b\},a,b) \cong (e,H,H,\{e,a\},a,b,\{a,b\},e,e) \\
 I_{11} & (e,\{e,a\},H,\{a,b\},e,e,H,b,b) \cong (e,H,\{e,b\},H,a,a,\{a,b\},e,e) \\
 I_{12} & (e,\{a,b\},\{e,b\},H,e,a,H,e,b) \cong (e,\{e,a\},\{a,b\},H,a,e,H,b,e) \\
 I_{13} & (e,\{a,b\},H,\{e,a\},e,b,H,e,b) \cong (e,H,\{a,b\},H,a,e,\{e,b\},a,e) \\
 I_{14} & (e,\{a,b\},\{e,b\},\{a,b\},e,a,\{e,b\},a,e) \cong (e,\{e,a\},\{a,b\},\{e,a\},e,b,\{a,b\},b,e) \\
 I_{15} & (e,H,\{a,b\},H,a,e,\{a,b\},e,a) \cong (e,\{a,b\},H,\{a,b\},b,e,H,e,b) \\
 I_{16} & (e,H,H,H,a,e,H,e,b) \cong (e,H,H,H,a,e,H,e,b) \\
 I_{17} & (e,a,b,a,b,e,b,e,a) \\
 I_{18} & (e,\{a,b\},\{e,b\},\{a,b\},e,e,\{e,b\},e,a) \cong (e,\{e,a\},\{a,b\},\{e,a\},b,e,\{a,b\},e,e) \\
 I_{19} & (e,\{a,b\},H,\{a,b\},e,e,H,e,b) \cong (e,H,\{a,b\},H,a,e,\{a,b\},e,e) \\
 I_{20} & (e,H,\{a,b\},\{a,b\},e,e,\{e,b\},a,e) \cong (e,\{a,b\},H,\{e,a\},e,b,\{a,b\},e,e)
 \end{aligned}$$

$$\begin{aligned} I_{21} & (e, \{e, a\}, \{a, b\}, \{a, b\}, e, e, H, b, e) \cong (e, \{a, b\}, \{e, b\}, H, e, a, \{a, b\}, e, e) \\ I_{22} & (e, \{a, b\}, \{a, b\}, \{a, b\}, e, e, \{a, b\}, e, e) \end{aligned}$$

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HYPER BCK-ALGEBRA

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Abstract: In this note the notion of a hyper BCK-algebra which is a generalization of a BCK-algebra introduced, and some basic properties for this algebra are proved. Then the notions of P -ideals and hypercongruence relations are introduced, and the relationship between them on a positive implicative hyper BCK-algebra with condition (S) , are specified.

1. Introduction and preliminaries

The theory of hypercompositional structures has been introduced by F.Marty, in 1934 [4]. Marty was led to the introduction of this structure from the study of several problems of the noncommutative algebra.

The notion of BCK-algebra first was introduced by, Y. Imai and K.Iseki, in 1966 [1]. This notion is a generalization of properties of the set-difference. Since then many mathematicians have worked in this area and obtained interesting results. For example see [2,3]. Now recall some definitions.

Definition 1.1. Let X be non-empty set with a binary operation $*$ and a constant 0 . Then X is called a BCK-algebra, if it satisfies the following conditions

$$(i) (x * y) * (x * z) \leq z * y$$

$$(ii) x * (x * y) \leq y$$

$$(iii) x \leq x$$

$$(iv) x \leq y \text{ and } y \leq x \text{ imply } x = y$$

$$(v) 0 \leq x \text{ for all } x \in X$$

where $x \leq y$ means $x * y = 0$.

Definition 1.2. A non-empty subset I of a BCK-algebra X is said to be an ideal if the following condition holds:

$$(i) 0 \in I$$

$$(ii) x * y \in I, y \in I \text{ imply } x \in I.$$

Definition 1.3. The non-empty subset A of a BCK-algebra X is called a implicative ideal if:

$$(i) 0 \in A$$

$$(ii) (y * x) * z \in A \text{ and } x * z \in A \text{ imply that } y * z \in A.$$

Definition 1.4. Let $B \neq \emptyset$. A hyperoperation on B is a function from $B \times B$ to $P^*(B) = P(B) \setminus \{\emptyset\}$.

2. Hyper BCK-Algebra

Definition 2.1. Let X be a set with a hyper operation $*$ and a constant 0 . Then X is called a hyper BCK-algebra, if it satisfies the following Conditions:

$$(H1) (x * y) * z = (x * z) * y,$$

$$(H2) t \in x * (x * y) \Rightarrow t \leq y,$$

$$(H3) x \leq y \Rightarrow x * z \leq y * z \text{ and } z * y \leq z * x,$$

$$(H4) x \leq x,$$

$$(H5) 0 \leq x,$$

$$(H6) x \leq y, y \leq x \Rightarrow x = y$$

for all $x, y, z \in X$ where $x \leq y$ is defined by $0 \in x * y$ and $A \leq B$ is defined by for all $b \in B$, there exists $a \in A$ such that $a \leq b$ for all $x, y \in X$ and $A, B \subseteq X$.

Henceforth $X = (X, *, 0)$ will be denoted a Hyper BCK-algebra.

Example 2.2. Let $X = R^+$ and

$$x * y = \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \\ \{0\} & \text{if } y = 0 \end{cases}$$

Then $(R^+, *, 0)$ is a BCK-algebra.

Notation. By $A \leq x$ and $x \leq A$ we mean $A \leq \{x\}$ and $\{x\} \leq A$, for all $x \in X$ and $A \subseteq X$.

Lemma 2.3. For all $x, y \in X$ and $A \subseteq X$,

(i) $(x * y) * A = (x * A) * y$,

(ii) $x * (x * 0) = \{0\}$.

Proof. The proof is simple.

Theorem 2.4. For all $x, y, z \in X$

(i) $(x * y) * (x * z) \leq z * y$,

(ii) $x \leq y, y \leq z \Rightarrow x \leq z$.

Proof. (i) is proved by Lemma 2.3 (i), (H2) and (H3).

(ii) The proof follows from (H3), (H5) and (H6).

Theorem 2.5. For all $x, y, z \in X$, and $A, B, C \subseteq X$

(i) $x * y \leq z \Rightarrow x * z \leq y$

(ii) $x * y \leq x$

(iii) $x \leq x * 0$

(iv) $A \leq B, B \leq C \Rightarrow A \leq C$

(v) $A * y \leq A$

(vi) $x * A \leq z \Leftrightarrow x * z \leq A$

(vii) $A \leq B \Rightarrow A * C \leq B * C$ and $C * B \leq C * A$

(viii) $A \leq A * 0$

(ix) $x \in x * 0$

(x) $t \in 0 * 0 \Leftrightarrow t = 0$

(xi) $x * x = \{x\} \Leftrightarrow x = 0$.

Proof. (i) $0 \in t * z \subseteq (x * y) * z = (x * z) * y$ for some $t \in x * y$. Thus there exists $w \in x * z$ such that $0 \in w * y$. In other words $x * z \leq y$.

(ii) $0 \in 0 * y \subseteq (x * x) * y = (x * y) * x$. Then there exists $t \in x * y$ such that $0 \in t * x$. Hence $x * y \leq x$.

(iii) The proof follows from Lemma 2.3 (ii).

(iv) Is proved by Theorem 2.4 (ii).

(v) Let $a \in A$ be an arbitrary element. Then, by (ii), $a * y \leq a$. Hence there is $b \in a * y \leq A * y$ such that $b \leq a$. That is $A * y \leq A$.

(vi) Since $x * A \leq z$, there exists $a \in A$ such that $x * a \leq z$. Hence by (i) $x * z \leq a \leq A$. Thus by (iv) $x * z \leq A$. The proof of converse is simple.

(vii) Let $A \leq B$ and $x \in B * C$. Then $x \in b * c$ for some $b \in B$, $c \in C$. Hence there is $a \in A$ such that $a \leq b$. So, by (H3), $a * c \leq b * c$.

Thus $t \leq x$, for some $t \in a * c \subseteq A * C$. That is $A * C \leq B * C$. Similarly $C * B \leq C * A$.

(viii) The proof follows from (iii).

(ix) Since $x * 0 \leq x$, there is $t \in x * 0$ such that $t \leq x$. Now since $x \leq x * 0$, we have $x \leq t$. Thus $t = x$.

(x) $t \in 0 * 0 \subseteq 0 * (0 * 0)$. Now, by (H2), $t \leq 0$. Hence by, (H5) and (H6), $t = 0$.

(xi) $\{x\} = x * x \subseteq x * (x * 0)$. Hence by (H5) $x \leq 0$. Thus $x = 0$. The proof of converse follows from (x).

3. Hyper relation

Definition 3.1. Let $A \neq \emptyset, B \neq \emptyset$. A hyper relation θ on $A \times B$ is a function from $A \times B$ to $P(B)$. If $C, D \in P^*(A)$, then by $\theta(C, D)$ we mean the set $\bigcup_{c \in C, d \in D} \theta(c, d)$. Obviously every relation is a hyper relation.

Definition 3.2. A hyper equivalence relation θ on a non-empty set Y is a hyper relation on $Y \times Y$ satisfying:

$$(i) \quad \theta(x, x) = \bigcup_{y, z \in Y} \theta(y, z) \quad (\text{reflexive})$$

$$(ii) \quad \theta(x, y) = \theta(y, x) \quad (\text{symmetric})$$

$$(iii) \quad \theta(x, z) \cap \theta(z, y) \subseteq \theta(x, y) \quad (\text{transitive})$$

for all $x, y, z \in Y$.

Example 3.3. Let $Y = R^+$ and

$$\theta(x, y) = \begin{cases} R^+ & \text{if } x = y \\ [0, \min(x, y)] & \text{if } x \neq y \end{cases}$$

Then θ is a hyper equivalence relation.

Definition 3.4. A hyper equivalence relation θ on X is called a hyper congruence relation on X iff

$$\theta(u * x, u * y) \supseteq \theta(x, y), \quad \forall x, y, u \in X. \quad (1)$$

Definition 3.5. Let $\emptyset \neq Y \subseteq X$. Then Y is called subalgebra of X if $y_1 * y_2 \subseteq Y$, for all $y_1, y_2 \in Y$.

Theorem 3.6. If θ is a hyper congruence relation on X and Y is a subalgebra of X , then the subset H of X which is defined as follows:

$$H = \{x \in X : \theta(x, y) = \theta(0, 0), \text{ for some } y \in Y\}$$

is a subalgebra of X .

Proof. The proof is simple.

Theorem 3.7. Let θ be a hyper equivalence relation on X and for all $x_1, x_2, y_1, y_2 \in X$,

$$\theta(x_1 * x_2, y_1 * y_2) \supseteq \theta(x_1, y_1) \cap \theta(x_2, y_2). \quad (2)$$

Then θ is a hyper congruence relation.

Proof. The proof follows from Definition 3.2 (i).

Notation. Let μ be a function of X to $P(X)$. Then by $\mu(A)$ we mean $\bigcup_{a \in A} \mu(a)$, for all $A \subseteq X$. Clearly

$$\bigcup_{a \in A} \mu(a * B) = \mu(A * B) = \bigcup_{b \in B} \mu(A * b)$$

Definition 3.8. A function μ of X to $P(X)$ is called a P -ideal $(\mu \Delta_P X)$ if

- (i) $\mu(0) \supseteq \mu(y)$
- (ii) $\mu(x) \supseteq \mu(x * y) \cap \mu(y)$

for all $x, y \in X$.

Definition 3.9. A non-empty subset I of X is said to be a hyper ideal of X . If

- (i) $0 \in I$
- (ii) $(x * y) \cap I \neq \emptyset$ and $y \in I \Rightarrow x \in I, \forall x, y \in X$.

Clearly each ideal is a hyper ideal.

Example 3.10. Let I be a hyper ideal of X . The function $\mu : X \rightarrow P(X)$ which is defined by

$$\mu(x) = \begin{cases} I & x \in I \\ \emptyset & x \notin I \end{cases}$$

is a P -ideal of X .

Definition 3.11. A function μ of X to $P(X)$ is called a P -implicative ideal $(\mu \Delta_{P-i} X)$ if:

- (i) $\mu(0) \supseteq \mu(y)$
- (ii) $\mu(x * z) \supseteq \mu((x * y) * z) \cap \mu(y * z)$

for all $x, y, z \in X$.

Definition 3.12. If in X , the equality

$$(x * z) * (y * z) = (x * y) * z, \forall x, y, z \in X$$

holds, then it is said to be positive implicative.

Definition 3.13. A non-empty subset I of X is called a hyper implicative ideal of X if for any $x, y, z \in X$

- (i) $0 \in I$
- (ii) $((x * y) * z) \cap I \neq \emptyset, (y * z) \cap I \neq \emptyset \Rightarrow (x * z) \cap I \neq \emptyset$.

Clearly each implicative ideal is a hyper implicative ideal.

Example 3.14. Let I be a hyper implicative ideal. Then the function which is defined in Example 3.10 is a P -implicative ideal.

Lemma 3.15. Let μ be a P -ideal of X . Then for all $\emptyset \neq A, B \subseteq X, x \in X$

- (i) $a \leq b \Rightarrow \mu(a) \supseteq \mu(b)$
- (ii) $A \leq B \Rightarrow \mu(A) \supseteq \mu(B)$
- (iii) $\mu(x * 0) = \mu(x)$

Proof. (i) $\mu(a) \supseteq \mu(a * b) \cap \mu(b) = \mu(0) \cap \mu(b) = \mu(b)$

(ii) Let b be an arbitrary element of B . Then there exist $a \in A$ such that $a \leq b$. Hence by (i), $\mu(A) \supseteq \mu(a) \supseteq \mu(b)$ for all $b \in B$. Thus $\mu(A) \supseteq \mu(B)$.

(iii) By Theorem 2.5, we have $x \leq x * 0 \leq x$. Hence $\mu(x) \supseteq \mu(x * 0) \supseteq \mu(x)$.

Theorem 3.16. (i) If $\mu\Delta_{p-i}X$, then $\mu\Delta_pX$

(ii) Let X be positive implicative and $\mu\Delta_pX$, then $\mu\Delta_{p-i}X$.

Proof. (i) The proof follows from Lemma 3.15 (iii).

(ii) Let $x, y, z \in X$. Then by Lemma 3.15 (ii) we have

$$\begin{aligned} \mu(x * z) &\supseteq \mu(t) \supseteq \mu(t * w) \cap \mu(w), \forall t \in x * z, \forall w \in y * z \\ &\supseteq \mu((x * z) * (y * z)) \cap \mu(y * z). \end{aligned}$$

Lemma 3.17. Let X be positive implicative then

$$x * y \leq (x * y) * y, \quad \forall x, y \in X.$$

Proof. For all $x, y \in X$, we have

$$(x * y) * (y * y) = (x * y) * y.$$

Now since $y * y \leq 0$, by Theorem 2.5 (viii), (vii), we get that

$$x * y \leq (x * y) * 0 \leq (x * y) * (y * y).$$

Definition 3.18. Let y, z be fixed elements of X . Consider

$$(x * y) \leq z \tag{3}$$

We suppose that there is a greatest element x satisfying (3). Then X is said satisfy the condition (S). In this case x is denoted by $yo z$.

Notation. Let $\emptyset \neq A \subseteq X$. Then by xoA and Aox we mean the sets $\{xoa : a \in A\}$ and $\{aox : a \in A\}$ respectively.

Lemma 3.19. Let X be a hyper BCK-algebra with the condition (S). Then for all $x, y \in X$, $\emptyset \neq A \subseteq X$, $(xoy) * A \leq y$ implies that $xoy \leq Aoy$.

Proof. Let $(xoy) * A \leq y$. Then from Theorem 2.5 (vi) we have $(xoy) * y \leq A$. Now let $t \in A$ be arbitrary. Then there is $w \in (xoy) * y$ such that $w \leq t$. In other words $(xoy) * y \leq t$. Hence, by Theorem 2.5 (i) $(xoy) * t \leq y$. Thus $xoy \leq toy$, for all $t \in A$. Consequently $xoy \leq Aoy$.

Definition 3.20. Let θ be a hyper relation on X . Then the function μ , from X to $P(X)$ which is defined by $\mu(x) = \theta(x, 0)$ is called the P -function induced by θ and it is denoted by $I(\theta)$.

Lemma 3.21. Let θ be a hyper congruence relation on X . Then

$$(i) \quad x_1 \leq x_2 \Rightarrow I(\theta)(x_1) \supseteq I(\theta)(x_2)$$

$$(ii) \quad A \leq B \Rightarrow I(\theta)(A) \supseteq I(\theta)(B)$$

Proof.

(i) Let $x_1 \leq x_2$. Then $0 \in x_1 * x_2$. Thus

$$\begin{aligned} I(\theta)(x_1) &= \theta(x_1, 0) \supseteq \theta(x_1 * 0, x_1 * x_2) && , \text{ by Theorem 2.5}(ix) \\ &= \theta(0, x_2) && , \text{ by Definition 3.7} \\ &= \theta(x_2, 0) && , \text{ since } \theta \text{ is symmetric} \\ &= I(\theta)(x_2) \end{aligned}$$

(ii) Let $b \in B$ be an arbitrary element. Then there is $a \in A$ such that $a \leq b$. So

$$\theta(A, 0) \supseteq \theta(a, 0) \supseteq \theta(b, 0), \quad \forall b \in B.$$

Definition 3.22. Let θ be a hyper equivalence relation on hyper BCK-algebra with the condition (S). Then θ is called O-congruence relation on X if

$$\theta(x_1oy_1, x_2oy_2) \supseteq \theta(x_1, x_2) \cap \theta(y_1, y_2)$$

If θ is a hyper congruence relation and O-congruence, then it is called a hyper O-congruence relation.

Lemma 3.23. If X is with the condition (S), then $0o0 = 0$.

Proof. By Theorem 2.5 (iii), $0o0 \leq 0o0 * 0 \leq 0$, therefore, $0o0 = 0$.

Lemma 3.24. Let θ be an O-congruence relation on a hyper BCK-algebra X with the condition (S), then

$$\theta(Aoy, 0) \supseteq \theta(A, 0) \cap \theta(y, 0) \tag{4}$$

Proof. Let $a \in A$, be an arbitrary element. Then

$$\theta(Aoy, \theta) \supseteq \theta(aoy, 0) = \theta(aoy, 0o0) \supseteq \theta(a, 0) \cap \theta(y, 0), \quad \forall a \in A$$

Hence, (4) holds.

Theorem 3.25. Let X be a positive implicative hyper BCK-algebra with the condition (S) and let θ be a hyper O-congruence relation on X . Then $I(\theta)\Delta_p X$.

Proof. Let $\mu = I(\theta)$. For all $x \in X$ we have

$$\mu(0) = \theta(0, 0) = \bigcup_{x, y \in X} \theta(x, y) \supseteq \theta(x, 0) = \mu(x).$$

Thus condition (i) of Definition 3.8 holds. Now, let $x, y \in X$. Since X is with the condition (S), $(xoy) * x \leq y$. Hence by Theorem 2.5 (i), $(xoy) * y \leq x$, which implies that $((xoy) * y) * y \leq x * y$ by Theorem 2.5 (vii). Hence by Lemma 3.17, $(xoy) * y \leq ((xoy) * y) * y \leq x * y$. Consequently $(xoy) * (x * y) \leq y$ by Theorem 2.5 (vi). So from Lemma 3.19 we conclude that

$$xoy \leq (x * y)oy, \quad \forall x, y \in X \quad (5)$$

From (5) and Lemma 3.21 (iii) we obtain that

$$\theta(xoy, 0) \supseteq \theta((x * y)oy, 0).$$

By Lemma 3.24 we have

$$\theta(xoy, 0) \supseteq \theta(x * y, 0) \cap \theta(y, 0). \quad (6)$$

Now since $0 \in x * x$, we conclude that $x * x \leq y$. Therefore $x \leq xoy$, by the condition (S). Hence

$$\theta(x, 0) \supseteq \theta(xoy, 0), \quad \forall x, y \in X, \quad \text{by Lemma 3.21(ii)}$$

Thus (6) implies that

$$\theta(x, 0) \supseteq \theta(x * y, 0) \cap \theta(y, 0)$$

In other words

$$\mu(x) \supseteq \mu(x * y) \cap \mu(y).$$

Consequently, Condition (ii) of Definition 3.8 holds.

Definition 3.26. Let $\mu : X \rightarrow P(X)$ be a function. Then for all $A \subseteq X$, by μ_A we mean the set

$$\mu_A = \{x \in X : \mu(x) \supseteq A\}.$$

Theorem 3.27. Let μ be a function from X to $P(X)$. Then

(i) If μ is a P -ideal, $A \subseteq \bigcup_{x \in X} \mu(x)$ and $\mu_A \neq \emptyset$ for all $A \subseteq X$, then

μ_A is a hyper ideal of X .

(ii) If μ_A is a hyperideal of X , for all $A \subseteq X$, then μ is a P -ideal of X .

Proof. (i) Since $A \subseteq \bigcup_{x \in X} \mu(x) = \mu(0)$, so $0 \in \mu_A$.

On the other hand, let $x * y \cap \mu_A \neq \emptyset$ and $y \in \mu_A$. Then there exists $t \in x * y$ such that $\mu(t) \supseteq A$. Thus we get

$$\mu(x * y) \supseteq \mu(t) \supseteq A, \mu(y) \supseteq A.$$

Hence $\mu(x) \supseteq \mu(x * y) \cap \mu(y) \supseteq A$ which implies that $x \in \mu_A$. In other words μ_A is a hyper ideal of X .

(ii) Let $x \in X$ be an arbitrary element. Then we put $A = \mu(x)$. Thus by hypothesis μ_A is a hyper ideal of X , so $0 \in \mu_A$. That is $\mu(0) \supseteq \mu(x)$, for all $x \in X$. On the other hand, let $x, y \in X$ and $t \in x * y$, be arbitrary. We put $\mu(t) = A$, $\mu(y) = B$ and $A \cap B = C$. Therefore μ_C is a hyper ideal, which implies that $x \in \mu_C$. That is $\mu(x) \supseteq C = \mu(t) \cap \mu(y)$ for all $t \in x * y$. Consequently

$$\mu(x) \supseteq \bigcup_{t \in x * y} (\mu(t) \cap \mu(y)) = \left(\bigcup_{t \in x * y} \mu(t) \right) \cap \mu(y) = \mu(x * y) \cap \mu(y).$$

Thus μ is a P -ideal.

Theorem 3.28. Let $\emptyset \neq A \subseteq X$. We define the function $\varphi_A : X \rightarrow P(X)$ by

$$\varphi_A(x) = \begin{cases} A & x \in A \\ \emptyset & x \notin A \end{cases}$$

Then A is a hyper ideal of X if and only if φ_A is a P -ideal of X .

Proof. Let φ_A be a P -ideal and $x \in A$. Since $\varphi_A(0) \supseteq \varphi_A(x) = A$, then we get that $0 \in A$. Now let $(x * y) \cap A \neq \emptyset$ and $y \in A$. Then there exists $t \in x * y$ such that $t \in A$. On the other hand we have

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y) \supseteq \varphi_A(t) \cap \varphi_A(y) = A.$$

Hence $x \in A$. In other words A is a hyper ideal of X .

Conversely, let A be a hyper ideal. We shall that $\varphi_A \Delta_P X$. Since $0 \in A$, we obtain that $\varphi_A(0) \supseteq \varphi_A(x)$, for all $x \in X$. Now let $x, y \in X$. If $(x * y) \cap A = \emptyset$ or $y \notin A$, then obviously

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y).$$

Now, if $(x * y) \cap A \neq \emptyset$ and $y \in A$, then $x \in A$. Hence

$$\varphi_A(x) \supseteq \varphi_A(x * y) \cap \varphi_A(y).$$

That is $\varphi_A \Delta_p X$.

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