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Polygonal Graceful Labeling of Some Simple Graphs

A. Rama Lakshmi [1]
M.P. Syed Ali Nisaya[2]

Abstract

Let $G = (V, E)$ be a graph with p vertices and q edges. Let $V(G)$ and$E(G)$ be the vertex set and edge set of Grespectively. A polygonal graceful labeling of a graph G is an injective function $\eta: V(G) \to Z^+$, where Z^+ is a set of all non-negative integers that induces a bijection $\eta^*: E(G) \to \{P_s(1), P_s(2), \ldots, P_s(q)\}$, where $P_s(q)$ is the q^{th} polygonal number such that $\eta^*(uv) = |\eta(u) - \eta(v)|$ for every edge $e = uv \in E(G)$. A graph which admits such labeling is called a polygonal graceful graph. For $s = 3$, the above labeling gives triangular graceful labeling. For $s = 4$, the above labeling gives tetragonal graceful labeling and so on. In this paper, polygonal graceful labeling is introduced and polygonal graceful labeling of some simple graphs is studied.

Keywords: Polygonal numbers, Polygonal graceful labeling, Polygonal graceful graph.

AMS Classification: 05C12[3]

[1]Research Scholar (Part Time-Internal), Department of Mathematics, The M.D.T Hindu college, Affiliated to Manonmaniam Sundaranar University, Tirunelveli–627010, Tamil Nadu, India.
Mail Id: guruji.ae@gmail.com

[2] Assistant Professor, Department of Mathematics, The M.D.T Hindu college, Affiliated to Manonmaniam Sundaranar University, Tirunelveli–627010, Tamil Nadu, India. Mail Id: syedalinisaya@mdthinducollege.org

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1. Introduction

We shall consider a simple, undirected and finite graph $G = (V, E)$ on $p = |V|$vertices and $q = |E|$edges. For all standard terminology, notations and basic definitions, we follow Harary [2] and for number theory, we follow Apostal [1]. A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/ total) labeling. Rosa [6] introduced β–valuation of a graph. Golomb [4] called it as graceful labeling. For a detailed survey of graph labeling, one can refer Gallian [3]. Ramesh and Syed Ali Nisaya [5] introduced some more polygonal graceful labeling of path. Here, we shall recall some definitions which are used in this paper.

2. Preliminaries

Definition 2.1. The **star** graph $K_{1,n}$of order $n + 1$ is a tree on n edges with one vertex having degree n and other vertices having degree 1.

Definition 2.2. A graph $S(G)$ which can be obtained from a given graph by breaking up each edge into one (or) more segment by inserting intermediate vertices between its two ends is called **subdivision graph**. It is denoted by $S(G)$.

Definition 2.3. A **path** $P_n$is obtained by joining$u_i$to the consecutive verticesu_{i+1} for $1 \leq i \leq n - 1$.

Definition 2.4. A **coconut tree** $CT(n, m)$ is a graph obtained from the path P_n by appending m new pendant edges at an end vertex.

Definition 2.5. Let P_2 be a path on two vertices and let u and v be the vertices of P_2. From u, there are m pendant vertices say $u_1, u_2, \ldots, u_m$and from v, there are n pendant vertices say$v_1, v_2, \ldots, v_n$. The resulting graph is a **bistar**$B_{m,n}$.

Definition 2.6. A **graceful labeling** of a graph G is an injective function $f: V(G) \rightarrow \{0,1,2,\ldots, q\}$ that induces a bijection $f^*: E(G) \rightarrow \{1,2,\ldots, q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|, \forall e = uv \in E(G)$. The graph which admits such a labeling is called a **graceful graph**.

Definition 2.7. A **polygonal number** is a number represented as dots (or) pebbles arranged in the shape of regular polygon. If s is the number of sides in a polygon, the formula for the $n^{th}s$–gonal number $P_s(n)$ is $P_s(n) = \frac{(s-2)(n)^2-(s-4)(n)}{2}$. For $s = 3$ gives triangular numbers. For $s = 4$ gives tetragonal numbers and so on.

3. Main results

Definition 3.1: Let $G = (V, E)$ be a graph with p vertices and q edges. Let $V(G)$ and$E(G)$ be the vertex set and edge set of Grespectively. A polygonal graceful labeling

of a graph G is an injective function $\eta: V(G) \to Z^+$, where Z^+ is a set of all non-negative integers that induces a bijection $\eta^*: E(G) \to \{P_s(1), P_s(2), \ldots, P_s(q)\}$, where $P_s(q)$ is the q^{th} polygonal number such that $\eta^*(uv) = |\eta(u) - \eta(v)|$ for every edge $e = uv \in E(G)$. A graph which admits such labeling is called a polygonal graceful graph. For $s = 3$, the above labeling gives triangular graceful labeling. For $s = 4$, the above labeling gives tetragonal graceful labeling and so on.

Example 3.2: The hexagonal graceful labeling of $K_{1,5}$is given in figure (1)

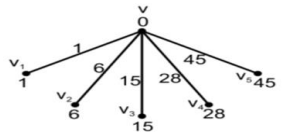

Figure (1)

Theorem 3.3: The star graph $K_{1,n}$ admits polygonal graceful labeling.
Proof: Consider the star graph $K_{1,n}$. Let $V(k_{1,n}) = \{v, v_i: 1 \le i \le n\}$ and$E(k_{1,n}) = \{vv_i: 1 \le i \le n\}$.
Then $K_{1,n}$ has $n + 1$ vertices and n edges. Define $\eta: V(K_{1,n}) \to \{0,1,2, \ldots, P_s(n)\}$ as follows.

$$\eta(v) = 0$$

$$\eta(v_i) = \frac{(s-2)i^2 - (s-4)i}{2}, \quad 1 \le i \le n$$

$= P_s(i), 1 \le i \le n$
Clearly η is injective and the induced edge labels are the first n polygonal numbers. Hence the star graph $K_{1,n}$ admits polygonal graceful graph.

Example 3.4: The pentagonal graceful labeling of $K_{1,7}$ is shown in the following figure (2)

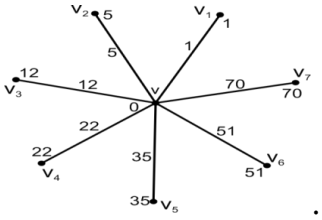

Figure (2)

Theorem 3.5: $S(K_{1,n})$, the subdivision of the star $K_{1,n}$ admits polygonal graceful labeling.
Proof: Consider the subdivision of the star graph $S(K_{1,n})$. Let $V(S(K_{1,n})) = \{v, v_i, u_i: 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iu_i: 1 \le i \le n\}$. Thus $S(K_{1,n})$ has $2n + 1$ vertices and $2n$ edges. Define $\eta: V(S(K_{1,n})) \to \{0,1,2, \ldots, P_s(2n)\}$ as follows
$\eta(v) = 0$

$$\eta(v_i) = \frac{(s-2)(2n-i+1)^2 - (s-4)(2n-i+1)}{2}, 1 \le i \le n$$
$$= P_s(2n-i+1), 1 \le i \le n$$
$$\eta(u_i) = \eta(v_i) + \frac{(s-2)(n-i+1)^2 - (s-4)(n-i+1)}{2}, 1 \le i \le n$$
$$= \eta(v_i) + P_s(n-i+1), 1 \le i \le n$$

Clearly η is injective and the induced edge labels are the first $2n$ polygonal numbers. Hence $S(K_{1,n})$ admits polygonal graceful graph.

Example 3.6: The triangular graceful labeling of $S(K_{1,3})$ is shown in the following figure (3).

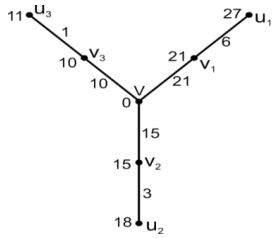

Figure (3)

Theorem 3.7: The path of length n is a polygonal graceful graph for all $n \ge 3$.

Proof. Let $G = (V, E)$ be the path of length n with the vertex set $V = \{v_1, v_2, \dots, v_n\}$and the edge set $E = \{v_i v_{i+1} / 1 \le i \le n-1\}$. Then G has n vertices and $n-1$ edges. Define $\eta: V(G) \to \{0,1,2,\dots, P_s(n-1)\}$ as follows.

$$\eta(v_{2i-1}) = (i-1)((s-2)n - (s-2)i + 1), i = 1,2,.., \left\lfloor \frac{n+1}{2} \right\rfloor$$
$$\eta(v_{2i}) = \frac{1}{2}(n-1)[(s-2)n - 2s + 6] - [(s-2)n - (s-2)i - (s-3)](i-1), i = 1,2,.., \left\lfloor \frac{n}{2} \right\rfloor$$

Clearly η is injective and the induced edge labels are the first $n-1$ polygonal numbers. Hence the path of length n is a polygonal graceful graph for all $n \ge 3$.

Example 3.8: The heptagonal graceful labeling of path of length 6 is given in figure (4).

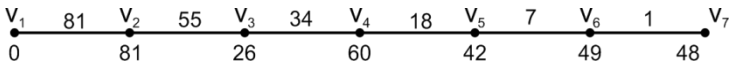

Figure (4)

Theorem 3.9: Coconut tree $CT(n,m)$ is a polygonal graceful labeling$\forall n \ge 1, m \ge 2$.

Proof: Consider the graph coconut tree $CT(n,m)$. Let $V(CT(n,m)) = \{v, v_i, u_j : 1 \le i \le n, 1 \le j \le m-1\}$ and $E(CT(n,m)) = \{vv_i, vu_1, u_j u_{j+1} : 1 \le i \le n, 1 \le j \le m-1\}$. Then $CT(n,m)$ has $m+n$ vertices and $m+n-1$ edges. Define$\eta: V(CT(n,m)) \to \{0,1,2,\dots, P_s(m+n-1)\}$ as follows.

$$\eta(v) = 0$$
$$\eta(v_i) = \frac{(s-2)(m+n-i)^2 - (s-4)(m+n-i)}{2}, 1 \le i \le n$$

$$= P_s(m+n-i), 1 \le i \le n$$

$$\eta(u_1) = \frac{(s-2)(m-1)^2 - (s-4)(m-1)}{2}$$

$$= P_s(m-1)$$

$$\eta(u_j) = \begin{cases} \eta(u_{j-1}) + \frac{(s-2)(m-j)^2 - (s-4)(m-j)}{2}, if\ j\ is\ odd\ and\ 2 \le j \le m-1 \\ \eta(u_{j-1}) - \frac{(s-2)(m-j)^2 - (s-4)(m-j)}{2}, if\ j\ is\ even\ and\ 2 \le j \le m-1 \end{cases}$$

$$\eta(u_j) = \begin{cases} \eta(u_{j-1}) + P_s(m-j), \text{ if } j \text{ is odd and } 2 \le j \le m-1 \\ \eta(u_{j-1}) - P_s(m-j), \text{ if } j \text{ is even and } 2 \le j \le m-1 \end{cases}$$

Clearly η is an injective function. Let $\eta \rightleftarrows^{*}$ be the induced edge labeling of η. Then

$$\eta^{*}(vv_i) = \frac{(s-2)(n+m-i)^2 - (s-4)(n+m-i)}{2}, 1 \le i \le n$$

$$= P_s(n+m-i), 1 \le i \le n$$

$$\eta^{*}(vu_1) = \frac{(s-2)(m-1)^2 - (s-4)(m-1)}{2}$$

$$= P_s(m-1)$$

$$\eta^{*}(u_j u_{j+1}) = \frac{(s-2)(m-j-1)^2 - (s-4)(m-j-1)}{2}, 1 \le j \le m-2$$

$$= P_s(m-j-1), 1 \le j \le m-2$$

Hence the induced edge labels are the first $m+n-1$polygonal numbers. Hence $CT(n,m)$ is a polygonal graceful graph.

Example 3.10. The octagonal graceful labeling of coconut tree $CT(5,6)$ is given in the following figure (5).

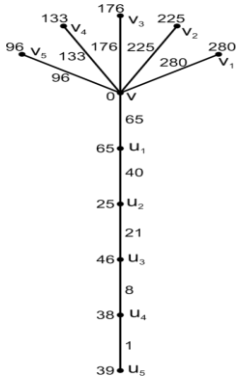

Figure (5)

Theorem 3.11. The bistar $B_{m,n}$ admits polygonal graceful labeling.

Proof.: Consider the graph $B_{m,n}$. Let $V(B_{m,n}) = \{u, v, u_i, v_j / 1 \le i \le m, 1 \le j \le n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j / 1 \le i \le m, 1 \le j \le n\}$. Then $B_{m,n}$ has $m+n+2$ vertices and $m+n+1$ edges. Define$\eta: V(B_{m,n}) \to \{0,1,2,\ldots, P_s(m+n+1)\}$ by $\eta(u) = 0$

$$\eta(v) = \frac{(s-2)(m+n+1)^2 - (s-4)(m+n+1)}{2}$$
$$= P_s(m+n+1)\eta(u_i) = \frac{(s-2)(m+n-i+1)^2 - (s-4)(m+n-i+1)}{2}, 1 \le i \le m$$
$$= P_s(m+n-i+1), 1 \le i \le m$$

$$\eta(v_j) = \eta(v) - \frac{(s-2)(n-j+2)^2 - (s-4)(n-j+2)}{2}, 1 \le j \le n$$
$$= \eta(v) - P_s(n-j+2), 1 \le j \le n$$

Clearly η is injective and the induced edge labels are the first $m+n+1$polygonal numbers. Hence $B_{m,n}$admits polygonal graceful graph.

Example 3.12: The hexagonal graceful labeling of $B_{5,4}$ is given in the following figure (6).

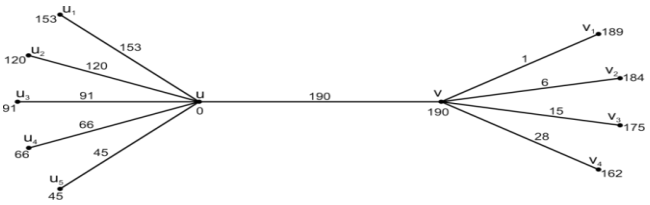

Figure (6)

4. Conclusion

In this paper, polygonal graceful labeling is introduced and polygonal graceful labeling of some simple graphs is studied. This work contributes several new results to the theory of graph labeling**.**

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Further Diversification of Nano Binary Open Sets

J. Jasmine Elizabeth[1]
G. Hari Siva Annam[2]

Abstract

The purpose of this paper is to introduce and study the nano binary exterior, nano binary border and nano binary derived in nano binary topological spaces. Also studied their characterizations.

Keywords: N_B- Derived, N_B- Exterior, N_B- Border.

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[1]Assistant Professor, Kamaraj College, Thoothukudi, (Part Time Research Scholar [Reg no. 19122102092008], Manonmaniam Sundaranar University, Tirunelveli-627012) Tamil Nadu, India. jasmineelizabeth89@gmail.com
[2]Assistant Professor, PG and Research Department of Mathematics, Kamaraj College, Thoothukudi-628003, Tamil Nadu, India. hsannam@yahoo.com. Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India.
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1. Introduction

M. Lellis Thivagar [1] introduced the concept of nano topological space with respect to a subset X of a universe U. S. Nithyanantha Jothi and P. Thangavelu [2] introduced the concept of binary topological spaces. By combining these two concepts Dr. G. Hari Siva Annam and J. Jasmine Elizabeth [3] introduced nano binary topological spaces. In this paper we have introduced the nano binary border, nano binary derived and nano binary exterior in nano binary topological spaces. Also studied their properties and characterizations with suitable examples.

2. Preliminaries

Definition 2.1: [3] Let (U_1, U_2) be a non-empty finite set of objects called the universe and R be an equivalence relation on (U_1, U_2)named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U_1, U_2, R)$is said to be the approximation space. Let $(X_1, X_2) \subseteq (U_1, U_2)$.

1. The lower approximation of (X_1, X_2) with respect to R is the set of all objects, which can be for certain classified as $(X_1, X_2)$with respect to R and it is denoted by $L_R(X_1, X_2)$.
That is,$L_R(X_1, X_2) = \cup_{(x_1,\ x_2)\epsilon(U_1,U_2)}\{R\ (x_1, x_2): R(x_1, x_2) \subseteq (X_1, X_2)\}$
Where $R(x_1, x_2)$denotes the equivalence class determined by (x_1, x_2).

2. The upper approximation of (X_1, X_2) with respect to R is the set of all objects, which can be possibly classified as $(X_1, X_2)$with respect to R and it is denoted by $U_R(X_1, X_2)$.
That is,$U_R(X_1, X_2) = \cup_{(x_1,x_2)\epsilon(U_1,U_2)}\{R\ (x_1, x_2): R(x_1, x_2) \cap (X_1, X_2) \neq \phi\}$.

3. The boundary region of (X_1, X_2) with respect to R is the set of all objects, which can be classified neither as (X_1, X_2) nor as not $-(X_1, X_2)$ with respect to R and it is denoted by $B_R(X_1, X_2)$.
That is, $B_R(X_1, X_2) = U_R(X_1, X_2) - L_R(X_1, X_2)$.

Definition 2.2: [3] Let (U_1, U_2) be the universe, R be an equivalence on (U_1, U_2) and $\tau_R(X_1, X_2) = \{(U_1, U_2), (\phi, \phi), L_R(X_1, X_2), U_R(X_1, X_2), B_R(X_1, X_2)\}$ where $(X_1, X_2) \subseteq (U_1, U_2)$. Then by the property R$(X_1, X_2)$satisfies the following axioms

1. $(U_1, U_2)$and $(\phi, \phi) \epsilon\ \tau_R(X_1, X_2)$.
2. The union of the elements of any sub collection of $\tau_R(X_1, X_2)$ is in $\tau_R(X_1, X_2)$.
3. The intersection of the elements of any finite sub collection of $\tau_R(X_1, X_2)$ is in $\tau_R(X_1, X_2)$.

That is,$\tau_R(X_1, X_2)$ is a topology on (U_1, U_2) called the nano binary topology on (U_1, U_2) with respect to(X_1, X_2).

We call $(U_1, U_2, \tau_R(X_1, X_2))$as the nano binary topological spaces. The elements of $\tau_R(X_1, X_2)$are called as nano binary open sets and it is denoted by N_B open sets. Their complement is called N_B closed sets.

Definition 2.3: [3] If $\big(U_1, U_2, \tau_R(X_1, X_2)\big)$is a nano binary topological spaces with respect to (X_1, X_2) and if $(H_1, H_2) \subseteq (U_1, U_2)$, then the nano binary interior of (H_1, H_2) is defined as the union of all $N_B$open subsets of (A_1, A_2) and it is defined by ${N_B}^{\circ}(H_1, H_2)$.

That is, ${N_B}^{\circ}(H_1, H_2)$ is the largest $N_B$open subset of(H_1, H_2). The nano binary closure of (H_1, H_2) is defined as the intersection of all $N_B$closed sets containing (H_1, H_2) and it is denoted by $\overline{N_B}(H_1, H_2)$.

That is, $\overline{N_B}(H_1, H_2)$ is the smallest $N_B$closed set containing(H_1, H_2).

3. Nano Binary Derived

Definition 3.1: A point $(x_1, x_2) \in (U_1, U_2)$ is said to be a N_B limit point of (A_1, A_2) if for each N_B-open set (K_1, K_2) containing (x_1, x_2) satisfies $(K_1, K_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$.

Definition 3.2: The set of all N_B limit points of (A_1, A_2) is said to be nano binary derived set and is denoted by $N_B_D(A_1, A_2)$.

Theorem 3.3: In $(U_1, U_2, \tau_R(X_1, X_2))$, let (A_1, A_2) and (B_1, B_2) be two subsets of (U_1, U_2). Then the following holds:
1) $N_B_D(\emptyset, \emptyset) = (\emptyset, \emptyset)$.
2) If $(x_1, x_2) \in N_B_D(A_1, A_2)$ then $(x_1, x_2) \in N_B_D((A_1, A_2) - (x_1, x_2))$.
3) If $(A_1, A_2) \subseteq (B_1, B_2)$, then $N_B_D(A_1, A_2) \subseteq N_B_D(B_1, B_2)$.
4) $N_B_D(A_1, A_2) \cup N_B_D(B_1, B_2) = N_B_D((A_1, A_2) \cup (B_1, B_2))$.
Proof: 1) Let $(x_1, x_2) \in (U_1, U_2)$ and (G_1, G_2) be a N_B-open set containing (x_1, x_2). Then $\big((G_1, G_2) - (x_1, x_2)\big) \cap (\emptyset, \emptyset) = (\emptyset, \emptyset) \Rightarrow (x_1, x_2) \notin N_B_D(\emptyset, \emptyset)$. Therefore, for any $(x_1, x_2) \in (U_1, U_2)$, (x_1, x_2) is not aN_B limit point of $(\emptyset, \emptyset)$. Hence $N_B_D(\emptyset, \emptyset) = (\emptyset, \emptyset)$.
2)Let $(x_1, x_2) \in N_B_D(A_1, A_2)$. Then $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$, for every N_B-open set (G_1, G_2) containing (x_1, x_2) implies every N_B-open set (G_1, G_2) of (x_1, x_2), contains at least one point other than (x_1, x_2) of (A_1, A_2). Therefore $(x_1, x_2) \in N_B_D((A_1, A_2) - (x_1, x_2))$.
3)Let $(x_1, x_2) \in N_B_D(A_1, A_2)$. Then $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$, for every N_B-open set (G_1, G_2) containing (x_1, x_2). Since $(A_1, A_2) \subseteq (B_1, B_2)$ implies $(G_1, G_2) \cap \big((B_1, B_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset) \Rightarrow (x_1, x_2) \in N_B_D(B_1, B_2)$. Thus $(x_1, x_2) \in N_B_D(A_1, A_2) \Rightarrow (x_1, x_2) \in N_B_D(B_1, B_2)$. Therefore $N_B_D(A_1, A_2) \subseteq N_B_D(B_1, B_2)$.
4)Since $(A_1, A_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$ and $(B_1, B_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$. By (3), $N_B_D(A_1, A_2) \subseteq N_B_D\big((A_1, A_2) \cup (B_1, B_2)\big)$ and $N_B_D(B_1, B_2) \subseteq N_B_D((A_1, A_2) \cup$

$(B_1, B_2))$. Therefore, $N_B_D(A_1, A_2) \cup N_B_D(B_1, B_2) \subseteq N_B_D((A_1, A_2) \cup (B_1, B_2))$…. (1). Let $(x_1, x_2) \notin N_B_D(A_1, A_2) \cup N_B_D(B_1, B_2)$. Then $(x_1, x_2) \notin N_{B_{D(A_1, A_2)}}$and $(x_1, x_2) \notin N_B_D(B_1, B_2)$. Therefore, there exists N_B-open sets (G_1, G_2) and (H_1, H_2) containing (x_1, x_2) such that $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) = (\emptyset, \emptyset)$ and $(H_1, H_2) \cap \big((B_1, B_2) - (x_1, x_2)\big) = (\emptyset, \emptyset)$. Since $(G_1, G_2) \cap (H_1, H_2) \subseteq (G_1, G_2)$ and (H_1, H_2), $((G_1, G_2) \cap (H_1, H_2)) \cap \big((A_1, A_2) - (x_1, x_2)\big) = (\emptyset, \emptyset)$ and $((G_1, G_2) \cap (H_1, H_2)) \cap \big((B_1, B_2) - (x_1, x_2)\big) = (\emptyset, \emptyset)$. Also $(G_1, G_2) \cap (H_1, H_2)$ is a N_B-open set containing(x_1, x_2). Therefore, $((G_1, G_2) \cap (H_1, H_2)) \cap \big(((A_1, A_2) \cup (B_1, B_2)) - (x_1, x_2)\big) = (\emptyset, \emptyset)$. That is, (x_1, x_2) is not a N_B limit point of $(A_1, A_2) \cup (B_1, B_2)$. Hence $(x_1, x_2) \notin N_B_D((A_1, A_2) \cup (B_1, B_2))$. Therefore, $N_B_D((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B_D(A_1, A_2) \cup N_B_D(B_1, B_2)$…. (2). From (1) and (2),$N_B_D(A_1, A_2) \cup N_B_D(B_1, B_2) = N_B_D((A_1, A_2) \cup (B_1, B_2))$.

Theorem 3.4: Let $(A_1, A_2)$and (B_1, B_2) be two subsets of N_B topological space $(U_1, U_2, \tau_R(X_1, X_2))$.Then $N_B_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B_D(A_1, A_2) \cap N_B_D(B_1, B_2)$.

Proof: Since $(A_1, A_2) \cap (B_1, B_2) \subseteq (A_1, A_2)$and $(A_1, A_2) \cap (B_1, B_2) \subseteq (B_1, B_2)$. By the previous theorem,$N_B_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B_D(A_1, A_2)$and $N_B_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B_D(B_1, B_2)$. Therefore,$N_B_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B_D(A_1, A_2) \cap N_B_D(B_1, B_2)$.

Remark 3.5: The reverse inclusion may not true as shown in the following example.

Example 3.6: $U_1 = \{a, b, c\}$, $U_2 = \{1, 2\}$ with $^{(U_1, U_2)}/_R = \{(\{a, b\}, \{2\}), (\{c\}, \{1\})\}$. Let (X_1, X_2) =$(\{b\}, \{2\})$. Then $\tau_R(X_1, X_2)$ =$\{(\Phi, \Phi), (U_1, U_2), (\{a, b\}, \{2\})\}$. Here $(A_1, A_2) = (\{a, b\}, \{1\})$and $(B_1, B_2) = (\{b, c\}, \{1,2\})$, $(A_1, A_2) \cap (B_1, B_2)$=$(\{b\}, \{1\})$ and hence $N_B_D((A_1, A_2) \cap (B_1, B_2))$=$\{(\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, \{2\}), (\{c\}, \{1\}), (\{c\}, \{2\})\}$. Also $N_B_D(A_1, A_2)$=$\{(\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, \{1\}), (\{b\}, \{2\}), (\{c\}, \{1\}), (\{c\}, \{2\})\}$ and $N_B_D(B_1, B_2)$=$\{(\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, \{1\}), (\{c\}, \{1\}), (\{c\}, \{2\})\}$. But $N_B_D(A_1, A_2) \cap N_B_D(B_1, B_2)$=$\{(\{a\}, \{1\}), (\{a\}, \{2\}), (\{b\}, \{1\}), (\{c\}, \{1\}), (\{c\}, \{2\})\}$.
Thus, $N_B_D(A_1, A_2) \cap N_B_D(B_1, B_2) \nsubseteq N_B_D((A_1, A_2) \cap (B_1, B_2))$.

Theorem 3.7: $\overline{N_B}(A_1, A_2) = (A_1, A_2) \cup N_B_D(A_1, A_2)$, where $(A_1, A_2) \subseteq (U_1, U_2)$.
Proof: If $(x_1, x_2) \in (A_1, A_2) \cup N_B_D(A_1, A_2)$, Then $(x_1, x_2) \in (A_1, A_2)$ or $(x_1, x_2) \in N_B_D(A_1, A_2)$. Let $(x_1, x_2) \notin (A_1, A_2)$. Then $(x_1, x_2) \in N_B_D(A_1, A_2)$. Therefore, for every N_B-open set (G_1, G_2)containing (x_1, x_2), $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$. Since $(x_1, x_2) \notin (A_1, A_2), (G_1, G_2) \cap (A_1, A_2) \neq (\emptyset, \emptyset)$. Therefore, $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$. Therefore, $(A_1, A_2) \cup N_B_D(A_1, A_2) \subseteq \overline{N_B}(A_1, A_2)$…. (1). Let $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$and $(x_1, x_2) \in (A_1, A_2)$. Then the result is obvious. If $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$and $(x_1, x_2) \notin (A_1, A_2)$. Therefore, $(G_1, G_2) \cap (A_1, A_2) \neq (\emptyset, \emptyset)$ for every N_B-open set (G_1, G_2) containing (x_1, x_2) and hence $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq$

$(\emptyset,\emptyset)$. Therefore, $(x_1,x_2)\in N_B_D(A_1,A_2)$ and hence $(x_1,x_2)\in(A_1,A_2)\cup N_B_D(A_1,A_2)$. Therefore, $\overline{N_B}(A_1,A_2)\subseteq(A_1,A_2)\cup N_B_D(A_1,A_2)\ldots$ (2). From (1) and (2), $\overline{N_B}(A_1,A_2)=(A_1,A_2)\cup N_B_D(A_1,A_2)$.

Result 3.8: ${N_B}^o(A_1,A_2)=(A_1,A_2)-N_B_D[(\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)]$, where $(A_1,A_2)\subseteq(\mathrm{U}_1,\mathrm{U}_2)$.

Proof: By the previous theorem, $\overline{N_B}(A_1,A_2)=(A_1,A_2)\cup N_B_D(A_1,A_2)\Rightarrow(\mathrm{U}_1,\mathrm{U}_2)-\overline{N_B}(A_1,A_2)=\big((\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)\big)\cap\big((\mathrm{U}_1,\mathrm{U}_2)-N_B_D(A_1,A_2)\big)\Rightarrow(\mathrm{U}_1,\mathrm{U}_2)-\overline{N_B}(A_1,A_2)=\big((\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)\big)-N_B_D(A_1,A_2)\Rightarrow{N_B}^o\big((\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)\big)=\big((\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)\big)-N_B_D(A_1,A_2)$. By replacing $(\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)$ by (A_1,A_2) and $(A_1,A_2)$by $(\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)$, ${N_B}^o(A_1,A_2)=(A_1,A_2)-N_B_D[(\mathrm{U}_1,\mathrm{U}_2)-(A_1,A_2)]$.

4. Nano Binary Exterior

Definition 4.1: For a subset$(A_1,A_2)\subseteq(U_1,U_2)$, the nano binary exterior of (A_1,A_2) is defined as${N_B}^o((U_1,U_2)-(A_1,A_2))$. It is denoted by $N_B_E(A_1,A_2)$.

Definition 4.2: For a subset$(A_1,A_2)\subseteq(U_1,U_2)$, the nano binary border of (A_1,A_2) is defined as $(A_1,A_2)-{N_B}^o(A_1,A_2)$. It is denoted by $N_B_B(A_1,A_2)$.

Theorem 4.3: Let $(A_1,A_2)$and (B_1,B_2) be two subsets of N_B topological space $(U_1,U_2,\tau_R(X_1,X_2))$. Then the following holds:
1) If $(A_1,A_2)\subseteq(B_1,B_2)$, then $N_B_E(B_1,B_2)\subseteq N_B_E(A_1,A_2)$.
2) $N_B_E\big((A_1,A_2)\cup(B_1,B_2)\big)\subseteq N_B_E(A_1,A_2)\cup N_B_E(B_1,B_2)$.
3) $N_B_E(A_1,A_2)\cap N_B_E(B_1,B_2)\subseteq N_B_E\big((A_1,A_2)\cap(B_1,B_2)\big)$.
Proof: 1) If $(A_1,A_2)\subseteq(B_1,B_2)$then $(U_1,U_2)-(B_1,B_2)\subseteq(U_1,U_2)-(A_1,A_2)\Rightarrow{N_B}^o\big((U_1,U_2)-(B_1,B_2)\big)\subseteq{N_B}^o\big((U_1,U_2)-(A_1,A_2)\big)\Rightarrow N_B_E(B_1,B_2)\subseteq N_B_E(A_1,A_2)$.
2) Since $(A_1,A_2)\subseteq(A_1,A_2)\cup(B_1,B_2)$and $(B_1,B_2)\subseteq(A_1,A_2)\cup(B_1,B_2)$. By (1), $N_B_E\big((A_1,A_2)\cup(B_1,B_2)\big)\subseteq N_B_E(A_1,A_2)$and $N_B_E\big((A_1,A_2)\cup(B_1,B_2)\big)\subseteq N_B_E(B_1,B_2)$. Therefore, $N_B_E\big((A_1,A_2)\cup(B_1,B_2)\big)\subseteq N_B_E(A_1,A_2)\cup N_B_E(B_1,B_2)$.
3) Since $(A_1,A_2)\cap(B_1,B_2)\subseteq(A_1,A_2)$and $(A_1,A_2)\cap(B_1,B_2)\subseteq(B_1,B_2)$. By (1) $N_B_E(A_1,A_2)\subseteq N_B_E\big((A_1,A_2)\cap(B_1,B_2)\big)$and $N_B_E(B_1,B_2)\subseteq N_B_E\big((A_1,A_2)\cap(B_1,B_2)\big)$. Therefore, $N_B_E(A_1,A_2)\cap N_B_E(B_1,B_2)\subseteq N_B_E\big((A_1,A_2)\cap(B_1,B_2)\big)$.

Remark 4.4: The inclusion may be strict. We can see in the following example.

Example 4.5: Let $\mathrm{U}_1=\{a,b,c\},\mathrm{U}_2=\{1,2\}$ with ${(\mathrm{U}_1,\mathrm{U}_2)}/_{\mathrm{R}}\{(\{\mathrm{a},\mathrm{b}\},\{2\}),(\{\mathrm{c}\},\{1\})\}$. Let $(\mathrm{X}_1,\mathrm{X}_2)=(\{\mathrm{b}\},\{2\})$. Then $\tau_{\mathrm{R}}(\mathrm{X}_1,\mathrm{X}_2)=\{(\Phi,\Phi),(\mathrm{U}_1,\mathrm{U}_2),(\{\mathrm{a},\mathrm{b}\},\{2\})\}$.
2) Take $(A_1,A_2)=(\{a,b\},\{2\})\ and\ (B_1,B_2)=(\{\mathrm{c}\},\{1\})$. $N_B_E(\{a,b\},\{2\})\cup N_B_E(\{\mathrm{c}\},\{1\})={N_B}^o(\{\mathrm{c}\},\{1\})-{N_B}^o(\{a,b\},\{2\})=(\Phi,\Phi)-(\{a,b\},\{2\})=(\{a,\ b\},\{2\})$. Also, $N_B_E((\{a,b\},\{2\})\cup(\{\mathrm{c}\},\{1\}))=N_B_E(\mathrm{U}_1,\mathrm{U}_2)={N_B}^o(\Phi,\Phi)=$

(Φ,Φ). Therefore, $(\{a,b\},\{2\}) \nsubseteq (\Phi,\Phi)$ and hence $N_B_E\big((A_1,A_2)\cup(B_1,B_2)\big) \subset N_B_E(A_1,A_2)\cup N_B_E(B_1,B_2)$.

3) Take $(A_1,A_2)=(\{c\},\{1,2\})\ and\ (B_1,B_2)=(\{a,c\},\{1\})$. $N_B_E((\{c\},\{1,2\})\cap(\{a,c\},\{1\}))=N_B_E(\{c\},\{1\})={N_B}^o(\{a,b\},\{2\})=(\{a,b\},\{2\})$and $N_B_E(\{c\},\{1,2\})\cap N_B_E(\{a,c\},\{1\})={N_B}^o(\{a,b\},\{\emptyset\})\cap {N_B}^o(\{b\},\{2\})=(\Phi,\Phi)\cap(\Phi,\Phi)=(\Phi,\Phi)$. Therefore, $(\{a,b\},\{2\})\nsubseteq(\Phi,\Phi)$ and hence $N_B_E(A_1,A_2)\cap N_B_E(B_1,B_2)\subset N_B_E((A_1,A_2)\cap(B_1,B_2))$.

Theorem 4.6: Let (A_1,A_2) and (B_1,B_2) be two subsets of N_B topological space $(U_1,U_2,\tau_R(X_1,X_2))$. Then the following holds:
1) $N_B_E(A_1,A_2)=(U_1,U_2)-\overline{N_B}(A_1,A_2)$
2)$N_B_E(N_B_E(A_1,A_2))={N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$
3) $N_B_E(U_1,U_2)=(\emptyset,\emptyset)andN_B_E(\emptyset,\emptyset)=(U_1,U_2)$
4) $N_B_E(A_1,A_2)=N_B_E[(U_1,U_2)-N_B_E(A_1,A_2)]$
5) ${N_B}^o(A_1,A_2)\subseteq N_B_E(N_B_E(A_1,A_2)$
6) ${N_B}^o(A_1,A_2),N_B_E(A_1,A_2),N_B_F(A_1,A_2)$ are mutually disjoint and $(U_1,U_2)={N_B}^o(A_1,A_2)\cup N_B_E(A_1,A_2)\cup N_B_F(A_1,A_2)$.
7) $(A_1,A_2)\cap N_B_E(A_1,A_2)=(\emptyset,\emptyset)$
8) $N_B_E(A_1,A_2)\subseteq(U_1,U_2)-(A_1,A_2)$
Proof: 1)$N_B_E(A_1,A_2)={N_B}^o\big((U_1,U_2)-(A_1,A_2)\big)=(U_1,U_2)-\overline{N_B}(A_1,A_2)$. Hence (1) is proved.

2)$N_B_E(N_B_E(A_1,A_2))=N_B_E\big[{N_B}^o\big((U_1,U_2)-(A_1,A_2)\big)\big]=N_B_E\big[(U_1,U_2)-\overline{N_B}(A_1,A_2)\big]={N_B}^o\left((U_1,U_2)-[(U_1,U_2)-\overline{N_B}(A_1,A_2)]\right)=$
${N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$.Therefore, $N_B_E(N_B_E(A_1,A_2))={N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$.

3)$N_B_E(U_1,U_2)={N_B}^o((U_1,U_2)-(U_1,U_2))={N_B}^o(\emptyset,\emptyset)=(\emptyset,\emptyset)$and $N_B_E(\emptyset,\emptyset)={N_B}^o((U_1,U_2)-(\emptyset,\emptyset))={N_B}^o(U_1,U_2)=(U_1,U_2)$. Therefore, $N_B_E(U_1,U_2)=(\emptyset,\emptyset)$and $N_B_E(\emptyset,\emptyset)=(U_1,U_2)$.

4)$N_B_E[(U_1,U_2)-N_B_E(A_1,A_2)]={N_B}^o\big((U_1,U_2)-[(U_1,U_2)-N_B_E(A_1,A_2)]\big)={N_B}^o(N_B_E(A_1,A_2))={N_B}^o[{N_B}^o\big((U_1,U_2)-(A_1,A_2)\big)]={N_B}^o\big((U_1,U_2)-(A_1,A_2)\big)=N_B_E(A_1,A_2)$. Therefore, $N_B_E(A_1,A_2)=N_B_E[(U_1,U_2)-N_B_E(A_1,A_2)]$.

5)Since $(A_1,A_2)\subseteq\overline{N_B}(A_1,A_2)\Rightarrow{N_B}^o(A_1,A_2)\subseteq{N_B}^o\left(\overline{N_B}(A_1,A_2)\right)={N_B}^o\big((U_1,U_2)-{N_B}^o[(U_1,U_2)-(A_1,A_2)]\big)=N_B_E(N_B_E[(U_1,U_2)-(A_1,A_2)])=$
$N_B_E(N_B_E(A_1,A_2))$. Therefore, ${N_B}^o(A_1,A_2)\subseteq N_B_E(N_B_E(A_1,A_2))$.

6)Assume that $N_B_E(A_1,A_2)\cap{N_B}^o(A_1,A_2)\neq(\emptyset,\emptyset)$. Then there exists $(x_1,x_2)\in N_B_E(A_1,A_2)\cap{N_B}^o(A_1,A_2)\Rightarrow(x_1,x_2)\in N_B_E(A_1,A_2)$ and $(x_1,x_2)\in{N_B}^o(A_1,A_2)\Rightarrow(x_1,x_2)\in(U_1,U_2)-(A_1,A_2)$and $(x_1,x_2)\in(A_1,A_2)$, which is not possible. Hence our

assumption is wrong. Therefore, $N_B_E(A_1, A_2) \cap {N_B}^o(A_1, A_2) = (\emptyset, \emptyset)$. In the same way we can prove the others. $N_B_E(A_1, A_2) = (U_1, U_2) - \overline{N_B}(A_1, A_2) = (U_1, U_2) - \overline{N_B}(A_1, A_2) = (U_1, U_2) - [{N_B}^o(A_1, A_2) \cup N_B_F(A_1, A_2)]$. Therefore, $(U_1, U_2) = N_B_E(A_1, A_2) \cup {N_B}^o(A_1, A_2) \cup N_B_F(A_1, A_2)$.

7)$(A_1, A_2) \cap N_B_E(A_1, A_2) = (A_1, A_2) \cap {N_B}^o((U_1, U_2) - (A_1, A_2)) \subseteq (A_1, A_2) \cap ((U_1, U_2) - (A_1, A_2)) = (\emptyset, \emptyset)$. Therefore, $(A_1, A_2) \cap N_B_E(A_1, A_2) = (\emptyset, \emptyset)$.

8)$N_B_E(A_1, A_2) = {N_B}^o((U_1, U_2) - (A_1, A_2)) \subseteq (U_1, U_2) - (A_1, A_2)$.

Note 4.7: If (A_1, A_2) is N_B closed, then equality holds in (5).

Theorem 4.8: In $(U_1, U_2, \tau_R(X_1, X_2))$, $(A_1, A_2)$and (B_1, B_2) be two subsets of (U_1, U_2). Then the following holds:
1) ${N_B}^o(A_1, A_2) \cap N_B_B(A_1, A_2) = (\emptyset, \emptyset)$
2) (A_1, A_2) is N_B-open if and only if $N_B_B(A_1, A_2) = (\emptyset, \emptyset)$
3) ${N_B}^o(N_B_B(A_1, A_2)) = (\emptyset, \emptyset)$
4) $N_B_B({N_B}^o(A_1, A_2)) = (\emptyset, \emptyset)$
5) $N_B_B(N_B_B(A_1, A_2)) = N_B_B(A_1, A_2)$
6) $N_B_B(A_1, A_2) = (A_1, A_2) - {N_B}^o(A_1, A_2) = (A_1, A_2) \cap \overline{N_B}((U_1, U_2) - (A_1, A_2))$.
7) If $(A_1, A_2) \subseteq (B_1, B_2)$, then $N_B_B(B_1, B_2) \subseteq N_B_B(A_1, A_2)$.
8) $N_B_B((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B_B(A_1, A_2) \cup N_B_B(B_1, B_2)$.
9) $N_B_B(A_1, A_2) \cap N_B_B(B_1, B_2) \subseteq N_B_B((A_1, A_2) \cap (B_1, B_2))$.
10) $N_B_B(A_1, A_2) = N_B_D((U_1, U_2) - (A_1, A_2))$and $N_B_D(A_1, A_2) = N_B_B((U_1, U_2) - (A_1, A_2))$.
11) $(A_1, A_2) = {N_B}^o(A_1, A_2) \cup N_B_B(A_1, A_2)$.
Proof: 1) ${N_B}^o(A_1, A_2) \cap N_B_B(A_1, A_2) = {N_B}^o(A_1, A_2) \cap [(A_1, A_2) - {N_B}^o(A_1, A_2)] = {N_B}^o(A_1, A_2) \cap [(A_1, A_2) \cap ((U_1, U_2) - {N_B}^o(A_1, A_2))] = {N_B}^o(A_1, A_2) \cap ((U_1, U_2) - {N_B}^o(A_1, A_2)) \cap (A_1, A_2) = (\emptyset, \emptyset) \cap (A_1, A_2) = (\emptyset, \emptyset)$.Therefore, ${N_B}^o(A_1, A_2) \cap N_B_B(A_1, A_2) = (\emptyset, \emptyset)$

2)Any subset (A_1, A_2) of N_B topological space $(U_1, U_2, \tau_R(X_1, X_2))$ is N_B-open $\Leftrightarrow (A_1, A_2) = {N_B}^o(A_1, A_2) \Leftrightarrow (A_1, A_2) - {N_B}^o(A_1, A_2) = (\emptyset, \emptyset) \Leftrightarrow N_B_B(A_1, A_2) = (\emptyset, \emptyset)$.

3)${N_B}^o(N_B_B(A_1, A_2)) = {N_B}^o((A_1, A_2) - {N_B}^o(A_1, A_2)) = {N_B}^o[(A_1, A_2) \cap \big((U_1, U_2) - {N_B}^o(A_1, A_2)\big)] \subseteq {N_B}^o(A_1, A_2) \cap {N_B}^o\big((U_1, U_2) - {N_B}^o(A_1, A_2)\big) \subseteq {N_B}^o(A_1, A_2) \cap \big((U_1, U_2) - {N_B}^o(A_1, A_2)\big) = (\emptyset, \emptyset)$. Therefore, ${N_B}^o(N_B_B(A_1, A_2)) = (\emptyset, \emptyset)$.

4)$N_B_B({N_B}^o(A_1, A_2)) = {N_B}^o(A_1, A_2) - {N_B}^o({N_B}^o(A_1, A_2)) = {N_B}^o(A_1, A_2) - {N_B}^o(A_1, A_2) = (\emptyset, \emptyset)$. Therefore, $N_B_B({N_B}^o(A_1, A_2)) = (\emptyset, \emptyset)$.

5)$N_B_B(N_B_B(A_1, A_2)) = N_B_B(A_1, A_2) - {N_B}^o(N_B_B(A_1, A_2)) = N_B_B(A_1, A_2) - (\emptyset, \emptyset)$ (By (3)) = $N_B_B(A_1, A_2)$. Therefore, $N_B_B(N_B_B(A_1, A_2)) = N_B_B(A_1, A_2)$.

6)$N_B_B(A_1, A_2) = (A_1, A_2) - {N_B}^o(A_1, A_2) = (A_1, A_2) \cap ((U_1, U_2) - {N_B}^o(A_1, A_2)) = (A_1, A_2) \cap \overline{N_B}((U_1, U_2) - (A_1, A_2))$. Therefore, $N_B_B(A_1, A_2) = (A_1, A_2) \cap \overline{N_B}((U_1, U_2) - (A_1, A_2))$.

7)If $(A_1, A_2) \subseteq (B_1, B_2)$, then ${N_B}^o(A_1, A_2) \subseteq {N_B}^o(B_1, B_2) \Rightarrow (U_1, U_2) - {N_B}^o(B_1, B_2) \subseteq (U_1, U_2) - {N_B}^o(A_1, A_2) \Rightarrow (A_1, A_2) \cap ((U_1, U_2) - {N_B}^o(B_1, B_2)) \subseteq (A_1, A_2) \cap ((U_1, U_2) - {N_B}^o(A_1, A_2)) \Rightarrow (A_1, A_2) - {N_B}^o(B_1, B_2) \subseteq (A_1, A_2) - {N_B}^o(A_1, A_2) \Rightarrow N_B_B(B_1, B_2) \subseteq N_B_B(A_1, A_2)$ (By (6))

8)Since $(A_1, A_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$and $(B_1, B_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$. By (4) $N_B_B((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B_B(A_1, A_2)$and $N_B_B((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B_B(B_1, B_2)$. Therefore, $N_B_B((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B_B(A_1, A_2) \cup N_B_B(B_1, B_2)$.

9)Since $(A_1, A_2) \cap (B_1, B_2) \subseteq (A_1, A_2)$and $(A_1, A_2) \cap (B_1, B_2) \subseteq (B_1, B_2)$. By (4), $N_B_B(A_1, A_2) \subseteq N_B_B((A_1, A_2) \cap (B_1, B_2))$and $N_B_B(B_1, B_2) \subseteq N_B_B((A_1, A_2) \cap (B_1, B_2))$. Therefore, $N_B_B(A_1, A_2) \cap N_B_B(B_1, B_2) \subseteq N_B_B((A_1, A_2) \cap (B_1, B_2))$.

10)$N_B_B(A_1, A_2) = (A_1, A_2) - {N_B}^o(A_1, A_2)$. By result 3.8, $(A_1, A_2) - {N_B}^o(A_1, A_2) = (A_1, A_2) - [(A_1, A_2) - N_B_D((U_1, U_2) - (A_1, A_2))] = N_B_D((U_1, U_2) - (A_1, A_2))$By replacing $(A_1, A_2)$by $(U_1, U_2) - (A_1, A_2)$, $N_B_D(A_1, A_2) = N_B_B((U_1, U_2) - (A_1, A_2))$.

11)${N_B}^o(A_1, A_2) \cup N_B_B(A_1, A_2) = {N_B}^o(A_1, A_2) \cup ((A_1, A_2) - {N_B}^o(A_1, A_2)) = {N_B}^o(A_1, A_2) \cup ((A_1, A_2) \cap \big((U_1, U_2) - {N_B}^o(A_1, A_2)\big)) = ({N_B}^o(A_1, A_2) \cup (A_1, A_2)) \cap ({N_B}^o(A_1, A_2) \cup \big((U_1, U_2) - {N_B}^o(A_1, A_2)\big)) = (A_1, A_2) \cap (U_1, U_2) = (A_1, A_2)$.
Therefore, $(A_1, A_2) = {N_B}^o(A_1, A_2) \cup N_B_B(A_1, A_2)$.

5. Conclusion

Nano binary derived, nano binary border and nano binary exterior in nano binary topological spaces were introduced and their properties were discussed. In future we will discuss generalized closed sets in nano binary topological spaces.

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Fuzzy Translation in Fuzzy d-Ideals and Fuzzy d-Subalgebra

R. G. Keerthana[1]
K. R. Sobha[2]

Abstract

In this paper, we discuss about some properties such as fuzzy translation in fuzzy d-ideals and fuzzy d-subalgebra.

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[1]Research Scholar, Reg.No.20213182092001, Sree Ayyappa College for Women, Chunkankadai, Nagercoil-62900. [Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627012, Tamilnadu, India.]. keerthanarajagopal.rg@gmail.com

[2]Assistant Professor, Department of Mathematics, Sree Ayyappa College for Women, Chunkankadai, Nagercoil. [Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627012, Tamilnadu, India.] vijayakumar.sobha9@gmail.com.

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1. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [6]. The study of fuzzy subsets and its applications to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics. It forms a branch of mathematics including fuzzy set theory and fuzzy logic.

Fuzzy set theory was guided by the assumption that the classical sets were not natural appropriate or useful notions in describing the real-life problems because every object encountered in the real physical world carries some degree of fuzziness. Hence fuzzy set has become strong area of research in engineering, medical science, graph theory etc.

Algebraic structures play an important role in mathematics with wide range of application in many disciplines such as computer sciences, control engineering, theoretical physics, information systems and topological spaces. Since these ideas have been applied to other algebraic structures such as group, semigroup, ring, modules, vector spaces and topologies. It gives enthusiasm to the researchers to view various concepts and results from the area of abstract algebra in the broader frame work of fuzzy setting.

Fuzzy algebra is an important branch of fuzzy mathematics. In 1996, J. Negger and H.S. Kim [3] introduced the class of d-algebra which is a generalization of BCK-algebras and investigated relation between d-algebra and BCK-algebra. M. Akram and K.H. Dar [1] introduced the concepts fuzzy d-algebra, fuzzy subalgebra and fuzzy d-ideals of d-algebra.

2. Preliminaries

Definition:2.1 A d-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfies the following axioms:

i. $x * x = 0$
ii. $0 * x = 0$
iii. $x * y = 0$ and $y * x = 0 \Rightarrow x = y, for\ all\ x, y \in X.$

Definition:2.2 A non-empty subset of a d-algebra X is called d-subalgebra of X if $x * y \in X, for\ all\ x, y \in X.$

Definition: 2.3 Let $(X,*,0)$ be a d-algebra and $a \in X$. Define $a * X = \{a * x / x \in X\}$. Then X is said to be edge if $a * X = \{0, a\}$ for all $a \in X$.

Definition: 2.4 Let X be a d-algebra and I be a subset of X, then I is called d-ideal of X if it satisfies the following conditions:

i. $0 \in I$

ii. $x * y \in I \text{ and } y \in I \Rightarrow x \in I$

iii. $x \in I \text{ and } y \in X \Rightarrow x * y \in I$.

Definition: 2.5 A fuzzy subset μ_A of X is called a fuzzy d-ideal of X if it satisfies the following condition:

i. $\mu_A(0) \geq \mu_A(x)$

ii. $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$

iii. $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\} \quad for\ all\ x, y \in X.$

3. Fuzzy Translation in Fuzzy d - Ideals

Definition: 3.1 Let μ_A be a fuzzy subset of X and $\alpha \in [0, T]$. A mapping $(\mu_A)^T_\alpha : X \to [0,1]$ is said to be a fuzzy-α-translation of μ_A if it satisfies $(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha, for\ all\ x \in X$.

Theorem: 3.2 If μ_A is a fuzzy d-ideal of X, then the fuzzy-α-translation $(\mu_A)^T_\alpha$ of μ_A is a fuzzy d-ideal of X, for all $\alpha \in [0,1]$.

Proof:

Let μ_A be a fuzzy d-ideal of X and let $\alpha \in [0,1]$

Then $(\mu_A)^T_\alpha(0) = \mu_A(0) + \alpha$

$\geq \mu_A(x) + \alpha$

$= (\mu_A)^T_\alpha(x)$

$(\mu_A)^T_\alpha(0) \geq (\mu_A)^T_\alpha(x)$

$(\mu_A)^T_\alpha(x) = \mu_A(x) + \alpha$

$\geq min\{\mu_A(x * y), \mu_A(y)\} + \alpha$

$= min\{\mu_A(x * y) + \alpha, \quad \mu_A(y) + \alpha\}$

$= min\{(\mu_A)^T_\alpha(x * y), (\mu_A)^T_\alpha(y)\}$

$(\mu_A)^T_\alpha(x) \geq min\{(\mu_A)^T_\alpha(x * y), (\mu_A)^T_\alpha(y)\}$

$(\mu_A)^T_\alpha(x * y) = \mu_A(x * y) + \alpha$

$\geq min\{\mu_A(x), \mu_A(y)\} + \alpha$

$= min\{\mu_A(x) + \alpha, \quad \mu_A(y) + \alpha\}$

$= min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$

$(\mu_A)^T_\alpha(x * y) \geq min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$

Hence $(\mu_A)^T_\alpha$ of μ_A is a fuzzy d-ideal of X for all $\alpha \in [0,1]$.

Theorem: 3.3 Let μ_A be a fuzzy subset of X such that the fuzzy-α-translation $(\mu_A)_\alpha^T$ of μ_A is a fuzzy d-ideal of $X$for some $\alpha \in [0,1]$, then μ_A is a fuzzy d-ideal of X.

Proof:

Let $(\mu_A)_\alpha^T$ is a fuzzy d-ideal of X for some $\alpha \in [0,1]$

Let $x, y \in X$

$\mu_A(0) + \alpha = (\mu_A)_\alpha^T(0)$

$\geq (\mu_A)_\alpha^T(x)$

$= \mu_A(x) + \alpha$

$\mu_A(0) \geq \mu_A(x)$

$\mu_A(x) + \alpha = (\mu_A)_\alpha^T(x)$

$\geq min\{(\mu_A)_\alpha^T(x * y), (\mu_A)_\alpha^T(y)\}$

$= min\{\mu_A(x * y) + \alpha, \mu_A(y) + \alpha\}$

$= min\{\mu_A(x * y), \mu_A(y)\} + \alpha$

$\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$

$\mu_A(x * y) + \alpha = (\mu_A)_\alpha^T(x * y)$

$\geq min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$

$= min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$

$= min\{\mu_A(x), \mu_A(y)\} + \alpha$

$\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$

Hence μ_A is a fuzzy d-ideal of X.

4. Fuzzy Translation in Fuzzy d - Subalgebra

Definition: 4.1 A fuzzy set $\mu_A$in d-algebra X is called fuzzy d-subalgebra of X if it satisfies**,** $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\} \quad for\ all\ x, y \in X$.

Theorem: 4.2 If μ_A is a fuzzy d-subalgebra of X, then the fuzzy-α-translation μ_α^T of μ_A is also a fuzzy d-subalgebra of X for all $\alpha \in [0,1]$.

Proof:

Let $x, y \in X$ and let $\alpha \in [0,1]$

Let μ_A be a fuzzy d-subalgebra of X .

Then $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$

$(\mu_A)_\alpha^T(xy) = \mu_A(xy) + \alpha$

$\geq min\{\mu_A(x), \mu_A(y)\} + \alpha$

$= min\{\mu_A(x) + \alpha, \quad \mu_A(y) + \alpha\}$

$= min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$

$(\mu_A)_\alpha^T(xy) \geq min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$

Hence $(\mu_A)_\alpha^T$ of μ_A is a fuzzy d-subalgebra of X for all $\alpha \in [0,1]$.

Theorem: 4.3 Let μ_A be a fuzzy subset of X such that the fuzzy-α-translation $(\mu_A)_\alpha^T$ of μ is a fuzzy d-subalgebra of X for some $\alpha \in [0,1]$, then μ_A is a fuzzy d-subalgebra of X.

Proof:

Let $(\mu_A)_\alpha^T$ is a fuzzy d-subalgebra of X for some $\alpha \in [0,1]$.

Let $x, y \in X$.

$\mu_A(xy) + \alpha = (\mu_A)_\alpha^T(xy)$

$\geq min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$

$= min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$

$= min\{\mu_A(x), \mu_A(y)\} + \alpha$

$\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$

Hence μ_A is a fuzzy d-subalgebra of X.

5. Conclusions

In this paper, we have given some ideas on fuzzy translation in fuzzy d-ideals and fuzzy d-subalgebras. Our further research will be focus on dot product and level set.

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Connected Hub Sets and Connected Hub Polynomials of the Lollipop Graph $L_{p,1}$

T. Angelinshiny[1]
T. Anitha Baby[2]

Abstract

Let G be a graph with vertex set $V(G)$. The number of vertices in G is the order of G and is denoted by $|V(G)|$. The connected hub polynomial of G denoted by $H_c(G,y)$ is defined as $H_c(G,y)=\sum_{k=\hbar_c(G)}^{|V(G)|} h_c(G,k)y^k$ where $h_c(G,k)$ denotes the number of connected hub sets of G of cardinality k and $\hbar_c(G)$ denotes the connected hub number of G. Let $L_{p,1}$ denotes the Lollipop graph with $p+1$ vertices. The connected hub polynomial of $L_{p,1}$ denoted by $H_c(L_{p,1},y)$ is defined as,$H_c(L_{p,1},y)=\sum_{k=\hbar_c(L_{p,1})}^{|V(L_{p,1})|} h_c(L_{p,1},k)\,y^k$where $h_c(L_{p,1},k)$ denotes the number of connected hub sets of $L_{p,1}$ of cardinality k, and $\hbar_c(L_{p,1})$ denotes the connected hub number of $L_{p,1}$.In this paper, we derive a recursive formula for $h_c(L_{p,1},k)$. From this recursive formula, we construct the connected hub polynomial of $L_{p,1}$ as,$H_c(L_{p,1},y)=\sum_{k=1}^{p+1} h_c(L_{p,1},k)\,y^k$Also we study some properties of this polynomial.

Keywords: Lollipop Graph, connected hub set, connected hub number, Connected hub polynomial.

Mathematics Subject Classification Code: 05C31, 05C99[3]

[1]Research Scholar (Reg. No. 20213282092009), Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India. Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012. Mail Id: angelinshinyt@gmail.com
[2] Assistant Professor, Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India. Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012. Mail Id: anithasteve@gmail.com
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1. Introduction

If any two distinct vertices of a graph G are adjacent, then G is a complete graph. If a tree has two nodes of vertex degree 1 and other nodes of vertex degree 2, then it is a path graph. A complete graph of order p is denoted by K_p and a path graph of order q is denoted by P_q. Join a complete graph K_p to a path graph P_q with a bridge. The resulting graph is a Lollipop graph $L_{p,q}$.

2. Connected Hub Sets of the Lollipop Graph $L_{p,1}$

In this section, we give the connected hub number of the Lollipop graph $L_{p,1}$ and some of the properties of the connected hub sets of the Lollipop graph $L_{p,1}$.

Definition 2.1 Join a complete graph K_p to a path graph P_1 with a bridge. The resulting graph is a Lollipop graph $L_{p,1}$.

Definition 2.2 Let $G = (V, E)$ be a connected graph. A subset H of V is called a hub set of G if for any two distinct vertices $u, v \in V - H$, there exists a $u - v$ path P in G, such that all the internal vertices of P are in H. The minimum cardinality of a hub set of G is called the hub number of G and is denoted by $\hbar(G)$.

Definition 2.3 A hub set H of G is called a connected hub set if the induced subgraph $< H >$ is connected. The minimum cardinality of a connected hub set of G is called connected hub number of G and is denoted by $\hbar_c(G)$.

Theorem 2.4

$$h_c(L_{p,1},k) = \begin{cases} \binom{p+1}{k} - \binom{p}{k} + 1 \; when \; k = 1 \; and \; p-1 \\ \binom{p+1}{k} - \binom{p}{k} \quad if \; 2 \le k \le p+1 \; and \; k \ne p-1 \end{cases}$$

Proof. Let $L_{p,1}$ be the Lollipop graph with $p+1$ vertices and $p \geq 4$. Let $v_1, v_2, v_3 \ldots v_p, v_{p+1}$ be the vertices of $L_{p,1}$, in which the degree of the vertices $v_1, v_2, v_3, \ldots, v_{p-1}$ is $p-1$, the degree of the vertex v_p is p and the degree of the vertex v_{p+1} is 1. Since, $L_{p,1}$ contains $p+1$ vertices, the number of subsets of $L_{p,1}$ with cardinality k is $\binom{p+1}{k}$. Also, Since, the subgraph with vertex set $\{v_1, v_2, v_3 \ldots v_{p-1}\}$ is not adjacent to v_{p+1} every hub set must contain the vertices v_p or v_{p+1}. Therefore, every time $\binom{p}{k}$ number of subsets of $L_{p,1}$ of cardinality k are not connected hub sets.
Thus, $L_{p,1}$ have $\binom{p+1}{k} - \binom{p}{k}$ number of connected hubs sets of cardinalities k.
When the cardinality is $p-1$, the set which contains v_{p-1} is also a connected hub set.
When the cardinality is 1, $\{v_p\}$ and $\{v_{p+1}\}$ are the only connected hub sets.

Hence, $h_c(L_{p,1},k) = \begin{cases} \binom{p+1}{k} - \binom{p}{k} + 1 \; when \; k = 1, \; p-1 \\ \binom{p+1}{k} - \binom{p}{k} \quad if \; 2 \leq k \leq p+1 \; and \; \; k \neq \; \; p-1 \end{cases}$

Theorem 2.5 Let $L_{p,1}$ be the Lollipop graph with $p \geq 4$.Then

(i) $h_c(L_{p,1},k) = \binom{p}{k-1}$ for all $2 \leq k \leq p+1 \; and \; k \neq \; p-1$

(ii) $h_c(L_{p,1},k) = \binom{p}{k-1} + 1$ when $k = 1 \; and \; p-1$.

(iii) $h_c(L_{p,1},k) =$

$\begin{cases} h_c(L_{p-1,1,}\, k) + h_c(L_{p-1,1,}k-1) \; if \; 1 \leq k \leq p+1 \; and \; k \neq 2\,, p-2 \\ \\ h_c(L_{p-1,1,}\, k) + \; h_c(L_{p-1,1,}k-1) - 1 \; if \; k = 2 \; and \; p-2 \end{cases}$

Proof: (i) From Theorem 2.4, we have

$h_c(L_{p,1,}k) = \binom{p+1}{k} - \binom{p}{k}$

We know that, $\binom{p+1}{k} - \binom{p}{k} = \binom{p}{k-1}$

Therefore, $h_c(L_{p,1,}k) = \binom{p}{k-1}$ for all $2 \leq k \leq p+1$ and $k \neq p-1$.

(ii) From Theorem 2.4, we have

$h_c(L_{p,1,}\, k) = \binom{p+1}{k} - \binom{p}{k} + 1 \; \; when \; k = 1 \; and \; p-1.$

We know that, $\binom{p+1}{k} - \binom{p}{k} = \binom{p}{k-1}$

Therefore, $h_c(L_{p,1,}k) = \binom{p}{k-1} + 1 \; when \; k = 1 \; and \; p-1.$

(iii) From (i) $h_c(L_{p,1,}k) = \binom{p}{k-1}$ for all $2 \leq k \leq p+1$ and $k \neq p-1$.

$h_c(L_{p-1,1,}\, k) = \binom{p-1}{k-1}$

and $h_c(L_{p-,1,}k-1) = \binom{p-1}{k-2}$

Consider, $h_c(L_{p-1,1,}\, k) + h_c(L_{p-,1,}k-1) = \binom{p-1}{k-1} + \binom{p-1}{k-2} = \binom{p}{k-1} = h_c(L_{p,1,}k)$

Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\, k) + h_c(L_{p-,1,}k-1)$ for $1 \leq k \leq p+1 \; and \; k \neq 2, p-2$

When $k = 2$,

$h_c(L_{p,1,}\, 2) = \binom{p}{1}$

$h_c(L_{p-1,1,}\, 2) = \binom{p-1}{1}$

and $h_c(L_{p-1,1,}\, 1) = \binom{p-1}{0} + 1$, by (ii)

Consider, $h_c(L_{p-1,1,}\, 2) + h_c(L_{p-1,1,}\, 1) = \binom{p-1}{1} + \binom{p-1}{0} + 1 = \binom{p}{1} + 1$

$= h_c(L_{p,1,}\ 2) + 1$
That is, $h_c(L_{p,1,}\ 2) = h_c(L_{p-1,1,}\ 2) + h_c(L_{p-1,1,}\ 1) - 1$.
Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\ k) + h_c(L_{p-,1,}k-1) - 1$ when $k = 2$.
When $k = p-2$,
$h_c(L_{p,1,}\ p-2) = \binom{p}{p-3}$
$h_c(L_{p-1,1,}\ p-2) = \binom{p-1}{p-3}$
and $h_c(L_{p-1,1,}\ p-3) = \binom{p-1}{p-4} + 1$, by (ii)
Consider, $h_c(L_{p-1,1,}\ p-2) + h_c(L_{p-1,1,}\ p-3) = \binom{p-1}{p-3} + \binom{p-1}{p-4} + 1 = \binom{p}{p-3} + 1 = h_c(L_{p,1,}\ p-2) + 1$
That is, $h_c(L_{p,1,}\ p-2) = h_c(L_{p-1,1,}\ p-2) + h_c(L_{p-1,1,}\ p-3) - 1$.

Therefore, $h_c(L_{p,1,}k) = h_c(L_{p-1,1,}\ k) + h_c(L_{p-,1,}k-1) - 1$ when $k = p-2$.
Hence,

$$h_c(L_{p,1,}k) = \begin{cases} h_c(L_{p-1,1,}\ k) + h_c(L_{p-1,1,}k-1)\ if\ 1 \le k \le p+1\ and\ k \ne 2\ , p-2 \\ \\ h_c(L_{p-1,1,}\ k) + h_c(L_{p-1,1,}k-1) - 1\ if\ k = 2\ and\ p-2 \end{cases}$$

3. Connected Hub Polynomials of the Lollipop Graph $L_{p,1}$.

Definition 3.1 Let $H_c(L_{p,1,}, k)$ denotes the family of connected hub sets of the Lollipop graph $L_{p,1,}$ of cardinality k and $h_c(L_{p,1,}, k) = |H_c(L_{p,1,}, k)|$. Then, the connected hub polynomial of $L_{p,1}$ denoted by $H_c(L_{p,1,}, y)$ is defined as $H_c(L_{p,1,}, y) = \sum_{k=\hbar_c(L_{p,1,})}^{p+1} h_c(L_{p,1,}, k)\, y^k$
where $\hbar_c(L_{p,1,})$ is connected hub number of $L_{p,1,}$.

Remark 3.2 $\hbar_c(L_{p,1,}) = 1$.
Proof: Label the vertices of $L_{p,1}$ by $v_1, v_{2,}v_3, \dots, v_p, v_{p+1}$ in which the degree of the vertices $v_1, v_2, v_3, \dots, v_{p-1}$ is $p-1$, the degree of the vertex v_p is p and the degree of the vertex v_{p+1} is 1. Since, any two vertices of $v_1, v_2, v_{3,} \dots, v_p$ are adjacent there is a path between any two vertices of $v_1, v_2, v_{3,} \dots, v_p$. Also, v_p is the internal vertex for all the path between the vertices of $\{v_1, v_{2,}v_3, \dots, v_{p-1}\}$ and v_{p+1}. Hence $\{v_p\}$ and $\{v_{p+1}\}$ are two connected hub sets of cardinalities 1.
Hence, $\hbar_c(L_{p,1}) = 1$.

Theorem 3.3 $H_c(L_{p,1},y) = (1+y)H_c(L_{p-1,1},y) - y^2 - y^{p-2}$ with initial value $H_c(L_{4,1},y) = 2y + 4y^2 + 7y^3 + 4y^4 + y^5$.

Proof. We have, $H_c(L_{p,1},y) = \sum_{k=1}^{p+1} h_c(L_{p,1},k)y^k$

$H_c(L_{p,1},y) = \sum_{\substack{k=1 \\ k\neq 2,p-2}}^{p+1} h_c(L_{p,1},k)y^k + h_c(L_{p,1},2)y^2 + h_c(L_{p,1},p-2)y^{p-2}$

$$= \sum_{\substack{k=1 \\ k\neq 2,p-2}}^{p+1} \left[h_c(L_{p-1,1},k) + h_c(L_{p-1,1},k-1)\right]y^k + \left[h_c(L_{p-1,1},2) + h_c(L_{p-1,1},1) - 1\right]y^2 + \left[h_c(L_{p-1,1},p-2) + h_c(L_{p-1,1},p-3) - 1\right]y^{p-2}$$

$= \sum_{k=1}^{p+1}\left[h_c(L_{p-1,1},k) + h_c(L_{p-1,1},k-1)\right]y^k - y^2 - y^{p-2}$

$= \sum_{k=1}^{p+1} h_c(L_{p-1,1},k)y^k + \sum_{k=1}^{p+1} h_c(L_{p-1,1},k-1)\,y^k - y^2 - y^{p-2}$

$= \sum_{k=1}^{p+1} h_c(L_{p-1,1},k)y^k + y\sum_{k=1}^{p+1} h_c(L_{p-1,1},k-1)\,y^{k-1} - y^2 - y^{p-2}$

$= H_c(L_{p-1,1},y) + yH_c(L_{p-1,1},y) - y^2 - y^{p-2}$

$= (1+y)H_c(L_{p-1,1},y) - y^2 - y^{p-2}$

Hence, $H_c(L_{p,1},y) = (1+y)H_c(L_{p-1,1},y) - y^2 - y^{p-2}$ with initial value $H_c(L_{4,1},x) = 2y + 4y^2 + 7y^3 + 4y^4 + y^5$.

Example 3.4 Consider the Lollipop graph $L_{5,1}$ be with order 6 given in Figure 1.

$L_{5,1}$:

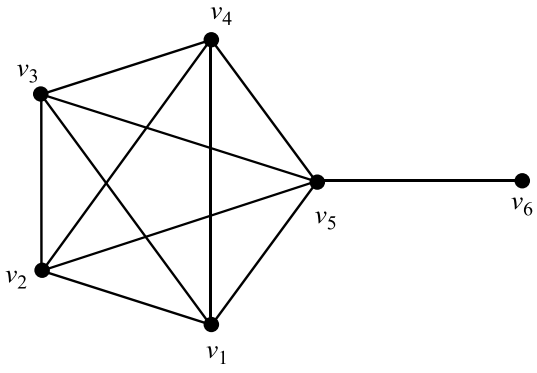

Figure 1

$H_c(L_{5,1},y) = 2y + 5y^2 + 10y^3 + 11y^4 + 5y^5 + y^6$

By Theorem 3.3, we have,

$H_c(L_{5,1},y) = (1+y)H_c(L_{4,1},y) - y^2 - y^3$

$= (1+y)(2y + 4y^2 + 7y^3 + 4y^4 + y^5) - y^2 - y^3$

$= 2y + 5y^2 + 10y^3 + 11y^4 + 5y^5 + y^6$

Theorem 3.5 Let $L_{p,1}$ be the Lollipop graph with $p \geq 4$.Then

(i) $H_c(L_{p,1},y) = \sum_{k=1}^{p+1}\binom{p+1}{k}y^k - \sum_{k=1}^{p+1}\binom{p}{k}y^k - y^2 - y^{p-2}$.

(ii) $H_c(L_{p,1,}\ y) = \sum_{k=1}^{p+1}\binom{p}{k-1}y^k - y^2 - y^{p-2}$.

Proof. Proof is obvious.

$h_c(L_{p,1,}\ k)$ for $4 \leq\ p \leq\ 14$ and $1 \leq\ k \leq\ 15$.

k / p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	2	4	7	4	1										
5	2	5	10	11	5	1									
6	2	6	15	20	16	6	1								
7	2	7	21	35	35	22	7	1							
8	2	8	28	56	70	56	29	8	1						
9	2	9	36	84	126	126	84	37	9	1					
10	2	10	45	120	210	252	210	120	46	10	1				
11	2	11	55	165	330	462	462	330	165	56	11	1			
12	2	12	66	220	495	792	924	792	495	220	67	12	1		
13	2	13	78	286	715	1287	1716	1716	1287	715	286	79	13	1	
14	2	14	91	364	100	2002	3003	3432	3003	2002	1001	364	92	14	1

Table 1

Theorem 3.6 The coefficients of $H_c(L_{p,1}, y)$ satisfy the following properties.

(i) $h_c(L_{p,1,}\ p+1) = 1$, for every $p \geq 4$.

(ii) $h_c(L_{p,1}, p) = p$, for every $p \geq 4$.

(iii) $h_c(L_{p,1}, p-1) = \frac{1}{2}(p^2 - p + 2)$, for every $p \geq 4$.

(iv) $h_c(L_{p,1}, p-2) = \frac{1}{6}(p^3 - 3p^2 + 2p)$, for every $p \geq 4$.

(v) $h_c(L_{p,1}, p-3) = \frac{1}{24}(p^4 - 6p^3 + 11p^2 - 6p)$, for every $p \geq 6$.

(vi) $h_c(L_{p,1}, 1) =\ 2$, for every $p \geq 4$.

(vii) $h_c(L_{p,1}, 2) =\ p$, for every $p \geq 4$.

4. Conclusion

In this paper, we identified the connected hub sets of $L_{p,1}$ and using the connected hub sets we derived the connected hub polynomial of $L_{p,1.}$ We can generalize this study to derive the connected hub polynomial of any Lollipop graph $L_{p,q}$.

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Markov Chain Model and its Application Yearly Rainfall Data in Nagapattinam District

Dr. S. Santha[1]
T. Subasini[2]

Abstract

A stochastic model expresses a sequence of possible events in which the possible event of each event depends on the previous event and is called a Markov chain. This paper has analyzed yearly rainfall in the Nagapattinam district and formulated three-state models. The first-order Markov chain to determine the long-term probability of rainfall in the following years and the steady-state. It can be used to make a forecast of the annual rainfall pattern. This model can give some information about rainfall to farmers and the government to plan strategies for high crop production in the Nagapattinam district.

Keywords: Markov chain, Yearly Rainfall, Transition probability

AMS Classification 2010: 37A30, 47D07[3].

[1]Assistant professor and Head, Department of Mathematics, Government Arts and Science College, Nagercoil, Affiliated by Manonmaniam Sundaranar University, Tirunelveli Tamil Nadu, India. santhawilliam14@gmail.com.
[2]Reg No.18121172092018, Research Scholar, Rani Anna Government Arts and Science College for Women, Tirunelveli, Affiliated by Manonmaniam Sundaranar University, Tirunelveli Tamil Nadu, India. subasinit@gmail.com.
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1. Introduction

Forecasting is a science of the future. Here based on the knowledge of the previous, predictions are made on the future. This involves a knowledge of all forecasting methods. The importance of forecasting is becoming necessary for prosperity. Preparation of a forecast needs mathematical formula and historical data into the future.

The most effective forecasting methods is to use mathematical techniques to routinely forecast demands. By combining mathematical techniques with informed judgement, they can serve checks on each other and tend to eliminate gross errors.

Agriculture largely depends on water resources. The variation and quantity of rainfall have two extreme impacts – bumper harvest or lean year. Rainfall modelling plays a prominent role for rainfall prediction. Apart from data generation, the application of rainfall modelling is vital for water resource management and in the field of hydrology and agriculture. Data relating to climate and rainfall needs a wide range of models in which time and spatial scales are involved.

Rainfall, its arrival, intensity, duration, was generally decided by the atmospheric factors which are available just before the onset. Hence, for forecasting the future rainfall it is difficult to estimate the probable environmental factors which are the causes of the rainfall. In view of this the researchers are left with the option of predicting it with the previous data set. Abubaker et al [10] have formulated a, four- state Markov model of annual rainfall in Minna with respect to crop production in the region. The present study emphasizes analyzing the annual rainfall data in a three-state model. In this paper the forecasting of annual rainfall pattern for the period of 20 years (2001 to 2020) in Nagapattinam District.

2. Methodology

Nagapattinam district is one of the 38 districts (a coastal district) of Tamilnadu state in southern India. The district lies between northern latitude 10.7906 degrees and 79.8428 degrees Eastern longitude. Since the study was forecasting of rainfall, the annual rainfall data was collected from the statistical department in Nagapattinam from the year 2001 to 2020.

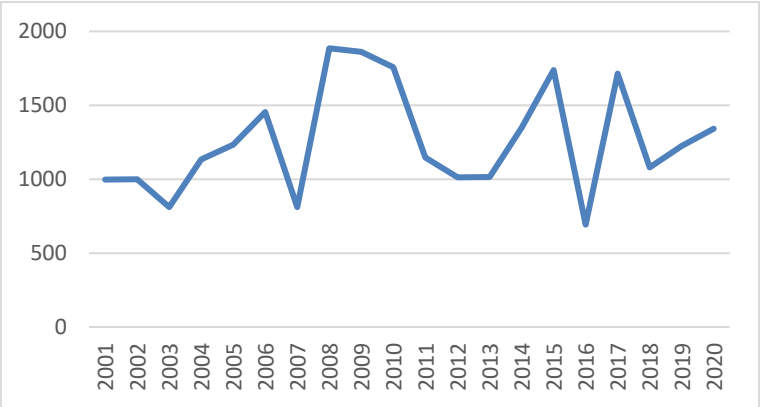

Figure: 1 Annual rainfall in Nagapattinam district from 2001-2020.

From Figure 1 time-series analysis of yearly rainfall is converted into rainfall states prepared by a suitable frequency distribution table. The frequency distribution of each class is specified states of rainfall and is denoted by S1, S2, and S3 in table 1.

SL. No	class interval	Frequency distribution	states
1.	below 1100	8	S_1
2	1100-1600	9	S_2
3	above 1600	3	S_3

Table: 1

3. Markov Chain Modeling

Definition: 3.1 A Markov chain is a random sequence (X_n, n Є N) for all X_o, X_1 ……. X_{n-1} Є I such that

$$P(X_n = J_n / X_0 = J_0, X_1 = J_1, X_2 = J_2 \ldots\ldots X_{n-1} = J_{n-1}) = P(X_n = J_n / X_{n-1} = J_{n-1}) \quad (1)$$

Definition: 3.2 If a Markov Chain (J_n, $n \geq 0$) is homogeneous.

We consider $P(X_n = j / X_{n-1} = i) = P_{ij}$ and we put the matrix P i.e., $P = [P_{ij}]$

The Markov process X has steady state transition probabilities if for any pair of states i, j: The first step transition probabilities P_{ij},

P_{ij}^n denote the n step transition probability.

That is, $P_{ij}^n = P\{X_{n+m} = J / X_m = i\}$ $n \geq 0$ all i, $j \geq 0$

We have $P_{ij}^1 = P_{ij}$.

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n \; P_{kj}^m \text{ for all } n, m \geq 0 \quad (2)$$

From (2)

$P^{n+m} = p^{(n)} \times p^{(m)}$

$P^{(2)} = P^{(1+1)} = P^2$

Hence by induction

$P^{(n)} = P^{(n-1+1)} = P^{(n-1)} P^1 = P^n$

We have $P^n = P^0 P^n$ (3)

Here P^0 denote the initial state vector of transition matrix

P^n denote limiting state probability.

Now, let $P^n = [P^n_1, P_2^n, P^n_3]$.

Also let $P^0 = [P^0_1 \;\; P^0_2 \;\; P^0_3]$.

Limiting state probabilities: 3.3

The probability distribution $\pi = [\pi_1 \pi_2 \ldots\ldots \pi_n]$ is called the limiting distribution of the continuous time Markov chain X (t)

if $\pi = (\pi_1 \pi_2 \pi_3)$ Howard [8]

let $n \to \infty$ in equation (3)

we get $\pi = \pi P$ (4)

also, $\pi = \sum_{i=1}^{3} \pi i = 1$ Abubakar et al [10]

These equations will be used to find the limiting state equilibrium probabilities.

4. Result and Discussion

Let the model for yearly rainfall is
S_1: less than 1100
S_2: in between 1100-1600
S_3: greater value of 1600.

Therefore, the transition probability matrix

$$P = \begin{bmatrix} P11 & P12 & P13 \\ P21 & P22 & 0 \\ P31 & 0 & 0 \end{bmatrix}$$

The probability of transition matrix

$$P = \begin{vmatrix} 4 & 3 & 1 \\ 4 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix} \quad (5)$$

i, e. $P_{ij} = \frac{fij}{\sum fij}$ i, j = 1,2,3. Tamil and Samuel (9)
Where f_{ij} → transition frequency from state i to state j, $0 \leq P_{ij} \leq 1$.
We get the probability matrix

$$P = \begin{vmatrix} 4/8 & 3/8 & 1/8 \\ 4/9 & 5/9 & 0 \\ 3/3 & 0 & 0 \end{vmatrix},$$

$$P = \begin{vmatrix} 0.5 & 0.375 & 0.125 \\ 0.44 & 0.56 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

n-step transition probability
we have

$$P^2 = \begin{bmatrix} 0.5415 & 0.396 & 0.0625 \\ 0.4689 & 0.4756 & 0.0555 \\ 0.5 & 0.375 & 0.0625 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.5091 & 0.4232 & 0.0442 \\ 0.5011 & 0.4402 & 0.0586 \\ 0.5415 & 0.396 & 0.0625 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.5101 & 0.4262 & 0.0519 \\ 0.5046 & 0.4326 & 0.0521 \\ 0.5091 & 0.4232 & 0.0442 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.5079 & 0.4282 & 0.0509 \\ 0.5071 & 0.4297 & 0.0519 \\ 0.5101 & 0.4262 & 0.0519 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} 0.5079 & 0.4284 & 0.0515 \\ 0.5074 & 0.4290 & 0.0514 \\ 0.5079 & 0.4282 & 0.0509 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} 0.5078 & 0.4286 & 0.0515 \\ 0.5076 & 0.4287 & 0.0515 \\ 0.5079 & 0.4284 & 0.0515 \end{bmatrix}$$

$$P^8 = \begin{bmatrix} 0.5078 & 0.4286 & 0.0515 \\ 0.5077 & 0.4286 & 0.0515 \\ 0.5078 & 0.4286 & 0.0515 \end{bmatrix}$$

$$p^{10} = \begin{bmatrix} 0.5078 & 0.4286 & 0.0515 \\ 0.5078 & 0.4286 & 0.0515 \\ 0.5078 & 0.4286 & 0.0515 \end{bmatrix} \quad (6)$$

$$p^{10} = \begin{bmatrix} 0.51 & 0.43 & 0.05 \\ 0.51 & 0.43 & 0.05 \\ 0.51 & 0.43 & 0.05 \end{bmatrix}$$ corrected to 2 decimal places

Limiting state probabilities

After n steps p^n gets the fixed value (6) i, e. n ≥10

Let us take $P^0 = (1 \quad 0 \quad 0)$

$$P^0 . P^n = (100)\begin{bmatrix} 0.507 & 0.4286 & 0.0515 \\ 0.5078 & 0.4286 & 0.0515 \\ 0.5078 & 0.4286 & 0.0515 \end{bmatrix} =$$

= (0.5078 0.4286 0.0515)

= (0.51 0.43 0.05)

From equation (4) and (6), n = nP = (0.51 0.43 0.05).

This result shows that the probabilities of yearly rainfall after ten years. In the first-year probability is (0.5 0.38 0.13) respectively. By the comparison of the probabilities, state 3 dropped slowly and the probability of state 1 and 2 increased in the above ten years. This shows that in the 51% annual rainfall in Nagapattinam will be state 1, 43%will be state 2 and 5% will be state 3.

5. Conclusion

This article show analyzes the yearly rainfall data used by the first-order Markov chain model. Yearly rainfall forecasting of the current year can be used to make a

forecast for the following year and in the long run. A yearly rainfall forecasting pattern used to give details about the production of the crops in the Nagapattinam district.

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On the effect of doubling of intervals on the 0 - distributive property of the lattice of weak congruences of chains

Gladys Mano Amirtha V[1]
D. Premalatha[2]

Abstract

Alan Day's doubling construction of intervals has been found to affect some properties of the lattice of weak congruences of chains. Here, in this paper, we study how it affects the property of 0-distributivity of the lattice of weak congruences of chains.

Keywords: doubling construction in lattices, weak congruence lattices, 0-distributive lattices.

2010 AMS subject classification: 06B10, 06D99[3].

[1]Reg. No.: 20111172092013, Ph.D. Research Scholar (Full time), PG and Research Department of Mathematics, (Rani Anna Government College for Women, Tirunelveli-627008, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.) gladyspeter3@gmail.com.
[2]Associate Professor, PG and Research Department of Mathematics, (Rani Anna Government College for Women, Tirunelveli-627008, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.) lathaaedward@gmail.com.
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1. Introduction

Vojvodić G. and Šešelja B. initiated the study of the concept of weak congruences of a lattice in the year 1988 [12]. J. C. Varlet [9] was the first to introduce the concept of 0-distributive lattices. Several other researchers made various contributions in different aspects of 0-distributivity. For example, one can refer to [1], [8], [11]. A. Veeramani [10] in his thesis, has studied about the lattice of weak congruences of a finite chain, Boolean lattices and the lattices C_n, M_3, N_5first by considering 0 and 1 as non-constants and then considering them as constants, again by considering the Boolean lattice as an algebra and he studied about some weaker properties like 0-distributivity, 0-modularity, consistency, etc.

G. Gratzer constructed a new lattice $L^U$from a given lattice L by adding an element a^U called the double of a $\neq 0$ or 1 in L where $L^U = L \cup \{a^U\}$with a new order denoted by $\leq^U$ [6]. Following that construction, A Day [3] introduced a similar construction L[I]by doubling an interval I of a given lattice L. After that it witnessed many developments, e.g., see [4], [5], [7]. Alan Day in [2] proved that a distributive lattice remains distributive when it is doubled by either a lower interval or an upper interval. In our present study, we analyse the effect of doubling of intervals on the property of 0-distributivity in the lattice of weak congruences of chains.

2. Preliminaries

Definition 2.1 [6] A lattice L satisfying the following identities

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$ is called a *distributive lattice*. If not, it is a non-distributive lattice.

Definition 2.2 [4] A lattice L is said to be *0 - distributive*, if for all $x, y, z \in L$, whenever $x \wedge y = 0$ and $x \wedge z = 0$, then$x \wedge (y \vee z) = 0$.

Lemma 2.3 [10] *If L_n is a chain of n elements, then $C_W(L_n)$ is 0 - distributive.*

Definition 2.4 [6] An equivalence relation θ on a lattice L is said to be a *congruence relation* on L, if it is compatible with both meet and join, that is, for all $a, b, c, d \in L$, $a \equiv b\ (\theta)$ and $c \equiv d\ (\theta)$ imply that $a \vee c \equiv b \vee d\ (\theta)$ and $a \wedge c \equiv b \wedge d\ (\theta)$.

Definition 2.5 [12] A *weak congruence relation* on an algebra A is a symmetric and transitive sub-universe of A^2.

Note 2.6 The lattice of all weak congruence relations of Lincluding ϕ with respect to the relation $\subseteq$ is denoted by $C_W(L)$. We consider 0 and 1 of L as non-constants in this paper.

Remark 2.7 [12] In $C_W(L)$, we have

- $[\phi, \Delta] \cong Sub(L)$, the lattice of all sublattices of L.

- $[\Delta, \tau\,] \cong Con(L)$, the lattice of all congruences of L.

Definition 2.8 [6] Let $I = [a, b]$ be an interval of a lattice L. The set $I \times C_2$ is formed using the two-element chain $C_2 = \{0,1\}$. The set $L[I] = (L \setminus I) \cup (I \times C_2)$ is the lattice given by the ordering: for $x, y \in L[I]$ and $i, j \in C_2$;
$x \leq y$ *if* $x \leq y$ *in* L;
$(x, i) \leq y$ *if* $x \leq y$ *in* L;
$x \leq (y, j)$ *if* $x \leq y$ *in* L;
$(x, i) \leq (y, j)$ *if* $x \leq y$ *in* L *and* $i \leq j$ *in* C_2.
$L[I]$ is the lattice got by doubling of the interval I in L. This is *Day's definition of doubling of intervals.*

2. Results and Discussions

In this section, we examine whether $[C_W(L_n)](I)$ is 0 - distributive or not. It turns out that $[C_W(L_n)](I)$ remains 0-distributive in case of lower and upper intervals in $C_W(L_n)$, whereas in the case of an intermediate interval, 0 – distributivity gets affected.

Theorem 3.1 If L_n is a chain of n elements, then $[C_W(L_n)](I)$ is 0 – distributive where I is a lower interval in $C_W(L_n)$.
Proof. Let $L_n = \{\, 0 \prec x_1 \prec x_2 \prec I \prec x_{n-1} = 1 \,\}$ be a chain of n elements. Let $C_W(L_n)$ be the lattice of all weak congruences of L_n. Let $I = [\phi, \theta]$ where θ is a proper congruence relation of L_n. Let $[C_W(L_n)](I)$ be the doubling of $C_W(L_n)$ by the interval I.
Let $A, B, C \in [C_W(L_n)](I)$, where I is a lower interval of $C_W(L_n)$ such that
$A \wedge B = (\phi, 0)$ and $A \wedge C = (\phi, 0)$.
To prove that, $[C_W(L_n)](I)$ is 0 – distributive. That is, we have to prove that
$A \wedge (B \vee C) = (\phi, 0)$ (3.1)
Suppose$(\{(x_i, x_i)\}, 0) \leq A \wedge (B \vee C) \Rightarrow (\{(x_i, x_i)\}, 0) \leq A$and
$(\{(x_i, x_i)\}, 0) \leq B \vee C \Rightarrow (\{(x_i, x_i)\}, 0) \leq A$ *and* $(\{(x_i, x_i)\}, 0) \leq B$ or
$(\{(x_i, x_i)\}, 0) \leq C$ or$(\{(x_i, x_i)\}, 0) \leq$ both B and C or incomparable with both.
Now,$A, B, C \in [C_W(L_n)](I) \Rightarrow A \in C_W(L_n) \setminus I$ or $A \in I \times C_2, B \in C_W(L_n) \setminus I$ or
$B \in I \times C_2, C \in C_W(L_n) \setminus I$ or$C \in I \times C_2$.

The following cases arise:
i. $A, B, C \in C_W(L_n) \setminus I$
ii. $A, B \in C_W(L_n) \setminus I$ *and* $C \in I \times C_2$
iii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in I \times C_2$
iv. $A, C \in C_W(L_n) \setminus I$ *and* $B \in I \times C_2$
v. $A, B, C \in I \times C_2$
vi. $A, B \in I \times C_2$ *and* $C \in C_W(L_n) \setminus I$
vii. $A, C \in I \times C_2$ *and* $B \in C_W(L_n) \setminus I$
viii. $A \in I \times C_2$ *and* $B, C \in C_W(L_n) \setminus I$

Case (i): Let $A, B, C \in C_W(L_n) \setminus I$. $A, B, C \in C_W(L_n) \setminus I$ implies that $A, B, C \in C_W(L_n)$. Therefore, (3.1) follows as $C_W(L_n)$ is 0- distributive.

Case (ii): Let $A, B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$, that is, $C = (\theta_3, j)$ where j = either 0 or 1.
(I) Let$(\{(x_i, x_i)\}, 0) \leq B \Rightarrow \{(x_i, x_i)\} \leq B \Rightarrow (x_i, x_i) \in B \Rightarrow (x_i, x_i) \in A \cap B$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_3$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \wedge \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow (x_i, x_i) \in B$ and$(x_i, x_i) \in \theta_3 \Rightarrow (x_i, x_i) \in A \cap B$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = B \vee (\theta_3, 0)$ implies $(x_i, x_i) \in B \vee \theta_3 \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in B \; or \; (x_i, x_k) \in \theta_3$. By symmetry and transitivity,
$(x_i, x_i) \in B$ or$(x_i, x_i) \in C$.Again we have $A \wedge B \neq (\phi, 0)$or $A \wedge C \neq (\phi, 0)$.This contradiction proves (3.1).

Case (iii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in I \times C_2$.Let $B = (\theta_2, j)$ and $C = (\theta_3, j)$ where j = either 0 or 1.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $(\{(x_i, x_i)\}, 0) \leq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, j) = A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_3$.Also, $(\{(x_i, x_i)\}, 0) \leq A$
$\Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$and $\{(x_i, x_i)\} \subseteq \theta_3$
$\Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2$ and$\{(x_i, x_i)\} \subseteq A \cap \theta_3$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, 0) = A \wedge B$ and $(\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$
This is a contradiction, since $A \wedge B = (\phi, 0)$ and $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (\theta_2 \vee \theta_3, j)$ implies $(x_i, x_i) \in \theta_2 \vee \theta_3$
$\Rightarrow$ there exists a $x_k \neq x_i$such that $(x_i, x_k) \in \theta_2 \; or \; (x_i, x_k) \in \theta_3$
$\Rightarrow (x_i, x_i) \in \theta_2 \; or \; (x_i, x_i) \in \theta_3$, by symmetry and transitivity in θ_2 and θ_3.
$\Rightarrow (\{(x_i, x_k)\}, 0) \leq (\theta_2, 0) \; or \; (\{(x_i, x_k)\}, 0) \leq (\theta_3, 0)$.
That is, $(\{(x_i, x_k)\}, 0) \leq B \; or \; (\{(x_i, x_k)\}, 0) \leq C$.
Therefore, we have $(\{(x_i, x_i)\}, 0) \leq B \; or \; (\{(x_i, x_i)\}, 0) \leq C$.
This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (iv): Let $A, C \in C_W(L_n) \setminus I$ and $B \in I \times C_2$, that is, $B = (\theta_2, j)$ where j = either 0 or 1.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C \Rightarrow \{(x_i, x_i)\} \leq C \Rightarrow (x_i, x_i) \in C \Rightarrow (x_i, x_i) \in A \cap C$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow (x_i, x_i) \in \theta_2$and $(x_i, x_i) \in C \Rightarrow (x_i, x_i) \in A \cap C$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.

(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (\theta_2 \vee C, 0)$ implies $(x_i, x_i) \in \theta_2 \vee C \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in \theta_2$ *or* $(x_i, x_k) \in C$. By symmetry and transitivity, $(x_i, x_i) \in B$ *or* $(x_i, x_i) \in C$. This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (v): Let $A, B, C \in I \times C_2$, that is, $A = (\theta_1, j)$, $B = (\theta_2, j)$, $C = (\theta_3, j)$ where $j =$ either 0 or 1.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \wedge \theta_2 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \wedge \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_3$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ *and* $C \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$ *and* $\{(x_i, x_i)\} \subseteq \theta_3$
$\Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2$ and $\{(x_i, x_i)\} \subseteq A \cap \theta_3$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, 0) = A \wedge B$ and $(\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge B = (\phi, 0)$ *and* $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (\theta_2 \vee \theta_3, 0)$ implies $(x_i, x_i) \in \theta_2 \vee \theta_3$
$\Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in \theta_2$ *or* $(x_i, x_k) \in \theta_3$
$\Rightarrow (\{(x_i, x_k)\}, 0) \leq (\theta_2, 0)$ *or* $(\{(x_i, x_k)\}, 0) \leq (\theta_3, 0)$.
That is, $(\{(x_i, x_k)\}, 0) \leq B$ *or* $(\{(x_i, x_k)\}, 0) \leq C$. By symmetry and transitivity in B and C, we have $(\{(x_i, x_i)\}, 0) \leq B$ *or* $(\{(x_i, x_i)\}, 0) \leq C$. This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (vi): Let $A, B \in I \times C_2$, that is, $A = (\theta_1, j)$, $B = (\theta_2, j)$ where j = either 0 or 1 and $C \in C_W(L_n) \setminus I$.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, i) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \wedge \theta_2 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \wedge \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C \Rightarrow \{(x_i, x_i)\} \leq C \Rightarrow (x_i, x_i) \in C \Rightarrow (x_i, x_i) \in A \cap C$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ *and* $C \Rightarrow (x_i, x_i) \in \theta_2$ *and* $(x_i, x_i) \in C$
$\Rightarrow \{(x_i, x_i)\} \leq A \cap C \Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (\theta_2 \vee C, 0)$ implies $(x_i, x_i) \in \theta_2 \vee C \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in \theta_2$ *or* $(x_i, x_k) \in C$. By symmetry and transitivity, we have $(x_i, x_i) \in \theta_2$ *or* $(x_i, x_i) \in C$. This is a contradiction to our assumption.Therefore, (3.1) holds.

Case (vii): Let $A, C \in I \times C_2$, that is, $A = (\theta_1, j)$, $C = (\theta_3, j)$ where $j =$ either 0 or 1 and $B \in C_W(L_n) \setminus I$.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B \Rightarrow \{(x_i, x_i)\} \leq B \Rightarrow (x_i, x_i) \in B \Rightarrow (x_i, x_i) \in A \cap B$

$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq C$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \wedge \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B \; and \; C \Rightarrow (x_i, x_i) \in B \; and \; (x_i, x_i) \in \theta_3$
$\Rightarrow (x_i, x_i) \in A \cap B \Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$.This is a contradiction, since $A \wedge B = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (B \vee \theta_3, 0)$ implies $(x_i, x_i) \in B \vee \theta_3$. $\Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in B \; or \; (x_i, x_k) \in \theta_3$. By symmetry and transitivity, we have $(x_i, x_i) \in B \; or \; (x_i, x_i) \in C$.This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (viii): Let $A \in I \times C_2$, B and $C \in C_W(L_n) \setminus I$. Let $A = (\theta_1, j)$where $j =$ either 0 or 1. This case follows, since $B \; and \; C \in C_W(L_n) \setminus I$ implies $B \; and \; C \in C_W(L_n)$ which is 0-distributive.
Hence, $[C_W(L_n)](I)$ is 0-distributive whenever I is a lower interval of $C_W(L_n)$.

Example 3.2 Consider the 0-distributive lattice $C_W(L_4)$ where L_4 is $\{0 \prec a \prec b \prec 1\}$.

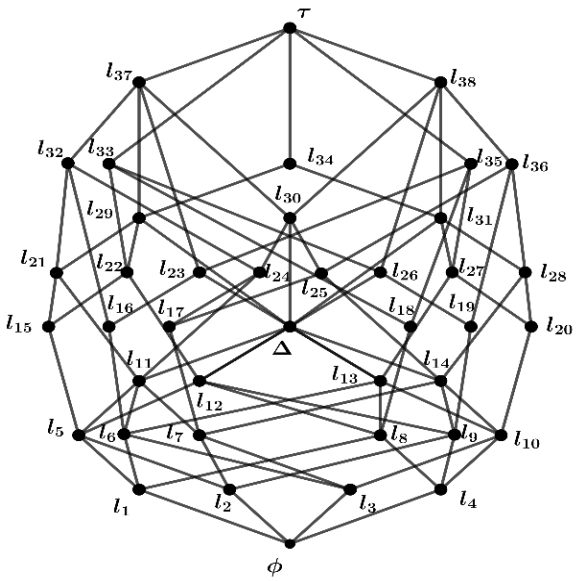

Figure 1. $C_W(L_4)$.

Consider the interval $I = [\phi, l_{32}]$ in the above lattice *Figure 1*. Let $C_2 = \{0,1\}$ be the two-element chain. We can form the new lattice $[C_W(L_4)](I) = \{C_W(L_4) \setminus I\} \cup (I \times C_2)$ given in *Figure 2*.

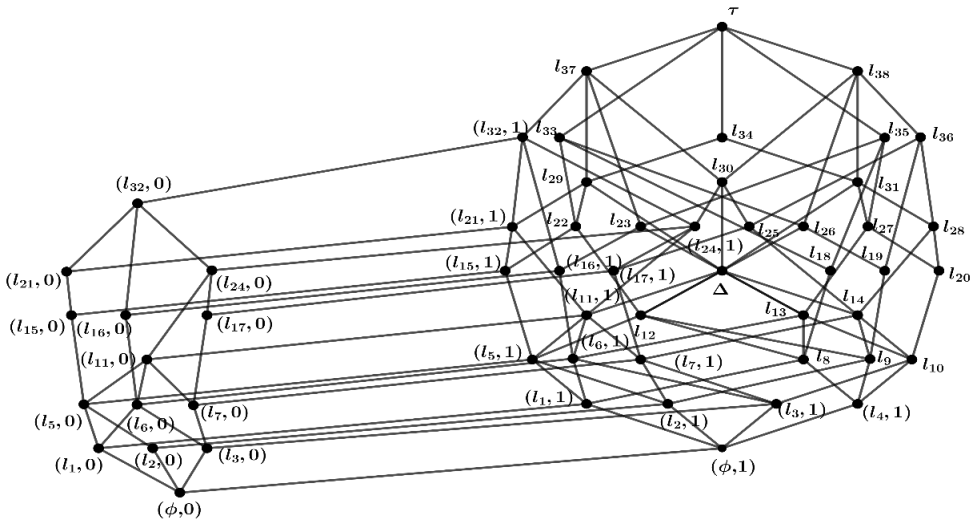

Figure 2. $[C_W(L_4)](I)$ where $I = [\phi, l_{32}]$.

Theorem 3.3 $[C_W(L_n)](I)$ *is 0 - distributive, when I is an upper interval of* $C_W(L_n)$.

Proof. Let $I = [\{(1,1)\}, \tau]$. Let $A, B, C \in [C_W(L_n)](I)$ such that $A \wedge B = \phi,\ A \wedge C = \phi$.

Claim: $A \wedge (B \vee C) = \phi$. (3.2)

There can be two possibilities, that is, A maybe in $I \times C_2$ or A maynot be in $I \times C_2$.

The following cases arise:

i. $A \in I \times C_2$ *and* $B, C \in C_W(L_n) \setminus I$
ii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in C_W(L_n) \setminus I$
iii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in I \times C_2$
iv. $A, B \in C_W(L_n) \setminus I$ *and* $C \in I \times C_2$

Case (i): Let $A \in I \times C_2$ and $B, C \in C_W(L_n) \setminus I$.

We note that either $A \wedge (B \vee C) \in I \times C_2$ or $A \wedge (B \vee C) \notin I \times C_2$.

Suppose $A \wedge (B \vee C) \neq \phi$. Therefore, there exists $\{(x_i, x_i)\} \leq A \wedge (B \vee C)$ $\Rightarrow \{(x_i, x_i)\} \leq A$ and$\{(x_i, x_i)\} \leq B \vee C \Rightarrow \{(x_i, x_i)\} \leq A$ and $\{(x_i, x_i)\} \leq B$ or $\{(x_i, x_i)\} \leq C$ or$\{(x_i, x_i)\} \leq$ both B and C or incomparable with both.

$\Rightarrow \{(x_i, x_i)\} \leq A$ *and* $\{(x_i, x_i)\} \leq B$ which is a contradiction, since $A \wedge B \neq \phi$.

Similarly, we get a contradiction, when $\{(x_i, x_i)\} \leq A$ *and* $\{(x_i, x_i)\} \leq C$ and when $\{(x_i, x_i)\} \leq A$ and $\{(x_i, x_i)\} \leq B$ & C.So, $A \wedge (B \vee C) = \phi$.

When $\{(x_i, x_i)\} \not\leq$ *both* B & C, then there exists x_k such that $\{(x_i, x_k)\} \leq B$ or $\{(x_i, x_k)\} \leq C \Rightarrow (x_i, x_k) \in B$ *or* $(x_i, x_k) \in C$. So, by symmetry and transitivity, we have $(x_i, x_i) \in B$.So, $A \wedge B \geq \{(x_i, x_i)\}$, a contradiction again. So, (3.2) holds.

Case (ii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in C_W(L_n) \setminus I$. As $C_W(L_n)$ is 0 - distributive, it follows that $A \wedge (B \vee C) = \phi$.

Case (iii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in I \times C_2$. Suppose $A \wedge (B \vee C) \neq \phi$.

Therefore, there exists $(x_i, x_i) \leq A \wedge (B \vee C)$.As in Case *(i)*,$A \wedge (B \vee C) = \phi$ follows.

Case (iv): Let $A, B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$. Suppose $A \wedge (B \vee C) \neq \phi$.
Therefore, there exists $(x_i, x_i) \leq A \wedge (B \vee C)$. As in Case *(i)*,$A \wedge (B \vee C) = \phi$ follows.
Hence, for an upper interval I, $[C_W(L_n)](I)$ is 0-distributive.

Example 3.4 Consider the interval $[l_1, \tau]$ in *Figure 1*. The lattice formed by the doubling of the interval is given in *Figure 3*.

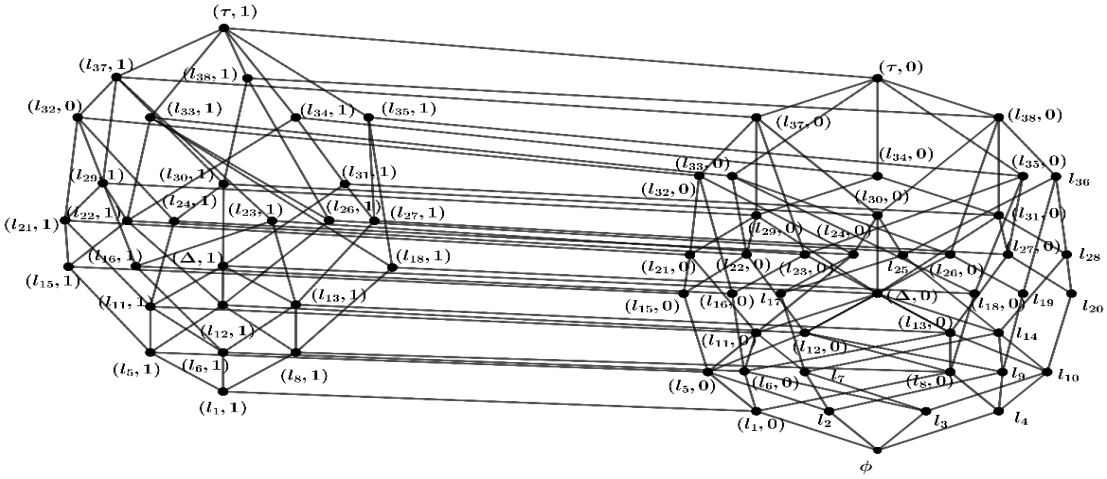

Figure 3. $[C_W(L_4)](I)$ where $I = [l_1, \tau]$.

Remark 3.5 The property of 0 - distributivity doesn't hold if we consider an intermediate interval I of $[C_W(L_n)]$.

Example 3.6 Consider the intermediate interval $I = [l_4, l_{20}]$ in Figure 1. Consider the elements $(l_4, 1)$ and $(l_{10}, 0)$ disjoint with l_3, that is, $l_3 \wedge (l_4, 1) = \phi$ and $l_3 \wedge (l_{10}, 0) = \phi$. Now, $l_3 \wedge [(l_4, 1) \vee (l_{10}, 0)] = l_3 \wedge (l_{10}, 1) = l_3 \neq \phi$.

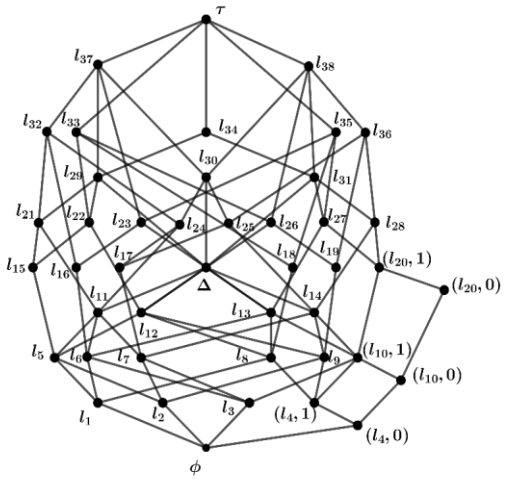

Figure 4. $[C_W(L_4)[I]$ where $I = [l_4, \ l_{20}]$.

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Connected $2-$ Dominating Sets and Connected $2-$ Domination Polynomials of the Complete Bipartite Graph $k_{2,m}$.

Y. A. Shiny[1]
T. Anithababy[2]

Abstract

Let $G=(V,E)$ be a simple graph. Let $D_{c_2}(G,j)$ be the family of connected $2-$dominating sets in G with cardinality j and $d_{c_2}(G,j)=|D_{c_2}(G,j)|$. Then the polynomial $D_{c_2}(G,x)=\sum_{j=\gamma_{c_2}(G)}^{|V(G)|} d_{c_2}(G,j)x^j$, is called the $2-$domination polynomial of G where $\gamma_{c_2}(G)$ is the connected $2-$ domination number of G.Let $D_{c_2}(k_{2,m},j)$ be the family of connected $2-$dominating sets of the Complete bipartite graph $k_{2,m}$ with cardinality j and let $d_{c_2}(k_{2,m},j)=|D_{c_2}(k_{2,m},j)|$. Then the connected $2-$ domination polynomial $D_{c_2}(k_{2,m},x)$ of $k_{2,m}$ is defined as $D_{c_2}(k_{2,m},x)=\sum_{j=\gamma_{c_2}(k_{2,m})}^{|V(k_{2,m})|} d_{c_2}(k_{2,m},j)x^j$, where $\gamma_{c_2}(k_{2,m},j)$ is the connected 2 – domination number of $k_{2,m,}$. In this paper, we obtain a recursive formula for $d_{c_2}(k_{2,m},j)$.Using this recursive formula, we construct the connected $2-$domination polynomial $D_{c_2}(k_{2,m},x)=\sum_{j=3}^{m+2} d_{c_2}(k_{2,m},j)x^j$, where $d_{c_2}(k_{2,m},j)$ is the number of connected $2-$dominating sets of $k_{2,m}$ of cardinality j and some properties of this polynomial have been studied.

Keywords: Dominating, Connected and cardinality

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[1]Reg. No.: 19213042092006, Research Scholar (Full time), Research Department of Mathematics, Women's Christian College, Nagercoil. Affiliated by Manonmaniam Sundaranar University, Tamil Nadu, India, shinyjebalin@gmail.com

[2]Assistant Professor, Research Department of Mathematics, Women's Christian College, Nagercoil. Affiliated by Manonmaniam Sundaranar University, Tamil Nadu, India. anithasteve@gmail.com

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1. Introduction

Let G = (V, E) be a simple graph of order, |V| = m. For any vertex vϵV, the open neighbourhood of v is the set N(v) = {uϵV/uvϵE} and the closed neighbourhood of V is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = U_{v \epsilon S} N(v)$ and the closed neighbourhood of S is $N(S) \cup S$.A set $D \subseteq V$ is a dominating set of G, if $N[D] = V$ or equivalently, every vertex in $V - D$ is adjacent to atleast one vertex in D.The domination number of a graph G is defined as the cardinality of a minimum dominating set D of vertices in G and is denoted by $\gamma(G)$. A dominating set D of G is called a connected dominating set if the induced sub-graph $< D >$ is connected. The connected domination number of a graph G is defined as the cardinality of a minimum connected dominating set D of vertices in G and is denoted by $\gamma_c(G)$.

A graph $G = (V, E)$is called a bipartite graph if its vertices V can be partitioned into two subsets V_1 and V_2 such that each edge of G connects a vertex of $V_1$to a vertex of V_2. If G contains every edge joining a vertex of V_1 and a vertex of V_2 then G is called a complete bipartite graph. It is denoted by $k_{m,n}$, where m and n are the numbers of vertices in V_1 and V_2 respectively. Let $k_{2,m}$ be the Complete bipartite graph with $m + 2$ vertices. Throughout this paper let us take $V(k_{2,m}) = \{v_1, v_2, v_3, \ldots, v_{m+1}, v_{m+2}\}$ and $E(k_{2,m}) = \{(v_1, v_3), (v_1, v_4), (v_1, v_5), \ldots, (v_1, v_{m+1}), (v_1, v_{m+2}), (v_2, v_3), (v_2, v_4), (v_2, v_5), \ldots, (v_2, v_{m+1}), (v_2, v_{m+2})$.

As usual we use $\binom{m}{j}$ for the combination m to j. Also, we denote the set $\{1, 2, \ldots \ldots \ldots, 2m-1, 2m\}$ by $[2m]$, throughout this paper.

2. Connected 2 – Dominating Sets of the Complete Bipartite Graph $k_{2,m}$

In this section, we state the connected 2 – domination number of the complete bipartite graph $k_{2,m}$ and some of its properties.

Definition 2.1. Let G be a simple graph of order m with no isolated vertices. A subset $D \subseteq V$ is a $2-$ dominating set of the graph G if every vertex $v \epsilon V - D$ is adjacent to atleast two vertices in D. A $2-$ dominating set is called a connected $2-$ dominating set if the induced subgraph $<D>$ is connected.

Definition 2.2. The cardinality of a minimum connected 2 – dominating sets of G is called the connected 2 – domination number of G and is denoted by $\gamma_{C_2}(G)$.

Lemma 2.3 For all $m \in z^+$, $\binom{m}{j} = 0$ if $j > m$ or $j < 0$.

Theorem 2.4 $d_{c2}(k_{2,m}, j) = \left\{\binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}\right.$ for $3 \leq j \leq m + 2$

Proof: Let the partite sets of $k_{2,m}$ be $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, \ldots, v_{m+1}, v_{m+2}\}$. Since the subgraph induced by the vertex set as $\{v_1, v_2\}$ is not connected, every connected 2 −dominating set of $k_{2,m}$ must contain the vertex $\{v_1\}$ or $\{v_2\}$ or $\{v_1, v_2\}$. When $3 \leq j \leq m$, every connected 2 −dominating set must contain $\{v_1, v_2\}$. Since, $|V(k_{2,m})| = m + 2$, $k_{2,m}$ contains $\binom{m+2}{j}$ number of subsets of cardinality j. Since, the subgraphs induced by $\{v_1, v_2\}$ and $\{v_3, v_4, \ldots, v_{m+1}, v_{m+2}\}$ are not connected, each time $\binom{m+1}{j}$ number of subsets of $k_{2,m}$ of cardinality j and $\binom{m}{j-1}$ number of subsets of $k_{2,m}$ of cardinality $j-1$ are not connected 2 −dominating sets. Hence, $k_{2,m}$ contains $\binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}$ number of subsets of connected 2 −dominating sets, when $3 \leq j \leq m$.

Therefore, $d_{c2}(k_{2,m}, j) = \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}$ for all $3 \leq j \leq m$.

When the cardinality is $m+1$, every subset of $k_{2,m}$ containing $\{v_1\}$ or $\{v_2\}$ are connected 2 −dominating sets. Therefore, two more sets are connected 2 −dominating sets when the cardinality is $m+1$.

Hence, $d_{c2}(k_{2,m}, j) = \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1} + 2$, when $j = m+1$.

Since, there is only one subset of $k_{2,m}$ with cardinality $m+2$ and that set is a connected 2 −dominating set. we get $d_{c2}(k_{2,m}, j) = \binom{m+2}{j}$ when $j = m+2$.

Theorem 2.5. Let $k_{2,m}$ be the complete bipartite graph with $m \geq 3$. Then

(i) $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1)$

(ii) $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + 1$ if $j = 3$.

(iii) $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1) - 2$ if $j = m$.

Proof:

(i) By Theorem 2.4, we have,

$d_{c2}(k_{2,m}, j) = \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}$ for all $3 \leq j \leq m+2$.

$d_{c2}(k_{2,m-1}, j) = \binom{m+1}{j} - \binom{m}{j} - \binom{m-1}{j-1}$

$d_{c2}(k_{2,m-1}, j-1) = \binom{m+1}{j-1} - \binom{m}{j-1} - \binom{m-1}{j-2}$.

Consider, $d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1) = \binom{m+1}{j} - \binom{m}{j} - \binom{m-1}{j-1} + \binom{m+1}{j-1} - \binom{m}{j-1} - \binom{m-1}{j-2}$.

$= \binom{m+1}{j} + \binom{m+1}{j-1} - \left[\binom{m}{j} + \binom{m}{j-1}\right] - \left[\binom{m-1}{j-1} + \binom{m-1}{j-2}\right]$

$= \binom{m+2}{j} - \binom{m+1}{j} - \binom{m}{j-1}.$
$= d_{c2}(k_{2,m}, j)$.
Therefore,
$d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1)$ for all $4 \leq j \leq m+2$ and $j \neq m$.
(i) When $j = 3, d_{c2}(k_{2,m}, 3) = \binom{m+2}{3} - \binom{m+1}{3} - \binom{m}{2}$ by Theorem 2.4
$= \binom{m+1}{2} - \binom{m}{2}$
$= \binom{m}{1}$
Consider, $d_{c2}(k_{2,m-1}, 3) = \binom{m+1}{3} - \binom{m}{3} - \binom{m-1}{2} = \binom{m}{2} - \binom{m-1}{2}$
$= \binom{m-1}{1}$
$= m - 1.$
That is, $d_{c2}(k_{2,m-1}, 3) = d_{c2}(k_{2,m}, 3) - 1.$
Therefore, $d_{c2}(k_{2,m}, 3) = d_{c2}(k_{2,m-1}, 3) + 1.$
Hence, $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + 1 \; if \; j = 3.$
(ii) When $j = m$,
$d_{c2}(k_{2,m}, m) = \binom{m+2}{m} - \binom{m+1}{m} - \binom{m}{m-1}$, by Theorem 2.2.
$= \binom{m+1}{m-1} - \binom{m}{m-1}.$
$= \binom{m}{m-2}.$
Consider, $d_{c2}(k_{2,m-1}, m) + d_{c2}(k_{2,m-1}, m-1) = \binom{m+1}{m} - \binom{m}{m} - \binom{m-1}{m-1} +$
$2 + \binom{m+1}{m-1} - \binom{m}{m-1} - \binom{m-1}{m-2}$
$= \binom{m+1}{m} - \binom{m+1}{m-1} - \left[\binom{m}{m} + \binom{m}{m-1}\right] - \left[\binom{m-1}{m-1} + \binom{m-1}{m-2}\right] + 2$
$= \binom{m+2}{m} - \binom{m+1}{m} - \binom{m}{m-1} + 2$
$= \binom{m+1}{m-1} - \binom{m}{m-1} + 2$
$= \binom{m}{m-2} + 2.$
$= d_{c2}(k_{2,m}, m) + 2.$
Therefore, $d_{c2}(k_{2,m}, m) = d_{c2}(k_{2,m-1}, m) + d_{c2}(k_{2,m-1}, m-1) + 2.$
Hence, $d_{c2}(k_{2,m}, j) = d_{c2}(k_{2,m-1}, j) + d_{c2}(k_{2,m-1}, j-1) - 2$ when $j = m$.

3. Connected 2 −Domination Polynomials of the Complete Bipartite Graph $k_{2,m}$.

Definition 3.1. Let $d_{c2}(k_{2,m}, j)$ be the number of connected 2 −dominating sets of the Complete bipartite Graph $k_{2,m}$ with cardinality j.Then, the connected 2 − domination

Connected 2 − Dominating Sets and Connected 2 − Domination Polynomials of the Complete Bipartite Graph $k_{2,m}$.

Polynomial of $k_{2,m}$ is defined as $D_{c_2}(k_{2,m},x) = \sum_{j=\gamma_{c_2}(k_{2,m})}^{|V(k_{2,m})|} d_{c_2}(k_{2,m},j)x^j$, where $\gamma_{c_2}(k_{2,m})$ is the connected 2 – domination number of $k_{2,m}$.

Remark 3.2 $\gamma_{c_2}(k_{2,m})$ =3.

Proof. Let $k_{2,m}$ be the complete bipartite graph with partite sets $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, ..., v_{m+1}, v_{m+2}\}$. Let $v_1, v_2 \epsilon$ V $(k_{2,m})$ and v_1, v_2 are adjacent to all the other vertices $v_3, v_4, ..., v_{m+1}, v_{m+2}$ of $k_{2,m}$. Also Since, v_1 and v_2 are not connected, every connected 2 –dominating set must contain the vertices v_1, v_2 and one more vertex from $\{v_3, v_4, ..., v_{m+1}, v_{m+2}\}$. Therefore, the minimum cardinality is 3. Hence, γ_{c_2} $(k_{2,m})$ =3.

Theorem 3.3 Let $k_{2,m}$ be the complete bipartite graph with $m \geq 3$.
Then $D_{c2}(k_{2,m},x)$=$(1+x)$ $D_{c2}(k_{2,m-1},x)$+$x^3 - 2x^m$.

Proof:

From the definition of connected 2 − domination Polynomial, we have,

$D_{c2}(k_{2,m},x) = \sum_{j=3}^{m+2} d_{c_2}(k_{2,m},j)x^j$.

$= d_{c2}(k_{2,m}, 3)x^3 + \sum_{j=4}^{m-1} d_{c_2}(k_{2,m},j)x^j + d_{c_2}(k_{2,m},m)x^m + \sum_{j=m+1}^{m+2} d_{c_2}(k_{2,m},j)x^j$.

$= \left[d_{c_2}(k_{2,m-1},3) + 1\right]x^3 + \sum_{j=4}^{m+2} \left[d_{c_2}(k_{2,m-1},j) + d_{c_2}k_{2,m-1,}\, j - 1\right]x^j$

$+[d_{c_2}(k_{2,m-1},m) + d_{c_2}(k_{2,m-1},m-1) - 2]x^m$, by Theorem 2.5

$= d_{c2}(k_{2,m-1}, 3)x^3 + x^3 + \sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1},j)\, x^j$

$+ \sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1},j-1)\, x^j - 2x^m$.

$= \sum_{j=3}^{m+2} d_{c_2}(k_{2,m-1},j)\, x^j + x \sum_{j=4}^{m+2} d_{c_2}(k_{2,m-1},j-1)\, x^{j-1} + x^3 - 2x^m$.

$= D_{c2}(k_{2,m-1},x) + xD_{c2}(k_{2,m-1},x) + x^3 - 2x^m$.

Hence, $D_{c2}(k_{2,m},x) = (1+x)D_{c2}(k_{2,m-1},x) + x^3 - 2x^m$, for every $m \geq 3$.

Example 3.4 Let $k_{2,7}$ be the complete bipartite graph with order 9 as given in Figure 2.1.

$k_{2,7}$:

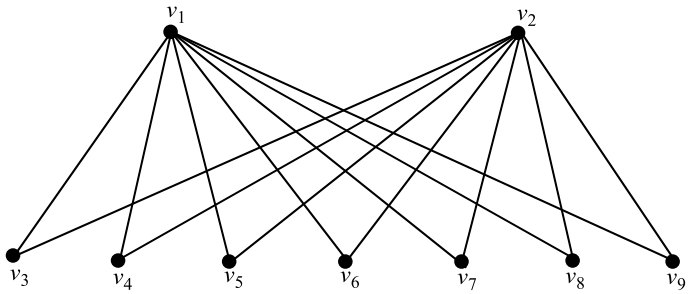

Figure 2.1

$D_{c_2}(K_{2,6},x) = 6x^3 + 15x^4 + 20x^5 + 15x^6 + 8x^7 + x^8$.

By Theorem 3.3, we have,
$D_{c_2}(K_{2,7},x) = (1+x)(\ 6x^3+15x^4+20x^5+15x^6+8x^7+x^8)+ x^3 - 2x^7.$
$= 6x^3+15x^4+20x^5+15x^6+8x^7+x^8 +6x^4+15x^5+20x^6+15x^7+8x^8+x^9 + x^3 - 2x^7.$
$= 7x^3+21x^4+35x^5+35x^6+21x^7+9x^8+x^9$
We obtain $d_{c_2}(k_{2,m},j)$ for $3\leq m \leq 15$ and $3\leq j \leq 15$ as shown in Table 1.

j / m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	0	3	5	1												
4	0	4	6	6	1											
5	0	5	10	10	7	1										
6	0	6	15	20	15	8	1									
7	0	7	21	35	35	21	9	1								
8	0	8	28	56	70	56	28	10	1							
9	0	9	36	84	126	126	84	36	11	1						
10	0	10	45	120	210	252	210	120	45	12	1					
11	0	11	55	165	330	462	462	330	165	55	13	1				
12	0	12	66	220	495	792	924	792	495	220	66	14	1			
13	0	13	78	286	715	1287	1716	1716	1287	715	286	78	15	1		
14	0	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	16	1	
15	0	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	107	17	1

Table 1. $d_{c_2}(k_{2,m},j)$, the number of connected $2-$ dominating sets of $k_{2,m}$ with cardinality j.

In the following Theorem, we obtain some properties of $d_{c2}\ (K_{2,m},j)$.

Theorem: 3.5 The following properties hold for the coefficients of $D_{c_2}(K_{2,m},x)$ for all m.
(i) $d_{c_2}(k_{2,m},3) = m$, for all $m \geq 3$.
(ii) $d_{c_2}(k_{2,m},m+2) = 1$, for all $m \geq 3$.
(iii) $d_{c_2}(k_{2,m},m+1) = m+2$, for all $m \geq 3$.
(iv) $d_{c_2}(k_{2,m},m) = \binom{m+2}{2} - \binom{m+1}{1} - m.$
(v) $d_{c_2}(k_{2,m},m-1) = \binom{m+2}{3} - \binom{m+1}{2} - \binom{m}{2}$, for all $m \geq 4$.
(vi) $d_{c_2}(k_{2,m},m-2) = \binom{m+2}{4} - \binom{m+1}{3} - \binom{m}{3}$, for all $m \geq 5$.
(vii) $d_{c_2}(k_{2,m},m-3) = \binom{m+2}{5} - \binom{m+1}{4} - \binom{m}{4}$, for all $m \geq 6$.

(viii) $d_{c_2}(k_{2,m,}m-i)=\binom{m+2}{i+2}-\binom{m+1}{i+1`}-\binom{m}{i}$, for all $m \geq 4$ and $i \geq 1$.

Proof:

(i) $d_{c_2}(k_{2,m,}3)=m$.

We prove this by induction on m.

When $m=3$, $d_{c_2}(k_{2,m,}3)=3$.

Therefore, the result is true for $m=3$.

Now, suppose that the result is true for all numbers less than 'm' and we prove it for m.

By Theorem 2.6,

$d_{c_2}(k_{2,m,}3)=d_{c_2}(k_{2,m-1,}3)+1=m-1+1\ =m$.

(ii) $d_{c_2}(k_{2,m,}m+2)=1$, for all $m \geq 3$.

Since, there is only one connected $2-$ dominating set of cardinalities $m+\ 2$, $d_{c_2}(k_{2,m,}m+2)=1$.

(iii) $d_{c_2}(k_{2,m,}m+1)=m+2$, for all m≥4.

Since, $d_{c_2}(k_{2,m,}m+1)=\{[m+2]-x/x\varepsilon[m+2]\}$,we have the result.

(iv), (v), (vi), (vii) and (viii) follows from Theorem 2.4.

4. Conclusion

In this paper, the connected $2-$ domination polynomials of the complete bipartite graph $K_{2,m}$ has been derived by identifying its connected $2-$ dominating sets. It also helps us to characterize the connected. Connected $2-$ dominating sets of cardinality j. We can generalize this study to any of complete bipartite graph $K_{n,m}$ and some interesting properties can be obtained.

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Medium Domination Decomposition of Graphs

Ebin Raja Merly E[1]
Saranya J[2]

Abstract

A set of vertices S in a graph G dominates G if every vertex in G is either in S or adjacent to a vertex in S. The size of any smallest dominating set is called domination number of G. The concept of Medium Domination Number was introduced by Vargor and Dunder which finds the total number of vertices that dominate all pairs of vertices and evaluate the average of this value. The Medium domination Number is a notation which uses neighbourhood of each pair of vertices. For *G = (V, E)* and $\forall u, v \in V$ *if u, v* are adjacent they dominate each other, then atleast *dom (u, v) = 1*. The total number of vertices that dominate every pair of vertices is defined as TDV(G)=$\sum$ dom (u, v), for every u, v$\in$ V(G). For any connected, undirected, loopless graph G of order p, the Medium Domination Number MD(G) = $\frac{TDV(G)}{pC_2}$. In this paper we have introduced the new concept Medium Domination Decomposition. A decomposition $(G_1, G_2, \dots, G_n)$ of a graph G is said to be Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i - 1, i = 1,2,\dots,n$.

Keywords: Domination Number, Medium Domination Number, Medium Domination Decomposition.

Mathematical classification Number: 05C69, 54D05[3]

[1] Associate Professor, Department of Mathematics, Nesamony Memorial Christian College, Marthandam. (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamilnadu, India), ebinmerly@gmail.com

[2] Research Scholar (Reg. No: 20113112092022), Department of Mathematics, Nesamony Memorial Christian College, Martha. (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, TamilNadu, India), saranyajohnrose@gmail.com

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1. Introduction

Graph is a mathematical representation of a network and it describes the relationship between vertices and edges. Let $G = (V, E)$ be a simple, connected, undirected, loopless graph with p vertices and q edges and G_i be the subgraph of G with p_i vertices and q_i edges, where $1 \leq i \leq n$, n is the number of subgraphs of G. The length of a shortest $u - v$ path in a connected graph G is called the distance from a vertex u to a vertex v. $d(u, v)$ denotes the distance between v and u. Two $u - v$ paths are internally disjoint if they have no vertices in common, other than u and v. The degree of a vertex v in a graph G is the number of edges of incident with v and denoted by $\deg(v)$. The minimum degree among the vertices of a graph G is denoted by $\delta(G)$. The maximum degree among vertices of a graph G is denoted by $\Delta(G)$. [1] The concept of Medium Domination Number was introduced by Vargor and Dunder which finds the total number of vertices that dominate all pairs of vertices and evaluate the average of this value.[5] T.I. Joel and E.E. Merly introduced the concept of Geodetic Decomposition of Graphs. Motivated by the above we have introduced the new concept of Medium Domination Decomposition of Graphs. For basic terminologies in graph theorem, we refer [2], [3] and [4]. The following are the basic definitions and results needed for the main section.

Definition 1.1. [1] For $G = (V, E)$ and $\forall\, u, v \in V$, if u and v are adjacent they dominate each other, then atleast $dom\,(u, v) = 1$.

Definition 1.2. [1] For $G = (V, E)$ and $\forall\, u, v \in V$, the total number of vertices that dominate every pair of vertices is defined as $TDV(G) = \Sigma_{\forall u,v \in V(G)} dom\,(u, v)$.

Definition 1.3. [1] For any connected, undirected, loopless graph G of order p the Medium Domination Number of G is defined as $MD(G) = \frac{TDV(G)}{\binom{P}{2}}$.

2. Medium Domination Decomposition of Graphs

Definition. 2.1 Let G be a simple connected (p, q) graph. A decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be a Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i - 1, i = 1,2,3, \ldots, n$.

Example. 2.2

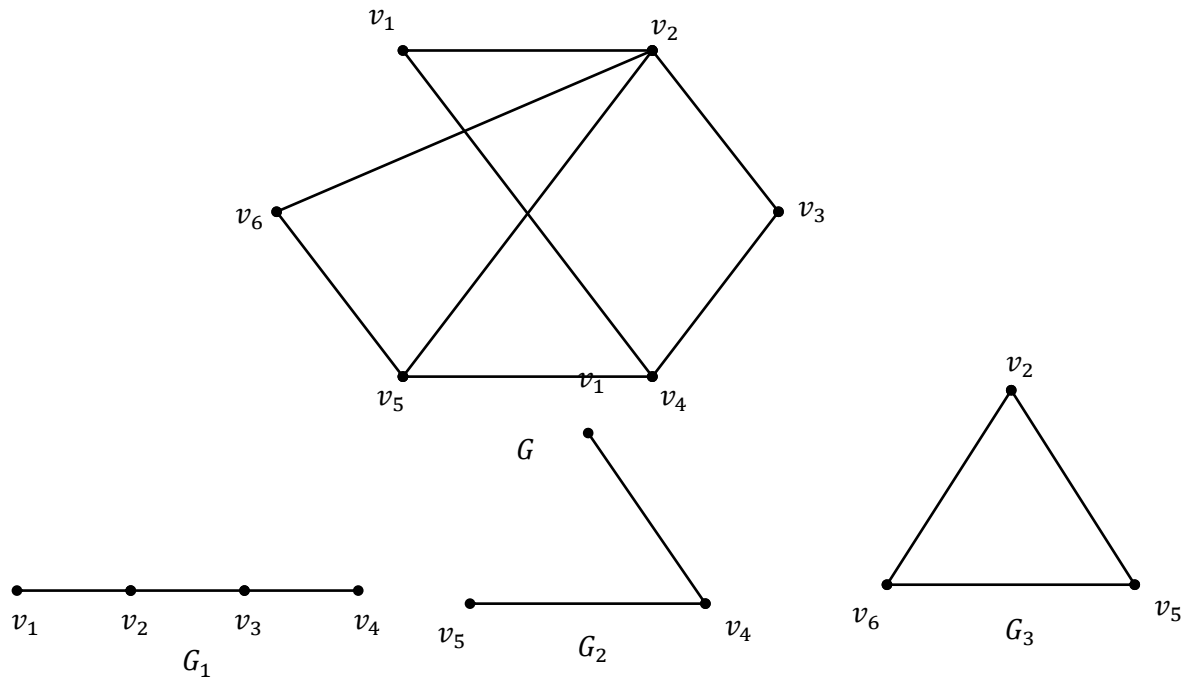

Figure 2.3

Here $MD(G_1) = 0.8, MD(G_2) = 1$ and $MD(G_3) = 2$
That is $\lfloor MD(G_1) \rfloor = 0, MD(G_2) = 1$ and $MD(G_3) = 2$

Remark. 2.4

i) Star Graph does not admit MDD
ii) $K_p, p \leq 4$, does not admit MDD

Theorem. 2.5 If a graph G admits $MDD(G_1, G_2, \dots, G_n)$, then $p \geq 4$ and $q \geq 3$.
Proof.

Since the Medium Domination Number is one, when $p \leq 3$ and $q \leq 2$ and the Medium Domination Number is two, when p and q is 3, we can't get any subgraph with $\lfloor MD(G_i) \rfloor = 0$.

Note: 2.6 The converse of the above theorem is need not be true. For example, the complete graph K_4.

Theorem: 2.7

Let G be a graph and G admits $MDD(G_1, G_2, \dots, G_n)$. Then
(i) $MD(G_2) = p_2 - \Delta(G_2)$ if and only if G_2 is $K_{1,m}$ for any m.
(ii) $MD(G_i) = \Delta(G_i)$ if and only if G_i is a complete graph, where $i = 2,3, \dots, n$
Proof:

Suppose G admits $MDD(G_1, G_2, \dots, G_n)$.

(i) Let G_2 be $K_{1,m}$ for any m.
$\Leftrightarrow G_2$ has p_2 vertices and $q_2(= p_2 - 1)$ edges and the maximum degree of $G_2 = p_2 - 1$
$\Leftrightarrow MD(G_2) = \frac{(p_2-1)+\binom{p_2-1}{2}}{\binom{p_2}{2}}$
$\Leftrightarrow MDD(G_2) = 1$
$\Leftrightarrow MD(G_2) = p_2 - \Delta(G_2)$
(ii) Let G_i be a complete graph, $i = 1,2,3, \dots, n$
$\Leftrightarrow G_i$ has p_i vertices and $q_i = \frac{p_i(p_i-1)}{2}$ edges and the maximum degree of $G_i = p_i - 1$.
$\Leftrightarrow MD(G_i) = \frac{\frac{p_i(p_i-1)}{2}+p_i[p_i-1)C_2]}{\binom{p_i}{2}}$
$\Leftrightarrow MD(G_i) = p_i - 1$
$\Leftrightarrow MD(G_i) = \Delta(G_i), i = 2,3, \dots, n$
Hence the proof.

Theorem: 2.8 Let G be a graph and G admits $MDD(G_1, G_2, \dots, G_n)$. Then $\sum_{i=1}^{n} \lfloor MD(G_i) \rfloor < \frac{n(n+1)}{2} \lceil MD(G) \rceil$, where n is the number of decompositions of G.

Proof:
We prove this theorem by induction on n.
When $n = 1$,
Then $\lfloor MD(G_1) \rfloor < \lceil MD(G) \rceil$,
Therefore, the result is true for $n = 1$
Assume that the theorem is true for $n - 1$
That is, $\sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor < \frac{n(n-1)}{2} \lceil MD(G) \rceil$
To prove: the theorem is true for n.
That is, $\sum_{i=1}^{n} \lfloor MD(G_i) \rfloor < \frac{n(n+1)}{2} \lceil MD(G) \rceil$.
Now,
$\sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor = \lfloor MD(G_1) \rfloor + \lfloor MD(G_2) \rfloor + \cdots + \lfloor MD(G_n) \rfloor$
$\Rightarrow \sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor + \lfloor MD(G_n) \rfloor < \frac{n(n+1)}{2} + \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$\Rightarrow \sum_{i=1}^{n} \lfloor MD(G_n) \rfloor < \frac{n^2-n}{2} + \lfloor MD(G) \rfloor + \lceil MD(G_n) \rceil$
$= \left(\frac{n^2 - n + n - n}{2}\right) \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$= \left(\frac{(n^2 + n)}{2} - \frac{2n}{2}\right) \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$= \left(\frac{n^2 + n}{2}\right) \lceil MD(G) \rceil - n \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$< \left(\frac{n^2+n}{2}\right) \lceil MD(G) \rceil$, since the value of $-n \lceil MD(G) \rceil + \lfloor MD(G) \rfloor$
is negative
$\sum_{i=1}^{n} \lfloor MD(G_n) \rfloor < \left(\frac{n(n+1)}{2}\right) \lfloor MD(G) \rfloor$

The result is true for n. Hence the proof.

Theorem: 2.8 If G admits $MDD(G_1, G_2, \dots, G_n)$ and $\Delta(G) = p - 1$ then
i) $\gamma(G) < \lceil MD(G) \rceil$
ii) $\gamma(G) \leq \gamma(G_i)$
iii) $\gamma(G) \leq \lceil MD(G_i) \rceil$
Proof:
Let G be a graph with p vertices and q edges and $\Delta(G) = p - 1$.
Since $\Delta(G) = p - 1, \gamma(G) = 1$. But the minimum value of $\lceil MD(G) \rceil = 1$.
Therefore $\gamma(G) < \lceil MD(G) \rceil$.
The proof of (ii) and (iii) is obvious.

Remark: 2.9 The equality holds in theorem 2.8, (ii) and (iii) when G_i is star.

Theorem: 2.10 If G admits $MDD(G_1, G_2, \dots, G_n)$ then
i) $\lceil MD(G) \rceil < \gamma(G) + \Delta(G)$
ii) $\gamma(G_i) < \gamma(G) + \Delta(G)$
The Proof is obvious.

Theorem: 2.11 If G admits $MDD(G_1, G_2, \dots, G_n)$ and H is a Spanning subgraph of a graph G, then
(i) $\gamma(G) \leq \gamma(H)$
(ii) $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$
Proof:
Let G be a simple connected graph and G admits MDD.
Case (i): $n = 1$
Then $\gamma(G) = \gamma(H)$ and $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$.Hence the result is obvious.
Case (ii): $n > 1$
Then, let $G_1, G_2, \dots G_i, \dots, G_n$ be the decomposition of G.
Let $H = G_i$ be the spanning subgraph of G.
Since the number of decompositions is more than one, $|E(H)| < |E(G)|$
Also $\Delta(G) \geq \Delta(H)$ and $\delta(G) \geq \delta(H)$.
Therefore $\gamma(H) \geq \gamma(G)$.
Obviously, since H is a spanning subgraph of G, the Medium Domination Number of H is less than the Medium Domination Number of G.
Therefore $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$.
Hence the proof.

3. Conclusion

In this paper, we calculated the number of vertices that are capable of dominating both of u and v. The total number of vertices that dominate every pair of vertices is examined and the average of this value is calculated which is called “the medium domination number” of graph. Some theorems and results on the Medium Domination Decomposition of a graph and basic graph classes are given. Further this concept can be extended to some family of graphs.

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On Intuitionistic Semi * Continuous Functions

G. Esther Rathinakani[*]
M. Navaneethakrishnan[†]

Abstract

In this paper we introduce intuitionistic semi * continuous and intuitionistic contra - semi * continuous functions via the concept of intuitionistic semi * open and intuitionistic semi * closed set respectively. Also, we investigate their properties and characterization.

Keywords: intuitionistic semi * open, intuitionistic semi * closed, intuitionistic semi * continuous, intuitionistic contra-semi * closed.

2010 AMS subject classification: 54C05[‡]

[*]Research Scholar, Reg. No. 19222102092011, PG and Research Department of Mathematics, Kamaraj College, Thoothukudi, Tamilnadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India; e-mail estherrathinakani@gmail.com.
[†]Associate Professor, PG and Research Department of Mathematics, Kamaraj College, Thoothukudi, Tamilnadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India; e-mail navaneethan65@yahoo.co.in.
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1. Introduction

The concept of intuitionistic set introduce by D. Coker [2] in 1996 and also he [1] has introduced the concept of intuitionistic topological space. In [5], we introduced the concept of intuitionistic generalized closure operator and defined a new intuitionistic topology τ^* and studied their properties. Also, in [6] we introduced a new open set namely intuitionistic semi * open and studied their properties. In this paper we define intuitionistic semi * continuous and intuitionistic contra - semi * continuous functions via the concept of intuitionistic semi * open and intuitionistic semi * closed set respectively. Also, we investigate their properties and characterization.

2.Preliminaries

Definition 2.1 [1] Let X be a nonempty fixed set. An intuitionistic set (IS in short) $\tilde{A}$ is an object having the form $\tilde{A} =< X, A^1, A^2 >$ where A^1 and A^2 are subsets of X such that $A^1 \cap A^2 = \emptyset$. The set A^1 is called the set of member of $\tilde{A}$, while A^2 is called the set of non member of $\tilde{A}$.

Definition 2.2 [1] An intuitionistic topology (IT in short) by subsets of a nonempty set X is a family τ of IS's satisfying the following axioms.

(a) $\tilde{\emptyset}_I , \tilde{X}_I \in \tau$,

(b) $\tilde{G}_1 \cap \tilde{G}_2 \in \tau$ for every $\tilde{G}_1 , \tilde{G}_2 \in \tau$, and

(c) $\cup \tilde{G}_i \in \tau$ for any arbitrary family $\{\tilde{G} : i \in J\} \subseteq \tau$.

The pair (X, τ) is called an intuitionistic topological space (ITS in short) and any IS $\tilde{A}$ in τ is called an intuitionistic open set (IOS). The complement of an IO set $\tilde{A}$ in is called an intuitionistic closed set (ICS).

Definition 2.3 [1] Let (X, τ) be an ITS and $\tilde{A} =< X, A^1, A^2 >$ be an IS in X. Then the interior and the closure of A are denoted by $Iint(\tilde{A})$ and $Icl(\tilde{A})$, and are defined as follows.

$Iint(\tilde{A}) = \cup \{\tilde{G} \mid \tilde{G}$ is an IOS and $\tilde{G} \subseteq \tilde{A}\}$ and

$Icl(\tilde{A}) = \cap \{\tilde{K} \mid \tilde{K}$ is an ICS and $\tilde{A} \subseteq \tilde{K}\}$.

Definition 2.4 [2] Let X be a nonempty set and $p \in X$ be a fixed element. Then the IS $\tilde{p}$ defined by $\tilde{p} =< X, \{p\}, \{p\}^c >$ is called an intuitionistic point (in short, IP).

Definition 2.5 [10] Let (X, τ) be an ITS and $\tilde{A} =< X, A^1, A^2 >$ be an IS in X, $\tilde{A}$ is said to be intuitionistic generalized closed set (briefly Ig – closed set) $Icl(\tilde{A}) \subseteq \tilde{U}$ whenever $\tilde{A} \subseteq \tilde{U}$ and $\tilde{U}$ is IOS in X.

Definition 2.6 [5] If $\tilde{A}$ is an IS of an ITS (X, τ), then the intuitionistic generalized closure of $\tilde{A}$ is defined as the intersection of all *Ig* – closed sets in *X* containing $\tilde{A}$ and is denoted by *Icl**$(\tilde{A})$.

Definition 2.7 [6] The IS $\tilde{A}$ of an ITS (X, τ) is called intuitionistic semi * open sets if there is an intuitionstic open set $\tilde{G}$ in *X* such that $\tilde{G} \subseteq \tilde{A} \subseteq Icl^*(\tilde{G})$.

Definition 2.8 [6] The intuitionistic semi * interior of $\tilde{A}$ is defined as the union of all intuitionistic semi * open sets of *X* contained in $\tilde{A}$. It is denoted by *IS*int*$(\tilde{A})$.

Definition 2.9 An intuitionistic set $\tilde{A}$ of a ITS (X, τ) is called an intuitionistic semi * closed set if $X - \tilde{A}$ is intuitionistic semi * open.

Definition 2.10 The semi * closure of an IS $\tilde{A}$ is defined as the intersection of all intuitionistic semi * closed sets in *X* that containing $\tilde{A}$. It is denoted by *IS*cl*$(\tilde{A})$.

Theorem 2.11 Let (X, τ) be an ITS and $\tilde{A}$ be an IS of *X*. Then

(i) $IS^*cl(X - \tilde{A}) = X - IS^*int(\tilde{A})$

(ii) $IS^*int(X - \tilde{A}) = X - IS^*cl(\tilde{A})$

Definition 2.12[9] A function $f : X \to Y$ is said to be intuitionistic semi continuous if $f^{-1}(\tilde{A})$ is ISO in *X* for every IOS $\tilde{A}$ in *Y*.

Theorem 2.13[6] Let (X, τ) be an ITS and $\tilde{A}$ be an IS of *X*. Then

(i) Every IOS is IS*O.

(ii) Every IS*O is ISO.

(iii) Every ICS is IS*C.

(iv) Every IS*C is ISC.

Theorem 2.14[6] Let (X, τ) be an ITS. Then

(i) If $\{\tilde{A}_\alpha\}$ is a collection of IS*O in *X* then $\bigcup \tilde{A}_\alpha$ is IS*O.

(ii) If $\tilde{A}$ is IS*O in *X* and $\tilde{B}$ is an IOS in *X*, then $\tilde{A} \cap \tilde{B}$ is IS*O in *X*.

Theorem 2.15[6] Let (X, τ) be an ITS and $\tilde{A}$ be an IS of *X*. Then

(i) $\tilde{A}$ is IS*O if and only if $IS^*int(\tilde{A}) = \tilde{A}$.

(ii) $\tilde{A}$ is IS*C if and only if $IS^*cl(\tilde{A}) = \tilde{A}$.

Theorem 2.16[6] Let (X, τ) be an ITS, $\tilde{A}$ be an IS of X and $\tilde{p} \in X$. Then $\tilde{p} \in IS^*cl(\tilde{A})$ if and only if every IS*O in X containing $\tilde{p}$ intersects $\tilde{A}$.

Definition 2.17[7] Let (X, τ) be an ITS and $\tilde{A}$ be an IS of X. Then the intuitionistic semi * frontier of $\tilde{A}$ (denoted by $IS^*Fr(\tilde{A})$) is defined by $IS^*Fr(\tilde{A}) = IS^*cl(\tilde{A}) - IS^*int(\tilde{A})$.

Theorem 2.18[7] Let (X, τ) be an ITS and $\tilde{A}$ be an IS of X. Then

$$IS^*Fr(\tilde{A}) = IS^*cl(\tilde{A}) \cap IS^*cl(\overline{\tilde{A}}).$$

3. Intuitionistic Semi * Continuous Functions

Definition 3.1 A function $f : X \to Y$ is said to be intuitionistic semi * continuous at $\tilde{p} \in X$ if for each intuitionistic open set $\widetilde{U}$ of Y containing $f(\tilde{p})$, there is an intuitionistic semi open set $\tilde{V}$ in X such that $\tilde{p} \in \tilde{V}$ and $f(\tilde{V}) \subseteq \widetilde{U}$.

Definition 3.2 A function $f : X \to Y$ is said to be intuitionistic semi * continuous if $f^{-1}(\widetilde{U})$ is IS*O in X for every IOS $\widetilde{U}$ in Y.

Theorem 3.3 Every intuitionistic continuous function is intuitionistic semi * continuous.

Proof: Let $f : X \to Y$ be intuitionistic continuous and $\widetilde{U}$ be IO in Y. Then $f^{-1}(\widetilde{U})$ is IO in X. Therefore by theorem 2.13(i), $f^{-1}(\widetilde{U})$ is IS*O in X. Hence f is intuitionistic semi * continuous function.

Remark 3.4 The converse of the above theorem need not be true as seen from the succeeding example

Example 3.5 Let $X = \{i, j, k\} = Y$ and $\tau_1 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{i\}, \{j, k\} >, < X, \{j\}, \{i, k\} >, < X, \{i, j\}, \{k\}>\}$, $\tau_2 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{j\}, \{i, k\} >, < X, \{i\}, \{j\} >, < X, \{i, j\}, \emptyset>\}$. Let f: $(X, \tau_1) \to (Y, \tau_2)$ be defined by $f(i) = j$, $f(j) = i$, $f(k) = k$. Then f is intuitionistic semi * continuous. Let $\widetilde{D} = < X, \{i, j\}, \emptyset>$. Then $f^{-1}(\widetilde{D}) = < X, \{k, i\}, \emptyset>$ is not IOS in τ_1. Therefore f is not an intuitionistic continuous.

Corollary 3.6 Every constant function is intuitionistic semi * continuous function.

Proof: We know that every constant function is intuitionistic continuous function. Therefore by theorem 3.3 every constant function is intuitionistic semi * continuous function.

Theorem 3.7 Let ß be the intuitionistic basis of the intuitionistic topological space *Y*. Then the function *f*: $X \to Y$ is intuitionistic semi * continuous if and only if inverse image of every basic IOS in *Y* under the function *f* is IS*O in *X*.

Proof: Let *f*: $X \to Y$ be intuitionistic semi * continuous. Then the inverse image of every IOS in *Y* is IS*O in *X*. In particular, the inverse image of every basic IOS in *Y* is IS*O in *X*. Coversely, assume that $\tilde{V}$ be an IOS in *Y*. Then $\tilde{V} = \bigcup \tilde{B}_\alpha$ where $\tilde{B}_\alpha \in$ ß. Now $f^{-1}(\tilde{V}) = f^{-1}(\bigcup \tilde{B}_\alpha) = \bigcup f^{-1}(\tilde{B}_\alpha)$. Therefore by hypothesis, $f^{-1}(\tilde{B}_\alpha)$ is IS*O for each α. Then by theorem 2.14(*i*), $f^{-1}(\tilde{V})$ is IS*O. Hence the function *f* is intuitionistic semi * continuous.

Theorem 3.8 Every intuitionistic semi * continuous function is intuitionistic semi continuous.

Proof: Let *f*: $X \to Y$ be intuitionistic semi * continuous function and $\tilde{V}$ be an IOS in *Y*. Then $f^{-1}(\tilde{V})$ is IS*O in *X*. Therefore by theorem 2.13(*ii*), $f^{-1}(\tilde{V})$ is ISO in *X*. Hence *f* is intuitionistic semi continuous.

Remark 3.9 The following example shows that the converse of the above theorem need not be true.

Example 3.10 Let $X = \{i, j, k\} = Y$ and $\tau_1 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{j\},\{i, k\} >, < X, \{i\}, \{j\} >, < X, \{i, j\}, \emptyset >\}$, $\tau_2 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{i\},\{j\} >, < X, \{i, j\}, \emptyset >\}$. Let *f*: $(X, \tau_1) \to (Y, \tau_2)$ be defined by $f(i) = j$, $f(j) = i$, $f(k) = k$. Then *f* is intuitionistic semi continuous. Let $\tilde{E} = < X, \{j\}, \{i\} >$. Then $f^{-1}(\tilde{E}) = < X, \{j\}, \{i\} >$ is not an IS*O in τ_1. Therefore *f* is not an intuitionistic semi * continuous.

Lemma 3.11 Let $(X. \tau)$ be an ITS and $\tilde{A}$ be an IS of *X*. Then

(i) $\tilde{A}$ is IS*O in *X* if and only if $Icl^*(Iint(\tilde{A})) = Icl^*(\tilde{A})$.

(ii) $\tilde{A}$ is IS*C in *X* if and only if $Iint^*(Icl(\tilde{A})) = Iint^*(\tilde{A})$.

Proof: (i) Let $\tilde{A}$ be an IS*O. Then by definition of IS*O we have $\tilde{A} \subseteq Icl^*(Iint(\tilde{A}))$. Hence $Icl^*(\tilde{A}) \subseteq Icl^*(Iint(\tilde{A}))$. Also we have $Iint(\tilde{A}) \subseteq \tilde{A}$, $Icl^*(Iint(\tilde{A})) \subseteq Icl^*(\tilde{A})$. Thus $Icl^*(Iint(\tilde{A})) = Icl^*(\tilde{A})$. On the other hand, let $Icl^*(Iint(\tilde{A})) = Icl^*(\tilde{A})$. Then by definition of IS*O, $\tilde{A}$ is IS*O.

(ii) $\tilde{A}$ is IS*C if and only if $X - \tilde{A}$ is *IS*O*. Then by (i) $\tilde{A}$ is IS*C if and only if $Icl^*(Iint(X - \tilde{A})) = Icl^*(X - \tilde{A})$. Hence $\tilde{A}$ is IS*C if and only if $Iint^*(Icl(\tilde{A})) = Iint^*(\tilde{A})$.

Theorem 3.12 Let $f: X \to Y$ be a function. Then the following are equivalent.

(i) f is intuitionistic semi * continuous.
(ii) f is intuitionistic semi * continuous at each IP of X.
(iii) $f^{-1}(\tilde{U})$ is IS*C in X for every ICS $\tilde{U}$ in Y.
(iv) $f(IS^*cl(\tilde{B})) \subseteq Icl(f(\tilde{B}))$ for every IS $\tilde{B}$ of X.
(v) $IS^*cl(f^{-1}(\tilde{A})) \subseteq f^{-1}(Icl(\tilde{A}))$ for every IS $\tilde{A}$ of Y.
(vi) $Iint^*(Icl(f^{-1}(\tilde{E}))) = Iint^*(f^{-1}(\tilde{E}))$ for every ICS $\tilde{E}$ in Y.
(vii) $Icl^*(Iint(f^{-1}(\tilde{F}))) = Icl^*(f^{-1}(\tilde{F}))$ for every IOS $\tilde{F}$ in Y.
(viii) $f^{-1}(Iint(\tilde{C}) \subseteq IS^*int(f^{-1}(\tilde{F}))$ for every IS $\tilde{C}$ in Y.

Proof: (i) ⟹ (ii). Let $f: X \to Y$ be an intuitionistic semi * continuous. Let $\tilde{p} \in X$ and $\tilde{V}$ be an IOS in Y containing $f(\tilde{p})$. Then $\tilde{p} \in f^{-1}(\tilde{V})$. Since f is intuitionistic semi * continuous, $\tilde{U} = f^{-1}(\tilde{V})$ is an IS*O in X containing $\tilde{p}$ such that $f(\tilde{U}) \subseteq \tilde{V}$. Hence f is intuitionistic semi * continuous at each IP of X.

(ii)⟹ (iii). Let $\tilde{U}$ be an ICS in Y. Then $\tilde{V} = Y - \tilde{U}$ is an IOS in Y. Let $\tilde{p} \in f^{-1}(\tilde{V})$. Then $f(\tilde{p}) \in \tilde{V}$. By hypothesis, there is a IS*O set $\tilde{A}_p$ in X containing $\tilde{p}$ such that $f(\tilde{p}) \in f(\tilde{A}_p) \subseteq \tilde{V}$. Therefore $\tilde{A}_p \subseteq f^{-1}(\tilde{V})$. Hence $f^{-1}(\tilde{V}) = \cup \{\tilde{A}_p : \tilde{p} \in f^{-1}(\tilde{V})\}$. By theorem 2.14(*i*), $f^{-1}(\tilde{V})$ is IS*O in X. Thus $f^{-1}(\tilde{U}) = f^{-1}(Y - \tilde{V}) = X - f^{-1}(\tilde{V})$ is IS*C in X. Hence $f^{-1}(\tilde{U})$ is IS*C in X for every ICS $\tilde{U}$ in Y.

(iii)⟹ (iv). Let $\tilde{B}$ be an IS of X and let $\tilde{U}$ be an ICS containing $f(\tilde{B})$. Then by (iii), $f^{-1}(\tilde{U})$ is *IS*C containing $\tilde{B}$. This implies that $IS^*cl(\tilde{B}) \subseteq f^{-1}(\tilde{U})$ and hence $f(IS^*cl(\tilde{B})) \subseteq \tilde{U}$. Thus $f(IS^*cl(\tilde{B})) \subseteq Icl(f(\tilde{B}))$.

(iv)⟹ (v).Let $\tilde{A}$ be an IS of Y. Let $\tilde{B} = f^{-1}(\tilde{A})$. By assumption, $f(IS^*cl(\tilde{B})) \subseteq Icl(f(\tilde{B}) \subseteq Icl(\tilde{A})$.This implies $(IS^*cl(\tilde{B})) \subseteq f^{-1}(Icl(\tilde{A}))$. Hence $IS^*cl(f^{-1}(\tilde{A})) \subseteq f^{-1}(Icl(\tilde{A}))$.

(v)⟹ (vi). Let $\tilde{E}$ be an ICS in Y. Then by (v), $IS^*cl(f^{-1}(\tilde{E})) \subseteq f^{-1}(Icl(\tilde{E})) = f^{-1}(\tilde{E})$. Also we have $f^{-1}(\tilde{E}) \subseteq IS^*cl(f^{-1}(\tilde{E}))$. Hence $IS^*cl(f^{-1}(\tilde{E})) = f^{-1}(\tilde{E})$. Thus by theorem 2.15(*ii*), $f^{-1}(\tilde{E})$ is closed. Therefore by lemma 3.11 (ii) $Iint^*(Icl(f^{-1}(\tilde{E}))) = Iint^*(f^{-1}(\tilde{E}))$.

(vi)⟹ (vii). Let $\tilde{F}$ be an IOS in Y. Then $Y - \tilde{F}$ is ICS in Y. Therefore by assumption, $Iint^*(Icl(f^{-1}(Y - \tilde{F}))) = Iint^*(f^{-1}(Y - \tilde{F}))$.This implies that $Icl^*(Iint(f^{-1}(\tilde{F}))) = Icl^*(f^{-1}(\tilde{F}))$.

(vii)⟹ (i). Let $\tilde{U}$ be an IOS in Y. Then by assumption, $Icl^*(Iint(f^{-1}(\tilde{U}))) = Icl^*(f^{-1}(\tilde{U}))$. Now by lemma 3.11 (i), $f^{-1}(\tilde{U})$ is IS*O in X. Hence f is intuitionistic semi * continuous.

(i)⟹ (viii). Let $\tilde{C}$ be any IS of Y. Then $Iint(\tilde{C})$ is IOS in Y. By intuitionistic semi * continuity of f, $f^{-1}(Iint(\tilde{C}))$ is IS*O in X and it is contained in $f^{-1}(\tilde{C})$. Hence $f^{-1}(Iint(\tilde{C})) \subseteq IS^*int(f^{-1}(\tilde{C}))$.

(viii)⟹ (i). Let $\widetilde{U}$ be an IOS in *Y*. Then *Iint*($\widetilde{U}$) = $\widetilde{U}$. By (viii) $f^{-1}(\widetilde{U}) \subseteq IS^*int(f^{-1}(\widetilde{U}))$ and hence $f^{-1}(\widetilde{U}) = IS^*int(f^{-1}(\widetilde{U}))$. Therefore by theorem 2.15(*i*), $f^{-1}(\widetilde{U})$ is IS*O in *X*. Thus *f* is intuitionistic semi * continuous.

Theorem 3.13 The function *f*: $X \to Y$ is not an intuitionistic semi * continuous at an IP $\tilde{p}$ in *X* if and only if $\tilde{p}$ belongs to the intuitionistic semi * frontier of the inverse image of some IOS in *Y* containing $f(\tilde{p})$.

Proof: Let *f* be not an intuitionistic semi * continuous at an IP $\tilde{p}$. Then there is an IOS $\widetilde{U}$ in *Y* containing $f(\tilde{p})$ such that $f(\tilde{V})$ is not an IS of $\widetilde{U}$ for every IS*O set $\tilde{V}$ in *X* containing $\tilde{p}$. Hence $\tilde{V} \cap (X - f^{-1}(\widetilde{U})) \neq \widetilde{\emptyset}_I$ for every IS*O set $\tilde{V}$ containing $\tilde{p}$. By theorem 2.16, $\tilde{p} \in IS^*cl(X - f^{-1}(\widetilde{U}))$. Also we have $\tilde{p} \in f^{-1}(\widetilde{U}) \subseteq IS^*cl(f^{-1}(\widetilde{U}))$. Thus $\tilde{p} \in IS^*cl(f^{-1}(\widetilde{U})) \cap IS^*cl(X - f^{-1}(\widetilde{U}))$. Hence by theorem 2.18, $\tilde{p} \in IS^*Fr(f^{-1}(\widetilde{U}))$. Conversely, let *f* be an intuitionistic semi * continuous at an IP $\tilde{p}$. Let $\widetilde{U}$ be any IOS in *Y* containing $f(\tilde{p})$. Then $f^{-1}(\widetilde{U})$ is an IS*O set in *X* containing $\tilde{p}$. Hence by theorem 2,15(*i*), $\tilde{p} \in IS^*int(f^{-1}(\widetilde{U}))$. Thus $\tilde{p} \notin IS^*Fr(f^{-1}(\widetilde{U}))$. This proves the theorem.

Theorem 3.14 Let *f*: $X \to \prod X_\alpha$ be an intuitionistic semi * continuous where $\prod X_\alpha$ is the intuitionistic product topology and $f(\tilde{p}) = (f_\alpha(\tilde{p}))$. Then each coordinate function $f_\alpha : X \to X_\alpha$ is an intuitionistic semi * continuous.

Proof: Let $\widetilde{U}$ be an IOS in X_α. Then $f_\alpha^{-1}(\widetilde{U}) = (P_\alpha \circ f)^{-1}(\widetilde{U}) = f^{-1}(P_\alpha^{-1}(\widetilde{U}))$, where $P_\alpha : \prod X_\alpha \to X$, the projection map. Since P_α is intuitionistic continuous, $P_\alpha^{-1}(\widetilde{U})$ is IOS in $\prod X_\alpha$. Since *f* is intuitionistic semi * continuous, $f_\alpha^{-1}(\widetilde{U}) = f^{-1}(P_\alpha^{-1}(\widetilde{U}))$ is IS*O in *X*. Thus each f_α is intuitionistic semi * continuous.

Remark 3.15 The converse of the above theorem is not true in general. However the converse is true if IS*O(*X*) is ICS under finite intersection as seen in the following theorem.

Theorem 3.16 Let *f*: $X \to \prod X_\alpha$ be defined by $f(\tilde{p}) = (f_\alpha(\tilde{p}))$ and $\prod X_\alpha$ be the intuitionistic product topology. Let IS*O(*X*) be ICS under finite intersection. If each coordinate function $f_\alpha : X \to X_\alpha$ is intuitionistic semi * continuous, then *f* is intuitionistic semi * continuous.

Proof: Let $\widetilde{U}$ be the basic IOS in $\prod X_\alpha$. Then $\tilde{V} = \cap P_\alpha^{-1}(\widetilde{U})$ where each $\widetilde{U}$ is IOS in X_α, the intersection being taken over finitely many α's and where $P_\alpha : \prod X_\alpha \to X$ is the

projection map. Now $f^{-1}(\widetilde{U}) = f^{-1}(\cap(P_\alpha^{-1}(\widetilde{U}_\alpha))) = \cap f^{-1}(P_\alpha^{-1}(\widetilde{U}_\alpha)) = \cap (P_\alpha \circ f)^{-1}(\widetilde{U}_\alpha) = \cap f_\alpha^{-1}(\widetilde{U}_\alpha)$ is IS*O, by hypothesis. Thus by theorem 3.7, *f* is intuitionistic semi * continuous.

Theorem 3.17 Let $f: X \to Y$ be intuitionistic continuous and $g: X \to Z$ be intuitionistic semi * continuous. Let $h: X \to Y \times Z$ be defined by $h(\tilde{p}) = (f(\tilde{p}), g(\tilde{p}))$ and $Y \times Z$ be the intuitionistic product topology. Then *h* is intuitionistic semi * continuous.

Proof: Using theorem 3.7, it is sufficient to prove that the inverse image under *h* of every basic IOS in $Y \times Z$ is IS*O in *X*. Let $\tilde{A} \times \tilde{B}$ be the basic IOS in $Y \times Z$. Then $h^{-1}(\tilde{A} \times \tilde{B}) = f^{-1}(\tilde{A}) \cap g^{-1}(\tilde{B})$. Now by intuitionistic continuity of *f*, $f^{-1}(\tilde{A})$ is IO in *X* and by intuitionistic semi * continuity of *g*, $g^{-1}(\tilde{B})$ is IS*O in *X*. Therefore by theorem 2.14(*ii*), $h^{-1}(\tilde{A} \times \tilde{B}) = f^{-1}(\tilde{A}) \cap g^{-1}(\tilde{B})$ is IS*O in *X*. Hence *h* is intuitionistic semi * continuous.

Theorem 3.18 Let $f: X \to Y$ be intuitionistic semi * continuous and $h: Y \to Z$ be an intuitionistic continuous. Then $h \circ f: X \to Z$ is intuitionistic semi * continuous.

Proof: Let $\widetilde{U}$ be an IOS in *Z*. Since *h* is intuitionistic continuous, $h^{-1}(\widetilde{U})$ is IOS in *Y*. Since *f* is intuitionistic semi * continuous, $f^{-1}(h^{-1}(\widetilde{U}))$ is IOS in *X*. Therefore $f^{-1}(h^{-1}(\widetilde{U})) = (h \circ f)^{-1})(\widetilde{U})$ is IS*O in *X*. Hence $h \circ f$ is intuitionistic semi * continuous.

Remark 3.19 From the above theorem it can be seen that the composition of two intuitionistic semi * continuous need not be intuitionistic semi * continuous.

Example 3.20 Let $X = Y = Z = \{i, j, k\}$ and $\tau_1 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{i\},\{j, k\} >, < X, \{k\}, \{i, j\} >, < X, \{i, k\}, \{b\}>\}$, $\tau_2 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{j\},\{i, k\} >, < X, \{i\},\{j\} >, < X, \{i, j\}, \emptyset>\}$ $\tau_3 = \{\widetilde{X}_I, \widetilde{\emptyset}_I, < X, \{j\},\{i\} >, < X, \{i, j\}, \emptyset>\}$. Let $f: (X, \tau_1) \to (Y, \tau_2)$ be defined by $f(i) = i$, $f(j) = k$, $f(k) = j$ and Let $g: (Y, \tau_2) \to (Z, \tau_3)$ be defined by $g(i) = j$, $g(j) = i$, $g(k) = k$. Then *f* and *g* are intuitionistic semi * continuous. Let $g \circ f: (X, \tau_1) \to (Z, \tau_3)$ and $\tilde{B} = < X, \{j\}, \{i\}>$. Then $(g \circ f)^{-1}(\tilde{B}) = g(f(< X, \{j\}, \{i\}>)) = g(<X, \{k\},\{i\}>) = <X, \{k\},\{i\}>$ is not an IS*O in (Z, τ_3). Therefore $g \circ f$ is not an intuitionistic semi * continuous.

Definition 3.21 A function $f: X \to Y$ is said to be intuitionistic contra semi * continuous if $f^{-1}(\widetilde{U})$ is IS*C in *X* for every IOS $\widetilde{U}$ in *Y*.

Remark 3.22 The concept of intuitionistic semi * continuity is free from intuitionistic contra semi * continuity.

Theorem 3.23 Let $f: X \to Y$ be the function. Then the following are equivalent

(i) f is intuitionistic contra semi * continuous.

(ii) For each $\tilde{p} \in X$ and each ISC $\tilde{E}$ in Y containing $f(\tilde{p})$, there exists a IS*O set $\tilde{V}$ in X containing $\tilde{p}$ such that $f(\tilde{V}) \subseteq \tilde{E}$.

(iii) The inverse image of each ISC in Y is IS*O in X.

(iv) $Icl^*(Iint(f^{-1}(\tilde{E}))) = Icl^*(f^{-1}(\tilde{E}))$ for every ICS $\tilde{E}$ in Y.

(v) $Iint^*(Icl(f^{-1}(\tilde{V}))) = Iint^*(f^{-1}(\tilde{V}))$ for every IOS $\tilde{V}$ in X.

Proof: (i) ⇒ (ii). Let f: $X \to Y$ be the intuitionistic contra semi * continuous function. Let $\tilde{p} \in X$ and $\tilde{E}$ be an ICS in Y containing $f(\tilde{p})$. Take $\tilde{U} = Y - \tilde{E}$. Then $\tilde{U}$ is an IOS in Y not containing $f(\tilde{p})$. Since f is intuitionistic contra semi * continuous, $f^{-1}(\tilde{U})$ is a intuitionistic semi * closed set in X not containing $\tilde{p}$. Therefore $f^{-1}(\tilde{U}) = X - f^{-1}(\tilde{E})$ is a IS*C in X not containing $\tilde{p}$. Thus $\tilde{V} = f^{-1}(\tilde{E})$ is a IS*O in X containing $\tilde{p}$ such that $f(\tilde{V}) \subseteq \tilde{E}$. Hence (i).

(ii) ⇒ (iii). Let $\tilde{E}$ be an ICS in Y and $\tilde{p} \in f^{-1}(\tilde{E})$. Then $f(\tilde{p}) \in \tilde{E}$. By assumption, there is an IS*O set $\tilde{V}_p$ in X containing $\tilde{p}$ such that $f(\tilde{p}) \in f(\tilde{V}_p) \subseteq \tilde{E}$. Therefore $\tilde{V}_p \subseteq f^{-1}(\tilde{E})$. Thus $f^{-1}(\tilde{E}) = \cup \{ \tilde{V}_p : \tilde{p} \in f^{-1}(\tilde{E})\}$. By theorem 2.14 (i) $f^{-1}(\tilde{E})$ is an IS*O in X. Hence (ii).

(iii) ⇒ (iv). Let $\tilde{E}$ be an ISC in Y. Then by hypothesis, $f^{-1}(\tilde{E})$ is IS*O in X. Hence from lemma 3.11(i), we have $Icl^*(Iint(f^{-1}(\tilde{E}))) = Icl^*(f^{-1}(\tilde{E}))$.

(iv) ⇒ (v). Let $\tilde{U}$ be an IOS in Y, Then $Y - \tilde{U}$ is an ICS in Y. By assumption, $Icl^*(Iint(f^{-1}(Y - \tilde{U}))) = Icl^*(f^{-1}(Y - \tilde{U}))$. Therefore $[Icl^*(Iint(f^{-1}(Y - \tilde{U})))]^c = [Icl^*(f^{-1}(Y - \tilde{U}))]^c$. Hence $Iint^*(Icl(f^{-1}(\tilde{U}))) = Iint^*(f^{-1}(\tilde{U}))$.

(v) ⇒ (i). Let $\tilde{U}$ be an IOS in Y. Then by assumption, $Iint^*(Icl(f^{-1}(\tilde{U}))) = Iint^*(f^{-1}(\tilde{U}))$. Therefore by lemma 3.11 (ii), $f^{-1}(\tilde{U})$ is IS*C in X. Thus f is an intuitionistic contra semi * continuous.

Theorem 3.24 Every intuitionistic contra continuous is intuitionistic contra semi * continuous.

Proof: Let f: $X \to Y$ be the intuitionistic contra continuous function and $\tilde{U}$ be an IOS in Y. Then $f^{-1}(\tilde{U})$ is an ICS in X. Hence f is intuitionistic contra semi * continuous.

Theorem 3.25 Every intuitionistic contra semi * continuous is intuitionistic contra semi continuous.

Proof: Let f: $X \to Y$ be the intuitionistic contra semi * continuous function and $\tilde{U}$ be an IOS in Y. Then $f^{-1}(\tilde{U})$ is an IS*C in X. Hence f is intuitionistic contra semi continuous.

Theorem 3.26 Let (X, τ_1) and (Y, τ_2) be an ITS.

(i) If f: $X \to Y$ is intuitionistic contra semi * continuous and g: $Y \to Z$ is intuitionistic contra continuous, then $g \circ f$: $X \to Z$ is intuitionistic semi * continuous.

(ii) If f: $X \to Y$ is intuitionistic semi * continuous and g: $Y \to Z$ is intuitionistic contra continuous, then $g \circ f$: $X \to Z$ is intuitionistic contra semi * continuous.

(iii) If f: $X \to Y$ is intuitionistic contra semi * continuous and g: $Y \to Z$ is intuitionistic continuous, then $g \circ f$: $X \to Z$ is intuitionistic contra semi * continuous.

Proof: Let $\widetilde{U}$ be an IOS in Z. Then $g^{-1}(\widetilde{U})$ is an ICS in Y. Since f is intuitionistic contra semi * continuous, $(g \circ f)^{-1}(\widetilde{U}) = f^{-1}(g^{-1}(\widetilde{U}))$ is IS*C in X. Thus $g \circ f$ is intuitionistic semi * continuous.

(ii) and (iii) can be proved in a similar way.

4. Conclusions

In this paper, we dealt with intuitionistic semi * continuous and intuitionistic contra semi * continuous. In future we wish to do our research work in intuitionistic semi * separated, intuitionistic semi * connected, intuitionistic semi * compact, intuitionistic semi * irresolute continuous function and so on.

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Remarks on Interiors and Closures of Weak Open Sets in Bigeneralized Topological Spaces

M. Anees Fathima*
R. Jamuna Rani†

Abstract

We establish the relationships between the interior and closure operators among the μ_{ij}-semiopen, μ_{ij}-preopen, $\alpha\mu_{ij}$-open, $\beta\mu_{ij}$-open sets in bigeneralized topological spaces.

Keywords: Generalized topology, bigeneralized topology, μ_i-open set, μ_i-closed set.
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*Research Scholar(Reg.No: 19121172092012), PG and Research Department of Mathematics, Rani Anna Government College for Women, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India. Email: afiseyan09@gmail.com.

†Assistant Professor, PG and Research Department of Mathematics, Rani Anna Government College for Women, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India. Email:jamunarani1977@gmail.com.

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1 Introduction

The study of generalized topological spaces (briefly GTS) was first initiated by A.Csaszar on 2002 (Császár, 2002). In that, he established the interior and closure operators in GTS. Later in 2010 (Boonpok, 2010)this study is extended to bigeneralized topological spaces (briefly BGTS) by C. Boonpak. In his paper, he found the conception of (m, n)-closed and (m, n)-open sets in BGTS. Also, he introduced $(m, n)g$-regular open, $(m, n)g$-semi open, $(m, n)g$-pre open, $(m, n)g - \alpha$-open . With these definitions, the properties of the weak open sets were studied by A. Jamuna Rani and M. Anees Fathima in (Fathima and Rani, 2019; Rani and Fathima, 2020) and some of its characterizations are also analysed.

The purpose of this paper is to prove the relationships between the interior and closure operators among μ_{ij}-semi open, μ_{ij}-preopen, $\alpha\mu_{ij}$-open and $\beta\mu_{ij}$- open sets in BGTS. Also we establish some of its Characterizations.

2 Preliminares

In this section, We provide some basic definitions and notations which are most essential to understand the subsequent section.

For any non empty set X and $\xi \in p(X)$, ξ is said to be GT (Császár, 2002) if $\phi \in \xi$ and ξ is closed under arbitrary union. Also, the function $\gamma \in \Gamma$ (where Γ denotes the collection of all mappings $\gamma : p(X) \to p(X)$ possessing the property of monotony. ie.,if $A \subset B \Rightarrow \gamma(A) \subset \gamma(B)$) is said to be μ-friendly (Császár, 2006)if $\gamma(A) \cap L \subset \gamma(A \cap L)$ for every $A \subset X$ and $L \in \mu$. If $\gamma \in \Gamma$ and $\mu = \{A \subset X / A \subset \gamma(A)\}$is the family of all γ- open sets, then μ is a GT (Császár, 2002) The pair (X, μ) is a called GTS. In (Sivagami, 2008) the family of all μ-friendly functions is denoted by Γ_4 and (X, γ) is called the γ-space. It is further proved that every γ-space is a quasi-topological space in (Császár, 2008) and all the results established in for γ-spaces are valid for quasi-topological spaces. Also for $\gamma \in \Gamma$, define $\gamma^* : p(X) \to p(X)$ by $\gamma^*(A) = X - \gamma(X - A)$ (Császár, 1997) for every subset A of X.

Let X be a non empty set and μ_1, μ_2 be generalized topologies on X. A triple (X, μ_1, μ_2) is said to be a bigeneralized topological space. Let A be subset of a bigeneralized topological space X. Then the closure of A and the interior of A with respect to μ_m are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$ respectively, for $m = 1, 2$. (Boonpok, 2010) A subset A of a bigeneralized topological space(X, μ_1, μ_2) is said to be μ_{ij}-semiopen (Fathima and Rani, 2019)(resp.μ_{ij}-preopen (Rani and Fathima, 2020) $\alpha\mu_{ij}$-open$\beta\mu_{ij}$-open (Jamuna Rani and Anees Fathima, 2021)) if $A \subset c_{\mu_i} i_{\mu_j}(A)$ where $i, j = 1, 2$ and $i \neq j$ (resp. $A \subset i_{\mu_i} c_{\mu_j}(A)$, $A \subset$

$i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$,$A \subset c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$).

Proposition 1.1. (Min, 2009)Let (X, μ) be a generalized topological space. For subsets A and B of X, the following properties holds.

(a) $c_\mu(X - A) = X - i_\mu(A)$ and $i_\mu(X - A) = X - c_\mu(A)$.

(b) If $(X - A) \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$.

(c) If $A \subseteq B$, then $c_\mu(A) \subseteq c_\mu(B)$ and $i_\mu(A) \subseteq i_\mu(B)$.

(d) If $A \subseteq c_\mu(A)$ and $i_\mu(A) \subseteq A$.

(e) $c_\mu\left(c_\mu\left(A\right)\right) = c_\mu\left(A\right)$ and $i_\mu\left(i_\mu\left(A\right)\right) = i_\mu\left(A\right)$.

Proposition 1.2. (Jamunarani et al., 2010) Let (X, γ) be a γ- space . Then $G \cap c_\gamma(A) \subset c_\gamma(G \cap A)$, for every $A \subset X$ and γ-open set G of X.

Proposition 1.3. Let (X, μ) be a quasi topological space and $A, B \subseteq X$, the following holds.

(a) If A and B are μ-open sets, then $A \cap B$ is μ-open (Sivagami, 2008)

(b) $i_\mu(A \cap B) = i_\mu(A) \cap i_\mu(B)$, for every subsets A and B of X (Császár, 2008)

(c) $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$, for every subsets A and B of X(Sivagami, 2008)

Proposition 1.4. (Fathima and Rani, 2019) Let (X, μ) be a generalized topological space. Let A be a subset of X. Then the following hold.

(a) $c_{\sigma_{ij}}(A)$ is the smallest μ_{ij}-semi closed set containing A.

(b) A is μ_{ij}-semi closed if and only if $A = c_{\sigma_{ij}}(A)$.

(c) $x \in c_{\sigma_{ij}}(A)$ if and only if for every μ_{ij}-semi open G containing x, $G \cap A \neq \phi$.

(d) $c_{\sigma_{ij}} \in \Gamma_{012+}$.

Proposition 1.5. (Fathima and Rani, 2019)Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X. Then the following hold.

(a) $\left(i_{\sigma_{ij}}\right)^* = c_{\sigma_{ij}}$.

(b) $\left(c_{\sigma_{ij}}\right)^* = i_{\sigma_{ij}}$.

(c) $i_{\sigma_{ij}}(X - A) = X - c_{\sigma_{ij}}(A)$ for every subset A of X.

(d) $c_{\sigma_{ij}}(X - A) = X - i_{\sigma_{ij}}(A)$ for every subset A of X.

Proposition 1.6. (Fathima and Rani, 2019)Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X. Then the following hold.

(a) A is μ_{ij}-semi open if and only if A is $c_{\mu_i} i_{\mu_j}$ - open if and only if $A = i_{c_{\mu_i} i_{\mu_j}}(A)$.

(b) $i_{\sigma_{ij}}(A) = i_{c_{\mu_i} i_{\mu_j}}$ and $c_{\sigma_{ij}} = c_{c_{\mu_i} i_{\mu_j}}$.

(c) $i_{\sigma_{ij}}(A) = A \cap c_{\mu_i} i_{\mu_j}(A)$.

(d) $c_{\sigma_{ij}}(A) = A \cap i_{\mu_i} c_{\mu_j}(A)$.

Similar results from (Jamuna Rani and Anees Fathima, 2021; Rani and Fathima, 2020; Jamuna Rani and Anees Fathima, 2020) are also used in the next section.

3 Relationship between the operators:

The following theorem gives some of the relationships between $i_{\sigma_{ij}}$, $c_{\sigma_{ij}}$, $i_{\alpha_{ij}}$, $c_{\alpha_{ij}}$, i_{μ_i} and c_{μ_i}.

Theorem 1.1.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\sigma_{ij}}(A) = A \cap c_{\mu_i} i_{\mu_j}(A)$

(b) $c_{\sigma_{ij}}(A) = A \cup i_{\mu_i} c_{\mu_j}(A)$

(c) $i_{\alpha_{ij}}(A) = A \cap i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$

(d) $c_{\alpha_{ij}}(A) = A \cup c_{\mu_i} i_{\mu_j} c_{\mu_i}(A)$

(e) $i_{\mu_i}(c_{\sigma_{ij}}(A)) = i_{\mu_i} c_{\mu_j}(A)$

(f) $c_{\mu_i}(i_{\sigma_{ij}}(A)) = c_{\mu_i} i_{\mu_j}(A)$

(g) $c_{\sigma_{ij}}(i_{\mu_i})(A) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$

(h) $i_{\sigma_{ij}}(c_{\mu_i})(A) = c_{\mu_i} i_{\mu_j} c_{\mu_i}(A)$

(i) $c_{\sigma_{ji}} i_{\sigma_{ij}}(A) = (A \cap c_{\mu_i} i_{\mu_j}(A)) \cup i_{\mu_j} c_{\mu_i} i_{\mu_j}(A)$ for every subset A of X.

(j) $i_{\sigma_{ij}}c_{\sigma_{ji}}(A) = (A \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)) \cup i_{\mu_j}c_{\mu_i}(A)$ for every subset A of X.

Proof. (a) By theorem 1.6(b), $i_{\sigma_{ij}} = i_{c_{\mu_i}i_{\mu_j}}$ implies $i_{\sigma_{ij}}(A) \subset A \cap c_{\mu_i}i_{\mu_j}(A)$. For the reverse part, $A \cap c_{\mu_i}i_{\mu_j} \subset c_{\mu_i}i_{\mu_j}(A) \subset c_{\mu_i}i_{\mu_j}\left(A \cap c_{\mu_i}i_{\mu_j}(A)\right)$. Thus $A \cap c_{\mu_i}i_{\mu_j} = i_{c_{\mu_i}i_{\mu_j}}(A \cap c_{\mu_i}i_{\mu_j}) \subset i_{\sigma_{ij}}(A)$.
(b) The result follows from (a).
(c) The proof is similar to the proof of (a).
(d) The result follows from (c).
(e) $i_{\mu_i}\left(c_{\sigma_{ij}}(A)\right) = i_{\mu_i}\left(A \cup i_{\mu_i}c_{\mu_j}(A)\right) \subset i_{\mu_i}\left(A \cup c_{\mu_j}(A)\right) = i_{\mu_i}c_{\mu_j}(A)$.
Also, by (b), $i_{\mu_i}c_{\sigma_{ij}}(A) \supset i_{\mu_i}i_{\mu_i}c_{\mu_j}(A) = i_{\mu_i}c_{\mu_j}(A)$.
(f) The result follows from (e).
(g) By (b), $c_{\sigma_{ij}}(i_{\mu_i}(A)) = i_{\mu_i}(A) \cup i_{\mu_i}c_{\mu_j}i_{\mu_i}(A) = i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$.
(h) By(a), $i_{\sigma_{ij}}(c_{\mu_i}(A)) = c_{\mu_i}(A) \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A) = c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$.
(i) By(a),$c_{\sigma_{ji}}i_{\sigma_{ij}}(A) = (A \cap c_{\mu_i}i_{\mu_j}(A)) \cup i_{\mu_j}c_{\mu_i}(A \cap c_{\mu_i}i_{\mu_j}(A))$.
Here, $c_{\mu_i}(A \cap c_{\mu_i}i_{\mu_j}(A)) \subset c_{\mu_i}i_{\mu_j}(A)$ and $c_{\mu_i}(A \cap c_{\mu_i}i_{\mu_j}(A)) \supset c_{\mu_i}\left(i_{\mu_j}(A) \cap c_{\mu_i}i_{\mu_j}(A)\right)$ $= c_{\mu_i}i_{\mu_j}(A)$.
(j) Now, $i_{\sigma_{ij}}c_{\sigma_{ji}}(A) = (A \cup i_{\mu_j}c_{\mu_i}(A)) \cap c_{\mu_i}i_{\mu_j}(A \cup i_{\mu_j}c_{\mu_i}(A))$ $= (A \cup i_{\mu_j}c_{\mu_i}(A)) \cap c_{\mu_j}i_{\mu_j}c_{\sigma_{ji}}(A) = \left(A \cup i_{\mu_j}c_{\mu_i}(A)\right) \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$ by (e).
Hence $i_{\sigma_{ij}}c_{\sigma_{ji}}(A) = \left(A \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)\right) \cup i_{\mu_j}c_{\mu_i}(A)$. □

The following theorem shows some results for the operators $i_{\pi_{ij}}$, $c_{\pi_{ij}}$, $i_{\beta_{ij}}$, $c_{\beta_{ij}}$.

Theorem 1.2.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\pi_{ij}}(A) = A \cap i_{\mu_i}c_{\mu_j}(A)$

(b) $c_{\pi_{ij}}(A) = A \cup c_{\mu_i}i_{\mu_j}(A)$

(c) $i_{\beta_{ij}}(A) = A \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$

(d) $c_{\beta_{ij}}(A) = A \cup i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$

(e) $i_{\mu_i}(i_{\sigma_{ji}}(A)) = i_{\mu_i}(A)$

(f) $c_{\mu_i}(c_{\sigma_{ji}}(A)) = c_{\mu_i}(A)$.

Proof. (a) In (Rani and Fathima, 2020), by theorem 3.2(e), we have $i_{\pi_{ij}} = i_{i_{\mu_i}c_{\mu_j}}$ and so $i_{\pi_{ij}}(A) \subset A \cap i_{\mu_i}c_{\mu_j}(A)$. Let $x \in i_{\mu_i}c_{\mu_j}(A)$. Let G be any μ_i open set containing x such that $G \cap i_{\mu_i}c_{\mu_j}(A)$ is a μ_i -open set containing x. since $x \in c_{\mu_j}(A)$ and $G \cap i_{\mu_i}c_{\mu_j}(A) \cap A \neq \phi$ and so $x \in c_{\mu_j}\left(i_{\mu_i}c_{\mu_j}(A) \cap A\right)$. Therefore, $i_{\mu_i}c_{\mu_j}(A) \subset c_{\mu_j}i_{\mu_i}c_{\mu_j}(A) \cap A$ and so $i_{\mu_i}c_{\mu_j}(A) \subset i_{\mu_i}c_{\mu_j}(A \cap i_{\mu_i}c_{\mu_j}(A))$, $A \cap i_{\mu_i}c_{\mu_j}(A) \subset i_{\mu_i}c_{\mu_j}(A) \subset i_{\mu_i}c_{\mu_j}\left(A \cap i_{\mu_i}c_{\mu_j}(A)\right), A \cap i_{\mu_i}c_{\mu j}(A)$

$= i_{i_{\mu_i}c_{\mu_j}}\left(A \cap i_{\mu_i}c_{\mu_j}(A)\right) \subset i_{\pi_{ij}}(A) \cap i_{\mu_i}c_{\mu_j}(A) \subset i_{\pi_{ij}}(A)$. Hence $i_{\pi_{ij}}(A) = A \cap i_{\mu_i}c_{\mu_j}(A)$.
(b) The result follows from (a).
(c) The proof is similar to the proof of (a).
(d) By theorem 3.10(i) in (Jamuna Rani and Anees Fathima, 2021), the proof is similar to the proof of (b).
(e) $i_{\mu_i}(i_{\sigma_{ji}}(A)) = i_{\mu_i}(A \cap c_{\mu_j}i_{\mu_i}(A))$ by theorem 1.1(a) and so $i_{\mu_i}(i_{\sigma_{ji}}(A)) = i_{\mu_i}(A) \cap i_{\mu_i}c_{\mu_j}i_{\mu_i}(A) = i_{\mu_i}(A)$ by proposition 1.3(b).
(f) The result follows from (e). □

The following theorem gives characterizations of μ_{ij}-semi open, μ_{ij}-preopen, $\alpha\mu_{ij}$-open, $\beta\mu_{ij}$-open sets using the interior and closure operators.

Theorem 1.3.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $A \in \pi_{ij}(\mu)$ if and only if $c_{\sigma_{ij}}(\mu) = i_{\mu_i}c_{\mu_j}(A)$

(b) A is μ_{ij}-preclosed if and only if $i_{\sigma_{ij}}(A) = c_{\mu_i}i_{\mu_j}(A)$.

(c) $A \in \sigma_{ij}(\mu)$ if and only if $c_{\pi_{ij}}(A) = c_{\mu_i}i_{\mu_j}(A)$.

(d) A is μ_{ij}-semiclosed if and only if $i_{\pi_{ij}}(A) = i_{\mu_i}c_{\mu_j}(A)$.

(e) $A \in \alpha_{ij}(\mu)$if and only if $c_{\beta_{ij}}(A) = i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$.

(f) A is $\alpha\mu_{ij}$ -closed if and only if $i_{\beta_{ij}}(A) = c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$.

(g) $A \in \beta_{ij}(\mu)$ if and only if $c_{\alpha_{ij}}(A) = c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$.

(h) A is $\beta_{\mu_{ij}}$-closed if and only if $i_{\alpha_{ij}}(A) = i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$.

Proof. The proof follows from theorem 1.1(b),1.2(b),1.2(d), 1.1(d).□

Theorem 1.4.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $c_{\mu_i}(c_{\pi_{ij}}(A)) = c_{\mu_i}(A)$.

(b) $i_{\mu_i}(i_{\pi_{ij}}(A)) = i_{\mu_i}(A)$.

(c) $c_{\mu_i}(c_{\beta_{ji}}(A)) = c_{\mu_i}(A)$.

(d) $i_{\mu_i}(i_{\beta_{ji}}(A)) = i_{\mu_i}(A)$.

(e) $i_{\mu_i}(c_{\pi_{ji}}(A)) \subset c_{\mu_j} i_{\mu_i}(A)$ and $c_{\mu_j} i_{\mu_i}(c_{\pi_{ji}}(A)) = c_{\mu_j} i_{\mu_i}(A)$.

(f) $i_{\mu_i}(c_{\pi_{ji}}(A)) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.

(g) $c_{\beta_{ij}}(i_{\mu_i}(A)) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A) = i_{\mu_i}(c_{\beta_{ij}}(A))$.

(h) $c_{\beta_{ij}}(i_{\beta_{ji}}(A)) = i_{\beta_{ji}}(c_{\beta_{ij}}(A)) = \big(A \cup i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)\big) \cap c_{\mu_j} i_{\mu_i} c_{\mu_j}(A)$.

(i) $i_{\beta_{ij}}\left(c_{\mu_i}(A)\right) = c_{\mu_i} i_{\mu_j} c_{\mu_i}(A) = c_{\mu_i} i_{\beta_{ij}}(A)$.

(j) $c_{\sigma_{ij}}\left(i_{\sigma_{ji}}(A)\right) \subset i_{\beta_{ji}}\left(c_{\beta_{ij}}(A)\right) \subset i_{\sigma_{ji}}\left(c_{\sigma_{ij}}(A)\right)$.

Proof. (a) Clearly, $c_{\mu_i}(A) \subset c_{\mu_i}\left(c_{\pi_{ij}}(A)\right)$.
Again, $c_{\mu_i}\left(c_{\pi_{ij}}(A)\right) \subset c_{\mu_i}\left(c_{\mu_i}(A)\right) = c_{\mu_i}(A)$.
(b) The proof follows from (a).
The proof of (c) and (d) are similar to (a) and (b).
(e) Similar proof, so omitted.
(f) By (e), $i_{\mu_i}\left(c_{\pi_{ji}}(A)\right) \subset i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$. Again, $i_{\mu_i}\left(c_{\pi_{ji}}(A)\right) = i_{\mu_i}\left(A \cup c_{\mu_j} i_{\mu_i}(A)\right) \supset i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$. Therefore, $i_{\mu_i}\left(c_{\pi_{ji}}(A)\right) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.
(g) By theorem 1.2(d), $c_{\beta_{ij}}\left(i_{\mu_i}(A)\right) = i_{\mu_i}(A) \cup i_{\mu_i} c_{\mu_j} i_{\mu_i}\left(i_{\mu_i}(A)\right) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.
Again, $i_{\mu_i}\left(c_{\beta_{ij}}(A)\right) = i_{\mu_i}\left(A \cup i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)\right) \supset i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.
For Converse part, $i_{\mu_i}\left(c_{\beta_{ij}}(A)\right) \subset i_{\mu_i}\left(c_{\pi_{ij}}(A)\right) \subset c_{\mu_j} i_{\mu_i}(A) \subset i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.
(h) Similar proof, so omitted.
(i) The proof follows from (g).
(j) The proof follows from 1.1(i) and 1.1(j).□

The following theorem gives a characterization of $\beta_{ij}(\mu)$-open sets.

Theorem 1.5.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following are equivalent.

(a) $A \in \beta_{ji}(\mu)$.

(b) $A \subset i_{\beta_{ji}}\left(c_{\beta_{ij}}(A)\right)$.

(c) $A \subset i_{\sigma_{ji}}\left(c_{\sigma_{ij}}(A)\right)$.

Proof. $(a) \Rightarrow (b)$ If $A \in \beta_{ji}(\mu)$, then $A = i_{\beta_{ji}}(A) \subset i_{\beta_{ji}}\left(c_{\beta_{ij}}(A)\right)$.
$(b) \Rightarrow (c)$ By theorem 1.4(j), the proof follows.
$(c) \Rightarrow (a)$ $A \subset i_{\sigma_{ji}}\left(c_{\sigma_{ij}}(A)\right) \Rightarrow A \subset \left(c_{\sigma_{ij}}(A) \cap c_{\mu_j} i_{\mu_i}\left(c_{\sigma_{ij}}(A)\right)\right) = \left(c_{\sigma_{ij}}(A) \cap c_{\mu_j} i_{\mu_i} c_{\mu_j}(A)\right)$ by theorem 1.1(e) and so $A \subset c_{\mu_j} i_{\mu_i} c_{\mu_j}(A)$. □

In the following theorem, we prove some relationships between $i_{\alpha_{ij}}$,$c_{\alpha_{ij}}$ with i_{μ_i}, c_{μ_i}.

Theorem 1.6.
Let (X, μ_1, μ_2) be a bigenralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $c_{\mu_i} c_{\alpha_{ij}}(A) = c_{\alpha_{ij}} c_{\mu_i}(A) = c_{\mu_i}(A)$.

(b) $i_{\mu_i} i_{\alpha_{ij}}(A) = i_{\alpha_{ij}} i_{\mu_i}(A) = i_{\mu_i}(A)$.

(c) $c_{\alpha_{ij}}\left(i_{\mu_j}(A)\right) = c_{\mu_i} i_{\mu_j}(A)$.

d) $c_{\mu_j}\left(i_{\alpha_{ij}}(A)\right) = c_{\mu_j} i_{\mu_i}(A)$.

(e) $i_{\alpha_{ij}}\left(c_{\mu_j}(A)\right) = i_{\mu_i} c_{\mu_j}(A)$.

(f) $i_{\mu_j}\left(c_{\alpha_{ij}}(A)\right) = i_{\mu_j} c_{\mu_i}(A)$.

Proof. (a) The proof follows from the theorem 1.1(d).
(b) The proof of (b) follows from (a).
(c) The proof follows from the theorem 1.1(d).
(d) $c_{\mu_j}\left(i_{\alpha_{ij}}(A)\right) \subset c_{\mu_j}\left(c_{\mu_j}(A) \cap i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)\right) = c_{\mu_j} i_{\mu_i}(A)$.
Again, $c_{\mu_j}\left(i_{\alpha_{ij}}(A)\right) \supset c_{\mu_j}\left(A \cap i_{\mu_i}(A)\right) = c_{\mu_j} i_{\mu_i}(A)$.
(e) The proof of (e) follows from (c).
(f) The proof of (f) follows from (d). □

The following theorem shows some relationships between $i_{\alpha_{ij}}$, $c_{\alpha_{ij}}$ with $i_{\sigma_{ij}}$, $c_{\sigma_{ij}}$, $i_{\pi_{ij}}$,$c_{\pi_{ij}}$, $i_{\beta_{ij}}$ and $c_{\beta_{ij}}$.

Theorem 1.7.
Let (X, μ_1, μ_2) be a bigenralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\alpha_{ij}}\left(c_{\sigma_{ij}}(A)\right) = i_{\mu_i} c_{\mu_j}(A)$.

(b) $i_{\alpha_{ij}}\left(c_{\pi_{ji}}(A)\right) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.

(c) $i_{\alpha_{ij}}\left(c_{\beta_{ji}}(A)\right) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.

(d) $c_{\alpha_{ij}}\left(i_{\sigma_{ij}}(A)\right)$=$c_{\mu_i}i_{\mu_j}(A)$.

(e) $c_{\alpha_{ij}}\left(i_{\pi_{ji}}(A)\right) = c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$.

(f) $c_{\alpha_{ij}}\left(i_{\beta_{ji}}(A)\right) = c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$.

Proof. (a) The proof follows from the theorem 1.1(c), 1.1(f).
(b) The proof follows from the theorem 1.1(c), 1.4(g) and 1.1(a).
(c) The proof follows from the theorem 1.1(c), 1.4(h).
(d) The proof of (d) follows from (a).
(e) The proof of (e) follows from (b).
(f) The proof of (f) follows from (c). □

Theorem 1.8.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\sigma_{ij}}c_{\alpha_{ij}}(A) = c_{\alpha_{ij}}(A) \cap c_{\mu_i}i_{\mu_j}c_{\mu_i}(A)$

(b) $c_{\sigma_{ij}}i_{\alpha_{ij}}(A) = i_{\alpha_{ij}}(A) \cap i_{\mu_i}c_{\mu_j}i_{\mu_i}(A)$

(c) $c_{\sigma_{ij}}c_{\alpha_{ji}}(A) =c_{\alpha_{ji}}(A) \cap c_{\sigma_{ij}}(A)$.

(d) $i_{\sigma_{ij}}i_{\alpha_{ji}}(A) =i_{\alpha_{ji}}(A) \cap i_{\sigma_{ij}}(A)$.

(e) $i_{\sigma_{ij}}i_{\beta_{ij}}(A) = i_{\sigma_{ij}}(A)$.

(f) $c_{\sigma_{ij}}c_{\beta_{ij}}(A) = c_{\sigma_{ij}}(A)$.

Proof. (a) The result follows from the theorem 1.1(a) and 1.6(f).
(b) The proof of (b) follows from (a).
(c) The result follows from the theorem 1.1(b) and 1.6(a).
(d) The proof of (d) follows from (c).
(e) The result follows from the theorem 1.1(a), 1.4(d) and 1.2(c).
(f) The proof of (f) follows from (e).

Theorem 1.9.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\pi_{ij}}c_{\alpha_{ji}}(A) = c_{\alpha_{ji}}(A) \cap i_{\mu_i}c_{\mu_j}(A)$.

(b) $i_{\pi_{ij}}i_{\beta_{ji}}(A) = i_{\pi_{ij}}(A)$.

(c) $c_{\pi_{ij}} i_{\alpha_{ji}}(A) = i_{\alpha_{ji}} \cap c_{\mu_i} i_{\mu_j}(A)$.

(d) $c_{\pi_{ij}} c_{\beta_{ji}}(A) = c_{\pi_{ij}}(A)$.

Proof. (a) The result follows from the theorem 1.2(a) and 1.6(a).
(b) The result follows from the theorem 1.2(a) and 1.4(i).
(c) The proof of (c) follows from (a).
(d) The proof of (d) follows from (b).□

Theorem 1.10.
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X and $\mu_i \in \Gamma_4$. Then the following hold.

(a) $i_{\beta_{ij}} c_{\alpha_{ij}}(A) = c_{\alpha_{ij}}(A) \cap c_{\mu_i} i_{\mu_j} c_{\mu_i}(A) = c_{\mu_i} i_{\mu_j} c_{\mu_i}(A)$.

(b) $c_{\beta_{ij}} i_{\alpha_{ij}}(A) = i_{\mu_i} c_{\mu_j} i_{\mu_i}(A) = i_{\alpha_{ij}}(A) \cap i_{\mu_i} c_{\mu_j} i_{\mu_i}(A)$.

Proof. (a) The result follows from the theorem 1.2(c) and 1.6(a).
(b) The proof of (b) follows from (a). □

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The Edge-To-Vertex Triangle Free Detor Distance in Graphs

S. Lourdu Elqueen[1]
G. Priscilla Pacifica[2]

Abstract

For every connected graph G, the triangle free detour distance $D_{\Delta f}(u, v)$ is the length of a longest u- v triangle free path in G, where u, v are the vertices of G. A u-v triangle free path of length $D_{\Delta f}(u, v)$ is called the u-v triangle free detour. In this article, the edge-to-vertex triangle free detour distance is introduced. It is found that the edge -to-vertex triangle free detour distance differs from the edge -to-vertex distance and edge-to-vertex detour distance. The edge-to-vertex triangle free detour distance is found for some standard graphs. Their bounds are determined and their sharpness is checked. Certain general properties satisfied by them are studied.

Keywords: connected graph, edge -to-vertex distance and edge-to-vertex detour distance

2010 AMS subject classification: 05C12, 05C69[3]

[1]Reg No: 19212212092009**,** Ph. D Research Scholar (Full Time) of Mathematics. Mary's College (Autonomous) Thoothukudi affiliated under Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, South India. sahayamelqueen@gmail.com.
[2]Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi, India, priscillamelwyn@gmail.com.
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1. Introduction

The facility location problem was introduced as edge-to-vertex distance by Santhakumaran [9], in 2010. For an edge e and a vertex v in a connected graph, the edge-to-vertex distance is defined by $d(e,v) = min\{d(u,v) : u \in e\}$. The edge-to-vertex eccentricity of e is defined by $e_2(e) = max\{d(e,v) : v \in V\}$. A vertex v of G such that $e_2(e) = d(e,v)$ is called an edge-to-vertex eccentric vertex of v. The edge-to-vertex radius r_2 of G is defined by $r_2 = min\{e_2(e) : e \in E\}$ and the edge-to-vertex diameter d_2 of G is defined by$d_2 = max\{e_2(e) : e \in E\}$. An edge e for which $e_2(e)$ is minimum is called an edge-to-vertex central edge of $G$and the set of all edge-tovertex central edges of G is the edge-to-vertex center $C_2(G)$ of$G$. An edge e for which $e_2(e)$ is maximum is called an edge-to-vertexperipheral edge of G and the set of all edge-to-vertex peripheraledges of G is the edge-to-vertex periphery $P_2(G)$ of G. If every edgeof $G$is an edge-to-vertex central edge then G is called the edge-to-vertex self-centered graph. This concept is useful in channel assignment problem in radio technology and security-based communication network design. The concept of edge-to-vertex detour distance was introduced by I. Keerthi Asir [6], Let e be an edge and v a vertex in a connected graph G. An edge-to-vertex $e-v$ path P is a $u-v$ path, where$u$ is a vertex in e such that P contains no vertices of e other than u. The edge-to-vertex detour distance $D(e,v)$ is the length of a longest $e-v$path in G. An$e-v$ path of length $D(e,v)$ is called an edge-to-vertex $e-v$ detour or simply $e-v$ detour. Forour convenience an$e-v$ path of length $d(e,v)$ is called an edge-to-vertex $e-v$ geodesic or simply $e-v$ geodesic.

The following theorems are used in the article.

Theorem: 1.1.[6] For any edge e and a vertex $v$in a non-trivial connected graph of order$n$, $0 \le d(e,v) \le D(e,v) \le n-2$.

Theorem: 1.2.[6] Let$K_{n,m}$ $(n < m)$ be a complete bipartite graph with partitionV_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let e be an edge and v a vertex such that $v \notin e$ in $K_{n,m}$, then

$$D(e,v) = \begin{cases} 2n-2 & if\, v \in V_1 \\ 2n-1 & if\, v \in V_2 \end{cases}$$

2. Edge-To-Vertex Triangle Free Detour Distance

Definition. 2.1 Let Gbe a connected graph. Let $e$be an edge and u a vertex in$G$. An edge-to-vertex $e-u$triangle free path $P$is a $u-v$ triangle free path, where$v$is a vertex in $e$such that Pcontains no vertices of e other than v. The edge-to-vertex triangle free detour distance is the length of the longest $e-u$ triangle free path in G.It is denoted by $D_{\Delta f}(e,v)$. An$e-u$ triangle free path of length$D_{\Delta f}(e,v)$is called an edge-to-vertex $e-u$triangle free detour.

Example: 2.1 Consider the graph Ggiven in the figure: 2.1. Let $e = \{u_6, u_7\}$and $v = u_4$. The paths between $e$and v are $P_1: u_6,\ u_5,\ u_4$;$P_2: u_7,\ u_2,\ u_4$;$P_3: u_7,\ u_2,\ u_3, u_4$;$P_4: u_7,\ u_8,\ u_9, u_1,\ u_2,\ u_4$; and $P_5: u_7,\ u_8,\ u_9, u_1,\ u_2,\ u_3, u_4$;The paths $P_1, P_2,\ P_4$are triangle free $e - v$ paths and P_3 and P_5 are not triangle free $e - v$ paths. Thus edge-to-vertex distance $d(e, v) = 2$, edge-to-vertex triangle free detour distance $D_{\Delta f}(e, v) = 5$ and edge-to-vertex detour distance $\mathrm{D(e, v)} = 6$.

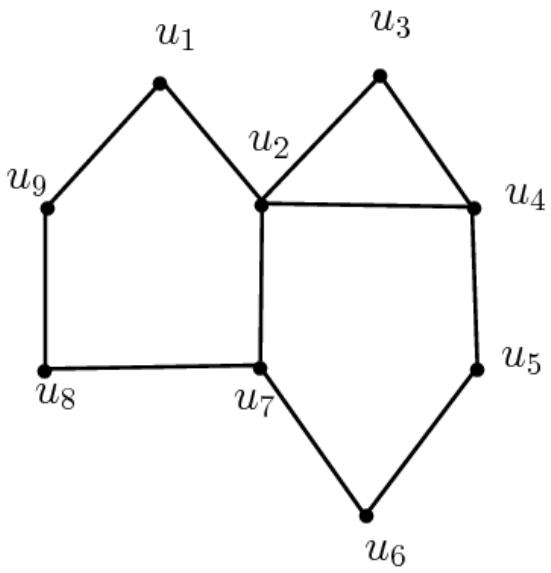

Figure: 2.1 G

Thus edge-to-vertex triangle free detour distance differs from the edge-to-vertex distance and edge-to-vertex detour distance.

Theorem. 2.1 Let G be a connected graph of order n. Let e be an edge and ua vertex of$G$, then $0 \leq d(e, v) \leq D_{\Delta f}(e, v) \leq D(e, v) \leq n - 2$.

Proof. By theorem 1.1 , we can conclude that $0 \leq d(e, v) \leq D(e, v) \leq n - 2$. It is enough to prove that (i)$d(e, v) \leq D_{\Delta f}(e, v)$ and (ii) $D_{\Delta f}(e, v) \leq D(e, v)$.

Thus (i) is true by the definition of edge-to-vertex distance and edge-to-vertex triangle free detour distance.

To prove :(ii)

Case(i): If the detour path does not induce a triangle in G, then$D_{\Delta f}(e, v) = D(e, v)$.

Case(ii): If the detour path induces a triangle in G, then $D_{\Delta f}(e, v) < D(e, v)$

Remark 2.1. The bounds in the theorem 2.1 are sharp. Let Gbe a graph and e be an edge, $d(e, u) = D_{\Delta f}(e, u) = D(e, u) = 0iffu \in e$.Let Gbe a path with vertices $\{v_1, v_2, \ldots . v_n\}$. Then$d(e, u) = D_{\Delta f}(e, u) = D(e, u) = n - 2$, where $e =$

$\{v_{n-1}, v_n\}$and $u = v_1$. Let Gbe a tree, $d(e,u) = D_{\Delta f}(e,u) = D(e,u)$ for every edge $e$and vertex $u$of G. For the graph G given in the figure:2.1, $e = \{u_6, u_7\}$ and $v = u_4$. The paths between $e$and v are $P_1: u_6,\ u_5,\ u_4$; $P_2: u_7,\ u_2,\ u_4$; $P_3: u_7,\ u_2,\ u_3, u_4$; $P_4: u_7,\ u_8,\ u_9, u_1,\ u_2,\ u_4$; and $P_5: u_7,\ u_8,\ u_9, u_1,\ u_2,\ u_3, u_4$;The paths $P_1, P_2,\ P_4$are triangle free $e - v$ paths and P_3 and P_5 are not triangle free $e - v$ paths. Thus edge-to-vertex distance $d(e,v) = 2$, edge-to-vertex triangle free detour distance $D_{\Delta f}(e,v) = 5$ and edge-to-vertex detour distanceD(e,v) = 6. Thus $0 < d(e,v) < D_{\Delta f}(e,v) < D(e,v) < n - 2$.

Theorem. 2.2 For a complete bipartite graph $G$with partitions $V_1$and V_2such that $|V_1| = n$and $|V_2| = m(n < m)$.Let e be an edge of G and ua vertex such that $u \notin e$ in G. Then, $D_{\Delta f}(e,u) = \begin{cases} 2n-2 & if u \in V_1 \\ 2n-1 & if u \in V_2 \end{cases}$

Proof. Since any vertex subset of G do not induce a cycle C_3in G. Thus edge-to-vertex triangle free detour distance is equal to edge-to-vertex detour distance. By theorem: 1.2, $D_{\Delta f}(e,u) = \begin{cases} 2n-2 & if u \in V_1 \\ 2n-1 & if u \in V_2 \end{cases}$

Corollary:2.1 Let G be a complete bipartite graph $K_{n,n}$ with partitions $V_1$and V_2 .Let e be an edge and u be a vertex such that $u \notin e$in$G$. Then$D_{\Delta f}(e,u) = 2n - 2$.

Theorem: 2.3 Let G be a tree, then for every edge e and a vertex $v$in G, $d(e,v) = D_{\Delta f}(e,v) = D(e,v)$.

Remark: 2.2 The converse of the theorem:2.3 need not be true. Consider the graph,$G = C_4$, where $d(e,v) = D_{\Delta f}(e,v) = D(e,v) = 1$ if $v \notin e$ and $d(e,v) = D_{\Delta f}(e,v) = D(e,v) = 0$ if $v \in e$.

Definition: 2.2 The edge-to-vertex triangle free detour eccentricity $e_{\Delta f2}(e)$ of an edge e in a connected graph G is defined as$e_{\Delta f2}(e) = max\{D_{\Delta f}(e,v) : v \in V\}$. A vertex v for which $e_{\Delta f2}(e) = D_{\Delta f}(e,v)$ is called an edge-to-vertex triangle free detour eccentric vertex of e. The edge-to-vertex triangle free detour radius of G is defined as $R_{\Delta f2} = rad_{\Delta f2}(G) = min\{e_{\Delta f2}(e): e \in E\}$. The edge-to-vertex triangle free detour diameter of G is defined as $D_{\Delta f2} = diam_{\Delta f2}(G) = max\{e_{\Delta f2}(e): e \in E\}$.

Definition: 2.3 An edge e is called an edge-to-vertex triangle free detour central edge if $e_{\Delta f2}(e) = R_{\Delta f2}$. The edge-to-vertex triangle free detour center of $G$is defined as $C_{\Delta f2}(G) = Cen_{\Delta f2}(G) = \{e \in E: e_{\Delta f2}(e) = R_{\Delta f2}\}$.

Definition: 2.4 An edge e is called an edge-to-vertex triangle free detour peripheral edge if $e_{\Delta f2}(e) = D_{\Delta f2}$. The edge-to-vertex triangle free detour periphery of G is defined as $P_{\Delta f2}(G) = Per_{\Delta f2}(G) = \{e \in E: e_{\Delta f2}(e) = D_{\Delta f2}\}$.

Definition. 2.5 If every edge of a graph G is a edge-to-vertex triangle free detour central edge, then G is called edge-to-vertex triangle free detour self centered graph.

Definition. 2.6 If G is the edge-to-vertex triangle free detour self centered graph, then$G$ is called edge-to-vertex triangle free detour periphery.

Example. 2.2 For the graph Ggiven in the figure: 2.2, $e_1 = \{u_1,\ u_2\},\ e_2 = \{u_2,\ u_3\}, e_3 = \{u_3,\ u_4\}, e_4 = \{u_4,\ u_5\}, e_5 = \{u_5,\ u_6\}, e_6 = \{u_6,\ u_7\}, e_7 = \{u_7,\ u_8\}, e_8 = \{u_1,\ u_8\}, e_9 = \{u_8,\ u_2\}, e_{10} = \{u_7,\ u_5\}, e_{11} = \{u_5,\ u_2\}, e_{12} = \{u_3,\ u_5\}$ are the edges of G.

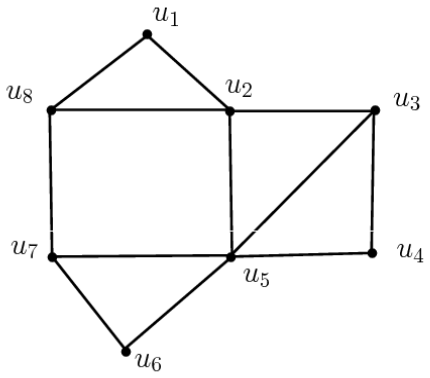

Figure:2.2 G

The edge-to- vertex triangle free detour distances of the graph G, are provided in the following table.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	$e_{\Delta f2}$
e_1	0	0	4	4	3	3	2	3	4
e_2	1	0	0	4	3	3	2	3	4
e_3	4	4	0	0	1	4	3	3	4
e_4	3	3	1	0	0	5	4	3	5
e_5	3	3	4	5	0	0	3	2	5
e_6	3	2	3	4	3	0	0	3	4
e_7	3	2	2	3	2	3	0	0	3
e_8	0	3	3	3	2	3	3	0	3
e_9	1	0	3	3	2	2	2	0	3
e_{10}	2	2	3	4	0	1	0	2	4
e_{11}	3	0	2	2	0	3	2	2	3
e_{12}	3	3	0	1	0	4	3	2	4

Table:2.1

The following table provides the edge-to- vertex distances, edge-to-vertex triangle free detour distances and edge-to- vertex detour distances of the graph Gin figure:2.2

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
e_2	2	2	2	2	2	2	2	3	2	2	1	2
$e_{\Delta f2}$	4	4	4	5	5	4	3	3	3	4	3	4
e_{D2}	6	6	6	6	6	6	6	6	5	5	4	5

Table: 2.2

The edge-to-vertex radius$r_2 = 1$, the edge-to-vertex triangle free detour radius $R_{\Delta f2} = 3$, the edge-to- vertex detour radius $R_2 = 4$. Thus, the edge-to-vertex triangle free detour radius is different from the edge-to- vertex radius and the edge-to- vertex detour radius. The edge-to-vertex diameter $d_2 = 3$, the edge-to- vertex triangle free detour diameter $D_{\Delta f2} = 6$, the edge-to- vertex detour diameter $D_2 = 6$. Thus, the edge-to- vertex triangle free detour diameter is different from the edge-to- vertex diameter and the edge-to- vertex detour diameter.

The edge-to-vertex center $C_2(G) = \{e_{11}\}$, the edge-to-vertex triangle free detour center $C_{\Delta f2}(G) = \{e_7,\ e_8,\ e_9,\ e_{11}\}$, the edge-to-vertex detour center $C_{D2}(G) = \{e_9,\ e_{10},\ e_{11}\}$Thus the edge-to- vertex triangle free detour center is different from the edge-to- vertex center and the edge-to- vertex detour center. The edge-to-vertex periphery $P_2(G) = \{e_8\}$, the edge-to-vertex triangle free detour periphery $P_{\Delta f2}(G) = \{e_4,\ e_5\}$, the edge-to-vertex detour periphery $P_{D2}(G) = \{e_1, e_2\ , e_3\ , e_4,\ e_5, e_6,\ e_7, e_8\}$. Thus, the edge-to- vertex triangle free detour periphery is different from the edge-to-vertex periphery and the edge-to- vertex detour periphery.

The edge-to-vertex triangle free detour radius $R_{\Delta f2}$ and the edge-to-vertex triangle free detour diameter $D_{\Delta f2}$ of some standard graphs are provided in the table:2.3

G	K_n	P_n	$C_n\ (n \geq 4)$	$W_n (n \geq 5)$	$K_{n,m} (n \geq m)$
$R_{\Delta f2}$	1	$\left\lfloor \frac{n-2}{n} \right\rfloor$	$n-2$	$n-2$	$\begin{cases} 2(n-1), if n = m \\ 2n-1,\ \ if n > m \end{cases}$
$D_{\Delta f2}$	1	$n-2$	$n-2$	$n-2$	$\begin{cases} 2(n-1), if n = m \\ 2n-1,\ \ if n > m \end{cases}$

Example: 2.3 The complete graph K_n, the Cycle graph $C_n\ (n \geq 4)$ and the wheel graph $W_n (n \geq 5)$ are the edge-to-vertex triangle free detour self centered graph.

Theorem:2.4 For a connected graph G of order n. Then
(i)$0 \leq e_2(e) \leq e_{\Delta f2}(e) \leq e_{D2}(e) \leq n-2$, for every edge $e$of G.
(ii)$0 \leq r_2 \leq R_{\Delta f2} \leq R_2 \leq n-2$.
(iii) $0 \leq d_2 \leq D_{\Delta f2} \leq D_2 \leq n-2$.

Remark: 2.3 The bounds in the theorem:2.4are sharp. If $G = P_2$, then $e_2(e) = e_{\Delta f2}(e) = e_{D2}(e) = 0$. If $G = C_n\ (n \geq 4)$, then $e_2(e) = e_{\Delta f2}(e) = e_{D2}(e) = n-2$. For the graph G given in the figure:2.2, $0 < e_2(e) < e_{\Delta f2}(e) < e_{D2}(e) < n-2$, for the edges $e = e_9, e_{10}, e_{11}, e_{12}$.

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The Forcing Geodetic Cototal Domination Number of a Graph

S. L. Sumi[1]
V. Mary Gleeta[2]
J. Befija Minnie[3]

Abstract

Let S be a geodetic cototal domination set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum geodetic cototal domination set containing T. The minimum cardinality T is the forcing geodetic cototal domination number of S is denotedby $f_{\gamma gct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing geodetic cototal domination number of G,denoted by $f_{\gamma gct}(S)$, is $f_{\gamma gct}(G) = min\{f_{\gamma gct}(S)\}$, where the minimum is takenover all γ_{gct}-sets S in G. Some general properties satisfied by this concept arestudied. It is shown that for every pair a, b of integers with $0 \leq a < b, b \geq 2$,there exists a connected graph G such that $f_{\gamma gct}(G) = a$ and $\gamma_{gct}(G) = b$. where$\gamma_{gct}(G)$ isthe geodetic cototal dominating number of G.

Keywords: geodetic set, cototal dominating set, geodetic cototal dominating set, geodetic cototal domination number, forcing geodetic cototal domination number.

AMS Subject Classification: 05C12, 05C69[4]

[1]Research Scholar, Register No.20123042092007, Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 629004, Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India. sumikrish123@gmail.com.
[2]Assistant Professor, Department of Mathematics, T.D.M.N.S College, T. Kallikulam - 627 113, Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India. gleetass@gmail.com.
[3]Assistant Professor, Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 629004, Affiliated by Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India, Mail Id: befija@gmail.com
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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by $m$and $n$respectively. For basic definitions and terminologies, we refer to [1,2]. For vertices u and v in a connected graph G, the distance$d(u, v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u, v)$ is called a $u-v$*geodesic*. The *eccentricity*$e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, $radG$ or$r(G)$ and the maximum eccentricity is its *diameter*, $diamG$of G. Let $x, y \in V$and let$I[x, y]$ be the set of all vertices that lies in $x - y$ geodesic including $x$and y. Let $S \subseteq V(G)$and $I[S] = \bigcup_{x,y \in S} I[x, y]$. Then S is said to be a *geodetic set* of G, if $I[S] = V$. The *geodetic number*$g(G)$ of $G$is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a g-*set*of G. A set $S \subseteq V\ (G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *domination number*, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied in [3]. A dominating set S of G is a *cototal dominating set* if every vertex $v \in V \setminus S$ is not an isolated vertex in the induced subgraph $\langle V \setminus S \rangle$. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set. The cototal domination number of a graph was studied in [4]. A set $S \subseteq V$ is said to be a geodetic cototal dominating set of G, IfS is both geodetic set and cototal dominating set of G. The geodetic cototal domination number of G is the minimum cardinality among all geodetic cototal dominatingsets in G and denoted by $\gamma_{gct}(G)$. A geodetic cototal dominating set of minimumcardinality is called the γ_{gct}-set of G. The geodetic cototal domination number of agraph was studied in [6]. The following theorems are used in the sequel.

Theorem 1.1. [6] Every end vertex of G belongs to every geodetic cototal dominating set of G.

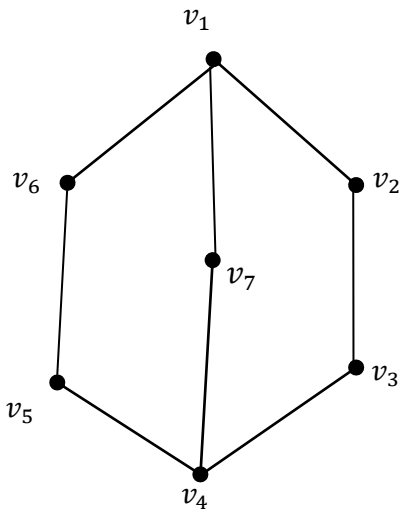
2. The forcing geodetic Cototal domination number of a graph

Even though every connected graph contains a minimum geodetic cototal dominating sets, some connected graph may contain several minimum geodetic cototal dominating sets. For each minimum geodetic cototal dominating set S in a connected graph there is always some subset T of S that uniquely determines S as the minimum geodetic cototal dominating set containing T such "forcing subsets" are considered in this section. The forcing concept was studied in [5]

Definition 2.1. Let S be a geodetic cototal domination set of G. A subset T ⊆ S is called a forcing subset for S if S is the unique minimum geodetic cototal domination set containing T. The minimum cardinality T is the forcing geodetic cototal domination

number of S is denoted by $f_{\gamma gct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing geodetic cototal domination number of G, denoted by$f_{\gamma gct}(S)$, is $f_{\gamma gct}(G)$ = min $\{f_{\gamma gct}(S)\}$, where the minimum is taken over all γ_{gct}-setsS in G.

Example 2.2. For the graph G of Figure 2.1, $S_1 = \{v_3, v_6, v_7\}$ and $S_2 = \{v_2, v_5, v_7\}$are the only two γ_{gct}-sets of G so that $\gamma_{gct}(G)$ = 3 and $f_{\gamma gct}(S_1)$ = $f_{\gamma gct}(S_2) = 1$ sothat $f_{\gamma gct}(G) = 1$.



G

Figure 2.1

The following result follows immediately from the definitions of the geodetic cototal domination number and the forcing geodetic cototal domination number of a connected graph G.

Theorem 2.3. For every connected graph G, $0 \leq f_{\gamma gct}(G) \leq \gamma_{gct}(G)$.

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the complete graph $G = K_n$,$S = V$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G)$ = 0. Also, the bounds in Theorem2.3 can be strict. For the graph G given in Figure 2.1, $\gamma_{gct}(G)$= 3 and $f_{\gamma gct}(G)$ = 1. Thus $0 < f_{\gamma gct}(G) < \gamma_{gct}(G)$.

Theorem 2.5. Let G be a connected graph. Then
(a) $f_{\gamma gct}(G)$ = 0 if and only if G has a unique minimum γ_{gct}-set.
(b) $f_{\gamma gct}(G)$ = 1 if and only if G has at least two minimum γ_{gct}-sets, one of which isa unique minimum γ_{gct}-set containing one of its elements and
(c) $f_{\gamma gct}(G) = \gamma_{gct}(G)$ if and only if no γ_{gct}-set of G is the unique minimum γ_{gct}-set containing any of its proper subsets.

Definition 2.6. A vertex v of a connected graph G is said to be a geodetic cototal dominating vertex of G if v belongs to every γ_{gct}-set of G.

Example 2.7. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two minimum γ_{gct}-sets of G so that $\{v_1, v_3\}$ is the geodeticco-total dominating vertex of G.Then $f_{\gamma gct}(G) \leq \gamma_{gct}(G) - |W|$.

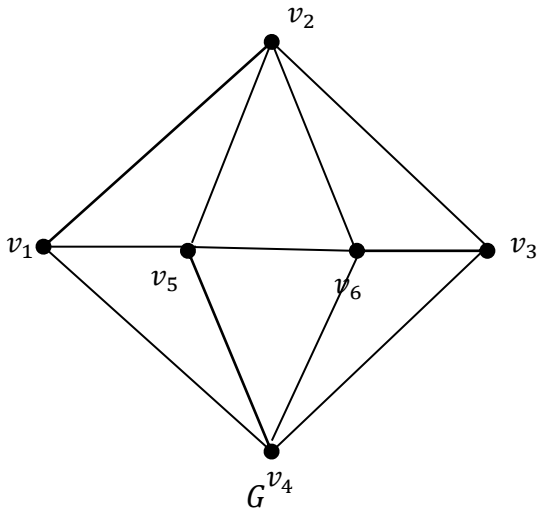


Figure 2.2

Remark 2.9. The bound in Corollary 2.7 is sharp. For the graph G of Figure 2.2, $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two minimum γ_{gct}-sets of G so that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$ so that $\gamma_{gct}(G) = 3$ and $f_{\gamma gct}(G) = 1$. Also, $W = \{v_1, v_3\}$ is theset of all geodetic cototal dominating vertices of G. Now,$\gamma_{gct}(G) - |W| = 3 - 2 = 1$. Thus $f_{\gamma gct}(G) < \gamma_{gct}(G) - |W|$. Also, the bounds in Theorem 2.7 can be strict.

Theorem 2.10. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$,

$$f_{\gamma gct}(G) = \begin{cases} 0, & if\ 1 \leq r \leq 3 \\ 4, & if 4 \leq r \leq s \end{cases}$$

Proof: Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the bipartite sets of G. For $1 \leq r \leq 3$. Let $S = U \cup W$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$. Let$1 \leq r \leq 3$. If $r \geq 4$, then every $\gamma_{gct}(G)$-set is of the form $S = \{u_{i_1}, u_{i_2}, w_{j_1}, w_{j_2}\}$where$1 \leq i_1 \leq i_2 \leq r$ and $1 \leq j_1 \leq j_2 \leq s$. Since S is not the unique geodetic cototaldominating set containing any of its proper subset, By Theorem $f_{\gamma gct}(G) = 4$.■

Theorem 2.11. For the wheel $G = K_n + C_{n-1}$ $(n \geq 5)$,

$$f_{\gamma gct}(G) = \begin{cases} 1, & if n is even \\ 2, & if n is odd \end{cases}$$

Proof: Let x be the central vertex of G and C_{n-1} be $v_1, v2, \ldots, v_{n-1}, v_n$.
Case 1: n is even.
Then $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-1}\}, S_2 = \{v_2, v_4, v_6, \ldots, v_{n-2}, v_n\}$ are the only twoγ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$ so that $f_{\gamma gct}(G) = 1$.
Case 2: n is odd.
Then $S_1 = \{v_1, v_3, v_5, \ldots, v_n\}, S_2 = \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\}, \ldots, S_{n/2} = \{v_{n/2}, v_{n/2+1}, \ldots, v_1, v_3, v_{n/2-1}\}$are the $n/2$ γ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = \ldots = 4f_{\gamma gct}(S_{n/2}) = 2$ sothat $f_{\gamma gct}(G) = 2$.■

Theorem 2.12. For the helm graph $G = H_r, G = T, f_{\gamma gct}(G) = 0$, for $n \geq 6$.
Proof: Let S be the set of end vertices and the cut vertices of G. Then S is theunique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.■

Theorem 2.13. For the Triangular snake graph $G = T_r$, $f_{\gamma gct}(G) = 0$.
Proof: Let S be the set of extreme vertices of G. Then S is the unique γ_{gct}-set of Gso that $f_{\gamma gct}(G) = 0$.■

Theorem 2.14. For the fan graph $F_n = K_1 + P_{n-1}$,
$f_{\gamma gct}(G) = \begin{cases} 0, & if n-1 \; is odd \\ 1, & if n is even \end{cases}$
Proof: Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.
Let $n-1$ is odd. Let $n-1 = 2k+1$. Then $S = \{v_1, v_3, v_5, \ldots, v_{2k+1}\}$ is the uniqueγ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.
Let $n-1$ be even. Let $n-1 = 2k$. Then $S_1 = \{v_1, v_3, v_5, \ldots, v_{2k-1}, v_{2k}\}, S_2 = \{v_1, v_3, v_5, \ldots, v_{2k-2}, v_{2k}, v_2\}$are the two γ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$.so that $f_{\gamma gct}(G) = 1$. ■

Theorem 2.15. For the Banana tree graph $G = B_{r,s}$, $f_{\gamma gct}(G) = 0$.
Proof: Let x be the centre vertex of G and the set of all end vertices of G. Then$S = Z \cup \{x\}$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.■

Theorem 2.16. For the sunflower graph $G = SF_n$, $f_{\gamma gct}(G) = 0$.
Proof: Let S be the set of extreme vertices of G. Then S is the unique γ_{gct}-set of G. So that $f_{\gamma gct}(G) = 0$. ■

Theorem 2.17. For every pair$a, b$of integers with $0 \leq a < b, b \geq 2$, there exists a connected graph G such that $f_{\gamma gct}(G) = a$ and $\gamma_{gct}(G) = b$.
Proof: Let $P : u, v, z$be a path of order three. Let P_i: u_i, v_i $(1 \leq i \leq a)$ be a copyof path on two vertices. Let Hbe a graph obtained from $P$and $P_i (1 \leq i \leq a)$ byjoining each u_i $(1 \leq i \leq a)$ with v and each v_i $(1 \leq i \leq a)$ with z. Let G be thegraph obtained from H by introducing new vertices$z_1, z_2, \ldots, z_{b-a+1}$ joining eachz_i $(1 \leq i \leq a)$ with z. The graph G is given in Figure 2.4.

First, we show that $\gamma_{gct}(G) = b$. Let $Z = \{u, z_1, z_2, \ldots, z_{b-a+1}\}$ be the set of endvertices of G. By Theorem 1.1, Z is a subset of every geodetic cototal dominating set of G. Let $H_i = \{u_i, v_i\}$. Then it is easily observed that every geodetic cototaldominating set containing at least one vertex from each $H_i (1 \leq i \leq a)$ and so $\gamma_{gct}(G) \geq b - a + a = b$. Let $S = Z \cup \{u_1, u_2, \ldots, u_a\}$. Then S is a minimum geodeticcototal dominating set of G so that $\gamma_{gct}(G) = b$.

Next, we prove that $f_{\gamma gct}(G) = a$. Since every geodetic co-total dominating set ofGcontains Z, it follows that$f_{\gamma gct}(G) \leq \gamma_{gct}(G) - |Z| = b - (b - a) = a$.Now, since $\gamma_{gct}(G) = b$ and every γ_{gct}-set of G contains Z, it is easily seen that everyγ_{gct}-set of G is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T beany proper subset of S with $|T| < a$. Then there exists an edge e_j $(1 \leq j \leq a)$ suchthat $e_j \notin T$. Let f_j be an edge of H_j distinct from e_j. Then $W_1 = (S - \{e_j\} \cup \{f_j\}$is a γ_{gct}-set properly containing T. Thus W is not the unique γ_{gct}-set containingT. Thus T is not a forcing subset of S. This is true for all minimum geodetic cototaldominating sets of G and so it follows that $f_{\gamma gct}(G) = a$.

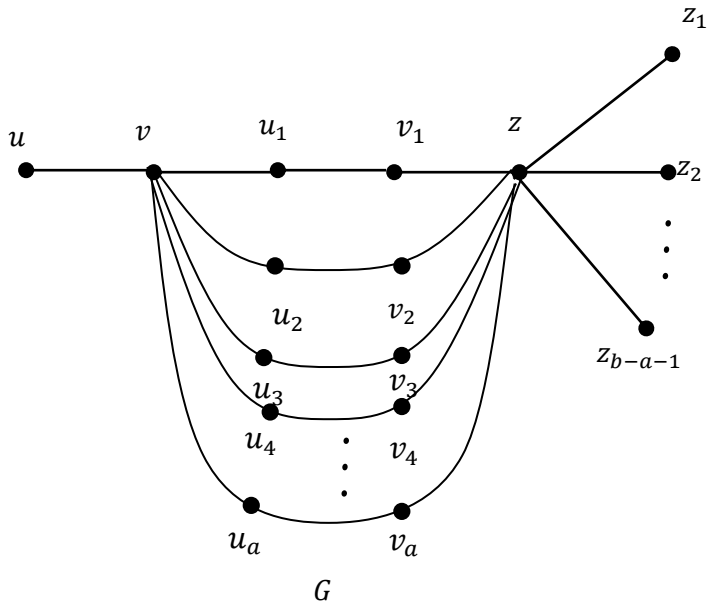


Figure 2.3

3. Conclusion

In this paper we studied the concept of forcing geodetic cototal domination number of some standard graphs some general properties satisfied by this concept are studied. In future studies, the same concept is applied for the other graph operations.

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Steiner domination decomposition number of graphs

Mahiba M[1]
Ebin Raja Merly E[2]

Abstract

In this paper, we introduce a new concept Steiner domination decomposition number of graphs. Let G be a connected graph with Steiner domination number$\gamma_s(G)$. A decomposition $\pi = \{G_1, G_2, \dots, G_n\}$ of G is said to be a Steiner Domination Decomposition (SDD) if $\gamma_s(G_i) = \gamma_s(G), 1 \leq i \leq n$. Steiner domination decomposition number of G is the maximum cardinality obtained for an SDD of G and is denoted as $\pi_{std}(G)$. Bounds on $\pi_{std}(G)$ are presented. Also, few characteristics of the subgraphs belonging to SDD of maximum cardinality are discussed.

Keywords: subgraphs; domination; decomposition number.

AMS subject classification: 05C12, 05C69[3]

[1]Research Scholar (Reg.No: 20213112092013), Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam-629165. Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India. mahibakala@gmail.com
[2]Associate Professor, Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam-629165. Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India. ebinmerly@gmail.com
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1. Introduction

Let G be a simple, connected and undirected graph with vertex set $V(G)$and edge set $E(G)$. The order and size of G are p and q respectively. For standard terminologies and notations, we refer to [1]. Steiner domination number of a graph is a concept introduced by John $et\ al.$ Further studies on this concept is found in [7], [8]. In [5], we introduced the concept of Steiner decomposition number of graphs and in [6] we presented the Steiner decomposition number of Complete $n-$ Sun graph. In this paper, a new decomposition concept called Steiner domination decomposition number of graphs is studied. The following are the basic definitions and results needed for the subsequent section.

Definition 1.1. [2] Let G be a connected graph. For a set $W \subseteq V(G)$, a tree T contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in G that lie on some Steiner tree with respect to W. The set W is a Steiner set for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number $s(G)$.

Definition 1.2. [3] A set $D \subseteq V(G)$in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Definition 1.3. [4] For a connected graph $G, W \subseteq V(G)$is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is said to be Steiner domination number and is denoted by $\gamma_s(G)$. A Steiner dominating set of cardinalities$\gamma_s(G)$ is said to be a $\gamma_s - set$ of G.

Definition 1.4. The decomposition π of a graph G is a collection of edge disjoint subgraphs $G_1, G_2, \dots, G_n$ such that each $G_i, 1 \leq i \leq n$is connected and $E(G) = E(G_1) \cup \mathrm{E}(G_2) \cup \dots \cup \mathrm{E}(G_n)$.

Definition 1.5. [5] For a connected graph G with Steiner number $s(G)$, a decomposition $\pi = \{G_1, G_2, \dots, G_n\}$of G is said to be a Steiner Decomposition(SD) if $s(G_i) = s(G)$ for all $i, (1 \leq i \leq n)$.The maximum cardinality obtained for the Steiner decomposition π of G is called the Steiner decomposition number of G and is denoted by $\pi_{st}(G)$. Steiner decomposition of cardinality $\pi_{st}(G)$ is denoted as SD_{max}.

Theorem 1.6. [4] For any connected graph G of order $p \geq 2$, $\gamma_s(G) = 2$ if and only if there exists a Steiner dominating set $W = \{u, v\}$of G such that $d(u, v) \leq 3$.

Theorem 1.7. [4] For a connected graph G of order $p \geq 2$, $\gamma_s(G) = p$ if and only if $G = K_p$.

Result 1.8. [7] For the path graph on p vertices $(p \geq 2), \gamma_s(P_p) =$

$$\begin{cases} \left\lceil \frac{p-4}{3} \right\rceil + 2 & if\ p \geq 5 \\ 2 & if\ p = 2,3,4 \end{cases}$$

Notation 1.9. $\mathcal{F}_p$ denotes the family of trees of order p with the property that each vertex is either a pendant vertex or a support vertex.

2. Steiner Domination Decomposition

Definition 2.1. A decomposition $\pi = \{G_1, G_2, \dots, G_n\}$of a graph G is called a Steiner Domination Decomposition (SDD) if $\gamma_s(G_i)$=$\gamma_s(G), (1 \leq i \leq n)$. The maximum cardinality obtained for π is called the Steiner domination decomposition number of G and is denoted by $\pi_{std}(G)$. An SDD of cardinality $\pi_{std}(G)$is denoted as SDD_{max}. A graph G with $\pi_{std}(G) = 1$ is said to be non-Steiner domination decomposable graph. If $\pi_{std}(G) \geq 2$ then G is said to be Steiner domination decomposable graph.

Example 2.2. Consider the graph G in figure 1.

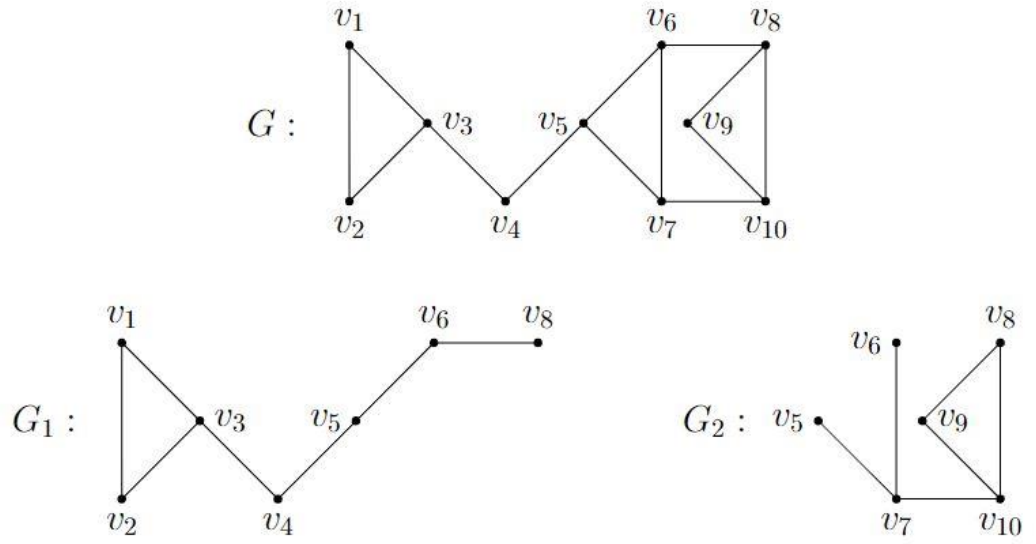

Figure 1. Graph G and its Steiner domination decomposition $\pi = \{G_1, G_2\}$

The set $W = \{v_1, v_2, v_5, v_9\}$is a $\gamma_s - set$ of G. Hence $\gamma_s(G) = 4$. Since $\gamma_s(G_1) = \gamma_s(G_2) = 4 = \gamma_s(G)$, $\pi = \{G_1, G_2\}$ is a SDD. It can be easily verified that π is a SDD_{max}. Thus $\pi_{std}(G) = 2$.

Theorem 2.3. If $\pi_{std}(G) = q$ then $diam\ G < 4$.

Proof. Steiner domination decomposition number of $G, \pi_{std}(G) = q \Leftrightarrow \pi = \{G_i \cong K_2 / 1 \leq i \leq q\}$ is a SDD_{max}. Steiner domination number of K_2 is 2, hence $\gamma_s(G) = 2$. Also, we have $\gamma_s(G) = 2$ implies $diam\ G < 4$. Therefore if $\pi_{std}(G) = q$ then $diam\ G < 4$. Hence proved.

Theorem 2.4. Let G be a connected graph with $\gamma_s(G) \geq 3$. Then $1 \leq \pi_{std}(G) \leq$

$\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$.

Proof. From definition 2.1, it is obvious that $\pi_{std}(G) \geq 1$. Let $\pi = \{G_i / 1 \leq i \leq n\}$ be a SDD of G. First to prove $|E(G_i)| \geq \gamma_s(G)$ for all i. Assume to the contrary that $|E(G_i)| < \gamma_s(G)$ for some i. Without loss of generality, let $|E(G_1)| < \gamma_s(G)$. Then $|V(G_1)| \leq \gamma_s(G)$.

Case (i):$|V(G_1)| < \gamma_s(G)$

If $|V(G_1)| < \gamma_s(G)$ then $\gamma_s(G_1) < \gamma_s(G)$. Therefore $G_1 \notin \pi$.

Case (ii): $|V(G_1)| = \gamma_s(G)$

In order to satisfy $\gamma_s(G_1) = \gamma_s(G), G_1$ must be a complete graph on $\gamma_s(G)$vertices. But we have $|V(G_1)| > |E(G_1)|$. Hence G_1 is non isomorphic to $K_{\gamma_s(G)}$. Therefore $G_1 \notin \pi$.

In both the cases, we arrive at a contradiction to our assumption that $G_1 \in \pi$. Hence $|E(G_1)| \geq \gamma_s(G)$. Since G_1 is chosen arbitrarily, we can conclude $|E(G_i)| \geq \gamma_s(G)$ for all i.Thus subgraphs of G belonging to any Steiner domination decomposition should have atleast $\gamma_s(G)$edges and so $\pi_{std}(G) \leq \left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$. Hence $1 \leq \pi_{std}(G) \leq \left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$.

Theorem 2.5. Let G be a Steiner domination decomposable graph with q edges. For $\gamma_s(G) = 3, \pi_{std}(G) = \frac{q}{3}$if and only if each $G_i \in SDD_{max}$is isomorphic to either $K_{1,3}$or K_3 and for $\gamma_s(G) > 3, \pi_{std}(G) = \frac{q}{\gamma_s(G)}$ if and only if each $G_i \in SDD_{max}$ is isomorphic to $K_{1,\gamma_s(G)}$.

Proof. Let G be a Steiner domination decomposable graph. Assume $\gamma_s(G) = 3$ and $\pi_{std}(G) = \frac{q}{3}$. Then for any $G_i \in SDD_{max}, |E(G_i)| = 3$ and hence$|V(G_i)| \leq 4$. If $|V(G_i)| \leq 3$ for some i, then the only graph that satisfies $\gamma_s(G_i) = 3$ is K_3. If $|V(G_i)| = 4$ for some i, then G_i is a tree. Star graph $K_{1,3}$ is the unique tree which satisfies the required properties. Thus if $\pi_{std}(G) = \frac{q}{3}$ then $G_i \cong K_{1,3}$or K_3 for all $G_i \in SDD_{max}$. Converse part is obvious. Now, assume$\gamma_s(G) > 3$and $\pi_{std}(G) = \frac{q}{\gamma_s(G)}$.Then$|E(G_i)| = \gamma_s(G)$ for every $G_i \in SDD_{max}$ and so $|V(G_i)| \leq \gamma_s(G) + 1$. There doesn't exist any graph G_i with the properties $|V(G_i)| \leq \gamma_s(G)$ and $\gamma_s(G_i) = \gamma_s(G)$. Since $K_{1,\gamma_s(G)}$ is the unique graph on $\gamma_s(G) + 1$ vertices that has Steiner domination number same as G, we have $|V(G_i)| = \gamma_s(G) + 1$ implies $G_i \cong K_{1,\gamma_s(G)}$. Hence if $\pi_{std}(G) = \frac{q}{\gamma_s(G)}$ then $G_i \cong K_{1,\gamma_s(G)}$ for all $G_i \in SDD_{max}$. Converse is obvious.

Theorem 2.6. Let G be a connected graph with $\gamma_s(G) \geq 3$ and $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. If $\pi_{std}(G) = m - n, (0 \leq n < m - 1)$ then $\gamma_s(G) \leq |E(G_i)| \leq (n + 2)\gamma_s(G) - 1$ for all $G_i \in SDD_{max}$.

Proof. Let G be a connected graph such that $\gamma_s(G) \geq 3$. Let $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. Assume $\pi_{std}(G) = m - n$ where $0 \leq n < m - 1$. Let $\pi = \{G_1, G_2, \dots, G_{m-n}\}$be a SDD_{max} of G. To prove $\gamma_s(G) \leq |E(G_i)| \leq (n + 2)\gamma_s(G) - 1$ for all $G_i \in \pi$. The requirement of edges in each subgraph belonging to any SDD of G is atleast $\gamma_s(G)$. Hence$|E(G_i)| \geq \gamma_s(G)$for every $G_i \in \pi$. Without loss of generality, assume

$|E(G_{m-n})| \geq |E(G_i)|, 1 \leq i \leq m-(n+1)$. Since $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m$, we get $m\gamma_s(G) \leq q \leq (m+1)\gamma_s(G) - 1$. We know that $\sum_{i=1}^{m-n} |E(G_i)| = q$ and $|E(G_i)| \geq \gamma_s(G)$ for $1 \leq i \leq m-(n+1)$.

Therefore,

$$\sum_{i=1}^{m-n} |E(G_i)| \leq (m+1)\gamma_s(G) - 1$$

$$(m-(n+1))\gamma_s(G) + |E(G_{m-n})| \leq (m+1)\gamma_s(G) - 1$$

$$\Rightarrow |E(G_{m-n})| \leq (n+2)\gamma_s(G) - 1$$

Thus, the possible number of edges in a subgraph belonging to SDD_{max} is atmost $(n+2)\gamma_s(G) - 1$. Hence $\gamma_s(G) \leq |E(G_i)| \leq (n+2)\gamma_s(G) - 1$ for all $G_i \in SDD_{max}$.

Theorem 2.7. Let G be a connected graph with $\gamma_s(G) \geq 5$ and $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. If $\pi_{std}(G) = m - n, (0 \leq n < m-1)$ then the number of path graphs belonging to SDD_{max} is strictly less than $n+1$.

Proof. Let G be a connected graph with q edges. Let $\gamma_s(G) = k+1$ where $k \geq 4$. Assume $\pi_{std}(G) = m - n, (0 \leq n < m-1)$. Let $\pi = \{G_i / 1 \leq i \leq m-n\}$ be a SDD_{max}. Let N denotes the number of path graphs belonging to π. First we try to prove $N \neq n+1$.

Suppose $N = n+1$. Consider $G_1, G_2, \ldots, G_{n+1} \in \pi$ as path graphs. Path graphs with Steiner domination number $k+1$ are P_{3k-1}, P_{3k} and P_{3k+1}. Therefore $3k-2 \leq |E(G_i)| \leq 3k$ for $1 \leq i \leq n+1$.

Now, $\sum_{i=1}^{m-n} |E(G_i)| = \sum_{i=1}^{n+1} |E(G_i)| + \sum_{i=n+2}^{m-n} |E(G_i)|$

$$\geq (n+1)(3k-2) + (m-2n-1)(k+1)$$

$$\sum_{i=1}^{m-n} |E(G_i)| \geq (n+2)k - (4n+3) + m(k+1)$$

For $k \geq 4, q \leq (m+1)(k+1) - 1 < (n+2)k - (4n+3) + m(k+1)$. .

This is a contradiction since $\sum_{i=1}^{m-n} |E(G_i)| = q$ and $q \leq (m+1)(k+1) - 1$. Hence $N \neq n+1$. If $N > n+1$ then $\sum_{i=1}^{m-n} |E(G_i)| > (n+2)k - (4n+3) + m(k+1)$. This again results in a contradiction. Thus $N < n+1$ and so number of path graphs belonging to π is strictly less than $n+1$. Hence the proof.

Theorem 2.8. If $T \in \mathcal{F}_p$ then $\pi_{std}(T) = 1$.

Proof. Assume $T \in \mathcal{F}_p$. Every vertex of T is either a pendant vertex or a support vertex. Let l and m be the number of pendant vertices and support vertices of T respectively. Clearly the set of all pendant vertices of T forms the $\gamma_s - set$. Hence $\gamma_s(T) = l$. Number of edges of T is $l + m - 1$. Also, $m \leq l$ for any graph. Hence by theorem 2.4, $\pi_{std}(T) = 1$.

Remark 2.9. If $s(G) = \gamma_s(G)$ then $\pi_{st}(G)$ need not be equal to $\pi_{std}(G)$.

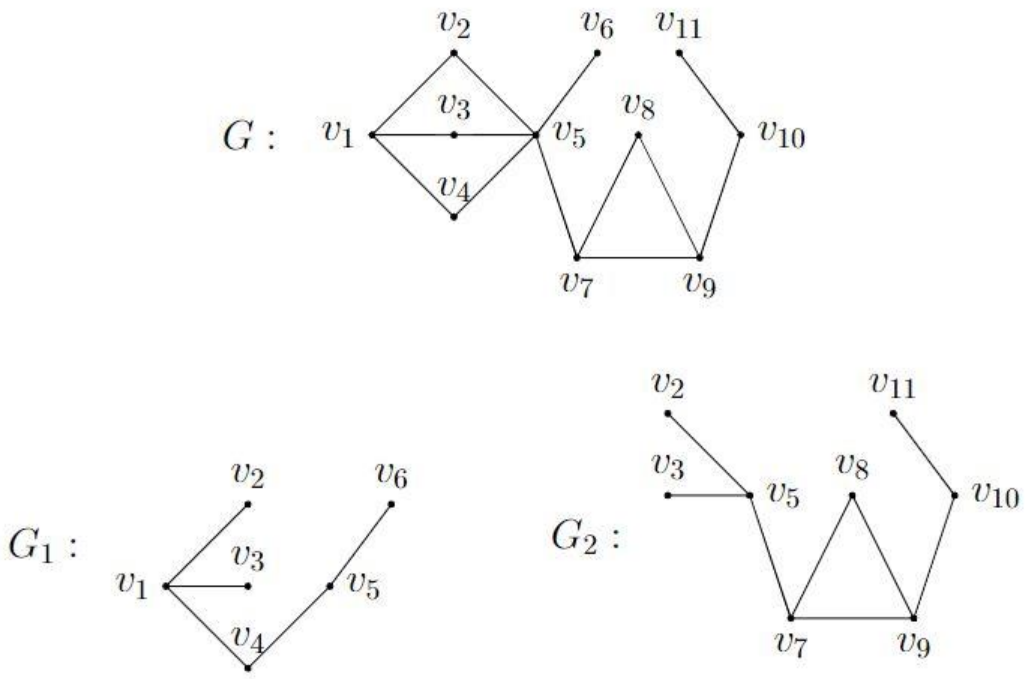

Figure 2. Graph G and its SDD_{max}, $\pi = \{G_1, G_2\}$

For the graph G in figure 2, minimum Steiner set$= \gamma_s - set = \{v_1, v_6, v_8, v_{11}\}$. Hence $s(G) = \gamma_s(G) = 4$. Steiner domination decomposition $\pi = \{G_1, G_2\}$is a SDD_{max} of G and so $\pi_{std}(G) = 2$. Also $\pi_{st}(G) = 1$. Therefore $\pi_{st}(G) \neq \pi_{std}(G)$.

Theorem 2.10. Let G be a connected graph such that $s(G) = \gamma_s(G) = k\ (say)$. If there exist some $SD_{max}$and SDD_{max} for G satisfying the condition that each subgraph in the decompositions is of order $k+1$ and has a cutvertex of degree k then $\pi_{st}(G) = \pi_{std}(G)$.

Proof. Consider a connected graph G with $s(G) = \gamma_s(G) = k$. Let π_1 and π_2 be the SD_{max} and SDD_{max}respectively which satisfies the condition that each subgraph in both the decompositions is of order $k+1$ and has a cutvertex of degree k. First to prove, $\pi_1$is a SDD. Let $\pi_1 = \{G_i\ /1 \leq i \leq n\}$. $\pi_1$is a SD implies $s(G_i) = k$ for all i. Each $G_i(1 \leq i \leq n)$is of order $k+1$ and has a cutvertex of degree k.Therefore minimum Steiner set of $G_i = \gamma_s - set$ of G_i for all i and so $\gamma_s(G_i) = k$. Thus π_1 is a SDD. In the similar way, we can prove π_2 is a SD. Now to prove, $\pi_{st}(G) = \pi_{std}(G)$. Suppose $\pi_{st}(G) > \pi_{std}(G)$ then $|\pi_1| > |\pi_2|$. Since π_1 is a SDD, we get a contradiction to π_2 is aSDD_{max}. Suppose $\pi_{st}(G) < \pi_{std}(G)$ then $|\pi_1| < |\pi_2|$. Since π_2 is a SD, we get a contradiction to π_1 is a SD_{max}. Therefore $\pi_{st}(G) = \pi_{std}(G)$.

3. Conclusion

In this paper, we initiate a study on Steiner domination decomposition number of graphs. It is quite interesting to investigate this new parameter and study the properties of the subgraphs belonging to SDD. Future works can be done on calculating the Steiner

domination decomposition number for families of graphs and finding the bounds in terms of other graph theoretical parameters.

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VLSI Implementation of Multi-Bit Error Detection and Correction Codes for Space Communications

Poongodi.S[1]
Asoda Sunayana Rani[2]

Abstract

Data transmission in advanced space communications are suffering with the different types of noises. Further, these noises causeburst errors indata. Thus, the error correction codes (ECC) plays the major role to detect and correct the errors. However, the conventional hamming encoders, decoderswere detected and corrected only one bit error. Therefore, this work implementation the Multi-Bit Error Detection and CorrectionCodes (MBE-DCC) for multiple bits error detection and correction. Initially, MBE-DCC encoding operation is implemented by using generator matrix, which contains both identity bits and parity bits. Then, encoded code word is transmitted into the channel of space communication, where encoded data corrupted by different types of noises, errors. Therefore, the MBE-DCC decoding operation performed at receiver side of space communications, which corrected all the errors using syndrome detection, error location detection, and error correction modules. The simulations revealed that the proposed MBE-DCC resulted in superior performance than conventional ECC methods.

Keywords: Multi-Bit Error Detection and Correction Codes, encoding, decoding, syndrome decoding, error analysis, error correction modules.

AMS Classification:05C12[3]

[1]Professor, Dept. of ECE, CMR Engineering College, Hyderabad. dr.poongodi@cmrec.ac.in
[2]PG student, CMR Engineering College, Hyderabad. sunayana7007@gmail.com
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1. Introduction

The space communicationscontain the channel and to store a value memory is used in channel. In which Random Access Memory (RAM) [1] has been the developing and most compatible device in particular for microprocessor and microcontroller. Main memory helps to keep the processor or controller busy and to perform any task in a minimum time [2]. To achieve this on-chipmemory termed as cache memory is used. This type of main memory is developed and placed at different levels close to the processor on same die, additionally to reduce the access time of the processor for the data that to be processed from the main memory (RAM) [3]. Moreover, the data saved in the cache memory need to be the actual that has been stored during the write operation. Nevertheless, stored value need not be the same as that was stored. Probability of change in stored value is increasing linearly because of rapid change in semiconductor processing technology. To overcome this issue of reliability in cache memory ECC is imperative [4].

Various works [5,6] proposed three models on memory with ECC, of which first model is data communication between memory and cache, second model is the same that includes the additional copy. Last model is the cross-switch communication between the original and additional memory with ECC. For all the model's memory designed is a simple single row address memory with ECC as a separate block [7]. ECC block for the memory is an additional hardware design which increases the delay in area, howeveradditional overhead in area and latency enhances the performance degradation of the system. Performance degradation of a system that includes cache memory can be improved, provided the decoding time for detection [8], correction and the ability to correct or detect with some failure should be a fraction of total memory design size. In recent studies memory with ECC is inherent for protecting the memory from errors in particular soft errors. Multiple Cell Upset (MCU) [9] denotes arbitrary number of cells which is repeatedly affected by soft error is random and most of the cases it is not adjacent. However due to scaling down of the gate width of transistors, density of transistors increases in this case occurrence of soft error has the tendency of dissemination. To mitigate this new and optimized coding algorithms are proposed and being proposed on adjacent error also called as burst errors. Most common adjacent errors are two, three termed as Double Adjacent [10], Triple Adjacent [11] respectively. However, the performance degradation of a system cannot be compromised for which optimized syndrome computation is considered. Therefore, the major contributions of this work are as follows:

- Implementation of MBE-DCC for multiple bits error-detection and correction in space communication applications.
- MBE-DCC encoder is developed with generator matrix, which generates the encoded code word.
- Implementationof MBE-DCC decoder with syndrome detection, error location detection, and error correction modules.

- Implementation of syndrome detection is introduced for detecting status of error, which results error presented or absented in encoded data.
- Implementation of error location detection module for identifying the number of error bits with their position.
- Implementation of error correction module for correcting all the errors in encoded data.

Rest of the article is organized as follows: section 2 deals with literature survey, section 3 deals with the proposed MBE-DCC implementation, section 4 deals with analysis of results with performance comparison, section 5 concludes the article with possible future directions.

2. Literature survey

Cosmic rays and IC packaging dye made from radioactive elements are few significant sources of soft error in cache memory. Furthermore, recent studies on cache memories have shown that reliability issues in cache memory for application that are used to store or during processing of data is significant concern. Especially graphical processing units [12] currently developed by NIVIDIA, AMD and INTEL are designed with high bandwidth cache memories and frequently subjected to reliability problems.

To substantive this, in [13] authors presented a flexible ECC designed for 32-byte cache memory which also consumes low energy for data fetching. In [14] authors compared different error detection techniques and developed a method which has error guard coverage of 97.9 % using tag in cache. The tags are used for index identification. A limitation is that the adjacent location tag in cache memory may be having the same bits that leads to a faulty read or write operation in the cache memory. The change in bits of tag is due to alpha particles. This can be reduced by deploying ECC, SEC – DED, In-Cache Replication (ICR), and in combination of SEC– DED [15], ICR and SEC – DED parity. However, the proposed ICR method has good fault coverage.

In [16] authors proposed counter, shifter, multiplexer and comparator were used and the additional peripherals added to the development of ICR contributed to additional overhead in area. Even after adding the addition peripherals, the delay and area are found to be less. This technique [17] lags because the disadvantage is if the size of the cache increases in terms of level 1, 2, the method proposed increases the complexity of peripherals in 11 proportion to the chance in the size of cache. Also, here only one error is possible to correct. Moreover, triple adjacent location is not addressed.

In [18] authors proposed a method to reduce energy overhead in DRAM (cache) achieved through ECC. DRAM is not modified in design instead, for the usual DRAM a new ECC that access or decodes only the error word is presented. Conventionally only encoded input data is stored, which is decoded to correct or detect the soft error using ECC in a memory. In contrary only error word is corrected using hamming code and error in a word is detected by parity codes. In [19] authors also proposed a DRAM (cache) with ECC, unlike other methods [20] in particular proposed a method that detects hard errors using a

BIST and build in self repair circuits on a chip. ECC that is capable to detect and correct physical fault and soft error is achieved through redundancy circuit or spare circuit and ECC respectively. However, study outlined above of DRAM with ECC [21] is limited to detect and correct two bits and single bit respectively and correcting soft error beyond one error is not possible. Most widely implemented memory circuit is made to store data in SRAM. There are few applications in which SRAM are neither used as cache nor main memory. For instance, SRAM is used as a configurable switch, programmed to connect the logic blocks, technically termed as routing this architecture [22] is implemented in commercial FPGA 's such as Xilinx virtex 4 and Altera stratix, moreover SRAM 's is used as configuration frames, which occupies more than 80% area in FPGA, such as Xilinx virtex 6 and FPGA of Altera family. However, both reconfigurable and configurable frames designed using SRAM are most probably sensitive to soft errors.

In [23] authors proposed a scheme to correct multiple bits upset in configurable frame of FPGA. To detect MBU by combining scrubbing and erasure code additionally to detect error Interleaving-n-Dimensional (InD) method is also proposed. This is also implemented in Virtex-6 XLV240T, which has less overhead in area. Since in proposed InD method [24], with reduced repeated parity bits in all dimensions at regular intervals helps for less area occupation in the SRAM. In [25] authors proposed also presented error correction for switch boxes that are built using SRAM, also mitigation in soft error is achieved through redundancy method. In which zero optimized SRAM are used for interconnection and one optimized SRAM are redundant interconnection termed short and open faults respectively. However major work presented is on optimized routing algorithm.

3. Proposed Method

This section gives the detailed analysis of proposed MBEC method. Figure 1 shows the flowchart of proposed MBEC method.Theproposed MBE-DCC is implemented for multiple bits error-detection and correction in space communication applications.Initially, MBE-DCC encoder is developed with generator matrix, which generates the encoded code word. Here, the matrix multiplication is operation is performed that the generator matrix and data input, which generates the code word. Then, the code-word is transmitted in channel of space engineering, where data bits are corrupted by different types of errors and noises. Further, MBE-DCC decoder is developed with syndrome detection, error location detection, and error correction modules.Here, syndrome detection is implemented for detecting status of error, which results error presented or absented in encoded data.Then, error location detection module is introduced for identifying the number of error bits with their position. Then, error correction module is developed for correcting all the errors in encoded data.

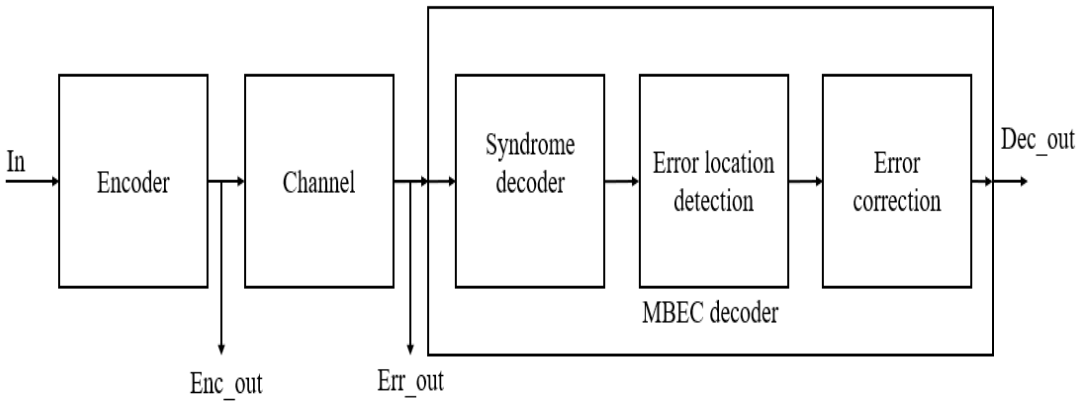


Figure 1.Proposed MBE-DCC flowchart

3.1. MBE-DCC Encoding

The operation of MBE – DCCencoding is achieved by performing the mathematicalmatrix multiplication between generator matrix and data input.

$$V = DG \quad (1)$$

Here, V is the encoded code word, G is the generator matrix, D is the input data. All of them are binary linear block codes. The process used to design these codes is based on some rules for linear block codes construction. In this paper, the proposed codes are also binary linear block codes and obey similar construction rules. Normally, the binary codes are described by the number of data-bits, k, redundancy bits, (n − k), and the block size of the encoded-word, n. An (n, k) code is defined by its generator matrix G or parity check matrix H in

$$G = [P_{k\times(n-k)} \cdot I_{k\times k}] \quad H = [P^T \cdot I_{(n-k)}] \quad (2)$$

where $I_{k\times k}$ is the identity matrix, P is the matrix with size k × (n − k), and P^T is the transpose of P. In the encoding process, the generator matrix G is used to encode the data bits through the process.

Table 1. Construction of generator matrix.

C 0	C 1	C 2	C 3	C 4	C 5	C 6	C 7	C 8	C 9	C 10	C 11	C 12	C 13	C 14	C 15	C 16	C 17	C 18	C 19	C 20	C 21	C 22
1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

1	1	0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1 shows the final constructed generator Matrix, which contains identity matrix and parity bits. Here, parity bits contain size as 16x7, which are ranged from all rows with C0-C6 columns. Further, the size of identity bits contains size as 16x16 i.e., all rows with C7-C22columns. The size of data is 16 bits as follows

$$D = [D1, D2, D3, D4, D5, D6, D7, D8, D9, D10, D11, D12, D13, D14, D15, D16] \quad (3)$$

Finally, encoding operation is achievedthrough calculation of check bits (C1 to C7)as follows:

$$C1 = D1 \oplus D4 \oplus D6 \oplus D8 \oplus D9 \oplus D10 \oplus D14 \quad (4)$$

$$C2 = D2 \oplus D4 \oplus D5 \oplus D7 \oplus D8 \oplus D11 \oplus D15 \quad (5)$$

$$C3 = D3 \oplus D7 \oplus D11 \oplus D13 \oplus D16 \oplus D10 \quad (6)$$

$$C4 = D1 \oplus D4 \oplus D8 \oplus D10 \oplus D12 \oplus D13 \quad (7)$$

$$C5 = D2 \oplus D5 \oplus D6 \oplus D7 \oplus D8 \oplus D13 \oplus D14 \quad (9)$$

$$C6 = D2 \oplus D6 \oplus D7 \oplus D11 \oplus D13 \oplus D16 \quad (10)$$

$$C7 = D3 \oplus D6 \oplus D9 \oplus D11 \oplus D12 \oplus D13 \oplus D15 \oplus D16 \quad (11)$$

Finally, encoding operation is achieved as follows:

$$V = \begin{bmatrix} C1, C2, C3, C4, C5, C6, C7, D1, D2, D3, D4, D5, D6, D7, D8, D9, \\ D10, D11, D12, D13, D14, D15, D16 \end{bmatrix} \quad (12)$$

3.2 MBE-DCC Decoding

The MBE-DCC decoding is consisting of syndrome detection, error location detection and error correction stages. The encoded output V is transmitted into space communication channel, where different types of noises were added.

$$R = V + E$$

$$RC = [C1, C2, C3, C4, C5, C6, C7] + [E1, E2, E3, E4, E5, E6, E7]$$

$$RD = [D1, D2, D3, D4, D5, D6, D7, D8, D9, D10, D11, D12, D13, D14, D15, D16] + [E8, E9, E10, E11, E12, E13, E14, E15, E16, E17, E18, E19, E20, DE21, E22, E23]$$

$$R = [RC, RD]$$

$$R = \begin{bmatrix} RC1, RC2, RC3, RC4, RC5, RC6, RC7, RD1, RD2, RD3, RD4, RD5, RD6, RD7, RD8, RD9, \\ RD10, RD11, RD12, RD13, RD14, RD15, RD16 \end{bmatrix}$$

Here, V is the encoded code word, which stores into memory or transmitted into channel. Further, E represents the error and R represents the received vector with error.

$$S = R.H^T \quad (6)$$

Here, S represents the syndrome value, H represents the parity check matrix and it is constructed from generator matrix as shown in Table 2. The size of identity bits (I) are 7x7 i.e., all rows with H1-H7columns and size of the parity bits (P) are 7x16 i.e., all rows with H8-H23columns. Further, if the S is zero, which indicates no errors in received data. Further, if the S is not equal to zero, which indicates errors presented in received data.

Table 2. Construction of parity check matrix.

H1	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12	H13	H14	H15	H16	H17	H18	H19	H20	H21	H22	H23
1	0	0	0	0	0	0	1	0	0	1	0	1	0	1	1	1	0	0	0	1	0	0
0	1	0	0	0	0	0	0	1	0	1	1	0	1	1	0	0	1	0	0	0	1	0
0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	1	0	1	0	0	1
0	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	1	0	1	1	0	0	0
0	0	0	0	1	0	0	0	1	0	0	1	1	1	1	0	0	0	0	1	1	0	0
0	0	0	0	0	1	0	0	1	0	0	0	1	1	0	0	0	1	0	1	0	0	1
0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	1	1	1	0	1	1
RC1	RC2	RC3	RC4	RC5	RC6	RC7	RD1	RD2	RD3	RD4	RD5	RD6	RD7	RD8	RD9	RD10	RD11	RD12	RD13	RD14	RD15	RD16

Finally, the syndrome calculation is simplified as follows:

$S1 = RC1 \oplus RD1 \oplus RD4 \oplus RD6 \oplus RD8 \oplus RD9 \oplus RD10 \oplus RD14$

$S2 = RC2 \oplus RD2 \oplus RD4 \oplus RD5 \oplus RD7 \oplus RD8 \oplus RD11 \oplus RD15$

$S3 = RC3 \oplus RD3 \oplus RD7 \oplus RD11 \oplus RD13 \oplus RD16 \oplus RD10$

$S4 = RC4 \oplus RD1 \oplus RD4 \oplus RD8 \oplus RD10 \oplus RD12 \oplus RD13$

$S5 = RC5 \oplus RD2 \oplus RD5 \oplus RD6 \oplus RD7 \oplus RD8 \oplus RD13 \oplus RD14$

$S6 = RC6 \oplus RD2 \oplus RD6 \oplus RD7 \oplus RD11 \oplus RD13 \oplus RD16$

$S7 = RC7 \oplus RD3 \oplus RD6 \oplus RD9 \oplus RD11 \oplus RD12 \oplus RD13 \oplus RD15 \oplus RD16$

Then, based on syndrome values error locations were identified using bit pattern matching process. It is explained by following example.

$D = [1,1,0,1,1,1,0,0,1,1,0,0,1,1,1,1]$

Then, Cvalues are calculated using Equations (4)-(11) and resulted as follows:

$C = [0,0,1,0,1,0,1]$

Then, encoded codeword V becomes,

$V = [0,0,1,0,1,0,1,1,1,0,1,1,1,0,0,1,1,0,0,1,1,1,1]$

Consider 4 bits are corrupted and error occurred in [D2, D3, D4, D5] positions of V. Then, R becomes

$R = [0,0,1,0,1,0,1,1,0,1,0,0,1,0,0,1,1,0,0,1,1,1,1]$

The error locations are identified performing all combinations of XOR in H-matrix columns and the XOR outcome will be matched with syndrome for anyone of the combination.The process is repeated until the syndrome is matched. Table 3 illustrates the error location identification process.

Table 2. Error location identification.

H9	H10	H11	H12	XOR (H9, H10, H11, H12)	S
0	0	1	0	1	1
1	0	1	1	1	1
0	1	0	0	1	1
0	0	1	0	1	1
1	0	0	1	0	0
1	0	0	0	1	1
0	1	0	0	1	1

The syndrome is matched with the XOR (H9, H10, H11, H12) combination according to Table 3, so error locations become $[D2, RD3, RD4, RD5]$.The error occurred positions in R vector is $[RD2, RD3, RD4, RD5]$, which are equivalent to H vector positions as [H9, H10, H11, H12]. Finally, error correction operation is implemented by performing the complement of error corrected bits. The resultant error corrected outcome is obtained as follows

$Out = [0,0,1,0,1,0,1,1,1,0,1,1,1,0,0,1,1,0,0,1,1,1,1]$

4. Results and discussions

Xilinx ISE software was used to create all of the MBE – DCCdesigns. This software programmed gives two types of outputs: simulation and synthesis. The simulation results provide a thorough examination of the MBE – DCCarchitecture in terms of input and output byte level combinations. Decoding procedure approximated simply by applying numerous combinations of inputs and monitoring various outputs through simulated study of encoding correctness. The use of area in relation to the transistor count will be accomplished as a result of the synthesis findings. In addition, a time summary will be obtained with regard to various path delays, and a power summary will be prepared utilizing the static and dynamic power consumption.

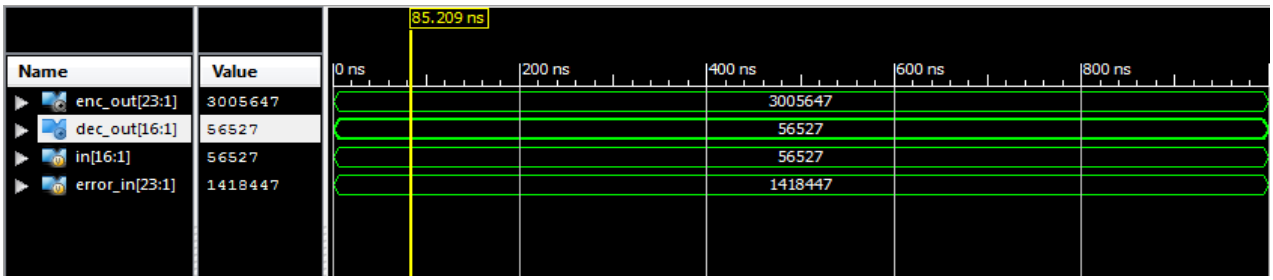


Figure 2. Simulation outcome of MBE-DCC.

Figure 5 represents the simulation outcome of MBE-DCC. Here, data denotes the initial input (in), error_in is the manual error input, and enc_out denotes the encoded operand as a whole. The dec_out is the decoded output data that was error-free and is identical to the input data.

Device Utilization Summary (estimated values)			[-]
Logic Utilization	**Used**	**Available**	**Utilization**
Number of Slice LUTs	55	17600	0%
Number of fully used LUT-FF pairs	0	55	0%
Number of bonded IOBs	77	100	77%

Figure 3. Design summary.

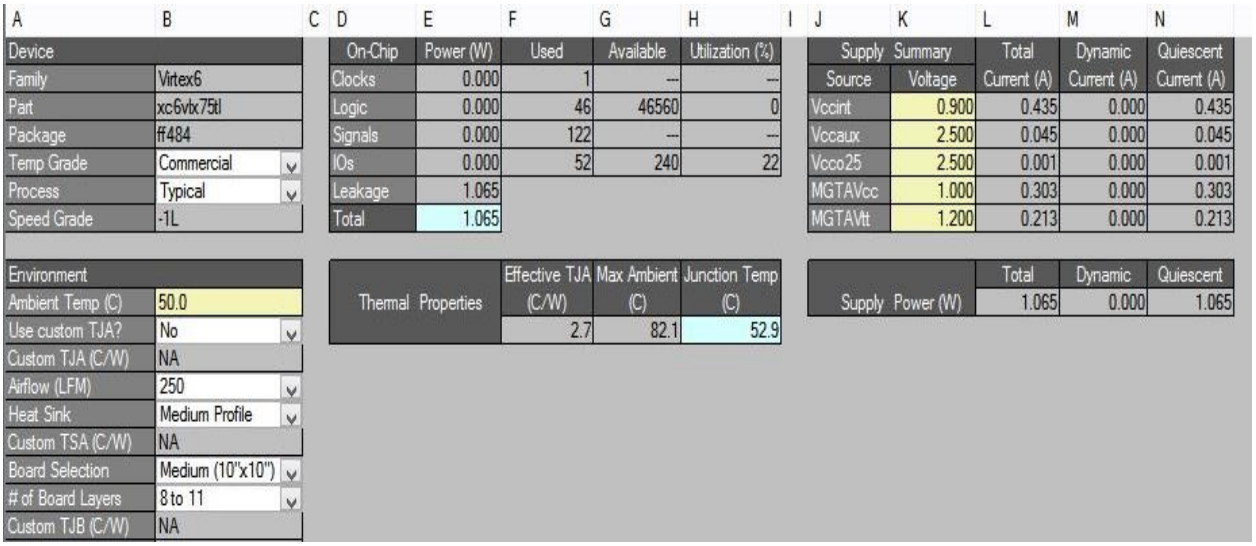
Figure 3 shows the design (area) summary of proposed method. Here, the proposed method utilizes the low area in terms of slice LUTs i.e., 55 out of available 17600.

```
Data Path: in<11> to D1/dec_out_11
                           Gate     Net
  Cell:in->out      fanout   Delay   Delay  Logical Name (Net Name)
  ----------------------------------------  ------------
   IBUF:I->O             10   0.000   0.321  in_11_IBUF (in_11_IBUF)
   LD:D                      -0.035          D1/dec_out_11
  ----------------------------------------
  Total                      0.321ns (0.000ns logic, 0.321ns route)
                                     (0.0% logic, 100.0% route)

==========================================================================
```

Figure 4. Time summary

Figure 4 shows the time summary of proposed method. Here, the proposed method consumed total 0.321ns of time delay, which is entirely route delay.



Device	
Family	Virtex6
Part	xc6vlx75tl
Package	ff484
Temp Grade	Commercial
Process	Typical
Speed Grade	-1L

On-Chip	Power (W)	Used	Available	Utilization (%)
Clocks	0.000	1	--	--
Logic	0.000	46	46560	0
Signals	0.000	122	--	--
IOs	0.000	52	240	22
Leakage	1.065			
Total	1.065			

Supply Summary		Total	Dynamic	Quiescent
Source	Voltage	Current (A)	Current (A)	Current (A)
Vccint	0.900	0.435	0.000	0.435
Vccaux	2.500	0.045	0.000	0.045
Vcco25	2.500	0.001	0.000	0.001
MGTAVcc	1.000	0.303	0.000	0.303
MGTAVtt	1.200	0.213	0.000	0.213

Environment	
Ambient Temp (C)	50.0
Use custom TJA?	No
Custom TJA (C/W)	NA
Airflow (LFM)	250
Heat Sink	Medium Profile
Custom TSA (C/W)	NA
Board Selection	Medium (10"x10")
# of Board Layers	8 to 11
Custom TJB (C/W)	NA

Thermal Properties	Effective TJA (C/W)	Max Ambient (C)	Junction Temp (C)
	2.7	82.1	52.9

Supply Power (W)	Total	Dynamic	Quiescent
	1.065	0.000	1.065

Figure 5. Power summary.

Figure 12 shows the power consumption report of propsoed MBE-DCC. Here, the proposed method consumed power as 1.065 watts. Table 4 compares the performance evaluation of various MBE-DCCapproches. Here, the propsoed MBE-DCCresulted in superior (reduced) performance in terms of LUTs, time-delay, and power consumption as compared to conventional approaches such as LBC [22], STBC [23], and Turbo [24].

Table 4. Performance evaluation.

Metric	**LBC[22]**	**STBC [23]**	**Turbo[24]**	**Proposed MBE-DCC**
LUTs	78	72	64	55
Time delay (ns)	3.28	2.284	1.453	0.321
Power consumption (w)	3.45	2.34	1.79	1.065

5. Conclusion

The MBE-DCC for multiple bits error detection and correction is implemented in this work. The initial implementation of MBE-DCC encoding employs a generator matrix that has both identity bits and parity bits. Then, encoded data that has been corrupted by various noises and errors is transmitted into the space communication channel. Thus, using error location detection, syndrome detection, and error correction modules, the MBE-DCC decoding operation was carried out at the receiver side of space communications. The simulations showed that the proposed MBE-DCC performed better than traditional ECC techniques.This work can be extended with advanced multiple bit error detection and correction codes for real time applications.

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Soft Semi*-δ-continuity in Soft Topological Spaces

Reena C[*]
Yaamini K S[†]

Abstract

In this paper, we introduce the concept of soft semi*δ-continuous functions and soft semi*δ-irresolute functions in soft topological spaces. Also, we investigate its properties and study its relation with other soft continuous functions.

Keywords: soft semi*δ-open, soft semi*δ-closed, soft semi*δ-continuous, soft semi*δ-irresolute.

2010 AMS subject classification: 54C05[‡]

[*]Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, Tamil Nadu, India; reenastephany@gmail.com
[†]Research Scholar, Reg. No. 21212212092002, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, Tamil Nadu, India; ksyaamini@gmail.com
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1. Introduction

The concept of soft set theory was first introduced by Molotov [8] in 1999 to deal with uncertainty. According to him, a soft set over the universe is a parameterized family of subsets of the universe. In 2011, Muhammad Shabir and Munazza Naz [10] introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters. Meanwhile, in 2010, Athar Kharal and B. Ahmad [4] defined the notion of soft mappings on soft classes. Later, in 2013, Aras and Sonmez[2] introduced and studied soft continuous mappings. Further, many authors defined and studied various forms of soft functions. Recently, the authors[12] of this paper introduced a new class of soft sets namely soft semi*δ-open sets and soft semi*δ-closed sets. In this paper, we introduce the concept of soft semi*δ-continuous functions and soft semi*δ-irresolute functions in soft topological spaces. We also investigate its properties and study its relation with other soft continuous functions.

2. Preliminaries

Throughout this work, $(X,\tilde{\tau},E),(Y,\tilde{\sigma},K)$ and $(Z,\tilde{\mu},L)$ are soft topological spaces. $S_t cl(F,A)$, $S_t int(F,A), S_t cl^*(F,A)$ and $S_t int^*(F,A)$ denote soft closure, soft interior, soft generalized closure and soft generalized interior of (F,A) respectively.

Definition 2.1. [10] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non – empty subset of E. A pair (F,A) is called a ***soft set*** over X where F is a mapping given by $F: A \to P(X)$.
The collection of all soft sets over X is called a soft class and denoted by $S_t(X,E)$.

Definition 2.2. [10] Let $\tilde{\tau}$ be the collection of soft set over X. Then $\tilde{\tau}$ is said to be a ***soft topology*** on X if
1) $\tilde{\phi}$, $\tilde{X}$ belongs to $\tilde{\tau}$
2) The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$
3) The intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(X,\tilde{\tau},E)$ is called a soft topological space. The members of $\tilde{\tau}$ are called soft open and its complements are called soft closed.

Definition 2.3. [4] Let $S_t(X,E)$ and $S_t(Y,K)$ be soft classes. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then a mapping $\tilde{f}: S_t(X,E) \to S_t(Y,K)$ is defined as: for a soft set (F,A) in $S_t(X,E)$, $(\tilde{f}(F,A),B), B = p(A) \subseteq K$ is a soft set in $S_t(Y,K)$ given by

$$\tilde{f}(F,A)(\beta) = \begin{cases} u\left(\bigcup_{\alpha \in p^{-1}(\beta)\cap A} F(\alpha)\right), & \text{if } p^{-1}(\beta)\cap A \neq \phi \\ \phi & \text{otherwise} \end{cases}$$

for $\beta \in B \subseteq K$. $(\tilde{f}(F,A),B)$ is called ***soft image*** of a soft set (F,A). If $B = K$, then $(\tilde{f}(F,A),K)$ is written as $\tilde{f}(F,A)$.

Definition 2.4. [4] Let $\tilde{f}: S_t(X,E) \to S_t(Y,K)$ be a mapping from a soft class $S_t(X,E)$ to $S_t(Y,K)$ and (G,C) be a soft set in $S_t(Y,K)$ where $C \subseteq K$. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then $(\tilde{f}^{-1}(G,C),D)$, $D = p^{-1}(C)$ is a soft set in $S_t(X,E)$ defined as

$$\tilde{f}^{-1}(G,C)(\alpha) = \begin{cases} u^{-1}(G(p(\alpha)), & \text{if } p(\alpha) \in C \\ \phi & \text{otherwise} \end{cases}$$

for $\alpha \in D \subseteq E$. $(\tilde{f}^{-1}(G,C),D)$ is called a ***soft inverse image*** of (G,C). We shall write $(\tilde{f}^{-1}(G,C),E)$ as $\tilde{f}^{-1}(G,C)$.

Definition 2.5. Let $(X,\tilde{\tau},E)$ and $(Y,\tilde{\sigma},K)$ be soft topological spaces. A soft function $\tilde{f}: (X,\tilde{\tau},E) \to (Y,\tilde{\sigma},K)$ is soft continuous[2] (respectively soft semi-continuous[5], soft pre-continuous[9], soft α-continuous[6], soft β-continuous [14], soft b-continuous[7], soft regular continuous[3], soft δ-continuous [11], soft generalized continuous[13], soft semi*-continuous, soft pre*-continuous, soft β*-continuous [1]) if $\tilde{f}^{-1}(G,B)$ is soft open (respectively soft semi-open, soft pre-open, soft α-open, soft β-open, soft b-open, soft regular open, soft δ-open, soft generalized open, soft semi*-open, soft pre*-open, soft β*-open) in $(X,\tilde{\tau},E)$ for every soft open set (G,B) in $(Y,\tilde{\sigma},K)$.

Definition 2.6.[12] A subset (F,A) of a soft topological space $(X,\tilde{\tau},E)$ is called ***soft semi*δ-open set*** if there exists a soft δ-open set (O,A) such that $(O,A) \tilde{\subseteq} (F,A) \tilde{\subseteq} S_t cl^*(O,A)$. The complement of soft semi*δ-open set is called soft semi*δ-closed. The class of soft semi*δ-open sets in $(X,\tilde{\tau},E)$ is denoted by $S_t S^* \delta O(X,\tilde{\tau},E)$.

Theorem 2.7.[12] In any soft topological space $(X,\tilde{\tau},E)$,

(i) Every soft δ-open set is soft semi*δ-open.
(ii) Every soft regular open set is soft semi*δ-open.
(iii) Every soft semi*δ-open set is soft semi-open.
(iv) Every soft semi*δ-open set is soft semi*-open.
(v) Every soft semi*δ-open set is soft β-open.
(vi) Every soft semi*δ-open set is soft β^*-open.
(vii) Every soft semi*δ-open set is soft b-open.

Remark 2.8:[12] The above theorem is also true for soft semi*δ-closed sets.

3. Soft semi*δ-continuous functions

Definition 3.1. Let $(X,\tilde{\tau},E)$ and $(Y,\tilde{\sigma},K)$ be soft topological spaces. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then the soft function $\tilde{f}: (X,\tilde{\tau},E) \to (Y,\tilde{\sigma},K)$ is said to be soft semi*δ-

continuous if $\tilde{f}^{-1}(G,B)$ is soft semi*δ-open in $(X,\tilde{\tau},E)$ for every soft open set (G,B) in $(Y,\tilde{\sigma},K)$.

The following soft sets are used in all examples

Let $X=\{a,b\}$ and $E=\{e_1,e_2\}$. Then the soft sets are

$F_1=\{(e_1,\{\phi\}),(e_2,\{\phi\})\}=\tilde{\Phi}$
$F_2=\{(e_1,\{\phi\}),(e_2,\{a\})\}$
$F_3=\{(e_1,\{\phi\}),(e_2,\{b\})\}$
$F_4=\{(e_1,\{\phi\}),(e_2,\{a,b\})\}$
$F_5=\{(e_1,\{a\}),(e_2,\{\phi\})\}$
$F_6=\{(e_1,\{a\}),(e_2,\{a\})\}$
$F_7=\{(e_1,\{a\}),(e_2,\{b\})\}$
$F_8=\{(e_1,\{a\}),(e_2,\{a,b\})\}$
$F_9=\{(e_1,\{b\}),(e_2,\{\phi\})\}$
$F_{10}=\{(e_1,\{b\}),(e_2,\{a\})\}$
$F_{11}=\{(e_1,\{b\}),(e_2,\{b\})\}$
$F_{12}=\{(e_1,\{b\}),(e_2,\{a,b\})\}$
$F_{13}=\{(e_1,\{a,b\}),(e_2,\{\phi\})\}$
$F_{14}=\{(e_1,\{a,b\}),(e_2,\{a\})\}$
$F_{15}=\{(e_1,\{a,b\}),(e_2,\{b\})\}$
$F_{16}=\{(e_1,\{a,b\}),(e_2,\{a,b\})\}=\tilde{X}$

Similarly, let $Y=\{x,y\}$ and $K=\{k_1,k_2\}$. Then the soft sets $G_1,G_2,\dots,G_{16}$ are obtained by replacing $a,b,e_1$and e_2 by $x,y,k_1$and k_2 respectively in the above sets.

Example 3.2. Let $X=\{a,b\},Y=\{x,y\},E=\{e_1,e_2\},K=\{k_1,k_2\}$. Define $u:X\to Y$ and $p:E\to K$ as $u(a)=x,u(b)=y,p(e_1)=k_2,p(e_2)=k_1$. Consider the soft topologies $\tilde{\tau}=\{\tilde{X},\tilde{\phi},F_4,F_5,F_8\}$ and $\tilde{\sigma}=\{\tilde{Y},\tilde{\phi},G_2,G_{13},G_{14}\}$. Let $\tilde{f}:(X,\tilde{\tau},E)\to(Y,\tilde{\sigma},K)$ be a soft mapping. Then, $\tilde{f}^{-1}(G_2)=F_5,\tilde{f}^{-1}(G_{13})=F_4$ and $\tilde{f}^{-1}(G_{14})=F_8$. Here, F_4,F_5,F_8 are soft semi*δ-open. Hence $\tilde{f}$ is soft semi*δ-continuous.

Theorem 3.3. Let $\tilde{f}:(X,\tilde{\tau},E)\to(Y,\tilde{\sigma},K)$ be a soft δ-continuous function. Then $\tilde{f}$ is soft semi*δ-continuous.

Proof: Let (G,B) be a soft open set in Y. Since $\tilde{f}$ is soft δ-continuous, $\tilde{f}^{-1}(G,B)$ is soft δ-open in X. Then by theorem 2.7(i), $\tilde{f}^{-1}(G,B)$ is soft semi*δ-open. Hence $\tilde{f}$ is soft semi*δ-continuous.

Remark 3.4. The converse of the above theorem need not be true.

Example 3.5. Let$X=\{a,b\},Y=\{x,y\},E=\{e_1,e_2\},K=\{k_1,k_2\}$. Define $u:X\to Y$ and $p:E\to K$ as $u(a)=x,u(b)=y,p(e_1)=k_1,p(e_2)=k_2$. Consider the soft topologies $\tilde{\tau}=\{\tilde{X},\tilde{\phi},F_2,F_{11},F_{12}\}$ and $\tilde{\sigma}=\{\tilde{Y},\tilde{\phi},G_6,G_{11}\}$. Let $\tilde{f}:(X,\tilde{\tau},E)\to(Y,\tilde{\sigma},K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_6)=F_6$ is soft semi*δ-open but not soft δ-open, $\tilde{f}$ is soft semi*δ-continuous but not soft δ-continuous.

Theorem 3.6. Let $\tilde{f}:(X,\tilde{\tau},E)\to(Y,\tilde{\sigma},K)$ be a soft regular continuous function. Then $\tilde{f}$ is soft semi*δ-continuous.

Proof. Similar to theorem 3.3, the proof follows from theorem 2.7(ii).

Remark 3.7. The converse of the above theorem need not be true.

Example 3.8. Let $X=\{a,b\},Y=\{x,y\},E=\{e_1,e_2\},K=\{k_1,k_2\}$. Define $u:X\to Y$ and $p:E\to K$ as $u(a)=x,u(b)=y,p(e_1)=k_2,p(e_2)=k_1$. Consider the soft

topologies $\tilde{\tau} = \{\tilde{X}, \widetilde{\phi}, F_5, F_9, F_{13}\}$ and $\tilde{\sigma} = \{\tilde{Y}, \widetilde{\phi}, G_2, G_3, G_4\}$.Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_4) = F_{13}$ is soft semi*δ-open but not soft regular open, $\tilde{f}$ is soft semi*δ-continuous but not soft regular continuous.

Theorem 3.9. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft function.

(i) If $\tilde{f}$ is soft semi*δ-continuous, then $\tilde{f}$ is soft semi-continuous.
(ii) If $\tilde{f}$ is soft semi*δ-continuous, then $\tilde{f}$ is soft semi*-continuous.
(iii) If $\tilde{f}$ is soft semi*δ-continuous, then $\tilde{f}$ is soft β-continuous.
(iv) If $\tilde{f}$ is soft semi*δ-continuous, then $\tilde{f}$ is soft β^*-continuous.
(v) If $\tilde{f}$ is soft semi*δ-continuous, then $\tilde{f}$ is soft b-continuous.

Proof. (i) Let (G, B) be a soft open set in Y. Since $\tilde{f}$ is soft semi*δ-continuous, $\tilde{f}^{-1}(G, B)$ is soft semi*δ-open in X. Then by theorem 2.7(iii), $\tilde{f}^{-1}(G, B)$ is soft semi-open. Hence $\tilde{f}$ is soft semi-continuous.
The other proofs are similar.

Remark 3.10. The converse of each of the statements in above theorem need not be true.

Example 3.11. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = x, u(b) = y, p(e_1) = k_2, p(e_2) = k_1$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \widetilde{\phi}, F_5, F_9, F_{13}\}$and $\tilde{\sigma} = \{\tilde{Y}, \widetilde{\phi}, G_6, G_{14}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_6) = F_6$ and $\tilde{f}^{-1}(G_{14}) = F_8$ are soft semi-open but not soft semi*δ-open, $\tilde{f}$ is soft semi-continuous but not soft semi*δ-continuous.

Example 3.12. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = y, u(b) = x, p(e_1) = k_1, p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \widetilde{\phi}, F_6, F_{14}\}$and $\tilde{\sigma} = \{\tilde{Y}, \widetilde{\phi}, G_{11}, G_{15}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{11}) = F_6$ and $\tilde{f}^{-1}(G_{15}) = F_{14}$ are soft semi*-open but not soft semi*δ-open, $\tilde{f}$ is soft semi*-continuous but not soft semi*δ-continuous.

Example 3.13. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = x, u(b) = y, p(e_1) = k_1, p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \widetilde{\phi}, F_4, F_9, F_{12}\}$ and $\tilde{\sigma} = \{\tilde{Y}, \widetilde{\phi}, G_2, G_6, G_{10}, G_{14}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_2) = F_2, \tilde{f}^{-1}(G_6) = F_6, \tilde{f}^{-1}(G_{10}) = F_{10}$ and $\tilde{f}^{-1}(G_{14}) = F_{14}$ are both soft β-open and soft β^*-open but not soft semi*δ-open, $\tilde{f}$ is both soft β-continuous and soft β^*-continuous but not soft semi*δ-continuous.

Example 3.14. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = y, u(b) = x, p(e_1) = k_1, p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \widetilde{\phi}, F_3, F_9, F_{11}\}$ and $\tilde{\sigma} = \{\tilde{Y}, \widetilde{\phi}, G_2, G_4, G_6, G_8\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_4) = F_4$ and $\tilde{f}^{-1}(G_8) = F_{12}$ are soft b-open but not soft semi*δ-open, $\tilde{f}$ is soft b-continuous but not soft semi*δ-continuous.

Remark 3.15. The concept of soft semi*δ-continuity and soft continuity are independent as shown in the following example.

Example 3.16. Let $X=\{a,b\}, Y=\{x,y\}, E=\{e_1,e_2\}, K=\{k_1,k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a)=x, u(b)=y, p(e_1)=k_2, p(e_2)=k_1$.Consider the soft topologies $\tilde{\tau}=\{\tilde{X}, \widetilde{\phi}, F_4, F_5, F_8\}$ and $\tilde{\sigma}=\{\tilde{Y}, \widetilde{\phi}, G_{15}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{15})=F_{12}$ is soft semi*δ-open but not soft open, $\tilde{f}$ is soft semi*δ-continuous but not soft continuous. Now, consider the soft topology $\tilde{\tau}=\{\tilde{X}, \widetilde{\phi}, F_4, F_8, F_{12}\}$ on X. Here, since $\tilde{f}^{-1}(G_{15})=F_{12}$ is soft open but not soft semi*δ-open, $\tilde{f}$ is soft continuous but not soft semi*δ-continuous.

Remark 3.17. The concept of soft semi*δ-continuity and soft generalized continuity are independent as shown in the following example.

Example 3.18. Let $X=\{a,b\}, Y=\{x,y\}, E=\{e_1,e_2\}, K=\{k_1,k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a)=x, u(b)=y,\ p(e_1)=k_1, p(e_2)=k_2$. Consider the soft topologies $\tilde{\tau}=\{\tilde{X}, \widetilde{\phi}, F_6, F_9, F_{14}\}$ and $\tilde{\sigma}=\{\tilde{Y}, \widetilde{\phi}, G_6, G_{11}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{11})=F_{11}$ is soft semi*δ-open but not soft generalized open, $\tilde{f}$ is soft semi*δ-continuous but not soft generalized continuous. Now, consider the soft topology $\tilde{\sigma}=\{\tilde{Y}, \widetilde{\phi}, G_{13}\}$ on Y. Here, since $\tilde{f}^{-1}(G_{13})=F_{13}$ is soft generalized open but not soft semi*δ-open, $\tilde{f}$ is soft generalized continuous but not soft semi*δ-continuous.

Remark 3.19. The concept of soft semi*δ-continuity and soft pre-continuity are independent as shown in the following example.

Example 3.20. Let $X=\{a,b\}, Y=\{x,y\}, E=\{e_1,e_2\}, K=\{k_1,k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a)=x, u(b)=y,\ p(e_1)=k_2, p(e_2)=k_1$. Consider the soft topologies $\tilde{\tau}=\{\tilde{X}, \widetilde{\phi}, F_2, F_{11}, F_{12}\}$and $\tilde{\sigma}=\{\tilde{Y}, \widetilde{\phi}, G_5, G_6, G_{15}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_6)=F_6$ is soft semi*δ-open but not soft pre-open, $\tilde{f}$ is soft semi*δ-continuous but not soft pre-continuous. Now, consider the mapping $u(a)=y, u(b)=x, p(e_1)=k_2, p(e_2)=k_1$. Here, since $\tilde{f}^{-1}(G_5)=F_3$ and $\tilde{f}^{-1}(G_{15})=F_8$ are soft pre-open but not soft semi*δ-open, $\tilde{f}$ is soft pre-continuous but not soft semi*δ-continuous.

Remark 3.21. The concept of soft semi*δ-continuity and soft pre*-continuity are independent as shown in the following example.

Example 3.22. Let $X=\{a,b\}, Y=\{x,y\}, E=\{e_1,e_2\}, K=\{k_1,k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a)=y, u(b)=x,\ p(e_1)=k_1, p(e_2)=k_2$. Consider the soft topologies $\tilde{\tau}=\{\tilde{X}, \widetilde{\phi}, F_6, F_9, F_{14}\}$ and $\tilde{\sigma}=\{\tilde{Y}, \widetilde{\phi}, G_{11}, G_{12}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$

be a soft mapping. Since $\tilde{f}^{-1}(G_{12}) = F_8$ is soft semi*δ-open but not soft pre*-open, $\tilde{f}$ is soft semi*δ-continuous but not soft pre*-continuous. Now, consider the soft topology $\tilde{\sigma} = \{\tilde{Y}, \tilde{\phi}, G_3, G_9, G_{11}\}$. Here, since $\tilde{f}^{-1}(G_3) = F_2$ and $\tilde{f}^{-1}(G_9) = F_5$ are soft pre*-open but not soft semi*δ-open, $\tilde{f}$ is soft pre*-continuous but not soft semi*δ-continuous.

Remark 3.23. The concept of soft semi*δ-continuity and soft α-continuity are independent as shown in the following example.

Example 3.24. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = y, u(b) = x$, $p(e_1) = k_1, p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, F_4, F_9, F_{12}\}$ and $\tilde{\sigma} = \{\tilde{Y}, \tilde{\phi}, G_4, G_8, G_{12}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{12}) = F_8$ is soft semi*δ-open but not soft α-open, $\tilde{f}$ is soft semi*δ-continuous but not soft α-continuous. Now, consider the soft topology $\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, F_4, F_8, F_{12}\}$ on X. Here, since $\tilde{f}^{-1}(G_4) = F_4$, $\tilde{f}^{-1}(G_8) = F_{12}$ and $\tilde{f}^{-1}(G_{12}) = F_8$ are soft α-open but not soft semi*δ-open, $\tilde{f}$ is soft α-continuous but not soft semi*δ-continuous.

From the above discussions, we have the following diagram

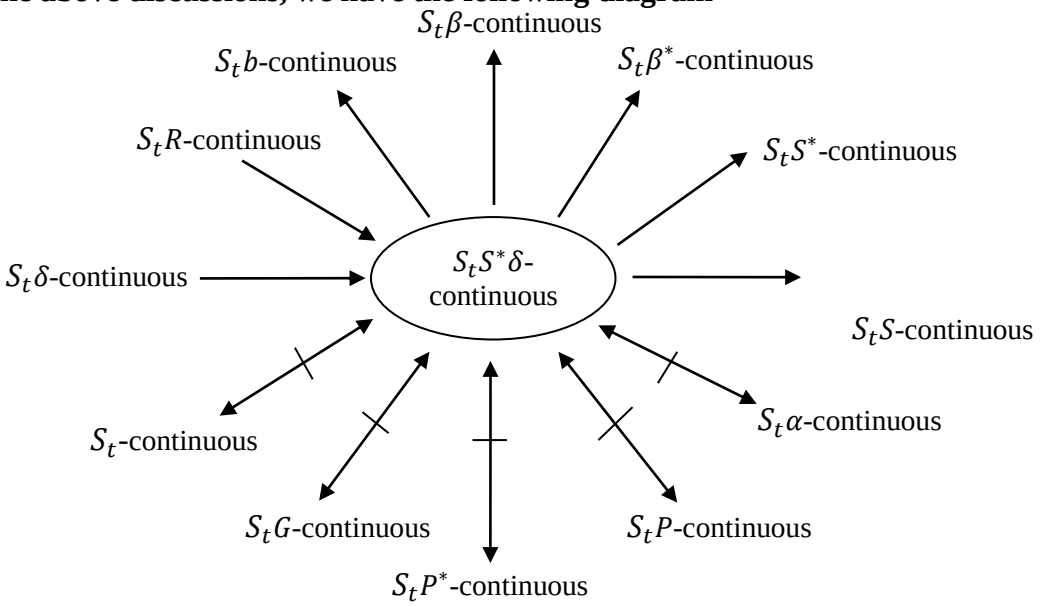

Figure 1

Theorem 3.25. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft function. Then the following statements are equivalent:

(i) $\tilde{f}$ is soft semi*δ-continuous.

(ii) The inverse image of every soft closed set in $(Y, \tilde{\sigma}, K)$ is soft semi*δ-closed in $(X, \tilde{\tau}, E)$.

(iii) $\tilde{f}(S_ts^*\delta cl(F, A)) \cong S_tcl(\tilde{f}(F, A))$ for every soft set (F, A) over X.

(iv) $S_t s^*\delta cl(\tilde{f}^{-1}(G,B)) \cong \tilde{f}^{-1}(S_t cl(G,B))$ for every soft set (G,B) over Y.

Proof.

(i) ⟹(ii) Let $\tilde{f}$ be a soft semi*δ-continuous function and (H,B) be a soft closed set in Y. Then $(H,B)^{\tilde{c}}$ is soft open in Y. Since $\tilde{f}$ is soft semi*δ-continuous, $\tilde{f}^{-1}((H,B)^{\tilde{c}})$ is soft semi*δ-open in X. That is, $(\tilde{f}^{-1}(H,B))^{\tilde{c}}$ is soft semi*δ-open in $(X,\tilde{\tau},E)$. Hence $\tilde{f}^{-1}(H,B)$ is soft semi*δ-closed in $(X,\tilde{\tau},E)$.

(ii)⟹(i) Let (G,B) be soft open in Y. Then $(G,B)^{\tilde{c}}$ be soft closed in Y. By assumption, $\tilde{f}^{-1}((G,B)^{\tilde{c}})$ is soft semi*δ-closed in X. That is, $(\tilde{f}^{-1}(G,B))^{\tilde{c}}$ is soft semi*δ-closed in X. Hence, $\tilde{f}^{-1}(G,B)$ is soft semi*δ-open in X. Therefore, $\tilde{f}$ is soft semi*δ-continuous.

(ii)⟹(iii) Let (F,A) be a soft set over X. Now, $(F,A) \cong \tilde{f}^{-1}(\tilde{f}(F,A))$ implies $(F,A) \cong \tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$. Since $S_t cl\left(\tilde{f}(F,A)\right)$ is soft closed in Y, by assumption $\tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$ is a soft semi*δ-closed set containing (F,A). Also, $S_t s^*\delta cl(F,A)$ is the smallest soft semi*δ-closed set containing (F,A). Hence, $S_t s^*\delta cl(F,A) \cong \tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$. Therefore, $\tilde{f}(S_t s^*\delta cl(F,A)) \cong S_t cl\left(\tilde{f}(F,A)\right)$.

(iii)⟹(ii) Let (H,B) be a soft closed set in Y. Then, by assumption, $\tilde{f}(S_t s^*\delta cl(\tilde{f}^{-1}(H,B))) \cong S_t cl(\tilde{f}(\tilde{f}^{-1}(H,B))) \cong S_t cl(H,B) = (H,B)$ implies $S_t s^*\delta cl\left(\tilde{f}^{-1}(H,B)\right) \cong \tilde{f}^{-1}(H,B)$. Also, $\tilde{f}^{-1}(H,B) \cong S_t s^*\delta cl\left(\tilde{f}^{-1}(H,B)\right)$. Hence, $\tilde{f}^{-1}(H,B)$ is soft semi*δ-closed in X.

(iii)⟹(iv) Let (G,B) be a soft set over Y and let $(F,A) = \tilde{f}^{-1}(G,B)$. By assumption, $\tilde{f}(S_t s^*\delta cl(F,A)) \cong S_t cl\left(\tilde{f}(F,A)\right) = S_t cl(G,B)$. This implies $S_t s^*\delta cl\left(\tilde{f}^{-1}(G,B)\right) \cong \tilde{f}^{-1}(S_t cl(G,B))$.

(iv)⟹(iii) Let $(G,B) = \tilde{f}(F,A)$. Then, by assumption $S_t s^*\delta cl(F,A) = S_t s^*\delta cl\left(\tilde{f}^{-1}(G,B)\right) \cong \tilde{f}^{-1}(S_t cl(G,B)) = \tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$. This implies $\tilde{f}(S_t s^*\delta cl(F,A)) \cong S_t cl\left(\tilde{f}(F,A)\right)$

(iv)⟹(i) Let (G,B) be soft open in Y. Then, $(G,B)^{\tilde{c}}$ is soft closed in Y. By assumption, $\tilde{f}^{-1}((G,B)^{\tilde{c}}) = \tilde{f}^{-1}(S_t cl(G,B)^{\tilde{c}}) \cong S_t s^*\delta cl(\tilde{f}^{-1}(G,B)^{\tilde{c}})$. Also, we know that $\tilde{f}^{-1}((G,B)^{\tilde{c}}) \cong S_t s^*\delta cl(\tilde{f}^{-1}(G,B)^{\tilde{c}})$. Hence $\tilde{f}^{-1}((G,B)^{\tilde{c}}) = S_t s^*\delta cl(\tilde{f}^{-1}(G,B)^{\tilde{c}})$. Therefore, $\tilde{f}^{-1}((G,B)^{\tilde{c}})$ is soft semi*δ-closed. That is, $(\tilde{f}^{-1}(G,B))^{\tilde{c}}$ is soft semi*δ-closed in X. Hence, $\tilde{f}^{-1}(G,B)$ is soft semi*δ-open in X. Therefore, $\tilde{f}$ is soft semi*δ-continuous.

Theorem 3.26. Let $\tilde{f}: (X,\tilde{\tau},E) \to (Y,\tilde{\sigma},K)$ be a soft function. Then $\tilde{f}$ is soft semi*δ-continuous if and only if $\tilde{f}^{-1}(S_t int(G,B)) \cong S_t s^*\delta int\left(\tilde{f}^{-1}(G,B)\right)$ for every soft set (G,B) over Y.

Proof. Let $\tilde{f}$ be a soft semi*δ-continuous function and (G,B) be a soft set over Y. Then $S_t int(G,B)$ is soft open set in Y. By assumption, $\tilde{f}^{-1}(S_t int(G,B))$ is soft

semi*δ-open in X. Now, $\tilde{f}^{-1}\left(S_t int(G,B)\right) \cong \tilde{f}^{-1}(G,B)$ and $S_t s^* \delta int\left(\tilde{f}^{-1}(G,B)\right)$ is the largest soft semi*δ-open set contained in $\tilde{f}^{-1}(G,B)$. Hence $\tilde{f}^{-1}\left(S_t int(G,B)\right) \cong S_t s^* \delta int\left(\tilde{f}^{-1}(G,B)\right)$. Conversely, Let (G,B) be soft open in Y. Then, $\tilde{f}^{-1}(G,B) = \tilde{f}^{-1}\left(S_t int(G,B)\right) \cong S_t s^* \delta int\left(\tilde{f}^{-1}(G,B)\right)$. Also, $S_t s^* \delta int\left(\tilde{f}^{-1}(G,B)\right) \cong \tilde{f}^{-1}(G,B)$. This implies $\tilde{f}^{-1}(G,B)$ is soft semi*δ-open in X. Hence $\tilde{f}$ is soft semi*δ-continuous.

Remark 3.27. The composition of two soft semi*δ-continuous functions need not be soft semi*δ-continuous.

Example 3.28. Let $X = \{a,b,c\}, Y = \{x,y\}, Z = \{m,n\}, E = \{e_1,e_2\}, K = \{k_1,k_2\}$, $L = \{l_1,l_2\}$. Consider the soft topologies $\tilde{\tau} = \left\{\tilde{X}, \widetilde{\phi}, F_5, F_{12}, F_{16}\right\}$, $\tilde{\sigma} = \left\{\tilde{Y}, \widetilde{\phi}, G_2, G_7, G_8\right\}$ and$\tilde{\mu} = \{\tilde{Z}, \widetilde{\phi}, H_{10}\}$ where $F_5 = \{(e_1,\{\phi\}),(e_2,\{a,b\})\}$, $F_{12} = \{(e_1,\{a\}),(e_2,\{c\})\}$, $F_{16} = \{(e_1,\{a\}),(e_2,\{a,b,c\})\}$, $G_2 = \{(k_1,\{\phi\}),(k_2,\{x\})\}$, $G_7 = \{(k_1,\{x\}),(k_2,\{y\})\}$, $G_8 = \{(k_1,\{x\}),(k_2,\{x,y\})\}$ and $H_{10} = \{(l_1,\{n\}),(l_2,\{m\})\}$. Define $u_1: X \longrightarrow Y$ and $p_1: E \longrightarrow K$ as $u_1(a) = u_1(b) = x, u_1(c) = y$, $p_1(e_1) = k_1, p_1(e_2) = k_2$. Then the soft mapping $\tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Y,\tilde{\sigma},K)$ is soft semi*δ-continuous. Also, define $u_2: Y \longrightarrow Z$ and $p_2: K \longrightarrow L$ as $u_2(x) = m, u_2(y) = n, p_2(k_1) = l_1$ and $p_2(k_2) = l_2$. Then the soft mapping $\tilde{g}: (Y,\tilde{\sigma},K) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-continuous. Now, let $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ be the composition of two soft semi*δ-continuous functions. Then $\tilde{g} \circ \tilde{f}$ is not soft semi*δ-continuous since $\left(\tilde{g} \circ \tilde{f}\right)^{-1}(H_{10}) = \tilde{f}^{-1}\left(\tilde{g}^{-1}(H_{10})\right) = \tilde{f}^{-1}(G_{10}) = \{(e_1,\{c\}),(e_2,\{a,b\})\}$ is not soft semi*δ-open.

Theorem 3.29. Let $(X,\tilde{\tau},E)$, $(Y,\tilde{\sigma},K)$ and $(Z,\tilde{\mu},L)$ be soft topological spaces and let $(Y,\tilde{\sigma},K)$ be a space in which every soft semi*δ-open set is soft open. Then the composition $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ of two soft semi*δ-continuous functions $\tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Y,\tilde{\sigma},K)$ and $\tilde{g}: (Y,\tilde{\sigma},K) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-continuous.

Proof. Let (H,C) be any soft open set in Z. Since $\tilde{g}$ is soft semi*δ-continuous, $\tilde{g}^{-1}(H,C)$ is soft semi*δ-open in Y. Then, by assumption $\tilde{g}^{-1}(H,C)$ is soft open in Y. Also, since $\tilde{f}$ is soft semi*δ-continuous, $\tilde{f}^{-1}\left(\tilde{g}^{-1}(H,C)\right) = \left(\tilde{g} \circ \tilde{f}\right)^{-1}(H,C)$ is soft semi*δ-open in X. Hence $\tilde{g} \circ \tilde{f}$ is soft semi*δ-continuous.

Theorem 3.30. Let $\tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Y,\tilde{\sigma},K)$ be a soft semi*δ-continuous function and $\tilde{g}: (Y,\tilde{\sigma},K) \longrightarrow (Z,\tilde{\mu},L)$ be a soft continuous function. Then their composition $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-continuous.

Proof. Let (H, C) be any soft open set in Z. Since $\tilde{g}$ is soft continuous, $\tilde{g}^{-1}(H, C)$ is soft open in Y. Also, since $\tilde{f}$ is soft semi*δ-continuous, $\tilde{f}^{-1}\left(\tilde{g}^{-1}(H, C)\right) = \left(\tilde{g} \circ \tilde{f}\right)^{-1}(H, C)$ is soft semi*δ-open in X. Hence $\tilde{g} \circ \tilde{f}$ is soft semi*δ-continuous.

4. Soft semi*δ-irresolute functions

Definition 4.1. Let $(X, \tilde{\tau}, E)$ and $(Y, \tilde{\sigma}, K)$ be soft topological spaces. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then the soft function $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ is said to be soft semi*δ-irresolute if $\tilde{f}^{-1}(G, B)$ is soft semi*δ-open in $(X, \tilde{\tau}, E)$ for every soft semi*δ-open set (G, B) in $(Y, \tilde{\sigma}, K)$.

Example 4.2. Let $X = \{a, b\}, Y = \{x, y\}, E = \{e_1, e_2\}, K = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: E \to K$ as $u(a) = y, u(b) = x$, $p(e_1) = k_1, p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, F_4, F_9, F_{12}\}$ and $\tilde{\sigma} = \{\tilde{Y}, \tilde{\phi}, G_4, G_5, G_8\}$.Here, $S_t S^* \delta O(X, \tilde{\tau}, E) = \{\tilde{X}, \tilde{\phi}, F_4, F_8, F_9, F_{12}, F_{13}\}$ and $S_t S^* \delta O(Y, \tilde{\sigma}, K) = \{\tilde{Y}, \tilde{\phi}, G_4, G_5, G_8, G_{12}, G_{13}\}$. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft mapping. Then, $\tilde{f}^{-1}(G_4) = F_4$, $\tilde{f}^{-1}(G_5) = F_9$ $\tilde{f}^{-1}(G_8) = F_{12}$, $\tilde{f}^{-1}(G_{12}) = F_8$and $\tilde{f}^{-1}(G_{13}) = F_{13}$. Hence, $\tilde{f}$ is soft semi*δ-irresolute.

Theorem 4.3. Let $(X, \tilde{\tau}, E)$ and $(Y, \tilde{\sigma}, K)$ be soft topological spaces and let $(Y, \tilde{\sigma}, K)$ be a space in which every soft semi*δ-open set is soft open. If $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ is soft semi*δ-continuous, then $\tilde{f}$ is soft semi*δ-irresolute.
Proof. Let (G, B) be soft semi*δ-open in Y. Then, by assumption (G, B) is soft open in Y. Since $\tilde{f}$ is soft semi*δ-continuous, $\tilde{f}^{-1}(G, B)$ is soft semi*δ-open in X. Hence $\tilde{f}$ is soft semi*δ-irresolute.

Theorem 4.4. Let $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ be a soft function. Then the following statements are equivalent:

(i) $\tilde{f}$ is soft semi*δ-irresolute.
(ii) The inverse image of every soft semi*δ-closed set in $(Y, \tilde{\sigma}, K)$ is soft semi*δ-closed in $(X, \tilde{\tau}, E)$.
(iii) $\tilde{f}(S_t s^* \delta cl(F, A)) \cong S_t s^* \delta cl(\tilde{f}(F, A))$ for every soft set (F, A) over X.
(iv) $S_t s^* \delta cl(\tilde{f}^{-1}(G, B)) \cong \tilde{f}^{-1}(S_t s^* \delta cl(G, B))$ for every soft set (G, B) over Y.

Proof. The proof is similar to theorem 3.25

Theorem 4.5. If $\tilde{f}: (X, \tilde{\tau}, E) \to (Y, \tilde{\sigma}, K)$ and $\tilde{g}: (Y, \tilde{\sigma}, K) \to (Z, \tilde{\mu}, L)$ are soft semi*δ-irresolute functions, then their composition $\tilde{g} \circ \tilde{f}: (X, \tilde{\tau}, E) \longrightarrow (Z, \tilde{\mu}, L)$ is also soft semi*δ-irresolute.
Proof. Let (H, C) be soft semi*δ-open in Z. Since $\tilde{g}$ is soft semi*δ-irresolute, $\tilde{g}^{-1}(H, C)$ is soft semi*δ-open in Y. Again, since $\tilde{f}$ is soft semi*δ-irresolute,

$\tilde{f}^{-1}(\tilde{g}^{-1}(H,C)) = \left(\tilde{g} \circ \tilde{f}\right)^{-1}(H,C)$ is soft semi*δ-open in X. Hence $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-irresolute.

Theorem 4.6. If $\tilde{f}: (X,\tilde{\tau},E) \to (Y,\tilde{\sigma},K)$ is soft semi*δ-irresolute and $\tilde{g}: (Y,\tilde{\sigma},K) \to (Z,\tilde{\mu},L)$ is soft semi*δ-continuous, then $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-continuous.

Proof. Let (H,C) be a soft open set in Z. Since $\tilde{g}$ is soft semi*δ-continuous, $\tilde{g}^{-1}(H,C)$ is soft semi*δ-open in Y. Now, since $\tilde{f}$ is soft semi*δ-irresolute, $\tilde{f}^{-1}(\tilde{g}^{-1}(H,C)) = \left(\tilde{g} \circ \tilde{f}\right)^{-1}(H,C)$ is soft semi*δ-open in X. Hence $\tilde{g} \circ \tilde{f}: (X,\tilde{\tau},E) \longrightarrow (Z,\tilde{\mu},L)$ is soft semi*δ-continuous.

5. Conclusions

We have studied the concept of continuity in soft topological spaces by means of soft semi*δ-open sets. We have also introduced the concept of soft semi*δ-irresolute functions. Further, we have compared it with other existing soft functions and we have also investigated the characterization of these functions.

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A New Class of Nano Generalized Closed Sets in Nano Topological Spaces

Anbarasi Rodrigo P[*]
Subithra P[†]

Abstract

In this paper, we introduce a new class of nano generalized closed sets in nano topological spaces namely nano generalized α^*-closed sets. Then we discuss some of its properties and investigate their relation with many other nano closed sets. Also, we define nano generalized α^*-open set and discuss its relation with other open sets. Finally, we define the properties of nano generalized α^*-interior and nano generalized α^*-closure.

Keywords: $\mathbb{N}g\alpha^*$- closed sets, $\mathbb{N}g\alpha^*$- open sets, $\mathbb{N}g\alpha^*$- int, $\mathbb{N}g\alpha^*$- cl.

2010 AMS subject classification: 54A05[‡]

[*]Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli) Thoothukudi-1, TamilNadu, India; anbu.n.u@gmail.com.

[†] Research Scholar, Reg.No. 21212212092003, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli) Thoothukudi-1, TamilNadu, India;p.subithra18@gmail.com.

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1. Introduction

The theory of nano topology proposed by Lellis Thivagar [4] and Carmel Richard is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The elements of a nano topological space are called the nano open set. The author has defined nano topological space in terms of lower and upper approximations and boundary region. He has defined nano closed sets, nano-interior and nano-closure of a set. He also introduced certain weak forms of nano open sets such as nano α-open set, nano semi-open sets and nano pre-open sets. Levine [5] introduced the class of g-closed sets in 1970. K. Bhuvaneswari introduced nano g-closed [1], nano gs-closed [3], nano αg-closed [6], nano gp-closed [2], nano gr-closed [12] sets and studied their properties. Nano g^*p-closed sets was introduced by Rajendran [10] and investigated. The aim of this paper is to introduce and study the properties of nano $g\alpha^*$-closed sets and nano $g\alpha^*$-open sets in nano topological spaces. Finally, we define the properties of nano generalized α^*-interior and nano generalized α^*-closure.

2. Preliminaries

Throughout this paper $(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$ represent nano topological spaces on which no separation axioms are assumed unless and otherwise mentioned. For a subset $\mathbb{S}$ of $(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$, $Ncl(\mathbb{S})$ and $Nint(\mathbb{S})$ denote the nano closure of $\mathbb{S}$ and nano interior of $\mathbb{S}$ respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1. [4] Let $\mathbb{U}$ be a non-empty finite set of objects called the universe and $\mathbb{R}$ be an equivalence relation on $\mathbb{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair ($\mathbb{U}$,$\mathbb{R}$) is said to be the approximation space. Let $\mathbb{X} \subseteq_{\mathbb{N}} \mathbb{U}$. Then

1) The lower approximation of $\mathbb{X}$ with respect to $\mathbb{R}$ is the set of all objects, which can be for certain classified as $\mathbb{X}$ with respect to $\mathbb{R}$ and it is denoted by $\mathbb{L}_{\mathbb{R}}(\mathbb{X})$.

$$\mathbb{L}_{\mathbb{R}}(\mathbb{X}) = \bigcup_{x \in \mathbb{U}} \{\mathbb{R}(x) : \mathbb{R}(x) \subseteq_{\mathbb{N}} \mathbb{X}\}$$

2) The upper approximation of $\mathbb{X}$ with respect to $\mathbb{R}$ is the set of all objects, which can be possibly classified as $\mathbb{X}$ with respect to $\mathbb{R}$ and it is denoted by $\mathbb{U}_{\mathbb{R}}(\mathbb{X})$.

$$\mathbb{U}_{\mathbb{R}}(\mathbb{X}) = \bigcup_{x \in \mathbb{U}} \{\mathbb{R}(x) : \mathbb{R}(x) \cap \mathbb{X} \neq \phi\}$$

3) The boundary region of $\mathbb{X}$ with respect to $\mathbb{R}$ is the set of all objects, which can be classified neither as $\mathbb{X}$ nor as not –$\mathbb{X}$ with respect to $\mathbb{R}$ and it is denoted by $\mathbb{B}_{\mathbb{R}}(\mathbb{X})$.

$$\mathbb{B}_{\mathbb{R}}(\mathbb{X}) = \mathbb{U}_{\mathbb{R}}(\mathbb{X}) - \mathbb{L}_{\mathbb{R}}(\mathbb{X}).$$

Definition 2.2. [4] Let $\mathbb{U}$ be the universe, $\mathbb{R}$ be an equivalence relation on $\mathbb{U}$ and $\tau_{\mathbb{R}}(\mathbb{X}) = \{\mathbb{U}, \phi, \mathbb{U}_{\mathbb{R}}(\mathbb{X}), \mathbb{L}_{\mathbb{R}}(\mathbb{X}), \mathbb{B}_{\mathbb{R}}(\mathbb{X})\}$ where $\mathbb{X} \subseteq_{\mathbb{N}} \mathbb{U}$. Then $\mathbb{R}(\mathbb{X})$ satisfies the following axioms:

1) $\mathbb{U}$ and $\phi \in \tau_{\mathbb{R}}(\mathbb{X})$,

2) The union of the elements of any sub collection of $\tau_{\mathbb{R}}(\mathbb{X})$ is in $\tau_{\mathbb{R}}(\mathbb{X})$,

3) The intersection of the elements of any finite sub collection of $\tau_{\mathbb{R}}(\mathbb{X})$ is in $\tau_{\mathbb{R}}(\mathbb{X})$.
That is, $\tau_{\mathbb{R}}(\mathbb{X})$is a topology on $\mathbb{U}$ called the nano topology on $\mathbb{U}$ with respect to $\mathbb{X}$. We call $\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ as the nano topological space(NTS). The elements of $\tau_{\mathbb{R}}(\mathbb{X})$ are called as nano open sets. The complement of nano-open sets is called nano closed sets.

Definition 2.3. [4] If $\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ is a NTS with respect to $\mathbb{X}$ and if $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{U}$, then

- The nano interior of $\mathbb{S}$ is defined as the union of all nano open subsets of $\mathbb{S}$ and it is denoted by $\mathbb{N}int(\mathbb{S})$. That is, $\mathbb{N}int(\mathbb{S})$ is the largest open subset of $\mathbb{S}$.
- The nano closure of $\mathbb{S}$ is defines as the intersection of all nano closed sets containing $\mathbb{S}$ and it is denoted by $\mathbb{N}cl(\mathbb{S})$. That is, $\mathbb{N}cl(\mathbb{S})$ is the smallest nano closed set containing $\mathbb{S}$.

Definition 2.4. A subset $\mathbb{S}$ of a NTS $\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ is called;

1) Nano pre-open [4] if $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}int(\mathbb{N}cl(\mathbb{S}))$
2) Nano semi-open [4] if $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}cl(\mathbb{N}int(\mathbb{S}))$
3) Nano α-open [4] if $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}int(\mathbb{N}cl(\mathbb{N}int(\mathbb{S})))$
4) Nano β-open [11] if $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}cl(\mathbb{N}int(\mathbb{N}cl(\mathbb{S})))$
5) Nano regular-open [4] if $\mathbb{S} = \mathbb{N}int(\mathbb{N}cl(\mathbb{S}))$

The complements of the above-mentioned sets are called their respective closed sets.

Definition 2.5. A subset $\mathbb{S}$ of a NTS $\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ is called;

1) $\mathbb{N}g$-closed [1] if $\mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
2) $\mathbb{N}gs$-closed [3] if $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
3) $\mathbb{N}\alpha g$-closed [6] if $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
4) $\mathbb{N}gp$-closed [2] if $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
5) $\mathbb{N}g\beta$-closed [7] if $\mathbb{N}\beta cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
6) $\mathbb{N}gr$-closed [12] if $\mathbb{N}rcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$.
7) $\mathbb{N}g^*$-closed [8] if $\mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.
8) $\mathbb{N}g^*s$-closed [9] if $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.
9) $\mathbb{N}g^*p$-closed [10] if $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.
10) $\mathbb{N}g^*r$-closed [13] if $\mathbb{N}rcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$, whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.

Theorem 2.6. [1] Every nano open set is $\mathbb{N}g$-open.

3. Nano generalized α^*-closed sets

Definition 3.1. A Nano generalized α^*(in short, $\mathbb{N}g\alpha^*$) closed set is a subset $\mathbb{S}$ of a $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ if$\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}and\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.

Instance 3.2. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ ={{p}, {r}, {q, s}} and $\mathbb{X}$ = {p, q} $\subseteq_{\mathbb{N}}$ $\mathbb{U}$. Then $\tau_R(\mathbb{X})$ = {$\mathbb{U}$, ϕ, {p}, {q, s}, {p, q, s}}. Here $\mathbb{N}g\alpha^*$-closed = {$\mathbb{U}$, ϕ, {r}, {p, r}, {q, r}, {r, s}, {p, q, r}, {p, r, s}, {q, r, s}}.

Instance 3.3. Let $\mathbb{U}$ = {p, q, r} with $\mathbb{U}/\mathbb{R}$ = {{p}, {p, q}} and $\mathbb{X}$ = {p} $\subseteq_{\mathbb{N}} \mathbb{U}$. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {$\mathbb{U}$, ϕ, {p}, {q}, {p, q}}. Here $\mathbb{N}g\alpha^*$- closed = {ϕ, $\mathbb{U}$, {r}, {p, r}, {q, r}}.

Theorem 3.4. Every nano closed set is $\mathbb{N}g\alpha^*$-closed.
Proof: Let $\mathbb{S}$ be a nano closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Then we have, $\mathbb{N}cl(\mathbb{S}) = \mathbb{S}$. Let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}cl(\mathbb{S}) = \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F} = \mathbb{N}int(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Thus $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Therefore $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed.

Remark 3.5 The invert of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.6. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{r}, {p, q, s}} and $\mathbb{X}$ = {q, s}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {p, q, s}}. Here {$\mathbb{U}$, ϕ, {r}, {p, r}, {q, r}, {r, s}, {p, q, r}, {p, r, s}, {q, r, s}} is $\mathbb{N}g\alpha^*$- closed set but the set is not nano closed.

Theorem 3.7. Every $\mathbb{N}g\alpha^*$- closed set is $\mathbb{N}g$ - closed.
Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$- closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ and let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Also, $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$.
Then, $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{F}$. Thus $\mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g$-closed.

Remark 3.8. The invert of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.9. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{p, q}, {r, s}} and $\mathbb{X}$ = {p, q}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {$\mathbb{U}$, ϕ, {p, q}}. Here {$\mathbb{U}$, ϕ, {r}, {s}, {p, r}, {p, s}, {q, r}, {q, s}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}, {q, r, s}} is $\mathbb{N}g$ – closed set but the set is not $\mathbb{N}g\alpha^*$- closed.

Theorem 3.10. Every $\mathbb{N}g\alpha^*$- closed set is $\mathbb{N}\alpha g$- closed.
Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ and let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Also, $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) = \mathbb{F}$. Thus $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}\alpha g$-closed.

Remark 3.11. The invert of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.12. Let $\mathbb{U}$ = {p, q, r} with $\mathbb{U}/\mathbb{R}$ = {{{r}, {p, q}} and $\mathbb{X}$ = {r}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {r}}. Here {ϕ, {p}, {q}, {p, q}, {p, r}, {q, r}} is $\mathbb{N}\alpha g$ - closed but the set is not $\mathbb{N}g\alpha^*$- closed.

Theorem 3.13. Every $\mathbb{N}g\alpha^*$-closed set is $\mathbb{N}gs$-closed.
Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Then $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) = \mathbb{F}$. Thus $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}gs$-closed.

Remark 3.14. The invert of the former theorem does not holds as witnessed in the succeeding instance.

Instance 3.15. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{p}, {q}, {r, s}} and $\mathbb{X}$ = {q, s}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {q}, {r, s}, {q, r, s}. Here {ϕ, $\mathbb{U}$, {p}, {q}, {r}, {s}, {p, q}, {p, r}, {p, s}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}} is $\mathbb{N}gs$ - closed but it is not $\mathbb{N}g\alpha^*$- closed.

Theorem 3.16. Every $\mathbb{N}g\alpha^*$-closed set is $\mathbb{N}gp$ - closed.
Proof: Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Then $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) = \mathbb{F}$. Thus $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}gp$-closed.

Remark 3.17. The invert of the former theorem does not holds as witnessed in the succeeding instance.

Instance 3.18. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {(p}, {r}, {q, s}} and $\mathbb{X}$ = {r, s}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {r}, {q, s}, {q, r, s}}. Here {ϕ, $\mathbb{U}$, {p}, {q}, {s}, {p, q}, {p, r}, {p, s}, {p, q, r}, {p, q, s} {p, r, s}} is $\mathbb{N}gp$-closed but it is not $\mathbb{N}g\alpha^*$- closed.

Theorem 3.19. Every $\mathbb{N}g\alpha^*$-closed set is $\mathbb{N}g\beta$-closed.
Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ and let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Then $\mathbb{N}\beta cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) = \mathbb{F}$. Thus $\mathbb{N}\beta cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g\beta$-closed.

Remark 3.20. The invert of the former theorem does not hold as witnessed in the succeeding instance.

Instance 3.21. Let $\mathbb{U}$ = {p, q, r} with $\mathbb{U}/\mathbb{R}$ = {{p}, {q, r}} and $\mathbb{X}$ = {p, q}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {p}, {q, r}}. Here {ϕ, $\mathbb{U}$, {p}, {q}, {r}, {p, q}, {p, r}, {q, r}} is $\mathbb{N}g\beta$-closed but it is not $\mathbb{N}g\alpha^*$-closed.

Theorem 3.22. Every$\mathbb{N}g\alpha^*$- closed set is $\mathbb{N}gr$ - closed.

Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. By theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed, $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}rcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) = \mathbb{F}$. Thus $\mathbb{N}rcl(A) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is nano open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}gr$-closed.

Remark 3.23. The transpose of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.24. Let $\mathbb{U}$ = {p, q, r} with $\mathbb{U}/\mathbb{R}$ = {{r}, {p, q}} and $\mathbb{X}$={r}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {r}}. Here {ϕ, {p}, {q}, {p, q}, {p, r}, {q, r}} is $\mathbb{N}gr$-closed which is not $\mathbb{N}g\alpha^*$-closed.

Theorem 3.25. Every $\mathbb{N}g^*$-closed set is $\mathbb{N}g\alpha^*$-closed.
Proof: Let $\mathbb{S}$ be a $\mathbb{N}g^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. Through theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g^*$-closed, $\mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F} = \mathbb{N}int(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Thus $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed.

Remark 3.26. The transpose of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.27. Let $\mathbb{U}$ = {p, q, r} with $\mathbb{U}/\mathbb{R}$ = {{p}, {q, r}} and $\mathbb{X}$ = {p}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {p}}. Here {ϕ, {q}, {r}, {q, r}} is $\mathbb{N}g\alpha^*$-closed which is not $\mathbb{N}g^*$-closed.

Theorem 3.28. Every $\mathbb{N}g\alpha^*$-closed set is $\mathbb{N}g^*s$-closed.
Proof. Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.Thus $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{F}$, we get $\mathbb{N}scl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g^*s$-closed.

Remark 3.29. The transpose of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.30. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{p}, {r}, {q, s}} and $\mathbb{X}$ = {q, r}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {r}, {q, s}, {q, r, s}}. Here {ϕ, $\mathbb{U}$, {p}, {r}, {p, q}, {p, r}, {p, s}, {q, s}, {p, q, r}, {p, q, s}, {p, r, s}} is $\mathbb{N}g^*s$-closed which is not $\mathbb{N}g\alpha^*$-closed.

Theorem 3.31. Every $\mathbb{N}g\alpha^*$-closed set is $\mathbb{N}g^*p$-closed.
Proof: Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$.Thus $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{F}$, we get $\mathbb{N}pcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F}$ whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g^*p$-closed.

Remark 3.32. The polar statement of the preceding theorem does not hold as witnessed in the succeeding instance.

Instance 3.33. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{p, q}, {r, s}} and $\mathbb{X}$ = {q, r, s}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {p, q}, {r, s}}. Here {ϕ, $\mathbb{U}$, {p}, {q}, {r}, {s}, {p, q}, {p, r}, {p, s}, {q, r}, {q, s}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}, {q, r, s}} is $\mathbb{N}g^{*}p$-closed but it is not $\mathbb{N}g\alpha^{*}$-closed.

Theorem 3.34. Every $\mathbb{N}g^{*}r$-closed set is $\mathbb{N}g\alpha^{*}$-closed.
Proof: Let $\mathbb{S}$ be a $\mathbb{N}g^{*}r$-closed set in $NTS\left(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X})\right)$ and let $\mathbb{F}$ be a nano open set in $\mathbb{U}$ such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and$\mathbb{F} = \mathbb{N}int(\mathbb{F})$. Through theorem 2.6, we have $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Since $\mathbb{S}$ is $\mathbb{N}g^{*}r$-closed, $\mathbb{N}rcl(A) \subseteq_{\mathbb{N}} \mathbb{F}$. Then $\mathbb{N}\alpha cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}rcl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{F} = \mathbb{N}int(\mathbb{F}) \subseteq_{\mathbb{N}} \mathbb{N}int^{*}(\mathbb{F})$whenever $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{S}$ is $\mathbb{N}g\alpha^{*}$-closed.

Remark 3.35. The polar statement of the preceding theorem does not hold as witnessed in the succeeding instance.

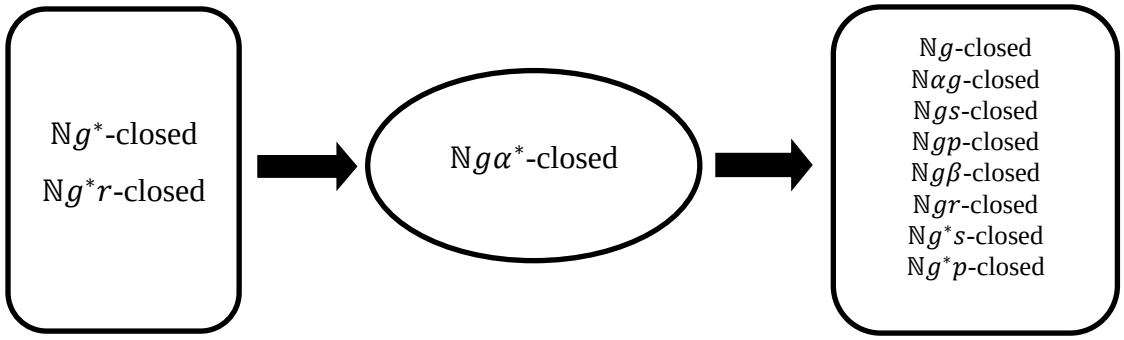
Instance 3.36. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{r}, {q, s}} and $\mathbb{X}$ = {p, r}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {ϕ, $\mathbb{U}$, {r}}. Here {ϕ, $\mathbb{U}$, {p}, {q}, {s}, {p, q}, {p, s}, {q, s}, {p, q, s}} is $\mathbb{N}g\alpha^{*}$-closed but it is not $\mathbb{N}g^{*}r$-closed.

Remark 3.37. The concepts of nano semi closed and $\mathbb{N}g\alpha^{*}$-closed are independent as witnessed in the succeeding instance.

Instance 3.38. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{p}, {r}, {q, s}} and $\mathbb{X}$ = {p, q} $\subseteq \mathbb{U}$. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {$\mathbb{U}$, ϕ, {p}, {q, s}, {p, q, s}}. The set {ϕ, $\mathbb{U}$, {p}, {r}, {p, r}, {q, s}, {q, r, s}} is nano semi closed yet not $\mathbb{N}g\alpha^{*}$-closed. The set {ϕ, $\mathbb{U}$, {r}, {p, r}, {q, r}, {r, s}, {p, q, r}, {p, r, s}, {q, r, s}} is $\mathbb{N}g\alpha^{*}$-closed but the set is not nano semi closed.

Remark 3.39. $\mathbb{N}g\alpha^{*}$-closed set lies between $\mathbb{N}g^{*}$-closed set and $\mathbb{N}g$-closed set. That is, $\mathbb{N}g^{*}$-closed$\subseteq_{\mathbb{N}}$ $\mathbb{N}g\alpha^{*}$-closed$\subseteq_{\mathbb{N}}$ $\mathbb{N}g$-closed.

Remark 3.40. The diagram that follows exhibit the relation between $\mathbb{N}g\alpha^{*}$-closed sets and other closed sets.



Theorem 3.41. If $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-closed sets in $NTS(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$, then $\mathbb{G} \cup \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-closed set.

Proof: Let $\mathbb{G}$and $\mathbb{H}$ be $\mathbb{N}g\alpha^*$-closed sets in a $NTS(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$ and let $\mathbb{F}$ be any $\mathbb{N}g$-open set in $\mathbb{U}$ containing $\mathbb{G}$ and $\mathbb{H}$. Then $\mathbb{G} \cup \mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$. Then $\mathbb{G} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$. Since $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-closed sets, $\mathbb{N}\alpha cl(\mathbb{G}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$and $\mathbb{N}\alpha cl(\mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Now, $\mathbb{N}\alpha cl(\mathbb{G} \cup \mathbb{H}) = \mathbb{N}\alpha cl(\mathbb{G}) \cup \mathbb{N}\alpha cl(H) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Thus, $\mathbb{N}\alpha cl(\mathbb{G} \cup \mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$ whenever $\mathbb{G} \cup \mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{G} \cup \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-closed.

Theorem 3.42. If $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-closed sets in $NTS\,(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$, then $\mathbb{G} \cap \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-closed set.

Proof: Let $\mathbb{G}$ and $\mathbb{H}$ be $\mathbb{N}g\alpha^*$-closed sets in a $NTS(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$ and let $\mathbb{F}$ be a $\mathbb{N}g$-open set in $\mathbb{U}$ such that $\mathbb{G} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$. Then $\mathbb{G} \cap \mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$. Since $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-closed sets, $\mathbb{N}\alpha cl(\mathbb{G}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$and $\mathbb{N}\alpha cl(\mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Now,$\mathbb{N}\alpha cl(\mathbb{G} \cap \mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}\alpha cl(\mathbb{G}) \cap \mathbb{N}\alpha cl(\mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$. Thus $\mathbb{N}\alpha cl(\mathbb{G} \cap \mathbb{H}) \subseteq_{\mathbb{N}} \mathbb{N}int^*(\mathbb{F})$ whenever $\mathbb{G} \cap \mathbb{H} \subseteq_{\mathbb{N}} \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{N}g$-open in $\mathbb{U}$. Hence $\mathbb{G} \cap \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-closed.

Corollary 3.43. If $\mathbb{G}$ is $\mathbb{N}g\alpha^*$-closed and $\mathbb{H}$ is nano closed in $\mathbb{U}$, then $\mathbb{G} \cap \mathbb{H}$ is $\mathbb{N}g\alpha^*$-closed.

Proof: Let $\mathbb{H}$ be nano closed in $\mathbb{U}$. Then by theorem 3.4, $\mathbb{H}$ is $\mathbb{N}g\alpha^*$-closed. $\mathbb{G}$ is also $\mathbb{N}g\alpha^*$-closed. By theorem 3.42, $\mathbb{G} \cap \mathbb{H}$ is $\mathbb{N}g\alpha^*$-closed.

Corollary 3.44. If $\mathbb{G}$ is $\mathbb{N}g\alpha^*$-closed and $\mathbb{H}$ is nano open in $\mathbb{U}$, then $\mathbb{G}\backslash\mathbb{H}$ is $\mathbb{N}g\alpha^*$-closed.

Proof: Let $\mathbb{G}\backslash\mathbb{H} = \mathbb{G} \cap (\mathbb{U}\backslash\mathbb{H})$. Since $\mathbb{H}$ is nano open in $\mathbb{U}$, $\mathbb{U}\backslash\mathbb{H}$ is nano closed in $\mathbb{U}$. Since $\mathbb{G}$ is $\mathbb{N}g\alpha^*$-closed and $\mathbb{U}\backslash\mathbb{H}$ is nano closed in $\mathbb{U}$, by corollary 3.43, $\mathbb{G} \cap (\mathbb{U}\backslash\mathbb{H})$is $\mathbb{N}g\alpha^*$-closed. Hence $\mathbb{G}\backslash\mathbb{H}$ is $\mathbb{N}g\alpha^*$-closed.

4. Nano generalized α^*-open sets

Definition 4.1. A subset $\mathbb{S}$ of a $NTS(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$ is called nano generalized α^*(in short, $\mathbb{N}g\alpha^*$) open set if its complement is $\mathbb{N}g\alpha^*$-closed.

Instance 4.2. Let $\mathbb{U}$ = {p, q, r, s} with $\mathbb{U}/\mathbb{R}$ = {{q}, {r}, {p, s}} and $\mathbb{X}$ = {p, r}. Then $\tau_{\mathbb{R}}(\mathbb{X})$ = {$\mathbb{U}$, ϕ, {r}, {p, s}, {p, r, s}}. Here {ϕ, $\mathbb{U}$, {p}, {r}, {s}, {p, r}, {p, s}, {r, s}, {p, r, s}} is $\mathbb{N}g\alpha^*$-open sets.

Theorem 4.3. Every nano open set is $\mathbb{N}g\alpha^*$-open but the invert may not be true.

Theorem 4.4. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}g$-open but the invert may not be true.

Theorem 4.5. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}\alpha g$-open but the invert may not be true.

Theorem 4.6. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}gs$-open but the invert may not be true.

Theorem 4.7. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}gp$-open but the invert may not be true.

Theorem 4.8. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}g\beta$-open but the invert may not be true.

Theorem 4.9. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}gr$-open but the invert may not be true.

Theorem 4.10. Every $\mathbb{N}g^*$-open set is $\mathbb{N}g\alpha^*$-open but the invert may not be true.

Theorem 4.11. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}g^*s$-open but the invert may not be true.

Theorem 4.12. Every $\mathbb{N}g\alpha^*$-open set is $\mathbb{N}g^*p$-open but the invert may not be true.

Theorem 4.13. Every $\mathbb{N}g^*r$-open set is $\mathbb{N}g\alpha^*$-open but the invert may not be true.

Theorem 4.14. If $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-open sets in NTS, then $\mathbb{G} \cup \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-open set.

Theorem 4.15. If $\mathbb{G}$ and $\mathbb{H}$ are $\mathbb{N}g\alpha^*$-open sets in NTS, then $\mathbb{G} \cap \mathbb{H}$ is a $\mathbb{N}g\alpha^*$-open set.

Corollary 4.16. If $\mathbb{G}$ is $\mathbb{N}g\alpha^*$-open and $\mathbb{H}$ is nano open in $\mathbb{U}$, then $\mathbb{G} \cap \mathbb{H}$ is $\mathbb{N}g\alpha^*$-open.

Corollary 4.17. If $\mathbb{G}$ is $\mathbb{N}g\alpha^*$-open and $\mathbb{H}$ is nano closed in $\mathbb{U}$, then $\mathbb{G}\backslash\mathbb{H}$ is $\mathbb{N}g\alpha^*$-open.

5. $\mathbb{N}g\alpha^*$-interior and $\mathbb{N}g\alpha^*$-closure

Definition 5.1. Let $\mathbb{U}$ be a NTS and let any point $a \in \mathbb{U}$. A subset $\mathbb{S}$ of $\mathbb{U}$ is called the $\mathbb{N}g\alpha^*$-nbhd of a if there exists a $\mathbb{N}g\alpha^*$-open set $\mathbb{K}$ such that $a \in \mathbb{K} \subseteq_{\mathbb{N}} \mathbb{S}$.

Definition 5.2. Let $\mathbb{S}$ be a subset of the $NTS\ (\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$. A point $a \in \mathbb{S}$is called $\mathbb{N}g\alpha^*$-interior point of $\mathbb{S}$ if $\mathbb{S}$ is a$\mathbb{N}g\alpha^*$-nbhd of a. The set which contains all $\mathbb{N}g\alpha^*$-interior points of $\mathbb{S}$ is called $\mathbb{N}g\alpha^*$-interior of $\mathbb{S}$ and symbolized as $\mathbb{N}g\alpha^*$-int($\mathbb{S}$).

Definition 5.3. Let $\mathbb{S}$ be a subset of the $NTS\ (\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$. Then the intersection of all $\mathbb{N}g\alpha^*$-closed sets containing $\mathbb{S}$ is called $\mathbb{N}g\alpha^*$-closure of $\mathbb{S}$. That is, $\mathbb{N}g\alpha^* - cl(\mathbb{S}) = \cap \{\mathbb{R}: \mathbb{R}\ is\ \mathbb{N}g\alpha^* - closed\ sets\ and\ \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R}\}$.

Theorem 5.4. If $\mathbb{S}$ be a subset of $\mathbb{U}$, then
$\mathbb{N}g\alpha^* - int(\mathbb{S}) = \cup \{\mathbb{R}: \mathbb{R}\ is\ \mathbb{N}g\alpha^* - open\ set\ and\ \mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}\}$.
Proof: Let $\mathbb{S}$ be a subset of $\mathbb{U}$.
$x \in \mathbb{N}g\alpha^* - int(\mathbb{S})$
$\Leftrightarrow x$ is a $\mathbb{N}g\alpha^*$-interior point of $\mathbb{S}$
$\Leftrightarrow \mathbb{S}$ is a $\mathbb{N}g\alpha^*$-nbhd of the point x
$\Leftrightarrow$There exists $\mathbb{N}g\alpha^*$-open set $\mathbb{R}$ such that $x \in \mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}$
$\Leftrightarrow x \in \cup \{\mathbb{R}: \mathbb{R}\ is\ \mathbb{N}g\alpha^* - open\ set\ and\ \mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}\}$
Hence $Ng\alpha^* - int(\mathbb{S}) = \cup \{\mathbb{R}: \mathbb{R}\ is\ \mathbb{N}g\alpha^* - open\ set\ and\ \mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}\}$.

Theorem 5.5. Let $\mathbb{R}$ and $\mathbb{S}$ be subsets of $\mathbb{U}$. Then
a) $\mathbb{N}g\alpha^* - int(\mathbb{U}) = \mathbb{U}$ and $\mathbb{N}g\alpha^* - int(\phi) = \phi$
b) $\mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{S}$
c) If $\mathbb{R}$ contains any $\mathbb{N}g\alpha^*$-open set $\mathbb{S}$, then $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R})$
d) If $\mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}$, then $\mathbb{N}g\alpha^* - int(\mathbb{R}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{S})$
Proof: a) Since $\mathbb{U}$ and ϕ are $\mathbb{N}g\alpha^*$-open sets, by theorem 5.4, $\mathbb{N}g\alpha^* - int(\mathbb{U}) = \cup \{\mathbb{R}: \mathbb{R}\ is\ \mathbb{N}g\alpha^* - open\ set\ and\ \mathbb{R} \subseteq_{\mathbb{N}} \mathbb{U}\}$.
$\Rightarrow Ng\alpha^* - int(\mathbb{U}) = \cup \{\mathbb{S}: \mathbb{S}\ is\ a\ \mathbb{N}g\alpha^* - open\ set\}$
$\Rightarrow Ng\alpha^* - int(\mathbb{U}) = \mathbb{U}$
Since ϕ is the only $\mathbb{N}g\alpha^*$-open set contained in ϕ, $\mathbb{N}g\alpha^* - int(\phi) = \phi$.
b) Let $x \in \mathbb{N}g\alpha^* - int(\mathbb{S})$
$\Rightarrow x$ is a $\mathbb{N}g\alpha^*$-interior point of $\mathbb{S}$.
$\Rightarrow \mathbb{S}$ is a $\mathbb{N}g\alpha^*$-nbhd of x.
$\Rightarrow x \in \mathbb{S}$
Thus $\mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{S}$.
c) Let $\mathbb{S}$ be any $\mathbb{N}g\alpha^*$-open set such that $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R}$ and let $x \in \mathbb{S}$.
Since $\mathbb{S}$ is a $\mathbb{N}g\alpha^*$-open set contained in $\mathbb{R}$, x is a $\mathbb{N}g\alpha^*$-interior point of $\mathbb{R}$.
That is, $x \in \mathbb{N}g\alpha^* - int(\mathbb{R})$
Hence $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R})$.
d) Let $\mathbb{R}$ and $\mathbb{S}$ be subsets of $\mathbb{U}$ such that $\mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}$.
Let $x \in \mathbb{N}g\alpha^* - int(\mathbb{R})$.
Then x is a$\mathbb{N}g\alpha^*$-interior point of $\mathbb{R}$ and so $\mathbb{R}$ is a$\mathbb{N}g\alpha^*$-nbhd of x contained in $\mathbb{S}$.
Therefore x is a$\mathbb{N}g\alpha^*$-interior point of $\mathbb{S}$.
Thus $x \in \mathbb{N}g\alpha^* - int(\mathbb{S})$.
Hence $\mathbb{N}g\alpha^* - int(\mathbb{R}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{S})$.

Theorem 5.6. If $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-open then $\mathbb{N}g\alpha^* - int(\mathbb{S}) = \mathbb{S}$.
Proof: Let $\mathbb{S}$ be a $\mathbb{N}g\alpha^*$-open in $\mathbb{U}$.
We know that $\mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{S}$.
Also, $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-open set contained in $\mathbb{S}$.
By theorem 5.5 c), $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{S})$.
Hence $\mathbb{N}g\alpha^* - int(\mathbb{S}) = \mathbb{S}$.

Theorem 5.7. If $\mathbb{R}$ and $\mathbb{S}$ are subsets of $\mathbb{U}$, then
$\mathbb{N}g\alpha^* - int(\mathbb{R}) \cup \mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R} \cup \mathbb{S})$.
Proof: We know that $\mathbb{R} \subseteq_{\mathbb{N}} \mathbb{R} \cup \mathbb{S}$ and $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R} \cup \mathbb{S}$.
Then by theorem 5.5 d), $\mathbb{N}g\alpha^* - int(\mathbb{R}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R} \cup \mathbb{S})$ and
$\mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R} \cup \mathbb{S})$.
Thus $\mathbb{N}g\alpha^* - int(\mathbb{R}) \cup \mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R} \cup \mathbb{S})$.

Theorem 5.8. If $\mathbb{R}$ and $\mathbb{S}$ subsets of $\mathbb{U}$, then
$\mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S}) = \mathbb{N}g\alpha^* - int(\mathbb{R}) \cap \mathbb{N}g\alpha^* - int(\mathbb{S})$.
Proof: We know that $\mathbb{R} \cap \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R}$ and $\mathbb{R} \cap \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{S}$.
By theorem 5.5 d), $\mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R})$ and
$\mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{S})$.

Then $\mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R}) \cap \mathbb{N}g\alpha^* - int(\mathbb{S})$(1)
Next, let $x \in \mathbb{N}g\alpha^* - int(\mathbb{R}) \cap \mathbb{N}g\alpha^* - int(\mathbb{S})$
Then $x \in \mathbb{N}g\alpha^* - int(\mathbb{R})$ and $x \in \mathbb{N}g\alpha^* - int(\mathbb{S})$
Hence x is a $\mathbb{N}g\alpha^*$-interior point of both sets $\mathbb{R}$ and $\mathbb{S}$.
It follows that $\mathbb{R}$ and $\mathbb{S}$ is a $\mathbb{N}g\alpha^*$-nbhd of x.
Thus $x \in \mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S})$
Hence $\mathbb{N}g\alpha^* - int(\mathbb{R}) \cap \mathbb{N}g\alpha^* - int(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S})$(2)
From (1) and (2), we get $\mathbb{N}g\alpha^* - int(\mathbb{R} \cap \mathbb{S}) = \mathbb{N}g\alpha^* - int(\mathbb{R}) \cap \mathbb{N}g\alpha^* - int(\mathbb{S})$.

Theorem 5.9. Let $\mathbb{S}$ be a subset of $\mathbb{U}$, then
a) $\mathbb{N}g\alpha^* - cl(\mathbb{S})$ is $\mathbb{N}g\alpha^*$-closed in $\mathbb{U}$ and $\mathbb{N}g\alpha^* - cl(\mathbb{S})$ is the smallest $\mathbb{N}g\alpha^*$-closed set in $\mathbb{U}$ containing $\mathbb{S}$.
b) $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed if and only if $\mathbb{N}g\alpha^* - cl(\mathbb{S}) = \mathbb{S}$.
Proof: a) Since the intersection of all $\mathbb{N}g\alpha^*$-closed subsets of $\mathbb{U}$ containing $\mathbb{S}$ is $\mathbb{N}g\alpha^* - cl(\mathbb{S})$, $\mathbb{N}g\alpha^* - cl(\mathbb{S})$ is $\mathbb{N}g\alpha^*$-closed. $\mathbb{N}g\alpha^* - cl(\mathbb{S})$ is contained in every $\mathbb{N}g\alpha^*$-closed set containing $\mathbb{S}$. Hence, the smallest $\mathbb{N}g\alpha^*$-closed set in $\mathbb{U}$ containing $\mathbb{S}$ is $\mathbb{N}g\alpha^* - cl(\mathbb{S})$.
b) Suppose $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed. By the definition of $\mathbb{N}g\alpha^*$-closure, $\mathbb{N}g\alpha^* - cl(\mathbb{S}) = \mathbb{S}$
Conversely, Suppose $\mathbb{N}g\alpha^* - cl(\mathbb{S}) = \mathbb{S}$. By theorem 3.42, $\mathbb{N}g\alpha^* - cl(\mathbb{S})$is the $\mathbb{N}g\alpha^*$-closed set. Therefore, $\mathbb{S}$ is $\mathbb{N}g\alpha^*$-closed.

Theorem 5.10 Let $\mathbb{R}$ and $\mathbb{S}$ be subsets of $\mathbb{U}$, then
a) $\mathbb{N}g\alpha^* - cl(\phi) = \phi$
b) $\mathbb{N}g\alpha^* - cl(\mathbb{U}) = \mathbb{U}$
c) $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{S})$
d) If $\mathbb{R} \subseteq_{\mathbb{N}} \mathbb{S}$ then $\mathbb{N}g\alpha^* - cl(\mathbb{R}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{S})$
e) $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S}) = \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S})$
f) $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cap \mathbb{N}g\alpha^* - cl(\mathbb{S})$
Proof: a), b), c), d) follows from the definition of $\mathbb{N}g\alpha^*$-closure.
e) We know that $\mathbb{R} \subseteq_{\mathbb{N}} \mathbb{R} \cup \mathbb{S}$ and $\mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R} \cup \mathbb{S}$.
By d), $\mathbb{N}g\alpha^* - cl(\mathbb{R}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S})$,$\mathbb{N}g\alpha^* - cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S})$.
Then $\mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S})$(1)
Next, we prove $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S})$
Let $x \notin \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S})$
$\Rightarrow x \notin \mathbb{N}g\alpha^* - cl(\mathbb{R})$ and $x \notin \mathbb{N}g\alpha^* - cl(\mathbb{S})$
By definition of $\mathbb{N}g\alpha^* - cl$, $\mathbb{N}g\alpha^* - cl(\mathbb{R}) = \cap \{F_i : \mathbb{R} \subseteq_{\mathbb{N}} F_i, F_i \; is \; \mathbb{N}g\alpha^* - closed\}$ and $\mathbb{N}g\alpha^* - cl(\mathbb{S}) = \cap \{F_i : \mathbb{S} \subseteq_{\mathbb{N}} F_i, F_i \; is \; \mathbb{N}g\alpha^* - closed\}$.
Then $x \notin F_i$ for some i.
Since $\mathbb{R} \subseteq_{\mathbb{N}} F_i$ and $\mathbb{S} \subseteq_{\mathbb{N}} F_i$, $\mathbb{R} \cup \mathbb{S} \subseteq F_i$.
Therefore $x \notin \mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S})$
Hence $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S})$(2)
From (1) and (2) we have, $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cup \mathbb{S}) = \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cup \mathbb{N}g\alpha^* - cl(\mathbb{S})$.
f) We know that $\mathbb{R} \cap \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{R}$ and $\mathbb{R} \cap \mathbb{S} \subseteq_{\mathbb{N}} \mathbb{S}$.

By d), $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R})$ and $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{S})$
Then $\mathbb{N}g\alpha^* - cl(\mathbb{R} \cap \mathbb{S}) \subseteq_{\mathbb{N}} \mathbb{N}g\alpha^* - cl(\mathbb{R}) \cap \mathbb{N}g\alpha^* - cl(\mathbb{S})$.

6. Conclusions

In this paper, we have introduced nano generalized α^*-closed sets and discussed some of its properties. Then we investigated its relation with many other nano closed sets. Further nano generalized α^*-open sets are defined and its properties and relations with other nano open sets are studied. Consequently, nano generalized α^*-interior and nano generalized α^*-closure are introduced and discussed.

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Soft Pre*-Generalized Continuous Functions in Soft Topological Spaces

C. Reena[1]
M. Karthika[2]

Abstract

The aim of this paper is to define a new class of generalized continuous functions called soft pre^*-generalized continuous functions and soft pre^*-generalized irresolute functions in soft topological spaces. We discuss several characterizations of soft pre^*-generalized continuous and irresolute functions and also investigate their relationship with other soft continuous functions.

Keywords: soft pre^*-generalized continuous functions and soft pre^*-generalized irresolute functions.

2010 AMS subject classification: 54A05, 54A10[3]

[1]Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, TamilNadu, India. Mail Id: reenastephany@gmail.com
[2]Research Scholar, Reg.No. 21112212092004, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, TamilNadu, India. Mail Id: karthikamarimuthu97@gmail.com
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1. Introduction

In 1999 Molostsov [6] initiated the study of soft set theory as a new mathematical tool to deal with uncertainties. Muhammad Shabir [7] and Munazza Naz (2011) introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters. Athar Kharal [2] and Ahmad introduced the concept of soft mapping. Aras [1] and Sonmez discussed the properties of soft continuous mappings. Akdag M [5] and Ozkan introduced soft pre-continuity in soft topological space. The authors [8] of this paper introduced a new glass of generalized closed set called soft pre^*-generalized closed sets in soft topological spaces. In this paper, we introduce soft pre^*-generalized continuous and soft pre^*-generalized irresolute functions in soft topological spaces. We investigate its fundamental properties and find its relation with other soft continuous functions.

2. Preliminaries

Throughout this paper, $(\mathrm{X}, \tilde{\tau}, \mathrm{A})$, $(\mathrm{Y}, \tilde{\sigma}, \mathrm{B})$ and $(\mathrm{Z}, \tilde{\mu}, \mathrm{C})$ are soft topological spaces. Let (F, A) be a subset of a soft topological space. Then$\mathrm{S_t cl}(\mathrm{F}, \mathrm{A})$, $\mathrm{S_t int}(\mathrm{F}, \mathrm{A})$, $\mathrm{S_t cl}^*(\mathrm{F}, \mathrm{A})$ and $\mathrm{S_t int}^*(\mathrm{F}, \mathrm{A})$ denote the soft closure, soft interior, soft generalized closure and soft generalized interior respectively.

Definition 2.1: [6] Let X be an initial universe, E be a set of parameters, $P(X)$ denote the power set of X and$A$ be a non-empty set of E. A pair (F, A) is called **soft set** over$X$, where F is a mapping given by $F: A \to P(X)$.

Definition 2.2: [7] Let $\tilde{\tau}$ be a collection of soft sets over X. Then $\tilde{\tau}$ is called a **soft topology** on X if

i. $\tilde{\emptyset}$,$\tilde{X}$ belons to $\tilde{\tau}$.
ii. The union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.
iii. The intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(X, \tilde{\tau}, A)$ is called soft topological space over X. The members of $\tilde{\tau}$ are called **soft open** and their complements are called **soft closed.**

Definition 2.3. A function $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ is said to be **soft continuous [1]** (respectively **soft semi continuous [4]**, **soft $semi^*$continuous**, **soft α continuous [5], soft regular continuous [3]**, **soft generalized continuous [9]**, **soft generalized α continuous** and **soft generalized pre continuous**) if inverse image of every soft closed set in $(Y, \tilde{\sigma}, B)$ is soft closed (respectively soft semi-closed, soft $semi^*$-closed, soft α closed, soft regular closed, soft generalized closed, soft generalized α closed and soft generalized pre closed) in $(X, \tilde{\tau}, A)$.

Definition 2.4. [8] A subset (F, A) of a soft topological space $(X, \tilde{\tau}, A)$ is said to be soft **pre^*- generalized closed** if $S_t pcl(F, A) \cong (U, A)$ whenever $(F, A) \cong (U, A)$ and (U, A)

is soft pre^*-open. The complement of soft pre^*-generalized closed is called **soft pre^*-generalized open**.

Theorem 2.5. [8] In any topological space $(X, \tilde{\tau}, A)$,

i. Every soft closed set is soft pre^*-generalized closed.
ii. Every soft regular-closed set is soft pre^*-generalized closed.
iii. Every soft α-closed set is soft pre^*-generalized closed.
iv. Every soft generalized α-closed set is soft pre^*-generalized closed.
v. Every soft generalized pre-closed set is soft pre^*-generalized closed.

Remark 2.6: The above theorem is true for soft pre^*-generalized open.

3. Soft PRE^*-Generalized Continuous Functions

Definition 3.1. Let $(X, \tilde{\tau}, A)$ and $(Y, \tilde{\sigma}, B)$ be soft topological spaces. Let $u: X \to Y$and $p: A \to B$ be mappings. The function $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ is said tobe soft pre^*-generalized continuous function if the inverse image of every soft closed set in $(Y, \tilde{\sigma}, B)$ is soft pre^*-generalized closed in $(X, \tilde{\tau}, A)$.

The following soft sets are used in all the examples:

Let $X = \{a, b\}$ and $A = \{e_1, e_2\}$. Then the soft sets are

$F_1 = \{(e_1, \{\emptyset\}), (e_2, \{\emptyset\})\} = \tilde{\emptyset}$ $F_2 = \{(e_1, \{\emptyset\}), (e_2, \{a\})\}$
$F_3 = \{(e_1, \{\emptyset\}), (e_2, \{b\})\}$ $F_4 = \{(e_1, \{\emptyset\}), (e_2, \{a, b\})\}$
$F_5 = \{(e_1, \{a\}), (e_2, \{\emptyset\})\}$ $F_6 = \{(e_1, \{a\}), (e_2, \{a\})\}$
$F_7 = \{(e_1, \{a\}), (e_2, \{b\})\}$ $F_8 = \{(e_1, \{a\}), (e_2, \{a, b\})\}$
$F_9 = \{(e_1, \{b\}), (e_2, \{\emptyset\})\}$ $F_{10} = \{(e_1, \{b\}), (e_2, \{a\})\}$
$F_{11} = \{(e_1, \{b\}), (e_2, \{b\})\}$ $F_{12} = \{(e_1, \{b\}), (e_2, \{a, b\})\}$
$F_{13} = \{(e_1, \{a, b\}), (e_2, \{\emptyset\})\}$ $F_{14} = \{(e_1, \{a, b\}), (e_2, \{a\})\}$
$F_{15} = \{(e_1, \{a, b\}), (e_2, \{b\})\}$ $F_{16} = \{(e_1, \{a, b\}), (e_2, \{a, b\})\} = \tilde{X}$

Similarly, let $Y = \{x, y\}$ and $B = \{k_1, k_2\}$ then the soft sets $G_1, G_2, \ldots, G_{16}$are obtained by replacing a, b, e_1 and e_2 by x, y, k_1 and k_2 respectively in the above sets.

Example 3.2. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, p $(e_1) = k_1$ and $p(e_2) = k_2$. Consider the soft topologies$\tilde{\tau}^{\tilde{c}} = \{F_1, F_2, F_8, F_{10}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_4, G_8, G_{12}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_4) = F_4$ and $\tilde{f}^{-1}(G_8) = F_8$ and $\tilde{f}^{-1}(G_{12}) = F_{12}$ are soft pre^*- generalized closed, $\tilde{f}$ is soft pre^*-generalized continuous.

Theorem 3.3. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft continuous function. Then $\tilde{f}$ is soft pre^*-generalized continuous.

Proof: Let (G, B) be a soft closed set in Y. Since $\tilde{f}$ is soft continuous, $\tilde{f}^{-1}(G, B)$ is soft closed. Then by theorem 2.5(i), $\tilde{f}^{-1}(G, B)$ is soft pre^*-generalized closed. Hence $\tilde{f}$ is soft pre^*-generalized continuous.

Remark 3.4. The converse of the above theorem need not be true as shown in the following example.

Example 3.5. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, p $(e_1) = k_1$ and $p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_5, F_8, F_{13}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_9, G_{12}, G_{13}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_9) = F_5$, $\tilde{f}^{-1}(G_{12}) = F_8$ and $\tilde{f}^{-1}(G_{13}) = F_{13}$ are soft pre^*-generalized closed but not soft closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft continuous.

Theorem 3.6. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft function.

i. If $\tilde{f}$ is soft regular continuous, then $\tilde{f}$ is soft pre^*-generalized continuous.
ii. If $\tilde{f}$ is soft α continuous, then $\tilde{f}$ is soft pre^*-generalized continuous.
iii. If $\tilde{f}$ is soft generalized α continuous, then $\tilde{f}$ is soft pre^*-generalized continuous.
iv. If $\tilde{f}$ is soft generalized pre continuous, then $\tilde{f}$ is soft pre^*-generalized continuous.

Proof: The proofs are similar to theorem 3.3.

Remark 3.7. The converse of each of the statements in above theorem need not be true.

Example 3. 8. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as$u(a) = x$, $u(b) = y$, $p(e_1) = k_2$ and $p(e_2) = k_1$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_5, F_9, F_{13}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_3, G_8, G_{11}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_3) = F_9$, $\tilde{f}^{-1}(G_8) = F_{14}$ and $\tilde{f}^{-1}(G_{11}) = F_{11}$are soft pre^*-generalized closed but not soft regular closed, $\tilde{f}$ is soft pre^*-generalized continuous but not regular continuous.

Example 3.9. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as$u(a) = x$, $u(b) = y$, $p(e_1) = k_2$ and $p(e_2) = k_1$. Consider the soft topologies$\tilde{\tau}^{\tilde{c}} = \{F_1, F_6, F_8, F_{14}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_3, G_{11}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_3) = F_9$ and $\tilde{f}^{-1}(G_{11}) = F_{11}$ are soft pre^*-generalized closed but not softα-closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft α continuous.

Example 3.10. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, $p(e_1) = k_2$ and $p(e_2) = k_1$. Consider the soft topologies$\tilde{\tau}^{\tilde{c}} = \{F_1, F_9, F_{10}, F_{12}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_5, G_9, G_{13}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_5) = F_2$, $\tilde{f}^{-1}(G_9) = F_3$ and $\tilde{f}^{-1}(G_{13}) = F_4$are soft pre^*-generalized closed but not soft generalized α-closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft generalized α continuous.

Example 3.11. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = y$, $u(b) = x$, $p(e_1) = k_1$ and $p(e_2) = k_2$. Consider the soft topologies$\tilde{\tau}^{\tilde{c}} = \{F_1, F_4, F_8, F_{12}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_{14}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{14}) = F_{15}$is soft pre^*-generalized closed but not soft generalized pre-closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft generalized pre continuous.

Remark 3.12. The concept of soft pre^*-generalized continuity and soft generalized continuity are independent as shown in the following example.

Example 3.13. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, $p(e_1) = k_1$, $p(e_2) = k_2$. Consider the soft topologies$\tilde{\tau}^{\tilde{c}} = \{F_1, F_2, F_3, F_4, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_5, G_6, G_8, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_5) = F_5$ and $\tilde{f}^{-1}(G_6) = F_6$are soft pre^*-generalized closed but not soft generalized closed,$\tilde{f}$ is soft pre^*-generalized continuous but not soft generalized continuous.

Example 3.14. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, $p(e_1) = k_2$, $p(e_2) = k_1$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_3, F_{11}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_{14}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{14}) = F_8$is soft generalized closed but not soft pre^*-generalized closed, $\tilde{f}$ is soft generalized continuous but not soft pre^*-generalized continuous.

Remark 3.15. The concept of soft pre^*-generalized continuity and soft $semi^*$-continuity are independent as shown in the following example.

Example 3.16. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = x$, $u(b) = y$, $p(e_1) = k_2$, $p(e_2) = k_1$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_5, F_9, F_{13}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_4, G_8, G_{12}, G_{16}\}$.Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_8) = F_{14}$ and $\tilde{f}^{-1}(G_{12}) = F_{15}$ are soft pre^*-generalized closed but not soft $semi^*$-closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft $semi^*$-continuous.

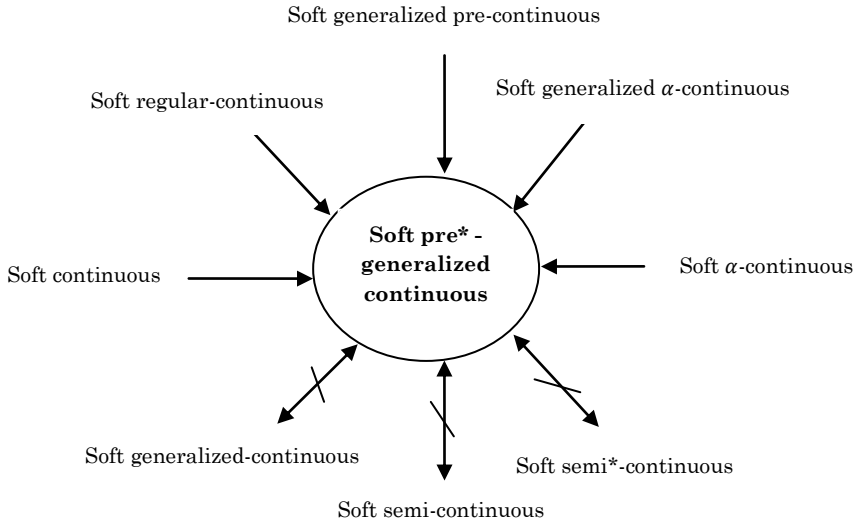
Example 3.17. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = y$, $u(b) = x$, $p(e_1) = k_1$, $p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_5, F_8, F_{13}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_4, G_9, G_{12}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_4) = F_4$ is soft $semi^*$-generalized closed but not soft pre^*-generalizedclosed, $\tilde{f}$ is soft $semi^*$-generalized continuous but not soft pre^*-generalized continuous.

Remark 3.18. The concept of soft pre^*-generalized continuity and soft semi continuity are independent as shown in the following example.

Example 3.19. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = y$, $u(b) = x$, $p(e_1) = k_1$, $p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_3, F_8, F_{11}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_2, G_9, G_{10}, G_{16}\}$. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_9) = F_5$ and $\tilde{f}^{-1}(G_{10}) = F_7$ are soft pre^*-generalized closed but not soft semi closed, $\tilde{f}$ is soft pre^*-generalized continuous but not soft semi continuous.

Example 3.20. Let $X = \{a, b\}$, $Y = \{x, y\}$, $A = \{e_1, e_2\}$ and $B = \{k_1, k_2\}$. Define $u: X \to Y$ and $p: A \to B$ as $u(a) = y$, $u(b) = x$, $p(e_1) = k_1$, $p(e_2) = k_2$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}} = \{F_1, F_6, F_8, F_{14}, F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_9, G_{10}, G_{12}, G_{16}\}$. Let$\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_{10}) = F_7$ is semi-closed but not soft pre^*-generalized closed, $\tilde{f}$ is soft semi continuous but not soft pre^*-generalized continuous.

From the above discussion we have the following diagram:



Theorem 3.21. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a function. Then the following are equivalent.

i. $\tilde{f}$ is soft pre^*-generalized continuous.
ii. The inverse image of every soft open set in $(Y, \tilde{\sigma}, B)$ is soft pre^*-generalized closed in $(X, \tilde{\tau}, A)$.
iii. For every subset (G, B) of Y, $S_t p^* gcl\left(\tilde{f}^{-1}(G, B)\right) \tilde{\subseteq} \tilde{f}^{-1}\left(S_t cl(G, B)\right)$.
iv. For every subset (F, A) of X, $\tilde{f}\left(S_t p^* gcl(F, A)\right) \tilde{\subseteq} S_t cl\left(\tilde{f}(F, A)\right)$.

Proof: (i)⇒(ii): Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be soft pre^*-generalized continuous and (G, B) be a soft open set in Y. Then $Y \setminus (G, B)$is soft closed in Y. Since $\tilde{f}$ is soft pre^*-

generalized continuous $\tilde{f}^{-1}(Y \setminus (G,B))$ is soft pre^*-generalized closed in X. But $\tilde{f}^{-1}(Y \setminus (G,B) = X \setminus \tilde{f}^{-1}(G,B)$. Hence $\tilde{f}^{-1}(G,B)$ is soft pre^*-generalized open in X.

(ii)⇒(i): Suppose the inverse image of every soft open set in Y is soft pre^*-generalized open in X. Let (G,B) be soft closed in Y. Then $Y \setminus (G,B)$ is open in Y. By assumption $\tilde{f}^{-1}(Y \setminus (G,B)$ is soft pre^*-generalized open. $\tilde{f}^{-1}(Y \setminus (G,B) = X \setminus \tilde{f}^{-1}(G,B)$. Therefore $\tilde{f}^{-1}(G,B)$ is soft pre^*-generalized closed in X. Hence $\tilde{f}$ is soft pre^*-generalized continuous.

(i)⇒(iii): Let (G,B) be a subset of Y. Since $\tilde{f}$ is soft pre^*-generalized continuous, $\tilde{f}^{-1}(S_t cl(G,B))$ is soft pre^*-generalized closed in X. Then $S_t p^* gcl\left(\tilde{f}^{-1}(S_t cl(G,B))\right) = \tilde{f}^{-1}(S_t cl(G,B))$.

Now$S_t p^* gcl\left(\tilde{f}^{-1}(G,B)\right) \tilde{\subseteq} S_t p^* gcl\left(\tilde{f}^{-1}(S_t cl(G,B))\right) = \tilde{f}^{-1}(S_t cl(G,B))$. This proves (ii).

(iii)⇒(iv): Let (F,A) be a subset of X. Then $\tilde{f}(F,A)$ is a subset of Y. By our assumption,$S_t p^* gcl\left(\tilde{f}^{-1}\left(\tilde{f}(F,A)\right)\right) \tilde{\subseteq} \tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$.But

$(S_t p^* gcl(F,A)) \tilde{\subseteq} S_t p^* gcl\left(\tilde{f}^{-1}\left(\tilde{f}(F,A)\right)\right)$.

Thus$S_t p^* gcl(F,A) \tilde{\subseteq} \tilde{f}^{-1}\left(S_t cl\left(\tilde{f}(F,A)\right)\right)$. Hence $\tilde{f}(S_t p^* gcl(F,A) \tilde{\subseteq} S_t cl(F,A)$. This proves (iii).

(iv)⇒(i): Let (F,A) be soft subset of X. Then $\tilde{f}^{-1}(F,A) = \tilde{f}^{-1}(S_t cl(F,A))$. By (iii)$\tilde{f}(S_t p^* gcl(F,A)) \tilde{\subseteq} S_t cl\left(\tilde{f}(F,A)\right) \tilde{\subseteq} S_t cl(F,A) = (F,A)$. That implies$S_t p^* gcl\left(\tilde{f}^{-1}(F,A)\right) \tilde{\subseteq} \tilde{f}^{-1}(F,A)$.But$\tilde{f}^{-1}(F,A) \tilde{\subseteq} S_t p^* gcl\left(\tilde{f}^{-1}(F,A)\right)$.

Thus$S_t p^* gcl\left(\tilde{f}^{-1}(F,A)\right) = \tilde{f}^{-1}(F,A)$ and so $\tilde{f}^{-1}(F,A)$ is soft pre^*-generalized closed. Hence $\tilde{f}$ is soft pre^*-generalized continuous.

Remark 3.22. The composition of two soft pre^*-generalized continuous functions need not be soft pre^*-generalized continuous as shown in the following example.

Example 3.23. Let$X = \{a,b\}$, $Y = \{x,y\}$, $Z = \{m,n\}$, $\tilde{\tau}^{\tilde{c}} = \{F_1, F_5, F_6, F_{15}, F_{16}\}$, $\tilde{\sigma}^{\tilde{c}} = \{G_1, G_3, G_8, G_{11}, G_{16}\}$ and $\tilde{\mu}^{\tilde{c}} = \{H_1, H_4, H_{16}\}$ where$H_4 = \{(l_1, \emptyset), (l_2, (m,n))\}$. Define $u_1: X \to Y$ and $p_1: A \to B$ as $u_1(a) = y$, $u_2(b) = x$, $p_1(e_1) = k_2$, $p_2(e_2) = k_1$. Then the soft mapping $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ is soft pre^*-generalized continuous. Also, define $u_2: Y \to Z$ and $p_2: B \to C$ as $u_1(x) = m$, $u_2(y) = n$, $p_1(k_1) = l_2$, $p_2(k_2) = l_1$. Then the soft mapping $\tilde{g}: (Y, \tilde{\sigma}, B) \to (Z, \tilde{\mu}, L)$ is soft pre^*-generalized continuous. Now let $\tilde{g} \circ \tilde{f}: (X, \tilde{\tau}, A) \to (Z, \tilde{\mu}, L)$ be the composition of two soft pre^*-generalized continuous functions. Since $(\tilde{g} \circ \tilde{f})^{-1}(H_4) = \tilde{f}^{-1}(\tilde{g}^{-1}(H_4)) = \tilde{f}^{-1}(G_{13}) = F_4$ is not soft pre^*-generalized closed, $\tilde{g} \circ \tilde{f}$ is not soft pre^*-generalized continuous.

Theorem 3.24. If $\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ is soft pre^*-generalized continuous and $\tilde{g}:(Y,\tilde{\sigma},B)\to(Z,\tilde{\mu},C)$ is soft continuous then their composition $\tilde{g}\circ\tilde{f}:(X,\tilde{\tau},A)\to(Z,\tilde{\mu},C)$ is also soft pre^*-generalized continuous.
Proof: Let (H,C) be soft closed set in Z. Since g is soft continuous, $\tilde{g}^{-1}(H,C)$ is closed in Y and since $\tilde{f}$ is soft pre^*-generalized continuous, $\tilde{f}^{-1}\left(\tilde{g}^{-1}(H,C)\right)$ is soft pre^*-generalized closed in X. This implies $\left(\tilde{g}\circ\tilde{f}\right)^{-1}(H,C)$ is soft pre^*-generalized closed in X. Thus $(\tilde{g}\circ\tilde{f})^{-1}$ is soft pre^*-generalized closed in X for every soft closed subset (H,C) of Z. Hence $\tilde{g}\circ\tilde{f}$ is soft pre^*-generalized continuous.

4. Soft Pre*-Generalized Irresolute Functions

Definition 4.1. A function $\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ is said to be soft pre^*-generalized irresolute if $\tilde{f}^{-1}(G,B)$ is soft pre^*-generalized closed in $(X,\tilde{\tau},A)$ for every soft pre^*-generalized closed set in $(Y,\tilde{\sigma},B)$.

Example 4.2. Let $X=\{a,b\}$, $Y=\{x,y\}$, $A=\{e_1,e_2\}$ and $B=\{k_1,k_2\}$. Define $u:X\to Y$ and $p:A\to B$ as $u(a)=y$, $u(b)=x$, p $(e_1)=k_1$ and $p(e_2)=k_2$. Consider the soft topologies $\tilde{\tau}^{\tilde{c}}=\{F_1,F_2,F_8,F_{10},F_{16}\}$ and $\tilde{\sigma}^{\tilde{c}}=\{G_1,G_4,G_8,G_{12},G_{16}\}$. Let $\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ be a soft mapping. Since $\tilde{f}^{-1}(G_4)=F_4$ and $\tilde{f}^{-1}(G_8)=F_{12}$ and $\tilde{f}^{-1}(G_{12})=F_8$ are soft pre^*- generalized closed, $\tilde{f}$ is soft pre^*-generalized irresolute.

Theorem 4.3. If$\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ is a soft pre^*-generalized irresolute function then $\tilde{f}$ is soft pre^*-generalized continuous.
Proof: Let (G,B) be soft closed in Y. By theorem 2.10(i), (G,B) is soft pre^*-generalized closed. Since $\tilde{f}$ is soft pre^*-generalized irresolute function, $\tilde{f}^{-1}(G,B)$ is soft pre^*-generalized closed in X.Hence $\tilde{f}$ is soft pre^*-generalized continuous.

Theorem 4.4. If$\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ and $\tilde{g}:(Y,\tilde{\sigma},B)\to(Z,\tilde{\mu},C)$ are soft pre^*-generalized irresolute functions then $\tilde{g}\circ\tilde{f}$ is soft pre^*-generalized irresolute.
Proof: Let (H,C) be soft pre^*-generalized closed in Z. Since $\tilde{g}$ is soft pre^*-generalized irresolute,$\tilde{g}^{-1}(H,C)$ is soft pre^*-generalized closed in Y. Also, since$\tilde{f}$ is soft pre^*-generalized irresolute, $\tilde{f}^{-1}\left(\tilde{g}^{-1}(H,C)\right)=\left(\tilde{g}\circ\tilde{f}\right)^{-1}(H,C)$ is soft pre^*-generalized closed in X. Hence $\tilde{g}\circ\tilde{f}$ is soft pre^*-generalized irresolute.

Theorem 4.5. Let $\tilde{f}:(X,\tilde{\tau},A)\to(Y,\tilde{\sigma},B)$ is soft pre^*-generalized irresolute and $\tilde{g}:(Y,\tilde{\sigma},B)\to(Z,\tilde{\mu},C)$ is soft pre^*-generalized continuous. Then $\tilde{g}\circ\tilde{f}:(X,\tilde{\tau},A)\to(Z,\tilde{\mu},C)$ is soft pre^*-generalized continuous.
Proof: Let (H,C) be soft closed set in Z. Since $\tilde{g}$ is soft pre^*-generalized continuous, $\tilde{g}^{-1}(H,C)$ is soft pre^*-generalized closed in Y. Also, since$\tilde{f}$ is soft pre^*-generalized irresolute, $\tilde{f}^{-1}\left(\tilde{g}^{-1}(H,C)\right)=\left(\tilde{g}\circ\tilde{f}\right)^{-1}(H,C)$ is soft pre^*-generalized closed in X.Hence$\tilde{g}\circ\tilde{f}$ is soft pre^*-generalized continuous.

Theorem 4.6. Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be a function. Then the following are equivalent.

i. $\tilde{f}$ is soft pre^*-generalized irresolute.
ii. The inverse image of every soft pre^*-generalized open set in $(Y, \tilde{\sigma}, B)$ is soft pre^*-generalized open in $(X, \tilde{\tau}, A)$.
iii. $S_t p^* gcl\left(\tilde{f}^{-1}(G, B)\right) \tilde{\subseteq} \tilde{f}^{-1}\left(S_t p^* gcl(G, B)\right)$ for every subset (G, B) of Y.
iv. $\tilde{f}\left(S_t p^* gcl(F, A)\right) \tilde{\subseteq} S_t p^* gcl\left(\tilde{f}(F, A)\right)$ for every subset (F, A) of X.

Proof: The proof is similar to theorem 3.21.

Theorem 4.7. A function $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ is soft pre^*-generalized iresolute if and only if $\tilde{f}^{-1}\left(S_t p^* gint(G, B)\right) \tilde{\subseteq} S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right)$ for every subset (G, B) of Y.

Proof: Let $\tilde{f}: (X, \tilde{\tau}, A) \to (Y, \tilde{\sigma}, B)$ be soft pre^*-generalized irresolute. Let (G, B) be a subset of Y. Then $S_t p^* gint(G, B)$ is soft pre^*-generalized open in Y. Since $\tilde{f}$ is soft pre^*-generalized irresolute, $\tilde{f}^{-1}\left(S_t p^* gint(G, B)\right)$ is soft pre^*-gene- ralized open in$X$. Then

$\tilde{f}^{-1}\left(S_t p^* gint(G, B)\right) = S_t p^* gint\left(\tilde{f}^{-1}\left(S_t p^* gint(G, B)\right)\right) \tilde{\subseteq} S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right)$.

Thus $\tilde{f}^{-1}\left(S_t p^* gint(G, B)\right) \tilde{\subseteq} S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right)$.

Conversely, let (G, B) be soft pre^*-generalized open in Y. Then by (iv), $\tilde{f}^{-1}(G, B) = \tilde{f}^{-1}\left(S_t p^* gint(G, B)\right) \tilde{\subseteq} S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right)$.But$S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right) \tilde{\subseteq} \tilde{f}^{-1}(G, B.$

Therefore $S_t p^* gint\left(\tilde{f}^{-1}(G, B)\right) = \tilde{f}^{-1}(G, B)$ and so $\tilde{f}^{-1}(G, B)$ is soft pre^*-generalized open. Hence $\tilde{f}$ is soft pre^*-generalized irresolute.

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Motor Imagery Classification Using Rough Neural Network

J. Anila Maily [1]
Dr. C. Velayutham[2]
Dr. M. Mohamed Sathik[3]

Abstract

Brain Computer Interface is a system which provides a communication channel between the user and a computer without using the normal neuromuscular pathways. With BCI a user will be able to communicate with the mind. In a BCI system the brain activities are measured using EEG acquisition system. The acquired brain signals are analyzed and classified to identify the user's intention. Motor imagery BCI works by making the user imagine their body parts without actually moving it. Prominent features are extracted from the acquired brain signals and the extracted features are classified to find the motor imagery performed by the user. This study uses datasets are provided by the Dr. Cichocki's Lab (Lab for Advanced Brain Signal Processing). We propose the Rough Neural Network (RNN) for Motor imagery classification. The experimental results show that RNN classifier gives higher accuracy than Backpropagation Classifier.

Keywords: BCI, EEG, Motor Imagery, RNN

2010 AMS subject classification: 06B10, 06D99[4].

[1] Research Scholar, (Part-time internal) (Register Number: 12336), Dept of Computer Science, Sadakathullah Appa College, Rahmath Nagar, Tirunelveli. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India
Associate Professor, St. Mary‟s College (Autonomous), Thoothukudi. Email: anilamaily@gmail.com

[2] Head and Associate Professor, Department of Computer Science, Aditanar College of Arts and Science Tiruchendur Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, TamilNadu, India. Email:cvsir22@gmail.com

[3] Principal, Sadakathullah Appa College, Rahmath Nagar, Tirunelveli -627011, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India. Email: mmadsadiq@gmail.com

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1. Introduction

Brain Computer Interfacing (BCI) is to provide a communication channel between man, and machine, BCI users brain signals are captured, analyzed into commands for communication and control (Birbaumer et. al., 2003, Wolpow et. al.,2002) [1, 2]. BCI is a boon for people with severe motor disorders like Amyotrophic Lateral Sclerosis (ALS), brain stem stroke, cerebral palsy and spinal cord injury. A user's intent, as reflected by brain signals, is translated by the BCI system into a desired output: computer-based communication or control of an external device (Joseph et. al., 2009) [3]. Motor Imagery works by making the people imagine the moving their body parts without actually doing it. The brain signals during the imagination are recorded and analyzed to identify the intent of the person.

In BCI systems the brain signals are recorded from multiple channels to preserve high spatial accuracy. The Rough Set Theory (RST) enables the discovery of data dependencies and the reduction of the number of attributes contained in the data set using the data alone requiring no additional information. Given a dataset with discretized attribute values, it is possible to find a subset (of the original attributes using Rough Set Theory that are the most informative. All other attributes can be removed from the dataset with minimal information loss. (Velayutham, C., et. Al., 2011) [4].

2. Classification by BPN and RNN

Neural networks are the classifier used mostly in BCI. Neural network integrated with Rough Set Theory (RST) is known as Rough Set Neural network (RNN). The rough set theory and neural network are the two important methods of intelligent information processing. Optimizing the net for correct responses to the training input data set is done by Backpropogation. More than one hidden layer may be beneficial for some applications, but one hidden layer is sufficient if enough hidden neurons are used ((R. Jensen, and Q. Shen, 2008, C. Velayutham, and K. Thangavel, 2011, Simon Haykin, 2005) [5 - 7].

Rough set and neural networks can solve complex and high-dimensional problems, which are called Rough Neural Networks (RNNs) (Weidong Zhao, and Guohua Chen. 2002) [8]. This paper investigates the Motor Imagery classification accuracy using Backpropagation network classifier and propose Rough neural network classifier. The rest of the paper is organized as follows. Section II briefly describes classification technique used with EEG. Section III presents the methodology adopted, Section IV is about results and discussion. Section V concludes the paper.

3. Literature Review

Hong K-S et al., (2018), [9] presented a brain-computer interface (BCI) framework for hybrid functional near-infrared spectroscopy (fNIRS) and electroencephalography (EEG) for locked-in syndrome (LIS) patients. For classification, linear discriminant analysis has been most widely used.

Siavash Sakhavi et al., (2018) [10] proposed a classification framework for Motor Imagery (MI) data by introducing a new temporal representation of the data and utilized a convolutional neural network (CNN) architecture for classification. The framework classified BCI competition IV-2a 4-class MI data set by 7% increase in average subject accuracy.

Vladimir A. Maksimenko et al., (2018) [11] applied Artificial Neural Network (ANN) for recognition and classification of EEG patterns associated with motor imagery in untrained subjects. ANN optimization is proposed by pre-processing the EEG signals with a low pass filter and it is shown filtration of high frequency spectral components enhances the classification performance up to 90±5%.

Han, C., Kim, Y., Kim, D.Y. et al., (2019) [12] investigated the possibility of using an EEG-based endogenous BCI paradigm for online binary communication by a patient in Complete Locked in Syndrome (CLIS). An online classification accuracy of 87.5% was achieved when Riemannian geometry-based classification was applied to real-time EEG data recorded while the patient was performing one of two mental-imagery tasks for 5 s.

Shiu Kumar et al., (2019) [13] introduced a novel scheme for classifying motor imagery (MI) tasks using electroencephalography (EEG) signal that can be implemented in real-time having high classification accuracy between different MI tasks. They proposed a new predictor, OPTICAL, that uses a combination of common spatial pattern (CSP) and long short-term memory (LSTM) network for obtaining improved MI EEG signal classification.

4. Methodology

The methodology followed for Motor Imagery classification is shown in figure 1.

Data sets of motor imagery EEG

The Datasets provided by the Dr. Cichocki's Lab (Lab. for Advanced Brain Signal Processing), is used for this study.

Data files and format

All data sets are stored in the MATLAB format (*.mat). The file name consists of subject ID, channel number, imagery tasks and session number. For example, *'SubC_6chan_3LRF_s1':* Subject C, 6 channels, 3-class imagery tasks of left hand, right hand and feet and session 1. Each file contains one session which consists of several runs separated by short breaks. Some subjects have many sessions conducted on different days. The detailed information of the dataset is given in table 1.

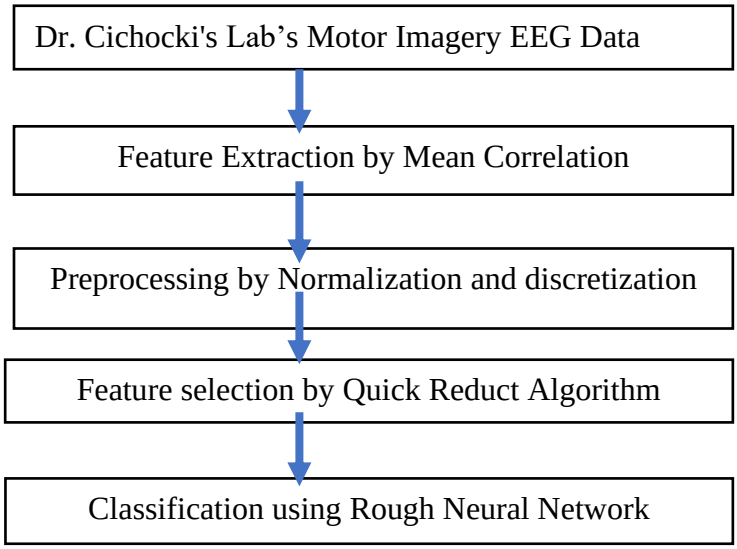


Figure 1: Methodology

Feature Extraction and Feature selection

Prominent features are extracted from the signals using statistical parameter mean correlation. The extracted features are preprocessed by applying minmax normalization and discretized with k-means algorithm. The preprocessed datasets are further reduced by using supervised feature selection algorithm Quick Reduct based on Rough Set Theory.

Classification by BPN and RNN

The reduced feature set selected from the feature selection algorithms are assigned to the input neurons. The number of hidden neurons is greater than or equal to the number of input neurons, and there is only one output neuron. Initial weights are assigned randomly. The output from each hidden neuron is calculated using the sigmoid function

$S_1 = \frac{1}{1+e^{-\lambda x}}$ where λ = 1 and $x = \sum_i w_{ih}\, k_i$ (1)

where w_{ih} is the weight assigned between input and hidden layer and *k* is the input value. The output from the output layer is calculated using the sigmoid function.

$S_2 = \frac{1}{1+e^{-\lambda x}}$, where λ = 1 and $x = \sum_i w_{ho}\, S_i$ (2)

where w_{ho} is the weight assigned between hidden and output layer and S_i is the output value from hidden neurons. S_2 is subtracted from the desired output. Using this error (*e*) value, the updating of weight is performed as:

$\delta = eS_2(1 - S_2)$ (3)

The weights assigned between the input and the hidden layer and the hidden and output layer are updated as:

Sl. No.	Dataset	Subject	Class	Channel	Duration (sec)	Trial	10x10 CV (Acc.±std.)	Sample rate	Device
1	SubA_5chan_3LRF	A	LH/RH/F	5	4s	270	0.92±0.004	256Hz	G. tec
2	SubB_5chan_3LRF	B	LH/RH/F	5	4s	174	0.86±0.01	250Hz	Neuroscan
3	SubB_6chan_3LRF			6		150	0.80±0.03		
4	SubC_5chan_3LRF	C	LH/RH/F	5	4s	180	0.86±0.01	256Hz	g.tec
5	SubC_6chan_3LRF_s1			6	3s	300	0.89±0.01		
6	SubC_6chan_3LRF_s2					300	0.84±0.01		
7	SubC_6chan_3LRF_s3					204	0.89±0.01		
8	SubC_5chan_3LRF _Day1	C	LH/RH/F	5	4s	210	0.72±0.02	256Hz	g.tec
9	SubC_5chan_3LRF _Day2					210	0.81±0.01		
10	SubC_5chan_3LRF _Day3					180	0.81±0.01		
11	SubC_5chan_3LRF _Day4					180	0.83±0.02		
12	SubC_5chan_3LRF _Day5					234	0.87±0.01)		
13	SubC_5chan_3LRF _Day6					150	0.88±0.01		
14	SubC_5chan_3LRF _Day7					180	0.88±0.01		
15	SubC_14chan_3LRR	C	LH/RH/R	14	4s	350	0.78±0.01	250Hz	Neuroscan

Table 1: The detailed information of three class dataset

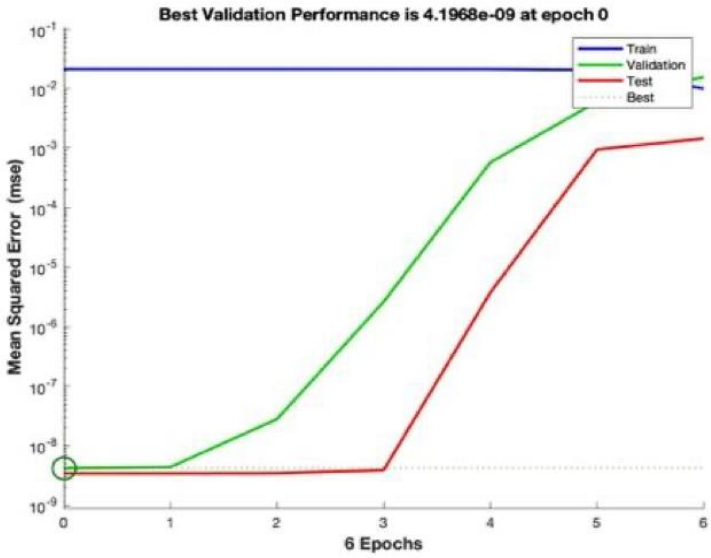
5.Results

The classification accuracy of three class datasets using Backpropagation classifier and Rough Neural Network classifier is tabulated in table 2. Figure 2 and figure 3 depicts the classification accuracy of SubA_5chan_3LRF. Classification accuracy of SubB_6chan_3LRF using BPN and RNN classifier is shown in Figure 4 and Figure 5. Figure 6 and figure 7 depicts the Regression plot of SubA_5chan_3LRF using BPN and RNN. It is observed that the performance of RNN classifier shows higher performance than the BPN classifier for all the datasets. Figure 8 shows the accuracy of all three class datasets using BPN and RNN classifier.

Sl No.	Dataset	Accuracy	
		BPN	RNN
	SubA_5chan_3LRF	98.80	99.70
	SubB_6chan_3LRF	97.20	99.00
	SubB_5chan_3LRF.	96.00	99.10
	SubC_5chan_3LRF.	98.00	99.50
	SubC_6chan_3LRF_s1	95.30	98.70
	SubC_6chan_3LRF_s2	99.40	99.70
	SubC_6chan_3LRF_s3	98.40	99.60
	SubC_5chan_3LRF_day1	99.60	100.0
	SubC_5chan_3LRF_day2	98.30	99.60
	SubC_5chan_3LRF_day3.	93.70	97.90
	SubC_5chan_3LRF_day4	97.80	99.50
	SubC_5chan_3LRF_day5	96.80	99.30
	SubC_5chan_3LRF_day6	97.50	99.90
	SubC_5chan_3LRF_day7	98.70	99.90
	SubC_14chan_3LRR	98.90	99.10

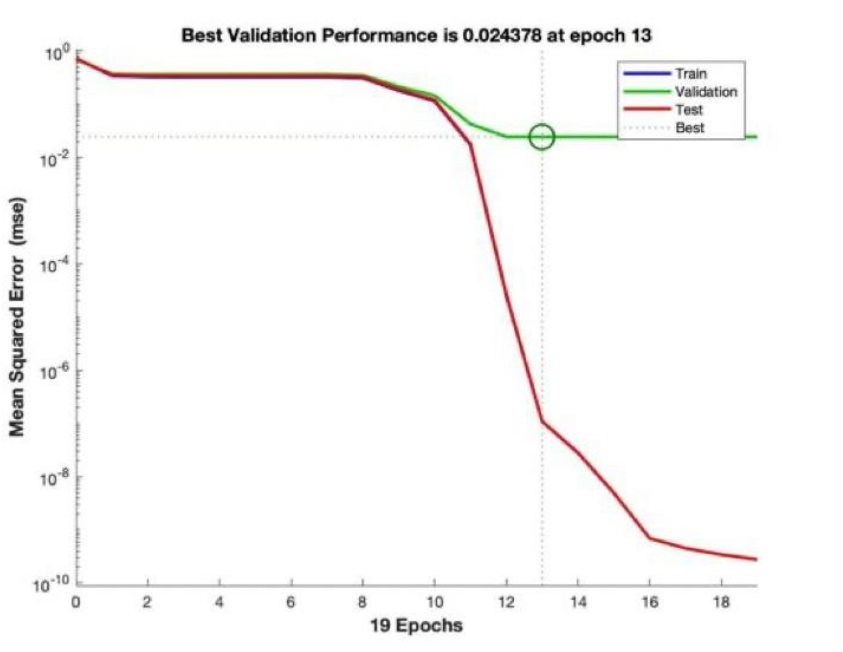
Table 2: Classification accuracy of three class data sets by BPN and RNN

Figure2:



Classification Performance of SubA_5chan_3LRF by BPN Classifier

Figure 3 :



Classification Performance of SubA_5chan_3LRF by RNN Classifier

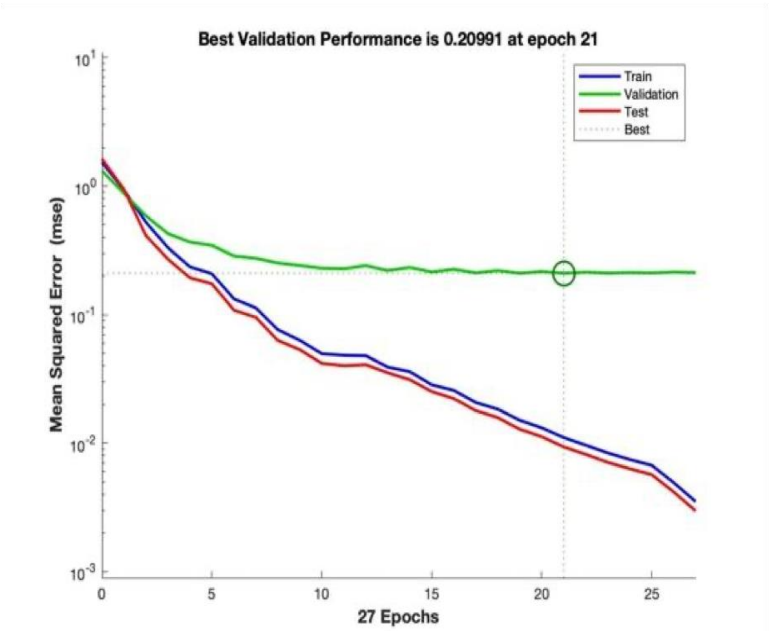


Figure 4: Classification Performance of SubB_6chan_3LRF BPN Classifier

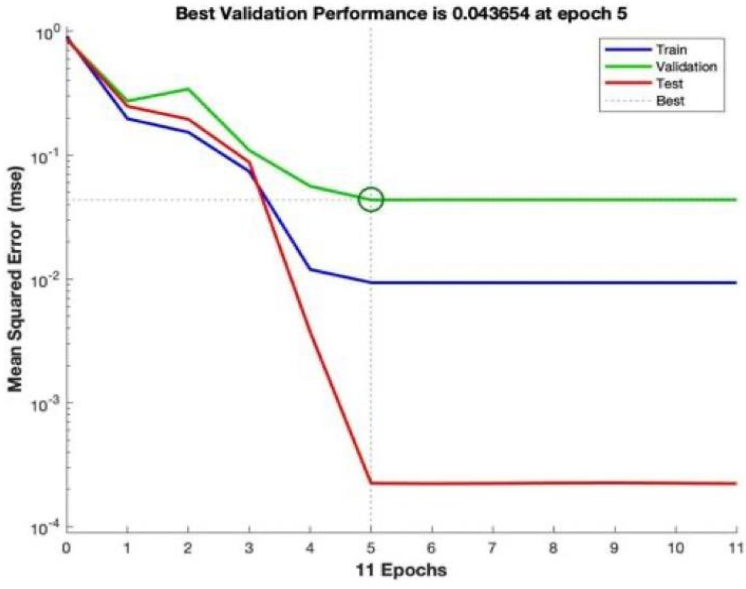


Figure 5: Classification Performance of SubB_6chan_3LRF RNN Classifier

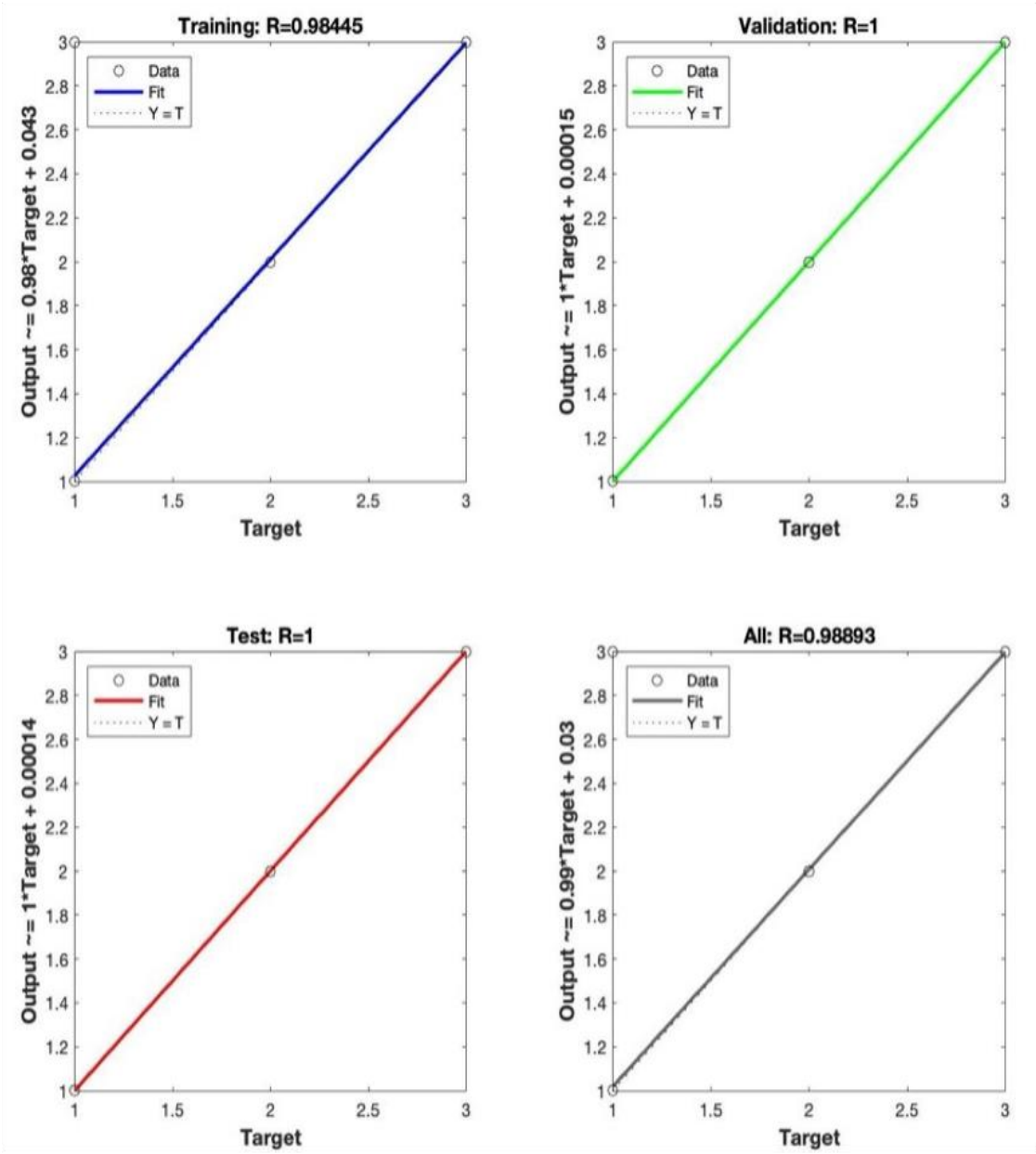


Figure 6: Regression plot of SubA_5chan_3LRF by BPN Classifier

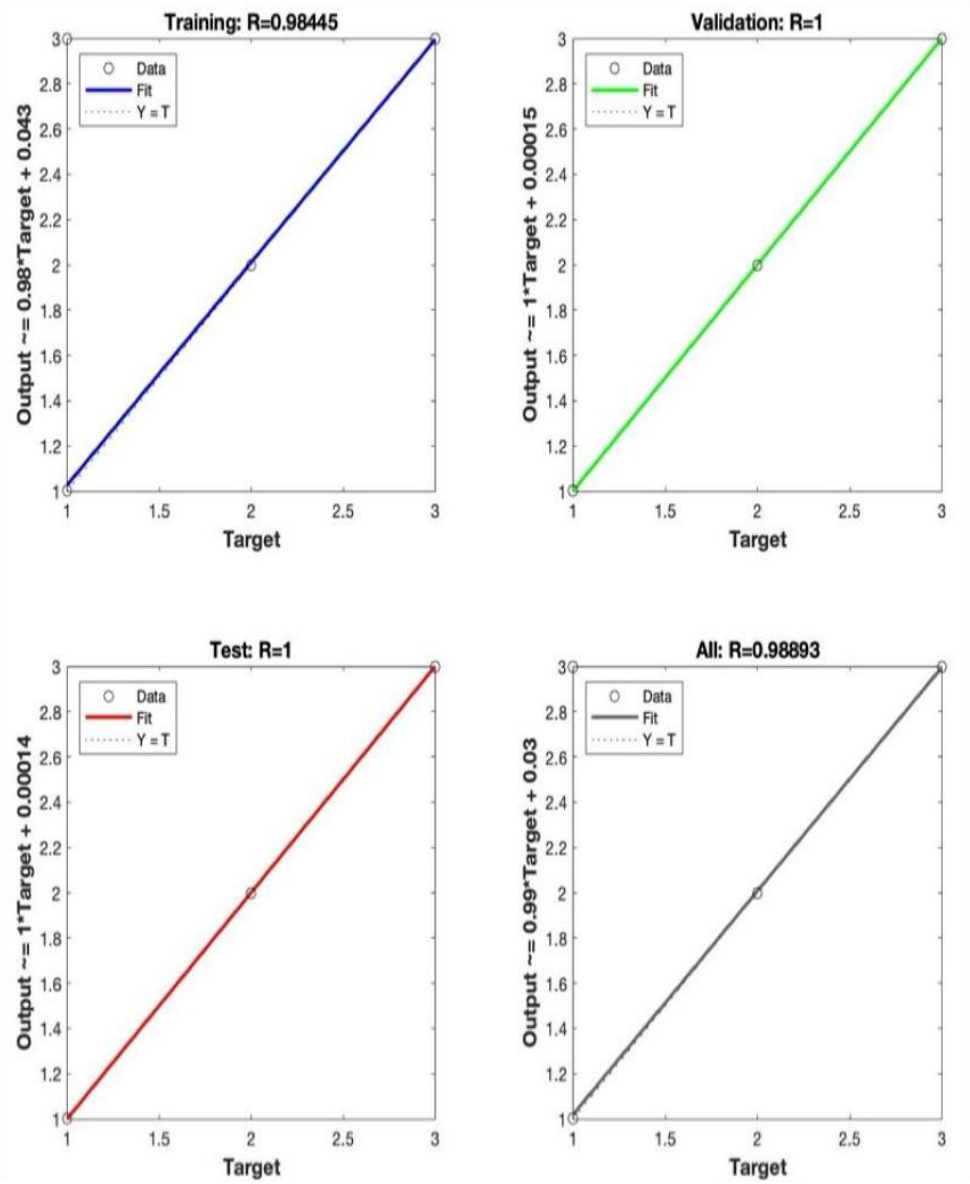


Figure 7: Regression plot of SubA_5chan_3LRF by RNN Classifier

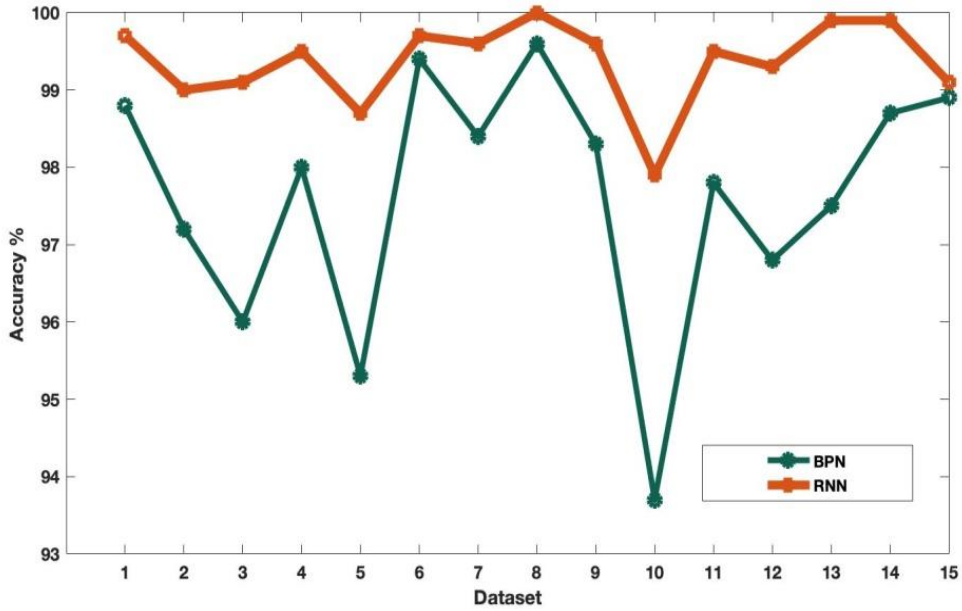


Figure 8: Classification Accuracy by BPN and RNN

5. Conclusion

The main aim of BCI is to identify the user's intention through the brain signals. Once the prominent features are extracted and selected, they have to be classified using the classification algorithm. The performance of BCI depends on the classification accuracy. Rough Neural Network (RNN) is proposed for Motor imagery classification. It is tested on three class datasets with feature extraction using mean correlation and supervised feature selection method using Quick Reduct algorithm. The RNN classifier gives higher accuracy for the three class datasets than the neural network back-propagation (BPN) classifier.

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The Upper and Forcing Fault Tolerant Geodetic Number of a Graph

T. Jeba Raj[1]
K. Bensiger[2]

Abstract

A fault tolerant geodetic $S \subseteq V$ is said to be minimal fault tolerant geodetic set of G if no proper subset of S is a fault tolerant geodetic set of G is called the upper fault tolerant geodetic number of G is denoted by $\gamma_{gft}^{+}(G)$. Some general properties satisfied by this concept are studied. For connected graphs of order $n \geq 3$ with $\gamma_{gft}^{+}(G)$to be $n - 1$ is given. It is shown that for every pair of $a, b$with $5 \leq a < b$, there exists a connected graph Gsuch that $\gamma_{gft}(G) = a$ and $\gamma_{gft}^{+}(G) = b$, where $\gamma_{gft}(G)$ is the fault tolerant geodetic number of G and $\gamma_{gft}^{+}(G)$ is the upper fault tolerant geodetic number of a graph. Let S be a gft-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique gft-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing fault tolerant geodetic number of S, denoted by$f_{gft}(S)$, is the cardinality of a minimum forcing subset of S. The forcing fault tolerant geodetic number of G, denoted by $f_{gft}(G)$, is $f_{gft}(G) = min\{f_{gft}(S)\}$, where the minimum is taken over all f_{gft}-sets in G. The forcing fault tolerant geodetic number of some standard graphs are determined. Some of its general properties are studied. It is shown that for every pair of positive integers a and b with $0 \leq a \leq b, b \geq 2$ and $b \geq 2a$, there exists a connected graph G such that $f_{gft}(G) = a$ and$g_{ft}(G) = b$.

Keywords: tolerant geodetic, connected graphs, minimum cardinality.

2010 AMS subject classification:05C12, 05C69[3]

[1]Assistant Professor, Department of Mathematics, Malankara Catholic College, Mariagiri, Kaliyakkavilai - 629 153, India, Email: jebaraj$_{math}$@gmail.com
[2]Register Number. 20123082091004, Research Scholar, Department of Mathematics, Malankara Catholic College, Mariagiri, Kaliyakkavilai - 629 153, India. Email: bensigerkm83@gmail.com
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India.)
[3]Received on June23rd, 2022. Accepted on Aug 10th, 2022. Published on Nov30th, 2022.doi: 10.23755/rm.v44i0.903. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.This paper is published under the CC-BY license agreement.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [9]. Two vertices u and v of said to be adjacent in G if $uv \in E(G)$. The neighborhood $N(v)$ of the vertex $v$in G is the set of vertices adjacent to v. The degree of the vertex v is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph $G$with $deg(u) = 1$ and $deg(v) > 1$, then we call e an end edge, u a leaf and v a support vertex. For any connected graph G, a vertex $v \in V(G)$ is called a cut vertex of G if $V(G) - v$ is disconnected. The subgraph induced by set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is called an extreme vertex of G if $\langle N(v) \rangle$ is *complete*.

A vertex x is an internal vertex of an $u - v$path P if x is a vertex of P and $x \neq u, v$. An edge e of G is an internal edge of an $u - v$ path P if e is an edge of P with both of its ends or in P. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An$u - v$ path of length $d(u, v)$ is called an $u - v$*geodesic*. A vertex x is said to lie on an $u - v$ geodesic P if x isa vertex of P including the vertices $u$and v. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $rad(G)$ and the maximum eccentricity is its diameter, $diam(G)$. We denote $rad(G)$ by r and $diam(G)$ by d. The closed interval $I[u, v]$ consists of u, v and all vertices lying on some $u - v$ geodesic of G. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \bigcup_{u,v \in S} I[u, v]$ is the closure of S. A set $S \subseteq V(G)$ is called a geodetic set if $I[S] = V(G)$. Thus every vertex of G is contained in a geodesic joining some pair of vertices in S. The minimum cardinality of a geodetic set of G is called the *geodetic number* of G and is denoted by $g(G)$. A geodetic set of minimum cardinalities is called g-set of G. For references on geodetic parameters in graphs see [4, 5, 6, 7, 10]. Let S be a geodetic set of G. Wbe the set of extreme vertices of G. Then S is said to be a fault tolerant geodetic set of G, if $S - \{v\}$ is also a geodetic set of G for every $v \in S \setminus W$. The minimum cardinality of a fault tolerant geodetic set is called fault tolerant geodetic number and is denoted by $g_{ft}(G)$. The minimum fault tolerant geodetic dominating set of G is denoted by g_{ft}-set of G.The following theorem is used in the sequel.

Theorem 1.1. [6] Each extreme vertex of a connected graph G belongs to every geodetic set of G.

2.The Upper Fault Tolerant Geodetic Number of a Graph

Definition 2.1. A fault tolerant geodetic $S \subseteq V$ is said to be minimal fault tolerant geodetic set of G if no proper subset of S is a fault tolerant geodetic set of G is called the upper fault tolerant geodetic number of G is denoted by $\gamma_{gft}^{+}(G)$.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_5, v_6\}$ is a γ_{gft}-setof G so that $\gamma_{gft}(G) = 5$. Let $S_1 = \{v_1, v_5, v_6, v_7, v_8, v_9\}$. Then S_1 is a minimal faulttolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq 6$.It is easily verified that there is no faulttolerant geodetic set of G with cardinality more than six. Therefore $\gamma_{gft}^{+}(G) = 6$.

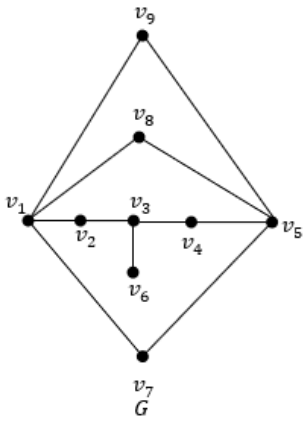

Figure 2.1

Observation 2.3. (i) For a connected graph of order $n \geq 2, 2 \leq \gamma_{gft}(G) \leq \gamma_{gft}^{+}(G) \leq n$.
(ii) No cut vertex of G belongs to any minimal fault tolerant geodetic set of G.
(iii) Each extreme vertex of G belong to any minimal fault tolerant geodetic set of G.

Theorem 2.4. For the complete graph $G = K_n n \geq 2, \gamma_{gft}^{+}(G) = n$.
Proof: This follows from Observation 2.3(iii). ■

Theorem 2.5. For any non-trivial tree, $\gamma_{gft}^{+}(G) =$ number of end vertices
Proof: This follows from Observation 2.3(ii) and (iii). ■

Theorem 2.6. For the cycle $G = C_n\ (n \geq 4), \gamma_{gft}^{+}(G) = \begin{cases} 4 & if n is odd \\ 5 & if n is even \end{cases}$
Proof: Let n be even. Let v be the antipodal vertex of u and y be the antipodal vertex of x, where $u \neq x$. Then $S = \{u, v, x, y\}$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq 4$. We prove that $\gamma_{gft}^{+}(G) = 4$. On the contrary, suppose that$\gamma_{gft}^{+}(G) \geq 5$.

Then there exists a geodetic set S' such that $|S'| \geq 5$. Then there exist at least two pair of antipodal vertices of G. Hence it follows that $S \subset S'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G)= 4$.

Let n be odd. Let u and x be two adjacent vertices of G. Let v and w be antipodal vertices of $u, w$and y be two antipodal vertices of x. Then $M = \{u, v, w, x, y\}$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G)\geq 5$. We prove that $\gamma_{gft}^{+}(G)= 5$. On the contrary, suppose that $\gamma_{gft}^{+}(G) \geq 6$. Then there exists a fault tolerant geodetic set M' of G such that $|M'| \geq 6$.ThenM' contains two pair of antipodal vertices. Which implies $M \subset M'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G)= 5$. ■

Theorem 2.7. Let G be the complete bipartite graph $K_{r,s}, (2 \leq r \leq s), \gamma_{gft}^{+}(G) = s + 2$.

Proof: Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the two bipartite sets of$G$. Let $S = Y \cup \{x_i, x_j\}$, where $2 \leq i \neq j \leq r$is a fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq s + 2$. We prove that $\gamma_{gft}^{+}(G)= s + 2$. On the contrary, suppose that $\gamma_{gft}^{+}(G)\geq s + 3$. Then there exists a fault tolerant geodetic set of G such that $|S'| \geq s + 3$. Which implies $S \subseteq S'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G) = s + 2$.■

Theorem 2.8. For every pair of $a, b$with $5 \leq a < b$, there exists a connected graph$G$ such that $\gamma_{gft}(G) = a$ and $\gamma_{gft}^{+}(G) = b$.

Proof: Let $P: v_1, v_2, v_3, v_4, v_5$ be a path on five vertices and $V(K_{b-a-2}) = \{z_1, z_2, \ldots, z_{b-a-2}\}$. Let H be a graph obtained from P and $V(K_{b-a-2})$ by joining each z_i $(1 \leq i \leq b - a + 2)$ with $v_1$and v_5. Let G be the graph obtained from H by introducing the vertices $h_1, h_2, \ldots, h_{a-4}$ and join each h_i $(1 \leq i \leq a - 4)$with v_3. The graph G is shown in Figure 2.2.

First, we prove that $\gamma_{gft}(G) = a$. Let $X = \{h_1, h_2, \ldots, h_{a-4}\}$ be the set of end vertices of G. By Observation 2.3(iii), X is a subset of every fault tolerant geodetic set of G. It is easily verified that there is no fault tolerant geodetic set of cardinality less than a and so $\gamma_{gft}(G) \geq a$. Let $S = X \cup \{v_1, v_2, v_4, v_5\}$ is a fault tolerant geodetic set of G so that $\gamma_{gft}(G) = a$.

Next, we prove that $\gamma_{gft}^{+}(G) = b$. Let $M = X \cup \{z_1, z_2, \ldots, z_{b-a-2}, v_1, v_4\}$. Then$M$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq b$. We prove that$\gamma_{gft}^{+}(G) = b$. On the contrary, suppose that $\gamma_{gft}^{+}(G) \geq b + 1$. Then there existsa fault tolerant geodetic set $M'$of G such that $|M'| \geq b + 1$. Since $v_3 \notin M'$ and $|V(G)| = b + 3$, it follows that either $S \subset M'$ or $M \subset M'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G) = b$.■

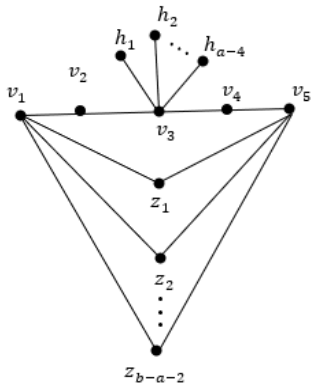

Figure 2.2

3. The Forcing Fault Tolerant Geodetic Number of a Graph

Definition 3.1. Let S be a g_{ft}-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique f_{gft}-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing fault tolerant geodetic number of S, denoted by $f_{gft}(S)$, is the cardinality of a minimum forcing subset of S. The forcing fault tolerant geodetic number of G, denoted by $f_{gft}(G)$, is $f_{gft}(G) = min\{f_{gft}(S)\}$, where the minimum is taken over all f_{gft}- sets in G.

Example 3.2. For the graph G given in Figure 3.1, $S_1 = \{v_1, v_3, v_4, v_5, v_6\}, S_2 = \{v_1, v_2, v_4, v_5, v_7\}$are the only two g_{ft}-sets of G so that $f_{gft}(S_1) = f_{gft}(S_2) = 1$ so that $g_{ft}(G) = 1$ and $f_{gft}(G) = 1$.

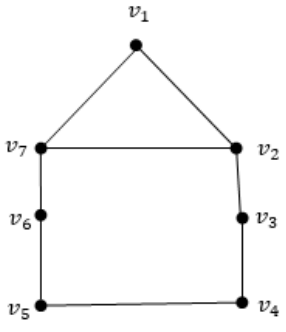

Figure 3.1

The next theorem follows immediately from the definition of the forcing fault tolerant geodetic number of the graph.

Theorem 3.3. For any connected graph $G, 0 \leq f_{gft}(G) \leq g_{ft}(G)$.
In the following we determine the forcing fault tolerant geodetic number of some standard graphs.

Theorem 3.4. For a non-trivial tree $T, f_{gft}(G) = 0$.
Proof: Since for a tree T, the set of end vertices of G is the unique g_{ft}-set of G. Hence it follows that $f_{gft}(G) = 0$. ■

Theorem 3.5. For the complete graph $G = K_n\ (n \geq 3), f_{gft}(G) = 0$.
Proof: Since $S = V\ (G)$ is the unique g_{ft}-set of G, $f_{gft}(G) = 0$.■

Theorem 3.6. For the cycle $G = C_n\ (n \geq 4), f_{gft}(G) = \begin{cases} 0 & if n = 4\ or\ 5 \\ 1\ if n is even and n \geq 6 \\ 3 & if n is odd and n \geq 7 \end{cases}$

Proof: Let $V(C_n) = \{v_1, v_2, \ldots, v_n, v_1\}$.
Case 1. Let n is even. For $n = 4, S = V\ (G)$ is the unique g_{ft}-set of G so that $f_{gft}(G) = 0$. So, let $n \geq 6$. let$n = 2k\ (k \geq 3)$. Let $S_1 = \{x, y, u, v\}$ be a g_{ft}-set of G, where y is the antipodal vertex of x and u is the antipodal vertex of u. Since $n \geq 6$, g_{ft}-set of G is not unique and so $f_{gft}(G) = 1$. Since S_1 is the unique g_{ft}-set of Gcontaining S_1, $f_{gft}(S_1) = 1$ so that $f_{gft}(G) = 1$.
Case 2. Let n is odd. For $n = 5, S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G) = 0$. So, let $n \geq 7$. let$n = 2k + 1\ (k \geq 3)$. Then it is easily verified that no singleton or two element subsets of a g_{ft}-set S_1 is not a forcing subset of S_2. Let $S_2 = \{v_1, v_{k+1}, v_{k+2}\} \cup \{v_2, v_{k+3}\}$. Then S_2 is the g_{ft}-set of G containing $\{v_1, v_{k+1}, v_{k+2}\}$. Therefore $f_{gft}(G) = 3$. ■

Theorem 3.7. For the fan graph $G = K_1 + P_{n-1}\ (n \geq 4), f_{gft}(G) = 0$.
Proof: Let $V(K_1) = \{x\}$ and $S = V(G) - \{x\}$ is the unique g_{ft}-set of G so that $f_{gft}(G) = 0$. ■

Theorem 3.8. For the wheel graph $G = K_1 + C_{n-1}\ (n \geq 4), f_{gft}(G) = 0$.
Proof: Let $V(K_1) = \{x\}$ and $S = V(G) - \{x\}$ is the unique g_{ft}-set of G so that $f_{gft}(G) = 0$. ■

Theorem 3.9. For the complete bipartite graph $G = K_{m,n}\ (2 \leq r \geq s)$,
$f_{gft}(G) = \begin{cases} 0 & 2 \leq r \leq s \leq 3 \\ 3 & r = 3, s \geq 4 \\ 4 & 4 \leq r \leq s \end{cases}$
Proof: Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the two partite sets of G.
Let $S = 2$. Then $G = C_4$. By Theorem, $f_{gft}(G) = 0$.

For $r = 2$and $s \geq 3$. Then $S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0. So let $r = 3, s = 3$. Then $S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0.For$r = 3, s \geq 4$, let S be a g_{ft}-set of G. Then any two element subsets of S is a forcing subset of G and so $f_{gft}(G) \geq 3$. Let $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Then $\{x_1, x_2, y_1, y_2\}$ is a forcing subset of S so that $f_{gft}(G)$= 3. Since this is true for al g_{ft}-set of G, $f_{gft}(G)$= 3. Let $m \geq 4$and $n \geq 4$. Let S be a g_{ft}-set of G. Then one or two or three element subsets of S is a forcing subset of$S$ and so $f_{gft}(G) \geq 4$. Let $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Then $\{x_1, x_2, y_1, y_2\}$ is a forcing subset of S so that $f_{gft}(G)$= 4. Since this is true for all g_{ft}-set $S$of G, $f_{gft}(G)$= 4. ■

Theorem 3.10. For every pair of positive integers $a$and b with $0 \leq a \leq b, b \geq 2$and $b \geq 2a$, there exists a connected graph G such that $f_{gft}(G)$= a and $g_{ft}(G) = b$.

Proof: Let $P: x, y, z$ be a path on three vertices. Let P_i: u_i, v_i, w_i $(1 \leq i \leq a)$ be a copy of path on three vertices. Let G be the graph obtained from P and P_i $(1 \leq i \leq a)$ by adding new vertices $z_1, z_2, \ldots, z_{b-2a}$ and introducing the edges xx_i $(1 \leq i \leq a), zv_i$ $(1 \leq i \leq a)$and zz_i $(1 \leq i \leq b - 2a)$. The graph is shown in Figure 3.2.

First, we prove that $g_{ft}(G) = b$. Let $Z = \{z_1, z_2, \ldots, z_{b-2a}\} \cup \{u_1, u_2, \ldots, u_a\}$. Then $Z$is a subset of every g_{ft}-set of G. Let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq a)$. Then every g_{ft}-set of Gcontains at least one vertex from each H_i $(1 \leq i \leq a)$ and so $g_{ft}(G) \geq b - 2a + a + a = b$. Let $S = Z \cup \{u_1, u_2, \ldots, u_a\}$. Then S is a g_{ft}-set of G so that $g_{ft}(G)$= b.

Next, we prove that $f_{gft}(G) = a$. By Theorem$f_{gft}(G) \leq g_{ft}(G) - |Z| = b - (b - a) = a$. Now since every Z is a subset of every g_{ft}-set of G and every g_{ft}-set contains at least one vertex from each H_i $(1 \leq i \leq a)$, every g_{ft}-set S is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be a forcing subset with $|T| < a$. Then there exists $H_i$for some $i$such that $T \cap H_i = \phi$. Therefore $f_{gft}(G) = a$.■

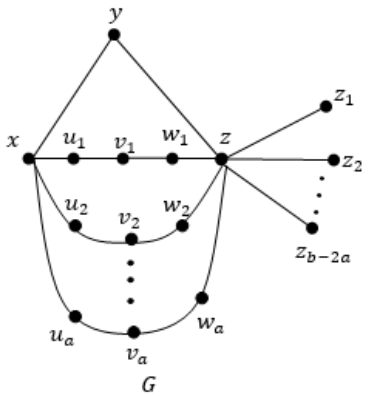

Figure 3.2

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Split Domination Decomposition of Path Graphs

E. Ebin RajaMerly[1]
Praisy B[2]

Abstract

A decomposition (G_1, G_2, G_3, …, G_n) of G is said to be a split domination decomposition (SDD), if the following conditions are satisfied:(i) each G_i is connected(ii)$\gamma_s(G_i) = i$, $1 \leq i \leq n$. In this paper, we prove that path, path corona and subdivision of path graph admit SDD.

Keywords: split domination, decomposition, split domination decomposition.

2010 AMS subject classification: 05C12, 05C69[3]

[1]Associate professor, Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam. Tamilnadu, India. Mail Id: ebinmerly@gmail.com
[2]Research Scholar, Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam, Tamilnadu, India. Mail. Id: praisyblessed@gmail.com
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1. Introduction

Graph theory in mathematics refers to the study of graphs. The theory of domination is one of the rapidly developing areas in graph theory. The concept of split domination was developed by Veerabhadrappa R. Kulli and Bidarahalli Janakiram [2]. Another important concept in graph theory is decomposition of graphs. Decompositions are imposed by applying several conditions on G_i in the decompositions by several authors based on their studies.

We introduce a new concept split domination decomposition of a graph which is motivated by the concepts of Linear path decomposition [3] and Connected Domination Decomposition [4]. We have considered here simple undirected graphs without loops or multiple edges. The order and size of the graph are indicated by p and q respectively. Terms not defined here are used in the sense of Frank Harary [1].

2. Preliminaries

Definition2.1: A dominating set D of a graph G = (V, E) is a set of vertices such that each vertex of G is either in D or has at least one neighbor in D.

Definition 2.2: A dominating set D of a graph G = (V, E) is a split dominating set if the induced sub graph <V-D> is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set.

Definition2.3: If G_1, G_2, G_3, …, G_n are edge disjoint sub graphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then ($G_1$, G_2, G_3,…, G_n) is said to be decomposition of G.

Definition2.4: The corona $P_p \odot K_1$ is the graph constructed from a copy of P_p, where for each vertex $u \in V(P_p)$, a new vertex u' and a pendent edge uu' are added. It is denoted by P_p^+ and is called comb.

Definition 2.5: A subdivision of a graph G is a graph obtained by inserting a new vertex in each edge of G and is denoted by S(G).

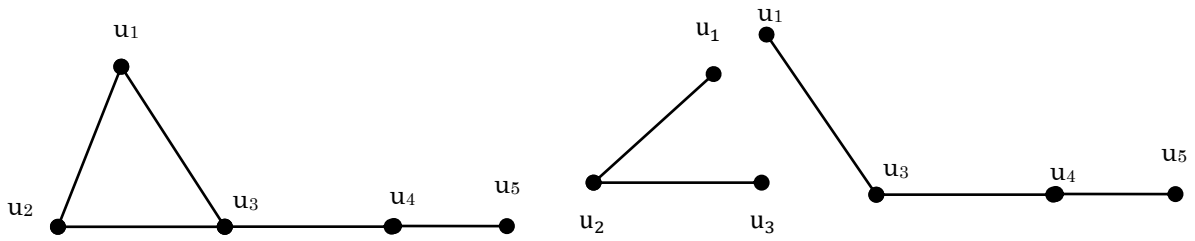
3. Split Domination Decomposition

Definition 3.1: A Decomposition (G_1, G_2, G_3, …, G_n) of G is said to be a split domination decomposition (SDD), if the following conditions are satisfied:

(i). each G_i is connected

(ii). $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Example 3.2:



G G_1 G_2

Figure 1: Graph G and its SDD (G_1, G_2)

Remark 3.3: A path with 3 vertices have split domination number 1 and path having 3k-2,3k-1 and 3k vertices have the split domination number k, k.≥2.

Theorem 3.4: A pathP_p, $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$ admits split domination decomposition (G_1, G_2, G_3, …, G_n) if and only if$\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$.

Proof: Let $P_p=u_1u_2 \dots u_p$ be a path of order p.

Assume that P_p , $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$ admits split domination decomposition (G_1, G_2, G_3,…, G_n).

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore $\gamma_s(G_1) + \gamma_s(G_2) + \cdots + \gamma_s(G_n) = 1 + 2 + \cdots + n$

$$\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$$

Conversely, assume that $\sum_{i=1}^{n} \gamma_s(G_i) = \frac{n(n+1)}{2}$.

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore P_p admits split domination decomposition (G_1, G_2, G_3, …,G_n).

Next, we have to find the bound for p.

By remark 3.3, the subgraphs G_1, G_2, G_3, …, $G_n$of P_p having minimum possible vertices are

$G_1 = u_1u_2u_3$

$G_2 = u_3u_4u_5u_6$

$G_3 = u_6u_7u_8u_9u_{10}u_{11}u_{12}$

⋮

$G_n = u_m u_{m+1} \dots u_p$ Where$m = \frac{3n^2-9n+12}{2}$, $p = \frac{3n^2-3n+6}{2}$

Clearly $|V(P_p)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_n)|$

$p = (3+4+\ldots+3n-2)-(n-1) = \frac{3n^2-3n+6}{2}$

Next, the maximum possible vertices of subgraphs $G_1, G_2, G_3, \ldots, G_n$ of P_p are

$G_1 = u_1u_2u_3$

$G_2 = u_3u_4u_5u_6u_7u_8$

$G_3 = u_8u_9u_{10}u_{11}u_{12}u_{13}u_{14}u_{15}u_{16}$

$\vdots$

$G_n = u_m u_{m+1} \ldots u_p$ Where $m = \frac{3n^2-5n+4}{2}$, $p = \frac{3n^2+n+2}{2}$

Clearly $|V(P_p)| = |V(G_1)| + |V(G_2)| + \cdots + |V(G_n)|$

$p = (3+6+\ldots+3n)-(n-1) = \frac{3n^2+n+2}{2}$

Therefore $\frac{3n^2-3n+6}{2} \leq p \leq \frac{3n^2+n+2}{2}$.

Corollary 3.5: If $\frac{3n^2+n+2}{2} < p < \frac{3n^2+3n+6}{2}$, then P_p does not admit split domination decomposition.

Theorem 3.6: P_p^+ admits split domination decomposition $(G_1, G_2, G_3, \ldots, G_n)$ if and only if P_p has $\frac{n^2+n}{2}(n>1)$ vertices.

Proof: Let $P_p = u_1u_2 \ldots u_p$ be a path with p vertices.

If we join the vertices $u_1', u_2', \ldots, u_p'$ to $u_1, u_2, \ldots, u_p$ respectively, then we get P_p^+.

Assume that P_p has $\frac{n^2+n}{2}(n>1)$ vertices.

To prove P_p^+ admits split domination decomposition $(G_1, G_2, G_3, \ldots, G_n)$.

Suppose $p = \frac{n^2+n}{2}$

$G_1 = <\{u_1, u_2, u_1'\}>$

$G_2 = <\{u_2, u_3, u_2', u_3'\}>$

$G_3 = <\{u_3, u_4, u_5, u_6, u_4', u_5', u_6'\}>$

$\vdots$

$G_n = <\{u_l, u_{l+1}, \ldots, u_p, u_{l+1}', \ldots, u_p'\}>$

Notice that the minimum split dominating set of G_n has n vertices and P_p has $1+2+3+\ldots+n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$ vertices.

Clearly $\gamma_s(G_i) = i$, $1 \leq i \leq n$.

Therefore $(G_1, G_2, G_3, \ldots, G_n)$ is a split domination decomposition of P_p^+.

Conversely, suppose P_p^+ admits split domination decomposition.

To prove P_p has $\frac{n^2+n}{2}$, $(n>1)$ vertices.

Suppose not,

Case (i): $|V(P_p)| > \frac{n^2+n}{2}$

We join m vertices in P_p where m=1, 2, 3, …, or n. Constructing $(G_1, G_2, G_3 \ldots, G_n)$ in the above, we have remaining m vertices where m=1, 2, 3, …, or n. We cannot arrange

the m vertices in the minimum split dominating set of G_i otherwise (G_1, G_2, G_3, …, G_n) would not be a split domination decomposition for P_p^+. If these m vertices alone to give a sub graph G_{k_m} , then (G_1,G_2, …,G_n , G_{k_m}) would not be a split domination decomposition for P_p^+which is a contradiction.

Case (ii):$|V(P_p)| < \frac{n^2+n}{2}$. We eliminate m vertices in P_p where m =1, 2, 3, … or n − 1.Constructing(G_1, G_2, G_3, …, G_n) in the above, we have remaining m vertices where $m = n - 1$, n-2, .., or n-(n-1). We cannot arrange the m vertices in the minimum split dominating set of G_i otherwise (G_1, G_2, G_3,…, G_n) would not be a split domination decomposition for P_p^+. If these m vertices alone to give a sub graph G_{k_m} , then(G_1,G_2 ,…,G_{n-1} , G_{k_m}) would not be a split domination decomposition for P_p^+which is a contradiction. Therefore P_p has$\frac{n^2+n}{2}$, (n > 1)vertices.

Note 3.7: In general, if P_p admits split domination decomposition, then S (P_p) need not admit split domination decomposition and vice-versa. So we cannot use the range of p as in theorem 3.4 to S (P_p).

Theorem 3.8: Let P_p be a (p, q) -path. Subdivision of the path graph S (P_p) admits split domination decomposition (G_1, G_2, G_3,…,G_n)if and only if $\frac{2n^2-6n+14}{2} \leq p \leq \frac{n^2+5n-8}{2}$.

Proof: Let P_p=u_1u_2 ... u_p be a path with p vertices. Then S (P_p) has $2p - 1$vertices.
Assume that S (P_p) admits split domination decomposition. Now we can find the range of p if and only if S (P_p) admits split domination decomposition.
From note 3.7, we can't apply the range of p in $P_p$as in theorem 3.4 toS (P_p). Hence using the range of p in P_p to S (P_p), the following table shows the probabilities for S (P_p) admits split domination decomposition.

No. of decompositions (n)	2	3	4	5	6	7
No. of vertices in$P_p(p)$	4	7 8	11 12 13 14	17 18 19 20 21	25 26 27 28 29	34 35 36 37 38 39
No. of vertices in $S(P_p)$	7	13 15	21 23 25 27	33 35 37 39 41	49 51 53 55 57	67 69 71 73 75 77

Table:1

Using Newton's forward difference formula, we have to find the upper and lower bound of p for P_p such that S (P_p) admitssplit domination decomposition.

To find the lower bound of p, using table-1.

By Newton's forward formula,$p = p_0 + \frac{u}{1!}\Delta p_0 + \frac{u(u-1)}{2!}\Delta^2 p_0 + \cdots$

Where $u = \frac{n-n_0}{h} = n - 3$ ($n_0 = 3$ and $h = 1$)

Here $p_0 = 7, \Delta p_0 = 4, \Delta^2 p_0 = 2, \Delta^3 p_0 = 0$

Therefore $p = \frac{2n^2-6n+14}{2}$

Next, to find the upper bound of p, using table-1.

By Newton's forward formula,$p = p_0 + \frac{u}{1!}\Delta p_0 + \frac{u(u-1)}{2!}\Delta^2 p_0 + \cdots$

Where $u = \frac{n-n_0}{h} = n - 3$ ($n_0 = 3$ and $h = 1$)

Here $p_0 = 8, \Delta p_0 = 6, \Delta^2 p_0 = 1, \Delta^3 p_0 = 0$

Therefore $p = \frac{n^2+5n-8}{2}$

Therefore S(P_p) admits split domination decomposition, if $\frac{2n^2-6n+14}{2} \leq p \leq \frac{n^2+5n-8}{2}$.

Conversely, Assume that $\frac{2n^2-6n+14}{2} \leq p \leq \frac{n^2+5n-8}{2}$.

To prove S (P_p) admit split domination decomposition.

Suppose not,

Consider the lower bound of p, if we eliminate one vertex fromP_p,then the corresponding S(P_p) will not admit split domination decomposition.

Hence $p = \frac{2n^2-6n+14}{2} - 1 < \frac{2n^2-6n+14}{2}$.which is a contradiction.

Consider the upper bound of p, if we join one vertex to P_p, then the corresponding S (P_p) will not admit split domination decomposition.

Hence$p = \frac{n^2+5n-8}{2} + 1 > \frac{n^2+5n-8}{2}$.which is a contradiction.

Therefore S (P_p) admits split domination decomposition.

4. Conclusion

In this paper, we deal that path, path corona and subdivision of path graph admits split domination decomposition. Further investigations could also be done to get the condition at which some graphs admit split domination decomposition.

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More Functions Related to ŠA* - Open Set in Soft Topological Spaces

P. Anbarasi Rodrigo[1]
S. Anitha Ruth[2]

Abstract

In this paper, we introduce some soft functions like Š Stronglyα^* - continuous function, Š Perfectly α^* - continuous function, Š Totally α^* - continuous function. We study the connections of these function with other Š function. Also, we establish the relationships in between the above functions and also investigate various aspects of these functions.

Keywords: soft functions, continuous function

2010 AMS subject classification: 54C05[3]

[1]Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi, Affiliated by Manonmaniam Sundaranar University Abishekapatti, Tirunelveli, India Email.id: anbu.n.u@gmail.com
[2]Research Scholar (Part Time),
Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi, Register Number: 19122212092001. Affiliated by Manonmaniam Sundaranar University Abishekapatti, Tirunelveli, India Email.id: anitharuthsubash@gmail.com
[*]Corresponding Author: anitharuthsubash@gmail.com
[3]Received on June 9th, 2022.Accepted on Sep 5st, 2022.Published on Nov 30th, 2022.doi: 10.23755/rm.v44i0.905. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.This paper is published under the CC-BY licence agreement.

1. Introduction

Molodtsov introduced the concept of soft sets from which the difficulties of fuzzy sets, intuitionistic fuzzy sets, vague sets, interval mathematics and rough sets have been rectified. Application of soft sets in decision making problems has been found by Maji et al. whereas Chen gave a parametrization reduction of soft sets and a comparison of it with attribute reduction in rough set theory. Further soft sets are a class of special information.

Shabir and Naz introduced soft topological spaces in 2011 and studied some basic properties of them. Meanwhile generalized closed sets in topological spaces were introduced by Levine in 1970 and recent survey of them is in which is extended to soft topological spaces in the year 2012. Further Kannan and Rajalakshmi have introduced soft g – locally closed sets and soft semi star generalized closed sets. Soft strongly g – closed sets have been studied by Kannan, Rajalakshmi and Srikanth. Chandrasekhara Rao and Palaiappan introduced generalized star closed sets in topological spaces and it is extended to the bitopological context by Chandrasekhara Rao and Kannan.

Recently papers about soft sets and their applications in various fields have increased largely. Modern topology depends strongly on the ideas of set theory. Any Research work should result in addition to the existing knowledge of a particular concept. Such an effort not only widens the scope of the concept but also encourages others to explore new and newer ideas. Therefore, in this work we introduce a new soft generalized set called Šα* open set and its related properties. This may be another starting point for the new soft set mathematical concepts and structures that are based on soft set theoretic operations.

2. Preliminaries

In this section, this project X be an initial universe and Ê be a set of parameters. Let P (X) denote the power set of X and A be a non – empty subset of ξ. A pair ($F^{š}$, A) denoted by $F^{š}{}_{a}$ is called a soft set over X, where Fš is a mapping given by $F^{š}$: A $\rightarrow$ P (X).

Definition2.1.1 [8] For two soft sets ($F^{š}$, A) and (G, B) over a common universe X, we say that ($F^{š}$, A) is a soft subset of (G, B) denoted by ($F^{š}$, A) $\subseteq_s$ (G, B), if

i. A $\subseteq_s$ B and

ii. $F^{š}$ (e) $\subseteq_s$ G (e) for all e $\in$ ξ

Definition2.1.2 [8] The complement of a soft set ($F^{š}$, A) denoted by ($F^{š}$, A) ^{c,} is defined by (($F^{š}$, A)) ^{c} = (($F^{šc}$, A), where $F^{šc}$: A $\rightarrow$ P (X) is a mapping given by $F^{šc}$ (e) = X - $F^{š}$ (e), for all e $\in$ ξ.

Definition2.1.[8] A Subset of a Štopological space (X, τ_s, ξ) is said to be

1. a Š Semi-Open set
2. if ($F^{š}$, Ê) $\subseteq_s$ ŠCl (Šint ($F^{š}$, Ê) and a Š Semi-Closed set if Š int (Š Cl ($F^{š}$, Ê) $\subseteq_s$ ($F^{š}$, Ê).
3. a Š Pre-Open set [1] if ($F^{š}$, Ê) $\subseteq_s$ Š Int (Š Cl ($F^{š}$, Ê)) and a ŠPre-Closed set if Š Cl (Š

int ($F^{š}$, Ê) ⊆ $_{s}$ ($F^{š}$, Ê) a Š α-Open set [1] if ($F^{š}$, Ê) $\subseteq_s$ Š In (ŠCl (int($F^{š}$, Ê) and a Š α-Closed set if ŠCl (Šint (ŠCl ($F^{š}$, Ê) $\subseteq_s$ ($F^{š}$, Ê)).

4. a Š β-Open set [1] if ($F^{š}$, Ê) $\subseteq_s$ Šcl (Šint (Š cl($F^{š}$, Ê))) and a Š β-Closed set if Š Int(ŠCl(int($F^{š}$, Ê)) $\subseteq_s$ ($F^{š}$, Ê)).

5. aŠ- generalized Closed set(briefly Šgs- Closed) if Š Cl($F^{š}$, Ê) $\subseteq_s$ (G , ξ) whenever ($F^{š}$, Ê)$\subseteq_s$ (G , ξ)and(G , ξ)is Š Open in (X, τ_s, ξ). The complement of a Š gs-Closed set is called a Šgs-Open set.

6. a Š Semi-generalized Closed set (briefly Š Sg-Closed) if Š Cl ($F^{š}$, Ê) $\subseteq_s$ (G, ξ) whenever ($F^{š}$, Ê) ⊆(G , ξ)and(G , ξ)is Šsemi Open in(X, τ_s, ξ).The complement of aŠSg-Closed set is called a ŠSg-Open set.

7. a generalized ŠSemi-Closed set (briefly gs-Closed) if ŠCl ($F^{š}$, Ê) $\subseteq_s$ (G, ξ) whenever ($F^{š}$, Ê) $\subseteq_s$ (G , ξ) and(G , ξ)is ŠOpenin(X, τ_s,ξ).Thecomplementofa Šgs-Closed setiscalled aŠgs-Open set.

8. aŠ – Closed [9] if Š Cl(F , ξ)⊈(G , ξ) whenever (F , ξ)⊆ $_{s}$ (G , ξ) and (G , ξ) isŠsemi Openin (X, τ_s, ξ)

9. aŠω-Closed [9] ifŠCl($F^{š}$, Ê) ⊆(G , ξ)whenever(F , ξ)⊆ (G , ξ)and(G , ξ) is Š semi Open

10. αa Š alpha-generalized Closed set (briefly Šαg-Closed) if α Š Cl($F^{š}$, Ê) ⊆ (G , ξ) whenever ($F^{š}$, Ê) ⊆(G , Ê)and(G, Ê) is ŠαOpen in (X, τ_s, Ê).The complement of a Šαg-Closed set is called a Šαg-Open set.

11. a Š generalized alpha Closed set (briefly Šgα-Closed) ifα Š Cl($F^{š}$, Ê) ⊈(G , Ê) whenever ($F^{š}$, Ê) ⊆(G , Ê)and(G , Ê) is Š Open in(X, τ_s, Ê).The complement of aŠ gα-Closed set is called a Šgα-Open set.

12. A Šgeneralized pre Closed set (briefly Šgp-Closed)[1] if p Š Cl($F^{š}$, Ê) ⊆(G , Ê) whenever a Š gp-Open set.

13. aŠgeneralized pre regular Closed set (briefly Šgpr-Closed) [5] if p ŠCl ($F^{š}$, Ê) ⊆ (G , Ê) whenever ($F^{š}$, Ê) ⊈(G , Ê) and (G , Ê) isŠ regular Open in (X, τ_s, Ê). The complement ofa Šgpr-Closedset is called a Š gpr – Open set.

3.1 Strongly Šα*-continuous function

Definition 3.1.1: A Š function f: (X, τ_s, Ê) ⟶ (Ý, τ_s, K) is said to be strongly Šα*-continuous function, if the inverse image of every Š α*- Ô(Ý)in (Ý, τ_s, K) is Š - Ô(X) in (X, τ_s, Ê).

Theorem 3.1.2: Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K) be strongly Šα*-continuous function, then it isŠ-continuous function.

Proof:

Let ($F^{š}$, Ê) be Š - Ô(X) in (Ý, τ_s, K). Since every Š - Ô(X) is Š α*- Ô(X), then ($F^{š}$, Ê) is Š α*- Ô(X) in (Ý, τ_s, K). Since, f is strongly Šα*-continuous function, f^{-1}($F^{š}$, Ê) is Š - Ô(X) in (X, τ_s, Ê).Therefore, f is Š-continuous.

Remark 3.1.3: The converse of the above theorem need not be true.

Example 3.1.4: Let X = Ý = { x_1, ,x_2}, τ_s = {$F^š_1$, $F^š_2$, $F^š_3$,$F^š_{15}$,$F^š_{16}$},and σ_s = {$F^š_3$, $F^š_{11}$, $F^š_{12}$,$F^š_{15}$,$F^š_{16}$},
Š α*- Ô(Ý)= {$F^š_1$,$F^š_2$, $F^š_3$,$F^š_7$,$F^š_8$,$F^š_9$,$F^š_{10}$,$F^š_{11}$,$F^š_{12}$,$F^š_{13}$,$F^š_{14}$,$F^š_{15}$,$F^š_{16}$ }. Let f : (X, τ_s, Ê) (Ý, τ_s, K)be defined by f($F^š_1$) = $F^š_3$, f($F^š_2$) = $F^š_{11}$,f($F^š_3$) = $F^š_{12}$,f($F^š_4$) = $F^š_1$,f($F^š_5$) = $F^š_2$,f($F^š_6$) = $F^š_{13}$,f($F^š_7$) = $F^š_4$f($F^š_8$)= $F^š_{14}$,f($F^š_9$) = f($F^š_{10}$) = f($F^š_{11}$)= f($F^š_{12}$)= $F^š_2$,f($F^š_{13}$) = $F^š_9$,f($F^š_{14}$) = $F^š_6$,f($F^š_{15}$) = $F^š_{15}$,f($F^š_{16}$) = $F^š_{16}$. Clearly f is Š– continuous but not strongly Šα*-continuous function, because $f^{-1}(F^š_1)$ = $F^š_4$is not Š - Ô(X) in (X, τ_s, Ê).

Theorem 3.1.5: Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K)be strongly Šα*-continuous function iff the inverse image of every Š α* - Ç(Ý) in (Ý, τ_s, K) is Š - Ç(X) in (X, τ_s, Ê)
Proof:
Assume that f is strongly Šα*-continuous function. Let ($F^š$, ξ) be any Š α* - Ç(X) in (Ý, τ_s, K). Then, ($F^š$, Ê)^{c}is Š α*- Ô(X) in (Ý, τ_s, K). Since f is strongly Šα*-continuous function.
f^{-1}(($F^š$, Ê) ^{c}) is Š - Ô(X) in (X, τ_s, Ê). But f^{-1}(($F^š$, ξ) ^{c}) = X - f^{-1}(($F^š$, ξ) is Š - Ô(X) in (X, τ_s, Ê)⇒ f^{-1}(($F^š$, Ê) is Š - Ç(X) in (X, τ_s, Ê). Conversely, assume that the inverse image of every Š α*- Ç(X) in (Ý, τ_s, Ê) is Š - Ç(X) in (X, τ_s, Ê). Let ($F^š$, Ê) be any Š α*- Ô(X) in (Ý, τ_s, K). Then, ($F^š$, Ê)^{c}is Š α*- Ç(X) in (Ý, τ_s, K). By assumption, f^{-1}(($F^š$, Ê) ^{c}) is Š - Ç(X) in (X, τ_s, Ê). But f^{-1}(($F^š$, Ê) ^{c}) = X - f^{-1}(($F^š$, Ê) is Š - Ç(X) in (X, τ_s, Ê)⇒ f^{-1}(($F^š$, Ê) is Š - Ô(X) in (X, τ_s, Ê). Hence f is strongly Šα*-continuous function.

Theorem 3.1.6: Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K)be strongly Š -continuous function then it is strongly Šα*-continuous function.
Proof: Let ($F^š$, Ê) be any Š - Ô(X) in (Ý, τ_s, K). Since every Š - Ô(X) is Š α*- Ô(X), Since f is strongly Š - continuous function, then f^{-1}(($F^š$, Ê) is both Š - Ô(X) and Š - Ç(X) in (X, τ_s, Ê).
⇒f^{-1}(($F^š$, Ê) is Š - Ô(X) in (X, τ_s, Ê). Hence f is strongly Š α*- continuous function.

Remark 3.1.7: The converse of the above theorem need not be true.

Example 3.1.8: Let X = Ý = { x_1, ,x_2}, τ_s = {$F^š_1$,$F^š_2$, $F^š_3$, $F^š_5$,$F^š_7$,$F^š_8$,$F^š_9$,$F^š_{10}$,$F^š_{12}$,$F^š_{13}$,$F^š_{14}$,$F^š_{15}$,$F^š_{16}$ }, τ_s^c = {$F^š_1$,$F^š_2$, $F^š_4$, $F^š_6$,$F^š_7$,$F^š_8$,$F^š_9$,$F^š_{10}$,$F^š_{13}$,$F^š_{14}$,$F^š_{15}$,$F^š_{16}$ }, and σ_s = {$F^š_1$, $F^š_{13}$$F^š_{15}$,$F^š_{16}$},
Š α*- Ô(Ý)= {$F^š_1$, $F^š_3$,$F^š_7$,$F^š_8$,$F^š_{11}$,$F^š_{12}$,$F^š_{13}$,$F^š_{15}$,$F^š_{16}$ }. Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K)be defined by f($F^š_1$) = $F^š_5$, f($F^š_2$) = $F^š_2$, f($F^š_3$) = $F^š_3$,f($F^š_4$) = $F^š_4$f($F^š_5$) = $F^š_5$, f($F^š_6$) = $F^š_6$f($F^š_7$) = $F^š_7$,f($F^š_8$)= $F^š_8$f($F^š_9$) =$F^š_9$,f($F^š_{10}$) = $F^š_{10}$,f($F^š_{11}$) =$F^š_{11}$,f($F^š_{12}$) = $F^š_{12}$,f($F^š_{13}$) = $F^š_{13}$,f($F^š_{14}$) = $F^š_4$,f($F^š_{15}$) = $F^š_{15}$,f($F^š_{16}$) = $F^š_{16}$. Clearly f is strongly Šα*-continuous function but not strongly Š-continuous function,Sincef^{-1}($F^š_1$) = $F^š_5$is Š - Ô(X) but not Š - Ç(X)

Theorem 3.1.13: Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K)be strongly Šα*-continuous function and g: (X, σ_s, Ê)⟶ (ž, η_s, K) be Šα*-continuous function, then

go f : (X, τ_s, Ê) ⟶ (ž, η_s, K) is Šα*-continuous.
Proof: Let($F^{š}$, Ê) be any Š - Ô(X) in (ž, η_s, K). Since g is Šα*-continuous, then $g^{-1}((F^{š}$, Ê)is Š α*- Ô(X) in (Ý, τ_s, K). Since f is strongly Šα*-continuous function, then $f^{-1}(g^{-1}((F^{š}, \xi))$ is Š - Ô(X) in (X, τ_s, Ê)⇒(go f) $^{-1}(F^{š}$, Ê) is Š - Ô(X) in (X, τ_s, Ê).)⇒(go f) $^{-1}(F^{š}$, Ê) is Šα* - Ô(X) in (X, τ_s, Ê). Hence gof isŠ α*-– continuous.

Theorem 3.1.14: Let f : (X, τ_s, Ê) ⟶ (Ý, σ_s, K)be strongly Š α*-continuous function and g: (Ý, σ_s, K) ⟶ (ž, η_s, K) be Šα*-irresolute, then
go f : (X, τ_s, Ê) ⟶(ž, η_s, K) is strongly Šα*– continuous.
Proof: Let ($F^{š}$, ξ) be any Šα*- Ô(X) in (ž, η_s, K). Since g is Šα*-irresolute, then $g^{-1}((F^{š}$, Ê)is Š α*- Ô(X) in (Ý, σ_s, K). Since f is strongly Šα*-continuous function, then $f^{-1}(g^{-1}((F^{š}$, Ê)) is Š - Ô(X) in (X, τ_s, Ê)⇒(go f) $^{-1}((F^{š}$, Ê) is Š - Ô(X) in (X, τ_s, Ê). Hence gof is strongly Šα*– continuous.

Theorem 3.1.15: Let f : (X, τ_s, Ê) ⟶ (Ý, σ_s, K)be Šα*-continuous and
g: (Ý, σ_s, K) ⟶ (ž, η_s, K) be strongly Š α*-continuous function, then
go f : (X, τ_s, Ê) ⟶ (ž, η_s, K) is Š α*–irresolute.
Proof: Let ($F^{š}$, Ê) be any Šα*- Ô(X) in (ž, η_s, K). Since g is strongly Šα*-continuous, then g $^{-1}((F^{š}$, Ê)isŠ - Ô(X) in (Ý, τ_s, K). Since f is Šα*-continuous function, then $f^{-1}(g^{-1}((F^{š}$, Ê)) is Šα*- Ô(X) in (X, τ_s, Ê)⇒(go f) $^{-1}((F^{š}$, Ê) is Šα*- Ô(X) in (X, τ_s, Ê). Hence gof is Š α*– irresolute.

Theorem 3.1.16: Let f : (X, τ_s, Ê) ⟶ (Ý, τ_s, K)be strongly Š α*-continuous function and g: (X, σ_s, Ê) ⟶ (ž, η_s, K) be strongly Šα*-continuous function, then
go f : (X, τ_s, Ê) ⟶ (ž, η_s, K) is strongly Š α*– continuous.
Proof: Let ($F^{š}$, Ê) be any Šα* - Ô(X) in (ž, η_s, K). Since g is strongly Šα*-continuous, then
($g^{-1}(F^{š}$, Ê))is Š Ô(X) in (Ý, τ_s, K). Since f is strongly Šα*-continuous function, then $f^{-1}(g^{-1}((F^{š}$, Ê)) is Š - Ô(X) in (X, τ_s, Ê)⇒(go f) $^{-1}((F^{š}$, Ê) is Š - Ô(X) in (X, τ_s, Ê). Hence gof isstrongly Š α*– continuous.

Theorem 3.1.17: Let f : (X, τ_s, Ê) ⟶ (Ý, σ_s, K)be Š -continuous function and g: (Ý, σ_s, K) ⟶(ž, η_s, K) be strongly Šα*-continuous function, then
go f : (X, τ_s, Ê) ⟶ (ž, η_s, K) is strongly Šα*– continuous.
Proof: Let ($F^{š}$, Ê) be any Šα*- Ô(X) in (ž, η_s, K). Since g is strongly Šα*-continuous, then
$g^{-1}((F^{š}$, Ê)is Š- Ô(X) in (Ý, τ_s, K). Since f is Š-continuous function, then $f^{-1}(g^{-1}((F^{š}, \xi))$ is Š - Ô(X) in (X, τ_s, Ê)⇒(go f) $^{-1}((F^{š}$, Ê) is Š - Ô(X) in (X, τ_s, Ê). Hence gof is strongly Š α*– continuous.

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Gaussian Tribonacci R-Graceful Labeling of Some Tree Related Graphs

Dr. Sunitha. K[1]
Sheriba. M[2]

Abstract

Let r be any natural number. An injective function $\phi : V(G) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all* k, where GT_{q+r-1} is the $(q+r-1)^{th}$ Gaussian Tribonacci number in the Gaussian Tribonacci sequence is said to be Gaussian Tribonacci r-graceful labeling if the induced edge labeling $\phi^* : E(G) \to \{GT_1, GT_2, ..., GT_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Gaussian Tribonacci r-graceful labeling, then G is called a Gaussian Tribonacci r-graceful graph. A graph G is said to be Gaussian Tribonacci arbitrarily graceful if it is Gaussian Tribonacci r-graceful for all r. In this paper we investigate the Path graph P_n, the Comb graph $P_m \Theta K_1$, the Coconut tree graph $CT(m,n)$, the regular caterpillar graph $P_m \Theta n K_1$, the Bistar graph $B_{m,n}$ and the Subdivision of Bistar graph $S[B_{m,n}]$ are Gaussian Tribonacci arbitrarily graceful.

Keywords: Gaussian Tribonacci sequence, Gaussian Tribonacci graceful labeling, Path graph, Comb graph, Coconut Tree graph, Regular caterpillar graph, Bistar graph and Subdivision of Bistar graph.

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[1] Assistant Professor, Department of Mathematics, Scott Christian College (Autonomous), Nagercoil-629003
Email: ksunithasam@gmail.com
[2] Part Time Research Scholar, Department of Mathematics, Scott Christian College (Autonomous), Nagercoil-629003.Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012
Email: sheribajerin@gmail.com
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1. Motivation and Main Results

Graphs considered throughout this paper are finite, simple, undirected and nontrivial. Labeling of graph is an assignment of values to vertices and edges or both subject to certain conditions. In 1967, Rosa [6] introduced the concept of graceful labeling. In 1982, Slater [4] introduced the concept of k-graceful labeling of graphs and is defined as follows: Let G be a simple graph with p vertices and q edges. Let k be any natural number. Define an injective mapping $\phi : V(G) \to \{0, 1, 2, ..., q+k-1\}$ that induces bijective mapping $\phi^* : E(G) \to \{k, k+1, ... , q+k-1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$ for all $uv \in E(G)$ and $u, v \in V(G)$, then ϕ is called k-graceful labeling while ϕ^* is called an induced edge k-graceful labeling and the graph G is called k-graceful graph. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful. Yuksel Soykan e tal. [11] introduced the concept of Gaussian Generalised Tribonacci Numbers. In this sequel, we introduce a new concept Gaussian Tribonacci r-graceful labeling of graphs. We follow D. B. West [10] and J. A. Gallian [2], for standard terminology and notations.

Definition 1.1 Let G be a graph with p vertices and q edges. Let r be any natural number. An injective function $\phi : V(G) \to \{0, ki, 1, 1+ki, 2, +ki, ..., GT_{q+r-1}\}$ *for all* k ,where GT_{q+r-1} is the $(q+r-1)^{th}$ Gaussian Tribonacci number in the Gaussian Tribonacci sequence is said to be Gaussian Tribonacci graceful labeling if the induced edge labeling $\phi^* : E(G) \to \{GT_1, GT_2, ..., GT_{q+r-1}\}$ such that $\phi^*(uv) = |\phi(u) - \phi(v)|$ is bijective. If a graph G admits Gaussian Tribonacci graceful labeling, then G is called a Gaussian Tribonacci graceful graph. A graph G is said to be Gaussian Tribonacci arbitrarily graceful if it is Gaussian Tribonacci r-graceful for all r.

Remark 1.1 [11] The Gaussian Tribonacci sequence is obtained as follows:
$GT_0 = 0, GT_1 = 1, GT_2 = 1+i$ *and* $GT_n = GT_{n-1} + GT_{n-2} + GT_{n-3} \ \forall \ n \geq 3$
$ie, \{0, 1, 1+i, 2+i, 4+2i, 7+4i, 13+7i, 24+13i, ...\}$ is the Gaussian Tribonacci sequence.

Definition 1.2 [9] The Comb graph $P_n \Theta K_1$ is obtained by joining a single pendent edge to each vertex of the path P_n.

Definition 1.3 [3] A regular caterpillar graph $P_m \Theta n K_1$ is obtained from the path P_m by joining nK_1 vertices to each vertices of the path P_m.

Definition 1.4 [8] The Bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.

Definition 1.5 [10] The Subdivision of Bistar graph $S(B_{m,n})$ is obtained by subdividing each edge of a Bistar graph $B_{m,n}$.

Definition 1.6[9] A Coconut Tree graph CT(m,n) is obtained from the path P_n by appending n new pendent edges at an end vertex of P_n.

2. Main Results

Theorem 2.1 The path graph P_n is Gaussian Tribonacci arbitrarily graceful for all $n \geq 2$.

Proof.

Let P_n be a path graph of length $n-1$ with vertex set $V(P_n) = \{u_i / 1 \leq i \leq n\}$ and edge set $E(P_n) = \{u_i u_{i+1} / 1 \leq i \leq n-1\}$ such that $|V(P_n)| = p = n$ *and* $|E(P_n)| = q = n-1$

Define $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all* k by

$\phi(u_1) = T_0$

$\phi(u_i) = \phi(u_{i-1}) + (-1)^i GT_{q+r-i+1},\ 2 \leq i \leq n,\ r \geq 1$

Thus ϕ admits Gaussian Tribonacci graceful labeling for all r .

Hence the path graph P_n is Gaussian Tribonacci arbitrarily graceful for all $n \geq 2$.

Example 2.1

The Gaussian Tribonacci 2-graceful labeling of path graph P_5 is given in Figure 2.1

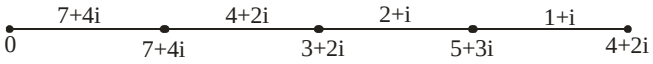

Figure 2.1

Theorem 2.2 The Comb graph $P_n \Theta K_1$ is Gaussian Tribonacci arbitrarily graceful for all $n \geq 2$.

Proof.

Let $u_i, 1 \leq i \leq n$ be the vertices of the path P_n and let $v_i, 1 \leq i \leq n$ be the vertices which are joined to each vertices of the path P_n .The resultant graph $P_n \Theta K_1$ whose vertex set is $V(P_n \Theta K_1) = \{u_i, v_i / 1 \leq i \leq n\}$ and edge set is $E(P_n \Theta K_1) = \{\{u_i v_i / 1 \leq i \leq n\} \cup \{u_i u_{i+1} / 1 \leq i \leq n-1\}\}$ such that $|V(P_n \Theta K_1)| = p = 2n$ and $|E(P_n \Theta K_1)| = q = 2n-1$

Case 1

For $n = 2$

Define $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all k* by

$\phi(u_1) = GT_0$, $\phi(u_2) = GT_{r+1}, r \geq 1$,

$\phi(v_1) = GT_{q+r-1}, r \geq 1$, $\phi(v_2) = \phi(u_2) - GT_r$, $r \geq 1$

Case 2

For $n \geq 3$

Define $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all k* by

$\phi(u_1) = GT_0$, $\phi(u_i) = \phi(u_{i-1}) + (-1)^i GT_{q+r-i+1}, 2 \leq i \leq n, r \geq 1, \phi(v_1) = GT_r, r \geq 1$

$\phi(v_i) = \phi(u_i) - GT_{r+i-1}, 2 \leq i \leq n, r \geq 1$

Thus ϕ admits Gaussian Tribonacci graceful labeling for all r.

Hence the comb graph $P_n \Theta K_1$ is Gaussian Tribonacci arbitrarily graceful for all $n \geq 2$.

Example2.2

The Gaussian Tribonacci 3-graceful labeling of Comb graph $P_2 \Theta K_1$ is given in Figure 2.2

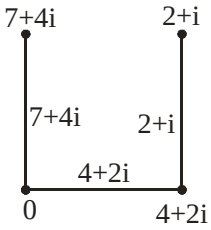

Figure 2.2

The Gaussian Tribonacci 2-graceful labeling of Comb graph $P_5 \Theta K_1$ is given in Figure 2.3

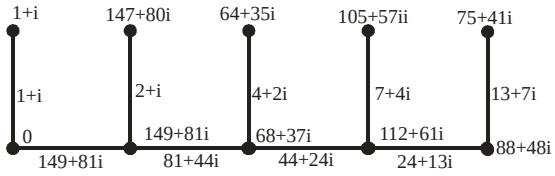

Figure 2.3

Theorem 2.3 The Coconut Tree graph CT (m, n) is Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 2$.

Proof. Let $u_i, 1 \leq i \leq m$ be the vertices of the path P_m and let $v_i, 1 \leq i \leq n$ be the new vertices which are attached to the m^{th} vertex of the path P_m. The resultant graph is CT (m, n) whose vertex set is $V[CT(m,n)] = \{\{u_i / 1 \leq i \leq m\} \cup \{v_i / 1 \leq i \leq n\}\}$ and edge set

is $E[CT(m,n)] = \{\{u_i u_{i+1} / 1 \le i \le m-1\} \cup \{u_m v_i / 1 \le i \le n\}\}$ such that $|V[CT(m,n)]| = p = m+n$ and $|E[CT(m,n)]| = q = m+n-1$

Define $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all k* by

$\phi(u_1) = GT_0$,

$\phi(u_i) = \phi(u_{i-1}) + (-1)^i GT_{q+r-i+1}, 2 \le i \le m, r \ge 1$

$\phi(v_i) = \phi(u_m) - GT_{r+i-1}, 1 \le i \le n, r \ge 1$

Thus ϕ admits Gaussian Tribonacci graceful labeling for all r.

Hence the Coconut Tree graph CT(m,n) is Gaussian Tribonacci arbitrarily graceful for all $m, n \ge 2$.

Example 2.3

The Gaussian Tribonacci1-graceful labeling of Coconut Tree graph $CT(5,5)$ is given in Figure 2.4

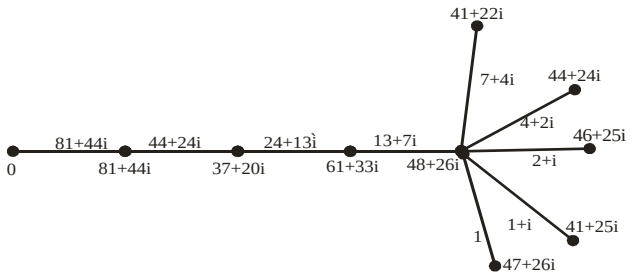

Figure 2.4

Theorem 2.4 The regular caterpillar graph $P_m \Theta nK_1$ is Gaussian Tribonacci arbitrarily graceful for all $m, n \ge 2$

Proof.

Let $v_i, 1 \le i \le m$ be the vertices of the path P_m and let $v_{ij}, 1 \le i \le m, 1 \le j \le n$ be the vertices attached to each vertices of the path P_m. The resultant graph is $P_m \Theta nK_1$ whose vertex set is $V[P_m \Theta nK_1] = \{\{v_i / 1 \le i \le m\} \cup \{v_{ij} / 1 \le i \le m, 1 \le j \le n\}\}$ and edge set is $E[P_m \Theta nK_1] = \{\{v_i v_{i+1} / 1 \le i \le m-1\} \cup \{v_i v_{ij} / 1 \le i \le m, 1 \le j \le n\}\}$ such that $|V[P_m \Theta nK_1]| = p = m + mn$ *and* $|E[P_m \Theta nK_1]| = q = m + mn - 1$

Define a function $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all k* by

$\phi(v_1) = GT_0$, $\phi(v_i) = \phi(v_{i-1}) + (-1)^i GT_{q+r-i+1}, 2 \le i \le m, r \ge 1$

$\phi(v_{1j}) = GT_{q+r-1-j-(m-2)}, 1 \le j \le n, r \ge 1$

$\phi(v_{ij}) = \phi(v_i) - GT_{q+r-1-(i-1)n-j-(m-2)}, 2 \le i \le m, 1 \le j \le n, r \ge 1$

Thus ϕ admits Gaussian Tribonacci r-graceful labeling for all r .

Hence the regular caterpillar graph $P_m \Theta nK_1$ are Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 2$.

Example 2.4 The Gaussian Tribonacci 2-graceful labeling of Regular caterpillar graph $P_4 \Theta 2K_1$ is given in Figure 2.5

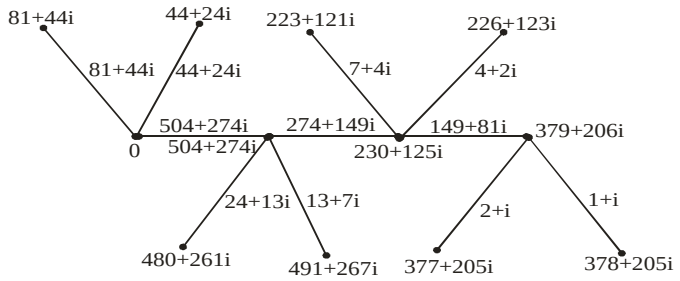

Figure 2.5

Theorem 2.5 The Bistar graph $B_{m,n}$ is Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 2$.

Proof.

Let u, v be the vertices of K_2 and let $u_i, 1 \leq i \leq m$ be the m vertices attached to one end of K_2 and $v_j, 1 \leq j \leq n$ be the n vertices attached to the other end of K_2. The resultant graph is $B_{m,n}$ whose vertex set is $V(B_{m,n}) = \{u_i, v_j, u, v / 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set is $E(B_{m,n}) = \{\{uu_i / 1 \leq i \leq m\} \cup \{vv_j / 1 \leq j \leq n\} \cup \{uv\}\}$ such that $|V(B_{m,n})| = p = m + n + 2$ and $|E(B_{m,n})| = q = m + n + 1$

Define a function $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all k* by

$\phi(u) = GT_0, \phi(v) = GT_{q+r-1}, \phi(v_i) = \phi(v) - GT_{q+r-i-1}, 1 \leq i \leq n, r \geq 1$

$\phi(u_i) = GT_{q+r-n-1-i}, 1 \leq i \leq m, r \geq 1$

Thus ϕ admits Gaussian Tribonacci r-graceful labeling for all r.

Hence $B_{m,n}$ is Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 2$.

Example 2.5

The Gaussian Tribonacci 1-graceful labeling of Bistar graph $B_{2,4}$ is given in Figure 2.6

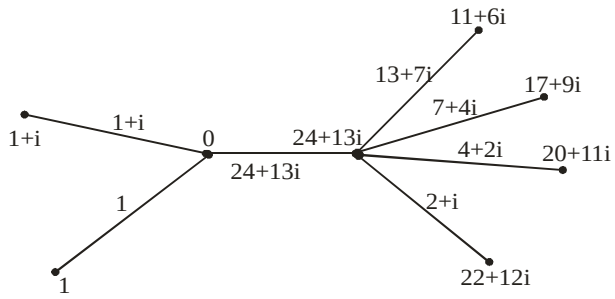

Figure 2.6

Theorem 2.6 The subdivision of the bistar graph $S(B_{m,n})$ is Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 1$

Proof.

Let u, v be the central vertices of the Bistar graph $B_{m,n}$ and let $u_i, 1 \leq i \leq m$ *and* $v_i, 1 \leq i \leq n$ be the vertices joined with u and v respectively. Let $s, u_i^1, 1 \leq i \leq m$ *and* $v_i^1, 1 \leq i \leq n$ be the new vertices obtained by subdividing the edges $uv, uu_i, 1 \leq i \leq m$ *and* $vv_i, 1 \leq i \leq n$ respectively. The resulting graph is $S(B_{m,n})$ whose vertex set is

$V[S(B_{m,n})] = \{ \{ u_i, u_i^1 / 1 \leq i \leq m \} \cup \{v_i, v_i^1 / 1 \leq i \leq n \} \cup \{u, v\} \cup \{s\} \}$ and edge set is $E[S(B_{m,n})] = \{ \{ u_i^1 u_i, uu_i^1 / 1 \leq i \leq m \} \cup \{us\} \cup \{sv\} \cup \{ vv_i^1, v_i^1 v_i / 1 \leq i \leq n \} \}$

such that $|V[S(B_{m,n})]| = p = 2(m+n)+3$ and $|E[S(B_{m,n})]| = q = 2(m+n)+2$

Define a function $\phi : V(P_n) \to \{0, ki, 1, 1+ki, 2, 2+ki, ..., GT_{q+r-1}\}$ *for all* k by

$\phi(s) = 0, \phi(v) = GT_{q+r-1}, r \geq 1, \phi(u) = GT_{q+r-2}$ $\phi(u_i^1) = \phi(u) - GT_{r+i-1}, 1 \leq i \leq m, r \geq 1$

$\phi(u_i) = \phi(u_i^1) - GT_{2m+r-i}, 1 \leq i \leq m, r \geq 1,$ $\phi(v_i^1) = \phi(v) - GT_{q+r-2-i}, 1 \leq i \leq n, r \geq 1$

$\phi(v_i) = \phi(v_i^1) - GT_{2m+i+r-1}, 1 \leq i \leq n, r \geq 1$

Thus ϕ admits Tribonacci r-graceful labeling for all r.

Hence the subdivision of the bistar graph $S(B_{m,n})$ is Gaussian Tribonacci arbitrarily graceful for all $m, n \geq 1$.

Example 2.6

The Gaussian Tribonacci 1-graceful labeling of Subdivision of Bistar graph S($B_{2,2}$) is given in Figure 2.7

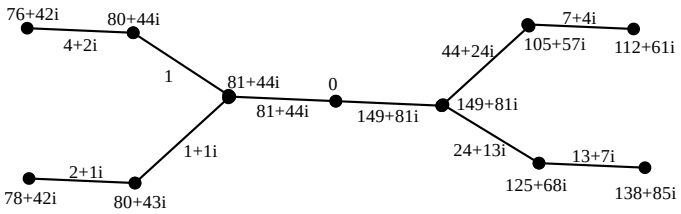

Figure 2.7

3. Conclusion

In this paper, we investigate the path graph, the Comb graph, the Coconut Tree graph, the Regular caterpillar graph, the Bistar graph and the Subdivision of Bistar graph are Gaussian Tribonacci arbitrarily graceful. In future, we investigate Gaussian Tribonacci arbitrarily graceful labeling of cycle related graphs.

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On The Study of Edge Monophonic Vertex Covering Number

K. A. Francis Jude Shini[*]
S. Durai Raj[†]
X. Lenin Xaviour[‡]
A. M. Anto[§]

Abstract

For a connected graph G of order n ≥ 2, a set S of vertices of G is an edge monophonic vertex cover of G if S is both an edge monophonic set and a vertex covering set of G. The minimum cardinality of an edge monophonic vertex cover of G is called the edge monophonic vertex covering number of G and is denoted by $\boldsymbol{m_{ea}(G)}$. Any edge monophonic vertex cover of cardinality $\boldsymbol{m_{ea}(G)}$ is a $\boldsymbol{m_{ea}(G)}$-set of G. Some general properties satisfied by edge monophonic vertex cover are studied.

Keywords: monophonic set; edge monophonic set; vertex coveringset; edgemonophonic vertex cover; edge monophonic vertex covering number.

2010 AMS subject classification: 05C12[**].

[*]Research Scholar, Reg No: 20213132092001, Department of Mathematics, (Pioneer Kumaraswamy College, Nagercoil-629003, Tamil Nadu, India.); shinishini111@gmail.com
[†]Associate Professor and Principal, Department of Mathematics, (Pioneer Kumaraswamy College, Nagercoil-629003, Tamil Nadu, India.); durairajsprinc@gmail.com
[‡]Assistant Professor, Department of Mathematics, (Nesamony Memorial Christian College, Marthandam-629165, Tamil Nadu, India.); leninxaviour93@gmail.com
[§]Assistant Professor, Department of Mathematics, (St. Alberts College(Autonomous), Ernakulam, Kochi, India.); antoalexam@gmail.comAffiliated to ManonmaniamSundaranar University, Abishekapatti, Tirunelveli – 627 012, TamilNadu, India.
[**] Received on June 10 th, 2022. Accepted on Sep 1st, 2022. Published on Nov30th, 2022. doi: 10.23755/rm.v44i0.907. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.This paper is published under the CC-BY licence agreement.

1. Introduction

By a graph G=(V,E) , we mean a finite undirected connected graph without loops and multiple edges. The order and size of G are denoted by n and m respectively. Also $\delta(G)$ is the minimum degree in a graph G. For basic graph theoretic terminology, we refer to Harary[7]. The distance d(u,v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G(1) For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest fromv. The minimum eccentricity among the vertices of G is the radius, rad G and the maximum eccentricity is its diameter, diamG. The neighbourhood of a vertex v of G is the set N(v) consisting of all vertices which are adjacent with v. A vertex v is a simplicial vertex or an extreme vertex of G if the subgraph induced by its neighbourhood N(v) is complete. A caterpillar is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar.

A diametralpath of a graph is a shortest path whose length is equal to the diameter of the graph. A tree containing exactly two non-pendent vertices is called a doublestar denoted by $S_{K_1 k_2}$ where k_1 and k_2 are the number of pendent vertices on these two non-pendent vertices. A graph G is called triangle free if it does not contain cycles of length 3. A set of vertices no two of which are adjacent is called an independentset. By a matching in a graph G, we mean an independent set of edges of G. A maximalmatching is a matching M of a graph G that is not a subset of any other matching. The independencenumber$\beta(G)$ of G is the maximum number of vertices in an independent set of vertices of G.

A geodeticset of G is a set $S \subseteq V$(G) such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodeticnumber$g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinalities$g(G)$ is a minimumgeodeticset or a geodeticbasis or a g-set of G. The geodetic number of a graph was introduced in [2, 8] and further studied in [3-5].

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A set S of vertices of G is a monophonic set of G if each vertex v of G lies on an $x-y$ monophonic path for some $x,y \in S$. The minimum cardinality of a monophonic set of G is the monophonicnumber of G and is denoted by $m(G)$. Any monophonic set of cardinalities$m(G)$ is a minimummonophonicsetoramonophonicbasisora$m$-set of G. The monophonic number of a graph was studied and discussed in [9, 12].

A set S of vertices in G is called an edgemonophonicset of G if every edge of G lies on a monophonic path joining some pair of vertices in S and the minimum cardinality of an edge monophonic set is the edgemonophonicnumber$m_e(G)$ of G. An edge monophonic set of cardinalities$m_e(G)$ is called an m_e-set of G. The edge monophonic number of a graph was introduced in and further studied in [10].

A subset $S \subseteq V(G)$ is said to be a *vertexcoveringset* of G if every edge has at least one end vertex in S. A vertex covering set of G with the minimum cardinality is called a

minimumvertexcoveringset of G. The *vertexcoveringnumber* of G is the cardinality of any minimum vertexcovering set of G. It is denoted by $\alpha(G).The$ vertex covering number was studied in [14].
For a connected graph G of order $n \geq 2$, a set S of vertices of G is an*monophonicvertexcover* of Gif$S$ is both a monophonic set and a vertex covering set of G. The minimum cardinality of ae monophonic vertex cover of G is called the *monophonicvertexcoveringnumber* of G and is denoted by $m_\alpha(G)$. Any monophonic vertex cover of cardinality $m_\alpha(G)$ is a m_α-set of G.
A subset $S \subseteq V(G)$ is a *dominatingset* if every vertex in V-S is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in a graph G is called the *dominating umber* of G and denoted by $\gamma(G)$. The dominating number of a graph was studied in [6]. A set of vertices of G is said to be *monophonicdominationset* if it is both a monophonic set and a dominating set of G. The minimum cardinality of a monophonic domination set of G is called a *monophonicdominationnumber* of G and denoted by $\gamma_m(G)$. The monophonic domination number was studied in [11]. A set of vertices of a graph G is an *edgemonophonicdominationset* if it is both edge monophonic set and a domination set of G. The minimum cardinality of an edge monophonic domination set of G is called an *edgemonophonicdominationnumber* of G and denoted by $\gamma_{me}(G)$. The edge monophonic domination number was studied in [13].

The following theorems will be used in the sequel.

Theorem 1.1. [10] Every extreme vertex of a connected graph G belongs to every edge monophonic set of G.
In particular, each end vertex of G belongs to every edge monophonic set of G.

Theorem 1.2. [10] Let G be a connected graph with cut-vertices and S be an edge monophonic set of G. If v is a cut-vertex of G, then every component of G-v contains an element of S.

2.The Edge Monophonic Vertex Coverofa Graph

Definition 2.1. Let G be a connected graph of order $n\geq$ 2. Aset S of vertices of G is an *edgemonophonicvertexcover* of Gif$S$ is both an edge monophonic set and a vertex cover of G. The minimum cardinality of an edge monophonic vertex cover of G is called the *edgemonophonicvertexcoveringnumber* of G and is denoted by $m_{e\alpha}(G)$. Any edge monophonic vertex cover of cardinality $m_{e\alpha}(G)$ is a $m_{e\alpha}$-set of G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1, v_4\}$ is a minimum
edge monophonic set of G so that $m_e(G) = 2$ and $S' = \{v_1, v_2, v_4, v_6\}$ is a minimum edge monophonic vertex cover of G so that $m_{e\alpha}(G) = 4$. Thus, the edge monophonic number and the edge monophonic vertex covering number of a graph are different.

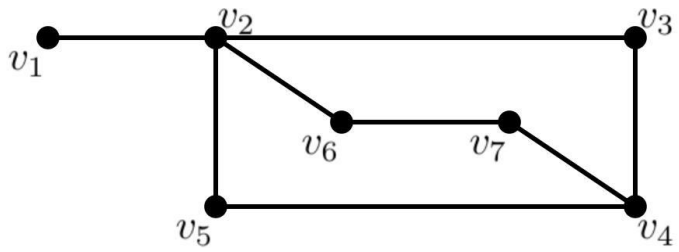

Figure 2.1: G

Remark 2.3. For the graph G given in Figure 2.2, $S = \{v_1, v_2, v_5\}$ is a minimum monophonic vertex cover of G so that $m_\alpha(G) = 3$ and $S' = \{v_1, v_2, v_3, v_4\}$ is a minimum edge monophonic vertex cover of G so that $m_{e\alpha}(G) = 4$. Hence the monophonic vertex covering number is different from the edge monophonic vertex covering number of a graph.

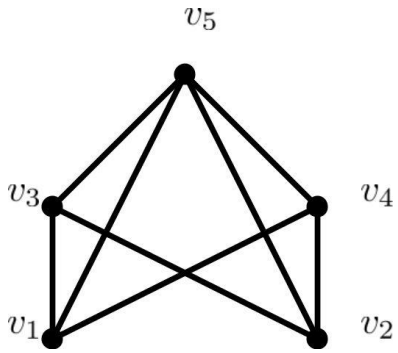

Figure 2.2: G

Remark 2.4. For the graph G given in Figure 2.3, $S = \{v_1, v_{11}\}$ is a minimum edge monophonic set of G so that $m_e(G) = 2. S' = \{v_1, v_8, v_{11}\}$ is a minimum edge monophonic dominating set of G so that $\gamma_{me}(G) = 3$ and $S'' = \{v_1, v_2, v_3, v_4, v_8, v_{11}\}$ is a minimum edge monophonic vertex cover of G so that $m_{e\alpha}(G) = 6$. Hence the edge monophonic vertex covering number of a graph is different from the edge monophonic number and the edge monophonic dominating number of a graph.

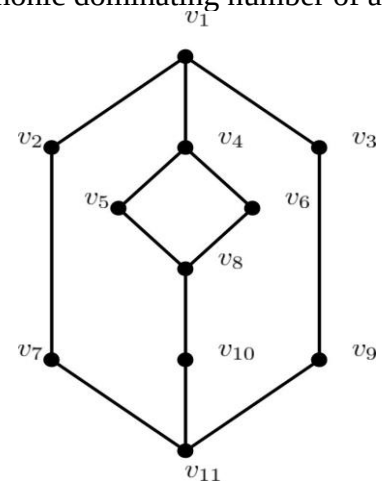

Figure 2.3: G

Theorem 2.5. For any connected graph G,$2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq n$.
Proof. Any edge monophonic set of G needs at least 2 vertices and so $2 \leq \max\{\alpha(G), m_e(G)\}$. From the definition of edge monophonic vertex cover of G, we have, $\max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G)$. Clearly V(G) is an edge monophonic vertex cover of G. Hence $m_{e\alpha}(G) \leq n$. Thus $2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq n$.□

Remark 2.6. The bounds in Theorem 2.5 are sharp. For the complete graph $K_n (n \geq 2)$,$m_{e\alpha}(K_n) = n$. In Remark 2.3, we have, $m_e(G) = m_{e\alpha}(G) = 4$
The bounds are strict in Example 2.2 as $\alpha(G) = 3$,$m_e(G) = 2$,$m_{e\alpha}(G) = 4$.Here $2 < 3 < 4 < 7$.

Remark 2.7. Clearly union of a vertex covering set and an edge monophonic set of G is an edge monophonic vertex cover of G. In Figure 2.1, $S = \{v_1, v_2, v_4, v_6\}$ is an edge monophonic vertex cover, in Figure 2.2, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is an edge monophonic vertex cover and in Figure 2.3, $S = \{v_1, v_2, v_3, v_4, v_8, v_{11}\}$ is an edge monophonic vertex cover.
Thus $2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq \min\{\alpha(G) + m_e(G), n\}$.

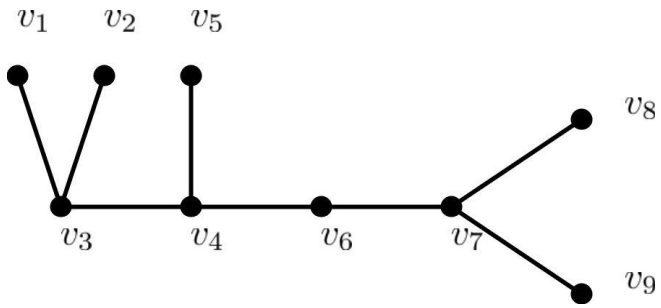

Figure 2.4: G

For the graph G in Figure 2.4, we observe that $S_1 = \{v_3, v_4, v_7\}$ is a minimum vertex cover of G so that $\alpha(G) = 3$,$S_2 = \{v_1, v_2, v_5, v_8, v_9\}$ is a minimum edge monophonic set of G so that $m_e(G) = 5$ and $S_3 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9\} = S_1 \cup S_2$ is a $m_{e\alpha}$-set of G and so $m_{e\alpha}(G) = 8 = \alpha(G) + m_e(G) < n = 9$.

Theorem 2.8. Each extreme vertex of G belongs to every edge monophonic vertex cover of G.
In particular, each end vertex of G belongs to every edge monophonic vertex cover of G.

Proof. From the definition of $m_{e\alpha}$-set, every $m_{e\alpha}$-set of G is a m_e-set of G. Hence the result follows from Theorem 1.1. □

Corollary 2.9. For any graph G with k extreme vertices, $\max\{2, k\} \leq m_{e\alpha}(G) \leq n$.

Proof. The result follows from Theorem 2.5 and Theorem 2.8. □

Corollary 2.10. Let $K_{1,n-1} (n \geq 3)$ be a star. Then $m_{e\alpha}(K_{1,n-1}) = n - 1$.

Proof. Let x be the centre and $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all extreme vertices of $K_{1,n-1}(n \geq 3)$. Clearly S is a minimum edge monophonic vertex cover of $K_{1,n-1}(n \geq 3)$ by Theorem 2.8. Hence $m_{e\alpha}(K_{1,n-1}) = n - 1$.□

Corollary 2.11. For the complete graph $K_n(n \geq 2)$,$m_{e\alpha}(K_n) = n$.
Proof. Since every vertex of the complete graph $K_n(n \geq 2)$ is an extreme vertex, by Theorem 2.8, the vertex set is the unique edge monophonic vertex cover of K_n. Thus $m_{e\alpha}(K_n) = n$.□

Remark 2.12. The converse of Corollary 2.11 need not be true.
For the graph G given in Figure 2.5, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is an $m_{e\alpha}$-set of G so that $m_{e\alpha}(G) = 5$ and G is not complete.

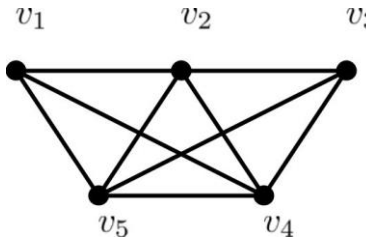

Figure 2.5: G

Theorem 2.13. Let G be a connected graph with cut-vertices and S be an edge monophonic vertex cover of G. If v is a cut-vertex of G, then every component of G-v contains an element of S.
Proof. From the definition of $m_{e\alpha}$-set, every $m_{e\alpha}$-set of G is a m_e-set of G. Hence the result follows from Theorem 1.2. □

Remark 2.14. The cut-vertex of G in Theorem 2.13 need not belong to S. For the graph G given in Figure 2.6, $S = \{v_1, v_3, v_5, v_6\}$ is aa edge monophonic vertex cover of G. Here 4 is a cut-vertex which does not belong to S and 3 is a cut-vertex which belong to S.

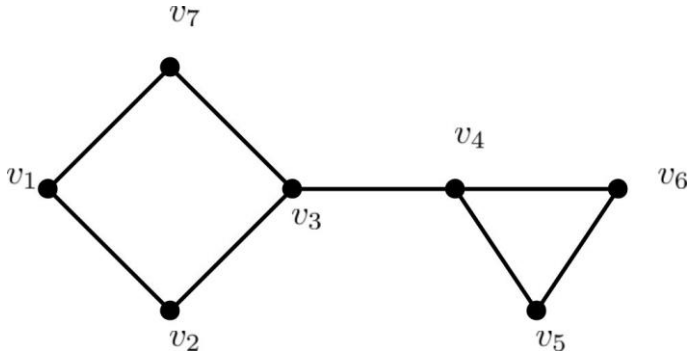

Figure 2.6: G

Theorem 2.15. If a and n are positive integers such that $2 \leq a \leq n$, then there exists a connected graph G of order n with $m_{e\alpha}(G) = a$.

Proof. We prove this theorem by considering two cases.

Case (i): $2 \leq a = n$. Let $G = K_n$. Then by Theorem 2.11, $m_{e\alpha}(G) = n = a$.

Case (ii): $2 \leq a < n$. Consider $H = K_{a-1}$, the complete graph on a-l vertices u_1,u_2,…, u_{a-1}. Add $n - a + 1$ new vertices v_1,v_2,... v_{n-a},x to H by joining the vertices v_1,v_2,…,v_{n-a} to both u_{a-1} and $x$and the graph G is shown in Figure 2.7.

Let $S = \{u_1, u_2, \dots, u_{a-2}\}$ be the set of all extreme vertices of G. Then by Theorem 1.1, they must belong to every edge monophonic set. Also, we observe that $S' = S \cup \{x\}$ is a minimum edge monophonic set. Also the edges of K_{a-1} and the edges $v_i x (1 \leq i \leq n - a)$ are covered by the vertices of S'. Now to cover the edges $u_{a-1}v_i (1 \leq i \leq n - a)$, we must include at least the vertex u_{a-1} to S'. Hence $S'' = \{u_1, u_2, \dots, u_{a-2}, u_{a-1}, x\}$ is a minimum edge monophonic vertex cover of G. Thus $m_{e\alpha}(G) = a < n$.□

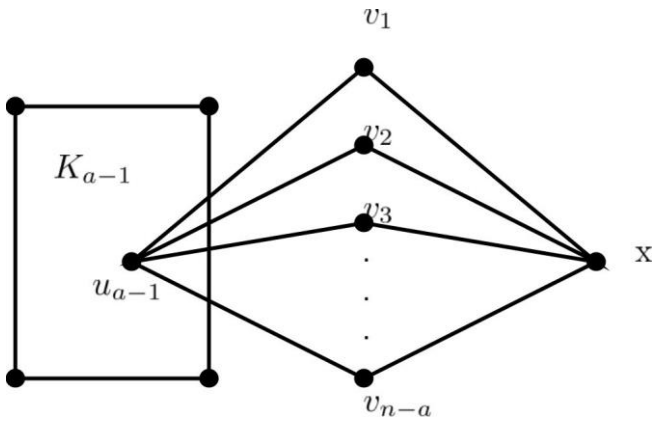

Figure 2.7: G

3.Conclusions

In this paper we analysed the edge monophonic vertex covering number of a graph. It is more interesting to continue my research in this area and it is very useful for further research.

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The Detour Domination and Connected Detour Domination values of a graph

R. V. Revathi[1]
M. Antony[2]

Abstract

The number of $\boldsymbol{\gamma dn}$-sets that $\boldsymbol{v}$ belongs to in G is defined as the detour domination value of $\boldsymbol{v}$, indicated by $\boldsymbol{\gamma D_V(v)}$, for each vertex $\boldsymbol{v} \in \boldsymbol{V(G)}$. In this article, we examined at the concept of a graph's detour domination value. The connected detour domination values of a vertex $\boldsymbol{v} \in \boldsymbol{V(G)}$, represented as $\boldsymbol{CD_V(G)}$, are defined as the number of $\boldsymbol{Cdn}$-sets to which a vertex belongs $\boldsymbol{v}$ to G. Some of the related detour dominating values in graphs' general characteristics are examined. This concept's satisfaction of some general properties is investigated. Some common graphs are established.

Keywords: domination number; detour number; detour domination value; connected detour domination value; etc.

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[1] Register Number 19133232092002, Research Scholar, Department of Mathematics, St. Jude's College, Thoothoor - 629 176, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India; revathi87gowri@gmail.com.
[2]Associate Professor, Department of Mathematics, St. Jude's College, Thoothoor - 629 176, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India; antony.michael@yahoo.com
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1. Introduction

Graph having the type G = (V, E) is a finite, undirected connected graph without loops or numerous edges. The order and size of the letter G are represented by the characters n and m, respectively. We refer to [3] for the fundamental terms used in graph theory. If uv is an edge of G, then two vertices u and v are said to be adjacent. If two edges of G connect a vertex, they are said to be adjacent. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is thelength of a shortest u-$v$path in G. A u-v geodesic is a u-v path of length $d(u,v)$.

The longest u-v path in G is also referred to as *detour distance* $D(u,v)$ between two vertices u and v in a linked graph G from u to v. A u-v detour is a u-v path of length $D(u,v)$. If x is a vertex of P that also contains the vertices u and v, then x is said to lie on a u-v detour. Every vertex of G is contained in a detour connecting some pair of vertices in S, which is the definition of a detour set of G. Any detour set of orderdn (G) is referred to as a minimum *detour set* of G or a dn -set of G. The *detour number* dn (G) of G is the minimum order of a detour set. These ideas have been researched in [4, 5, 6]. If for every v ∈ V \ D is adjacent to a vertex in D, then the set D⊆ V is a dominant set of G. If no subset of a dominating set D is a dominating set of G's, then D is said to be minimal. The symbol$\gamma(G)$ denotes the domination number of G, which is the least cardinality of a minimal set of G dominating sets. In [4], the graph's domination number was studied. If a set S is both a detour and a dominating set of G's, then it is referred to as a detour dominating set of G. Any detour dominating set of order $\gamma_d(G)$is referred to as a γ_d– set of G. The *detour domination number*$\gamma_d(G)$ of G is the minimal order of its detour dominating set. In [8], the detour domination number of a graph $\gamma_d(G)$studied. If a set S is a detour dominating set of G and its induction by S is connected, the set S ⊆ V(G) is referred to as a *connected detour dominating set* of G . Any connected detour dominating set with order $\gamma_{cd}(G)$ is referred to as aγ_{cd} - set of G. The *connected detour domination number* of $\gamma_{cd}(G)$ of G is the maximum order of its connected detour dominating sets. In [8,9], the connected detour domination number of a graph was investigated. The subsequent theorem is applied thereafter.

Theorem 1.1[3] Every detour set of a connected graph G contains each end vertex.

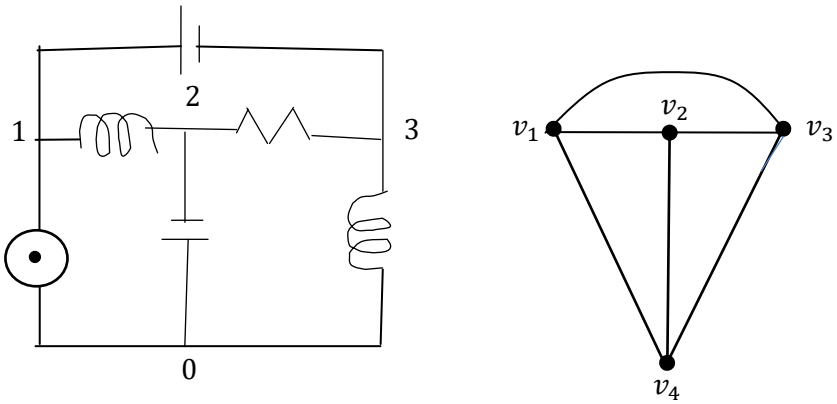
Theorem 1.2[3] Let G be a connected graph $n \geq 2$. Then $dn(G) = n$ if and only if $G = K_2$.

Theorem 1.3[3] Let G be a connected graph of order $n \geq 4$. Then $dn(G) = n - 1$ if and only if $G = K_{1,n-1}$.

2.The Detour Domination Value of a Graph

Definition 2.1. For each vertex$v \in V(G)$, we define the detour domination value of v, denoted by $\gamma D_V(v)$, to be the number of γdn-sets to which v belongs to G.

Example 2.2. In relation to the graph G in Figure 2.1, S_1= $\{v_1, v_3\}$,S_2= $\{v_1, v_2\}$, S_3= $\{v_1, v_4\}$, S_4= $\{v_2, v_3\}$, S_5= $\{v_2, v_4\}$, S_6= $\{v_3, v_4\}$are the onlysix minimum γdn-sets of G such that $\gamma D_V(v_1) = 3, \gamma D_V(v_2) = 3, \gamma D_V(v_3) = 3, \gamma D_V(v_4) = 3,\ \gamma\tau(G) = 6$.



G

Figure 2.1

Theorem 2.3. For the complete graph$G$= $K_n(n \geq 2)\gamma D_V(v) = n - 1,\ \gamma\tau(G) = nC_2$for each $v \in V(G)$.

Proof. Since any two sets of G's vertices is the γdn-set of G, thus $\gamma\tau(G) = nC_2$. Since each vertex of G belongs to exactly $n - 1\gamma dn$-sets, it follows that $\gamma D_V(v) = n - 1$ for each $v \in V(G)$.■

Theorem 2.4. For a star G= $K_{1,n-1}(n \geq 3)\ \gamma D_V(v) = 1, \gamma\tau(G) = 1$ for each $v \in V(G)$.

Proof. We have $G = K_{1,n-1}$. Let S represent the set of all of the end vertices in G. Then S is the unique γdn-set of G. Thus $\gamma\tau(G) = 1$. Therefore $\gamma D_V(v) = 1$ for each $v \in V(G)$.■

Theorem 2.5. For the complete bipartite graph $G = K_{m,n}$with bipartite sets X and Y.

and $\gamma\tau(G) = \begin{cases} mn \quad if\ m, n \geq 2 \\ 1\ if\ m = n = 1 \\ 1 \quad if\ \{m, n\} = \{1, x\} where\ x > 1 \end{cases}$

If $m, n \geq 2$ then $\gamma D_V(v) = \begin{cases} n, \quad if\ v \in X \\ m, \quad if\ v \in Y \end{cases}$

If $m = n = 1 = \gamma D_V(v) = 1$ for any v in $K_{1,1}$.

If $\{m,n\}=\{1,x\}$ with $x>1$ then$G=K_{1,x}.\gamma D_V(v)=\begin{cases}1, & if\ v\in X\\ 0, & if\ v\in Y\end{cases}$

Proof. Let $X=\{x_1,x_2,\dots,x_m\}$ and $Y=\{y_1,y_2,\dots,y_n\}$ be the two bipartite sets of G. Since any two adjacent vertices of G is a γdn-sets of G, it follows that $\gamma\tau(G)=mn$ if $m,n\geq 2$.

If $m,n=1$ then it has only one a γdnset of G such that $\gamma\tau(G)=1$.

If $\{m,n\}=\{1,x\}$ then it only one γdn-set of G such that $\gamma\tau(G)=1$.

If $v\in X$ then any vertex in Y belongs to aγdn-set of G hence $\gamma D_V(v)=n$. Also if $v\in Y$,then any vertex in X belongs to a γdn-set thus $\gamma D_V(v)=m$ for $m,n\geq 2$. If $m=n=1$, then $G=K_2,\gamma D_V(v)=1$ for any v in $K_{1,1}$. If $\{m,n\}=\{1,x\}$ with $x>1$, then$G=K_{1,x}$, $\gamma D_V(v)=\begin{cases}1, & if\ v\in X\\ 0, & if\ v\in Y\end{cases}$.■

Theorem 2.6. For the wheel graph $G=K_1+C_{n-1}\,(n\geq 5)$, $\gamma\tau(G)=\begin{cases}10, & n=5\\ 2n-2, & n\geq 6\end{cases}$ and $\gamma D_V(v)=\begin{cases}4, & if\ n=5\\ 3, & if\ n\geq 6\ \text{and}\ v\in V(C_{n-1})\\ n-1, & if\ n\geq 5\ , v\in V(K_1)\end{cases}$

Proof. Let $V(K_1)=x$ and $V(C_{n-1})=\{v_1,v_2,\dots,v_{n-1}\}$. Let $n=5$. Then $S_1=\{v_1,v_2\}$, $S_2=\{v_2,v_3\}$, $S_3=\{v_3,v_4\}$,$S_4=\{v_4,v_1\}$, $S_5=\{v_1,x\}$, $S_6=\{v_2,x\}$, $S_7=\{v_3,x\}$,$S_8=\{v_4,x\}$,$S_9=\{v_2,v_4\}$,$S_{10}=\{v_1,v_3\}$ are γdn-sets of G such that $\gamma D_V(v_1)=4,\gamma D_V(v_2)=4,\gamma D_V(v_3)=4,\gamma D_V(v_4)=4,\ \gamma D_V(x)=4$.

Let $n\geq 6$. Then any two adjacent vertices of G is a γdn-set of G so that $\gamma\tau(G)=(n-1)+(n-1)=2n-2$ for $v\in V(C_{n-1})$, hence v lies in exactly three γdn-set of G so that $\gamma D_V(v)=3$ for all $v\in V(C_{n-1})$. Since x is adjacent to $n-1$ vertices of $G,\gamma D_V(x)=n-1$.■

Theorem 2.7. For the cycle graph $G=C_n(n\geq 3)$, $\gamma\tau(C_n)=\begin{cases}3 & if\ n\equiv 0(mod\ 3)\\ n\left(1+\frac{1}{2}\left\lfloor\frac{n}{3}\right\rfloor\right) & if\ n\equiv 1(mod\ 3)\\ n & if\ n\equiv 2(mod\ 3)\end{cases}$

Proof. Let $V(C_n)=\{v_1,v_2,\dots,v_n\}$. Let $n=3k$, where $k\geq 1$, Here $\gamma dn(C_n)=k$,a γdn-set $\mathcal{F}$ comprises k K_1's and $\mathcal{F}$ is fixed by the choice of the first K_1. There exists exactly one $\gamma dn(C_n)$-set containing the vertex v_1, and there are two $\gamma dn(C_n)$-sets omitting the vertex v_1 such as $\mathcal{F}$ containing the vertex v_2 and $\mathcal{F}$ containing the vertex v_n. Thus $\gamma\tau(C_n)=3$.

Let $n=3k+1$, where $k\geq 1$. Here $\gamma dn(C_n)=k+1$, aγdn-set $\mathcal{F}$is constituted in exactly one of the following two ways.

i) $\mathcal{F}$comprises $(k-1)K_1$'s and one K_2.

ii) $\mathcal{F}$comprises$(k+1)K_1$'s.

Case(i) $\langle \mathcal{F} \rangle \cong (k-1)K_1 \cup K_2$: Note that $\mathcal{F}$ is fixed by the choice of the single K_2 choosing a K_2 in the same as choosing its initial vertex in the counter clockwise order. Hence $\tau = 3k+1$.

Case(ii)$\langle \mathcal{F} \rangle \cong (k+1)K_1$:It is clear that each$K_1$ dominates three vertices, exactly there are two vertices, say x and y, each of whom is adjacent to two distinct K_1's in $\mathcal{F}$. And $\mathcal{F}$ is fixed by the placements of x and y.There are $n = 3k+1$ ways of choosing x. Consider the P_{3k-2} (a sequence of $3k-2$ slots) obtained as a result of cutting from C_n the P_3 centered about x vertex. y may be placed in the first slot of any of the $\left\lceil \frac{3k-2}{3} \right\rceil = k$. As the order of selecting the two vertices x and y is immaterial $\tau = \frac{(3k+1)}{2}k$.

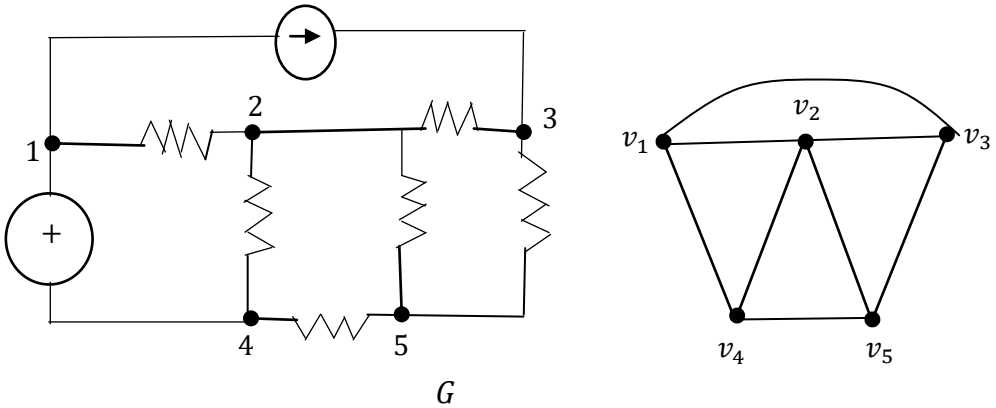
Summing over the two disjoint cases, we get $\gamma\tau(C_n) = (3k+1) + \frac{(3k+1)}{2}k = (3k+1)\left(1+\frac{k}{2}\right) = n\left(1+\frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor\right)$

Let $n = 3k+2$, where $k \geq 1$, Here$\gamma dn(C_n) = k+1$, a $\gamma dn(C_n)$-set $\mathcal{F}$ comprises of only $K_1's$ and is fixed by the placement of the only vertex which is adjacent to two distinct K_1'sin $\mathcal{F}$. Hence $\gamma\tau(C_n) = n$.■

3. The Connected Detour Domination Value of a Graph

Definition 3.1. For each vertex $v \in V(G)$,we define the connected detour domination values of v, denoted by $CD_V(G)$to be the number of Cdn-sets to which v belongs to G.

Example 3.2. For the graph G given in Figure 3.1, S_1= $\{v_1, v_2\}$,S_2= $\{v_1, v_3\}$, S_3= $\{v_1, v_4\}$, S_4= $\{v_2, v_4\}$, S_5= $\{v_2, v_3\}$, S_6= $\{v_2, v_5\}$, S_7=$\{v_3, v_5\}$, S_8= $\{v_4, v_5\}$are the only eight minimum Cdn-sets of G such that $CD_V(v_1) = 3, CD_V(v_2) = 4, CD_V(v_3) = 3$, $CD_V(v_4) = 3, CD_V(v_5) = 3and\tau_c(G) = 8$.



G

Figure 3.1

Proposition 3.3. Let G be a graph with n vertices without cut vertices and $\Delta = n-1$. Then $Cdn(G) = 2$ and $CD_V(v) \leq n-1 \; \forall v \in V(G)$ and equality holds if and only if $\deg(v) = n-1$.

Proof. Let x be a universal vertex of G. Let $y \in N(x)$. Then $S = \{x, y\}$ is a Cdn-set of G so that $Cdn(G) = 2$. Since x is a universal vertex of G x belongs to every Cdn-set of G. Since G contains at most $n-1$ Cdn-sets, $CD_V(v) \leq n-1$. Let $CD_V(v) = n-1$. Hence it follows that v belongs to every Cdn-set of G. Therefore $CD_V(v) = n-1$. The converse is clear. ■

Theorem3.4. For $n \geq 3, \tau_c(C_n) = n$ and $CD_V(v) = n-2 \; \forall v \in V(G)$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Then $S_i = V(C_n) - \{v_i, v_{i+1}\} (1 \leq i \leq n-1)$ and $S = V(C_n) - \{v_1, v_n\}$ are the n, Cdn-sets of G, so that $\tau_c(C_n) = n$. As C_n is vertex transitive $CD_V(v) = CD_V(v_1)$ for all $v \in V(C_n)$. Since v_1 belongs to $n-2$ Cdn-sets of C_n, it follows that $CD_V(v) = n-2$ for all $v \in V(C_n)$. ■

Theorem3.5. For $n \geq 2, \tau_c(P_n) = 1$ and $CD_V(v) = 1$ for each vertex $\forall v \in V(P_n)$.

Proof. Since $S = V(G)$ is the unique Cdn-sets of G the results follow theorem. ■

Theorem3.6. For the complete graph $G = K_n (n \geq 4), CD_V(v) = n-1, \tau_c(G) = nC_2$ for each vertex $v \in V(G)$.

Proof. Since any two set of vertices of G is the $\underline{Cdn}$-set of G, it follows that $\tau_c(G) = nC_2$. Since each vertex of G belongs to exatly $n-1$ Cdn-sets, it follows that $CD_V(v) = n-1$, for each vertex $v \in V(G)$. ■

Theorem3.7. For the wheel graph $G = K_1 + C_{n-1} (n \geq 5), \tau_c(G) = \begin{cases} 10, & n \geq 5 \\ 2n-2, & n \geq 6 \end{cases}$ and $CD_V(v) = \begin{cases} 4, & if \; n = 5 \\ 3, & if \; n \geq 6 \; and \; v \in V(C_{n-1}). \\ n-1, & if \; n \geq 5, \; v \in V(K_1) \end{cases}$

Proof. Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Let $n = 5$. Then $S_1 = V(v_1, v_2), S_2 = V(v_2, v_3), S_3 = V(v_3, v_4), S_4 = V(v_4, v_1), S_5 = V(v_1, x), \; S_6 = V(v_2, x), \; S_7 = V(v_3, x), S_8 = V(v_4, x), S_9 = V(v_2, v_4), S_{10} = V(v_1, v_3)$ are the Cdn-sets of G, such that $CD_V(v_1) = 4, CD_V(v_2) = 4, CD_V(v_3) = 4, CD_V(v_4) = 4, CD_V(x) = 4$ and $\tau_c(G) = 10$. Let $n \geq 6$. Then any two adjacent vertices of G is a Cdn-sets of G so that $\tau_c(G) = (n-1) + (n-1) = 2n-2$ for $v \in V(C_{n-1}), v$ lies in excatly three Cdn-sets of G so that $CD_V(v) = 3$ for all $v \in V(C_{n-1})$. Since x is adjacent to $n-1$ vertices of $G, CD_V(x) = n-1$. ■

Theorem3.8. Let $G = K_1 + P_{n-1}$ and $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.

Then for $n-1$ is odd $\tau_c(G)=3$ and $CD_V(v)=\begin{cases}2, & if\ v=x,v_1,v_{n-1}\\ 0, & otherwise\end{cases}$ and for $n-1$ is even $\tau_c(G)=4$ and $CD_V(v)\begin{cases}3, & if\ v=x\\ 2, & if\ v=v_1\ or\ v_{n-1}\\ 1, & if\ v=v_{\frac{n-1}{2}}\\ 0, & otherwise\end{cases}$.

Proof. Let $V(K_1)=x$ and $V(P_{n-1})=\{v_1,v_2,\dots,v_{n-1}\}$.

Case (i) $n-1$ is odd. $S_1=\{x,v_1\},S_2=\{x,v_{n-1}\},S_3=\{v_1,v_{n-1}\}$ are the only three Cdn-sets of G, such that $CD_V(x)=2,CD_V(v_1)=2,CD_V(v_{n-1})=2$ so that $\tau_c(G)=3$.

Case (ii) $n-1$ is even. $M_1=\{x,v_1\},M_2=\{x,v_{n-1}\},M_3=\left\{x,v_{\frac{n-1}{2}}\right\},M_4=\{v_1,v_{n-1}\}$ arethe only four Cdn-sets of G, so that $CD_V(x)=3,CD_V(v_1)=2,CD_V(v_{n-1})=2$, $CD_V\left(v_{\frac{n-1}{2}}\right)=1$ and$\tau_c(G)=4$. ■

Theorem3.9. $\tau_c(P_2\times P_n)=\begin{cases}4, & if\ n=2\\ 1, & if\ n=3\\ 8, & if\ n\geq 4\end{cases}$

Proof. Let S be a Cdn-sets of $P_2\times P_n$ of cardinality n where $n\geq 2$ if $n=2$, then $P_2\times P_n\cong C_4$ and any two adjacent vertices form a Cdn-set i.e.$\{u_1,v_1\},\{u_1,u_2\},\{v_1,v_2\},\{u_2,v_2\}$are all possible$\underline{Cdn}$-sets of $P_2\times P_2$. If $n=3$, there is a unique$\underline{Cdn}$-set$\{u_2,v_2\}$. So let $n\geq 4$. By lemma 2.2 either $\{u_3,u_4,\dots,u_{n-3},u_{n-2}\}\subset S$ or $\{v_3,v_4,\dots,v_{n-3},v_{n-2}\}\subset S$ (and not both). Let $\{u_3,u_4,\dots,u_{n-3},u_{n-2}\}\subset S$. As $v_3\notin S$, to maintain connectedness of $\langle S\rangle$ and to dominate u_1, we have $u_2\in S$. In the same way, $u_{n-1}\in S$. Thus $\{u_2,u_3,\dots,u_{n-2},u_{n-1}\}\subset S$. Since S contains n elements, let the other 2 vertices in S be l,m. To dominate u_1 and v_1, one of l and m (say l) must be either $u_1$or v_2. Similarly m is either u_n or v_{n-1}.Since there are two choices each for l and m such that S forms a Cdn-set, the number of Cdn-sets containing$u_3,u_4,\dots,u_{n-3},u_{n-2}$ is 4. Similarlythe number of Cdn-sets containing$v_3,v_4,\dots,v_{n-3},v_{n-2}$ is 4. Hence by lemma 2.2, we get $\tau_c(P_2\times P_n)=8$ for $n\geq 4$.■

Theorem3.10. Let$P_2\times P_n$ be a rectangular grid with $n\geq 2$ and let $a_i=u_i$ or v_i. If $n=2,$then $CD_V(v)=2$ for all $v\in V(P_2\times P_n)$. If $n=3$, then $CD_V(a_1)=CD_V(a_3)=0$ and $CD_V(a_2)=1$, If $n\geq 4$ then $CD_V(a_i)=\begin{cases}2, & if\ i=1\ or\ n\\ 6, & if\ i=2\ or\ n-1\\ 4, & otherwise\end{cases}$

Proof. The proof is clear for $n=2$ and theorem 2.10, so we assume that $n\geq 4$. Let v be a vertex in $P_2\times P_n$.

Case 1: $[v\in\{u_1,v_1,u_n,v_n\}]$. Let $v=u_1$, then using the line of proof of Theorem 3.10, theCdn-sets containing u_1 are precisely those where $l=u_1$ and m is either u_n or v_{n-1} i.e,$CD_V(v)=2$. Same for the case when $v=V_1$ or $v=u_n$ or $v=v_n$.

Case 2: $[v \in \{u_2, v_2, u_{n-1}, v_{n-1}\}]$. Let $v = u_2$. Note that any connected dominating set contains either u_2, v_2. Also total number of minimum connected dominating sets is 8, out of which only two does not contain u_2, namely $\{v_1, v_2, \dots, v_n\}$ and $\{v_1, v_2, \dots, v_{n-1}, u_{n-1}\}$. Thus $CD_V(u_2) = 8 - 2 = 6$. Now,as there exist isomorphisms which maps u_2 to v_2, u_{n-1}, v_{n-1}respectively, by proposition 2.2, we have $CD_V(u_2) = CD_V(v_2) = CD_V(u_{n-1}) = CD_V(v_{n-1}) = 6$.

Case 3: $[v \notin \{u_1, v_1, u_2, v_2, u_{n-1}, v_{n-1}, u_n, v_n\}]$. In this case, from the proof of Theorem 2.10 we have $CD_V(v) = 4$.

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Extraction of aspects from Online Reviews Using a Convolution Neural Network

Vidya Kamma[1]
Sridevi Gutta[2]
Teja Santosh Dandibhotla[3]

Abstract

The quality of the product is measured based on the opinions gathered from product reviews expressed on a product. Opinion mining deals with extracting the features or aspects from the reviews expressed by the users. Specifically, this model uses a deep convolutional neural network with three channels of input: a semantic word embedding channel that encodes the semantic content of the word, a part of speech tagging channel for sequential labelling and domain embedding channel for domain specific embeddings which is pooled and processed with a Softmax function. This model uses three input channels for aspect extraction. Experiments are conducted on amazon review dataset. This model achieved better results.

Keywords: Neural network, softmax function and extraction.

AMS Classification: 68T07[4]

[1] Research Scholar, Department of Computer Science and Engineering, KoneruLakshmaiah Education Foundation, Vaddeswaram, A.P., India. & Assistant Professor, Department of Computer Science, Neil Gogte Institute of Technology. Mail Id: kammavidya@gmail.com

[2] Professor, KoneruLakshmaiah Education Foundation, Vaddeswaram, A.P., India. Mail Id:sridevi.gutta2012@gmail.com

[3] Associate Professor, Sreenidhi Institute of Science and Technology, Yamnampet, Ghatkesar, Telangana, India. Mail Id:tejasantoshd@gmail.com

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1. Introduction

Customers are expressing views and opinions of products purchased. Customers can leverage the information contained in such reviews to identify the best products. These reviews help organizations to identify customer needs. The rise of reviews is gaining greater attention in research community. When it comes to traditional sentiment analysis, talk about opinion on entire sentence. Customer reviews often express opinion on different aspects of product rather than opinion on whole product Schouten, K., & Frasincar, F. (2015). This raised the problem of aspect extraction for sentiment analysis. Many challenge to be addressed, identifying implicit aspects, multiple aspects in single statement.

The major task of AE (Aspect Extraction) to Extracting aspects of customer reviews Cheung, D. 2009, et al. Jakob, N., & Gurevych, I. (2010), extracted aspects of a spatial dataset using conditional random fields, the limitation is it need to work of large datasets and is linear in nature. Hu, M., & Liu, B. (2004), used linguistic models for extracting aspects from reviews. The limitation is rules to be drafted manually. Efficiency based on grammatical accuracy of sentences.

Neural network models are widely used for aspect extraction. Existing CNN based models are lacking in extracting context-level features. Ye, H. et al., (2017) and Dong, F. et al., (2017), used CNN integrated with LSTM for aspect extraction achieved better performance with increase the complexity of the model parameters. In this paper, to overcome above mentioned issues we contributed the following.

To encode contextual information, three different input channels were used: i) Word embeddings + lexicon ii) Domain embeddings iii) POS Tag embeddings. Previous models used either pre-trained embeddings or review embeddings Poria, S. et al., (2016). To increase performance of model we included lexicons in word embedding layer. The first two channels are used to capture semantic and syntactic information. Third channel is used for sequential labelling of aspects.

2. Related work

Among the many tasks involved in aspect level sentiment analysis, a major one is aspect extraction (Liu B, 2012). Current approaches to extracting features are based on deep learning. Macháček, J. (2016) implemented traditional supervised machine learning method which uses biagram (BOW) bag of words. Li and Lam, (2017) performed aspect extraction by considering annotated data and achieved better results. Luo et al., (2018) integrated word embedding with CRF&BiLSTM for better extracting the aspects. Toh, Z., &Su, J. (2016) combined CNN with binary classifier and implemented anhybrid approach. Approach was top rated in SemEval 2016 Task 5 competition. Jihan, N. et al., (2017) used pre-processing pipeline for normalizing the data. They implemented multi-domain feature extraction and predicted aspect category using SVM. Khalil, T.et al (2016) initialized pre-trained word vectors for CNN and bag of words as features to ensemble classifier. To improve classification, they used secondary classifier. Stéphan Tulkens et al., (2020), implemented (CAt) Contrastive

Attention mechanism an unsupervised approach based on RBF kernel for extraction aspects. Ruidan He et al., (2017), extracted coherent aspects using neural approach. They used attention mechanism in the training phase to minimize irrelevant words. Poria, S. et al., (2016), used Deep CNN with 7 layers plus Linguistics Patterns for extracting aspects in product reviews. Wang et al., (2016) proposed the method which combined both CRF and dependency tree for better aspect extraction. Wang et al., (2014), based on seeding words implemented the extraction of product aspects was done using two semi-supervised models. Collobert et al., (2011) initialized CNN with word embeddings to solve semantic role labelling and named entity recognition problem. YinYang et al., (2017), proposed (CAT-LDA) a two-layer topic model to extract hierarchical aspects i.e parent and child category. Lin Wang (2015), used Restricted Boltzmann Machines to extract aspects. Xu et al., (2017) and Li & Lam (2018), used deep learning model for extracting aspect and opinion items.

3. Model

The proposed model has three input layers. Three embedding layers word-embedding layer enriched word vectors using lexicons, domain embedding layer, POS embedding layer, CNN layers with ReLu-nonlinear activation function, fully connected layer, pooling layer for
multiclass labels a softmax classifier is used for labelling y={B,I,O,E,S},with beginning, inside, outside, end and single of the aspect term. Figure 1. Shows the proposed CNN model

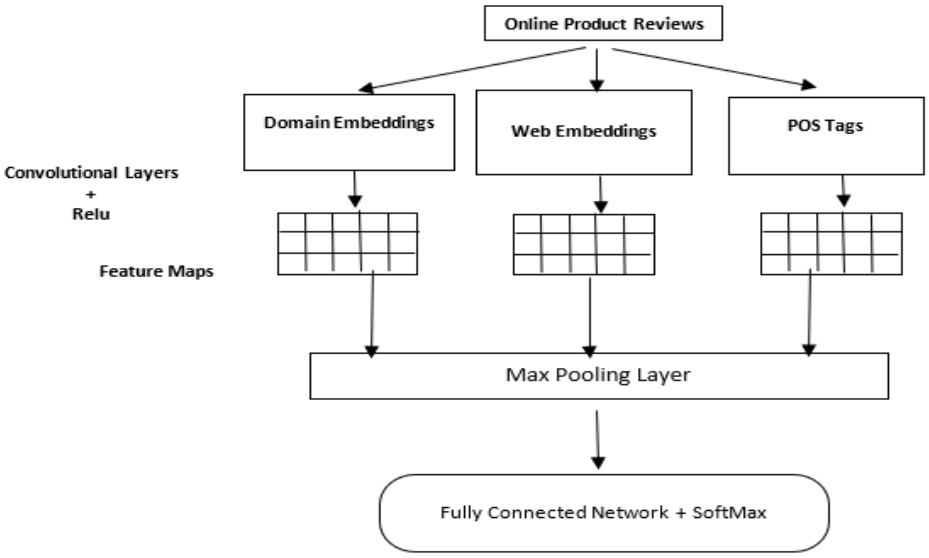

Figure 1. Proposed Convolution Neural Network model

Pre-processing

Mainly pre-processing is performed to clean and convert the reviews for further processing. We used python with NLP libraries are used for removal of numbers, converting acronyms, white spaces, tabs, stop words, words with length of 1 and special characters. Repeated letter words i.e "goooooood" or replaced with original form. All the review sentences are converted to lower case.

Word Embeddings

The three inputs google corpus, amazon reviews, POS tagging are mapped to vectors. Padding is performed to maintain the sentences of equal length. Let the input $w.x_{i+h}$ $x = (x_1, \ldots\ldots, x_{n)}$. The three embedding layers representations are W^{gl} for google + lexicon embeddings, W^d for domain embeddings and W^p for POS Tag embeddings are fed to CNN.

Convolution

Convolution layer is mainly used to extract features. Filters of different sizes (3,4,5) are used on each matrix. The stride used in each layer if 1 to tag each word. The Relu is the activation function is used with the convolution operation. The features produced will be of the form.

$$G_i = \max(0, (\sum_{i=1}^{n} w.x_{i+h} + b)) \quad (1)$$

Where $w.x_{i+h}$ is weight and vector of h-gram in a sentence.

Max Pooling Layer

Pooling operation is applied on the feature maps. The purpose of this layer is to extract maximum efficient features so max pooling is applied. The feature map obtained is the concatenation of three feature maps. Where n, m, o represents the filter sizes for google +lexicon features, domain features and POS features.

$$f = f_{gl}^1 \oplus \ldots\ldots\ldots \oplus f_{gl}^n \oplus f_d^1 \oplus \ldots\ldots\ldots \oplus f_d^m \oplus f_p^1 \oplus \ldots\ldots\ldots \oplus f_p^o \quad (2)$$

To choose hyperparameters we applied cross-validation strategy. Overfitting is the major problem with deep neural networks if the parameters are high. Dropout regularization is added. 50% is considered as the dropout rate to avoid over fitting. And features are added to fully connected layer $G_i = \max(0, (\sum_{i=1}^{n} w.x_{i+h} + b))$.

$$K = \alpha(w * f + b) \quad (3)$$

Where w is the weight matrix, b is bias and α is Relu activation function.

Softmax Classifier

This is the final layer used for classification of aspects into classes with highest probability.

$\hat{y} = \arg\max_j P(y = j / x, w, a) = \arg\max_j (e^{xwj+aj} / \sum_k = 1^k xe^{xwk+aj})$

w_j: weights of class j and a_j: bias of class j. The above-mentioned process is repeated for the complete training samples.

4. Experiments

These datasets are a collection of Amazon reviews of four categories of products. Product categories included in the analysis of reviews include laptops, smartphones, cameras and wristwatches. From the E-commerce applications, 100 products are considered for each product category. Dataset details are presented in Table 1.

Table.1: Dataset details

Document attributes	**Values**
Documents reviewed	4,13,841
Each review should include one sentence	1
Max. sentences per review	43
Customer review average	5.98
Review count for the product on average	49.47

Amazon reviews are used to train the domain embeddings. Electronic product reviews are collected from amazon. In this work, word embeddings are obtained from the Google News Corpus Word2Vec embeddings. We remove all stop words and non-English words from the data during pre-processing. POS tags are applied to the obtained words. The table 2 below shows the experimental setup for the modified CNN model, which covers a variety of settings.

Table.2: Experimental setup

Parameters	**Value**
Filter size	(3,4,5)
No. of feature maps	100
Pooling Function	MAX
Regularization (Dropout Rate)	50%
Activation Function	ReLu
Number of epochs	10

By carefully adjusting the settings of the modified CNN, good accuracy was achieved. According to Pedro Domingos, "Bigger data are better than smart algorithms" which is also the case for our modified CNN model which is clear from accuracy metrics when considering the size of the reviews collection.

Figure-2 and Figure-3 shows that the model has different performances across the four different domains.

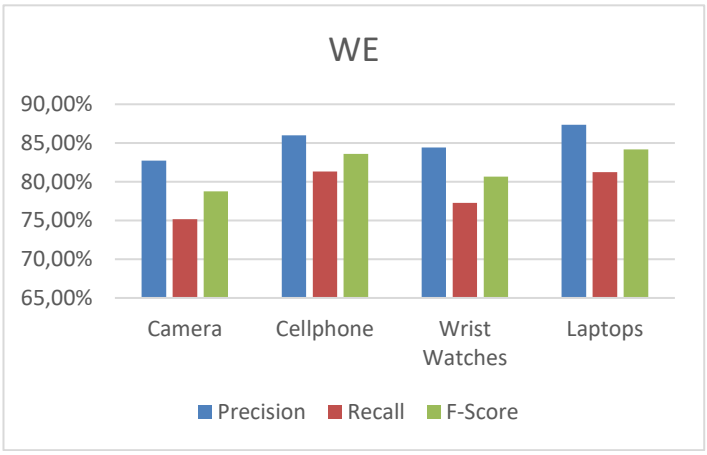

Figure 2: Comparison of the performance with in four domains

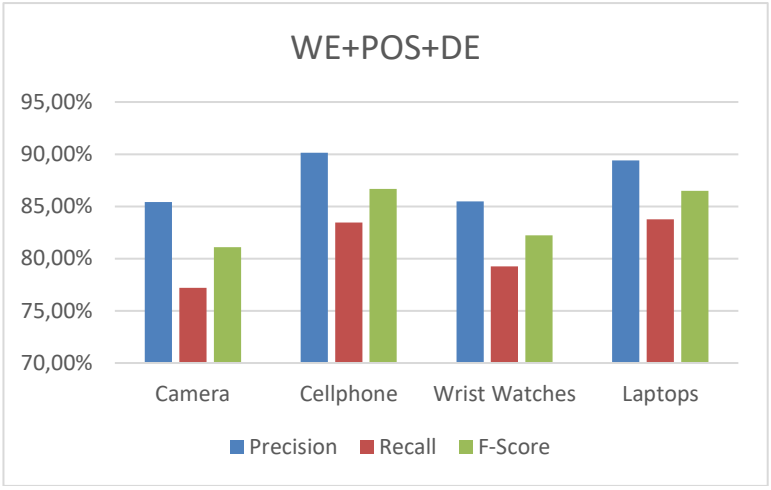

Figure 3: Comparison of the performance with in four domains

Table.3: Impact of the POS feature on the dataset.

Domain	**Classifiers**	**Precision**	**Recall**	**F-Score**
Camera	WE	80.74%	74.15%	74.76%
Camera	WE+POS+DE	84.42%	75.21%	80.10%
Cell phone	WE	84.01%	80.32%	81.59%
Cell phone	WE+POS+DE	89.05%	83.47%	84.68%
Wrist Watches	WE	82.41%	73.26%	79.67%
Wrist Watches	WE+POS+DE	83.48%	77.25%	81.24%
Laptops	WE	86.35%	80.23%	82.17%
Laptops	WE+POS+DE	86.40%	80.77%	83.49%

POS features were used in conjunction with word embedding features to achieve the

highest accuracy on this dataset. POS play an important role in extraction of aspects as well as word embedding, as demonstrated here. Impact of the POS feature on the dataset is presented in Table.3.

5. Conclusion

In this paper, three channel-based CNN for aspect extraction is introduced. Experiments are conducted on five different domains smartphones, camera, laptop, wrist watches, books. Python is used as acomputational environment. We determined the efficiency of using lexicon in word embedding layer. These carefully configured settings have helped the modified CNN architecture achieve good accuracy.

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Properties of Intuitionistic Multi-Anti Fuzzy Normal Ring

Dr. R. Muthuraj[1]
S. Yamuna[2]

Abstract

In this paper, we discuss the properties of an intuitionistic multi-anti fuzzy normal ring of a ring is defined and discussed its properties. some results based on cartesian product, homomorphism and anti homomorphism of an intuitionistic multi-anti fuzzy normal ring of a ring are also discussed.

Keywords –R - Intuitionistic multi-anti fuzzy ring, H - Intuitionistic multi-anti fuzzy normal subring, R_1, R_2- rings.

Mathematics Subject Classification:03E72, 47S40,08A05,08A72,16Y30,08A20N25[3].

[1]PG and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai - 622001, TamilNadu, India. E-mail: rmr1973@yahoo.co.i; rmr1973@gmail.com

[2] Part time Research Scholar, PG and Research, Department Mathematics, H.H. The Rajah's College, Pudukkottai - 622001, TamilNadu, India. E - mail: bassyam1@gmail.com (Affiliated to Bharathidasan University, Tiruchirappalli - 24)

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1. Introduction

The idea of fuzzy sets introduced by L. A. Zadeh 1965 [19] is an approach to mathematical representation of vagueness in everyday curriculum, The idea of fuzzy set is welcome because it handles uncertainty and vagueness which ordinary set could not address. In fuzzy set theory membership function of an element is single value between 0 and 1. Therefore, a generalization of fuzzy set was introduced by Attanassov [1], 1983 called intuitionistic fuzzy set (IFS) which deals with the degree of non-membership function and the degree of hesitation. After several year, Sabu Sebastian [13] introduced the theory of multi-fuzzy sets in terms of multi-dimensional membership function. R. Muthuraj and S. Balamurugan [15] introduced the concept of multi-anti fuzzy subgroup and discussed some of its properties. R. Muthuraj and S. Balamurugan [17] introduced the concept of multi-anti fuzzy ideal of a ring under homomorphism.
In this paper, we discuss the properties of an intuitionistic multi-anti fuzzy normal ring of a ring is defined and discussed its properties. some results based on cartesian product, homomorphism and anti homomorphism of an intuitionistic multi-anti fuzzy normal ring of a ring.

2. Preliminaries

2.1 Definition [1, 2, 4] A fuzzy subset A of a ring R is called a fuzzy subring of R if for all x, y $\in$R
i. A (x– y) $\geq$ min {A(x), A (y)} and
ii. A (xy) $\geq$ min {A (x), A (y)}.

2.2 Definition [2, 7] A fuzzy subset A of a ring R is called an anti-fuzzy subring of R if for all x, y $\in$R
i. A (x– y) $\leq$ max {A(x), A (y)} and
ii. A (xy) $\leq$ max {A (x), A (y)}.

2.3 Proposition [7] Let R_1 and R_2be rings and let f be a homomorphism from R_1 onto R_2. If A is a anti fuzzy ideal of R_2 then $f^{-1}(A)$ is a anti fuzzy ideal of R_1.

2.4 Definition [2] Let R be a ring. Let G ={ $\langle$ x, A(x), B(x) $\rangle$ / x$\in$R} be an intuitionistic fuzzy set defined on a ring R, where A: R$\rightarrow$[0,1], B: R$\rightarrow$[0,1] such that 0 $\leq$ A(x) + B(x) $\leq$ 1. An intuitionistic fuzzy subset G of R is called an intuitionistic fuzzy ring on R if the following conditions are satisfied. For all x, y $\in$R,
i. A (x – y) $\geq$ min {A (x), A(y)} ,
ii. A (xy) $\geq$ min {A(x), A(y)},
iii. B (x– y) $\leq$ max {B (x), B (y)},
iv. B (xy) $\leq$ max {B (x), B (y)}.

2.5 Definition [7,17] Let R be a ring. Let G = {⟨x, A(x), B(x)⟩ / x∈R} be an intuitionistic fuzzy set defined on a ring R, where A: R→[0,1], B: R→[0,1] such that $0 \le A(x) + B(x) \le 1$. An intuitionistic fuzzy subset G of R is called an intuitionistic anti-fuzzy ring on R if the following conditions are satisfied. For all x, y ∈R,

i. $A(x - y) \le \max\{A(x), A(y)\}$,
ii. $A(xy) \le \max\{A(x), A(y)\}$,
iii. $B(x - y) \ge \min\{B(x), B(y)\}$,
iv. $B(xy) \ge \min\{B(x), B(y)\}$.

2.6 Definition [18] An intuitionistic multi-anti fuzzy ring G = {⟨x, A(x), B(x)⟩ / x∈R} on a ring R is said to be an intuitionistic multi-anti fuzzy normal ring on R if for every x, y ∈ R, A (xy) = A (yx) and B(xy) = B(yx).

2.7 Example [18] Consider the intuitionistic fuzzy sets, G = {⟨x, A(x), B(x)⟩ / x∈R} of dimension 2 on Z is defined as,

$A_1(x) = 0.2$ if $x = 0$; $A_1(x) = 0.7$ if $x \neq 0$ and $A_2(x) = 0.3$ if $x = 0$; $A_2(x) = 0.9$ if $x \neq 0$.

$B_1(x) = 0.7$ if $x = 0$; $B_1(x) = 0.2$ if $x \neq 0$ and $B_2(x) = 0.6$ if $x = 0$; $B_2(x) = 0.1$ if $x \neq 0$.

Then the intuitionistic multi-fuzzy set G = (A, B) of dimension 2 on Z is defined as,

$$A(x) = (A_1(x), A_2(x)) = \begin{cases} (0.2,0.3) & \text{if } x = 0 \\ (0.7,0.9) & \text{if } x \neq 0 \end{cases} . B(x) = (B_1(x), B_2(x)) = \begin{cases} (0.7,0.6) & \text{if } x = 0 \\ (0.2,0.1) & \text{if } x \neq 0 \end{cases}$$

Clearly, G is an intuitionistic multi-anti fuzzy normal ring on Z.

3. Properties of Intuitionistic multi-anti fuzzy normal ring

In this section, the properties of an intuitionistic multi-anti fuzzy normal ring is discussed.

3.1 Theorem Let G = {⟨x, A(x), B(x)⟩ / x∈R} and H = {⟨x, C(x), D(x)⟩ / x∈R} be any two intuitionistic multi-anti fuzzy normal subrings of rings R_1 and R_2 respectively. Then their anti cartesian product G × H is an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$.

Proof

Let G and H be any two intuitionistic multi-anti fuzzy normal subrings of rings R_1 and R_2 respectively. Then, by Theorem 2.2.5, G × H is an intuitionistic multi-anti fuzzy subring of $R_1 \times R_2$.

Let $(p, q), (r, s) \in R_1 \times R_2$.For each i = 1, 2, ..., k,

Now,

$(A \cup C)((p, q)(r, s)) = ((A_i \cup C_i)(pr, qs))$

$= (\max \{A_i(pr), C_i(qs)\})$
$= (\max \{A_i(rp), C_i(sq)\})$
$= ((A_i \cup C_i)(rp, sq))$
$= (A \cup C)((r, s)(p, q))$
Therefore, $(A \cup C)((p, q)(r, s)) = (A \cup C)((r, s)(p, q))$ and
$(B \cap D)((p, q)(r, s)) = ((B_i \cap D_i)(pr, qs))$
$= (\min \{B_i(pr), D_i(qs)\})$
$= (\min \{B_i(rp), D_i(sq)\})$
$= ((B_i \cap D_i)(rp, sq))$
$= (B \cap D)((r, s)(p, q))$
$(B \cap D)((p, q)(r, s)) = (B \cap D)((r, s)(p, q))$
Hence, $(G \times H)((p, q)(r, s) = (G \times H)((r, s)(p, q))$.
Hence, the anti cartesian product $G \times H$ is an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$.

3.2 Theorem Let $G = \{\langle x, A(x), B(x)\rangle / x \in R\}$ and $H = \{\langle x, C(x), D(x)\rangle / x \in R\}$ be intuitionistic multi-fuzzy subsets of R_1 and R_2 respectively, such that $C(0_2) \leq A(x)$ and $D(0_2) \geq B(x)$ for all x in R_1, where 0_2 is the additive identity element of R_2. The anti cartesian product $G \times H$ is an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$, and then G is an intuitionistic multi-anti fuzzy normal subring of R_1.

Proof
Let $p, r \in R_1$ and $0_2 \in R_2$. Let $G \times H$ be an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$.
Then, by Theorem 2.2.7, G is an intuitionistic multi-anti fuzzy subring of R_1.
For each i = 1, 2,..., k,
$A(pr) = (A_1(pr), A_2(pr), \ldots, A_k(pr))$
$= (\max \{A_1(pr), C_1(0_2 0_2)\}, \ldots,$
$\max\{A_k(pr), C_k(0_2 0_2)\})$
$A(pr) = (\max (A_i(pr), C_i(0_2 0_2)))$ and
$B(pr) = (B_1(pr), B_2(pr), \ldots, B_k(pr))$
$= (\min \{B_1(pr), D_1(0_2 0_2)\}, \ldots,$
$\min\{B_k(pr), D_k(0_2 0_2)\})$
$B(pr) = (\min (B_i(pr), D_i(0_2 0_2)))$
That is, $A(pr) = (\max (A_i(pr), C_i(0_2 0_2)))$ and
$B(pr) = (\min (B_i(pr), D_i(0_2 0_2)))$.
Hence, $(G \times H)(pr, 0_2 0_2) = (G \times H)(pr, 0_2 0_2)$
$= (G \times H)((p, 0_2) \cdot (r, 0_2))$
$= (G \times H)((r, 0_2) \cdot (p, 0_2))$
$(G \times H)(pr, 0_2 0_2) = (G \times H)(pr, 0_2 0_2)$.
That is, $A(pr) = A(rp)$ and
$B(pr) = B(rp)$.

3.3 Theorem. Let G = {⟨x, A(x), B(x)⟩ / x∈R}and H = {⟨x, C(x), D(x)⟩ / x∈R} be intuitionistic multi-fuzzy subsets of R_1 and R_2 respectively, such that A $(0_1) \leq$ C(y) and B $(0_1) \geq$ D(y)for all y in R_2, where 0_1 is the additive identity element of R_1. The anti cartesian product G × An intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$, then His a multi-anti fuzzy normal subring of R_2.

Proof

Let q, s ∈ R_2 and $0_1 \in R_1$.Let G × H is an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$.

Then, by Theorem 2.2.8, An intuitionistic multi-anti fuzzy subring of R_1.

For each i = 1, 2, k,

$C(qs) = (C_1(qs), C_2(qs), ..., C_k(qs))$

$= (\max \{C_1(qs), A_1(0_1 0_1)\}, ...,$

$\max \{C_k(qs), A_k(0_1 0_1)\})$

$C(qs) = (\max (C_i(qs), A_i(0_1 0_1)))$ and

$D(qs) = (D_1(qs), D_2(qs), ..., D_k(qs))$

$= (\min \{D_1(qs), B_1(0_1 0_1)\}, ...,$

$\min \{D_k(qs), B_k(0_1 0_1)\})$

$D(qs) = (\min (Bi(0_1 0_1), D_i(qs)))$

That is, $C(qs) = (\max (A_i(0_1 0_1), C_i(qs)))$ and

$D(qs) = (\min (B_i(0_1 0_1), D_i(qs)))$.

Hence, $(G \times H)(0_1 0_1, qs) = (G \times H)(0_1 0_1, qs)$

$= (G \times H)(0_1, q) \cdot (0_1, s))$

$= (G \times H)((0_1, s) \cdot (0_1, p))$

$(G \times H)(0_1 0_1, qs) = (G \times H)(0_1 0_1, sq)$.

That is, C(qs) = C(sq) and

D(qs) = D(sq).

Hence, H is an intuitionistic multi-anti fuzzy normal subring of R_1.

3.4 Remark Let G = {⟨x, A(x), B(x)⟩ / x∈R}and H = {⟨x, C(x), D(x)⟩ / x∈R}be intuitionistic multi-fuzzy subsets of rings R_1 and R_2 respectively. The anti cartesian product G × H is an intuitionistic multi-anti fuzzy normal subring of $R_1 \times R_2$, then it is not necessarily that both G and H are intuitionistic multi-anti fuzzy normal subrings of R_1 andR_2 respectively.

4. Properties of intuitionistic multi-anti fuzzy normal subring of a ring under homomorphism and anti homomorphism

In this section, the properties of intuitionistic multi-anti fuzzy normal subring of a ring under homomorphism and anti homomorphism are discussed.

4.1 Theorem Let R_1 and R_2 be any two rings. Let f: $R_1 \to R_2$ be an onto homomorphism. If G = $\{\langle x, A(x), B(x)\rangle / x \in R_1\}$ is an intuitionistic multi-anti fuzzy normal subring of R_1, then f(G) is an intuitionistic multi-anti fuzzy normal subring of R_2, if G has inf property and G is f-invariant.

Proof
Let G be an intuitionistic multi-anti fuzzy normal subring of R_1.
Then, by Theorem 2.3.2, f(G) is an intuitionistic multi-anti fuzzy subring of R_2.
Then if $x, y \in R_1$, then $f(x), f(y) \in R_2$.
Now, f(A)(f(x)f(y)) = f(A)(f(xy))
= A(xy)
= A(yx)
= f(A)(f(yx))
= f(A)(f(y)f(x))
There fore, f(A)(f(x)f(y)) = f(A)(f(y)f(x)) and
f(B)(f(x)f(y)) = f(B)(f(xy))
= B(xy)
= B(yx)
= f(B)(f(yx))
= f(B)(f(y)f(x))
There fore, f(B)(f(x)f(y)) = f(B)(f(y)f(x)).
Hence, G(f(x)f(y) = G(f(y)f(x)).
Hence, f(G) is an intuitionistic multi-anti fuzzy normal subring of R_2.

4.2 Theorem Let R_1 and R_2 be any two rings. Let f: $R_1 \to R_2$ be a homomorphism. If H = $\{ \langle x, C(x), D(x)\rangle / x \in R_1\}$ is an intuitionistic multi-anti fuzzy normal subring of R_2, then $f^{-1}(H)$ is an intuitionistic multi-anti fuzzy normal subring of R_1.
Proof
Let H be an intuitionistic multi-anti fuzzy normal subring of R_2.
Then, by Theorem 2.3.3, $f^{-1}(H)$ is an intuitionistic multi-anti fuzzy subring of R_1.
For any $x, y \in R_1$,
Now, $f^{-1}(C)(xy) = C(f(xy))$
= C(f(x)f(y))
= C(f(y)f(x))

= C(f(yx))
= f^{-1}(C)(yx)
Therefore, f^{-1}(C)(xy) = f^{-1}(C)(yx) and
f^{-1}(D)(xy) = D(f(xy))
= D(f(x)f(y))
= D(f(y)f(x))
= D(f(yx))
= f^{-1}(D)(yx)
Therefore, f^{-1}(D)(xy) = f^{-1}(D)(yx).
Hence, f^{-1}(H)(xy) = f^{-1}(H)(yx).
Hence, f^{-1}(H) is an intuitionistic multi-anti fuzzy normal subring of R_1.

4.3 Theorem Let R_1 and R_2 be any two rings. Let f: $R_1 \to R_2$ be an onto anti homomorphism. If G = { ⟨x, A(x), B(x)⟩ / x∈R_1} is an intuitionistic multi-anti fuzzy normal subring of R_1, then f(G) is an intuitionistic multi-anti fuzzy normal subring of R_2, if G has inf property and G is f-invariant.

Proof
Let G be an intuitionistic multi-anti fuzzy normal subring of R_1.
Then, by Theorem 2.3.4, f(G) is an intuitionistic multi-anti fuzzy subring of R_2.
Then if x, y∈R_1, then f(x), f(y)∈R_2.
Now, f(A)(f(x)f(y)) = f(A)(f(yx))
= A(yx)
= A(xy)
= f(A)(f(xy))
= f(A)(f(y)f(x))
There fore, f(A)(f(x)f(y)) = f(A)(f(y)f(x)) and
f(B)(f(x)f(y)) = f(B)(f(yx))
= B(yx)
= B(xy)
= f(B)(f(xy))
= f(B)(f(y)f(x))
There fore, f(B)(f(x)f(y) = f(B)(f(y)f(x)).
Hence, G(f(x)f(y)) = G(f(y)f(x)).
Hence, f(G) is an intuitionistic multi-anti fuzzy normal subring of R_2.

4.4 Theorem Let R_1 and R_2 be any two rings. Let f: $R_1 \to R_2$ be an anti homomorphism. If H = {⟨x, A(x), B(x)⟩ / x∈R_1} is an intuitionistic multi-anti fuzzy normal subring of R_2, then f^{-1}(H) is an intuitionistic multi-anti fuzzy normal subring of R_1.
Proof
Let B be an intuitionistic multi-anti fuzzy normal subring of R_2.
Then, by Theorem 2.3.5, f^{-1}(H) is an intuitionistic multi-anti fuzzy subring of R_1.

For any $x, y \in R_1$,
Now, $f^{-1}(C)(xy) = C(f(xy))$
$= C(f(y)f(x))$
$= C(f(x)f(y))$
$= C(f(yx))$
$= f^{-1}(C)(yx)$
Therefore, $f^{-1}(C)(xy) = f^{-1}(C)(yx)$ and
$f^{-1}(D)(xy) = D(f(xy))$
$= D(f(y)f(x))$
$= D(f(x)f(y))$
$= D(f(yx))$
$= f^{-1}(D)(yx)$
Therefore, $f^{-1}(D)(xy) = f^{-1}(D)(yx)$.
$f^{-1}(H)(xy) = f^{-1}(H)(yx)$.
Hence, $f^{-1}(H)$ is an intuitionistic multi-anti fuzzy normal subring of R_1.

5. Conclusion

In this paper, we discuss the properties of an intuitionistic multi-anti fuzzy normal ring of a ring is defined and discussed its properties. Homomorphism and anti homomorphism of an intuitionistic multi-anti fuzzy normal ring of a ring.

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Various Product on Multi Fuzzy Graphs

R. Muthuraj[1]
K. Krithika[2]
S. Revathi[3]

Abstract

In this paper, the definition of complement of multi fuzzy graph, direct sum of two multi fuzzy graphs are given and derived some theorems related to them. Also, we examine the different product on multi fuzzy graphs such as Direct product, Cartesian product, Strong product, Composition, Corona product and some properties are analyzed.

Key Words: Multi fuzzy graph, complement of multi fuzzy graph, direct sum, direct product, cartesian product, strong product, composition, corona product,

AMS Subject Classification: 03E72, 05C72, 05C07[4]

[1] PG and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai -622001, Affiliated to Bharathidasan University, Tiruchirappalli, Tamilnadu, India. E-mail: rmr1973@yahoo.co.in;rmr1973@gmail.com

[2] Part time Research Scholar, PG and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai, 622001, (Affiliated to Bharathidasan University, Tiruchirappalli), Tamilnadu, India.
Department of Mathematics, Dhaanish Ahmed College of Engineering, Chennai – 601301, Tamilnadu, India E-mail: krithika.cv1982@gmail.com

[3] Department of Mathematics, Saranathan College of Engineering, Trichy – 620012, Tamilnadu, India. E-mail: revathi.soundar@gmail.com

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1. Introduction

The notion of fuzzy set and fuzzy relations were proposed by L.A Zadeh [18] in 1965 for representing uncertainty. The concept of fuzzy graph was first introduced by Kauffman [2] from the concept fuzzy relation introduced by L.A Zadeh in 1973. In 1975, Rosenfeld [14] developed the theory of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. Thereafter in 1987, Bhattacharya [1] defined some remarks on fuzzy graphs. The operations of union, join, cartesian product and composition of two fuzzy graphs were defined by Mordeson. J.N, and Prem Chand S. Nair, [3] in 2000. After that M.S. Sunitha and A. Vijayakumar [17] extended the concept of operations on fuzzy graph in 2002.Sebu Sebastian, T.V. Ramakrishnan [15] defined Multi fuzzy set in 2010. Radha. K and Arumugam. S [11, 12] defined the direct sum of two fuzzy graphs in 2013 and strong product of two fuzzy graphs in 2014.OzgeColakogluHavare and Hamza Menken [10] defined the Corona Product of Two Fuzzy Graphs in 2016.In 2020R.Muthuraj and S. Revathi [5] introduced the concept of multi fuzzy graph which is the extension of a fuzzy graph with single phenomenon into a multi-phenomenon which suits to describe the real-life problems in a better manner than fuzzy graph. Later on, and Multi anti fuzzy graph defined by Muthuraj. Ret.al [6]. In this paper complement of multi fuzzy graph, direct sum of two multi fuzzy graphs and various product on multi fuzzy graphs are defined and proved some theorems related to them.

2. Preliminaries

Definition 2.1 [2] A fuzzy graph $G=(\sigma,\mu)$ defined on the underlying crisp graph $G^*=(V,E)$ where $E\subseteq V\times V$ is a pair of functions $\sigma:V\to[0,1]$ and $\mu:V\times V\to[0,1]$, μ is a symmetric fuzzy relation on σ such that $\mu(uv)\le\min\{\sigma(u),\sigma(v)\}$ for $u,v\in V$

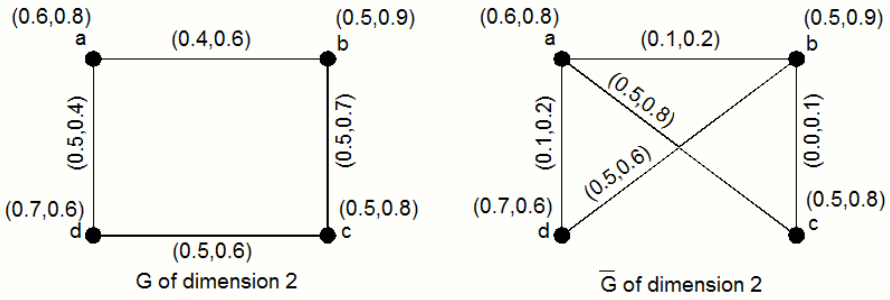
Definition 2.2 [15] Let X be a non-empty set. A Multi Fuzzy Set A in X is defined as a set of ordered sequences: $A=\{(x,\mu_1(x),\mu_2(x),.....\mu_i(x)...):x\in X\}$ where $\mu_i:X\to[0,1]$ for all i.

Definition 2.3 [5] A Multi fuzzy Graph (MFG) of dimension m defined on the underlying crisp graph $G^*=(V,E)$ where $E\subseteq V\times V$, is denoted as $G=((\sigma_1,\sigma_2,...\sigma_m),(\mu_1,\mu_2,...\mu_m))$ and $\sigma_i:V\to[0,1]$ and $\mu_i:V\times V\to[0,1]$, μ_i is a symmetric fuzzy relation on σ_i such that $\mu_i(uv)\le\min\{\sigma_i(u),\sigma_i(v)\}$ for all $i=1,2,3....m$ where $u,v\in V$ and $uv\in E$

3. Complement of Multi Fuzzy Graph

Definition 3.1 The complement of a multi-fuzzy graph $G=((\sigma_1,\sigma_2,....\sigma_m),(\mu_1,\mu_2,....\mu_m))$ of dimension m is a multi-fuzzy graph $\overline{G}=((\overline{\sigma_1},\overline{\sigma_2},...\overline{\sigma_m}),(\overline{\mu_1},\overline{\mu_2},...\overline{\mu_m}))$ of dimension m where $\overline{\sigma_i}=\sigma_i$ and $\overline{\mu_i}(u,v)=(\sigma_i(u)\wedge\sigma_i(v))-\mu_i(u,v)$ for all $u,v\in V$ and for all $i=1,2,3....m$

Example 3.2



Theorem 3.3 If G is a strong multi fuzzy graph then $\overline{G}$ is also strong multi fuzzy graph.

Proof: Let $u,v\in E$. Then

$\overline{\mu_i}(u,v)=(\sigma_i(u)\wedge\sigma_i(v))-\mu_i(u,v)$

$=(\sigma_i(u)\wedge\sigma_i(v))-(\sigma_i(u)\wedge\sigma_i(v))=0$ since G is strong.

Let $u,v\notin E$. Then

$\overline{\mu_i}(u,v)=(\sigma_i(u)\wedge\sigma_i(v))-\mu_i(u,v)=(\sigma_i(u)\wedge\sigma_i(v))-0=(\sigma_i(u)\wedge\sigma_i(v)$

Theorem 3.4 The complement of complete multi fuzzy graph is a null graph.

Proof: Let $G=(V,E)$ be a multi-fuzzy graph with the underlying crisp graph $G^*=(V,E)$ is complete. ie., $\mu_i(u,v)=(\sigma_i(u)\wedge\sigma_i(v))\ \forall u,v\in V$ & $uv\in E$

Let $u,v\in E$

$\overline{\mu_i}(u,v)=(\sigma_i(u)\wedge\sigma_i(v))-\mu_i(u,v)$

$=(\sigma_i(u)\wedge\sigma_i(v))-(\sigma_i(u)\wedge\sigma_i(v))=0$ since G is complete.

So, we have the edge set of $\overline{G}$ is empty when G is a complete multi fuzzy graph. Hence the complement of complete multi fuzzy graph is a null graph.

4. Various Product On Multi Fuzzy Graphs

In this section $G_1=((\sigma_1,\sigma_2,...\sigma_m),(\mu_1,\mu_2,...\mu_m))$ denotes the multi fuzzy graph with dimension m with the underlying crisp graph $G_1^*=(V_1,E_1)$ and $G_2=((\alpha_1,\alpha_2,...\alpha_n),(\beta_1,\beta_2,...\beta_n))$ denotes the multi fuzzy graph with dimension n with the underlying crisp graph $G_2^*=(V_2,E_2)$

Definition 4.1 The operation **Direct sum** between two MFG G_1 and G_2 is defined as follows, $G_1 \oplus G_2 = ((\sigma_1 \oplus \alpha_1, \sigma_2 \oplus \alpha_2, ...\sigma_k \oplus \alpha_k), (\mu_1 \oplus \beta_1, \mu_2 \oplus \beta_2, ...\mu_k \oplus \beta_k))$ with the underlying crisp graph $G_1^* \oplus G_2^* = (V_1 \oplus V_2, E_1 \oplus E_2)$, $(\sigma_i \oplus \alpha_i)(u) = \begin{cases} \sigma_i(u) \ if \ u \in V_1 - V_2 \\ \alpha_i(u) \ if \ u \in V_2 - V_1 \\ \max\{\sigma_i(u), \alpha_i(u)\} \ if \ u \in V_1 \cap V_2 \end{cases}$

for all $i = 1,2,3....k$ and

$(\mu_i \oplus \beta_i)(u,v) = \begin{cases} \mu_i(u,v) & if \ (u,v) \in E_1 \\ \beta_i(u,v) & if \ (u,v) \in E_2 \end{cases}$ for all $i = 1,2,3....k$

If m ≠ n, let k = max (m, n). Suppose $m < n$, then let us introduce n – m membership values of multi fuzzy graph G_1 into 0 so as to convert the multi fuzzy graphs G_1 and G_2 have the same dimension as k.

Theorem 4.2 The direct sum of two multi fuzzy graph is also a multi-fuzzy graph,

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To prove: $G = G_1 \oplus G_2$ is also multi fuzzy graph with dimension k where k=max (m, n)

$(\sigma_i \oplus \alpha_i)(u) = \begin{cases} \sigma_i(u) \ if \ u \in V_1 - V_2 \\ \alpha_i(u) \ if \ u \in V_2 - V_1 \\ \max\{\sigma_i(u), \alpha_i(u)\} \ if \ u \in V_1 \cap V_2 \end{cases}$

Case (i): Let $(u,v) \in E_1$

$(\mu_i \oplus \beta_i)(u,v) = \mu_i(u,v)$

$\leq \min(\sigma_i(u), \sigma_i(v))$

$= \min((\sigma_i \oplus \alpha_i)(u), (\sigma_i \oplus \alpha_i)(v))$

$\therefore (\mu_i \oplus \beta_i)(u,v) \leq \min((\sigma_i \oplus \alpha_i)(u), (\sigma_i \oplus \alpha_i)(v))$

Case (ii): Let $(u,v) \in E_2$

$(\mu_i \oplus \beta_i)(u,v) = \beta_i(u,v)$

$\leq \min(\alpha_i(u), \alpha_i(v))$

$= \min((\sigma_i \oplus \alpha_i)(u), (\sigma_i \oplus \alpha_i)(v))$

$\therefore (\mu_i \oplus \beta_i)(u,v) \leq \min((\sigma_i \oplus \alpha_i)(u), (\sigma_i \oplus \alpha_i)(v))$

Definition4.3 The operation **Direct Product** between two MFG G_1 and G_2 is defined as follows, $G_1 * G_2 = ((\sigma_1 * \alpha_1, \sigma_2 * \alpha_2, ...\sigma_k * \alpha_k), (\mu_1 * \beta_1, \mu_2 * \beta_2, ...\mu_k * \beta_k))$ with the underlying crisp graph $G_1^* * G_2^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{(u_1, v_1), (u_2, v_2) / (u_1, u_2) \in E_1 \ \& \ (v_1, v_2) \in E_2\}$ with

$(\sigma_i * \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \ \alpha_i(v_1)\} \ for \ all \ u_1 \in V_1, v_1 \in V_2 \ \& \ (u_1, v_1) \in V_1 \times V_2$

$(\mu_i * \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\} \ for \ all \ (u_1, u_2) \in E_1 \ \& \ (v_1, v_2) \in E_2$ for all I = 1, 2, 3, … k. If $m \neq n$, let k = min (m, n). Suppose $m < n$, then we take first m dimensions for G_2 so as to convert the MFG G_1 and G_2 have the same dimension k.

Example 4.4

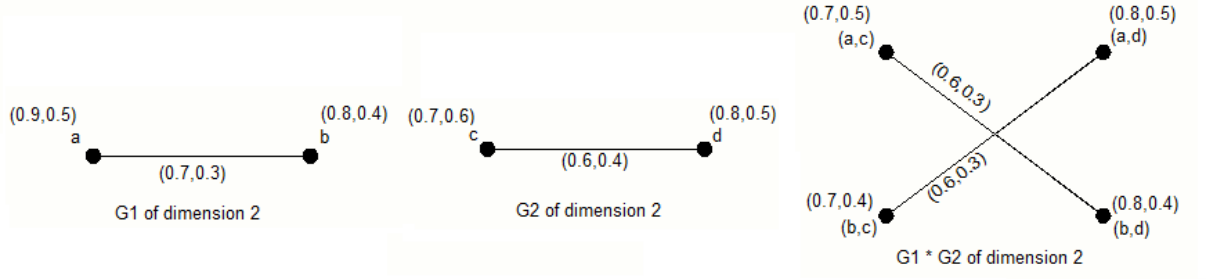


Figure1

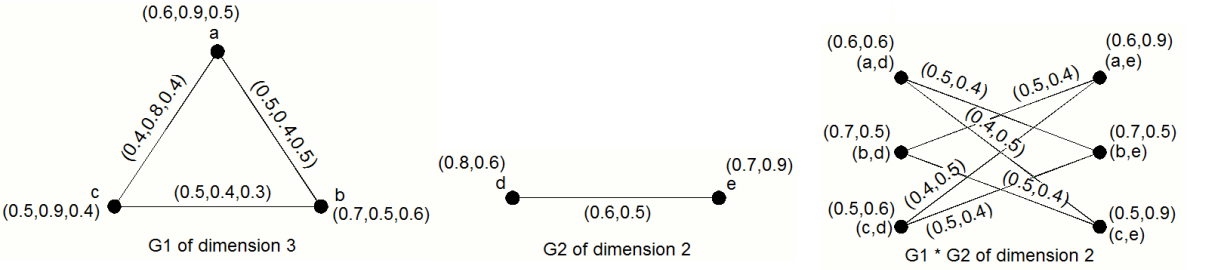


Figure2

Theorem 4.5 Direct product of two multi fuzzy graph is also a multi-fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G = G_1 * G_2$ is a multi-fuzzy graph of dimension k where k=min (m, n)

$(\sigma_i * \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1),\ \alpha_i(v_1)\}$

$(\mu_i * \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\}$

$\leq \min\{\min(\sigma_i(u_1), \sigma_i(u_2)), \min(\alpha_i(v_1), \alpha_i(v_2))\}$

$= (\sigma_i(u_1) \wedge \sigma_i(u_2)) \wedge ((\alpha_i(v_1) \wedge \alpha_i(v_2))$

$= (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge ((\sigma_i(u_2) \wedge \alpha_i(v_2))$

$= ((\sigma_i * \alpha_i)(u_1, v_1)) \wedge ((\sigma_i * \alpha_i)(u_2, v_2))$

$\therefore (\mu_i * \beta_i)((u_1, v_1), (u_2, v_2)) \leq \min((\sigma_i * \alpha_i)(u_1, v_1), (\sigma_i * \alpha_i)(u_2, v_2))$.

Theorem 4.6 If G_1 and G_2 are strong multi fuzzy graphs then $G_1 * G_2$ is also a strong multi fuzzy graph.

Proof:

$(\sigma_i * \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1),\ \alpha_i(v_1)\}$

$(\mu_i * \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\}$

$= \min\{\min(\sigma_i(u_1), \sigma_i(u_2)), \min(\alpha_i(v_1), \alpha_i(v_2)\}$

$= (\sigma_i(u_1) \wedge \sigma_i(u_2)) \wedge ((\alpha_i(v_1) \wedge \alpha_i(v_2))$

$= (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge ((\sigma_i(u_2) \wedge \alpha_i(v_2))$

$= ((\sigma_i * \alpha_i)(u_1, v_1)) \wedge ((\sigma_i * \alpha_i)(u_2, v_2))$

$\therefore (\mu_i * \beta_i)((u_1, v_1),(u_2, v_2)) = \min((\sigma_i * \alpha_i)(u_1, v_1),(\sigma_i * \alpha_i)(u_2, v_2))$

Remark: If G_1 and G_2 are complete multi fuzzy graphs then $G_1 * G_2$ is not a complete multi fuzzy graph.

Definition 4.7 The operation **Cartesian Product** between two MFG G_1 and G_2 as follows, $G_1 \times G_2 = ((\sigma_1 \times \alpha_1, \sigma_2 \times \alpha_2, ... \sigma_k \times \alpha_k),(\mu_1 \times \beta_1, \mu_2 \times \beta_2, ... \mu_k \times \beta_k))$ with the underlying crisp graph $G_1^* \times G_2^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{(u_1, v_1)(u_2, v_2) / u_1 = u_2, (v_1, v_2) \in E_2 \text{ or } v_1 = v_2, (u_1, u_2) \in E_1\}$ with

$(\sigma_i \times \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$ *for all* $u_1 \in V_1$ *and* $v_1 \in V_2$ & $(u_1, v_1) \in V_1 \times V_2$

$$(\mu_i \times \beta_i)((u_1, v_1),(u_2, v_2)) = \begin{cases} \min\{\sigma_i(u), \beta_i(v_1, v_2)\} & u_1 = u_2 = u, \text{ for all } u \in V_1 \ \& (v_1, v_2) \in E_2 \\ \min\{\mu_i(u_1, u_2), \alpha_i(v)\} & v_1 = v_2 = v, \text{ for all } v \in V_2 \ \& (u_1, u_2) \in E_1 \end{cases}$$

for all i= 1, 2, 3, ... k

If $m \neq n$ let k = min (m, n). Suppose $m < n$ then we take first m dimensions for G_2 so as to convert the MFG G_1 and G_2 have the same dimension k.

Example 4.8

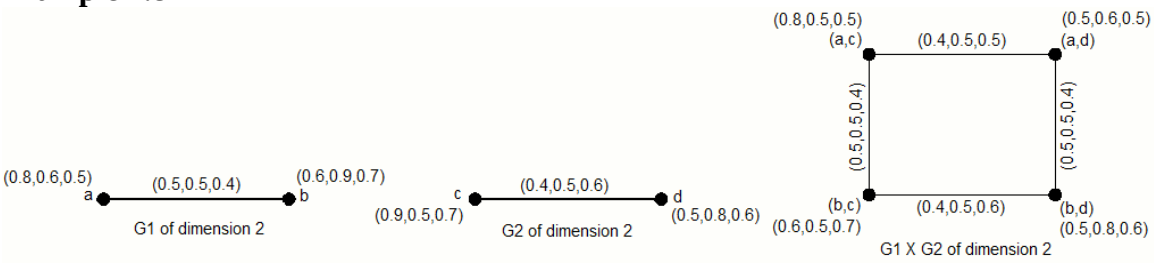


Figure 3

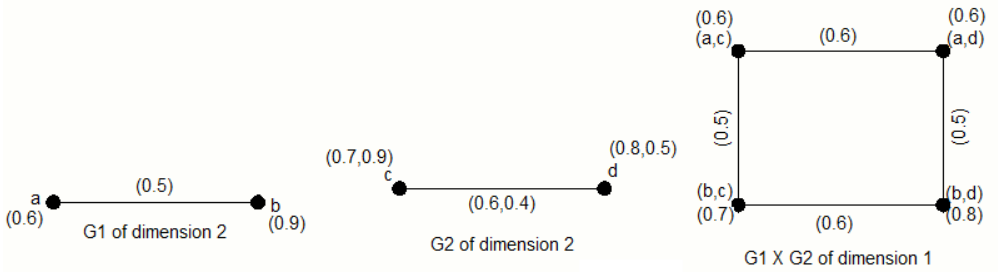


Figure 4

Theorem 4.9 Cartesian product of two multi fuzzy graph is also a multi-fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ... \sigma_m),(\mu_1, \mu_2, ... \mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ... \alpha_n),(\beta_1, \beta_2, ... \beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G = G_1 \times G_2$ is a multi-fuzzy graph of dimension k where k=min (m, n)

$(\sigma_i \times \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$

$(\mu_i \times \beta_i)((u, v_1),(u, v_2)) = \min\{\sigma_i(u), \beta_i(v_1, v_2)\}$

$\leq \min\{\sigma_i(u), \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$
$= \min\{\min\{\sigma_i(u), \alpha_i(v_1)\}, \min\{\sigma_i(u), \alpha_i(v_2)\}\}$
$= \min\{(\sigma_i \times \alpha_i)(u, v_1), (\sigma_i \times \alpha_i)(u, v_2)\}$
$(\mu_i \times \beta_i)((u_1, v), (u_2, v)) = \min\{\mu_i(u_1, u_2), \alpha_i(v)\}$
$\leq \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \alpha_i(v)\}$
$= \min\{\min\{\sigma_i(u_1), \alpha_i(v)\}, \min\{\sigma_i(u_2), \alpha_i(v)\}\}$
$= \min\{(\sigma_i \times \alpha_i)(u_1, v), (\sigma_i \times \alpha_i)(u_2, v)\}$

Theorem 4.10 Cartesian product of two strong multi fuzzy graph is also a strong multi fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G = G_1 \times G_2$ is a multi-fuzzy graph of dimension k where k=min (m,n)

$(\sigma_i \times \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$
$(\mu_i \times \beta_i)((u, v_1), (u, v_2)) = \min\{\sigma_i(u), \beta_i(v_1, v_2)\}$
$= \min\{\sigma_i(u), \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$
$= \min\{\min\{\sigma_i(u), \alpha_i(v_1)\}, \min\{\sigma_i(u), \alpha_i(v_2)\}\}$
$= \min\{(\sigma_i \times \alpha_i)(u, v_1), (\sigma_i \times \alpha_i)(u, v_2)\}$
$(\mu_i \times \beta_i)((u_1, v), (u_2, v)) = \min\{\mu_i(u_1, u_2), \alpha_i(v)\}$
$= \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \alpha_i(v)\}$
$= \min\{\min\{\sigma_i(u_1), \alpha_i(v)\}, \min\{\sigma_i(u_2), \alpha_i(v)\}\}$
$= \min\{(\sigma_i \times \alpha_i)(u_1, v), (\sigma_i \times \alpha_i)(u_2, v)\}$

Remark: If G_1 and G_2 are complete multi fuzzy graphs then $G_1 \times G_2$ is not a complete multi fuzzy graph.

Theorem 4.11 If $G_1 \times G_2$ is a strong multi fuzzy graph then at least one G_1 or G_2 is a strong multi fuzzy graph.

Proof: Suppose assume that the contrary that G_1 and G_2 are not strong fuzzy graphs.

$\mu_i(u_1, v_1) < \sigma_i(u_1) \wedge \sigma_i(v_1)$ and $\beta_i(u_2, v_2) < \alpha_i(u_2) \wedge \alpha_i(v_2)$ (1)

Without loss of generality, we assume that

$\beta_i(u_2, v_2) \leq \mu_i(u_1, v_1) < \sigma_i(u_1) \wedge \sigma_i(v_1) \leq \sigma_i(u_1)$

Let $E = \{(u_1, v_1)(u_2, v_2) / u_1 = u_2, (v_1, v_2) \in E_2 \text{ or } v_1 = v_2, (u_1, u_2) \in E_1\}$

Consider $(u_1, v_1)(u_2, v_2) \in E$, by definition of $G_1 \times G_2$ & inequality (1)

$(\mu_i \times \beta_i)((u_1, v_1), (u_1, v_2)) = \sigma_i(u_1) \wedge \beta_i(v_1, v_2) < \sigma_i(u_1) \wedge \alpha_i(v_1) \wedge \alpha_i(v_2)$ (2)

$(\sigma_i \times \alpha_i)(u_1, v_1) = \sigma_i(u_1) \wedge \alpha_i(v_1)$ & $(\sigma_i \times \alpha_i)(u_1, v_2) = \sigma_i(u_1) \wedge \alpha_i(v_2)$

$(\sigma_i \times \alpha_i)(u_1, v_1) \wedge (\sigma_i \times \alpha_i)(u_1, v_2) = \sigma_i(u_1) \wedge \alpha_i(v_1) \wedge \sigma_i(u_1) \wedge \alpha_i(v_2)$
$= \sigma_i(u_1) \wedge \alpha_i(v_1) \wedge \alpha_i(v_2)$ (3)

From (2) and (3),

$(\mu_i \times \beta_i)((u_1,v_1),(u_1,v_2)) < \sigma_i(u_1) \wedge \alpha_i(v_1) \wedge \alpha_i(v_2) = (\sigma_i \times \alpha_i)(u_1,v_1) \wedge (\sigma_i \times \alpha_i)(u_1,v_2)$

$\therefore (\mu_i \times \beta_i)((u_1,v_1),(u_1,v_2)) < (\sigma_i \times \alpha_i)(u_1,v_1) \wedge (\sigma_i \times \alpha_i)(u_1,v_2)$

This implies that $G_1 \times G_2$ is not a strong multi fuzzy graph.

This gives a contradiction. So, if $G_1 \times G_2$ is a strong multi fuzzy graph then atleast one G_1 or G_2 is a strong multi fuzzy graph.

Definition 4.12 The operation **Strong Product** between two MFG G_1 and G_2 is defined as follows, $G_1 \bullet G_2 = ((\sigma_1 \bullet \alpha_1, \sigma_2 \bullet \alpha_2, ...\sigma_k \bullet \alpha_k), (\mu_1 \bullet \beta_1, \mu_2 \bullet \beta_2, ...\mu_k \bullet \beta_k))$ with the underlying crisp graph $G_1^* \bullet G_2^* = (V,E)$ where $V = V_1 \times V_2$ and

$E = \{(u_1,v_1)(u_2,v_2) / u_1 = u_2, (v_1,v_2) \in E_2 \text{ or } v_1 = v_2, (u_1,u_2) \in E_1 \text{ or } (u_1,u_2) \in E_1 \ \& (v_1,v_2) \in E_2\}$

with $(\sigma_i \bullet \alpha_i)(u_1,v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$ *for all* $u_1 \in V_1$ *and* $v_1 \in V_2$ & $(u_1,v_1) \in V_1 \times V_2$

$$(\mu_i \bullet \beta_i)((u_1,v_1),(u_2,v_2)) = \begin{cases} \min\{\sigma_i(u), \beta_i(v_1,v_2)\} & u_1 = u_2 = u, \text{ for all } u \in V_1 \ \& (v_1,v_2) \in E_2 \\ \min\{\mu_i(u_1,u_2), \alpha_i(v)\} & v_1 = v_2 = v, \text{ for all } v \in V_2 \ \& (u_1,u_2) \in E_1 \\ \min\{\mu_i(u_1,u_2), \beta_i(v_1,v_2)\} & \text{for all } (u_1,u_2) \in E_1 \ \& (v_1,v_2) \in E_2 \end{cases}$$

for all i= 1, 2, 3, ... k

If $m \neq n$, let k = min (m, n). Suppose $m < n$, then we take first m dimensions for G_2 so as to convert the MFG G_1 and G_2 have the same dimension k.

Example 4.13

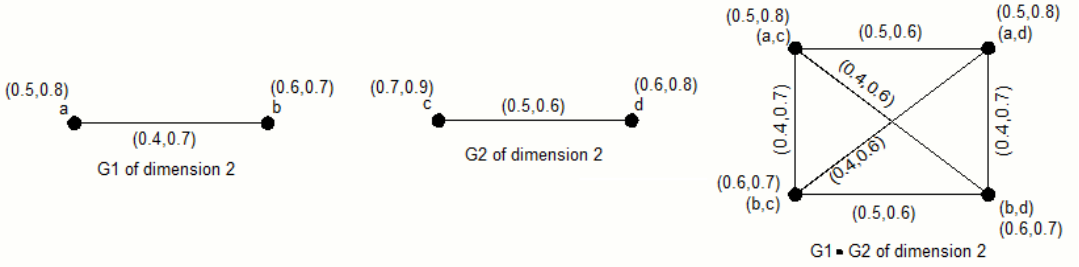


Figure 5

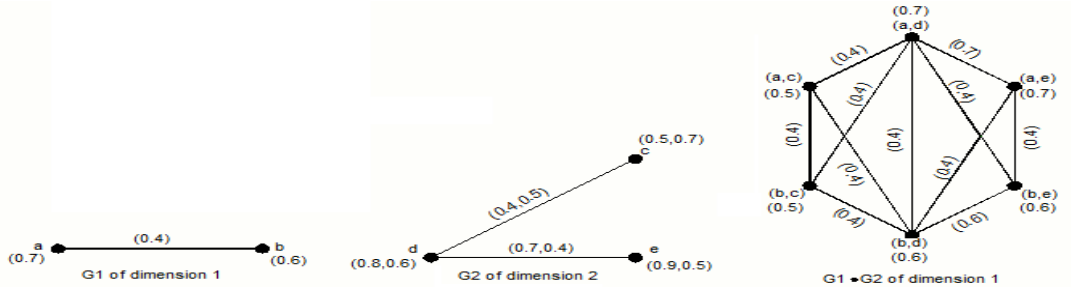


Figure 6

Theorem 4.14 Strong product of two multi fuzzy graph is also a multi-fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G = G_1 \bullet G_2$ is a multi-fuzzy graph of dimension k where k=min (m, n)

$(\sigma_i \bullet \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$ *for all* $u_1 \in V_1$ *and* $v_1 \in V_2$ & $(u_1, v_1) \in V_1 \times V_2$

$(\mu_i \bullet \beta_i)((u, v_1), (u, v_2)) = \min\{\sigma_i(u), \beta_i(v_1, v_2)\}$

$\leq \min\{\sigma_i(u), \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$

$= \min\{\min\{\sigma_i(u), \alpha_i(v_1)\}, \min\{\sigma_i(u), \alpha_i(v_2)\}\}$

$= (\sigma_i(u) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u) \wedge \alpha_i(v_2)) \quad = \min\{(\sigma_i \bullet \alpha_i)(u, v_1), (\sigma_i \bullet \alpha_i)(u, v_2)\}$

$(\mu_i \bullet \beta_i)((u_1, v), (u_2, v)) = \min\{\mu_i(u_1, u_2), \alpha_i(v)\}$

$\leq \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \alpha_i(v)\}$

$= \min\{\min\{\sigma_i(u_1), \alpha_i(v)\}, \min\{\sigma_i(u_2), \alpha_i(v)\}\} \quad = (\sigma_i(u_1) \wedge \alpha_i(v)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v))$

$= \min\{(\sigma_i \bullet \alpha_i)(u_1, v), (\sigma_i \bullet \alpha_i)(u_2, v)\}$

$(\mu_i \bullet \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\}$

$\leq \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$

$= (\sigma_i(u_1) \wedge \sigma_i(u_2)) \wedge (\alpha_i(v_1) \wedge \alpha_i(v_2))$

$= (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v_2))$

$= \min\{(\sigma_i \bullet \alpha_i)(u_1, v_1), (\sigma_i \bullet \alpha_i)(u_2, v_2)\}$

Theorem 4.15 If G_1 and G_2 are strong multi fuzzy graphs then $G_1 \bullet G_2$ is also a strong multi fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G_1 \bullet G_2$ is a strong multi fuzzy graph of dimension k where k=min (m, n)

$(\sigma_i \bullet \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$ *for all* $u_1 \in V_1$ *and* $v_1 \in V_2$ & $(u_1, v_1) \in V_1 \times V_2$

$(\mu_i \bullet \beta_i)((u, v_1), (u, v_2)) = \min\{\sigma_i(u), \beta_i(v_1, v_2)\}$

$= \min\{\sigma_i(u), \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$

$= \min\{\min\{\sigma_i(u), \alpha_i(v_1)\}, \min\{\sigma_i(u), \alpha_i(v_2)\}\} \quad = (\sigma_i(u) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u) \wedge \alpha_i(v_2))$

$= \min\{(\sigma_i \bullet \alpha_i)(u, v_1), (\sigma_i \bullet \alpha_i)(u, v_2)\}$

$(\mu_i \bullet \beta_i)((u_1, v), (u_2, v)) = \min\{\mu_i(u_1, u_2), \alpha_i(v)\}$

$= \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \alpha_i(v)\}$

$= \min\{\min\{\sigma_i(u_1), \alpha_i(v)\}, \min\{\sigma_i(u_2), \alpha_i(v)\}\} \quad = (\sigma_i(u_1) \wedge \alpha_i(v)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v))$

$= \min\{(\sigma_i \bullet \alpha_i)(u_1, v), (\sigma_i \bullet \alpha_i)(u_2, v)\}$

$(\mu_i \bullet \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\}$

$= \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$

$= (\sigma_i(u_1) \wedge \sigma_i(u_2)) \wedge (\alpha_i(v_1) \wedge \alpha_i(v_2))$

$= (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v_2))$

$= \min\{(\sigma_i \bullet \alpha_i)(u_1, v_1), (\sigma_i \bullet \alpha_i)(u_2, v_2)\}$

Theorem 4.16 If G_1 and G_2 are complete multi fuzzy graphs then $G_1 \bullet G_2$ is also a complete multi fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the two complete multi fuzzy graphs with dimension m and n respectively. Then G_1 and G_2

are strong multi fuzzy graphs where $G_1^* \text{ and } G_2^*$ are complete graphs. Therefore, $G_1 \bullet G_2$ is a strong multi fuzzy graph by the theorem (4.15) with $G_1^* \text{ and } G_2^*$ are complete graphs. Hence $G_1 \bullet G_2$ is a complete multi fuzzy graph.

Theorem 4.17 The strong product of two multi fuzzy graphs G_1 and G_2 is the direct sum of the cartesian product of G_1 and G_2 and the direct product of G_1 and G_2.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively. Let $G_1 \times G_2$ and $G_1 * G_2$ be the cartesian product and direct product of G_1 and G_2 with dimension k where k = min(m,n)

To Prove: $G_1 \bullet G_2 = (G_1 \times G_2) \oplus (G_1 * G_2)$

$(\sigma_i \times \alpha_i)(u_1, v_1) = (\sigma_i * \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1),\ \alpha_i(v_1)\}\ \forall (u_1, v_1) \in V_1 \times V_2$

So, $(\sigma_i \times \alpha_i)(u_1, v_1) \oplus (\sigma_i * \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1),\ \alpha_i(v_1)\}\ \forall (u_1, v_1) \in V_1 \times V_2$

$(\sigma_i \bullet \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$

$\therefore (\sigma_i \bullet \alpha_i)(u_1, v_1) = (\sigma_i \times \alpha_i)(u_1, v_1) \oplus (\sigma_i * \alpha_i)(u_1, v_1)$

$$((\mu_i \times \beta_i)(u_1, v_1), (u_2, v_2)) = \begin{cases} \sigma_i(u_1) \wedge \beta_i(v_1, v_2) & f\ u_1 = u_2\ and (v_1, v_2) \in E_2 \\ \alpha_i(v_1) \wedge \mu_i(u_1, u_2) & if\ v_1 = v_2\ and\ (u_1, u_2) \in E_1 \end{cases}$$

$(\mu_i * \beta_i)(u_1, v_1), (u_2, v_2)) = \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\},\ if\ (u_1, u_2) \in E_1\ \&\ (v_1, v_2) \in E_2$

$$((\mu_i \times \beta_i) \oplus (\mu_i * \beta_i))(u_1, v_1), (u_2, v_2)) = \begin{cases} \sigma_i(u_1) \wedge \beta_i(v_1, v_2) & if\ u_1 = u_2\ and (v_1, v_2) \in E_2 \\ \alpha_i(v_1) \wedge \mu_i(u_1, u_2) & if\ v_1 = v_2\ and\ (u_1, u_2) \in E_1 \\ \min\{\mu_i(u_1, u_2), \beta_i(v_1, v_2)\} & if\ (u_1, u_2) \in E_1\ \&\ (v_1, v_2) \in E_2 \end{cases}$$

$= (\mu_i \bullet \beta_i)((u_1, v_1), (u_2, v_2))$

Result: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be two strong multi fuzzy graph with dimension m and n respectively and $G_1 \times G_2 \,\&\, G_1 * G_2$ be the cartesian product and direct product of G_1 and G_2 with dimension k where k = min(m,n) and $\overline{G_1 \times G_2}$ and $\overline{G_1 * G_2}$ be the complement of two multi fuzzy graphs then $\overline{G_1 \times G_2} \oplus \overline{G_1 * G_2} = G_1 \times G_2 \oplus G_1 * G_2$.

Definition 4.18 The operation **Composition** between two MFG G_1 and G_2 as follows $G_1[G_2] = ((\sigma_1 \circ \alpha_1, \sigma_2 \circ \alpha_2, ...\sigma_k \circ \alpha_k), (\mu_1 \circ \beta_1, \mu_2 \circ \beta_2, ...\mu_k \circ \beta_k))$ with the underlying crisp graph $G_1^*[G_2^*] = (V, E)$ where $V = V_1 \times V_2$ and

$E = \{(u_1, v_1)(u_2, v_2) / u_1 = u_2, (v_1, v_2) \in E_2\ or\ v_1 = v_2, (u_1, u_2) \in E_1\ or\ v_1 \neq v_2, (u_1, u_2) \in E_1\}$ with

$(\sigma_i \circ \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}\ for\ all\ u_1 \in V_1\ and\ v_1 \in V_2\ \&\ (u_1, v_1) \in V_1 \times V_2$

$$(\mu_i \circ \beta_i)((u_1, v_1), (u_2, v_2)) = \begin{cases} \min\{\sigma_i(u), \beta_i(v_1, v_2)\} & u_1 = u_2 = u,\ for\ all\ u \in V_1\ \&\ (v_1, v_2) \in E_2 \\ \min\{\mu_i(u_1, u_2), \alpha_i(v)\} & v_1 = v_2 = v,\ for\ all\ v \in V_2\ \&\ (u_1, u_2) \in E_1 \\ \min\{\alpha_i(v_1), \alpha_i(v_2), \mu_i(u_1, u_2)\} & for\ all\ (u_1, u_2) \in E_1 \end{cases}$$

for all i= 1, 2, 3, ... k.

If $m \neq n$, let k = min (m, n). Suppose $m < n$, then we take first m dimensions for G_2 so as to convert the MFG G_1 and G_2 have the same dimension k

Example 4.19

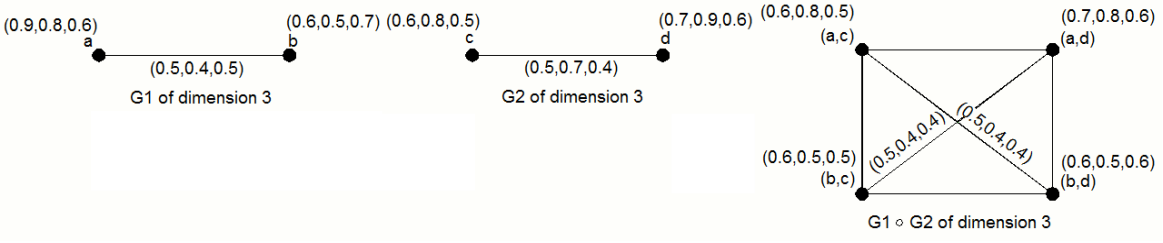


Figure 7

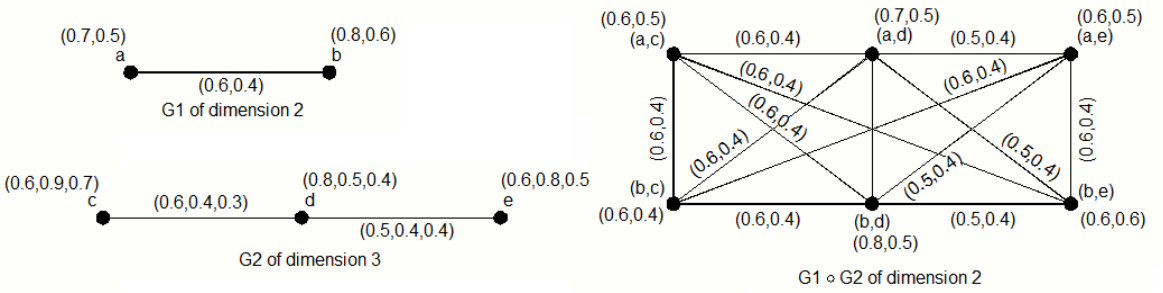


Figure 8

Theorem 4.20 Composition of two multi fuzzy graph is also a multi-fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $\boldsymbol{G = G_1 \circ G_2}$ is a multi-fuzzy graph of dimension k where k = min (m, n)

$(\sigma_i \circ \alpha_i)(u_1, v_1) = \min\{\sigma_i(u_1), \alpha_i(v_1)\}$ *for all* $u_1 \in V_1$ *and* $v_1 \in V_2$ & $(u_1, v_1) \in V_1 \times V_2$

$(\mu_i \circ \beta_i)((u, v_1), (u, v_2)) = \min\{\sigma_i(u), \beta_i(v_1, v_2)\}$

$\leq \min\{\sigma_i(u), \min\{\alpha_i(v_1), \alpha_i(v_2)\}\}$

$= \min\{\min\{\sigma_i(u), \alpha_i(v_1)\}, \min\{\sigma_i(u), \alpha_i(v_2)\}\} \quad = (\sigma_i(u) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u) \wedge \alpha_i(v_2))$

$= \min\{(\sigma_i \circ \alpha_i)(u, v_1), (\sigma_i \circ \alpha_i)(u, v_2)\}$

$(\mu_i \circ \beta_i)((u_1, v), (u_2, v)) = \min\{\mu_i(u_1, u_2), \alpha_i(v)\}$

$\leq \min\{\min\{\sigma_i(u_1), \sigma_i(u_2)\}, \alpha_i(v)\}$

$= \min\{\min\{\sigma_i(u_1), \alpha_i(v)\}, \min\{\sigma_i(u_2), \alpha_i(v)\}\} \quad = (\sigma_i(u_1) \wedge \alpha_i(v)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v))$

$= \min\{(\sigma_i \circ \alpha_i)(u_1, v), (\sigma_i \circ \alpha_i)(u_2, v)\}$

$(\mu_i \circ \beta_i)((u_1, v_1), (u_2, v_2)) = \min\{\alpha_i(v_1), \alpha_i(v_2), \mu_i(u_1, u_2)\}$

$\leq \min\{\alpha_i(v_1), \alpha_i(v_2), \min\{\sigma_i(u_1), \sigma_i(u_2)\}\}$

$= \min\{\alpha_i(v_1), \alpha_i(v_2), (\sigma_i(u_1) \wedge \sigma_i(u_2))\}$

$= (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge (\sigma_i(u_2) \wedge \alpha_i(v_2))$

$= \min\{(\sigma_i \circ \alpha_i)(u, v_1), (\sigma_i \circ \alpha_i)(u, v_2)\}$

Theorem 4.21 If $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ are two strong multi fuzzy graphs with dimension m and n respectively and $G_1 \circ G_2$ is a strong multi fuzzy graph of dimension k where k = min (m, n). Prove that $\overline{G_1 \circ G_2} = \overline{G_1} \circ \overline{G_2}$

Proof: Let $G = G_1 \circ G_2 = ((\sigma_1 \circ \alpha_1, \sigma_2 \circ \alpha_2, ...\sigma_k \circ \alpha_k), (\mu_1 \circ \beta_1, \mu_2 \circ \beta_2, ...\mu_k \circ \beta_k))$

$\overline{G_1 \circ G_2} = ((\sigma_1 \circ \alpha_1, \sigma_2 \circ \alpha_2, ...\sigma_k \circ \alpha_k), (\overline{\mu_1 \circ \beta_1}, \overline{\mu_2 \circ \beta_2}, ...\overline{\mu_k \circ \beta_k}))$

$\overline{G_1} = ((\sigma_1, \sigma_2, ...\sigma_m), (\overline{\mu_1}, \overline{\mu_2}, ...\overline{\mu_m}))\ \overline{G_2} = ((\alpha_1, \alpha_2, ...\alpha_n), (\overline{\beta_1}, \overline{\beta_2}, ...\overline{\beta_n}))$

$\overline{G_1} \circ \overline{G_2} = ((\sigma_1 \circ \alpha_1, \sigma_2 \circ \alpha_2, ...\sigma_k \circ \alpha_k), (\overline{\mu_1} \circ \overline{\beta_1}, \overline{\mu_2} \circ \overline{\beta_2}, ...\overline{\mu_k} \circ \overline{\beta_k}))$

To prove $\overline{G_1 \circ G_2} = \overline{G_1} \circ \overline{G_2}$ It is enough to prove $\overline{\mu_i \circ \beta_i} = \overline{\mu_i} \circ \overline{\beta_i}$ for all i= 1, 2, 3, ... k.

To prove the above result, there are different cases may arise depending upon the edges joining the vertices

Case(i): Consider the edge $e = ((u, v_1), (u, v_2)), (v_1, v_2) \in E_2$

Then $e \in E$ and G is a strong multi fuzzy graph, so $\overline{\mu_i \circ \beta_i}(e) = 0$

Also $(\overline{\mu_1} \circ \overline{\beta_1})(e) = 0$ since $(v_1, v_2) \notin \overline{E_2}$

If $e = ((u, v_1), (u, v_2))$, $v_1 \neq v_2$ *and* $(v_1, v_2) \notin E_2$ then $e \notin E$ $(\mu_i \circ \beta_i)((u, v_1), (u, v_2)) = 0$

Now $\overline{\mu_i \circ \beta_i}(e) = ((\sigma_i \circ \alpha_i)(u, v_1)) \wedge ((\sigma_i \circ \alpha_i)(u, v_2))$

$= (\sigma_i(u) \wedge \alpha_i(v_1)) \wedge ((\sigma_i(u) \wedge \alpha_i(v_2))$

$= \sigma_i(u) \wedge \alpha_i(v_1) \wedge \alpha_i(v_2)$

$(\overline{\mu_1} \circ \overline{\beta_1})(e) = \sigma_i(u) \wedge \overline{\beta_i}(v_1, v_2) = \sigma_i(u) \wedge \alpha_i(v_1) \wedge \alpha_i(v_2)$

$\therefore\ \overline{\mu_i \circ \beta_i} = \overline{\mu_i} \circ \overline{\beta_i}$ for all i= 1, 2, 3, ... k.

Case(ii): Consider the edge $e = ((u_1, v), (u_2, v)), (u_1, u_2) \in E_1$

Then $e \in E$ and G is a strong multi fuzzy graph, so $\overline{\mu_i \circ \beta_i}(e) = 0$

Also $(\overline{\mu_1} \circ \overline{\beta_1})(e) = 0$ since $(u_1, u_2) \notin \overline{E_1}$

If $e = ((u_1, v), (u_2, v))$, $(u_1, u_2) \notin E_1$ then $e \notin E$ $(\mu_i \circ \beta_i)((u, v_1), (u, v_2)) = 0$

Now $\overline{\mu_i \circ \beta_i}(e) = ((\sigma_i \circ \alpha_i)(u_1, v)) \wedge ((\sigma_i \circ \alpha_i)(u_2, v))$

$= (\sigma_i(u_1) \wedge \alpha_i(v)) \wedge ((\sigma_i(u_2) \wedge \alpha_i(v))$

$= \sigma_i(u_1) \wedge \sigma_i(u_2) \wedge \alpha_i(v)$ since $(u_1, u_2) \in \overline{E_1}$

$(\overline{\mu_1} \circ \overline{\beta_1})(e) = \overline{\mu_i}(u_1, u_2) \wedge \alpha_i(v) = \sigma_i(u_1) \wedge \sigma_i(u_2) \wedge \alpha_i(v)$

$\therefore\ \overline{\mu_i \circ \beta_i} = \overline{\mu_i} \circ \overline{\beta_i}$ for all i= 1, 2, 3, ... k.

Case(iii): Consider the edge $e = ((u_1, v_1), (u_2, v_2)), (u_1, u_2) \in E_1$ & $v_1 \neq v_2$

Then $e \in E$ and G is a strong multi fuzzy graph, So $\overline{\mu_i \circ \beta_i}(e) = 0$

since $(u_1, u_2) \notin \overline{E_1}$, $(\overline{\mu_1} \circ \overline{\beta_1})(e) = 0$

If $e = ((u_1, v_1), (u_2, v_2)), (u_1, u_2) \notin E_1$ & $v_1 \neq v_2$

Then $e \notin E$ $(\mu_i \circ \beta_i)(e) = 0$

$\overline{\mu_i \circ \beta_i}(e) = ((\sigma_i \circ \alpha_i)(u_1, v_1)) \wedge ((\sigma_i \circ \alpha_i)(u_2, v_2)) = (\sigma_i(u_1) \wedge \alpha_i(v_1)) \wedge ((\sigma_i(u_2) \wedge \alpha_i(v_2))$

Since $(u_1,u_2)\in\overline{E_1}$ we have

$(\overline{\mu_1}\,o\,\overline{\beta_1})(e)=\overline{\mu_i}(u_1,u_2)\wedge\alpha_i(v_1)\wedge\alpha_i(v_2)=\sigma_i(u_1)\wedge\sigma_i(u_2)\wedge\alpha_i(v_1)\wedge\alpha_i(v_2)=\overline{\mu_i\,o\beta_i}(e)$

$\therefore\ \overline{\mu_i\,o\beta_i}=\overline{\mu_i}\,o\overline{\beta_i}$ for all i= 1, 2, 3, ... k.

Case(iv): Consider the edge $e=((u_1,v_1),(u_2,v_2)),\ (u_1,u_2)\notin E_1$ & $(v_1,v_2)\notin E_2$

Since $e\in E$, $(\mu_i\,o\,\beta_i)(e)=0$

$\overline{\mu_i\,o\beta_i}(e)=((\sigma_i\,o\alpha_i)(u_1,v_1))\wedge((\sigma_i\,o\alpha_i)(u_2,v_2))=(\sigma_i(u_1)\wedge\alpha_i(v_1))\wedge((\sigma_i(u_2)\wedge\alpha_i(v_2))$

If $(u_1,u_2)\in\overline{E_1}$ and if $v_1=v_2$ then we have case (ii)

If $(u_1,u_2)\notin\overline{E_1}$ and if $v_1\neq v_2$ then we have case (iii)

In all the cases we have, $\overline{\mu_i\,o\beta_i}=\overline{\mu_i}\,o\overline{\beta_i}$ for all i= 1, 2, 3, ... k.

Definition 4.22 The operation **Corona Product** between two MFG G_1 and G_2 is defined as follows, $G_1\otimes G_2=((\sigma_1\otimes\alpha_1,\sigma_2\otimes\alpha_2,\ldots\sigma_k\otimes\alpha_k),(\mu_1\otimes\beta_1,\mu_2\otimes\beta_2,\ldots\mu_k\otimes\beta_k))$ with the underlying crisp graph $G_1^*\otimes G_2^*=(V,E)=(V_1\cup V_2,E_1\cup E_2)$

$(\sigma_i\otimes\alpha_i)(u)=\begin{cases}\sigma_i(u),u\in V_1\\ \alpha_i(u),u\in V_2\end{cases}$ and $(\mu_i\otimes\beta_i)(u,v)=\begin{cases}\mu_i(u,v), & (u,v)\in E_1\\ \beta_i(u,v), & (u,v)\in E_2\\ \min\{\sigma_i(u),\alpha_i(v)\}, & uv\in E'\end{cases}$ where E' is the set of all edges joining by an edge the ith vertex of G_1 to every vertex in the ith copy of G_2

If $m\neq n$, let k = max (m, n). Suppose $m<n$ then let us introduce n – m membership values of multi fuzzy graph G_1 into 0 so as to convert the multi fuzzy graphs G_1 and G_2 have the same dimension as k.

Example 4.23

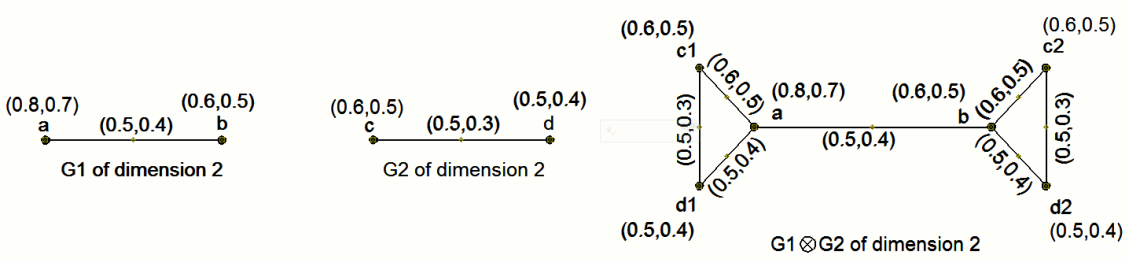


Figure 9

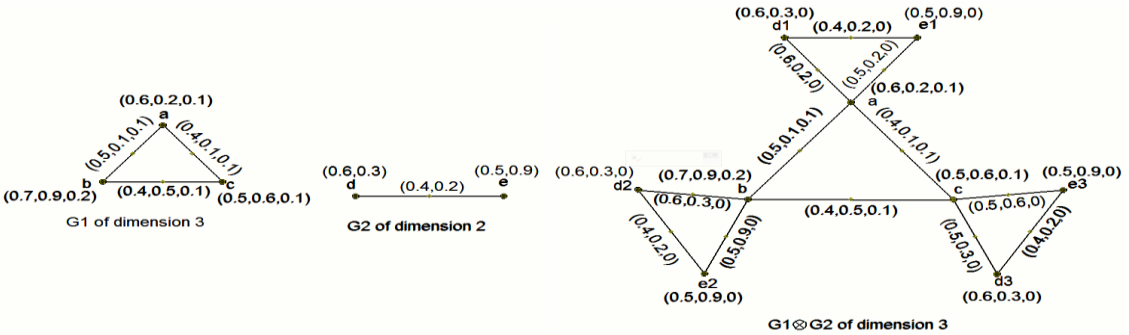


Figure 10

Theorem 4.24 Corona product of two multi fuzzy graph is also a multi-fuzzy graph.

Proof: Let $G_1 = ((\sigma_1, \sigma_2, ...\sigma_m), (\mu_1, \mu_2, ...\mu_m))$ and $G_2 = ((\alpha_1, \alpha_2, ...\alpha_n), (\beta_1, \beta_2, ...\beta_n))$ be the multi fuzzy graph with dimension m and n respectively

To Prove: $G = G_1 \otimes G_2$ is a multi-fuzzy graph of dimension k where k= max (m, n)

$$(\sigma_i \otimes \alpha_i)(u) = \begin{cases} \sigma_i(u), u \in V_1 \\ \alpha_i(u), u \in V_2 \end{cases}$$

Case(i): If $(u,v) \in E_1$

$(\mu_i \otimes \beta_i)(u,v) = \mu_i(u,v)$

$\leq \min\{\sigma_i(u), \sigma_i(v)\}$

$= \min\{(\sigma_i \otimes \alpha_i)(u), (\sigma_i \otimes \alpha_i)(v)\}$

Case(ii): If $(u,v) \in E_2$

$(\mu_i \otimes \beta_i)(u,v) = \beta_i(u,v)$

$\leq \min\{\alpha_i(u), \alpha_i(v)\}$

$= \min\{(\sigma_i \otimes \alpha_i)(u), (\sigma_i \otimes \alpha_i)(v)\}$

Case(iii): If $(u,v) \in E'$

$(\mu_i \otimes \beta_i)(u,v) = \min\{\sigma_i(u), \alpha_i(v)\} = \min\{(\sigma_i \otimes \alpha_i)(u), (\sigma_i \otimes \alpha_i)(v)\}$

5. Conclusion

In this paper, the complement of multi fuzzy graph and direct sum of two multi fuzzy graphs are defined and proved some results connected to them. Also defined various product on multi fuzzy graphs such as direct product, strong product, cartesian product, composition, corona product and proved some properties related to them.

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Soft $I_{g\delta s}$-Closed Functions

Y. Rosemathy[1]
Dr. K. Alli[2]

Abstract

In this paper, we have introduced a new class of open and closed functions called soft $I_{g\delta s}$-closed and soft $I_{g\delta s}$-open functions in ideal topological spaces and also investigated some of its characterizations and properties with the existing sets.

Key words and phrases. soft sets, soft topological spaces, soft regular open, soft δ-cluster point, soft $I_{g\delta s}$-closed functions, soft strongly $I_{g\delta s}$-closed functions.

Mathematics Subject Classification. 54A10, 54A20, 54C08[3].

[1] Research Scholar (Reg.No-18111072092002) The M.D.T Hindu College, Tirunelveli-627010, TamilNadu, India. (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India). Email: ravimathy18@gmail.com

[2]Assistant Professor, Department of Mathematics, The M.D.T Hindu College, Tirunelveli-627010, Tamil Nadu, India. (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India)

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1. Introduction

The concept of soft sets was first introduced by Molodtsov [12] in 1999 as a general mathematical tool for dealing with uncertain objects. In [12, 13], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [11], the properties and applications of soft set theory have been studied increasingly [3, 8, 13]. In [14] O.Ravi et all Decompositions of Ï g-Continuity via Idealization and In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [1, 2, 4, 9, 10, 11, 13]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [5]. Recently, in 2011, Shabir and Naz [15] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft interior, soft closure, soft neighborhood of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Hussain and Ahmad [6] investigated the properties of soft open, soft closed, soft interior, soft closure, soft neighborhood of a point. They also defined and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology and will strengthen the foundations of the theory of soft topological spaces. In [16] S. Tharmar and R. Senthilkumar Introduced Soft Locally Closed Sets in Soft Ideal Topological Spaces.

In this paper, we have introduced a new class of open and closed functions called soft $I_{g\delta s}$-closed and soft $I_{g\delta s}$-open functions in ideal topological spaces and also investigated some of its characterizations and properties with the existing sets.

2. Preliminaries

In this section, we present some basic definitions and results which are needed in further study of this paper which may found in earlier studies. Throughout this paper, X refers to an initial universe, E is a set of parameters, $\wp(X)$ is the power set of X, and $A \subset E$

Definition 2.1. [12] A soft set F_A over the universe X is defined by the set of ordered pairs $F_A=\{(e, F_A(e)) : e \in E, F_A(e)\in\wp(X)\}$ where $F_A : E\to\wp(X)$, such that $F_A(e)\neq\emptyset$, if $e\in A\subset E$ and $F_A(e)=\emptyset$ if $e\notin A$. The family of all soft sets over X is denoted by SS(X).

Definition 2.2. [11] The soft set $F_\emptyset$ over a common universe set X is said to be null soft set, denoted by $\emptyset$. Here $F_\emptyset(e)=\emptyset$, $\forall e\in E$.

Definition 2.3. [11] A soft set F_A over X is called an absolute soft set, denoted by $\tilde{A}$, if $e\in A$, $F_A(e)=X$.

Definition 2.4. [11] Let F_A, G_B be soft sets over a common universe set X. Then F_A is a soft subset of G_B, denoted $F_A\subset G_B$ if $F_A(e)\subset G_B(e)$, $\forall e\in E$.

Definition 2.5. [11] Let F_A, G_B be soft sets over a common universe set X. The union of F_A and G_B, is a soft set H_C defined by $H_C(e)=F_A(e)\cup G_B(e)$, $\forall e\in E$, where $C=A\cup B$.
That is, $H_C=F_A\cup G_B$.

Definition 2.6. [11] Let F_A, G_B be soft sets over a common universe set X. The intersection of F_A and G_B, is a soft set H_C defined by $H_C(e)=F_A(e)\cap G_B(e)$, $\forall e\in E$, where $C=A\cap B$.
That is, $H_C=F_A\cap G_B$.

Definition 2.7. *[16] The complement of the soft set F_A over X_A denoted by F^c is defined by* $A^C(e)=X-F_A(e)$, $\forall e\in E$.

Definition 2.8. [16] Let F_A be a soft set over X and $x\in X$. We say that $x\in F_A$ if $x\in F_A(e)$, $\forall e\in A$. For any $x\in X$, $x\notin F_A$ if $x\notin F_A(e)$ for some $e\in A$.

Definition 2.9. [20] The soft set $F_A\in SS(X)$ is called a soft point in SS(X) if there exist $x\in X$ and $e\in E$ such that $F(e)=\{x\}$ and $F(e^c)=\emptyset$ for each $e^c\in E-\{e\}$ and the soft point F_A is denoted by x_ε.

Definition 2.10. [16] A soft topology τ is a family of soft sets over X satisfying the following properties.

(1) $\emptyset$, $\tilde{X}$ belong to τ.

(2) The union of any number of soft sets in τ belongs to τ.

(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space.

Definition 2.11. [15] Let (X, τ, E) be a soft topological space over X. Then

(1) The members of τ are called soft open sets in X.

(2) A soft set F_A over X is said to be a soft closed set in X if $F^c\in\tau$.

(3) A soft set F_A is said to be a soft neighborhood of a point $x\in X$ if $x\in F_A$ and F_A is soft open in (X, τ, E)

(4) The soft interior of a soft set F_A is the union of all soft open subsets of F_A. The

soft
interior of F_A is denoted by int(F_A).

(5) The soft closure of F_A is the intersection of all soft closed super sets of F_A. The soft closure of F_A is denoted by cl(F_A) or $\overline{F_A}$.

Definition 2.12. [18] A soft set F_A in a soft topological space (X, τ, E) is said to be a soft regular open (resp. soft regular closed) if F_A=int(cl(F_A)) (resp. F_A=cl(int(F_A))).

Definition 2.13. Let I be a non-null collection of soft sets over a universe X with the same set of parameters E. Then I⊂SS(X) is called a soft ideal on X with the same set E if
(1) $F_A \in I$ and $G_A \in I \Rightarrow F_A \cup G_A \in I$.
(2) $F_A \in I$ and $G_A \subset F_A \Rightarrow G_A \in I$.

Definition 2.14. Let (X, τ , E) be a soft topological space and I be a soft ideal over X with the same set of parameters E. Then $F^*_A = \cup\{x_e \in X : O_{xe} \cap F_A \notin I$, for all $O_{xe} \in \tau\}$ is called the soft local function of F_A with respect to I and τ, where O_{xe} is a τ-open set containing x_e.

Theorem 2.15. Let I and J be any two soft ideals with the same set of parameters E on a soft topological space (X, τ, E). Let F_A, $G_A \in SS(X)$. Then
(1) $(\emptyset)^* = \emptyset$.
(2) $F_A \subset G_A \Rightarrow F^*_A \subset G^*_A$.
(3) $I \subset J \Rightarrow F^*_A(J) \subset FA^*(I)$.
(4) $F^*_A \subset cl(F_A)$, where cl is the soft closure w.r.t τ.
(5) F^*_A is τ-closed soft set.
(6) $(F^*_A)^* \subset F^*_A$.
(7) $(F_A \cup G_A)^* = F^*_A \cup GA^*$.

Definition 2.16. Let (X, τ , E) be a soft topological space, I be a soft ideal over X with the same set of parameters E and $cl^* : SS(X) \to SS(X)$ be the soft closure operator. Then there exists a unique soft topology over X with the same set of parameters E, finer than τ , called the ⋆-soft topology, defined by τ^*, given by $\tau^* = \{ F_A \in SS(X) : cl^*(X - F_A) = X - F_A\}$.

Definition 2.17. [7] Let F_A be a soft subset of soft topological space (X, τ, E). Then

(1) x_ε is called a soft δ-cluster point of F_A if $F_A \cap int(cl(U_A)) \neq \emptyset$ for every soft open set U_A containing x_ε.

(2) The family of all soft δ-cluster point of F_A is called the soft δ-closure of F_A and is denoted by $cl_\delta(F_A)$.

(3) A soft subset F_A is said to be soft δ-closed if $cl_\delta(F_A) = F_A$. The complement of a soft
δ-closed set of X is said to be soft δ-open.

Lemma 2.18. [7] Let F_A be a soft subset of soft topological space (X, τ, E). Then, the following properties hold:

(1) Int (cl (F_A)) is soft regular open,

(2) Every soft regular open set is soft δ-open,

(3) Every soft δ-open set is the union of a family of soft regular open sets.

(4) Every soft δ-open set is soft open.

Proposition 2.19. [7] Intersection of two soft regular open sets is soft regular open.

Lemma 2.20. [7] Let F_A and G_A be soft subsets of soft topological space (X, τ, E). Then, the following properties hold.

(1) $F_A \subset cl_\delta(F_A)$,

(2) If $F_A \subset G_A$, then $cl_\delta(F_A) \subset cl_\delta(G_A)$,

(3) $cl_\delta(F_A) = \cap\{G_A \in SS(X): F_A \subset G_A \text{ and } G_A \text{ is soft } \delta\text{-closed}\}$,

(4) If $(F_A)_\alpha$ is a soft δ-closed set of X for each $\alpha \in \triangle$, then $\cap\{(F_A)_\alpha: \alpha \in \triangle\}$ is soft δ-closed,

(5) $cl_\delta(F_A)$ is soft δ-closed.

Theorem 2.21. [7] Let (X, τ, E) be a soft topological space and $\tau_\delta = \{F_A \in SS(X) : F_A \text{ is a soft } \delta\text{-open set}\}$. Then τ_δ is a soft topology weaker than τ.

Definition 2.22. A soft subset F_A of a soft ideal topological space (X, τ, E, I) is said to be

(1) soft pre-I-open if $F_A \subset$ int (cl (F_A)),

(2) soft semi-I-open if $F_A A \subset$ cl (int (F_A)),

(3) soft α-I-open if $F_A \subset$ int (cl (int (F_A))).

The complement of soft pre-I-open (resp. soft semi-I-open, soft α-I-open) set is called a soft pre-I-closed (resp. soft semi-I-closed, soft α-I-closed).

Definition 2.23. The soft semi-I-closure of F_A is defined by the intersection of all soft semi-I- closed sets containing F_A and is denoted by SI_{scl}-(F_A)

Definition 2.24. A soft set F_A of soft ideal topological space X is called soft generalized δ semi-closed (briefly soft $I_{g\delta s}$-closed) set if SI_{scl} (F_A) $\subset G_A$ whenever $F_A \subset G_A$ and G_A are soft δ-open over X.

A soft set F_A of X is called soft generalized δ semi-open (briefly soft $SI_{g\delta s}$-open) set if F_A^c is soft $SI_{g\delta s}$-closed.

The family of all soft $SI_{g\delta s}$-closed subsets of the space X is denoted by $SI_{g\delta s}$-C(X) and soft $SI_{g\delta s}$-open subsets of the space X is denoted by $SI_{g\delta s}$-O(X).

3. soft SIgδs-closed and soft SIgδs-open functions

Definition 3.1. A function f: (X, τ, E) → (Y, σ, K, I) is said to be soft $SI_{g\delta s}$-closed (resp. soft

$SI_{g\delta s}$-open) if f (V_A) is soft $SI_{g\delta s}$-closed (resp. sost $SI_{g\delta s}$-open) over Y for every soft closed (resp. soft open) set V_A over X.

Definition 3.2. (1) A function f: (X, τ, E, I) → (Y, σ, K, J) is soft $I_{g\delta s}$-irresolute if $f^{-1}(V_A)$ is soft $SI_{g\delta s}$-closed over X for every soft $SI_{g\delta s}$-closed set V_A over Y.
(2) A function f: $(X,\tau,E,I) \rightarrow (Y,K,\sigma)$ is soft $SI_{g\delta s}$-continuous if $f^{-1}(V_A)$ is soft $SI_{g\delta s}$-closed over X for every soft closed set V_A over Y.

Theorem 3.3. A function f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{g\delta s}$-closed if and only if f (V_A) is soft $SI_{g\delta s}$-open over Y for every soft open set V_A over X.
Proof: Suppose f: $(X,\tau,E) \rightarrow (Y,\sigma,K,I)$ is soft $SI_{g\delta s}$-closed function and V_A is a soft open set over X. Then $\tilde{X} - V_A$ is soft closed over X. By hypothesis $f(\tilde{X} - V_A) = \tilde{Y} - f(V_A)$ is a soft $SI_{g\delta s}$-closed set over Y and hence $f(V_A)$ is soft $SI_{g\delta s}$-open set over Y. On the other hand, if F_A is soft closed set over X, then $\tilde{X} - F_A$ is a soft open set over X. By hypothesis $f(\tilde{X} - F_A) = \tilde{Y} - f(F_A)$ is soft $SI_{g\delta s}$-open set over Y, implies f (F_A) is soft $SI_{g\delta s}$-closed set over Y. Therefore, f is soft $SI_{g\delta s}$-closed function.

Definition 3.4. A soft ideal topological space X is said to be soft $TI_{g\delta s}$-space if every soft $SI_{g\delta s}$-closed set is soft closed over X.

Definition 3.5. A soft ideal topological space X is said to be soft $SI_{g\delta s}$-T_2 space if every soft $SI_{g\delta s}$-closed set is soft semi-closed over X.

Theorem 3.6. If f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{g\delta s}$-closed function and Y is soft $TI_{g\delta s}$- space, then f is a soft closed function.
Proof: Let V_A be a soft closed set over X. Since f is a soft $SI_{g\delta s}$-closed function, implies f (V_A) is soft $SI_{g\delta s}$-closed over Y. Now Y is soft $TI_{g\delta s}$-space, implies f (V_A) is a soft closed set over Y. Therefore, f is a soft closed function.

Theorem 3.7. If f: $(X,\tau,E) \rightarrow (Y,\sigma,K,I)$ is soft $SI_{g\delta s}$-closed function and Y is soft $SI_{g\delta s}$-T_2 space, then f is soft semi-closed function.
Proof: Let V_A be a soft closed set over X. Since f is a soft $SI_{g\delta s}$-closed function, f (V_A) soft is $SI_{g\delta s}$-closed set over Y. Now Y is soft $SI_{g\delta s}$-T_2 space, implies f (V_A) is a soft semi-closed set over Y. Therefore, f is a soft semi-closed function.
Theorem 3.8. For the function f: (X, τ, E) → (Y, σ, K, I), the following statements are equivalent.

(1) f is a soft $SI_{g\delta s}$-open function.

(2) For each soft subset F_A of X, $f(\text{int}(F_A)) \subset SI_{g\delta s} - \text{int}(f(F_A))$

(3) For each $x_e \in \tilde{X}$, the image of every soft nhd of x_e is soft $SI_{g\delta s}$-nhd of $f(x_e)$.

Proof: (1) → (2) Suppose (1) holds and $F_A \subset X$. Then int (F_A) is a soft open set over X. By (1), f (int (F_A)) is a soft $SI_{g\delta s}$-open set over Y.

Therefore $SI_{g\delta s}$ – int (f (int (F_A))) = f (int (F_A)). Since f (int (F_A)) ⊂ f (F_A), implies $SI_{g\delta s}$ – int (f (int (F_A))) ⊂ $SI_{g\delta s}$ – int (f (F_A)). That is f(int (F_A)) ⊂ $SI_{g\delta s}$ – int (f(F_A)).

(2) → (3) Suppose (2) holds. Let $x_e \in \tilde{X}$ and F_A be an arbitrary soft nhd of x_e over X. Then there exists a soft open set H_A in X such that $x_e \in H_A \subset F_A$. By (2), f(H_A) = f(int (H_A)) ⊂ $SI_{g\delta s}$–int (f(H_A)). But $SI_{g\delta s}$–int (f(H_A)) ⊂ f(H_A) is always true. Therefore, f(H_A) = $SI_{g\delta s}$–int (f (H_A)) and hence f (H_A) is soft $SI_{g\delta s}$-open set over Y. Further f (x_e) ∈ f (H_A) ⊂ f (F_A), this implies, f(F_A) is a soft $SI_{g\delta s}$-nhd of f(x_e) over Y. Hence (3) holds.

(3) → (1) Suppose (3) holds. Let V_A be any soft open set over X and $x_e \in V_A$. Then y_e = f (x_e) ∈ f (V_A). By (3), for each $y_e \in$ f (V_A), there exists a soft $SI_{g\delta s}$-nhd $(Z_A)_{ye}$ of y_e over Y. Since $(Z_A)_{ye}$ is a soft $SI_{g\delta s}$-nhd of y_e, there exists a soft $SI_{g\delta s}$-open set $(V_A)_{ye}$ in V_A such that $y_e \in (V_A)_{ye} \subset (Z_A)_{ye}$. Therefore f ($V_A$) = ∪{$(V_A)_{ye}$: $y_e \in$ f (V_A)}, which is union of soft $SI_{g\delta s}$-open sets and hence soft $SI_{g\delta s}$-open set over Y. Therefore, f is soft $SI_{g\delta s}$-open function.

Theorem 3.9. A function f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{g\delta s}$-closed if and only if for each soft subset H_A over Y and for each soft open set U_A over X containing $f^{-1}(H_A)$, there exists a soft $SI_{g\delta s}$-open set V_A over Y such that $H_A \subset V_A$ and $f^{-1}(V_A) \subset U_A$.

Proof: Assume that f is soft $SI_{g\delta s}$-closed function. Let $H_A \subset Y$ and U_A be a soft open set over X containing $f^{-1}(H_A)$. Since f is a soft $SI_{g\delta s}$-closed function and $\tilde{X} - U_A$ is soft closed over X, implies f ($\tilde{X} - U_A$) is soft $SI_{g\delta s}$-closed set over Y. Then $V_A = \tilde{Y} -$ f ($\tilde{X} - U_A$) is soft $SI_{g\delta s}$-open set over Y such that $H_A \subset V_A$ and $f^{-1}(V_A) \subset U_A$.

Conversely, let F_A be a soft closed set over X, then $\tilde{X} - F_A$ is a soft open set over X and

$f^{-1}(\tilde{Y} - f(F_A)) \subset \tilde{X} - F_A$. By hypothesis, there is a soft $SI_{g\delta s}$-open set V_A over Y such that $\tilde{Y}$ −f (F_A) ⊂ V_A and $f^{-1}(V_A) \subset \tilde{X} - F_A$. Therefore, $\tilde{Y} - V_A \subset f(F_A) \subset f(\tilde{X} - F^{-1}(V_A)) \subset \tilde{Y} - V_A$, this implies f ($F_A$) = $\tilde{Y} - V_A$. Since V_A is a soft $SI_{g\delta s}$-open set over Y and so f (F_A) is soft $SI_{g\delta s}$- closed over Y. Hence f is soft $SI_{g\delta s}$-closed function.

Theorem 3.10. If f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{g\delta s}$-closed, then for each soft $SI_{g\delta s}$-closed set H_A over Y and each soft open set G_A over X containing $f^{-1}(H_A)$, there exists soft $SI_{g\delta s}$-open set V_A containing H_A such that $f^{-1}(V_A) \subset U_A$.

Proof: Suppose f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{g\delta s}$-closed function. Let H_A be any soft $SI_{g\delta s}$-closed set over Y and U_A is a soft open set over X containing $f^{-1}(H_A)$, by theorem 3.9, there exists a soft $SI_{g\delta s}$-open set G_A over Y such that $H_A \subset G_A$ and $f^{-1}(G_A) \subset U_A$. Since H_A is soft $SI_{g\delta s}$-closed set and G_A is soft $SI_{g\delta s}$-open set containing H_A implies $H_A \subset I_{g\delta s}$-int ($G_A$). Put $V_A = I_{g\delta s}$-int (G_A), then $H_A \subset V_A$ and V_A are soft $SI_{g\delta s}$-open set over Y and $f^{-1}(V_A) \subset U_A$.

Theorem 3.11. A function f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ is soft $SI_{g\delta s}$-closed, if and only if $SI_{g\delta s}$- $cl(f(F_A)) \subset f(cl(F_A))$, for every soft subset F_A over X.

Proof: Suppose f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ is a soft $SI_{g\delta s}$-closed and $F_A \subset X$. Then f $(cl(F_A))$ is soft $SI_{g\delta s}$-closed over Y. Since $f(F_A) \subset f(cl(F_A))$, implies $SI_{g\delta s}$-$cl(f(F_A)) \subset SI_{g\delta s}$-$cl(f(cl(F_A))) = f(cl(F_A))$. Hence $SI_{g\delta s}$-$cl(f(F_A)) \subset f(cl(F_A))$.

Conversely, let F_A be any soft closed set over X. Then $cl(F_A) = F_A$. Therefore, $f(F_A) = f(cl(F_A))$. By hypothesis, $SI_{g\delta s}$-$cl(f(F_A)) \subset f(cl(F_A)) = f(F_A)$ implies $SI_{g\delta s}$-$cl(f(F_A)) \subset f(F_A)$. But $f(F_A) \subset SI_{g\delta s}$-$cl(f(F_A))$ is always true. This shows, $f(F_A) = SI_{g\delta s}$-$cl(f(F_A))$. Therefore $f(F_A)$ is soft $SI_{g\delta s}$-closed set over Y and hence f is soft $SI_{g\delta s}$-closed.

Theorem 3.12. Let $f : (X, \tau, E) \rightarrow (Y, \sigma, K, I)$ and $g : (Y, \sigma, K, I) \rightarrow (Z, \mu, L, J)$ be any two functions. Then $(g \circ f) : (X, \tau, E) \rightarrow (Z, \mu, L, J)$ is soft $SI_{g\delta s}$-closed function if f and g satisfy one of the following conditions

(1) f, g are soft $SI_{g\delta s}$-closed functions and Y is soft $TI_{g\delta s}$-space.

(2) f is soft closed and g is soft $SI_{g\delta s}$-closed function.

Proof: (1) Suppose F_A is soft closed set over X. Since f is soft $SI_{g\delta s}$-closed function $f(F_A)$ is soft $SI_{g\delta s}$-closed set over Y. Now Y is soft $TI_{g\delta s}$-space, implies f (F_A) is soft closed set over Y. Also, g is soft $SI_{g\delta s}$-closed function, implies g (f $(F_A)) = (g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{g\delta s}$-closed function.

(2) Suppose F_A is soft closed set over X. Since f is soft closed function $f(F_A)$ is soft closed set over Y. Now g is soft $SI_{g\delta s}$-closed function, implies $g(f(F_A)) = (g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{g\delta s}$-closed function.

Theorem 3.13. Let f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ and $g : (Y, \sigma, K, I) \rightarrow (Z, \mu, L, J)$ be any two functions such that $(g \circ f) : X \rightarrow Z$ be soft $SI_{g\delta s}$-closed function. Then following results hold

(1) If f is soft continuous surjection, then g is soft $SI_{g\delta s}$-closed function.

(2) If g is soft $SI_{g\delta s}$-irresolute and injective, then f is soft $SI_{g\delta s}$-closed function.

Proof: (1) Suppose F_A is a soft closed set over Y. Since f is soft continuous and surjective, $f^{-1}(F_A)$ is a soft closed set over X. Therefore, $(g \circ f)(f^{-1}(F_A)) = g(F_A)$ is soft $SI_{g\delta s}$-closed set over Z and hence g is soft $SI_{g\delta s}$-closed function.

(2) Suppose H_A is soft closed set over X. Then $(g \circ f)(H_A)$ is soft $SI_{g\delta s}$-closed set over Z. Since g is soft $SI_{g\delta s}$-irresolute, $g^{-1}((g \circ f)(H_A)) = f(H_A)$ is soft $SI_{g\delta s}$-closed set over Y. Hence f is soft $SI_{g\delta s}$-closed function.

Theorem 3.14. For any bijection f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$, the following statements are equivalent:

(1) f^{-1} is soft $SI_{g\delta s}$-continuous.

(2) f is a soft $SI_{g\delta s}$-open function.

(3) f is a soft $SI_{g\delta s}$-closed function.

Proof: (1) → (2) Suppose F_A is a soft open set over X, then by (1), $(f^{-1})^{-1}(F_A) = f(F_A)$ is soft $SI_{g\delta s}$-open set over Y and hence f is soft $SI_{g\delta s}$-open function.

(2) → (3) Suppose F_A is a soft closed set over X, then $\tilde{X} - F_A$ is a soft open set over X. By (2), $f(\tilde{X} - F_A) = \tilde{Y} - f(F_A)$ is soft $SI_{g\delta s}$-open over Y, implies $f(F_A)$ is soft $SI_{g\delta s}$-closed over Y. Therefore, f is soft $SI_{g\delta s}$-closed function.

(3) → (1) Let F_A be a soft closed set over X. By (3), $f(F_A) = (f^{-1})^{-1}(F_A)$ is soft $SI_{g\delta s}$-closed over Y. Therefore f^{-1} is soft $SI_{g\delta s}$ continuous function.

4. soft SIpgδs-closed functions

Definition 4.1. A function f: (X, τ, E) → (Y, σ, K, I) is said to be soft $SI_{pg\delta s}$ closed (resp. soft $SI_{pg\delta s}$ open) if $f(V_A)$ is soft $SI_{g\delta s}$-closed (resp. soft $SI_{g\delta s}$-open) over Y for every soft semi-closed (resp. soft semi-open) set V_A over X.

Definition 4.2. (1) A function f: (X, τ, E) → (Y, σ, K, I) is said to be soft semi-closed if $f(V_A)$ is soft semi-closed over Y for every soft semi-closed set V_A over X.

(2) A function f: (X, τ, E) → (Y, σ, K) is said to be soft pre-closed if $f(V_A)$ is soft closed over Y for every soft semi-closed set V_A over X.

(3) A function f: (X, τ, E) → (Y, σ, K) is said to be soft δ-continuous if $f^{-1}(V_A)$ is soft
δ-closed over X for every soft δ-closed set V_A over Y.

Theorem 4.3. A function f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{pg\delta s}$-closed if and only if $f(V_A)$
is soft $SI_{g\delta s}$-open over Y for every soft semi-open set V_A over X.

Proof: Similar to the proof of theorem 3.3.

Remark 4.4. Every semi-open function is $I_{g\delta s}$-open function.

Theorem 4.5. If f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{pg\delta s}$-closed function and Y is soft $I_{g\delta s}$-$T_{1/2}$
space, then f is soft semi closed function.

Proof: Suppose V_A is a soft semi-closed set over X. Since f is a soft $SI_{pg\delta s}$-closed function $f(V_A)$ is soft $SI_{g\delta s}$-closed set over Y. Now Y is soft $I_{g\delta s}$-$T_{1/2}$ space $f(V_A)$ is a soft semi-closed set over Y. Therefore, f is a soft semi-closed function.

Theorem 4.6. A function f: (X, τ, E) → (Y, σ, K, I) is soft $SI_{pg\delta s}$-closed if and only if for each soft subset H_A over Y and for each soft semi-open set U_A over X containing $f^{-1}(H_A)$, there exists

a soft $SI_{g\delta s}$-open set V_A over Y such that $H_A \subset V_A$ and $f^{-1}(V_A) \subset U_A$.
Proof: Similar to the proof of theorem 3.9.

Theorem 4.7. If f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ is soft $SI_{pg\delta s}$-closed, then for each soft $SI_{g\delta s}$-closed set H_A over Y and each soft semi-open set G_A over X containing $f^{-1}(H_A)$, there exists soft $SI_{g\delta s}$-open set V_A over Y containing H_A such that $f^{-1}(V_A) \subset U_A$.
Proof: Similar to the proof of theorem 3.10.

Theorem 4.8. If f is soft δ-continuous, soft $SI_{pg\delta s}$-closed, then $f(H_A)$ is soft $SI_{g\delta s}$-closed over Y for each soft $SI_{g\delta s}$-closed H_A over X, with X is soft $I_{g\delta s}$-$T_{1/2}$ space.
Proof: Suppose H_A is any soft $SI_{g\delta s}$-closed set over X and V_A is a soft δ-open set over Y containing $f(H_A)$. This implies $H_A \subset f^{-1}(V_A)$. Since f is soft δ-continuous, $f^{-1}(V_A)$ is a soft δ- open set containing H_A, therefore, $SI_{g\delta s}$-cl $(H_A) \subset f^{-1}(V_A)$ and hence f $(I_{g\delta s}$-cl $(H_A)) \subset V_A$. Since f is soft $SI_{pg\delta s}$-closed, implies f $(I_{g\delta s}$-cl $(H_A))$ is soft $SI_{g\delta s}$-closed set contained over Y, implies $SI_{g\delta s}$-cl $(f(SI_{g\delta s}$-cl $(H_A))) \subset V_A$. Thus, $SI_{g\delta s}$-cl$(f(H_A)) \subset SI_{g\delta s}$-cl $(f\ (SI_{g\delta s}$-cl $(H_A))) \subset V_A$. That is, $SI_{g\delta s}$-cl$(f(H_A)) \subset V_A$. This shows that $f(H_A)$ is soft $SI_{g\delta s}$-closed over Y.

Theorem 4.9. Let f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ and g: $(Y, \sigma, K, I) \rightarrow (Z, \mu, L, J)$ be any two functions. Then $(g \circ f)$: $X \rightarrow Z$ is soft $SI_{pg\delta s}$-closed function if f and g satisfy one of the following conditions:

(1) f, g are soft $SI_{pg\delta s}$-closed functions and Y is soft $I_{g\delta s}$-$T_{1/2}$ space.

(2) f is soft pre-closed and g is soft $SI_{g\delta s}$-closed function.

(3) f is soft semi-closed and g is soft $SI_{pg\delta s}$-closed function.

(4) f is soft $SI_{pg\delta s}$-closed function and g is soft δ-continuous, soft $SI_{pg\delta s}$-closed function and Y is soft $I_{g\delta s}$-$T_{1/2}$-space.
Proof: (1) Suppose F_A is soft semi-closed set over X. Since f is soft $SI_{pg\delta s}$-closed function $f(F_A)$ is soft $SI_{pg\delta s}$-closed set over Y. Now Y is soft $I_{g\delta s}$-$T_{1/2}$-space, therefore $f(F_A)$ is soft semi closed set over Y. Also g is soft $SI_{pg\delta s}$-closed function, implies $g(f(F_A)) = (g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{pg\delta s}$-closed function.

(2) Suppose F_A is soft semi-closed set over X. Since f is soft pre-closed, $f(F_A)$ is soft closed set over Y. Now g is soft $SI_{g\delta s}$-closed function, implies $g(f(F_A)) = (g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{pg\delta s}$-closed function.

(3) Suppose F_A is soft semi-closed set over X. Since f is soft semi-closed function, $f(F_A)$ is soft semi-closed set over Y. Now g is soft $SI_{pg\delta s}$-closed function, implies $g(f(F_A)) = (g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{pg\delta s}$-closed function.

(4) Suppose H_A is a soft semi-closed set over X. Since f is soft $SI_{pg\delta s}$-closed function $f(H_A)$ is soft $SI_{g\delta s}$-closed set over Y. Since g is soft δ-continuous, soft $SI_{pg\delta s}$-closed function by Theorem 4.8, $g(f(H_A)) = (g \circ f)(H_A)$ is soft $SI_{g\delta s}$-closed set over Z. Hence $(g \circ f)$ is soft $SI_{pg\delta s}$-closed function.

5. Strongly soft SIgδs-closed and soft quasi SIgδs-closed functions

Definition 5.1. A function f: (X, τ, E) → (Y, σ, K, I) is said to be strongly soft $SI_{g\delta s}$-closed (resp. strongly soft $SI_{g\delta s}$-open), if f (F_A) is soft $SI_{g\delta s}$-closed (resp. soft $SI_{g\delta s}$-open) set over Y for every soft $SI_{g\delta s}$-closed (resp. soft $SI_{g\delta s}$-open) set F_A over X.

Remark 5.2. Every strongly soft $SI_{g\delta s}$-closed function is soft $SI_{g\delta s}$-closed function.

Theorem 5.3. A surjective function f: (X, τ, E) → (Y, σ, K, I) is strongly soft $SI_{g\delta s}$-closed (resp. strongly soft $SI_{g\delta s}$-open), if and only if for any soft subset G_A over V_A and each soft $SI_{g\delta s}$- open (resp. soft $SI_{g\delta s}$-closed) set U_A over X containing $f^{-1}(G_A)$, there exists a soft $SI_{g\delta s}$-open (resp. soft $SI_{g\delta s}$-closed) set V_A over Y containing G_A and $f^{-1}(V_A) \subset U_A$.
Proof: Similar to the proof of theorem 3.9.

Theorem 5.4. If a function f: (X, τ, E) → (Y, σ, K, I) is a strongly soft $SI_{g\delta s}$ closed function, then for each soft $SI_{g\delta s}$-closed set H_A over Y and each soft $SI_{g\delta s}$-open set U_A over X containing $f^{-1}(H_A)$, there exists soft $SI_{g\delta s}$-open set V_A over Y containing H_A such that $f^{-1}(V_A) \subset U_A$.
Proof: Similar to the proof of theorem 3.10.

Theorem 5.5. A function f: (X, τ, E) → (Y, σ, K, I) is strongly soft $SI_{g\delta s}$-closed, if and only if $SI_{g\delta s}-cl\ (f(F_A)) \subset f(SI_{g\delta s}\text{-}cl\ (F_A))$ for every soft subset F_A over X.
Proof: Let f be strongly soft $SI_{g\delta s}$-closed function and $F_A \subset X$. Then f ($SI_{g\delta s}$-cl (F_A)) is soft $SI_{g\delta s}$-closed over Y. Since f (F_A) ⊂ f ($SI_{g\delta s}$-cl (F_A)), implies $SI_{g\delta s}$-cl(f(F_A)) ⊂ $SI_{g\delta s}$-cl (f ($SI_{g\delta s}$- cl (F_A))) = f ($SI_{g\delta s}$-cl (F_A)). Therefore, $SI_{g\delta s}$-cl(f(F_A)) ⊂ f($SI_{g\delta s}$-cl (F_A)).
Conversely, F_A is any soft $SI_{g\delta s}$-closed set over X. Then $SI_{g\delta s}$-cl (F_A) = F_A, implies, f (F_A) = f ($SI_{g\delta s}$-cl (F_A)). By hypothesis, $SI_{g\delta s}$-cl(f(F_A)) ⊂ f ($SI_{g\delta s}$-cl (F_A)) = f (F_A). Hence $SI_{g\delta s}$- cl(f(F_A)) ⊂ f (F_A). But f (F_A) ⊂ $SI_{g\delta s}$-cl(f(F_A)) is always true. This shows, f (F_A) = $SI_{g\delta s}$- cl(f(F_A)). Therefore, f (F_A) is soft $SI_{g\delta s}$-closed set over Y. Hence f is strongly soft $SI_{g\delta s}$-closed- closed function.

Theorem 5.6. Let f: (X, τ, E) → (Y, σ, K, I) and g: (Y, σ, K, I) → (Z, μ, L, J) be two functions, such that (g ◦ f) : X → Z is strongly soft $SI_{g\delta s}$-closed function. Then

(1) f is soft $SI_{g\delta s}$-irresolute and surjective implies g is strongly soft $SI_{g\delta s}$-closed.

(2) g is soft $SI_{g\delta s}$-irresolute and injective implies f is strongly soft $SI_{g\delta s}$-closed.

Proof: (1) Let F_A be soft $SI_{g\delta s}$-closed set over Y. Since f is soft $SI_{g\delta s}$ irresolute and surjective, $f^{-1}(F_A)$ is soft $SI_{g\delta s}$-closed set over X. Also since (g ◦ f) is strongly soft $SI_{g\delta s}$-closed function, implies (g ◦ f)($f^{-1}(F_A)$) = g(F_A) is soft $SI_{g\delta s}$-closed over Z. Therefore, g is strongly soft $SI_{g\delta s}$- closed.

(2) Let F_A be soft $SI_{g\delta s}$-closed set over X. Since (g ◦ f) is strongly soft $SI_{g\delta s}$-

closed function $(g \circ f)(F_A)$ is soft $SI_{g\delta s}$-closed over Z. Also, since g is soft $SI_{g\delta s}$-irresolute and injective, $g^{-1}(g \circ f)(F_A) = f(F_A)$ is soft $SI_{g\delta s}$-closed set over Y. Therefore, f is strongly soft $SI_{g\delta s}$ closed.

Theorem 5.7. For any bijection, f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ the following statements are equivalent.

(1) f^{-1} is soft $SI_{g\delta s}$-irresolute.

(2) f is a strongly soft $SI_{g\delta s}$-open function.

(3) f is a strongly soft $SI_{g\delta s}$-closed function.

Proof: Similar to the proof of theorem 3.14.

Definition 5.8. A function f: $(X, \tau, E) \rightarrow (Y, \sigma, K, I)$ is said to be soft quasi $SI_{g\delta s}$-closed (resp. soft quasi $SI_{g\delta s}$-open), if for each soft $SI_{g\delta s}$-closed (resp. soft $SI_{g\delta s}$-open) set F_A over X, $f(F_A)$ is soft closed (resp. open) set over Y.

Remark 5.9. Every soft quasi $SI_{g\delta s}$-closed function is soft closed, strongly soft $SI_{g\delta s}$ closed and soft $SI_{g\delta s}$-closed function.

Remark 5.10. Every soft quasi $SI_{g\delta s}$-closed function is soft $SI_{pg\delta s}$-closed.

Remark 5.11. Following diagram is obtained from the Definitions.

soft $SI_{pg\delta s}$-closed

↗ ↑ ↘

soft quasi $SI_{g\delta s}$-closed → Strongly soft $SI_{g\delta s}$-closed → soft $SI_{g\delta s}$-closed

↘ ↗

sot closed

Theorem 5.12. A surjective function f: $(X, \tau, E, I) \rightarrow (Y, \sigma, K)$ is soft quasi $SI_{g\delta s}$-closed (resp. soft quasi $SI_{g\delta s}$-open), if and only if for any soft subset G_A over Y and each soft $SI_{g\delta s}$-open (resp. soft $SI_{g\delta s}$-closed) set U_A over X containing $f^{-1}(G_A)$, there exists a soft open (resp. soft closed) set V_A over Y containing G_A and $f^{-1}(V_A) \subset U_A$.

Proof: Similar to the proof of theorem 3.9.

Theorem 5.13. A function f: $(X, \tau, E, I) \rightarrow (Y, \sigma, K)$ is soft quasi $SI_{g\delta s}$-closed if and only if

$Cl(f(F_A)) \subset f(SI_{g\delta s}\text{-}cl(F_A))$ for every soft subset F_A over X.

Proof: Suppose that f is soft quasi $SI_{g\delta s}$-closed function and $F_A \subset X$. Then $SI_{g\delta s} - cl(F_A)$ is soft $SI_{g\delta s}$-closed set over X. Therefore $f(SI_{g\delta s}\text{-}cl(F_A))$ is soft closed over Y. Since $f(F_A) \subset f(SI_{g\delta s}\text{-}cl(F_A))$, implies $cl(f(F_A)) \subset cl(f(SI_{g\delta s}\text{-}cl(F_A))) = f(SI_{g\delta s}\text{-}cl(F_A))$. This im- plies, $cl(f(F_A)) \subset f(SI_{g\delta s}\text{-}cl(F_A))$.

Conversely, F_A is any soft $SI_{g\delta s}$-closed set over X. Then $SI_{g\delta s}\text{-}cl(F_A) = F_A$. Therefore, $f(F_A) = f(SI_{g\delta s}\text{-}cl(F_A))$. By hypothesis, $cl(f(F_A)) \subset f(SI_{g\delta s}\text{-}cl(F_A)) = f(F_A)$. Hence $cl(f(F_A)) \subset f(F_A)$. But $f(F_A) \subset cl(f(F_A))$ is always

true. This shows, $f(F_A) = cl(f(F_A))$. This implies $f(F_A)$ is soft closed set over Y. Therefore, f is soft quasi $SI_{g\delta s}$-closed function.

Theorem 5.14. Let $f: (X, \tau, E, I) \rightarrow (Y, \sigma, K, J)$ be a function from a space X to a soft
$TI_{g\delta s}$-space Y. Then following are equivalent

(1) f is strongly soft $S\text{-}I_{g\delta s}$-closed function.

(2) f is soft quasi-$S\text{-}I_{g\delta s}$-closed function.

Proof: (1) ⇒ (2) Suppose (1) holds. Let F_A be a soft $SI_{g\delta s}$-closed set over X. Then $f(F_A)$ is soft $S\text{-}I_{g\delta s}$-closed over Y. Since Y is soft $TI_{g\delta s}$-space, $f(F_A)$ is soft closed over Y. Therefore, f is soft quasi $SI_{g\delta s}$-closed function.
(2) ⇒ (1) Suppose (2) holds. Let F_A be a soft $S\text{-}I_{g\delta s}$-closed set over X. Then f (F_A) is soft closed and hence soft $SI_{g\delta s}$-closed over Y. Therefore, f is strongly soft $S\text{-}I_{g\delta s}$-closed. function.

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$G_{\alpha,\beta}$ Antagonistic Intuitionistic Fuzzy Sub Commutative Ideals of Subtraction G-Algebra

B. Lena[1]
C. Ragavan[2]

Abstract

The notions of $G_{\alpha,\beta}$ over antagonistic intuitionistic fuzzy sub commutative ideals of subtraction G-algebras are introduced. The characterization properties of antagonistic intuitionistic fuzzy sub commutative ideals are obtained. We investigate the relations between antagonistic intuitionistic fuzzy sub implicative ideals and antagonistic intuitionistic fuzzy sub commutative ideals of subtraction G-algebra.

Keywords: Subtraction G-algebra, AIFSI ideals, AIFSC ideals.

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[1]Department of Mathematics, Trinity College for Women, Namakkal, Tamil Nadu, India;lenasenthil1977@gmail.com
[2]Department of Mathematics, SVM College, Uthangarai, Krishnagiri, Tamil Nadu, India.
[3]Received on APRIL 12th, 2022.Accepted on Sep 1st, 2022.Published on Nov 30th, 2022.doi: 10.23755/rm.v44i0.913. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.This paper is published under the CC-BY license agreement.

1. Introduction

The introduction of fuzzy sets by Zadeh [17], there have been a number of generalizations of this fundamental concept. BCK-algebras and BCI-algebras are two classes of logical algebra. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others which were initiated by K. Iseki [3, 4]. The notion of fuzzy sets, invented by L. A. Zadeh [17], has been applied to many fields. Since then, fuzzy BCI/BCK algebras have been extensively investigated by several researchers. For BCK-algebras, Y. B. Jun et al. [7, 10] introduced the notions of fuzzy positive implicative ideals. Bandaru and Rafi [1] introduced a new notion called G-algebra, since the fuzzy G-algebras have been extensively investigated by several researches. The properties of fuzzy sub commutative ideals and fuzzy sub implicative ideals are obtained. Ragavan and Solairaj [14] some new results on intuitionistic fuzzy H-ideals in BCI algebra.

2. Preliminaries

Definition 2.1. [8] A Subtraction G-algebra we mean a nonempty set X with a binary operation - and a constant 0 satisfying the following conditions:
(F1) x- x = 0,
(F2) x – (x - y) = y, for all x, y ϵ X.

Definition 2.2. [15] (G-subalgebra) A non-empty subset S of a subtraction G-algebra X is called a subtraction G-subalgebra of X if x-y $\in$S [0,1].

Definition 2.3. [16] A fuzzy set f of a universe X is a function from X to the unit closed interval [0, 1], that is f: X $\rightarrow$ [0, 1].

Definition 2.4. [2] An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, x_3, x_4, \cdots x_n\}$ is given by $A = \{<x, \Psi_A(x), \Omega_A(x) >: x \in X\}$, Where Ψ_A: X $\rightarrow$ [0, 1] and Ω_A : X $\rightarrow$ [0, 1] such that $0 \leq \Psi_A(x) + \Omega_A(x) \leq 1$. The number $\Psi_A(x)$ and $\Omega_A(x)$ denote the degree of membership and non-membership of $x \in X$ to A, respectively. For each IFS A in X, if $\pi_A(x) = 1 - \Psi_A(x) - \Omega_A(x)$,for all $x \in X$.

Definition 2.5. A nonempty subset I of X is called an ideal of X if (I_1): 0 $\in$ I, and (I_2): x - y $\in$ I and y $\in$ I imply x $\in$ I.

Definition 2.6. A fuzzy subset Ψ_A of X is said to be a fuzzy ideal of X if it satisfies
(F_3) $\Psi_A(0) \geq \Psi_A(x)$ for all x $\in$ X,
(F_4) $\Psi_A(x) \geq \min \{\Psi_A(x - y), \Psi_A(y)\}$ for all x, y $\in$ X.

Definition 2.7. An intuitionistic fuzzy set $A = \{x, \Psi_A(x), \Omega_A(x): x \in X\}$ in X is called an intuitionistic fuzzy ideal of X if it satisfies
(F_5) $\Psi_A(0) \geq \Psi_A(x)$, $\Omega_A(0) \leq \Omega_A(x)$,
(F_6) $\Psi_A(x) \geq \min\{\Psi_A(x - y), \Psi_A(y)\}$ and
(F_7) $\Omega_A(x) \leq \max\{\Omega_A(x - y), \Omega_A(y)\}$ for all $x, y \in X$.

Definition 2.8. An intuitionistic fuzzy set $A = \{x, \Psi_A(x), \Omega_A(x): x \in X\}$ in X is called an antagonistic -intuitionistic fuzzy ideal of X if it satisfies
(F_8) $\Psi_A(0) \leq \Psi_A(x)$, $\Omega_A(0) \geq \Omega_A(x)$,
(F_9) $\Psi_A(x) \leq \max\{\Psi_A(x - y), \Psi_A(y)\}$ and
(F_{10}) $\Omega_A(x) \geq \min\{\Omega_A(x - y), \Omega_A(y)\}$ for all $x,y \in X$.

Definition 2.9. [9, 11] A nonempty subset I of X is called a positive implicative ideal (i.e., weakly positive implicative ideal) of X if it satisfies (I_1) $0 \in I$ and (I_3): $((x - z) - z) - (y - z) \in I$ and $y \in I$ imply $x - z \in I$.

Definition 2.10. [9, 14] A nonempty subset I of X is called a sub-implicative ideal of X if it satisfies (I_1) $0 \in I$ and (I_3): $((x - (x - y)) - (y - x)) - z \in I$ and $z \in I$ imply $y - (y - x) \in I$.

Definition 2.11. [9, 13] A nonempty subset I of X is called a sub-commutative ideal of X if it satisfies (I_1) $0 \in I$ and (I_4): $(y - (y - (x - (x - y)))) - z \in I$ and $z \in I$ imply $x - (x - y) \in I$.

Definition 2.12. An anti-Fuzzy Subset $\Psi_A$of X is called an antagonistic -fuzzy sub-implicative ideal of X if it satisfies
(F_{11}) $\Psi_A(0) \leq \Psi_A(x)$ for all $x \in X$, and
(F_{12}) $\Psi_A(y - (y - x)) \leq \max\{\Psi_A(((x - (x - y)) - (y - x)) - z), \Psi_A(z)\}$ for all $x, y, z \in X$.

Definition 2.13. An anti-fuzzy subset Ψ_A of X is called an antagonistic -fuzzy sub-commutative ideal (briefly, AFSC-ideals) of X if it satisfies (F_1) and
(F_{13}) $\Psi_A(x - (x - y)) \leq \max\{\Psi_A((y - (y - (x - (x - y)))) - z), \Psi_A(z)\}$ for all $x, y, z \in X$.

Definition 2.14. An anti-Fuzzy Subset (Ψ_A, Ω_A) of X is called an antagonistic -intuitionistic fuzzy sub-implicative ideal of X if it satisfies
(F_{16}) $\Psi_A(0) \leq \Psi_A(x)$, $\Omega_A(0) \geq \Omega_A(x)$ for all $x \in X$, and
(F_{17}) $\Psi_A(y - (y - x)) \leq \max\{\Psi_A(((x - (x - y)) - (y - x)) - z), \Psi_A(z)\}$ for all $x, y, z \in X$.
(F_{18}) $\Omega_A(y - (y - x)) \geq \min\{\Omega_A(((x - (x - y)) - (y - x)) - z), \Omega_A(z)\}$ for all $x, y, z \in X$.

Definition 2.15. An anti-fuzzy subset (Ψ_A, Ω_A) of X is called an antagonistic intuitionistic fuzzy sub-commutative ideal (briefly, AFSC-ideals) of X if it satisfies
(F_{19}) $\Psi_A(0) \leq \Psi_A(x)$, $\Omega_A(0) \geq \Omega_A(x)$ for all $x \in X$, and
(F_{20}) $\Psi_A(x - (x - y)) \leq \max\{\Psi_A((y - (y - (x - (x - y)))) - z), \Psi_A(z)\}$ for all $x, y, z \in X$.

(F_{21}) $\Omega_A (x - (x - y)) \geq \min \{\Omega_A ((y - (y - (x - (x - y)))) - z), \Omega_A(z))\}$ for all x, y, z ∈ X.

3. $G_{\alpha,\beta}$ Over Antagonistic-Intuitionistic Fuzzy Sub Commutative Ideal of subtraction G-algebras

Theorem 3.1. If $G_{\alpha,\ \beta}(A)$ and $G_{\alpha,\ \beta}(B)$ are antagonistic -intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra then $G_{\alpha,\ \beta}(A+B)$ is also antagonistic -intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\ \beta}(A)$ and $G_{\alpha,\ \beta}(B)$ are antagonistic -intuitionistic fuzzy sub-commutative ideal of X.

(1) $\Psi_A (0) \leq \Psi_A(x)$, $\Omega_A (0) \geq \Omega_A (x)$ for all x ∈ X, and

(2) $\Psi_A (x - (x - y)) \leq \max \{\Psi_A ((y - (y - (x - (x - y)))) - z), \Psi_A (z)\}$ for all x, y, z ∈ X.

(3) $\Omega_A (x - (x - y)) \geq \min \{ \Omega_A ((y - (y - (x - (x - y)))) - z), \Omega_A (z))\}$ for all x, y, z ∈ X.

(4) $\Psi_B (0) \leq \Psi_B (x)$, $\Omega_B (0) \geq \Omega_B (x)$ for all x ∈ X, and

(5) $\Psi_B (x - (x - y)) \leq \max \{\Psi_B ((y - (y - (x - (x - y)))) - z), \Psi_B (z)\}$ for all x, y, z ∈ X.

(6) $\Omega_B (x - (x - y)) \geq \min \{\Omega_B ((y - (y - (x - (x - y)))) - z), \Omega_B (z))\}$ for all x, y, z ∈ X.

Case1:

$\Psi_A (0) + \Psi_B (0) - \Psi_A (0). \Psi_B (0) \leq \Psi_A (x) + \Psi_B (x) - \Psi_A (x). \Psi_B (x)$

$G_{\alpha,\ \beta} \{\Psi_A (0) + \Psi_B (0) - \Psi_A (0). \Psi_B (0)\} \leq G_{\alpha,\ \beta} \{\Psi_A (x) + \Psi_B (x) - \Psi_A (x). \Psi_B (x)\}$

$\{\alpha\Psi_A (0) + \alpha\Psi_B (0) - \alpha\Psi_A (0). \alpha \Psi_B (0)\} \leq \{\alpha \Psi_A(x) + \alpha \Psi_B (x) - \alpha \Psi_A (x). \alpha \Psi_B (x)\}$

$\{\alpha\Psi_A (0) + \alpha\Psi_B (0) - \alpha^2\Psi_A (0). \Psi_B (0)\} \leq \{\alpha\Psi_A (x)+ \alpha\Psi_B (x)- \alpha^2\Psi_A (x). \Psi_B (x)\}$

$G_{\alpha,\ \beta} (A + B) (0) \leq G_{\alpha,\ \beta} (A + B) (x)$

$\Omega_A (0). \Omega_B (0) \geq \Omega_A (x). \Omega_B (x)$

$G_{\alpha,\ \beta} \{\Omega_A (0). \Omega_B (0)\} \geq G_{\alpha,\ \beta} \{\Omega_A (x). \Omega_B (x)\}$

$\{\beta \Omega_A (0). \beta \Omega_B (0)\} \geq \{\beta \Omega_A(x). \beta \Omega_B(x)\}$

$\{\beta^2 \Omega_A (0). \Omega_B (0)\} \geq \{\beta^2 \Omega_A (x). \Omega_B (x)\}$

$G_{\alpha,\ \beta} (A + B) (0) \geq G_{\alpha,\ \beta} (A + B) (x)$

Case 2:

$\{ \Psi_A (x - (x - y)) + \Psi_B (x - (x - y)) - \Psi_A (x - (x - y)). \Psi_B (x - (x - y))\} \leq \max \{[\Psi_A ((y -(y - (x - (x - y))) - z), \Psi_A(z)]\} + \max [\Psi_B ((y - (y - (x - (x - y))) - z), \Psi_B (z)]- \max \{\Psi_A ((y - (y - (x - (x - y))) - z), \Psi_A (z)\}. \max \{\Psi_B ((y - (y - (x - (x - y))) - z), \Psi_B (z)\}\}$

$G_{\alpha,\ \beta} \{\Psi_A (x - (x - y)) + \Psi_B (x - (x - y)) - \Psi_A (x - (x - y)). \Psi_B (x - (x - y))\} \leq G_{\alpha,\ \beta} \max\{\{\mu_A ((y - (y - (x - (x - y))) - z) + \mu_B ((y - (y - (x - (x - y))), \mu_A(z) + \mu_B (z)\}- \max\{ \mu_A ((y - (y - (x - (x - y))) - z).\mu_B ((y - (y - (x - (x - y))), \mu_A (z).\mu_B (z)\}\}$

$\{\alpha\Psi_A(x-(x-y)) + \alpha\Psi_B(x-(x-y)) - \alpha\Psi_A(x-(x-y)). \alpha\Psi_B(x-(x-y))\} \leq \max \{\{\alpha \Psi_A ((y-(y-(x-(x-y)))-z) + \alpha \Psi_B ((y-(y-(x-(x-y)))-z), \alpha \Psi_A (z) + \alpha \Psi_B(z)\}- \max \{\alpha \Psi_A ((y-(y-(x-(x-y)))-z). \alpha \Psi_B((y-(y-(x-(x-y)))-z)-, \alpha \Psi_A (z). \alpha \Psi_B (z)\}\}$

$\{\alpha\Psi_A(x-(x-y)) + \alpha\Psi_B(x-(x-y)) – \alpha^2\Psi_A(x-(x-y)). \Psi_B(x-(x-y))\} \leq \max \{\{ \alpha \Psi_A ((y-(y-(x-(x-y)))-z) + \alpha\Psi_B ((y-(y-(x-(x-y)))-z) – \alpha^2\Psi_A ((y-(y-(x-(x-y)))-z). \Psi_B ((y-(y-(x-(x-y)))-z)\}, \{\alpha \Psi_A (z) + \alpha\Psi_B (z)- \alpha^2\Psi_A (z). \Psi_B (z)\}\}$

$G_{\alpha,\ \beta}(A+B) (x-(x-y)) \leq \max \{G_{\alpha,\ \beta} (A+B) ((y-(y-(x-(x-y)))-z), G_{\alpha,\ \beta} (A+B) (z)\}$

$\Omega_A (x - (x - y)). \Omega_B (x - (x - y)) \geq \min \{\Omega_A ((y - (y - (x - (x - y)))) - z), \Omega_A (z)\}. \min \{\Omega_B ((y - (y - (x - (x - y)))) - z), \Omega_B (z)\}$

Ω_A (x - (x - y)). Ω_B (x - (x - y)) ≥ min {Ω_A ((y - (y - (x - (x - y)))) - z). Ω_B ((y - (y - (x - (x - y))))-z}, min {Ω_A (z). Ω_B(z)}
$G_{\alpha,\beta}${ Ω_A(x - (x - y)). Ω_B (x - (x - y))}≥ $G_{\alpha,\beta}${ min { Ω_A ((y - (y - (x - (x - y)))) - z). Ω_B ((y - (y - (x - (x - y))))-z}, min{ Ω_A (z). Ω_B(z)}}
{β Ω_A(x - (x - y)). β Ω_B (x - (x - y))} ≥min {β Ω_A ((y - (y - (x - (x - y)))) - z). β Ω_B ((y - (y - (x - (x - y))))-z}, min {α Ω_A(z)).α Ω_B (z)}
{ β^2 Ω_A (x - (x - y)). Ω_B (x - (x - y))} ≥min {{β^2 Ω_A((y - (y - (x - (x - y)))) - z). Ω_B ((y - (y - (x - (x - y))))-z},{ β^2 Ω_A(z)).α Ω_B (z}}
$G_{\alpha,\beta}$(A+B)(x- (x - y))≥min{ $G_{\alpha,\beta}$(A+B)((y - (y - (x - (x - y)))), $G_{\alpha,\beta}$(A+B) (A+B)(z)}
Therefore, the two numbers of A+B is an antagonistic- intuitionistic fuzzy sub-commutative ideal in subtraction G-algebra of X.

Theorem 3.2. If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic -intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A•B) is also antagonistic -intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra.
Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic -intuitionistic fuzzy sub-commutative ideal of X.
(1) Ψ_A (0)≤ Ψ_A (x), Ω_A(0)≥Ω_A(x) for all x ∈ X, and
(2) Ψ_A(x - (x - y)) ≤ max { Ψ_A ((y - (y - (x - (x - y)))) - z), Ψ_A(z)} for all x, y, z ∈ X.
(3) Ω_A (x - (x - y)) ≥min { Ω_A((y - (y - (x - (x - y)))) - z), Ω_A (z))} for all x, y, z ∈ X.
(4) Ψ_B (0)≤ Ψ_B (x), Ω_B(0)≥ Ω_B(x) for all x ∈ X, and
(5) Ψ_B (x - (x - y)) ≤ max { Ψ_B ((y - (y - (x - (x - y)))) - z), Ψ_B (z)} for all x, y, z ∈ X.
(6) Ω_B (x - (x - y))≥ min { Ω_B ((y - (y - (x - (x - y)))) - z), Ω_B (z))} for all x, y, z ∈ X.
Case1:
{Ψ_A(0). Ψ_B(0)} ≤ {Ψ_A(x). Ψ_B (x)}
$G_{\alpha,\beta}${Ψ_A(0). Ψ_B(0)}≤ $G_{\alpha,\beta}${Ψ_A (x). Ψ_B (x)}
{αΨ_A(0). αΨ_B(0)}≤ { αΨ_A (x). αΨ_B (x)}
{$\alpha^2\Psi_A$(0). Ψ_B(0)}≤ { $\alpha^2\Psi_A$ (x). Ψ_B (x)}
$G_{\alpha,\beta}$(A•B)(0)≤ $G_{\alpha,\beta}$(A•B)(x)
{Ω_A(0)+ Ω_B(0) - Ω_A(0). Ω_B(0)} ≥ {Ω_A(x)+ Ω_B (x)- Ω_A (x). Ω_B (x)}
$G_{\alpha,\beta}${ Ω_A(0)+ Ω_B(0) - Ω_A(0). Ω_B(0)} ≥ $G_{\alpha,\beta}${ Ω_A(x)+ Ω_B (x)- Ω_A (x). Ω_B (x)}
{β Ω_A(0)+ β Ω_B(0) - β Ω_A(0). β Ω_B(0)} ≥{ β Ω_A(x)+ β Ω_B (x)- β Ω_A (x). β Ω_B (x)}
{β Ω_A(0)+ β Ω_B(0) – $\beta^2\Omega_A$(0). Ω_B(0)} ≥ { β Ω_A(x)+ β Ω_B (x)- $\beta^2\Omega_A$ (x). Ω_B (x)}
$G_{\alpha,\beta}$(A•B)(0) ≥ $G_{\alpha,\beta}$(A•B)(x)
Case2:
{Ψ_A(x- (x- y)). Ψ_B (x- (x- y))} ≤ max {Ψ_A ((y-(y-(x-(x-y)))-z), Ψ_A (z)}. max {Ψ_B ((y-(y-(x-(x-y)))-z), Ψ_B (z)}}
$G_{\alpha,\beta}${ Ψ_A(x-(x-y)). Ψ_B(x-(x-y))} ≤ $G_{\alpha,\beta}$ max{ Ψ_A ((y-(y-(x-(x-y)))-z). Ψ_B((y-(y-(x-(x-y)))), Ψ_A(z). Ψ_B(z)}}
{αΨ_A(x-(x-y)). αΨ_B(x-(x-y))} ≤ max {α Ψ_A ((y-(y-(x-(x-y)))-z). α Ψ_B((y-(y-(x-(x-y)))-z)-, α Ψ_A (z). α Ψ_B (z)}}
{$\alpha^2\Psi_A$(x-(x-y)). Ψ_B(x-(x-y))} ≤ max {{$\alpha^2\Psi_A$ ((y-(y-(x-(x-y)))-z). Ψ_B ((y-(y-(x-(x-y)))-z)}, {α Ψ_A (z) + αΨ_B (z)- $\alpha^2\Psi_A$ (z). Ψ_B (z)}}

$G_{\alpha,\beta}(A\bullet B)(\ x\text{-}(x\text{-}y))\leq \max\{\ G_{\alpha,\beta}\ (A\bullet B)\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ G_{\alpha,\beta}\ (A\bullet B)\ (z)\}$

$\{\ \Omega_A(x\text{-}\ (x\text{-}\ y))\ +\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\ -\ \Omega_A(x\text{-}\ (x\text{-}\ y)).\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\}\ \geq \min\ \{[\ \Omega_A((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \Omega_A(z)]\}\ +\ \max\ [\Omega_B((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \Omega_B\ (z)]\text{-}\ \max\ \{\Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \Omega_A\ (z)\}.\ \max\ \{\ \Omega_B\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \Omega_B\ (z)\}\}$

$\{\ \Omega_A(x\text{-}\ (x\text{-}\ y))\ +\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\ -\ \Omega_A(x\text{-}\ (x\text{-}\ y)).\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\}\ \geq\{\min\ \{\ \Omega_A((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ +\Omega_B((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ -\ \Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z).\ \Omega_B\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\},\ \min\{\ \Omega_A\ (z)+\ \Omega_B\ (z)-\ \Omega_A\ (z).\ \Omega_B\ (z)\}\}$

$\Omega_A(x\text{-}\ (x\text{-}\ y))\ +\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\ -\ \Omega_A(x\text{-}\ (x\text{-}\ y)).\ \Omega_B\ (x\text{-}\ (x\text{-}\ y))\}\ \geq\{\min\ \{\ \Omega_A((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ +\Omega_B((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ -\ \Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z).\ \Omega_B\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\},\ \{\Omega_A\ (z)+\ \Omega_B\ (z)-\ \Omega_A\ (z).\ \Omega_B\ (z)\}\}$

$G_{\alpha,\beta}\{\Omega_A(x\text{-}(x\text{-}y))+\ \Omega_B(x\text{-}(x\text{-}y))\ -\ \Omega_A(x\text{-}(x\text{-}y)).\ \Omega_B(x\text{-}(x\text{-}y))\}\ \geq\min\ G_{\alpha,\beta}\ \{\{\Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ +\ \Omega_B\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y))),\ \Omega_A\ (z)\ +\ \Omega_B\ (z)\}\text{-}\ \max\{\Omega_A((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z).\ \Omega_B((y\text{-}(y\text{-}(x\text{-}(x\text{-}y))),\ \Omega_A(z).\ \Omega_B(z)\}\}$

$\{\alpha\Omega_A(x\text{-}(x\text{-}y))+\ \alpha\Omega_B(x\text{-}(x\text{-}y))\ -\ \alpha\Omega_A(x\text{-}(x\text{-}y)).\ \alpha\Omega_B(x\text{-}(x\text{-}y))\}\ \geq\min\{\{\alpha\Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z)\ +\ \alpha\Omega_B\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \alpha\Omega_A\ (z)\ +\ \alpha\Omega_B(z)\}\text{-}\ \max\ \{\alpha\Omega_A\ ((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z).\ \alpha\Omega_B((y\text{-}(y\text{-}(x\text{-}(x\text{-}y)))\text{-}z),\ \alpha\Omega_A\ (z).\ \alpha\Omega_B\ (z)\}\}$

$\{\alpha\ \Omega_A\ (x\ -\ (x\ -\ y))\ +\ \alpha\ \Omega_B\ (x\ -\ (x\ -\ y))\ -\ \alpha^2\ \Omega_A\ (x\ -\ (x\ -\ y)).\ \Omega_B\ (x\ -\ (x\ -\ y))\}\ \geq\ \min\ \{\{\ \alpha\ \Omega_A\ ((y\ -\ (y\ -\ (x\ -\ (x\ -\ y)))\ -\ z)\ +\ \alpha\ \Omega_B\ ((y\ -\ (y\ -\ (x\ -\ (x\ -\ y)))\ -\ z)\ -\ \alpha^2\Omega_A\ ((y\ -\ (y\ -\ (x\ -\ (x\ -\ y)))\ -\ z).\ \Omega_B\ ((y\ -\ (y\ -\ (x\ -\ (x\ -\ y)))\ -\ z)\},\ \{\alpha\Omega_A\ (z)\ +\ \alpha\Omega_B\ (z)\text{-}\ \alpha^2\Omega_A\ (z).\ \Omega_B\ (z)\}\}$

$G_{\alpha,\ \beta}(A\ \bullet\ B)\ (x\ -\ (x\ -\ y))\geq \min\ \{G_{\alpha,\ \beta}\ (A\ \bullet\ B)\ ((y\ -\ (y\ -\ (x\ -\ (x\ -\ y)))\ -\ z),\ G_{\alpha,\beta}\ (A\ \bullet\ B)\ (z)\}$

Therefore, the two numbers of A. B is an antagonistic- intuitionistic fuzzy sub-commutative ideal in subtraction G-algebra of X.

Example 3.1. Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic -intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	b	c
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B > by routine calculation B is an antagonistic -intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	b	C
Ψ_B	.51	.53	.61	.72
Ω_B	.47	42	.31	.26

If A and B are antagonistic -intuitionistic fuzzy sub-implicative ideal of X. obviously the union A. B, Since x = b, y = c, z = a, α =.41, β=.46. $\{\alpha^2\Psi_A(0).\ \Psi_B(0)\} \leq \{\alpha^2\Psi_A(x).\ \Psi_B(x)\}$. 0343 ≤ .0636

$\{\beta\Omega_A(0) + \beta\Omega_B(0) - \beta^2\Omega_A(0).\ \Omega_B(0)\} \geq \{\beta\Omega_A(x) + \beta\ \Omega_B(x) - \beta^2\ \Omega_A(x).\ \Omega_B(x)\}$. 4605 ≥ .2569

$\{\alpha^2\ \Psi_A(x - (x - y)).\ \Psi_B(x - (x - y))\} \leq \max\{\{\alpha^2\Psi_A((y - (y - (x - (x - y)))) - z).\ \Psi_B((y - (y - (x - (x - y)))) - z)\}, \{\alpha^2\Psi_A(z).\ \Psi_B(z)\}\}$. 0920 ≱ .0636

$\{\beta\Omega_A(b - (b - c)) + \beta\Omega_B(b - (b - c)) - \beta^2\ \Omega_A(b - (b - c)).\ \Omega_B(b - (b - c)) \geq \min\{\{\beta\Omega_A((c - (c - (b - (b - c)))) - a) + \beta\ \Omega_B((c - (c - (b - (b - c)))) - a) - \beta^2\ \Omega_A((c - (c - (b - (b - c)))) - a).\ \Omega_B((c - (c - (b - (b - c)))) - a)\}, \{\beta\Omega_A(a)) + \beta\Omega_B(a)) - \beta^2\ \Omega_A(a)).\ \Omega_B(a))\}\}$. 2047 ≱. 3603

Therefore, the given two products of antagonistic -intuitionistic fuzzy sub-implicative ideal of X is not an antagonistic -intuitionistic fuzzy sub-implicative ideal of X.

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Relationship Between Weight Function and 1 – Norm

M. Melna Frincy[*]
J. Robert Victor Edward[†]

Abstract

The δ function on a subset E of $\mathbb{R}$ is the function defined by $\delta(x) = \begin{cases} 0 \, if \, x = 0 \\ 1 \, if \, x \neq 0 \, . \end{cases}$

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define $\delta(x) = (\delta(x_1), \delta(x_2), \ldots, \delta(x_n))$. The Hamming weight $w(x)$ of x is the number of non – zero coordinates of x, where $x \in \mathbb{R}^n$. From this one could see that $w(x) = \|\delta(x)\|_1$, where $\| \ \|_1$ is the 1 – norm of x given by $\|x\|_1 = \sum_{j=1}^{n} |x_j|$, where $= (x_1, x_2, \ldots, x_n)$. This gives a relationship between the weight function and the 1 – norm.In this paper we establish certain properties of the weight function using the properties of norms.

Keywords:mininorm, mininormed space, 1-norm, weight function.

2010 AMS subject classification: 90B06[‡]

[*]Research Scholar, Department of Mathematics, Scott Christian College(Autonomous) Nagercoil- 629003, TamilNadu, India. Affiliated to ManonmaniumSundaranar University, Tirunelveli – 627012, Tamilnadu, India.E- mail: melnabensigar84@gmail.com.
[†]Department of Mathematics, Scott Christian College(Autonomous)Nagercoil629003,TamilNadu,IndiaAffiliated to ManonmaniumSundaranar University, Tirunelveli – 627012, Tamilnadu, India.E- mail: jrvedward@gmail.com.
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1. Introduction

Let $X = \mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers. Then, X is a vector space over $\mathbb{R}$ of dimension n. The Hamming weight function on X is the function $w_H : X \to \mathbb{R}$ given by $w_H(x) =$ number of non –zero co-ordinates of x,
For $x = (x(1), x(2), \dots, x(n)) \in \mathbb{R}^n$.
Thus w_H satisfies the conditions:
$w_H(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $w_H(x) = 0$ if and only if $x = 0$.(1)
$w_H(\alpha x) = w_H(x)$ for all $x \in \mathbb{R}^n$ and $0 \neq \alpha \in \mathbb{R}$. (2)
$w_H(x + y) \leq w_H(x) + w_H(y)$ for all $x, y \in \mathbb{R}^n$.(3)

A norm on $\mathbb{R}^n$ is a function $\| \ \| : X \to \mathbb{R}$ satisfying $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$ and if and only if $x = 0$(4)
$\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$. (5)
and $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.(6)
We see that w_H satisfies the condition of a norm except the condition (5). Instead, it satisfies (2). We may call such a function a mininorm. Let us formalize the definition.

Definition:1.1 Let X be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A mininorm on X is a function $p: X \to \mathbb{R}$ satisfying the following conditions:

$p(x) \geq 0$ for all $x \in X$ and $p(x) = 0$ if and only if $x = 0$(7)

$p(\alpha x) = p(x)$ for all $x \in X$ and $0 \neq \alpha \in K$. (8)

$p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$. (9)

a vector space with a mininorm defined on it is called a mininormed spaces.
It is clear that w_H is a mininorm on $\mathbb{R}^n$.

2.The weight function and the 1- norm

The 1- norm or $\| \ \|_1$ on $\mathbb{R}^n$ is defined by

$\|x\|_1 = \sum_{j=1}^{n} |x(j)|$, where $x = (x(1), x(2), \dots, x(n))$.(10)

We cannot connect the weight function with the 1- norm using the δ - function.

The δ – function on $\mathbb{R}$ is defined by $\delta(x) = \begin{cases} 0 \ if \ x = 0 \\ 1 \ if \ x \neq 0. \end{cases}$(11)

The δ – function can be extended to $\mathbb{R}^n$ in the following way:

$\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n))$, (12)

where $x = (x(1), x(2), \dots, x(n))$.

This δ – function satisfies the following [1] :

$\delta(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\delta(x) = 0$ if and only if $x = 0$(13)

$\delta(\alpha x) = \delta(x)$ for all $x \in \mathbb{R}^n$ and $0 \neq \alpha \in \mathbb{R}$. (14)

and $\delta(x + y) \leq \delta(x) + \delta(y)$ for all $x, y \in \mathbb{R}^n$. (15)

Hence the partial order relation $\leq$ on $\mathbb{R}^n$ is defined as follows:

For $x = (x(1), x(2), \ldots, x(n))$ and $y = (y(1), y(2), \ldots, y(n))$ in $\mathbb{R}^n$,

$x \leq y$ if and only if $x(j) \leq y(j)$, $j = 1, 2, \ldots, n$.(16)

Now let $x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^n$.

Then, $\delta(x) = (\delta(x_1), \delta(x_2), \ldots, \delta(x_n))$

Now, $\delta(x_j) = \begin{cases} 0 \; if \; x_j = 0 \\ 1 \; if x_j \neq 0. \end{cases}$

Hence $\| \delta(x) \|_1 = \sum_{j=1}^{n} |\delta(x_{(j)})|^2$ = number of non- zero components of x.

Thus, $\| \delta(x) \|_1 = w_H(x)$. (17)

This gives the connection between the Hamming weight function and the 1- norm, via the
δ – function.

3.Topological Properties of the δ – function

Proposition:3.1 The δ – function on $\mathbb{R}^n$ is bounded.

Proof: Let $x, \in \mathbb{R}^n$.

$\| \delta(x) \|_1 = \| \delta(x_1), \delta(x_2), \ldots, \delta(x_n) \|_1$

$= \sum_{j=1}^{n} |x(j)|$

$\leq n$, since $|\delta(x_{(j)})| \leq 1$ for all j.

Hence δ is bounded.

Proposition:3.2 The δ – function on $\mathbb{R}^n$ is not continuous.

Proof: First we show that w_H is not continuous,

Let $x, \in \mathbb{R}^n$.

Then $\left\| \frac{1}{n} x \right\|_1 = \frac{1}{n} \|x\|_1 \to 0$ as $n \to \infty$.

That is, $\frac{1}{n} x \to 0$ in $\mathbb{R}^n$ with $\|x\|_1$.

But $\left\| w_H\left(\frac{1}{n}\, x\right)\right\|_1 = \| w_H(x)\|$ for all n.

So, $w_H\left(\frac{1}{n}\, x\right) \nrightarrow 0$.

Hence w_H is not continuous.

Now $w_H(x) = \| \delta(x)\|_1$ for all $x \in \mathbb{R}^n$.

Thus, $w_H = \| \ \|_1 \circ \delta$, where $\circ$ denotes the composition of functions.

$\| \ \|_1$ is continuous [3].

Suppose δ is continuous.

Hence $w_H$is continuous, since the composition of two continuous functions is continuous.
This is not possible.
Hence $\delta$is not continuous ■

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On Forgotten Index of Stolarsky-3 Mean Graphs

Sree Vidya.M[1]
Sandhya. S. S[2]

Abstract

The Forgotten index of a graph G is defined as F(G) = $\sum_{uv \in E(G)}({d_u}^2 + {d_v}^2)$ over all edges uv of G,where d_u ,d_v are the degrees of the vertices u and v in G, respectively. In this paper, we introduced Forgotten index of some standard Stolarsky-3 Mean Graphs.

Keywords: Forgotten index, Stolarsky-3 Mean Graphs.

AMS Subject Classification: 05C12[3]

[1] Research Scholar, Sree Ayyappa College for Women, Chunkankadai
[2] Research Supervisor, Department of Mathematics, Sree Ayyappa College for Women, Chunkankadai. [Affiliated to Manonmaniam Sundaranar University, Abishekapatti – Tirunelveli - 627012, Tamilnadu, India] Email: witvidya@gmail.com & sssandhya2009@gmail.com

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1. Introduction

Let G be a simple graph corresponding to a drug structure with vertex (atom) set V(G) and edge (bond) set E(G). The edge joining the vertices u and v is denoted by uv. Thus, if $u, v \in E(G)$ then u and v are adjacent in G. The degree of a vertex u, denoted by d(u), is the number of edge incident to u. Several topological indices such as Estrada index, Zagreb index, PI index, eccentric index, and wiener index have been introduced in the literature to study the chemical and pharmacological properties of molecules.

The forgotten topological index of a graph G is defined as the sum of weights ${d_u}^2 + {d_v}^2$over all edges uv of G,where d_u andd_v are the degrees of the vertices u and v in G, respectively. In this paper, we characterize the external properties of F-index (forgotten topological index). We first introduce some graph transformation which increase or decrease this index. Recently in 2015 Furtula and Gutman was introduced another topological index called index or F-index as F(G) = $\sum_{uv \in E(G)}({d_u}^2 + {d_v}^2)$. On the basis of this work, we introduce a new concept Forgotten index ofStolarsky3Meangraphs.Inthispaperweinvestigate Forgotten index of some standard graphs which admit Forgotten Mean graphs. We will provide a brief summary of definitions and other information which are necessary for our present investigation.

Definition:1.1 A graph G with p vertices and q edges is called a Stolarsky-3 Mean graph, if each vertex $x \in V$ with distinct labels $f(x)$ from $1,2,\ldots\ldots,q+1$ and eachedge $e = uv$ is assigned the distinct labels $f(e = uv) =$

$\left\lfloor \sqrt{\frac{f(u)^2 + f(u)f(v) + f(v)^2}{3}} \right\rfloor$ *or* $\left\lceil \sqrt{\frac{f(u)^2 + f(u)f(v) + f(v)^2}{3}} \right\rceil$**t**hen the resulting edge labels are distinct. In this case f is called **Stolarsky-3 Mean labeling** of G.

Definition:1.2 Let G be a Stolarsky-3 Mean graph. The **Forgotten index** of a graph F(G) is defined by F(G) = $\sum_{uv \in E(G)}({d_u}^2 + {d_v}^2)$, where d(u) is the degree of vertex u in G.

Theorem 1.3: Any Path Pn is a Stolarsky-3 mean graph.

Theorem 1.4: Any Cycle Cn is a Stolarsky-3 mean graph.

Theorem 1.5: Any Comb $Pn \odot K1$is a Stolarsky 3 mean graph.

Theorem 1.6: The ladder graph $L_n$is a Stolarsky-3 mean graph.

Theorem 1.7: ATriangular Snake graph $T_n$is a Stolarsky-3 mean graph.

Theorem 1.8: AQuadrilateral Snake graph $Q_n$is a Stolarsky-3 mean graph.

Remark 1.9: If G is a Stolarsky 3 mean graph, then '1' must be a label of one of the vertices of G, Since, an edge should get label '1'.

Remark1.10: If u gets label '1', then any edge incident with must get label 1 (or) 2 (or) 3. Hence this vertex must have a degree ≤ 3.

2. Main results

Theorem 2.1: Let G = P_n be a Stolarsky-3 mean graph. Then the Forgotten index of a path P_n is F(P_n) =5n+4.

Proof. Let G = P_n be a Stolarsky-3 mean graph.

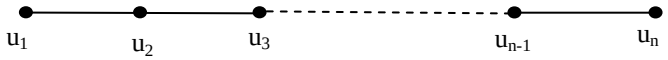

Figure: 1 Path P_n

We have $|V| = n$ and $|E| = n - 1$.

Therefore, by the definition of forgotten topological index, we obtain

F(G) = $\sum_{uv \in E(G)}({d_u}^2 + {d_v}^2)$

= $[(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2)]$

= $(1^2 + 2^2) + (2^2 + 2^2) + \cdots + (2^2 + 1^2)$

= $(n-1) \times 2 + (n+2) \times 3 = 2n - 2 + 3n + 6$

F(G) = $5n + 4$

Example 2.2. Forgotten index of P_6 is given below

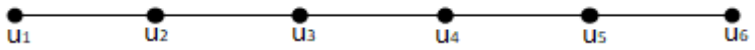

Figure: 2 Path P_6

F(P_6) = $[(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + (d(u_3)^2 + d(u_4)^2) + (d(u_4)^2 + d(u_5)^2) + (d(u_5)^2 + d(u_6)^2)]$ = $(1^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 1^2)$

= $5 + 8 + 8 + 8 + 5$

= $5 \times 2 + 8 \times 3$

= 34

Theorem 2.3. The Forgotten index of cycle C_n is $F(C_n) = 8n$.

Proof. Let $G = C_n$ be a Stolarsky-3 mean graph

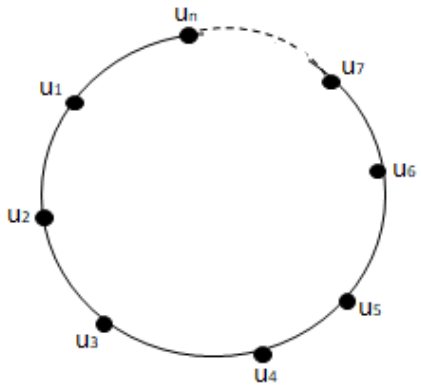

Figure: 3 $G = C_n$

We have $|V| = n$ and $|E| = n - 1$. Therefore, by the definition of forgotten topological index, we obtain

F(G) = $\sum_{uv \in E(G)}({d_u}^2 + {d_v}^2)$

= $[(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2) + (d(u_n)^2 + d(u_1)^2)]$

= $(2^2 + 2^2) + (2^2 + 2^2) + \cdots + (2^2 + 2^2) + (2^2 + 2^2)$

= $(8 + 8 + \cdots + n\ times)$

F(G) = $8n$

Example 2.4. Forgotten index of C_6 is given below

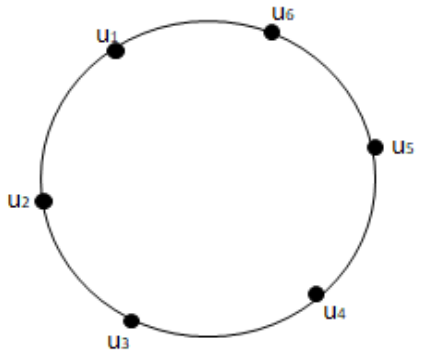

Figure: 4 $G = C_6$

F(C_6) = $[(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + (d(u_3)^2 + d(u_4)^2) + (d(u_4)^2 + d(u_5)^2) + (d(u_5)^2 + d(u_6)^2) + (d(u_6)^2 + d(u_1)^2)]$

= $(2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2)$

= 8×6 = 48

Theorem 2.5. The Forgotten index of Comb graph $F(P_n \odot K_1) = 21n - 65$.

Proof. Let G = P_n Θ K_1 be a Stolarsky – 3 Mean graph.

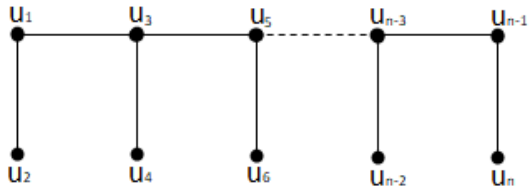

Figure: 5 Comb $P_n \Theta K_1$

We have $|V| = n$ and $|E| = n - 1$. Therefore, by the definition of forgotten topological index, we obtain

F(G) = $\sum_{uv \in E(G)} (d_u{}^2 + d_v{}^2)$

$$= \left[\begin{matrix}(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2) + (d(u_1)^2 + d(u_3)^2) \\ + \cdots + (d(u_{n-3})^2 + d(u_{n-1})^2)\end{matrix}\right]$$

$$=[(2^2 + 1^2) + (3^2 + 1^2) + \cdots + (2^2 + 1^2) + (2^2 + 3^2) + (3^2 + 3^2) + \cdots + (3^2 + 2^2)]$$

$$= 18 \times 3 + 13 \times 2 + 10 \times 4 + 5 \times 2$$

$$= 9n + 4n + 6n + 16$$

$$F(G) = 19n + 16$$

Example 2.6: Forgotten index of $P_6 \Theta K_1$ is given below.

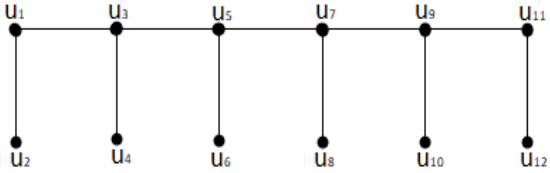

Figure: 6 Comb $P_6 \Theta K_1$

$F(P_6 \Theta K_1) = [(2^2 + 1^2) + (2^2 + 3^2) + (3^2 + 1^2) + (3^2 + 3^2) + (3^2 + 1^2) + (3^2 + 3^2) + (3^2 + 1^2) + (3^2 + 3^2) + (3^2 + 1^2) + (3^2 + 2^2) + (2^2 + 1^2)]$

$$= 18 \times 3 + 13 \times 2 + 10 \times 4 + 5 \times 2 = 130$$

Theorem 2.7: Forgotten index of ladder graph L_n is $H(L_n) = \begin{cases} 8n & if \quad n = 2 \\ 32n + 12 & if \quad n > 2 \end{cases}$

Proof. Let G = $P_n \times P_2$ be a Stolarsky- 3 Mean graph

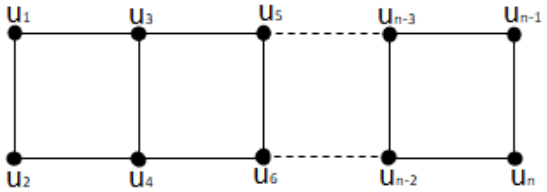

Figure: 7 G = $P_n \times P_2$

Case (i) if n = 2

$F(L_2) = [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_4)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2)]$

$= (2^2 + 2^2) + (2^2 + 2^2) + \cdots + (2^2 + 2^2)$
$= 8 + 8 + \cdots + n\ times$
$= 8n$

Case (ii) if n>2

$F(L_4) = \left[\begin{matrix}(d(u_1)^2 + d(u_2)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2) + (d(u_1)^2 + d(u_3)^2) + \cdots + \\ (d(u_{n-3})^2 + d(u_{n-1})^2) + (d(u_2)^2 + d(u_4)^2) + \cdots + (d(u_{n-2})^2 + d(u_n)^2)\end{matrix}\right]$

$= [(2^2 + 2^2) + (3^2 + 2^2) + \cdots + (2^2 + 2^2) + (2^2 + 3^2) + (3^2 + 3^2) + \cdots + (3^2 + 2^2) + (2^2 + 3^2) + \cdots + (3^2 + 2^2)]$
$= 8 \times 2 + 13 \times 4 + 18 \times 4$
$= 2n \times 2 + (3n + 1) \times 4 + (4n + 2) \times 4$
$= 4(8n + 3)$
$F(L_4) = 32n + 12$

Example 2.8. Forgotten index of L_4 is given below.

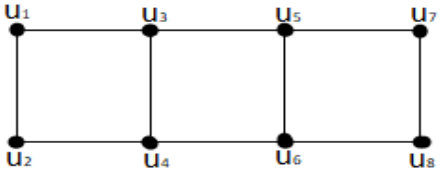

Figure: 8 L_4

$F(L_4) = \left[\begin{matrix}(d(u_1)^2 + d(u_2)^2) + (d(u_3)^2 + d(u_4)^2) + \cdots + (d(u_7)^2 + d(u_8)^2) + \\ (d(u_1)^2 + d(u_3)^2) + \cdots + (d(u_5)^2 + d(u_7)^2) + (d(u_2)^2 + d(u_4)^2) + \cdots + (d(u_6)^2 + d(u_8)^2)\end{matrix}\right]$

$= [(2^2 + 2^2) + (3^2 + 3^2) + (3^2 + 3^2) + (2^2 + 2^2) + (2^2 + 3^2) + (3^2 + 3^2) + (3^2 + 2^2) + (2^2 + 3^2) + (3^2 + 3^2) + (3^2 + 2^2)]$
$= 8 \times 2 + 13 \times 4 + 18 \times 4$
$= 16 + 52 + 72 = 140$

Theorem 2.9. Forgotten index of Triangular Snake graph T_n is 58n-6.

Proof. Let us consider a Stolarsky-3 Mean graph $G = T_n$ be a Stolarsky-3 mean graph.

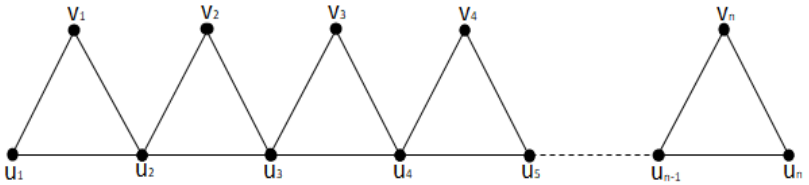

Figure: 9 $G = T_n$

$F(G) = \sum_{uv \in E(G)} (d_u{}^2 + d_v{}^2)$

$= [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2) + (d(u_1)^2 + d(v_1)^2) + (d(u_2)^2 + d(v_2)^2) + \cdots + (d(u_n)^2 + d(v_n)^2) + (d(v_1)^2 + d(u_2)^2) + (d(v_2)^2 + d(u_3)^2) + \cdots (d(v_n)^2 + d(u_{n+1})^2)]$

$= [(2^2 + 4^2) + (4^2 + 4^2) + \cdots + (4^2 + 2^2) + (2^2 + 2^2) + (4^2 + 2^2) + \cdots + (2^2 + 4^2) + (2^2 + 4^2) + (2^2 + 4^2) + \cdots + (2^2 + 2^2)]$

$= 8 \times 2 + 20 \times 6 + 32$

$= (7n - 1) \times 6 + (3n - 1) \times 2 + 32$

$= 42n - 6 + 6n - 2 + 32$

$F(G) = 48n + 24$

Example 2.10. Forgotten index of T_3 is given below.

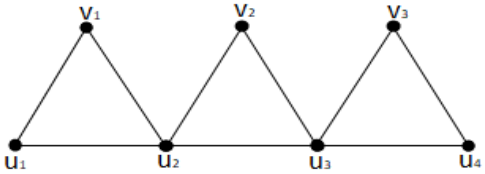

Figure: 10 $G = T_3$

$F(T_3) = [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + (d(u_3)^2 + d(u_4)^2) + (d(u_1)^2 + d(v_1)^2) + (d(u_2)^2 + d(v_2)^2) + (d(u_3)^2 + d(v_3)^2) + (d(v_1)^2 + d(u_2)^2) + (d(v_2)^2 + d(u_3)^2) + (d(v_3)^2 + d(u_4)^2)]$

$= [(2^2 + 4^2) + (4^2 + 4^2) + (4^2 + 2^2) + (2^2 + 2^2) + (4^2 + 2^2) + (2^2 + 4^2) + (2^2 + 4^2) + (2^2 + 4^2) + (2^2 + 2^2)]$

$= 8 \times 2 + 20 \times 6 + 32$ = 120+16+32

= 168

Theorem 2.11: Forgotten index of Quadrilateral Snake graph Q_n is 80n-48.

Proof. Consider $G = Q_n$ be a Stolarsky-3 mean graph.

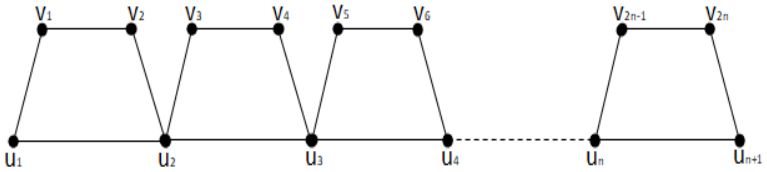

Figure:11$G = Q_n$

$F(G) = \sum_{uv \in E(G)} (d_u{}^2 + d_v{}^2)$

$= [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_n)^2 + d(u_{n+1})^2) + (d(u_1)^2 + d(v_1)^2) + (d(u_2)^2 + d(v_3)^2) + \cdots + (d(u_n)^2 + d(v_{2n-1})^2) + (d(u_2)^2 + d(v_2)^2) + (d(u_3)^2 + d(v_4)^2) + \cdots + (d(u_{n+1})^2 + d(v_{2n})^2) + (d(v_1)^2 + d(v_2)^2) + (d(v_3)^2 + d(v_4)^2) + \cdots + (d(v_{2n-1})^2 + d(v_{2n})^2)$

$= [(2^2 + 4^2) + (4^2 + 4^2) + \cdots + (4^2 + 2^2) + (2^2 + 2^2) + (4^2 + 2^2) + \cdots + (4^2 + 2^2) + (4^2 + 2^2) + (4^2 + 2^2) + \cdots + (2^2 + 2^2) + (2^2 + 2^2) + \cdots + (2^2 + 2^2)]$

$= 6(6n + 2) + 5(2n + 2) + (n - 2)(9n + 5)$

$= 36n + 12 + 10n + 10 + 9n^2 + 5n - 18n - 10$
F (G)= $9n^2 + 33n + 12$

Example 2.12. Forgotten index of Q_3 is given below.

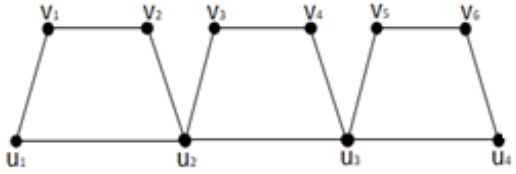

Figure: 12G $= Q_3$

F(Q_3) $= [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + (d(u_3)^2 + d(u_4)^2) + (d(u_1)^2 + d(v_1)^2) + (d(u_2)^2 + d(v_3)^2) + (d(u_3)^2 + d(v_5)^2) + (d(u_2)^2 + d(v_2)^2) + (d(u_3)^2 + d(v_4)^2) + (d(u_4)^2 + d(v_6)^2) + (d(v_1)^2 + d(v_2)^2) + (d(v_3)^2 + d(v_4)^2) + (d(v_5)^2 + d(v_6)^2)]$
$= [(2^2 + 4^2) + (4^2 + 4^2) + (4^2 + 2^2) + (2^2 + 2^2) + (4^2 + 2^2) + (4^2 + 2^2) + (4^2 + 2^2) + (4^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2) + (2^2 + 2^2)]$
$= [(20 \times 6) + (8 \times 5) + 32 \times 1$
$= 120 + 40 + 32$
$= 192$

Theorem 2.13: Forgotten index of Crown graph $C_n \odot K_1$ is 28n.
Proof. Consider G $= C_n \odot K_1$ be a Stolarsky-3 mean graph.
F (G) $= \sum_{uv \in E(G)} (d_u{}^2 + d_v{}^2)$
$= [(d(u_1)^2 + d(u_2)^2) + (d(u_2)^2 + d(u_3)^2) + \cdots + (d(u_{n-1})^2 + d(u_n)^2) + (d(u_n)^2 + d(u_1)^2) + (d(u_1)^2 + d(v_1)^2) + (d(u_2)^2 + d(v_2)^2) + \cdots + (d(u_n)^2 + d(v_n)^2)]$
$= [(3^2 + 3^2) + (3^2 + 3^2) + \cdots + (3^2 + 3^2) + (3^2 + 1^2) + \cdots + (3^2 + 1^2)]$
$= [(18 + 18 + \cdots + 18) + (10 + 10 + \cdots + 10)$
$= 18n + 10n$
F (G)= $28n$

Example 2.14. Forgotten index of $C_n \odot K_1$ is given below.

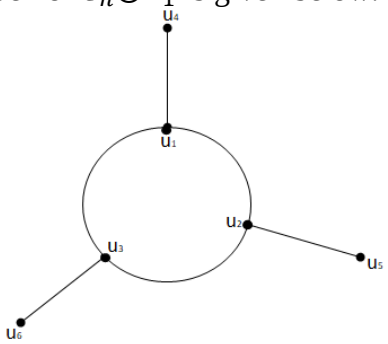

Figure: 13G $= C_n \odot K_1$

$F(C_n \odot K_1) = [(3^2 + 3^2) + (3^2 + 3^2) + (3^2 + 3^2) + (3^2 + 1^2) + (3^2 + 1^2) + (3^2 + 1^2)]$
$= [(18 \times 3) + (10 \times 3)]$
$= 54 + 30$
$= 84$

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Ratio Mathematica Volume 44, 2022

Odd Fibonacci Stolarsky-3 Mean Labeling of Some Special Graphs

Sree Vidya. M[1] ,
Sandhya S. S[2]

Abstract

Let G be a graph with p vertices and q edges and an injective function $f: V(G) \to \{1,3,5,13,21,55,89, \dots\}$ where each f_i is a odd Fibonacci number and the induced edge labeling $f: V(G) \to N$ are defined by $f(e = uv) = \left\lfloor \sqrt{\frac{f(u)^2 + f(u)f(v) + f(v)^2}{3}} \right\rfloor$ or $\left\lfloor \sqrt{\frac{f(u)^2 + f(u)f(v) + f(v)^2}{3}} \right\rfloor$ and all these edge labeling are distinct is called Odd Fibonacci Stolarsky-3 Mean Labeling. A graph which admits a Odd Fibonacci Stolarsky-3 Mean Labeling is called a Odd Fibonacci Stolarsky-3 mean graph.

Keywords: Stolarsky-3 Mean Labeling of Graphs, Odd Fibonacci Stolarsky-3 Mean Labeling of Graphs, Bull graph, Wheel graph, (m, n)-tadpole graph, Fire Cracker graph, Pan graph, Gear graph, Star graph.

AMS Subject Classification: 05C78[3]

[1]Research Scholar, Sree Ayyappa College for Women, Chunkankadai.
[2] Assistant Professor, Department of Mathematics, Sree Ayyappa College for Women, Chunkankadai.
[Affiliated to Manonmaniam Sundaranar University, Abishekapatti – Tirunelveli - 627012, Tamilnadu, India] Email: witvidya@gmail.com[1] sssandhya2009@gmail.com[2]
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1. Introduction

The graph considered here will be finite, undirected and simple graph with p vertices and q edges. For all detailed survey of graph labeling, we refer to Galian [1]. For all other standard terminology and notations, we follow Harary [2]. S. S. Sandhya, S. Somasundaram and S. Kavitha introduced the concept of Stolarsky 3 Mean labeling of graphs in [3]. In this paper, we introduced a new concept namely Odd Fibonacci Mean Labeling of graphs.

Definition: 1.1 The Fibonacci numbers can be defined by linear recurrence $\boldsymbol{F_n = F_{n-1} + F_{n-2}}$; $\boldsymbol{(n \geq 2)}$. This generates the infinite sequence of integer beginning 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,144, 233, ….

Definition: 1.2 Let G be a graph with p vertices q edges. An injective function $f: V(G) \to \{1,3,5,13,21,55, \dots\}$ where each f_i isa odd Fibonacci number and the induced edge labeling $f: E(G) \to N$ defined by edge e =uv is labeled with f=(e=uv)=$\left\lfloor \sqrt{\frac{f(u)^2+f(u)f(v)+f(v)^2}{3}} \right\rfloor$ (or)$\left\lfloor \sqrt{\frac{f(u)^2+f(u)f(v)+f(v)^2}{3}} \right\rfloor$, then all the edge labels are distinct and are from Odd Fibonacci

number $F_1, F_2, \dots, F_q$ where $F_1 = 1, F_2 = 3, F_3 = 5, \dots$. A graph that admits Odd Fibonacci Stolarsky-3 Mean labeling is called Odd Fibonacci Stolarsky-3 Mean graph.

Definition: 1.3 The Bull graph is a planar undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendent edges.

Definition: 1.4 The Wheel graph $W_n, n \geq 3$ is join of the graphs $C_n \& K_1$, $W_n = C_n + K_1$.

Definition: 1.5 The (m, n)-tadpole graph is a graph is a special consisting of a graph on m (at least 3) vertices and a path graph on n vertices connected with a bridge.

Definition: 1.6 An Firecracker is a graph obtained by the concatenation of stars by linking one leaf.

Definition: 1.7 Gear graph is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle. The gear graph G_n has 2n+1 vertices and 3n edges.

Definition:1.8 The pan graph is the graph obtained by joining a cycle graph to a singleton graph with a bridge.

Definition: 1.9A Star graph with n vertices is a tree with one vertex having degree n-1

and other n-1 vertices having degree 1. A star graph with n+1 vertices $K_{1,n}$.

2. Main Results

Theorem 2.1: The Bull graph is a Odd Fibonacci Stolarsky-3 mean labeling.
Proof. Let G be a Bull graph.
Here $|V| = n,\ |E| = n - 1$
Define a function $f: V(G) \to \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number.
$f(u_i) = 2i - 1, 1 \leq i \leq 3$
$f(u_4) = 3i + 4, i = 4$
$f(u_5) = 4i + 1, i = 5$
Then the induced edge labeling is all distinct.
Hence, we proved a Bull graph admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.2: Odd Fibonacci Stolarsky-3 mean labeling of Bull graph is shown below.

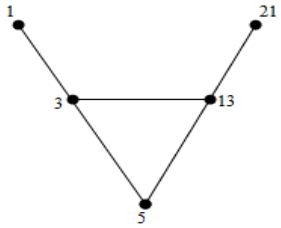

Figure 2.1: Bull graph

Theorem 2.3: The Wheel graph W_n admits Odd Fibonacci Stolarsky-3 mean labeling.
Proof. Let $G = W_n$ be a Wheel graph.
Here $|V| = n,\ |E| = n - 1$
Define a function $f: V(G) \to \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number.
$f(u_i) = 2i - 1, 1 \leq i \leq 3$
$f(u_4) = 3i + 4, i = 4$
$f(u_5) = 4i + 1, i = 5$
$f(u_6) = 9i + 1, i = 6$
Then the induced edge labeling are all distinct.
Hence, we proved a wheel graph W_n admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.4: Odd Fibonacci Stolarsky-3 mean labeling of Wheel graph W_5 is shown below.

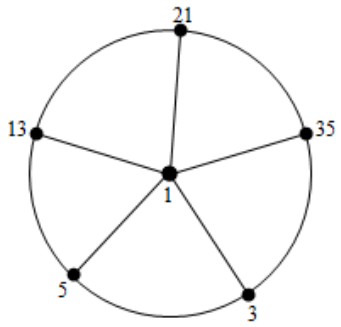

Figure 2.2 Wheel graph W_5

Theorem 2.5: The (m, n)- tadpole graph admits Odd Fibonacci Stolarsky-3 mean labeling.

Proof. Let G be a (m, n)- tadpole graph.

Here $|V| = n,\ |E| = n - 1$

Define a function $f: V(G) \rightarrow \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number.

$f(u_i) = 2i - 1, 1 \leq i \leq 3$

$f(u_4) = 3i + 4, i = 4$

$f(u_5) = 4i + 1, i = 5$

$f(u_6) = 9i + 1, i = 6$

Then the induced edge labeling is all distinct.

Hence, we proved a (m, n)- tadpole graph admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.6: Odd Fibonacci Stolarsky-3 mean labeling of (5,1)- tadpole graph is shown below.

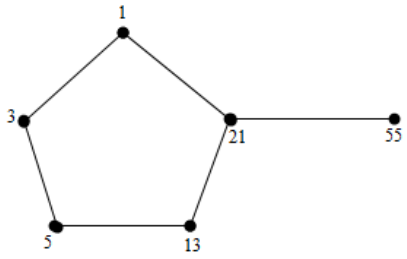

Figure 2.3 (5, 1)- tadpole graph

Theorem 2.7: A Fire Cracker graph admits Odd Fibonacci Stolarsky-3 mean labeling.

Proof. Let G be a Fire Cracker graph.

Here $|V| = n,\ |E| = n - 1$

Define a function $f: V(G) \rightarrow \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number and assignment of vertex labeling are

$f(u_i) = 2i - 1, 1 \leq i \leq 3$

$f(u_4) = 3i + 4, i = 4$

$f(u_5) = 4i + 1, i = 5$

$f(u_6) = 9i + 1, i = 6$
$f(u_7) = 12i + 5, i = 7$
$f(u_8) = 29i + 1, i = 8$
Then the induced edge labeling is all distinct.
Hence, we proved a Fire Cracker admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.8: Odd Fibonacci Stolarsky-3 mean labeling of Fire Cracker graph is shown below.

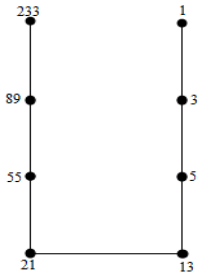

Figure 2.4: Fire Cracker graph

Theorem 2.9: The Pan graph admits an Odd Fibonacci Stolarsky-3 mean labeling.
Proof. Let G be a Pan graph.
Here $|V| = n,\ |E| = n - 1$
Define a function $f: V(G) \to \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number and assignment of vertex labeling are
$f(u_i) = 2i - 1, 1 \leq i \leq 3$
$f(u_4) = 3i + 4, i = 4$
$f(u_5) = 4i + 1, i = 5$
Then the induced edge labeling are all distinct.
Hence, we proved a Pan admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.10: Odd Fibonacci Stolarsky-3 mean labeling of Pan graph is shown below.

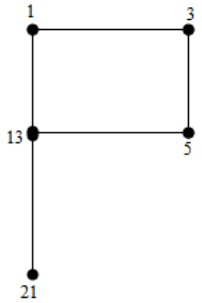

Figure 2.5: Pan graph

Theorem 2.11: The Gear graph G_n admits Odd Fibonacci Stolarsky-3 mean labeling.
Proof. Let $G = G_n$ be Gear graph.

Here $|V| = n,\ |E| = n - 1$
Define a function $f: V(G) \to \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number.
$f(u_i) = 2i - 1, 1 \leq i \leq 3$
$f(u_4) = 3i + 4, i = 4$
$f(u_5) = 4i + 1, i = 5$
$f(u_6) = 9i + 1, i = 6$
$f(u_7) = 12i + 1, i = 7$
$f(u_8) = 29i + 1, i = 8$
$f(u_9) = 42i - 1, i = 9$
Then the induced edge labeling is all distinct.
Hence, we proved a Gear graph G_n admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.12: Odd Fibonacci Stolarsky-3 mean labeling of Gear graph G_4 is shown below.

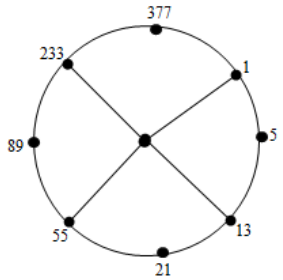

Figure 2.6 Gear graph

Theorem 2.13: The Star graph admits an Odd Fibonacci Stolarsky-3 mean labeling.
Proof. Let G be a Star graph.
Here $|V| = n,\ |E| = n - 1$
Define a function $f: V(G) \to \{1,3,5,13,21,55, \ldots\}$ where each f_i is a odd Fibonacci number and assignment of vertex labeling are
$f(u_i) = 2i - 1, 1 \leq i \leq 3$
$f(u_4) = 3i + 4, i = 4$
$f(u_5) = 4i + 1, i = 5$
$f(u_6) = 9i + 1, i = 6$
$f(u_7) = 12i + 5, i = 7$
Then the induced edge labeling is all distinct.
Hence, we proved a Star graph admits Odd Fibonacci Stolarsky-3 mean labeling.

Example 2.14: Odd Fibonacci Stolarsky-3 mean labeling of Star graph is shown below.

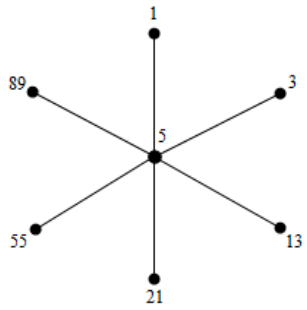

Figure 2.7 Star graph

3. Conclusion

we have introduced a new labeling namely Odd Fibonacci Stolarsky-3 Mean Labeling of graphs. We prove that Bull graph, Wheel graph, (m,n)-tadpole graph, Fire Cracker graph, Pan graph, Gear graph, Star graph. Extending the study to other families of graphs is an open area of research.

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Nano Semi*α-open sets

Reena C[*]
Kanaga M[†]

Abstract

In this paper, we introduce a new class of sets called nano semi*α-open sets and discuss some of its properties in nano topological space. We also, present nano semi*α-interior, nano semi*α-closure and study some of its fundamental properties.

Keywords: nano semi*α-open, nano semi*α-closed, nano semi*α-interior, nano semi*α–closure.

AMS subject classification: 54A05 [‡]

[*] Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, TamilNadu, India; reenastephany@gmail.com.

[†] Sec Research Scholar, Reg.No. 21122212092007, Department of Mathematics, St. Mary's College (Autonomous), (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli), Thoothukudi-1, TamilNadu, India; kanahaspm@gmail.com.

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1. Introduction

The notion of Nano topology was introduced by Lellis Thivagar [8] which was defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it and also defined Nano closed sets, Nano-interior and Nano-closure. He also introduced the weak forms of Nano open sets namely Nano α-open sets, Nano semi-open sets and Nano pre-open sets. In 2017, the concept of nano semi α open sets was introduced by Qays Hatem Imran [3]. In 2014, A. Robert and S. Pious Missier [7] have introduced and studied semi*α-open sets in general topology. In this paper we introduce nano semi*α-open sets and nano semi *α-closed sets in nano topological spaces. We investigate its fundamental properties and find its relation with other Nano sets and study some of its properties.

2. Preliminaries

Throughout this chapter (U, τ_R(X)) is a nano topological space with respect to X where X ⊆ U, R is an equivalence relation on U, U/R denotes the family of equivalence classes of U by R.

Definition 2.1 [8]: Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be discernible with one another. The pair (U, R) is said to be the approximation space. Let X⊆U

1. The **lower approximation** of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is L_R(X) = $\bigcup_{x\in U}${R(x) / R(x) ⊆ X} where R(x) denotes the equivalence class determined by X.
2. The **upper approximation** of X with respect to R is the set of all objects which can be possibly defined as X with respect to R and it is denoted by U_R(X). That is U_R(X)= $\bigcup_{x\in U}${R(x) / R(x) ∩ X ≠ ϕ}
3. The **boundary region** of X with respect to R is the set of all objects which can be classified neither as X nor as not X with respect to R and is denoted by B_R(X). That is B_R(X) = U_R(X) – L_R(X).

Proposition 2.2 [8] If (U, R) is an approximation space and X, Y ⊆ U, then

1. L_R(X) ⊆ X ⊆ U_R(X)
2. L_R (∅) = U_R (∅) = ∅ and L_R (U) = U_R(U) = U
3. U_R (X ∪Y) = U_R(X) ∪U_R(Y)
4. U_R (X ∩ Y) ⊆ U_R(X) ∩ U_R(Y)
5. L_R (X ∪Y) ⊇ L_R(X) ∪ L_R(Y)
6. L_R (X ∩ Y) = L_R(X) ∩ L_R(Y)
7. L_R(X) ⊆L_R(Y) and U_R(X) ⊆U_R(Y) whenever X ⊆Y
8. $U_R(X^c)$ = $[L_R(X)]^c$ and $L_R(X^c)$ = $[U_R(X)]^c$
9. U_RU_R(X) = L_R U_R(X) = U_R(X)
10. L_RL_R(X) = U_R L_R(X) = L_R(X)

Definition 2.3 [8]: Let U be the universe, R be an equivalence relation on U andτ_R(X) = {U, ∅, L_R(X), U_R(X), B_R(X)} where X ⊆ U. Then by the proposition 2.2, R(X) satisfies the following axioms:
1. U and ∅ ∈ τ_R(X)
2. The union of the elements of any subcollection of τ_R(X) is in τ_R(X).
3.The intersection of the elements of any finite subcollection of τ_R(X) is in τ_R(X).
That is τ_R(X) is a topology on U called the nano topology on U with respect to X. We call (U, τ_R(X)) as the **nano topological space**. The elements of τ_R(X) are called as **nano-open sets**.

Definition 2.4 [8]: If (U,τ_R(X)) is a nano topological space with respect to X and if A ⊆ U, then (i) ***nano interior*** of A is defined as the union of all nano-open sets contained in A and is denoted by NInt (A). That is, NInt(A) is the largest nano-open subset of A.
(ii) n***ano closure*** of A is defined as the intersection of all nano-closed sets containing A and it is denoted by NCl(A). That is, NCl(A) is the smallest nano-closed set containing A.

Definition 2.5 [2]: Let (U, τ_R(X)) be a nano topological space. A subset A of (U, τ_R(X)) is called ***nano generalized-closed*** (briefly Ng- closed) if NCl(A) ⊆V where A ⊆V and V is Nano-open.
The complement of nano generalized -closed set is called as ***nano generalized-open***.

Definition 2.6 [2]: For every set A ⊆U, the ***nano generalized closure of A*** is defined as the intersection of all Ng- closed sets containing A and is denoted by NCl*(A).

Definition 2.7 [2]: For every set A ⊆U, the ***nano generalized interior of A*** is defined as the union of all Ng- open sets contained in A and is denoted by NInt*(A).

Definition 2.8: Let (U,τ_R(X)) be a nano topological space and A ⊆U. Then A is said to be
(i) **nano α-open [8]** if A ⊆ N Int (NCl (NInt (A)))
(ii) **nano semi*-open [1]** if A ⊆ NCl*(NInt(A))
(iii) **nano semi α -open [3]** if A ⊆ NCl (NInt (NCl (NInt A)))
(iv) **nano semi pre-open [6]** if A ⊆ NCl (NInt (NCl (A)))
(v) **nano regular-open** [**8**] if A = NInt (NCl (A))
(vi) **nano regular *-open [5]** if A = $\mathrm{NInt}(\mathrm{NCl}^*(\mathrm{A}))$
(vii) **nano pre *-open [4]** if A ⊆ $\mathrm{NInt}^*(\mathrm{NCl}\ (\mathrm{A}))$
(viii) **nano pre-open [8]** If A ⊆ NInt (NCl (A))
(ix) **nano θ-open [9]**, if for each x ∈ A, there exists a nano open set G such that x ∈ G ⊆ Ncl (A) ⊆ A.
The complements of the above-mentioned sets are called their respective ***nano-closed*** sets.

3. Nano Semi*α-open sets

Definition 3.1: A subset A of a nano topological space $(U, \tau_R(X))$ is called **nano semi*α- open** if there is a nano α-open set G in U such that $G \subseteq A \subseteq N_o Cl^*(G)$.
The collection of all nano semi*α-open sets is denoted by $\boldsymbol{N_o S^* \alpha\, O}\big(\boldsymbol{U, \tau_R(X)}\big)$.

Example 3.2: Let $U = \{a, b, c, d\}$ with U/R= {{a},{b,c,d}}. Let $X = \{b, c, d\}$. Then $\tau_R(X) = \{U, \emptyset, \{b, c, d\}\}$.The nano-closed sets are $\{U, \emptyset, \{a\}\}$.The nano generalized-closed sets are$\{U, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$**.**
The nano generalized-open sets are $\{U, \emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$.
$\mathrm{N_o S^* \alpha\, O}\big(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X})\big) = \{U, \phi, \{b, c, d\}\}$.

Theorem 3.3: For a subset A of a nano topological space $(U, \tau_R(X))$ the following statements are equivalent:
(i)A is nano semi*α-open.
(ii) $A \subseteq N_o Cl^*(\alpha \mathrm{N_o Int}\,(\mathrm{A}))$
(iii) $\mathrm{N_o Cl^*}\big(\alpha \mathrm{N_o Int}\,(\mathrm{A})\big) = \mathrm{N_o Cl^*}(\mathrm{A})$
Proof: (i)⟹(ii) If A is a nano semi*α-open, then there is a nano α-open set G in U such that $G \subseteq A \subseteq N_o cl^*(G)$. Now $G \subseteq A \Longrightarrow$G=$\alpha \mathrm{N_o Int}\,(\mathrm{G}) \subseteq \alpha \mathrm{N_o Int}\,(\mathrm{A}) \implies \mathrm{A} \subseteq N_o Cl^*(G) \subseteq N_o Cl^*(\alpha \mathrm{N_o Int}\,(\mathrm{A}))$.
(ii) ⟹(iii)By assumption, A$\subseteq N_O Cl^*(N_o \alpha Int\,(A))$. we have $N_o Cl^*$(A)$\subseteq N_o Cl^*(N_o Cl^*(\alpha \mathrm{N_o Int}\,(\mathrm{A})))= N_o Cl^*\big(\alpha \mathrm{N_o Int}\,(\mathrm{A})\big)$. Now $\alpha \mathrm{N_o Int}\,(\mathrm{A}) \subseteq A$ implies that $N_o Cl^*\big(\alpha \mathrm{N_o Int}\,(\mathrm{A})\big) \subseteq N_o Cl^*(A)$.Therefore $N_o Cl^*\big(\alpha \mathrm{N_o Int}\,(\mathrm{A})\big) = N_o Cl^*(A)$
(iii) ⟹(i) Take G=$\alpha \mathrm{N_o Int}\,(\mathrm{A})$.Then G is a nano α-open set in U such that
G$\subseteq A \subseteq N_O Cl^*(G) = N_O Cl^*\big(\alpha \mathrm{N_o Int}\,(\mathrm{A})\big) = N_O Cl^*(G)$.Therefore by definition 3.1, A is nano semi*α- open.

Theorem 3.4: Arbitrary union of nano semi *α -open set is nano semi*α-open.
Proof: Let $\{A_\alpha\}$ be a collection of nano semi*α-open sets in nano topological space U. Then there exists a nano α-open set G_α such that $G_\alpha \subseteq A_\alpha \subseteq N_O Cl^*(G_\alpha)$ for each α. Hence $\cup\, G_\alpha \subseteq \cup A_\alpha \subseteq \cup N_O Cl^*(G_\alpha) \subseteq N_O Cl^*(\cup\, G_\alpha)$.Since $\cup\, G_\alpha$is nano α-open , by definition 3.1 $\cup\, A_\alpha$ is nano semi *α -open.

Remark 3.5: The intersection of two nano semi*α- open sets need not be a nano semi*α-open as seen from the following example.

Example 3.6: Let $U = \{a, b, c, d\}$,$U/R = \{\{a\}, \{c\}, \{b, d\}\}$.Let $X = \{a, b\}$.Then $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $N_o\, S^* \alpha\, O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Here the subsets $A = \{a, c\}\ and\ B = \{b, c, d\}$ are nano semi*α-open sets, but A$\cap B = \{c\}$ is not nano semi*α- open.

Theorem 3.7: If A is nano semi$^*\alpha$-open in U and B is nano open in X ,then $A \cap B$ is nano semi$^*\alpha$-open in U.
Proof: Since A is nano semi$^*\alpha$-open in U,there is an nano α-open sets G such that $G \subseteq A \subseteq N_O\, Cl^*(G)$.Since B is nano open ,$G \cap B \subseteq A \cap B \subseteq N_O\, Cl^*(G) \cap B \subseteq N_O\, Cl^*(G \cap B)$.Since $G \cap B$is nano α-open,by definition 3.1,$A \cap B$ is nano semi$^*\alpha$-open.

Theorem 3.8: Every nano α-open set is nano semi$^*\alpha$-open.
Proof: Let A be a nano α-open set in U. Then $N_o\, \alpha\, intA = A$ and hence $A \subseteq N_o Cl^*(A) = N_O Cl^*(N_o \alpha Int\ (A))$.Hence A is nano semi$^*\alpha$-open.

Remark 3.9: The converse of the above theorem is not true as shown in the following example.

Example 3.10: Let $U = \{a, b, c, d\}$,$U/R = \{\{a\}, \{c\}, \{b, d\}\}$.Let $X = \{a, b\}$. Then $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $N_O \alpha O\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{b, d\}, \{a, b, d\}\}$. Clearly the sets$\{a, c\}$and $\{b, c, d\}$ are nano semi* α-open but not nano α-open.

Theorem 3.11: Every nano open set is nano semi$^*\alpha$-open.
Proof: Let A be any nano open set. Since every nano open set is nano α-open and hence by theorem 3.8, A is nano semi$^*\alpha$-open.

Remark 3.12: The converse of the above theorem is not true as shown in the following example.

Example 3.13: Let $U = \{a, b, c, d\}$,$U/R = \{\{a\}, \{c\}, \{b, d\}\}$.Let $X = \{a, b\}$.
Then $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $N_o\, S^*\alpha\ O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$.Clearly the sets $\{a, c\}$and $\{b, c, d\}$ is nano semi$^*\alpha$- open but not nano open.

Theorem 3.14: Every nano semi*-open set is nano semi$^*\alpha$-open.
Proof: Let A be any nano semi* open set. Then there is a nano open set G in U such that $G \subseteq A \subseteq N_o Cl^*(G)$.Since every nano open set is nano α-open ,A is nano semi$^*\alpha$-open.

Remark 3.15: The converse of the above theorem is not true as shown in the following example.

Example 3.16: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{b\}\}$. Let X = $\{a, c, d\}$.Then $\tau_R(X) = \{U, \emptyset, \{a\}\}$. $N_o\, S^*\alpha\ O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. $N_o\, S^*\ O\big(U, \tau_R(X)\big) = \big\{U, \emptyset, \{a\}\big\}$. Clearly the sets $\{a, b\}$,$\{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ are nano semi$^*\alpha$-open but not nano semi*-open.

Theorem 3.17: Every nano semi$^*\alpha$ open set is nano semi α-open.

Proof: Let A be any nano semi* α open set. Then there is a nano α-open set G in U such that $G \subseteq A \subseteq N_oCl^*(G)$.Since $N_oCl^*(G) \subseteq N_oCl(G)$,we have $G \subseteq A \subseteq N_oCl(G)$.Hence A is nano semi α-open.

Remark 3.18: The converse of the above theorem is not true as shown in the following example.

Example3.19: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{d\}, \{b, c\}\}$. Let $X = \{a, c\}$ $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$.$N_o\, S^*\alpha\, O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. $N_o\, S\alpha\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$.Clearly the subset $\{a, d\}$ and $\{b, c, d\}$are semi nano α-open but not nano semi*α-open.

Theorem 3.20: Every nano semi*α open set is nano semi pre-open.
Proof: Let A be any nano semi*α-open set. Then there is a nano α-open set G such that $G \subseteq A \subseteq N_oCl^*(G)$.Since every nano α-open set is nano pre-open and $N_oCl^*(G) \subseteq N_oCl(G)$, A is nano semi pre-open.

Remark 3.21: The converse of the above theorem is not true as shown in the following example.

Example 3.22: Let U={ $a, b, c, d\}$, U/R= $\{\{a\}, \{d\}, \{b, c\}\}$.Let X= $\{a, c\}$. $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. $N_oS^*\alpha\, O\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $N_oSPO(U, \tau_R(X))$={ $U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}$, $\{b, d\}, \{c, d\}$, $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Clearly the subsets $\{b\}$,$\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}, \{a, b, d\}, \{a, c, d\}$ are nano semi pre-open but not nano semi*α-open.

Theorem 3.23: Every nano regular open set is nano semi*α-open.
Proof: Let A be any nano regular open set. Since every nano regular open set is nano open and by theorem 3.11, we have A is nano semi*α-open.

Remark 3.24: The converse of the above theorem is not true as shown in the following example.

Example 3.25: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{d\}, \{b, c\}\}$. Let $X = \{a, c\}$. $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$.$N_o\, S^*\, \alpha\, O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. $N_o\, R\, O\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{b, c\}\}$.Clearly the subset $\{a, b, c\}$ is nano semi*α-open but not nano regular open.

Theorem 3.26: Every nano regular*-open set is nano semi*α-open.
Proof: Let A be any nano regular*-open set. Since every nano regular*-open set is nano open and by theorem 3.11, we have A is nano semi*α-open.

Remark 3.27: The converse of the above theorem is not true as shown in the following example.

Example 3.28: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{b\}\}$. Let X = $\{a, c, d\}$.Then $\tau_R(X) = \{U, \emptyset, \{a\}\}$, $N_o S^* \alpha O(U, \tau_R(X)) = \{ U, \emptyset, \{a\}, \{a, b\}, \{a, c\}$, $\{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. $N_o R^* O\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}\}$.Clearly the subsets$\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ are nano semi$^*\alpha$-open but not nano regular*-open.

Theorem 3.29: Every nano semi$^*\alpha$-open set is nano pre *-open.
Proof: Let A be any nano semi$^*\alpha$-open set. Then there is a nano α-open set G in U such that $G \subseteq A \subseteq N_o Cl^*(G)$ .Since every nano α-open set is nano pre*-open, we have A is nano pre*- open.
Remark 3.30: The converse of the above theorem is not true as shown in the following example.

Example3.31: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{b\}\}$. Let X = $\{a, c, d\}$.Then $\tau_R(X) = \{U, \emptyset, \{a\}\}$. $N_o S^* \alpha O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{a, b\}$, $\{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ $\{ a, c, d \}$ and $N_o P^* O(U, \tau_R(X)) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}$, $\{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, $\{ a, b, c\}$, $\{a, b, d\}$,$\{b, c, d\}, \{a, c, d\}\}$.Clearly the subsets $\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}$are nano pre*-open but not nano semi$^*\alpha$-open.

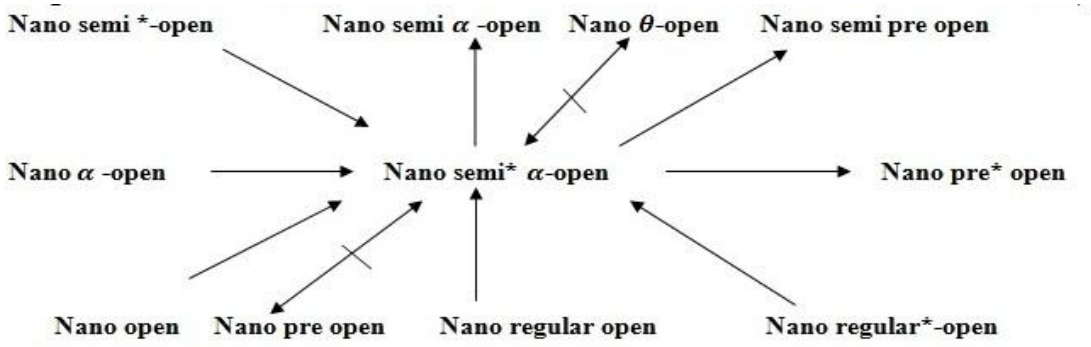
Remark 3.32: The concept of nano semi$^*\alpha$-open sets and nano pre-open sets are independent as shown in the following example.

Example 3.33: Let $U = \{a, b, c, d, e\} U/R = \big\{\{a\}, \{d\}\{b, c\}\big\}$. Let X = $\{a, c\}$.Then $\tau_R(X) = \big\{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\big\}$. $N_o S^* \alpha O\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{a, e\}$, $\{b, c\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, e\}\}$ and $N_o PO(U, \tau_R(X))$={ $U, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}, \{a, c, e\}$, $\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}\}$. Clearly the subset $\{a, e\}$ is nano semi$^*\alpha$-open but not nano pre-open and the subsets $\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}$, $\{a, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$ are nano pre-open but not nano semi *α-open.

Remark 3.34: The concept nano semi$^*\alpha$-open and nano θ-open sets are independent as shown in the following example.

Example 3.35: Let $U = \{a, b, c, d\}$ $U/R = \{\{a\}, \{d\}, \{b, c\}\}$. Let $X = \{a, c\}$. $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$.$N_o S^* \alpha O(U, \tau_R(X)) = \big\{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\big\}$ and $N_o \theta O(U, \tau_R(X)) = \{U, \emptyset, \{d\}, \{a, d\}, \{b, c, d\}\}$.Clearly the subsets $\{a\}, \{b, c\}, \{a, b, c\}$ are nano semi$^*\alpha$-open but not nano θ-open and the subsets $\{d\}, \{a, d\}, \{b, c, d\}$ are nano θ-open but not semi$^*\alpha$-open.

Diagram 3.36: From the above discussions we have the following diagram.



4. Nano semi* α - closed sets

Definition 4.1: The complement of nano semi$^*\alpha$-open set is called as **nano semi$^*\alpha$-closed**. The collection of all nano semi$^*\alpha$-open sets is denoted by $\boldsymbol{N_o\, S^*\alpha\, C(U, \tau_R(X))}$**.**

Example 4.2: Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b\}, \{c\}\}$.Let $X = \{b, c\}$. Then $\tau_R(X) = \{U, \emptyset, \{b, c\}\}$. The nano-closed sets are {U,$\emptyset$,$\{a\}\}$. The nano generalized – closed sets are $\{U, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. The nano generalized open sets are $\{U, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}\}$.

Theorem 4.3: Arbitrary intersection of nano semi$^*\alpha$-closed sets is nano semi$^*\alpha$- closed.
Proof: Let $\{A_\alpha\}$ be a collection of nano semi$^*\alpha$-closed sets in U. Since each A_α is nano semi$^*\alpha$- closed,$U\backslash A_\alpha$ is a nano semi$^*\alpha$-open. Since $U\backslash(\cap A_\alpha) = \cup (U\backslash A_\alpha)$ and hence by thm 3.4, $U\backslash(\cap A_\alpha)$is nano semi$^*\alpha$-open. Hence $\cap A_\alpha$ is nano semi$^*\alpha$-closed.

Remark 4.4: Union of two nano semi$^*\alpha$-closed sets need not be nano semi$^*\alpha$-closed as shown in the following example.

Example 4.5: Consider $U = \{a, b, c, d\}$, U/R = $\{\{a\}, \{c\}, \{b, d\}\}$. $X = \{a, b\}$. Then $\tau_R^c(X) = \{U, \emptyset, \{c\}, \{a, c\}, \{b, c, d\}\}$ and $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{\, U, \emptyset, \{a\}, \{c\}\, , \{a, c\}, \{b, d\}, \{b, c, d\}\}$. The sets $\{a\}$ and $\{b, d\}$ are nano semi$^*\alpha$-closed but their union $\{a\} \cup \{b, d\} = \{a, b, d\}$ is not nano semi$^*\alpha$-closed.

Theorem 4.6: In any nano topological space.
(i)Every nano α-closed set is nano semi$^*\alpha$-closed.
(ii) Every nano-closed set is nano semi$^*\alpha$-closed.
(iii) Every nano semi*-closed set is nano semi$^*\alpha$-closed.
(iv) Every nano semi$^*\alpha$-closed set is nano semi α-closed.
(v) Every nano semi$^*\alpha$- closed set is nano semi pre-closed.
(vi) Every nano regular closed set is nano semi$^*\alpha$-closed.

(vii) Every nano regular*- closed set is nano semi$^*\alpha$-closed.
(viii) Every nano semi$^*\alpha$- closed set is nano pre *-closed.
Proof: (i)Let A be any nano α-closed set in U, then U\A is nano α-open. By theorem 3.8, U\A is nano semi$^*\alpha$-open. Hence A is nano semi$^*\alpha$-closed. (ii) Let A be any nano-closed set in U. Then U\A is nano open. By theorem 3.11, U\A is nano semi$^*\alpha$-open. Hence A is nano semi$^*\alpha$-closed. (iii)Let A be any nano semi*-closed set in U, then U\A is nano semi*-open. By theorem 3.14, U\A is nano semi$^*\alpha$-open. Hence A is nano semi$^*\alpha$-closed.(iv) Let A be a nano semi$^*\alpha$-closed set in U. Then U\A is nano semi$^*\alpha$-open. By theorem 3.16, U\A is nano semi α-open. Hence A is nano semi α-closed. (v) Let A be a nano semi$^*\alpha$-closed set in U, then U\A is nano semi$^*\alpha$-open. By theorem 3.20, U\A is nano semi pre-open. Hence A is nano semi pre-closed. (vi) Let A be a nano regular closed set in U. Then U\A is nano regular open. By theorem 3.23, U\A is nano semi$^*\alpha$-open. Hence A is nano semi$^*\alpha$-closed. (vii) Let A be a nano regular*-closed set in U. Then U\A is nano regular*-open. By theorem 3.26, U\A is nano semi$^*\alpha$- open .Hence A is nano semi$^*\alpha$-closed .(viii) Let A be a nano semi$^*\alpha$-closed set in U. Then U\A is nano semi$^*\alpha$-open set. By theorem 3.28, U\A is nano pre*-open. Hence A is nano pre*-closed.

Remark 4.7: The converse of each of the statements in theorem 4.6 is not true as shown in the following examples.

Example 4.8: Let $U = \{a, b, c, d\}$ with U/R=$\{\{a\},\{c\},\{b, d\}\}$.Let $X = \{a, b\}$.Then $\tau_R^c(X) = \{U, \emptyset, \{c\}, \{a, c\}, \{b, c, d\}\}$ and $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{c\}, \{a, c\},$ $\{b, d\}, \{b, c, d\}\}$ and $N_o\, \alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{c\}, \{a, c\}, \{b, c, d\}\}$.Clearly the subsets $\{a\}$and$\{b, d\}$ are nano semi$^*\alpha$-closed but not nano α closed .

Example 4.9: Let $U = \{a, b, c\}$ with U/R = $\{\{a\},\{b\}\}$. Let X = $\{a, c\}$.Then ${\tau_R}^c(X) = \{U, \emptyset, \{b, c\}\}$and $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{b\}, \{c\}, \{b, c\}\}$.Clearly the subsets $\{b\}, \{c\}$ nano semi$^*\alpha$-closed but not nano-closed.

Example 4.10: Let $U = \{a, b, c, d\}$ with U/R = $\{\{a\},\{b\}\}$. Let X =$\{a, c, d\}$. Then ${\tau_R}^c(X) = \{U, \emptyset, \{b, c, d\}\}$ and $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{\ U, \emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\},$ $\{c, d\}, \{b, c, d\}\}$. $N_oS^*C(U, \tau_R(X) = \{\ U, \emptyset, \{b, c, d\}\}$. Clearly the subsets $\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}$ are nano semi$^*\alpha$-closed but not nano semi*-closed.

Example 4.11: Let $U = \{a, b, c, d\}$, $U/R = \{\{a\}, \{d\}, \{b, c\}\}$ $X = \{a, c\}$.Then ${\tau_R}^c(X) = \{U, \emptyset, \{a, d\}, \{b, c, d\}\}$ and $N_o\, S^*\alpha C\big(U, \tau_R(X)\big) = \big\{U, \emptyset, \{d\}, \{a, d\}, \{b, c, d\}\big\}$
$N_o\, S\, \alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}, \{d\}, \{a, d\}, \{b, c\}, \{b, c, d\}\}$.Clearly the subsets $\{a\}, \{b, c\}$ are nano semi α- closed but not nano semi$^*\alpha$-closed .

Example 4.12: Let $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b\}, \{c\}\}$.Let $X = \{b, c\}$. Then $\tau_R^c(X) = \big\{U, \emptyset, \{a\}\big\}$ and $N_o\, S^*\alpha\, C\big(U, \tau_R(X)\big) = \{U, \emptyset, \{a\}\}$. $N_oSPC\big(U, \tau_R(X)\big) = \{U, \emptyset,$

$\{a\},\{b\},\{c\},\{a,c\},\{a,b\}\}$. Clearly the subsets {b}, {c} $\{a,c\},\{a,b\}$are nano semi pre-closed but not nano semi$^*\alpha$-closed.

Example 4.13: Let $U=\{a,b,c\}$ with $U/R=\{\{a\},\{b\},\{c\}\}$.Let $X=\{b,c\}$. Then $\tau_R^c(X)=\{U,\emptyset,\{a\}\}$, $N_o\,S^*\alpha\,C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{a\}\}$.$N_o\,R\,C\big(U,\tau_R(X)\big)=\{U,\emptyset\}$. Clearly the subset $\{a\}$ is nano semi$^*\alpha$-closed but not nano regular-closed.

Example 4.14: Let $U=\{a,b,c\}$ withU/R= $\{\{a\},\{b\},\{c\}\}$.Let $X=\{b,c\}$. Then $\tau_R^c(X)=\{U,\emptyset,\{a\}\}$, $N_o\,S^*\alpha\,C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{a\}\}$.$N_o\,R^*\,C\big(U,\tau_R(X)\big)=\{U,\emptyset\}$ Clearly the subset $\{a\}$is nano semi$^*\alpha$- closed but not nano regular*- closed.

Example 4.15: Let $U=\{a,b,c\}$ with $U/R=\{\{a\},\{b\},\{c\}\}$.Let $X=\{b,c\}$. Then $\tau_R^c(X)=\{U,\emptyset,\{a\}\}$, $N_o\,S^*\alpha\,C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{a\}\}$. $N_o\,P^*C\big(U,\tau_R(X)\big)=\{U,\emptyset,$ $\{a\},\{b\},\{c\},\{a,c\},\{a,b\}\}$. Clearly the subsets $\{b\},\{c\},\{a,c\},\{a,b\}$ are nano pre* -closed but not nano semi$^*\alpha$-closed.

Remark 4.16: The concept of nano semi$^*\alpha$-closed sets and nano pre-closed sets are independent as shown in the following example.

Example 4.17: Let $U=\{a,b,c,d,e\}U/R=\{\{a\},\{d\}\{b,c\}\}$. Let X = $\{a,c\}$. Then ${\tau_R}^c(X)=\{U,\emptyset,\{d,e\},\{a,d,e\},\{b,c,d,e\}\}$. $N_o\,S^*\alpha\,C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{d\},\{e\},\{d,e\},$ $\{a,d,e\},\{b,c,d\},\{b,c,d,e\}\}$. $N_o\,P\,C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{b\},\{c\},\{d\},\{e\},\{b,d\},$ $\{b,e\},\{c,d\},\{c,e\},\{d,e\},\{a,d,e\},\{b,c,e\},\{b,d,e\},\{c,d,e\},\{a,b,d,e\},\{a,c,d,e\}$ $\{b,c,d,e\}\}$. Clearly the subset $\{b,c,d\}$ is nano semi$^*\alpha$-closed but not nano pre closed and the subsets $\{b\},\{c\},\{b,d\},\{b,e\},\{c,d\},\{c,e\},\{b,c,e\},\{b,d,e\},\{c,d,e\},\{a,b,d,e\},$ $\{a,c,d,e\}$ are nano pre- closed but not nano semi$^*\alpha$-closed.

Remark 4.18: The concept of nano semi$^*\alpha$-closed sets and nano θ- closed sets are independent as shown in the following example.

Example 4.19: Let U=$\{a,b,c,d\}$,U/R = $\{\{a\},\{d\},\{b,c\}\}$. Let X= $\{a,c\}$. ${\tau_R}^c(X)$ = {U,$\emptyset$, {d},$\{a,d\},\{b,c,d\}\}$. Then $N_o\,S^*\,\alpha\,C(U,\tau_R(X))=\{U,\emptyset,\{d\},\{a,d\},\{b,c,d\}\}$. $N_o\,\theta C\big(U,\tau_R(X)\big)=\{U,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$.Clearly the subsets $\{d\},\{a,d\},\{b,c,d\}$ are nano semi$^*\alpha$-closed but not nano θ- closed and the subsets $\{a\},\{b,c\},\{a,b,c\}$ are nano θ-closed but not semi$^*\alpha$-closed.

5.Nano semi* α- interior and nano semi* α-closure

Definition 5.1: The **nano semi*α- interior** of A is defined as the union of all nano semi*****α - open sets contained in A. It is denoted by s*αN_oInt(A).

Definition 5.2: Let A be a subset of U. A point u in U is called a **nano semi*α-interior point** of A if A contains a nano semi* α-open set containing u.

Theorem 5.3: If A is any subset of a nano topological space $(U, \tau_R(X))$,then
(i) s*αN_oInt(A) is the largest nano semi**α**-open set contained in A.
(ii)A is nano semi**α**-open if and only if s*αN_oInt(A)=A.
Proof: (i) Being the union of all nano semi**α**-open subsets of A, by theorem 3.4, s*αN_oInt(A) is nano semi**α**-open and contains every nano semi**α**-open subsets of A. (ii) A is nano semi**α**-open implies s*αN_oInt(A)=A is obvious from definition 5.1. On the other hand, suppose s*αN_oInt(A)=A. Hence by (i) s*αN_oInt(A) is nano semi**α**-open and hence A is nano semi**α**-open.
Theorem 5.4: In any nano topological space (U,τ_R(X), if A and B are subsets of U, then the following results hold:
i) s*αN_oInt (∅)=∅
ii) s*αN_oInt(U)=U
iii) s*αN_oInt(A)⊆A
iv) A⊆B⇒ s*αN_oInt(A)⊆s*αN_oInt(B)
v) s*αN_oInt(s*αN_oInt(A)) = s*αN_oInt(A).
vi) N_oInt(A)⊆s*αN_oInt(A)⊆sαN_oInt(A)⊆A
vii) s*αN_oInt(A∪ B)⊇s*αN_oInt(A)∪s*αN_oInt(B)
viii) s*αN_oInt(A∩B) ⊆ s*αN_oInt(A)∩s* αN_oInt(B)
Proof: (i), (ii), (iii) and (iv) follows from definition 5.1. By theorem 5.3(i), s*αN_oInt(A) is nano semi**α**-open and by theorem 5.3(ii), s*αN_oInt(s*αN_oInt(A)) = s*αN_oInt(A). Thus (v) proved. (vi) follows from theorem 3.11 and 3.17. (vii)Since *A*⊆*A*∪*B*, from statement (iv) we have s*αN_oInt(A) ⊆ s*αN_oInt(*A*∪*B*). Similarly, s*αN_oInt(B) ⊆ s*αN_oInt(*A*∪*B*). Then s*αN_oInt(A∪ B) ⊇ s*αN_oInt(A) ∪ s*αN_oInt(B).
(viii) Since $A \cap B \subseteq A$,from statement (iv) we have s*αN_oInt($A \cap B$) ⊆ s*αN_oInt(A).Similarly s*αN_oInt($A \cap B$) ⊆ s*αN_oInt(B). Therefore s*αN_oInt($A \cap B$) ⊆ s*αN_oInt(A)∩ αN_oInt(B).

Remark 5.5: In Theorem 5.4(vi), each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 5.6: Let $U = \{a, b, c, d, e\}$,$U\backslash R = \{\{a\}, \{d\}, \{b, c\}\}$,$X = \{a, c\}$.Then $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}andN_o\, S^*\alpha\, O\big(U, \tau_R(X)\big) = \{U, \emptyset,\ \{a\},\ \{a,\ e\},\ \{b,\ c\},$ $\{a,\ b,\ c\}, \{a, b, c, d\}, \{a, b, c, e\}\}$ Let $A = \{b, c\}$.Then $N_o\, int\ (A) =$ s*α N_o int(A) = s α N_o int(A) = {b, c} = A. Let $B = \{a, e\}$.Then N_o int (B)={a}, s*α N_o int(B) = {a, e}, s α N_o int(B) = {a, e}. Here $N_o\ int(B) \subsetneq$ s*α N_o int(B) = s α N_o int(B) = B. Let $C = \{b, d\}$. Then $N_o\, int\ (C) = \emptyset$,s*α N_o int(C) = ∅, s α N_o int(C) = ∅. Let $D = \{a, c, d, e\}$,Then $N_o\, int\ (D) = \{a\}$,s*α N_o int(D) = {a, e},s α N_o int(D) = {a, d, e}. Here $N_o\, int\ (D) \subsetneq$ s*α N_o int(D) ⊊ s α N_o int(D) ⊊ D. Let $E = \{b, c, d, e\}$.Then $N_o\, int\ (E) = \{b, c\}$,s*α N_o int(E) = {b, c},s α N_o int(E) = {b, c, d, e}. Here $N_o\, int\ (E) =$ s*α N_o int(E) ⊊ s α N_o int(E) = E.

Remark 5.7: In Theorem 5.5(vii) and (viii), each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 5.8: Let $U = \{a, b, c, d\}$,$U\backslash R = \{\{a\}, \{c\}, \{b, d\}\}$,X=$\{a, b\}$. Then $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $N_o\, S^*\, \alpha O\left(U, \tau_R(X)\right)$ = $\{U, \emptyset, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$.
Let$A = \{a, b\}$, B = $\{b, d\}$.Then $A \cup B = \{a, b, d\}$ and $s^*\alpha N_o\text{int(A)} = \{a\}$, $s^*\alpha\,\text{Nint(B)} = \{b, d\}$, $s^*\alpha N_o\text{int(A} \cup \text{B)} = \{a, b, d\}$. Therefore $s^*\alpha N_o\text{int(A)} \cup s^*\alpha N_o\text{int(B)} = \{a, b, d\} = s^*\alpha N_o\text{int(A} \cup \text{B)}$.
LetC= $\{a, b\}$,$D = \{a, d\}$. Then $C \cup D = \{a, b, d\}$ and $s^*\alpha N_o\text{int(C)} = \{a\}$, $s^*\alpha N_o\text{int(D)} = \{a\}$, $s^*\alpha N_o\text{int(C} \cup \text{D)} = \{a, b, d\}$
Therefore $s^*\alpha N_o\text{int(C)} \cup s^*\alpha N_o\text{int(D)} \subsetneq s^*\alpha N_o\text{int(C} \cup \text{D)}$.
Let $E = \{a, b, d\}$,$F = \{a, c, d\}$.Then $E \cap F = \{a, d\}$ and $s^*\alpha N_o\text{int(E)} = \{a, b, d\}$, $s^*\alpha N_o\text{int(F)} = \{a, c\}$,$s^*\alpha N_o\text{int(E} \cap \text{F)} = \{a\}$.
Therefore $s^*\alpha N_o\text{int(E} \cap \text{F)} = s^*\alpha N_o\text{int(E)} \cap s^*\alpha N_o\text{int(F)}$.
Let G= $\{a, b, c\}$, $H = \{b, c, d\}$.Then $G \cap H = \{b, c\}$ and $s^*\alpha N_o\text{int(G)} = \{a, c\}$, $s^*\alpha N_o\text{int(H)} = \{b, c, d\}$, $s^*\alpha N_o\text{int(G} \cap \text{H)} = \emptyset$.
Therefore $s^*\alpha N_o\text{int(G} \cap \text{H)} \subsetneq s^*\alpha N_o\text{int(G)} \cap s^*\alpha N_o\text{int(H)}$.

Definition 5.9: If A is a subset of a nano topological space U, the Nano semi$^*\alpha$-closure of A is defined as the intersection of all nano semi$^*\alpha$-closed sets in U containing A. It is denoted by $s^*\alpha\,\text{Ncl(A)}$.

Theorem 5.10: If A is any subset of a nano topological space (U, $\tau_R(X)$) ,then
(i)$s^*\alpha\,\text{Ncl(A)}$is the smallest nano semi*α-closed set in U containing A.
(ii)A is nano semi*α-closed if and only if $s^*\alpha N_o\text{cl(A)} = \text{A}$.
Proof: (i) Since $s^*\alpha\,\text{Ncl(A)}$ is the intersection of all nano semi* α-closed subsets of U containing A, by theorem 4.3, it is nano semi* α-closed and it is contained in every nano semi*α-closed set containing A and hence it is the smallest nano semi*α-closed set in U containing A. (ii)If A is nano semi*α-closed, then $s^*\alpha N_o\text{cl(A)} = \text{A}$ is obvious. Conversely, let $s^*\alpha N_o\text{cl(A)} = \text{A}$, By (i) $s^*\alpha N_o\text{cl(A)}$ is nano semi*α-closed and hence A is nano semi*α-closed.

Theorem 5.11: In any nano topological space (U,τ_R(X), if A and B are subsets of U, then the following results hold:
(i)$s^*\alpha N_o\text{cl}(\emptyset) = \emptyset$
(ii)$s^*\alpha N_o\text{cl(U)} = \text{U}$
(iii)$\text{A} \subseteq s^*\alpha N_o\text{cl(A)}$
(iv)$\text{A} \subseteq \text{B} \Rightarrow s^*\alpha N_o\text{cl(A)} \subseteq s^*\alpha N_o\text{cl(B)}$
(v)$s^*\alpha N_o cl(s^*\alpha N_o cl(\text{A})) = s^*\alpha N_o cl(\text{A})$.
(vi)$\text{A} \subseteq s\alpha N_o\text{cl(A)} \subseteq s^*\alpha N_o\text{cl(A)} \subseteq \alpha N_o\text{cl(A)}$
(vii)$s^*\alpha N_o\text{cl(A} \cup B) \supseteq s^*\alpha N_o\text{cl(A)} \cup s^*\alpha N_o\text{cl(B)}$
(viii)$s^*\alpha N_o\text{cl(A} \cap \text{B)} \subseteq s^*\alpha N_o\text{cl(A)} \cap s^*\alpha N_o\text{cl(B)}$

Proof: (i), (ii), (iii) and (iv) follows from definition 5.7.
From theorem 5.10(i) s*αN_ocl(A) is the nano semi*α-closed and from theorem5.10(ii) s*$\alpha N_o cl$(s*$\alpha N_o cl$(A))= s*$\alpha N_o cl$(A).This proves (v).
(vi) follows from theorem 4.6 and 4.9. (vii) Since $A \subseteq A \cup B$, from statement (iv) we have s*Ncl(A) ⊆ s*αNcl(A∪B). Similarly, s*αN_ocl(B) ⊆ s*αN_ocl(A∪ B). Then s*αN_ocl(A∪B) ⊇s*αN_ocl(A) ∪ s*αN_ocl(B)
(viii) Since $A \cap B \subseteq A$,from statement (iv) we have s* αN_ocl$(A \cap B) \subseteq$ s* αN_ocl(A).Similarly s* αN_ocl$(A \cap B) \subseteq$ s* αN_ocl(B). Therefore s* αN_ocl$(A \cap B) \subseteq$ s* αN_oInt(A)∩ s*αN_ocl(B).

Remark 5.12: In Theorem 5.11(vi), each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 5.13: Let $U = \{a, b, c, d, e\}$, $U \backslash R = \{\{a\}, \{d\}, \{b, c\}\}$, $X = \{a, c\}$,
Then $\tau_R(X) = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and
$N_o S^*\ \alpha C\ \left(U, \tau_R(X)\right) = \{U, \emptyset, \{d\}, \{e\}, \{d, e\}, \{a, d, e\}$,
$\{b, c, d\}, \{b, c, d, e\}\}$. $N_o S\ \alpha C\ \left(U, \tau_R(X)\right) = \{U, \emptyset, \{a\}, \{a, e\}, \{b, c\}$, $\{a, b, c\}, \{a, b, c, d\}$, $\{a, b, c, e\}\}$.
Let $A = \{a, b, c\}$.Then s*αN_o cl(A)= sαN_o cl(A)= αN_o cl(A)=U.
Let $B = \{a, e\}$.Then s*αN_o cl(B)= $\{a, d, e\}$, sαN_o cl(B)= $\{a, e\}$,αN_o cl(A) = {a, d, e}
Here $B = s\alpha N_o$ cl(B) ⊊ s $* \alpha N_o$ cl(B) = αN_o cl(A).
Let$C = \{b, c, d\}$.Thens*αN_o cl(C)= $\{b, c, d\}$,sαN_o cl(C) = {b, c, d},αN_o cl(C) = {b, c, d, e}.
Here $C = s\alpha N_o$ cl(C) = s $* \alpha N_o$ cl(C) ⊊ αN_o cl(C).
Let $D = \{c, d, e\}$. Then s*αN_o cl(D)= $\{b, c, d, e\}$, sαN_o cl$(D) = b, c, d, e\}$, αN_o cl(A)= {b, c, d, e}.
Here $D \subsetneq s\alpha N_o$ cl(D) = s $* \alpha N_o$ cl(D) = αN_o cl(D).
Let E= $\{a\}$.Then s*αN_o cl(E)= $\{a, d, e\}$, sαN_o cl(E) = $\{a\}$, αN_o cl(E)= {a, d, e}.
Here $E = s\alpha N_o$ cl(E) ⊊ s $* \alpha N_o$ cl(E) = αN_o cl(E).

6. Conclusions

In this article, we have introduced nano semi*α-open sets and nano semi *α-closed sets in nano topological spaces and studied their characterizations with other nano open sets. A diagramatic explanation gives a clear explanation of this article.

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The Detour Monophonic Convexity Number of a Graph

M. Sivabalan[*]
S. Sundar Raj[†]
V. Nagarajan[‡]

Abstract

A set S is detour monophonic convexif$J_{dm}[S\} = S$. The detour monophonic convexity number is denoted by $C_{dm}(G)$, is the cardinality of a maximum proper detour monophonic convex subset of V.Some general properties satisfied by this concept are studied. The detour monophonic convexity number of certain classes of graphs are determined. It is shown that for every pair of integers a and b with $3 \leq a < b$, there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$, where$C_m(G)$ is the monophonic convexity number of G.

Keywords: convex, detour, chord, detour monophonic path, monophonic convexity number, detour monophonic, convexity number.

AMS subject classification: 05C12,05C38[§].

[*]Register Number.12567, Research Scholar, Department of Mathematics, S.T. Hindu College, (Nagercoil – 629002, India); e-mail: sivabalanvkc@gmail.com
[†]Department of Mathematics, Vivekananda College, (Agasteeswaram – 629701, India);
e-mail: sundarrajvc@gmail.com
[‡]Department of Mathematics, S.T. Hindu College, (Nagercoil – 629002, India);
e-mail: sthcrajan@gmail.com Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India
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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to [1]. A vertex v is adjacent to another vertex u if and only if there exists an edge $e = uv \in E(G)$. If $uv \in E(G)$, we say that u is a *neighbor* of v and denote by $N_G(v)$, the set of neighbors of v. A vertex v is said to be *universal vertex* if $\deg_G(v) = p - 1$. A vertex v is called an *extreme vertex* if the subgraph induced by v iscomplete.

The length of a path is the number of its edges. Let u and v be vertices of a connected graph G. A shortest u-v path is also called a u-vgeodesic. The (shortest path) distance is defined as the length of a u-v geodesic in G and is denoted by$d_G(u, v)$ or $d(u, v)$ for short if the graph is clear from the context. For a set S of vertices, let $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set $S \subset V$ is called a *convex set* of G if $I[S] = S$. These concepts were studied in [1, 3]

A chord of a path P is an edge which connects two non-adjacent vertices of P. A u-v path is called a monophonic path if it is a chordless path. For two vertices u and v, the closed interval J[u, v] consists of all the vertices lying in a $u - v$ monophonic path including the vertices u and v. If u and v are adjacent, then $J[u, v] = \{u, v\}$. For a set M of vertices, let $J[M] = \cup_{u,v \in M} J[u, v]$. Then certainly $M \subseteq J[M]$. A set $M \subseteq V(G)$ is called a *monophonic se*t of G if $J[M] = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is called a m-set of G. A set $M \subseteq V(G)$ is called a monophonic convex set of G if $J(M) = M$. The monophonic convexity number $C_m(G)$ of G is the cardinality of a maximum proper monophonic convex subset of V. These concepts were studied in [5-10].

The detour distance $D(u, v)$ between two vertices u and v in a connected graph G from u to v is defined as the length of a longest u-v path in G. An u-v path of length $D(u, v)$ is called an u-v detour. The detour monophonic distance dm(u,v) between two vertices u and v is the length of a longest u-v monophonic path in G, Any monophonic path of length dm(u,v) is called u-v detour monophonic path. For two vertices $u, v \in V$, let $J_{dm}[u, v]$ denotes the set of all vertices that lies in u-$v$detour monophonic path including u and v, and $J_{dm}(u, v)$denotes the set of all internal vertices that lies in $u - v$detour monophonic path . For $M \subseteq V$, let $J_{dm}[M] = \bigcup_{u,v \in M} J_{dm}[u, v]$.A set $M \subseteq V$ is a *detour monophonic set* if $J_{dm}[M] = V$.The minimum cardinality of a detour monophonic set of *G* is the *detour monophonic number* of *G* and is denoted by $dm(G)$. The detour monophonic set of cardinality $dm(G)$ is called dm-set. These concepts were studied in [2, 4, 11].

2.The detour monophonic convexity number of a Graph

Definition 2.1. A set S is *detour monophonic convex* if$J_{dm}[S\} = S$. Clearly $S = \{v\}$ or $S = V$ then S is detour monophonic convex. The detour monophonic convexity number

is denoted by $C_{dm}(G)$, is the cardinality of a maximum proper detour mono-phonic convex subset of V.

Example 2.2. For the graph Gin Figure 2.1, $M_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$is a C_{dm}-set of G so that $C_{dm}(G) = 8$. Also $M_2 = \{v_1, v_2\}$is a C_m-set of G so that $C_m(G) = 2$.

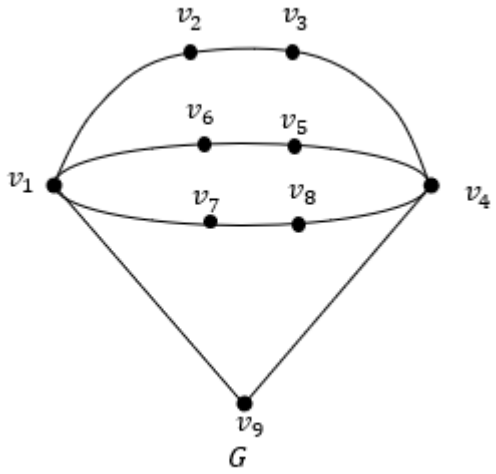

Figure 2.1

Observation 2.3. Let G be a connected graph of order $p \geq 3$. Then $2 \leq C_{dm}(G) \leq p - 1$.

Theorem 2.4. Let G be a connected graph of order p and G contains an extreme vertex. Then $C_{dm}(G) = p - 1$.
Proof. Let G contain an extreme vertex, say v. Then $S = V(G) - \{v\}$ is a dm-convex set of G so that $C_{dm}(G) = p - 1$.

Theorem 2.5. Let G be a connected graph of order $p \geq 3$. Then$2 \leq \omega(G) \leq C_{dm}(G) \leq p - 1$, where $\omega(G)$ is the clique number of G.
Proof. Since G is a connected graph of order $p \geq 3, \omega(G) \geq 2$. Let H be a subgraph of G such that $\langle V(H)\rangle$ is a maximal complete subgraph of G so that $C_{dm}(G) \geq |v(H)| = \omega(G)$. Let S be a dm-convex set of G. Then S is a convex set of G so that $C_{dm}(G) \leq C(G)$. Since every convex set of G is a proper subset of $G, C(G) \leq p - 1$. Therefore $2 \leq \omega(G) \leq C_{dm}(G) \leq p - 1$..

Corollary 2.6. (i) For the complete graph $G = K_p$ $(p \geq 3)$, $C_{dm}(G) = p - 1$.
(ii) For a trivial tree G of order $p \geq 3$, $C_{dm}(G) = p - 1$.
(iii) For the fan graph $G = K_1 + P_{p-1}$ $(p \geq 4)$, $C_{dm}(G) = p - 1$.

Theorem 2.7. For the cycle $G = C_p, (p \geq 3), C_{dm}(G) = 2$.

Proof. Let$S = \{x, y\}$be a set of two adjacent vertices of G. Then $J_{dm}[S] = S$, it follows that S is a dm-convex set of G so that $C_{dm}(G) \geq 2$. We prove that $C_{dm}(G) = 2$. Suppose that $C_{dm}(G) \geq 3$. Then there exists a dm-convex set S_1 such that $|S_1| \geq 3$. Hence it follows that S_1 contains two independent vertices of G. Then $J_{dm}[S_1] \neq S_1$. Therefore $C_{dm}(G) = 2$.■

Theorem 2.8. For the complete bipartite graph $G = K_{m,n}, C_{dm}(G) = 2$.

Proof: Let (V_1, V_2) be a partition of G. Since $\omega(G) = 2, C_{dm}(G) \geq 2$. We prove that $C_{dm}(G) = 2$. Suppose that $C_{dm}(G) \geq 3$. Then there exists two vertices x and y belong to the same partite V_1 (or V_2). Since $d(x, y) = 2$ in G, every vertex in V_1 (or V_2) lie on $x - y$ detour monophonic. Hence it follows that $J_{dm}[S] \neq S$. Therefore $C_{dm}(G) = 2$.

■

Theorem 2.9. For the wheel graph G= $W_p = K_1 + C_{p-1}(p \geq 4), C_{dm}(G) = 3$.

Proof. Let$V(K_1) = x$and$V(C_{p-1}) = \{v_1, v_2, \ldots v_{p-1}\}$. Then $S = \{x, v_1, v_2\}$ is a det-our monophonic convex set of G so that $C_{dm}(G) \geq 3$.We prove that $C_{dm}(G) = 3$. Suppose that $C_{dm}(G) \geq 4$. Then there exists dm-convex set S_1 such that $|S_1| \geq 4$. Hence it follows that S_1 contains two independent vertices of G. Then $J_{dm}[S_1] \neq S_1$. Therefore$C_{dm}(G) = 3$.

■

Theorem 2.10. For any two positive integers such that $2 \leq a \leq b$, there exists a connected graph G such that $\omega(G) = a$ and $C_{dm}(G) = b$.

Proof. For $a = b$,let $G = K_{a+1} - \{e\}$. Then $\omega(G) = C_{dm}(G) = a$.For $a < b$, let K_a be the complete graph with vertices $v_1, v_2, \ldots, v_a$. Let $P: u_1, u_2, \ldots, u_{b-a}, u_{b-a+1}, \ldots, u_c$ where $c > b - a$ a path on c vertices. Let G be the graph obtained from K_a and P by joining u_1 with $v_{a-1}$and v_a each$u_i (2 \leq i \leq b - a)$ with v_{a-1} and u_c with v_{a-1}. The graph G is shown in Figure 2.2.

First, we prove that $\omega(G) = a$. Let $S = \{v_1, v_2, \ldots, v_a\}$. It is clear that S is a maximal complete subgraph of G such that $\omega(G) = a$.

Next, we prove that $C_{dm}(G) = b$. Let $W = \{v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_{b-a}\}$. It is clear that W is a dm-convex set of G so that$C_{dm}(G) \geq G$. We prove that $C_{dm}(G) = b$. Suppose that $C_{dm}(G) > b$. Let S_1 be a dm-convex set with $|S_1| \geq b + 1$.Then there exists a vertex$u_i (b - a + 1 \leq i \leq c)$such that $u_i \in S_1$. Then $J_{dm}[S_1] \neq S_1$.Therefore$C_{dm}(G) = b$. ■

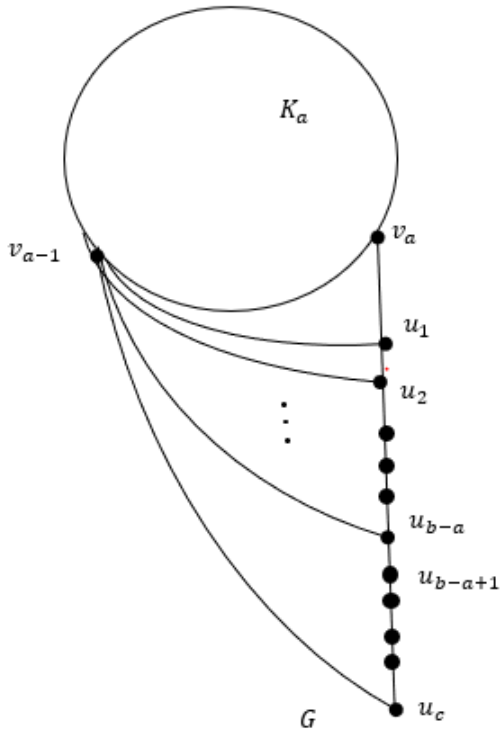

Figure 2.2

Theorem 2.11. For every pair of integers a and b with $3 \leq a < b$, there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$.

Proof. Let $V(\overline{K}_2) = \{x, y\}$. Let $P_i: u_i, v_i (1 \leq i \leq b)$be a copy of path of order two. Let G be the graph obtained from $\overline{K}_2, P_i (1 \leq i \leq b)$ and K_{a-1} by joining x with each $u_i (1 \leq i \leq b)$ and y with each $v_i (1 \leq i \leq b)$and x and y with each vertex of K_a. The graph G is shown in Figure 2.3.

First, we prove that $C_m(G) = a$. Let $M = V(K_a) \cup \{x\}$. Then M is a monophonic convex set of G and so $C_m(G) \geq a$. We prove that $C_m(G) = a$. Suppose that $C_m(G) \geq a + 1$. Let M_1 be m-convex set with $|S| \geq a + 1$. Then there exists at least one vertex, say x such that $x \in M_1$ and $x \notin M$. Hence it follows that $x = u_i$ or v_i or y for some i $(1 \leq i \leq b)$. Then $J_m[M_1] \neq M_1$, which is a contradiction. Therefore $C_m(G) = a$.

Next we prove that $C_{dm}(G) = 2(b+1)$. Let $S = V(G) - V(K_a)$. Then S is a detour monophonic convex set of G and so $C_{dm}(G) \geq 2(b+1)$. We prove that $C_{dm}(G) = 2(b+1)$. On the contrary $C_{dm}(G) > 2(b+1)$. Let S_1 be a dm-convex set with $|S_1| \geq 2(b+1) + 1$. Then there exists a vertex $x \in S_1$ such that $x \notin S$. Hence it follows that $x \in K_a$. Then $J_{dm}[S_1] \neq S_1$. Therefore $C_{dm}(G) = 2(b+1)$.■

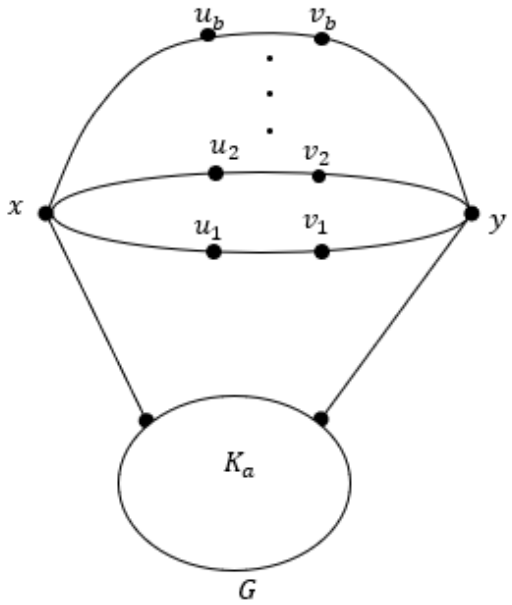

Figure 2.3

3. Conclusions

In this paper, we investigated the detour monophonic convexity number of some standard graphs. Also, we proved for every pair of integers a and b with $3 \leq a < b$, there exists a connected graph G such that $C_m(G) = a$ and $C_{dm}(G) = 2(b+1)$.

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Outer independent square free detour number of a graph

K. Christy Rani [*]
G. Priscilla Pacifica [†]

Abstract

For a connected graph $G = (V, E)$, a set S of vertices is called an outer independent square free detour set if S is a square free detour set of G such that either $V = S^*$ or $V - S^*$ is an independent set. The minimum cardinality of an outer independent square free detour set of G is called an outer independent square free detour number of G and is denoted by $dn_{\square f}^{oi}(G)$. We determine the outer independent square free detour number of some graphs. We characterize the graph which realizes the result that for any pair of integers α and β with $2 \leq \alpha \leq \beta$, there exists a connected graph G of order $\beta + 3$ with square free detour number α and outer independent square free detour number β.

Keywords: square free detour set; outer independent square free detour set; outer independent square free detour number.

2010 subject classification: 05C12, 05C38[‡]

[*] Research Scholar, Reg. No.: 20122212092002, PG and Research Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628 001, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, India; christy.agnes@gmail.com.

[†]Assistant Professor, Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628001, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, India; priscillamelwyn@gmail.com.

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1. Introduction

In this article, a graph G is considered to be a finite, undirected and connected graph of order $n(n \geq 2)$ with neither loops nor multiple edges. Let $D(u, v)$ be the longest path in G and a $u - v$ path of $D(u, v)$ is called $u - v$ detour. The parameters on detour concept were developed by Chartrand [2]. The detour concept was extended to triangle free detour concept by S. Athisayanathan et al. [1, 7]. The detour concept was applied in domination by number of authors. The detour domination number was studied and extended to outer independent detour domination by number of authors in [5, 6].

For any two vertices u, v in a connected graph G, the $u - v$ path P is called $u - v$ triangle free path if no three vertices of P induce a triangle. The triangle free detour distance $D_{\Delta f}(u - v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\Delta f}(u, v)$ is called a $u - v$ triangle free detour. A set $S \subseteq V$ of G is called a triangle free detour set of G if every vertex of G lies on a $u - v$ triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\Delta f}(G)$ of $G$is the minimum order of its triangle free detour sets. This triangle free detour number was studied by S. Athisayanathan and S. Sethu Ramalingam in [8]. This concept was extended to square free detour number by K. Christy Rani and G. Priscilla Pacifica [4]. A square free detour number of G denoted by $dn_{\square f}(G)$ is defined as the minimum order of square free detour set S consisting of every pair of vertices of all the square free detours in which every vertex of G lies on.

In this article, we introduce the outer independent square free detour number denoted by $dn_{\square f}^{oi}(G)$. The outer independent square free detour number of some standard graphs and cycle related graphs are determined. For the basic terminologies we refer to Chartrand [2].

2. Preliminaries

The following theorems are used in the sequel.

Theorem 2.1 [3] For any connected graph $G, 2 \leq dn(G) \leq n$.

Theorem 2.2 [3] Every end-vertex of a non-trivial connected graph G belongs to every detour set of G.

Theorem 2.3 [3] If T is a tree with k end-vertices, then $dn(T) = k$.

Theorem 2.4 [4] If G is the cycle $C_n(\ n \geq 3)$, then $dn_{\square f}(G) = 2$.

3. Outer independent square free detour number of a graph

Definition 3.1 Let $G = (V, E)$ be a simple connected graph of order $n \geq 2$. A set of vertices S^* is called an outer independent square free detour set in G if S^* is a square free detour set such that either $V = S^*$ or $V - S^*$ is independent. The minimum cardinality of an outer independent square free detour number of G is called outer connected square free detour number of G and is denoted by $dn_{\square f}^{oi}(G)$.

Example 3.2 For the graph G shown in Figure 1, the set $S_1^* = \{a, d, e, f, g, h, k\}$ is a minimum outer independent square free detour set and $S = \{a, d, e, g, h, k\}$ is a minimum square free detour set for G and so $dn_{\square f}^{oi}(G) = 7$ and $dn_{\square f}(G) = 6$. Here we find that $dn_{\square f}(G) < dn_{\square f}^{oi}(G)$. Moreover, the sets $S_2^* = \{a, d, e, f, h, i, k\}$, $S_3^* = \{a, d, e, f, i, j, k\}$ and $S_4^* = \{a, b, c, e, f, g, k\}$ are also the minimum outer independent square free detour sets of G. Hence there can be more than one minimum outer independent square free detour set for a graph G.

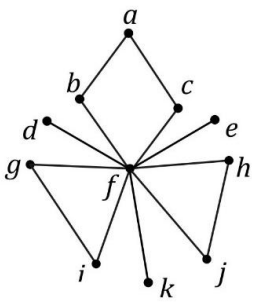

Figure 1: G

Theorem 3.3 *For any connected graph G, every end-vertex of G belongs to every outer independent detour set of G.*
Proof. Since every outer independent square free detour set is also a detour set of G, the proof follows from Theorem 2.2.

Theorem 3.4 *For any connected graph G, $2 \leq dn(G) \leq dn_{\square f}^{oi}(G) \leq n$.*
Proof. The result follows from Theorems 2.1 and 3.3.

Remark 3.5 The bounds in Theorem 3.4 are sharp. The set of two end-vertices of a path P_2 is its minimum outer independent square free detour set so that $dn(P_2) = dn_{\square f}^{oi}(P_2) = 2 = n$. The bounds in Theorem 3.4 are also strict. For the graph G of order 11 given in Figure 1, $2 < dn(G) < dn_{\square f}^{oi}(G) < n$.

Theorem 3.6 *If G is a Path P_n, then $dn_{\square f}^{oi}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

Proof. Let $G = P_n$ be a path of order n and S^* be a set of $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices. Then we consider two cases.

Case 1. Let n be odd. Let $S^* = \left\{ x_{2i+1} : 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$ be the square free detour set such that $V - S^*$ is independent. Hence S^* is an outer independent square free detour set. Thus $dn_{\square f}^{oi}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case 2. Let n be even. Let $S^* = \left\{ x_1, x_{2j} : 1 \leq j \leq \frac{n}{2} \right\}$ be a square free detour set such that $V - S^*$ is independent. Thus $dn_{\square f}^{oi}(G) = \frac{n}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

The following corollary is immediate.

Corollary 3.7 For any connected graph G, $dn_{\square f}^{oi}(G) = n$ if and only if $G = P_2$.

Theorem 3.8 *If G is a star S_n, then $dn_{\square f}^{oi}(G) = dn_{\square f}(G)$.*

Proof. Let $G = S_n$ be a star with $n - 1$ end-vertices. Then by Theorem 3.3, $S^* = \{x_i : 1 \leq i \leq n - 1\}$ where $x_i, x_2, \ldots, x_{n-1}$ are the end-vertices of S_n such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = n - 1$. By Theorem 2.3, S^* is also a minimum detour set and so $dn_{\square f}(G) = n - 1$. Hence $dn_{\square f}^{oi}(G) = dn_{\square f}(G)$.

Theorem 3.9 If G is a complete bipartite graph $K_{m,n}(2 \leq m \leq n)$, then $dn_{\square f}^{oi}(G) = m$.

Proof. Let $G = K_{m,n}$ be a complete bipartite graph of order with two partitions X and Y where $|X| = m$ and $|Y| = n$. Let S^* be a set of m vertices of X. Now, it is easy to verify that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = m$.

Remark 3.10 Due to the connectivity of the complete graph K_n, it is not possible to find the outer independent square free detour number for K_n.

Theorem 3.11 *If G is a cycle C_n, then $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{dn_{\square f}(G)} \right\rceil$.*

Proof. Let $G = C_n$ be a cycle of order n. Let S^* be any set of $\left\lceil \frac{n}{2} \right\rceil$ vertices of G. we consider two cases.

Case 1. Let n be odd. Let $S^* = \left\{ x_{2i+1} : 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$ be a square free detour set such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Case 2. Let n be even. Let $S^* = \left\{ x_{2j} : 1 \leq j \leq \frac{n}{2} \right\}$ be a square free detour set such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = \frac{n}{2}$.

From the above cases, we observe that $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{2} \right\rceil$. Then by Theorem 2.4, it follows that $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{dn_{\square f}(G)} \right\rceil$.

Theorem 3.12 *If G is a Wheel, then* $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let $G = W_n$ be a Wheel of order n. Then $S^* = \left\{ x_0, x_{2i+1} : 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$ is a square free detour set such that $V - S^*$ is independent. Thus $dn_{\square f}^{oi}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.13 *If G is a Flower graph Fl_n, then* $dn_{\square f}^{oi}(G) = \begin{cases} n+1 & if\ n\ is\ even \\ n+2 & if\ n\ is\ odd \end{cases}$

Proof. Let $G = Fl_n$ be a Flower graph of order n. Let x_0 be the hub, $x_i (1 \leq j \leq n)$ be the vertices on the inner rim and $y_j (1 \leq j \leq n)$ be the vertices at square free detour distance 3 from the hub. Then we have two cases.

Case 1. Let n be even. Let $S^* = \left\{ x_0, x_{2i+1}, y_{2j} : 0 \leq i \leq \frac{n}{2} : 1 \leq j \leq \frac{n}{2} \right\}$ is the square free detour set such that $V - S^*$ is independent. Thus $dn_{\square f}^{oi}(G) = n + 1$.

Case 2. Let n be odd. Then $S^* = \left\{ y_1, x_{2i+1}, y_{2j} : 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j \leq \frac{n}{2} \right\}$ is a square free detour set such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = n + 2$.

Theorem 3.14 *If G is a Helm H_n, then* $dn_{\square f}^{oi}(G) = \begin{cases} \frac{3(n+1)}{2} & if\ n\ is\ odd \\ \frac{3n}{2} + 1 & if\ n\ is\ even \end{cases}$

Proof. Let $G = H_n$ be a Helm of order n. Let S^* be any set in G. Then we have the following two cases.

Case 1. Let n be odd. Then $S^* = \left\{ x_0, x_{2i+1}, y_j : 0 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, 1 \leq j \leq n \right\}$ is a square free detour set of $\frac{3(n+1)}{2}$ vertices where x_0 is the hub, $x_i (1 \leq j \leq n)$ are the vertices on the rim and $y_j (1 \leq j \leq n)$ are the pendent vertices of H_n such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = \frac{3(n+1)}{2}$.

Case 2. Let n be even. Then $S^* = \left\{ x_0, x_{2i+1}, y_j : 0 \leq i \leq \frac{n}{2}, 1 \leq j \leq n \right\}$ is a square free detour set of $\frac{3n+2}{2}$ vertices where x_0 is the hub, $x_i (1 \leq j \leq n)$ are the vertices on the rim and $y_j (1 \leq j \leq n)$ are the pendent vertices of H_n such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = \frac{3n}{2} + 1$.

Theorem 3.15 *If G is a Closed Helm CH_n, then* $dn_{\square f}^{oi}(G) = \begin{cases} n+1 & if\ n\ is\ even \\ n+2 & if\ n\ is\ odd \end{cases}$

Proof. Let $G = CH_n$ be a closed Helm of order n. Let x_0 be the hub, $x_i(1 \le j \le n)$ and $y_j(1 \le j \le n)$ are the vertices of inner and outer rim of CH_n respectively. Then we consider two cases.

Case 1. Let n be even. Let $S^* = \left\{x_0, x_{2i+1}, y_{2j} : 0 \le i \le \frac{n}{2} : 1 \le j \le \frac{n}{2}\right\}$ is the square free detour set such that $V - S^*$ is independent. Thus $dn_{\square f}^{oi}(G) = n + 1$.

Case 2. Let n be odd. Then $S^* = \left\{x_0, y_1, x_{2i+1}, y_{2j}, y_1 : 0 \le i \le \left\lfloor\frac{n}{2}\right\rfloor : 1 \le j \le \frac{n-1}{2}\right\}$ is a square free detour set such that $V - S^*$ is independent. Hence $dn_{\square f}^{oi}(G) = n + 2$.

Theorem 3.16 *For any pair of integers α and β with $2 \le \alpha \le \beta$, there exists a connected graph G of order $\beta + 3$ with square free detour number α and outer independent square free detour number β.*

Proof. We consider two cases.

Case 1. $2 \le \alpha = \beta$. Any star S_n with $n - 1$ end-vertices has the desired property.

Case 2. $2 \le \alpha < \beta$. Let H_1 be a graph obtained from $C_6 : x_1, x_2, x_3, x_4, x_5, x_6, x_1$ by adding $\alpha - 1$ new vertices $y_1, y_2, \ldots, y_{\alpha-1}$ to x_6. Let H_2 be the graph derived from H_1 by adding $\beta - \alpha - 1$ new vertices $z_1, z_2, \ldots, z_{\beta-\alpha-1}$ and identifying z_1 with x_3. Let G be the graph derived from H_2 by joining the remaining $\beta - \alpha - 2$ vertices $z_2, z_3, \ldots, z_{\beta-\alpha-1}$ to x_2 and x_4 of C_6. The resulting graph G of order $\beta + 3$ is shown in Figure 2. By Theorem 3.3, it is verified that $S_1 = \{x_1, y_1, y_2, \ldots, y_{\alpha-1}\}$ and $S_2 = \{x_5, y_1, y_2, \ldots, y_{\alpha-1}\}$ are the square free detour sets of G and so $dn_{\square f}^{oi}(G) = \alpha$.

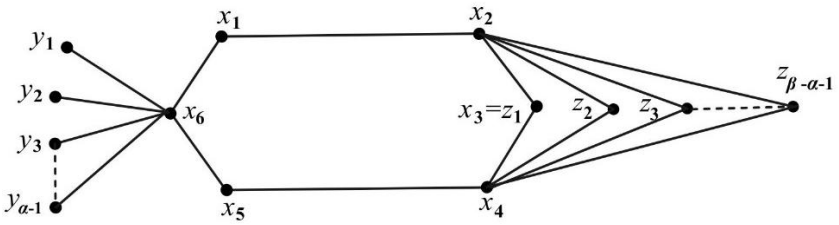

Figure 2: G

Now, consider $S' = \{z_1, z_2, \ldots, z_{\beta-\alpha-1}\}$, the set of all vertices finally added to C_6 to obtain G. It is easy to verify that $S^* = S' \cup (S_1 \cup S_2)$ is the outer independent square free detour set of G and so $dn_{\square f}^{oi}(G) = \beta$.

4. Conclusion

In this paper, we determined the outer independent square free detour number of some standard graphs and cycle related graphs. The relationship between the square free detour number and the outer independent square free detour number has been established. Further investigation is open for any other class of graphs.

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Totally magic d-lucky number of graphs

N. Mohamed Rilwan [*]
A. Nilofer [†]

Abstract

In this paper we introduce a new labeling named as, totally magic d-lucky labeling, find the totally magic d-lucky number of some standard graphs like wheel, cycle, bigraph etc. and find the totally magic d-lucky number of some zero divisor graphs. A totally magic d-lucky labeling $t: V \to \{1,2,\dots,p\}$ of a graph G = (V, E) is a labeling of vertices and label the graph's edges using the total label of its incident vertices in such a way that for any two different incident vertices u and v, their colors $d_t(u) = \sum_{v \in N(u)} t(v) +$ dgu, dtv= u∈Nvtu+ dg v are distinct and for any different edges in a graph, their weights $t(u) + t(v) + t(uv) \equiv 0 \pmod 2$ are same Where $d_g(u)$ represents the degree of u in a graph and $N(u)$ represents the open neighbourhood of u in a graph.

Keywords: Totally magic d-lucky labeling, totally magic d-lucky number, zero divisor graphs.

2010 AMS subject classification: 05C78[‡]

[*]Assistant Professor, Department of Mathematics, Sadakathullah Appa College (Autonomus), Rahmath nagar, Tirnelveli-627011, India; e-mail- rilwan2020@gmail.com.
[†]Research scholar, register number 18221192092017, Research center, Department of Mathematics, Sadakathullah Appa College (Autonomus), Rahmath nagar, Tirunelveli 627011, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli 627012, India; e-mail- lourdhunilofer@gmail.com
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1. Introduction

In [2], The idea of lucky labeling was first proposed by Czerwinski, Grytczuk, and Zelezny. In [1], The idea of "d-lucky labeling" was developed by Indira Rajasingh, D. Ahima Emilet, and D. Azhubha Jemilet. [1] Let l: V(G) → N is a vertex labeling. If for each pair of incident vertices of u and v, c(u) ≠ c(v) holds where c(u) = $d_g(u) + \sum_{v\in N(u)} l(v)$, c(v) = $d_g(u) + \sum_{u\in N(v)} l(u)$, $d_g(u)$ represents the degree of u and N(u) represents the open neighbourhood of the vertex u in a graph, then the labeling l is a d-lucky labeling. A graph's d-lucky number is the smallest value of labeling required to label the graph. Motivated by this labeling, we introduce Totally magic d-lucky labeling. A graph's total labeling is a mapping from the union of the vertex set and the edge set to positive integers. If the sum of the edge label and the label of the edge's end points has the same constant, the total labeling is said to be totally magic labeling. In [5,] we learned about the totally magic labeling. A totally magic d-lucky labeling $t: V \to \{1,2,\dots,p\}$ of a graph G = (V, E) is a labeling of vertices and label the edges of the graph by the sum of the labels of its incident vertices in such a way that for any two different incident vertices u and v, their colors $d_t(u) = \sum_{v\in N(u)} t(v) + d_g(u)$, $d_t(v) = \sum_{u\in N(v)} t(u) + d_g(v)$ are distinct and for any different edges in a graph, their weights $t(u) + t(v) + t(uv) \equiv 0 \pmod 2$ are same Where $d_g(u)$ represents the degree of u in a graph and $N(u)$ represents the open neighborhood of u in a graph.

2. Totally magic d-lucky labeling

In this section we introduce a new labeling named as the totally magic d-lucky labeling and apply it on the cycle, path, complete graph, bigraph, and wheel.

Definition 2.1 Define t: V(G) → {1, 2, …, p}and label the edges of E(G) as the label of the edge's incident vertices added together. The labeling is said to be Totally magic d-lucky labeling if $d_t(u) \neq d_t(v)$ and t(u) + t(v) + t(uv) ≡ 0 (mod 2) where u, v ∈ V(G) $d_t(u) = \sum_{v\in N(u)} t(v) + d_G(u)$ and $d_t(v) = \sum_{u\in N(v)} t(u) + d_G(v)$. The totally magic d-lucky number of G, tdln(G) is defined as the lowest value of p for which the graph G has totally magic d-lucky labeling.

Theorem 2.2. For a cycle graph C_n, tdln (C_n) = $\begin{cases} n & \text{if } n \text{ is odd} \\ 2 & \text{otherwise} \end{cases}$

Proof. Let G be the cycle graph.
Let V(G) = $\{v_i: 1\leq i \leq n\}$ and E(G) = $\{v_i v_{i+1}: 1\leq i \leq n-1\} \cup \{v_n v_1\}$
Case. (i).
When $n \equiv 1 \pmod 2$.
Let t: $V(C_n) \to \{1,2,\dots,p\}$ defined by for $1\leq i\leq n-1$
$t(v_i)=i$ for $1\leq i \leq n$
Then the induced edge labelling is,
$t(v_i v_{i+1}) = 2i+1$ for $1\leq i \leq n-1$

$t(v_nv_1) = n+1$
We observe that,
$d_t(v_1) = n+4$
$d_t(v_i) = 2i+2$, for $2 \le i \le n-1$
$d_t(v_n) = n+2$
$d_t(v_i) \neq d_t(v_{i+1})$ and $d_t(v_1) \neq d_t(v_n)$
$t(v_i)+ t(v_{i+1}) + t(v_iv_{i+1})=4i+2 \equiv 0 \pmod 2$ for $2 \le i \le n-1$ and
$t(v_1)+ t(v_n)+t(v_1vn) = 2n+2 \equiv 0 \pmod 2$
case (ii).
When $n \equiv 0 \pmod 2$
Define $t: V(C_n) \rightarrow \{1,2, \ldots, p\}$as follows,
for $1 \le i \le n$

$$t(v_i) = \begin{cases} 1, & i \equiv 1 \pmod 2 \\ 2, & i \equiv 0 \pmod 2 \end{cases}$$

Then the induced edge labelling is,
$t(v_iv_{i+1}) = 3$ for $2 \le i \le n-1$
$t(v_nv_1) = 3$
we observe that,
$d_t(v_i) = 6$ if i is odd
$d_t(v_i) = 4$ if i is even
$d_t(v_i) \neq d_t(v_{i+1})$
and $t(v_i)+ t(v_{i+1}) + t(v_iv_{i+1})= 6 \equiv 0 \pmod 2$
It can be easily verified that weights of the incident vertices are pair wise distinct and have the common totally magic d-lucky constant for its edges. Thus, the totally magic d-lucky number of cycle graph is 2. ■

Theorem 2.3 Every path P_n has $tdln(P_n)=2$
Proof Let P_n be the path graph, $V(P_n)=\{v_i: 1 \le i \le n\}$ and
$E(P_n) = \{v_iv_{i+1}: \text{for } 1 \le i \le n\}$
Define $t: V(P_n) \rightarrow \{1,2,\ldots,p\}$as follows:

$$t(v_i) = \begin{cases} 1, & i \equiv 1 \pmod 2 \\ 2, & i \equiv 0 \pmod 2 \end{cases}$$

Then the induced edge labelling is,
$t(v_i\, v_{i+1}) =3$ for all edges in P_n
we observe that,
when n is even,
$d_t(v_1) = 3$
$d_t(v_i) = 4$ if $i \equiv 0 \pmod 2$
$d_t(v_i) = 6$ if $i \equiv 1 \pmod 2$
$d_t(v_n) = 2$
$d_t(v_i) \neq d_t(v_{i+1})$ for all i
and $t(v_i) + t(v_{i+1}) + t(v_iv_{i+1}) = 6 \equiv 0 \pmod 2$.
When n is odd
$d_t(v_1) = 3 = d_t(v_n)$

$d_t(v_i) = 4$ if $i \equiv 0(\text{mod } 2)$
$d_t(v_i) = 6$ if $i \equiv 1 (\text{mod } 2)$
$d_t(v_i) \neq d_t(v_{i+1})$ for all i
and $t(v_i) + t(v_{i+1}) + t(v_iv_{i+1}) = 6 \equiv 0 (\text{mod } 2)$.
Hence tdln $(P_n) = 2$. ■

Theorem 2.4 For a complete graph k_n, tdln $(k_n) = n$
Proof In complete graph K_n, Each and every pair of vertices are close together.
Define $t:V(K_n) \rightarrow \{1,2,\ldots,p\}$ as follows:
$t(v_i) = i : 1 \leq i \leq n$
Then the induced edge labelling is,
$t(v_iv_j) = i+j$ for all edges in P_n
we observe that,
for $1 \leq i \leq n$
$d_t(v_i) = \frac{(n^2+3n-2i-2)}{2}$
$d_t(v_i) \neq d_t(v_j)$
and $t(v_i) + t(v_j) + t(v_iv_j) = 2(i+j) \equiv 0(\text{mod } 2)$.
tdln $(K_n) = n$. It is simple to confirm that the colors of the pair wise incident vertices are distinct and that the sum of the labels for each edge and the incident vertices of its edges is even. ■

Theorem 2.5 For a bigraph $K_{m,n}$, tdln $(K_{m,n}) = 1$.
Proof A bigraph's vertices can be divided into two separate subsets, V_1 and V_2, and each edge of the bigraph connects a point on each subset. $K_{m,n}$ indicates a bigraph.
Let $V(K_{m,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ and $E(K_{m,n}) = \{u_iv_j : u_i \in V_1\ v_j \in V_2, 1 \leq i \leq m, 1 \leq j \leq n\}$.
Define $t: V(K_{m,n}) \rightarrow \{1, 2, \ldots, p\}$ as follows: $t(u_i) = 1$, $t(v_j) = 1$
Then the induced edge labeling is
$t(u_iv_j) = 2$ for all edges in $K_{m,n}$
We observe that,
$d_t(u_i) = 2n$,
$d_t(v_j) = 2m$,
$t(u_i) + t(v_j) + t(u_iv_j) = 2(i+j) \equiv 0 (\text{mod } 2)$.
It is obvious that all incident vertices have pair wise different colors and that all of the edges in the $K_{m,n}$ graph have the same totally magic d-lucky constant.
Hence tdln $(K_{m,n}) = 1$. ■

Theorem 2.6 For a wheel graph W_n, tdln $(W_n) = \begin{cases} n-1 & \text{if } n \text{ is odd} \\ n & \text{otherwise} \end{cases}$.
Proof A wheel graph is obtained by joining a vertex to all the vertices of a cycle graph. It is denoted by W_n for n>3, where n is the number of vertices in the graph.
Let $V(W_n) = \{u_i : 1 \leq i \leq n\}$ and $E(W_n) = \{u_1u_i : 2 \leq i \leq n\} \cup \{u_iu_{i+1} : 2 \leq i \leq n\}$
Case(i) When $n \equiv 1(\text{mod } 2)$

Define a labeling $t:V(W_n) \rightarrow \{1,2,\ldots,p\}$as follows:
$t(u_1) = 1$, $t(u_i) = i-1$ for $2 \leq i \leq n$
Then the induced edge labelling is
$t(u_1u_i) = i$ for $2 \leq i \leq n$
We observe that,
$d_t(u_1) = \frac{n^2+n-2}{2}$
$d_t(u_2) = n+5$
$d_t(u_i) = 2i+2$ for $3 \leq i \leq n-1$
$d_t(u_n) = n+3$
$d_t(u_1) \neq d_t(u_i)$ for $2 \leq i \leq n$
$d_t(u_i) \neq d_t(u_{i+1})$ for $2 \leq i \leq n-1$
$d_t(u_n) \neq d_t(u_1)$
$t(u_1)+t(u_i)+t(u_1u_i) = 2i \equiv 0 \pmod 2$;
$t(u_i)+t(u_{i+1})+t(u_iu_{i+1}) = 4i-2 \equiv 0 \pmod 2$, for $2 \leq i \leq n-1$;
$t(u_1)+t(u_n)+t(u_n) = 2n \equiv 0 \pmod 2$
Hence tdln $(W_n) = n-1$
Case (ii) When $n \equiv 0 \pmod 2$
Define $t:V(W_n) \rightarrow \{1,2,\ldots,n\}$ as follows:
$t(u_i) =$ i for $1 \leq i \leq n$
Then the induced edge labelling is
$t(u_1u_i) = 1 + i$ for $2 \leq i \leq n$;
$t(u_iu_{i+1}) = 2i+1$ for $2 \leq i \leq n-1$
$t(u_nu_2)=n+2$
we observe that,
$d_t(u_1) = \frac{n^2+3n-4}{2}$
$d_t(u_2) = n+7$
$d_t(u_i) = 2i+4$ for $3 \leq i \leq n-1$
$d_t(u_n) = n+5$
$d_t(u_1) \neq d_t(u_i)$ for $2 \leq i \leq n$
$d_t(u_i) \neq d_t(u_{i+1})$ for $2 \leq i \leq n$
$d_t(u_n) \neq d_t(u_1)$
for $2 \leq i \leq n$
$t(u_1) + t(u_{i+1}) + t(u_1u_{i+1}) = 2i+2 \equiv 0 \pmod 2$,
$t(u_i) + t(u_{i+1}) + t(u_iu_{i+1}) = 4i+2 \equiv 0 \pmod 2$,
$t(u_2) + t(u_n) + t(u_2u_n) = 2n+4 \equiv 0 \pmod 2$.
Hence tdln $(W_n) = n$
It is simple to confirm that all incident vertices' colors are pairwise different and preserve the totally magic d-lucky constant for all of the graph's edges W_n. ■

3. Totally magic d-lucky number of some zero divisor graphs

In this part, the totally magic d-lucky number of some zero divisor graphs is examined.

Theorem. 3.1 For $R = Z_k$, $k = mn$, $m=2,3$ and $n>3$ be a prime number, $tdln(\Gamma(R)) = 1$
Proof Consider $G_0=\Gamma(R)$ where $R=Z_k$, $k=mn$
Case(i)
when $m=2$,$n >3$be a prime. By the definition of zero divisor graph,
Assume $V(G_0) = \{2, 4... 2 (n-1), n\} = \{v_i: 1 \leq i\leq n\}$, $E(G_0) = \{v_i v_n: v_i \in V (\Gamma (R))-\{v_n\}\}$.
We have $d_g(v_i) = m-1$, $d_g(v_n) = n-1$, $1 \leq i \leq 2(n-1)$
Define $t:V(G_0)\rightarrow\{1,2,\ldots,p\}$ as follows:
$t(v_i) = 1$for $1\leq i\leq n$
Then the induced edge labeling is,
$t(e) = 2$ for all edges e in G_0
we observe that,
$d_t(v_i) = 2m-2$,
$d_t(v_n) = 2n-2$
$d_t(v_i) \neq d_t(v_n)$ for $1 \leq i \leq n-1$ and
$t(v_i)+ t(v_n) + t(v_iv_n) = 4 \equiv 0 \pmod 2$
Hence $tdln(G_0) = 1$
Case(ii)
when $m=3$, $n>3$ be a prime.
In this graph, we have $V(G_0) = V_1(G_0) \cup V_2(G_0)$ where $V_1(G_0) = \{n,2n\}$,
$V_2(G_0) = \{3i: 1\leq i\leq n-1\}$ and $E(G_0) = \{uv: u\in V_1(G_0), v\in V_2(G_0)\}$.
Hence $d_g(u) = n-1$ for all $u\in V_1(G_0)$, $d_g(v) = m-1$, for all $v\in V_2(G_0)$ and $|E(G_0)| = 2n-2$
Define a labeling $t: V(G_0)\rightarrow\{1,2,\ldots,p\}$ as follows:
$t(u) = 1$ for all $u\in V_1(G_0)$
$t(v) = 1$ for all $v\in V_2(G_0)$
Then the induced edge labeling is,
$t(uv) = 2$ for all $uv \in E(G_0)$
We observe that,
$d_t(u) = 2n-2$,
$d_t(v) = 2m-2$
$d_t(u) \neq d_t(v)$ for all $u \in V_1$, for all $v \in V_2$ and
$t(u) + t(v) + t(uv_i) = 4 \equiv 0 \pmod 2$ for all $uv \in E(G_0)$
Hence $tdln(G_0) = 1$. ■

Theorem 3.2 For $R = Z_k$, $k =m^2 n$, $n >3$ be a prime number,
$tdln(\Gamma(R)) = \begin{cases}1 \text{ when } m = 2\\2 \text{ when } m = 3\end{cases}$
Proof Assume $G_0 = \Gamma(R)$

Case(i) When m=2, In this case(G_0) has partitioned into two sets $V_1(G_0)$,$V_2(G_0)$. $V_1(G_0)$ contains the multiples of n in Z_k, $V_2(G_0)$ contains the multiples of m excluding 2n in Z_k.
Let $V_1(G_0) = \{r_1, r_2, r_3\}$ and $V_2(G_0) = \{s_1, s_2, \dots, s_{n-1}, s_{n+1}, \dots, s_{2n-1}\}$
$|V(G_0)| = 2n+1$
$E(G_0) = \{r_i s_j : i \in \{1,3\}, s_j = \{4,8, \dots, (4m-4)\}\} \cup \{r_2 s_j : \text{for all } s_j \in V_2(G_0)\}$.
$|E(G_0)| = 4n-4$
Hence $d_g(r_i) = n-1$, for $i \in \{1,3\}$;
$d_g(r_2) = 2n-2$;
$d_g(s_j) = m+1$, $s_j \in \{4,8, \dots, 4n-4\}$;
$d_g(s_j) = m-1$, $s_j \in V_2(G_0) - \{4, 8, \dots, 4n-4\}$
Define a labeling $t: V(G_0) \rightarrow \{1, 2, \dots, p\}$ as follows:
$t(r_i) = 1$ for $1 \leq i \leq 3$;
$t(s_j) = 1$ for $1 \leq j \leq 2n-2$;
Then the induced edge labelling is,
$t(r_i s_j) = 2$ for all $r_i s_j \in E(G_0)$
We observe that,
$d_t(r_i) = 2n-2$, $i \in \{1,3\}$,
$d_t(r_2) = 4n-4$,
$d_t(s_j) = 2m-2$, $s_j \in V(\Gamma(R))-\{4,8,\dots,4n-4\}$,
$d_t(s_j) = 2m+2$, $s_j \in \{4,8,\dots,4n-4\}$
$d_t(r_i) \neq d_t(s_j)$ for all $r_i \in V_1(G_0)$, $s_j \in V_2(G_0)$ and
$t(r_i) + t(s_j) + t(r_i s_j) = 4 \equiv 0 \pmod 2$ for all edges in G_0
Hence $tdln(G_0)=1$.
Case(ii) when m = 3,
In this case, the vertex set of G_0 partitioned into two sets V_1 and V_2.
Where $V_1 = \{n, 2n, 3n, 4n, 5n, 6n, 7n, 8n\} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and
$V_2 = \{3, 6, 9, \dots, 9n-3\} - \{n, 2n\} = \{v_1, v_2, v_3, \dots, v_{3n-1}\}$.
$E(G_0) = \{u_i v_i : \text{for all } u_i \in V_1 , v_i \in \{9,18,27,\dots,9(n-1)\} \cup \{u_i v_i : u_i \in \{3n,6n\}\ v_i \in V_2\} \cup \{u_3,u_6\}$.
Hence $d_g(u_i) = n-1$ for all $u_i \in V_1-\{3n,6n\}$;
$d_g(u_i) = 3n-2$, $i=\{3,6\}$;
$d_g(v_i) = 8$ for all $v_i \in \{9,18,\dots,9(n-1)\}$;
$d_g(v_i) = 2$, $v_i \in V_2-\{9,18,\dots,9(n-1)\}$.
Define the labelling $t: V(G_0) \rightarrow \{1,2,\dots,p\}$ as follows:
$t(u_i) = 1$, for $1 \leq i \leq 8, u_i \in V_1$;
$t(u_6) = 2$, $u_6 \in V_1$;
$t(v_i) = 1$, for all $v_i \in V_2$.
Then the induced edge labellings are,
$t(u_i v_i) = 2$ for all $u_i v_i \in E(G_0)$
$t(u_3 u_6) = 3$
$t(u_6 v_i) = 3$ for all $v_i \in V_2$
We observe that,
$d_t(u_i) = 2n-2$,

$d_t(u_3)$ = 6n-3,
$d_t(u_6)$ = 6n-4,
$d_t(v_i)$ = 5 for all $v_i \in V_2(G_0)$-{9,8,…,9(n-1)}
$d_t(v_i)$ = 17, $v_i \in$ {9,8,…,9(n-1)}
$d_t(u_i) \neq d_t(v_i)$,
$d_t(u_3) \neq d_t(u_6)$ and
$t(u_i)+t(v_i)+t(u_iv_i) = 4 \equiv 0 \pmod 2$ for all edges in G_0
$t(u_3) + t(u_6) + t(u_3u_6) = 6 \equiv 0 \pmod 2$
$t(u_6) + t(v_i) + t(u_6v_i) = 3$ for all $v_i \in V_2$
It can be easily verified that weights of all the incident vertices are distinct and all the edges of the graph have common totally magic d-lucky constant.
Hence tdln (G_0) = 2. ■

Theorem 3.3 Let R = $\prod_{i=1}^{k} Z_{m_i{}^{n_i}}$ be a commutative ring with unity. For the zero-divisor graph Γ(R), tdln(Γ(R)) = M-1 where M = (m_1, m_2, m_3, …, m_k), m_i's are distinct prime numbers, n_i's are positive integers.
Proof Consider G_0 = Γ(R) be a zero-divisor graph of commutative ring R = $\prod_{i=1}^{k} Z_{m_i{}^{n_i}}$ where m_i's are prime numbers and n_i's are positive integers.
The vertex set of G_0 consists of different blocks**,**
$V(G_0) = \cup B_{x_1,x_2,\ldots,x_k}$
where $(x_1,x_2, \ldots, x_k) \neq (0, 0\ldots, 0)$ and $(x_1,x_2, \ldots, x_k) \neq (n_1, n_2, \ldots, n_k)$.
$B_{x_1,x_2,\ldots,x_k}$ = {$(u_1,u_2, \ldots, u_k)$: u_i=0 if $x_i = n_i$ and $m_i{}^{x_i} \mid u_i$ and $m_i{}^{x_i+1} \nmid u_i$
if $x_i \in$ {0,1,2,…,n_i-1}}
All the vertices in $B_{x_1,x_2,\ldots,x_k}$ are adjacent to all the vertices in $B_{y_1,y_2,\ldots,y_k}$ if $x_i + y_i \geq n_i$ for all i= 1,2,…,k.
The vertices in $B_{x_1,x_2,\ldots,x_k}$ form a clique in G_0 if $2x_i \geq n_i$ for all i= 1,2,..,k
Hence we have, for each u ∈ $B_{x_1,x_2,\ldots,x_k}$,
$d_g(u) = -2 + \prod_{i=1}^{k} m_i{}^{x_i}$ if $B_{x_1,x_2,\ldots,x_k}$ is clique ;
$d_g(u) =$ -1 + $\prod_{i=1}^{k} m_i{}^{x_i}$ if $B_{x_1,x_2,\ldots,x_k}$ is not a clique.
Define a labeling t: $V(G_0) \rightarrow \{1, 2, \ldots, p\}$ as follows,
Label the vertices of the block as 1 if the block is not form a clique. If the block is form a clique, label the vertices of clique u_j as $1 \leq j \leq (|B_{x_1,x_2,\ldots,x_k}| = q)$,
where $|B_{x_1,x_2,\ldots,x_k}| = \prod_{i=1}^{k} \varphi(m_i{}^{n_i - x_i})$,
Then the induced edge labellings are,
if the block is not form a clique,
t(e) = 2 for all edge e in this block
if the block is form a clique,
$t(u_ju_{j+1}) = 2j+1$ for all $1 \leq j \leq q-1$,
$t(u_qu_1) = q+1$

Let T = Max ($|B_{x_1,x_2,\ldots,x_k}|$) if $B_{x_1,x_2,\ldots,x_k}$ form a clique
We observe that, for each u∈ $B_{x_1,x_2,\ldots,x_k}$,
we have, ,

$$d_t(u) = \begin{cases} -1 + \prod_{i=1}^{k} m_i^{x_i} + \sum_{v\in N(u)} t(v) & \text{if } B_{x_1,x_2,\ldots,x_k} \text{ is not a clique} \\ -2 + \prod_{i=1}^{k} m_i^{x_i} + \sum_{v\in N(u)} t(v) & \text{if } B_{x_1,x_2,\ldots,x_k} \text{ forms a clique} \end{cases}$$

$t(u)+t(v)+t(uv) \equiv 0 \pmod 2$ for all $uv \in E(G_0)$
Hence $tdln(G_0) = T = M-1$ where $M=Max(m_1, m_2, \ldots, m_k)$.
It can be easily verified that colors of all the incident vertices are pairwise distinct and have the common constant for all the edges of the our given graph. ■

4. Conclusions

In this paper, we introduced a new labeling, totally magic d-lucky labeling, found the totally magic d-lucky number of some standard graphs and some zero divisor graphs. In future, we use this labeling in some other graphs**.**

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The Outer Connected Detour Monophonic Number of a Graph

N.E. Johnwin Beaula[1]
S. Joseph Robin[2]

Abstract

For a connected graph $G = (V, E)$ of order $n \geq 2$, a set $M \subseteq V$ is called a *monophonic set* of Gif every vertex of $G$is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum cardinality of its monophonic sets. If $M = V$ or the subgraph $G[V - M]$is connected, then a detour monophonic set M of a connected graph G is said to be an *outer connected detour monophonic set*of G.The *outer connecteddetourmonophonic number* of G, indicated by the symbol $ocd_m(G)$, is the minimum cardinality of an outer connected detour monophonic set of G. The outer connected detour monophonic number of some standard graphs are determined. It is shown that for positive integers r_m, $d_m$and $l \geq 2$ with $r_m < d_m \leq 2r_m$,there exists a connected graph $G$with$rad_mG = r_m$, $diam_mG = d_m$and $ocd_m(G) = l$. Also, it is shown that for every pair of integers $a$and b with $2 \leq a \leq b$, there exists a connected graph $G$with$dm(G) = a$ and$ocd_m(G) = b$.

Keywords: chord, monophonic path, monophonic number, detour monophonic path, detour monophonic number, outer connected detour monophonic number.

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[1] Register Number.20123162092018, Research Scholar. Scott Christian College (Autonomous), Nagercoil – 629003, India. beaulajohnwin@gmail.com
[2] Department of Mathematics, Scott Christian College (Autonomous), Nagercoil – 629003, India prof.robinscc@gmail.com Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India
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1. Introduction

A finite, undirected connected graph with no loops or many edges is referred to as a graph $G = (V, E)$. By n and m, respectively, we indicate the order and size of G. We refer to [1] for the fundamental terms used in graph theory. If uv is an edge of G, then two vertices u and v are said to be adjacent. If two edges of G share a vertex, they are said to to be adjacent. Let $S \subset V$ be any subset of vertices of G. Then the graph with S as its vertex set and all of its edges in E having both of their end points in S is the *induced subgraph* $G[S]$. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete.

The length of the shortest path in a connected graph G is equal to the distance $\mathrm{d(u, v)}$ between two vertices u and v. An $\mathrm{u - v}$geodesic is a $\mathrm{u - v}$ path with length $\mathrm{d(u, v)}$. An edge that connects two non-adjacent vertices of a path P is called the chord of P . A chordless$\mathrm{u - v}$ path is referred to as a *monophonic path*. The *monophonic distance* $d_m(u, v)$for two vertices u and v in a connected graph G is the length of a longest $u - v$ monophonic path in G. An $\mathrm{u - v}$detour monophonic path is one that has a length of $\mathrm{d_m(u, v)}$.The monophonic eccentricity of a vertex v, denoted by $\mathrm{e_m(v)}$ is the monophonic distance between v and a vertex farthest from v. The monophonic radius, $rad_m(G)$, and the monophonic diameter, $diam_m(G)$ are the vertices respective minimum and maximum monophonic eccentricities.

The closed interval $J_{dm}[u, v]$ for two vertices u and v is consists of all the vertices along an $u - v$ detour monophonic path, including the vertices u and v. If $v \in E$, then$J_{dm}[u, v] = \{u, v\}$. For a set M of vertices, let $J_{dm}[M] = \cup_{u,v \in M} J[u, v]$. Then certainly $M \subseteq J_{dm}[M]$. If $J_{dm}[M] = V$,a set $M \subseteq V(G)$ is referred to as a *detour monophonic set* of G.The *detour monophonic* number $dm(G)$ of G is the minimum order of its detour monophonic sets.Any detour monophonic set of order $d_m(G)$ is referred to as an d_m-set of G. In [2-4], these concepts were investigated. The following theorem is used in sequel.

Theorem 1.1. [4] Each extreme vertex of a connected graph G belongs to everydetour monophonic set of G.

2. The Outer Connected Detour Monophonic Number of a Graph

Definition 2.1. If $M = V$ or the subgraph $G[V - M]$is connected, then a detour monophonic set M of a connected graph G is said to be an *outer connected detour monophonic set*of G.The *outer connecteddetourmonophonic number* of G, indicated by the symbol $ocd_m(G)$, is the minimum cardinality of an outer connected detour monophonic set of G. The ocd_m*-set* of G is a minimum cardinality of an outer connected detour monophonic setof G.

Example 2.2. $M = \{v_2, v_4\}$ is a dm-set of the graph G in Figure 2.1 such that $dm(G) = 2$. M is not an outer connected detour monophonic set of G because $G[V - M]$ is not connected, and as a result, $ocd_m(G) \geq 3$. Now since $M_1 = \{v_2, v_3, v_4\}$, is a ocd_m-set of G, and
$ocd_m(G) = 3$ as a result.

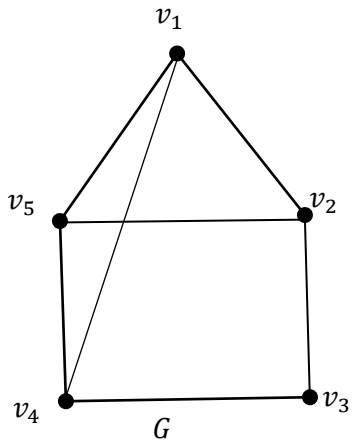

Figure 2.1

Observation 2.3.
(i) Each extreme vertex of a connected graph G belongs to every outer connected detour monophonic set of G.
(ii) No cut vertex of G belongs to any ocd_m-set of G.
(iii)For any connected graph G of order$n$, $2 \leq dm(G) \leq ocd_m(G) \leq n$.

Theorem 2.4. $ocd_m(G) = n$, for the complete graph $G = K_n$ $(n \geq 2)$.
Proof. The vertex set of $K_n$is the unique outer connected detour monophonic set of K_n since every vertex of the complete graph $K_n(n \geq 2)$is the extreme vertex. Therefore, $ocd_m(G) = n$. ■

Theorem 2.5. $ocd_m(T) = k$, for any tree T with k end vertices.
Proof. Let M represent the collection of $T's$ end vertices. According to Observation 2.3(i) and (ii), $ocd_m(T) \geq |M|$. M is the unique outer connected detour monophonic set of T,since the subgraph $G[V - M]$ is connected. Consequently, $ocd_m(T) = |M| = k$.■

Corollary2.6. $ocd_m(P_n) = 2$ for the non-trivial path P_n $(n \geq 3)$.

Corollary2.7. $ocd_m(K_{1,n-1}) = n - 1$ for star $K_{1,n-1}(n \geq 3)$.

Theorem 2.8. $ocd_m(G) = 3$, for the cycle $G = C_n(n \geq 4)$.

Proof. Set the cycle C_n to be $v_1, v_2, \dots, v_n, v_1$. Then, $M = \{v_1, v_2, v_3\}$ is a $G's$ outer connected detour monophonic set, resulting in $ocd_m(G) \leq 3$. We establish$ocd_m(G) = 3$. Assume that $ocd_m(G)$= 2. Then $G'socd_m$-set is $M_1 = \{x, y\}$. It is obvious that x and y are not adjacent. A contradiction results since $G[V - M_1]$is not connected and M_1 is not a $G's\ ocd_m -$ set. ,Consequently, $ocd_m(G) = 3$. ■

Theorem 2.9. $ocd_m(G) = \begin{cases} s, & if\ r = 1, s \geq 2 \\ 3, & if\ r = s = 2 \\ 4, & if\ 2 < r \leq s \end{cases}$for the complete bipartite graph G= $K_{r,s}$

Proof. $G = K_{r,s}$is a tree with s end vertices when $r = 1$ and $s \geq 2$.Therefore,$ocd_m(K_{1,s}) = s$ as per Corollary2.7.$G = K_{2,2}$is the cycleC_4 when $r = s = 2$, Thus, according to Theorem 2.8, $ocd_m(K_{2,2}) = 3$. Let $2 < r \leq s$.Let $X = \{x_1, x_2, \dots, x_m\}$,$Y = \{y_1, y_2, \dots, y_n\}$ be the bipartitions of G. Let $M = \{x_i, x_j, y_k, y_l\}$, where$i \neq j, k \neq l$.Then M is adetour monophonic set of G. M is anouter connected detour monophonic set of G because the subgraph$G[V - M]$is connected, and as a result, $ocd_m(G) \leq 4$. We demonstrate that $ocd_m(G) = 4$. Let's assume that $ocd_m(G) \leq 3$.Then $|M| \leq 3$ and there exists a$ocd_m(G)$-set M. If $M \subseteq X$ or $M \subseteq Y$ then $G[V - M]$ is not connected. Consequently, $M \subset XUY$.Which suggests M is not aouter connected detour monophonic set of G, which is in contrast with the statement made earlier. Thus$ocd_m(G) = 4$. ■

Theorem 2.10 $ocd_m(G) = \begin{cases} 2\ if\ n = 5 \\ 3\ if\ n \geq 6 \end{cases}$for the wheel $G = K_1 + C_{n-1}$ $(n \geq 5)$.

Proof. Let's say that $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \dots\dots\dots, v_{n-1}\}$. $M = \{v_1, v_3\}$ is an $ocd_m - set$ of G for $n = 5$. Therefore, $ocd_m(G) = 2$. So, let $n \geq 6$. Hence it follows that $ocd_m(G) \geq 3$ Let $M_1 = \{v_1, v_2, v_3\}$. Then M_1 is an outer connected detour monophonic set of G. Consequently, $ocd_m(G) = 3$. ■

Theorem 2.11 $ocd_m(G) = \begin{cases} 2\ if\ n = 4 \\ 3\ if\ n > 4 \end{cases}$,for the graph $G = K_1 + P_{n-1}$.

Proof. Let's say that $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_2, \dots\dots\dots, v_{n-1}\}$. $M_1 = \{v_1, v_3\}$ is a$ocd_m - set\ of\ G$ for n = 4 , and $ocd_m(G) = 2$. So, let $n \geq 5$. Let $M = \{v_1, v_{n-1}\}$ be the extreme vertices of G.By observation 2.3(i)M is a subset of every $ocd_m - set\ of\ G$. Since M is not a outer connected detour monophonic set of G, $ocd_m(G) \geq 3$. Now $M_2 = M \cup \{x\}$ is a $ocd_m - set\ of\ G.so\ that\ ocd_m(G) = 3$. ■

Theorem 2.12. Consider the connected graph G, where $dm(G) = 2$. If deg $(x) \geq 3$ for every $x \in V$, then $ocd_m(G) = 2$.

Proof. Let the detour monophonic set of G be $M\{u, v\}$,$deg(x) \geq 3$ for$x \in V$, so $G[V - M]$is connected. As a result, M is an outer connected detour monophonic set of G so that

$ocd_m(G) = 2$. ■

Theorem 2.13. Suppose G is a connected graph with d$m(G) = 2$. If G has 2 possible outermost vertices $u, v \in V$ that are not relatedand $\Delta[\langle V - \{u, v\}\rangle] = n - 3$.Then$ocd_m(G)$= 2.

Proof. Let u and v represent the outermost vertices of G. Let $M = \{u, v\}$. M is therefore a G detour monophonic set. Given that $\Delta\,[\langle V - \{u, v\}\rangle] = n - 3$, and that u and v are not adjacent outermost vertices of G, $G[V - M]$is connected. As a result, M is a outer connected detour monophonic set of G, and$ocd_m(G) = 2$. ■

Theorem 2.14. Let G be a connected graph of order n that has precisely one vertex that is not a cut vertex and has a degree of n - 1. Then$ocd_m(G) \leq n - 3$.

Proof. Let x represent the non-cut vertex of G at the vertex of degreen– 1.Since $N(x)$is not complete, there exist at least two non-adjacent vertices, say y and z that are both members of $N(x)$ .There are at least two vertices say $x_1$and x_2 because $N(x)$is the unique vertex of degree $n - 1$, and they are both located on the $y - z$detour monophonic path such that $x_1 \neq x$, $x_2 \neq x$. $M = V(G) - \{x, x_1, x_2\}$ is a detour monophonic set of G. M is an outer connected detour monophonic set of G since the subgraph $G[V–M]$is connected, which causes$ocd_m(G) \leq n - 3$. ■

Theorem 2.15. Let G be an order $n \geq 3$. $ocd_m(G) \leq n - 1$ if G contains a cut vertex of degree n – 1.

Proof. Let M be a minimum outer connected detour monophonic set and v be the cut vertex of degree $n - 1$ in G. Observation 2.3(ii) says that $v \notin M$. It is obvious that$ocd_m(G) \leq n - 1$, $M = V(G) - \{v\}$ is a detour monophonic set of G. M is an outer connected detour monophonic set of G Since v is a universal cut vertex of G, the subgraph $G\langle V - M\rangle$is connected. As a result, M is an outer connected detour monophonic set of G. which causes $ocd_m(G) \leq n - 1$. ■

Theorem 2.16 There exists a connected graph G with $rad_m G = r_m, dia_m G = d_m$and $ocd_m(G) = l$ for positive integers $r_m, d_m$and $l \geq 2$ with $r_m < d_m \leq 2r_m$.

Proof. We make the convenient assumptions that $r_m = r$ and $d_m = d$. Let $G = K_{1,l}$ when $r = 1$. Theorem 2.5 states that $ocd_m(G) = l$. Let $r_m \geq 2$. Let C_{r+2}: $v_1, v_2, \ldots v_{r+2}$be a cycle of length $r + 2$ and let $P_{d_m - r_m + 1}$: $u_0, u_1, u_2, \ldots, u_{d_m - r_m}$be that cycle. By locating v_1 in $C_{r+2}$and $u_0$in $p_{d_m - r_m + 1}$, we may construct the graph H.The graph shown in Figure 2.2 is then created by joining each of the $w_i$vertices ($1 \leq i \leq l - 3$) to the vertex $u_{d_m - r_m - 1}$ and adding new vertices$w_1, w_2, \ldots, w_{l-3}$to $u_{d_m - r_m - 1}$.So, $rad_m G = r_m$, $dia_m G = d_m$. The set of all $G's$ end vertices, $W = \{w_1, w_2, \ldots, w_{l-3}, u_{d_m - r_m}\}$shall be defined. W is then contained in every detour monophonic detour set of G according to Observation 2.3(i). Since$J_{dm}[M] \neq V, W$ is not an outer connected detour monophonic set of G and so $ocd_m(G) \geq l - 1$. $ocd_m(G) \geq l$ because it is obvious that M is not an outer connected detour

monophonic set of G, where$M = W \cup\{u_0\}$ and$u_0 \notin M$. $ocd_m(G) = l$ because it is obvious that M is an outer connected detour monophonic set of G, where $M = W \cup\{v_2, v_3\}$.

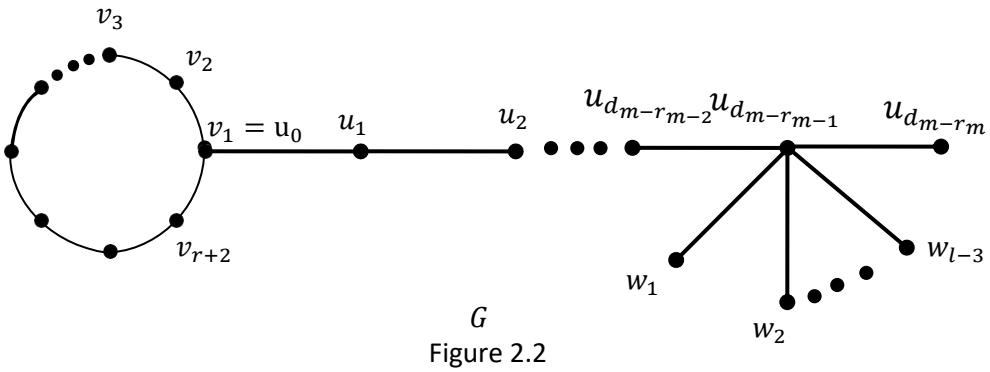

Figure 2.2

Theorem 2.17. There is a connected graph G with $dm(G) = a$ and $ocd_m(G) = b$ for every pair of positive integers a and b such that $2 \leq a \leq b$.

Proof. Let $V(\overline{K_2}) = \{x, y\}$. Let a graph be created by adding additional vertices to $(\overline{K_2})$ as follows: $z_1, z_2, \dots, z_{a-1}, v_1, v_2, \dots, v_{b-a}$, and connecting each z_i $(1 \leq i \leq a - 1)$ with x and y. Graph G is displayed in Figure 2.3. First, we demonstrate that $dm(G) = a$. Assume that $Z = \{z_1, z_2, \dots, z_{a-1}\}$ is the collection of all end vertices of G. According to Theorem1.1, every detour monophonic set of G has Z as a subset. Since it is obvious that Z is not a monophonic detour set of $G, dm(G) \geq a$. Now that $Z\cup\{y\}$ is a monophonic set, $dm(G) = a$.

We then demonstrate that $ocd_m(G) = b. Z \cup \{y\}$is not an outer connected detour monophonic set of G because $G[V - (Z \cup \{y\})]$ is not connected. According to Observation 2.3(i), each outer connected detour monophonic set of G has the vertex z_i $(1 \leq i \leq a - 1)$. Additionally, it is simple to see that any outer connected detour monophonic set of G contains each $v_i(1 \leq i \leq b - a)$,which means that $ocd_m(G) \geq a - 1 + b - a = b - 1$. Let $M = Z \cup \{v_1, v_2, \dots, v_{b-a}\}$. Since M is not an outer connected detour monophonic set of G, then $ocd_m(G) \geq b$. Now that $M \cup \{x\}$ is an outer connected detour monophonic set of G $ocd_m(G) = b$.

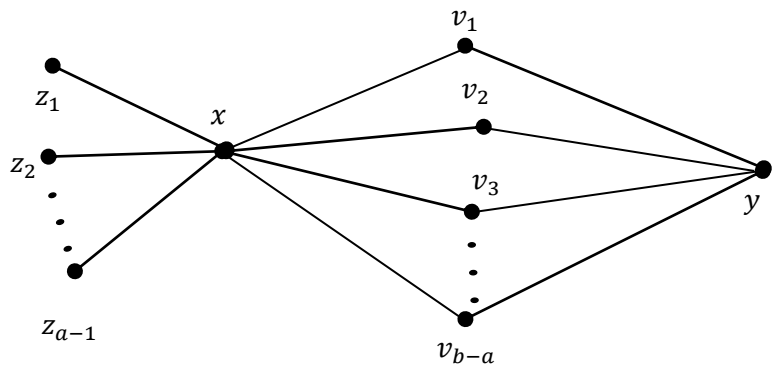

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Figure 2.3

3. Conclusion

This article established a novel detour monophonic distance parameter called the outer connected detour monophonic number of graphs. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

Acknowledgements

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Co-even Geodetic Number of a Graph

T. Jebaraj [*]
Ayarlin Kirupa.M [†]

Abstract

Let $G = (V, E)$ be a graph with vertex set V and edge set E. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. If $I[S] = V(G)$, then S is a geodetic set for G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set. A geodetic set S is called co- even geodetic set if the degree of vertex v is even number for all $v \in V - S$. The cardinality of a smallest co- even geodetic set of G, denoted by $g_{coe}(G)$ is the co- even geodetic number of G. In this paper, we find the co-even geodetic number of certain graphs and complement graphs.

Keywords: geodetic set, co-even geodetic set, co-even geodetic number

2010 AMS subject classification: 05C12.[‡]

[*]Assistant professor**,** Research Department of Mathematics, Malankara Catholic College, Mariagiri, Kaliakkavilai, India; jebaraj.math@gmail.com
[†]Research Scholar, Reg. No.20113082092003, Research Department of Mathematics, Malankara Catholic College, Mariagiri, Kaliakkavilai, India; ayarlin.kirupa19@gmail.com.
[‡]Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India; Received on June 6 th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.922. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors.This paper is published under the CC-BY licence agreement.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The minimum and maximum degree $\delta(G)$ and $\Delta(G)$, respectively. In case where $\Delta(G) = \delta(G)$, G is called a regular graph. The distance $d(x, y)$ is the length of a shortest $x - y$ path in G. It is known that the distance is a metric on the vertex set of G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. For any vertex u of G,the eccentricity of u is $e(u) = max\{d(u, v) : v \in V\}$. A vertex v is an eccentric vertex of u if $e(u) = d(u, v)$. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an extreme vertex of G if the subgraph induced by its neighbors is complete. The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$ and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. In this paper, we study the co-even geodetic number and is denoted by $g_{coe}(G)$ also we discuss the co-even geodetic number of some standard graphs.

2. Co-even geodetic number of a graph

Definition 2.1 A geodetic set $\boldsymbol{S}$ is called co-even geodetic set if the degree of vertex $\boldsymbol{v}$ is even number for all $\boldsymbol{v} \in \boldsymbol{V} - \boldsymbol{S}$. The cardinality of a smallest co-even geodetic set of $\boldsymbol{G}$, denoted by $\boldsymbol{g_{coe}(G)}$ is the co-even geodetic number of $\boldsymbol{G}$.

Example 2.2

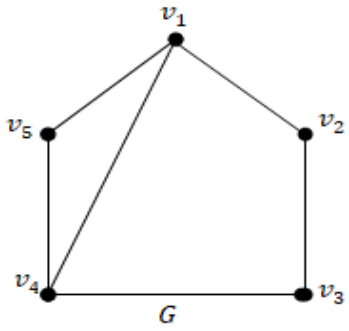

Figure 2.1

In figure 2.1, $S = \{v_1, v_3, v_4, v_5\}$ is a co-even geodetic set. Here, the vertices v_1 and v_4 has odd degree. These two vertices do not make a geodetic set and no 3- element subset of G is a co-even geodetic set. Then it is clear that $g_{coe}(G) = 4$.

Remark In figure 2.1, $S = \{v_1, v_3, v_5\}$ is the minimum geodetic set of G. ie) $g(G) = 3$. Thus, the geodetic number and co-even geodetic number of a graph G can be different.

Proposition 2.3 Let G be a graph and S is a co-even geodetic set . Then,
i) All vertices of odd degrees belong to every co-even geodetic set.
ii) $deg(v) \geq 2$ for all $v \in V - S$.

Proposition 2.4 If G is p- regular graph, then $g_{coe}(G) = \begin{cases} n & if \ \ p \ \ is \ odd \\ g(G) & if \ \ p \ \ is \ \ even \end{cases}$

Theorem 2.5 If G be a graph of order n, then $2 \leq g(G) \leq g_{coe}(G) \leq n$.
Proof: A geodetic set needs atleast two vertices. Therefore, $g(G) \geq 2$. Clearly, every co-even geodetic set is a geodetic set of G, $g(G) \leq g_{coe}(G)$. Also, all the vertices of G is the co-even geodetic set of G.ie) $g_{coe}(G) \leq n$.

Remark 2.6 The bounds of the theorem 2.5 are sharp. The co-even geodetic number of paths P_n with n vertices is 2. In this case, the smallest bounds is obtained. Also, K_n with n vertices have the co-even geodetic number is n. Then the upper bound is obtained.

Theorem 2.7 If G is a non trivial connected graph with $n \geq 2$.If $g_{coe}(G) = 2$ then $g(G) = 2$.
Proof. It is follows from theorem 2.5.

Remark 2.8 The converse part of above theorem is need not be true for all graphs. In Figure 2.2, The minimum geodetic number is 2 and the minimum co-even geodetic number is 3.

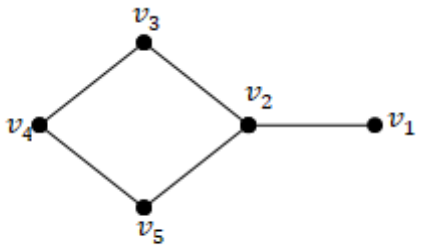

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Figure 2.2

Corollary 2.9 Let G be the non-trivial connected graph, $g(G) = 2$ then $g_{coe}(G) = 2$.
Proof. Case (i) If $G = K_2$
It is easy to see $g(K_2) = 2$ then $g_{coe}(K_2) = 2$.
Case (ii) All the vertices of G should be even degree.
Consider the even Cycle C_{2n}. All vertices have even degree for C_{2n}.We know that $g(C_{2n}) = 2$. Further more, $g_{coe}(C_{2n}) = 2$.

Case (iii) A graph with exactly two odd degree vertices which only belongs to the minimum geodetic set.

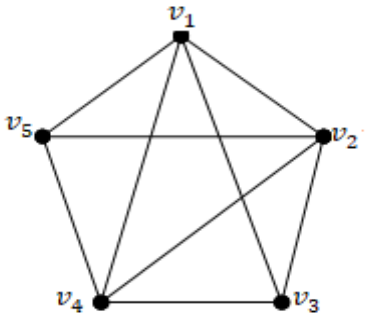

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Figure 2.3

For example, In Figure 2.3, the vertices v_3 and v_5 have odd degree and v_1, v_2, v_4 have even degree. The minimum geodetic number of G is 2. Also, it is easily seen that $g_{coe}(G) = 2$.

Remark All the graphs are not satisfied for the corollary 2.9 except the above three type graphs.

Observation. 2.10 $g_{coe}(C_n) = g(C_n)$, where C_n is a cycle of order n.
Proof. Every cycle is the 2- regular graph .by the proposition 2.4, we get $g_{coe}(C_n) = g(C_n)$.

Theorem 2.11 For the Wheel graph W_n $(n \geq 4)$, then

$$g_{coe}(W_n) = \begin{cases} n-1 & if\ \ n\ \ is\ \ odd \\ n & if\ \ n\ \ is\ \ even \end{cases}$$

Proof. Case (i) n is odd

Let $W_n = K_1 + C_{n-1}$ and u be the vertex of K_1. It is easy to see that the $n-1$ vertices has odd degree except the vertex u. By the proposition 2.3, $n-1$ vertices belong to the co-even geodetic set S. Also, the vertex $u \in V - S$, which has even degree. Hence $|S| = n-1$.

Case (ii) n is even.

Every vertex of W_n has odd degree. By the proposition 2.3, All the vertices of W_n belongs to the co-even geodetic set. Therefore, $g_{coe}(W_n) = n$.

Corollary 2.12 For the wheel graph with $n \geq 4$ then $g_{coe}(W_n) = 2\alpha_0(W_n) - 2$.

Proof. We prove this theorem by two cases.

Case (i) n is even

We have $g_{coe}(W_n) = n$ if n is even and $\alpha_0(W_n) = \frac{n+2}{2}$.

We have $g_{coe}(W_n) = n$.

$g_{coe}(W_n) + 2 = n + 2$. Then $\frac{g_{coe}(W_n)+2}{2} = \frac{n+2}{2}$

$$\frac{g_{coe}(W_n)}{2} + 1 = \alpha_0(W_n)$$

$$g_{coe}(W_n) = 2\alpha_0(W_n) - 2$$

Case (ii) n is odd

Since $g_{coe}(W_n) = n - 1$ if n is odd and $\alpha_0(W_n) = \frac{n+1}{2}$

We have $g_{coe}(W_n) = n - 1$

$$\frac{g_{coe}(W_n) + 1}{2} = \frac{n - 1 + 1}{2}$$

$$\frac{g_{coe}(W_n)}{2} = \frac{n + 1}{2} - 1$$

$$g_{coe}(W_n) = 2\alpha_0(W_n) - 2.$$

Theorem 2.13 If G is the double fan graph $F = P_n + \overline{K_2}$ with $n \geq 5$, then $g_{coe}(G) = 4$.

Proof

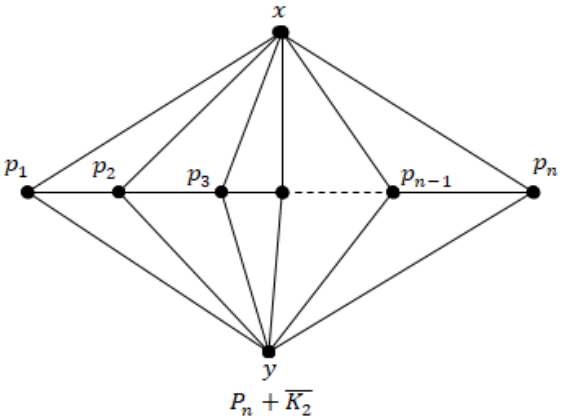

Figure 2.4

Let $p_1, p_2, \ldots, p_n$ be the vertices of path P_n and let x and y be the two vertices of $\overline{K_2}$. All the vertices of path P_n is adjacent to x and y. Now, the double fan graph $F = P_n + \overline{K_2}$ have the $n + 2$ vertices. We prove this theorem by two cases.

Case (i) n is odd

If n is odd then the end vertices of P_n and the vertices of $\overline{K_2}$ have the odd degree. By the proposition 2.3, these four vertices p_1, p_n, x, y belongs to co-even geodetic set. Also all the vertices of F lies on any geodesic of the co-even geodetic set. Thus $g_{coe}(P_n + \overline{K_2}) = 4$.

Case (ii) n is even

If n is even then all the vertices of F is even degree except the vertices p_1 and p_n belongs to co-even geodetic set. All the vertices of F does not lies the $p_1 - p_n$ geodesic. So we chosen the vertices x and y in the co-even geodetic set. Now the set $S = \{p_1, p_n, x, y\}$ is the co-even geodetic set as well as all the vertices of $V - S$ has even degree. Therefore, $g_{coe}(P_n + \overline{K_2}) = 4$.

Corollary 2.14 For the double fan graph $F = P_n + \overline{K_2}$ with $n \geq 5$ then,

$$g_{coe}(P_n + \overline{K_2}) = \begin{cases} 2\alpha_0(P_n + \overline{K_2}) - n + 1 & if\ n\ is\ odd \\ 2\alpha_0(P_n + \overline{K_2}) - n & if\ n\ is\ even \end{cases}$$

Theorem 2.15 For the ladder graph L_n then, $g_{coe}(L_n) = 2n - 2$.

Proof

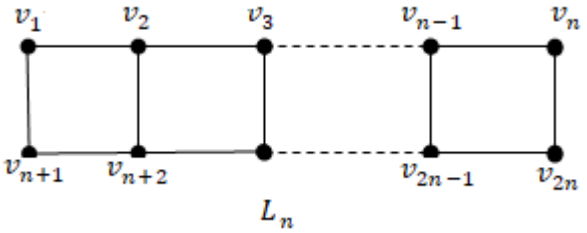

Figure 2.5

The ladder graph $L_n$with $2n$ vertices. The geodetic number of L_n is 2. $S = \{v_1, v_{2n}\}$ or $\{v_n, v_{n+1}\}$ is the minimum geodetic set of L_n, which is not a co-even geodetic set. Because some vertices of $V - S$ has odd degree. Therefore, the odd degree vertices $\{v_2, v_3, \ldots, v_{n-1}, v_{n+2}, \ldots, v_{2n+1}\}$ is belong to the co-even geodetic set of L_n. Therefore, all the vertices of L_n except two vertices make the co-even geodetic set. Hence $g_{coe}(L_n) = 2n - 2$.

Theorem2.16 For the Cone graph $C_m + \overline{K}_n$ then $g_{coe}(C_m + \overline{K}_n) = \begin{cases} n\ if\ n\ is\ even, & m \geq 5 \\ m\ if\ m\ is\ even, & n\ is\ odd \\ m + n\ if\ m\ is\ odd, & n\ is\ odd \end{cases}$

Proof. The Cone graph $C_m + \overline{K}_n$ is adding with cyclic graph C_m and empty graph $\overline{K}_n$. The cone graph has $m + n$ vertices. We prove this theorem by three cases.

Case (i) If n is even

In this case, we prove with two subcases.

Sub Case (i) If n is even, m is odd

For the Cone graph $C_m + \overline{K}_n$, only n vertices have odd degree. By the proposition 2.3, n- vertices belongs to the co-even geodetic set. Now, every vertex belongs to any geodesic of the co-even geodetic set. Hence $g_{coe}(C_m + \overline{K}_n) = n$.

Sub Case (ii) If n is even, m is even

Both the vertices of $C_m + \overline{K}_n$ has even degree. Now, n- vertices forms a co-even geodetic set of $C_m + \overline{K}_n$. Hence $g_{coe}(C_m + \overline{K}_n) = n$.

Case (ii) If m is even and n is odd

Let m is even number of vertices and n is odd number of vertices. Here, $C_m + \overline{K}_n$ has m- even vertices have odd degree and n-odd vertices have even degree. Then it follows from the sub case (i) we get $g_{coe}(C_m + \overline{K}_n) = m$.

Case (iii) If both m and n are odd

For all the vertices of $C_m + \overline{K}_n$ have odd degree. Then it follows from the subcase (i). Thus, we get, $g_{coe}(C_m + \overline{K}_n) = m + n$. Hence proved.

3.Co-even geodetic number of Complement of a graph

Theorem 3.1 If P_n is a path graph with $n \geq 5$,then $g_{coe}(\overline{P_n}) = \begin{cases} 4 & if \ \ n \ \ is \ \ odd \\ n-2 & if \ \ n \ \ is \ \ even \end{cases}$

Proof. Let u and v be the end vertices of P_n. The vertices u and v are adjacent to $n-2$ vertices in $\overline{P_n}$. The remaining vertices are adjacent to $n-3$ vertices in $\overline{P_n}$.

Case (i) If n is odd

Since u and v are adjacent to $n-2$ vertices in $\overline{P_n}$. Clearly, u and v are odd vertices. Therefore $\{u, v\} \in S$. Also, $\{u, v\}$ is not a geodetic set. Consider a vertex x, which is adjacent to v and non adjacent to u. Obviously, $n-3$ vertices lie on the $x-u$ geodesic. Choose a vertex y there exist $y \in V(\overline{P_n})$ such that $y \notin I[x, u]$. Also no 3-element subset contains the co-even geodetic set. Hence, $S = \{u, v, x, y\}$ is the minimum co-even geodetic set.

Case (ii) If n is even

For n is even, clearly, u and v are even degree vertices. Remaining $n-2$ vertices are adjacent to $n-3$ vertices. Obviously, $n-2$ vertices is odd vertices. Also, every vertex lies on the any geodesic of $n-2$ vertices. Therefore, the minimum co-even geodetic number is $n-2$. ie) $g_{coe}(\overline{P_n}) = n-2$.

Theorem 3.2 For any Gear graph G_n with $n \geq 3$ then $g_{coe}(\overline{G_n}) = n+1$.

Proof. For the Gear graph G_n ,if n is odd, then $\overline{G_n}$ has $n+1$ odd vertices. By the proposition 2.3, $n+1$ vertices belong to co-even geodetic set. Moreover, if n is even, then the graph $\overline{G_n}$ has n vertices have odd degree. These n vertices containing the co-even geodetic set. It is easy to see that all vertices do not lies any geodesic of co-even geodetic set. So we add one more vertex in co-even geodetic set. Obviously, $g_{coe}(\overline{G_n}) = n+1$.

Theorem 3.3 For the complement of the cycle $\overline{C_n}$ with $n \geq 5$, then $g_{coe}(\overline{C_n}) = \begin{cases} 3 \ \ if \ \ n \ \ is \ odd \\ n \ \ if \ \ n \ \ is \ \ even \end{cases}$

Proof. This theorem follows from the Theorem 3.1

4. Conclusions

In this paper, we obtained co-even geodetic number of some kind of graphs and complement of some graphs. Also, we see the relation between vertex covering and co-even geodetic number of some graphs.

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Product Signed Domination in Graphs

T. M. Velammal[*]
A. Nagarajan[†]
K. Palani[‡]

Abstract

Let $G = (V, E)$ be a simple graph. The closed neighborhood of v, denoted by $N[v]$, is the set $\{u: uv \in E\} \cup \{v\}$. A function $f: V \to \{-1,1\}$ is a product signed dominating function, if for every vertex $v \in V$, $f[v] = 1$where $f[v] = \prod_{u \in N[v]} f(u)$. The weight of f, denoted by $f(G)$, is the sum of the function values of all the vertices in G. $(i.e.)$ $f(G) = \sum_{v \in V} f(v)$. The product signed domination number of G, $\gamma^*_{sign}(G)$ is the minimum positive weight of a product signed dominating function. In this paper, we establish bounds on the product signed domination number and estimate product signed domination number for some standard graphs.

Keywords: graphs, product signed dominating function, product signed domination number.

2010 AMS subject classification: 05C69[§].

[*]Research Scholar (Reg. No. 21212232092010), PG & Research Department of Mathematics, V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.avk.0912@gmail.com
[†]Associate Professor (Retd.), V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.nagarajan.voc@gmail.com
[‡]Associate Professor, A.P.C. Mahalaxmi College For Women, Thoothukudi-628002, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India. palani@apcmcollege.ac.in
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1. Introduction

The fundamental thought of graphs was first presented in eighteenth era by Swiss Mathematician Leonhard Euler. It has numerous applications in Natural Sciences, Technology, Information System Research and so on. The quickest developing region in theory of graph is domination. Ore introduced the terms "Dominating Set" and "Domination Number". Dunbar et al. introduced signed domination number [1],[2],[4],[5]. Hosseini gave a lower and upper bound for the signed domination number of any graph [3]. In this paper, we introduce the concept of product signed domination number and find bounds on product signed domination number.

2. Preliminaries

Definition 2.1: A comb graph $P_n \circ K_1$ is a graph obtained by joining a pendant edge to each vertex of a path.

Definition 2.2: A star graph $K_{1,n-1}$ is a tree on n vertices with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1.

Definition 2.3: A tree containing exactly two non-pendant vertices is called a double star. It is denoted by $D_{m,n}$

3. Main Results

Definition 3.1: Let $G=(V,E)$ be a simple graph. The closed neighborhood of v, denoted by $N[v]$, is the set $\{u: uv \in E\} \cup \{v\}$. A function $f: V \to \{-1,1\}$ is a product signed dominating function, if for every vertex $v \in V,\ f[v]=1$
where $f[v]=\prod_{u\in N[v]} f(u)$. The weight of f, denoted by $f(G)$, is the sum of the function values of all the vertices in G.

$(i.e.)\ f(G)=\sum_{v\in V} f(v)$

The product signed domination number of G, $\gamma^*_{sign}(G)$ is the minimum positive weight of a product signed dominating function.

Observation 3.2. (i) In a graph G, a pendant vertex u and its corresponding support vertex v get the same functional values (i.e.) either $+1$ or -1 since otherwise$f[u]=\prod_{x\in N[u]} f(x)=-1$.

(ii) In a product signed dominating function, all the vertices of a graph should not be assigned -1 since product signed domination number is positive.

(iii) In a product signed dominating function, for every vertex $v \in V, N[v]$ contains either zero or even number of vertices with functional value -1, since otherwise $f[v] = \prod_{x \in N[v]} f(x) = -1$.
(iv) If $N_f[1]$ and $N_f[-1]$ denote the number of vertices with functional values 1 and -1 respectively, then $N_f[1] - N_f[-1] \geq 1$.

Theorem 3.3:
$For\ p \leq 4,\ \gamma^*_{sign}(K_p) = p$, the total number of vertices.

$$For\ p > 4,\ \gamma^*_{sign}(K_p) = \begin{cases} 1\ if\ p\ is\ odd\ and\ \dfrac{p-1}{2}\ is\ even \\ 2\ if\ p\ is\ even\ and\ \dfrac{p}{2}\ is\ odd \\ 3\ if\ p\ and\ \dfrac{p-1}{2}\ are\ odd \\ 4\ if\ p\ and\ \dfrac{p}{2}\ are\ even \end{cases}$$

Proof:
Let K_p be a complete graph on p vertices.
Let $V(K_p) = \{v_1, v_2, \dots, v_p\}$and $E(K_p) = \{v_i v_j | i \neq j, 1 \leq i, j \leq p\}$.
Since each pair of vertices is connected by an edge, in a product signed dominating function the number of vertices with functional value -1 must be even.
Case 1: $p \leq 4$
Define a function $f: V(G) \to \{-1, +1\}$ as follows.
When $p \leq 4$, every vertex should be assigned $+1$ under f, since otherwise f would not be a product signed dominating function with a positive weight.

$$Therefore, \quad \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = p$$

=the total number of vertices.
Case 2: $p > 4$ and p is odd
Subcase 2.1: $\frac{p-1}{2}$ is even
Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+1}{2}$, $|V_2| = \frac{p-1}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1\ \forall\ v \in V_1 \\ -1\ \forall\ v \in V_2 \end{cases}$
Obviously, for every $v \in V$, $f[v] = 1$ and hence f is a product signed dominating function.

$$Therefore, \quad \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v)$$

$$= \left(\frac{p+1}{2}\right).1 + \left(\frac{p-1}{2}\right).(-1) = 1$$

Subcase 2.2. $\frac{p-1}{2}$ is odd

Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+3}{2}$, $|V_2| = \frac{p-3}{2}$ and $V_1 \cap V_2 = \emptyset$. Here $|V_1|$ is odd and $|V_2|$ is even. Define $f: V(G) \to \{-1, +1\}$ as

$$f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$$

Clearly, for every $v \in V$, $f[v] = 1$.Hence f is a product signed dominating function. Also, $N_f[1] - N_f[-1] = 3$. Since $\frac{p-1}{2}$ is odd. This function f gives the minimum value for product signed domination number. $Therefore,\ \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = \left(\frac{p+3}{2}\right).1 + \left(\frac{p-3}{2}\right).(-1) = 3$

Case 3: $p > 4$and p is even

Subcase 3.1:$\frac{p}{2}$ is even

If we partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p}{2}$,$|V_2| = \frac{p}{2}$ and $V_1 \cap V_2 = \emptyset$ and assign $+1$ to all the vertices in V_1 and -1 to all the vertices in V_2, then the function would be a product signed dominating function but the weight would be zero.

Since $\frac{p}{2}$ is even, $\frac{p-2}{2}$ is odd. Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+4}{2}$, $|V_2| = \frac{p-4}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$

Therefore f is a product signed dominating function. Also $N_f[1] - N_f[-1] = 4$. Since $\frac{p}{2}$ is even, this function f gives the minimum value for product signed domination number as before. $Therefore,\ \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = \left(\frac{p+4}{2}\right).1 + \left(\frac{p-4}{2}\right).(-1)$
$= 4$

Subcase 3.2: $\frac{p}{2}$ is odd

We have $\frac{p-2}{2}$ is even.

Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+2}{2}$, $|V_2| = \frac{p-2}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$

Correspondingly, for every $v \in V$, $f[v] = 1$. Hence f is a product signed dominating function. Also $N_f[1] - N_f[-1] = 2$. Proceeding as above, this function f gives the minimum value for product signed domination number.

$$Therefore, \quad \gamma_{sign}^{*}(K_p) = \sum_{v \in V(K_p)} f(v) = \left(\frac{p+2}{2}\right).1 + \left(\frac{p-2}{2}\right).(-1) = 2$$

$Therefore,\ p \leq 4,\ \gamma_{sign}^{*}(K_p) = p$, the total number of vertices.

$$For\ p > 4, \gamma_{sign}^{*}(K_p) = \begin{cases} 1\ if\ p\ is\ odd\ and\ \dfrac{p-1}{2}\ is\ even \\ 2\ if\ p\ is\ even\ and\ \dfrac{p}{2}\ is\ odd \\ 3\ if\ p\ and\ \dfrac{p-1}{2}\ are\ odd \\ 4\ if\ p\ and\ \dfrac{p}{2}\ are\ even \end{cases}$$

Theorem 3.4: For the comb graph, $P_n \circ K_1$, the product signed domination number $\gamma_{sign}^{*}(P_n \circ K_1) = 2n$, the total number of vertices.

Proof:

Let G be a comb graph $P_n \circ K_1$.

Let $V = \{v_i, u_i | 1 \leq i \leq n\}$ be the vertex set with u_i's representing the pendant vertices and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n\}$ be the edge set.

Since u_i is the pendant vertex to v_i, both $f(u_i)$ and $f(v_i)$ must be either $+1$ or -1 for $1 \leq i \leq n$ (by 3.2(i)).

Case 1: n is odd

Define $f: V(G) \to \{-1, +1\}$ as follows.

If u_1 and v_1 are both assigned -1, then u_2 and v_2 should be assigned $+1$ since otherwise $f[v_1]$ would be -1. Further if $f(u_3) = f(v_3) = 1$, again $f[v_2] = -1$. Hence $f(u_1) = f(v_1) = -1,\ f(u_2) = f(v_2) = 1,\ f(u_3) = f(v_3) = -1,\ f(u_4) = f(v_4) = 1$ and so on. Then $f(u_n) = f(v_n) = -1$ and f is a product signed dominating function. Correspondingly, the weight of the graph is $-2 =$ a negative integer which is a contradiction to the weight is positive.

Hence, let us start with $f(u_1) = f(v_1) = 1$.

Then $f(u_i) = f(v_i) = 1\ \forall\ i = 2\ to\ n$, since otherwise f is not a product signed dominating function.

Hence $f(u_i) = f(v_i) = 1\ \forall\ i = 1\ to\ n$ is the only product signed dominating function having a positive weight. Hence it is the unique product signed dominating function.

$Therefore,\ \gamma_{sign}^{*}(G) = \sum_{v \in V(G)} f(v) = 2n =$ the total number of vertices of G

Case 2: n is even

Define $f: V(G) \to \{-1, +1\}$ as follows.

If u_1 and v_1 are both assigned -1, then u_2 and v_2 should be assigned $+1$ since otherwise $f[v_1]$ would be -1. Further if $f(u_3) = f(v_3) = 1$, again $f[v_2] = -1$. Hence

$f(u_1) = f(v_1) = -1,\ f(u_2) = f(v_2) = 1,\ f(u_3) = f(v_3) = -1,\ f(u_4) = f(v_4) = 1$ and so on. Then $f(u_n) = f(v_n) = 1$ and hence $f[v_n] = -1$.
Therefore, this f is not a product signed dominating function.
Hence, let us start with $f(u_1) = f(v_1) = 1$. Then $f(u_i) = f(v_i) = 1\ \forall\ i = 2\ to\ n$, since otherwise f is not a product signed dominating function.
Hence $f(u_i) = f(v_i) = 1\ \forall\ i = 1\ to\ n$ is the unique product signed dominating function.
$Therefore,\ \gamma^*_{sign}(G) = \sum_{v \in V(G)} f(v) = 2n =$ the total number of vertices of G
By cases 1 and 2, $\gamma^*_{sign}(G) = 2n$.

Observation 3.5: For any graph G, $1 \leq \gamma^*_{sign}(G) \leq p\ ; p =$ total number of vertices of G. Here the bounds are sharp since $\gamma^*_{sign}(K_5) = 1$ and $\gamma^*_{sign}(P_n \circ K_1) = 2n =$ total number of vertices.

Theorem 3.6:
The product signed domination number of a path on n vertices is equal to n.
Proof:
Let P_n be a path on n vertices.
If $f(v_i) = -1\ (2 \leq i \leq n-1)$
Then $f(v_{i-1}) = -1\ and\ f(v_{i+1}) = +1$
(or)
$f(v_{i-1}) = +1\ and\ f(v_{i+1}) = -1$
If $f(v_i) = +1\ (2 \leq i \leq n-1)$
Then $f(v_{i-1}) = f(v_{i+1}) = -1$
(or) $f(v_{i-1}) = f(v_{i+1}) = +1$
By the above observation, if $v_1$is assigned -1, then $v_2$must be assigned -1 so that $f[v_1] = +1$. Then v_3 must be assigned $+1$ so that $f[v_2] = +1$. So v_4 must be assigned -1 so that $f[v_3] = +1$.
Proceeding like this, we define a function $f: V(G) \to \{-1, +1\}$ as follows.
For $1 \leq i \leq n,\ f(v_i) = \begin{cases} +1\ if\ i \equiv 0(mod 3) \\ -1\ otherwise \end{cases}$
So when $n = 3k, f$ is not a product signed dominating function since $f[v_n] = -1$.
When $n = 3k+1, f$ is not a product signed dominating function since $f[v_n] = -1$
When $n = 3k+2,\ f$ is a product signed dominating function having a negative weight.
So let us try with $+1$ assigned to v_1.
If v_1 is assigned $+1$, v_2 must be assigned $+1$ so that $f[v_1] = +1$. Again v_3 must be assigned $+1$ so that $f[v_2] = +1$. Again v_4 must be assigned $+1$ so that $f[v_3] = +1$ and so on.
Therefore, $f(v_i) = +1\ \forall\ i = 1\ to\ n$. And f is a minimum positive weight product signed dominating function.

The weight of this function $= n$, the total number of vertices.
Therefore, $\gamma^*_{sign}(P_n) = n$, the total number of vertices.

Theorem 3.7:
The product signed domination number of a cycle on n vertices is equal to n.

Proof:
Let C_n be a path on n vertices.
If $f(v_i) = -1$ $(2 \leq i \leq n-1)$
Then $f(v_{i-1}) = -1$ and $f(v_{i+1}) = +1$
(or)
$f(v_{i-1}) = +1$ and $f(v_{i+1}) = -1$
If $f(v_i) = +1$ $(2 \leq i \leq n-1)$
Then $f(v_{i-1}) = f(v_{i+1}) = -1$
(or)
$f(v_{i-1}) = f(v_{i+1}) = +1$
By the above observation, if $v_1$is assigned -1, then $v_2$must be assigned -1 so that $f[v_1] = +1$. Then v_3 must be assigned $+1$ so that $f[v_2] = +1$. So v_4 must be assigned -1 so that $f[v_3] = +1$.
Proceeding like this, we define a function $f: V(G) \to \{-1, +1\}$ as follows.
For $1 \leq i \leq n, f(v_i) = \begin{cases} +1 \; if \; i \equiv 0(mod 3) \\ -1 \; otherwise \end{cases}$
So, when $n = 3k$, f is a product signed dominating function having negative weight.
When $n = 3k+1$, f is not a product signed dominating function since
$f[v_1] = -1$.
When $n = 3k+2$, f is not a product signed dominating function since
$f[v_1] = f[v_n] = -1$.
So let us try with $+1$ assigned to v_1.
If v_1 is assigned $+1$, v_2 must be assigned $+1$ so that $f[v_1] = +1$. Again v_3 must be assigned $+1$ so that $f[v_2] = +1$. Again v_4 must be assigned $+1$ so that $f[v_3] = +1$ and so on.
Therefore, $f(v_i) = +1 \; \forall \; i = 1 \; to \; n$. And f is a minimum positive weight product signed dominating function.
The weight of this function $= n$, the total number of vertices.
Therefore, $\gamma^*_{sign}(C_n) = n$, the total number of vertices.

Theorem 3.8:
The product signed domination number of a star graph $K_{1,n-1}$on n vertices is equal to n.
Proof:
Let $K_{1,n-1}$ be a star graph on n vertices.
Let $V(K_{1,n-1}) = \{v, v_1, v_2, \dots, v_{n-1}\}$and $E(K_{1,n-1}) = \{vv_i | 1 \leq i \leq n-1\}$.
By 3.2(i), v and $v_i (1 \leq i \leq n-1)$ should be assigned same functional value.

If $f(v) = -1$, then the weight of f is negative.
Therefore $f(v)$ must be equal to $+1$ and hence define $f: V(G) \to \{-1, +1\}$ as $f(v_i) = 1\ for\ 1 \le i \le n-1$. And obviously f is the minimum positive weight product signed dominating function.
Therefore, $\gamma^*_{sign}(K_{1,n-1}) = n$, the total number of vertices.

Theorem 3.9:
The product signed domination number of a double star graph $D_{m,n}$is equal to $m + n$.
Proof:
Let $D_{m,n}$ be a double star graph on $m + n$ vertices.
Let $V(D_{m,n}) = \{v, v_1, v_2, \dots, v_{n-1}, u, u_1, u_2, \dots, u_{m-1}\}$ and
$E(D_{m,n}) = \{vv_i | 1 \le i \le n-1\} \cup \{uu_j | 1 \le j \le m-1\} \cup \{uv\}$.
Case 1: Number of pendant vertices to atleast one of u, v is odd.
Without loss of generality, assume that number of pendant vertices to u is odd.
If we assign $-1$to u, then all the pendant vertices to u must be assigned -1 (by 3.2(i)). Since number of pendant vertices to u is odd, v must be assigned $+1$. Hence again by 3.2(i), all the pendant vertices to $v$get $+1$. But here $f[v] = -1$. So this f is not a product signed dominating function.
Hence define $f: V(G) \to \{-1, +1\}$as $f(v) = +1\ \forall\ v \in V(D_{m,n})$
Clearly, f is the minimum positive weight product signed dominating function.

$$Hence, \sum_{v \in V(D_{m,n})} f(v) = \sum_{v \in V(D_{m,n})} 1 = m + n$$

Therefore, $\gamma^*_{sign}(D_{m,n}) = m + n$, the total number of vertices.
Case 2: Number of pendant vertices to both u and v is even.
If we assign $-1$to u, then all the pendant vertices to u must be assigned -1 (by 3.2(i)). Since number of pendant vertices to u is even, v must be assigned -1. Hence again by 3.2(i), the pendant vertices to $v$get -1. Here this f is a product signed dominating function having a negative weight.
So, the only possible positive weight product signed dominating function $f: V(G) \to \{-1, +1\}$is $f(v) = +1\ \forall\ v \in V(D_{m,n})$

$$Hence, \sum_{v \in V(D_{m,n})} f(v) = \sum_{v \in V(D_{m,n})} 1 = m + n$$

Therefore, $\gamma^*_{sign}(D_{m,n}) = m + n$, the total number of vertices.

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Independent Restrained k - Rainbow Dominating Function

M. Esakki Dharani*
A. Nagarajan†
K. Palani‡

Abstract

Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set {1, 2…, k} that is f: V(G) → P [1, 2, …, k]. If for each vertex v ∈ V(G) such that f(v) = ∅ .we have $\bigcup_{u \in N[v]} f(u) = \{1, 2, .., k\}$ then f is called the k – Rainbow Dominating Function (KRDF) of G. A k – Rainbow Dominating Function is said to be Independent Restrained k - Rainbow Dominating function if (i)The set of vertices assigned with non – empty set is independent. (ii)The induced subgraph of G, by the vertices with label ∅ has no isolated vertices. The weight w(f) of a function f is defined as w(f) = $\sum_{v \in V(G)} |f(v)|$ .The Independent Restrained k – Rainbow Domination number is the minimum weight of G. In this paper we introduce Independent Restrained k – Rainbow Domination and find for some graphs

Keywords: Independent, Restrained, Rainbow domination number, weight.

2010 AMS subject classification: 05C69§

*Research Scholar (Reg.No.21212232092011), PG & Research Department of Mathematics, V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.dhanushiya1411@gmail.com
†Associate Professor (Retd.), V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.nagarajan.voc@gmail.com
‡Associate Professor, A.P.C. Mahalaxmi College for Women, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.palani@apcmcollege.ac.in
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1. Introduction

Domination in graphs originates from location problems in operations research. As a variation of domination in graphs, rainbow domination was introduced by Bresar et al. [2]. Shao et al. [7] gave bounds for the k – rainbow domination number on an arbitrary graph. Hao et al. [4] studied the k – rainbow domination number of directed graphs. Independent rainbow domination was introduced by Zehui Shao et al. [8]. Amjadi et al. [1] was investigated the rainbow restrained domination number. In this paper we introduce Independent Restrained k – Rainbow Domination and find for some graphs.

2. Preliminaries

A graph G consists of pair(V(G), E(G)) where V(G) is a non-empty finite set whose elements are called points or vertices and E(G) is a set of unordered pair of distinct elements of V(G). The elements of E(G) are called lines or edges of the graph G. For any vertex u in G, the open neighbourhood of u, is denoted by N(u) is the set of vertices adjacent to u and the closed neighbourhood of u, is denoted by N[u] = N(u) ∪ {u}. A set of vertices in a graph is said to be an independent set of vertices or simply an independent if no two vertices in the set are adjacent. The splitting graph of G is denoted by S[G]For each vertex v of a graph G, take a new vertex v' and join v' to all vertices adjacent to v in G.The corona product of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. It is denoted as $G \circ H$. Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set {1, 2, …, k} that is f : V(G) → P [1,2,…,k]. If for each vertex v∈ V(G) such that f(v) = ∅ .we have $\bigcup_{u \in N[v]} f(u) = \{1,2,..,k\}$ then f is called the k – Rainbow Dominating Function (KRDF) of G. A k – Rainbow dominating function is called independent k – Rainbow Domination if vertices assigned with non – empty sets are pairwise non-adjacent. A k – Rainbow dominating function is called Rainbow Restrained Domination function if vertices assigned with empty sets has no isolated vertex.

3. Main Results

Definition 3.1. Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set {1, 2, …, k} that is f : V(G) → P [1,2,…,K]. If for each vertex v∈ V(G) Such that f(v) = ∅ . we have $\bigcup_{u \in N[v]} f(u) = \{1,2,..,k\}$ then f is called the k – Rainbow Dominating Function (KRDF) of G. A k – Rainbow Dominating Function is said to be Independent Restrained k- Rainbow Dominating function if

i. The set of vertices assigned with non – empty set is independent.
ii. The induced subgraph of G, by the vertices with label ∅ has no isolated vertices.

The weight w(f) of a function f is defined as w(f) = $\sum_{v\in V(G)}|f(v)|$. The Independent Restained k – Rainbow Domination number is the minimum weight of G.

Observation 3.2.

1. Let u be a pendant vertex to v. If one of the vertex u and v is assigned {1, 2, … k} then the other may be assigned ∅. (i.e) if the end vertex is assigned {1, 2, … k} then the support vertex may be assigned ∅
2. Always $k \leq \gamma_{irkr} \leq nk$
3. If a graph G has a clique as a subgraph, then one of the vertices of the clique should be labelled {1, 2, … k} and all the remaining vertices are labelled ∅.

Theorem 3.3. Let G be a graph and let v be a full degree vertex in G. Suppose G – {v} has no isolated vertex then $\gamma_{irkr}(G) = k$

Proof:

Define f: V(G) → P [1,2,…,k] by

f(x) = $\begin{Bmatrix} \{1,2,\dots,k\} if x = v \\ \emptyset \quad \text{otherwise} \end{Bmatrix}$

Then clearly f is an independent restrained rainbow dominating function.

Further w(f) = k

As $\gamma_{irkr}(G) \geq k$, f is a minimum independent restrained k rainbow dominating function. Therefore, $\gamma_{irkr}(G) = k$

Corollary 3.4: Complete graph, Wheel graph, Fan graph, Flower graph all have independent restrained k rainbow domination number is equal to k.

Theorem 3.5: The Independent Restrained k- Rainbow Dominating function exists for path graph P_n if and only if $n \equiv 1\ mod\ 3$ and then $\gamma_{irkr}(P_n) = \left\lceil \frac{n}{3} \right\rceil$ k, where $n \equiv 1\ mod\ 3$

Proof: When $n \equiv 0\ or\ 2\ \ mod\ 3$, the graph G does not admit any Independent Restrained k Rainbow dominating function. Since any k Rainbow Dominating function f fails to satisfy either Independent or Restrained condition.

let $v_1, v_2, v_3, \dots v_n$ be the vertices of the path graph.

Define f: V(G) → P [1, 2, …, k] by

$f(x) = \begin{Bmatrix} \{1,2,\dots,k\} if x = v_i \\ where i \equiv 1\ mod\ 3\ \ and\ \ 0 \leq i \leq n \\ \emptyset \quad otherwise \end{Bmatrix}$

Then clearly f is an Independent Restrained k Rainbow Dominating function.

Therefore, w(f) = $\sum|f(v_i)| = |f(v_1)| + |f(v_4)| + |f(v_7)| + \cdots + |f(v_n)|$

Therefore, $\gamma_{irkr}(P_n) = \left\lceil \frac{n}{3} \right\rceil$ k

Theorem 3.6: The Independent Restrained k- Rainbow Dominating function exists for cycle graph C_n if and only if $n \equiv 0\ mod\ 3$ and then $\gamma_{irkr}(C_n) = \frac{nk}{3}$, $n \equiv 0\ mod\ 3$

Proof: When $n \equiv 1\ or\ 2\ mod\ 3$, the graph G does not admit any Independent Restrained k Rainbow dominating function. Since any k Rainbow Dominating function f fails to satisfy either Independent or Restrained condition.

let $v_1, v_2, v_3, \ldots v_n$ be the vertices of the cycle graph.

Define f: V(G) → P [1, 2, …, k] by

$$f(x) = \begin{Bmatrix} \{1,2,\ldots,k\} if x = v_i \\ where i \equiv 0\ mod\ 3\ and\ 0 \le i \le n \\ \emptyset \quad otherwise \end{Bmatrix}$$

Then clearly f is an Independent Restrained k Rainbow Dominating function.

Therefore, w(f) = $\sum |f(v_i)| = |f(v_3)| + |f(v_6)| + |f(v_9)| + \cdots + |f(v_n)|$

Therefore, $\gamma_{irkr}(C_n) = \left\lceil \frac{n}{3} \right\rceil$ k

Theorem 3.7: Let H_n be the helm graph obtained from the wheel by attaching a pendant vertex to each rim vertex. Then $\gamma_{irkr}(H_n)$ = k + n - 2 for n ≥ 4

Proof: Let $v_1, v_2, v_3, \ldots v_{n-1}$ be the rim vertex. Let u be the apex vertex. Let $w_1, w_2, \ldots w_{n-1}$ such that w_i is adjacent to v_i for $1 \le i \le n-1$.

Define f: V(G) → P [1, 2, …, k] by

$$f(x) = \begin{Bmatrix} \{1\} if x = w_i \text{ where } 1 \le i \le n-1 \\ \{2,3,\ldots,k\} if x = u \\ \emptyset \quad otherwise \end{Bmatrix}$$

Obviously, every vertex v with labelled $\emptyset$ satisfies the condition $\bigcup_{u \in N[v]} f(u) = \{1,2,..,k\}$

∴f is a k Rainbow Dominating Function.

Let S be the set of vertices assigned $\emptyset$ labelled then S = $\{v_1, v_2, v_3, \ldots v_{n-1}\}$. Here <S> has no isolated vertices.

Let S' be the set of vertices assigned non empty labelled. Then S' = $\{u, w_1, w_2, \ldots w_{n-1}\}$ is Independent.

Then clearly f is an Independent Restrained k Rainbow Dominating function.

∴ w(f) = $\sum$ ($|f(u)| + |f(w_i)|$) where i = 1 to n – 1

Therefore, $\gamma_{irkr}(H_n)$ = k + n – 2

Theorem 3.8: Let $(C_n \circ K_1)$be the crown graph obtained by joining a pendant edge to each vertex of C_n. Then for n ≥ 1, $\gamma_{i_{rkr}}(C_n \circ k_1)$ = nk

Proof: Let $v_1, v_2, v_3, \ldots v_n$ be the vertices of the cycle. Let $w_1, w_2, w_3, \ldots w_n$ be the set of end vertices of the crown graph, where $1 \le i \le n$.

Define f: V(G) → P [1, 2, …, k] by

$$f(x) = \begin{Bmatrix} \{1,2,\ldots,k\} if x = w_i ;\ 1 \le i \le n \\ \emptyset \quad \text{otherwise} \end{Bmatrix}$$

Obviously, every vertex v with labelled $\emptyset$ satisfies the condition $\bigcup_{w\in N[v]} f(w) = \{1,2,..,k\}$
∴f is a k Rainbow Dominating Function.
By Observation 3.2(1), we assigned $\{1,2,\dots,k\}$ to the end vertices, which is independent and we assigned$\emptyset$ to all the support vertices. Then the induced subgraph of empty set is also connected.
Hence it satisfies both Independent and Restrained condition.
Then clearly f is an Independent Restrained k Rainbow Dominating function.
∴ w(f) = $\sum$ ($|f(w_i)|$) where i = 1 to n = $|f(w_1)| + |f(w_2)| + |f(w_3)| + \cdots + |f(w_n)|$ = nk
Therefore, $\gamma_{i_{rkr}}(C_n \circ k_1)$ = nk

Remark 3.9:

(i) In Observation 2.2 (2), Equality holds. Since $\gamma_{i_{rkr}}(K_n)$ = nk and $\gamma_{i_{rkr}}(C_n \circ k_1)$ = nk.
(ii) The inequality is also strict. Since $\gamma_{irkr}(H_n)$ = k + n – 2

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Semi generalization of δI*-closed sets in ideal topological space

Dr K Palani [1]
M Karthigai Jothi[2]

Abstract

In this paper we introduce the notion of semi generalized δI*-closed sets or gsδI*-closed sets using semi open sets and investigate its basic properties and characterizations in an ideal topological space. This class of sets is properly lies between the class of δI*-closed sets and the class of g-closed sets. Also, study the relationship with various existing closed sets in ideal topological spaces. Moreover, we introduce and study the concept of maximal gsδI*-closed sets.

Keywords: ideal topological space, δI*-closed sets, gsδI*-closed sets.

2010AMS subject classification: 05C69[3]

[1]Associate Professor and Head, PG & Research Department of Mathematics, A.P.C Mahalaxmi College for Women, Thoothukudi-2. Tamilnadu, India.E-mail: palani@apcmcollege.ac.in
[2]Research Scholar, Reg. No: 21212012092005, PG & Research Department of Mathematics, A.P.C Mahalaxmi College for Women, Thoothukudi-[2]. Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -12. Tamilnadu, India.E-mail: jothiperiyasamy05@gmail.com.
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1. Introduction and Preliminaries

An ideal I is a non-empty collection of subsets of X which satisfies: (i) A∈I and B ⊆ A implies B ∈I, and (ii) A∈I and B ∈I implies A ∪ B ∈I. Given a topological space (X, τ) with an ideal I on X called ideal topological space denoted by (X, τ, I). Kuratowski [5] and vaidhyanathaswamy [18] was studied the notion of ideal topological spaces, J. Dontchev, M. Ganster [3], Navaneethakrishnan, P. Paulraj Joseph [13], D. Jankovic, T. R. Hamlett [4], M. N. Mukherjee, R. Bishwambhar, R. Sen [10], A. A. Nasef, R. A. Mahmond [12] etc., were investigated applications to various fields of ideal topology. If P(X) is the collection of all subsets of X a set operator (.)*: P(X) → P(X) called a local function [5] for any subset A of X with respect to I and τ is defined as, A*(I, τ) = {x ∈X: U ∩ A ∉ I for every U ∈ τ(x)}, where τ(x) = {U ∈ τ / x ∈ U}. A kuratowski closure operator cl*(A) for a topology τ*(I, τ) called *-topology finer than τ is defined by cl*(A) = A ∪A*(I, τ). A subset A of X is said to be δ-closed [19] set if $cl_\delta(A) = A$, where $cl_\delta(A)$ = {x ∈X: Int(cl(U)) ∩ A ≠ϕ, for every U ∈τ(x)}. The complement of δ-closed set is δ-open set. A subset A subset A of a space (X, τ) is an α-open [14] (resp. semi open [7]) set if A ⊂int(cl(int(A))) (resp. A ⊂ cl(int(A))). The complement of a semi open (resp.α-open) set is called a semi closed (resp.α-open).

Definition 1.1. Let (X,τ) be a topological space. A subset A of X is said to be
(i) a generalized closed (briefly, g-closed) set [6] if cl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).
(ii) a generalized semi closed (briefly, gs-closed) set [1] if scl(A) ⊂ U whenever A ⊂ U and U is open set in (X, τ).
(iii) a semi-generalized closed (briefly, sg-closed) set [2] if scl(A) ⊂ U whenever A ⊂ U and U is semi open set in (X, τ).
(iv)anα-generalized closed (briefly,αg-closed) se [8]t if αcl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).
(v) a generalized α-closed (briefly, gα-closed) set [9] if αcl(A) ⊂ U whenever A ⊂ U and U is α-open in (X, τ).
(vi) a $\hat{g}$ (or) w-closed set [20] if cl(A) ⊂ U whenever A ⊂ U and U is semi open set in (X, τ).

Definition 1.2. [21] Let (X, τ, I) be an ideal topological space. A subset A of X is said to be an I_g-closed set if A* ⊂ U whenever A ⊂ U and U is open in X.

Definition 1.3. [21] Let (X, τ, I) be an ideal topological space, A a subset of X and x is a point of X. Then

(1) x is called a δ-I-cluster point of A if $A \cap int(cl^*(U)) \neq \phi$, for each open neighborhood U of x.
(2) the family of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by $[A]_{\delta\text{-}I}$.
(3) a subset A is said to be δ-I-closed if $[A]_{\delta\text{-}I} = A$. The complement of a δ-I-closed set of X is said to be δ-I-open.

Lemma 1.4. [21] Let A and B be subsets of an ideal topological space (X, τ, I). Then, the following properties hold.
(1) $A \subset [A]_{\delta\text{-}I}$.
(2) If $A \subset B$, then $[A]_{\delta\text{-}I} \subset [B]_{\delta\text{-}I}$.
(3) $[A]_{\delta\text{-}I} = \cap \{F \subset X / A \subset F \text{ and } F \text{ is } \delta\text{-I-closed}\}$.
(4) If A_α is δ-I-closed set of Xs for each $\alpha \in \Delta$, then $\cap \{A_\alpha / \alpha \in \Delta\}$ is δ-I-closed.
(5) $[A]_{\delta\text{-}I}$ is δ-I-closed.

Lemma 1.5. [21] Let (X, τ, I) be an ideal topological space and $\tau_{\delta\text{-}I} = \{A \subset X / A \text{ is } \delta\text{-I-open subset of } (X, \tau, I)\}$. Then $\tau_{\delta\text{-}I}$ is a topology such that $\tau_S \subset \tau_{\delta\text{-}I} \subset \tau$, where τ_S is the collection of
δ-open sets.

Definition 1.6. [16] Let (X, τ, I) be an ideal topological space and A a subset of X. Then $[A]^*(I, \tau) = \{x \in X: int[U]_{\delta\text{-}I} \cap A \neq \phi \text{ for every } U \in \tau(x)\}$ is called local δI-closure function of A with respect to the ideal I and topology τ, where $\tau(x) = \{U \in \tau / x \in U\}$. A subset A is said to be δI-closed if $[A]^* = A$. The complement of δI-closed set is called δI-open set.

Remark 1.7.[16] Always, (i) $[A]^*$ is closed, (ii) $[\phi]^* = \phi$ and $[X]^* = X$, (iii) $A \subseteq [A]^*$.

Lemma.1.8. [16] Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then for local δI-closure functions the following properties hold.
(i) If $A \subseteq B$ then $[A]^* \subseteq [B]^*$.
(ii) $[A \cup B]^* = [A]^* \cup [B]^*$.
(iii) $[A \cap B]^* \subseteq [A]^* \cap [B]^*$.
(iv) $[[A]^*]^* = [A]^*$.

Lemma 1.9.[16] (i) $cl(A) \subseteq [A]^*$,
(ii) $A^* \subseteq [A]^*$,
(iii) $cl_\delta(A) \subseteq [A]^*$,
(iv) $[A]_{\delta\text{-}I} \subseteq [A]^*$.

Definition 1.10. [17] A subset A of an ideal space (X, τ, I) is called gδI*-closed if [A]* ⊆ U whenever A ⊆ U and U is open in (X, τ, I). The complement of a gδI*-closed set in (X, τ, I) is called gδI*open set in (X, τ, I).

2. gsδI*- closed Sets

In this section we introduce gsδI*-closed sets and discuss the relationship with some existing sets.

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is called gsδI*-closed if [A]*⊆ U whenever A ⊆ U and U is semi open set in (X, τ, I). The complement of gsδI*-closed set in (X, τ, I), is called gsδI*-open set in (X, τ, I).

Theorem 2.2. EveryδI*-closed set is gsδI*-closed.
Proof. Let A be any δI*-closed set and U be any semi open set containing A. Since A is δI*-closed, [A]* = A. Therefore, A is gsδI*-closed set in (X, τ, I).

Remark 2.3. The converse of the above Theorem 2.2 is need not be true as shown in the following Example 2.4.

Example 2.4. Let X = {a, b, c}, τ = {X, ϕ, {b}, {c, d}, {b, c}, {b, c, d}}, I = {ϕ, {d}}. Let A = {a, b, c}. Then, A is gsδI*-closed but not δI*-closed.

Theorem 2.5. In an ideal topological space (X, τ, I), every gsδI*-closed set is
(i) ĝ -closed set in (X, τ).
(ii) g-closed (resp. gα, αg, sg, gs) -closed set in (X, τ).
(iii) I_g -closed set in (X, τ, I).
Proof. (i) Let A be a gsδI*-closed set and U be any semi open set in (X, τ, I) containing A. Since A is gsδI*-closed, [A]*⊆ U. Then cl(A) ⊆ U and hence A is ĝ -closed in (X, τ, I), by Lemma 1.9.
(ii) By [20], every ĝ-closed set is g-closed (resp. gα-closed, αg-closed, sg-closed, gs-closed) set in (X, τ, I). Therefore, it holds.
(iii) Since every g-closed set is I_g-closed, it holds.

Remark 2.6. The following Example 2.7 shows that, the converse of the above Theorem 2.5 (i) is not always true.
Example 2.7. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {b}}. Let A = {c, d}. Then A is ĝ-closed set but not gsδI*-closed.

Remark 2.8. The following Examples shows that, the converse of Theorem 2.5 (ii) is not true.

Example 2.9. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {c}, {b, c,}} and I = {ϕ, {d}}. Let A = {d}. Then A is g-closed, αg-closed, gα-closed but not gsδI*-closed.

Example 2.10. Let X = {a, b, c, d}, τ = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d}} and I = {ϕ, {a}}. Let A = {a, b}. Then A is gs-closed and sg-closed but not gsδI* closed.

Remark 2.11. The following Example 2.12 shows that, the converse of Theorem 2.5 (iii) is not always true.

Example 2.12. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {b}}. Let A = {b}. Then A is I_g -closed but not gsδI*-closed.

3. Characterizations

In this section we study some of the basic properties and characterizations of gsδI*-closed sets.

Theorem 3.1. Let (X, τ, I) be an ideal space and A a subset of X. Then [A]* is semi closed.
Proof. By Remark 1.7, [A]* is closed and hence it is semi closed.

Theorem 3.2. Let (X, τ, I) be an ideal space and A ⊆ X. If A ⊆ B ⊆[A]*, then [A]* = [B]*.
Proof. Since A ⊆ B, [A]*⊆[B]* and since B ⊆[A]*, [B]*⊆[[A]*] * = [A]*, By Lemma 1.8 and Lemma 1.9. Therefore, [A]* = [B]*.

Theorem 3.3. Let (X, τ, I) be an ideal space. Then [A]* is always gsδI*-closed for every subset A of X.
Proof. Let [A]*⊆ U, where U is semi open. Always, [[A]*] * = [A]*. Hence [A]* is gsδI*-closed.

Theorem 3.4. Let (X, τ, I) be an ideal space and A ⊆ X. If sker(A) is gsδI*-closed, then A is also gsδI*-closed.
Proof. Suppose that, sker(A) is a gsδI*-closed set. If A ⊆ U and U is semi open, then sker(A) ⊆ U. Since sker(A) is gsδI*-closed, [sker(A)]*⊆ U. Always, [A]*⊆[sker(A)]*. Thus, A is gsδI*-closed.
The following Example 3.5 shows that, the converse of the above Theorem 3.4 is not always hold.

Example 3.5. In Example 2.12, let A = {a, b}. Then A is gsδI*-closed. But, sker(A) = {a ,b, c} is not gsδI*-closed.

Theorem 3.6. If A is gsδI*-closed subset in (X, τ, I), then [A]* – A does not contain any nonempty closed set in (X, τ, I).
Proof. Let F be any closed set in (X, τ, I) such that F ⊆[A]* – A then A ⊆ X – F and X – F is open and hence semiopen in (X, τ, I). Since A is gsδI*-closed, [A]*⊆X – F. Hence, F ⊆ X – [A]*. Therefore, F ⊆ ([A]* – A) ∩ (X – [A]*) = ϕ.

Remark 3.7. The converse of the above Theorem 3.6 is not always true as shown in the following Example 3.8.

Example 3.8. Let X = {a, b, c}, τ = {X,ϕ, {a}, {b}, {a, b}} and I = {ϕ, {c}, {d}, {c, d}}. Let A = {a, b, c}. Then [A]* – A = X – {a, b, c} = {d} does not contain any nonempty closed set. But A is not a gsδI*-closed subset of (X, τ, I).

Theorem 3.9. For a subset A of an ideal space (X, τ, I), cl(A) – A is gsδI*-closed if and only if A ∪ (X – cl(A)) is gsδI*-open.
Proof. Necessity - Let F = cl(A) – A. By hypothesis, F is gsδI*-closed and X – F = X ∩ (X – F) = X ∩ (X – (cl(A) – A)) = A ∪ (X – cl(A)). Since X – F is gsδI*-open, A ∪(X– cl(A)) is gsδI*-open.
Sufficiency-Let U = A ∪ (X – cl(A)). By hypothesis, U is gsδI*-open. Then X – U is gsδI*-closed and X – U = X – (A ∪ (X – cl(A))) = cl(A) ∩ (X – A) = cl(A) – A. Hence proved.

Theorem 3.10. Let (X, τ, I) be an ideal space. Then every subset of X is gsδI*-closed if and only if every semiopen subset of X is δI*-closed.
Proof. Necessity - Suppose every subset of X is gsδI*-closed. If U is a semiopen subset of X, then U is gsδI*-closed and so [U]* = U. Hence, U is δI*-closed.
Sufficiency - Suppose A⊆ U and U is semiopen. By hypothesis, U is δI*-closed. Therefore, [A]*⊆[U]* = U and hence A is gsδI*-closed.

Theorem 3.11. Let (X, τ, I) be an ideal space. If every subset of X is gsδI*-closed, then every open subset of X is δI*-closed.
Proof. Suppose every subset of X is gsδI*-closed. If U is an open subset of X, then U is gsδI*-closed and so [U]*⊆ U, since every open set is semiopen. Hence, U is δI*-closed.

Theorem 3.12. Intersection of a gsδI*-closed set and aδI*-closed set is always gsδI*-closed.
Proof. Let A be a gsδI*-closed set and G be any δI*-closed set of an ideal space (X, τ, I). Suppose A ∩ G ⊆ U and U is semiopen set in X. Then, A⊆ U ∪ (X – G). Now, X – G is δI*-open and hence open and so semiopen set. Therefore, U ∪ (X – G) is a semiopen set containing A. But A is gsδI*-closed and therefore, [A]*⊆ U ∪(X – G).

Therefore, $[A]^* \cap G \subseteq U$ which implies that, $[A \cap G]^* \subseteq U$. Hence, $A \cap G$ is gsδI*-closed.

Theorem3.13. In an ideal space (X, τ, I), for each $x \in X$, either $\{x\}$ is semiclosed or $\{x\}^c$ is gsδI*-closed.
Proof. Suppose that $\{x\}$ is not a semiclosed set, then $\{x\}^c$ is not a semiopen set and hence X is the only semiopen set containing $\{x\}^c$. Therefore, $[\{x\}^c]^* \subseteq X$ and hence $\{x\}^c$ is gsδI*-closed in (X, τ, I).

Theorem 3.14. Every gsδI*-closed, semiopen set is δI*-closed.
Proof. Let A be a gsδI*-closed, semiopen set in (X, τ, I). Since A is semiopen such that $A \subseteq A$, by hypothesis, $[A]^* \subseteq A$. Thus, A is δI*-closed.

Corollary 3.15. Every gsδI*-closed; open set is δI*-closed set.

Theorem 3.16. If A and B are gsδI*-closed sets in an ideal topological space (X, τ, I), then $A \cup B$ is a gsδI*-closed set in (X, τ, I).
Proof. Suppose that $A \cup B \subseteq U$, where U is semi open set in (X, τ, I). Then $A \subseteq U$ and $B \subseteq U$. Since A and B are gsδI*-closed sets in (X, τ, I), $[A]^* \subseteq U$ and $[B]^* \subseteq U$. Always, $[A \cup B]^* = [A]^* \cup [B]^*$. Therefore, $[A \cup B]^* \subseteq U$, whenever U is semi open. Hence, $A \cup B$ is gsδI*-closed set in (X, τ, I).

Theorem 3.17. Let (X, τ, I) be an ideal space. If A is a gsδI*-closed subset of X and $A \subseteq B \subseteq [A]^*$, then B is also gsδI*-closed.
Proof. The proof is clear.

Theorem 3.18. A subset A of an ideal space (X, τ, I) is gsδI*-closed if and only if $[A]^* \subseteq \ker(A)$.
Proof. Necessity - Suppose A is gsδI*-closed and $x \in [A]^*$. If $x \notin \ker(A)$, then there exist a semiopen set U such that $A \subseteq U$ but $x \notin U$. Since A is gsδI*-closed, $[A]^* \subseteq U$ and so $x \notin [A]^*$, a contradiction. Therefore, $[A]^* \subset \ker(A)$.
Sufficiency - Suppose that $[A]^* \subseteq \ker(A)$. If $A \subseteq U$ and U is semiopen then $\ker(A) \subseteq U$ and so $[A]^* \subseteq U$. Therefore, A is gsδI*-closed.

Theorem3.19. Let A be a semi$\wedge$ - set of an ideal space (X, τ, I). Then A is gsδI*-closed if and only if A is δI*-closed.
Proof. Necessity - Suppose A is gsδI*-closed. Then by Theorem 3.18, $[A]^* \subseteq \ker(A) = A$, since A is semi$\wedge$ - set. Therefore, A is δI*-closed.
Sufficiency - The proof is follows from the Theorem 2.2.

Definition 3.20. A proper nonempty gsδI*-closed subset A of an ideal space (X, τ, I) is said to be maximal gsδI*-closed if any gsδI*-closed set containing A is either X or A.

Example 3.21. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {c, d}, {b, c, d}} and I = {ϕ, {d}}. Then {a, b, c} is a maximal gsδI*-closed set.

Theorem 3.22. In an ideal space (X, τ, I), the following are true.
(i) Let F be a maximal gsδI*-closed set and G be a gsδI*-closed set. Then F ∪ G = X or G ⊆F.
(ii) Let F and G be maximal gsδI*-closed sets. Then F ∪ G = X or F = G.
Proof. (i)Let F be a maximal gsδI*-closed set and G be a gsδI*-closed set. If F ∪ G = X, then there is nothing to prove. Assume that, F ∪ G ≠ X. Now, F ⊆ F ∪ G. By Theorem 3.16, F ∪ G
is a gsδI*-closed set. Since F is maximal gsδI*-closed, we have F ∪ G = X or F ∪ G = F. Hence, F ∪ G = F and so G ⊆ F.
(ii) Let F and G be maximal gsδI*-closed sets. If F ∪ G = X, then there is nothing to prove. Assume that, F ∪ G ≠ X. Then by (i), F ⊆ G and G ⊆ F, which implies that, F = G.

Theorem 3.23. A subset A of an ideal space (X, τ, I) is gsδI*-open if and only if F ⊆$[A]_{int}$* whenever F is semiclosed andF ⊆ A.
Proof. Necessity - Suppose A is gsδI*-open and F be a semiclosed set contained in A. Then X – A ⊆ X – F and hence [X – A]*⊆ X – F. Thus, F ⊆ X – [X – A]* = $[A]_{int}$*.
Sufficiency - Suppose X – A ⊆ U, where U is semiopen. Then X – U ⊆ A and X – U is semiclosed. Then X – U ⊆$[A]_{int}$*, which implies [X – A]*⊆U. Therefore, X – A is gsδI*-closed and hence A is gsδI*-open.

Theorem 3.24. If A is a gsδI*-open subset of an ideal space (X, τ, I) and $[A]_{int}$*⊆ B ⊆A. Then B is also a gsδI*-open subset of (X, τ, I).
Proof. Suppose F ⊆ B, where F is semiclosed set. Then, F ⊆ A. Since A is gsδI*-open, F ⊆$[A]_{int}$*. Since $[A]_{int}$*⊆$[B]_{int}$*, we have F ⊆$[B]_{int}$*. By the above Theorem 3.23, B is gsδI*-open.

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Reach Energy of Digraphs

V. Mahalakshmi[*]
B. Vijaya Praba[†]
K. Palani[‡]

Abstract

A Digraph D consists of two finite sets $(V, \mathcal{A})$, where V denotes the vertex set and $\mathcal{A}$ denotes the arc set. For vertices $u, v \in V$, if there exists a directed path from u to v then v is said to be reachable from u and vice versa. The Reachability matrix of D is the $n \times n$ matrix $R(D) = [r_{ij}]$, where $r_{ij} = 1$, if v_j is reachable from v_i and $r_{ij} = 0$ otherwise. The eigen values corresponding to the reachability matrix are called reach eigen values. The reach energy of a digraph is defined by $E_R(D) = \sum_{i=1}^{n} |\lambda_i|$where λ_i is the eigen value of the reachability matrix. In this paper we introduce the reach spectrum of a digraph and study its properties and bounds. Moreover, we compute reach spectrum for some digraphs.

Keywords: Reachable, Reachability matrix, reach eigen values, reach spectrum, Reach energy.

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[*]Assistant Professor (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - mahalakshmi@apcmcollege.ac.in
[†]Research Scholar Reg.No: 21212012092002 (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - vijayapraba.b@gmail.com
[‡] Third Author's Affiliation (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - palani@apcmcollege.ac.in
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India;
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1. Introduction

In this paper we considered simple and connected graph. A directed graph or digraph D consists of two finite sets $(V, \mathcal{A})$ where V denotes the vertex set and $\mathcal{A}$ denotes the arc set. For two vertices u and v, an arc from u to v is denoted by uv. Two vertices u and v is said to be adjacent if either $uv \in \mathcal{A}$ or $vu \in \mathcal{A}$. In 1978 Gutman [4] defined the energy of a simple graph as the sum of the absolute values of its eigen values and it is denoted by $E(G)$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The concept of graph energy was extended to digraph by Pena and Rada [8] and Adiga et al. [1]. Khan et al. [5] defined a new notion of energy of digraph called iota energy. In this paper, we investigate the properties and some bounds on reach energy.

Definition 1.1. A path is said to be directed path in which all the edges are directed either in clockwise or in anticlockwise direction and it is denoted as $\overrightarrow{P_n}$. Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of a directed path. Then the set $\{v_i v_{i+1} |\ i = 1,2,\dots, n-1\}$ is the arc set of $\overrightarrow{P_n}$.

Definition 1.2. A path is said to be alternate path in which the edges are given alternate direction and it is denoted as $\overrightarrow{AP_n}$

Definition 1.3. A star graph $K_{1,n}$ in which all the edges are directed towards the root vertex is called an instar and is denoted as $\overrightarrow{\iota K_{1,n}}$

Definition 1.4. A star graph $K_{1,n}$ in which all the edges are directed away from the root vertex is called an outstar and is denoted as $\overrightarrow{oK_{1,n}}$

2. Reach Energy

Definition 2.1. Let $D = (V, A)$ be a directed graph with n vertices. The reachability matrix [2] $R(D) = [r_{ij}]$ is the $n \times n$ matrix with $r_{ij} = 1$, if v_j is reachable from v_i and $r_{ij} = 0$ otherwise. We assume that each vertex is reachable from itself. The characteristic polynomial of $R(D) = [r_{ij}]$ is denoted by $f(D, \lambda) = det(R(D) - I\lambda)$. Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the reach eigen values of D. The reach eigen values of the graph D are the eigen values of $R(D)$ and is called as reach spectrum of D. The spectrum of D is denoted by

$$spec\ D = \begin{Bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m_1 & m_2 & \dots & m_n \end{Bmatrix}$$

where m_i is the algebraic multiplicity of the eigen values λ_i, for $1 \le i \le n$

Then the reach energy of D is defined as the sum of absolute values of reach spectrum of D.

i.e., $E_R(D) = \sum_{i=1}^{n} |\lambda_i|$

Example 2.2.

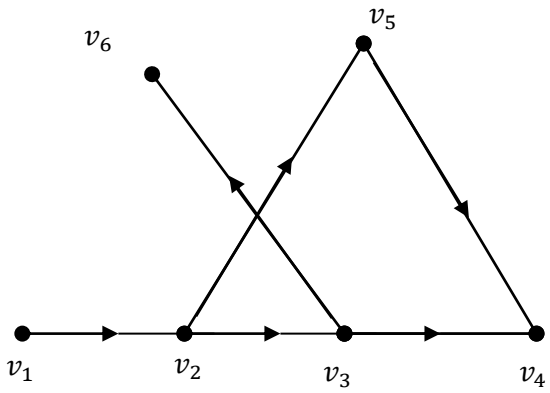

Figure 1.

Reachability matrix is

$$R(D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Characteristic polynomial of $R(D)$ is given by
$f(D,\lambda) = \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 15\lambda^2 - 6\lambda + 1.$
Hence, the reach spectrum is $\begin{Bmatrix} 1 \\ 6 \end{Bmatrix}$.
Therefore, the reach energy of D is $E_R(D) = 6$.

3. Reach Energy of Some Graphs

Theorem 3.1. Directed path and alternate path attains same Reach Energy.
Proof: Let $P_n(D)$ be the directed path with vertex set $V = \{v_1, v_2, \dots, v_n\}$
Let $AP_n(D)$ be the alternate path with vertex set $V = \{v_1', v_2', \dots, v_n'\}$.
The reachability matrix of $P_n(D)$ is in the upper triangular matrix form with the entries 1.
The reachability matrix of $AP_n(D)$ is of the form

$$R(AP_n(D)) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Characteristic polynomial of P_n is $f(P_n(D), \lambda) = \det(R(P_n(D)) - \lambda\, I_n)$

$$f(P_n(D), \lambda) = \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 0 & 1-\lambda & 1 & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\, (\lambda - 1)^n.$

Characteristic polynomial of AP_n is $f(AP_n(D), \lambda) = \det(R(AP_n(D)) - \lambda\, I_n)$

$$f(AP_n(D), \lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\, (\lambda - 1)^n.$
Clearly, $f(P_n(D), \lambda) = f(AP_n(D), \lambda).$

Since the characteristic polynomial of $P_n(D)$ and $AP_n(D)$ are same, Spectrum of $R(P_n)$ and $R(AP_n)$ are same.
Hence, the Reach Spectrum of $R(P_n(D))$ and $R(AP_n(D))$ are $\begin{Bmatrix} 1 \\ n \end{Bmatrix}$ and its Reach Energy is

$$E_R(P_n(D)) = E_R(AP_n(D)) = \sum_{1}^{n} 1 = n$$

Therefore, the directed path and alternate path attains same Reach Energy.

Theorem 3.2. Reach energy of directed star is independent of its orientation.
Proof: Let $K_{1,n-1}(D)$ be the directed instar with vertex set $v_1, v_2, \ldots, v_{n-1}$
Let $K'_{1,n-1}$(D) be the directed outstar with vertex set $v_1', v_2', \ldots, v_{n-1}'$
The reachability matrix of $K_{1,n-1}(D)$ is of the form

$$R\left(K_{1,n-1}(D)\right) = I_n + \begin{pmatrix} 0 & 0_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1\times n-1} \end{pmatrix}$$

The reachability matrix of $K'_{1,n-1}(D)$ is of the form

$$R\left(K'_{1,n-1}(D)\right) = I_n + \begin{pmatrix} 0 & J_{1\times n-1} \\ 0_{n-1\times 1} & 0_{n-1\times n-1} \end{pmatrix}$$

Characteristic polynomial of $K_{1,n-1}(D)$ is

$f\left(K_{1,n-1}(D),\lambda\right) = \det(R\left(K_{1,n-1}(D)\right) - \lambda\, I_n)$

$$f\left(K_{1,n-1}(D),\lambda\right) = \begin{vmatrix} 1-\lambda & 0 & \cdots & 0 \\ 1 & 1-\lambda & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 1 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda-1)^n$

Characteristic polynomial of $K'_{1,n-1}(D)$ is

$f\left(K'_{1,n-1}(D),\lambda\right) = \det(R\left(K'_{1,n-1}(D)\right) - \lambda\, I_n)$

$$f\left(K'_{1,n-1}(D),\lambda\right) = \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 0 & 1-\lambda & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda-1)^n$.
Clearly, $f\left(K_{1,n-1}(D)\,,\lambda\right) = f\left(K'_{1,n-1}(D)\,,\lambda\right)$.

Since the characteristic polynomial of $K_{1,n-1}(D)$ and $K'_{1,n-1}(D)$ are same, Spectrum of $R(K_{1,n-1}(D)\,)$ and $R(K'_{1,n-1}(D)\,)$ are same.
Hence, the Reach Spectrum of $R(K_{1,n-1}(D)\,)$ and $R(K'_{1,n-1}(D)\,)$ are $\begin{Bmatrix}1\\n\end{Bmatrix}$ and its Reach Energy

$$E_R\left(K_{1,n-1}(D)\right) = E_R\left(K'_{1,n-1}(D)\right) = \sum_{1}^{n} 1 = n$$

Therefore, the Reach Energy of directed star is independent of its orientation.

4. Properties of Reach Eigen values

Theorem 4.1: Let D be any digraph. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the reach eigen values of $R(D)$, then the following condition holds.

i. $\sum_{i=1}^{n} \lambda_i = n$
ii. $\sum_{i=1}^{n} {\lambda_i}^2 = n + \alpha + \beta$;

where $\alpha = \sum_{i>j} r_{ij} r_{ji}$ and $\beta = \sum_{i<j} r_{ij} r_{ji}$

Proof:

i. Sum of eigen values of $R(D)$ is same as the trace of $R(D)$.
i.e., $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} r_{ii}$;
Since each vertex is reachable from itself, all the diagonal entries must be 1.
$= 1 + 1 + 1 + \cdots + 1$ (n times)

Therefore,

$$\sum_{i=1}^{n} \lambda_i = n$$

ii. Since, the sum of squares of the eigen values of R is the trace of $[R(D)]^2$

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{i=1}^{n} r_{ij}\, r_{ji}$$

$$= \sum_{i=j=1}^{n} r_{ii}r_{ii} + \sum_{i\neq j=1}^{n} r_{ij}r_{ji}$$

$$= \sum_{i=j} (r_{ii})^2 + \sum_{i>j} r_{ij}r_{ji} + \sum_{i<j} r_{ij}r_{ji}$$

$= n + \alpha + \beta$; where $\alpha = \sum_{i>j} r_{ij}r_{ji}$ and $\beta = \sum_{i<j} r_{ij}r_{ji}$

Therefore,

$$\sum_{i=1}^{n} {\lambda_i}^2 = n + \alpha + \beta$$

5. Bounds for Reach Energy

Theorem 5.1: Let D be a directed graph. Let Z be the absolute value of determinant of the reachability matrix R of D i.e., $Z = |\det R(D)|$

Then

$$n\sqrt{n+\alpha+\beta} \leq E_R(D) \leq \sqrt{(n+\alpha+\beta) + n(n-1)Z^{2/n}}$$

Proof:

We know that Cauchy Schwarz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

Put $a_i = 1$, $b_i = |\lambda_i|$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)^2 \left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$[E_R(D)]^2 \leq n^2(n+\alpha+\beta)$$

$$E_R(D) \leq n\sqrt{n+\alpha+\beta} \qquad (1)$$

Since arithmetic mean is not smaller than geometric mean, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \prod_{i=1}^{n}|\lambda_i|^{\frac{2}{n}}$$

$$= \left|\prod_{i=1}^{n}\lambda_i\right|^{\frac{2}{n}}$$

$$= |\det R(D)|^{\frac{2}{n}} = Z^{\frac{2}{n}}$$

Therefore,

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq n(n-1)\, Z^{\frac{2}{n}} \qquad (2)$$

Now consider,

$$[E_R(D)]^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i||\lambda_i|$$

$$\geq (n+\alpha+\beta) + n(n-1)\, Z^{\frac{2}{n}});\ by\ (2)$$

Hence, $E_R(D) \geq \sqrt{(n+\alpha+\beta) + n(n-1)\, Z^{\frac{2}{n}}}$

From (1) and (2),

$$n\sqrt{n+\alpha+\beta} \leq E_R(D) \leq \sqrt{(n+\alpha+\beta) + n(n-1)Z^{2/n}}$$

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Some Graph Parameters of Clique graph of Cyclic Subgroup graph on certain Non-Abelian Groups

S. Ragha[*]
R. Rajeswari[†]

Abstract

The aim of this paper is to examine various graph parameters of clique graph of cyclic subgroup graph on certain non-abelian groups and also we obtain some theorems and results in detail.

Keywords: Cyclic Subgroup graph, Clique graph, Hub number, Topological Indices

2010 AMS subject classification: 05C09,05C25,05C12,05C50[‡]

[*]Research Scholar (Reg. No. 20212012092008), PG & Research Department of Mathematics, A.P.C Mahalaxmi College for Women, Thoothukudi-628002, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.raghasankar810@gmail.com

[†]Assistant Professor, PG &Research Department of Mathematics, A.P.C Mahalaxmi College for Women, Thoothukudi-628002, Tamil Nadu, India. rajimuthuram@gmail.com

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1. Introduction

Algebraic Graph theory is a branch of mathematics in which graphs are constructed from the algebraic structures such as groups, rings etc. J. John Arul Singh and S. Devi [3] have introduced the notion of Cyclic Subgroup Graph of a finite group. The concept of Clique graphs was discussed at least as early as 1968 by Hamelink and Ronald. C. After that, Roberts and Spencer have given the concept of A Characterization of Clique Graphs [2] in 1971. Gutman introduced the concept of Energy in 1978. A topological index of a graph G is a numerical parameter mathematically derived from the graph structure. It is a graph invariant which does not depend on the labeling or pictorial representation of the graph and it is the graph invariant number calculated from a graph representing a molecule. Our present work is provoked by the above study. In section 2, we discuss some graph parameters for the clique graph of cyclic subgroup graph on certain non-abelian groups. In section 3, we examine some energies on clique graph of cyclic subgroup graph for dihedral group and generalised quaternion group. In section 4, we give some topological indices for the clique graph of cyclic subgroup graph on certain non-abelian groups. In this paper, p represents prime number and pq represents distinct primes where $q > p$. Before entering, let us look into some necessary definitions and notations. The *cyclic subgroup graph*$\Gamma_z(G)$ for a finite group G is a simple undirected graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. The *clique graph*$\mathcal{K}(G)$ of an undirected simple graph G, is a graph with a node for each maximal cliques in G. Two vertices in $\mathcal{K}(G)$ are adjacent when their corresponding maximal cliques in G share at least one vertex in common. For an integer $n \geq 3$, the *dihedral group*D_{2n} of order $2n$ is $D_{2n} =< r, f: r^n = f^2 = 1, frf = r^{-1} >$. The generalized quaternion group $Q_{4n} =< a, b: b^2 = a^n, a^{2n} = e, bab^{-1} = a^{-1} >$, where e is the identity element.A *hub set* in a graph G is a set H of vertices in G, such that any two vertices outside H are connected by a path whose all internal vertices lie in H. The *hub number* of G, denoted by $h(G)$, is the minimum cardinality of a hub set in G. Let G be a simple undirected graph, note that $u_i \sim u_j$ denotes that u_i is adjacent to u_j for $1 \leq i \neq j \leq n$. The *adjacency matrix* of G denoted by $A(G) = (a_{ij})$ is an $n \times n$ matrix defined as $a_{ij} = 1$ when $u_i \sim u_j$ and 0 otherwise. The sum of the absolute values of the eigen values of its *adjacency matrix* is called the energy i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The *closed neighborhood matrix*$N = [n_{i,j}] = A + I_n$ has $n_{i,j} = 1$ if and only if $u_i \in N[u_j]$. The sum of the absolute values of the eigen values of its closed neighborhood matrix is called the *closed neighborhood energy*. The *Laplacian matrix* and *Signless Laplacian matrix* is defined as $L(G) = D(G) - A(G)$ and $SL(G) = D(G) + A(G)$ where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees.

2. Some Graph Parameters of Clique graph of Cyclic Subgroup graph on certain Non-Abelian Groups

Theorem 2.1: If $\mathcal{K}(\Gamma_z(G))$ be a clique graph of cyclic subgroup graph on a non-abelian group, then $\mathcal{K}(\Gamma_z(G))$ is biconnected.
Proof: Let $\mathcal{K}(\Gamma_z(G))$ be the clique graph of cyclic subgroup graph on a non-abelian group. Now, $W_1, W_2, \dots, W_n$ be the maximal cliques of $\Gamma_z(G)$. By the definition of clique graph, $V(U) = W_1, W_2, \dots, W_n$ and $(W_i, W_j) \in E(U)$ if only if $i \neq j$ and $W_i \cap W_j \neq \emptyset$ and take $U = \mathcal{K}(\Gamma_z(G))$. For $\Gamma_z(G)$, there exists a universal vertex which is adjacent to rest of its vertices. Now, any two vertices in $\mathcal{K}(\Gamma_z(G))$ are adjacent only when their corresponding maximal cliques in $\Gamma_z(G)$ have atleast one vertex in common. It is clear that, there is a path in between every starting vertex and ending vertex. Even after removing any vertex, the graph remains connected. Now concluding that G is connected and it does not contain any articulation point which results to a biconnected graph.

Theorem 2.2: For a clique graph of cyclic subgroup graph on any non-abelian group, the hub number is 0.
Proof: Based on the proof of 2.1, for $\mathcal{K}\left(\Gamma_z(G)\right)$, every pair of vertices are adjacent. Hence, there does not exist an intermediate vertex lies in the hub set. In this case, the minimum hub set is a null set. Hence, $h\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = 0$.

Theorem 2.3: Let $\mathcal{G} = \mathcal{K}(\Gamma_z(G))$ be a clique graph of cyclic subgroup graph on any non-abelian group and $|V(\mathcal{G})| = m$. Then $g(\mathcal{G}) = m(\mathcal{G}) = m$.
Proof: Let $\mathcal{G} = \mathcal{K}(\Gamma_z(G))$ be a clique graph of cyclic subgroup graph on any non-abelian group. The geodetic closure of a vertex set $S \subset \mathcal{V}$ is the set of all vertices $y \in \mathcal{V}$which lies in some geodesic in $\mathcal{G}$ joining two vertices u and v of S. Clearly for $\mathcal{K}\left(\Gamma_z(G)\right)$, there exists m maximal cliques which is connected by at least one vertex in common. Now, the resulting graph consists of m independent vertices in it. Hence $g(\mathcal{G}) = m$.
Consider, a set D of vertices of $\mathcal{K}(\Gamma_z(G))$ is a monophonic set of $\mathcal{K}\left(\Gamma_z(G)\right)$, if each vertex $v \in \mathcal{K}(\Gamma_z(G))$ lies on an $u - w$ monophonic path in $\mathcal{K}\left(\Gamma_z(G)\right)$ for some $u, w \in D$ and the minimum cardinality of a monophonic set of $\mathcal{K}\left(\Gamma_z(G)\right), m(\mathcal{G}) = m$.

Theorem 2.4: For any non-abelian group, $\mathcal{K}\left(\Gamma_z(G)\right)$ is non-planar.
Proof directly follows from theorem 2.1.

Theorem 2.5: Let $\mathcal{G} = \mathcal{K}(\Gamma_z(G))$ be a clique graph of cyclic subgroup graph on any non-abelian group and $|V(\mathcal{G})| = m$. Then $\kappa(\mathcal{G}) = m - 1$.
Proof: For $\mathcal{K}\left(\Gamma_z(G)\right)$, by removing $m - 1$ vertices which makes the graph disconnected.

Theorem 2.6: The independence number of $\mathcal{K}(\Gamma_z(G))$ is 1.
Proof follows from direct computation.

Theorem 2.7: For any non-abelian group, $h\left(\mathcal{K}(\Gamma_z(G))\right) \neq \gamma\left(\mathcal{K}(\Gamma_z(G))\right)$.

Proof: Let $\mathcal{K}(\Gamma_z(G))$ be a clique graph of cyclic subgroup graph on any non-abelian group. Here, every pair of vertices are adjacent. Choose any one vertex $u_1 \in \mathcal{K}(\Gamma_z(G))$ as a dominating set, which is adjacent to all other vertices. Hence, the domination number is 1. By theorem 2.2, this theorem can be proved.

3.Various Graph Energies on $\mathcal{K}(\Gamma_z(D_{2n}))$ and $\mathcal{K}(\Gamma_z(Q_{4n}))$

Theorem 3.1: The adjacency energy on clique graph of cyclic subgroup graph for a dihedral group of order $2n, n \in \mathbb{N}$ and $n > 2$ is

$E\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{cases} 2n & if\ n = p, p^2 \\ 2(n+1) & if\ n = pq \end{cases}$

Proof: Case i: For $n = p, p^2$
The vertex set of $\mathcal{K}(\Gamma_z(D_{2n})) = \{u_1, u_2, \dots, u_{n+1}\}$.
The Adjacency matrix can be written as $A = J_{n+1} - I_{n+1}$.
The obtained characteristic polynomial is $(x-n)(x+1)^n$.
The spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} n & -1 \\ 1 & n \end{Bmatrix}$.
For $n = p, p^2, E\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = 2n$

Case ii: For $n = pq$
The vertex set of $\mathcal{K}(\Gamma_z(D_{2n})) = \{u_1, u_2, \dots, u_{n+2}\}$.
The Adjacency matrix can be written as $A = J_{n+2} - I_{n+2}$.
The obtained characteristic polynomial is $(x-(n+1))(x+1)^{n+1}$.
The spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} n+1 & -1 \\ 1 & n+1 \end{Bmatrix}$.
For $n = pq, E\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = 2(n+1)$.

Theorem 3.2: If $n = p$, then the eigen values of $A\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right)$ are $n+1$ with multiplicity 1 &-1 with multiplicity $n+1$ and $E\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right) = 2(n+1)$.

Proof: The vertex set of $\mathcal{K}(\Gamma_z(Q_{4n})) = \{u_1, u_2, \dots, u_{n+2}\}$.
The Adjacency matrix can be written as $A = J_{n+2} - I_{n+2}$.
The obtained characteristic polynomial is $-(x-(n+1))(x+1)^{n+1}$.
The spectrum of $\mathcal{K}(\Gamma_z(Q_{4n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right) = \begin{Bmatrix} n+1 & -1 \\ 1 & n+1 \end{Bmatrix}$.
For $n = p, E\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right) = 2(n+1)$.

Theorem 3.3: The closed neighborhood energy on clique graph of cyclic subgroup graph for a dihedral group of order $2n, n \in \mathbb{N}$ and $n > 2$ is $E_N\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = \begin{cases} n+1 & if\ n = p, p^2 \\ n+2 & if\ n = pq \end{cases}$

Proof: Case i: For $n = p, p^2$

The vertex set of $\mathcal{K}\left(\Gamma_z(D_{2n})\right) = \{u_1, u_2, \dots, u_{n+1}\}$.

The Closed Neighborhood matrix can be written as $N = J_{n+1}$.

The obtained characteristic polynomial is $(x - (n+1))x^n$.

The Closed Neighborhood spectrum of $\mathcal{K}\left(\Gamma_z(D_{2n})\right)$will be written as,

$spec\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = \begin{Bmatrix} n+1 & 0 \\ 1 & n \end{Bmatrix}$.

For $n = p, p^2, E_N\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = n + 1$.

Case ii: For $n = pq$

The vertex set of $\mathcal{K}\left(\Gamma_z(D_{2n})\right) = \{u_1, u_2, \dots, u_{n+2}\}$.

The Closed Neighborhood matrix can be written as $N = J_{n+2}$.

The obtained characteristic polynomial is $(x - (n+2))x^{n+1}$.

The Closed Neighborhood spectrum of $\mathcal{K}\left(\Gamma_z(D_{2n})\right)$will be written as,

$spec\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = \begin{Bmatrix} n+2 & 0 \\ 1 & n+1 \end{Bmatrix}$.

For $n = pq, E_N\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = n + 2$.

Theorem 3.4: If $n = p$, then the eigen values of $N\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right)$ are $n+2$ with multiplicity 1 &0 with multiplicity $n+1$ and $E_N\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right) = n + 2$.

Proof: The vertex set of $\mathcal{K}\left(\Gamma_z(Q_{4n})\right) = \{u_1, u_2, \dots, u_{n+2}\}$.

The Closed Neighbourhood matrix can be written as $N = J_{n+2}$.

The obtained characteristic polynomial is $-(x - (n+2))x^{n+1}$.

The Closed Neighborhood spectrum of $\mathcal{K}\left(\Gamma_z(Q_{4n})\right)$will be written as, $spec\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right) = \begin{Bmatrix} n+2 & 0 \\ 1 & n+1 \end{Bmatrix}$.

For $n = p, E_N\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right) = n + 2$.

Theorem 3.5: The Laplacian spectrum on clique graph of cyclic subgroup graph for a dihedral group of order $2n, n \in \mathbb{N}$ and $n > 2$ is

(i) For $n = p, p^2, spec\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = \begin{Bmatrix} n+1 & 0 \\ n & 1 \end{Bmatrix}$

(ii) For $n = pq, spec\left(\mathcal{K}\left(\Gamma_z(D_{2n})\right)\right) = \begin{Bmatrix} n+2 & 0 \\ n+1 & 1 \end{Bmatrix}$.

Proof: Case i: For $n = p, p^2$

The vertex set of $\mathcal{K}\left(\Gamma_z(D_{2n})\right) = \{u_1, u_2, \dots, u_{n+1}\}$.

The Laplacian matrix can be written as $L = (n+1)I_{n+1} - J_{n+1}$.

The obtained characteristic polynomial is $(x - (n+1))^n x$.

The Laplacian spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} n+1 & 0 \\ n & 1 \end{Bmatrix}$.

Case ii: For $n = pq$

The vertex set of $\mathcal{K}(\Gamma_z(D_{2n})) = \{u_1, u_2, \dots, u_{n+2}\}$.

The Laplacian matrix can be written as $L = (n+2)I_{n+2} - J_{n+2}$.

The obtained characteristic polynomial is$(x-(n+2))^{n+1}x$.

The Laplacian spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} n+2 & 0 \\ n+1 & 1 \end{Bmatrix}$.

Theorem 3.6: If $n = p$, then the eigen values of $L\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right)$ are $n+2$ with multiplicity $n+1$ &0 with multiplicity 1.

Proof: The vertex set of $\mathcal{K}(\Gamma_z(Q_{4n})) = \{u_1, u_2, \dots, u_{n+2}\}$.

The Laplacian matrix can be written as $L = (n+2)I_{n+2} - J_{n+2}$.

The obtained characteristic polynomial is $-(x-(n+2))^{n+1}x$.

The Laplacian spectrum of $\mathcal{K}(\Gamma_z(Q_{4n}))$will be written as, $spec\left(\mathcal{K}(\Gamma_z(Q_{4n}))\right) = \begin{Bmatrix} n+2 & 0 \\ n+1 & 1 \end{Bmatrix}$.

Theorem 3.7: The signless Laplacian spectrum on clique graph of cyclic subgroup graph for a dihedral group of order $2n, n \in \mathbb{N}$ and $n > 2$ is

$(i) For\ n = p, p^2, spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} 2n & n-1 \\ 1 & n \end{Bmatrix}$

$(ii) For\ n = pq, spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} 2n+2 & n \\ 1 & n+1 \end{Bmatrix}$

Proof: Case i: For $n = p, p^2$

The vertex set of $\mathcal{K}(\Gamma_z(D_{2n})) = \{u_1, u_2, \dots, u_{n+1}\}$.

The Signless Laplacian matrix can be written as $SL = J_{n+1} + (n-1)I_{n+1}$.

The obtained characteristic polynomial is $(x-2n)(x-(n-1))^n$.

The Signless Laplacian spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as,

$spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} 2n & n-1 \\ 1 & n \end{Bmatrix}$.

Case ii: For $n = pq$

The vertex set of $\mathcal{K}(\Gamma_z(D_{2n})) = \{u_1, u_2, \dots, u_{n+2}\}$.

The Signless Laplacian matrix can be written as $SL = J_{n+2} + nI_{n+2}$.

The obtained characteristic polynomial is $(x-(2n+2))(x-n)^{n+1}$.

The Signless Laplacian spectrum of $\mathcal{K}(\Gamma_z(D_{2n}))$will be written as,

$spec\left(\mathcal{K}(\Gamma_z(D_{2n}))\right) = \begin{Bmatrix} 2n+2 & n \\ 1 & n+1 \end{Bmatrix}$.

Theorem 3.8: If $n = p$, then the eigen values of $SL\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right)$ are $2n+2$ with multiplicity 1 &n with multiplicity $n+1$.
Proof: The vertex set of $\mathcal{K}\left(\Gamma_z(Q_{4n})\right) = \{u_1, u_2, \dots, u_{n+2}\}$.
The Signless Laplacian matrix can be written as $SL = J_{n+2} + nI_{n+2}$.
The obtained characteristic polynomial is $(x-(2n+2))(x-n)^{n+1}$.
The Signless Laplacian spectrum of $\mathcal{K}\left(\Gamma_z(Q_{4n})\right)$will be written as,
$spec\left(\mathcal{K}\left(\Gamma_z(Q_{4n})\right)\right) = \left\{\begin{smallmatrix} 2n+2 & n \\ 1 & n+1 \end{smallmatrix}\right\}$.

4.Some Topological Indices on Clique graph of Cyclic Subgroup graph for certain Non-Abelian Groups

Theorem 4.1: If $\mathcal{K}\left(\Gamma_z(G)\right)$ be a clique graph of cyclic subgroup graph on a non-abelian group and $\left|V\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)\right| = m$, then the balaban index is $J\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{m^3-m^2}{2(m^2-3m+4)}$

Proof: $J\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{y}{y-x+2}\sum_{uv\in E\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)}\frac{1}{\sqrt{w(u).w(v)}}$, where the sum is taken over all edges of a connected graph G, x and y are the cardinalities of the vertex and the edge set of $G, w(u)$ and $w(v)$ denoted the sum of distances from u (resp.v) to all other vertices of G. $J\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{m^2-m}{m^2-m-2m+4}\left(\frac{m(m-1)}{2(m-1)}\right)$
$= \frac{m^2-m}{m^2-3m+4}\left(\frac{m}{2}\right) = \frac{m^3-m^2}{2(m^2-3m+4)}$

Theorem 4.2: If $\mathcal{K}\left(\Gamma_z(G)\right)$ be a clique graph of cyclic subgroup graph on a non-abelian group and $\left|V\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)\right| = m$, then atom bond connectivity status index is $ABCS\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{1}{\sqrt{2}}m\sqrt{m-2}$.

Proof: $ABCS\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \sum_{uv\in E\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)}\sqrt{\frac{\sigma(u)+\sigma(v)-2}{\sigma(u)\sigma(v)}}$
$= \left(\sqrt{\frac{(m-1)+(m-1)-2}{(m-1)(m-1)}}\right)\frac{m(m-1)}{2}$
$= \frac{1}{\sqrt{2}}m\sqrt{m-2}$.

Theorem 4.3: If $\mathcal{K}\left(\Gamma_z(G)\right)$ be a clique graph of cyclic subgroup graph on a non-abelian group and $\left|V\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)\right| = m$, then the arithmetic-geometric status index is $AGS\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{m(m-1)}{2}$.

Proof: $AGS\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \sum_{uv\in E\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)} \frac{\sigma(u)+\sigma(v)}{2\sqrt{\sigma(u)\sigma(v)}}$

$= \frac{m-1+m-1}{2\sqrt{(m-1)(m-1)}} \times \frac{m(m-1)}{2}$

$= \frac{m(m-1)}{2}$

Theorem 4.4: If $\mathcal{K}\left(\Gamma_z(G)\right)$ be a clique graph of cyclic subgroup graph on a non-abelian group and $\left|V\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)\right| = m$, then

(i)The First Zagrebdegree eccentricityindex, $DE_1\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = m^3$

(ii)The Second Zagreb degree eccentricity index, $DE_2\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \frac{m^3(m-1)}{2}$.

Proof: (i) $DE_1\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \sum_{v_i\in V\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)} (e_i + d_i)^2$ (where e_i be the eccentricity and d_i be the degree)

$= m(1+m-1)^2 = m^3$

$(ii) DE_2\left(\mathcal{K}\left(\Gamma_z(G)\right)\right) = \sum_{v_iv_j\in E\left(\mathcal{K}\left(\Gamma_z(G)\right)\right)} (e_i + d_i)(e_j + d_j)$

$= \frac{m(m-1)}{2}(1+m)(1+m)$

$= \frac{m^3(m-1)}{2}$.

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Conjunction Weighted Average Method with Fuzzy Expert System for Weather Event Forecasting – A Monthly Outlook

U. Ramya Devi[1]
K. Uma[2]

Abstract

Fuzzy logic as a limiting case of approximate reasoning is viewed in exact reasoning, consider everything in a matter of degree. A collection of elastic or equivalently interpreted to knowledge, a collection of variables in fuzzy constraint. Inference is process as a propagation of elastic constraints. Every logical system is fuzzified in fuzzy logic. Fuzzy logic is fascinating area of research, it trading off between significance and precision. It is convenient way to map space of input to a space of output. Fuzzy logic as so far as the laws of Mathematics refers to reality, they are not certain and so far, as they are certain as complexity rises, precise statements lose meaning and meaningful statements lose precision. Most meteorological infrastructure is surprisingly versatile. For example, the same radar system that can detect oncoming storms will also be useful for gathering general rainfall data for the farming sector. Being able to predict and forecast the weather also allows for data to be gathered to build up a more detailed picture of a nation's climate, and trends within it.

Keywords: Fuzzy logic, rainfall and weather forecasting

Mathematical classification: 94D05[3]

[1] PG and Research Department of Mathematics, Poompuhar College, (Autonomous), Melaiyur Nagapattinam Affiliated by Bharathidasan University-trained, India, E-mail: ramyadeviu@yahoo.com E-mail: umak197630@gmail.com

[2] PG and Research Department of Mathematics, Poompuhar College, (Autonomous),Melaiyur Nagapattinam Affiliated by Bharathidasan University Tamil Nadu, India, E-mail: ramyadeviu@yahoo.com E-mail:umak197630@gmail.com

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1. Introduction

Fuzzy logic models have been developed as a solution forecasting method, because the weather parameters can be easily classified unlike the other techniques mentioned above [4, 6]. Also, it does not require a computational mapping of inputs to outputs or no need for precise inputs. Fuzzy logic is simply a means of representing human reasoning. The main components of fuzzy logic are fuzzy set, membership function and fuzzy IF-THEN rule base. IF – THEN rule base is used to convert the fuzzy input into the fuzzy output [8]. In this research, the temperature, pressure, wind speed, humidity and the precipitation will serve as the input parameters for the Month forecasting. The aim of this work is to develop fuzzy logic methodology for weather forecasting with the following objectives: to formulate fuzzy logic membership function that will facilitate the monthly weather forecasting. Also, proposed monthly weather forecasting using fuzzy logic based on weather historical data (temperature, humidity, wind speed, pressure and precipitation) for the year 2021. Weather data used is from State Tamil Nadu, India. The work focused on month wise for the year 2021. Rainfall forecasts have significant value for resources planning and management e.g., reservoir operations, agricultural practices and flood emergency responses. To mitigate this, effective planning and management of water resources is necessary. In the short term, this requires a good idea of the upcoming season. In the long term, it needs realistic projections of scenarios of future variability and change [3].

2. Monthly Weather Forecasting

Monthly forecasting has the potential to play a significant role in enhancing end users' resilience to the impacts of climate change and variability. Smallholder farmers, for example, are vital to the economies and food security within the majority of the developing world, yet they are confronted with increasingly scarce resources, changing weather patterns, and extreme events that pose significant threats to the stability of both production and income. Monthly forecasts can provide this key stakeholder group with information to support their decision making regarding which crops they should plant, when to plant and harvest, and when to apply fertilizer and other inputs to maximize their yields and mitigate their losses.

2.1 Data and Area of Study

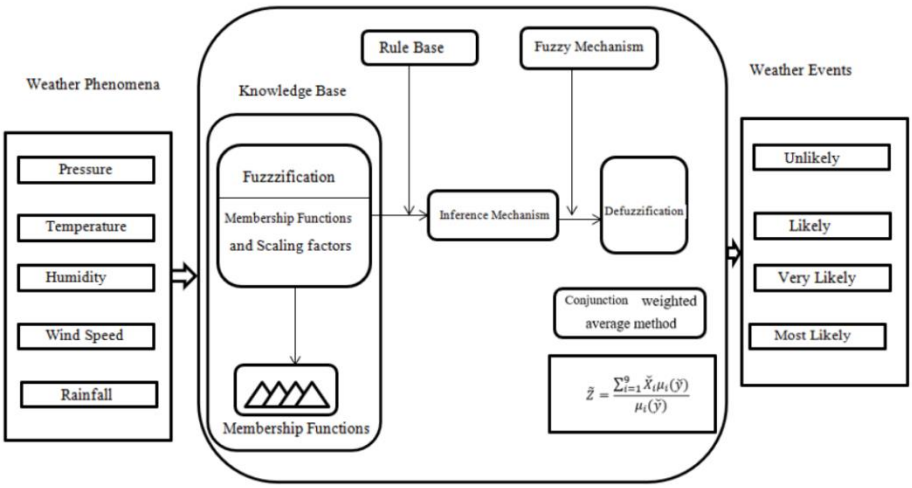
The study area is Tamil Nadu, which is one of the states in India where the climate is influenced mainly by the rain-bearing southwest monsoon winds from the ocean and the dry northwest winds from the Sahara Desert. When using Monthly forecasting as part of a Forecast-based Action system, it is important to comprehend how the prediction relates to the harm you are trying to mitigate. For instance, even a "above normal" rainfall season may have less flooding than a "normal" rainfall season depending on the type of rainfall (i.e., whether it falls gradually over several days or weeks versus all at once during a period of several hours). However, if there is a

forecast for below-average rainfall and a humanitarian organization wants to take proactive measures to address a drought, they may do so by using the monthly outlook.

Seasons of Tamil Nadu	
Season	**Period**
Winter Season	January-February
Summer Season	March- May
Southwest Monsoon	June-September
Northeast Monsoon	October –December

Table 1: Seasons of Tamil Nadu

3. Architecture of Proposed Model



4. Methodology

A fuzzy set is a function that transfers the universe object y on to the interval [0, 1]. a set b's fuzzy membership function represented mathematically, where the functional mapping is provided.

$\mu_{\widetilde{M}}$ (y)ϵ [0, 1]

similarly, the symbol $\mu_{\widetilde{M}}$ (y) in the degree of membership element y in the fuzzy set $\widetilde{M}$. A membership function that maps a component of a domain, space, or universe of discourse to the unit interval [0, 1] defines a fuzzy set. A fuzzy set $\widetilde{w}$ in a universe of discourse y is defined as following set of pairs

$\widetilde{w} = \{ y, \mu_{\widetilde{w}}$ (y); y $\in Y$ }

Here, $\mu_{\widetilde{w}}$: y→ [0,1] is a mapping called degree of membership function of fuzzy set $\widetilde{w}$ and $\mu_{\widetilde{w}}$ (y) is called the membership value of y ∈ Y in the fuzzy set $\widetilde{w}$. These membership levels are frequently expressed as real numbers between [0, 1].

4.1 Algorithm

Step 1:

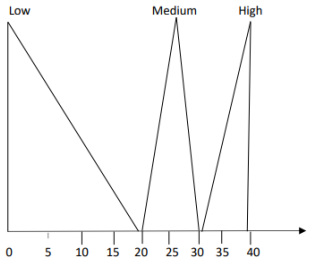
Construct a set$\sum_{i=1}^{12} \widetilde{W_i} = \sum_{i=1}^{12} \sum_{j=1}^{5} \widetilde{M}_i \, \tilde{O}_j$

Step 2:

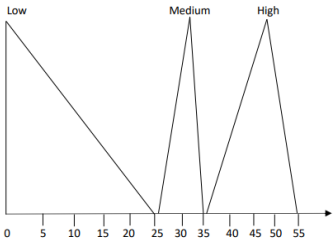
Create a triangular fuzzy membership function with respect to the decision makers parameters of their own choice.

Step 3:

Construct the Membership function for Pressure, Temperature, Humidity, Wind Speed, Rainfall.



Membership function of Wind Speed



Membership function of Temperature

Step 4:

Determine the product fuzzy conjunction

$\mu(\widetilde{W}) = \mu(\widetilde{M}_1\tilde{O}_1) \wedge \mu(\widetilde{M}_1\tilde{O}_2) \wedge \mu(\widetilde{M}_1\tilde{O}_3) \wedge \mu(\widetilde{M}_1\tilde{O}_4) \wedge \mu(\widetilde{M}_1\tilde{O}_5)$

Step 5:

Calculate the product fuzzy weighted average defuzzification method $\tilde{X} = \sum_{i=1}^{12}\sum_{j=1}^{5}\frac{\widetilde{W}_i \cdot \mu(\widetilde{W}_J)}{\mu(\widetilde{W}_J)}$

5. Case study

Consider the set $\widetilde{M} = \{\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3 \ldots\ldots\ldots\ldots \widetilde{M}_{12}\}$ as a universal sets where $\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3 \ldots\ldots\ldots\ldots \widetilde{M}_{12}$ represent the month from January to December for the Year 2021 and let the set $\tilde{O} = \{\tilde{O}_1, \tilde{O}_2, \tilde{O}_3, \tilde{O}_4, \tilde{O}_5\}$ where

$\tilde{O}_1 -$ Pressure
$\tilde{O}_2 -$ Temperature
$\tilde{O}_3 -$ Humidity
$\tilde{O}_4 -$ Wind Speed
$\tilde{O}_5 -$ Rainfall

The set $\tilde{O}$represent the parameter Environmental factors exposure to Monthly Weather forecasting outlook. It gives the relationship $\widetilde{W}$ called the set month and parameter data. Here, following steps made for weather forecasting in January 2021 in Tamil nadu.

Step 1& 2:

$\widetilde{W_1} = \{(\widetilde{M}_1\tilde{O}_1), (\widetilde{M}_1\tilde{O}_2), (\widetilde{M}_1\tilde{O}_3), (\widetilde{M}_1\tilde{O}_4), (\widetilde{M}_1\tilde{O}_5)\}$
= {(97.86), (22.85), (86.88), (2.27), (100.2)}

Step 3:

When $\widetilde{M}_1\tilde{O}_1 = 97.86$

$$\mu_{Low}(\widetilde{M}_1\tilde{O}_1) = \begin{cases} 0 & \widetilde{M}_1\tilde{O}_1 \geq 98.10 \\ 1 & \widetilde{M}_1\tilde{O}_1 = 96.01 \\ \dfrac{(98.10 - \widetilde{M}_1\tilde{O}_1)}{(98.10 - 96.01)} & 96.01 < \widetilde{M}_1\tilde{O}_1 < 98.10 \end{cases}$$

= 0.11

$$\mu_{Moderate}(\widetilde{M}_1\tilde{O}_1) = \begin{cases} 0 & \widetilde{M}_1\tilde{O}_1 \leq 98.11 \; or \widetilde{M}_1\tilde{O}_1 \geq 101.5 \\ \dfrac{(\widetilde{M}_1\tilde{O}_1 - 98.11)}{(99.13 - 101.5)} & 98.11 < \widetilde{M}_1\tilde{O}_1 < 99.13 \\ 1 & \widetilde{M}_1\tilde{O}_1 = 101.5 \\ \dfrac{(101.5 - \widetilde{M}_1\tilde{O}_1)}{(101.5 - 99..13)} & 101.5 < \widetilde{M}_1\tilde{O}_1 < 120.0 \end{cases}$$

$$\mu_{High}(\widetilde{M}_1\tilde{O}_1) = \begin{cases} 0 & \widetilde{M}_1\tilde{O}_1 \leq 101.6 \\ 1 & \widetilde{M}_1\tilde{O}_1 = 120.0 \\ \dfrac{(\widetilde{M}_1\tilde{O}_1 - 101.6)}{(120.0 - 101.6)} & 101.6 < \widetilde{M}_1\tilde{O}_1 < 120.0 \end{cases}$$

Step 4:

$= \mu_{Low}(\tilde{M}_1\tilde{O}_1) \wedge \mu_{Low}(\tilde{M}_1\tilde{O}_2) \wedge \mu_{High}(\tilde{M}_1\tilde{O}_3) \wedge \mu_{Low}(\tilde{M}_1\tilde{O}_4) \wedge \mu_{Low}(\tilde{M}_1\tilde{O}_5)$

$= 0.11 \wedge 0.086 \wedge 0.43 \wedge 0.9294 \wedge 0.0457$

Step 5:

$$= \frac{0.11\ X\ 97.86+0.086\ X\ 22.85+0.43\ X\ 86.88+0.9294\ X\ 2.27+0.0457\ X\ 100.2}{0.11+0.086+0.43+0.9294+0.0451}$$

$$= \frac{56.7769}{1.6011}$$

= 35.46 %

(i.e) January Month got 35.46 % that it is to be chance of possibility of to get Likely weather event.

6. Result and Justification

In this effort, Pressure, Temperature, Humidity, Wind Speed and Rainfall are taken as a important parameter of weather forecasting for monthly outlook based on Fuzzy Expert System with Product conjunction. The result appear for the month of February is 15.09% (i.e.,) Very less Rainfall and high temperature so it is considered to be unlikely weather event, meanwhile July 92.24% and November 97.67% (i.e.) more rainfall and less temperature so it is considered to be a most likely weather event month. For, remaining month weather event are given in table 2. This method is used to determination of monthly outlook weather forecasting with high accuracy.

Table 2: Weather Event prediction – A Monthly outlook.

Terminologies of Likely(L), Unlikely (UL), Most Likely (ML) are justified with standard operation procedure – weather forecasting and warning services given by Indian metrological department also discussed with expert metrologiest.

7.Conclusion

The effect of the shape of membership functions upon the solution is very important. Broader input membership functions, those with an extended domain with membership of 100%, have a larger weighting during the rule evaluation. This will be reflected in the final solution. In this work, fuzzy methodology for one-year 2021of Tamil Nadu weather event forecasting is discussed.

In this model, that is month wise weather Event prediction model, we applied the notion of Fuzzy Triangular membership function in Fuzzy Expert system. The benefit of this

model is that if the weather parameter and Triangular Fuzzy number are known, it is possible to find out monthly outlook of weather event as forecasting.

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