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Gaussian twin Neighborhood Prime Labeling on Fan Digraphs

Palani K^{*}
Shunmugapriya A[†]

Abstract

Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and $i^2 = -1$. The set of Gaussian integers is usually denoted by $\mathbb{Z}[i]$. A Gaussian integer $\gamma = a + ib \in \mathbb{Z}[i]$ is prime if and only if either $\gamma = \pm(1 \pm i)$, $N(\gamma) = a^2 + b^2$ is a prime integer congruent to 1 (mod 4), or $\gamma = p + 0i$ or $= 0 + pi$ where p is a prime integer and $|p| \equiv 3 \pmod{4}$. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \rightarrow [\gamma_n]$ is said to be a Gaussian twin neighborhood prime labeling of D , if it is both Gaussian in and out neighborhood prime labeling. A digraph which admits a Gaussian twin neighborhood prime labeling is called a Gaussian twin neighborhood prime digraph. In this paper, we introduce some definitions of fan digraphs. Further, we establish the Gaussian twin neighborhood prime labeling in fan digraphs using Gaussian integers.

Keywords: Gaussian integers, neighborhood prime, labeling, digraphs.

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1. Introduction

Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in [1]. An useful survey on graph labeling by J.A. Gallian can be found in [2]. Spiral ordering of the Gaussian integer was first introduced by Hunter Lehmann and Andrew Park in [4]. T. J. Rajesh Kumar and T. K. Mathew Varkey [7] introduced the concept of Gaussian neighborhood prime labeling of a graph. K. Palani [6] et.al, introduced the concept of Gaussian twin neighborhood prime labeling in digraphs.

Let $D = (V, A)$ be a digraph of order n . Then V is the set of vertices of D with $|V| = n$, and A is the set of arcs of D consisting of ordered pairs of distinct vertices. The in-degree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The out-degree $d^+(v)$ of v is the number of arcs having v as its initial vertex [2]. Throughout this article we use only digraphs.

In this article, we introduce the definition of fan and double fan digraphs by orienting fan and double fan graphs. Also, we investigate the existence of Gaussian twin neighborhood prime labeling in fan digraphs.

2. Preliminaries

The following basic definitions and properties are from [4]

Definition 2.1. Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and $i^2 = -1$. The set of Gaussian integers is usually denoted by $\mathbb{Z}[i]$.

A Gaussian integer is even if $1 + i$ divides γ . Otherwise it is an odd Gaussian integer.

Definition 2.2. A Gaussian integer $\gamma = a + ib \in \mathbb{Z}[i]$ is prime if and only if either

- (i) $\gamma = \pm(1 \pm i)$
- (ii) $N(\gamma) = a^2 + b^2$ is a prime integer congruent to 1 (mod 4), or
- (iii) $\gamma = p + 0i$ Or $= 0 + pi$ where p is a prime integer and $|p| \equiv 3(\text{mod } 4)$.

Definition 2.3. The spiral ordering of the Gaussian integers is recursively defined ordering of the Gaussian integers. We denote the n th Gaussian integer in the spiral ordering by γ_n . The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + i & \text{if } \text{Re}(\gamma_n) \equiv 1 \pmod{2}, \text{Re}(\gamma_n) > \text{Im}(\gamma_n) + 1 \\ \gamma_n - 1 & \text{if } \text{Im}(\gamma_n) \equiv 0 \pmod{2}, \text{Re}(\gamma_n) \leq \text{Im}(\gamma_n) + 1, \text{Re}(\gamma_n) > 1 \\ \gamma_n + 1 & \text{if } \text{Im}(\gamma_n) \equiv 1 \pmod{2}, \text{Re}(\gamma_n) < \text{Im}(\gamma_n) + 1 \\ \gamma_n + i & \text{if } \text{Im}(\gamma_n) \equiv 0 \pmod{2}, \text{Re}(\gamma_n) = 1 \\ \gamma_n - i & \text{if } \text{Re}(\gamma_n) \equiv 0 \pmod{2}, \text{Re}(\gamma_n) \geq \text{Im}(\gamma_n) + 1, \text{Im}(\gamma_n) > 0 \\ \gamma_n + 1 & \text{if } \text{Re}(\gamma_n) \equiv 0 \pmod{2}, \text{Im}(\gamma_n) = 0 \end{cases}$$

and $[\gamma_n]$ denotes the set of first n Gaussian integers in the spiral ordering.

Properties 2.4. 1. A Gaussian integer $\gamma = x + iy$ is called a prime Gaussian integer if its only divisors are $\pm 1, \pm i, \pm \gamma$ or $\pm i\gamma$.

2. Two Gaussian integers x and y are relatively prime if their only common divisors are the units in $\mathbb{Z}[i]$.
3. Let γ be a Gaussian integer and let u be a unit. Then γ and $\gamma + u$ are relatively prime.
4. In the spiral ordering, consecutive Gaussian integers are relatively prime.
5. In the spiral ordering, consecutive odd Gaussian integers are relatively prime.
6. Let α be a prime Gaussian integer and γ be a Gaussian integer. Then γ and $\gamma + \alpha$ are relatively prime if and only if $\alpha \nmid \gamma$.
7. Let γ be an odd Gaussian integer, let t be a positive integer and u be a unit. Then γ and $\gamma + u(1 + i)^t$ are relatively prime.

The following definitions are taken from [6]

Definition 2.5. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \rightarrow [\gamma_n]$ is called **Gaussian in-neighborhood prime labeling** of D , if for every vertex $v \in V(D)$ with $d^-(v) > 1$, the Gaussian integers in the set $\{f(u): u \in N^-(v)\}$ are relatively prime where $N^-(v) = \{u \in V(D) : \overrightarrow{uv} \in A(D)\}$.

Definition 2.6. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \rightarrow [\gamma_n]$ is called **Gaussian out-neighbourhood prime labeling** of D , if for every vertex $v \in V(D)$ with $d^+(v) > 1$, the Gaussian integers in the set $\{f(u): u \in N^+(v)\}$ are relatively prime where $N^+(v) = \{u \in V(D) : \overrightarrow{vu} \in A(D)\}$.

Definition 2.7. Let $D = (V, A)$ be a digraph with $|V| = n$. A function $f: V(D) \rightarrow [\gamma_n]$ is said to be a **Gaussian twin neighbourhood prime labeling** of D , if it is both Gaussian in and out neighborhood prime labeling. A digraph which admits Gaussian twin neighborhood prime labeling is called a Gaussian twin neighborhood prime digraph.

Observation 2.8.

1. If D is a digraph such that $N^+(v)$ or $N^-(v)$ are either $\varnothing$ or singleton set, then D admits a Gaussian twin neighborhood prime labeling.
2. A neighborhood prime digraph D in which every vertex is such that either its in-degree or out-degree at most 1 is Gaussian twin neighborhood prime.

The following definitions are referred from [8].

Definition 2.9. Fan graph is defined as the graph $P_n + K_1$, $n \geq 2$ where K_1 is the empty graph on one vertex and P_n , a path graph on n vertices.

Definition 2.10. A double fan DF_n consists of two fan graphs with a common path. In other words $DF_n = P_n + \overline{K_2}$, $n \geq 2$.

3. Fan Digraphs

In this section, some new digraphs are introduced by orienting fan graphs in different possible ways and named accordingly. Also we investigate the existence of the Gaussian twin neighborhood prime labeling of those digraphs.

Definition 3.1. In a $\text{fan}F_n = P_n + K_1$, orient the edges of the path P_n clockwise or anticlockwise and the spoke edges towards the central vertex. Call the resulting digraph as **in-fan** and denote it by $i\overrightarrow{F_n}$.

Definition 3.2. In a $\text{fan}F_n = P_n + K_1$, orient the edges of the path P_n clockwise or anticlockwise and the spoke edges away from the central vertex. Call the resulting digraph as **out-fan** and denote it by $o\overrightarrow{F_n}$.

Definition 3.3. A $\text{fan}F_n = P_n + K_1$, is said to be an **alternating fan** ($A\overrightarrow{F_n}$) if the edges of the path P_n are oriented clockwise or anticlockwise and the spoke edges alternately.

Definition 3.4. In a $\text{fan}F_n = P_n + K_1$, orient the path edges alternately and the spoke edges towards the central vertex. Call the resulting digraph as **alternating in-fan** and denote it by $Ai\overrightarrow{F_n}$.

Definition 3.5. In a $\text{fan}F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges away from the central vertex. Call the resulting digraph as **alternating out-fan** and denote it by $Ao\overrightarrow{F_n}$.

Definition 3.6. In a $\text{fan}F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges such that either $d^+(v) > 0$ or $d^-(v) > 0 \forall v \in V(P_n)$. Call the resulting digraph as **sole double alternating fan** and denote it by $SDA\overrightarrow{F_n}$.

Definition 3.7. In a $\text{fan}F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges such that neither $d^+(v) > 0$ nor $d^-(v) > 0 \forall v \in V(P_n)$. Call the resulting digraph as **di-double alternating fan** and denote it by $DDA\overrightarrow{F_n}$.

Theorem 3.8. In-fan ($i\overrightarrow{F_n}$) admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

Then $A(i\overrightarrow{F_n}) = \{\overrightarrow{v_i v_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_i u} | 1 \leq i \leq n\}$ is the arc set.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Define an injective function $f: V(i\overrightarrow{F_n}) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$ and $f(v_i) = \gamma_{i+1}$ for $1 \leq i \leq n$.

Here, $d^-(u) > 1$. Further, the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

$N^-(v_1) = \emptyset$ And $N^-(v_i) = \{v_{i-1}\}$ for $2 \leq i \leq n$.

Therefore, f is a Gaussian in-neighborhood prime labeling.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Gaussian twin neighborhood prime labeling of fan digraphs

Now $d^+(v_i) > 1$ for $1 \leq i \leq n-1$. Here, the out-neighborhood vertices of v_i ($1 \leq i \leq n-1$) contains the Gaussian integer $\gamma_1 = 1$ and γ_1 is relatively prime to all the Gaussian integers.

Further, $N^+(v_n) = \{u\}$ and $N^+(u) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling.

Which implies f is a Gaussian twin neighborhood prime labeling.

Hence, in-fan $(i\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Theorem 3.9. Out-fan $(o\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\vec{P}_n$ and u be the apex vertex.

Then $A(o\vec{F}_n) = \{\overrightarrow{v_i v_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{u v_i} | 1 \leq i \leq n\}$ is the arc set.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Define an injective function $f: V(o\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$ and $f(v_i) = \gamma_{i+1}$ for $1 \leq i \leq n$.

Now $d^-(v_i) > 1$ for $2 \leq i \leq n$. In the above labeling, the in-neighborhood vertices of v_i contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all Gaussian integers.

Further, $N^-(v_1) = \{u\}$ and $N^-(u) = \phi$.

Therefore, f is a Gaussian in-neighborhood prime labeling.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and the labels of the out-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_i) = \{v_{i+1}\}$ for $1 \leq i \leq n-1$ and $N^+(v_n) = \phi$

Therefore, f is a Gaussian out-neighborhood prime labeling.

Which implies f is a Gaussian twin neighborhood prime labeling.

Hence, out-fan $(o\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Theorem 3.10. Alternating fan $(A\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\vec{P}_n$ and u be the apex vertex.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Case (i): n is odd

$A(\overrightarrow{AF_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}v_{2i+1}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} \mid 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\}$ is the corresponding arc set.

Define $f: V(\overrightarrow{AF_n}) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$; $f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Clearly $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $N^-(u) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and hence are relatively prime.

Further, $N^-(v_{2i}) = \{u, v_{2i-1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of the in-neighbors of v_{2i} contains $\gamma_1 = 1$.

Also, $N^-(v_1) = \emptyset$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

$N^+(u) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the out-neighborhood vertices of u are labeled with the consecutive Gaussian integers in the spiral ordering and so by the result 1.4(4), the labels are relatively prime.

Now $N^+(v_{2i-1}) = \{u, v_{2i}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $N^+(v_n) = \{u\}$.

Also, the out-neighborhood vertices of v_{2i-1} ($1 \leq i \leq \frac{n-1}{2}$) contains the Gaussian integer $\gamma_1 = 1$.

Further, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(\overrightarrow{AF_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}v_{2i+1}} \mid 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} \mid 1 \leq i \leq \frac{n}{2}\}$ is the corresponding arc set.

Define $f: V(\overrightarrow{AF_n}) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$; $f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Clearly $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Now $N^-(u) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n}{2}$ and the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so are relatively prime.

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Further, $N^-(v_{2i}) = \{u, v_{2i-1}\}$ for $1 \leq i \leq \frac{n}{2}$ and the label set of in-neighbors of v_{2i} contains the Gaussian integer $\gamma_1 = 1$.

Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \leq i \leq \frac{n}{2}$

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

$N^+(u) = \{v_2, v_4, \dots, v_{2i}\}$ For $1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of u are labeled with the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^+(v_{2i-1}) = \{u, v_{2i}\}$ for $1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of v_{2i-1} ($1 \leq i \leq \frac{n}{2}$) contains the Gaussian integer $\gamma_1 = 1$.

Also, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and $N^+(v_n) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

f is a Gaussian twin neighborhood prime labeling when n is even.

Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.

Hence, alternating fan $(A\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Theorem 3.11. Alternating in-fan $(Ai\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Proof. Let $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\vec{P}_n$ and u be the apex vertex.

Case (i): n is odd

$A(Ai\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} \mid 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u} \mid 1 \leq i \leq \frac{n-1}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(Ai\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of in-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Now $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the labels of the in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering

Further, $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n+1}{2}$

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$ and the labels of vertices in $N^+(v_{2i-1})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Further, $N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n-1}{2}$ and $N^+(u) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

By (1) and (2), f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(Ai\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} \mid 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}u} \mid 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(Ai\overrightarrow{F_n}) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Here $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$.

Clearly, the label set of in-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Now $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and the label set of in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering.

Also, $N^-(v_n) = \{v_{n-1}\}$ and $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Now $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of v_{2i-1} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Further, $N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

By (3) and (4), f is a Gaussian twin neighborhood prime labeling when n is even.

Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.

Thus, an alternating in-fan $(Ai\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Theorem 3.12. Alternating out-fan $(Ao\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Proof: Let $n \geq 2$.

Let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

This digraph has $n + 1$ vertices and $2n - 1$ arcs.

Case (i): n is odd

$A(Ao\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} \mid 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} \mid 1 \leq i \leq \frac{n-1}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(Ao\overrightarrow{F_n}) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

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$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of in-neighbors of v_{2i} are the consecutive Gaussian integers in the spiral ordering.

Further, $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n+1}{2}$ and $N^-(u) = \phi$.

Therefore, f is a Gaussian in- neighborhood prime labeling when n is odd.

Next to prove f is a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of out-neighborhood vertices of u contains all the vertices of the path $\vec{P}_n$ which are labeled with the consecutive Gaussian integers in the spiral ordering.

Further, $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$ and the labels of the out-neighborhood vertices of v_{2i-1} are consecutive Gaussian integers in the spiral ordering.

Also, $N^+(v_1) = \{v_2\}$, $N^+(v_n) = \{v_{n-1}\}$ and $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

(1) and (2) imply f is a Gaussian twin neighborhood prime labeling if n is odd.

Case (ii): n is even

$A(Ao\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} \mid 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{uv_{2i-1}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} \mid 1 \leq i \leq \frac{n}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(Ao\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Now $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and the label set of the in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^-(u) = \phi$ and $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Here $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$.

Clearly, the labels out-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Further, $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n}{2}$ and the labels of out-neighborhood vertices of v_{2i-1} are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_1) = \{v_2\}$ and $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out- neighborhood prime labeling when n is even.

(3) and (4) imply f is a Gaussian twin neighborhood prime labeling if n is even.

From the cases (i) and (ii), f is a Gaussian twin neighborhood prime labeling.

Thus, an alternating outfanned $(Ao\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Theorem 3.13. Sole double alternating fan $(SDA\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Proof: Let $V(SDA\vec{F}_n) = \{u, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the i th vertex of the directed path $\vec{P}_n$ and u is the apex vertex.

This digraph has $n + 1$ vertices and $2n - 1$ arcs.

Case (i): n is odd

$A(SDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(SDA\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$$f(v_{2i-1}) = \gamma_{i+1} \text{ for } 1 \leq i \leq \frac{n+1}{2} \text{ and } f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1} \text{ for } 1 \leq i \leq \frac{n-1}{2}.$$

Now $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

The label set of in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Also, the in-neighborhood vertices of v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Further, $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

The label set of out-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Also, the label set of out-neighborhood vertices of v_{2i-1} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

$N^+(v_{2i}) = \emptyset$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(SDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(SDA\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$$f(v_{2i-1}) = \gamma_{i+1} \text{ for } 1 \leq i \leq \frac{n}{2} \text{ and } f(v_{2i}) = \gamma_{\frac{n}{2}+i+1} \text{ for } 1 \leq i \leq \frac{n}{2}.$$

Now $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

The label set of in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

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Further, the label set of in-neighborhood vertices of v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^-(v_{2i-1}) = \emptyset$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

The label set of out-neighborhood vertices of u are consecutive Gaussian integers and so they are relatively prime. Then the out-neighborhood vertices of v_{2i-1} contains the Gaussian integer $\gamma_1 = 1$.

Also, $N^+(v_{2i}) = \emptyset$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

Case (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.

Hence, the Sole-double alternating fan $(SDA\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Theorem 3.14. Di-double alternating fan $DDA\vec{F}_n$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Proof: Let $n \geq 2$ and let $V(DDA\vec{F}_n) = \{u, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the i th vertex of the directed path $\vec{P}_n$ and u is the apex vertex.

This digraph has $n + 1$ vertices and $2n - 1$ arcs.

Case (i): n is odd

$A(DDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n+1}{2}\}$ is the arc set.

Define an injective function $f: V(DDA\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $d^-(u) > 1$. The label set of in-neighborhood vertices of u are consecutive Gaussian integers in the in the spiral ordering and so they are relatively prime.

Also, $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$ and the in-neighborhood vertices of v_{2i} are labeled with the consecutive Gaussian integers.

Further, $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Now $d^+(u) > 1$ and the out-neighborhood vertices of u are consecutive Gaussian integers in the labeling and so they are relatively prime.

Also, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$ and the label set of the out-neighborhood vertices of v_{2i-1} contains the consecutive Gaussian integers in the spiral ordering.

Further, $N^+(v_1) = \{v_2\}$ and $N^+(v_n) = \{v_{n-1}\}$.

$N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(DDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} \mid 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i}u} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i-1}} \mid 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(DDA\vec{F}_n) \rightarrow [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Now $d^-(u) > 1$ and labeling of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and they are relatively prime.

Also, $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$ and the in-neighborhood vertices of v_{2i} are labeled with the consecutive Gaussian integers and so are relatively prime.

Further, $N^-(v_n) = \{v_{n-1}\}$ and $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and the labeling of the out-neighborhood vertices of u are consecutive Gaussian integers.

Also, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$ and the label set of the out-neighborhood vertices of v_{2i-1} ($2 \leq i \leq \frac{n}{2}$) are consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Also, $N^+(v_1) = \{v_2\}$ and $N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.

Thus the di-double alternating fan $(DDA\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

4. Double fan digraphs

In this section, some new digraphs are introduced by orienting double fan graphs in different possible ways and named accordingly. Also, the Gaussian twin neighborhood prime labeling is proved for those digraphs.

Definition 4.1. In a double fan $DF_n = P_n + \overline{K_2}$, orient the edges of the common path P_n clockwise or anticlockwise and the spoke edges are towards the central vertex. Call the resulting digraph as **double in-fan** and denote it by $Di\vec{F}_n$.

Definition 4.2. In a double fan $DF_n = P_n + \overline{K_2}$, orient the edges of the common path P_n clockwise or anticlockwise and the spoke edges away from the central vertex. Call the resulting digraph as **double out-fan** and denote it by $Do\overrightarrow{F_n}$.

Definition 4.3. A double fan $DF_n = P_n + \overline{K_2}$, is said to be a **double alternating fan** ($DA\overrightarrow{F_n}$) if the edges of the common path P_n are oriented in clockwise or anticlockwise and the spoke edges alternately.

Definition 4.4. In a double fan DF_n , orient the edges of the common path P_n alternately and the spoke edges are towards the central vertices. Call the resulting digraph as **double alternating in-fan** and denote it by $DAo\overrightarrow{F_n}$.

Definition 4.5. In a double fan DF_n , orient the edges of the common path P_n alternately and the spoke edges are away the central vertices. Call the resulting digraph is called a **double alternating out-fan** and denote it by $DAi\overrightarrow{F_n}$.

Definition 4.6. In a double fan DF_n , orient the edges of the common path P_n alternately and the spoke edges such that either $d^+(v) = 0$ or $d^-(v) = 0 \forall v \in V(P_n)$. The resulting digraph is called a **double sole-double alternating fan** and denoted it as $DSDA\overrightarrow{F_n}$.

Definition 4.7. In a double fan DF_n , orient the edges of the common path P_n alternately and the spoke edges such that neither $d^+(v) = 0$ nor $d^-(v) = 0 \forall v \in V(P_n)$. The resulting digraph is called a **double di-double alternating fan** and denote it by $DDDA\overrightarrow{F_n}$.

Theorem 4.8. Double in-fan ($Di\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph.

Proof: Let $V(Di\overrightarrow{F_n}) = \{u, w, v_i | 1 \leq i \leq n\}$ where u and w are the apex vertices and v_i represent the i th vertex of the directed path $\overrightarrow{P_n}$.

Then $A(Di\overrightarrow{F_n}) = \{\overrightarrow{v_i v_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_i u} | 1 \leq i \leq n\} \cup \{\overrightarrow{v_i w} | 1 \leq i \leq n\}$ is the arc set.

This digraph graph has $n+2$ vertices and $3n-1$ edges.

Define an injective function $f: V(Di\overrightarrow{F_n}) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$, $f(w) = \gamma_2$ and $f(v_i) = \gamma_{i+2}$ for $1 \leq i \leq n$.

Here $d^-(u) > 1$, $d^-(w) > 1$.

By the definition of f , the in-neighborhood vertices of u and w are labeled by consecutive Gaussian integers γ_1 and γ_2 in the spiral ordering and so are relatively prime.

Now, $N^-(v_1) = \emptyset$ and $N^-(v_i) = \{v_{i-1}\}$ for $2 \leq i \leq n$.

Therefore, f is a Gaussian in-neighborhood prime labeling.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Now, $d^+(v_i) > 1$ for $1 \leq i \leq n$.

Further, $N^+(v_i) = \{u, v_{i+1}, w\}$ for $1 \leq i \leq n-1$ and the labels of vertices in $N^+(v_i)$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^+(v_n) = \{u, w\}$ and labels of u and w are consecutive Gaussian integers.

Further, $N^+(u) = N^+(w) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling.

f is a Gaussian twin neighborhood prime labeling.

Thus, double fan($Di\vec{F}_n$) is a Gaussian twin neighborhood prime digraph.

Theorem 4.9. Double out-fan($Do\vec{F}_n$) admits Gaussian twin neighborhood prime labeling.

Proof: Let $V(Do\vec{F}_n) = \{u, w, v_i \mid 1 \leq i \leq n\}$ where u and w are the apex vertices and v_i represent the i th vertex of the directed path $\vec{P}_n$.

Then $A(Do\vec{F}_n) = \{\overrightarrow{v_i v_{i+1}} \mid 1 \leq i \leq n-1\} \cup \{\overrightarrow{u v_i} \mid 1 \leq i \leq n\} \cup \{\overrightarrow{w v_i} \mid 1 \leq i \leq n\}$ is the arc set.

This digraph has $n+2$ vertices and $3n-1$ edges.

Define an injective function $f: V(Do\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by

$f(u) = \gamma_1$, $f(w) = \gamma_2$ and $f(v_i) = \gamma_{i+2}$ for $1 \leq i \leq n$.

Now, $d^-(v_i) > 1$ for $1 \leq i \leq n$.

$N^-(v_1) = \{u, w\}$ and $N^-(v_i) = \{u, v_{i-1}, w\}$ for $2 \leq i \leq n$.

Clearly, label set of vertices in $N^-(v_i)$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^-(u) = N^-(w) = \phi$.

Therefore, f is a Gaussian in-neighborhood prime labeling.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Here $d^+(u) > 1$, $d^+(w) > 1$.

By the definition of f , the set of out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers in the spiral ordering and which are relatively prime.

Further, $N^+(v_i) = \{v_{i+1}\}$ for $1 \leq i \leq n-1$ and $N^+(v_n) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling.

f is a Gaussian twin neighborhood prime labeling.

Hence, double out-fan ($Do\vec{F}_n$) admits Gaussian twin neighborhood prime labeling.

Theorem 4.10. Double alternating fan ($DA\vec{F}_n$) is a Gaussian twin neighborhood prime digraph.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the path $\vec{P}_n$ and u, w be the apex vertices.

Let $V(DA\vec{F}_n) = \{u, w, v_i \mid 1 \leq i \leq n\}$ be the vertex set.

This digraph has $n+2$ vertices and $3n-1$ arcs.

Case (i): n is even

$A(DA\vec{F}_n) = \{\overrightarrow{v_{2i-1} v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i} v_{2i+1}} \mid 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1} u} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i-1} w} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{u v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{w v_{2i}} \mid 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(DA\overrightarrow{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$; $f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+2}$ for $1 \leq i \leq \frac{n}{2}$.

Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Clearly, the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering.

Further, $N^-(v_{2i}) = \{v_{2i-1}, u, w\}$ for $1 \leq i \leq \frac{n}{2}$. By the definition of f , the vertex u is labeled as $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.

Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \leq i \leq \frac{n}{2}$.

Therefore, f is Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Clearly, the label set of out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering.

Further, $N^+(v_{2i-1}) = \{v_{2i}, u, w\}$ for $1 \leq i \leq \frac{n}{2}$ and by the definition of f , the label of u is $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.

Also, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and $N^+(v_n) = \phi$.

Therefore, f is a Gaussian out neighborhood prime labeling when n is even.

f is a Gaussian twin neighborhood prime labeling when n is even.

Case (ii): n is odd.

Then, $A(DA\overrightarrow{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}v_{2i+1}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V(DA\overrightarrow{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$; $f(w) = \gamma_2$ and

$f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{(\frac{n+1}{2})+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering.

Further, $N^-(v_{2i}) = \{v_{2i-1}, u, w\}$ for $1 \leq i \leq \frac{n-1}{2}$ and by the definition of f , the vertex u has the label $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.

Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in- neighborhood prime labeling when n is odd.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

By the definition of f , the label set of out-neighborhood vertices of u and w consecutive Gaussian integers in the spiral ordering.

Also, $N^+(v_{2i-1}) = \{v_{2i}, u, w\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label of u is $\gamma_1 = 1$ is relatively prime to the Gaussian integers.

$N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$.

Further, $N^+(v_n) = \{u, w\}$. The vertices u and v are labeled by γ_1 and γ_2 respectively. Since γ_1 and γ_2 are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling.

From both the cases, f is a Gaussian twin neighborhood prime labeling.

Hence, double alternating fan $(DA\vec{F}_n)$ is a Gaussian twin neighborhood prime digraph.

Theorem 4.11. Double alternating in-fan $(DAi\vec{F}_n)$ admits Gaussian twin neighborhood prime labeling.

Proof. Let $V(DAi\vec{F}_n) = \{u, w, v_i | 1 \leq i \leq n\}$ where u, w the apex vertices are and v_i represent the i th vertex of the common path $\vec{P}_n$.

This digraph has $n + 2$ vertices and $3n - 1$ arcs.

Case (i): n is odd.

$A(DAi\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \leq i \leq \frac{n-1}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(DAi\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$, $f(w) = \gamma_2$,
 $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n+1}{2})+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

$N^-(u) = N^-(w) = \{v_1, v_2, \dots, v_n\}$. The vertices $v_1, v_2, \dots, v_n$ are labeled with the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the labels of $N^-(v_{2i})$ are consecutive Gaussian integers in the spiral ordering.

Also, $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Here $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$ and $d^+(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$

Further, $N^+(v_1) = \{v_2, u, w\}$ and $N^+(v_{2i-1}) = \{u, w, v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$.

and the label set of out-neighborhood vertices $v_{2i-1} (1 \leq i \leq \frac{n-1}{2})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^+(v_{2i}) = \{u, w\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the vertices u and w are labeled by the Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively.

Now, $N^+(u) = N^+(w) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling when n is odd

Case (ii): n is even

$A(DAi\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(DAi\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$; $f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+2}$ for $1 \leq i \leq \frac{n}{2}$.

Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$.

By the definition of f , the label set of the in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and the vertices in $N^-(v_{2i})$ are labeled by consecutive Gaussian integers in the spiral ordering.

Also, $N^-(v_n) = \{v_{n-1}\}$ and $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f admits a Gaussian in-neighborhood prime labeling when n is even.

Now $d^+(v_{2i-1}) > 1$ and $d^+(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Here, $N^+(u) = N^+(w) = \phi$.

Also, $N^+(v_1) = \{v_2, u, w\}$ and $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \leq i \leq \frac{n}{2}$ and the labels of $N^+(v_{2i-1})$ ($1 \leq i \leq \frac{n}{2}$) contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

$N^+(v_{2i}) = \{u, w\}$ for $1 \leq i \leq \frac{n}{2}$ and the vertices u and w are labelled by the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. So the labels of vertices in $N^+(v_{2i})$ are relatively prime.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

f is a Gaussian twin neighborhood prime labeling.

Hence, double alternating in-fan ($DAi\vec{F}_n$) admits Gaussian twin neighborhood prime labeling.

Theorem 4.12. Double alternating out-fan ($DAo\vec{F}_n$) is a Gaussian twin neighborhood prime digraph.

Proof. Let $V(DAo\vec{F}_n) = \{u, w, v_i | 1 \leq i \leq n\}$ where u, w are the apex vertices and v_i represent the i th vertex of the common path $\vec{P}_n$.

This digraph has $n + 2$ vertices and $3n - 1$ arcs.

Case (i): n is odd.

$A(DAo\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V(DAo\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1$, $f(w) = \gamma_2$,

$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n+1}{2})+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

The vertices u and w has no in-neighbors.

That is, $N^-(u) = N^-(w) = \phi$.

Now $N^-(v_{2i-1}) = \{u, w\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the vertices u and w are labelled by the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. So the labels of vertices in $N^-(v_{2i})$ are relatively prime.

Further, $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of in-neighborhood vertices v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Now, $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$.

The out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$ and the vertices v_{2i-2} and v_{2i} are labeled by $\gamma_{(\frac{n+1}{2})+i+1}$ and $\gamma_{(\frac{n+1}{2})+i+2}$ which are consecutive Gaussian integers in the spiral ordering.

$N^+(v_1) = \{v_2\}$ and $N^+(v_n) = \{v_{n-1}\}$.

Then, $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling when n is odd

Case (ii): n is even

$A(DAo\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the corresponding arc set.

Define an injective function $f: V(DAo\overrightarrow{F_n}) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1, f(w) = \gamma_2$,

$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n}{2})+i+2}$ for $1 \leq i \leq \frac{n}{2}$.

Here, $d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Further, $N^-(u) = N^-(w) = \phi$.

Also, $N^-(v_{2i-1}) = \{u, w\}$ for $1 \leq i \leq \frac{n}{2}$ and the vertices u and w are labeled with the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively.

Here, $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and $N^-(v_n) = \{u, w, v_{n-1}\}$.

Further, the label of the in-neighborhood vertices of v_{2i} ($1 \leq i \leq \frac{n}{2}$) contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now, $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$.

Clearly, the label of the out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Also, $N^+(v_1) = \{v_2\}$.

$N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ For $2 \leq i \leq \frac{n}{2}$. The vertices v_{2i-2} and v_{2i} are labeled by $\gamma_{\frac{n}{2}+i+1}$ and $\gamma_{\frac{n}{2}+i+2}$ which are consecutive Gaussian integers in the labeling in spiral ordering.

Also, $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

Therefore, f is a Gaussian twin neighborhood prime labeling when n is even.

Thus, double alternating out-fan ($DAo\vec{F}_n$) is a Gaussian twin neighborhood prime digraph.

Theorem 4.13. Double sole double alternating fan ($DSDA\vec{F}_n$) admits Gaussian twin neighborhood prime labeling.

Proof: Let $V(DSDA\vec{F}_n) = \{u, w, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the i th vertex of the common path $\vec{P}_n$ and u, w be the apex vertices.

This digraph has $n + 2$ vertices and $3n - 1$ arcs.

Case (i) n is odd.

$A(DSDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \leq i \leq \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V(DSDA\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1, f(w) = \gamma_2$,

$f(v_{2i-1}) = \gamma_{i+2}$ For $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n+1}{2})+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

Further, $N^-(u) = N^-(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the labels of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering and hence are relatively prime.

Also $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Here, $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Here, $d^+(u) > 1, d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

Now $N^+(u) = N^+(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_1) = \{u, w, v_2\}$, $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \leq i \leq \frac{n-3}{2}$ and

$N^+(v_n) = \{u, w, v_{n-1}\}$.

Further, the label of vertices in $N^+(v_{2i-1})$ for $1 \leq i \leq \frac{n-1}{2}$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

Therefore, f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(DSDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(DSDA\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1, f(w) = \gamma_2,$

$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n}{2})+i+2}$ for $1 \leq i \leq \frac{n}{2}.$

Here, $d^-(u) > 1, d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}.$

Now $N^-(u) = N^-(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n}{2}$ and the label set of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering.

Further, $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and $N^-(v_n) = \{u, w, v_{n-1}\}.$

By the definition of f , the label of the vertex u is γ_1 which is relatively prime to all the Gaussian integers.

$N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n}{2}.$

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Here $d^+(u) > 1, d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}.$

Now $N^+(u) = N^+(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers $\gamma_{(\frac{n}{2})+3}, \gamma_{(\frac{n}{2})+4}, \dots, \gamma_{(\frac{n}{2})+i+2}$ and so they are relatively prime.

Further, $N^+(v_1) = \{u, w, v_2\}$ and $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \leq i \leq \frac{n}{2}.$

and the labels of vertices in $N^+(v_1)$ and $N^+(v_{2i-1})$ for $2 \leq i \leq \frac{n}{2}$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n}{2}.$

Therefore, f is a Gaussian out-neighborhood prime labeling when n even.

From both the cases, f is a Gaussian twin neighborhood prime labeling.

Hence double sole double alternating fan $(DSDA\vec{F}_n)$ admits a Gaussian twin neighborhood prime labeling.

Theorem 4.14. Double di-double alternating fan $DDDA\vec{F}_n$ is a Gaussian twin neighborhood prime digraph.

Proof: Let $V(DDDA\vec{F}_n) = \{u, w, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the i th vertex of the common path $\vec{P}_n$ and u, w be the apex vertices.

This digraph has $n + 2$ vertices and $3n - 1$ arcs.

Case (i): n is odd

$A(DDDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \leq i \leq \frac{n-1}{2}\}$ is the arc set.

Gaussian twin neighborhood prime labeling of fan digraphs

Define an injective function $f: V(DDDA\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1; f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n+1}{2})+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1, d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $N^-(u) = N^-(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Also, $N^-(v_{2i-1}) = \{u, w\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the vertices u and w are labeled with consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$. Since the consecutive Gaussian integers in the spiral ordering are relatively prime.

Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the labels of the vertices in $N^-(v_{2i})$ are consecutive Gaussian integers and so are relatively prime.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd. Here $d^+(u) > 1, d^+(w) > 1, d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$ and $d^+(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $N^+(u) = N^+(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n+1}{2}$.

By the definition of f , the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers $\gamma_3, \gamma_4, \dots, \gamma_{i+2}$. Since the consecutive Gaussian integers in the spiral ordering are relatively prime.

$N^+(v_1) = \{v_2\}$ and $N^+(v_n) = \{v_{n-1}\}$.

Now $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$. The vertices v_{2i-2} and v_{2i} are labeled by the consecutive Gaussian integers and which are relatively prime.

Also, $N^+(v_{2i}) = \{u, w\}$ for $1 \leq i \leq \frac{n-1}{2}$. Since the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

(I) and (II) imply, f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(DDDA\vec{F}_n) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(DDDA\vec{F}_n) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1, f(w) = \gamma_2$, $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{(\frac{n}{2})+i+2}$ for $1 \leq i \leq \frac{n}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1, d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$.

Now $N^-(u) = N^-(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n}{2}$. Further, the labels of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Also, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and the vertices in $N^-(v_{2i})$ are labeled with the consecutive Gaussian integers and so those are relatively prime.

$N^-(v_n) = \{v_{n-1}\}$.

$N^-(v_{2i-1}) = \{u, w\}$ for $1 \leq i \leq \frac{n}{2}$. Since the vertices u and w are labeled with the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. Then γ_1 and γ_2 are relatively prime.

Therefore, f is a Gaussian in- neighborhood prime labeling when n is even.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Here $d^+(u) > 1$, $d^+(w) > 1$, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$ and $d^+(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

Now $N^+(u) = N^+(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n}{2}$.

By the definition of f , the label set of the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers $\gamma_3, \gamma_4, \dots, \gamma_{i+2}$ and so those are relatively prime.

Also, $N^+(v_1) = \{v_2\}$.

Now $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n}{2}$ and the vertices v_{2i-2} and v_{2i} are labeled by the consecutive Gaussian integers $\gamma_{\frac{n}{2}+i+1}$ and $\gamma_{\frac{n}{2}+i+2}$.

$N^+(v_{2i}) = \{u, w\}$ for $1 \leq i \leq \frac{n}{2}$. Since the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the consecutive Gaussian integers.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

(III) and (IV) imply f is a Gaussian twin neighborhood prime labeling when n is even.

From cases (i) and (ii), f is a Gaussian twin neighborhood prime labeling.

Hence, double di-double alternating fan $DDDA\vec{F}_n$ is a Gaussian twin neighborhood prime digraph.

5 Conclusions

In this article, we established Gaussian twin neighborhood prime labeling in fan and double fan digraphs. In this way, we extend our thoughts to different types of labeling in digraphs.

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Radio Heronian Mean k -Graceful Labeling on Degree Splitting of Graphs

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Abstract

A mapping $g: V(G) \rightarrow \{k, k+1, \dots, k+N-1\}$ is a radio heronian mean k -labeling such that if for any two distinct vertices s and t of G , $d(s, t) + \left\lceil \frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3} \right\rceil \geq 1 + D$, for every $s, t \in V(G)$, where D is the diameter of G . The radio heronian mean k -number of G , $rrhmn_k(G)$, is the maximum number assigned to any vertex of G . The radio heronian mean number of G , $rhmn_k(G)$, is the minimum value of $rhmn_k(G)$ taken over all radio heronian mean labelings g of G . If $rhmn_k(G) = |V(G)| + k - 1$, we call such graphs as radio heronian mean k -graceful graphs. In this paper, we investigate the radio heronian mean k -graceful labeling on degree splitting of graphs such as comb graph $P_n \odot K_1$, rooted tree graph $RT_{n,n}$, hurdle graph Hd_n and twig graph TW_n .

Keywords: Radio heronian mean labeling, k -graceful labeling, degree splitting graphs, comb graph, rooted tree graph, hurdle graph and twig graph.

AMS Subject Classification 2020: 05C78[‡].

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1. Introduction

Throughout this paper we consider the graphs are finite, simple, undirected and connected. Let $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G . A labeling of a graph G is an assignment of integers to the vertices or edges or both subject to certain conditions. Radio labeling or multilevel distance labeling is motivated by the channel assignment problem for radio transmitters [5]. Ponraj et al [8] introduced the notion of radio mean labeling of graphs and investigated radio number of some graphs. The radio mean number of, $rmn(G)$, is the lowest span taken overall radio mean labelings of the graph G . The span of a labelling is the maximum integer that g maps to a vertex of G . S. S. Sandya, et al [10] introduced heronian mean labeling of graphs. Y. Lavanya et al [6], raised the ideas about radio mean graceful graphs. In this sequel, we introduced radio heronian mean k -graceful labelling on degree splitting of graphs. For standard terminology and notations, we follow Harary [4] and Gallian [3].

2. Preliminaries

Definition 2.1. [1] The distance $d(s, t)$ from a vertex s to a vertex t in a connected graph G is the minimum length of the $s - t$ paths in G .

Definition 2.2. [1] The eccentricity $e(t)$ of a vertex t in a connected graph G is the distance between t and a vertex farthest from t in G .

Definition 2.3. [1] The diameter D is the greatest eccentricity among the vertices of G .

Definition 2.4. [8][9] A radio mean labeling of a connected graph G is a one-to-one map $g : V(G) \rightarrow \mathbb{N}$ such that for any two distinct vertices s and t of G , $d(s, t) + \left\lfloor \frac{g(s) + g(t)}{2} \right\rfloor \geq 1 + D$ for every $s, t \in V(G)$, where D is the diameter of G . The radio mean number of G is denoted by $rmn(G)$.

Definition 2.5. [10] A graph $G = (V, E)$ with p vertices and q edges is said to be a heronian mean graph if it is possible to label the vertices $s \in V$ with distinct labels $g(s)$ from $1, 2, \dots, q + 1$ in such way that when each edge $e = st$ is labelled with, $g(e = s, t) = \left\lfloor \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rfloor$ (or) $\left\lfloor \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rfloor$ then the edge labels are distinct. In this case g is called a heronian mean labelling of G .

Definition 2.6. [1] Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \dots \cup S_t \cup T$, where each S_i is a set of vertices having atleast two vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining w_i to each vertex of S_i ($1 \leq i \leq t$).

Definition 2.7. A radio heronian mean k -labeling of a connected graph G is a one to one map $g : V(G) \rightarrow \{k, k + 1, k + 2, \dots, k + N - 1\}$ such that for any two

distinct vertices s and t of G , $d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 1 + D$, for every $s, t \in V(G)$ where D is the diameter of G . The Radio heronian mean k -number of G is denoted by $rhmn_k(g)$. If $rhmn_k(g) = |V(G)| + k - 1$, we call such graphs as radio heronian mean k -graceful graphs.

Definition 2.8. [1] The comb graph $P_n \odot K_1$ is obtained by joining a single pendent edge to each vertex of a path P_n .

Definition 2.9. [2] A tree in which one vertex is distinguished from all the other is called a rooted tree and the vertex is called the root of the tree. A rooted tree with n vertices is denoted by $RT_{n,n}$.

Definition 2.10. [7] A graph obtained from a path P_n by attaching a pendant edge to every internal vertex of the path is called the hurdle graph with $n - 2$ hurdles and is denoted by Hd_n .

Definition 2.11. [7] A twig graph TW_n is obtained from a path P_n by attaching exactly two pendant edges to each internal vertices of the path P_n .

3. Main Results

Theorem 3.1. The degree splitting of comb graph $DS(P_n \odot K_1)$ is radio heronian mean k -graceful for $n \geq 4$ and $k \geq 1$.

Proof. Let $s_i, 1 \leq i \leq n$ be the vertices of the path K_1 . Join the pendent vertex t_i to the vertex s_i of the path $P_n, 1 \leq i \leq n$. The resultant graph is $P_n \odot K_1$. Introduce three new vertices s, s' and t . Join s' to s_1, s_n and s to $s_i, 2 \leq i \leq n - 1$ and t to $t_i, 1 \leq i \leq n$ respectively. The resulting graph is $DS(P_n \odot K_1)$ whose edge set is $E(DS(P_n \odot K_1)) = \{s_i s_{i+1} / 1 \leq i \leq n - 1\} \cup \{s_i t_i / 1 \leq i \leq n\} \cup \{s s_i / 2 \leq i \leq n - 1\} \cup \{s' s_1, s' s_n\}$ and diameter of $DS(P_n \odot K_1)$ is $D = 3$.

Define a function $g : V(DS(P_n \odot K_1)) \rightarrow \{k, k + 1, \dots, k + N - 1\}$ by

$$\begin{aligned} g(s') &= k, k \geq 1; \\ g(s) &= k + 1, k \geq 1; \\ g(t) &= k + 2, k \geq 1; \\ g(s_i) &= n + k + i + 2, k \geq 1, 1 \leq i \leq n; \\ g(t_i) &= k + i + 2, k \geq 1, 1 \leq i \leq n; \end{aligned}$$

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s, s')

$$d(s, s') + \left\lceil \frac{g(s) + g(s') + \sqrt{g(s)g(s')}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k + 1 + \sqrt{k(k + 1)}}{3} \right\rceil \geq 4 = 1 + D$$

Case b: Consider the pair (t, s')

$$d(t, s') + \left\lfloor \frac{g(t) + g(s') + \sqrt{g(t)g(s')}}{3} \right\rfloor \geq 3 + \left\lfloor \frac{2k + 2 + \sqrt{(k+2)(k+1)}}{3} \right\rfloor \geq 4$$

Case c: Consider the pair (s, t)

$$d(s, t) + \left\lfloor \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rfloor \geq 3 + \left\lfloor \frac{2k + 3 + \sqrt{(k+2)(k+1)}}{3} \right\rfloor \geq 4$$

Case d: Consider the pair (s', s_i) , $1 \leq i \leq n$

$$\begin{aligned} d(s', s_i) + \left\lfloor \frac{g(s') + g(s_i) + \sqrt{g(s')g(s_i)}}{3} \right\rfloor \\ \geq 1 + \left\lfloor \frac{n + 2k + i + 2 + \sqrt{k(n+k+i+1)}}{3} \right\rfloor \geq 4 \end{aligned}$$

Case e: Consider the pair (s', t_i) , $1 \leq i \leq n$

$$d(s', t_i) + \left\lfloor \frac{g(s') + g(t_i) + \sqrt{g(s')g(t_i)}}{3} \right\rfloor \geq 2 + \left\lfloor \frac{2k + i + 2 + \sqrt{k(k+i+1)}}{3} \right\rfloor \geq 4$$

Case f: Consider the pair (s, s_i) , $1 \leq i \leq n$

$$\begin{aligned} d(s, s_i) + \left\lfloor \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rfloor \\ \geq 1 + \left\lfloor \frac{n + 2k + i + 3 + \sqrt{(k+1)(n+k+i+2)}}{3} \right\rfloor \geq 4 \end{aligned}$$

Case g: Consider the pair (s, t_i) , $1 \leq i \leq n$

$$\begin{aligned} d(s, t_i) + \left\lfloor \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rfloor &\geq 2 + \left\lfloor \frac{2k + i + 3 + \sqrt{(k+1)(k+i+2)}}{3} \right\rfloor \\ &\geq 4 \end{aligned}$$

Case h: Consider the pair (t, s_i) , $1 \leq i \leq n$

$$\begin{aligned} d(t, s_i) + \left\lfloor \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rfloor \\ \geq 2 + \left\lfloor \frac{n + 2k + i + 4 + \sqrt{(k+2)(n+k+i+2)}}{3} \right\rfloor \geq 4 \end{aligned}$$

Case i: Consider the pair (t, t_i) , $1 \leq i \leq n$

$$\begin{aligned} d(t, t_i) + \left\lfloor \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rfloor &\geq 1 + \left\lfloor \frac{2k + i + 4 + \sqrt{(k+2)(k+i+2)}}{3} \right\rfloor \\ &\geq 4 \end{aligned}$$

Case j: Consider the pair (s_i, s_j) , $i \neq j, 1 \leq i \leq n$

$$d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil \\ \geq 1 + \left\lceil \frac{2n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(n + k + j + 2)}}{3} \right\rceil \geq 4$$

Case k: Consider the pair (t_i, t_j) , $i \neq j, 1 \leq i \leq n$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil \\ \geq 2 + \left\lceil \frac{2k + i + j + 4 + \sqrt{(k + i + 2)(k + j + 2)}}{3} \right\rceil \geq 4$$

Case j: Consider the pair (s_i, t_j) , $1 \leq i \leq n$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil \\ \geq 1 + \left\lceil \frac{n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(k + j + 2)}}{3} \right\rceil \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence g is a valid radio heronian mean k -labeling of $DS(P_n \odot K_1)$

Therefore, $rhmn_k(DS(P_n \odot K_1)) = 2n + k + 2, n \geq 4, k \geq 1$.

Clearly $|V(DS(P_n \odot K_1))| = 2n + 3$.

Thus $|V(DS(P_n \odot K_1))| + k - 1 = 2n + k + 2, n \geq 4, k \geq 1$.

Therefore, $rhmn_k(DS(P_n \odot K_1)) = |V(DS(P_n \odot K_1))| + k - 1$.

Hence the degree splitting of comb graph $DS(P_n \odot K_1)$ is radio heronian mean k -graceful for $n \geq 4$ and $k \geq 1$. ■

Theorem 3.2. The degree splitting of rooted tree graph $DS(RT_{n,n})$ is radio heronian mean k -graceful for $n \geq 2$ and $k \geq 1$.

Proof. Let s be the root of the tree and $s_i, 1 \leq i \leq n$ be the vertices which are joined to the vertex s of the tree. Let $t_i, 1 \leq i \leq n$ be the vertices which are joined to the vertex $s_i, 1 \leq i \leq n$. Introduce two new vertices s' and t' . Join s' to s_i and t' to t_i the resultant graph is $DS(RT_{n,n})$ whose edge set is $E(DS(RT_{n,n})) = \{ss_i, s_i t_i, s' s_i, t' t_i / 1 \leq i \leq n\}$ and diameter of $DS(RT_{n,n})$ is $D = 3$.

Define a function $g : V(DS(RT_{n,n})) \rightarrow \{k, k + 1, \dots, k + N - 1\}$ by

$$g(s) = k + 2, k \geq 1;$$

$$g(s') = k + 1, k \geq 1;$$

$$g(s_i) = k + i + 2, k \geq 1, 1 \leq i \leq n$$

$$g(si) = k + i + 2, k \geq 1, 1 \leq i \leq n;$$

$$g(t') = k, k \geq 1$$

$$g(t_i) = n + k + i + 2, k \geq 1, 1 \leq i \leq n;$$

Now we verify the radio heronian mean condition for g .

Case a: Consider the pair (s, s')

$$d(s, s') + \left\lfloor \frac{g(s) + g(s') + \sqrt{g(s)g(s')}}{3} \right\rfloor \geq 2 + \left\lfloor \frac{2k + 3 + \sqrt{(k+2)(k+1)}}{3} \right\rfloor \geq 4$$

$$= 1 + D$$

Case b: Consider the pair (s, s_i) , $1 \leq i \leq n$

$$d(s, s_i) + \left\lfloor \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rfloor \geq 1 + \left\lfloor \frac{2k + i + 4 + \sqrt{(k+2)(k+i+2)}}{3} \right\rfloor$$

$$\geq 4$$

Case c: Consider the pair (s, t')

$$d(s, t') + \left\lfloor \frac{g(s) + g(t') + \sqrt{g(s)g(t')}}{3} \right\rfloor \geq 3 + \left\lfloor \frac{2k + 2 + \sqrt{(k+2)(k)}}{3} \right\rfloor \geq 4$$

Case d: Consider the pair (s, t_i) , $1 \leq i \leq n$

$$d(s, t_i) + \left\lfloor \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rfloor$$

$$\geq 2 + \left\lfloor \frac{n + 2k + i + 4 + \sqrt{(k+2)(n+k+i+2)}}{3} \right\rfloor \geq 4$$

Case e

Consider the pair (s_i, s_j) , $i \neq j, 1 \leq i \leq n$

$$d(s_i, s_j) + \left\lfloor \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rfloor$$

$$\geq 2 + \left\lfloor \frac{2k + i + j + 4 + \sqrt{(k+i+2)(k+j+2)}}{3} \right\rfloor \geq 4$$

Case f: Consider the pair (t_i, t_j) , $i \neq j, 1 \leq i \leq n$

$$d(t_i, t_j) + \left\lfloor \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rfloor$$

$$\geq 2 + \left\lfloor \frac{2n + 2k + i + j + 4 + \sqrt{(n+k+i+2)(n+k+j+2)}}{3} \right\rfloor \geq 4$$

Case g: Consider the pair (s_i, t_j) , $1 \leq i, j \leq n$

$$d(s_i, t_j) + \left\lfloor \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rfloor$$

$$\geq 1 + \left\lfloor \frac{n + 2k + i + j + 4 + \sqrt{(k+i+2)(n+k+j+2)}}{3} \right\rfloor \geq 4$$

Case h: Consider the pair (s_i, t') , $1 \leq i \leq n$

$$d(s_i, t') + \left\lfloor \frac{g(s_i) + g(t') + \sqrt{g(s_i)g(t')}}{3} \right\rfloor \geq 2 + \left\lfloor \frac{2k + i + 2 + \sqrt{(k+i+2)(k)}}{3} \right\rfloor \geq 4$$

Case i: Consider the pair (s', t_i) , $1 \leq i \leq n$

$$d(s', t_i) + \left\lfloor \frac{g(s') + g(t_i) + \sqrt{g(s')g(t_i)}}{3} \right\rfloor$$

$$\geq 2 + \left\lfloor \frac{n + 2k + i + 3 + \sqrt{(k+1)(n+k+i+2)}}{3} \right\rfloor \geq 4$$

Case j: Consider the pair (s', s_i) , $1 \leq i \leq n$

$$d(s', s_i) + \left\lfloor \frac{g(s') + g(s_i) + \sqrt{g(s')g(s_i)}}{3} \right\rfloor$$

$$\geq 1 + \left\lfloor \frac{2k + i + 3 + \sqrt{(k+1)(k+i+2)}}{3} \right\rfloor \geq 4$$

Case k: Consider the pair (s', t')

$$d(s', t') + \left\lfloor \frac{g(s') + g(t') + \sqrt{g(s')g(t')}}{3} \right\rfloor \geq 3 + \left\lfloor \frac{2k + 1 + \sqrt{k(k+1)}}{3} \right\rfloor \geq 4$$

Case l: Consider the pair (s, t')

$$d(s, t') + \left\lfloor \frac{g(s) + g(t') + \sqrt{g(s)g(t')}}{3} \right\rfloor \geq 3 + \left\lfloor \frac{2k + 2 + \sqrt{(k+2)(k+1)}}{3} \right\rfloor \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence g is a valid radio heronian mean k -labeling of $DS(RT_{n,n})$.

Therefore, $rhmn_k(DS(RT_{n,n})) = 2n + k + 2, n \geq 2, k \geq 1$.

Clearly, $|V(DS(RT_{n,n}))| = 2n + 3$.

Thus, $rhmn_k(DS(RT_{n,n})) = |V(DS(RT_{n,n}))| + k - 1$.

Hence the degree splitting of rooted tree graph $DS(RT_{n,n})$ is radio heronian mean $k -$ graceful for $\geq 2, k \geq 1$. ■

Theorem 3.3. The degree splitting of hurdle graph $DS(Hd_n)$ is radio heronian mean k -graceful for $n \geq 4$ and $k \geq 1$.

Proof. Let $s_i, 1 \leq i \leq n$ be the vertices of the path P_n by attaching a pendant edge to each internal vertices s_i of the path $P_n, 2 \leq i \leq n-1$. The resultant graph is Hd_n . Introduce two new vertices s and t . Join s to $s_i, 2 \leq i \leq n-1$ and t to $s_1, s_n, t_i, 2 \leq i \leq n-1$. The resultant graph is $DS(Hd_n)$ whose edge set is $E(DS(Hd_n)) = \{s_i s_{i+1}, 1 \leq i \leq n-1\} \cup \{s_i t_{i-1}, t t_{i-1}, s s_i, 2 \leq i \leq n-1\} \cup \{t s_1, t s_n\}$ and diameter of $DS(Hd_n)$ is $D = 3$. Define a function $g : V(DS(Hd_n)) \rightarrow \{k, k+1, \dots, k+N-1\}$ by

$$g(s) = k, k \geq 1;$$

$$g(t) = k+1, k \geq 1;$$

$$g(s_i) = n+k+i-1, k \geq 1, 1 \leq i \leq n;$$

$$g(t_i) = k+i+1, k \geq 1, 1 \leq i \leq n-2;$$

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k+1 + \sqrt{k(k+1)}}{3} \right\rceil \geq 4 = 1 + D$$

Case b: Consider the pair $(s, s_i), 1 \leq i \leq n$

$$\begin{aligned} d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \\ \geq 1 + \left\lceil \frac{n+2k+i-1 + \sqrt{k(n+k+i-1)}}{3} \right\rceil \geq 4 \end{aligned}$$

Case c: Consider the pair $(s, t_i), 1 \leq i \leq n-2$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k+i+1 + \sqrt{k(k+i+1)}}{3} \right\rceil \geq 4$$

Case d: Consider the pair $(t, s_i), 1 \leq i \leq n$

$$\begin{aligned} d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rceil \\ \geq 1 + \left\lceil \frac{n+2k+i + \sqrt{(k+1)(n+k+i-1)}}{3} \right\rceil \geq 4 \end{aligned}$$

Case e: Consider the pair $(t, t_i), 1 \leq i \leq n-2$

$$\begin{aligned} d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k+i+2 + \sqrt{(k+1)(k+i+1)}}{3} \right\rceil \\ \geq 4 \end{aligned}$$

Case f: Consider the pair $(s_i, s_j), i \neq j, 1 \leq i \leq n$

$$d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil$$

$$\geq 1 + \left\lfloor \frac{2n + 2k + i + j - 2 + \sqrt{(n + k + i - 1)(n + k + j - 1)}}{3} \right\rfloor \geq 4$$

Case g: Consider the pair (t_i, t_j) , $i \neq j, 1 \leq i \leq n - 2$

$$d(t_i, t_j) + \left\lfloor \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rfloor$$

$$\geq 2 + \left\lfloor \frac{2k + i + j + 2 + \sqrt{(k + i + 1)(k + j + 1)}}{3} \right\rfloor \geq 4$$

Case h: Consider the pair (s_i, t_j) , $1 \leq i \leq n, 1 \leq j \leq n - 2$

$$d(s_i, t_j) + \left\lfloor \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rfloor$$

$$\geq 1 + \left\lfloor \frac{n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(k + j + 2)}}{3} \right\rfloor \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence g is a valid radio heronian mean k -labeling of $DS(Hd_n)$.

Therefore, $rhmn_k(DS(Hd_n)) = 2n + k - 1, n \geq 4, k \geq 1$. Clearly $|V(DS(Hd_n))| = 2n$. Thus $|V(DS(Hd_n))| + k - 1 = 2n + k - 1, n \geq 4, k \geq 1$.

Therefore, $(rhmn_k(DS(Hd_n)) = |V(DS(Hd_n))| + k - 1$. Hence the degree splitting of hurdle graph $DS(Hd_n)$ is radio heronian mean k -graceful for $n \geq 4$ and $k \geq 1$ ■

Theorem 3.4. The degree splitting of twig graph $DS(TW_n)$ is radio heronian mean k -graceful for $n \geq 4$ and $k \geq 1$.

Proof. Let $s_i, 1 \leq i \leq n$ be the vertices of the path P_n by attaching exactly two pendent edges to each internal vertices s_i of the path $P_n, 2 \leq i \leq n - 1$. The resultant graph is TW_n . Introduce two new vertices s and t . Join s to $s_i, 2 \leq i \leq n - 1$ and t to $s_1, s_n, t_i, t'_i, 2 \leq i \leq n - 2$. the resultant graph is $DS(TW_n)$ whose edge set is $E(DS(TW_n)) = \{s_i s_{i+1} / 1 \leq i \leq n - 1\} \cup \{s_i t_{i-1}, s_i t'_{i-1}, t t_{i-1}, t t'_{i-1}, s s_i / 2 \leq i \leq n - 1\}$ and diameter of $DS(TW_n)$ is $D = 3$.

Define a function $g : V(DS(TW_n)) \rightarrow \{k, k + 1, \dots, k + N - 1\}$ by

$$g(s) = k, k \geq 1;$$

$$g(t) = k + 1, k \geq 1;$$

$$g(s_i) = 2n + k + i - 3, k \geq 1, 1 \leq i \leq n;$$

$$g(t_i) = k + i + 1, k \geq 1, 1 \leq i \leq n - 2;$$

$$g(t'_i) = n + k + i - 1, k \geq 1, 1 \leq i \leq n - 2;$$

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s, t)

$$d(s, t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k + 1 + \sqrt{k(k+1)}}{3} \right\rceil \geq 4 = 1 + D$$

Case b: Consider the pair (s, s_i) , $1 \leq i \leq n$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{2n + 2k + i - 3 + \sqrt{k(2n + k + i - 3)}}{3} \right\rceil \geq 4$$

Case c: Consider the pair (s, t_i) , $1 \leq i \leq n - 2$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + i + 1 + \sqrt{k(k+i+1)}}{3} \right\rceil \geq 4$$

Case d: Consider the pair (s, t'_i) , $1 \leq i \leq n - 2$

$$d(s, t'_i) + \left\lceil \frac{g(s) + g(t'_i) + \sqrt{g(s)g(t'_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{n + 2k + i - 1 + \sqrt{k(k+i-1)}}{3} \right\rceil$$

$$\geq 4$$

Case e: Consider the pair (t, s_i) , $1 \leq i \leq n$

$$d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{2n + 2k + i - 2 + \sqrt{(k+1)(2n + k + i - 3)}}{3} \right\rceil \geq 4$$

Case f: Consider the pair (t, t_i) , $1 \leq i \leq n - 2$

$$d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 2 + \sqrt{(k+1)(k+i+1)}}{3} \right\rceil$$

$$\geq 4$$

Case g: Consider the pair (t, t'_i) , $1 \leq i \leq n - 2$

$$d(t, t'_i) + \left\lceil \frac{g(t) + g(t'_i) + \sqrt{g(t)g(t'_i)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{n + 2k + i + \sqrt{(k+1)(n + k + i + 1)}}{3} \right\rceil \geq 4$$

Case h: Consider the pair (s_i, s_j) , $i \neq j, 1 \leq i \leq n$

$$\begin{aligned}
 d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil \\
 \geq 1 + \left\lceil \frac{4n + 2k + i + j - 6 + \sqrt{(2n + k + i - 3)(2n + k + j - 3)}}{3} \right\rceil \\
 \geq 4
 \end{aligned}$$

Case i: Consider the pair (t_i, t_j) , $i \neq j, 1 \leq i \leq n - 2$

$$\begin{aligned}
 d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil \\
 \geq 2 + \left\lceil \frac{2k + i + j + 2 + \sqrt{(k + i + 1)(k + j + 1)}}{3} \right\rceil \geq 4
 \end{aligned}$$

Case j: Consider the pair (t'_i, t'_j) , $i \neq j, 1 \leq i \leq n - 2$

$$\begin{aligned}
 d(t'_i, t'_j) + \left\lceil \frac{g(t'_i) + g(t'_j) + \sqrt{g(t'_i)g(t'_j)}}{3} \right\rceil \\
 \geq 2 + \left\lceil \frac{2n + 2k + i + j - 2 + \sqrt{(n + k + i - 1)(n + k + j - 1)}}{3} \right\rceil \geq 4
 \end{aligned}$$

Case k: Consider the pair (s_i, t_j) , $1 \leq i \leq n, 1 \leq j \leq n - 2$

$$\begin{aligned}
 d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil \\
 \geq 1 + \left\lceil \frac{2n + 2k + i + j - 2 + \sqrt{(2n + k + i - 3)(k + j + 1)}}{3} \right\rceil \geq 4
 \end{aligned}$$

Case l: Consider the pair (s_i, t'_j) , $1 \leq i \leq n, 1 \leq j \leq n - 2$

$$\begin{aligned}
 d(s_i, t'_j) + \left\lceil \frac{g(s_i) + g(t'_j) + \sqrt{g(s_i)g(t'_j)}}{3} \right\rceil \\
 \geq 1 + \left\lceil \frac{3n + 2k + i + j - 4 + \sqrt{(2n + k + i - 3)(n + k + j - 1)}}{3} \right\rceil \geq 4
 \end{aligned}$$

Case m: Consider the pair (t_i, t'_j) , $1 \leq i, j \leq n - 2$

$$d(t_i, t'_j) + \left\lceil \frac{g(t_i) + g(t'_j) + \sqrt{g(t_i)g(t'_j)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{n + 2k + i + j + \sqrt{(k+i1)(n+k+j-1)}}{3} \right\rceil \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence, g is a valid radio heronian mean k -labeling of $DS(TW_n)$.

Therefore, $rhmn_k(DS(TW_n)) = 3n + k - 3, n \geq 4, k \geq 1$.

Clearly, $|V(DS(TW_n))| = 3n - 2$.

Thus, $|V(DS(TW_n))| + k - 1 = 3n + k - 3, n \geq 4, k \geq 1$.

Therefore, $rhmn_k(DS(TW_n)) = |V(DS(TW_n))| + k - 1$.

Hence the degree splitting of twig graph $DS(TW_n)$ is radio heronian mean k -graceful for $n \geq 4, k \geq 1$.

4. Conclusion

In this paper, we investigate degree splitting of comb graph, rooted tree graph, hurdle graph and twig graph are radio heronian mean k -graceful. In future, we investigate degree splitting on m -graceful labeling of graphs.

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On the edge covering transversal edge Domination in Graphs

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Abstract

Let $G = (V, E)$ be any graph with n vertices and m edges. An edge dominating set which intersects every minimum edge covering set in a graph G is called an edge covering transversal edge dominating set of G . The minimum cardinality of an edge covering transversal edge dominating set is called an edge covering transversal edge domination number of G and is denoted by $\gamma_{ect}(G)$. Any edge covering transversal edge dominating set of cardinalities $\gamma_{ect}(G)$ is called a γ_{ect} -set of G . The edge covering transversal edge domination number of some standard graphs are determined. Some properties satisfied by this concept are studied.

Keywords: domination number, edge domination number, edge covering, edge covering number, edge covering transversal edge domination number.

Mathematics Subject Classification Code: 05C69, 05C70³.

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1. Introduction

Let $G = (V, E)$ be any graph with n vertices and m edges. For any graph theoretic terminologies not defined here, refer to the book of Bondy and Murthy [3]. One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering and matching. By a graph G , we mean a non-trivial, finite, undirected graph with neither loops nor multiple edges. Two vertices u and v are said to be *adjacent* if uv is an edge of G . The *open neighbourhood* of a vertex v in a graph G is defined as the set $N_G(v) = \{v \in V(G) : uv \in E(G)\}$, while the *closed neighbourhood* of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. For any vertex v in a graph G , the number of vertices adjacent to v is called the *degree* of v in G , denoted by $\deg_G(v)$. If the degree of a vertex is 0, it is called an *isolated vertex*, while if the degree is 1, it is called an *end-vertex*. The *minimum degree* of vertices in G is defined by $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. The *maximum degree* of vertices in G is defined by $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$. A vertex v is called a *universal vertex* if $\deg_G(v) = n - 1$. Two edges are said to be *adjacent edges* if they have a common vertex. For any set S of vertices of G , the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S .

A subset $S \subseteq V(G)$ is called a *dominating set* [3, 6, 7, 8] if every vertex $v \in V(G) \setminus S$ is adjacent to a vertex $u \in S$. The *domination number*, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G . A minimum dominating set of a graph G is hence often called as a γ -set of G . A subset $S \subseteq E(G)$ is called an *edge dominating set* [1, 2] if every edge $f \in E(G) \setminus S$ is adjacent to an edge $h \in S$. The *edge domination number*, $\gamma_e(G)$, of a graph G denotes the minimum cardinality of such edge dominating sets of G . A minimum edge dominating set of a graph G is hence often called as a γ_e -set of G . An *edge cover* [5] of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. A minimum edge covering is an edge covering of smallest possible size. The *edge covering number* $\rho(G)$ is the size of a minimum edge covering. Given a graph G and a collection of subsets of its vertices, a subset of $V(G)$ is called a *transversal* of G if it intersects each subset of the collection [4]. In this paper, we studied the concept of the edge covering transversal edge domination number of G .

2. On the edge covering transversal edge domination number of a graph

Definition 2.1. An edge dominating set which intersects every minimum edge covering set in a graph G is called an *edge covering transversal edge dominating set* of G . The minimum cardinality of an edge covering transversal edge dominating set is called an *edge covering transversal edge domination number* of G and is denoted by $\gamma_{ect}(G)$. Any edge covering transversal edge dominating set of cardinalities $\gamma_{ect}(G)$ is called a γ_{ect} -set of G .

Example 2.2. For the graph G given in Fig. 2.1, $D_1 = \{v_2v_3, v_4v_5\}$ and $D_2 = \{v_2v_5, v_3v_4\}$ are the only two minimum edge dominating sets of G .

Also, $S_1 = \{v_1v_5, v_2v_3, v_3v_4\}$, $S_2 = \{v_1v_2, v_2v_3, v_4v_5\}$, $S_3 = \{v_1v_2, v_2v_5, v_3v_4\}$, $S_4 = \{v_1v_5, v_2v_3, v_4v_5\}$ are the only 4 minimum edge covering sets of G . Since $D_1 \cap S_3 = \emptyset$ and $D_2 \cap S_4 = \emptyset$, D_1 and D_2 are not an edge covering transversal edge dominating sets of G and so $\gamma_{eect}(G) = 3$. Now, $D_3 = \{v_1v_2, v_2v_3, v_4v_5\}$ is a γ_{eect} -set of G , so that $\gamma_{eect}(G) = 3$.

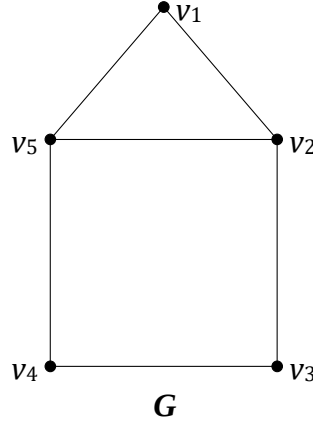


Figure 2.1

Theorem 2.3. A set S of edges of $G = K_{r,r}$ ($r \geq 2$) is a minimum edge covering transversal edge domination of G if and only if S consists of n -independent edges.

Proof. Let S be any set of r -independent edges of $G = K_{r,s}$ ($2 \leq r \leq s$). Then S is a minimum edge covering set of G . Since $S \cap D = \emptyset$ for any edge covering transversal edge dominating set D of G , S is an edge covering transversal edge dominating set of G . Hence, it follows that $\gamma_{eect}(G) \leq r$. If $\gamma_{eect}(G) < r$, then there exists an edge covering transversal edge dominating set S' , such that $|S'| < r$. Therefore, there exists at least one vertex v of G , such that v is not incident with any edge of S' and so S' is not an edge covering transversal edge dominating set of G , which is a contradiction. Hence, S is a minimum edge covering transversal edge dominating set of G .

Conversely, let S be a minimum edge covering set of G . Let S' be any set of n -independent set of edges of G . Then as in first part of this theorem, S' is a minimum edge covering set of G . Therefore, $|S'| = r$. Hence, $|S| = r$. If S is not independent, then there exists a vertex v of G such that, v is not independent with any edge of S . Hence S is not an edge covering transversal edge dominating set of G , which is a contradiction. Thus, S consists of r -independent edges.

Theorem 2.4. A set S of edges of $G = K_{r,s}$ ($2 \leq r \leq s$) is a minimum edge covering transversal edge domination of G , if and only if S consists of $r - 1$ independent edges of G and $s - r + 1$ adjacent edges of G .

Proof. Let $X = \{u_1, u_2, \dots, u_r\}$ and $Y = \{v_1, v_2, \dots, v_s\}$ be a bipartition of G . Let S be any set of $r - 1$ independent edges of G and $s - r + 1$ adjacent edges of G . Since each vertex of G is incident with an edge of S , it follows that $\gamma_{eect}(G) < s$.

If $\gamma_{eect}(G) < s$, then there exists an edge covering transversal edge dominating set S' of G such that $|S'| < s$. Therefore, there exists at least one vertex v of G such that v is not incident with any edge of S' and so S' is not an edge covering transversal edge dominating set of G , which is a contradiction. Hence, S is a minimum edge covering transversal edge dominating set of G .

Conversely, let S be a minimum edge covering transversal edge dominating set of G . Let S' be any set of $r - 1$ independent edges of G and $r - s + 1$ adjacent edges of G . Then as in the first part of this theorem, S is a minimum edge covering transversal edge dominating set of G . Therefore $|S'| = s$. Hence $|S| = s$. Let us assume that $S = S_1 \cup S_2$, where S_1 consists of independent edges and S_2 consists of adjacent edges of G . If $|S_1| \leq s - 2$, then S_2 must contain at most $s - r$ edges. Then there exists at least one vertex v of G , such that v is not incident with any edge of S and so S is not an edge covering transversal edge dominating set of G , which is a contradiction. Therefore S consists of $s - 1$ independent edges of G and $s - r + 1$ adjacent edges of G .

Corollary 2.5. For the complete bipartite graph, $K_{r,s} (2 \leq r \leq s)$, $\gamma_{eect}(G) = s$.

Theorem 2.6. For the complete graph $G = K_n (n \geq 4)$ with n even, a set S of edges of G is a minimum edge covering transversal edge domination of G if and only if S consists of $\frac{n}{2}$ independent edges.

Proof. The proof is similar to the proof of the Theorem 2.4.

Theorem 2.7. For the complete graph $G = K_n (n \geq 5)$ with n odd, a set S of edges of G is a minimum edge covering transversal edge dominating set of G if and only if S consists of $\frac{n-3}{2}$ independent edges and two adjacent edges of G .

Proof. The proof is similar to the proof of the Theorem 2.4.

Corollary 2.8. For the Complete graph

$$G = K_n (n \geq 4), \gamma_{eect}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.9. For the Cycle $G = C_n (n \geq 4)$,

$$\gamma_{eect}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $C_n: \{v_1, v_2, v_3, \dots, v_n, v_1\}$ be the cycle. We consider the following two cases

Case (i). n is even.

Let $n = 2k (k \geq 2)$, then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}\}$ and $S_2 = \{v_2v_3, v_4v_5, v_6v_7, \dots, v_{2k-2}v_{2k-1}, v_{2k}v_1\}$ are the only two minimum edge covering sets of G . Let D be an edge dominating sets of G . Then it is clear that $D \cap S_1 \neq \emptyset$ and $D \cap$

On the Edge Covering Transversal Edge Domination in Graphs

$S_2 \neq \phi$. Therefore S_1 and S_2 are the only two minimum edge covering transversal edge dominating sets of G , so that $\gamma_{ect}(G) = k = \frac{n}{2}$

Case (ii). n is odd

Let. $n = 2k + 1 (k \geq 2)$, then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}, v_{2nk}v_{2k+1}\}$ is a minimum edge covering transversal edge dominating set of G , so that $\gamma_{ect}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.10. For the path $G = P_n (n \geq 4)$, $\gamma_{ect}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

Proof. The proof is similar to the proof of the Theorem 2.9.

Theorem 2.11. For the star graph, $G = K_{1,n-1}$, $\gamma_{ect}(G) = n - 1$.

Theorem 2.12. For the wheel graph $G = K_1 + C_{n-1} (n \geq 5)$,

$$\gamma_{ect}(G) = \begin{cases} \frac{n-1}{2} & \text{if } n-1 \text{ is even} \\ \frac{n+1}{2} & \text{if } n-1 \text{ is odd} \end{cases}$$

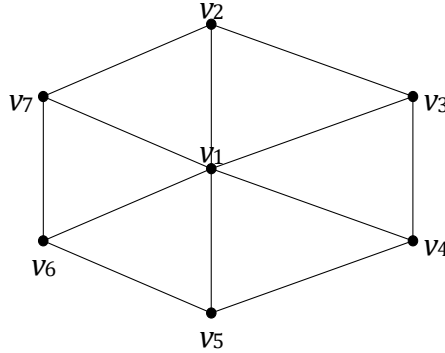


Figure 2.2

Proof. Let $V(K_1) = \{x\}$ and C_{n-1} be $v_1, v_2, v_3, \dots, v_{n-1}, v_1$.

We consider the following cases.

Case (i). $n - 1$ is even.

Let $n - 1 = 2k (k \geq 3)$. Then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}\}$ and $S_2 = \{v_2v_3, v_4v_5, v_6v_7, \dots, v_{2k-2}v_{2k-1}, v_{2k}v_1\}$ are the only two minimum edge covering sets of G . Then it is clear that $D \cap S_1 \neq \phi$ and that $D \cap S_2 \neq \phi$. Therefore S_1 and S_2 are the only two minimum edge covering transversal edge dominating sets of G , so that $\gamma_{ect}(G) = k = \frac{n-1}{2}$.

Case (ii). $n - 1$ is odd.

Let $n - 1 = 2k + 1$ ($k \geq 2$). Then $S_1 = \{v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}, v_{2k}v_{2k+1}\}$ is a minimum edge covering transversal edge dominating set of G , so that $\gamma_{eect}(G) = \left\lceil \frac{n-1}{2} \right\rceil$.

Theorem 2.13. Let G be a connected graph with m edges ($m \geq 2$), then $1 \leq \gamma_{eect}(G) \leq m$.

Theorem 2.14. For any graph G , $\gamma_e(G) \leq \gamma_{eect}(G) \leq \gamma_e(G) + \Delta(G)$.

Proof. Let D be a γ_{eect} -set of G . Then D itself is an edge dominating set. Therefore $\gamma_e(G) \leq |D| = \gamma_{eect}(G)$. Now, let S be a γ_e -set in G and let v be a vertex of maximum degree. Therefore $\deg(v) = \Delta(G)$. Then every minimum edge covering transversal edge dominating set of G contains an edge of $\langle E[N(v)] \rangle$ so $S \cup \langle E[N(v)] \rangle$ is an edge covering transversal edge dominating set. Also, since S intersects $\langle E[N(v)] \rangle$ $|S \cup \langle E[N(v)] \rangle| \leq \gamma_e(G) + \Delta(G)$. Then $\gamma_e(G) \leq \gamma_{eect}(G) \leq \gamma_e(G) + \Delta(G)$.

Remark 2.15. The bound in the Theorem 2.14 is sharp.

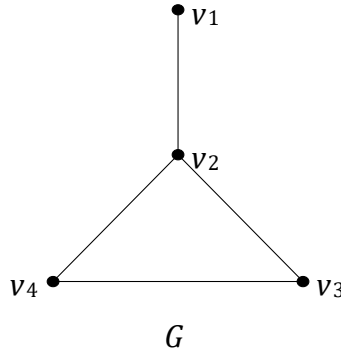


Figure 2.3

For the graph given in Figure 2.3, $S_1 = \{v_1v_2, v_3v_4\}$ is a γ_e -set of G and γ_{eect} -set of G so that $\gamma_e(G) = \gamma_{eect}(G)$.

3. Conclusions

In this paper, we obtain some results related to edge covering transversal domination in graphs and this work gives the scope for an extensive study of various domination parameters of these graphs.

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The Connected Vertex Strong Geodetic Number of a Graph

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Abstract

In this paper we introduce the concept of connected vertex strong geodetic number $cg_{sx}(G)$ of a graph G at a vertex x and investigate its properties. We determine bounds for it and find the same for some special classes of graphs. We prove that $sg_x(G) \leq csg_x(G)$ for any vertex x in G . Necessary conditions for $sg_x(G)$ to be n or $n - 1$ are given for some vertex x in G . It is shown for every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $csg_x(G) = b$ for some vertex x in G .

Keywords: strong geodetic number; vertex strong geodetic number; connected strong geodetic number.

2010 AMS subject classification: 05C15[‡].

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be *adjacent* if uv is an edge of G . Two edges of G are said to be adjacent if they have a common vertex. The *distanced*(u, v) between two vertices u and v in a connected graph G is the length of a shortest u - v path in G .

An u - v path of length $d(u, v)$ is called an u - v geodesic. An x - y path of length $d(x, y)$ is called geodesic. A vertex v is said to lie on a geodesic P if v is an internal vertex of P . The closed interval $I[x, y]$ consists of x, y and all vertices lying on some x - y geodesic of G and for a non-empty set $S \subseteq V(G)$, $I[S] = \cup_{x, y \in S} I[x, y]$. A set $S \subseteq V(G)$ in a connected graph G is a geodetic set of G if $I[S] = V(G)$. The geodetic number of G , denoted by $g(G)$, is the minimum cardinality of a geodetic set of G . The geodetic concept were studied in [1, 3, 4]. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex $y \in S, x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x - y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y) : y \in S\}$ and let $V(\tilde{I}_x[S]) = \cup_{p \in \tilde{I}_x[S]} V(p)$. If $V(\tilde{I}_x[S]) = V$ for some

$\tilde{I}_x[S]$ then the set S is called a vertex strong geodetic set of G . The minimum cardinality of a vertex strong geodetic set of G is called the vertex strong geodetic number of G and is denoted by $sg_x(G)$. The following theorem is used in sequel.

Theorem 1.1[4] Each extreme vertex of a connected graph belong to every geodetic set of G .

2. The connected vertex strong geodetic number of a graph

Definition 2.1. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex $y \in S, x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x - y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y) : y \in S\}$ and let $V(\tilde{I}_x[S]) = \cup_{p \in \tilde{I}_x[S]} V(p)$. If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$ then the set S is called a

vertex strong geodetic set of G . A vertex strong geodetic set S of x of G is called a connected vertex strong geodetic set of G if $G[S]$ is connected. The minimum cardinality of a connected vertex strong geodetic set of G is called the connected vertex strong geodetic number of G and is denoted by $csg_x(G)$.

Example 2.2. For the graph G given in Figure 2.1, csg_x -sets and $csg_x(G)$ for each vertex x is given in the following Table 2.1.

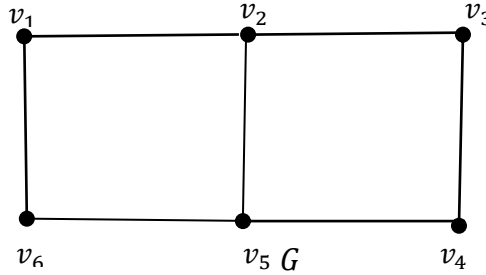


Figure 2.1

Table 2.1

Vertex	csg_x -sets	$csg_x(G)$
v_1	$\{v_3, v_4\}, \{v_4, v_5\}$	2
v_2	$\{v_4, v_5, v_6\}$	3
v_3	$\{v_1, v_6\}, \{v_5, v_6\}$	2
v_4	$\{v_1, v_6\}, \{v_1, v_2\}$	2
v_5	$\{v_1, v_2, v_3\}$	3
v_6	$\{v_3, v_4\}, \{v_2, v_3\}$	2

Observation 2.3. Let x be any vertex of a connected graph G .

(i) If $y \neq x$ be a simplicial vertex of G , then y belongs to every connected x -vertex strong geodetic set of G .

(ii) The eccentric vertices of x belong to every connected x -vertex strong geodetic set of G .

In the following we determine the connected vertex strong geodetic number of some standard graphs G for each vertex in G .

Theorem 2.4. For the path $G = P_n (n \geq 3)$,

$$csg_x(G) = \begin{cases} 1 & \text{if } x \text{ is an end vertex of } G \\ n & \text{if } x \text{ is a cut vertex of } G \end{cases}$$

Proof. Let P_n be $v_1, v_2, \dots, v_n$.

If $x = v_1$, then $S = \{v_n\}$ is a csg_x -set of G so that $csg_x(G) = 1$. Similarly if $x = v_n$, then $csg_x(G) = 1$. Let x be a cut vertex of G . Then by Observation 2.3 (i)

$\{v_1, v_n\}$ is a subset of every $cs g_x$ -set of G . Let S be a $cs g_x$ -set of G . Since $G[S]$ is connected, it follows that $S = V(G)$ is the unique $cs g_x$ -set of G so that $cs g_x(G) = n$. ■

Theorem 2.5. For the cycle $G = C_n (n \geq 4)$, $cs g_x(G) = 2$, for every $x \in G$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Without loss of generality let us assume that $x = v_1$.

Case (i) Let n be even. Let $n = 2k (k \geq 2)$. Then v_{k+1} is the eccentric vertex of G . By Observation 2.3(ii) since $\{v_{k+1}\}$ is not a sg_x -set of G so that $cs g_x(G) \geq 2$. Let $S = \{v_{k+1}, v_{k+2}\}$. Then S is a $cs g_x$ -set of G so that $cs g_x(G) = 2$.

Case (ii) Let n be odd. Let $n = 2k + 1 (k \geq 2)$. Then $S = \{v_{k+1}, v_{k+2}\}$ is the eccentric vertices of G . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq 2$. Since S is a sg_x -set of G and $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = 2$. ■

Theorem 2.6. For the complete graph $G = K_n (n \geq 4)$, $cs g_x(G) = n - 1$, for every $x \in G$.

Proof. Let x be a vertex of G . Let $S = V(G) - \{x\}$. Since every vertex of G is an extreme vertex of G , it follows from Observation 2.3(i), S is the unique $cs g_x$ -set of G so that $cs g_x(G) \geq n - 1$ for every vertex x in G . ■

Theorem 2.7. For the fan graph $G = K_1 + P_{n-1} (n \geq 5)$.

$$cs g_x(G) = \begin{cases} n - 1 & \text{if } x \in V(K_1) \\ n - 3 & \text{if } x \text{ is extreme vertex of } P_{n-1} \\ n - 2 & \text{if } x \text{ is internal vertex of } P_{n-1} \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.

Case (i) Let $x = y$, Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = n - 1$. Let $x \in V(P_{n-1})$. Let $x = v_1$. Then $S = \{v_3, v_4, \dots, v_{n-1}\}$ are eccentric vertices of G . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq n - 3$. Now S is a sg_x -set of G and $G[S]$ is connected. Therefore S is a $cs g_x$ -set of G so that $cs g_x(G) = n - 3$. If $x = v_{n-1}$, by the similar way we can prove that $cs g_x(G) = n - 3$. Let $x \in \{v_2, v_3, \dots, v_{n-2}\}$. Without loss of generality let us assume that $x = v_2$. Then $\{v_1, v_{n-1}\}$ is set of extreme vertices of G . By Observation 2.3 (i) $\{v_1, v_{n-1}\}$ is a subset of every $cs g_x$ -set of G . $\{v_4, v_5, \dots, v_{n-2}\}$ is the set of eccentric vertices of v_2 . Then $\{v_4, v_5, \dots, v_{n-2}\}$ is a subset of every $cs g_x$ -set of G . Let $S' = \{v_1, v_4, v_5, \dots, v_{n-2}, v_{n-1}\}$. Then S' is a sg_x -set of G but $G[S']$ is not connected. Therefore $S' \cup \{y\}$ is a $cs g_x$ -set of G so that $cs g_x(G) = n - 2$. ■

Theorem 2.8. For the wheel graph $G = K_1 + C_{n-1} (n \geq 5)$.

$$cs g_x(G) = \begin{cases} n - 1 & \text{if } x \in v_1 \\ n - 3 & \text{if } x \in V(C_{n-1}) \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.

Case(i) Let $x = y$, Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = n - 1$.

Case (ii) Let $x \in V(C_{n-1})$. Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_3, v_4, \dots, v_{n-1}\}$ are eccentric vertices of G . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq n - 3$. Now S is a sg_x -set of G and $G[S]$ is connected. Therefore S is a $cs g_x$ -set of G so that $cs g_x(G) = n - 3$. ■

Theorem 2.9. For the star graph $G = K_{1,n-1}$ ($n \geq 3$), $cs g_x(G) = n - 1$ for every $x \in G$.

Proof. Let y be the cut vertex of G and $\{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices of G . Let $x = y$, Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = n - 1$. Let $x \in \{v_1, v_2, \dots, v_{n-1}\}$ Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_2, v_3, \dots, v_{n-1}\}$ are set of eccentric vertices of v_1 . By Observation 2.3 (ii) S is a subset of every sg_x -set of G and so $cs g_x(G) \geq n - 2$. Now S is a $cs g_x$ -set of G but $G[S]$ is not a $cs g_x$ -set of G and so $cs g_x(G) \geq n - 1$. Let $S' = S \cup \{x\}$. Then S' is a $cs g_x$ -set of G so that $cs g_x(G) = n - 1$. ■

Theorem 2.10. For the Peterson graph G , $cs g_x(G) = 6$ for every $x \in G$.

Proof.

Case (i) Let $x \in \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality let us assume that $x = v_1$. Then $S = \{v_2, v_5, v_7, v_8, v_9, v_{10}\}$ is the set of all eccentric vertices for x . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq 6$. Since S is a sg_x -set of G and $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = 6$.

Case (ii) Let $x \in \{v_6, v_7, v_8, v_9, v_{10}\}$. Without loss of generality let us assume that $x = v_6$. Then $S = \{v_2, v_3, v_4, v_5, v_8, v_9\}$ is the set of all eccentric vertices for x . By Observation 2.3 (ii) S is a subset of every $cs g_x$ -set of G and so $cs g_x(G) \geq 6$. Since S is a sg_x -set of G and $G[S]$ is connected, S is a $cs g_x$ -set of G so that $cs g_x(G) = 6$. ■

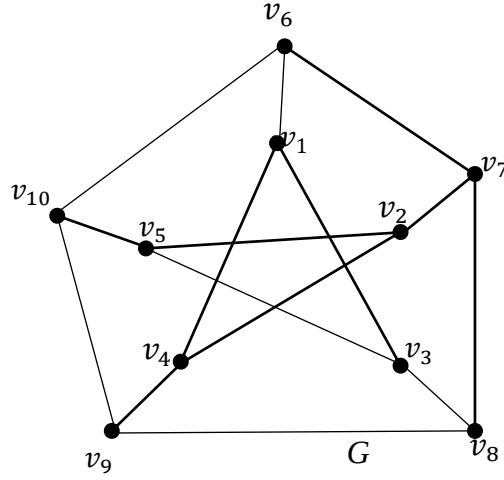


Figure 2.2

Theorem 2.11. Let G be a connected graph. Then $1 \leq sg_x(G) \leq csg_x(G) \leq n$ for every vertex x in G .

Proof. Let x be a vertex of G . Since every sg_x -set of G needs at least one vertex $sg_x(G) \geq 1$. Since every connected strong vertex geodetic set of G is a strong vertex geodetic set of G , $sg_x(G) \leq csg_x(G)$. Since $V(G)$ is a connected strong vertex geodetic set of G , $csg_x(G) \leq n$. Therefore $1 \leq sg_x(G) \leq csg_x(G) \leq n$. ■

Theorem 2.12. Let G be a connected graph. Then $csg_x(G) = 1$ if and only if x is an end vertex of P_n ($n \geq 2$).

Proof. Let x be an end vertex of P_n . Then by Theorem 2.4, $csg_x(G) = 1$. Conversely let $csg_x(G) = 1$. Let $S = \{y\}$ be the csg_x -set of x . We prove that x is an end vertex of P_n . On the contrary suppose that x is not an end vertex of P_n . Then there are at least two $x - y$ geodesics, which is a contradiction to S a csg_x -set of G . Therefore x is an end vertex of P_n . ■

Theorem 2.13. Let G be a connected graph and $x \in G$. If x is a universal vertex of G . Then $csg_x(G) = n - 1$.

Proof. Let x be a universal vertex of G . Then $V(G) - \{x\}$ is set of all eccentric vertices for x . By Observation 2.3 (ii), S is a subset of every csg_x -set of G and so $csg_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a csg_x -set of G so that $csg_x(G) = n - 1$. ■

Theorem 2.14. Let G be a connected graph and $x \in G$. If x is a cut vertex and universal vertex of G . Then $csg_x(G) = n$.

Proof. Since x is a universal vertex of G , then $V(G) - \{x\}$ is set of all eccentric vertices for x . By Observation 2.3 (ii), S is a subset of every csg_x -set of G and so

$cs g_x(G) \geq n - 1$. Since $G[S]$ is not connected, S is not a $cs g_x$ -set of G . Therefore $S = V(G)$ is the unique $cs g_x$ -set of G . Hence $cs g_x(G) = n$. ■

Theorem 2.15. For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $cs g_x(G) = b$ for some vertex x in G .

Proof. For $a = b$, let $G = K_{a+1}$. Then by Theorem 2.11 $sg_x(G) = cs g_x(G) = a$ for every vertex x in G . For $b = a + 1$, let $G = K_{1,a}$. Let x be a universal vertex of G . Then by Theorem 2.14, $sg_x(G) = a$ and $cs g_x(G) = a + 1$. So, let $b \geq a + 2$. Let $P_0: u_0, u_1, u_2, \dots, u_{b-a}, u_{b-a+1}$ be a path of order $b - a + 2$. Let G be the graph obtained from P by adding the new vertices $z_1, z_2, \dots, z_{a-1}$ and introducing the edges $z_i u$ ($1 \leq i \leq b - a + 1$). The graph G is shown in Figure 2.3. Let $x = u_{b-a+1}$.

First we prove that $sg_x(G) = a$. Let $S = \{u_0, z_1, z_2, \dots, z_{a-1}, u_{b-a+1}\}$ be the end vertices of G . By Observation 2.3(i), $S_1 = S - \{u_{b-a+1}\}$ is a subset of every sg_x -set of G and so $sg_x(G) \geq a$. Since S_1 is a sg_x -set of G , $sg_x(G) = a$.

Next we prove that $cs g_x(G) = b$. By Observation, S_1 is a subset of every $cs g_x$ -set of G . Since $G[S_1]$ is not connected S_1 is not a $cs g_x$ -set of G . let $S_2 = S_1 \cup \{u_1, u_2, \dots, u_{b-a}\}$. Then S_2 is a $cs g_x$ -set of G and $G[S_2]$ is connected. Therefore S_2 is a $cs g_x$ -set of G so that, $cs g_x(G) = b$. ■

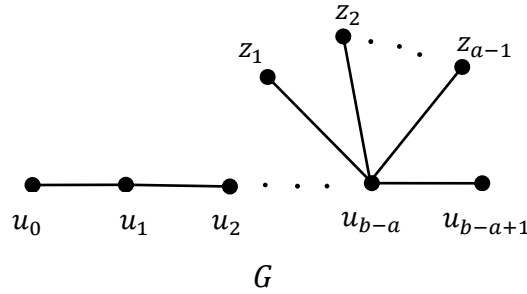


Figure 2.3

3. Conclusions

In this article we explore the concept of the forcing strong geodetic number of a graph. We extend this concept to some other distance related parameters in graphs.

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Monophonic Distance Energy for Join of Some Graphs

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Abstract

Let G be a connected graph with n vertices and m edges. Let $\mu_1, \mu_2, \dots, \mu_n$ be the Eigen values of distance matrix of G . The distance energy of a graph $E_D(G) = \sum_{i=1}^n |\mu_i|$, was already studied. We now define and investigate the monophonic distance energy as $E_M(G) = \sum_{i=1}^n |\mu_i^M|$, where $\mu_1^M, \mu_2^M, \dots, \mu_n^M$ are the eigen values of monophonic distance matrix of graphs. In this paper we find the monophonic distance energy for join of some graphs.

Keywords: Join of graphs; Monophonic distance matrix; Monophonic distance energy.

2010 Mathematics Subject Classification: 05C12, 05B20³.

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1. Introduction

In this paper we considered simple, connected and undirected graphs. The concept of energy of a graph was introduced by I. Gutman [2] in the year 1978. Let G be a connected graph with n vertices and m edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A , assumed in non-increasing order, are the eigen values of the graph G . The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigen values of G . ie., $E(G) = \sum_{i=1}^n |\lambda_i|$. Also, Distance energy of a graph was introduced by I. Gutman and others [4] in the year 2008. A. P. Santhakumaran and others introduced the monophonic number of a graph in 2014 [8]. For any two vertices u and v in a connected graph G , a $u - v$ path is a monophonic path if it contains no chords, and the monophonic distance $d_m(u, v)$ is the length of a longest $u - v$ monophonic path in G . Based on these we introduce a new concept monophonic distance energy of a graph. Based on these we introduce a new concept monophonic distance energy of a graph. In this paper we investigate the monophonic distance energy of $K_{1,n} + K_{1,n}$, $K_{n,n} + K_n$, $K_{n,n} + K_{n,n}$ and $K_n + K_{1,n}$.

2. Definitions

Definition 2.1. Let G be a connected graph with vertex set $v_1, v_2, \dots, v_n$. The monophonic distance matrix of G is defined as

$$M = M[G] = (d_{m_{ij}})_{n \times n} \text{ Where } d_{m_{ij}} = \begin{cases} d_m(v_i, v_j), & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$$

Here $d_m(v_i, v_j)$ is the monophonic distance of v_i to v_j . The eigen values of Monophonic distance matrix $M(G)$ are denoted by $\mu_1^M, \mu_2^M, \dots, \mu_n^M$ and are said to be M -Eigen values of G and to form the M -spectrum of G , denoted by $spec_M(G)$. We note that since the Monophonic distance matrix is symmetric, its eigen values are real and can be ordered as $\mu_1^M \leq \mu_2^M \leq \dots \leq \mu_n^M$. We can define the monophonic distance energy of a graph as $E_M(G) = \sum_{i=1}^n |\mu_i^M|$.

3. Main Results

Definition 3.1. [3] The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2 .

Theorem 3.2. For the Star graph $K_{1,n}$, $E_M(K_{1,n} + K_{1,n}) = 5(n - 1) + \sqrt{9n^2 - 2n + 9}$.

Proof. From the definition of join, the monophonic distance matrix of $K_{1,n} + K_{1,n}$ can be written as

Monophonic Distance Energy for Join of Some Graphs

$$M(K_{1,n} + K_{1,n}) = \begin{bmatrix} M(K_{1,n}) & J_n \\ J_n & M(K_{1,n}) \end{bmatrix}$$

where $M(K_{1,n})$ be a monophonic distance matrix of $K_{1,n}$.

We have $M(K_{1,n} + K_{1,n}) = (d_{m_{ij}})_{2(n+1) \times 2(n+1)}$.

The monophonic distance spectrum $Spec_M(K_{1,n} + K_{1,n})$ is

$$\left(\begin{array}{ccccc} -2 & -1 & \frac{(3n-1) - \sqrt{9n^2 - 2n + 9}}{2} & (n-2) & \frac{(3n-1) + \sqrt{9n^2 - 2n + 9}}{2} \\ (n-1) & 1 & 1 & 1 & 1 \end{array} \right).$$

Thus the monophonic distance energy of $K_{1,n} + K_{1,n}$ is

$E_M(K_{1,n} + K_{1,n}) = \sum_{i=1}^{2(n+1)} |\mu_i^M|$, where $\mu_1^M, \mu_2^M, \dots, \mu_{2(n+1)}^M$ are the eigen values of monophonic distance matrix $M(K_{1,n} + K_{1,n})$.

For $n > 1$,

$$\begin{aligned} E_M(K_{1,n} + K_{1,n}) &= |-2| + |-2| + \dots + |-2| + |-1| \\ &\quad + \left| \frac{(3n-1) - \sqrt{9n^2 - 2n + 9}}{2} \right| + |(n-2)| \\ &\quad + \left| \frac{(3n-1) + \sqrt{9n^2 - 2n + 9}}{2} \right| \\ &= 2(2(n-1) + 1) - \frac{(3n-1) + \sqrt{9n^2 - 2n + 9}}{2} + (n-2) \\ &\quad + \frac{(3n-1) + \sqrt{9n^2 - 2n + 9}}{2} \\ &= 5(n-1) + \sqrt{9n^2 - 2n + 9}. \end{aligned}$$

Theorem 3.3. For the complete bipartite graph $K_{n,n}$ and complete graph K_n ,

$$E_M(K_{n,n} + K_n) = \begin{cases} (6n-7) + \sqrt{12n^2 - 4n + 1} & \text{if } 1 \leq n \leq 4 \\ 10(n-1) & \text{if } n \geq 5 \end{cases}.$$

Proof. From the definition of join, the monophonic distance matrix of $K_{n,n} + K_n$ can be written as

$$M(K_{n,n} + K_n) = \begin{bmatrix} M(K_{n,n}) & J_{2n,n} \\ J_{n,2n} & M(K_n) \end{bmatrix}$$

where $M(K_{n,n})$ be a monophonic distance matrix of $K_{n,n}$ and $M(K_n)$ be a monophonic distance matrix of K_n .

We have $M(K_{n,n} + K_n) = (d_{m_{ij}})_{3n \times 3n}$.

The monophonic distance spectrum $Spec_M(K_{n,n} + K_n)$ is

$$\left(\begin{array}{ccccc} -2 & -1 & \frac{(4n-3) - \sqrt{12n^2 - 4n + 1}}{2} & \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} & n-2 \\ 2(n-1) & (n-1) & 1 & 1 & 1 \end{array} \right).$$

Thus the monophonic distance energy of $K_{n,n} + K_n$ is

$E_M(K_{n,n} + K_n) = \sum_{i=1}^{3n} |\mu_i^M|$, where $\mu_1^M, \mu_2^M, \dots, \mu_{3n}^M$ are the eigen values of monophonic distance matrix $M(K_{n,n} + K_n)$.

For $1 \leq n \leq 4$,

$$\begin{aligned} E_M(K_{n,n} + K_n) &= |-2| + |-2| + \dots + |-2| + |-1| + |-1| + \dots + |-1| \\ &\quad + \left| \frac{(4n-3) - \sqrt{12n^2 - 4n + 1}}{2} \right| + \left| \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} \right| + (n-2) \\ &= 2(2(n-1) + (n-1) - \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} \\ &\quad + \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} + (n-2) \\ &= (6n-7) + \sqrt{12n^2 - 4n + 1} \end{aligned}$$

For $n \geq 5$,

$$\begin{aligned} E_M(K_{n,n} + K_n) &= |-2| + |-2| + \dots + |-2| + |-1| + |-1| + \dots + |-1| \\ &\quad + \left| \frac{(4n-3) - \sqrt{12n^2 - 4n + 1}}{2} \right| + \left| \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} \right| + (n-2) \\ &= 2(2(n-1) + (n-1) + \frac{(4n-3) - \sqrt{12n^2 - 4n + 1}}{2} \\ &\quad + \frac{(4n-3) + \sqrt{12n^2 - 4n + 1}}{2} + (n-2) \\ &= 10(n-1). \end{aligned}$$

Theorem 3.4. For the complete bipartite graph $K_{n,n}$,

$$E_M(K_{n,n} + K_{n,n}) = \begin{cases} 2(5n-2) & \text{if } n = 1 \\ 16(n-1) & \text{if } n \geq 2 \end{cases}$$

Proof. From the definition of join, the monophonic distance matrix of $K_{n,n} + K_{n,n}$ can be written as

$$M(K_{n,n} + K_{n,n}) = \begin{bmatrix} M(K_{n,n}) & J_{2n} \\ J_{2n} & M(K_{n,n}) \end{bmatrix}$$

where $M(K_{n,n})$ be a monophonic distance matrix of $K_{n,n}$

We have $M(K_{n,n} + K_{n,n}) = (d_{m_{ij}})_{4n \times 4n}$.

The monophonic distance spectrum $Spec_M(K_{n,n} + K_{n,n})$ is

$$\begin{pmatrix} -2 & (n-2) & (5n-2) \\ 4(n-1) & 3 & 1 \end{pmatrix}.$$

Thus the monophonic distance energy of $K_{n,n} + K_{n,n}$ is

$E_M(K_{n,n} + K_{n,n}) = \sum_{i=1}^{4n} |\mu_i^M|$, where $\mu_1^M, \mu_2^M, \dots, \mu_{4n}^M$ are the eigen values of monophonic distance matrix $M(K_{n,n} + K_{n,n})$.

For $n = 1$,

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$$\begin{aligned}
 E_M(K_{n,n} + K_{n,n}) &= |-2| + |-2| + \cdots + |-2| + |n-2| + \cdots + |n-2| + |5n-2| = \\
 &= 2(4(n-1) + 3(-n+2) + 5n-2) = 8n-8-3n+6+5n-2 \\
 &= 10n-4 \\
 &= 2(5n-2) \\
 \text{For } n \geq 2, \\
 E_M(K_{n,n} + K_{n,n}) &= |-2| + |-2| + \cdots + |-2| + |n-2| + \cdots + |n-2| + |5n-2| = \\
 &= 2(4(n-1) + 3(n-2) + 5n-2) = 8n-8+3n-6+5n-2 \\
 &= 16n-16 \\
 &= 16(n-1).
 \end{aligned}$$

Theorem 3.5. For the star graph $K_{1,n}$ and complete graph K_n ,

$$E_M(K_n + K_{1,n}) = \begin{cases} (3n-2) + \sqrt{5n^2+4} & \text{if } n \leq 2 \\ 2(3n-2) & \text{if } n \geq 3 \end{cases}$$

Proof. . From the definition of join, the monophonic distance matrix of $K_n + K_{1,n}$ can be written as

$$M(K_n + K_{1,n}) = \begin{bmatrix} M(K_n) & J_{n,n+1} \\ J_{n+1,n} & M(K_{1,n}) \end{bmatrix}$$

where $M(K_{1,n})$ be a monophonic distance matrix of $K_{1,n}$ and $M(K_n)$ be a monophonic distance matrix of K_n .

We have $M(K_n + K_{1,n}) = (d_{m_{ij}})_{(2n+1) \times (2n+1)}$.

The monophonic distance spectrum $Spec_M(K_{n,n} + K_n)$ is

$$\begin{pmatrix} -2 & -1 & \frac{(3n-2) - \sqrt{5n^2+4}}{2} & \frac{(3n-2) + \sqrt{5n^2+4}}{2} \\ (n-1) & n & 1 & 1 \end{pmatrix}.$$

Thus the monophonic distance energy of $K_n + K_{1,n}$ is

$E_M(K_n + K_{1,n}) = \sum_{i=1}^{2n+1} |\mu_i^M|$, where $\mu_1^M, \mu_2^M, \dots, \mu_{2n+1}^M$ are the eigen values of monophonic distance matrix $M(K_n + K_{1,n})$.

For $n \leq 2$,

$$E_M(K_n + K_{1,n}) = |-2| + |-2| + \cdots + |-2| + |-1| + |-1| + \cdots + |-1|$$

$$\begin{aligned}
 &+ \left| \frac{(3n-2) - \sqrt{5n^2+4}}{2} \right| + \left| \frac{(3n-2) + \sqrt{5n^2+4}}{2} \right| \\
 &= 2(n-1) + n - \frac{(3n-2) + \sqrt{5n^2+4}}{2} + \frac{(3n-2) + \sqrt{5n^2+4}}{2} \\
 &= 2n-2 + n + \sqrt{5n^2+4} \\
 &= (3n-2) + \sqrt{5n^2+4}.
 \end{aligned}$$

For $n \geq 5$,

$$E_M(K_n + K_{1,n}) = |-2| + |-2| + \cdots + |-2| + |-1| + |-1| + \cdots + |-1|$$

$$\begin{aligned}
 & + \left| \frac{(3n-2) - \sqrt{5n^2+4}}{2} \right| + \left| \frac{(3n-2) + \sqrt{5n^2+4}}{2} \right| \\
 = & 2(n-1) + n + \frac{(3n-2) - \sqrt{5n^2+4}}{2} + \frac{(3n-2) + \sqrt{5n^2+4}}{2} \\
 = & 2n - 2 + n + 3n - 2 \\
 = & 6n - 4 \\
 = & 2(3n - 2).
 \end{aligned}$$

4. Conclusions

In this paper we discussed the concept of monophonic distance energy and obtained some results on monophonic distance energy for certain Join graphs. In future we planned to extend our research for various graph operations.

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Strong perfect cobondage number of standard graphs

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Abstract

Let G be a simple graph. A subset $S \subseteq V(G)$ is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$ ($|N_w(u) \cap S| = 1$) for every $u \in V(G) - S$ where $N_s(u) = \{v \in V(G) / uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G) / uv \in E(G), \deg v \leq \deg u\}$). The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)$ ($\gamma_{wp}(G)$). The strong perfect cobondage number $bc_{sp}(G)$ of a nonempty graph G is defined to be the minimum cardinality among all subsets of edges $X \subseteq E(G)$ for which $\gamma_{sp}(G + X) < \gamma_{sp}(G)$. If $bc_{sp}(G)$ does not exist, then $bc_{sp}(G)$ is defined as zero. In this paper study of strong perfect cobondage number of standard graphs and some special graphs are determined.

Keywords: Strong perfect dominating set, strong perfect domination number and strong perfect cobondage number.

AMS subject classification: 05C69³

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set D of G is a subset of $V(G)$ such that every vertex in $V - D$ is adjacent to at least one vertex in D [6]. A set $D \subseteq V(G)$ is a strong(weak) dominating set of G [5] if every vertex in $V - D$ is strongly(weakly) dominated by at least one vertex in D . The strong (weak) domination number $\gamma_s(G)$ ($\gamma_w(G)$) is the minimum cardinality of a strong (weak) dominating set of G . A dominating set S is a perfect dominating set of G [1] if $|N(v) \cap S| = 1$ for each $v \in V - S$. Motivated by these definitions, the authors defined strong perfect domination in graphs [2]. In [4], V.R. Kulli and B. Janakiram introduced the concept of cobondage number of a graph. The cobondage number of graph G is the minimum cardinality among the set of edges X such that $\gamma(G+X) < \gamma(G)$. In this paper strong perfect cobondage number is defined and strong perfect cobondage number of paths, cycle, bistar, complete bipartite graph and some special graphs are determined. For all graph theoretic terminologies and notations Harary [3] is followed.

2. Preliminaries

Definition 2.1. [2] Let G be a simple graph. A subset $S \subseteq V(G)$ is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$ ($|N_w(u) \cap S| = 1$) for every $u \in V(G) - S$ where $N_s(u) = \{v \in V(G)/uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G)/uv \in E(G), \deg v \leq \deg u\}$).

Remark 2.2. [2] The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)$ ($\gamma_{wp}(G)$).

Definition 2.3. Bi star is the graph obtained by joining the apex vertices of two copies of star $K_{1, n}$.

Definition 2.4. The wheel W_n is defined to be the graph $C_{n-1} + K_1$, $n \geq 4$.

Definition 2.5. The helm H_n is the graph obtained from the wheel W_n with n spokes by adding n pendant edges at each vertex on the wheel's rim.

Definition 2.6. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 (where G_1 has p_1 points and q_1 lines) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2 , and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 .

Definition 2.7. The lily graph L_n , $n \geq 2$ can be constructed by two-star graphs $2K_{1, n}$, $n \geq 2$ joining two path graphs $2P_n$, $n \geq 2$ with sharing a common vertex.

Theorem 2.8. [2] For any path P_m ,

$$\text{Then } \gamma_{\text{sp}}(P_m) = \begin{cases} n & \text{if } m = 3n, n \in \mathbb{N} \\ n + 1 & \text{if } m = 3n + 1, n \in \mathbb{N} \\ n + 2 & \text{if } m = 3n + 2, n \in \mathbb{N} \end{cases}$$

Theorem 2.9 [2] For any cycle C_m ,

$$\text{Then } \gamma_{\text{sp}}(C_m) = \begin{cases} n & \text{if } m = 3n, n \in \mathbb{N} \\ n + 1 & \text{if } m = 3n + 1, n \in \mathbb{N} \\ n + 2 & \text{if } m = 3n + 2, n \in \mathbb{N} \end{cases}$$

Theorem 2.10. [2] Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{\text{sp}}(G \odot K_1) = n$.

Remark 2.11. [2]

- (i) $\gamma_{\text{sp}}(D_{r,s}) = 2, r, s \in \mathbb{N}$
- (ii) $\gamma_{\text{sp}}(K_{m,n}) = \begin{cases} 2 & \text{if } m = n, m, n \geq 2 \\ m + n & \text{if } m \neq n, m, n \geq 2 \end{cases}$
- (iii) $\gamma_{\text{sp}}(H_n) = n, n \geq 5$ and $\gamma_{\text{sp}}(H_4) = 3$.

3. Main Results

Definition 3.1. The strong perfect cobondage number $bc_{\text{sp}}(G)$ of a graph is defined to be the minimum cardinality among all subsets of edges $X \subseteq E(G)$ for which $\gamma_{\text{sp}}(G + X) < \gamma_{\text{sp}}(G)$. If $bc_{\text{sp}}(G)$ does not exist, then $bc_{\text{sp}}(G)$ is defined as zero.

Example 3.2. Consider the graphs G and $G + e$ in the following figures 1 and 2 respectively.

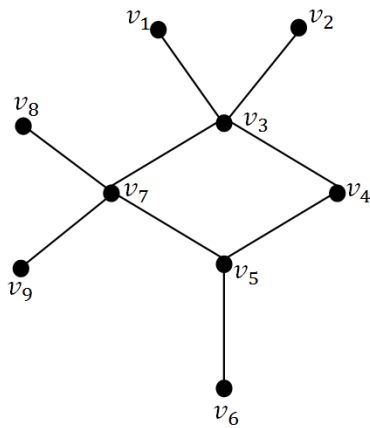


Fig. 1.

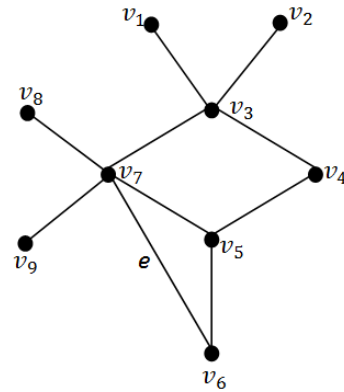


Fig. 2.

$\{v_3, v_7, v_6\}$, $\{v_3, v_7, v_5, v_4\}$ and $\{v_3, v_8, v_9, v_4, v_5\}$ are some strong perfect dominating sets of G . Hence $\gamma_{sp}(G) \leq 3$. Since there is no full degree vertex in G , $\gamma_{sp}(G) \geq 2$. There are five pendent vertices. Any strong perfect dominating sets contains either pendent vertices or their support vertices. v_3, v_7, v_5 are the support vertices adjacent with pendent vertices. Therefore $\gamma_{sp}(G) \geq 3$. Hence $\gamma_{sp}(G) = 3$. Let $e = v_6v_7$. $\{v_3, v_7\}$, $\{v_7, v_4, v_1, v_2\}$ are some strong perfect dominating sets of $G + e$. Therefore $\gamma_{sp}(G + e) \leq 2$. Since there is no full degree vertex in $G + e$, $\gamma_{sp}(G) \geq 2$. Hence $\gamma_{sp}(G + e) = 2 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Theorem 3.3. $bc_{sp}(P_m) = \begin{cases} 1 & \text{if } m = 3n + 1, n \geq 1 \\ 1 & \text{if } m = 3n + 2, n \geq 1 \\ 3 & \text{if } m = 3n, n \geq 2 \end{cases}$

Proof. Let $G = P_m$, $m \geq 4$. Let $V(G) = \{v_i / 1 \leq i \leq m\}$.

Case (1). Let $m = 3n + 1$, $n \geq 1$. $\gamma_{sp}(G) = n + 1$. Let $e = v_{3n-1}v_{3n+1}$, $\deg v_{3n-1} = \Delta(G + e) = 3$. $T = \{v_2, v_5, v_8, \dots, v_{3n-1}\}$ is a strong perfect dominating set of $G + e$. $|T| = n$. Therefore $\gamma_{sp}(G + e) \leq n$. Let S be any strong perfect dominating set of $G + e$. Since v_{3n-1} is the unique maximum degree vertex in $G + e$, v_{3n-1} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3n-2}$ is P_{3n-3} . Therefore $\gamma_{sp}(P_{3n-3}) = n - 1$. $|S| = n$. Hence $\gamma_{sp}(G + e) \geq n$. Therefore $\gamma_{sp}(G + e) = n < n + 1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let $m = 3n + 2$, $n \geq 1$. $\gamma_{sp}(G) = n + 2$. Let $e = v_{3n}v_{3n+2}$, $\deg v_{3n} = \Delta(G + e) = 3$. $T_1 = \{v_1, v_3, v_6, \dots, v_{3n}\}$, $T_2 = \{v_2, v_5, \dots, v_{3n-4}, v_{3n-3}, v_{3n}\}$ are some strong perfect dominating sets of $G + e$. $|T_1| = |T_2| = n + 1$. Therefore $\gamma_{sp}(G + e) \leq n + 1$. Let S be any strong perfect dominating set of $G + e$. v_{3n} strongly dominates v_{3n+2} . Therefore $v_{3n} \in S$. The subgraph induced by the vertices $v_1, v_2, \dots, v_{3n-2}$ is P_{3n-2} . Therefore $\gamma_{sp}(P_{3n-2}) = \gamma_{sp}(P_{3(n-1)+1}) = n$. Therefore $|S| = n + 1$. $\gamma_{sp}(G + e) \geq n + 1$. Therefore $\gamma_{sp}(G + e) = n + 1 < n + 2 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (3). Let $G = P_{3n}$, $n \geq 2$. $\gamma_{sp}(G) = n$. When one or two edges added with P_{3n} strong perfect domination number of the resulting graph either increases or remains same. Hence $bc_{sp}(G) \geq 3$. Let $e_1 = v_1v_3$, $e_2 = v_3v_5$, $e_3 = v_3v_6$. Let $X = \{e_1, e_2, e_3\}$. $T = \{v_3, v_8, v_{11}, \dots, v_{3n-1}\}$ is a strong perfect dominating set of $G + X$. $|T| = n - 1$. Therefore $\gamma_{sp}(G + X) \leq n - 1$. Let S be any strong perfect dominating set of $G + X$. $\deg v_3 = 5$. v_3 is the unique maximum degree vertex of $G + X$. v_3 belongs to S . The subgraph induced by the vertices $v_7, v_8, \dots, v_{3n}$ is a P_{3n-6} . Therefore $\gamma_{sp}(P_{3n-6}) = n - 2$. Therefore $|S| = n - 2 + 1 = n - 1$. $\gamma_{sp}(G + X) \geq n - 1$. Therefore $\gamma_{sp}(G + X) = n - 1 < n = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 3$.

Remark. Let $m = 3$. $\gamma_{sp}(G) = 1$. Hence $bc_{sp}(G) = 0$.

Theorem 3.4. $bc_{sp}(C_m) = \begin{cases} 1 & \text{if } m = 3n + 1, n \geq 1 \\ 1 & \text{if } m = 3n + 2, n \geq 1 \\ 3 & \text{if } m = 3n, n \geq 2 \end{cases}$

Proof. Let $G = C_m$, $m \geq 4$. Let $V(G) = \{v_i / 1 \leq i \leq m\}$.

Case (1). Let $m = 3n + 1$, $n \geq 1$. $\gamma_{sp}(G) = n + 1$. Let $e = v_1v_3$. $T = \{v_1, v_5, v_8, \dots, v_{3n-1}\}$ is a strong perfect dominating set of $G + e$. $|T| = n$. Therefore $\gamma_{sp}(G + e) \leq n$. Let S be any strong perfect dominating set of $G + e$. Since v_1, v_3 are maximum degree vertices in

$G + e$, either v_1 belongs to S or v_3 belongs to S . Suppose v_1 belongs to S (Proof is similar if v_3 belongs to S). The subgraph induced by the vertices $v_4, v_5, \dots, v_{3n}$ is the path P_{3n-3} . $\gamma_{sp}(P_{3n-3}) = n - 1$. Hence $\gamma_{sp}(G + e) \geq n$. Therefore $\gamma_{sp}(G + e) = n < n+1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let $m = 3n+2, n \geq 2$. $\gamma_{sp}(G) = n+2$. Let $e = v_1v_{3n+1}$. $T = \{v_1, v_4, v_7, \dots, v_{3n-2}, v_{3n-1}\}$ is a strong perfect dominating set of $G + e$. $|T| = n+1$. Therefore $\gamma_{sp}(G + e) \leq n+1$. Let S be any strong perfect dominating set of $G + e$. Since v_1, v_{3n+1} are maximum degree vertices in $G + e$, either v_1 belongs to S or v_{3n+1} belongs to S . Suppose v_1 belongs to S (Proof is similar if v_{3n+1} belongs to S). The subgraph induced by the vertices $v_3, v_4, \dots, v_{3n}$ is the path P_{3n-2} . Therefore $\gamma_{sp}(P_{3n-2}) = \gamma_{sp}(P_{3(n-1)+1}) = n$. Hence $\gamma_{sp}(G + e) \geq n+1$. Therefore $\gamma_{sp}(G + e) = n+1 < n+2 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Subcase (2a). If $n = 1$, $\gamma_{sp}(C_5) = 3$. Let $e = v_1v_4$. $\{v_1, v_2\}$ is the unique γ_{sp} - set of $G + e$. $\gamma_{sp}(G + e) = 2 < \gamma_{sp}(G)$. Hence $bc_{sp}(C_5) = 1$.

Case (3). Let $m = 3n, n \geq 2$. $\gamma_{sp}(G) = n$. When one or two edges added with C_{3n} strong perfect domination number of the resulting graph either increase or remain same. Hence $bc_{sp}(G) \geq 3$. Let $e_1 = v_1v_3, e_2 = v_1v_4, e_3 = v_1v_5$. $X = \{e_1, e_2, e_3\}$. $T = \{v_1, v_7, v_{10}, \dots, v_{3n-2}\}$ is a strong perfect dominating set of $G + X$. $|T| = n - 1$. Therefore $\gamma_{sp}(G + X) \leq n - 1$. Let S be any strong perfect dominating set of $G + X$. Since v_1 is the unique maximum degree vertex in $G + X$, v_1 belongs to S . The subgraph induced by the vertices $v_6, v_7, \dots, v_{3n-1}$ is the path P_{3n-6} . Therefore $\gamma_{sp}(P_{3n-6}) = \gamma_{sp}(P_{3(n-2)}) = n - 2$. Hence $\gamma_{sp}(G + X) \geq n - 1$. Therefore $\gamma_{sp}(G + X) = n - 1 < n = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 3$.

Theorem 3.5. $bc_{sp}(D_{r,s}) = s, r \geq s, r, s \geq 1$.

Proof. Let $G = D_{r,s}, r, s \geq 1$. Let $V(G) = \{u, v, u_i, v_j / 1 \leq i \leq r, 1 \leq j \leq s\}$, $E(G) = \{uv, uu_i, vv_j / 1 \leq i \leq r, 1 \leq j \leq s\}$. $\gamma_{sp}(G) = 2$. Let $r \geq s, r, s \geq 1$, $\gamma_{sp}(G + X) = 1$ if and only if $G + X$ must have a full degree vertex. It is possible only if u is adjacent with $v_1, v_2, \dots, v_s$ or v is adjacent with $u_1, u_2, \dots, u_r$. Since $r \geq s$, $X = \{uv_j / 1 \leq j \leq s\}$. u is the unique full degree vertex of $G + X$. Therefore $\gamma_{sp}(G + X) = 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = s$.

Theorem 3.6. $bc_{sp}(H_n) = 1, n \geq 4$.

Proof: Let $G = H_n, n \geq 4$. Let $V(G) = \{v, v_i, u_i / 1 \leq i \leq n - 1\}$, $E(G) = \{vv_i, v_iu_i / 1 \leq i \leq n - 1\} \cup \{v_n v_1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n - 2\}$. $\gamma_{sp}(G) = n, n \geq 5$. $\deg v = n - 1, \deg v_i = 4, \deg u_i = 1, 1 \leq i \leq n - 1$.

Case (1). Let $n \geq 5$. Let $e = u_1u_{n-1}$. $T = \{v, u_1, u_2, \dots, u_{n-2}\}$ and $\{v, u_2, u_3, \dots, u_{n-1}\}$ are some strong perfect dominating sets of $G + e$. $|T| = n - 1$. Therefore $\gamma_{sp}(G + e) \leq n - 1$. Let S be any strong perfect dominating set of $G + e$. There are $n - 3$ pendent vertices, either $n - 3$ pendent vertices belong to S or their support vertices belong to S . v is the unique maximum degree vertex of $G + e$. Hence v belongs to S . Since u_1 and u_{n-1} are adjacent either u_1 or u_{n-1} belong to S . Therefore $\gamma_{sp}(G + e) \geq n - 1$. Hence $\gamma_{sp}(G + e) = n - 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let $n = 4$. $\gamma_{sp}(G) = 3$. Let $e = v_1u_3$. $\{v_1, u_2\}$ is the unique γ_{sp} - set of $G + e$. Therefore $\gamma_{sp}(G + e) = 2 < 3 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

$$\textbf{Theorem 3.8. } bc_{sp}(K_{m,n}) = \begin{cases} n-1 & \text{if } m=n, m, n \geq 2 \\ 1 & \text{if } m=2, n \geq 3, m \neq n, m < n \\ n-m & \text{if } m \neq n, m < n, m \geq 3, n \geq 4 \end{cases}$$

Proof. Let $G = K_{m,n}$, $m, n \geq 2$. $V(G) = \{v_i, u_j / 1 \leq i \leq m, 1 \leq j \leq n\}$.

Case (1). Let $m = n$, $m, n \geq 2$. $\gamma_{sp}(G) = 2$. $\gamma_{sp}(G + X) = 1$ if and only if $G + X$ must have a full degree vertex. It is possible only if any u_i is adjacent with all other u_j or v_i is adjacent with all other v_j , $i \neq j, 1 \leq j \leq n$. Therefore, u_i or v_i is a full degree vertex of $G + X$. Let $X = \{u_i u_j / 1 \leq j \leq n \text{ and } j \neq i\}$ or $\{v_i v_j / 1 \leq j \leq n \text{ and } i \neq j\}$. $|X| = n - 1$. Therefore $\gamma_{sp}(G + X) = 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = n - 1$.

Case (2). Let $m \neq n$, $m < n$, $m = 2$, $n \geq 2$. $G = K_{2,n}$. $\gamma_{sp}(G) = n + 2$. Let $e = v_1 v_2$. v_1 and v_2 are full degree vertices of $G + e$. Therefore $\gamma_{sp}(G + e) = 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (3). Let $m \neq n$, $m < n$, $m \geq 3$, $n \geq 4$. $\gamma_{sp}(G) = m + n$. If any two vertices v_i and v_j are adjacent then G' is the new graph $\gamma_{sp}(G') = m + n$, otherwise the vertices u_j are strongly dominated by two vertices v_i and v_j , a contradiction. Suppose degree of the vertex u_j is increased so that $\deg v_i = \deg u_j$. u_j strongly dominates the vertices v_i , $1 \leq i \leq m$ and also strongly dominates $(n - m)$ u_i 's. Without loss of generality, let u_1 be adjacent with $u_2, u_3, \dots, u_{n-m+1}$. Let $X = \{u_1 u_2, u_1 u_3, \dots, u_1 u_{n-m+1}\}$. $|X| = n - m$. $T = \{u_1, u_{n-m+2}, u_{n-m+3}, \dots, u_n\}$ is a strong perfect dominating set of $G + X$. $\gamma_{sp}(G + X) \leq m$. Let S be any strong perfect dominating set of G . v_i 's, $1 \leq i \leq m$ are maximum degree vertices and they are mutually non adjacent if they belong to S then u_1 is strongly dominated by more than one vertex in S , a contradiction. If u_1 is included along with v_i 's then u_j , $j \neq i$, is dominated by more than one vertex, a contradiction. Hence S contains all the vertices of $G + X$. $|S| = n + m$. Any γ_{sp} -set does not contain $v_1, v_2, \dots, v_m$. Therefore, T is the minimum strong perfect dominating set of $G + X$. Therefore $\gamma_{sp}(G + X) = m < m + n = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = n - m$.

Theorem 3.9. Let $G \odot K_1 = G'$ be a connected graph. Then $bc_{sp}(G') = 1$.

Proof. Let $V(G') = \{v_i, u_i / 1 \leq i \leq n\}$. $\gamma_{sp}(G') = n$. $\deg v_i \geq 2$, $\deg u_i = 1, 1 \leq i \leq n$. Let $T = \{v_i / 1 \leq i \leq n\}$ be a strong perfect dominating set of G' . $|T| = n$. Join any vertex v_i with u_j , without loss of generality, let $e = v_2 u_1$. v_2 strongly dominates v_1, v_3, u_1 and u_2 . The subgraph induced by the remaining vertices $v_i, u_i, 3 \leq i \leq n$, is $H \odot K_1$, where $|V(G')| = n - 2$. Therefore $\gamma_{sp}(H \odot K_1) = n - 2$. These $n - 2$ vertices along with v_1 form a strong perfect dominating set of $G \odot K_1$. $\gamma_{sp}(G') \geq n - 2 + 1 = n - 1$. Let $T = \{v_i / 2 \leq i \leq n\}$ is a strong perfect dominating set of the resulting graph $G' + e$. Therefore $\gamma_{sp}(G' + e) < \gamma_{sp}(G')$. Hence $bc_{sp}(G') = 1$.

$$\textbf{Theorem 3.10. } \gamma_{sp}(L_n) = \begin{cases} 2k + 1 & \text{if } n = 3k, 3k + 2, k \geq 1 \\ 2k + 3 & \text{if } n = 3k + 1, k \geq 1 \end{cases}$$

Proof. Let $G = L_n$, $n \geq 3$. $V(G) = \{u_i, v_j / 1 \leq i \leq 2n, 1 \leq j \leq 2n - 1\}$. $|V(G)| = 4n - 1$. $\deg v_n = 2n + 2 = \Delta(G)$, $\deg u_i = 1, 1 \leq i \leq 2n$, $\deg v_1 = \deg v_{2n-1} = 1$, $\deg v_j = 2, 2 \leq j \leq 2n - 2$.

Case (1). Let $n = 3k$, $k \geq 1$. $T = \{v_2, v_3, v_6, \dots, v_{3k-3}, v_{3k}, v_{3k+3}, v_{3k+6}, \dots, v_{6k-6}, v_{6k-3}, v_{6k-2}\}$ is a strong perfect dominating set of G . $|T| = 1 + \frac{3k-3-3}{3} + 2 + \frac{6k-6-3k-3}{3} +$

$3 = 2k + 1$. $\gamma_{sp}(G) \leq 2k + 1$. Let S be any strong perfect dominating set of G . Since v_{3k} is unique the maximum degree vertex in G . v_{3k} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3k-2} \cup v_{3k+2}, v_{3k+3}, \dots, v_{6k-1}$ is $2P_{3k-2}$. $\gamma_{sp}(2P_{3k-2}) = 2\gamma_{sp}(P_{3(k-1)+1}) = 2k$. $|S| = 2k + 1$. Therefore $\gamma_{sp}(G) \geq 2k + 1$. Hence $\gamma_{sp}(G) = 2k + 1$.

Case (2). Let $n = 3k+1$, $k \geq 1$. $T = \{v_2, v_3, v_4, v_7, \dots, v_{3k-2}, v_{3k+1}, v_{3k+4}, v_{3k+7}, \dots, v_{6k-2}, v_{6k-1}, v_{6k}\}$ is a strong perfect dominating set of G . $|T| = 1 + \frac{3k-2-4}{3} + 3 + \frac{6k-2-3k-4}{3} + 3 = 2k + 3$. $\gamma_{sp}(G) \leq 2k + 3$. Let S be any strong perfect dominating set of G . Since v_{3k+1} is unique the maximum degree vertex in G . v_{3k+1} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3k-1} \cup v_{3k+3}, v_{3k+4}, \dots, v_{6k+1}$ is $2P_{3k-1}$. $\gamma_{sp}(2P_{3k-1}) = 2\gamma_{sp}(P_{3(k-1)+2}) = 2k + 2$. $|S| = 2k + 3$. Therefore $\gamma_{sp}(G) \geq 2k + 3$. Hence $\gamma_{sp}(G) = 2k + 3$.

Case (3). Let $n = 3k+2$, $k \geq 1$. $T = \{v_2, v_5, \dots, v_{3k-1}, v_{3k+2}, v_{3k+5}, v_{3k+8}, \dots, v_{6k+2}\}$ is a strong perfect dominating set of G . $|T| = 1 + \frac{3k-1-2}{3} + 1 + \frac{6k+2-3k-5}{3} + 1 = 2k + 1$. $\gamma_{sp}(G) \leq 2k + 1$. Let S be any strong perfect dominating set of G . Since v_{3k+2} is unique the maximum degree vertex in G . v_{3k+2} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3k} \cup v_{3k+4}, v_{3k+5}, \dots, v_{6k+3}$ is $2P_{3k}$. $\gamma_{sp}(2P_{3k}) = 2\gamma_{sp}(P_{3k}) = 2k$. $|S| = 2k + 1$. Therefore $\gamma_{sp}(G) \geq 2k + 1$. Hence $\gamma_{sp}(G) = 2k + 1$.

Remark. $L_2 = K_{1,6}$. Therefore $\gamma_{sp}(L_2) = 1$.

Theorem 3.11. $bc_{sp}(L_n) = \begin{cases} 1 & \text{if } n = 3k, 3k + 1, k \geq 1 \\ 3 & \text{if } n = 3k + 2, k \geq 1 \end{cases}$

Proof. Let $G = L_n$, $n \geq 3$. $V(G) = \{u_i, v_j / 1 \leq i \leq 2n, 1 \leq j \leq 2n - 1\}$.

Case (1). Let $n = 3k$, $k \geq 1$. Let $e = v_{3k-2}v_{3k}$. $\deg v_{3k} = 6k + 3 = \Delta(G + e)$, $\deg u_i = 1$, $1 \leq i \leq 6k$, $\deg v_1 = \deg v_{6k-1} = 1$, $\deg v_j = 2$, $2 \leq j \leq 6k - 2$, $j \neq 3k - 2$, $\deg v_{3k-2} = 3$. $T = \{v_2, v_5, \dots, v_{3k-4}, v_{3k}, v_{3k+3}, v_{3k+6}, \dots, v_{6k-3}, v_{6k-2}\}$ is a strong perfect dominating set of $G + e$. $|T| = \frac{3k-4-2}{3} + 2 + \frac{6k-3-3k-3}{3} + 2 = 2k$. $\gamma_{sp}(G + e) \leq 2k$. Let S be any strong perfect dominating set of $G + e$. Since v_{3k} is unique the maximum degree vertex in $G + e$. v_{3k} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3k-3} \cup v_{3k+2}, v_{3k+3}, \dots, v_{6k-1}$ is $P_{3k-3} \cup P_{3k-2}$. $\gamma_{sp}(P_{3(k-1)}) + \gamma_{sp}(P_{3(k-1)+1}) = 2k - 1$. $|S| = 2k$. Therefore $\gamma_{sp}(G + e) \geq 2k$. Hence $\gamma_{sp}(G + e) = 2k$. Therefore $\gamma_{sp}(G + e) < 2k + 1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let $n = 3k+1$, $k \geq 1$. Let $e = v_{3k-1}v_{3k+1}$. $\deg v_{3k+1} = 6k + 5 = \Delta(G + e)$, $\deg u_i = 1$, $1 \leq i \leq 6k + 2$, $\deg v_1 = \deg v_{6k+1} = 1$, $\deg v_j = 2$, $2 \leq j \leq 6k$, $j \neq 3k - 1$, $\deg v_{3k-1} = 3$. $T = \{v_2, v_3, v_6, \dots, v_{3k-3}, v_{3k+1}, v_{3k+4}, \dots, v_{6k-2}, v_{6k-1}, v_{6k}\}$ is a strong perfect dominating set of $G + e$. $|T| = 1 + \frac{3k-3-3}{3} + 2 + \frac{6k-2-3k-4}{3} + 3 = 2k + 2$. $\gamma_{sp}(G + e) \leq 2k + 2$. Let S be any strong perfect dominating set of $G + e$. Since v_{3k+1} is unique the maximum degree vertex in $G + e$. v_{3k+1} belongs to S . The subgraph induced by the vertices $v_1, v_2, \dots, v_{3k-2} \cup v_{3k+3}, v_{3k+4}, \dots, v_{6k+1}$ is $P_{3k-2} \cup P_{3k-1}$. $\gamma_{sp}(P_{3(k-1)+1}) + \gamma_{sp}(P_{3(k-1)+2})$

$= 2k + 1$. $|S| = 2k + 2$. Therefore $\gamma_{sp}(G + e) \geq 2k + 2$. Hence $\gamma_{sp}(G + e) = 2k + 2 < 2k + 3 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (3). Let $n = 3k + 2$, $k \geq 1$. $\gamma_{sp}(G) = 2k + 1$. To reduce the strong perfect domination number, at least one edge must be added. Since v_{3k+2} is the unique maximum degree vertex, it belongs to any strong perfect dominating set. Make v_{3k+2} adjacent with v_{3k} , v_{3k-1} , v_{3k-2} . So that strong perfect domination number decreases by at least one. Let $X = \{v_{3k}v_{3k+2}, v_{3k-1}v_{3k+2}, v_{3k-2}v_{3k+2}\}$. $T = \{v_2, v_5, \dots, v_{3k-4}, v_{3k+2}, v_{3k+5}, v_{3k+8}, \dots, v_{6k+2}\}$ is a strong perfect dominating set of $G + X$. Hence $\gamma_{sp}(G + X) = 2k < 2k + 1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 3$.

4. Conclusion

In this paper, strong perfect cobondage number of some standard graphs and some special graphs are studied. Further study can be observed for path related graphs, cycle related graphs, subdivision graphs, middle graphs and product graphs. Bounds for strong perfect cobondage number of graphs and Nordhaus Gaddum type results can be determined.

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Modular Coloring and Switching in Some Planar Graphs

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Abstract

For a connected graph G , let $c: V(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G . The color sum $\sigma(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$ that is $\sigma(v) = \sum_{u \in N(v)} c(u) \pmod{k}$. The coloring c is called a modular k -coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs of adjacent vertices $x, y \in G$. The modular chromatic number or simply the mc-number of G is the minimum k for which G has a modular k -coloring. A switching graph is an ordinary graph with switches. For many problems, switching graphs are a remarkable straight forward and natural model, but they have hardly been studied. A vertex switching G_v of a graph G is obtained by taking a vertex V of G , removing the entire edges incident with V and adding edges joining V to every vertex which are not adjacent to V in G . In this paper we determine the modular chromatic number of Wheel graph, Friendship graph and Gear graph after switching on certain vertices. Here, we first define switching of graphs. Next, we investigating several problems on finding the $mc(G)$ after switching of graphs and provide their characterization in terms of complexity.

Keywords. Modular coloring, Modular chromatic number, Switching, Wheel Graph, Friendship graph, Gear graph.

MSC AMS Classification 2020: 05C15³

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1. Introduction

We are encouraged by the modular colorings and the modular chromatic number of different graphs, where the chromatic number is defined as the color sum of all the neighboring vertices in $\mathbb{Z}_k$. At this point of view, to the curiosity for minimizing the modular chromatic number, determined to switching in certain vertices in some graphs.

For a vertex v of a graph G , let $N(v)$ denote the neighborhood of v (the set of adjacent vertices to vertex v). For a graph G without isolated vertices, let $c: V(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. The color sum $\sigma(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$, that is $\sigma(v) = \sum_{u \in N(v)} c(u)$ [1, 2, 3]. The coloring c is called a modular sum k -coloring or simply a modular k -coloring of G , if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G . A coloring c is called modular coloring if c is a modular k -coloring for some integer $k \geq 2$. The modular chromatic number $mc(G)$ is the minimum k for which G has a modular k -coloring. This concept was introduced by Okamoto, Salehi and Zhang [4, 5, 6, 8]. In order to distinguish the vertices of a connected graph and to differentiate the adjacent vertices of a graph with the minimum number of colors, the concept of modular coloring was put forward by Okamoto, Salehi and Zhang [6].

A graph H is the switching of a graph G with respect to the vertex v of G if $V(H) = V(G)$ and $E(H) = (E(G) \setminus \{uv : u \in N_G(v)\}) \cup \{uv : u \in N'_G(v)\}$ [7,9]. The switching of G with respect to v is denoted $G_s(v)$. The operation of creating $G_s(v)$ is called switching on v in G . In other words, switching on a vertex v of a graph has the effect of removing all edges incident with the vertex v and joining the vertex v to all vertices to which it was formerly non-adjacent. Here, we first define switching of graphs. Next, we investigating several problems on finding the $mc(G)$ after switching of graphs and provide their characterization in terms of complexity. In this paper we find the modular chromatic number of wheel graph, friendship graph and gear graph after switching on certain vertices at different levels.

2. Modular coloring of wheel graph after switching

The switching of a vertex in a wheel having n vertices is denoted by $W_s(n)$.

In Wheel switching is not possible in $W(3)$ since it is a complete graph. Switching in a wheel is obtained in two ways. They are

1) The switching at the vertex $u \in \ell_0$.

2) The switching at the vertex $v_i \in \ell_1$. be the vertices

Let $u \in \ell_0$ be the central vertex and $v_1, v_2, \dots, v_n \in \ell_1$ be the vertices which are adjacent to $u \in \ell_0$. The switching at $u \in \ell_0$ makes the graph $W(n)$ is a cycle having n vertices with a central vertex $u \in \ell_0$.

Theorem 2.1

The modular coloring of a graph obtained after the switching of a vertex $v_i \in \ell_1$ is $W_s(n)$ = 3 for $n = 4k, 4k+1, [4k+2; k > 1]$; $W_s(6) = 4$; $W_s(n) = 4$ for $n = 4k+3, k \geq 1$.

Proof:

For a wheel $W(n)$, let the vertex $u \in \ell_0, v_i \in \ell_1$ for $i = 1, 2, \dots, n$ be the vertices.

Case (i) $mc[W_s(4)] = 3$.

Let $v_1, v_2, v_3, v_4 \in \ell_1$ be the vertices at level $\ell=1$. Switching is taken for $v_1 \in \ell_1$.

Consider the modular coloring $c(v):v[W_s(4)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 0 & \text{for } v_1 \in \ell_1, u \in \ell_0 \\ 1 & \text{otherwise} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0 \\ 2 & \text{for } v_3 \in \ell_1 \\ 1 & \text{otherwise} \end{cases} \text{ Here } \sigma(x) \neq \sigma(y) \forall x, y \text{ of adjacent vertices in } W_s(4).$$

$\therefore mc[W_s(4)] = 3$. Hence the proof.

Case (ii) $mc[W_s(5)] = 3$.

Let $v_1, v_2, v_3, v_4, v_5 \in \ell_1$ be the vertices at level $\ell=1$. Switching is taken for $v_1 \in \ell_1$.

$$\text{Consider the modular coloring } c(v):v[W_s(5)] \rightarrow \mathbb{Z}_3 \text{ defined by } c(v) = \begin{cases} 1 & \text{for } v_3 \in \ell_1 \\ 2 & \text{for } v_4 \in \ell_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0, v_1 \in \ell_1 \\ 1 & \text{for } v_2, v_4 \in \ell_1 \\ 2 & \text{otherwise} \end{cases} \text{ Here } \sigma(x) \neq \sigma(y) \forall x, y \text{ of adjacent vertices in } W_s(5).$$

$\therefore mc[W_s(5)] = 3$. Hence the proof.

Case (iii) $mc[W_s(6)] = 4$. Let $v_1, v_2, v_3, v_4, v_5, v_6 \in \ell_1$ be the vertices at level $\ell=1$. Switching is taken for $v_1 \in \ell_1$. Consider the modular coloring $c(v):v[W_s(6)] \rightarrow \mathbb{Z}_3$ defined by $c(v) =$

$$\begin{cases} 1 & \text{for } v_3 \in \ell_1 \\ 2 & \text{for } v_4 \in \ell_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 3 & \text{for } u \in \ell_0, v_1 \in \ell_1 \\ 1 & \text{for } v_2, v_4 \in \ell_1 \\ 2 & \text{for } v_3, v_5 \in \ell_1 \\ 0 & \text{otherwise} \end{cases} \text{ Here } \sigma(x) \neq \sigma(y) \forall x, y \text{ of adjacent vertices in } W_s(6).$$

$W_s(6)$.

$\therefore mc[W_s(6)] = 4$. Hence the proof.

Case (iv) $mc[W_s(4k+3)] = 4$.

Let $u \in \ell_0$ be the central vertex. Let $v_1, v_2, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{4k+3} \in \ell_1$ be the vertices at level $\ell=1$. Switching is taken for $v_i \in \ell_1$. After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+3}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} . The $4k$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k}$.

Subcase (i) $mc[W_s(4k+3)] = 4$ for $k = 1 + 4j, j = 0, 1, 2, \dots$ [Eg: $W_s(7), W_s(23), W_s(39), \dots$

Consider the modular coloring $c(v):v[W_s(4k+3)] \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 2 & \text{if } R_{4k} \in \ell_1 \\ 1 & \text{if } R_{1+4j} \in \ell_1, j = 0, 1, 2, \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 3 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 2 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 1 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots (2k-1) \\ 0 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+3)$. $\therefore mc[w_s(4k+3)] = 4$ for $k = 1 + 4j, j = 0, 1, 2, \dots$. Hence the proof.

Eg:

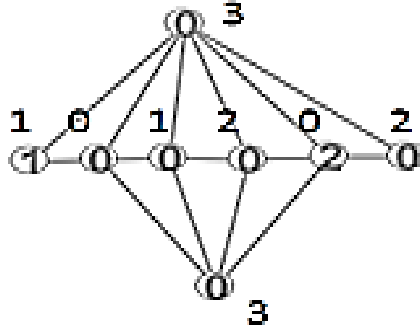


Figure1. Switching with modular coloring in $W_s(7)$

Subcase(ii) $mc[W_s(4k+3)] = 4$ for $k = 2 + 4j, j = 0, 1, 2, \dots$ [$W_s(11), W_s(27), W_s(43), \dots$]

Consider the modular coloring $c(v):v[W_s(4k+3)] \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 2 & \text{if } R_{4k} \in \ell_1 \\ 1 & \text{if } u \in \ell_0, R_{1+4j} \in \ell_1, j = 0, 1, 2, \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 3 & \text{for } R_{4k-1} \in \ell_1 \\ 2 & \text{for } v_{i+1}, v_{i-1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots (2k-1) \\ 1 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+3)$. $\therefore mc[w_s(4k+3)] = 4$ for $k = 2 + 4j, j = 0, 1, 2, \dots$. Hence the proof.

Eg:

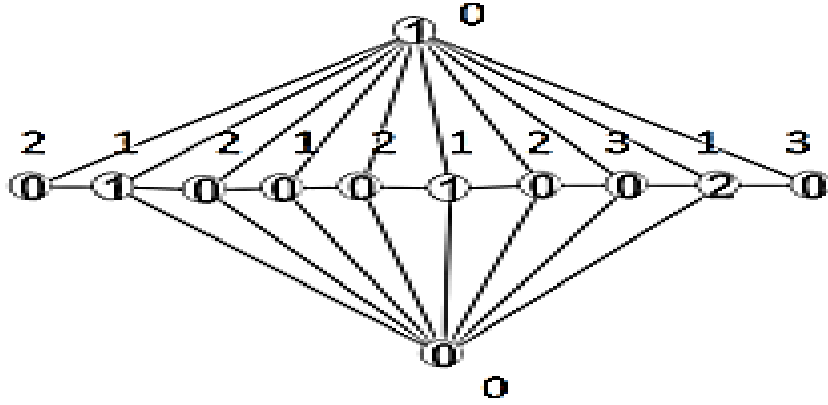


Figure 2. Switching with modular coloring in $W_s(11)$

Subcase(iii) $mc[W_s(4k+3)]=4$ for $k=3+4j, j=0,1,2,...$ [Eg: $W_s(15), W_s(31), W_s(47), ...$]

Consider the modular coloring $c(v):v[W_s(4k+3)] \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 2 & \text{if } u \in \ell_0, R_{4k} \in \ell_1 \\ 1 & \text{if } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 1 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 3 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots (2k-1) \\ 2 & \text{otherwise} \end{cases}$$

adjacent vertices in $W_s(4k+3)$. $\therefore mc[w_s(4k+3)]=4$ for $k=3+4j, j=0,1,2,...$. Hence the proof.

Eg:

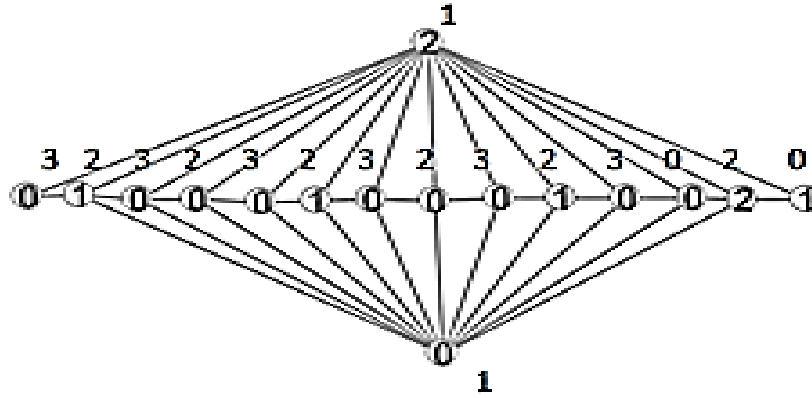


Figure 3. Switching with modular coloring in $W_s(15)$

Subcase(iv) $mc[W_s(4k+3)]=4$ for $k=4+4j, j=0,1,2,...$ [$W_s(19), W_s(35), W_s(51), ...$]

Consider the modular coloring $c(v):v[W_s(4k+3)] \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 3 & \text{if } u \in \ell_0 \\ 1 & \text{if } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 2 & \text{if } R_{4k} \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 1 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 2 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 0 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots (2k-1) \\ 3 & \text{otherwise} \end{cases} \text{ Here } \sigma(x) \neq \sigma(y) \forall x, y \text{ of}$$

adjacent vertices in $W_s(4k+3)$. $\therefore mc[w_s(4k+3)] = 4$ for $k=4+4j, j=0, 1, 2, \dots$. Hence the proof.

Eg:

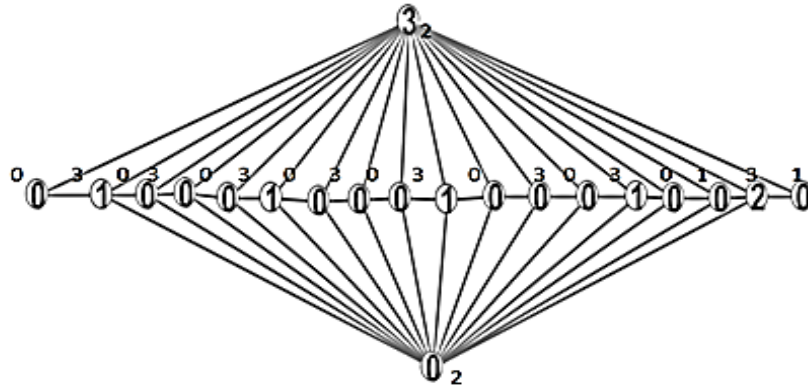


Figure 4. Switching with modular coloring in $W_s(19)$

Case(v) $mc[W_s(4k)] = mc[W_s(4k+1)] = mc[W_s(4k+2)] = 3$.

Subcase(i) $mc[W_s(4k)] = 3$ for $k=2+3j, j=0, 1, 2, \dots [W_s(8), W_s(20), W_s(32), \dots]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} . Let the $4k-3$ vertices which are adjacent to v_i be renamed as $R_1, R_2, \dots, R_{4k-3}$ respectively. Consider the modular coloring $c(v):v[W_s(4k)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{if } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 2 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 1 & \text{for } v_{i+1}, v_{i-1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots, 2(k-1) \\ 0 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k)$. $\therefore mc[w_s(4k)] = 3$ for $k=2+3j, j=0, 1, 2, \dots$. Hence the proof.

Eg:

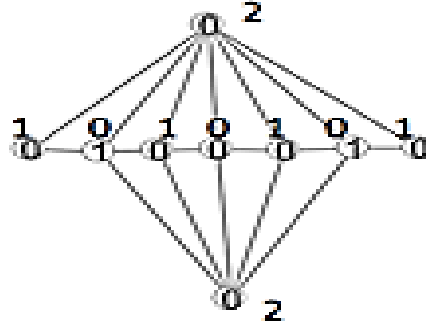


Figure 5. Switching with modular coloring in $Ws(8)$

Subcase(ii) $mc[W_s(4k)] = 3$ for $k=3+3j, j=0,1,2,...[Ws(12),Ws(24),Ws(36),...]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} .

Let the $4k-3$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-3}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{if } u \in \ell_0, R_{1+4j} \in \ell_1, j = 0, 1, 2, \dots, (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 1 & R_{1+2j} \in \ell_1, \text{ for } j = 0, 1, 2, \dots, (k+1) \\ 2 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k)$. $\therefore mc[w_s(4k)] = 3$ for $k=3+3j, j=0,1,2,...$. Hence the proof.

Eg:

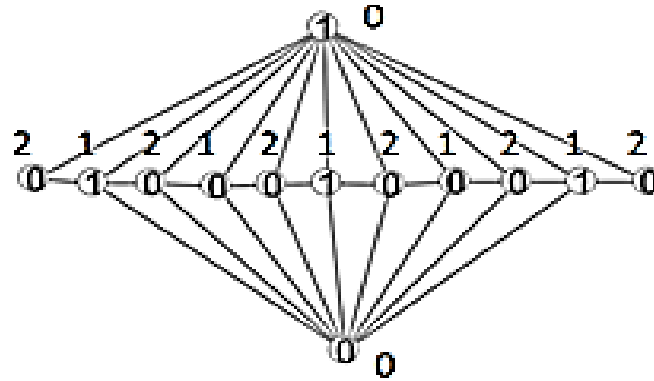


Figure 6. Switching with modular coloring in $Ws(12)$

Subcase(iii) $mc[W_s(4k)] = 3$ for $k=4+3j, j=0,1,2,...[Ws(16),Ws(28),Ws(40),...]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} .

Let the $4k-3$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-3}$ respectively.

Consider the modular coloring $c(v):v[W_s(4k)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 2 & \text{if } u \in \ell_0 \\ 1 & \text{for } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 1 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 2 & R_{1+2j} \in \ell_1, \text{ for } j = 0, 1, 2, \dots, 2(k-1) \\ 0 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k)$.

$\therefore mc[W_s(4k)] = 3$ for $k = 4 + 3j, j = 0, 1, 2, \dots$

Hence the proof.

Eg:

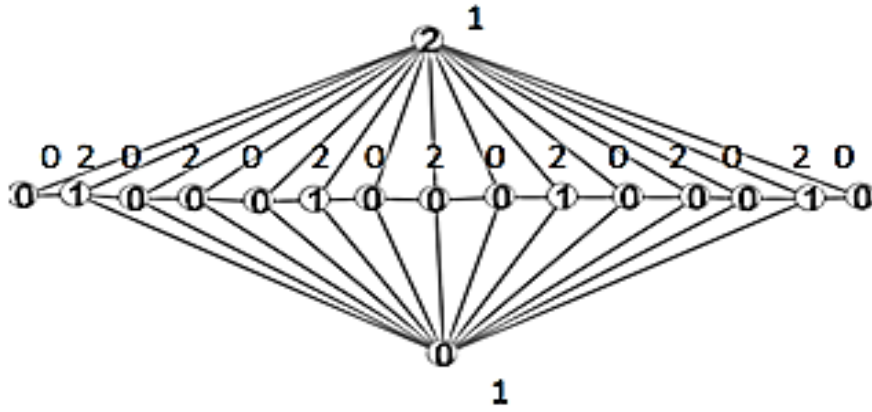


Figure7. Switching with modular coloring in $W_s(16)$

Subcase(iv) $mc[W_s(4k+1)] = 3$ for $k = 2 + 3j, j = 0, 1, 2, \dots [W_s(9), W_s(21), W_s(33), \dots]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+1}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} .

Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-2}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k+1)] \rightarrow \mathbb{Z}_3$ defined by $c(v) =$

$$\begin{cases} 1 & \text{for } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 2 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 1 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots, (2k-1) \\ 0 & \text{otherwise} \end{cases}$$

of adjacent vertices in $W_s(4k+1)$. $\therefore mc[W_s(4k+1)] = 3$ for $k = 2 + 3j, j = 0, 1, 2, \dots$

Hence the proof.

Eg:

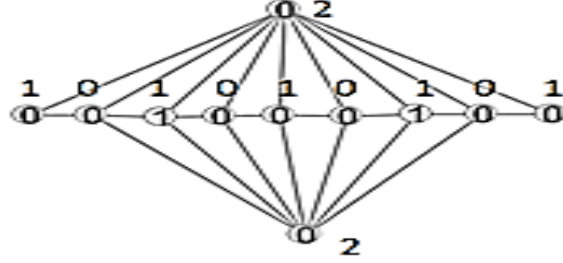


Figure 8. Switching with modular coloring in $W_s(9)$

Subcase (v) $mc[W_s(4k+1)] = 3$ for $k = 3+3j, j = 0, 1, 2, \dots$ [$W_s(13), W_s(25), W_s(37), \dots$]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+1}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} .

Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-2}$ respectively.

Consider the modular coloring $c(v):v[W_s(4k+1)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{for } u \in \ell_0; R_{1+4j} \in \ell_1, j = 0, 1, 2, \dots, (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 2 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots, (2k-1) \\ 1 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+1)$.

$\therefore mc[w_s(4k+1)] = 3$ for $k = 3+3j, j = 0, 1, 2, \dots$

Hence the proof.

Eg:

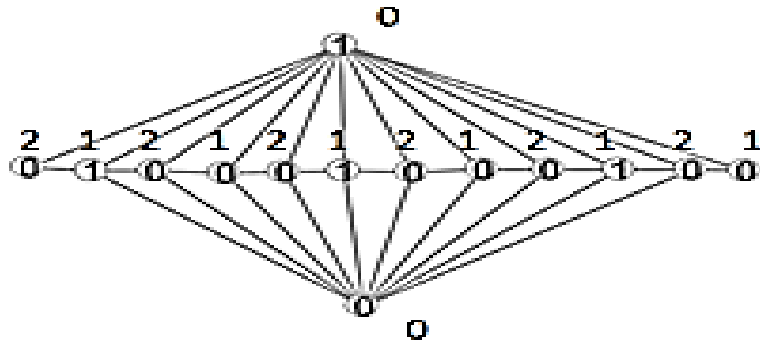


Figure 9. Switching with modular coloring in $W_s(13)$

Subcase (vi) $mc[W_s(4k+1)] = 3$ for $k = 4+3j, j = 0, 1, 2, \dots$ [$W_s(17), W_s(29), W_s(41), \dots$]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+1}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} . Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-2}$ respectively.

Consider the modular coloring $c(v):v[W_s(4k+1)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 & \text{for } R_{1+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v) = \begin{cases} 1 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 0 & \text{for } v_{i+1}, R_{2j} \in \ell_1, \text{ for } j = 1, 2, \dots, (2k-1) \\ 2 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+1)$. $\therefore mc[w_s(4k+1)] = 3$ for $k=4+3j, j=0, 1, 2, \dots$. Hence the proof.

Eg:

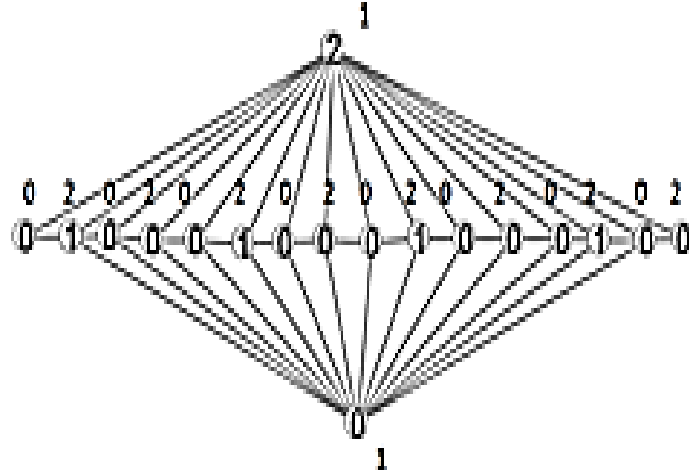


Figure 10. Switching with modular coloring in $W_s(17)$

Subcase(vii) $mc[W_s(4k+2)] = 3$ for $k=2+3j, j=0, 1, 2, \dots [W_s(10), W_s(22), W_s(34), \dots]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+2}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} .

Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-1}$ respectively.

Consider the modular coloring $c(v):v[W_s(4k+2)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{for } R_{2+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v) = \begin{cases} 2 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 1 & \text{for } R_{1+2j} \in \ell_1, \text{ for } j = 0, 1, 2, \dots, (2k-1) \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+2)$. $\therefore mc [w_s(4k+2)] = 3$ for $k = 2+3j, j = 0, 1, 2, \dots$. Hence the proof.

Eg:

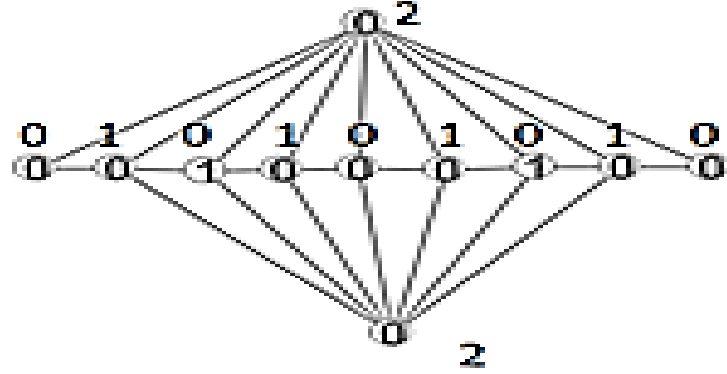


Figure 11. Switching with modular coloring in $Ws(10)$

Subcase(viii) $mc[Ws(4k+2)] = 3$ for $k=3+3j, j=0,1,2,\dots, [Ws(14), Ws(26), Ws(38), \dots]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+2}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} . Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-1}$ respectively.

Consider the modular coloring $c(v): v[Ws(4k+2)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{for } u \in \ell_0, R_{2+4j} \in \ell_1, j = 0, 1, 2, \dots, (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 2 & \text{for } R_{1+2j} \in \ell_1, \text{ for } j = 0, 1, 2, \dots, (2k-1) \\ 1 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $Ws(4k+2)$.

$\therefore mc[Ws(4k+2)] = 3$ for $k=3+3j, j=0,1,2,\dots$. Hence the proof.

Eg:

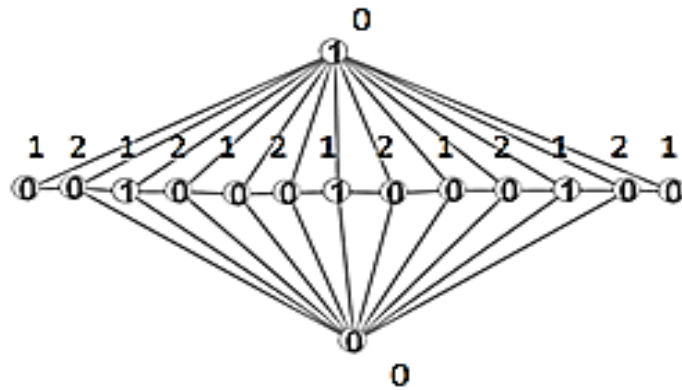


Figure 12. Switching with modular coloring in $Ws(14)$

Subcase(ix) $mc[Ws(4k+2)] = 3$ for $k=4+3j, j=0,1,2,\dots, [Ws(18), Ws(30), Ws(42), \dots]$

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \dots, v_{4k+2}, v_1, v_2, \dots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1} . Let the $4k-2$ vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-1}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k+2)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 & \text{for } R_{2+4j} \in \ell_1, j = 0, 1, 2 \dots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v) = \begin{cases} 1 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 0 & \text{for } R_{1+2j} \in \ell_1, \text{ for } j = 0, 1, 2, \dots, (2k-1) \\ 2 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $W_s(4k+2)$. $\therefore mc[w_s(4k+2)] = 3$ for $k=4+3j, j=0, 1, 2, \dots$. Hence the proof.

Eg:

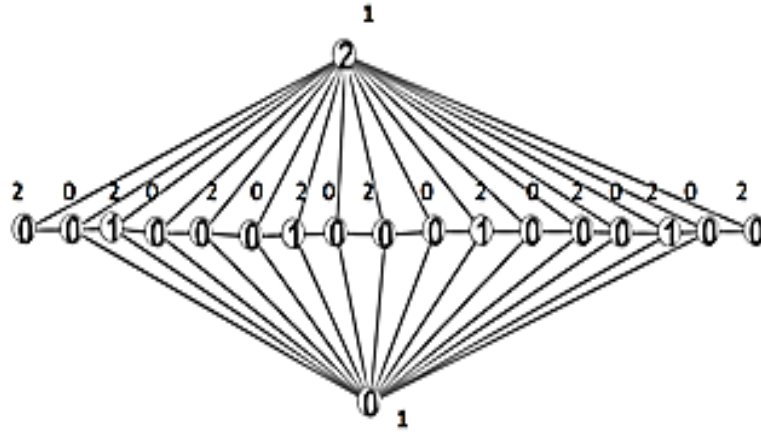


Figure 13. Switching with modular coloring in $W_s(18)$

3. Modular colorings after switching on Friendship graph

Let $u \in \ell_0$ be the center $v_1, v_2, v_3, \dots, v_{2n}$ be the vertices in ℓ_1 where each of 2 consecutive vertices forms an edge for the respective cycles since a friendship graph is constructed by joining n copies of the cycle C_3 with a common vertex. The vertices in ℓ_1 is taken in the clockwise direction. Modular coloring after switching of a friendship graph with n vertices is denoted by $mc[FS_s(n)]$. Here switching can be taken only for $v_i \in \ell_1$ for any $i=1, 2, \dots, 2n$. We cannot form a switching with $u \in \ell_0$ since it is adjacent to all vertices of ℓ_1 .

Theorem 3.1.

The modular coloring of a friendship graph after switching a vertex in ℓ_1 then (i) $mc[FS_s(2)] = 3$. (ii) $mc[FS_s(n)] = 4; n \geq 3$.

Proof:

Case (i) $mc[FS_s(2)] = 3$.

Modular Coloring and Switching In Some Planar Graphs

Switching is taken for $v_1 \in \ell_1$. Therefore v_1 is adjacent to v_3 and v_4 after switching in $FS_s(2)$.

Consider a modular coloring $c(v):v[FS_s(2)] \rightarrow \mathbb{Z}_3$ defined by $C(v) = \begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 & \text{for } v_4 \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$

then $\sigma(v) = \begin{cases} 1 & \text{for } u \in \ell_0, v_1 \in \ell_1 \\ 2 & \text{for } v_4, v_2 \in \ell_1 \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $FS_s(2)$

$\therefore mc[FS_s(2)] = 3$. Hence the proof.

Case(ii) $mc[FS_s(n)] = 4; n \geq 3$.

Subcase (i) $mc[FS_s(n)] = 4; n \geq 3$ for n is odd.

Switching is taken for $v_1 \in \ell_1$. Therefore v_1 is adjacent to $v_3, v_4, \dots, v_{2n}$ after switching in $FS_s(n)$.

Consider a modular coloring $c(v):v[FS_s(n)] \rightarrow \mathbb{Z}_4$ defined by

$$C(v) = \begin{cases} 3 & \text{for } u \in \ell_0 \\ 2 & \text{for } v_{2j} \in \ell_1; j = 1, 2, \dots, n \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v) = \begin{cases} 2 & \text{for } u \in \ell_0 \\ 3 & \text{for } v_{2j} \in \ell_1; j = 1, 2, \dots, n \\ 0 & \text{for } v_1 \in \ell_1 \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $FS_s(n)$

$\therefore mc[FS_s(n)] = 4$ for $n \geq 3$ for n is odd. Hence the proof.

Eg:

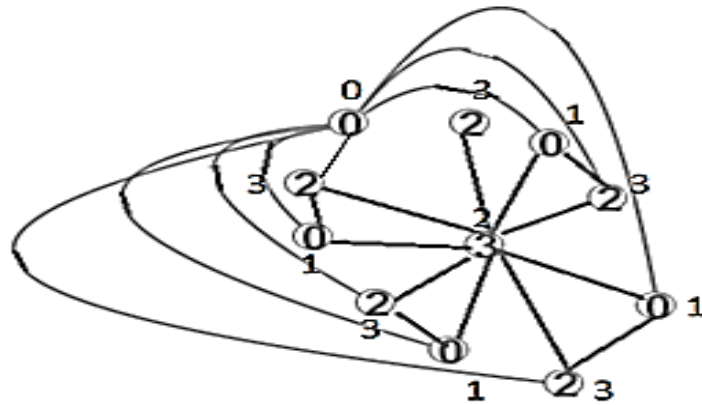


Figure 14. Switching with modular coloring in $FS_s(5)$

Subcase (ii). $mc[FS_s(n)]=4; n \geq 3$ for n is even.

Switching is taken for $v_1 \in \ell_1$. Therefore v_1 is adjacent to $v_3, v_4, \dots, v_{2n}$ after switching in $FS_s(n)$.

Consider a modular coloring $c(v): v[FS_s(n)] \rightarrow \mathbb{Z}_4$ defined by

$$C(v) = \begin{cases} 3 & \text{for } u \in \ell_0 \\ 2 & \text{for } v_{2j} \in \ell_1; j = 1, 2, \dots, n \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{then } \sigma(v) = \begin{cases} 0 & \text{for } u \in \ell_0 \\ 3 & v_{2j} \in \ell_1; j = 1, 2, \dots, n \\ 2 & \text{for } v_1 \in \ell_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{Here } \sigma(x) \neq \sigma(y) \forall x, y \text{ of adjacent vertices}$$

in $FS_s(n) \therefore mc[FS_s(n)]=4$ for $n \geq 3$ for n is even. Hence the proof.

Eg:

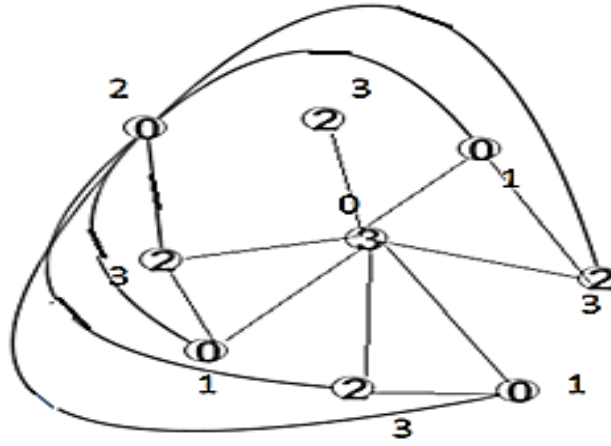


Figure 15. Switching with modular coloring in $FS_s(4)$

4. Modular colorings after switching on Gear graph.

Let $u \in \ell_0$ be the center of a gear graph. Let $v_1, v_3, v_5, \dots, v_{2n-1}$ be the vertices in ℓ_1 are adjacent to $u \in \ell_0$ and $v_2, v_4, v_6, \dots, v_{2n}$ be the vertices in ℓ_1 . The switching in $G(n)$ is denoted by $G_s(n)$. Switching in gear graph is obtained in two ways.

That is (i) switching of the vertex $u \in \ell_0$ and (ii) switching of a vertex $v_i \in \ell_1; i = 1, 2, \dots, 2n$.

(i) By switching of the vertex $u \in \ell_0$ in a gear graph $G(n)$ result in another gear graph $G'(n)$ in which vertices in ℓ_1 which are not adjacent with $u \in \ell_0$ in $G(n)$ become adjacent with $G'(n)$. Therefore $G_s(n) = G(n) = G'(n)$. Hence $mc[G_s(n)] = mc[G(n)] = mc[G'(n)]$.

(ii) switching of a vertex $v_i \in \ell_1; i = 1, 2, \dots, 2n$. Here specifying the vertex $v_i \in \ell_1$ which are adjacent with $u \in \ell_0$ is taken for switching. In general take v_i as v_1 .

Theorem 4.1

The modular coloring of the graph obtained after the switching of a vertex in Gear graph in ℓ_1 (which are adjacent with $u \in \ell_0$).ie (i) $mc[G_s(2)]=2$.(ii) $mc[G_s(n)]=3; n>2$.

Proof.

Case (i) $mc[G_s(2)]=2$.

Switching is taken for $v_1 \in \ell_1$ in $G(2)$. Therefore v_1 is adjacent to v_3 after switching in $G_s(n)$.

Consider a modular coloring $c(v):v[G_s(2)] \rightarrow \mathbb{Z}_2$ defined by $C(v) = \begin{cases} 1 & \text{for } v_3 \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$

Then $\sigma(v) = \begin{cases} 0 & v_3 \in \ell_1 \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $G_s(2)$
 $\therefore mc[G_s(2)]=2$. Hence the proof.

Case (ii) $mc[G_s(n)]=3$ for $n>2$.

Switching is taken for $v_1 \in \ell_1$ in $G(n)$. Therefore v_1 is adjacent to $v_3, v_4, \dots, v_{2n-1}$ after switching in $G_s(n)$ and not adjacent to the remaining vertices in $G_s(n)$.

Consider a modular coloring $c(v):v[G_s(n)] \rightarrow \mathbb{Z}_3$ defined by

$$C(v) = \begin{cases} 1 & \text{for } u \in \ell_0; v_1 \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v) = \begin{cases} 2 & \text{for } v_{3+2j} \in \ell_1; j = 0, 1, 2, \dots, (n-2). \\ 1 & \text{for } v_{4+2j} \in \ell_1; j = 0, 1, 2, \dots, (n-3) \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y) \forall x, y$ of adjacent vertices in $G_s(n)$ $\therefore mc[G_s(n)]=3$ for $n>2$. Hence the proof.

Eg:

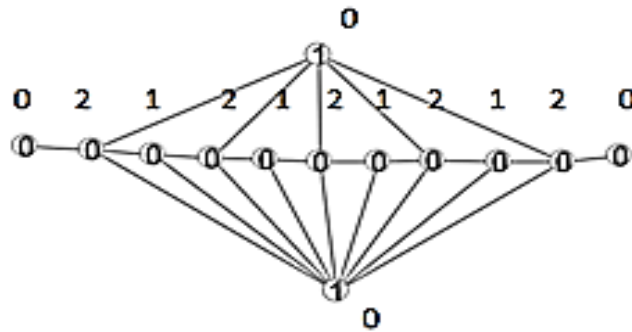


Figure 16. Switching with modular coloring in $G_s(6)$.

5. Conclusions

In a Wheel Graph the modular coloring of a graph obtained after the switching of a vertex $v_i \in \ell_1$ is $W_s(n)=3$ for $n=4k, 4k+1, [4k+2; k>1]$; $W_s(6)=4$; $W_s(n)=4$ for $n=4k+3, k \geq 1$. The labeling is quite similar to one other and differs according to the change in number of vertices. Also in a Friendship Graph the modular coloring after switching a vertex in ℓ_1 then (i) $mc[FS_s(2)]=3$. (ii) $mc[FS_s(n)]=4; n \geq 3$. Similarly in Gear Graph the modular coloring obtained after the switching of a vertex in ℓ_1 (which are adjacent with $u \in \ell_0$).ie (i) $mc[G_s(2)]=2$. (ii) $mc[G_s(n)]=3; n > 2$. Altogether It is explicitly clear that after switching in different levels of the graphs, the modular chromatic number varies in between two to four. We cannot expect a higher level of modular chromatic number after switching in vertices at different levels. Studying this problem and related problems in the context of switching graphs may help in answering the long open question whether all of these problems have a polynomial algorithm. We conclude this paper by listing a number of switching graph problems of which we do not know the complexity

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Micro S_p -Open Sets in Micro Topological Spaces

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Abstract

In this paper, a new class of open sets called Micro S_p - Open sets in Micro topological spaces are introduced and its fundamental properties are analyzed. Also, some operations on Micro S_p -open sets are investigated.

Keywords: Micro-open, Micro-Semi open, Micro-Pre-open, Micro-Pre closed and Micro S_p -open, Micro S_p -closed.

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1. Introduction

The concept of Nano topology was introduced by Lellis Thivagar [3] which is in terms of the lower and upper approximations and the boundary region of a subset of an universe. The notion of approximations and boundary region of a set was originally proposed by Pawlak [4] in order to introduce the concept of rough set theory. Chandrasekar [5] introduced the concept of micro topology which is a simple extension of Nano topology and he also studied the concepts of Micro pre-open and Micro semi-open sets. In 2010, Shareef introduced the class of semi open sets called S_p -open sets. Chandrasekar and Swathi [1] introduced Micro α -open in micro topological space. In this paper a new class of sets in Micro topological spaces called Micro S_p -Open set is introduced and some of its properties are derived.

2. Preliminaries

Definition 2.1. [4] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space, Let $X \subseteq U$, Then

(i) The Lower approximation of X with respect to R is the set of all objects which can be certain classified as X with respect to R and is denoted by $L_R(X)$. That is $L_R(X) = \bigcup_{X \in U} \{R(X) : R(X) \subseteq X\}$ where $R(X)$ denotes the equivalence class determined by $X \in U$.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is $U_R(X) = \bigcup_{X \in U} \{R(X) : R(X) \cap X \neq \emptyset\}$.

(iii) The Boundary region of X with respect to R is the set of all objects, which can be classified as neither as X nor as not- X with respect to R and is denoted by $B_R(X)$. That is $B_R(X) = U_R(X) - L_R(X)$.

Definition2.2. [3] Let R be an equivalence relation on the universe U and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ and $x \in U$. Then $\tau_R(X)$ satisfies the following axioms.

(i) U and $\emptyset \in \tau_R(X)$.

(ii) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$. That is, $\tau_R(X)$ is a topology on U called the Nano topology on U with respect to X . Thus $(U, \tau_R(X))$ as called as Nano topological space. The elements of $\tau_R(X)$ are called as Nano open sets. A subset F of U is nano closed if its complement is nano open.

Definition 2.3. [5] Let $(U, \tau_R(X))$ be a Nano topological space. Then $\mu_R(X) = \{N \cup (N' \cap \mu) : N, N' \in \tau_R(X)\}$ and $\mu \notin \tau_R(X)$ is called the Micro topology in U with respect to X . The triplet $(U, \tau_R(X), \mu_R(X))$ is called Micro topological space and then elements of

$\mu_R(X)$ are called Micro open sets and the complement of a Micro open set is called a Micro closed set.

Definition 2.4. [1] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and $A \subseteq U$. Then A is called

- (i) Micro α -open if $A \subseteq \text{Mic-Int}(\text{Mic-Cl}(\text{Mic-Int}(A)))$
- (ii) Micro pre-open if $A \subseteq \text{Mic-Int}(\text{Mic-Cl}(A))$.
- (iii) Micro semi-open if $A \subseteq \text{Mic-Cl}(\text{Mic-Int}(A))$.

Definition 2.5. [5] Let $(U, \tau_R(X), \mu_R(X))$ be a Micro topological space. Let A and B be any two subsets of U . Then

- (i) A is Micro open set if and only if $\text{Mic-Int}(A) = A$.
- (ii) A is micro closed set if and only if $\text{Mic-Cl}(A) = A$.
- (iii) $\text{Mic-Int}(U \setminus A) = U \setminus \text{Mic-Cl}(A)$.
- (iv) $\text{Mic-Cl}(U \setminus A) = U \setminus \text{Mic-Int}(A)$.

3. Micro S_p -Open Sets

Definition 3.1. Let $(U, \tau_R(X), \mu_R(X))$ be a Micro topological space and $A \subseteq U$. Then A is said to be Micro S_p -open (briefly Mic S_p -open) if for each $x \in A \in \text{Mic-SO}(U, X)$, there exists a Micro pre-closed set F such that $x \in F \subseteq A$. The set of all Micro S_p -open sets is denoted by $\text{Mic } S_p\text{-O}(U, X)$.

Definition 3.2. Let $(U, \tau_R(X), \mu_R(X))$ be a Micro topological space. A subset B of U is called Micro S_p -closed (briefly Mic S_p -closed) if and only if its complement is Micro S_p -open and $\text{Mic } S_p\text{-CL}(U, X)$ denotes the collection of all Micro S_p -closed sets.

Example 3.3. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$, $X = \{b, d\} \subseteq U$ then $\tau_R(X) = \{U, \phi, \{b, d\}\}$. If $\mu = \{a\}$. Then $\mu_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$ and Micro S_p -open sets are $\{U, \phi, \{a, c\}, \{b, d\}, \{b, c, d\}\}$

Remark 3.4. Every Micro S_p -open set is a Micro semi open set but the converse need not always be true as shown from the following example.

Example 3.5. In example 3.3, $\text{Mic-SO}(U, X) = \{U, \phi, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. $\text{Mic } S_p\text{-O}(U, X) = \{U, \phi, \{a, c\}, \{b, d\}, \{b, c, d\}\}$ the set $\{a, b, d\}$ is Micro semi-open but not Micro S_p -open.

Theorem 3.6. An arbitrary union of any family of Micro S_p -open sets is Micro S_p -open.

Proof: Let $\{A_i: i \in \Delta\}$ be a family of Mic S_p -open sets. If $x \in A$, then for each $x \in \bigcup_{i \in \Delta} A_i \subseteq \text{Mic-SO}(U, X)$ there exists a Micro pre-closed set F such that $x \in F \subseteq A_i \subseteq \bigcup_{i \in \Delta} A_i$ which

implies $x \in F \subseteq \bigcup_{i \in \Delta} A_i$. Therefore $\bigcup_{i \in \Delta} A_i$ is Micro S_p -open.

Remark 3.7. From the above Theorem 3.6, arbitrary intersection of Micro S_p -closed sets of a Micro topological space is Micro S_p -closed as shown by the following example.

Example 3.8. In example 3.3, $\text{Mic } S_p\text{-O}(U, X) = \{U, \phi, \{a, c\}, \{b, d\}, \{b, c, d\}\}$ and $\text{Mic } S_p\text{-CL}(U, X) = \{U, \phi, \{b, d\}, \{a, c\}, \{a\}\}$. Here $\{b, d\} \cap \{b, c, d\} = \{b, d\}$ which is a Micro S_p -closed set.

Remark 3.9. The intersection of any two Micro S_p -open sets need not be a Micro S_p -open set.

From example 3.3, $\{a, c\}, \{b, c, d\}$ are Micro S_p -open sets but $\{a, c\} \cap \{b, c, d\} = \{c\}$ is not a Micro S_p -open set.

Proposition 3.10. If a subset A of a Micro topological space $(U, \tau_R(X), \mu_R(X))$ is Micro S_p -open. Then A is a Micro semi-open set and A is a union of Micro pre-closed sets.

Proof: Let $A \in \text{Mic } S_p\text{-O}(U, X)$ and $A \subseteq U$. Then for each $x \in A \in \text{Mic-SO}(U, X)$, there exists a Micro pre-closed set F containing x such that $x \in F \subseteq A$. Thus A is a Micro semi-open set and also $F \subseteq A$ which implies A is the union of Micro pre-closed sets.

Remarks 3.11. The converse of the above Proposition 3.10 need not be true as shown in the following example.

Example 3.12. In example 3.3, $\text{Mic-SO}(U, X) = \{U, \phi, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$ and $\text{Mic-PCL}(U, X) = \{U, \phi, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. Then $\{a, b, d\} \in \text{Mic-SO}(U, X)$ and $\{a, b, d\}$ is not in the union of Micro Pre-closed sets, also $\{a, b, d\} \notin \text{Mic } S_p\text{-O}(U, X)$.

4. Operations on Micro S_p -open Sets

Definition 4.1. A point $x \in U$ is said to be a Micro S_p -interior point of A if there exists a Micro S_p -open set V containing x such that $V \subseteq A$.

The set of all Micro S_p -interior points of A is said to be Micro S_p -interior of A and is denoted by $\text{Mic } S_p\text{-Int}(A)$.

Definition 4.2. Let A be any subset of a Micro topological space $(U, \tau_R(X), \mu_R(X))$. Then a point $x \in U$ is in the Micro S_p -closure of A if and only if $A \cap H \neq \phi$ for every $H \in \text{Mic } S_p\text{-O}(U, X)$ containing x .

The intersection of all Micro S_p -closed sets containing H is called the Micro S_p -closure of F and is denoted by $\text{Mic } S_p\text{-Cl}(A)$.

Theorem 4.3. Let A be any subset of a Micro topological space $(U, \tau_R(X), \mu_R(X))$. If a point $x \in \text{Mic } S_p\text{-Int } (A)$, then there exists $F \in \text{Mic-PCl } (U, X)$ containing x such that $F \subseteq A$.

Proof: Suppose that $x \in \text{Mic } S_p\text{-Int } (A)$. Then $V \in \text{Mic } S_p\text{-O } (U, X)$ containing x such $V \subseteq A$. since $V \in \text{Mic } S_p\text{-O } (U, X)$, then there exists $F \in \text{Mic-PCl } (U, X)$ containing x such that $F \subseteq V \subseteq A$. Hence $x \in F \subseteq A$.

Theorem 4.4. Let A be a subset of a Micro topological space $(U, \tau_R(X), \mu_R(X))$. If $A \cap F \neq \emptyset$ for every $F \in \text{Mic-PCl } (U, X)$ containing x , then $x \in \text{Mic } S_p\text{-Cl } (A)$.

Proof: Let $V \in \text{Mic } S_p\text{-O } (U, X)$ containing x , then there exists a Micro pre-closed set F containing x such that $F \subseteq V$. Since $A \cap F \neq \emptyset$, $x \in \text{Mic } S_p\text{-Cl } (A)$.

Theorem 4.5. For any two subsets A and B of a Micro topological space $(U, \tau_R(X), \mu_R(X))$, the following properties are true.

- (i) $\text{Mic } S_p\text{-Int } (\text{Mic } S_p\text{-Int } (A)) = \text{Mic } S_p\text{-Int } (A)$.
- (ii) $\text{Mic } S_p\text{-Int } (A) = U - \text{Mic } S_p\text{-Cl } (U - A)$.
- (iii) If $A \subseteq B$, then $\text{Mic } S_p\text{-Int}(A) \subseteq \text{Mic } S_p\text{-Int } (B)$
- (iv) $\text{Mic } S_p\text{-Int } (A) \cup \text{Mic } S_p\text{-Int } (B) \subseteq \text{Mic } S_p\text{-Int } (A \cup B)$.
- (v) $\text{Mic } S_p\text{-Int } (A \cap B) \subseteq \text{Mic } S_p\text{-Int } (A) \cap \text{Mic } S_p\text{-Int } (B)$.

Proof: Obvious.

The converse of (iii), (iv) and (v) of Theorem 4.5 need not be true as shown in the following example.

Example 4.6. Consider $U = \{p, q, r, s, t\}$ with $U/R = \{\{p, q, r\}, \{s\}, \{t\}\}$, $X = \{p, q\} \subseteq U$, then $\tau_R(X) = \{U, \emptyset, \{p, q, r\}\}$. If $\mu = \{t\}$, then $\mu_R(X) = \{U, \emptyset, \{t\}, \{p, q, r\}, \{p, q, r, s\}\}$ and $\text{Mic } S_p\text{-O}(U, X) = \{U, \emptyset, \{s, t\}, \{p, q, r\}, \{p, q, r, s\}\}$.

(iii) If $A = \{s\}$ and $B = \{q, r, t\}$, then $\text{Mic } S_p\text{-Int } (\{s\}) = \emptyset = \text{Mic } S_p\text{-Int } (\{q, r, t\})$ but $A \not\subseteq B$.

(iv) Let $A = \{p, q\}$ and $B = \{r, s\}$, then $\text{Mic } S_p\text{-Int } (\{p, q\}) \cup \text{Mic } S_p\text{-Int } (\{r, s\}) = \emptyset \cup \emptyset = \emptyset$. But $\text{Mic } S_p\text{-Int } (\{p, q\} \cup \{r, s\}) = \text{Mic } S_p\text{-Int } (\{p, q, r, s\}) = \{p, q, r, s\}$ which implies that $\text{Mic } S_p\text{-Int } (A \cup B) \not\subseteq \text{Mic } S_p\text{-Int } (A) \cup \text{Mic } S_p\text{-Int } (B)$.

(v) Consider $A = \{p, q, r, s\}$ and $B = \{p, s, t\}$, then $\text{Mic } S_p\text{-Int } (\{p, q, r, s\}) \cap \text{Mic } S_p\text{-Int } (\{p, s, t\}) = \{p, q, r, s\} \cap \{s, t\} = \{s\}$. But $\text{Mic } S_p\text{-Int } (\{p, s, t\} \cap \{p, q, r, s\}) = \text{Mic } S_p\text{-Int } (\{p, s\}) = \emptyset$. Therefore, $\text{Mic } S_p\text{-Int } (A) \cap \text{Mic } S_p\text{-Int } (B) \not\subseteq \text{Mic } S_p\text{-Int } (A \cap B)$.

Theorem 4.7. For any two subsets A and B of a Micro topological space $(U, \tau_R(X), \mu_R(X))$, the following properties are true.

- (i) $\text{Mic } S_p\text{-Cl } (\text{Mic } S_p\text{-Cl } (A)) = \text{Mic } S_p\text{-Cl } (A)$.
- (ii) $\text{Mic } S_p\text{-Cl } (A) = U - \text{Mic } S_p\text{-Int } (U - A)$.
- (iii) If $A \subseteq B$, then $\text{Mic } S_p\text{-Cl } (A) \subseteq \text{Mic } S_p\text{-Cl } (B)$

(iv) $\text{Mic } S_p\text{-Cl}(A) \cup \text{Mic } S_p\text{-Cl}(B) \subseteq \text{Mic } S_p\text{-Cl}(A \cup B)$.

(v) $\text{Mic } S_p\text{-Cl}(A \cap B) \subseteq \text{Mic } S_p\text{-Cl}(A) \cap \text{Mic } S_p\text{-Cl}(B)$.

Proof: Obvious, The converse of (iii), (iv) and (v) of Theorem 4.7 need not be true as shown in the following example.

Example 4.8. Consider $U = \{p, q, r, s, t\}$ with $U/R = \{\{p, q, r\}, \{s\}, \{t\}\}$, $X = \{p, q\} \subseteq U$, then $\tau_R(X) = \{U, \phi, \{p, q, r\}\}$. If $\mu = \{t\}$, then $\tau_R(X) = \{U, \phi, \{t\}, \{p, q, r\}, \{p, q, r, t\}\}$ and $\text{Mic } S_p\text{-O}(U, X) = \{U, \phi, \{s, t\}, \{p, q, r\}, \{p, q, r, s\}\}$, $\text{Mic } S_p\text{-Cl}(U, X) = \{U, \phi, \{p, q, r\}, \{s, t\}, \{t\}\}$.

(iii) Let $A = \{p, t\}$ and $B = \{q, s, t\}$, then $\text{Mic } S_p\text{-Cl}(\{p, t\}) = U$ and $\text{Mic } S_p\text{-Cl}(\{q, s, t\}) = U$ but $A \not\subseteq B$.

(iv) Let $A = \{p, q, r\}$ and $B = \{t\}$, then $\text{Mic } S_p\text{-Cl}(\{p, q, r\}) \cup \text{Mic } S_p\text{-Cl}(\{t\}) = \{p, q, r\} \cup \{t\} = \{p, q, r, t\}$. But $\text{Mic } S_p\text{-Cl}(\{p, q, r\} \cup \{t\}) = \text{Mic } S_p\text{-Cl}(\{p, q, r, t\}) = U$. Therefore $\text{Mic } S_p\text{-Cl}(A \cup B) \not\subseteq \text{Mic } S_p\text{-Cl}(A) \cup \text{Mic } S_p\text{-Cl}(B)$.

(v) In general, for any closure operator $Cl(F) \cup Cl(E) = Cl(F \cup E)$ and for most of the closure operators $Cl(F) \cap Cl(E) \neq Cl(F \cap E)$. In the case of $\text{Mic } S_p$ -closure operator, the equality sign need not hold for both the cases and it is justified by the following example. This obviously leads to the conclusion that $\text{Mic } S_p$ -closure operator is not a Kuratowski's operator. For, let $A = \{r\}$ and $B = \{p, t\}$ then $\text{Mic } S_p\text{-Cl}(\{r\}) \cap \text{Mic } S_p\text{-Cl}(\{p, t\}) = \{p, q, r\} \cap U = \{p, q, r\}$. But $\text{Mic } S_p\text{-Cl}(\{r\} \cap \{p, t\}) = \text{Mic } S_p\text{-Cl}(\phi) = \phi$. Hence $\text{Mic } S_p\text{-Cl}(A) \cap \text{Mic } S_p\text{-Cl}(B) \not\subseteq \text{Mic } S_p\text{-Cl}(A \cap B)$.

5. Conclusion

In this paper $\text{Mic } S_p$ -open sets and $\text{Mic } S_p$ -closed sets are defined and some of their properties are discussed. This shall be extended in the future Research with some applications.

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Ng^* s-Continuous functions in Nano Topological Spaces

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Abstract

The aim of this paper is to introduces Ng^* s-continuous function in nano topological spaces and we also study the relation between Ng^* s-irresolute functions and Ng^* s-continuous functions in different closed sets.

Keywords: g^* s-closed set, g^* s-continuous functions, Ng^* s-irresolute.

2010 AMS subject classification: 54C05[‡]

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1. Introduction

Topology which is a branch of mathematics, is formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation, especially those properties that are invariant under a certain kind of equivalence. The term topology was introduced by a German Mathematician called Johnn Benedict Listing in 1847. The modern topology largely based on the idea of set theory was developed by George Cantor in the later part of 19th century. Since its inception the topic has been growing in different level and in various fields.

Nano topology is one of the latest feathers in topology that applies to real life situations. Lellis Thivagar was the main brain behind developing the concept of nano topology. It is constructed in terms of lower and upper approximations and boundary region of a subset of a universe. The term “Nano” can be ascribed to any unit of measure.

The concept of continuity plays a very major role in general topology and they are now the research topics of many topologists worldwide. Indeed, a significant theme in general topology concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. N. Levine [7] introduced the concept of generalized closed sets in 1970. The concept of g^* s –closed sets was introduced by M. Anto [12]. In 2013, M. Lellis Thivagar [6] has introduced nano topological space with respect to a subset X of a universe U , which is defined in terms of lower and upper approximation of X . He has also defined nano-closed sets, nano-interior and nano-closure of a set. He has also introduced, among other, some certain weak form of nano open sets such as nano α -open sets, nano semi open sets and nano pre-open sets. The aim of this paper is to introduce a new class of sets on nano topological spaces called Ng^* s –closed sets. Further, we investigate and discuss the relation of this new sets with existing ones.

2. Preliminaries

Definition 2.1 Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the in-discernibility relation. Then U is divided into disjoint equivalence class, Elements belonging to the same equivalence class are said to be in discernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

- The lower approximation of X with respect to R is the set of all object which can be for certain classifies as X with respect to R and it is denoted by $L_R(X)$. That is $L_R(X) = \bigcup_{x \in U} \{R(x): R(x) \subseteq X\}$, where $R(X)$ denotes the equivalence classes determined by $X \in U$.
- The upper approximation of X with respect to R is the of all objects, which can be for certain classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x): R(x) \cap X \neq \emptyset\}$.

- The boundary of the region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 If (U, R) is an approximation space and $X, Y \subseteq U$, then

1. $L_R(X) \subseteq X \subseteq U_R(X)$
2. $L_R(\emptyset) = U_R(\emptyset) = \emptyset$ and $L_R(U) = U_R(U) = U$
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$
5. $U_R(XY) \supseteq U_R(U_R(Y))$
6. $U_R(X \cap Y) = U_R(X) \cap U_R(Y)$
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$
8. $U_R((X^c)) = [L_R(X)]^c$ and $L_R((X^c)) = [U_R(X)]^c$
9. $U_R(U_R(X)) = L_R(L_R(X)) = L_R(X)$
10. $L_R(L_R(X)) = U_R(U_R(X)) = U_R(X)$

Definition 2.3 Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. $\tau_R(X)$ satisfies the following axioms:

1. U and $\emptyset \in \tau_R(X)$
2. The union of elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$. That is, $\tau_R(X)$ forms a topology on U is called the nano topology on U with respect to X . We call $(U, \tau_R(X))$ is called the nano topological space.

Definition 2.4 If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then the nano interior of A is defined as the union of all nano open subsets of A and it is denoted by $Nint(A)$. That is, $Nint(A)$ is the largest nano open subset of A . The nano closure of A is defined as the intersection of all nano closed sets containing A and its denoted by $Ncl(A)$. $Ncl(A)$ is the smallest nano closed set containing A .

REMARK 2.5 If $\tau_R(X)$ is the nano topology on U with respect to X , then the set $B = \{U, L_R(X), U_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.6 A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is called

- Nano-continuous [2] if $f^{-1}(A)$ is nano-closed in (X, τ) for every nano-closed set A in (Y, σ) .
- Nano α -continuous [7] if $f^{-1}(A)$ is nano α -closed in (X, τ) for every nano-closed set A in (Y, σ) .
- Nano semi-continuous [4] if $f^{-1}(A)$ is nano semi closed in (X, τ) for every nano-closed set A in (Y, σ) .
- Nano regular-continuous if $f^{-1}(A)$ is nano regular closed in (X, τ) for every nano-closed set A in (Y, σ) .

Definition 2.7 If $(U, \tau_R(X))$ is a nano topological space if $A \subseteq U$, then A is said to be

- $Ng^\#\alpha$ -closed if $N\alpha cl(A) \subseteq V$ whenever $A \subseteq V$ and V is Ng -open in $(U, \tau_R(X))$
- Ng^* -closed if $Ncl(A) \subseteq V$ whenever $A \subseteq V$ and V is Ng -open in $(U, \tau_R(X))$ • Ng^*s -closed if $Nscl(A) \subseteq V$ whenever $A \subseteq V$ and V is Ng -open in $(U, \tau_R(X))$.
- Nsg -closed if $Ncl(A) \subseteq U$ whenever $A \subseteq U$ and U is Nsg -open in $(U, \tau_R(X))$.
- $Ng\alpha$ -closed if $N\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is Ng -open in $(U, \tau_R(X))$ • $Ng\alpha^*s$ -closed if $Nscl(A) \subseteq U$ whenever $A \subseteq U$ and U is $Ng\alpha$ -open in $(U, \tau_R(X))$.
- $Ng\alpha g$ - closed if $Ncl(A) \subseteq U$ whenever $A \subseteq U$ and U is $Ng\alpha$ -open in $(U, \tau_R(X))$

Definition 2.8 If $(U, \tau_R(X))$ is a nano topological space if $A \subseteq U$, then A is said to be

- $Ng^\#\alpha$ -continuous [9] if $f^{-1}(A)$ is $Ng^\#\alpha$ -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- Ng^* -continuous [12] if $f^{-1}(A)$ is Ng^* -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- Ng^*s -continuous [12] if $f^{-1}(A)$ is Ng^*s -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- Nsg -continuous [6] if $f^{-1}(A)$ is Nsg -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- Nsg -continuous [11] if $f^{-1}(A)$ is Nsg -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- $Ng\alpha$ -continuous [10] if $f^{-1}(A)$ is $Ng\alpha$ -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- $Ng\alpha^*s$ -continuous [8] if $f^{-1}(A)$ is $Ng\alpha^*s$ -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$
- $Ng\alpha g$ -continuous [5] if $f^{-1}(A)$ is $Ng\alpha g$ -closed in $(U, \tau_R(X))$ for every nano-closed set A in $(V, \sigma_R(Y))$

Definition 2.9[1] A subset A of a nano topological space $(U, \tau_R(X))$ is called Ng^*s - closed set if $Nscl(A) \subseteq U$ whenever $A \subseteq U$ and U is Ng -open in $(U, \tau_R(X))$.

3. Ng^*s -Continuous Functions

In this section we define Ng^*s -Continuous functions and discuss some of their properties.

Definition 3.1 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. Then a mapping $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is called Ng^*s -continuous if $f^{-1}(S)$ is Ng^*s -closed in $(U, \tau_R(X))$ for every nano-closed set S in $(V, \sigma_R(Y))$.

Definition 3.2 A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is Ng^*s -irresolute if $f^{-1}(S)$ is Ng^*s closed in $(U, \tau_R(X))$ for each Ng^*s -closed set S in $(V, \sigma_R(Y))$.

Example 3.3. Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. Let $U = \{a, b, c, d\}$ with $U/R = \{\{b, c\}, \{a\}, \{d\}\}$ and $X = \{a\}$. Then the nano topology $\tau_R(X)$

$=\{\emptyset, U, \{a\}\}$. Then $N\hat{g}^*$ -C($U, \tau_R(X)$) = $\{\emptyset, U, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $V = \{a, b, c, d\}$ with $V/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $Y = \{a, b, c\}$. $\sigma_R(Y) = \{\emptyset, V, \{a, c\}, \{b, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, c\}, \{b, d\}\}$. $N\hat{g}^*$ -C($V, \sigma_R(Y)$) = $\{\emptyset, U, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define $f: U \rightarrow V$ as $f(a) = b, f(b) = a, f(c) = d, f(d) = c$. We have $f^{-1}(a, c) = \{b, d\}, f^{-1}(b, d) = \{a, c\}$. Thus, the inverse image of every N -closed set in V is $N\hat{g}^*$ -closed U . Therefore, f is $N\hat{g}^*$ -continuous.

4. Main Results

Proposition 4.1 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is Nano-continuous then f is $N\hat{g}^*$ -continuous.

Proof: Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is Nano continuous, $f^{-1}(A)$ is nano closed in $(U, \tau_R(X))$. Since every nano closed set is $N\hat{g}^*$ -closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*$ -closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*$ -continuous.

Remark 4.2 The converse of the above theorem need not be true, as proved by the following example.

Example 4.3 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, b\}$. $\tau_R(X) = \{\emptyset, U, \{b\}, \{a, d\}, \{a, b, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, c, d\}, \{c\}, \{b, c\}\}$. $N\hat{g}^*$ -C($U, \tau_R(X)$) = $\{\emptyset, U, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. $NSC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{a, c, d\}\}$.

Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{b\}, \{d\}, \{a, c\}\}$ and $Y = \{a, d\}$. Let $\sigma_R(Y) = \{\emptyset, V, \{d\}, \{a, c\}, \{a, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, b, c\}, \{b\}, \{b, d\}\}$. Define $f(a) = d, f(b) = a, f(c) = b, f(d) = c$. Let $f^{-1}(b) = \{c\}, f^{-1}(b, d) = \{a, c\}, f^{-1}(a, b, c) = \{b, c, d\}$. Here $\{b, c, d\}$ is $N\hat{g}^*$ -closed but not N -closed.

Proposition 4.4 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $N\alpha$ -continuous then f is $N\hat{g}^*$ -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is α -continuous, $f^{-1}(A)$ is nano α -closed in $(U, \tau_R(X))$. Since every nano α -closed set is $N\hat{g}^*$ -closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*$ -closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*$ -continuous.

Remark 4.5 The converse of the above theorem need not be true, as proved by the following example.

From the example 3.6, the sub set $A = \{b, c, d\}$ is not nano α -closed set in $(U, \tau_R(X))$. Hence f is not nano α -continuous.

Proposition 4.6 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is N semi-continuous then f is $N\hat{g}^*$ -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is semi-continuous, $f^{-1}(A)$ is nano semi-closed in $(U, \tau_R(X))$. Since every nano semi-closed set is $N\hat{g}^*s$ -closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*s$ -closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*s$ -continuous.

Remark 4.7 The converse of the above theorem need not be true, as proved by the following example.

From the example 3.6, the sub set $A = \{b, c, d\}$ is not nano semi-closed set in $(U, \tau_R(X))$. Hence f is not nano semi-continuous.

Proposition 4.8 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $Ng^\#\alpha$ -continuous then f is $N\hat{g}^*s$ -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is $Ng^\#\alpha$ -continuous, $f^{-1}(A)$ is $Ng^\#\alpha$ -closed in $(U, \tau_R(X))$. Since every $Ng^\#\alpha$ -closed set is $N\hat{g}^*s$ -closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*s$ -closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*s$ -continuous.

Remark 4.9 The converse of the above theorem need not be true, as proved by the following example.

Example 4.10 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a\}$. $\tau_R(X) = \{\emptyset, U, \{a\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}\}$. $Ng^\#\alpha C(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $Y = \{a, b\}$. $\sigma_R(Y) = \{\emptyset, V, \{a\}, \{b, d\}, \{a, b, d\}, (\sigma_R(Y)) = \{\emptyset, V, \{c\}, \{a, c\}, \{b, c, d\}\}$. Define $f(a) = b, f(b) = a, f(c) = c, f(d) = d$. Let $f^{-1}(c) = \{c\}$. $f^{-1}(a, c) = \{b, c\}$, $f^{-1}(b, c, d) = \{a, c, d\}$. Here $\{a, c, d\}$ is $N\hat{g}^*s$ closed but not $Ng^\#\alpha$ -closed.

Proposition 4.11 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $N\hat{g}\alpha$ -continuous then f is $N\hat{g}^*s$ -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is $N\hat{g}\alpha$ -continuous, $f^{-1}(A)$ is $N\hat{g}\alpha$ -closed in $(U, \tau_R(X))$. Since every $N\hat{g}\alpha$ -closed set is $N\hat{g}^*s$ -closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*s$ -closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*s$ -continuous.

Remark 4.12 The converse of the above theorem need not be true, as proved by the following example.

Example 4.13 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{b, d\}$. $\tau_R(X) = \{\emptyset, U, \{b\}, \{c, d\}, \{b, c, d\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}\}$. $N\hat{g}\alpha C(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $Y = \{a, b, d\}$. $\sigma_R(Y) = \{\emptyset, V, \{a, b, d\}\}$, $(\sigma_R(Y)) = \{\emptyset, V, \{c\}\}$.

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Define $f(a) = b, f(b) = c, f(c) = d, f(d) = a$. Let $f^{-1}(c) = \{b\}$. Here $\{b\}$ is $N\hat{g}^*$ -s-closed but not $N\hat{g}\alpha$ -closed.

Proposition 4.14 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is Ng^* -continuous then f is $N\hat{g}^*$ -s-continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is Ng^* -continuous, $f^{-1}(A)$ is Ng^* -closed in $(U, \tau_R(X))$. Since every Ng^* -closed set is $N\hat{g}^*$ -s-closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*$ -s-closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*$ -s-continuous.

Remark 4.15 The converse of the above theorem need not be true, as proved by the following example.

Example 4.16 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{b, c\}$. $\tau_R(X) = \{\emptyset, U, \{c\}, \{b, d\}, \{b, c, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, b, d\}, \{a\}, \{a, c\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}\}$. $Ng^*C(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $Y = \{a, c\}$. $\sigma_R(Y) = \{\emptyset, V, \{c\}, \{a, d\}, \{a, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, b, d\}, \{b\}, \{b, c\}\}$. Define $f(a) = c, f(b) = d, f(c) = b, f(d) = a$. Let $f^{-1}(a, b, d) = \{a, b, c\}, f^{-1}(b, c) = \{a, c\}, f^{-1}(b) = \{c\}$. Here $\{c\}$ is $N\hat{g}^*$ -s-closed but not Ng^* -closed.

Proposition 4.17 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is Ng^* -s-continuous then f is $N\hat{g}^*$ -s-continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is Ng^* -s-continuous, $f^{-1}(A)$ is Ng^* -s-closed in $(U, \tau_R(X))$. Since every Ng^* -s-closed set is $N\hat{g}^*$ -s-closed. Therefore $f^{-1}(A)$ is $N\hat{g}^*$ -s-closed in $(U, \tau_R(X))$. Hence f is $N\hat{g}^*$ -s-continuous.

Remark 4.18 The converse of the above theorem need not be true, as proved by the following example.

Example 4.19

Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $X = \{a, c\}$. $\tau_R(X) = \{\emptyset, U, \{a\}, \{c, d\}, \{a, c, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{b, c, d\}, \{b\}, \{a, b\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \{b, c, d\}\}$. $Ng^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $Y = \{a, b\}$. $\sigma_R(Y) = \{\emptyset, V, \{a\}, \{b, d\}, \{a, b, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{b, c, d\}, \{c\}, \{a, c\}\}$. Define $f(a) = d, f(b) = c, f(c) = b, f(d) = a$. Let $f^{-1}(b, c, d) = \{a, b, c\}, f^{-1}(a, c) = \{b, d\}, f^{-1}(c) = \{b\}$. Here $\{b\}$ is $N\hat{g}^*$ -s-closed but not Ng^* -s-closed.

Proposition 4.20 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $\text{Ns}\hat{g}$ -continuous then f is Ng^*s -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is $\text{Ns}\hat{g}$ -continuous, $f^{-1}(A)$ is $\text{Ns}\hat{g}$ -closed in $(U, \tau_R(X))$. Since every $\text{Ns}\hat{g}$ -closed set is Ng^*s -closed. Therefore $f^{-1}(A)$ is Ng^*s -closed in $(U, \tau_R(X))$. Hence f is Ng^*s -continuous.

Remark 4.21 The converse of the above theorem need not be true, as proved by the following example.

Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, b\}$. $(U) = \{\emptyset, U, \{b\}, \{a, d\}, \{a, b, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, c, d\}, \{c\}, \{b, c\}\}$. $\text{Ng}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c, d\}\}$.

$\text{Ns}\hat{g}C(U, \tau_R(X)) = \{\emptyset, U, \{c\}, \{b, c\}, \{a, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R = \{\{a\}, \{b\}, \{c, d\}\}$ and $Y = \{b, c\}$. $\sigma_R(Y) = \{\emptyset, V, \{b\}, \{c, d\}, \{b, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, c, d\}, \{a\}, \{a, b\}\}$. Define $f(a) = b, f(b) = a, f(c) = d, f(d) = c$. Let $f^{-1}(a, c, d) = \{b, c, d\}, f^{-1}(a, b) = \{a, b\}, f^{-1}(a) = \{b\}$. Here $\{b\}$ is $\text{Ns}\hat{g}$ -closed but not Ng^*s -closed.

Proposition 4.22 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $\text{Ng}\alpha g$ -continuous then f is Ng^*s -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is $\text{Ng}\alpha g$ -continuous, $f^{-1}(A)$ is $\text{Ng}\alpha g$ -closed in $(U, \tau_R(X))$. Since every $\text{Ng}\alpha g$ -closed set is Ng^*s -closed. Therefore $f^{-1}(A)$ is Ng^*s -closed in $(U, \tau_R(X))$. Hence f is Ng^*s -continuous.

Remark 4.23 The converse of the above theorem need not be true, as proved by the following example.

Example 4.24 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, b\}$. $\tau_R(X) = \{\emptyset, U, \{b\}, \{a, d\}, \{a, b, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, c, d\}, \{c\}, \{b, c\}\}$. $\text{Ng}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c, d\}\}$. $\text{Ns}\hat{g}C(U, \tau_R(X)) = \{\emptyset, U, \{c\}, \{b, c\}, \{a, c, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R' = \{\{a\}, \{b\}, \{c, d\}\}$ and $Y = \{b, c\}$. $\sigma_R(Y) = \{\emptyset, V, \{b\}, \{c, d\}, \{b, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, c, d\}, \{a\}, \{a, b\}\}$. Define $f(a) = b, f(b) = a, f(c) = d, f(d) = c$. Let $f^{-1}(a, c, d) = \{b, c, d\}, f^{-1}(a, b) = \{a, b\}, f^{-1}(a) = \{b\}$. Here $\{b\}$ is $\text{Ns}\hat{g}$ -closed but not Ng^*s -closed.

Proposition 4.25 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be two nano topological spaces. A function $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $\text{N}\alpha g^*s$ -continuous then f is Ng^*s -continuous.

Proof Let A be any nano closed set in $(V, \sigma_R(Y))$. Since f is $\text{N}\alpha g^*s$ -continuous, $f^{-1}(A)$ is $\text{N}\alpha g^*s$ -closed in $(U, \tau_R(X))$. Since every $\text{N}\alpha g^*s$ -closed set is Ng^*s -closed. Therefore $f^{-1}(A)$ is Ng^*s -closed in $(U, \tau_R(X))$. Hence f is Ng^*s -continuous.

Remark 4.26 The converse of the above theorem need not be true, as proved by the following example.

Example 4.27 Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{b, c\}$. $\tau_R(X) = \{\emptyset, U, \{b, c\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, d\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c, d\}\}$.

$N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{d\}, \{a, d\}\}$. Let $(V, \sigma_R(Y))$ be a nano topological space where $V = \{a, b, c, d\}$ with $V/R' = \{\{b\}, \{c\}, \{a, d\}\}$ and $Y = \{a, c\}$. $\sigma_R(Y) = \{\emptyset, V, \{c\}, \{a, d\}, \{a, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, b, d\}, \{b\}, \{b, c\}\}$. Define $f(a) = b, f(b) = c, f(c) = d, f(d) = a$. Let $f^{-1}(a, b, d) = \{a, c, d\}, f^{-1}(b, c) = \{a, b\}, f^{-1}(b) = \{a\}$. Here $\{a, b\}$ is $N\hat{g}^*$ -closed but not Ng^* -closed.

Proposition 4.28

Composition of two $N\hat{g}^*$ -continuous function need not be $N\hat{g}^*$ -continuous.

Let $(U, \tau_R(X))$ be a nano topological space where $U = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, b\}$. $\tau_R(X) = \{\emptyset, U, \{b\}, \{a, b, d\}, \{a, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, c, d\}, \{c\}, \{b, c\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$.

Define $f(a) = d, f(b) = a, f(c) = b, f(d) = c$. Let $f^{-1}(a, b, c) = \{b, c, d\}, f^{-1}(b, d) = \{a, c\}, f^{-1}(b) = \{c\}$. Here $\{a, b\}$ is $N\hat{g}^*$ -but not Ng^* -closed.

Let $(V, \sigma_R(Y))$ be a nano topological space Let $V = \{a, b, c, d\}$ with $V/R = \{\{b\}, \{d\}, \{a, c\}\}$ and $Y = \{a, d\}$. $\sigma_R(Y) = \{\emptyset, V, \{d\}, \{a, c\}, \{a, c, d\}\}$. $(\sigma_R(Y))^c = \{\emptyset, V, \{a, b, c\}, \{b\}, \{b, d\}\}$. $N\hat{g}^*sC(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.

Let $(W, (Z))$ be a nano topological space Let $W = \{a, b, c, d\}$ with $W/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $Z = \{b, c\}$. $(Z) = \{\emptyset, W, \{c\}, \{b, d\}, \{b, c, d\}\}$. $((Z))^c = \{\emptyset, V, \{a, b, d\}, \{a\}, \{a, c\}\}$. Define $g(a) = b, g(b) = a, g(c) = d, g(d) = c$. Let $g^{-1}(a, b, d) = \{a, b, c\}, g^{-1}(a, c) = \{b, d\}, g^{-1}(a) = \{b\}$.

Now, $\gamma : (U, \tau_R(X)) \rightarrow (W, \mu_R(Z))$ by $g(f(a) = c, g(f(b) = b, g(f(c) = a, g(f(d) = d, g^{-1}(f^{-1}(a)) = a, g^{-1}(f^{-1}(a, c)) = (a, c), g^{-1}(f^{-1}(a, b, d)) = (a, b, d)$ is not $N\hat{g}^*$ -closed in U but $\{a, b, d\}$ is closed in Z . Therefore, is not $N\hat{g}^*$ -continuous.

Proposition 4.29 Let $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ be $N\hat{g}^*$ be a function. Then following are equivalent.

- (i) f is $N\hat{g}^*$ -continuous.
- (ii) $f^{-1}(A)$ is $N\hat{g}^*$ -open for each open set A in Y .

Proof: (i) $\Rightarrow$ (ii)

Suppose that f is a $N\hat{g}^*$ -continuous. Let A be N -open in U . Then A^c is N -closed in V . Since f is $N\hat{g}^*$ -continuous, we have $f^{-1}(A^c)$ is $N\hat{g}^*$ -closed in U . But $f^{-1}(A^c) = [f^{-1}(A)]^c$. Hence $f^{-1}(A)$ is $N\hat{g}^*$ -open in U .

(ii) $\Rightarrow$ (i) Suppose that $f^{-1}(A)$ is $N\hat{g}^*s$ -open for each N -open set A in V . Let S be N -closed in V . Then S^c is nano open in V . By assumption, $f^{-1}(S^c)$ is $N\hat{g}^*s$ -open in U and hence $f^{-1}(S)$ is $N\hat{g}^*s$ -closed in U . Hence f is $N\hat{g}^*s$ -continuous.

Proposition 4.30 Let $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ be function
(i) f is $N\hat{g}^*s$ -continuous.

(ii) For each u in U and for each N -open set B containing $f(u)$, there is a $N\hat{g}^*s$ -open set A containing u such that $f(A) \subseteq B$.

(iii) $f(N\hat{g}^*s \text{ cl}(A)) \subseteq Ncl(f(A))$ for each subset A of U .

(iv) $N\hat{g}^*scl f^{-1}(B) \subseteq f^{-1}(Ncl(f(B)))$ for each subset B of V . Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

Proof: (i) $\Rightarrow$ (ii)

Let $u \in U$ and B be an open set containing $f(u)$. Then, by (i) $f^{-1}(B)$ is $N\hat{g}^*s$ -open set of U containing u . If $A = f^{-1}(B)$, then $f(A) = f(f^{-1}(B)) \subseteq B$.

(ii) $\Rightarrow$ (iii) Let A be a subset of a space U and $f(u) \notin Ncl(f(A))$. Then there exists N -open set B of V containing $f(u)$ such that $B \cap f(A) = \emptyset$. Now, by (ii), there is a $N\hat{g}^*s$ -open set G containing u such that $f(u) \in f(G) \subseteq B$. Hence $f(A) \cap f(G) = \emptyset$. That is, $f(A \cap G) = \emptyset$. i.e., $A \cap G = \emptyset$. Therefore, $u \notin N\hat{g}^*scl(A)$ and also $(u) \notin N\hat{g}^*scl f(A)$. Therefore $f(N\hat{g}^*s \text{ cl}(A)) \subseteq Ncl(f(A))$

(iii) $\Rightarrow$ (iv) Let B be a subset of V such that $A = f^{-1}(B)$. By (iii), $f(N\hat{g}^*scl(A)) \subseteq Ncl(f(A))$ for each subset A of U . Therefore, $(N\hat{g}^*scl f^{-1}(B)) \subseteq Ncl(f(f^{-1}(B)))$. i.e., $f(N\hat{g}^*scl f^{-1}(B)) \subseteq Ncl(B)$. i.e., $N\hat{g}^*scl f^{-1}(B) \subseteq f^{-1}(Ncl(B))$.

Lemma 4.31 A subset A of a nano topological space $(U, \tau_R(X))$ is $N\hat{g}^*s$ -open iff $F \subseteq Nsint(A)$ whenever $F \subseteq A$ and F is $N\hat{g}$ -closed.

Proposition 4.32 Let B be a $N\hat{g}^*s$ open (or $N\hat{g}^*s$ -closed) subset of $(V, \tau_R(Y))$ (satisfying

$Nsint(B) = Nint(B)$). Then $f^{-1}(B)$ is $N\hat{g}^*s$ -open (or $N\hat{g}^*s$ -closed) in $(U, \tau_R(X))$. If $f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ is $N\hat{g}^*s$ -continuous and if image of a $N\hat{g}$ -closed set in U under f is $N\hat{g}$ closed set in V .

Proof. Let B be a $N\hat{g}^*s$ -open set in V . Let $F \subseteq f^{-1}(B)$ where F is a $N\hat{g}$ -closed set in U . Then $f(F) \subseteq B$ holds. By our assumption, $f(F)$ is $N\hat{g}$ -closed set in V and B be a $N\hat{g}^*s$ -open set in V . Therefore, by lemma 3.7 $f(F) \subseteq Nsint(B)$ holds. Again, by our assumption, $f(F) \subseteq Nint(B)$ and hence $F \subseteq f^{-1}(Nint(B))$ holds. Since f is $N\hat{g}^*s$ -continuous and $Nint(B)$ is N open in V , $f^{-1}(Nint(B))$ is $N\hat{g}^*s$ -open in U . So, by lemma 3.7, $F \subseteq Nsint(f^{-1}(Nint(B)))$ holds. i.e., $F \subseteq Nsint(f^{-1}(Nint(B))) \subseteq Nsint(f^{-1}(B))$ holds. Therefore $f^{-1}(B)$ is $N\hat{g}^*s$ -open. By taking complements, we can show that if B is $N\hat{g}^*s$ -closed in V , then $f^{-1}(B)$ is $N\hat{g}^*s$ -closed in U .

Proposition 4.33 Let $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ be a function. Then the following are equivalent.

(i) f is $N\hat{g}^*s$ -irresolute.

(ii) $f^{-1}(B)$ is $N\hat{g}^*s$ -open for each $N\hat{g}^*s$ -open set B in V .

Proof: (i) $\Rightarrow$ (ii)

Suppose that f is $N\hat{g}^*$ -irresolute. Let B be $N\hat{g}^*$ -open in V . Then B^c is $N\hat{g}^*$ -closed in V . Since f is $N\hat{g}^*$ -irresolute, we have $f^{-1}(B^c)$ is $N\hat{g}^*$ -closed in U . But $f^{-1}(B^c) = [f^{-1}(B)]^c$. Therefore $f^{-1}(B)$ is $N\hat{g}^*$ -open in U .

(ii) $\Rightarrow$ (i) $N\hat{g}^*$

Suppose that $f^{-1}(B)$ is $N\hat{g}^*$ -open for each $N\hat{g}^*$ -open set B in V . Let H be $N\hat{g}^*$ -closed in V . Then H^c is $N\hat{g}^*$ -open in V . Therefore $f^{-1}(H^c)$ is $N\hat{g}^*$ -open in U . Therefore $f^{-1}(H)$ is $N\hat{g}^*$ -closed in U . Therefore, f is $N\hat{g}^*$ -irresolute.

Proposition 4.34 If a function $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $N\hat{g}^*$ -irresolute, then f is $N\hat{g}^*$ -continuous.

Proof: Let B be a N -closed set of V . But every N -closed set is $N\hat{g}^*$ -closed. Therefore, B is a $N\hat{g}^*$ -closed set of V . Since f is $N\hat{g}^*$ -irresolute, $f^{-1}(B)$ is $N\hat{g}^*$ -closed in U . Hence, by definition 3.2 f is $N\hat{g}^*$ -continuous.

Remark 4.35 The converse of Proposition 3.10 need not be true as seen from the following example.

Example 4.36 Let $(U, \tau_R(X))$ and $(V, \sigma_R(Y))$ be a nano topological spaces where $U = V = \{a, b, c, d\}$ with $U/R = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, c\}$. Then the nano topology $\tau_R(X) = \{\emptyset, U, \{c\}, \{a, c, d\}, \{a, d\}\}$. $(\tau_R(X))^c = \{\emptyset, U, \{a, b, d\}, \{b\}, \{b, c\}\}$. Then $N\hat{g}^*C(U, \tau_R(X)) = \{\emptyset, U, \{b\}, \{c\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Let $V/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $Y = \{a, c\}$. Then the nano topology $\sigma_R(Y) = \{\emptyset, V, \{a\}, \{a, b, c\}, \{b, c\}\}$. $(\sigma_R(Y))^c = \{\emptyset, U, \{b, c, d\}, \{d\}, \{a, d\}\}$. Then $N\hat{g}^*C(U, \tau_R(X)) = \{\emptyset, U, \{a\}, \{d\}, \{b, d\}, \{c, d\}, \{a, d\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}\}$. Define $f : U \rightarrow V$ as $f(a) = c, f(b) = d, f(c) = a, f(d) = b$. We have $f^{-1}(a) = \{c\}, f^{-1}(d) = \{b\}, f^{-1}(b, d) = \{a, d\}, f^{-1}(c, d) = \{a, b\}, f^{-1}(a, d) = \{b, c\}, f^{-1}(a, b, d) = \{b, c, d\}, f^{-1}(a, c, d) = \{a, b, c\}, f^{-1}(b, c, d) = \{a, b, d\}$. $f^{-1}(B)$ is $N\hat{g}^*$ -closed for each nano-closed set B in V . Hence $f : U \rightarrow V$ is $N\hat{g}^*$ -continuous. But, $f^{-1}(a, c, d) = \{a, b, c\}$ is not $N\hat{g}^*$ -closed. Therefore $f : U \rightarrow V$ is not $N\hat{g}^*$ -irresolute.

Proposition 4.37 If a function $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $N\hat{g}^*$ -irresolute, then for every subset A of U . Then $f(N\hat{g}^*scl(A)) \subseteq Nscl(f(A))$.

Proof: Let $A \subseteq U$. We know that every N -closed set is $N\hat{g}^*$ -closed set in V . Therefore, we have $Nscl(f(A))$ is $N\hat{g}^*$ -closed in V . Since f is $N\hat{g}^*$ -irresolute, then $f^{-1}(Nscl(f(A)))$ is $N\hat{g}^*$ -closed in U . Also $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(Nscl(f(A)))$. Since $f^{-1}(Nscl(f(A)))$ is $N\hat{g}^*$ -closed, we have $N\hat{g}^*scl(A) \subseteq f^{-1}(Nscl(f(A)))$. Therefore $f(N\hat{g}^*scl(A)) \subseteq f\{f^{-1}(Nscl(f(A)))\} \subseteq Nscl(f(A))$.

Proposition 4.38 If a function $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is bijective, $N\hat{g}^*$ -continuous, $Nscl(A) = Ncl(A)$ for all subsets B in V and if image of a $N\hat{g}$ -open set is $N\hat{g}$ -open under f , then f is $N\hat{g}^*$ -irresolute.

Proof: Let B be $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed set of V . Let $f^{-1}((B)) \subseteq A$ where A is $\mathcal{N}\hat{\mathcal{G}}$ -open in U . Then $f^{-1}(f(B)) \subseteq f(A)$. Since f is surjective, $B \subseteq f(A)$. Since $f(A)$ is $\mathcal{N}\hat{\mathcal{G}}$ -open and since B is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in V , we have $\text{Nscl}(B) \subseteq f(A)$. By our assumption, $\text{Ncl}(B) \subseteq f(A)$. Since f is injective, $f^{-1}(\text{Ncl}(B)) \subseteq A$. Since f is $\mathcal{N}\hat{\mathcal{G}}^*$ s -continuous and since $\text{Ncl}(B)$ is N -closed in V , $f^{-1}(\text{Ncl}(B))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s-closed in U . Therefore $f^{-1}((B))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in U and hence f is $\mathcal{N}\hat{\mathcal{G}}^*$ s -irresolute.

Definition 4.39 A function $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is called a $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed map if $f(A)$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in V for every N -closed set A of U

Definition 4.40 A function $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is called a $\mathcal{N}\hat{\mathcal{G}}^*$ s-open map if $f(A)$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s-open in V for every N -open set A of U

Proposition 4.41 If $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -irresolute and A is a $\mathcal{N}\hat{\mathcal{G}}^*$ s closed subset of U , then $f(A)$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in V .

Proof: Let $f(A) \subseteq B$ and B is $\mathcal{N}\hat{\mathcal{G}}$ -open in V . Then $f^{-1}(f(A)) \subseteq f^{-1}(B)$. i.e., $A \subseteq f^{-1}(B)$. Since f is $\mathcal{N}\hat{\mathcal{G}}$ -irresolute, $f^{-1}(B)$ is $\mathcal{N}\hat{\mathcal{G}}$ -open in U . Since A is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed, $\text{Ncl}(A) \subseteq f^{-1}(B)$. So, $f(\text{Ncl}(A)) \subseteq f(f^{-1}(B))$. i.e., $f(\text{Ncl}(A)) \subseteq B$. Since f is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed and $\text{Ncl}(A)$ is N closed in U , $f(\text{Ncl}(A))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in V . Therefore $\text{Nscl}(f(\text{Ncl}(A))) \subseteq B$. Since $f(A) \subseteq f(\text{Ncl}(A))$, we have $\text{Nscl}(f(A)) \subseteq \text{Nscl}(f(\text{Ncl}(A))) \subseteq B$. Therefore $f(A)$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in V .

Proposition 4.42 If $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is N -closed and: $V \rightarrow W$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed, then $g \circ f$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed.

Proof: Let A be a N -closed set of U . Since f is N -closed, $f(A)$ is N -closed in V . Since g is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed, $g(f(A))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed in W . Hence $g \circ f : U \rightarrow W$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed.

Proposition 4.43 Let $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is N -closed and $g : (V, \sigma_R(Y)) \rightarrow (W, \mu_R(Z))$ be two maps such that $g \circ f : (U, \tau_R(X)) \rightarrow (W, \mu_R(Z))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -open map, if f is N continuous and surjective.

Proof: Let B be a N -open V . Since f is N -continuous, $f^{-1}(B)$ is N -open in U . Since $f^{-1}(B)$ is N -open in U , $g \circ f(f^{-1}(B))$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -open in W . i.e., $g(B)$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s -open in W . Therefore, g is a $\mathcal{N}\hat{\mathcal{G}}^*$ s -open map.

Proposition 4.44 For any bijection $f : (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$, the following are equivalent:

- (i) $f^{-1} : (V, \sigma_R(Y)) \rightarrow U$ is $\mathcal{N}\hat{\mathcal{G}}^*$ s-continuous.
- (ii) f is $\mathcal{N}\hat{\mathcal{G}}^*$ s -open.
- (iii) f is $\mathcal{N}\hat{\mathcal{G}}^*$ s -closed.

Proof:

- (i) $\Rightarrow$ (ii) Let A be N -open in U . Then $U-A$ is N -closed in U . Since f^{-1} is $N\hat{g}^*$ -continuous, $(f^{-1})^{-1}(U-A) = f(U-F) = V-f(F)$ is $N\hat{g}^*$ -closed in V . Then $f(F)$ is $N\hat{g}^*$ -open in V . Hence f is $N\hat{g}^*$ -open.
- (ii) $\Rightarrow$ (iii) Let A be N -closed in U . Then $U-A$ is N -open in U . Since f is $N\hat{g}^*$ -open, $f(U-F) = V-f(F)$ is $N\hat{g}^*$ -open in V . Then $f(F)$ is $N\hat{g}^*$ -closed in V . Hence f is $N\hat{g}^*$ -closed.
- (iii) $\Rightarrow$ (i) Let A be N -closed in U . Since $f: (U, \tau_R(X)) \rightarrow (V, \sigma_R(Y))$ is $N\hat{g}^*$ -closed. $f(A)$ is $N\hat{g}^*$ -closed in V . i.e., $(f^{-1})^{-1}(A)$ $N\hat{g}^*$ -closed in V . Therefore f^{-1} is $N\hat{g}^*$ -continuous.

5. Conclusion

In this paper, we introduced and studied the concepts of $N\hat{g}^*$ -continuous and $N\hat{g}^*$ -irresolute in nano topological spaces and we compare it with other nano-continuous and irresolute function and proved that composition of two $N\hat{g}^*$ -continuous functions need not be a $N\hat{g}^*$ -continuous functions. We also investigate some of its properties and give suitable examples for the reverse which is not true. In future this work will be extended with some real-life applications.

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A study on $\hat{g}^{**}s - R_0$ and $\hat{g}^{**}s - R_1$ spaces In Topological spaces

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Abstract

In this paper, we introduces the concept of R_0 -space, R_1 -space, door space, submaximal using $\hat{g}^{**}s$ -closed set and investigate its properties. We have also study their relationship with some other higher separation axioms. We have also, introduced a new definition $S\hat{g}^*$ -space by using semi-closed and $\hat{g}^*$ -closed sets and study its relationship with other closed sets using $\hat{g}^{**}s$ -closed set.

Keywords. $\hat{g}^{**}s - R_0$, $\hat{g}^{**}s - R_1$ space, $\hat{g}^{**}s$ -door spaces, $\hat{g}^{**}s$ -submaximal, $S\hat{g}^*$ -space.

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1. Introduction

Topology is a major area of mathematics concerned with spatial properties that are preserved under continuous deformations of objects. It emerged through the development of concepts from geometry and set theory, such as space, dimension, and transformation. Ideas that are now classified as topology were expressed as early as 1736. By the middle of the 20th century, topology had become an important area of study within mathematics. The word topology is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Topology includes many subfields namely point-set topology, algebraic topology and geometric topology.

In the literature of General Topology, the concept of semi open sets was introduced by Levine in 1963 [11] and g -closed sets in 1970 [12]. M.K.R.S. Veera Kumar defined $\hat{g}$ -closed sets in 2001 [10] and $\hat{g}^*$ -closed sets in 1996 [9]. In the year 1995, J. Dontchev [2, 3], defined on door spaces and submaximal spaces.

The notion of R_0 topological space is introduced by N. A. Shanin [13] in 1943. Later, A. S. Davis [13] rediscovered it and studied some properties of this weak separation axiom. In 2010, S. Balasubramanian defined separation axioms on generalized sets [14]. Several topologists further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2 . M. G. Murdeshwar and S. A. Naimpally [8] studied some of the fundamental properties of the class of R_1 topological spaces. In 1963, N. Levine [11] offered a new notion to the field of general topology by introducing semi-open sets. He defined this notion by utilizing the known notion of closure of an open set, i.e., a subset of a topological space is semi-open if it is contained in the closure of its interior. Since the advent of this notion, several new notions are defined in terms of semi-open sets of which two are semi- R_0 and semi- R_1 introduced by S. N. Maheshwari and R. Prasad [15] and C. Dorsett [1], respectively. These two notions are defined as natural generalizations of the separation axioms R_0 and R_1 by replacing the closure operator with the semi closure operator and openness with semi-openness. Since then, this notion received wide usage in general topology.

In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of $\hat{g}^{**}s$ -closed sets [7] called $\hat{g}^{**}s$ - R_0 and $\hat{g}^{**}s$ - R_1 . We have also applied the definition in [7] to door space and submaximal space. In this paper we introduced $S\hat{g}^*$ -space using semi-closed and $\hat{g}^*$ -closed sets.

2. Preliminaries

Throughout this paper (X, τ) represent the non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $cl(A)$ and $int(A)$ denote the closure and interior of A respectively.

Definition. 2.1

- i) A space (X, τ) is R_0 [13] if for each open set U of X , $x \in U$ implies $cl(\{x\}) \subseteq U$.
- ii) A topological space (X, τ) is R_1 [13] if for $x, y \in X$ such that $cl(\{x\}) \neq cl(\{y\})$, there are disjoint open sets U and V such that $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$.
- iii) A topological space (X, τ) is a door space [3] if every subset of X is either open or closed in X .
- iv) A topological space (X, τ) is a submaximal space [4] if every dense subset of X is open in X .

3. $\hat{g}^{**}s - R_0$ And $\hat{g}^{**}s - R_1$ spaces

Definition. 3.1. A space (X, τ) is said to be $\hat{g}^{**}s - R_0$ if for any open set U , $x \in U$, then $\hat{g}^{**}scl\{x\} \subseteq U$.

Definition. 3.2. A space (X, τ) is said to be $\hat{g}^{**}s - R_1$ if for x and y in X , with $\hat{g}^{**}scl(\{x\}) \neq \hat{g}^{**}scl(\{y\})$, there exist two disjoint $\hat{g}^{**}s$ -open sets U and V such that $\hat{g}^{**}scl(\{x\}) \subseteq U$ and $\hat{g}^{**}scl(\{y\}) \subseteq V$

Theorem. 3.3. Every $\hat{g}^{**}s - R_0$ -space is $\hat{g}^{**}s - T_0$ -space.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$ -space. Let x and y be two distinct points of X . By hypothesis, for any open set U and $x \in U \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq U$. Also, every open set is $\hat{g}^{**}s$ -open, which implies U is $\hat{g}^{**}s$ -open and $x \in U$. Therefore, for any distinct points x and y , there exist a $\hat{g}^{**}s$ -open set U containing x and not y . Hence, (X, τ) is a $\hat{g}^{**}s - T_0$ -space.

Theorem. 3.4. Every $\hat{g}^{**}s - R_0$ -space is $\hat{g}^{**}s - T_2$ -space.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$ -space. Let x and y be two distinct points of X . By hypothesis, for any open set U and $x \in U \Rightarrow \hat{g}^{**}scl\{x\} \subseteq U$. If $\hat{g}^{**}scl(\{x\}) \not\subseteq U$, there exist another open set V not containing x , so $y \in V$. Also, $U \cap V = \emptyset$. Therefore, for any distinct points x and y , there exist two disjoint $\hat{g}^{**}s$ -open sets U and V . Hence, (X, τ) is a $\hat{g}^{**}s - T_2$ -space.

Theorem. 3.5. If (X, τ) is $\hat{g}^{**}s - R_0$, then for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}s$ -open set G such that $U \subseteq G$ and $x \notin G$.

Proof. Suppose (X, τ) is $\hat{g}^{**}s - R_0$. Let U be any closed set and $x \notin U \Rightarrow X \setminus U$ is open and $x \in X \setminus U$. By assumption, $\hat{g}^{**}scl(\{x\}) \subseteq X \setminus U \Rightarrow U \subseteq X \setminus \hat{g}^{**}scl(\{x\})$. Put $G =$

$X \setminus \hat{g}^{**}scl\{x\}$. Since $\hat{g}^{**}scl\{x\}$ is $\hat{g}^{**}s$ -open. Also, $U \subseteq G$ and $x \notin G$. Therefore, for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}s$ -open set G such that $U \subseteq G$ and $x \notin G$. Conversely, suppose for any closed set U and $x \notin U$, there exist an $\hat{g}^{**}s$ -open set G such that $U \subseteq G$ and $x \notin G$.

Theorem. 3.6. Every $\hat{g}^{**}s - R_0$ -space is $\hat{g}^{**}s$ -regular.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$ -space. Let F be a closed set and $x \in X - F$. By theorem: 3.3, there exist an $\hat{g}^{**}s$ -open set G such that $U \subseteq G$ and $x \notin G$. Put $H = \{x\}$. Since every singleton set is open, so H is open. Also, we know that, every open set is $\hat{g}^{**}s$ -open. Also, $G \cap H = G \cap \{x\} = \emptyset$. Thus, G and H are disjoint $\hat{g}^{**}s$ -open sets containing x and H respectively. Therefore, (X, τ) is $\hat{g}^{**}s$ -regular.

Theorem. 3.7. A topological space (X, τ) is $\hat{g}^{**}s - R_0$ iff for any points x and y in X , $x \neq y$ implies $\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \emptyset$.

Proof. Let X be $\hat{g}^{**}s - R_0$ and $x \neq y$ in $X \Rightarrow \{x\}$ be an open set and $y \notin \{x\}$. Since, $x \in \{x\}$, we have $\hat{g}^{**}scl(\{x\}) \subseteq \{x\}$. Thus $\hat{g}^{**}scl(\{x\}) = \{x\}$. Now, $\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \{x\} \cap \{y\} = \emptyset$. Conversely, Suppose for any points x and y in X , $x \neq y \Rightarrow \hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \emptyset$. Let V be an open set and $x \in V$ and let $y \in \hat{g}^{**}scl(\{x\})$ (1). Suppose $y \notin V$. By assumption, $\hat{g}^{**}scl(\{x\}) \cap \hat{g}^{**}scl(\{y\}) = \emptyset \Rightarrow y \notin \hat{g}^{**}scl(\{x\})$ this is a contradiction to (1). Therefore, $y \in V$ and $\hat{g}^{**}scl(\{x\}) \subseteq V$. Thus (X, τ) is $\hat{g}^{**}s - R_0$.

Corollary. 3.8. A topological space (X, τ) is $\hat{g}^{**}s - R_0$ iff for any points x and y in X , $x \neq y \Rightarrow cl(\{x\}) \cap cl(\{y\}) = \emptyset$.

Proof. Directly follows from theorem.3.5

Theorem. 3.9. If (X, τ) is $\hat{g}^{**}s - R_0$, then it is $\hat{g}^{**}s - T_1$.

Proof. Let (X, τ) be a $\hat{g}^{**}s - R_0$ -space and $x \in X \Rightarrow \{x\}$ is open. By hypothesis, $\hat{g}^{**}scl(\{x\}) \subseteq \{x\} \Rightarrow \hat{g}^{**}scl(\{x\}) = \{x\} \Rightarrow \{x\}$ is $\hat{g}^{**}s$ -closed $\Rightarrow$ every singleton set is $\hat{g}^{**}s$ -closed $\Rightarrow (X, \tau)$ is $\hat{g}^{**}s - T_1$.

Theorem. 3.10. Every $\hat{g}^{**}s - R_0$ -space is $\hat{g}^{**}s - R_1$ -space.

Proof. Let (X, τ) is $\hat{g}^{**}s - R_0$. Let $x, y \in X$, with $\hat{g}^{**}scl(\{x\}) \neq \hat{g}^{**}scl(\{y\})$. By above theorem, $\hat{g}^{**}scl(\{x\}) = \{x\}$ and $\hat{g}^{**}scl(\{y\}) = \{y\} \Rightarrow \{x\}$ and $\{y\}$ are $\hat{g}^{**}s$ -open sets and $\{x\} \cap \{y\} = \emptyset \Rightarrow (X, \tau)$ is $\hat{g}^{**}s - R_1$ -space.

Theorem. 3.11. For any $\hat{g}^{**}s$ -closed set H , $\hat{g}^{**}scl(\{x\}) \cap H = \emptyset$, for every $x \in X \setminus H$, then (X, τ) is $\hat{g}^{**}s - R_0$ -space.

Proof. Assume that for any $\hat{g}^{**}s$ -closed set H , $\hat{g}^{**}scl(\{x\}) \cap H = \emptyset$, for every $x \in X \setminus H$. Let G be any open set and $x \in G$. Then $x \in G \Rightarrow x \in X \setminus (X \setminus G)$ and $X \setminus G$ is closed. Therefore, by assumption, $\hat{g}^{**}scl(\{x\}) \cap (X \setminus G) = \emptyset \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq G$. Hence proved.

Corollary. 3.12. For any $\hat{g}^{**}s$ -closed set $H, cl(\{x\}) \cap H = \emptyset$, for every $x \in X \setminus H$, then (X, τ) is $\hat{g}^{**}s - R_0$ -space.

Proof. Assume that for any $\hat{g}^{**}s$ -closed set $H, cl(\{x\}) \cap H = \emptyset$, for every $x \in X \setminus H$. Let V be any open set in X and $x \in V$. Then, $x \in V = X \setminus (X \setminus V)$ and $X \setminus V$ is closed. Since every closed set is $\hat{g}^{**}s$ -closed. Since, every closed set is $\hat{g}^{**}s$ -closed which implies that $X \setminus V$ is $\hat{g}^{**}s$ -closed. By assumption, $cl(\{x\}) \cap (X \setminus V) = \emptyset \Rightarrow cl(\{x\}) \subseteq V \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq cl(\{x\}) \subseteq V \Rightarrow \hat{g}^{**}scl(\{x\}) \subseteq V \Rightarrow (X, \tau)$ is $\hat{g}^{**}s - R_0$ -space.

Theorem. 3.13. If a topological space, (X, τ) is $\hat{g}^{**}s - R_1$ and $\hat{g}^{**}s - T_0$ -space.

Proof. Since every $\hat{g}^{**}s - T_1$ is $\hat{g}^{**}s - T_0$. Then, the result is obvious.

Theorem. 3.14. If (X, τ) is $\hat{g}^{**}s - R_0$ and $\hat{g}^{**}s - T_0$ -space, then it is also a $\hat{g}^{**}s - T_1$ -space.

Proof. Let $x \neq y$ be any two points of X . Since (X, τ) is $\hat{g}^{**}s - R_0$, there exist an open set U such that $x \in U$ and $\hat{g}^{**}scl(\{x\}) \subseteq U \Rightarrow x \notin X \setminus U$. Since, $y \notin U$, there exist another $\hat{g}^{**}s$ -open set $X \setminus U = V$ (say) containing y but not x . Therefore, for any two distinct points x and y , there exist two distinct $\hat{g}^{**}s$ -open sets U and V . Hence, (X, τ) is $\hat{g}^{**}s - T_1$ -space.

Theorem. 3.15. If a topological space (X, τ) is $\hat{g}^{**}s - R_1$, then either $cl(\{x\}) = X$, for each $x \in X$ or $cl(\{x\}) \neq X$, for each $x \in X$.

Proof. Assume that (X, τ) is $\hat{g}^{**}s - R_1$. If $cl(\{x\}) = X$, for each $x \in X$, then the theorem is obvious. If not, then there exist $y \in X$ such that $cl(\{y\}) \neq X$. Suppose not, there exist $z \in X$ such that $cl(\{z\}) = X$. Now, $cl(\{y\}) \neq X = cl(\{z\})$. Since (X, τ) is $\hat{g}^{**}s - R_1$, there exist disjoint $\hat{g}^{**}s$ -open sets U and V containing $cl(\{y\})$ and $cl(\{z\})$ respectively. Since $cl(\{z\}) = X$, we have $V = X \Rightarrow U \cap V = U \neq \emptyset$ which a contradiction is to $U \cap V = \emptyset$. Hence proved.

4. $\hat{g}^{**}s$ -door space

Definition. 4.1. A topological space (X, τ) is called a $\hat{g}^{**}s$ -door space if every subset is either $\hat{g}^{**}s$ -open or $\hat{g}^{**}s$ -closed.

Definition. 4.2. A topological space (X, τ) is called $\hat{g}^{**}s$ -submaximal if every dense subset of X is $\hat{g}^{**}s$ -open.

Definition. 4.3. A topological space (X, τ) is called $\hat{g}^{**}s$ -extremally disconnected space in which the $\hat{g}^{**}s$ -closure of every $\hat{g}^{**}s$ -open subset is $\hat{g}^{**}s$ -open.

Definition. 4.4. A topological space is called $S\hat{g}^*$ -space if the intersection of a semi-closed set with $\hat{g}^*$ -closed set is $\hat{g}^*$ -closed.

Definition. 4.5. A topological space (X, τ) is called $\hat{g}^{**}$ -hyperconnected if every non-empty $\hat{g}^{**}$ -open subset of X is $\hat{g}^{**}$ -dense.

Theorem. 4.6. Every subspace of $\hat{g}^{**}$ -s-door space is a $\hat{g}^{**}$ -s-door space.

Proof. Let S be a subspace of X and $A \subseteq S$ is a subset of X . Since X is $\hat{g}^{**}$ -s-door space, A is either $\hat{g}^{**}$ -s-open or $\hat{g}^{**}$ -s-closed in X . Hence A is either $\hat{g}^{**}$ -s-open or $\hat{g}^{**}$ -s-closed in S . Therefore, S is $\hat{g}^{**}$ -s-door space.

Theorem. 4.7. In a $\hat{g}^{**}$ -hyperconnected space, every $\hat{g}^{**}$ -s-submaximal space is a $\hat{g}^{**}$ -s-door space.

Proof. Let $A \subset X$ and if A is dense in X , so A is $\hat{g}^{**}$ -s-open. (1)

If A is not dense, there exist a non-empty open set $B \subset A^c$. Since every open set is $\hat{g}^{**}$ -s-open, there exist a non-empty $\hat{g}^{**}$ -s-open set $B \subset A^c$. Since X is $\hat{g}^{**}$ -s-hyperconnected, B is dense and A^c is also dense. Again by definition of $\hat{g}^{**}$ -s-submaximal, A^c is $\hat{g}^{**}$ -s-open which implies A is $\hat{g}^{**}$ -s-closed. (2)

From (1) and (2), A is either $\hat{g}^{**}$ -s-open or $\hat{g}^{**}$ -s-closed. Therefore, (X, τ) is a $\hat{g}^{**}$ -s-door space.

Theorem. 4.8. In (X, τ) , every door space is $\hat{g}^{**}$ -s-door space.

Proof. Let A be a door space. Every subset of X is either open or closed. Since every closed set is $\hat{g}^{**}$ -s-closed and every open set is $\hat{g}^{**}$ -s-open, every subset of X is either $\hat{g}^{**}$ -s-open or $\hat{g}^{**}$ -s-closed. Therefore, A is a $\hat{g}^{**}$ -s-door space.

Remark. 4.9. Converse of above theorem is not true.

Example. 4.10. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\tau^c = \{\emptyset, X, \{b, c\}\}$, $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Hence (X, τ) is a $\hat{g}^{**}$ -s-door space but it is not a door space.

Theorem. 4.11. For a subset A of a $S\hat{g}^*$ -space (X, τ) the following are equivalent:

- i) A is $\hat{g}^{**}$ -s-closed
- ii) $\hat{g}^*cl(\{x\}) \cap A \neq \emptyset$, for each $x \in scl(A)$
- iii) $scl(A) - A$ Contains no non-empty $\hat{g}^*$ -closed set.

Proof. i) $\Rightarrow$ ii) Let $x \in scl(A)$. Suppose $\hat{g}^*cl(\{x\}) \cap A = \emptyset$. Then $A \subseteq X - \hat{g}^*cl(\{x\})$. Since A is $\hat{g}^{**}$ -s-closed, $scl(A) \subseteq X - \hat{g}^*cl(\{x\})$ which is a contradiction to $x \in scl(A)$. Therefore, $\hat{g}^*cl(\{x\}) \cap A \neq \emptyset$

ii) $\Rightarrow$ iii) Let F be a $\hat{g}$ -closed set such that $F \subseteq \text{scl}(A) - A$. Suppose $x \in F$. Then $\hat{g}^*\text{cl}(\{x\}) \subseteq F$. Therefore $\emptyset \neq \hat{g}^*\text{cl}(\{x\}) \cap A \subseteq F \cap A \subseteq [\text{scl}(A) - A] \cap A = \emptyset$ which is a contradiction.

iii) $\Rightarrow$ i) Let $A \subseteq G$ and G be $\hat{g}^*$ -open in X , Suppose $\text{scl}(A)$ not a subset of G . Then $\text{scl}(A) \cap (X - G)$ is non-empty $\hat{g}^*$ -closed subset of $\text{scl}(A) - A$ which is a contradiction. Therefore, $\text{scl}(A) \subseteq G$ and hence A is $\hat{g}^{**}s$ -closed.

Theorem. 4.12. If B is a cl open subset of (X, τ) , then B is a $\hat{g}^{**}s$ -closed set.

Proof. Since B is cl open, B is both open and closed. Let U be a $\hat{g}^*$ -open set in X and $B \subseteq U$. Since B is clopen, we have $\text{int}(\text{cl}(B)) = B \Rightarrow \text{int}(\text{cl}(B)) \subseteq B$ and $B \subseteq \text{int}(\text{cl}(A))$, (ie) $\text{int}(\text{cl}(B)) \subseteq B \Rightarrow B$ is semi-closed and $\text{scl}(B) - B$. Therefore, $\text{scl}(B) \subseteq U$. Hence B is $\hat{g}^{**}s$ -closed in X .

Theorem. 4.13. In (X, τ) , every $\hat{g}^{**}s$ -dense is dense.

Proof. Let $A \subseteq X$ be $\hat{g}^{**}s$ -dense in $(X, \tau) \Rightarrow \hat{g}^{**}\text{scl}(A) = X$. Since, $X = \hat{g}^{**}\text{scl}(A) \subseteq \text{cl}(A) \Rightarrow X \subseteq \text{cl}(A)$. Always, $\text{cl}(A) \subseteq X \Rightarrow \text{cl}(A) = X \Rightarrow A \subseteq X$ be dense in (X, τ) .

Remark. 4.14. Converse of above theorem is not true.

Example. 4.15. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\tau^c = \{\emptyset, X, \{c\}\}$, $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$ and $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ is a dense subset of X but it is not $\hat{g}^{**}s$ -dense.

Remark. 4.16. Every $\hat{g}^{**}s$ -door space need not be a $\hat{g}^{**}s$ -submaximal.

Example. 4.17. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$ and $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Here (X, τ) is a $\hat{g}^{**}s$ -door space but it is not $\hat{g}^{**}s$ -submaximal.

Remark. 4.18. Every $\hat{g}^{**}s$ -submaximal space need not be $\hat{g}^{**}s$ -door space.

Example. 4.19. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}\}$,
 $sc(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}\}$,
 $\hat{g}^*c(X, \tau)$
 $= \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}$,
 $\hat{g}^{**}sc(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}\}$,
 $\hat{g}^{**}so(X, \tau) = \{\emptyset, X, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$.

Remark. 4.20. Every $S\hat{g}^*$ -space need not be a door space.

Example. 4.21. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\tau^c = \{\emptyset, X, \{a\}, \{b, c\}\}$, $sc(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\hat{g}^*c(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$. Here $sc \cap \hat{g}^*c = \hat{g}^*c$ but it is not a door space.

Theorem. 4.22. In $S\hat{g}^*$ -space

- i) $(sg)^*$ -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- ii) gs^{**} -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- iii) $\hat{\eta}^*$ -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- iv) αg -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- v) ps -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- vi) gsp -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- vii) sp -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- viii) sg -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- ix) ga -Closed set coincide with $\hat{g}^{**}s$ -closed set.
- gs -Closed set coincide with $\hat{g}^{**}s$ -closed set.

Theorem. 4.23. In $S\hat{g}^*$ -space, semi-closed and $\hat{g}$ -closed are independent.

Theorem. 4.24. The property of being $\hat{g}^{**}s$ -door space is a topological property.

Proof. Let (X, τ) be a $\hat{g}^{**}s$ -door space and let $f: X \rightarrow Y$ be a homeomorphism. Let $A \subseteq Y$, consider $f^{-1}(A) \subseteq X$, since X is a $\hat{g}^{**}s$ -door space, then $f^{-1}(A)$ is either $\hat{g}^{**}s$ -open or $\hat{g}^{**}s$ -closed in X . Now, $f(f^{-1}(A)) = A$. Then A is either $\hat{g}^{**}s$ -open or $\hat{g}^{**}s$ -closed in Y . Therefore, Y is a $\hat{g}^{**}s$ -door space.

Theorem. 4.25. The property of being a $\hat{g}^{**}s$ -door space is an expensive property.

Proof. Suppose (X, τ) is a $\hat{g}^{**}s$ -door space. Let $\tau \subset \tau_1$ and $A \subseteq X$. Since, (X, τ) is a $\hat{g}^{**}s$ -door space, then A is either $\hat{g}^{**}s - \tau$ -open or $\hat{g}^{**}s - \tau$ -closed. Since $\tau_1 \subset \tau$, A is either $\hat{g}^{**}s - \tau_1$ -open or $\hat{g}^{**}s - \tau_1$ -closed. Then (X, τ_1) is a $\hat{g}^{**}s$ -door space.

Theorem. 4.26. Let (X, τ) be a $\hat{g}^{**}s$ -door space and $Y \subseteq X$ be a clopen subset of Y , then (Y, τ_Y) is also a $\hat{g}^{**}s$ -door space.

Proof. Let $A \subseteq Y$ be a subset of Y . Now, $A \subseteq X$. By hypothesis, (X, τ) is a $\hat{g}^{**}s$ -door space which implies that A is either $\hat{g}^{**}s$ -open or $\hat{g}^{**}s$ -closed in X . Since Y is both open and closed. Then A is either $\hat{g}^{**}s$ -open or $\hat{g}^{**}s$ -closed in Y . Therefore, Y is also a $\hat{g}^{**}s$ -door space.

Proposition. 4.27. Let A and B be $\hat{g}^{**}s$ -closed subsets of X such that $cl(A) = scl(A)$ and $cl(B) = scl(B)$. Thus $A \cup B$ is $\hat{g}^{**}s$ -closed.

Proof. Let U be $\hat{g}^*$ -open such that $A \cup B \subseteq U \Rightarrow A \subseteq U$ and $B \subseteq U \Rightarrow scl(A) \subseteq U$ and $scl(B) \subseteq U \Rightarrow cl(A) \subseteq U$ and $cl(B) \subseteq U \Rightarrow cl(A \cup B) = cl(A) \cup cl(B) \subseteq U \Rightarrow cl(A \cup B) \subseteq U \Rightarrow scl(A \cup B) \subseteq cl(A \cup B) \subseteq U$. Therefore, $A \cup B$ is $\hat{g}^{**}s$ -closed.

Proposition. 4.28. If B is a regular open subset of a topological space (X, τ) , then B is $\hat{g}^{**}s$ -closed.

Proof. Let U be $\hat{g}^*$ -open set such that $B \subseteq U$. Since B is regular open, $int(cl(B)) = B \Rightarrow int(cl(B)) \subseteq B \subseteq U \Rightarrow B$ is semi-closed $\Rightarrow scl(B) = B \subseteq U \Rightarrow scl(B) \subseteq U$. Therefore, B is $\hat{g}^{**}s$ -closed.

Theorem. 4.29. If A and B are subsets of X such that $A \subseteq B$ and A is g^* -closed, then B is $\hat{g}^{**}s$ -closed.

Proof. Assume $B \subseteq U$ and U is $\hat{g}^*$ -open. Therefore, $A \subseteq B \Rightarrow A \subseteq U$. Also, every $\hat{g}^*$ -open set is g -open which implies U is g -open in X . Therefore, $A \subseteq U$ and U is g -open $\Rightarrow cl(A) \subseteq U \Rightarrow cl(B) \subseteq U \Rightarrow scl(B) \subseteq cl(B) \subseteq U \Rightarrow scl(B) \subseteq U$. Therefore B is $\hat{g}^{**}s$ -closed.

Theorem. 4.30. If A and B are subsets of X such that $A \subseteq B$ and A is g^* -closed, then B is $\hat{g}^{**}s$ -closed.

Proof. Assume that $B \subseteq U$ and U is $\hat{g}^*$ -open. Since $A \subseteq B \Rightarrow A \subseteq U$. Also, every $\hat{g}^*$ -open set is g -open which implies U is g -open in X . Therefore, $A \subseteq U$ and U is g -open $\Rightarrow scl(A) \subseteq U \Rightarrow scl(A) \subseteq scl(B) \subseteq U \Rightarrow scl(B) \subseteq U$. Hence proved.

5. Conclusion

We have studied the concept of R_0 -space, R_1 -space, door space, and submaximal space via $\hat{g}^{**}s$ -closed set and examples are provided to state the converse doesn't implies. It shows that every $\hat{g}^{**}s - R_0$ -space is $\hat{g}^{**}s - T_2$ -space, $\hat{g}^{**}s$ -regular and $\hat{g}^{**}s - R_1$ -space. In addition we have shown that every $\hat{g}^{**}s$ -door space need not be a $\hat{g}^{**}s$ -submaximal and suitable examples are given for it. A new concept of $S\hat{g}^*$ -space is also introduced. Further, we have compared $\hat{g}^{**}s$ -closed set with other existing closed sets and we have also investigated the characterization of these spaces.

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New Characterization Of $(1, 2)S_p$ -Kernel in Bitopological Spaces

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Abstract

In this paper, the concept of $(1,2)S_p$ -Kernal in bitopological spaces is introduced. Also the closure and kernel are defined in terms of $(1,2)S_p$ -separation using $(1,2)S_p$ -open sets in bitopological spaces and some of its properties are studied.

Keywords: $(1,2)$ semi-open, $(1,2)$ pre-open, $(1,2)$ pre-closed, $(1,2)S_p$ -open sets, $(1,2)S_p$ -closed sets, $(1,2)S_p$ -kernel sets, $(1,2)S_p$ -derived sets, $(1,2)S_p$ -shell sets.

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1. Introduction

In 1963, Kelly [3] initiated the study of a triplet (X, τ_1, τ_2) , where X is a non-empty set and τ_1, τ_2 are the two topologies on X . The notion of $(1,2)\alpha$ -open set [4] in a bitopological space was introduced by Lellis Thivagar in 1991. Raja Rajeswari [6] defined and studied the concepts of ultra-kernel in bitopological spaces. In 2017, $(1,2)S_p$ -open set [2] in bitopological spaces was introduced by Hardi Ali Shareef et.al. In this paper, a new class of sets in bitopological spaces called $(1,2)S_p$ -kernel is introduced and some of its properties are derived.

2. Preliminaries

Definition 2.1. [5] A subset A of a bitopological space X is called a

- (i) $(1, 2)$ semi-open if $A \subseteq \tau_1 \tau_2 \text{Cl}(\tau_1 \text{Int}(A))$.
- (ii) $(1, 2)$ pre-open if $A \subseteq \tau_1 \text{Int}(\tau_1 \tau_2 \text{Cl}(A))$.
- (iii) $(1, 2)$ regular-open if $A = \tau_1 \text{Int}(\tau_1 \tau_2 \text{Cl}(A))$.

The collection of all $(1, 2)$ semi-open, $(1, 2)$ pre-open and $(1, 2)$ regular-open sets are denoted by $(1, 2) \text{SO}(X)$, $(1, 2) \text{PO}(X)$ and $(1, 2) \text{RO}(X)$ respectively.

Definition 2.2 [5] A subset A of a bitopological space X is called a

- (i) $(1, 2)$ α -closed if $\tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(\tau_1 \text{Cl}(A))) \subseteq A$.
- (ii) $(1, 2)$ semi-closed if $\tau_1 \tau_2 \text{Int}(\tau_1 \text{Cl}(A)) \subseteq A$.
- (iii) $(1, 2)$ pre-closed if $\tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(A)) \subseteq A$.
- (iv) $(1, 2)$ regular-closed if $A = \tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(A))$.

The set of all $(1, 2)$ α -closed, $(1, 2)$ semi-closed, $(1, 2)$ pre-closed and $(1, 2)$ regular-closed sets are denoted as $(1, 2) \alpha \text{CL}(X)$, $(1, 2) \text{SCL}(X)$, $(1, 2) \text{PCL}(X)$ and $(1, 2) \text{RCL}(X)$ respectively. Also, for any subset A of X , the $(1, 2)$ α -closure, $(1, 2)$ semi-closure, $(1, 2)$ pre-closure and $(1, 2)$ regular-closure of A is denoted as $(1, 2) \alpha \text{Cl}(A)$, $(1, 2) \text{SCL}(A)$, $(1, 2) \text{PCL}(A)$ and $(1, 2) \text{RCL}(A)$ respectively.

Definition 2.3. [2] A $(1, 2)$ semi-open set A of a bitopological space X is called $(1, 2) S_p$ -open set if for each $x \in A$, there exists a $(1, 2)$ pre-closed set F such that $x \in F \subseteq A$.

Remark 2.4. If A and B are $(1, 2)$ α -open sets of X , then $A \cup B$ is also a $(1, 2)$ α -open set.

Definition 2.5. A subset A of a bitopological space X is said to be $(1, 2) S_p$ -locally closed (briefly $(1, 2) S_p \text{LC}$) if $A = C \cap D$, where C is a $(1, 2) S_p$ -open set and D is a $(1, 2) S_p$ -closed set in X . The family of $(1, 2) S_p$ -locally-closed sets is denoted by $(1, 2) S_p \text{LC}(X)$.

3. (1, 2) S_p -Kernel in Bitopological Spaces

Definition 3.1. Let A be a non-empty subset of a bitopological spaces X . Then (1, 2) S_p -kernel of A is denoted by (1, 2) S_p -Ker ($\{A\}$) and it is defined as (1, 2) S_p -Ker ($\{A\}$) = $\cap \{G \in (1, 2) S_p O(X) / A \subseteq G\}$.

Definition 3.2. Let $x \in X$. Then the (1, 2) S_p -kernel of x is defined by (1, 2) S_p -Ker ($\{x\}$) = $\cap \{G \in (1, 2) S_p O(X) / x \in G\}$.

Definition 3.3. A subset N of a bitopological space X is said to be (1, 2) S_p -neighborhood ((1, 2) S_p -nbhd) of a point $x \in X$, if there exists a (1, 2) S_p -open set U such that $x \in U \subseteq N$.

Lemma 3.4. Let X be a bitopological space. Then for any non-empty subset A of X , (1, 2) S_p -Ker ($\{A\}$) = $\{x \in X / (1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi\}$.

Proof: Let $x \in (1, 2)S_p\text{-Ker}(\{A\})$ and $(1, 2)S_p\text{-Cl}(\{x\}) \cap A \neq \phi$. Then $A \subseteq [X - (1, 2) S_p\text{-Cl}(\{x\})]$ and $[X - (1, 2) S_p\text{-Cl}(\{x\})]$ is a (1, 2) S_p -open set containing A but not x , which is a contradiction. Hence $(1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi$.

Also, let $x \notin (1, 2) S_p\text{-Ker}(\{A\})$ and $(1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi$. Then there exists a (1, 2) S_p -open set D containing A but not x and there exists an element $y \in (1, 2)S_p\text{-Cl}(\{x\}) \cap A$. Hence we get a (1, 2) S_p -nbhd of y , say D with $x \notin D$, which is a contradiction. Hence $x \in (1, 2) S_p\text{-Ker}(A)$.

Definition 3.5. In a bitopological space X , a subset A of X is said to be weakly (1, 2) S_p -separated from a subset B of X if there exists a (1, 2) S_p -open set G of X such that $A \subseteq G$ and $G \cap B = \phi$ or $A \cap (1, 2)S_p\text{-Cl}(B) = \phi$. It is shown in the following example.

Example 3.6. Let $X = \{a, b, c, d\}$. $\tau_1 = \{\phi, X, \{a, c\}, \{a, c, d\}\}$. $\tau_2 = \{\phi, X\}$. $\tau_1\tau_2 Cl = \{X, \phi, \{b, d\}, \{b\}\}$. (1, 2) $SO(X) = \{\phi, X, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. (1, 2) $PCL(X) = \{X, \phi, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{d\}, \{c\}, \{b\}, \{a\}\}$. (1, 2) $S_p O(X) = \{\phi, X, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. (1, 2) $S_p CL(X) = \{X, \phi, \{b, d\}, \{d\}, \{b\}\}$. Let $A = \{b\}$, $B = \{d\}$ and $G = \{a, b, c\} \in (1, 2)S_p O(X)$. Here $A \subseteq G$ and $G \cap B = \phi$ or $A \cap (1, 2)S_p\text{-Cl}(B) = \phi$. Hence a subset A of X is weakly (1, 2) S_p -separated from a subset B of X .

Theorem 3.7. Suppose X is a (1, 2) S_p -space and $A, B \in (1, 2) S_p LC(X)$. If A and B are weakly (1, 2) S_p -separated, then $A \cup B \in (1, 2) S_p LC(X)$.

Proof: Assume $A, B \in (1, 2) S_p LC(X)$ and let $A = G \cap (1, 2) S_p\text{-Cl}(A)$ and $B = E \cap (1, 2) S_p\text{-Cl}(B)$, where G and E are (1, 2) S_p -open sets of X . Put $U = G \cap [X - (1, 2) S_p\text{-Cl}(B)]$ and $V = E \cap [X - (1, 2) S_p\text{-Cl}(A)]$. Then $U \cap (1, 2)S_p\text{-Cl}(A) = G \cap [X - (1, 2) S_p\text{-Cl}(B)] \cap (1, 2) S_p\text{-Cl}(A) = [G \cap X - (1, 2) S_p\text{-Cl}(A)] \cap [X - (1, 2)S_p\text{-Cl}(B)] = A \cap [X - (1, 2) S_p\text{-Cl}(B)] = A$.

Similarly, $V \cap (1, 2) S_p\text{-Cl}(B) = [E \cap (X - (1, 2) S_p\text{-Cl}(\{A\}))] \cap (1, 2) S_p\text{-Cl}(\{A\})$

$= [E \cap [X - (1, 2) S_p\text{-Cl}(\{B\})]] \cap [(X - (1, 2) S_p\text{-Cl}(\{A\}))] = B \cap [(X - (1, 2) S_p\text{-Cl}(\{A\}))] = B$ and hence $U \cap (1, 2) S_p\text{-Cl}(B) = V \cap (1, 2) S_p\text{-Cl}(A) = \phi$. Moreover U and V are $(1, 2) S_p$ -open in X . By Remark 2.4, $(U \cup V)$ is also $(1, 2) S_p$ -open. Then $(U \cup V) \cap (1, 2) S_p\text{-Cl}(A \cup B) = [U \cap (1, 2) S_p\text{-Cl}(A)] \cup [U \cap (1, 2) S_p\text{-Cl}(B)] \cup [V \cap (1, 2) S_p\text{-Cl}(A)] \cup [V \cap (1, 2) S_p\text{-Cl}(B)] = A \cup B$. Hence $A \cup B$ is $(1, 2) S_p$ -locally closed set.

Lemma 3.8. In view of Lemma 3.4 and Definition 3.5, we have for x, y in X of a bitopological space,

- (i) $(1, 2) S_p\text{-Cl}(\{x\}) = \{y: y \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.
- (ii) $(1, 2) S_p\text{-Ker}(\{x\}) = \{y: x \text{ is not weakly } (1, 2) S_p\text{-separated from } y\}$.

Definition 3.9. For any point x of a bitopological space X ,

- (i) The derived set of x is denoted by $(1, 2) S_p\text{-d}(\{x\})$ and is defined to be the set $(1, 2) S_p\text{-d}(\{x\}) = (1, 2) S_p\text{-Cl}(\{x\}) - \{x\} = \{y: y \neq x \text{ and } y \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.
- (ii) the shell of a point x of X is denoted by $(1, 2) S_p\text{-shl}(\{x\})$ and is defined to be the set $(1, 2) S_p\text{-shl}(\{x\}) = (1, 2) S_p\text{-Ker}(\{x\}) - \{x\} = \{y: y \neq x \text{ and } x \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.

Definition 3.10: Let X be a bitopological space. Then we define

- (i) $(1, 2) S_p\text{-N-D} = \{x: x \in X \text{ and } (1, 2) S_p\text{-d}(\{x\}) = \phi\}$.
- (ii) $(1, 2) S_p\text{-N-Shl} = \{x: x \in X \text{ and } (1, 2) S_p\text{-shl}(\{x\}) = \phi\}$.
- (iii) $(1, 2) S_p\text{-N} = (1, 2) S_p\text{-Cl}(\{x\}) \cap (1, 2) S_p\text{-Ker}(\{x\})$.

Theorem 3.11. Let $x, y \in X$. Then the following conditions hold good:

- (i) $y \in (1, 2) S_p\text{-Ker}(\{x\})$ if and only if $x \in (1, 2) S_p\text{-Cl}(\{y\})$.
- (ii) $y \in (1, 2) S_p\text{-shl}(\{x\})$ if and only if $x \in (1, 2) S_p\text{-d}(\{y\})$.
- (iii) $y \in (1, 2) S_p\text{-Cl}(\{x\})$ implies $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{x\})$.
- (iv) $y \in (1, 2) S_p\text{-Ker}(\{x\})$ implies $(1, 2) S_p\text{-Ker}(\{y\}) \subseteq (1, 2) S_p\text{-Ker}(\{x\})$.

Proof: The proof of (i) and (ii) are obvious from the Lemma 3.8.

(iii) Let $z \in (1, 2) S_p\text{-Cl}(\{y\})$. Then z is not weakly $(1, 2) S_p$ -separated from y which implies there exists a $(1, 2) S_p$ -open set G containing x such that $G \cap \{y\} \neq \phi$. Hence $y \in G$ and by assumption $G \cap \{x\} \neq \phi$. Hence z is not weakly $(1, 2) S_p$ -separated from x which implies $z \in (1, 2) S_p\text{-Cl}(\{x\})$. Therefore $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{x\})$.

(iv) Let $z \in (1, 2) S_p\text{-Ker}(\{y\})$. Then y is not weakly $(1, 2) S_p$ -separated from z which implies $y \in (1, 2) S_p\text{-Cl}(\{z\})$. Hence $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{z\})$. By assumption $y \in (1, 2) S_p\text{-Ker}(\{x\})$ that implies $x \in (1, 2) S_p\text{-Cl}(\{y\})$. Then $(1, 2) S_p\text{-Cl}(\{x\}) \subseteq (1, 2) S_p\text{-Cl}(\{y\})$. Ultimately, $(1, 2) S_p\text{-Cl}(\{x\}) \subseteq (1, 2) S_p\text{-Cl}(\{z\})$ which implies $x \in (1, 2) S_p\text{-Cl}(\{z\})$, that is $z \in (1, 2) S_p\text{-Ker}(\{x\})$. Therefore $(1, 2) S_p\text{-Ker}(\{y\}) \subseteq (1, 2) S_p\text{-Ker}(\{x\})$.

Theorem 3.12. Let x, y be in X . Then

(i) for every $x \in X$, $(1,2)S_p\text{-shl}(\{x\})$ is a degenerate set if and only if for all $x, y \in X$, $x \neq y$, $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) = \emptyset$.

(ii) for every $x \in X$, $(1,2)S_p\text{-d}(\{x\})$ is a degenerate set if and only if for every $x, y \in X$, $x \neq y$, $(1,2)S_p\text{-shl}(\{x\}) \cap (1,2)S_p\text{-shl}(\{y\}) = \emptyset$.

Proof. (i) Let $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) \neq \emptyset$. Then there exists a $z \in X$ such that $z \in (1,2)S_p\text{-d}(\{x\})$ and that $z \in (1,2)S_p\text{-d}(\{y\})$. Then $z \neq y \neq x$ and $z \in (1,2)S_p\text{-Cl}(\{x\})$ and $z \in (1,2)S_p\text{-Cl}(\{y\})$. That is $x, y \in (1,2)S_p\text{-Ker}(\{z\})$. Hence $(1,2)S_p\text{-Ker}(\{z\})$ implies $(1,2)S_p\text{-shl}(\{z\})$ is not a degenerate set, which is a contradiction. Hence $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) = \emptyset$.

Also, let $x, y \in (1,2)S_p\text{-shl}(\{z\})$ is not a degenerate set. Then $x \neq y \neq z$ and $x, y \in (1,2)S_p\text{-Ker}(\{z\})$. Then z is an element of both $(1,2)S_p\text{-Cl}(\{x\})$ and $(1,2)S_p\text{-Cl}(\{y\})$ which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Cl}(\{y\}) \neq \emptyset$ which is a contradiction. Hence $(1,2)S_p\text{-shl}(\{z\})$ is a degenerate set.

The proof of (ii) is similar that of (i).

Theorem 3.13. If $y \in (1,2)S_p\text{-}\langle x \rangle$, then $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$,

Proof. If $y \in (1,2)S_p\text{-}\langle x \rangle$, then $y \in (1,2)S_p\text{-Cl}(\{x\})$ and $y \in (1,2)S_p\text{-Ker}(\{x\})$ and by Theorem 3.10, $(1,2)S_p\text{-Cl}(\{y\}) \subseteq (1,2)S_p\text{-Cl}(\{x\})$ and $(1,2)S_p\text{-Ker}(\{y\}) \subseteq (1,2)S_p\text{-Ker}(\{x\})$. which implies $(1,2)S_p\text{-Cl}(\{y\}) \cap (1,2)S_p\text{-Ker}(\{y\}) \subseteq (1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Ker}(\{x\})$. Thus $(1,2)S_p\text{-}\langle y \rangle \subseteq (1,2)S_p\text{-}\langle x \rangle$. Now, $y \in (1,2)S_p\text{-Cl}(\{x\})$ implies $x \in (1,2)S_p\text{-Ker}(\{y\})$ and $y \in (1,2)S_p\text{-Ker}(\{x\})$ which implies $x \in (1,2)S_p\text{-Cl}(\{y\})$. Which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Ker}(\{x\}) \subseteq (1,2)S_p\text{-Ker}(\{y\}) \cap (1,2)S_p\text{-Cl}(\{y\})$. Thus $(1,2)S_p\text{-}\langle x \rangle \subseteq (1,2)S_p\text{-}\langle y \rangle$. Hence $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Theorem 3.14. For all $x, y \in X$, either $(1,2)S_p\text{-}\langle x \rangle \cap (1,2)S_p\text{-}\langle y \rangle = \emptyset$ or $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Proof. Let $(1,2)S_p\text{-}\langle x \rangle \cap (1,2)S_p\text{-}\langle y \rangle \neq \emptyset$, then there exists $z \in X$ such that $z \in (1,2)S_p\text{-}\langle x \rangle$ and $z \in (1,2)S_p\text{-}\langle y \rangle$. By theorem 3.13, $(1,2)S_p\text{-}\langle z \rangle = (1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$. Hence $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Theorem 3.15. For any two points x, y in X , the following statements are equivalent.

(i) $(1,2)S_p\text{-Ker}(\{x\}) \neq (1,2)S_p\text{-Ker}(\{y\})$.

(ii) $(1,2)S_p\text{-Cl}(\{x\}) \neq (1,2)S_p\text{-Cl}(\{y\})$

Proof. (i) $\Rightarrow$ (ii) Let us assume that $(1,2)S_p\text{-Ker}(\{x\}) \neq (1,2)S_p\text{-Ker}(\{y\})$. Then there exists a point $z \in (1,2)S_p\text{-Ker}(\{x\})$ but $z \notin (1,2)S_p\text{-Ker}(\{y\})$. As $z \in (1,2)S_p\text{-Ker}(\{x\})$, $x \in (1,2)S_p\text{-Cl}(\{z\})$ and $(1,2)S_p\text{-Cl}(\{x\}) \subseteq (1,2)S_p\text{-Cl}(\{z\})$. Also since $z \notin (1,2)S_p\text{-Ker}(\{y\})$. By Lemma 3.4, $(1,2)S_p\text{-Cl}(\{z\}) \cap \{y\} = \emptyset$ which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap \{y\} = \emptyset$ and y is weakly $(1,2)S_p$ -separated from x , that is $y \notin (1,2)S_p\text{-Cl}(\{x\})$. Hence $(1,2)S_p\text{-Cl}(\{y\}) \neq (1,2)S_p\text{-Cl}(\{x\})$.

(ii) $\Rightarrow$ (i) Suppose $(1,2)S_p\text{-Cl}(\{x\}) \neq (1,2)S_p\text{-Cl}(\{y\})$. Then there exists a point

$z \in (1, 2)S_p\text{-Cl}(\{x\})$ but $z \notin (1, 2)S_p\text{-Cl}(\{y\})$. Also a $(1, 2)S_p$ -open set containing z and x but not y implies $y \notin (1, 2)S_p\text{-Ker}(\{x\})$. Hence $(1, 2)S_p\text{-Ker}(\{y\}) \neq (1, 2)S_p\text{-Ker}(\{x\})$.

4. Conclusions

In this paper, the new characterization of $(1, 2)S_p$ -kernel was introduced and some of its properties are discussed. Later on Research be reached out with certain applications.

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Translations of Bipolar Valued Multi Fuzzy Subnearring of a Nearing

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Abstract

In this paper, some translations of bipolar valued multi fuzzy subnearring of a nearing are introduced and using these translations, some theorems are stated and proved.

Key Words. Bipolar valued fuzzy subset, bipolar valued multi fuzzy subset, bipolar valued multi fuzzy subnearring, translations, intersection.

Subject Classification. 97H40, 03B52, 03E72³.

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1. Introduction

In 1965, Zadeh [9] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then, it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc.

W. R. Zhang [10, 11] introduced an extension of fuzzy sets named bipolar valued fuzzy sets in 1994 and bipolar valued fuzzy set was developed by Lee [2, 3]. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0, 1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter property. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [3].

Vasanthakandasamy. W. B [7] introduced the basic idea about the fuzzy group and fuzzy bigroup. M.S. Anitha et.al [1] introduced the bipolar valued fuzzy subgroup. Sheena. K. P and K. Uma Devi [6] have introduced the bipolar valued fuzzy subgroup of a bigroup. Shanthi. V.K and G. Shyamala [5] have introduced the bipolar valued multi fuzzy subgroups of a group.

Yasodara. S, KE. Sathappan [8] defined the bipolar valued multi fuzzy subsemirings of a semiring. Bipolar valued multi fuzzy subnearing of a nearing has been introduced by S. Muthukumaran and B. Anandh [4]. In this paper, the concept of translations of bipolar valued multi fuzzy subnearing of a nearing is introduced and established some results.

Definition 1.1. ([11]) A bipolar valued fuzzy set (BVFS) B in X is defined as an object of the form $B = \{ \langle x, B^+(u), B^-(u) \rangle / x \in X \}$, where $B^+: X \rightarrow [0, 1]$ and $B^-: X \rightarrow [-1, 0]$. The positive membership degree $B^+(u)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set B and the negative membership degree $B^-(u)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set B .

Definition 1.2. ([8]) A bipolar valued multi fuzzy set (BVMFS) A in X is defined as an object of the form $B = \{ \langle x, B_1^+(u), B_2^+(u), \dots, B_n^+(u), B_1^-(u), B_2^-(u), \dots, B_n^-(u) \rangle / x \in X \}$, where $B_i^+: X \rightarrow [0, 1]$ and $B_i^-: X \rightarrow [-1, 0]$, for all i . The positive membership degrees $B_i^+(u)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar valued multi fuzzy set B and the negative membership degrees $B_i^-(u)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued multi fuzzy set B .

Translations of Bipolar Valued Multi Fuzzy Subnearring of a Nearring

Definition 1.3. ([4]) Let $(N, +, \cdot)$ be a nearring. A BVMFS B of N is said to be a bipolar valued multi fuzzy subnearring of N (BVMFSNR) if the following conditions are satisfied, for all i ,

- (i) $B_i^+(u-v) \geq \min \{B_i^+(u), B_i^+(v)\}$
- (ii) $B_i^+(uv) \geq \min \{B_i^+(u), B_i^+(v)\}$
- (iii) $B_i^-(u-v) \leq \max \{B_i^-(u), B_i^-(v)\}$
- (iv) $B_i^-(uv) \leq \max \{B_i^-(u), B_i^-(v)\}, \forall u, v \in N$.

Definition 1.4. ([8]) Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^- \rangle$ and $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ be two bipolar valued multi fuzzy subsets with degree n of a set X . We define the following relations and operations:

- (i) $A \subset B$ if and only if for all i , $A_i^+(u) \leq B_i^+(u)$ and $A_i^-(u) \geq B_i^-(u), \forall u \in X$.
- (ii) $A \cap B = \{ \langle u, \min(A_1^+(u), B_1^+(u)), \min(A_2^+(u), B_2^+(u)), \dots, \min(A_n^+(u), B_n^+(u)), \max(A_1^-(u), B_1^-(u)), \max(A_2^-(u), B_2^-(u)), \dots, \max(A_n^-(u), B_n^-(u)) \rangle / u \in X \}$.

Definition 1.5. Let $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_n^- \rangle$ be a bipolar valued multi fuzzy subnearring of a nearring R and $s \in R$. Then the pseudo bipolar valued multi fuzzy coset $(sC)^p = \langle (sC_1^+)^{p_1^+}, (sC_2^+)^{p_2^+}, \dots, (sC_n^+)^{p_n^+}, (sC_1^-)^{p_1^-}, (sC_2^-)^{p_2^-}, \dots, (sC_n^-)^{p_n^-} \rangle$ is defined by $(sC_i^+)^{p_i^+}(a) = p_i^+(s) C_i^+(a)$ and $(sC_i^-)^{p_i^-}(a) = -p_i^-(s) C_i^-(a)$, for all i and every $a \in R$ and $p \in P$, where P is a collection of bipolar valued multi fuzzy subsets of R .

Definition 1.6. [8] Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X . Then the height $H(A) = \langle H(A_1^+), H(A_2^+), \dots, H(A_n^+), H(A_1^-), H(A_2^-), \dots, H(A_n^-) \rangle$ is defined for all i as $H(A_i^+) = \sup A_i^+(x)$ for all $x \in X$ and $H(A_i^-) = \inf A_i^-(x)$ for all $x \in X$.

Definition 1.7. [6] Let $A = \langle A_1^+, A_2^+, \dots, A_i^+, A_1^-, A_2^-, \dots, A_i^- \rangle$ be a bipolar valued multi fuzzy subset of X . Then ${}^0A = \langle {}^0A_1^+, {}^0A_2^+, \dots, {}^0A_n^+, {}^0A_1^-, {}^0A_2^-, \dots, {}^0A_n^- \rangle$ is defined for all i as ${}^0A_i^+(x) = A_i^+(x) H(A_i^+)$ for all $x \in X$ and ${}^0A_i^-(x) = -A_i^-(x) H(A_i^-)$ for all $x \in X$.

Definition 1.8. [6] Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X . Then ${}^\Delta A = \langle {}^\Delta A_1^+, {}^\Delta A_2^+, \dots, {}^\Delta A_n^+, {}^\Delta A_1^-, {}^\Delta A_2^-, \dots, {}^\Delta A_n^- \rangle$ is defined for all i as ${}^\Delta A_i^+(x) = A_i^+(x) / H(A_i^+)$ for all $x \in X$ and ${}^\Delta A_i^-(x) = -A_i^-(x) / H(A_i^-)$ for all $x \in X$.

Definition 1.9. [6] Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X . Then ${}^\oplus A = \langle {}^\oplus A_1^+, {}^\oplus A_2^+, \dots, {}^\oplus A_n^+, {}^\oplus A_1^-, {}^\oplus A_2^-, \dots, {}^\oplus A_n^- \rangle$ is defined for all i as ${}^\oplus A_i^+(x) = A_i^+(x) + 1 - H(A_i^+)$ for all $x \in X$ and ${}^\oplus A_i^-(x) = A_i^-(x) - 1 - H(A_i^-)$ for all $x \in X$.

Definition 1.10. [6] Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^- \rangle$ be a bipolar valued multi fuzzy subset of X . Then A is called bipolar valued normal multi fuzzy subset of X if $H(A_i^+) = 1$ and $H(A_i^-) = -1$ for all i .

2. Properties

Theorem 2.1. ([4]) If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_n^- \rangle$ are two bipolar valued multi fuzzy subnearrings with degree n of a nearring R , then their intersection $B \cap C$ is a bipolar valued multi fuzzy Subnearring of R .

Theorem 2.2. Let $K = \langle K_1^+, K_2^+ \dots K_n^+, K_1^-, K_2^- \dots K_n^- \rangle$ be a bipolar valued multi fuzzy subnearring with degree n of a nearring R . Then the pseudo bipolar valued multi fuzzy coset $(a_1 K)^m$ is a bipolar valued multi fuzzy subnearring of the nearring R , for every a_1 in R and m in M , where M is a collection of bipolar valued multi fuzzy subset of R .

Proof. Let b_1, c_1 in R and $a_1 \in R$. For each i , then $(a_1 K_i^+)^{m_i^+} (b_1 - c_1) = m_i^+(a_1) K_i^+(b_1 - c_1) \geq m_i^+(a_1) \min\{K_i^+(b_1), K_i^+(c_1)\} = \min\{m_i^+(a_1) K_i^+(b_1), m_i^+(a_1) K_i^+(c_1)\} = \min\{(a_1 K_i^+)^{m_i^+}(b_1), (a_1 K_i^+)^{m_i^+}(c_1)\}$. Therefore $(a_1 K_i^+)^{m_i^+} (b_1 - c_1) \geq \min\{(a_1 K_i^+)^{m_i^+}(b_1), (a_1 K_i^+)^{m_i^+}(c_1)\}$, for $b_1, c_1 \in R$. And for each i , then $(a_1 K_i^+)^{m_i^+} (b_1 c_1) = m_i^+(a_1) K_i^+(b_1 c_1) \geq m_i^+(a_1) \min\{K_i^+(b_1), K_i^+(c_1)\} = \min\{m_i^+(a_1) K_i^+(b_1), m_i^+(a_1) K_i^+(c_1)\} = \min\{(a_1 K_i^+)^{m_i^+}(b_1), (a_1 K_i^+)^{m_i^+}(c_1)\}$. Therefore $(a_1 K_i^+)^{m_i^+} (b_1 c_1) \geq \min\{(a_1 K_i^+)^{m_i^+}(b_1), (a_1 K_i^+)^{m_i^+}(c_1)\}$, for all $b_1, c_1 \in R$. For each i , $(a_1 K_i^-)^{m_i^-} (b_1 - c_1) = m_i^-(a_1) K_i^-(b_1 - c_1) \leq m_i^-(a_1) \max\{K_i^-(b_1), K_i^-(c_1)\} = \max\{m_i^-(a_1) K_i^-(b_1), m_i^-(a_1) K_i^-(c_1)\} = \max\{(a_1 K_i^-)^{m_i^-}(b_1), (a_1 K_i^-)^{m_i^-}(c_1)\}$. Therefore $(a_1 K_i^-)^{m_i^-} (b_1 - c_1) \leq \max\{(a_1 K_i^-)^{m_i^-}(b_1), (a_1 K_i^-)^{m_i^-}(c_1)\}$, for $b_1, c_1 \in R$. Also for each i , then $(a_1 K_i^-)^{m_i^-} (b_1 c_1) = m_i^-(a_1) K_i^-(b_1 c_1) \leq m_i^-(a_1) \max\{K_i^-(b_1), K_i^-(c_1)\} = \max\{m_i^-(a_1) K_i^-(b_1), m_i^-(a_1) K_i^-(c_1)\} = \max\{(a_1 K_i^-)^{m_i^-}(b_1), (a_1 K_i^-)^{m_i^-}(c_1)\}$. Therefore $(a_1 K_i^-)^{m_i^-} (b_1 c_1) \leq \max\{(a_1 K_i^-)^{m_i^-}(b_1), (a_1 K_i^-)^{m_i^-}(c_1)\}$, for all $b_1, c_1 \in R$. Hence $(a_1 K)^m$ is a bipolar valued multi fuzzy subnearring of the nearring R .

Theorem 2.3. If $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring with degree n of a nearring R , then ${}^\oplus K = \langle {}^\oplus K_1^+, {}^\oplus K_2^+, \dots, {}^\oplus K_n^+, {}^\oplus K_1^-, {}^\oplus K_2^-, \dots, {}^\oplus K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring of R .

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Proof. Let a_1, b_1 in R . For each i , $\oplus K_i^+(a_1 - b_1) = K_i^+(a_1 - b_1) + 1 - H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\} + 1 - H(K_i^+) = \min\{K_i^+(a_1) + 1 - H(K_i^+), K_i^+(b_1) + 1 - H(K_i^+)\} = \min\{\oplus K_i^+(a_1), \oplus K_i^+(b_1)\}$ implies $\oplus K_i^+(a_1 - b_1) \geq \min\{\oplus K_i^+(a_1), \oplus K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , $\oplus K_i^+(a_1 b_1) = K_i^+(a_1 b_1) + 1 - H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\} + 1 - H(K_i^+) = \min\{K_i^+(a_1) + 1 - H(K_i^+), K_i^+(b_1) + 1 - H(K_i^+)\} = \min\{\oplus K_i^+(a_1), \oplus K_i^+(b_1)\}$ which implies $\oplus K_i^+(a_1 b_1) \geq \min\{\oplus K_i^+(a_1), \oplus K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. Also for all i , $\oplus K_i^-(a_1 - b_1) = K_i^-(a_1 - b_1) - 1 - H(K_i^-) \leq \max\{K_i^-(a_1), K_i^-(b_1)\} - 1 - H(K_i^-) = \max\{K_i^-(a_1) - 1 - H(K_i^-), K_i^-(b_1) - 1 - H(K_i^-)\} = \max\{\oplus K_i^-(a_1), \oplus K_i^-(b_1)\}$ implies $\oplus K_i^-(a_1 - b_1) \leq \max\{\oplus K_i^-(a_1), \oplus K_i^-(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , $\oplus K_i^-(a_1 b_1) = K_i^-(a_1 b_1) - 1 - H(K_i^-) \leq \max\{K_i^-(a_1), K_i^-(b_1)\} - 1 - H(K_i^-) = \max\{K_i^-(a_1) - 1 - H(K_i^-), K_i^-(b_1) - 1 - H(K_i^-)\} = \max\{\oplus K_i^-(a_1), \oplus K_i^-(b_1)\}$ implies $\oplus K_i^-(a_1 b_1) \leq \max\{\oplus K_i^-(a_1), \oplus K_i^-(b_1)\}$ for all $a_1, b_1 \in R$. Hence $\oplus K$ is a bipolar valued multi fuzzy subnearring of R .

Corollary 2.4. Let $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring with degree n of a nearring R .

- (i) If $e \in R$, then for each i , $\oplus K_i^+(e) = 1$ and $\oplus K_i^-(e) = -1$, where e is an Identity element of R ;
- (ii) For each i , there exists $e \in R$ such that $K_i^+(e) = 1$ and $K_i^-(e) = -1$ if and only if $\oplus K_i^+(a_1) = K_i^+(a_1)$ and $\oplus K_i^-(a_1) = K_i^-(a_1)$ for all $a_1 \in R$;
- (iii) For each i , there exists $a_1 \in R$ such that $K_i^+(a_1) = K_i^+(e)$ and $K_i^-(a_1) = K_i^-(e)$ if and only if $\oplus K_i^+(a_1) = 1$ and $\oplus K_i^-(a_1) = -1$, for some $a_1 \in R$;
- (iv) For each i , if there exists $a_1 \in R$ such that $K_i^+(a_1) = 1$ and $K_i^-(a_1) = -1$, then $\oplus K_i^+(a_1) = 1$ and $\oplus K_i^-(a_1) = -1$;
- (v) For each i , if $K_i^+(e) = 1$, $K_i^-(e) = -1$, $\oplus K_i^+(a_1) = 0$ and $\oplus K_i^-(a_1) = 0$, then $K_i^+(a_1) = 0$, $K_i^-(a_1) = 0$;
- (vi) $\oplus (\oplus K) = \oplus K$,
- (vii) $\oplus K$ is a bipolar valued normal multi fuzzy subnearring of R containing K ;
- (viii) K is a bipolar valued normal multi fuzzy subnearring of R if and only if $\oplus K = K$;
- (ix) If there exists a bipolar valued multi fuzzy subnearring P of R satisfying $\oplus P \subseteq K$; then K is a bipolar valued normal fuzzy subnearring of R ;
- (x) If there exists a bipolar valued multi fuzzy subnearring P of R satisfying $\oplus P \subseteq K$, then $\oplus K = K$.

Proof. (i), (ii), (iii), (iv), (v) and (x) are trivial. (vi) Let $a_1, b_1 \in R$. For each i , then $\oplus (\oplus K_i^+) (a_1) = \oplus K_i^+(a_1) + 1 - \oplus K_i^+(e) = \{K_i^+(e) + 1 - K_i^+(e)\} + 1 - \{K_i^+(e) + 1 - K_i^+(e)\} = K_i^+(a_1) + 1 - K_i^+(e) = \oplus K_i^+(a_1)$. Also for each i , $\oplus (\oplus K_i^-) (a_1) = \oplus K_i^-(a_1) - 1 - \oplus K_i^-(e) = \{K_i^-(e) - 1 - K_i^-(e)\} - 1 - \{K_i^-(e) - 1 - K_i^-(e)\} = K_i^-(a_1) - 1 - K_i^-(e) = \oplus K_i^-(a_1)$. Hence $\oplus (\oplus K) = \oplus K$. (vii) Let $e \in R$. Clearly $K_i^+(e) = 1$ and $K_i^-(e) = -1$. Thus $\oplus K$ is a bipolar valued normal multi fuzzy subnearring of R and $K \subseteq \oplus K$. (viii) If $K^* = K$, then it is obvious that K is a bipolar valued normal multi fuzzy subnearring of R . Assume that K is a bipolar valued normal multi fuzzy subnearring of R . Let $a_1 \in R$. Then $\oplus K_i^+(a_1) = K_i^+(a_1) + 1 - K_i^+(e) = K_i^+(a_1)$ and

$\oplus K_i^-(a_1) = K_i^-(a_1) - 1 - K_i^-(e) = K_i^-(a_1)$. Hence $\oplus K = K$. (ix) Suppose there exists a bipolar valued multi fuzzy subnearring P of H such that $\oplus P \subseteq K$. Then $1 = \oplus P_i^+(e) \leq K_i^+(e)$ and $-1 = \oplus P_i^-(e) \geq K_i^-(e)$. Hence $K_i^+(e) = 1$ and $K_i^-(e) = -1$.

Theorem 2.5. If $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring with degree n of a nearring R , then ${}^0K = \langle {}^0K_1^+, {}^0K_2^+, \dots, {}^0K_n^+, {}^0K_1^-, {}^0K_2^-, \dots, {}^0K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring of R .

Proof. Let a_1, b_1 in R . For each i , ${}^0K_i^+(a_1 - b_1) = K_i^+(a_1 - b_1)H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\}H(K_i^+) = \min\{K_i^+(a_1)H(K_i^+), K_i^+(b_1)H(K_i^+)\} = \min\{{}^0K_i^+(a_1), {}^0K_i^+(b_1)\}$ implies ${}^0K_i^+(a_1 - b_1) \geq \min\{{}^0K_i^+(a_1), {}^0K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , ${}^0K_i^+(a_1 b_1) = K_i^+(a_1 b_1)H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\}H(K_i^+) = \min\{K_i^+(a_1)H(K_i^+), K_i^+(b_1)H(K_i^+)\} = \min\{{}^0K_i^+(a_1), {}^0K_i^+(b_1)\}$. Thus ${}^0K_i^+(a_1 b_1) \geq \min\{{}^0K_i^+(a_1), {}^0K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. Also for all i , ${}^0K_i^-(a_1 - b_1) = -K_i^-(a_1 - b_1)H(K_i^-) \leq -\max\{K_i^-(a_1), K_i^-(b_1)\}H(K_i^-) = \max\{-K_i^-(a_1)H(K_i^-), -K_i^-(b_1)H(K_i^-)\} = \max\{{}^0K_i^-(a_1), {}^0K_i^-(b_1)\}$ implies ${}^0K_i^-(a_1 - b_1) \leq \max\{{}^0K_i^-(a_1), {}^0K_i^-(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , ${}^0K_i^-(a_1 b_1) = -K_i^-(a_1 b_1)H(K_i^-) \leq -\max\{K_i^-(a_1), K_i^-(b_1)\}H(K_i^-) = \max\{-K_i^-(a_1)H(K_i^-), -K_i^-(b_1)H(K_i^-)\} = \max\{{}^0K_i^-(a_1), {}^0K_i^-(b_1)\}$. Therefore ${}^0K_i^-(a_1 b_1) \leq \max\{{}^0K_i^-(a_1), {}^0K_i^-(b_1)\}$ for all $a_1, b_1 \in R$. Hence ${}^0K$ is a bipolar valued multi fuzzy subnearring of R .

Theorem 2.6. If $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring with degree n of a nearring R , then ${}^\Delta K = \langle {}^\Delta K_1^+, {}^\Delta K_2^+, \dots, {}^\Delta K_n^+, {}^\Delta K_1^-, {}^\Delta K_2^-, \dots, {}^\Delta K_n^- \rangle$ is a bipolar valued multi fuzzy subnearring of R .

Proof. Let a_1, b_1 in R . For each i , then ${}^\Delta K_i^+(a_1 - b_1) = K_i^+(a_1 - b_1) / H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\} / H(K_i^+) = \min\{K_i^+(a_1) / H(K_i^+), K_i^+(b_1) / H(K_i^+)\} = \min\{{}^\Delta K_i^+(a_1), {}^\Delta K_i^+(b_1)\}$ implies ${}^\Delta K_i^+(a_1 - b_1) \geq \min\{{}^\Delta K_i^+(a_1), {}^\Delta K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , ${}^\Delta K_i^+(a_1 b_1) = K_i^+(a_1 b_1) / H(K_i^+) \geq \min\{K_i^+(a_1), K_i^+(b_1)\} / H(K_i^+) = \min\{K_i^+(a_1) / H(K_i^+), K_i^+(b_1) / H(K_i^+)\} = \min\{{}^\Delta K_i^+(a_1), {}^\Delta K_i^+(b_1)\}$. Therefore ${}^\Delta K_i^+(a_1 b_1) \geq \min\{{}^\Delta K_i^+(a_1), {}^\Delta K_i^+(b_1)\}$ for all $a_1, b_1 \in R$. Also for all i , ${}^\Delta K_i^-(a_1 - b_1) = -K_i^-(a_1 - b_1) / H(K_i^-) \leq -\max\{K_i^-(a_1), K_i^-(b_1)\} / H(K_i^-) = \max\{-K_i^-(a_1) / H(K_i^-), -K_i^-(b_1) / H(K_i^-)\} = \max\{{}^\Delta K_i^-(a_1), {}^\Delta K_i^-(b_1)\}$ implies ${}^\Delta K_i^-(a_1 - b_1) \leq \max\{{}^\Delta K_i^-(a_1), {}^\Delta K_i^-(b_1)\}$ for all $a_1, b_1 \in R$. And for all i , ${}^\Delta K_i^-(a_1 b_1) = -K_i^-(a_1 b_1) / H(K_i^-) \leq -\max\{K_i^-(a_1), K_i^-(b_1)\} / H(K_i^-) = \max\{-K_i^-(a_1) / H(K_i^-), -K_i^-(b_1) / H(K_i^-)\} = \max\{{}^\Delta K_i^-(a_1), {}^\Delta K_i^-(b_1)\}$. Therefore ${}^\Delta K_i^-(a_1 b_1) \leq \max\{{}^\Delta K_i^-(a_1), {}^\Delta K_i^-(b_1)\}$, for all $a_1, b_1 \in R$. Hence ${}^\Delta K$ is a bipolar valued multi fuzzy subnearring of R .

Corollary 2.7. Let $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ be a bipolar valued multi fuzzy subnearring with degree n of a nearring R .

- (i) If for each i , $H(K_i^+) < 1$, then ${}^0K_i^+ < K_i^+$;
- (ii) If for each i , $H(K_i^-) > -1$, then ${}^0K_i^- > K_i^-$;
- (iii) If for each i , $H(K_i^+) < 1$ and $H(K_i^-) > -1$, then ${}^0K < K$;

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- (iv) If for each i , $H(K_i^+) < 1$, then ${}^\Delta K_i^+ > K_i^+$;
- (v) If for each i , $H(K_i^-) > -1$, then ${}^\Delta K_i^- < K_i^-$;
- (vi) If for each i , $H(K_i^+) < 1$ and $H(K_i^-) > -1$, then ${}^\Delta K > K$;
- (vii) If for each i , $H(K_i^+) < 1$ and $H(K_i^-) > -1$, then ${}^\Delta K$ is a bipolar valued normal multi fuzzy subnearring of R .

Proof. (i), (ii), (iii), (iv), (v), (vi) and (vii) are trivial.

Corollary 2.8. If K is a bipolar valued normal multi fuzzy subnearring of a nearring R , then (i) ${}^0K = K$, (ii) ${}^\Delta K = K$.

Proof. The proof follows from Definitions 1.8, 1.9 and 1.11.

Theorem 2.9. Let $K = \langle K_1^+, K_2^+ \dots K_n^+, K_1^-, K_2^- \dots K_n^- \rangle$ be a bipolar valued multi fuzzy subnearring with degree n of a nearring R . If $(a_1K)^m$ and $(b_1K)^m$ are two pseudo bipolar valued multi fuzzy coset of K , then their intersection $(a_1K)^m \cap (b_1K)^m$ is also a bipolar valued multi fuzzy subnearring of the nearring R , for every $a_1, b_1 \in R$ and m in M , where M is a collection of bipolar valued multi fuzzy subset of R .

Proof. The Proof follows from the Theorem 2.1 and 2.2.

Theorem 2.10. Let $K = \langle K_1^+, K_2^+ \dots K_n^+, K_1^-, K_2^- \dots K_n^- \rangle$ be a bipolar valued multi fuzzy subnearring with degree n of a nearring R . If $(a_1K)^m$ and $(b_1K)^m$ are two pseudos bipolar valued multi fuzzy coset of K and $m(a_1) \leq m(b_1)$ or $m(a_1) \geq m(b_1)$, then their union $(a_1K)^m \cup (b_1K)^m$ is also a bipolar valued multi fuzzy subnearring of the nearring R , for every $a_1, b_1 \in R$ and m in M , where M is a collection of bipolar valued multi fuzzy subset of R .

Proof. The proof follows from the Theorem 2.2.

Theorem 2.11. Let $K = \langle K_1^+, K_2^+, \dots, K_n^+, K_1^-, K_2^-, \dots, K_n^- \rangle$ be a bipolar valued multi fuzzy subnearring with degree n of a nearring R . Then K is a bipolar valued multi fuzzy subnearring of R if and only if each (K_i^+, K_i^-) is a bipolar valued fuzzy subnearring of R .

Proof. Let a_1, b_1 in R . Suppose K is a bipolar valued multi fuzzy subnearring of R , for each i , $K_i^+(a_1 - b_1) \geq \min \{K_i^+(a_1), K_i^+(b_1)\}$, $K_i^+(a_1 b_1) \geq \min \{K_i^+(a_1), K_i^+(b_1)\}$, $K_i^-(a_1 - b_1) \leq \max \{K_i^-(a_1), K_i^-(b_1)\}$ and $K_i^-(a_1 b_1) \leq \max \{K_i^-(a_1), K_i^-(b_1)\}$. Hence each (K_i^+, K_i^-) is bipolar valued fuzzy subnearring of R . Conversely, assume that each (K_i^+, K_i^-) is bipolar valued fuzzy subnearring of R . As per the definition of bipolar valued multi fuzzy subnearring of R , K is a bipolar valued multi fuzzy subnearring of R .

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Inverse Domination Parameters of Jump Graph

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Abstract

Let $G = (V, E)$ be a connected graph. Let D be a minimum dominating set in G . If $V - D$ contains a dominating set D' of G , then D' is called an inverse dominating set with respect to D . The minimum cardinality of an inverse dominating set of G is called inverse domination number of G . In this article, we determine inverse domination parameters of jump graph of a graph.

Keywords: domination number, inverse domination number, non-split inverse domination number, connected inverse domination number, jump graph.

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1.Introduction

The *undirected graph* $G = (V, E)$ discussed in this paper is simple and connected. The *order* and *size* are denoted by n and m respectively. For basic graph theoretical reference, we refer [2]. If uv is an edge of G , then two vertices u and v are said to be adjacent. If $uv \in E(G)$, then u is v 's neighbour, and the set of v 's neighbours is denoted by $N(v)$. Vertex $v \in V$ has a degree of $\deg(v) = |N(v)|$. If $\deg(v) = n - 1$, a vertex v is referred to as a universal vertex.

Globally, the domination conception in graphs start off its root in 1850s along the concern of certain chess players. Domination has diverse function consists of the morphological analysis, social network theory, CCTV installation and the most generally argued is the computer network communication. The aforementioned network comprises communication links among a firm number of slots. The trouble is to prime a least number of location where transmitters are installed such that a network as a whole united by a direct communication link to the transmitter site. In alternative words, the issue is to locate the least dominant set in the graph that corresponds to this network.

V.R. Kulli and C. Sigarkanthi [10] were the first to propose the idea of an inverse domination number. A set D of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . If $V - D$ contains a dominating set D' of G , then D' is called an inverse dominating set with respect to D . The inverse domination number $\gamma'(G)$ of G is the minimum cardinality of an inverse dominating set of G .

The dominating set S of G is connected dominating set of G if induced sub graph $\langle S \rangle$ is connected. The connected domination number $\gamma_c(G)$ of G is referred to as a minimum cardinality of connected dominating set.

The dominating set S of G is a non-split dominating set of G if induced sub graph $\langle V - S \rangle$ is connected. The non-split domination number $\gamma_{ns}(G)$ of G is referred to as a minimum cardinality of non-split dominating set.[7]

The n -sunlet graph is a graph on $2n$ vertices is obtained by attaching n -pendant edges to the cycle C_n and it is denoted by S_n .

Let P_n be a path graph in n vertices. The comb graph is defined as $P_n \odot K_1$. It has $2n$ vertices and $2n - 1$ edges.

Fan graph F_n $n \geq 2$ determined by joining all vertices of a path P_n to a different vertex, called centre. Thus F_n has $n + 1$ vertices, such as $u, u_1, u_2, u_3, \dots, u_n$ and $(2n - 1)$ edges, such as $uu_i, 1 \leq i \leq n - 1$. [12]

The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G . We call the complement of line graph $L(G)$ as the jump graph $J(G)$ of G , found in [11]. The jump graph $J(G)$ of a graph G is the graph defined on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in G . Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G .

2. Main Results

Inverse domination number of jump graph of n -sunlet graph

Theorem 2.1. For the graph $G = J(S_n)$ ($n \geq 4$), $\gamma'(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of n -sunlet graph is $2n$. Then it has $2n$ edges. Let $G = J(S_n)$. The number of vertices of jump graph of the n -sunlet graph is $2n$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a inverse dominating set of G so that $\gamma'(G) = 2$. ■

Theorem 2.2. For the graph $G = J(S_n)$ ($n \geq 4$), $\gamma'_{ns}(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of n -sunlet graph is $2n$. Then it has $2n$ edges. Let $G = J(S_n)$. The number of vertices of jump graph of the n -sunlet graph is $2n$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_{ns}(G) = 2$. ■

Theorem 2.3. For the graph $G = J(S_n)$ ($n \geq 4$), $\gamma'_c(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of n -sunlet graph is $2n$. Then it has $2n$ edges. Let $G = J(S_n)$. The number of vertices of jump graph of the n -sunlet graph is $2n$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 2$. ■

Inverse domination number of jump graph of comb graph

Theorem 2.4. For the graph $G = J(P_n \odot K_1)$ ($n \geq 4$), $\gamma'(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of combgraph is $2n$. Then it has $2n - 1$ edges. Let $G = J(P_n \odot K_1)$. The number of vertices of jump graph of the comb graph is $2n - 1$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_{n-1}, u_1, u_2, u_3, \dots, u_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, v_1\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a inverse dominating set of G so that $\gamma'(G) = 2$. ■

Theorem 2.5. For the graph $G = J(P_n \odot K_1)$ ($n \geq 4$), $\gamma'_{ns}(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of comb graph is $2n$. Then it has $2n - 1$ edges. Let $G = J(P_n \odot K_1)$. The number of vertices of jump graph of the comb graph is $2n - 1$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_{n-1}, u_1, u_2, \dots, u_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, v_1\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a non-split inverse dominating

set of G so that $\gamma'_{ns}(G) = 2$. ■

Theorem 2.6. For the graph $G = J(P_n \odot K_1)$ ($n \geq 4$), $\gamma'_c(G) = 2$.

Proof. For $n \geq 4$, the number of vertices of comb graph is $2n$. Then it has $2n - 1$ edges. Let $G = J(P_n \odot K_1)$. The number of vertices of jump graph of the comb graph is $2n - 1$. Let the vertices of the graph is labeled as $\{v_1, v_2, \dots, v_{n-1}, u_1, u_2, \dots, u_n\}$, Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, v_1\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{u_2, u_4\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 2$. ■

Inverse domination number of jump graph of fan graph

Theorem 2.7. For fan graph $G = J(F_n)$ ($n \geq 5$), $\gamma'(G) = \begin{cases} 3 & \text{if } n = 3, 4 \\ 2 & \text{if } n \geq 5 \end{cases}$.

Proof. The number of vertices of fan graph is $n + 1$. Then it has $2n - 1$ edges. Let $G = J(F_n)$. The number of vertices of jump graph of the fan graph is $2n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}\}$, Since G contains no universal vertices, $\gamma(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(F_n)$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_1, v_3, v_4\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{v_2, u_2, u_3\}$. Then D' is a γ' -set of G . Since $\gamma(G) = 3$, we have $\gamma'(G) = 3$.

Let $n = 4$. Let $D = \{v_1, v_4\}$. Then D is a γ -set of G so that $\gamma(G) = 2$. Let $D' = \{v_2, v_3, u_3\}$. Then D' is a γ' -set of G and so $\gamma'(G) \leq 3$. It is easily observed that no two element subsets of G is not a γ' -set of G . Therefore $\gamma'(G) = 3$.

Let $n \geq 5$. Let $D = \{v_1, v_4\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{v_2, v_5\}$. Then D' is a inverse dominating set of G and so $\gamma'(G) = 2$. ■

Theorem 2.8. For the graph $G = J(F_n)$ ($n \geq 5$), $\gamma'_{ns}(G) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n \geq 5 \end{cases}$.

Proof. The number of vertices of fan graph is $n + 1$. Then it has $2n - 1$ edges. Let $G = J(F_n)$. The number of vertices of jump graph of the fan graph is $2n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}\}$, Since G contains no universal vertices, $\gamma(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(F_n)$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_2, u_2, u_3\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{v_1, v_3, u_4\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_{ns}(G) = 3$.

Let $n = 4$. Let $D = \{v_1, v_4\}$. Then D is a γ -set of G so that $\gamma(G) = 2$. Let $D' = \{v_2, v_3, u_3\}$. Then D' is a not a non-split inverse dominating set of G .

Let $n \geq 5$. Let $D = \{v_1, v_4\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{v_2, v_5\}$. Then D' is a non-split inverse dominating set of G and so $\gamma'_{ns}(G) = 2$. ■

Theorem 2.9. For the graph $G = J(F_n)$ ($n \geq 5$), $\gamma'_c(G) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n \geq 5 \end{cases}$.

Proof. The number of vertices of fan graph is $n + 1$. Then it has $2n - 1$ edges. Let $G =$

$J(F_n)$. The number of vertices of jump graph of the fan graph is $2n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(F_n)$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_2, u_2, u_3\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{v_1, v_3, u_4\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 3$.

Let $n = 4$. Let $D = \{v_1, v_4\}$. Then D is a γ -set of G so that $\gamma(G) = 2$. Let $D' = \{v_2, v_3, u_3\}$. Then D' is a not a connected inverse dominating set of G .

Let $n \geq 5$. Let $D = \{v_1, v_4\}$. Then D is a dominatingset of G and so $\gamma(G) = 2$. Let $D' = \{v_2, v_5\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 2$. ■

Inverse domination number of jump graph of $C_n \odot \overline{K}_2$

Theorem 2.10. For the graph $G = J(C_n \odot \overline{K}_2)$ ($n \geq 4$), $\gamma'(G) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}$

Proof. The number of vertices of $C_n \odot \overline{K}_2$ is $3n$. Then it has $3n$ edges. Let $G = J(C_n \odot \overline{K}_2)$. The number of vertices of jump graph of the $C_n \odot \overline{K}_2$ is $3n$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(C_n \odot \overline{K}_2)$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_2, w_2, w_3\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{u_1, u_2, u_3\}$. Then D' is a γ' -set of G . Since $\gamma(G) = 3$, we have $\gamma'(G) = 3$.

Let $n \geq 4$. Let $D = \{u_2, u_4\}$. Then D is a dominatingset of G and so $\gamma(G) = 2$. Let $D' = \{w_2, w_4\}$. Then D' is a inverse dominating set of G so that $\gamma'(G) = 2$. ■

Theorem 2.11. For the graph $G = J(C_n \odot \overline{K}_2)$ ($n \geq 4$), $\gamma'_{ns}(G) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}$

Proof. The number of vertices of $C_n \odot \overline{K}_2$ is $3n$. Then it has $3n$ edges. Let $G = J(C_n \odot \overline{K}_2)$. The number of vertices of jump graph of the $C_n \odot \overline{K}_2$ is $3n$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(C_n \odot \overline{K}_2)$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_2, w_2, w_3\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{u_1, u_2, u_3\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_{ns}(G) = 3$.

Let $n \geq 4$. Let $D = \{u_2, u_4\}$. Then D is a dominating set of G so that $\gamma(G) = 2$. Let $D' = \{w_2, w_4\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_{ns}(G) = 2$.

Theorem 2.12. For the graph $G = J(C_n \odot \overline{K}_2)$ ($n \geq 4$), $\gamma'_c(G) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}$

Proof. The number of vertices of $C_n \odot \overline{K}_2$ is $3n$. Then it has $3n$ edges. Let $G =$

$J(C_n \odot \overline{K_2})$. The number of vertices of jump graph of the $C_n \odot \overline{K_2}$ is $3n$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma'_c(G) \geq 2$.

Let $n = 3$. It is easily verified that no two element subsets of $J(C_n \odot \overline{K_2})$ is not a γ -set of G and so $\gamma(G) \geq 3$. Let $D = \{v_2, w_2, w_3\}$. Then D is a γ -set of G so that $\gamma(G) = 3$. Let $D' = \{u_1, u_2, u_3\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_c(G) = 2$.

Let $n \geq 4$. Let $D = \{u_2, u_4\}$. Then D is a dominating set of G so that $\gamma(G) = 2$. Let $D' = \{w_2, w_4\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 2$. ■

Inverse domination number of jump graph of $P_n \odot \overline{K_2}$

Theorem 2.13. For the graph $G = J(P_n \odot \overline{K_2})$ ($n \geq 3$), $\gamma'(G) = 2$.

Proof. For $n \geq 3$, the number of vertices of $P_n \odot \overline{K_2}$ is $3n$. Then it has $3n - 1$ edges. Let $G = J(P_n \odot \overline{K_2})$. The number of vertices of jump graph of the $P_n \odot \overline{K_2}$ is $3n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a γ -set of G so that $\gamma(G) = 2$. Let $D' = \{w_1, w_3\}$. Then D' is a inverse dominating set of G so that $\gamma'(G) = 2$. ■

Theorem 2.14. For the graph $G = J(P_n \odot \overline{K_2})$ ($n \geq 3$), $\gamma'_{ns}(G) = 2$.

Proof. For $n \geq 3$, the number of vertices of $P_n \odot \overline{K_2}$ is $3n$. Then it has $3n - 1$ edges. Let $G = J(P_n \odot \overline{K_2})$. The number of vertices of jump graph of the $P_n \odot \overline{K_2}$ is $3n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{w_1, w_3\}$. Then D' is a non-split inverse dominating set of G so that $\gamma'_{ns}(G) = 2$. ■

Theorem 2.15. For the graph $G = J(P_n \odot \overline{K_2})$ ($n \geq 3$), $\gamma'_c(G) = 2$.

Proof. For $n \geq 3$, the number of vertices of $P_n \odot \overline{K_2}$ is $3n$. Then it has $3n - 1$ edges. Let $G = J(P_n \odot \overline{K_2})$. The number of vertices of jump graph of the $P_n \odot \overline{K_2}$ is $3n - 1$. Let the vertices of the graph is labeled as $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n-1}, w_1, w_2, \dots, w_n\}$. Since G contains no universal vertices, $\gamma(G) \geq 2$. Let $D = \{u_1, u_3\}$. Then D is a dominating set of G and so $\gamma(G) = 2$. Let $D' = \{w_1, w_3\}$. Then D' is a connected inverse dominating set of G so that $\gamma'_c(G) = 2$. ■

3. Conclusions

In this article, we determined some inverse domination parameter for jump graph of some special graphs. We will determine some more inverse domination parameters for jump graph of some special graph in future work.

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Tri- $b\hat{g}$ Closed sets in Tri- Topological Spaces

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Abstract

In this paper, we introduce a new class of sets called tri- $b\hat{g}$ closed sets and tri- $b\hat{g}$ open sets via the concept of tri- $\hat{g}$ closed sets in tri topological spaces. Also, we investigate the relationship with other existing closed sets in tri-topological space.

Keywords: Tri- $b\hat{g}$ closed sets, tri- $b\hat{g}$ open sets, tri- $b\hat{g}$ closure, tri- $b\hat{g}$ interior.

2010 AMS subject classification: 54A40[‡].

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1. Introduction

The concept of tri- topological space was first initiated by M. Kovar [6] in 2000, in 2003, R. Subasree and M. Maria Singam [10] defined $b\hat{g}$ - closed sets in topological spaces. In [3], we introduced tri- $\hat{g}$ closed sets in tri- topological spaces and studied their properties. In this paper, we define tri- $b\hat{g}$ closed sets and tri- $b\hat{g}$ open sets via the concept of tri- $\hat{g}$ closed sets. Also, we investigate the relationship with other existing closed sets in tri- topological space.

2. Preliminaries

Throughout this paper $(X, \tau_1, \tau_2, \tau_3)$ (or simply X) represents tri- topological spaces on which no separation axioms are assumed unless other wised mentioned. For a subset A of $(X, \tau_1, \tau_2, \tau_3)$, $\text{tri-cl}(A)$, $\text{tri-int}(A)$ and A^c denote the tri- closure of A , tri- interior of A and compliment of A respectively.

Definition 2.1 Let X be a non-empty set. A family τ of subsets of X is said to be a topology on X , if τ satisfies the following axioms.

- a) $\phi, X \in \tau$,
- b) If $A_i \in \tau$ for $i = 1, 2, \dots, n$, then $\bigcap_{i=1}^n A_i \in \tau$,
- c) If $A_\alpha \in \tau$ for $\alpha \in I$, then $\bigcup_{\alpha} A_\alpha \in \tau$.

The pair (X, τ) is called a topological space and any set A in τ is called an open set. The complement of an open set A is called closed set.

Definition 2.2 Let X be a non-empty set. A family G of subsets of X is said to be a generalized topology on X , if G satisfies the followings axioms.

- a) $\phi \in G$,
- b) If $A_\alpha \in G$ for $\alpha \in I$, then $\bigcup_{\alpha} A_\alpha \in G$.

The pair (X, G) is called a generalized topological space.

Definition 2.3 Let X be a non-empty set. A family τ^* of subsets of X is said to be a Supra topology on X , if τ^* satisfies the following axioms.

- a) $\phi, X \in \tau^*$,
- b) If $A_\alpha \in \tau^*$ for $\alpha \in I$, then $\bigcup_{\alpha} A_\alpha \in \tau^*$.

The pair (X, τ^*) is called a Supra topological space.

Definition 2.4 Let X be a non-empty set. A family τ_{iX} of subsets of X is said to be a Infra topology on X , if τ_{iX} satisfies the following axioms.

- a) $\phi, X \in \tau_{iX}$,
- b) If $A_i \in \tau_{iX}$ for $i = 1, 2 \dots n$, then $\bigcap_{i=1}^n A_i \in \tau_{iX}$.

The pair (X, τ_{iX}) is called Infra topological space.

Definition 2.5 Let (X, τ) be a topological space then τ is said to be indiscrete topology if τ is a collection of only X and ϕ . Indiscrete topology is also known as trivial topology.

Definition 2.6 Let (X, τ) be a topological space then τ is said to be discrete topology if τ is a collection of all subsets of X .

Definition 2.7 Let (X, τ) be a topological space then a subset A of X is said to be $b\hat{g}$ -closed set if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$, U is $\hat{g}$ -open in X .

Definition 2.8 Let X be a nonempty set and τ_1, τ_2 and τ_3 are topologies on X . Then a subset A of X is said to be tri- open set if $A \in \tau_1 \cup \tau_2 \cup \tau_3$ and its complement is said to be tri- closed set and X with three topologies called tri- topological spaces $(X, \tau_1, \tau_2, \tau_3)$.

Definition 2.9 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space and let $A \subseteq X$. The union of all tri- open sets contained in A is called the tri- interior of A . The intersection of all tri- closed sets containing A is called the tri- closure of A .

Definition 2.10 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space. $A \subseteq X$ is said to be

- 1) A tri- α open set if $A \subseteq \text{tri- int}(\text{tri- cl}(\text{tri- int}(A)))$.
- 2) A tri- b open set if $A \subseteq [\text{tri- cl}(\text{tri- int}(A))] \cup [\text{tri- int}(\text{tri- cl}(A))]$.
- 3) A tri- semi closed set if $\text{tri- int}(\text{tri- cl}(A)) \subseteq A$.
- 4) A tri- g closed set if $\text{tri- cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is tri- open set in X .
- 5) A tri- gs closed set if $\text{tri- scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is tri- open set in X .
- 6) A tri- $b\tau$ closed set if $\text{tri- cl}_b(A) \subseteq U$ whenever $A \subseteq U$ and U is tri- open set in X .
- 7) A tri- g^*bw closed set if $\text{tri- bcl}(A) \subseteq U$ whenever $A \subseteq U$, U is tri- gs open in X .
- 8) A tri- $\hat{g}$ closed set if $\text{tri- cl}(A) \subseteq U$ whenever $A \subseteq U$, U is tri- semi open in X .

The complement of tri- α open set, tri- b open set, tri- semi closed set, tri- g closed set, tri- gs closed set, tri- $b\tau$ closed set, tri- g^*bw closed set and tri- $\hat{g}$ closed set is called tri- α closed set, tri- b closed set, tri- semi open set, tri- g open set, tri- gs open set, tri- $b\tau$ open set, tri- g^*bw open set and tri- $\hat{g}$ open set respectively.

Theorems 2.11

- 1) Every tri- closed set is tri- semi closed.
- 2) Every tri- closed set is tri- b closed.
- 3) Every tri- closed set is tri- gs closed.
- 4) Every tri- closed set is tri- $b\tau$ closed.
- 5) Every tri- closed set is tri- $g^*b\omega$ closed.
- 6) Every tri- closed set is tri- $\hat{g}$ closed set.
- 7) Every tri- semi closed set is tri- gs closed.

- 8) Every tri- semi closed set is tri- b closed.
- 9) Every tri- semi closed set is tri- $g^*b\omega$ closed.
- 10) Every tri- b closed set is tri- $b\tau$ closed.
- 11) Every tri- semi closed set is tri- $b\tau$ closed.
- 12) Every tri- α closed set is tri- b closed set.
- 13) Every tri- $g^*b\omega$ closed set is tri- $b\tau$ closed.
- 14) Every tri- $\hat{g}$ closed set is tri- g closed.
- 15) Every tri- $\hat{g}$ closed set is tri- gs closed.

3. Tri- $b\hat{g}$ Closed Sets in Tri- Topological Space

We introduce the following definitions

Definition 3.1 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space then a subset A of X is said to be tri- $b\hat{g}$ closed set if $\text{tri- bcl}(A) \subseteq U$ whenever $A \subseteq U$, U is tri- $\hat{g}$ open in X. The family of all tri- $b\hat{g}$ closed sets of X is denoted by tri- $b\hat{g}C(X)$.

Example 3.2 Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{b, c\}\}$, $\tau_3 = \{X, \phi, \{a, c\}\}$, Open sets in tri- topological spaces are union of all three topologies. $\tau_1 \cup \tau_2 \cup \tau_3 = \{X, \phi, \{a, b\}, \{b, c\}, \{a, c\}\}$; Tri- $\hat{g}O(X) = \{X, \phi, \{a, b\}, \{b, c\}, \{a, c\}\}$; Hence tri- $b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}\}$.

Remark 3.3 ϕ and X are always tri- $b\hat{g}$ closed set.

Remark 3.4 Intersection of tri- $b\hat{g}$ closed sets need not be tri- $b\hat{g}$ closed set.

Example 3.5 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \tau_3 = \{X, \phi, \{a\}\}$, tri- $b\hat{g}C(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Here, $\{a, b\}$, $\{a, c\}$ are tri- $b\hat{g}$ closed sets but $\{a, b\} \cap \{a, c\} = \{a\}$ is not a tri- $b\hat{g}$ closed set.

Remark 3.6 Union of tri- $b\hat{g}$ closed sets need not be tri- $b\hat{g}$ closed set.

Example 3.7 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{b\}, \{b, c\}\}$, $\tau_3 = \{X, \phi, \{c\}, \{a, b\}\}$, tri- $b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Here, $\{b\}$, $\{c\}$ are tri- $b\hat{g}$ closed sets but $\{b\} \cup \{c\} = \{b, c\} \notin \text{tri- } b\hat{g}C(X)$.

Remark 3.8 Difference of two tri- $b\hat{g}$ closed sets need not be tri- $b\hat{g}$ closed set.

Example 3.9 In previous example – 3.7, tri- $b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let $A = X$ and $B = \{a\}$, Also A and B are tri- $b\hat{g}$ closed sets. But $A \setminus B = X \setminus \{a\} = \{b, c\}$ is not a tri- $b\hat{g}$ closed set.

Remark 3.10

- 1) $(X, \text{Tri- } b\hat{g} C(X))$ need not be Topological space.
- 2) $(X, \text{Tri- } b\hat{g} C(X))$ need not be Generalized topological space.
- 3) $(X, \text{Tri- } b\hat{g} C(X))$ need not be Supra topological space.
- 4) $(X, \text{Tri- } b\hat{g} C(X))$ need not be Infra topological space.

Example 3.11 In examples – 3.5, 3.7 we get the results.

Definition 3.12 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space. The intersection of all tri- $b\hat{g}$ closed sets of X containing a subset A of X is called tri- $b\hat{g}$ closure of A and is denoted by tri- $b\hat{g} cl(A)$. (i.e) tri- $b\hat{g} cl(A) = \cap \{B \subseteq X: B \supseteq A \text{ and } B \text{ is tri- } b\hat{g} \text{ closed set}\}$.

Remark 3.13

- 1) tri- $b\hat{g} cl(\phi) = \phi$,
- 2) tri- $b\hat{g} cl(X) = X$,
- 3) $A \subseteq \text{tri- } b\hat{g} cl(A)$,
- 4) tri- $b\hat{g} cl(A) = \text{tri- } b\hat{g} cl(\text{tri- } b\hat{g} cl(A))$.

Proposition 3.14 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space. Let $A \subseteq X$, Then $A = \text{tri- } b\hat{g} cl(A)$ if A is tri- $b\hat{g}$ closed set.

Proof. Suppose A is a tri- $b\hat{g}$ closed set in X then, tri- $bcl(A) \subseteq U$ whenever $A \subseteq U$, U is tri- $\hat{g}$ open in X . Since, $A \supseteq A$ and A is tri- $b\hat{g}$ closed set. Let $B \subseteq X$ then $A \in \{B \subseteq X : B \supseteq A \text{ and } B \text{ is tri- } b\hat{g} \text{ closed}\} \Rightarrow A = \cap \{B \subseteq X : B \supseteq A \text{ and } B \text{ is tri- } b\hat{g} \text{ closed}\}$. Hence $A = \text{tri- } b\hat{g} cl(A)$.

Remark 3.15 The tri- $b\hat{g}$ closure of a set A is not always tri- $b\hat{g}$ closed set.

Example 3.16 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \tau_3 = \{X, \phi, \{a\}\}$, tri- $b\hat{g} C(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Here, tri- $b\hat{g} cl(\{a\}) = \{a\}$ is not a tri- $b\hat{g}$ closed set.

Proposition 3.17 Every tri- b closed set is tri- $b\hat{g}$ closed set.

Proof: Let A be any tri- b closed set in X and U be any tri- $\hat{g}$ open set in X such that $A \subseteq U$. Since, A is tri- b closed then tri- $bcl(A) = A$ for every subset A of X . tri- $bcl(A) = A \subseteq U$. Hence A is tri- $b\hat{g}$ closed set.

Converse of the above proposition need not be true as seen in the following example.

Example 3.18 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \tau_3 = \{X, \phi, \{a\}\}$, tri- $b C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}$; tri- $b\hat{g} C(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$; here $\{a, b\}$, $\{a, c\}$ are tri- $b\hat{g}$ closed sets but not a tri- b closed set.

Proposition 3.19 Every tri- closed set is tri- $b\hat{g}$ closed set.

Proof: Let A be any tri- closed set in X . Since every tri- closed set is tri- b closed set. Therefore, A is tri- b closed set in X . By proposition 3.17, A is tri- $b\hat{g}$ closed set. Converse of the above proposition need not be true as seen in the following example.

Example 3.20 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\tau_3 = \{X, \phi, \{a, c\}\}$, tri- $C(X) = \{X, \phi, \{b\}, \{a, c\}, \{b, c\}\}$; tri- $b\hat{g}C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; here $\{c\}$ is tri- $b\hat{g}$ closed set but not a tri- closed set.

Proposition 3.21 Every tri- semi closed set is tri- $b\hat{g}$ closed set.

Proof: Let A be any tri- semi closed set in X . Since every tri- semi closed set is tri- b closed set. Therefore, A is tri- b closed set in X . By proposition 3.17, A is tri- $b\hat{g}$ closed set. Converse of the above proposition need not be true as seen in the following example.

Example 3.22 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$, $\tau_3 = \{X, \phi, \{b, c\}\}$, tri- $sC(X) = \{X, \phi, \{a\}, \{c\}, \{b, c\}\}$; tri- $b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; here $\{b\}, \{a, c\}$ are tri- $b\hat{g}$ closed sets but not a tri-semi closed set.

Proposition 3.23 Every tri- α closed set is tri- $b\hat{g}$ closed set.

Proof: Let A be any tri- α closed set in X . Since every tri- α closed set is tri- b closed set. Therefore, A is tri- b closed set in X . By proposition 3.17, A is tri- $b\hat{g}$ closed set. Converse of the above proposition need not be true as seen in the following example.

Example 3.24 Let $X = \{a, b, c\}$, $\tau_1 = \tau_2 = \{X, \phi, \{a\}\}$, $\tau_3 = \{X, \phi, \{b, c\}\}$, tri- $\alpha C(X) = \{X, \phi, \{a\}, \{b, c\}\}$; tri- $b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$; here $\{b\}, \{c\}, \{a, b\}, \{b, c\}$ are tri- $b\hat{g}$ closed sets but not a tri- α closed set.

Proposition 3.25 Every tri- $g^*b\omega$ closed set is tri- $b\hat{g}$ closed set.

Proof: Let A be any tri- $g^*b\omega$ closed set in X and $A \subseteq U$, where U is tri- $\hat{g}$ open set in X . Since, every tri- $\hat{g}$ open set is tri- g s open. Therefore, U is tri- g s open in X . Since, A is tri- $g^*b\omega$ closed set in X then tri- $bcl(A) \subseteq U$. Hence A is tri- $b\hat{g}$ closed set in X . Converse of the above proposition need not be true as seen in the following example.

Example 3.26 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi, \{a\}\}$, $\tau_3 = \{X, \phi, \{a, b\}\}$, tri- $g^*b\omega C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$; tri- $b\hat{g}C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; here $\{a, c\}$ is tri- $b\hat{g}$ closed set but not a tri- $g^*b\omega$ closed set.

Proposition 3.27 Every tri- $b\hat{g}$ closed set is tri- $b\tau$ closed set.

Proof: Let A be any tri- $b\hat{g}$ closed set in X and $A \subseteq U$, where U is tri- open set in X . Since, every tri- open set is tri- $\hat{g}$ open. Therefore, U is tri- $\hat{g}$ open in X . Since, A is tri- $b\hat{g}$ closed set in X then tri- $bcl(A) \subseteq U$. Hence A is tri- $b\tau$ closed set in X . Converse of the above proposition need not be true as seen in the following example.

Tri- $b\hat{g}$ Closed Sets in Tri- Topological Spaces

Example 3.28 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\tau_3 = \{X, \phi, \{a, c\}\}$, $\text{tri-}b\hat{g}C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; $\text{tri-}b\tau C(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$; here $\{a, b\}$ is tri- $b\tau$ closed set but not a tri- $b\hat{g}$ closed set.

Remark 3.29 Tri- g closed sets and tri- $b\hat{g}$ closed sets are independent.

Example 3.30 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\tau_3 = \{X, \phi, \{a, b\}\}$, $\text{tri-}gC(X) = \{X, \phi, \{c\}, \{b, c\}, \{a, c\}\}$; $\text{tri-}b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; here $\{a\}$ and $\{b\}$ are tri- $b\hat{g}$ closed sets but not a tri- g closed sets.

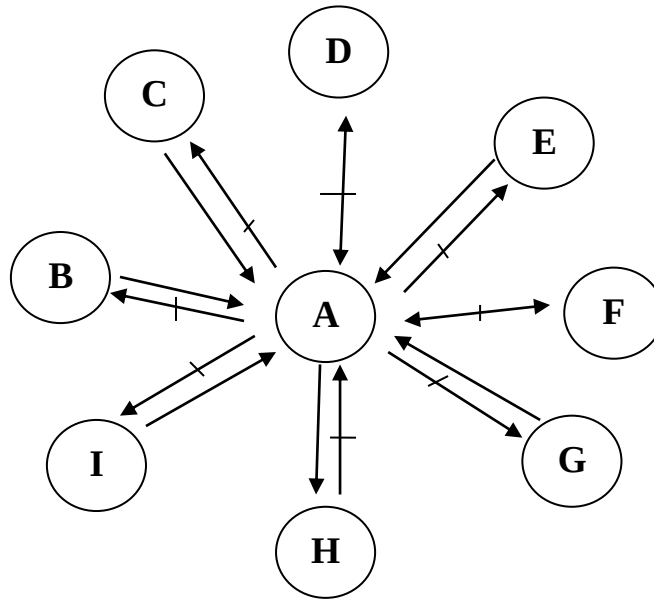
Example 3.31 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\tau_3 = \{X, \phi, \{a, c\}\}$, $\text{tri-}b\hat{g}C(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; $\text{tri-}gC(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$; here $\{a, b\}$ is tri- g closed set but not a tri- $b\hat{g}$ closed set.

Remark 3.32 Tri- gs closed sets and tri- $b\hat{g}$ closed sets are independent.

Example 3.33 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$, $\tau_3 = \{X, \phi, \{b, c\}\}$, $\text{tri-}gsC(X) = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$; $\text{tri-}b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, c\}\}$; here $\{b\}$ is tri- $b\hat{g}$ closed set but not a tri- gs closed set.

Example 3.34 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi, \{a\}\}$, $\tau_3 = \{X, \phi, \{b\}\}$, $\text{tri-}b\hat{g}C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; $\text{tri-}gsC(X) = P(X)$; here $\{a, b\}$ is tri- gs closed set but not a tri- $b\hat{g}$ closed set.

Remark 3.35 The following diagram shows the relationship of tri- $b\hat{g}$ closed sets with other known existing closed sets in tri- topological space.



$A \rightarrow \text{Tri- } b\hat{g} \text{ closed set}$	$B \rightarrow \text{Tri- closed set}$	$C \rightarrow \text{Tri- } b \text{ closed set}$
$D \rightarrow \text{Tri- } g \text{ closed set}$	$E \rightarrow \text{Tri- } \alpha \text{ closed set}$	$F \rightarrow \text{Tri- } gs \text{ closed set}$
$G \rightarrow \text{Tri- } g^*bw \text{ closed set}$	$H \rightarrow \text{Tri- } b\tau \text{ closed set}$	$I \rightarrow \text{Tri- semi closed set}$

Remark 3.36 If $(X, \text{Tri- } C(X))$ is indiscrete topology then $(X, \text{Tri- } b\hat{g} C(X))$ is discrete topology but converse part need not be true.

Example 3.37 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \tau_3 = \{X, \phi, \{b, c\}\}$; $\text{Tri- } C(X) = \{X, \phi, \{a\}, \{b, c\}\}$; $\text{Tri- } b\hat{g} C(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} = P(X)$. Here, $(X, \text{Tri- } b\hat{g} C(X))$ is discrete topology but $(X, \text{Tri- } C(X))$ is not an indiscrete topology.

Remark 3.38 If $(X, \text{Tri- } C(X))$ is discrete topology then $(X, \text{Tri- } b\hat{g} C(X))$ is discrete topology but converse part need not be true.

Example 3.39 In example – 3.7, $(X, \text{Tri- } b\hat{g} C(X))$ is discrete topology but $(X, \text{Tri- } C(X))$ is not a discrete topology.

Remark 3.40 If $(X, \text{Tri- } C(X))$ is indiscrete topology then,

- 1) Every tri- $b\hat{g}$ closed set is tri- b closed set.
- 2) Every tri- $b\hat{g}$ closed set is tri- g closed set.
- 3) Every tri- $b\hat{g}$ closed set is tri- gs closed set.
- 4) Every tri- $b\hat{g}$ closed set is tri- g^*bw closed set.
- 5) Every tri- g closed set is tri- $b\hat{g}$ closed set.
- 6) Every tri- gs closed set is tri- $b\hat{g}$ closed set.
- 7) Every tri- $b\tau$ closed set is tri- $b\hat{g}$ closed set.

Example 3.41 Let X be any non-empty set, $\tau_1 = \tau_2 = \tau_3 = \{X, \phi\}$ are topologies of X . $\text{Tri- } C(X) = \{X, \phi\}$; $\text{Tri- } bC(X) = \text{Tri- } gC(X) = \text{Tri- } gsC(X) = \text{Tri- } b\tau C(X) = \text{Tri- } g^*bwC(X) = \text{Tri- } b\hat{g} C(X) = P(X)$.

Remark 3.42 If $(X, \text{Tri- } C(X))$ is discrete topology then,

- 1) Every tri- $b\hat{g}$ closed set is tri- closed set.
- 2) Every tri- $b\hat{g}$ closed set is tri- semi closed set.
- 3) Every tri- $b\hat{g}$ closed set is tri- α closed set.
- 4) Every tri- $b\hat{g}$ closed set is tri- b closed set.
- 5) Every tri- $b\hat{g}$ closed set is tri- g closed set.
- 6) Every tri- $b\hat{g}$ closed set is tri- gs closed set.
- 7) Every tri- $b\hat{g}$ closed set is tri- g^*bw closed set.
- 8) Every tri- g closed set is tri- $b\hat{g}$ closed set.
- 9) Every tri- gs closed set is tri- $b\hat{g}$ closed set.
- 10) Every tri- $b\tau$ closed set is tri- $b\hat{g}$ closed set.

Example 3.43 Let X be any non-empty set, $\tau_1 = \tau_2 = \tau_3 = P(X)$ are topologies of X .
 Tri- $C(X) = \text{Tri- } sC(X) = \text{Tri- } \alpha C(X) = \text{Tri- } bC(X) = \text{Tri- } gC(X) = \text{Tri- } gsC(X) =$
 Tri- $b\tau C(X) = \text{Tri- } g^*b\omega C(X) = \text{Tri- } b\hat{g} C(X) = P(X)$.

4. Tri- $b\hat{g}$ Open Sets In Tri- Topological Space

Definition 4.1 The complement of a tri- $b\hat{g}$ closed set is called the tri- $b\hat{g}$ open set. The family of all tri- $b\hat{g}$ open sets of X is denoted by tri- $b\hat{g} O(X)$.

Example 4.2 In example 3.2, tri- $b\hat{g} O(X) = \{X, \phi, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Remark 4.3 ϕ and X are always tri- $b\hat{g}$ open set.

Remark 4.4 Intersection of tri- $b\hat{g}$ open sets need not be tri- $b\hat{g}$ open set.

Example 4.5 In example – 3.2, tri- $b\hat{g} O(X) = \{X, \phi, \{a, b\}, \{b, c\}, \{a, c\}\}$. Here, $\{a, b\}$, $\{b, c\}$ are tri- $b\hat{g}$ open sets but $\{a, b\} \cap \{b, c\} = \{b\} \notin \text{tri- } b\hat{g} O(X)$.

Remark 4.6 Union of tri- $b\hat{g}$ open sets need not be tri- $b\hat{g}$ open set.

Example 4.7 In example – 3.16, tri- $b\hat{g} O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Here, $\{b\}$ and $\{c\}$ are tri- $b\hat{g}$ open sets but $\{b\} \cup \{c\} = \{b, c\} \notin \text{tri- } b\hat{g} O(X)$.

Remark 4.8 Difference of two tri- $b\hat{g}$ open sets need not be tri- $b\hat{g}$ open set.

Example 4.9 In previous example – 4.7, tri- $b\hat{g} O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let $A = X$ and $B = \{a\}$, Also A and B are tri- $b\hat{g}$ open sets. But $A \setminus B = X \setminus \{a\} = \{b, c\}$ is not a tri- $b\hat{g}$ open set.

Definition 4.10 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri- topological space. The union of all tri- $b\hat{g}$ open sets of X contained in A is called the tri- $b\hat{g}$ interior of A and is denoted by tri- $b\hat{g} \text{ int}(A)$. (i.e) tri- $b\hat{g} (A) = \cup \{B \subseteq X / B \subseteq A \text{ and } A \text{ is tri- } b\hat{g} \text{ open set}\}$.

Remark 4.11

- 1) tri- $b\hat{g} \text{ int}(\phi) = \phi$,
- 2) tri- $b\hat{g} \text{ int}(X) = X$,
- 3) tri- $b\hat{g} \text{ int}(A) \subseteq A$,
- 4) tri- $b\hat{g} \text{ int}(A) = \text{tri- } b\hat{g} \text{ int}(\text{tri- } b\hat{g} \text{ int}(A))$.

Proposition 4.12 For any $A \subseteq X$, $(\text{tri- } b\hat{g} \text{ int}(A))^c = \text{tri- } b\hat{g} \text{ cl}(A^c)$.

Proof: $(\text{tri- } b\hat{g} \text{ int}(A))^c = [\cup \{G / G \subseteq A \text{ \& } G \text{ is tri- } b\hat{g} \text{ open set}\}]^c = \cap \{G^c / G^c \supseteq A^c \text{ \& } G^c \text{ is tri- } b\hat{g} \text{ closed set}\} = \cap \{F / F \supseteq A^c \text{ \& } F \text{ is tri- } b\hat{g} \text{ closed set}\}$ where $F = G^c$. Hence, $(\text{tri- } b\hat{g} \text{ int}(A))^c = \text{tri- } b\hat{g} \text{ cl}(A^c)$.

Proposition 4.13 Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri-topological space. Let $A \subseteq X$. Then $\text{tri- } b\hat{g} \text{ int}(A) = A$ if A is tri- $b\hat{g}$ open set.

Proof: Suppose A is a tri- $b\hat{g}$ open set in X , then A^c is tri- $b\hat{g}$ closed set in X . (i.e) $\text{tri- } b\hat{g} \text{ cl}(A^c) \subseteq A^c$. By the definition, $A^c \subseteq \text{tri- } b\hat{g} \text{ cl}(A^c)$. Therefore $\text{tri- } b\hat{g} \text{ cl}(A^c) = A^c \Rightarrow (\text{tri- } b\hat{g} \text{ int}(A))^c = A^c \Rightarrow \text{tri- } b\hat{g} \text{ int}(A) = A$.

Remark 4.14 The tri- $b\hat{g}$ interior of a set A is not always tri- $b\hat{g}$ open set.

Example 4.15 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \tau_3 = \{X, \phi, \{a\}\}$, $\text{tri- } b\hat{g} C(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$; $\text{tri- } b\hat{g} O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Here, $\text{tri- } b\hat{g} \text{ int}(\{b, c\}) = \{b, c\}$ is not a tri- $b\hat{g}$ open set.

Proposition 4.16

- 1) Every tri- open set is tri- $b\hat{g}$ open set.
- 2) Every tri- b open set is tri- $b\hat{g}$ open set.
- 3) Every tri- semi open set is tri- $b\hat{g}$ open set.
- 4) Every tri- α open set is tri- $b\hat{g}$ open set.
- 5) Every tri- $g^*b\omega$ open set is tri- $b\hat{g}$ open set.
- 6) Every tri- $b\hat{g}$ open set is tri- $b\tau$ open set.

Proof: By proposition – 3.17, 3.19, 3.21, 3.23, 3.25, 3.27 we get the results.

5. Conclusions

In this paper, we dealt with tri- $b\hat{g}$ closed sets and tri- $b\hat{g}$ open sets. In future we wish to do our research work in tri- $b\hat{g}$ continuous functions, tri- $b\hat{g}$ separated, tri- $b\hat{g}$ connected sets, tri- $b\hat{g}$ compact and so on.

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Forecasting of Annual Rainfall using Fuzzy Logic Interval Based Partitioning in Different Intervals

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Abstract

Fuzzy time series models have been proposed by many researchers around the world for rainfall forecasting, but the forecasting has not been as accurate as existing methods. Frequency density or ratio-based segmentation methods have been used to represent discourse segmentation. In this paper, to make such predictions, we used interval-based segmentation as the discourse segmentation and the urban mean rainfall in the Trichy district as the discourse universe. Fuzzy models are used for forecasting in many fields such as admissions prediction, stock price analysis, agricultural production, horticultural production, marine production, weather forecasting, and more.

Keywords. Mean Square Error; Fuzzy time series; Average Forecast Error Rate.

AMS Subject Classification: 05C78[‡].

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1. Introduction

A forecasting process issued to predict future outcomes. Relevant data and figures are carefully analyzed to make accurate forecasts and make optimal choices regarding the future. There are two main reasons for choosing time series forecasting. First, most of the data that exists in the real world, such as economics, business, and finance, are time series. Second, evaluation of time series data is easy, and many techniques are available for evaluation of time series forecasts. If the future is in doubt, the forecasting process is mandatory. Implement using the fuzzy time series method to forecast precipitation and compare results with other existing techniques. Accurate rainfall information is essential for planning and managing water resources. Moreover, in urban areas, rainfall has a strong impact on transportation, sewerage, N. Q. Hung (nguyenquang.hung@ait.ac.th) systems, and other human activities. Nevertheless, precipitation is best understood and modelled for hydrological cycles because of the complexity of the atmospheric processes that generate precipitation and the tremendous range of variability over a wide range of scales in both space and time. It is one of the complex and difficult factors (French et al., 1992). Accurate rainfall forecasting is thus one of the greatest challenges in operational hydrology, despite many advances in weather forecasting in recent decades (Gwangseob and Ana, 2001). The data for prediction has all the uncertainties.

For this purpose, we generate rainfall simulations that reproduce in a distributional sense the set of key rainfall statistics obtained from the observational dataset (Benoit and Mariethoz, 2017). The practical interest of probabilistic rainfall models is, among other things, to complement numerical weather models for simulating rainfall heterogeneity at fine scales and to quantify uncertainties associated with rainfall reconstructions. . Indeed, numerical weather models face the challenge of reproducing spatial and temporal rainfall heterogeneity, especially at fine scales (Bauer et al., 2015; Bony et al., 2015). Some of the main applications of probabilistic rainfall models are therefore for local impact studies, e.g. related to hydrology (Paschalis et al., 2014; Caseri et al., 2016).

Fuzzy time series forecasting is a smart method in areas where information is explicit, imprecise, and approximate. Fuzzy time series can also tackle situations that do not provide trend investigation and analysis, or visualization of time series patterns. In-depth research work has been done on forecasting problems using this concept. Vikas [1] proposed various techniques for predicting crop yields and used artificial neural networks to predict wheat yields. Did. Adesh [2] conducted a comparative study of

Various techniques, including neural networks and fuzzy models. Askar [3] also tried to predict yield using time series models. Sachin [4-5] specifically worked on rice yield prediction using fuzzy time series models. Narendra [6] attempted to predict wheat yields. Pankaj [7] used an adaptive neuro-fuzzy system for predicting wheat yield. W. Qiu, X. Liu, and H. Li proposed a generalized method of forecasting based on fuzzy time series models [30].

The concept and definition of fuzzy time series was devised and published by Song & Chissom. They also delineated concepts and notions of variant and invariant time series [8-9]. First, the time-series data for the University of Alabama were obtained, the

admission prediction was performed, and after a few years [10] average auto, then Chen [11-12] was the max-min configuration operation previously used by Song & Chissom. We depicted simplified arithmetic operations instead of using, and used higher-order fuzzy time series to organize our forecasting models. Huarng [13-14], Hwang and Chen [15], Lee Wang and Chen [16], Li and Kozma [17] all produced a number of fuzzy prediction methods with slight variations in each. Lee et al. Subsequently, a multivariate heuristic model was designed and implemented to obtain highly complex and complex matrix computations [20]. Research work has been done to ascertain the interval length of fuzzy time series [21].

Wong et al. (2003) used SOM and back propagation neural networks to build a fuzzy rule base and used the rule base to develop a forecast model for Swiss rainfall using spatial interpolation. Bardossy et al. (1995) implemented fuzzy logic for classification of atmospheric circulation patterns. Ozerkan et al. (1996) compared the performance of regression analysis and fuzzy logic in studying the relationship between monthly atmospheric circulation patterns and precipitation. Pesti et al. (1996) implemented fuzzy logic for drought assessment. Baum et al. (1997) developed a cloud classification model using fuzzy logic. Fujibe (1989) classified precipitation patterns in Honshu using the fuzzy C-means method. Garanboshi et al. (1999) Using fuzzy logic, we investigated the effects of ENSO and macro circulation patterns on precipitation over Arizona. Vivekanandan et al. (1999) developed and implemented a fuzzy logic algorithm for water meteor particle identification that is simple and efficient enough to run in real time for operational use. Wuwardi et al. (2006) use a neural fuzzy system to model tropical rainfall during the rainy season. The model results in lower RMSE values, indicating that the forecast model is reliable in representing recent inter annual variability in tropical rainfall during the wet season.

2. Basic Concepts of Fuzzy time Series Modeling

Definition 1.1. (Song and Chissom, 1993a, 1994; Liaw, 1997). A fuzzy number on the real line $\mathfrak{R}$ is a fuzzy subset of $\mathfrak{R}$ that is normal and convex.

Definition 1.2. (Song and Chissom, 1993a, 1994). Suppose that $R_1 = \cup_{ij} R_1 ij(t, t - 1)$ and $R_2(t, t - 1) = \cup_{ij} R_2 ij(t, t - 1)$ are two fuzzy relations between $F(t)$ and $F(t - 1)$. If, for any $f_j(t) \in F(t)$, where $j \in J$, there exists $f_i(t - 1) \in F(t - 1)$, where $i \in I$, and fuzzy relations $R_1 ij(t, t - 1)$ and $R_2 ij(t, t - 1)$ such that $f_i(t) = f_i(t - 1) \circ R_1 ij(t, t - 1)$ and $f_i(t) = f_i(t - 1) \circ R_2 ij(t, t - 1)$, then define $R_1(t, t - 1) = R_2(t, t - 1)$.

Definition 1.3. (Song and Chissom, 1993a, 1994). Suppose that $F(t)$ is only caused by $F(t - 1), F(t - 2), \dots$, or $F(t - m)$ ($m > 0$). This relation can be expressed as the following fuzzy relational equation: $F(t) = F(t - 1) \circ R_0(t, t - m)$, which is called the first-order model of $F(t)$.

Definition 1.4 (Song and Chissom, 1993a, 1994). Suppose that $F(t)$ is simultaneously caused by $F(t-1), F(t-2), \dots$, and $F(t-m)$ ($m > 0$). This relation can be expressed as the following fuzzy relational equation: $F(t) = (F(t-1) \times F(t-2) \times \dots \times F(t-m)) \circ Ra(t, t-m)$, which is called the m th-order model of $F(t)$.

Definition 1.5 (Chen, 1996). $F(t)$ is fuzzy time series if $F(t)$ is a fuzzy set. The transition is denoted as $F(t-1) \rightarrow F(t)$.

Definition 1.6 (Chou, 2011). Let $d(t)$ be a set of real numbers $d(t) \subseteq R$. A lower interval $d(t)$ is a number b such that $x \geq b$ for all $x \in d(t)$. The set $d(t)$ is said to be an interval below if $d(t)$ has a lower interval. A number, $\min$, is the minimum of $d(t)$ if $\min$ is a lower interval $d(t)$ and $\min \in d(t)$.

Definition 1.7 (Chou, 2011). Let $d(t)$ be a set of real numbers $d(t) \subseteq R$. An upper interval $d(t)$ is a number b such that $x \leq b$ for all $x \in d(t)$. The set $d(t)$ is said to be an interval higher if $d(t)$ has an upper interval. A number, $\max$, is the maximum of $d(t)$ if $\max$ is an upper interval $d(t)$ and $\max \in d(t)$.

3. Proposed Method

In this section, we use real-world rainfall as the universe of discourse and propose a method for forecasting using interval-based segmentation. Relevant concepts and definitions regarding this can be found by referring to a previously published paper [29]. Another method for predicting the values provided in this paper is clearly explained in the following lines. The forecasting process follows these steps:

Step 1:

First, obviously analysis descriptive statistics. It helps in facilitating data visualization. Next, we describe the discourse universe U and the parcel U in intervals of equal length. Here, according to the data, 379.79 is the minimum value and 1163.59 is the maximum value. We need to specify the discourse universe, the intervals in which all given values of rainfall exist. So in this case the discourse universe would be [300, 1200]. Descriptive statistics and block-by-block rainfall data are shown in Tables I and II.

Step 2:

Fuzzy partitioning is a methodology for generating fuzzy sets that represent the underlying data. The techniques can be classified into three categories: grid partitioning, tree partitioning, and distributed partitioning. Among the various fuzzy partitioning methods, grid partitioning is the most commonly used in practice, especially in system control applications. Grid partitioning forms partitions by dividing the input space into several fuzzy slices.

Next, divide Universe of discourse in 6, 9 and 18 equal intervals these are as following. The discourse universe can be defined by $U = [300, 1200]$. U is then divided into 6 equal length intervals and the midpoint of the 6th interval is calculated as shown below.

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Table 3.1. Annual rainfall in Trichy district (From 2004 to 2010)

Blocks	Rainfall(mm)
Andanallur	1085.23
Lalgudi	1136.79
Manachanallur	569.44
Manapparai	837.43
Manikandam	379.79
Marungapuri	942.16
Musiri	794.31
Pullambadi	1163.59
T.pet	710.16
Thiruverumbur	960.9
Thottiam	866.73
Thuraiur	942.11
Uppiliyapuram	473.35
Vaiyampatty	922.27

Table 3.2. Descriptive statistics

Minimum	= 379.79
Maximum	= 1163.59
Range	= 783.8
Count	= 14
Sum	= 11784.26
Mean	= 841.733
Median	= 894.5
Mode	= No mode
Standard Deviation	= 237.65
Variance	= 56479.54

Table 3.3. a) 6 equal intervals

u_1	[300 – 450]	375
u_2	[450 – 600]	525
u_3	[600 – 750]	675
u_4	[750 – 900]	825
u_5	[900 – 1050]	975
u_6	[1050 – 1200]	375

Here, U is partitioned into 9 equal length intervals and calculated mid points of 9th intervals given below:

Table 3.4. b) 9 equal Intervals with Midpoints

v_1	[300 – 400]	350
v_2	[400 – 500]	450
v_3	[500 – 600]	550
v_4	[600 – 700]	650
v_5	[700 – 800]	750
v_6	[800 – 900]	850
v_7	[900 – 1000]	950
v_8	[1000 – 1100]	1050
v_9	[1100 – 1200]	1150

Here, U is partitioned into 18 equal length intervals and calculated mid points of 18th intervals given below.

Table 3.5. c) 18 equal intervals with Midpoints

w_1	[300 – 350]	325
w_2	[350 – 400]	375
w_3	[400 – 450]	425
w_4	[450 – 500]	475
w_5	[500 – 550]	525
w_6	[550 – 600]	575
w_7	[600 – 650]	625
w_8	[650 – 700]	675
w_9	[700 – 750]	725
w_{10}	[750 – 800]	775
w_{11}	[800 – 850]	825
w_{12}	[850 – 900]	875
w_{13}	[900 – 950]	925
w_{14}	[950 – 1000]	975

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w_{15}	[1000 – 1050]	1025
w_{16}	[1050 – 1100]	1075
w_{17}	[1100 – 1150]	1125
w_{18}	[1150 – 1200]	1175

Step 3. Define a fuzzy set based on 6, 9, and 18 intervals to fuzz the historical data.

a. 6 Equal intervals

Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the world of discourse. The number of intervals depends on the number of considered linguistic variables (fuzzy sets) $A_1, A_2, A_3, A_4, A_5, A_6$. Define 6 fuzzy sets $A_1, A_2, \dots, A_6$ as linguistic variables in the discourse world U. These fuzzy variables are defined as:

Table 3.6. Label Linguistic value of enrolments

Fuzzified	Linguistic Value
A_1	very few
A_2	very very few
A_3	Moderate
A_4	High
A_5	very High
A_6	very very High

b. 9 equal Intervals

Let U be the universe of discourse, where $U = \{v_1, v_2, v_3, v_4, v_5, v_6 \dots v_9\}$. The number of intervals will be in accordance with the number of linguistic variables (fuzzy sets) $B_1, B_2, B_3, B_4, \dots B_9$, to be considered.

Define 9 fuzzy sets $B_1, B_2, B_3, B_4, \dots B_9$, as linguistic variables on the universe of discourse U. These fuzzy variables are being defined as:

Table 3.7. Label Linguistic value of enrolments

Fuzzified	Linguistic Value
B_1	very ³ few
B_2	very ² few
B_3	very ¹ few
B_4	Few
B_5	Moderate
B_6	High

B ₇	Very ¹ High
B ₈	very ² High
B ₉	very ³ High

c. 18 Equal intervals

Let U be the universe of discourse, and let $U = \{w_1, w_2, w_3, \dots, w_{18}\}$. The number of intervals depends on the number of considered linguistic variables (fuzzy sets) $C_1, C_2, C_3, \dots, C_{18}$. Define 18 fuzzy sets $C_1, C_2, C_3, \dots, C_{18}$ as linguistic variables in the universe of discourse U . These fuzzy variables are defined as:

Table 3.8. Label Linguistic value of enrolments

Fuzzified	Linguistic Value
C ₁	very ⁸ few
C ₂	very ⁷ few
C ₃	very ⁶ few
C ₄	very ⁵ few
C ₅	very ⁴ few
C ₆	very ³ few
C ₇	very ² few
C ₈	very ¹ few
C ₉	Few
C ₁₀	High
C ₁₁	Very ¹ High
C ₁₂	very ² High
C ₁₃	very ³ High
C ₁₄	very ⁴ High
C ₁₅	very ⁵ High
C ₁₆	very ⁶ High
C ₁₇	very ⁷ High
C ₁₈	very ⁸ High

Step 4:

Fuzzy set defined by U (all intervals).

A fuzzy set A_i is represented as

$$\begin{aligned} A_1 &= 1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\ A_2 &= 0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\ &\dots \\ &\dots \\ A_{10} &= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0.5/u_5 + 1/u_6 \end{aligned}$$

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A fuzzy set B_i are expressed as follows:

$$\begin{aligned} B_1 &= 1/u_1 + 0.5/u_2 + 0/u_3 + \dots + 0/u_9 \\ B_2 &= 0.5/u_1 + 1/u_2 + \dots + 0/u_9 \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \\ B_9 &= 0/u_1 + 0/u_2 + \dots + 0.5/u_5 + 1/u_9 \end{aligned}$$

A fuzzy set C_i are expressed as follows:

$$\begin{aligned} C_1 &= 1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\ C_2 &= 0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \\ C_{18} &= 0/u_1 + 0/u_2 + 0/u_3 + \dots + 0.5/u_{17} + 1/u_{18} \end{aligned}$$

Step 5: Fuzzify historical data.

Table 3.9. Linguistic values for the enrolments from 2004 to 2010

Block	Rainfall (mm)	Linguistic value 6th Interval	Linguistic value 9th Interval	Linguistic value 18th Interval
Andanallur	1085.23	C ₆	D ₈	E ₁₆
Lalgudi	1136.79	C ₆	D ₉	E ₁₇
Manachanallur	569.44	C ₂	D ₃	E ₆
Manapparai	837.43	C ₄	D ₆	E ₁₂
Manikandam	379.79	C ₁	D ₁	E ₂
Marungapuri	942.16	C ₅	D ₇	E ₁₃
Musiri	794.31	C ₄	D ₅	E ₁₀
Pullambadi	1163.59	C ₆	D ₉	E ₁₈
T.pet	710.16	C ₄	D ₅	E ₉
Thiruverumbur	960.9	C ₅	D ₇	E ₁₄
Thottiam	866.73	C ₄	D ₆	E ₁₂
Thuraiur	942.11	C ₅	D ₇	E ₁₃
Uppiliyapuram	473.35	C ₂	D ₂	E ₄
Vaiyampatty	922.27	C ₅	D ₇	E ₁₃

Step 6:

Calculate predicted registrations for 6, 9, and 18 intervals given below:

Table 3.10. Forecasted value for all intervals

Blocks	Rainfall (mm)	Forecasted value (6 intervals)	Forecasted value (9 intervals)	Forecasted value (18 intervals)
Andanallur	1085.23	-	-	-
Lalgudi	1136.79	-	-	-
Manachanallur	569.44	-	-	-
Manapparai	837.43	900	900	912.5
Manikandam	379.79	712.5	725	737.5
Marungapuri	942.16	675	675	687.5
Musiri	794.31	750	725	737.5
Pullambadi	1163.59	825	800	812.5
T.pet	710.16	937.5	900	900
Thiruverumbur	960.9	937.5	900	912.5
Thottiam	866.73	937.5	925	937.5
Thuraiur	942.11	900	875	875
Uppiliyapuram	473.35	825	800	812.5
Vaiyampatty	922.27	825	800	800

Step 7:

Calculate MSE and AFER values for 6, 9, and 18 intervals given below:

Mean Squared Error (MSE) measures the amount of error in a statistical model. Evaluate the mean squared difference between observed and predicted values. If the model has no errors, MSE is equal to zero.

$$\text{MSE formula} = \left(\frac{1}{n}\right) * \Sigma(\text{Actual} - \text{Forecast})^2$$

Mean Absolute Percentage Error (MAPE), also known as Mean Absolute Percentage Deviation (MAPD), is a measure of the predictive accuracy of a forecasting method in statistics. Accuracy is usually expressed as a ratio defined by the following formula:

$$\frac{|\text{Actual} - \text{predicted}|}{\text{Actual}} \times 100\% = \text{Average Forecasting Error Rate (AFER)}$$

4. Performance Evaluation and Comparative Studies

A. Performance rating:

Two parameters are used to compare the results of the proposed method with existing methods. These are MSE & AFER. MSE & AFER are the calculated values for intervals 6, 9 and 18 as shown in Tables XI, XII and XIII. Interval-based partitioning is calculated in Table XI. MSE indicates the deviation error from the actual value to the predicted value. The deviation is shown in Figure 4.1 in the form of a graphical representation for better visualization. As we can see the proposed algorithm gives values very close to what is the actual rainfall value. The same is done for the 9th and 18th intervals, as shown in Tables XII & XIII and Fig. 4.2 and 4.3.

Table 3.11. MSE and AFER Values (6 Intervals)

Block	A_i (Rainfall mm)	F_i	MSE $(A_i - F_i)^2$	AFER $ A_i - F_i /A_i$
Andanallur	1085.23	-	-	-
Lalgudi	1136.79	-	-	-
Manachanallur	569.44	-	-	-
Manapparai	837.43	900	3915.005	0.074717
Manikandam	379.79	712.5	110695.9	0.876037
Marungapuri	942.16	675	71374.47	0.283561
Musiri	794.31	750	1963.376	0.055784
Pullambadi	1163.59	825	114643.2	0.290987
T.pet	710.16	937.5	51683.48	0.320125
Thiruverumbur	960.9	937.5	547.56	0.024352
Thottiam	866.73	937.5	5008.393	0.081652
Thuraiur	942.11	900	1773.252	0.044698
Uppiliyapuram	473.35	825	123657.7	0.742896
Vaiyampatty	922.27	825	9461.453	0.105468
			MSE = 35337.42	AFER = 20.72%

Table 3.12. MSE AND AFER VALUES (9 INTERVALS)

Block	A_i (Rainfall mm)	F_i	MSE $(A_i - F_i)^2$	AFER $ A_i - F_i /A_i$
Andanallur	1085.23			
Lalgudi	1136.79			
Manachanallur	569.44			
Manapparai	837.43	912.5	5635.505	0.089643
Manikandam	379.79	737.5	127956.4	0.941863
Marungapuri	942.16	687.5	64851.72	0.270294
Musiri	794.31	737.5	3227.376	0.071521
Pullambadi	1163.59	812.5	123264.2	0.30173
T.pet	710.16	900	36039.23	0.26732
Thiruverumbur	960.9	912.5	2342.56	0.050369
Thottiam	866.73	937.5	5008.393	0.081652
Thuraiur	942.11	875	4503.752	0.071234
Uppiliyapuram	473.35	812.5	115022.7	0.716489
Vaiyampatty	922.27	800	14949.95	0.132575
			MSE=35914.42	AFER=21.39%

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Table 3.13. MSE AND AFER values (18 intervals)

Block	A_i (Rainfall mm)	F_i	MSE $(A_i - F_i)^2$	AFER $ A_i - F_i /A_i$
Andanallur	1085.23	-	-	-
Lalgudi	1136.79	-	-	-
Manachanallur	569.44	-	-	-
Manapparai	837.43	900	3915.005	0.074717
Manikandam	379.79	725	119169.9	0.90895
Marungapuri	942.16	675	71374.47	0.283561
Musiri	794.31	725	4803.876	0.087258
Pullambadi	1163.59	800	132197.7	0.312473
T.pet	710.16	900	36039.23	0.26732
Thiruverumbur	960.9	900	3708.81	0.063378
Thottiam	866.73	925	3395.393	0.06723
Thuraiur	942.11	875	4503.752	0.071234
Uppiliyapuram	473.35	800	106700.2	0.690081
Vaiyampatty	922.27	800	14949.95	0.132575
			MSE=35768.45	AFER=21.13%

The following figures (Fig.4.1, Fig 4.2 and Fig 4.3) are compared in Forecasted and Actual rainfall for the corresponding intervals respectively 6,9&18.And also Fig. 4.4 compares the MSE for all intervals.

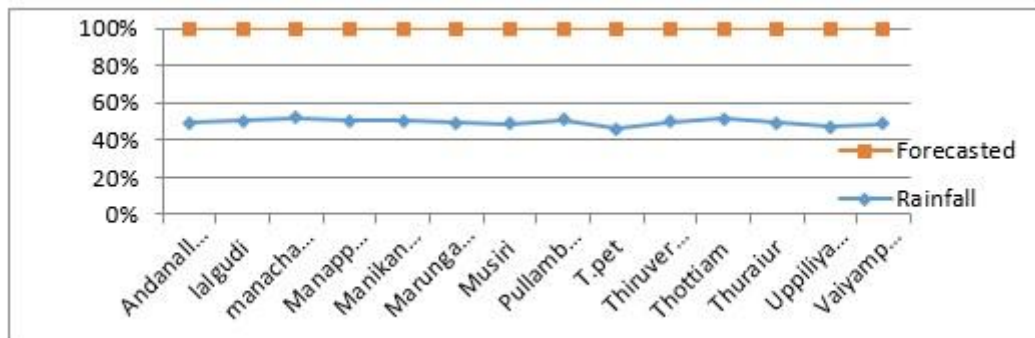


Fig. 4.1. Forecasted vs. Rainfall (6 intervals)

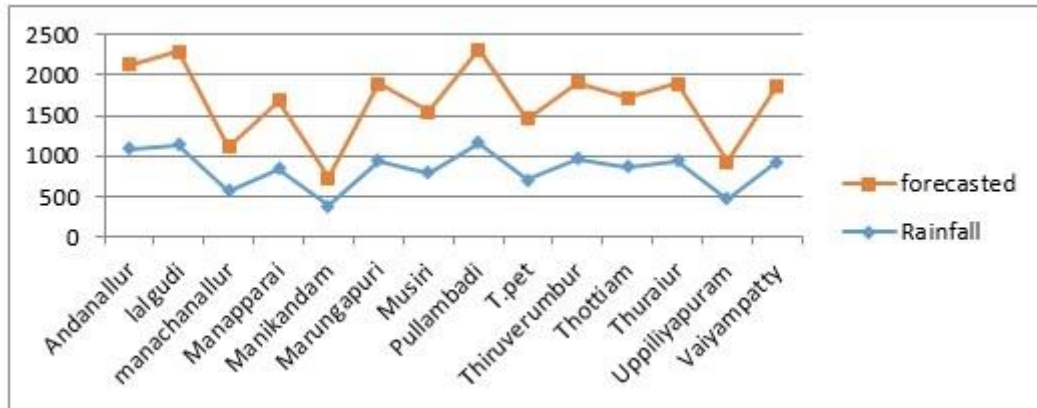


Fig. 4.2. Forecasted vs. Rainfall (9 intervals)

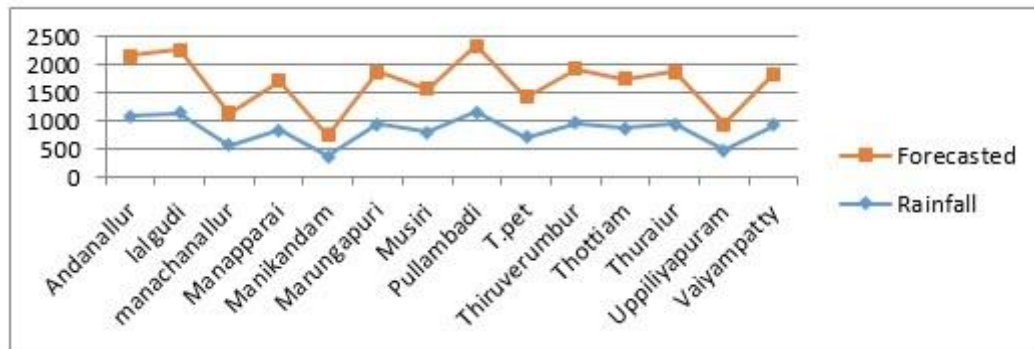


Fig. 4.3. Forecasted vs. Rainfall (18 intervals)

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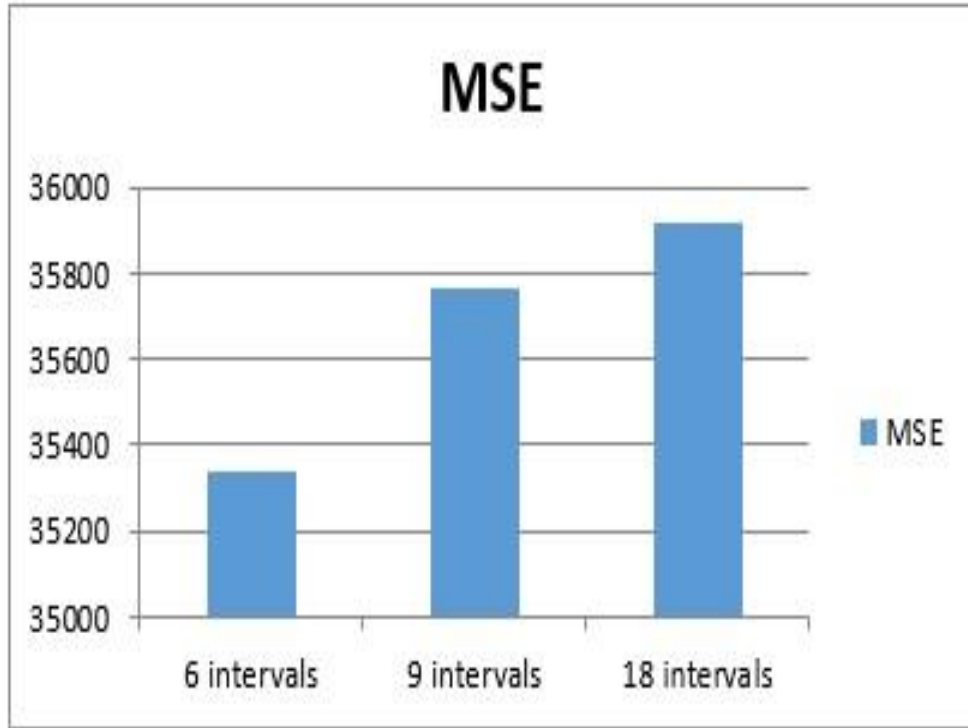


Fig. 4.4. Graph of MSE 6, 9 and 18 intervals

B. Results and Discussion.

The MSE and AFER calculated in Tables 3.11, 3.12 and 3.13 above were analyzed. The paper shows working with different intervals such as 6 , 9 and 18 . The majority of recently published papers work in one of these intervals. The focus of this paper was to propose a new algorithm and check its predicted variability over all these intervals. The results show that prediction works best on the 6th interval among all other intervals. All results are presented in an easy-to-understand bar chart format to reduce the complexity of this study and present it in a more understandable manner. To prove that this algorithm is efficient, a comparison is made with other existing methods proposed by Heuristic and FTS first order in Table XIV. As can be seen from Fig. 4.5, the proposed algorithm was able to achieve significantly lower MSE compared to other methods. This model not only provides a lower MSE, but also explains why researchers making fuzzy logic predictions choose the 6th interval for their work. All other intervals do not give better results than the 6th interval partitioning. A possible reason for increasing the number of intervals is that the data is overcrowded. As such, the relevant data between intervals is not included in the forecasting algorithm and affects forecasting results. Keeping the interval lower than the 6th interval will spread the data too much. Therefore, the 6th interval partitioning seems to be the best overall for fuzzy logic-based predictive models.

Table 4.1. Comparison to prove efficiency

Method	MSE	AFER
<i>Heuristic</i>	109728.7	44.87%
<i>Proposed Method</i>	35337.42	20.72%
<i>First order</i>	151459.9	46.90%

5. Conclusion

First, divide the data set into 6, 9, and 18 intervals and compute the values for each block. Use these midpoint values to compute rainfall forecasts for all blocks. The results were then validated by comparing with other existing methods using accuracy, precision and robustness of the proposed model. We found the new method to be the best advanced algorithm. The proposed model can be extended by working with more intervals. You can also apply frequency-based partitioning to intervals to fine-tune the distribution.

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Rainfall Forecast in Different Methods of Trend Equations by Fuzzy Time Series

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Abstract

Fuzzy time series models have been put forward for Rainfall Prediction from many researchers around the globe. Fuzzy time series methods do not require any assumptions valid for classic time series approaches. The most important disadvantage of fuzzy time series approaches is that it needs subjective decisions, especially in fuzzification stage. This paper proposes a novel improvement of forecasting approach based on using first order fuzzy time series. In contrast to traditional forecasting methods, fuzzy time series can be also applied to problems, in which historical rainfall data of Trichy district. In this study reveals some feature of FTS predicting Rainfall and the results have been compared with other methods.

Keywords: Predicting Rainfall, Fuzzy Time Series; Time Variant, Fuzzy Membership Grade; Fuzzy Logical Relations, Fuzzy Relation.

AMS Subject Classification: 05C78³.

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1. Introduction

Forecasting activities play an important role in our daily life. We often forecast the weather, earthquakes, stock market, and everything which people want to foresee. People can plan or prevent beforehand by forecasting activities. It is impossible to make a one hundred percent forecast, but we can do our best to increase the accuracy of forecasts. Traditional forecasting methods can deal with many forecasting cases, but they cannot solve forecasting problems in which the historical data are linguistic values. In [9, 10], Song and Chissom presented the definition of fuzzy time series and outlined its modeling by means of fuzzy relational equations and approximate reasoning [18, 25].

Fuzzy time series is a new concept which can be used to deal with forecasting problems in which historical data are linguistic values. In [10], Song and Chissom have applied the fuzzy time series model to forecast the enrollments of the University of Alabama, where a first-order time invariant model is developed and a step-by-step procedure is provided. In [10], they also pointed out that Bintley [1] has successfully applied fuzzy logic and approximate reasoning to a practical case of forecasting, but the concept of fuzzy time series was not applied on the method presented in [1].

The method presented by Song and Chissom [10] uses the following model for forecasting university enrollments: $A_i = A_{i-1} \circ R$, (1) where A_{i-1} is the enrollment of year $i-1$ represented by a fuzzy set [8, 23, 24], A_i is the forecasted enrollment of year i represented by a fuzzy set, " $\circ$ " is the max-min composition operator, and R is a fuzzy relation indicating fuzzy relationships between fuzzy time series. However, the forecasted method presented in [3] requires a large amount of computations to derive the fuzzy relation R of (1), and the max-min composition operations of (1) will take a large amount of computation time when the fuzzy relation R of (1) is very big. Thus, we must develop a new method to forecast the enrollments of the university in a more efficient manner. In [10], Sullivan et al. reviewed Making decisions on forecasting is a complicated process. Due to the complexity of meteorological phenomena, accurate prediction of rainfall is challenging one. To plan our day-to-day activities accurate weather predictions are important. Farmers need information about the weather to plan for planting and harvesting their crops.

Weather forecasting helps to keep us out of danger. The first-order time-invariant fuzzy time-series model and the first-order time-invariant model proposed by Song and Chissom, where these models are compared with each other and with a time-variant Markov model using linguistic labels with probability distributions. In 1993, Song and Chissom presented the theory of fuzzy time series. Fuzzy time series model deal with the both linguistic and numerical values.

In this paper, we review the concept of fuzzy time series and the forecasting methods presented in [12, 18, 23, 24]. A time series represents a collection of values of certain events or tasks which are obtained with respect to time. Future prediction of time series events has been attracted people from its beginning. Song and Chissom developed a model, based on uncertainty and imprecise knowledge contained in time series data. They initially used the fuzzy sets concept to represent or manage all these uncertainties and referred this concept as Fuzzy Time Series (FTS). Researchers have developed

numerous models based on the FTS concept to deal with the forecasting problems of short term as well as long term events. The following are some definitions related with fuzzy time series.

In section 2 some basic definitions of fuzzy time series are discussed. In section 3 rainfall predicting model and computational procedures are discussed.

2. Basic Definitions of Fuzzy Time Series

2.1. Time Series. A time series is a sequence of observations collected at regular time intervals when there is correlation among successive observations.

2.2. Fuzzy Time Series. Let $X(t)$ ($t = 0, 1, 2, \dots$) be the universe of discourse and the fuzzy set defined on $X(t)$ be $f_i(t)$ ($t = 0, 1, 2, \dots$). Then $F(t) = f_i(t)$ $t = 0, 1, 2, \dots$, $i = 1, 2, \dots$ the collection of all fuzzy sets defined on $X(t)$ is called a fuzzy time series of $X(t)$ ($t = 0, 1, 2, \dots$).

2.3. Fuzzy Relation. If $F(t)$ is caused by $F(t - 1)$, denoted by $F(t) \rightarrow F(t - 1)$, then this relationship can be represented by $F(t) = F(t - 1) \circ R(t, t - 1)$, where $\circ$ denotes the composition operator and $R(t, t - 1)$ is a fuzzy relation between $F(t)$ and $F(t - 1)$.

2.4. First Order Model. The model in which the relation $R(t, t - 1)$ is a fuzzy relation between $F(t)$ and $F(t - 1)$ is called the first order model of $F(t)$.

2.5. Time Invariant Fuzzy Time Series. If in first order model of $F(t)$ relation $R(t, t - 1) = R(t - 1, t - 2)$ for any time t , then $F(t)$ is called time invariant fuzzy time series.

2.6. Time Variant Fuzzy Time Series. If in first order model of $F(t)$ relation $R(t, t - 1) \neq R(t - 1, t - 2)$ for any time t , then $F(t)$ is called time invariant fuzzy time series.

2.7. n-order Fuzzy time series. Let $F(t)$ be a fuzzy time series. If $F(t)$ is caused by $F(t - 1), F(t - 2), \dots, F(t - n)$, then this fuzzy relationship is represented by $F(t) = F(t - 1), F(t - 2), \dots, F(t - n) \rightarrow F(t)$ and is called an n- order fuzzy time series.

2.8. Fuzzy variables. Imprecise data at equally spaced discrete time points are modeled as fuzzy variables. The set of this discrete fuzzy data forms a fuzzy time series.

Theorem 2.9.1. If $F(t)$ is a fuzzy time series, for any t , $F(t)$ has only finite elements $f_i(t)$ ($t = 1, 2, \dots, n$) and $F(t) = F(t - 1)$ then $F(t)$ is a time -invariant fuzzy time series.

Given that, $F(t)$ is a fuzzy time series, for any t , $F(t)$ has only finite elements $f_i(t)$ ($t = 1, 2, \dots, n$) and $F(t) = F(t - 1)$.

To prove that $F(t)$ is a time –invariant fuzzy time series.

Proof. Generality, rearrange the subscript of each element of $F(t)$ ($t = 1, 2, \dots, n$) such that these with the same values at different time t 's have the same subscript. For simplicity, double subscripts for each element. According to definition, “suppose $F(t)$ is a fuzzy time series ($t = \dots, 0, 1, 2, 3, 4, \dots$) and $t_1 \neq t_2$. If for any $f_i(t_1) \in F(t_1)$ there exists an $f_j(t_2) \in F(t_2)$ such that $f_i(t_1) = f_j(t_2)$ and vice versa, then define $F(t_1) = F(t_2)$ ”, the following is true.

$$\begin{aligned} \dots &= f_{10}(t) = f_{11}(t-1) = f_{12}(t-2) = f_{13}(t-3) = f_{14}(t-4) = \dots = f_{1n}(t-n) = \dots \\ \dots &= f_{20}(t) = f_{21}(t-1) = f_{22}(t-2) = f_{23}(t-3) = f_{24}(t-4) = \dots = f_{2n}(t-n) = \dots \\ \dots &= f_{30}(t) = f_{31}(t-1) = f_{32}(t-2) = f_{33}(t-3) = f_{34}(t-4) = \dots = f_{3n}(t-n) = \dots \quad (1) \\ \dots &= f_{40}(t) = f_{41}(t-1) = f_{42}(t-2) = f_{43}(t-3) = f_{44}(t-4) = \dots = f_{4n}(t-n) = \dots \\ \dots &= f_{n0}(t) = f_{n1}(t-1) = f_{n2}(t-2) = f_{n3}(t-3) = f_{n4}(t-4) = \dots = f_{nn}(t-n) = \dots \end{aligned}$$

Let $R(t, t-1) = \cup_{i,j} R_{ij}(t, t-1)$ and, $R(t-1, t-2) = \cup_{i,j} R_{ij}(t-1, t-2)$. For any $j \in J$ and $i \in I$, suppose $R_{ij}(t, t-1) = f_{i1}(t-1) \times f_{j0}(t)$ which is equivalent to $f_{j0}(t) = f_{i1}(t-1) \circ R_{ij}(t, t-1)$. According to (1), there exist an $f_{i2}(t-2)$ and $f_{j1}(t-1)$ such that $f_{j0}(t) = f_{j1}(t-1)$ and $f_{i1}(t-1) = f_{i2}(t-2)$. Therefore,

$$\begin{aligned} R_{ij}(t, t-1) &= f_{i1}(t-1) \times f_{j0}(t) = f_{i2}(t-2) \times f_{j1}(t-1) \\ &= R_{ij}(t-1, t-2) \end{aligned}$$

This means that $R_{ij}(t, t-1) \subseteq R_{ij}(t-1, t-2)$. Suppose $R_{ij}(t-1, t-2) = f_{i2}(t-2) \times f_{j1}(t-1)$. According to (1), there exist an $f_{j0}(t)$ and $f_{i1}(t-1)$ such that $f_{j0}(t) = f_{j1}(t-1)$ and $f_{i1}(t-1) = f_{i2}(t-2)$. Therefore,

$$R_{ij}(t-1, t-2) = f_{i2}(t-2) \times f_{j1}(t-1) = f_{i1}(t-2) \times f_{j0}(t) = R_{ij}(t, t-1).$$

This implies that $R_{ij}(t-1, t-2) \subseteq R_{ij}(t, t-1)$. Hence, $R_{ij}(t, t-1) = R_{ij}((t-1), (t-1)-1)$, $R_{ij}(t, t-2) = R_{ij}((t-1), (t-1)-2)$, $R_{ij}(t, t-3) = R_{ij}((t-1), (t-1)-3)$, $R_{ij}(t, t-4) = R_{ij}((t-1), (t-1)-4)$... Similarly, we get for any t , $R_a(t, t-m) = R_{ij}((t-1), (t-1)-m)$ and $R_0(t, t-m) = R \circ ((t-1), (t-1)-m)$. According to definition, “If $F(t) = F(t-1) \circ R(t, t-1)$ or $F(t) = (F(t-1) \cup F(t-2) \cup \dots \cup F(t-m)) \circ R_0(t, t-m)$ or $F(t) = (F(t-1) \times F(t-2) \times \dots \times F(t-m)) \circ R_a(t, t-m)$, the fuzzy relation $R(t, t-1)$, $R_a(t, t-m)$ or $R_0(t, t-m)$ of $F(t)$ is independent of time t , that is to say for different times t_1 and t_2 , $R(t_1, t_1-1) = R(t_2, t_2-1)$, or $R_a(t_1, t_1-m) = R_a(t_2, t_2-m)$, or $R_0(t_1, t_1-m) = R_0(t_2, t_2-m)$, then $F(t)$ is called a time –invariant fuzzy time series. Otherwise, it is called a time –variant fuzzy time series.

In the case of time –invariant fuzzy time series,

$$\begin{aligned} R(t, t-1) &= R \\ R_a(t, t-m) &= R_a(m), \end{aligned}$$

$$R_0(t, t - m) = R_0(m).$$

It should be noted that generally at different times t_1 and t_2 , $R(t_1, t_1 - 1) \neq R(t_2, t_2 - 1)$ or $R_a(t_1, t_1 - m) \neq R_a(t_2, t_2 - m)$ and $R_0(t_1, t_1 - m) \neq R_0(t_2, t_2 - m)$. There are two reasons for this: first the universe of discourse on which the fuzzy sets are defined may be different at different times; second, the value of $F(t)$ at different times may be different. Therefore, the classifications of fuzzy time series are meaningful." Hence, proved that $F(t)$ is called a time – invariant fuzzy time series.

Theorem 2.9.2 If $F(t)$ is a fuzzy time series, for any t , $F(t) = F(t - 1)$ and $F(t)$ has only finite elements $f_i(t)$ ($i = 1, 2, \dots, n$) then $R(t, t - 1)$ can be expressed as $R = R(t, t - 1) = \dots \cup f_{i1}(t - 1) \times f_{j0}(t) \cup f_{i2}(t - 2) \times f_{j1}(t - 1) \cup \dots \cup f_{im}(t - m) \times f_{jm} - 1(t - m + 1) \cup \dots$ where $m > 0$ and each pair of fuzzy sets are different.

Given that: $F(t)$ is a fuzzy time series, for any t , $F(t) = F(t - 1)$ and $F(t)$ has only finite elements $f_i(t)$ ($t = 1, 2, \dots, n$).

To prove that: $R = R(t, t - 1) = \dots \cup f_{i1}(t - 1) \times f_{j0}(t) \cup f_{i2}(t - 2) \times f_{j1}(t - 1) \cup \dots \cup f_{im}(t - m) \times f_{jm} - 1(t - m + 1) \cup \dots$ where $m > 0$ and each pair of fuzzy sets are different.

Proof. We know that, "If $F(t)$ is a fuzzy time series, for any t , $F(t)$ has only finite elements $f_i(t)$ ($i = 1, 2, \dots, n$) and $F(t) = F(t - 1)$ then $F(t)$ is a time – invariant fuzzy time series." Hence, $F(t)$ is a time – invariant fuzzy time series. According to definition, "If $F(t) = F(t - 1) \circ R(t, t - 1)$ or $F(t) = (F(t - 1) \cup F(t - 2) \cup \dots \cup F(t - m)) \circ R_0(t, t - m)$ or $F(t) = (F(t - 1) \times F(t - 2) \times \dots \times F(t - m)) \circ R_a(t, t - m)$, the fuzzy relation $R(t, t - 1)$, $R_a(t, t - m)$ or $R_0(t, t - m)$ of $F(t)$ is independent of time t , that is to say for different times t_1 and t_2 , $R(t_1, t_1 - 1) = R(t_2, t_2 - 1)$, or $R_a(t_1, t_1 - m) = R_a(t_2, t_2 - m)$, or $R_0(t_1, t_1 - m) = R_0(t_2, t_2 - m)$, then $F(t)$ is called a time – invariant fuzzy time series. Otherwise, it is called a time – variant fuzzy time series. In the case of time – invariant fuzzy time series, $R(t, t - 1) = R$, $R_a(t, t - m) = R_a(m)$, $R_0(t, t - m) = R_0(m)$ ".

It should be noted that generally at different times t_1 and t_2 , $R(t_1, t_1 - 1) \neq R(t_2, t_2 - 1)$ or $R_a(t_1, t_1 - m) \neq R_a(t_2, t_2 - m)$ and $R_0(t_1, t_1 - m) \neq R_0(t_2, t_2 - m)$. There are two reasons for this: first the universe of discourse on which the fuzzy sets are defined may be different at different times; second, the value of $F(t)$ at different times may be different. Therefore, the classifications of fuzzy time series are meaningful.

$R(t, t - 1)$ is independent of t and therefore, $R(t, t - 1) = R$.

Because of this, the following holds:

$$\dots = R(t + 1, t) = R(t, t - 1) = R(t - 1, t - 2) = R(t - 2, t - 3) = \dots$$

Since $R(t, t-1) = \cup_{ij} (R_{ij}(t, t-1)) = f_i(t-1) \times f_j(t)$ ($t = \dots 0, 1, 2, \dots n$) and $F(t)$ has only n elements, the maximum number of different components, $(R_{ij}(t, t-1))$'s, of $R(t, t-1)$, is n^2 . Clearly, all $(R_{ij}(t, t-1))$'s have the same components.

Let us consider a selection process. First at an arbitrary and fixed time, select a pair of elements of $F(t)$ and $F(t-1)$, say, $f_{j_0}(t)$ and $f_{i_1}(t-1)$, and then calculate $f_{i_1}(t-1) \times f_{j_0}(t)$. At $t-1$, select a different pair of elements of $F(t-1)$ and $F(t-2)$, say, $f_{i_1}(t-1)$ and $f_{i_2}(t-2)$, and then calculate $f_{i_2}(t-2) \times f_{j_1}(t-1)$. Suppose at time $t-k$ ($k > 1$), we have selected a pair of elements, say, $f_{j_k-1}(t-k+1)$ and $f_{i_k}(t-k)$, of $F(t-k+1)$ and $F(t-k)$ which are different from any pairs selected before, and then calculate $f_{i_k}(t-k) \times f_{j_k-1}(t-k+1)$. Since R has at most n^2 components, the above procedure, at most n^2 steps are needed to select all the different components needed to calculate R . Hence, proved that $R = R(t, t-1) = \dots \cup f_{i_1}(t-1) \times f_{j_0}(t) \cup f_{i_2}(t-2) \times f_{j_1}(t-1) \cup \dots \cup f_{i_m}(t-m) \times f_{j_m-1}(t-m+1) \cup \dots$ where $m > 0$ and each pair of fuzzy sets are different.

3. Fuzzy Time Series Rainfall Predicting Model

3.1. General description of the model

First, we tested the model [2] with rainfall data to forecast the distribution. Then we applied the methods of [6], [3] and [4]. Steps for finding forecasting fuzzy time series (Chen's Method) Chen [2] proposed a simple method to forecast the enrollment in fuzzy time series. The seven steps involved in the method is given below:

- Step 1: Define the universe of discourse to accommodate the time series.
- Step 2: Define the universe of discourse and partition it into equally lengthy intervals.
- Step 3: Fuzzify historical data.
- Step 4: Identify fuzzy logical relationship (FLR's).
- Step 5: Establish fuzzy logical relationship groups (FLRG's).
- Step 6: Computing the predicted rainfall (fuzzy output)
- Step 7: Defuzzification of the fuzzy output for crisp predicting

3.2. Computational Procedures

The descriptive statistics and block wise rainfall data are given in Table I & II.

Table 3.1. Actual Enrollments and Linguistic of Historical data

Blocks	Actual Enrollments	Blocks	Actual Enrollments
Andanallur	1085.23	Pullambadi	1163.59
Lalgudi	1136.79	T.pet	710.16
Manachanallur	569.44	Thiruverumbur	960.9
Manapparai	837.43	Thottiam	866.73
Manikandam	379.79	Thuraiur	942.11
Marungapuri	942.16	Uppiliyapuram	473.35
Musiri	794.31	Vaiyampatty	922.27

Table 3.2. Descriptive Statistics of the Rainfall Data from 2004 - 2010

Minimum(min)	= 379.79
Maximum(max)	= 1163.59
Range(R)	= 783.8
Count(n)	= 14
Sum	= 11784.26
Mean($\bar{x}$)	= 841.733
Median($\tilde{x}$)	= 894.5
Mode	= No mode
Standard Deviation(s)	= 237.65
Variance(s^2)	= 56479.51

3.2.1. Specification of the universe of discourse U (Step 1)

Generally, to define the universe of discourse U, find the minimum enrollment D_{min} , and the maximum enrollment D_{max} from the known historical data. Based on D_{min} , and D_{max} Define U as $[D_{min} - D_1, D_{max} + D_2]$ where D_1 and D_2 are two proper positive numbers. So far, we have collected historical enrollment data from 2004 to 2010. The minimum enrollment D_{min} and the maximum enrollment D_{max} are 379.79 and 1163.59

respectively. Therefore, we choose $D_1 = 79.79$ and $D_2 = 36.41$ to make $U = [300, 1200]$.

3.2.2. Partitioning U into Several Even Length Intervals (Step 2)

The process is the same as in [3]. We have six intervals which are given below.

Table 3.3. Length of Intervals

u_1	[300 - 450]
u_2	[450 - 600]
u_3	[600 - 750]
u_4	[750 - 900]
u_5	[900 - 1050]
u_6	[1050 - 1200]

3.2.3. Fuzzify historical data (Step 3).

Consider six fuzzy sets which are $A_1 =$ (very few), $A_2 =$ (very very few), $A_3 =$ (Moderate), $A_4 =$ (High), $A_5 =$ (very High), and $A_6 =$ (very very High). To define these fuzzy sets, we can use the same process as in [3]. Thus, all the fuzzy sets A_i ($i = 1$ to 6) can be defined as follows

The fuzzy sets A_i are expressed as follows:

$$\begin{aligned}
 A_1 &= 1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\
 A_2 &= 0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 \\
 A_3 &= 0/u_1 + 0.5/u_2 + 1/u_3 + 0.5/u_4 + 0/u_5 + 0/u_6 \\
 A_4 &= 0/u_1 + 0/u_2 + 0.5/u_3 + 1/u_4 + 0.5/u_5 + 0/u_6 \\
 A_5 &= 0/u_1 + 0/u_2 + 0/u_3 + 0.5/u_4 + 1/u_5 + 0.5/u_6 \\
 A_6 &= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0.5/u_5 + 1/u_6
 \end{aligned}$$

Where u_i ($i = 1$ to 6) is the element and the number below / is the membership of u_i to A_j ($j = 1$ to 6). For simplicity, we will also use A_1, A_2, A_3, A_4, A_5 and A_6 as row vectors whose elements are the memberships of the corresponding fuzzy sets.

3.2.4. Identify fuzzy logical relationship (step 4)

To fuzzify the historical data, we can apply the same method as employed in [3]. For the block Andanallur and Lalgudi, the enrollments are 1085.23 and 1136.79 respectively, which are almost equal to those of Pullambadi and Thiruverumbur respectively and therefore the fuzzified enrollments for Andanallur and Lalgudi are A_6

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and A_6 ([3] for details of the fuzzification process). All the fuzzified historical data are listed in Table III where A_i ($i = 1$ to 6) are defined in Section 4.4

Table 3.4. Fuzzified Historical Enrollments Based on Variations

Blocks	Rainfall (mm)	Fuzzified	Blocks	Rainfall (mm)	Fuzzified
Andanallur	1085.23	A_6	Pullambadi	1163.59	A_6
Lalgudi	1136.79	A_6	T.pet	710.16	A_4
Manachanallur	569.44	A_2	Thiruverumbur	960.9	A_5
Manapparai	837.43	A_4	Thottiam	866.73	A_4
Manikandam	379.79	A_1	Thuraiur	942.11	A_5
Marungapuri	942.16	A_5	Uppiliyapuram	473.35	A_2
Musiri	794.31	A_4	Vaiyampatty	922.27	A_5

3.2.5. Establish fuzzy logical relationship groups (FLRG's) (step 5).

The fuzzy logical relationships are as follows (the repeated relationships are omitted)

Table 3.5. Predictions of Fuzzy Logical Relationships

No.	Relationship	No.	Relationship
1	$A_6 \rightarrow A_6$	8	$A_6 \rightarrow A_4$
2	$A_6 \rightarrow A_2$	9	$A_4 \rightarrow A_5$
3	$A_2 \rightarrow A_4$	10	$A_5 \rightarrow A_4$
4	$A_4 \rightarrow A_1$	11	$A_4 \rightarrow A_5$
5	$A_1 \rightarrow A_5$	12	$A_5 \rightarrow A_2$
6	$A_5 \rightarrow A_4$	13	$A_2 \rightarrow A_5$
7	$A_4 \rightarrow A_6$		

3.2.6. Computing the predicted rainfall (fuzzy output) (step6)

Generally, to determine the fuzzy relations, we need to find two consecutive blocks' data, say, t and $t - 1$. If the data for t is A_t , and A_{t-1} for $t - 1$, then the fuzzy logical relationship will be $A_{t-1} \rightarrow A_t$, and the fuzzy relation between A_{t-1} and A_t will be $A_{t-1}^T \times A_t$. The fuzzy logical relationships are as follows (the repeated relationships are

omitted).

Table 3.6. Predictions of Fuzzy Logical Relationships Groups

No.	Group of Relationships
1	$A_1 \rightarrow A_5$
2	$A_2 \rightarrow A_4, A_5$
3	$A_3 \rightarrow 0$
4	$A_2 \rightarrow A_1, A_5, A_5, A_6$
5	$A_5 \rightarrow A_2, A_4$
6	$A_6 \rightarrow A_2, A_4, A_6$

Combine the fuzzy relationship into fuzzy logical relation group starting from identical left-hand sides (Table VI).After that calculate R_i , $i=1, 2, 3 \dots 6$, as a union of logical relationship in each group.

$$R_1 = A_1^T \times A_5$$

$$R_2 = A_2^T \times A_4 \cup A_2^T \times A_5$$

.....

.....

.....

$$R_6 = A_6^T \times A_2 \cup A_6^T \times A_2 \cup A_6^T \times A_6$$

Table 3.7. Fuzzification of Time Series Data of Rainfall

Blocks	Actual	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	Fuzzified
Andanallur	1085.23	0	0	0.2	0.5	0.8	1	A ₆
Lalgudi	1136.79	0	0	0	0.4	0.7	1	A ₆
Manachanallur	569.44	0	1	0.5	0.2	0	0	A ₂
Manapparai	837.43	0	0.3	0.7	1	0.5	0.2	A ₄
Manikandam	379.79	1	0.5	0.5	0	0	0	A ₁
Marungapuri	942.16	0	0	0.3	0.5	1	0.2	A ₅
Musiri	794.31	0	0.2	0.7	1	0.5	0	A ₄
Pullambadi	1163.59	0	0	0.3	0.5	0.8	1	A ₆
Thathaingarpet	710.16	0	0.3	0.7	1	0.5	0	A ₄
Thiruverumbur	960.9	0	0	0.2	0.5	1	0.5	A ₅
Thottium	866.73	0	0.2	0.7	1	0.5	0.3	A ₄
Thurayur	942.11	0	0	0.4	0.5	1	0.5	A ₅
Uppiliyapuram	473.35	0	1	0.5	0.2	0	0	A ₂

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Vaiyampaty	922.27	0	0.2	0.4	05	1	0.3	A ₅

Table 3.8. Forecasted Outputs and Enrollments from 2004 to 2010

No.	Blocks	Actual	Fuzzy Outputs					
1.	Andanallur	1085.23	0.5	1	0.5	1	0.5	1
2.	Lalgudi	1136.79	0.5	1	0.5	1	0.5	1
3.	Manachanallur	569.44	0.5	0.5	0.5	1	1	0.5
4.	Manapparai	837.43	1	1	0.5	0.5	1	1
5.	Manikandam	379.79	0.5	0.5	0.5	0.5	0.5	0.5
6.	Marungapuri	942.16	0.5	0.5	0.5	1	0.5	0.5
7.	Musiri	794.31	1	1	0.5	0.5	1	1
8.	Pullambadi	1163.59	0.5	1	0.5	1	0.5	1
9.	Thathaingarpet	710.16	1	1	0.5	0.5	1	1
10.	Thiruverumbur	960.9	0.5	1	0.5	1	0.5	0.5
11.	Thottium	866.73	1	1	0.5	0.5	1	1
12.	Thurayur	942.11	0.5	1	0.5	1	0.5	0.5
13.	Uppiliyapurm	473.35	0.5	0.5	0.5	1	1	0.5
14.	Vaiyampaty	922.27	0.5	1	0.5	1	0.5	0.5

The prediction rainfall obtained by the four models have been place in the following table and presented in figure1.

3.2.7. Defuzzification of the fuzzy output for crisp predicting (step7)

Defuzzification is the process by which fuzzy output of model is transformed to crisp values for getting the predicted values. The output of each of the four methods are defuzzified in the following ways.

Table 3.9. Rainfall Forecast in Different Methods

Blocks	Actual	Mid. value of 16 th Interval (Mehod1)	Max. value of 9 th Interval (Method2)	Tsaur et al. (Method 3)	Chen (Method 4)
Andanallur	1085.23	1125	1100	1050	1150
Lalgudi	1136.79	1125	1200	1150	550
Manachanallur	569.44	525	600	550	850
Manapparai	837.43	825	900	850	950
Manikandam	379.79	375	400	400	683.30
Marungapuri	942.16	975	1000	950	1150
Musiri	794.31	825	800	800	1050
Pullambadi	1163.59	1125	1200	1150	950
Thathaingarpet	710.16	825	800	750	683.30
Thiruverumbur	960.9	975	1000	950	662.10
Thottium	866.73	825	900	850	950
Thurayur	942.11	975	950	950	450

Uppiliyapurm	473.35	525	500	500	950
Vaiyampaty	922.27	975	950	950	950

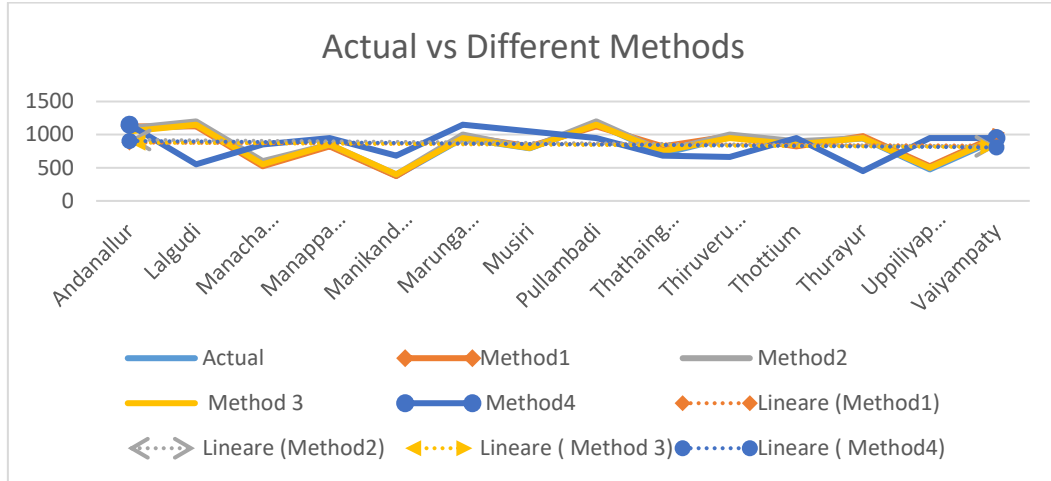


Figure 3.1. Graph of Actual vs. Different methods

Table 3.10. Rainfall Forecast in Different Methods of Trend equations with r^2 value

Methods	Trend Equations	R^2 value
Actual	$y = -7.2052x + 895.77$	0.0161
Method1	$y = -3.6264x + 884.34$	0.0041
Method2	$y = -7.6923x + 929.12$	0.0175
Method 3	$y = -5.1648x + 885.16$	0.0088
Method4	$y = -7.4859x + 911.77$	0.0209

Methods and its R square value

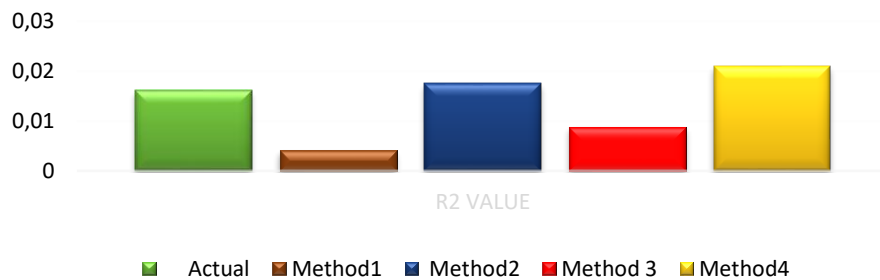


Figure 3.2. Graph of Methods and its R square value

4. Conclusions

In this paper, we have proposed a new method (Method1) for handling forecasting problems based on fuzzy time series, where the data of historical enrollments of the Trichy district rainfall data are adopted to illustrate the forecasting process. We have also shown that the method1 is more efficient than another Method. Furthermore, the linear trend line and R^2 value (Method 1) is smaller than another ones. The forecasting ability is better than traditional forecasting methods of Chen and tsaur et al., in the area of forecasting average rainfall. It has been successfully implemented to the forecasting of the annual average rainfall of a district of Trichy.

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On δ -open sets in ideal nano topological spaces

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Abstract

Aim of this paper, the new notions of introduce δ -open sets in ideal nano topological spaces and investigate some of their properties. A comparison between these types of ideal nano continuity will be discussed. Finally, we introduce application examples in ideal nano topological spaces.

Keywords: nano-open (resp. n -open), semi- n -open (resp. ns -open) semi- nI -open, pre- nI -open and strong β - nI -open and nI_δ -open.

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1 Introduction

An ideal I (16) on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1. $A \in I$ and $B \subset A$ imply $B \in I$ and
2. $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a space (X, τ) with an ideal I on X if $\wp(X)$ is the set of all subsets of X , a set operator $(.)^* : \wp(X) \rightarrow \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ (2). The closure operator defined by $cl^*(A) = A \cup A^*(I, \tau)$ (15) is a Kuratowski closure operator which generates a topology $\tau^*(I, \tau)$ called the $\star$ -topology which is finer than τ . We will simply write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X , then (X, τ, I) is called an ideal topological space or an ideal space.

Rajasekaran and Nethaji, introduced pre- nI -open sets and α - nI -open sets in the concept of ideal nano topological spaces.

In this paper, we introduce the notions of δ -open sets in ideal nano topological spaces and investigate some of their properties. A comparison between these types of ideal nano continuity will be discussed. Finally, we introduce application examples in ideal nano topological spaces.

2 Preliminaries

Definition 2.1. (9) Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x .
2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$.
3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. (3) Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$,
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n -open sets). The complement of a n -open set is called n -closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset O of U are denoted by $I_n(O)$ and $C_n(O)$, respectively.

A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called (6) an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes (6) the family of nano open sets containing x .

In future an ideal nano topological spaces $(U, \mathcal{N}, I)$ is referred as a space.

Definition 2.3. (6) Let $(U, \mathcal{N}, I)$ be a space with an ideal I on U . Let $(\cdot)_n^*$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U).

For a subset $O \subseteq U$, $O_n^*(I, \mathcal{N}) = \{x \in U : G_n \cap O \notin I, \text{ for every } G_n \in G_n(x)\}$ is called the nano local function (briefly, n -local function) of A with respect to I and $\mathcal{N}$. We will simply write O_n^* for $O_n^*(I, \mathcal{N})$.

Theorem 2.1. (6) Let $(U, \mathcal{N}, I)$ be a space and O and B be subsets of U . Then

1. $O \subseteq B \Rightarrow O_n^* \subseteq B_n^*$,
2. $O_n^* = C_n(O_n^*) \subseteq C_n(O)$ (O_n^* is a n -closed subset of $C_n(O)$),
3. $(O_n^*)_n^* \subseteq O_n^*$,
4. $(O \cup B)_n^* = O_n^* \cup B_n^*$,
5. $V \in \mathcal{N} \Rightarrow V \cap O_n^* = V \cap (V \cap O)_n^* \subseteq (V \cap O)_n^*$,
6. $J \in I \Rightarrow (O \cup J)_n^* = O_n^* = (O - J)_n^*$.

Theorem 2.2. (6) Let $(U, \mathcal{N}, I)$ be a space with an ideal I and $O \subseteq O_n^*$, then $O_n^* = C_n(O_n^*) = C_n(O)$.

Definition 2.4. (8) A subset A of a space $(U, \mathcal{N}, I)$ is $n\star$ -dense in itself (resp. $n\star$ -perfect and $n\star$ -closed) if $O \subseteq O_n^*$ (resp. $O = O_n^*$, $O_n^* \subseteq O$).

The complement of a $n\star$ -closed set is said to be $n\star$ -open.

Definition 2.5. (5) A subset O of U in a nano topological space $(U, \mathcal{N})$ is called nano-codense (briefly n -codense) if $U - O$ is n -dense.

Theorem 2.3. (6) Let $(U, \mathcal{N}, I)$ be an ideal nano space. Then $\mathcal{I}$ is n -codense $\iff O \subseteq O^*$ for every n -open set O .

Definition 2.6. (6) Let $(U, \mathcal{N}, I)$ be a space. The set operator C_n^* called a nano $\star$ -closure is defined by $C_n^*(O) = O \cup O_n^*$ for $O \subseteq U$.

It can be easily observed that $C_n^*(O) \subseteq C_n(O)$.

Theorem 2.4. (7) In a space $(U, \mathcal{N}, I)$, if O and B are subsets of U , then the following results are true for the set operator $n\text{-cl}^*$.

1. $O \subseteq C_n^*(O)$,
2. $C_n^*(\phi) = \phi$ and $C_n^*(U) = U$,
3. If $O \subset B$, then $C_n^*(O) \subseteq C_n^*(B)$,
4. $C_n^*(O) \cup C_n^*(B) = C_n^*(O \cup B)$.
5. $C_n^*(C_n^*(O)) = C_n^*(O)$.

Definition 2.7. A subset O of a nano space $(U, \mathcal{N})$, is called a

1. nano pre-open (resp. np -open) set (3) if $O \subseteq I_n(C_n(O))$.
2. nano semi-open (resp. ns -open) set (3) if $O \subseteq C_n(I_n(O))$.
3. nano ε -open (resp. $n\varepsilon$ -open) set (13) if $I_n(C_n(O)) \subseteq C_n(I_n(O))$.
4. nano nowhere dense (resp. n -nowhere dense) (4) if $I_n(C_n(O)) = \phi$.

Definition 2.8. A subset O of an ideal nano space $(U, \mathcal{N}, I)$, is called a

1. nano pre- I -open (resp. pre- nI -open) (10) if $O \subseteq I_n(C_n^*(O))$.
2. nano semi- I -open (resp. semi- nI -open) (10) if $O \subseteq C_n^*(I_n(O))$.
3. nano α - I -open (resp. α - nI -open) (10) if $O \subseteq I_n(C_n^*(I_n(O)))$.
4. strongly nano β - I -open (resp. $S\beta$ - nI -open) (11) if $O \subseteq C_n^*(I_n(C_n^*(O)))$.

Theorem 2.5. (10) In a nano space $(U, \mathcal{N}, I)$, if O is α - nI -open, then O is semi- nI -open.

Definition 2.9. (1) A function $f : (U, \mathcal{N}, I) \longrightarrow (V, \mathcal{N}')$ is said to be

1. α - nI -continuous if $f^{-1}(H)$ is α - nI -open set in $(U, \mathcal{N}, I)$ for every n -open set H in $(V, \mathcal{N}')$.
2. pre- nI -continuous if $f^{-1}(H)$ is pre- nI -open set in $(U, \mathcal{N}, I)$ for every n -open set H in $(V, \mathcal{N}')$.
3. semi- nI -continuous if $f^{-1}(H)$ is semi- nI -open set in $(U, \mathcal{N}, I)$ for every n -open set H in $(V, \mathcal{N}')$.

3 On δ -open sets in ideal nano space

Definition 3.1. A subset O of an ideal nano space $(U, \mathcal{N}, I)$, is called a nano I_δ -open (resp. nI_δ -open) set if $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$.

Example 3.1. Let $U = \{A_1, A_2, A_3, A_4\}$ with $U/R = \{\{A_2\}, \{A_4\}, \{A_1, A_3\}\}$ and $X = \{A_3, A_4\}$.

Then the nano topology $\mathcal{N} = \{\phi, \{A_4\}, \{A_1, A_3\}, \{A_1, A_3, A_4\}, U\}$ and $I = \{\phi, \{A_3\}\}$. Clear that $\{\phi, \{A_2\}, \{A_3\}, \{A_4\}, \{A_1, A_3\}, \{A_2, A_3\}, \{A_2, A_4\}, \{A_1, A_2, A_3\}, \{A_1, A_3, A_4\}, U\}$ is nI_δ -open.

Proposition 3.1. Let $(U, \mathcal{N}, I)$ be an ideal nano space. Then a subset of U is semi- nI -open $\iff$ if it is both nI_δ -open and $\mathcal{S}\beta$ - nI -open.

Proof.

Necessity. Let O be a semi- nI -open, then we have $O \subseteq C_n^*(I_n(O)) \subseteq C_n^*(I_n(C_n^*(O)))$. This show that O is $\mathcal{S}\beta$ - nI -open.

Moreover, $I_n(C_n^*(O)) \subseteq C_n^*(O) \subseteq C_n^*(C_n^*(I_n(O))) = C_n^*(I_n(O))$. Therefore O is nI_δ -open.

Sufficiency. Let O be nI_δ -open and $\mathcal{S}\beta$ - nI -open, then we have $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$. Thus we obtain that $C_n^*(I_n(C_n^*(O))) \subseteq C_n^*(C_n^*(I_n(O))) = C_n^*(I_n(O))$. Since O is $\mathcal{S}\beta$ - nI -open, we have $O \subseteq C_n^*(I_n(C_n^*(O))) \subseteq C_n^*(I_n(O))$ and $O \subseteq C_n^*(I_n(O))$. Hence O is semi- nI -open. $\square$

Proposition 3.2. Let $(U, \mathcal{N}, I)$ be an ideal nano space. Then a subset of U is α - nI -open $\iff$ if it is both nI_δ -open and pre- nI -open.

Proof.

Necessity. Let O be a α - nI -open, since every α - nI -open set is semi- nI -open (10), by Proposition 3.1 O is a nI_δ -open set. Now we prove that $O \subseteq I_n(C_n^*(O))$. Since O is α - nI -open set, we have $O \subseteq I_n(C_n^*(I_n(O))) \subseteq I_n(C_n^*(O))$. Hence O is a pre- nI -open set.

Sufficiency. Let O be a nI_δ -open and pre- nI -open. Then we have $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$ and hence $I_n(C_n^*(O)) \subseteq I_n(C_n^*(I_n(O)))$. Since O is pre- nI -open, we have $O \subseteq I_n(C_n^*(O))$. Therefore we obtain that $O \subseteq I_n(C_n^*(I_n(O)))$ and hence O is α - nI -open. $\square$

Remark 3.1. In $(U, \mathcal{N}, I)$ ideal nano space.

1. $\mathcal{S}\beta$ - nI -open and nI_δ -open are independent.
2. pre- nI -open and nI_δ -open are independent.

Example 3.2. In Example 3.1,

1. the set $\{A_1\}$ is $\mathcal{S}\beta$ - nI -open set but not nI_δ -open.
2. the set $\{A_2\}$ is not $\mathcal{S}\beta$ - nI -open but nI_δ -open.
3. the set $\{A_1, A_4\}$ is pre- nI -open but not nI_δ -open.
4. the set $\{A_2, A_3\}$ is not pre- nI -open but nI_δ -open.

Proposition 3.3. Let O, P be subsets of an ideal nano space $(U, \mathcal{N}, I)$. If $O \subseteq P \subseteq C_n^*(O)$ and O is nI_δ -open, then P is nI_δ -open.

Proof.

Suppose that $O \subseteq P \subseteq C_n^*(O)$ and O is nI_δ -open. Then, since O is nI_δ -open, we have $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$. Since, $O \subseteq P$, $C_n^*(I_n(O)) \subseteq C_n^*(I_n(P))$ and $I_n(C_n^*(O)) \subseteq C_n^*(I_n(P))$. Since $P \subseteq C_n^*(O)$, we have $C_n^*(P) \subseteq C_n^*(C_n^*(O)) = C_n^*(O)$ and $I_n(C_n^*(P)) \subseteq I_n(C_n^*(O))$. Therefore, we obtain that $I_n(C_n^*(P)) \subseteq C_n^*(I_n(P))$. This show that P is a nI_δ -open. $\square$

Proposition 3.4. let O, P and Q be subsets of an ideal nano space $(U, \mathcal{N}, I)$. If O is nI_δ -open, then $O = P \cup Q$, where P is α - nI -open, $I_n(C_n^*(Q)) = \phi$ and $P \cap Q = \phi$.

Proof.

Suppose that O is nI_δ -open. Then we have $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$ and $I_n(C_n^*(O)) \subseteq I_n(C_n^*(I_n(O)))$.

Now we have $O = (I_n(C_n^*(O)) \cap O) \cup (O - I_n(C_n^*(O)))$. Now, we set $P = I_n(C_n^*(O)) \cap O$ and $Q = O - I_n(C_n^*(O))$.

We first show that P is α - nI -open, that is, $P \subseteq I_n(C_n^*(I_n(P)))$. Now we have $I_n(C_n^*(I_n(P))) = I_n(C_n^*(I_n(I_n(C_n^*(O)) \cap O))) = I_n(C_n^*(I_n(C_n^*(O)) \cap I_n(O))) = I_n(C_n^*(I_n(O)))$. Since O is nI_δ -open, $I_n(C_n^*(I_n(O))) \supseteq I_n(C_n^*(O)) \supseteq P$ and thus P is α - nI -open.

Next we show that $I_n(C_n^*(Q)) = \phi$. Since $\mathcal{N} \subseteq \mathcal{N}^*$, $C_n^*(K) \subseteq C_n(K)$ for any subset K of U . Therefore, we have $I_n(C_n^*(Q)) = I_n(C_n^*(O \cap (U - I_n(C_n^*(O))))) \subseteq I_n(C_n^*(O)) \cap I_n(C_n^*(U - I_n(C_n^*(O)))) \subseteq I_n(C_n^*(O)) \cap I_n(C_n(U - I_n(C_n^*(O)))) \subseteq I_n(C_n^*(O)) \cap (U - I_n(C_n^*(O))) = \phi$.

It is obvious that $P \cap Q = (I_n(C_n^*(O)) \cap O) \cap (O - I_n(C_n^*(O))) = \phi$. $\square$

Remark 3.2. In an ideal nano space $(U, \mathcal{N}, I)$, $n\epsilon$ -open and nI_δ -open sets are independent.

Example 3.3. In Example 3.1,

1. the set $\{A_1, A_2\}$ is $n\epsilon$ -open but not nI_δ -open.
2. the set $\{A_3\}$ is not $n\epsilon$ -open but nI_δ -open.

Proposition 3.5. *Let $(U, \mathcal{N}, I)$ be an ideal nano space and $O \subseteq U$. If O is both nI_δ -open and ns -closed, then O is a $n\varepsilon$ -open.*

Proof.

Since O is ns -closed, $I_n(C_n(O)) \subseteq O$ and hence $I_n(C_n(O)) = I_n(O)$. Thus, $I_n(C_n(O)) \subseteq I_n(O) \subseteq I_n(C_n^*(O)) \subseteq C_n^*(I_n(O)) = I_n(O) \cup (I_n(O))^*_n \subseteq C_n(I_n(O))$ and hence we obtain $I_n(C_n(O)) \subseteq C_n(I_n(O))$. This show that O is $n\varepsilon$ -open. $\square$

Proposition 3.6. *Let $(U, \mathcal{N}, I)$ be an ideal nano space. Let $I = \{\phi\}$ or $I = H$, where H is the ideal of n -nowhere dense. Then a subset O of U is nI_δ -open $\iff$ O is a $n\varepsilon$ -open.*

Proof.

1. Let $I = \{\phi\}$. Then for every subset O of U , $O_n^* = C_n(O)$ and $C_n^*(O) = O \cup O_n^* = O \cup C_n(O) = C_n(O)$. Therefore, the statement holds obviously.
2. Let $I = H$, we have $O_n^* = C_n(I_n(C_n(O)))$.
First, let O be a $n\varepsilon$ -open. Then $I_n(C_n(O)) \subseteq C_n(I_n(O))$ and $I_n(C_n^*(O)) = I_n(O_n^* \cup O) \subseteq I_n(C_n(O) \cup O) = I_n(C_n(O)) \subseteq C_n(I_n(O)) = C_n(I_n(C_n(I_n(O)))) = (I_n(O))^*_n \subseteq C_n^*(I_n(O))$.
Therefore, O is nI_δ -open set. Next, let O be a nI_δ -open set. Then $I_n(C_n^*(O)) \subseteq C_n^*(I_n(O))$. We have $I_n(C_n(O)) = I_n(C_n(I_n(C_n(O)))) = I_n(O_n^*) \subseteq I_n(O_n^* \cup O) = I_n(C_n^*(O)) \subseteq C_n^*(I_n(O)) = (I_n(O))^*_n \cup I_n(O) = C_n(I_n(C_n(I_n(O)))) \cup I_n(O) = C_n(I_n(O))$. Therefore, $n\varepsilon$ -open. $\square$

4 On semi- δ - nI -continuous

Definition 4.1. A function $f : (U, \mathcal{N}, I) \longrightarrow (V, \mathcal{N}')$ is called a

1. semi- δ - nI -continuous if every $H \in \mathcal{N}'$, $f^{-1}(H) \in nI_\delta$ -open.
2. strong β - nI -continuous (resp. $\mathcal{S}\beta$ - nI -continuous) if every $H \in \mathcal{N}'$, $f^{-1}(H) \in \mathcal{S}\beta$ - nI -open.

Theorem 4.1. *For a function $f : (U, \mathcal{N}, I) \longrightarrow (V, \mathcal{N}')$, the following properties are equivalent.*

1. f is semi- nI -continuous.
2. f is $\mathcal{S}\beta$ - nI -continuous and semi- δ - nI -continuous.

Proof.

The proof is obvious by Proposition 3.1. $\square$

Theorem 4.2. For a function $f : (U, \mathcal{N}, I) \longrightarrow (V, \mathcal{N}')$, the following properties are equivalent.

1. f is α - nI -continuous.
2. f is pre- nI -continuous and semi- nI -continuous.
3. f is pre- nI -continuous and semi- δ - nI -continuous.

Proof.

The proof is obvious by Proposition 3.1 and Proposition 3.2. □

5 Conclusion

Because of the spaces is stripped of the geometric form and its is used to measure thing that are difficult to measure, such as intelligence, beauty and goodness. In this paper, different type of ideal nano continuity and ideal nano closed sets are introduced and studied. Also, we introduce an applications example in ideal nano topology. Some applications on them are given in some real life branches such as medicine and physics.

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Analytical Study of Mixed Convective Flow and Heat Transfer in Vertical Channel Filled with Immiscible Viscous Fluids

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Abstract

In this paper investigation of mixed convective flow and heat transfer in vertical channel filled with immiscible viscous fluids has been carried out. The governing differential equations are solved analytically by regular perturbation method. The impact of governing parameters on velocity and temperature fields namely Grash of number, Brinkman number, perturbation parameter, viscosity ratio, width ratio, conductivity ratio, Nusselt number are investigated and represented graphically.

Keywords: mixed convective flow, heat transfer, perturbation method.

2010 AMS subject classification: 76D05, 35Q30[‡]

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1. Introduction

Mixed convective flow and heat transfer has importance from researchers due to their diverse application in engineering, automobile sector and various technical fields. This includes geothermal mining, nuclear reactors, heat and cold storage, welding equipment, aircrafts Desai and Vafai [5]. The study of laminar fully developed mixed convection in a vertical channel with uniform wall temperature was one of the first attempt of Tao [10]. Recently Hamadah and Wirtz [11], Bratetta [3], Aung and Worku [1] assumed symmetric and asymmetric heating of walls of the vertical channel. Prathap kumar et al [7] studied the chemical reaction effects on mixed convection flow in vertical channel with immiscible fluids analytical as well as numerically. Umavati and Chamkha [6] analyzed mixed convection in presence of heat source or heat sink in a vertical channel. Jha and Oni [2] assumed temperature dependent viscosity to study the mixed convection flow in a vertical channel where they found that increase in viscosity parameter increases fluid velocity. With hall and ion-slip effects Srinivasacharya and Shafeeurrahman [4] analyzed mixed convection flow in a vertical channel filled with nanofluid where they found with the increment in magnetic parameter decrease in temperature, velocity, nanoparticle concentration were occurred. Prathap kumar et al [8] analyzed and found impact of different governing parameter on mixed convective flow in a vertical channel using third kind of boundary condition where channel is filled with porous media using differential transfer method as well as perturbation method. In fluid dynamics regular perturbation method is most preferably used, Rashidi and Ganji [9]. Keeping in view of several applications of mixed convective flow in vertical channel, the aim of this paper is to investigate laminar fully developed mixed convection flow and heat transfer in vertical channel with immiscible viscous fluid analytically to extend the studies available in the literature. The governing differential equations are solved using regular perturbation method valid for small values of perturbation parameter. The thermal buoyancy force, viscous dissipation, viscosity ratio, width ratio, conductivity ratio, Nusselt number are considered to investigate their impact on flow field.

Notation:

b- ratio of thermal expansion coefficients (β_2/β_1)

Br - Brinkman number ($\mu_1 U_0^2/k_1 \Delta T$)

g- acceleration due to gravity

Gr - Grashof number ($g\beta_1 D^3 \Delta T/\nu^2$)

GR - mixed convective parameter (Gr/Re)

D - width ratio (D_2/D_1)

D1, D2 - width of regions

k - thermal conductivities (k_1/k_2)

m - ratio of viscosities (μ_1/μ_2)

n - ratio of densities (ρ_2/ρ_1)

p - pressure (assuming $p_1=p_2=p$)

P- $=p+\rho_0 gX$, difference between the pressure and hydrostatic pressure

Re - Reynolds number ($D_1 U_0(1)/\nu_1$)

$U_0(i)$ - reference velocity

Greek Symbols

α_1, α_2 - Thermal diffusivities

β_1, β_2 - Coefficients of thermal expansion

ΔT - Difference in temperature ($T_2 - T_1$)

ε - Dimensionless parameter/perturbation parameter (GR Br)

Θ - Dimensionless temperature

θ_1, θ_2 - Temperatures

μ_1, μ_2 - Viscosities

ν_1, ν_2 - Kinematics viscosities

ρ_1, ρ_2 - Densities

Subscripts $i = 1, 2$ corresponding to region-I and region-II respectively.

2. Preliminaries

Consider a steady two dimensional laminar fully developed mixed convection flow in open ended vertical channel filled with immiscible viscous fluids. The X axis is taken upward and parallel to the walls and Y axis is normal on it, shown in Fig 1. We consider fluid to be incompressible, and temperature between the plate and fluid is small, so that the fluid properties taken as constant except the density in the buoyancy term of equation of motion.

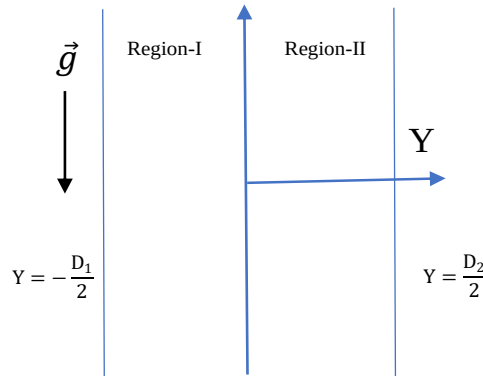


Fig.1. Physical configuration

Region –I

$$g\beta_1(T_1 - T_0) - \frac{1}{\rho_1} \frac{dP}{dX} + \frac{\mu_1}{\rho_1} \frac{d^2 U_1}{dY^2} = 0 \quad (2.1)$$

$$\alpha_1 \frac{d^2 T_1}{dY^2} + \frac{\nu_1}{C_p} \left(\frac{dU_1}{dY} \right)^2 = 0 \quad (2.2)$$

Region –II

$$g\beta_2(T_2 - T_0) - \frac{1}{\rho_2} \frac{dP}{dX} + \frac{\mu_2}{\rho_2} \frac{d^2U_2}{dY^2} = 0 \quad (2.3)$$

$$\alpha_2 \frac{d^2T_2}{dY^2} + \frac{\nu_2}{C_p} \left(\frac{dU_2}{dY} \right)^2 = 0 \quad (2.4)$$

P depends only on X

$$\frac{dP}{dY} = 0 \quad (2.5)$$

In the presence of viscous dissipation, the energy balance equation can be written as
Region –I

$$\frac{d^2T_1}{dY^2} = \frac{-\nu_1}{\beta_1 g} \frac{d^4U_1}{dY^4} \quad (2.6)$$

From Eq. (2.2) and Eq. (2.6)

$$\frac{d^4U_1}{dY^4} = \frac{\beta_1 g \rho_1}{k_1} \left(\frac{dU_1}{dY} \right)^2 \quad (2.7)$$

Region –II

$$\frac{d^2T_2}{dY^2} = \frac{-\nu_2}{\beta_2 g} \frac{d^4U_2}{dY^4} \quad (2.8)$$

From Eq. (2.4) and Eq. (2.8)

$$\frac{d^4U_2}{dY^4} = \frac{\beta_2 g \rho_2}{k_2} \left(\frac{dU_2}{dY} \right)^2 \quad (2.9)$$

On account of Eq. (2.1) and Eq. (2.3) there exists a constant A such that

$$\frac{dP}{dX} = A \quad (2.10)$$

The boundary and interface conditions are

$$\begin{aligned} U_1 \left(\frac{-D_1}{2} \right) = 0 = U_2 \left(\frac{-D_2}{2} \right), \quad U_1(0) = U_2(0), \quad T_0 = \frac{T_1 + T_2}{2}, \\ \left. \frac{d^2U_1}{dY^2} \right|_{Y=-\frac{D_1}{2}} = \frac{A}{\mu_1} + \frac{\beta_1 g [T_2 - T_1]}{2\nu_1}, \quad \left. \frac{d^2U_2}{dY^2} \right|_{Y=\frac{D_2}{2}} = \frac{A}{\mu_2} - \frac{\beta_2 g [T_2 - T_1]}{2\nu_2}, \\ \mu_1 \frac{dU_1}{dY} = \mu_2 \frac{dU_2}{dY}, \text{ at } Y = 0, \quad \frac{d^3U_1}{dY^3} = \frac{1}{mnbk} \frac{d^3U_2}{dY^3}, \text{ at } Y = 0 \\ \frac{d^2U_1}{dY^2} = \frac{1}{mnb} \frac{d^2U_2}{dY^2} + \frac{A}{\mu_1} \left[1 - \frac{1}{nb} \right] \text{ at } Y = 0 \\ T_1(0) = T_2(0), \quad k_1 \frac{dT_1}{dY} = k_2 \frac{dT_2}{dY}, \text{ at } Y = 0 \end{aligned} \quad (2.11)$$

Equations (2.7), (2.9) and (2.11) determine the velocity distribution. They can be written in a non-dimensional form by means of following dimensionless variables

$$\begin{aligned} u_1 = \frac{U_1}{U_0^{(1)}}; \quad u_2 = \frac{U_2}{U_0^{(2)}}; \quad y_1 = \frac{Y_1}{D_1}; \quad y_2 = \frac{Y_2}{D_2}; \quad Gr = \frac{g\beta_1 \Delta T D_1^3}{\nu_1^2}; \quad Re = \frac{U_0^{(1)} D_1}{\nu_1} \\ Br = \frac{\mu_1 U_0^{(1)2}}{k_1 \Delta T}; \quad U_0^{(1)} = \frac{-AD_1^2}{48\mu_1}; \quad U_0^{(2)} = -\frac{AD_2^2}{48\mu_2}; \end{aligned}$$

$$G_R = \frac{G_r}{Re} ; \quad \theta_1 = \frac{T_1 - T_0}{\Delta T} ; \theta_2 = \frac{T_2 - T_0}{\Delta T} ; R_T = \frac{T_2 - T_1}{\Delta T} \quad (2.12)$$

Eqs. (2.7), (2.9) becomes

Region –I

$$\frac{d^4 u_1}{dy^4} = G_R B_r \left(\frac{du_1}{dy} \right)^2 \quad (2.13)$$

Region –II

$$\frac{d^4 u_2}{dy^4} = G_R B_r m n b k D^4 \left(\frac{du_2}{dy} \right)^2 \quad (2.14)$$

The boundary and interface conditions are

$$\begin{aligned} u_1 U_0^{(1)} = 0, \text{ at } y = -\frac{1}{4} ; \quad u_1 \left(-\frac{1}{4} \right) = 0 = u_2 \left(\frac{1}{4} \right) ; \quad u_1(0) = m D^2 u_2(0) ; \\ \frac{d^2 u_1}{dy^2} = -48 + \frac{G_R R_T}{2}, \text{ at } y = -\frac{1}{4} ; \quad \frac{d^2 u_2}{dy^2} = -48 - \frac{G_R n b R_T}{2}, \text{ at } y = \frac{1}{4} ; \\ \frac{du_1}{dy} = D \frac{du_2}{dy}, \text{ at } y = 0 ; \quad \frac{d^2 u_1}{dy^2} = \frac{1}{n b} \left[\frac{d^2 u_2}{dy^2} + 48(1 - n b) \right], \text{ at } y = 0 ; \\ \frac{d^3 u_1}{dy^3} = \frac{1}{n b k D} \frac{d^3 u_2}{dy^3}, \text{ at } y = 0 \end{aligned} \quad (2.15)$$

Where $D = \frac{D_2}{D_1}$, $m = \frac{\mu_1}{\mu_2}$, $n = \frac{\rho_2}{\rho_1}$, $b = \frac{\beta_2}{\beta_1}$, $k = \frac{k_1}{k_2}$

Solutions

Case of Negligible of Viscous Dissipation ($B_r = 0$)

The solution of Eqs. (2.13) and (2.14) can be obtained using Eq. (2.15) in the absence of viscous dissipation, that is, when the parameter, ($B_r = 0$) is given by

Region –I

$$u_1 = E_1 + E_2 y + E_3 y^2 + E_4 y^3 \quad (3.1)$$

Region –II

$$u_2 = E_5 + E_6 y + E_7 y^2 + E_8 y^3 \quad (3.2)$$

Using Eq. (2.12) in Eqs. (2.1) and (2.3) the energy balance equations are

Region –I

$$\theta_1 = -\frac{1}{G_R} \left[48 + \frac{d^2 u_1}{dy^2} \right] \quad (3.3)$$

Region –II

$$\theta_2 = -\frac{1}{n b G_R} \left[48 + \frac{d^2 u_2}{dy^2} \right] \quad (3.4)$$

Using the expressions obtained in Eqs. (3.1) and (3.2) the energy balance Eqs. (3.3) and (3.4) becomes

Region –I

$$\theta_1 = -\frac{1}{G_R} [48 + 2E_3 + 6E_4y] \quad (3.5)$$

Region –II

$$\theta_2 = -\frac{1}{nbG_R} [48 + 2E_7 + 6E_8y] \quad (3.6)$$

Case of Negligible buoyancy Force ($G_R = 0$)

When the buoyancy forces are negligible ($G_R = 0$) and viscous dissipation is dominating ($B_r \neq 0$), so that purely forced convection occurs. For this case, the solutions of Eqs. (2.13) and (2.14) can be obtained using the Eq. (2.15), the velocities are given by

Region –I

$$u_1 = F_1 + F_2y + F_3y^2 + F_4y^3 \quad (3.7)$$

Region –II

$$u_2 = F_5 + F_6y + F_7y^2 + F_8y^3 \quad (3.8)$$

The energy balance Eqs. (2.6) and (2.8) in non-dimensional form can also be written as

Region –I

$$\frac{d^2\theta_1}{dy^2} = -B_r \left(\frac{du_1}{dy} \right)^2 \quad (3.9)$$

Region –II

$$\frac{d^2\theta_2}{dy^2} = -B_r m k D^4 \left(\frac{du_2}{dy} \right)^2 \quad (3.10)$$

The boundary and interface conditions are

$$\begin{aligned} \theta_1 \left(-\frac{1}{4} \right) &= -\frac{R_T}{2} ; & \theta_2 \left(\frac{1}{4} \right) &= \frac{R_T}{2} ; \\ \theta_1(0) &= \theta_2(0) ; & \frac{d\theta_1}{dy} &= \frac{1}{kD} \frac{d\theta_2}{dy}, \text{ at } y = 0 ; \end{aligned} \quad (3.11)$$

Solving Eqs. (3.9) and (3.10), using Eqs (3.7) and (3.8) we obtain

Region –I

$$\theta_1 = -B_r \left(\frac{G_3y^2 + G_4y^3 + G_5y^4}{G_6y^5 + G_7y^6} \right) + G_2y + G_1 \quad (3.12)$$

Region –II

$$\theta_2 = -m k D^4 B_r \left(\frac{G_{10}y^2 + G_{11}y^3 + G_{12}y^4}{G_{13}y^5 + G_{14}y^6} \right) + G_9y + G_8 \quad (3.13)$$

Combine Effects of Buoyancy Force and Viscous Dissipation

By using the perturbation method, we solve Eqs. (2.13) and (2.14) with a dimensionless parameter $|\varepsilon| (< 1)$ defined as

$$\varepsilon = G_R B_r \quad (3.14)$$

Which is independent of the reference temperature difference ΔT . The solutions are assumed in the form

$$u(y) = u_0(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) + \dots = \sum_{n=0}^{\infty} \varepsilon^n u_n(y) \quad (3.15)$$

Substituting Eq. (3.15) in Eqs. (2.13) and (2.14) and the coefficients of like powers of ε to obtain the zeroth and first order equations as follows

Region –I (Zeroth -order equation)

$$\frac{d^4 u_{10}}{dy^4} = 0 \quad (3.16)$$

First-order equation

$$\frac{d^4 u_{11}}{dy^4} = \left(\frac{du_{10}}{dy} \right)^2 \quad (3.17)$$

Region –II (Zeroth-order equation)

$$\frac{d^4 u_{20}}{dy^4} = 0 \quad (3.18)$$

First-order equation

$$\frac{d^4 u_{21}}{dy^4} = mnbkD^4 \left(\frac{du_{20}}{dy} \right)^2 \quad (3.19)$$

The corresponding boundary and interface conditions for the zeroth and first order by Eq. (2.15) reduces to

$$\begin{aligned} u_{10} \left(-\frac{1}{4} \right) &= 0 = u_{20} \left(\frac{1}{4} \right); \quad u_{11} \left(-\frac{1}{4} \right) = 0 = u_{21} \left(\frac{1}{4} \right); \\ u_{10}(0) &= mD^2 u_{20}(0); \quad u_{11}(0) = mD^2 u_{21}(0); \\ \frac{d^2 u_{10}}{dy^2} &= -48 + \frac{G_R R_T}{2}, \text{ at } y = -\frac{1}{4}; \quad \frac{d^2 u_{11}}{dy^2} = 0, \text{ at } y = -\frac{1}{4}; \\ \frac{d^2 u_{20}}{dy^2} &= -48 - \frac{nbG_R R_T}{2}, \text{ at } y = \frac{1}{4}; \quad \frac{d^2 u_{21}}{dy^2} = 0, \text{ at } y = \frac{1}{4}; \\ \frac{du_{10}}{dy} &= D \frac{du_{20}}{dy}, \text{ at } y = 0; \quad \frac{du_{11}}{dy} = D \frac{du_{21}}{dy}, \text{ at } y = 0; \\ \frac{d^2 u_{10}}{dy^2} &= \frac{1}{nb} \left[\frac{d^2 u_{20}}{dy^2} + 48(1 - nb) \right], \text{ at } y = 0; \\ \frac{d^2 u_{11}}{dy^2} &= \frac{1}{nb} \left[\frac{d^2 u_{21}}{dy^2} \right], \text{ at } y = 0; \quad \frac{d^3 u_{10}}{dy^3} = \frac{1}{nbkD} \frac{d^3 u_{20}}{dy^3}, \text{ at } y = 0; \\ \frac{d^3 u_{11}}{dy^3} &= \frac{1}{nbkD} \frac{d^3 u_{21}}{dy^3}, \text{ at } y = 0; \end{aligned} \quad (3.20)$$

Solutions of zeroth-order Eqs. (3.16) and (3.18) using Eq. (3.20) are

$$u_{10} = C_1 + C_2 y + C_3 y^2 + C_4 y^3 \quad (3.21)$$

$$u_{20} = B_1 + B_2 y + B_3 y^2 + B_4 y^3 \quad (3.22)$$

Solutions of first-order Eqs. (3.17) and (3.19) using Eq. (3.20) are

$$\begin{aligned} u_{11} &= P_5 y^8 + P_6 y^7 + P_7 y^6 + P_8 y^5 \\ &\quad + P_9 y^4 + \frac{P_1}{6} y^3 + \frac{P_2}{2} y^2 + P_3 y + P_4 \end{aligned} \quad (3.23)$$

$$\begin{aligned} u_{21} &= Q_5 y^8 + Q_6 y^7 + Q_7 y^6 + Q_8 y^5 \\ &\quad + Q_9 y^4 + \frac{Q_1}{6} y^3 + \frac{Q_2}{2} y^2 + Q_3 y + Q_4 \end{aligned} \quad (3.24)$$

Using the velocities given by Eqs. (3.21) - (3.24) the energy balance Eqs. (3.3) and (3.4) becomes

Region –I

$$\theta_1 = -\frac{1}{Gr} \left[\frac{48 + 2C_3 + 6C_4y}{56P_5y^6 + 42P_6y^5 + 30P_7y^4 + 20P_8y^3 + 12P_9y^2 + P_1y + P_2} \right] + \varepsilon \left(\frac{48 + 2C_3 + 6C_4y}{56P_5y^6 + 42P_6y^5 + 30P_7y^4 + 20P_8y^3 + 12P_9y^2 + P_1y + P_2} \right) \quad (3.25)$$

Region –II

$$\theta_2 = -\frac{1}{nbGr} \left[\frac{48 + 2B_3 + 6B_4y}{56Q_5y^6 + 42Q_6y^5 + 30Q_7y^4 + 20Q_8y^3 + 12Q_9y^2 + Q_1y + Q_2} \right] + \varepsilon \left(\frac{48 + 2B_3 + 6B_4y}{56Q_5y^6 + 42Q_6y^5 + 30Q_7y^4 + 20Q_8y^3 + 12Q_9y^2 + Q_1y + Q_2} \right) \quad (3.26)$$

Heat Transfer

The wall heat transfer expression in terms of the Nusselt number is

$$\begin{aligned} Nu_- &= (1 + D) \frac{d\theta_1}{dy}, \quad \text{at } y = -\frac{1}{4} \\ Nu_+ &= \left(1 + \frac{1}{D}\right) \frac{d\theta_1}{dy}, \quad \text{at } y = \frac{1}{4} \\ Nu_- &= -\frac{(1+D)}{Gr} \left[6C_4 - \varepsilon \left(\frac{21}{64}P_5 - \frac{105}{128}P_6 + \frac{15}{8}P_7 - \frac{15}{4}P_8 + 6P_9 - P_1 \right) \right] \end{aligned} \quad (3.27)$$

$$Nu_+ = -\frac{\left(1 + \frac{1}{D}\right)}{nbGr} \left[6B_4 + \varepsilon \left(\frac{21}{64}Q_5 + \frac{105}{128}Q_6 + \frac{15}{8}Q_7 + \frac{15}{4}Q_8 + 6Q_9 - Q_1 \right) \right] \quad (3.28)$$

3. Results and discussions

Investigation of laminar mixed convection flow in vertical channel filled with immiscible viscous fluid has been done analytically by using a regular perturbation method taking the product of the thermal Grashof number ($GR=Gr/Re$) and Brinkman number Br as perturbation parameter. And solution are valid only for small values of perturbation parameter $\varepsilon(<1)$. Viscous dissipation term is also included in the energy equations.

The flow fields are evaluated in case of asymmetric heating ($RT=1$) and are represented graphically in Fig 2-8. The velocity and temperature fields in the absence of Brinkman number ($Br=0$) are obtained for different values of thermal Grashof number (GR) and are depicted in Fig.2a, Fig.2b respectively. For negative value of GR velocity field increases in region-I and decreases in region II whereas for positive values of GR velocity decreases in region I and increases in region II. One can also observe that flow was an increasing function for value $GR(>0)$ and decreasing function for $GR(<0)$. But temperature field decreases in both the regions for all different values of GR .

The dimensionless temperature field θ is obtained and is shown in Fig.3 for different values of Brinkman number Br in case of negligible Buoyancy force ($GR=0$). As Brinkman number increases temperature field is also increases in both regions.

The velocity and temperature fields are obtained at $GR=\pm 500$ for different values of ε and are shown in Fig.4a, Fig.4b respectively. The velocity field is an increasing function of ε for upward flow $\varepsilon (>0)$ and decreasing function of ε for downward flow $\varepsilon (<0)$. Whereas temperature field increases for both $\varepsilon (>0)$ and $\varepsilon (<0)$. The perturbation parameter ε is more effective on velocity field than temperature field. From Fig.4a it can also pointed out that at cold (left) and hot (right) walls reversal flow occurs for upward and downward flow respectively.

The effect of viscosity ratio m , width ratio D and conductivity ratio k on flow field evaluated for the values of $GR=100$, and $\varepsilon=0.01$ in case of asymmetric heating ($RT=1$). The effect of viscosity ratio m on the velocity and temperature fields are shown in Fig.5a, Fig.5b respectively. As the viscosity ratio m increases the flow field increases in both regions. Temperature increases from cold to hot walls for all values of m .

The effect of width ratio D on the velocity and temperature fields are shown in Fig.6a, Fig.6b respectively. Effect of the width ratio is similar to effect of viscosity ratio on flow field as D increases both velocity and temperature fields increases.

The effect of conductivity ratio k on the fields of velocity and temperature are depicted in Fig.7a, Fig.7b respectively. As k increases, the velocity and temperature fields reduce in both the regions, this means larger the conductivity ratio smaller the flow rate is. The Nusselt number at the cold wall (Nu_-) and hot wall (Nu_+) for $|\varepsilon|$ is shown in Fig.8. The Nu_- is an increasing function of $|\varepsilon|$ and Nu_+ is a decreasing function of $|\varepsilon|$

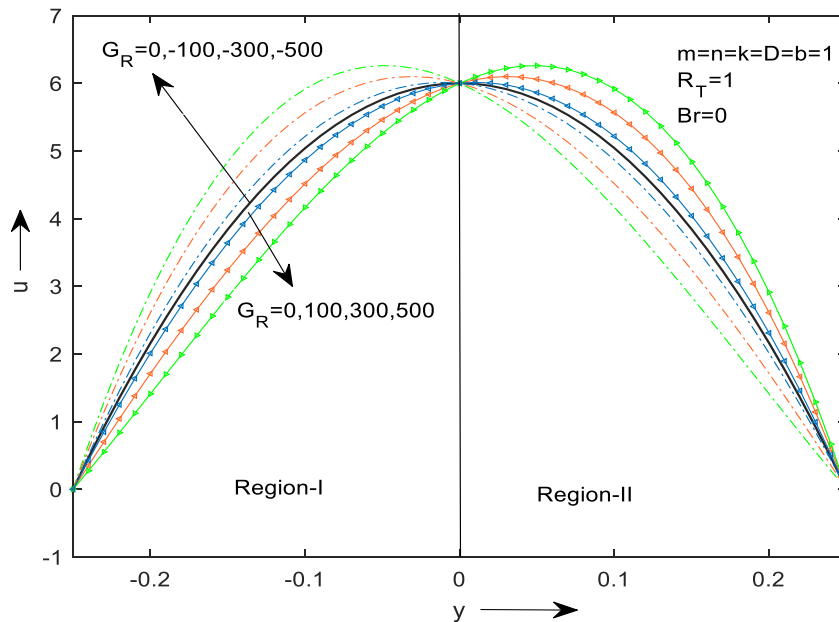


Fig. 2a. Velocity profiles for different values of G_R

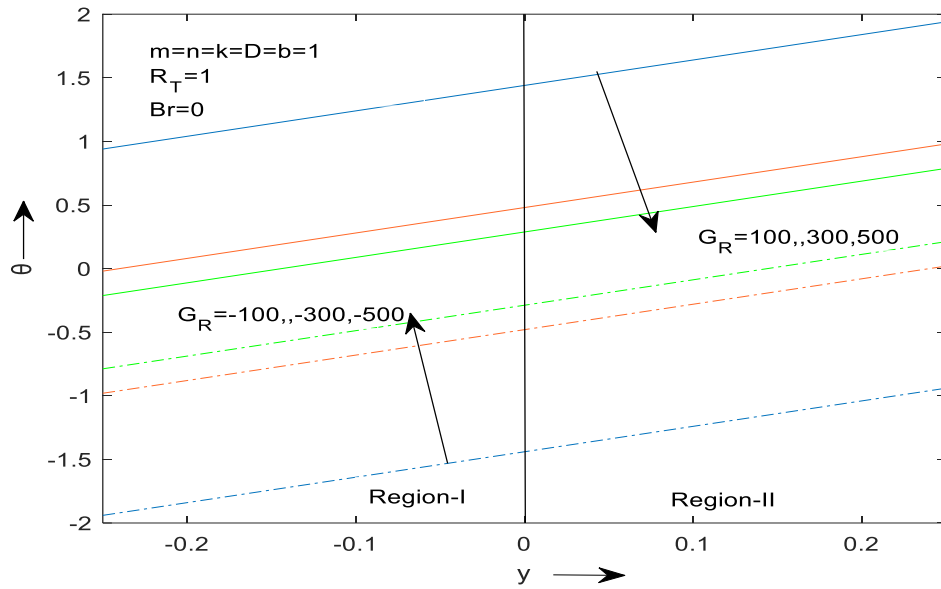


Fig. 2b. Temperature profiles for different values of G_R

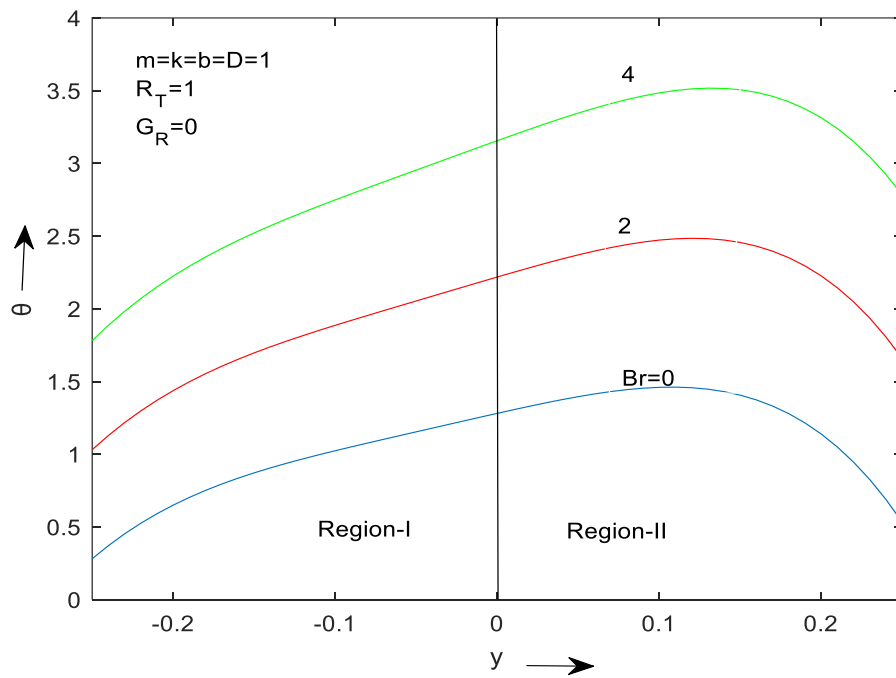


Fig. 3. Temperature profile for different values of Br

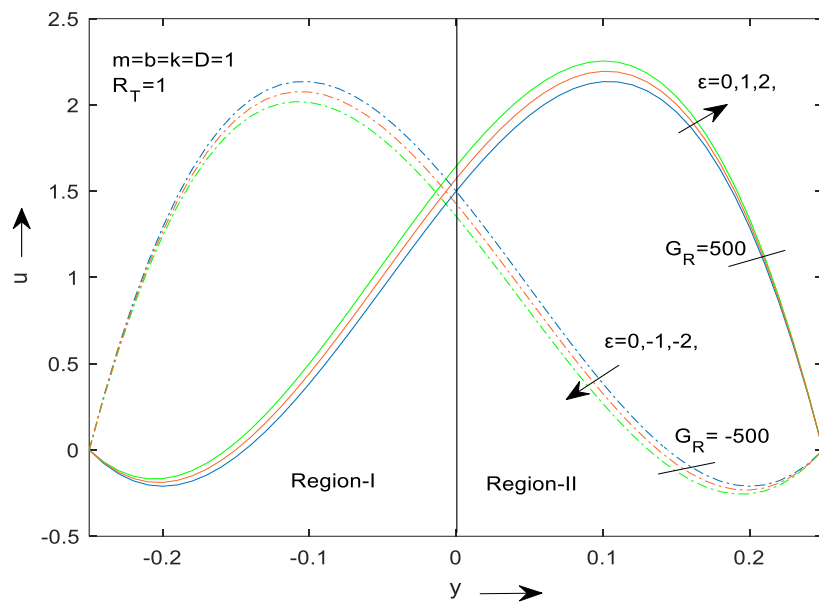


Fig. 4a. Velocity profile for different values of ε .

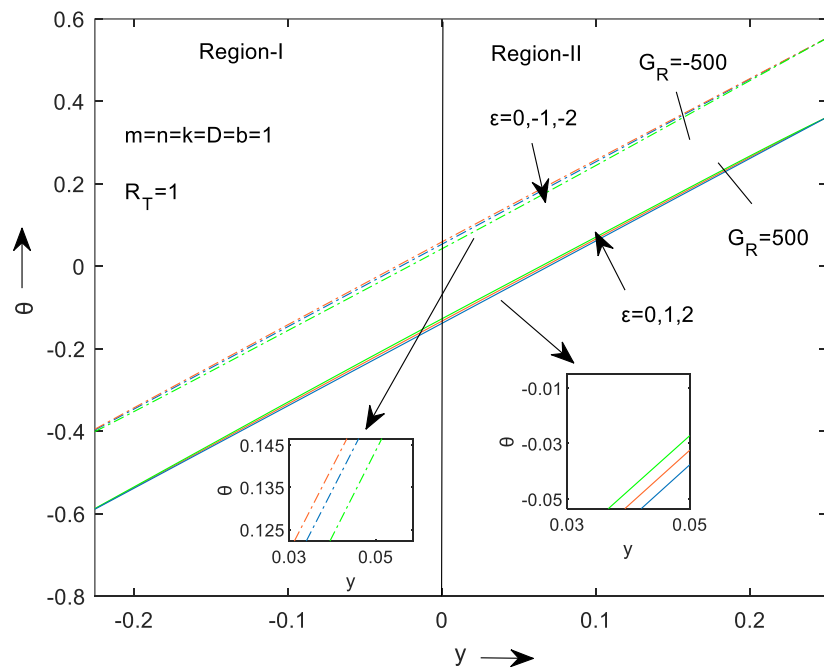


Fig. 4b. Temperature profile for different values of ε .

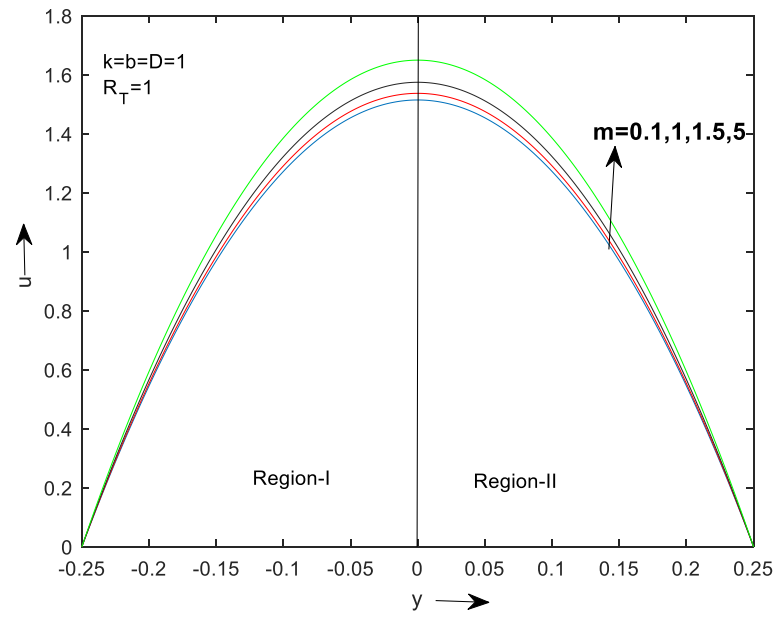


Fig. 5a. Velocity profile for different values of m .

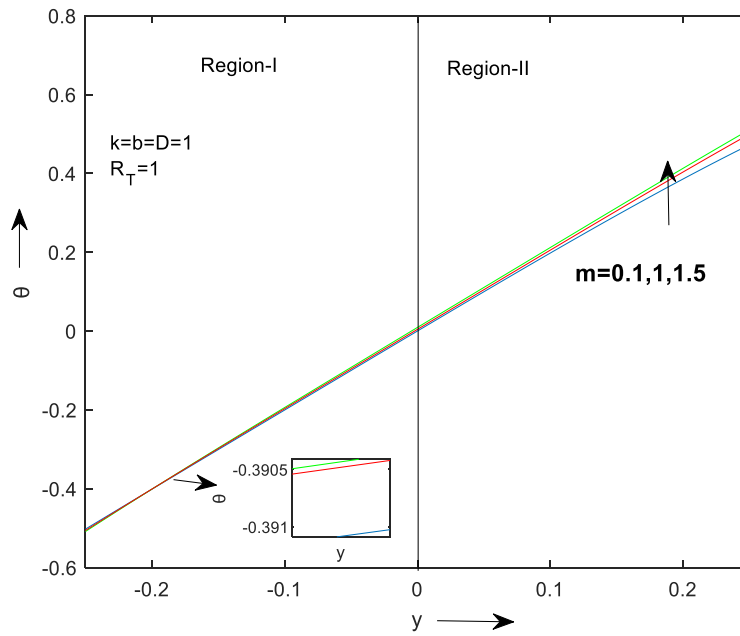


Fig. 5b. Temperature profile for different values of m .

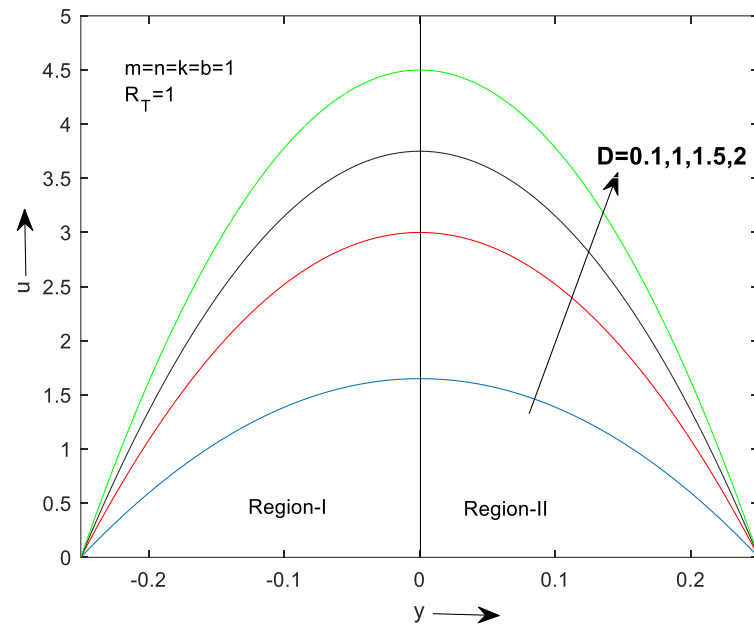


Fig. 6a. Velocity profile for different values of D .

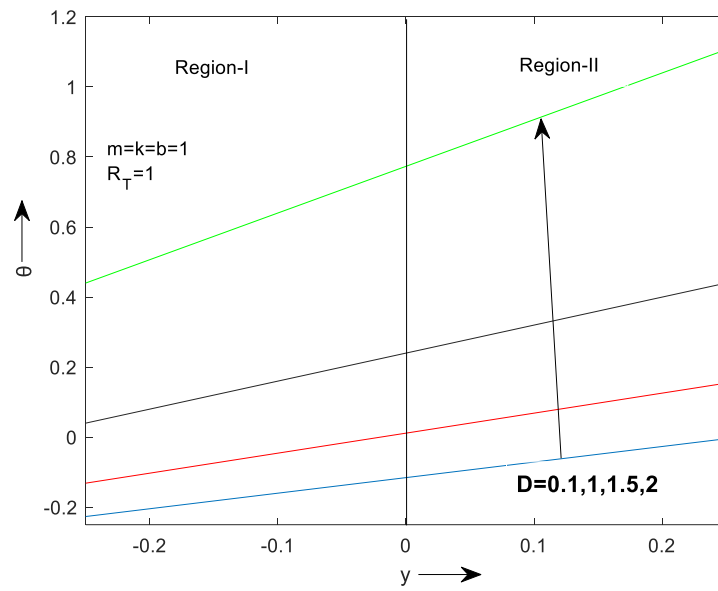


Fig. 6b. Temperature profile for different values of D .

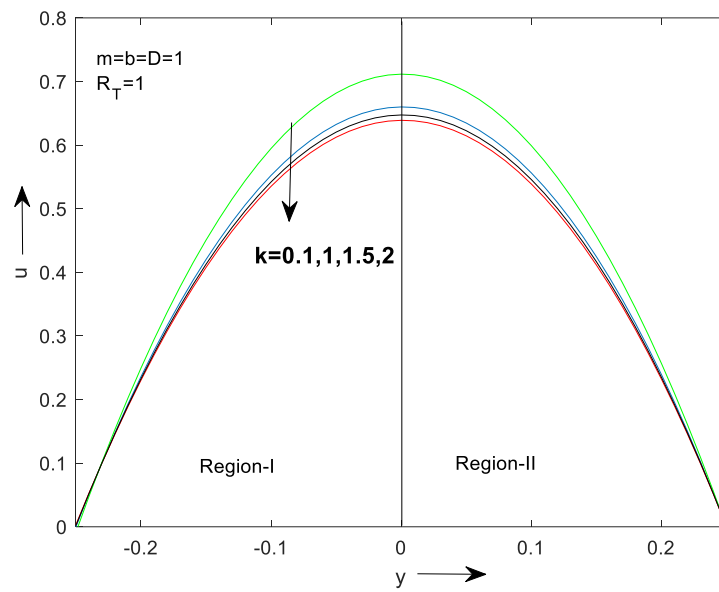


Fig. 7a. Velocity profile for different values of k

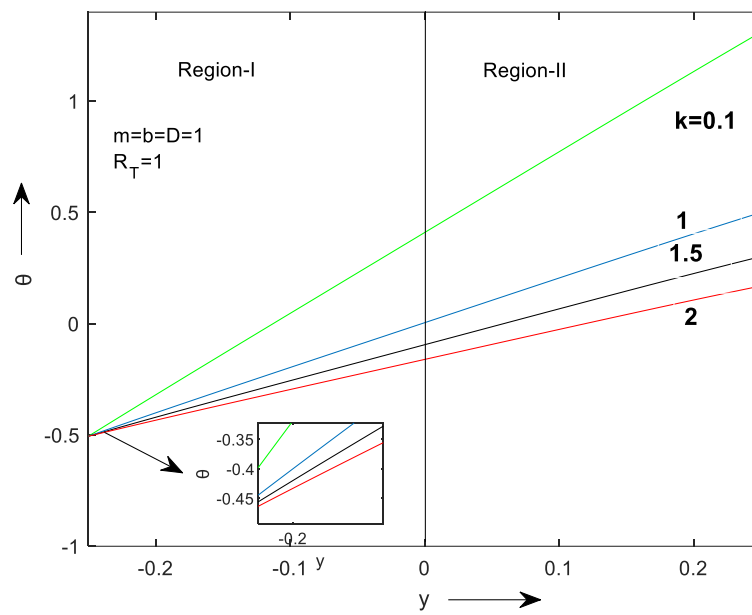


Fig. 7b. Temperature profile for different values of k

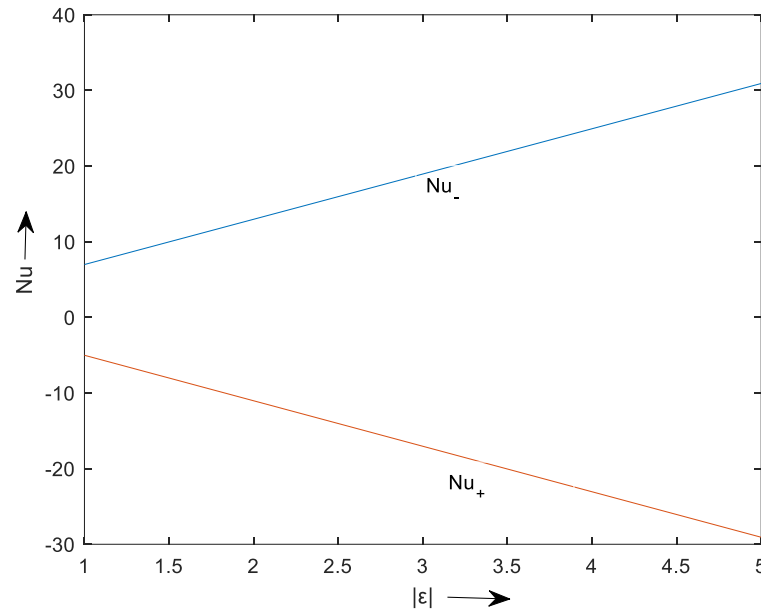


Fig. 8. Nusselt number Vs $|\epsilon|$.

4. Conclusions

The problem of mixed convective flow and heat transfer in vertical channel filled with immiscible viscous fluids was analyzed analytically by regular perturbation method and represented graphically. The conclusions made are, the flow was an increasing function of perturbation parameter ϵ for upward flow and decreasing function of ϵ for downward flow, viscosity ratio m and width ratio D enhance the velocity and temperature fields where as the larger the value of conductivity ratio k , smaller the fields of velocity and temperature.

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Weakly $\check{\theta}$ - $\mathcal{I}$ -Closed Sets and Weakly $\check{\theta}$ - $\mathcal{I}$ -continuous functions with respect to an Ideal Topological Spaces

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Abstract

In this paper, we introduce $\check{\theta}$ - $\mathcal{I}$ -closed sets, $\check{\theta}$ - $\mathcal{I}_\alpha$ -closed sets, $\check{\theta}$ - $\mathcal{I}$ -continuous functions and $\check{\theta}$ - $\mathcal{I}_\alpha$ -continuous functions and investigate their properties and its characterizations. After that we introduce weakly $\check{\theta}$ - $\mathcal{I}$ -continuous functions and study the relationship between other types of continuous functions with suitable examples.

Keywords: $\check{\theta}$ - $\mathcal{I}$ -cld, $w\check{\theta}$ - $\mathcal{I}$ -cld, $w\check{\theta}$ - $\mathcal{I}$ -openfunction, $w\check{\theta}$ - $\mathcal{I}$ -continuous.

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1.Introduction

The advent of generalized closed sets, a new trend was created by Sundaram and Sheik John. Accordingly, a new notion called weakly closed sets were introduced by them. In this direction, many modifications of weakly closed sets are being introduced by the modern topologists due to their requirements. In this way, Ravi et al. [18] introduced weakly πg -closed sets and Sundaram and Nagaveni [24] introduced weakly g -closed sets in topological spaces. In 1961 Levine [14] obtained a decomposition of continuity. Later Professor Rose improved Levine's decomposition. In 1986 Tong [26] obtained a decomposition of continuity and proved that his decomposition is independent of Levine's. In 1989, Tong [27] improved upon his earlier decomposition and obtained yet another decomposition of continuity. In 1990, Ganster and Reilly [8] obtained a decomposition of continuity improving the first result of Tong. A. Acikgoz and et al. [1], introduced on α -I-continuous and α -I-open functions. J. Antony Rex Rodrigo and et al. [2], the introduced the mildly-I-locally closed sets and decompositions of $\star$ -continuity .

K. Kuratowski [11], introduced topology. S. Jafari and N. Rajesh [12], introduced the generalized closed sets with respect to an ideal. N. Levine [13], introduced the generalized closed sets in topology. O. Njstad [17], introduced the on some classes of nearly open sets. R. Devi, K. Balachandran and H. Maki [4], introduced the semi-generalized closed maps and generalized semi-closed maps. Devi, R., Balachandran, K. and Maki, H.[5], introduced the on generalized α -continuous maps and α -generalized continuous maps. Devi, R., Balachandran, K. and Maki, H.[6], introduced the semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces.

Dontchev, J.[7], introduced the on generalizing semi-preopen sets. Levine, N.[15], introduced the semi-open sets and semi-continuity in topological spaces. A. S. Mashhour, et al. [16], introduced the α -continuous and α -open mappings. Rajamani, M. and Viswanathan, K.[19], introduced the on α gs-continuous maps in topological spaces. V. Renukadevi [20], introduced the note on IR-closed and AIR-sets. Sundaram, P. and et al. [22], introduced the semi-generalized continuous maps and semi- $T_{1/2}$ -spaces. Sundaram, P.[23], introduced the study on generalizations of continuous maps in topological spaces. P. Sundaram and M. Rajamani [25], introduced some decompositions of regular generalized continuous maps in topological spaces. Veera Kumar, M. K. R. S.[28], introduced the between semi-closed sets and semi pre-closed sets.

In this paper, we introduce $\check{\theta}$ - $\mathcal{J}$ -closed sets, $\check{\theta}$ - $\mathcal{I}_\alpha$ -closed sets, $\check{\theta}$ - $\mathcal{I}$ -continuous functions and $\check{\theta}$ - $\mathcal{I}_\alpha$ -continuous functions and investigate their properties and its characterizations. After that we introduce weakly $\check{\theta}$ - $\mathcal{J}$ -continuous functions and study the relationship between other types of continuous functions with suitable examples.

2. Preliminaries

An ideal $\mathcal{J}$ on a topological space (briefly, TPS) (X, τ) is a nonempty collection of subsets of X which satisfies

- (1) $A \in \mathcal{J}$ and $B \subseteq A \Rightarrow B \in \mathcal{J}$ and
- (2) $A \in \mathcal{J}$ and $B \in \mathcal{J} \Rightarrow A \cup B \in \mathcal{J}$.

Given a topological space (X, τ) with an ideal $\mathcal{J}$ on X if $\wp(X)$ is the set of all subsets of X , a set operator $(\bullet)^*$: $\wp(X) \rightarrow \wp(X)$, called a local function [10] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{J}, \tau) = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(\bullet)$ for a topology $\tau^*(\mathcal{J}, \tau)$, called the $\star$ -topology and finer than τ , is defined by $cl^*(A) = A \cup A^*(\mathcal{J}, \tau)$ [10]. We will simply write A^* for $A^*(\mathcal{J}, \tau)$ and τ^* for $\tau^*(\mathcal{J}, \tau)$. If $\mathcal{J}$ is an ideal on X , then $(X, \tau, \mathcal{J})$ is called an ideal topological space (briefly, ITPS). A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is $\star$ -closed (briefly, $\star$ -cld) [10] if $A^* \subseteq A$. The interior of a subset A in $(X, \tau^*(\mathcal{J}))$ is denoted by $int^*(A)$.

Definition 2.1 A subset K of a TPS X is called:

- (i) semi-open set [9] if $K \subseteq cl(int(K))$;
- (ii) regular open set [21] if $K = int(cl(K))$;

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.2 A subset K of a TPS X is called

- (i) g -closed set (briefly, g -cld) [13] if $cl(K) \subseteq V$ whenever $K \subseteq V$ and V is open.
- (ii) semi-generalized closed (briefly, sg -cld) [8] if $scl(K) \subseteq V$ whenever $K \subseteq V$ and V is semi-open.
- (iii) generalized semi-closed (briefly, gs -cld) [29] if $scl(K) \subseteq V$ whenever $K \subseteq V$ and V is open.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3 A subset K of a ITPS X is called

- (i) $\mathcal{J}_g$ -closed (briefly, $\mathcal{J}_g$ -cld) set [9] if $K^* \subseteq V$ whenever $K \subseteq V$ and V is open.

The complements of the above-mentioned closed sets are called their respective open sets.

Definition 2.4 A subset A of a topological space X is called:

- (i) a weakly g -closed (briefly, wg -cld) set [24] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X .
- (ii) a weakly πg -closed (briefly, $w\pi g$ -cld) set [18] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is π -open in X .

(iii) a regular weakly generalized closed (briefly, rwg-cld) set [18] if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X .

3. Weakly $\check{\theta}$ - $\mathcal{I}$ -Closed Sets

We introduce the following definition:

Definition 3.1 A subset K of X is called

(i) $\check{\theta}$ - $\mathcal{I}$ -closed (briefly, $\check{\theta}$ - $\mathcal{I}$ -cld) if $K^* \subseteq V$ whenever $K \subseteq V$ and V is sg-open.

The complement of $\check{\theta}$ - $\mathcal{I}$ -cld is called $\check{\theta}$ - $\mathcal{I}$ -open.

The family of all $\check{\theta}$ - $\mathcal{I}$ -cld in X is denoted by $\check{\theta}\mathcal{IC}(X)$.

(ii) $\check{\theta}$ - $\mathcal{I}_\alpha$ -closed (briefly, $\check{\theta}$ - $\mathcal{I}_\alpha$ -cld) if $\alpha \text{cl}(K^*) \subseteq V$ whenever $K \subseteq V$ and V is sg-open.

The complement of $\check{\theta}$ - $\mathcal{I}_\alpha$ -cld is called $\check{\theta}$ - $\mathcal{I}_\alpha$ -open.

(iii) A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called $\check{\theta}$ - $\mathcal{I}$ -continuous if the inverse image of every closed set in Y is $\check{\theta}$ - $\mathcal{I}$ -cld set in X .

(iv) $\star$ -continuous if $f^{-1}(V)$ is a $\star$ -cld set in X , for every closed set in Y .

(v) A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called $\check{\theta}$ - $\mathcal{I}_\alpha$ -continuous if $f^{-1}(V)$ is a $\check{\theta}$ - $\mathcal{I}_\alpha$ -cld set in X , for every closed set in Y .

(vi) A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called a weakly $\check{\theta}$ - $\mathcal{I}$ -closed (briefly, $w\check{\theta}$ - $\mathcal{I}$ -cld) set if $(\text{int}(A))^* \subseteq V$ whenever $A \subseteq V$ and V is sg-open in X .

Theorem 3.2 Every $\check{\theta}$ - $\mathcal{I}$ -cld set is $w\check{\theta}$ - $\mathcal{I}$ -cld but not conversely.

Example 3.3 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{p, q\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then the set $\{p\}$ is $w\check{\theta}$ - $\mathcal{I}$ -cld set but it is not a $\check{\theta}$ - $\mathcal{I}$ -cld in X .

Theorem 3.4 Every $w\check{\theta}$ - $\mathcal{I}$ -cld set is wg-cld but not conversely.

Proof

Let H be any $w\check{\theta}$ - $\mathcal{I}$ -cld set and V be any open set containing H . Then V is a sg-open set containing H . We have $(\text{int}(H))^* \subseteq V$. Thus, H is wg-cld.

Example 3.5 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{p\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then the set $\{p, q\}$ is wg-cld but it is not a $w\check{\theta}$ - $\mathcal{I}$ -cld.

Theorem 3.6 Every $w\check{\theta}$ - $\mathcal{I}$ -cld set is $w\pi$ g-cld but not conversely.

Proof

Let H be any $w\check{\theta}$ - $\mathcal{I}$ -cld set and V be any π -open set containing H . Then V is a sg-open set containing H . We have $(\text{int}(H))^* \subseteq V$. Thus, H is $w\pi$ g-cld.

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Example 3.7 In Example 3.5, the set $\{p, r\}$ is $w\pi g$ -cld but it is not a $w\check{\theta}$ - $\mathcal{J}$ -cld.

Theorem 3.8 Every $w\check{\theta}$ - $\mathcal{J}$ -cld set is rwg -cld but not conversely.

Proof

Let H be any $w\check{\theta}$ - $\mathcal{J}$ -cld set and V be any regular open set containing H . Then V is a sg-open set containing H . We have $(\text{int}(H))^* \subseteq V$. Thus, H is rwg -cld.

Example 3.9 In Example 3.5, the set $\{p\}$ is rwg -cld but it is not a $w\check{\theta}$ - $\mathcal{J}$ -cld.

Theorem 3.10 If a subset H of an ideal topological space X is both $\star$ -cld and α g-cld, then it is $w\check{\theta}$ - $\mathcal{J}$ -cld in X .

Proof

Let H be a α g-cld set in X and V be any open set containing H . Then $V \supseteq \alpha \text{ cl}(H) = H \cup (\text{int}(H^*))^*$. Since H is $\star$ -cld, $V \supseteq (\text{int}(H))^*$ and hence $w\check{\theta}$ - $\mathcal{J}$ -closed in X .

Theorem 3.11 If a subset H of an ideal topological space X is both open and $w\check{\theta}$ - $\mathcal{J}$ -cld, then it is $\star$ -cld.

Proof

Since H is both open and $w\check{\theta}$ - $\mathcal{J}$ -cld, $H \supseteq (\text{int}(H))^* = H^*$ and hence H is $\star$ -cld in X .

Corollary 3.12 If a subset H of an ideal topological space X is both open and $w\check{\theta}$ - $\mathcal{J}$ -cld, then it is both regular open and regular closed in X .

Theorem 3.13 Let X be an ideal topological space and $H \subseteq X$ be open. Then, H is $w\check{\theta}$ - $\mathcal{J}$ -cld if and only if H is $\check{\theta}$ - $\mathcal{J}$ -cld.

Proof Let H be $\check{\theta}$ - $\mathcal{J}$ -cld. By Proposition 3.2, it is $w\check{\theta}$ - $\mathcal{J}$ -cld.

Conversely, let H be $w\check{\theta}$ - $\mathcal{J}$ -cld. Since H is open, by Theorem 3.11, H is $\star$ -cld. Hence H is $\check{\theta}$ - $\mathcal{J}$ -cld.

Theorem 3.14 A set H is $w\star$ -cld if and only if $(\text{int}(H))^* - H$ contains no non-empty sg-cld set.

Proof

Necessity. Let G be a sg-cld set such that $G \subseteq (\text{int}(H))^* - H$. Since G^c is sg-open and $H \subseteq G^c$, from the definition of $w\check{\theta}$ - $\mathcal{J}$ -closedness it follows that $(\text{int}(H))^* \subseteq G^c$. i.e., $G \subseteq ((\text{int}(H))^*)^c$. This implies that $G \subseteq ((\text{int}(H))^*) \cap ((\text{int}(H))^*)^c = \phi$.

Sufficiency. Let $H \subseteq J$, where J is $\star$ -cld and sg-open set in X . If $(\text{int}(H))^*$ is not contained in J , then $(\text{int}(H))^* \cap J^c$ is a non-empty sg-cld subset of $(\text{int}(H))^* - H$, we obtain a contradiction. This proves the sufficiency and hence the theorem.

Theorem 3.15 Let X be an ideal topological space and $H \subseteq Y \subseteq X$. If H is $w\check{\theta}$ - $\mathcal{J}$ -cld in X , then H is $w\check{\theta}$ - $\mathcal{J}$ -cld relative to Y .

Proof

Let $H \subseteq Y \cap J$ where J is sg-open in X . Since H is $w\check{\theta}\text{-}\mathcal{J}$ -cld in X , $H \subseteq J$ implies $(\text{int}(H))^* \subseteq J$. That is $Y \cap ((\text{int}(H))^*)^c \subseteq Y \cap J$ where $Y \cap (\text{int}(H))^*$ is closure of interior of H in Y . Thus, H is $w\check{\theta}\text{-}\mathcal{J}$ -cld relative to Y .

Theorem 3.16 If a subset H of an ideal topological space X is nowhere dense, then it is $w\check{\theta}\text{-}\mathcal{J}$ -cld.

Proof

Since $\text{int}(H) \subseteq \text{int}(H^*)$ and H is nowhere dense, $\text{int}(H) = \emptyset$. Therefore $(\text{int}(H))^* = \emptyset$ and hence H is $w\check{\theta}\text{-}\mathcal{J}$ -cld in X .

The converse of Theorem 3.16 need not be true as seen in the following example.

Example 3.17 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{p\}, \{q, r\}, X\}$ with $\mathcal{J} = \{\emptyset\}$. Then the set $\{p\}$ is $w\check{\theta}\text{-}\mathcal{J}$ -cldset but not nowhere dense in X .

Remark 3.18

The following examples show that $w\check{\theta}\text{-}\mathcal{J}$ -closedness and semi-closedness are independent.

Example 3.19

In Example 3.3, we have the set $\{p, r\}$ is $w\check{\theta}\text{-}\mathcal{J}$ -cldset but not semi-cld in X .

Example 3.20

Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{p\}, \{q\}, \{p, q\}, X\}$ with $\mathcal{J} = \{\emptyset\}$. Then the set $\{p\}$ is semi-cld set but not $w\check{\theta}\text{-}\mathcal{J}$ -cld in X .

Remark 3.21

From the above discussions and known results in [18]. We obtain the following diagram, where $A \rightarrow B$ represents A implies B but not conversely.

Diagram

$$\star\text{-cld} \rightarrow w\check{\theta}\text{-}\mathcal{J}\text{-closed} \rightarrow \text{wg-closed} \rightarrow w\pi\text{g-closed} \rightarrow \text{rwg-closed}$$

Definition 3.22 A subset H of an ideal topological space X is called $w\check{\theta}\text{-}\mathcal{J}$ -open set if H^c is $w\check{\theta}\text{-}\mathcal{J}$ -cld in X .

Proposition 3.23

- (i) Every $\check{\theta}\text{-}\mathcal{J}$ -open set is $w\check{\theta}\text{-}\mathcal{J}$ -open but not conversely.
- (ii) Every g-open set is $w\check{\theta}\text{-}\mathcal{J}$ -open but not conversely.

Theorem 3.24 A subset H of an ideal topological space X is $w\check{\theta}\text{-}\mathcal{J}$ -open if $J \subseteq \text{int}(H^*)$ whenever $J \subseteq H$ and J is sg-cld.

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Proof

Let H be any $w\check{\theta}$ - $\mathcal{J}$ -open. Then H^c is $w\check{\theta}$ - $\mathcal{J}$ -cld. Let J be a sg-cld set contained in H . Then J^c is a sg-open set containing H^c . Since H^c is $w\check{\theta}$ - $\mathcal{J}$ -cld, we have $(\text{int}(H^c))^* \subseteq J^c$. Therefore $J \subseteq \text{int}(H^*)$.

Conversely, we suppose that $J \subseteq \text{int}(H^*)$ whenever $J \subseteq H$ and J is sg-cld. Then J^c is a sg-open set containing H^c and $J^c \supseteq (\text{int}(H^*))^c$. It follows that $J^c \supseteq (\text{int}(H^c))^*$. Hence H^c is $w\check{\theta}$ - $\mathcal{J}$ -cld and so A is $w\check{\theta}$ - $\mathcal{J}$ -open.

4. Weakly $\check{\theta}$ - $\mathcal{J}$ -continuous functions

Definition 4.1 Let X and Y be two an ideal topological space. A function $f : X \rightarrow Y$ is called weakly $\check{\theta}$ - $\mathcal{J}$ -continuous (briefly, $w\check{\theta}$ - $\mathcal{J}$ -continuous) if $f^{-1}(V)$ is a $w\check{\theta}$ - $\mathcal{J}$ -open set in X for each open set V of Y .

Example 4.2 Let $X = Y = \{p, q, r\}$, $\tau = \{\emptyset, \{p\}, \{q, r\}, X\}$ with $\mathcal{J} = \{\emptyset\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. The function $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ defined by $f(p) = q$, $f(q) = r$ and $f(r) = p$ is $w\check{\theta}$ - $\mathcal{J}$ -continuous, because every subset of Y is $w\check{\theta}$ - $\mathcal{J}$ -cld in X .

Theorem 4.3 Every $\check{\theta}$ - $\mathcal{J}$ -continuous function is $w\check{\theta}$ - $\mathcal{J}$ -continuous.

Proof

It follows from Proposition 3.23 (i).

The converse of Theorem 4.3 need not be true as seen in the following example.

Example 4.4 Let $X = Y = \{p, q, r\}$, $\tau = \{\emptyset, \{p\}, \{q, r\}, X\}$ with $\mathcal{J} = \{\emptyset\}$ and $\sigma = \{\emptyset, \{q\}, Y\}$. Let $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ be the identity function. Then f is $w\check{\theta}$ - $\mathcal{J}$ -continuous but not $\check{\theta}$ - $\mathcal{J}$ -continuous.

Theorem 4.5 A function $f : X \rightarrow Y$ is $w\check{\theta}$ - $\mathcal{J}$ -continuous if and only if $f^{-1}(V)$ is a $w\check{\theta}$ - $\mathcal{J}$ -cld set in X for each closed set V of Y .

Proof

Let V be any closed set of Y . According to the assumption $f^{-1}(V^c) = X \setminus f^{-1}(V)$ is $w\check{\theta}$ - $\mathcal{J}$ -open in X , so $f^{-1}(V)$ is $w\check{\theta}$ - $\mathcal{J}$ -cld in X .

The converse can be proved in a similar manner.

Definition 4.6 An ideal topological space X is said to be locally $\check{\theta}$ - $\mathcal{J}$ -indiscrete if every $\check{\theta}$ - $\mathcal{J}$ -open set of X is $\star$ -cld in X .

Theorem 4.7 Let $f : X \rightarrow Y$ be a function. If f is $\check{\theta}$ - $\mathcal{J}$ -continuous and X is locally $\check{\theta}$ - $\mathcal{J}$ -indiscrete, then f is $\star$ -continuous.

Proof. Let W be an open in Y . Since f is $\check{\theta}$ - $\mathcal{J}$ -continuous, $f^{-1}(W)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X . Since X is locally $\check{\theta}$ - $\mathcal{J}$ -indiscrete, $f^{-1}(W)$ is $\star$ -cld in X . Hence f is $\star$ -continuous.

Theorem 4.8 Let $f : X \rightarrow Y$ be a function. If f is contra $\check{\theta}\text{-}\mathcal{I}$ -continuous and X is locally $\check{\theta}\text{-}\mathcal{I}$ -indiscrete, then f is $w\check{\theta}\text{-}\mathcal{I}$ -continuous.

Proof

Let $f : X \rightarrow Y$ be contra $\check{\theta}\text{-}\mathcal{I}$ -continuous and X is locally $\check{\theta}\text{-}\mathcal{I}$ -indiscrete. By Theorem 4.7, f is $\star$ -continuous, then f is $w\check{\theta}\text{-}\mathcal{I}$ -continuous.

Proposition 4.9 If $f : X \rightarrow Y$ is perfectly continuous and $w\check{\theta}\text{-}\mathcal{I}$ -continuous, then it is R-map.

Proof

Let W be any regular open subset of Y . According to the assumption, $f^{-1}(W)$ is both open and $\star$ -cld in X . Since $f^{-1}(W)$ is $\star$ -cld, it is wg-closed. We have $f^{-1}(V)$ is both open and wg-closed. Hence, it is regular open in X , so f is R-map.

Definition 4.10 An ideal topological space X is called $\check{\theta}\text{-}\mathcal{I}$ -compact if every cover of X by $\check{\theta}\text{-}\mathcal{I}$ -open sets has finite subcover.

Definition 4.11 An ideal topological space X is weakly $\check{\theta}\text{-}\mathcal{I}$ -compact (briefly, $w\check{\theta}\text{-}\mathcal{I}$ -compact) if every $w\check{\theta}\text{-}\mathcal{I}$ -open cover of X has a finite subcover.

Remark 4.12 Every $w\check{\theta}\text{-}\mathcal{I}$ -compact space is $\check{\theta}\text{-}\mathcal{I}$ -compact.

Theorem 4.13 Let $f : X \rightarrow Y$ be surjective $w\check{\theta}\text{-}\mathcal{I}$ -continuous function. If X is $w\check{\theta}\text{-}\mathcal{I}$ -compact, then Y is compact.

Proof

Let $\{A_i : i \in I\}$ be an open cover of Y . Then $\{f^{-1}(A_i) : i \in I\}$ is a $w\check{\theta}\text{-}\mathcal{I}$ -open cover in X . Since X is $w\check{\theta}\text{-}\mathcal{I}$ -compact, it has a finite subcover, say $\{f^{-1}(A_1), f^{-1}(A_2), \dots, f^{-1}(A_n)\}$. Since f is surjective $\{A_1, A_2, \dots, A_n\}$ is a finite subcover of Y and hence Y is compact.

Definition 4.14 An ideal topological space X is weakly $\check{\theta}\text{-}\mathcal{I}$ -connected (briefly, $w\check{\theta}\text{-}\mathcal{I}$ -connected) if X cannot be written as the disjoint union of two non-empty $w\check{\theta}\text{-}\mathcal{I}$ -open sets.

Theorem 4.15 If an ideal topological space X is $w\check{\theta}\text{-}\mathcal{I}$ -connected, then X is almost connected (resp. $\check{\theta}\text{-}\mathcal{I}$ -connected).

Proof

It follows from the fact that each regular open set (resp. $\check{\theta}\text{-}\mathcal{I}$ -open set) is $w\check{\theta}\text{-}\mathcal{I}$ -open.

Theorem 4.16 For an ideal topological space X , the following statements are equivalent:

- i. X is $w\check{\theta}\text{-}\mathcal{I}$ -connected.
- ii. The empty set ϕ and X are only subsets which are both $w\check{\theta}\text{-}\mathcal{I}$ -open and $w\check{\theta}\text{-}\mathcal{I}$ -cld.

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iii. Each $w\check{\theta}$ - $\mathcal{J}$ -continuous function from X into a discrete space Y which has at least two points is a constant function.

Proof

(i) $\Rightarrow$ (ii). Let $S \subseteq X$ be any proper subset, which is both $w\check{\theta}$ - $\mathcal{J}$ -open and $w\check{\theta}$ - $\mathcal{J}$ -cld. Its complement $X \setminus S$ is also $w\check{\theta}$ - $\mathcal{J}$ -open and $w\check{\theta}$ - $\mathcal{J}$ -cld. Then $X = S \cup (X \setminus S)$ is a disjoint union of two non-empty $w\check{\theta}$ - $\mathcal{J}$ -open sets which is a contradiction with the fact that X is $w\check{\theta}$ - $\mathcal{J}$ -connected. Hence, $S = \emptyset$ or X .

(ii) $\Rightarrow$ (i). Let $X = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B are $w\check{\theta}$ - $\mathcal{J}$ -open. Since $A = X \setminus B$, A is $w\check{\theta}$ - $\mathcal{J}$ -closed. According to the assumption $A = \emptyset$, which is a contradiction.

(ii) $\Rightarrow$ (iii). Let $f : X \rightarrow Y$ be a $w\check{\theta}$ - $\mathcal{J}$ -continuous function where Y is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is $w\check{\theta}$ - $\mathcal{J}$ -closed and $w\check{\theta}$ - $\mathcal{J}$ -open for each $y \in Y$ and $X = \cup \{f^{-1}(\{y\}) \mid y \in Y\}$. According to the assumption, $f^{-1}(\{y\}) = \emptyset$ or $f^{-1}(\{y\}) = X$. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, f will not be a function. Also there is no exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence, there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \emptyset$ where $y \neq y_1 \in Y$. This shows that f is a constant function.

(iii) $\Rightarrow$ (ii). Let $S \neq \emptyset$ be both $w\check{\theta}$ - $\mathcal{J}$ -open and $w\check{\theta}$ - $\mathcal{J}$ -closed in X . Let $f : X \rightarrow Y$ be a $w\check{\theta}$ - $\mathcal{J}$ -continuous function defined by $f(S) = \{a\}$ and $f(X \setminus S) = \{b\}$ where $a \neq b$. Since f is constant function, we get $S = X$.

Theorem 4.17 Let $f : X \rightarrow Y$ be a $w\check{\theta}$ - $\mathcal{J}$ -continuous surjective function. If X is $w\check{\theta}$ - $\mathcal{J}$ -connected, then Y is connected.

Proof

We suppose that Y is not connected. Then $Y = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B are open sets in Y . Since f is $w\check{\theta}$ - $\mathcal{J}$ -continuous surjective function, $X = f^{-1}(A) \cup f^{-1}(B)$ are disjoint union of two non-empty $w\check{\theta}$ - $\mathcal{J}$ -open subsets. This is contradiction with the fact that X is $w\check{\theta}$ - $\mathcal{J}$ -connected.

5. Weakly $\check{\theta}$ - $\mathcal{J}$ -Open Functions and Weakly $\check{\theta}$ - $\mathcal{J}$ -Closed Functions

Definition 5.1 Let X and Y be an ideal topological space. A function $f : X \rightarrow Y$ is called weakly $\check{\theta}$ - $\mathcal{J}$ -open (briefly, $w\check{\theta}$ - $\mathcal{J}$ -open) if $f(V)$ is a $w\check{\theta}$ - $\mathcal{J}$ -open set in Y for each open set V of X .

Remark 5.2 Every $\check{\theta}$ - $\mathcal{J}$ -open function is $w\check{\theta}$ - $\mathcal{J}$ -open but not conversely.

Example 5.3 Let $X = Y = \{p, q, r, s\}$, $\tau = \{\emptyset, \{p\}, \{p, q, s\}, X\}$ with $\mathcal{J} = \{\emptyset\}$ and $\sigma = \{\emptyset, \{p\}, \{q, r\}, \{p, q, r\}, Y\}$. Let $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ be the identity function. Then f is $w\check{\theta}$ - $\mathcal{J}$ -open but not $\check{\theta}$ - $\mathcal{J}$ -open.

Definition 5.4 Let X and Y be an ideal topological space. A function $f : X \rightarrow Y$ is called weakly $\check{\theta}$ - $\mathcal{J}$ -closed (briefly, $w\check{\theta}$ - $\mathcal{J}$ -cld) if $f(V)$ is a $w\check{\theta}$ - $\mathcal{J}$ -cld set in Y for each closed set V of X .

It is clear that an open function is $w\check{\theta}$ - $\mathcal{J}$ -open and a closed function is $w\check{\theta}$ - $\mathcal{J}$ -cld.

Theorem 5.5 Let X and Y be an ideal topological space. A function $f : X \rightarrow Y$ is $w\check{\theta}$ - $\mathcal{J}$ -closed if and only if for each subset B of Y and for each open set G containing $f^{-1}(B)$ there exists a $w\check{\theta}$ - $\mathcal{J}$ -open set F of Y such that $B \subseteq F$ and $f^{-1}(F) \subseteq G$.

Proof

Let B be any subset of Y and let G be an open subset of X such that $f^{-1}(B) \subseteq G$. Then $F = Y \setminus f(X \setminus G)$ is $w\check{\theta}$ - $\mathcal{J}$ -open set containing B and $f^{-1}(F) \subseteq G$.

Conversely, let U be any closed subset of X . Then $f^{-1}(Y \setminus f(U)) \subseteq X \setminus U$ and $X \setminus U$ is open. According to the assumption, there exists a $w\check{\theta}$ - $\mathcal{J}$ -open set F of Y such that $Y \setminus f(U) \subseteq F$ and $f^{-1}(F) \subseteq X \setminus U$. Then $U \subseteq X \setminus f^{-1}(F)$. From $Y \setminus F \subseteq f(U) \subseteq f(X \setminus f^{-1}(F)) \subseteq Y \setminus F$ it follows that $f(U) = Y \setminus F$, so $f(U)$ is $w\check{\theta}$ - $\mathcal{J}$ -cld in Y . Therefore, f is a $w\check{\theta}$ - $\mathcal{J}$ -cld function.

Remark 5.6 The composition of two $w\check{\theta}$ - $\mathcal{J}$ -cld functions need not be a $w\check{\theta}$ - $\mathcal{J}$ -cld as we can see from the following example.

Example 5.7 Let $X = Y = Z = \{p, q, r\}$, $\tau = \{\phi, \{p\}, \{p, q\}, X\}$ with $\mathcal{J} = \{\phi\}$ and $\sigma = \{\phi, \{p\}, \{q, r\}, Y\}$ and $\eta = \{\phi, \{p, q\}, Z\}$ with $\mathcal{J} = \{\phi\}$. We define $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ by $f(p) = r$, $f(q) = q$ and $f(r) = p$ and let $g : (Y, \sigma) \rightarrow (Z, \eta, \mathcal{J})$ be the identity function. Hence both f and g are $w\check{\theta}$ - $\mathcal{J}$ -cld functions. Hence the composition of two $w\check{\theta}$ - $\mathcal{J}$ -cld functions need not be a $w\check{\theta}$ - $\mathcal{J}$ -cld.

Theorem 5.8 Let X , Y and Z be an ideal topological space. If $f : X \rightarrow Y$ is a $\star$ -cld function and $g : Y \rightarrow Z$ is a $w\check{\theta}$ - $\mathcal{J}$ -cld function, then $g \circ f : X \rightarrow Z$ is a $w\check{\theta}$ - $\mathcal{J}$ -cld function.

Theorem 5.9 A set K of X is $\check{\theta}$ - $\mathcal{J}$ -open if and only if $F \subseteq \text{int}(K)$ whenever F is sg-cld and $F \subseteq K$.

Proof

Suppose that $F \subseteq \text{int}(K)$ such that F is sg-cld and $F \subseteq K$. Let $K^c \subseteq G$ where G is sg-open. Then $G^c \subseteq K$ and G^c is sg-cld. Therefore $G^c \subseteq \text{int}(K)$ by hypothesis. Since $G^c \subseteq \text{int}(K)$, we have $(\text{int}(K))^c \subseteq G$. i.e., $(K^c)^* \subseteq G$, since $(K^c)^* = (\text{int}(K))^c$. Thus, K^c is $\check{\theta}$ - $\mathcal{J}$ -cld. i.e., K is $\check{\theta}$ - $\mathcal{J}$ -open.

Conversely, suppose that K is $\check{\theta}$ - $\mathcal{J}$ -open such that $F \subseteq K$ and F is sg-cld. Then F^c is sg-open and $K^c \subseteq F^c$. Therefore, $(K^c)^* \subseteq F^c$ by definition of $\check{\theta}$ - $\mathcal{J}$ -cld and so $F \subseteq \text{int}(K)$, since $(K^c)^* = (\text{int}(K))^c$.

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Lemma 5.10 For an $x \in X$, $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ if and only if $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x .

Proof

Let $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ for any $x \in X$. To prove $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x . Prove the result by contradiction. Suppose there exists a $\check{\theta}$ - $\mathcal{J}$ -open set V containing x such that $V \cap K = \emptyset$. Then $K \subseteq V^c$ and V^c is $\check{\theta}$ - $\mathcal{J}$ -cld. We have $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) \subseteq V^c$. This shows that $x \notin \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ which is a contradiction. Hence $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x .

Conversely, let $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x . To prove $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$. We prove the result by contradiction. Suppose $x \notin \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$. Then there exists a $\check{\theta}$ - $\mathcal{J}$ -cld set F containing K such that $x \notin F$. Then $x \in F^c$ and F^c is $\check{\theta}$ - $\mathcal{J}$ -open. Also, $F^c \cap K = \emptyset$, which is a contradiction to the hypothesis. Hence $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$.

Proposition 5.11

A function $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ is $\check{\theta}$ - $\mathcal{J}$ -continuous if and only if $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X , for every open set U in Y .

Proof

Let $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ be $\check{\theta}$ - $\mathcal{J}$ -continuous and U be an open set in Y . Then U^c is closed in Y and since f is $\check{\theta}$ - $\mathcal{J}$ -continuous, $f^{-1}(U^c)$ is $\check{\theta}$ - $\mathcal{J}$ -cld in X . But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X .

Conversely, assume that $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X , for each open set U in Y . Let F be a closed set in Y . Then F^c is open in Y and by assumption, $f^{-1}(F^c)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X . Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $\check{\theta}$ - $\mathcal{J}$ -cld in X and so f is $\check{\theta}$ - $\mathcal{J}$ -continuous.

Theorem 5.12 If $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ is $\check{\theta}$ - $\mathcal{J}$ -continuous and pre-sg-closed and if A is an $\check{\theta}$ - $\mathcal{J}$ -open (or $\check{\theta}$ - $\mathcal{J}$ -cld) subset of Y , then $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{J}$ -open (or $\check{\theta}$ - $\mathcal{J}$ -cld) in X .

Proof

Let H be an $\check{\theta}$ - $\mathcal{J}$ -open set in Y and F be any sg-closed set in X such that $F \subseteq f^{-1}(H)$. Then $f(F) \subseteq H$. By hypothesis, $f(F)$ is sg-closed and H is $\check{\theta}$ - $\mathcal{J}$ -open in Y . Therefore, $f(F) \subseteq \text{int}(H)$ and so $F \subseteq f^{-1}(\text{int}(H))$. Since f is $\check{\theta}$ - $\mathcal{J}$ -continuous and $\text{int}(H)$ is open in Y , $f^{-1}(\text{int}(H))$ is $\check{\theta}$ - $\mathcal{J}$ -open in X . Thus $F \subseteq \text{int}(f^{-1}(\text{int}(H))) \subseteq \text{int}(f^{-1}(H))$. i.e., $F \subseteq \text{int}(f^{-1}(H))$ and $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X . By taking complements, we can show that if H is $\check{\theta}$ - $\mathcal{J}$ -cld in Y , $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{J}$ -cld in X .

6. Conclusions

The new class of generalized closed sets called weakly $\check{\theta}$ - $\mathcal{J}$ -closed sets and $w\check{\theta}$ - $\mathcal{J}$ -continuous functions are very useful for new research in topological spaces. This may leads some new applications in real life problems.

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Some New $\check{\theta}$ - $\mathcal{I}$ -Locally Closed sets with Respect to an Ideal Topological Spaces

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Abstract

In this paper, we introduce the new notions called $\check{\theta}$ - $\mathcal{I}$ -locally closed sets, $\check{\theta}$ - $\mathcal{I}$ -locally closed sets and $\check{\theta}$ - $\mathcal{I}$ -closed functions and investigated their properties and also we have studied their relations to the other types of locally closed sets with suitable examples. Finally we introduce the notion $\check{\theta}$ - $\mathcal{I}$ -submaximal spaces and also investigated the properties with examples.

Keywords: $\check{\theta}$ - $\mathcal{I}$ -cld, $\check{\theta}$ - $\mathcal{I}$ -lc, $\check{\theta}$ - $\mathcal{I}$ -lc*, $\check{\theta}$ - $\mathcal{I}$ -lc**.

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1. Introduction

In 1970, Levine [11] introduced the generalized closed sets and after that many authors introduced and studied many types of generalized closed sets in topological spaces. In the continuity, locally closedness was done by Bourbaki [4]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [16] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [6] to define LC-continuity and LC-irresoluteness. Balachandran et al [3] introduced the concept of generalized locally closed sets. Veera Kumar [18] (Sheik John [15]) introduced $\hat{g}$ -locally closed sets (ω -locally closed sets) respectively.

In this paper, we introduce the new notions called $\check{\theta}$ - $\mathcal{J}$ -locally closed sets, $\check{\theta}$ - $\mathcal{J}$ -locally closed sets and $\check{\theta}$ - $\mathcal{J}$ -closed functions and investigated their properties and also we have studied their relations to the other types of locally closed sets with suitable examples. Finally we introduce the notion $\check{\theta}$ - $\mathcal{J}$ -submaximal spaces and also investigated the properties with examples.

2. Preliminaries

An ideal $\mathcal{J}$ on a topological space (briefly, TPS) (X, τ) is a nonempty collection of subsets of X which satisfies

- (1) $A \in \mathcal{J}$ and $B \subseteq A \Rightarrow B \in \mathcal{J}$ and
- (2) $A \in \mathcal{J}$ and $B \in \mathcal{J} \Rightarrow A \cup B \in \mathcal{J}$.

Given a topological space (X, τ) with an ideal $\mathcal{J}$ on X if $\wp(X)$ is the set of all subsets of X , a set operator $(\bullet)^*$: $\wp(X) \rightarrow \wp(X)$, called a local function [10] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{J}, \tau) = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(\bullet)$ for a topology $\tau^*(\mathcal{J}, \tau)$, called the $\star$ -topology and finer than τ , is defined by $cl^*(A) = A \cup A^*(\mathcal{J}, \tau)$ [10]. We will simply write A^* for $A^*(\mathcal{J}, \tau)$ and τ^* for $\tau^*(\mathcal{J}, \tau)$. If $\mathcal{J}$ is an ideal on X , then $(X, \tau, \mathcal{J})$ is called an ideal topological space (briefly, ITPS). A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is $\star$ -closed (briefly, $\star$ -cld) [10] if $A^* \subseteq A$. The interior of a subset A in $(X, \tau^*(\mathcal{J}))$ is denoted by $int^*(A)$.

Definition 2.1 A subset K of a TPS X is called:

- (i) semi-open set [9] if $K \subseteq cl(int(K))$;
- (ii) α -open set [9] if $K \subseteq int(cl(int(K)))$;
- (iii) regular open set [12] if $K = int(cl(K))$;

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.2 A subset K of a TPS X is called

- (i) g -closed set (briefly, g -cld) [11] if $cl(K) \subseteq V$ whenever $K \subseteq V$ and V is open.

- (ii) semi-generalized closed (briefly, sg-cld)[7] if $scl(K) \subseteq V$ whenever $K \subseteq V$ and V is semi-open.
- (iii) generalized semi-closed (briefly, gs-cld)[18] if $scl(K) \subseteq V$ whenever $K \subseteq V$ and V is open.
- (iv) $\hat{g}$ -closed set [18] (ω -cld set [18]) if $cl(K) \subseteq V$ whenever $K \subseteq V$ and V is semi-open in X .

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3 A subset K of a ITPS X is called

- (i) $\mathcal{J}_g$ -closed (briefly, $\mathcal{J}_g$ -cld) set [9] if $K^* \subseteq V$ whenever $K \subseteq V$ and V is open.
- The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.4 A subset K of a space X is called a regular generalized closed (briefly, rg-closed) set [12] if $cl(K) \subseteq V$ whenever $K \subseteq V$ and V is regular open in X . The complement of rg-closed set is called rg-open set;

Remark 2.5

The collection of all rg-closed sets in X is denoted by $RG C(X)$.

The collection of all rg-open sets in X is denoted by $RG O(X)$.

Definition 2.6 A subset K of a space X is called

- i. generalized locally closed (briefly, glc) [2] if $A = V \cap F$, where V is g-open and F is g-closed in X .
- ii. semi-generalized locally closed (briefly, sglc) [13] if $K = V \cap F$, where V is sg-open and F is sg-closed in X .
- iii. regular-generalized locally closed (briefly, rg-lc) [1] if $K = V \cap F$, where V is rg-open and F is rg-closed in X .
- iv. generalized locally semi-closed (briefly, glsc) [8] if $K = V \cap F$, where V is g-open and F is semi-closed in X .
- v. locally semi-closed (briefly, lsc) [8] if $K = V \cap F$, where V is open and F is semi-closed in X .
- vi. α -locally closed (briefly, α -lc) [8] if $K = V \cap F$, where V is α -open and F is α -closed in X .
- vii. ω -locally closed (briefly, ω -lc) [15] if $K = V \cap F$, where V is ω -open and F is ω -closed in X .

The class of all generalized locally closed (resp. generalized locally semi-closed, locally semi-closed, ω -locally closed) sets in X is denoted by $GLC(X)$ (resp. $GLSC(X)$, $LSC(X)$, ω -LC(X)).

Definition 2.7 A topological space X is called

- i. submaximal [5, 18] if every dense subset is open.

- ii. $\hat{g}$ (or ω)-submaximal [15, 18] if every dense subset is ω -open.
- iii. g -submaximal [2] if every dense subset is g -open.
- iv. rg -submaximal [12] if every dense subset is rg -open.

Theorem 2.8 Let X be a topological space

- i. If X is submaximal, then X is $\hat{g}$ -submaximal.
- ii. If X is $\hat{g}$ -submaximal, then X is g -submaximal.
- iii. If X is g -submaximal, then X is rg -submaximal.

3. $\check{\theta}$ - $\mathcal{J}$ -Locally Closed Sets

We introduce the following definition.

Definition 3.1 A subset K of X is called

- (i) $\check{\theta}$ - $\mathcal{J}$ -closed (briefly, $\check{\theta}$ - $\mathcal{J}$ -cld) if $K^* \subseteq V$ whenever $K \subseteq V$ and V is sg -open. The complement of $\check{\theta}$ - $\mathcal{J}$ -cld is called $\check{\theta}$ - $\mathcal{J}$ -open. The family of all $\check{\theta}$ - $\mathcal{J}$ -cld in X is denoted by $\check{\theta}\mathcal{J}C(X)$.
- (ii) $\check{\theta}$ - $\mathcal{J}$ -locally closed (briefly, $\check{\theta}$ - $\mathcal{J}$ -lc) if $K = S \cap G$, where S is $\check{\theta}$ - $\mathcal{J}$ -open and G is $\check{\theta}$ - $\mathcal{J}$ -cld.
- (iii) A function $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma)$ is called $\check{\theta}$ - $\mathcal{J}$ -continuous if the inverse image of every closed set in Y is $\check{\theta}$ - $\mathcal{J}$ -cld set in X . The class of all $\check{\theta}$ - $\mathcal{J}$ -locally closed sets in X is denoted by $\check{\theta}\mathcal{J}LC(X)$.

Proposition 3.2 Every $\check{\theta}$ - $\mathcal{J}$ -cld (resp. $\check{\theta}$ - $\mathcal{J}$ -open) set is $\check{\theta}$ - $\mathcal{J}$ -lc set but not conversely.

Proof

It follows from Definition 3.1 (i) and (ii).

Example 3.3 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{q\}, X\}$ with $\mathcal{J} = \{\emptyset\}$. Then the set $\{q\}$ is $\check{\theta}$ - $\mathcal{J}$ -lc set but it is not $\check{\theta}$ - $\mathcal{J}$ -closed and the set $\{p, r\}$ is $\check{\theta}$ - $\mathcal{J}$ -lc set but it is not $\check{\theta}$ - $\mathcal{J}$ -open in X .

Proposition 3.4 Every lc set is $\check{\theta}$ - $\mathcal{J}$ -lc set but not conversely.

Proof

It follows from Proposition 3.2.

Example 3.5 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{q, r\}, X\}$ with $\mathcal{J} = \{\emptyset\}$. Then the set $\{q\}$ is $\check{\theta}$ - $\mathcal{J}$ -lc set but it is not lc set in X .

Proposition 3.6 Every $\check{\theta}$ - $\mathcal{J}$ -lc set is a (i) ω -lc set, (ii) glc set and (iii) $sglc$ set. However the separate converses are not true.

Proof

It is obviously.

Example 3.7 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{p\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then the set $\{b\}$ is glc set but it is not $\check{\theta}$ - $\mathcal{I}$ -lc set in X . Moreover, the set $\{r\}$ is sglc set but it is not $\check{\theta}$ - $\mathcal{I}$ -lc set in X .

Example 3.8 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{q\}, \{p, r\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then the set $\{p\}$ is ω -lc set but it is not $\check{\theta}$ - $\mathcal{I}$ -lc set in X .

Remark 3.9 The concepts of α -lc sets and $\check{\theta}$ - $\mathcal{I}$ -lc sets are independent of each other.

Example 3.10 The set $\{q, r\}$ in Example 3.3 is α -lc set but it is not a $\check{\theta}$ - $\mathcal{I}$ -lc set in X and the set $\{p, q\}$ in Example 3.5 is $\check{\theta}$ - $\mathcal{I}$ -lc set but it is not an α -lc set in X .

Remark 3.11 The concepts of lsc sets and $\check{\theta}$ - $\mathcal{I}$ -lc sets are independent of each other.

Example 3.12 The set $\{p\}$ in Example 3.3 is lsc set but it is not a $\check{\theta}$ - $\mathcal{I}$ -lc set in X and the set $\{p, q\}$ in Example 3.5 is $\check{\theta}$ - $\mathcal{I}$ -lc set but it is not a lsc set in X .

Remark 3.13 The concepts of $\check{\theta}$ - $\mathcal{I}$ -lc sets and glsc sets are independent of each other.

Example 3.14 The set $\{q, r\}$ in Example 3.3 is glsc set but it is not a $\check{\theta}$ - $\mathcal{I}$ -lc set in X and the set $\{p, q\}$ in Example 3.5 is $\check{\theta}$ - $\mathcal{I}$ -lc set but it is not a glsc set in X .

Remark 3.15 The concepts of $\check{\theta}$ - $\mathcal{I}$ -lc sets and $sglc^*$ sets are independent of each other.

Example 3.16 The set $\{q, r\}$ in Example 3.3 is $sglc^*$ set but it is not a $\check{\theta}$ - $\mathcal{I}$ -lc set in X and the set $\{p, q\}$ in Example 3.5 is $\check{\theta}$ - $\mathcal{I}$ -lc set but it is not a $sglc^*$ set in X .

Theorem 3.17 For a $T\check{\theta}$ - $\mathcal{I}$ -space X , the following properties hold

- (i) $\check{\theta}$ - $\mathcal{I}$ -LC(X) = $LC(X)$.
- (ii) $\check{\theta}$ - $\mathcal{I}$ -LC(X) $\subseteq$ $GLC(X)$.
- (iii) $\check{\theta}$ - $\mathcal{I}$ -LC(X) $\subseteq$ $GLSC(X)$.
- (iv) $\check{\theta}$ - $\mathcal{I}$ -LC(X) $\subseteq$ ω - $LC(X)$.

Proof

- (i) Since every $\check{\theta}$ - $\mathcal{I}$ -open set is open and every $\check{\theta}$ - $\mathcal{I}$ -closed set is closed in (X, τ) , $\check{\theta}$ - $\mathcal{I}$ -LC(X) $\subseteq$ $LC(X)$ and hence $\check{\theta}$ - $\mathcal{I}$ -LC(X) = $LC(X)$.
- (ii), (iii) and (iv) follows from (i), since for any space X , $LC(X) \subseteq GLC(X)$, $LC(X) \subseteq GLSC(X)$ and $LC(X) \subseteq \omega$ - $LC(X)$.

Corollary 3.18 If $G O(X) = \tau$, then $\check{\theta}$ - $\mathcal{I}$ -LC(X) $\subseteq$ $GLSC(X) \subseteq LSC(X)$.

Proof

$GO(X) = \tau$ implies that X is a $T\check{\theta}\mathcal{J}$ -space and hence by Theorem 3.17, $\check{\theta}\mathcal{J}\text{-LC}(X) \subseteq GLSC(X)$. Let $K \in GLSC(X)$. Then $K = V \cap F$, where V is g -open and F is semi-closed. By hypothesis, V is open and hence K is a lsc-set and so $K \in LSC(X)$.

Definition 3.19 A subset K of a space X is called:

- (i) $\check{\theta}\mathcal{J}\text{-lc}^*$ set if $K = S \cap G$, where S is $\check{\theta}\mathcal{J}$ -open in X and G is closed in X .
 - (ii) $\check{\theta}\mathcal{J}\text{-lc}^{**}$ set if $K = S \cap G$, where S is open in X and G is $\check{\theta}\mathcal{J}$ -closed in X .
- The class of all $\check{\theta}\mathcal{J}\text{-lc}^*$ (resp. $\check{\theta}\mathcal{J}\text{-lc}^{**}$) sets in a ideal topological space X is denoted by $\check{\theta}\mathcal{J}\text{-LC}^*(X)$ (resp. $\check{\theta}\mathcal{J}\text{-LC}^{**}(X)$).

Proposition 3.20 Every lc-set is $\check{\theta}\mathcal{J}\text{-lc}^*$ set but not conversely.

Proof

It follows from Definition 3.19 (i) and Definition of locally closed set.

Example 3.21 The set $\{q\}$ in Example 3.5 is $\check{\theta}\mathcal{J}\text{-lc}^*$ set but it is not a lc set in X .

Proposition 3.22 Every lc-set is $\check{\theta}\mathcal{J}\text{-lc}^{**}$ set but not conversely.

Proof

It follows from Definition 3.19 (ii) and Definition of locally closed set.

Example 3.23 The set $\{p, r\}$ in Example 3.5 is $\check{\theta}\mathcal{J}\text{-lc}^{**}$ set but it is not a lc set in X .

Proposition 3.24 Every $\check{\theta}\mathcal{J}\text{-lc}^*$ set is $\check{\theta}\mathcal{J}\text{-lc}$ set but not conversely.

Proof

It follows from Definitions 3.1 and 3.19 (i).

Example 3.25 The set $\{p, q\}$ in Example 3.5 is $\check{\theta}\mathcal{J}\text{-lc}$ set but it is not a $\check{\theta}\mathcal{J}\text{-lc}^*$ set in X .

Proposition 3.26 Every $\check{\theta}\mathcal{J}\text{-lc}^{**}$ set is $\check{\theta}\mathcal{J}\text{-lc}$ set but not conversely.

Proof

It follows from Definitions 3.1 and 3.19 (ii).

Remark 3.27 The concepts of $\check{\theta}\mathcal{J}\text{-lc}^*$ sets and lsc sets are independent of each other.

Example 3.28 The set $\{r\}$ in Example 3.5 is $\check{\theta}\mathcal{J}\text{-lc}^*$ set but it is not a lsc set in X and the set $\{p\}$ in Example 3.3 is lsc set but it is not a $\check{\theta}\mathcal{J}\text{-lc}^*$ set in X .

Remark 3.29 The concepts of $\check{\theta}\mathcal{J}\text{-lc}^{**}$ sets and α -lc sets are independent of each other.

Example 3.30 The set $\{p, q\}$ in Example 3.5 is $\check{\theta}$ - $\mathcal{I}$ - lc^{**} set but it is not a α - lc set in X and the set $\{p, q\}$ in Example 3.3 is α - lc set but it is not a $\check{\theta}$ - $\mathcal{I}$ - lc^{**} set in X .

Remark 3.31 From the above discussions we have the following implications where $A \rightarrow B$ (resp. $A \nrightarrow B$) represents A implies B but not conversely (resp. A and B are independent of each other).

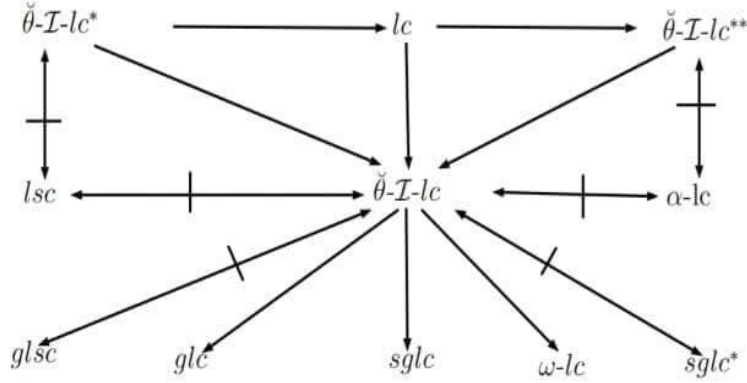


Figure 1: Relations between some of generalized closed sets

Proposition 3.32 If $G O(X) = \tau$, then $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^*(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$.

Proof

For any space (X, τ) , $\tau \subseteq \check{\theta}\text{-}\mathcal{I}\text{-}O(X) \subseteq G O(X)$. Therefore by hypothesis, $\check{\theta}\text{-}\mathcal{I}\text{-}O(X) = \tau$. i.e., (X, τ) is a $\check{\theta}\text{-}\mathcal{I}$ -space and hence $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^*(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$.

Remark 3.33 The converse of Proposition 3.32 need not be true.

For the ideal topological space X in Example 3.3. $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^*(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$. However, $G O(X) = \{\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, X\} \neq \tau$.

Proposition 3.34 Let X be an ideal topological space. If $G O(X) \subseteq LC(X)$, then $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$.

Proof

Let $K \in \check{\theta}\text{-}\mathcal{I}\text{-}LC(X)$. Then $K = S \cap G$ where S is $\check{\theta}\text{-}\mathcal{I}$ -open and G is $\check{\theta}\text{-}\mathcal{I}$ -closed. Since $\check{\theta}\text{-}\mathcal{I}\text{-}O(X) \subseteq G O(X)$ and by hypothesis $G O(X) \subseteq LC(X)$, S is locally closed. Then $S = P \cap Q$, where P is open and Q is $*$ -closed. Therefore, $K = P \cap (Q \cap G)$. We have, $Q \cap G$ is $\check{\theta}\text{-}\mathcal{I}$ -closed and hence $K \in \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$. i.e., $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) \subseteq \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$. For any ideal topological space, $\check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X) \subseteq \check{\theta}\text{-}\mathcal{I}\text{-}LC(X)$ and so $\check{\theta}\text{-}\mathcal{I}\text{-}LC(X) = \check{\theta}\text{-}\mathcal{I}\text{-}LC^{**}(X)$.

Remark 3.35 The converse of Proposition 3.34 need not be true in general.

For the ideal topological space X in Example 3.3, then $\check{\theta}\text{-}\mathcal{I}\text{-LC}(X) = \check{\theta}\text{-}\mathcal{I}\text{-LC}^{**}(X) = \{\emptyset, \{q\}, \{p, r\}, X\}$. But $G O(X) = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, X\} \not\subseteq LC(X) = \{\emptyset, \{q\}, \{p, r\}, X\}$.

Corollary 3.36 Let X be an ideal topological space. If $\omega O(X) \subseteq LC(X)$, then $\check{\theta}\text{-}\mathcal{I}\text{-LC}(X) = \check{\theta}\text{-}\mathcal{I}\text{-LC}^{**}(X)$.

Proof

It follows from the fact that $\omega O(X) \subseteq G O(X)$ and Proposition 3.34.

Remark 3.37 The converse of Corollary 3.36 need not be true in general.

For the ideal topological space X in Example 3.8, then $\check{\theta}\text{-}\mathcal{I}\text{-LC}(X) = \check{\theta}\text{-}\mathcal{I}\text{-LC}^{**}(X) = \{\emptyset, \{q\}, \{p, r\}, X\}$. But $\omega O(X) = P(X) \not\subseteq LC(X) = \{\emptyset, \{q\}, \{p, r\}, X\}$.

The following results are characterizations of $\check{\theta}\text{-}\mathcal{I}\text{-lc}$ sets, $\check{\theta}\text{-}\mathcal{I}\text{-lc}^*$ sets and $\check{\theta}\text{-}\mathcal{I}\text{-lc}^{**}$ sets.

Theorem 3.38 Assume that $\check{\theta}\text{-}\mathcal{I}\text{-C}(X)$ is closed under finite intersection. For a subset K of X , the following statements are equivalent:

- (i) $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}(X)$.
- (ii) $K = S \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$ for some $\check{\theta}\text{-}\mathcal{I}$ -open set S .
- (iii) $\check{\theta}\text{-}\mathcal{I}\text{-cl}(K) - K$ is $\check{\theta}\text{-}\mathcal{I}$ -closed.
- (iv) $K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c$ is $\check{\theta}\text{-}\mathcal{I}$ -open.
- (v) $K \subseteq \check{\theta}\text{-}\mathcal{I}\text{-int}(K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c)$.

Proof

(i) $\Rightarrow$ (ii). Let $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}(X)$. Then $K = S \cap G$ where S is $\check{\theta}\text{-}\mathcal{I}$ -open and G is $\check{\theta}\text{-}\mathcal{I}$ -closed. Since $K \subseteq G$, $\check{\theta}\text{-}\mathcal{I}\text{-cl}(K) \subseteq G$ and so $S \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K) \subseteq K$. Also $K \subseteq S$ and $K \subseteq \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$ implies $K \subseteq S \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$ and therefore $K = S \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$.

(ii) $\Rightarrow$ (iii). $K = S \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$ implies $\check{\theta}\text{-}\mathcal{I}\text{-cl}(K) - K = \check{\theta}\text{-}\mathcal{I}\text{-cl}(K) \cap S^c$ which is $\check{\theta}\text{-}\mathcal{I}$ -closed since S^c is $\check{\theta}\text{-}\mathcal{I}$ -closed and $\check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$ is $\check{\theta}\text{-}\mathcal{I}$ -closed.

(iii) $\Rightarrow$ (iv). $K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c = (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K) - K)^c$ and by assumption, $(\check{\theta}\text{-}\mathcal{I}\text{-cl}(K) - K)^c$ is $\check{\theta}\text{-}\mathcal{I}$ -open and so is $K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c$.

(iv) $\Rightarrow$ (v). By assumption, $K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c = \check{\theta}\text{-}\mathcal{I}\text{-int}(K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c)$ and hence $K \subseteq \check{\theta}\text{-}\mathcal{I}\text{-int}(K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c)$.

(v) $\Rightarrow$ (i). By assumption and since $K \subseteq \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$, $K = \check{\theta}\text{-}\mathcal{I}\text{-int}(K \cup (\check{\theta}\text{-}\mathcal{I}\text{-cl}(K))^c) \cap \check{\theta}\text{-}\mathcal{I}\text{-cl}(K)$. Therefore, $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}(X)$.

Theorem 3.39 For a subset K of X , the following statements are equivalent:

- i. $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}^*(X)$.
- ii. $K = S \cap K^*$ for some $\check{\theta}\text{-}\mathcal{I}$ -open set S .
- iii. $K^* - K$ is $\check{\theta}\text{-}\mathcal{I}$ -closed.
- iv. $K \cup (K^*)^c$ is $\check{\theta}\text{-}\mathcal{I}$ -open.

Proof

- (i) $\Rightarrow$ (ii). Let $K \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^*(X)$. There exist an $\check{\theta}$ - $\mathcal{J}$ -open set S and a $*$ -closed set G such that $K = S \cap G$. Since $K \subseteq S$ and $K \subseteq K^*$, $K \subseteq S \cap K^*$. Also, since $K^* \subseteq G$, $S \cap K^* \subseteq S \cap G = K$. Therefore $K = S \cap K^*$.
- (ii) $\Rightarrow$ (i). Since S is $\check{\theta}$ - $\mathcal{J}$ -open and K^* is a $*$ -closed set, $K = S \cap K^* \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^*(X)$.
- (ii) $\Rightarrow$ (iii). Since $K^* - K = K^* \cap S^c$, $K^* - K$ is $\check{\theta}$ - $\mathcal{J}$ -closed.
- (iii) $\Rightarrow$ (ii). Let $S = (K^* - K)^c$. Then by assumption S is $\check{\theta}$ - $\mathcal{J}$ -open in X and $K = S \cap K^*$.
- (iii) $\Rightarrow$ (iv). Let $G = K^* - K$. Then $G^c = K \cup (K^*)^c$ and $K \cup (K^*)^c$ is $\check{\theta}$ - $\mathcal{J}$ -open.
- (iv) $\Rightarrow$ (iii). Let $S = K \cup (K^*)^c$. Then S^c is $\check{\theta}$ - $\mathcal{J}$ -closed and $S^c = K^* - K$ and so $K^* - K$ is $\check{\theta}$ - $\mathcal{J}$ -closed.

Theorem 3.40 Let K be a subset of X . Then $K \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^{**}(X)$ if and only if $K = S \cap \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ for some open set S .

Proof Let $K \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^{**}(X)$. Then $K = S \cap G$ where S is open and G is $\check{\theta}$ - $\mathcal{J}$ -closed. Since $K \subseteq G$, $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) \subseteq G$. We obtain $K = K \cap \check{\theta}\text{-}\mathcal{J}\text{-cl}(K) = S \cap G \cap \check{\theta}\text{-}\mathcal{J}\text{-cl}(K) = S \cap \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$.

Converse part is trivial.

Corollary 3.41 Let K be a subset of X . If $K \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^{**}(X)$, then $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) - K$ is $\check{\theta}$ - $\mathcal{J}$ -closed and $K \cup (\check{\theta}\text{-}\mathcal{J}\text{-cl}(K))^c$ is $\check{\theta}$ - $\mathcal{J}$ -open.

Proof

Let $K \in \check{\theta}\text{-}\mathcal{J}\text{-LC}^{**}(X)$. Then by Theorem 3.40, $K = S \cap \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ for some open set S and $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) - K = \check{\theta}\text{-}\mathcal{J}\text{-cl}(K) \cap S^c$ is $\check{\theta}$ - $\mathcal{J}$ -closed in X . If $G = \check{\theta}\text{-}\mathcal{J}\text{-cl}(K) - K$, then $G^c = K \cup (\check{\theta}\text{-}\mathcal{J}\text{-cl}(K))^c$ and G^c is $\check{\theta}$ - $\mathcal{J}$ -open and so is $K \cup (\check{\theta}\text{-}\mathcal{J}\text{-cl}(K))^c$.

4. $\check{\theta}$ - $\mathcal{J}$ -Submaximal Spaces

Definition 4.1

- i. A subset K of a space X is called $\mathcal{J}$ -dense if $K^* = X$.
- ii. A subset K of a space X is called $\mathcal{J}_g$ -submaximal if every $\mathcal{J}$ -dense subset is $\mathcal{J}_g$ -open.

Proposition 4.2 Every $\check{\theta}$ - $\mathcal{J}$ -dense set is $\mathcal{J}$ -dense.

Proof Let K be an $\check{\theta}$ - $\mathcal{J}$ -dense set in X . Then $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) = X$. Since $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) \subseteq \text{cl}(K)$, we have $K^* = X$ and so K is $\mathcal{J}$ -dense.

The converse of Proposition 4.2 need not be true as can be seen from the following example.

Example 4.3 The set $\{p, r\}$ in Example 3.5 is a $\mathcal{J}$ -dense in X but it is not $\check{\theta}$ - $\mathcal{J}$ -dense in X .

We introduce the following definition.

Definition 4.4 An ideal topological space X is called $\check{\theta}$ - $\mathcal{J}$ -submaximal if every $\mathcal{J}$ -dense subset in it is $\check{\theta}$ - $\mathcal{J}$ -open in X .

Proposition 4.5 Every submximal space is $\check{\theta}$ - $\mathcal{I}$ -submaximal.

Proof Let X be a submximal space and K be a $\mathcal{I}$ -dense subset of X . Then K is open. But every open set is $\check{\theta}$ - $\mathcal{I}$ -open and so K is $\check{\theta}$ - $\mathcal{I}$ -open. Therefore, X is $\check{\theta}$ - $\mathcal{I}$ -submaximal. The converse of Proposition 4.5 need not be true as can be seen from the following example.

Example 4.6 For the ideal topological space X of Example 3.5, every $\mathcal{I}$ -dense subset is $\check{\theta}$ - $\mathcal{I}$ -open and hence X is $\check{\theta}$ - $\mathcal{I}$ -submaximal. However, the set $K = \{p, q\}$ is $\mathcal{I}$ -dense in X , but it is not open in X . Therefore, X is not submaximal.

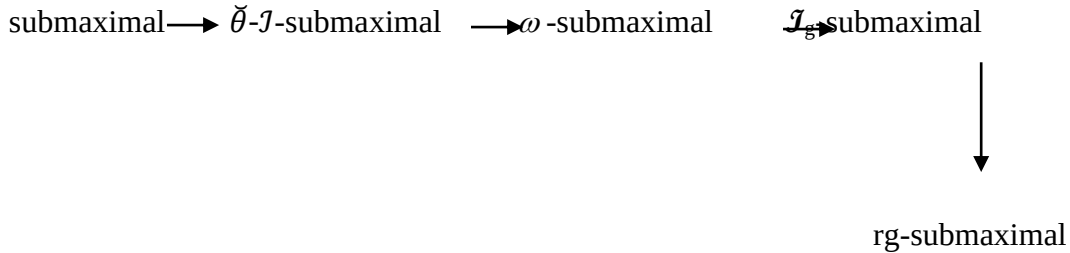
Proposition 4.7 Every $\check{\theta}$ - $\mathcal{I}$ -submaximal space is ω -submaximal.

Proof Let X be an $\check{\theta}$ - $\mathcal{I}$ -submaximal space and K be a $\mathcal{I}$ -dense subset of X . Then K is $\check{\theta}$ - $\mathcal{I}$ -open. But every $\check{\theta}$ - $\mathcal{I}$ -open set is ω -open and so K is ω -open. Therefore, X is ω -submaximal.

The converse of Proposition 4.7 need not be true as can be seen from the following example.

Example 4.8 Consider the ideal topological space X in Example 3.8. Then X is ω -submaximal but it is not $\check{\theta}$ - $\mathcal{I}$ -submaximal, because the set $K = \{q, r\}$ is a $\mathcal{I}$ -dense set in X but it is not $\check{\theta}$ - $\mathcal{I}$ -open in X .

Remark 4.9 From Propositions 4.5, 4.7, we have the following diagram:



Theorem 4.10 A space X is $\check{\theta}$ - $\mathcal{I}$ -submaximal if and only if $P(X) = \check{\theta}\text{-}\mathcal{I}\text{-LC}^*(X)$.

Proof

Necessity. Let $K \in P(X)$ and let $V = K \cup (K^*)^c$. This imply that $V^* = K^* \cup (K^*)^c = X$. Hence $V^* = X$. Therefore, V is a dense subset of X . Since X is $\check{\theta}$ - $\mathcal{I}$ -submaximal, V is $\check{\theta}$ - $\mathcal{I}$ -open. Thus $K \cup (K^*)^c$ is $\check{\theta}$ - $\mathcal{I}$ -open and by Theorem 3.39, we have $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}^*(X)$.

Sufficiency. Let K be a $\mathcal{I}$ -dense subset of X . This implies $K \cup (K^*)^c = K \cup X^c = K \cup \emptyset = K$. Now $K \in \check{\theta}\text{-}\mathcal{I}\text{-LC}^*(X)$ implies that $K = K \cup (K^*)^c$ is $\check{\theta}$ - $\mathcal{I}$ -open by Theorem 3.39. Hence X is $\check{\theta}$ - $\mathcal{I}$ -submaximal.

Remark 4.11 Union of two $\check{\theta}$ - $\mathcal{I}$ -lc sets (resp. $\check{\theta}$ - $\mathcal{I}$ -lc* sets, $\check{\theta}$ - $\mathcal{I}$ -lc** sets) need not be an $\check{\theta}$ - $\mathcal{I}$ -lc set (resp. $\check{\theta}$ - $\mathcal{I}$ -lc* set, $\check{\theta}$ - $\mathcal{I}$ -lc** set) as can be seen from the following examples.

Example 4.12 Let $X = \{p, q, r\}$ with $\tau = \{\emptyset, \{p\}, \{p, q\}, X\}$. Then $\check{\theta}$ - $\mathcal{I}$ -LC(X) = $\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, X\}$. Then the sets $\{p\}$ and $\{r\}$ are $\check{\theta}$ - $\mathcal{I}$ -lc sets, but their union $\{p, r\} \notin \check{\theta}$ - $\mathcal{I}$ -LC(X).

Example 4.13 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{q\}, \{p, q\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then $\check{\theta}$ - $\mathcal{I}$ -LC*(X) = $\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, X\}$. Then the sets $\{q\}$ and $\{r\}$ are $\check{\theta}$ - $\mathcal{I}$ -lc* sets, but their union $\{q, r\} \notin \check{\theta}$ - $\mathcal{I}$ -LC*(X).

Example 4.14 Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, \{q\}, \{q, r\}, X\}$ with $\mathcal{I} = \{\emptyset\}$. Then $\check{\theta}$ - $\mathcal{I}$ -LC**(X) = $\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, r\}, \{q, r\}, X\}$. Then the sets $\{p\}$ and $\{q\}$ are $\check{\theta}$ - $\mathcal{I}$ -lc** sets, but their union $\{p, q\} \notin \check{\theta}$ - $\mathcal{I}$ -LC**(X).

We introduce the following definition.

Definition 4.15 Let K and B be subsets of X . Then K and B are said to be $\check{\theta}$ - $\mathcal{I}$ -separated if $K \cap \check{\theta}$ - $\mathcal{I}$ -cl(B) = $\emptyset$ and $\check{\theta}$ - $\mathcal{I}$ -cl(K) $\cap$ B = $\emptyset$.

Example 4.16 For the ideal topological space X of Example 3.5. Let $K = \{q\}$ and let $B = \{r\}$. Then $\check{\theta}$ - $\mathcal{I}$ -cl(K) = $\{p, q\}$ and $\check{\theta}$ - $\mathcal{I}$ -cl(B) = $\{p, r\}$ and so the sets K and B are $\check{\theta}$ - $\mathcal{I}$ -separated.

Proposition 4.17 Assume that $\check{\theta}$ - $\mathcal{I}$ -O(X) forms an ideal topology. For the ideal topological space X , the followings are true

- i. Let $K, B \in \check{\theta}$ - $\mathcal{I}$ -LC(X). If K and B are $\check{\theta}$ - $\mathcal{I}$ -separated then $K \cup B \in \check{\theta}$ - $\mathcal{I}$ -LC(X).
- ii. Let $K, B \in \check{\theta}$ - $\mathcal{I}$ -LC*(X). If A and B are separated (i.e., $K \cap B^* = \emptyset$ and $K^* \cap B = \emptyset$), then $K \cup B \in \check{\theta}$ - $\mathcal{I}$ -LC*(X).
- iii. Let $K, B \in \check{\theta}$ - $\mathcal{I}$ -LC**(X). If K and B are $\check{\theta}$ - $\mathcal{I}$ -separated then $K \cup B \in \check{\theta}$ - $\mathcal{I}$ -LC**(X).

Proof

(i) Since $K, B \in \check{\theta}$ - $\mathcal{I}$ -LC(X), by Theorem 3.38, there exist $\check{\theta}$ - $\mathcal{I}$ -open sets U and V of X such that $K = U \cap \check{\theta}$ - $\mathcal{I}$ -cl(A) and $B = V \cap \check{\theta}$ - $\mathcal{I}$ -cl(B). Now $G = U \cap (X - \check{\theta}$ - $\mathcal{I}$ -cl(B)) and $H = V \cap (X - \check{\theta}$ - $\mathcal{I}$ -cl(K)) are $\check{\theta}$ - $\mathcal{I}$ -open subsets of X . Since $K \cap \check{\theta}$ - $\mathcal{I}$ -cl(B) = $\emptyset$, $K \subseteq (\check{\theta}$ - $\mathcal{I}$ -cl(B))^c. Now $K = U \cap \check{\theta}$ - $\mathcal{I}$ -cl(K) becomes $K \cap (\check{\theta}$ - $\mathcal{I}$ -cl(B))^c = $G \cap \check{\theta}$ - $\mathcal{I}$ -cl(K). Then $K = G \cap \check{\theta}$ - $\mathcal{I}$ -cl(K). Similarly $B = H \cap \check{\theta}$ - $\mathcal{I}$ -cl(B). Moreover $G \cap \check{\theta}$ - $\mathcal{I}$ -cl(B) = $\emptyset$ and $H \cap \check{\theta}$ - $\mathcal{I}$ -cl(K) = $\emptyset$. Since G and H are $\check{\theta}$ - $\mathcal{I}$ -open sets of X , $G \cup H$ is $\check{\theta}$ - $\mathcal{I}$ -open. Therefore $K \cup B = (G \cup H) \cap \check{\theta}$ - $\mathcal{I}$ -cl(K $\cup$ B) and hence $A \cup B \in \check{\theta}$ - $\mathcal{I}$ -LC(X).

(ii) and (iii) are similar to (i), using Theorems 3.39 and 3.40.

Remark 4.18 The assumption that K and B are $\check{\theta}$ - $\mathcal{J}$ -separated in (i) of Proposition 4.17 cannot be removed. In the ideal topological space X in Example 4.12, the sets $\{p\}$ and $\{r\}$ are not $\check{\theta}$ - $\mathcal{J}$ -separated and their union $\{p, r\} \notin \check{\theta}\text{-}\mathcal{J}\text{-LC}(X)$.

Lemma 4.19 For an $x \in X$, $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ if and only if $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x .

Proof

Let $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ for any $x \in X$. To prove $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x . Prove the result by contradiction. Suppose there exists a $\check{\theta}$ - $\mathcal{J}$ -open set V containing x such that $V \cap K = \emptyset$. Then $K \subseteq V^c$ and V^c is $\check{\theta}$ - $\mathcal{J}$ -cld. We have $\check{\theta}\text{-}\mathcal{J}\text{-cl}(K) \subseteq V^c$. This shows that $x \notin \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$ which is a contradiction. Hence $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x .

Conversely, let $V \cap K \neq \emptyset$ for every $\check{\theta}$ - $\mathcal{J}$ -open set V containing x . To prove $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$. We prove the result by contradiction. Suppose $x \notin \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$. Then there exists a $\check{\theta}$ - $\mathcal{J}$ -cld set F containing K such that $x \notin F$. Then $x \in F^c$ and F^c is $\check{\theta}$ - $\mathcal{J}$ -open. Also, $F^c \cap K = \emptyset$, which is a contradiction to the hypothesis. Hence $x \in \check{\theta}\text{-}\mathcal{J}\text{-cl}(K)$.

Theorem 4.20 Suppose that A is $\check{\theta}$ - $\mathcal{J}$ -open in X and that B is $\check{\theta}$ - $\mathcal{J}$ -open in Y . Then $A \times B$ is $\check{\theta}$ - $\mathcal{J}$ -open in $X \times Y$.

Proof

Suppose that F is $\star$ -cld and hence sg-cld in $X \times Y$ and that $F \subseteq A \times B$. It suffices to show that $F \subseteq \text{int}(A \times B)$.

Let $(x, y) \in F$. Then, for each $(x, y) \in F$, $(\{x\})^* \times (\{y\})^* = (\{x\} \times \{y\})^* = (\{x, y\})^* \subseteq F^* = F \subseteq A \times B$. Two $\star$ -cld sets $(\{x\})^*$ and $(\{y\})^*$ are contained in A and B respectively. It follows from the assumption that $(\{x\})^* \subseteq \text{int}(A)$ and that $(\{y\})^* \subseteq \text{int}(B)$. Thus $(x, y) \in (\{x\})^* \times (\{y\})^* \subseteq \text{int}(A) \times \text{int}(B) \subseteq \text{int}(A \times B)$. It means that, for each $(x, y) \in F$, $(x, y) \in \text{int}(A \times B)$ and hence $F \subseteq \text{int}(A \times B)$. Therefore, $A \times B$ is $\check{\theta}$ - $\mathcal{J}$ -open in $X \times Y$.

Theorem 4.21 The following are equivalent for a function $f : (X, \tau, I) \rightarrow (Y, \sigma)$

- i. f is $\check{\theta}$ - $\mathcal{J}$ -continuous.
- ii. The inverse image of a regular closed set of Y is $\check{\theta}$ - $\mathcal{J}$ -open in X .
- iii. $f^{-1}(\text{int}(V^*))$ is $\check{\theta}$ - $\mathcal{J}$ -closed in X for every open subset V of Y .
- iv. $f^{-1}((\text{int}(F))^*)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X for every closed subset F of Y .
- v. $f^{-1}(U^*)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X for every $U \in \beta O(Y)$.
- vi. $f^{-1}(U^*)$ is $\check{\theta}$ - $\mathcal{J}$ -open in X for every $U \in SO(Y)$.
- vii. $f^{-1}(\text{int}(U^*))$ is $\check{\theta}$ - $\mathcal{J}$ -closed in X for every $U \in PO(Y)$.

Proof

(i) $\Leftrightarrow$ (ii). Obvious.

(i) $\Leftrightarrow$ (iii). Let V be an open subset of Y . Since $\text{int}(V^*)$ is regular open, $f^{-1}(\text{int}(V^*))$ is $\check{\theta}$ - $\mathcal{J}$ -closed. The converse is similar.

(ii) $\Leftrightarrow$ (iv). Similar to (i) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (v). Let U be any β -open set of Y . We have U^* is regular closed. Then by (ii) $f^{-1}(U^*)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X .

(v) $\Rightarrow$ (vi). Obvious from the fact that $SO(Y) \subseteq \beta O(Y)$.

(vi) $\Rightarrow$ (vii). Let $U \in PO(Y)$. Then $Y \setminus \text{int}(U^*)$ is regular closed and hence it is semi-open. Then, we have $X \setminus f^{-1}(\text{int}(U^*)) = f^{-1}(Y \setminus \text{int}(U^*)) = f^{-1}((Y \setminus \text{int}(U^*))^*)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X . Hence $f^{-1}(\text{int}(U^*))$ is $\check{\theta}$ - $\mathcal{I}$ -closed in X .

(vii) $\Rightarrow$ (i). Let U be any regular open set of Y . Then $U \in PO(Y)$ and hence $f^{-1}(U) = f^{-1}(\text{int}(U^*))$ is $\check{\theta}$ - $\mathcal{I}$ -closed in X .

Proposition 4.22 A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is $\check{\theta}$ - $\mathcal{I}$ -continuous if and only if $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X , for every open set U in Y .

Proof

Let $f : (X, \tau, I) \rightarrow (Y, \sigma)$ be $\check{\theta}$ - $\mathcal{I}$ -continuous and U be an open set in Y . Then U^c is closed in Y and since f is $\check{\theta}$ - $\mathcal{I}$ -continuous, $f^{-1}(U^c)$ is $\check{\theta}$ - $\mathcal{I}$ -cld in X . But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X .

Conversely, assume that $f^{-1}(U)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X , for each open set U in Y . Let F be a closed set in Y . Then F^c is open in Y and by assumption, $f^{-1}(F^c)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X . Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $\check{\theta}$ - $\mathcal{I}$ -cld in X and so f is $\check{\theta}$ - $\mathcal{I}$ -continuous.

Theorem 4.23 If $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is $\check{\theta}$ - $\mathcal{I}$ -continuous and pre-sg-closed and if A is an $\check{\theta}$ - $\mathcal{I}$ -open (or $\check{\theta}$ - $\mathcal{I}$ -cld) subset of Y , then $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{I}$ -open (or $\check{\theta}$ - $\mathcal{I}$ -cld) in X .

Proof

Let H be an $\check{\theta}$ - $\mathcal{I}$ -open set in Y and F be any sg-closed set in X such that $F \subseteq f^{-1}(H)$. Then $f(F) \subseteq H$. By hypothesis, $f(F)$ is sg-closed and H is $\check{\theta}$ - $\mathcal{I}$ -open in Y . Therefore, $f(F) \subseteq \text{int}(H)$ and so $F \subseteq f^{-1}(\text{int}(H))$. Since f is $\check{\theta}$ - $\mathcal{I}$ -continuous and $\text{int}(H)$ is open in Y , $f^{-1}(\text{int}(H))$ is $\check{\theta}$ - $\mathcal{I}$ -open in X . Thus $F \subseteq \text{int}(f^{-1}(\text{int}(H))) \subseteq \text{int}(f^{-1}(H))$. i.e., $F \subseteq \text{int}(f^{-1}(H))$ and $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{I}$ -open in X . By taking complements, we can show that if H is $\check{\theta}$ - $\mathcal{I}$ -cld in Y , $f^{-1}(H)$ is $\check{\theta}$ - $\mathcal{I}$ -cld in X .

5. Conclusion

The notions of sets and functions in ideal topological spaces and fuzzy topological spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences.

By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all topological functions defined in this thesis will have many possibilities of applications in digital topology and computer graphics.

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Near mean labeling in dicyclic snakes

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Abstract

K. Palani et al. defined the concept of near mean labeling in digraphs. Let $D = (V, A)$ be a digraph where V the vertex is set and A is the arc set. Let $f: V \rightarrow \{0, 1, 2, \dots, q\}$ be a 1-1 map. Define $f^*: A \rightarrow \{1, 2, \dots, q\}$ by $f^*(e = \overrightarrow{uv}) = \left\lfloor \frac{f(u) + f(v)}{2} \right\rfloor$. Let $f^*(v) = |\sum_{w \in V} f^*(\overrightarrow{vw}) - \sum_{w \in V} f^*(\overrightarrow{wv})|$. If $f^*(v) \leq 2 \forall v \in A(D)$, then f is said to be a near mean labeling of D and D is said to be a near mean digraph. In this paper, different dicyclic snakes are defined and the existence of near mean labeling in them is checked.

Keywords: Near mean labeling, digraphs, di-cyclic, di-quadrilateral, di-pentagonal, snake.

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1. Introduction

Graph theory has applications in many areas of the computing, social and natural science. The theory is also intimately related to many branches of mathematics, including matrix theory, numerical analysis, probability, topology and combinatory. Over the last 50 year graph theory has evolved into an important mathematical tool in the solution of a wide variety of problems in many areas of society. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions have been motivated by practical problems, labelled graphs serve useful mathematical models for a broad range of applications such as: coding theory, including the design of good types codes, synch-set codes, missile guidance codes and convolutional codes with optimal auto correlation properties. The concept of graph labeling was introduced by Rosa in 1967 [6]. A useful survey on graph labeling by J.A. Gallian (2014) can be found in [2]. Somasundaram and Ponraj [4] have introduced the notion of mean labeling of graphs. A directed graph or digraph D consists of a finite set V of vertices and a collection of ordered pairs of distinct vertices. Any such pair (u, v) is called an arc or directed line and will usually be denoted by $\overrightarrow{uv}$. A digraph D with p vertices and q arcs is denoted by $D(p, q)$. The indegree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d^+(v)$ of v is the number of arcs having v as its initial vertex [1].

K. Palani et al. introduced the concepts of mean and near mean digraphs in [3]. In this paper, different di-cyclic snakes are introduced and the existence of near mean labeling is investigated.

2. Preliminaries

The following definition and theorem are basics which are needed for the subsequent section.

Definition 2.1: [3] Let $f: V \rightarrow \{0, 1, 2, \dots, q\}$ be a 1-1 map. Define $f^*: A \rightarrow \{1, 2, \dots, q\}$ by $f^*(e = \overrightarrow{uv}) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Let $f^*(v) = |\sum_{w \in V} f^*(\overrightarrow{vw}) - \sum_{w \in V} f^*(\overrightarrow{wv})|$. If $f^*(v) \leq 2 \forall v \in A(D)$, then f is said to be a near mean labeling of D and D is said to be a near mean digraph.

Definition 2.2: [5] A cyclic snake mC_n is obtained by replacing every edge of P_m by C_n .

Theorem 2.3: [3] The directed cycle $\overrightarrow{C_n}$ is a near mean digraph.

3. Main Results

In this section, the different dicyclic snakes are defined and the near mean labeling existence is verified.

Definition 3.1: In cyclic snake mC_n , orient the edges of each cycle clockwise. The resulting graph is called **Di-Cyclic Snake** and it is denoted $asm\overrightarrow{C_n}$. For $n = 3, 4, 5$, Di-Cyclic Snakes are called Di-Triangular Snake $\overrightarrow{TS_n}$, Di-Quadrilateral Snake $\overrightarrow{QS_n}$ and Di-Pentagonal Snake $\overrightarrow{PS_n}$ respectively.

Definition 3.2: In mC_n when n is even, orient the edges of the cycle alternately and call resulting graph as **Alternating Di-Cyclic Snake**. Denote it as $Am\overrightarrow{C_n}$.

Theorem 3.3: Di-Quadrilateral Snake $\overrightarrow{QS_n}$ admits near mean labeling.

Proof: Let $V(\overrightarrow{QS_n}) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n-1\} \cup \{w_i | 1 \leq i \leq n-1\}$ be the vertex set and let $A(\overrightarrow{QS_n}) = \{\overrightarrow{u_i v_i} \cup \overrightarrow{v_i u_{i+1}} \cup \overrightarrow{u_{i+1} w_i} \cup \overrightarrow{w_i u_i} | 1 \leq i \leq n-1\}$ be the arc set.

Define $f: V(\overrightarrow{QS_n}) \rightarrow \{0, 1, 2, \dots, (4n-4)\}$ by

$$f(u_i) = 4(i-1) \quad \text{for } 1 \leq i \leq n$$

$$f(v_i) = 4i-2 \quad \text{for } 1 \leq i \leq n-1$$

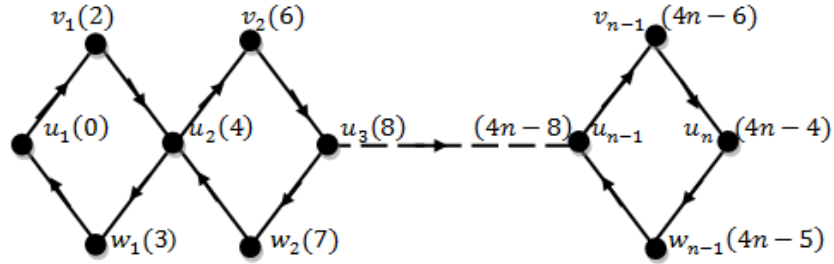


Figure 3.1: The labeling of a Di-Quadrilateral snake

For $i = 1$ to $n-1$

$$f^*(\overrightarrow{u_i v_i}) = \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil = \left\lceil \frac{[4(i-1)] + [4i-2]}{2} \right\rceil = \left\lceil \frac{8i-6}{2} \right\rceil = 4i-3. \quad (3.3.1)$$

$$f^*(\overrightarrow{v_i u_{i+1}}) = \left\lceil \frac{f(v_i) + f(u_{i+1})}{2} \right\rceil = \left\lceil \frac{[4i-2] + [4(i+1-1)]}{2} \right\rceil = \left\lceil \frac{8i-2}{2} \right\rceil = 4i-1. \quad (3.3.2)$$

$$f^*(\overrightarrow{u_{i+1} w_i}) = \left\lceil \frac{f(u_{i+1}) + f(w_i)}{2} \right\rceil = \left\lceil \frac{[4(i+1-1)] + [4i-1]}{2} \right\rceil = \left\lceil \frac{8i-1}{2} \right\rceil = \frac{8i}{2} = 4i. \quad (3.3.3)$$

$$f^*(\overrightarrow{w_i u_i}) = \left\lceil \frac{f(w_i) + f(u_i)}{2} \right\rceil = \left\lceil \frac{[4i-1] + [4(i-1)]}{2} \right\rceil = \left\lceil \frac{8i-5}{2} \right\rceil = \frac{8i-4}{2} = 4i-2. \quad (3.3.4)$$

$$\begin{aligned} \text{Now } f^*(u_1) &= |f^*(\overrightarrow{u_1 v_1}) - f^*(\overrightarrow{w_1 u_1})| \\ &= |[4(1)-3] - [4(1)-2]| \quad [\text{by (3.3.1) \& (3.3.4)}] \\ &= |1-2| = |-1| < 2 \end{aligned}$$

$$\text{Therefore, } f^*(u_1) < 2 \quad (3.3.5)$$

For $i = 2$ to $n-1$

$$\begin{aligned} f^*(u_i) &= |[f^*(\overrightarrow{u_i v_i}) + f^*(\overrightarrow{u_i w_{i-1}})] - [f^*(\overrightarrow{w_i u_i}) + f^*(\overrightarrow{v_{i-1} u_i})]| \\ &= |[4i-3 + 4(i-1)] - [4i-2 + 4(i-1)-1]| \quad [\text{by (1), (3), (4) \& (2)}] \\ &= |8i-7-8i+7| = |0| < 2 \end{aligned}$$

$$\text{Therefore, } f^*(u_i) < 2 \quad \text{for } i = 2 \text{ to } n-1 \quad (3.3.6)$$

$$\begin{aligned} f^*(u_n) &= |f^*(\overrightarrow{u_n w_{n-1}}) - f^*(\overrightarrow{v_{n-1} u_n})| \\ &= |4(n-1) - [4(n-1)-1]| \quad [\text{by (3.3.3) \& (3.3.2)}] \end{aligned}$$

$$= |1| < 2$$

Therefore, $f^*(u_n) < 2$ (3.3.7)

For $i = 1$ to $n - 1$

$$\begin{aligned} f^*(v_i) &= |f^*(\overrightarrow{v_i u_{i+1}}) - f^*(\overrightarrow{u_i v_i})| \\ &= |(4i - 1) - (4i - 3)| \quad [\text{by (3.3.2) \& (3.3.1)}] \\ &= |4i - 1 - 4i + 3| = |2| = 2 \end{aligned}$$

Therefore, $f^*(v_i) = 2$ for $i = 1$ to $n - 1$ (3.3.8)

For $i = 1$ to $n - 1$

$$\begin{aligned} f^*(w_i) &= |f^*(\overrightarrow{w_i u_i}) - f^*(\overrightarrow{u_{i+1} w_i})| \\ &= |(4i - 2) - (4i)| \quad [\text{by (3.3.4) \& (3.3.3)}] \\ &= |-2| \leq 2 \end{aligned}$$

Therefore, $f^*(w_i) \leq 2$ for $i = 1$ to $n - 1$ (3.3.9)

From equations (5) to (9), $f^*(u) \leq 2 \forall u \in V(\overrightarrow{QS_n})$

Hence Di-Quadrilateral Snake $\overrightarrow{QS_n}$ is a near mean digraph.

Theorem 3.4. Di-Pentagonal Snake $\overrightarrow{PS_n}$ is a near mean digraph.

Proof:

Let $V(\overrightarrow{PS_n}) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n - 1\} \cup \{w_i | 1 \leq i \leq n - 1\} \cup \{x_i | 1 \leq i \leq n - 1\}$ be the vertex set and let $A(\overrightarrow{PS_n}) = \{\overrightarrow{u_i v_i} \cup \overrightarrow{v_i w_i} \cup \overrightarrow{w_i u_{i+1}} \cup \overrightarrow{u_{i+1} x_i} \cup \overrightarrow{x_i u_i} | 1 \leq i \leq n - 1\}$ be the arc set.

Define $f: V(\overrightarrow{PS_n}) \rightarrow \{0, 1, 2, \dots, (5n - 5)\}$ by

$$\begin{aligned} f(u_i) &= 5(i - 1) \quad \text{for } 1 \leq i \leq n \\ f(v_i) &= 5i - 3 \quad \text{for } 1 \leq i \leq n - 1 \\ f(w_i) &= 5i - 1 \quad \text{for } 1 \leq i \leq n - 1 \\ f(x_i) &= 5i - 2 \quad \text{for } 1 \leq i \leq n - 1 \end{aligned}$$

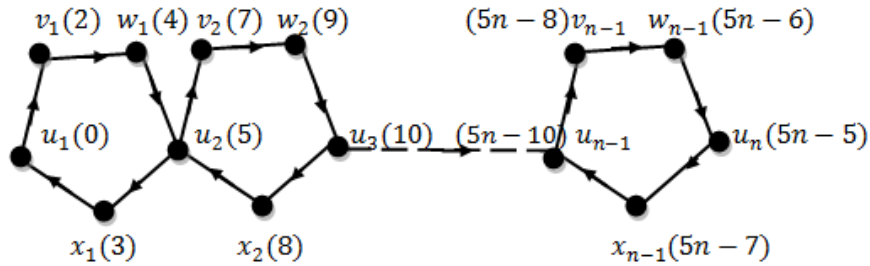


Figure 3.2. The labeling of a Di-Pentagonal snake

For $i = 1$ to $n - 1$

$$f^*(\overrightarrow{u_i v_i}) = \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil = \left\lceil \frac{[5(i-1)] + [5i-3]}{2} \right\rceil = \left\lceil \frac{10i-8}{2} \right\rceil = 5i - 4 \quad (3.4.1)$$

$$f^*(\overrightarrow{v_i w_i}) = \left\lceil \frac{f(v_i) + f(w_i)}{2} \right\rceil = \left\lceil \frac{[5i-3] + [5i-1]}{2} \right\rceil = \left\lceil \frac{10i-4}{2} \right\rceil = 5i - 2 \quad (3.4.2)$$

$$f^*(\overrightarrow{w_i u_{i+1}}) = \left\lceil \frac{f(w_i) + f(u_{i+1})}{2} \right\rceil = \left\lceil \frac{[5i-1] + [5(i+1-1)]}{2} \right\rceil = \left\lceil \frac{10i-1}{2} \right\rceil = \frac{10i}{2} = 5i \quad (3.4.3)$$

$$f^*(\overrightarrow{u_{i+1} x_i}) = \left\lceil \frac{f(u_{i+1}) + f(x_i)}{2} \right\rceil = \left\lceil \frac{[5(i+1-1)] + [5i-2]}{2} \right\rceil = \left\lceil \frac{10i-2}{2} \right\rceil = 5i - 1 \quad (3.4.4)$$

$$f^*(\overrightarrow{x_i u_i}) = \left\lceil \frac{f(x_i) + f(u_i)}{2} \right\rceil = \left\lceil \frac{[5i-2] + [5(i-1)]}{2} \right\rceil = \left\lceil \frac{10i-7}{2} \right\rceil = \frac{10i-6}{2} = 5i - 3 \quad (3.4.5)$$

Next to find $f^*(u_i)$

$$\begin{aligned} \text{Now } f^*(u_i) &= |f^*(\overrightarrow{u_i v_i}) - f^*(\overrightarrow{x_i u_i})| \\ &= |[5(1) - 4] - [5(1) - 3]| \quad [\text{by (3.4.1) \& (3.4.5)}] \\ &= |1 - 2| = |-1| < 2 \end{aligned}$$

$$\text{Therefore, } f^*(u_i) < 2 \quad (3.4.6)$$

For $i = 2$ to $n - 1$

$$\begin{aligned} f^*(u_i) &= |[f^*(\overrightarrow{u_i v_i}) + f^*(\overrightarrow{u_i x_{i-1}})] - [f^*(\overrightarrow{x_i u_i}) + f^*(\overrightarrow{w_{i-1} u_i})]| \\ &= |[5i - 4 + 5(i - 1) - 1] - [5i - 3 + 5(i - 1)]| \quad [\text{by (3.4.1), (3.4.4), (3.4.5)}] \\ &\& (3.4.3)] \\ &= |10i - 10 - 10i + 8| = |-2| \leq 2 \end{aligned}$$

$$\text{Therefore, } f^*(u_i) \leq 2 \text{ for } i = 2 \text{ to } n - 1 \quad (3.4.7)$$

$$\begin{aligned} f^*(u_n) &= |f^*(\overrightarrow{u_n x_{n-1}}) - f^*(\overrightarrow{w_{n-1} u_n})| \\ &= |5(n - 1) - 1 - 5(n - 1)| \quad [\text{by (3.4.4) \& (3.4.3)}] \\ &= |-1| < 2 \end{aligned}$$

$$\text{Therefore, } f^*(u_n) < 2 \quad (3.4.8)$$

For $i = 1$ to $n - 1$

$$\begin{aligned} f^*(v_i) &= |f^*(\overrightarrow{v_i w_i}) - f^*(\overrightarrow{u_i v_i})| \\ &= |(5i - 2) - (5i - 4)| \quad [\text{by (3.4.2) \& (3.4.1)}] \\ &= |2| = 2 \end{aligned}$$

$$\text{Therefore, } f^*(v_i) = 2 \text{ for } i = 1 \text{ to } n - 1 \quad (3.4.9)$$

For $i = 1$ to $n - 1$

$$\begin{aligned} f^*(w_i) &= |f^*(\overrightarrow{w_i u_{i+1}}) - f^*(\overrightarrow{v_i w_i})| \\ &= |(5i) - (5i - 2)| \quad [\text{by (3.4.3) \& (3.4.2)}] \\ &= |2| = 2 \end{aligned}$$

$$\text{Therefore, } f^*(w_i) = 2 \text{ for } i = 1 \text{ to } n - 1 \quad (3.4.10)$$

For $i = 1$ to $n - 1$

$$\begin{aligned} f^*(x_i) &= |f^*(\overrightarrow{x_i u_i}) - f^*(\overrightarrow{u_{i+1} x_i})| \\ &= |(5i - 3) - (5i - 1)| \quad [\text{by (3.4.3) \& (3.4.2)}] \\ &= |-2| \leq 2 \end{aligned}$$

$$\text{Therefore, } f^*(x_i) \leq 2 \text{ for } i = 1 \text{ to } n - 1 \quad (3.4.11)$$

From equations (6) to (11), $f^*(u) \leq 2 \forall u \in V(\overrightarrow{PS_n})$

Hence, Di-Pentagonal Snake $\overrightarrow{PS_n}$ is a near mean digraph.

Theorem 3.5. Di-Cyclic snake $m\overrightarrow{C_n}$ is a near mean digraph for $m \geq 1$ and $n \geq 3$.

Proof: Let u_{ij} denote the j th vertex in the i th copy of $\overrightarrow{C_n}$

Here, $V(m\overrightarrow{C_n}) = \{u_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$A(m\overrightarrow{C_n}) = \{\overrightarrow{u_{ij} u_{i(j+1)}} | 1 \leq i \leq m, 1 \leq j \leq n - 1\} \cup \{\overrightarrow{u_{in} u_{i1}} | 1 \leq i \leq m\}$$

Following the procedure of near mean labeling of $\overrightarrow{C_n}$ in [3], define

$f: V(m\overrightarrow{C_n}) \rightarrow \{0, 1, 2, \dots, mn\}$ as below

$$f(u_{ij}) = \begin{cases} (i-1)n + 2(j-1) & \text{for } 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, 1 \leq i \leq m \\ (i-1)n + 2(n - (j-1)) + 1 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n, 1 \leq i \leq m \end{cases}$$

where $|V(m\vec{C}_n)| = mn - m + 1$ and $|A(m\vec{C}_n)| = mn$

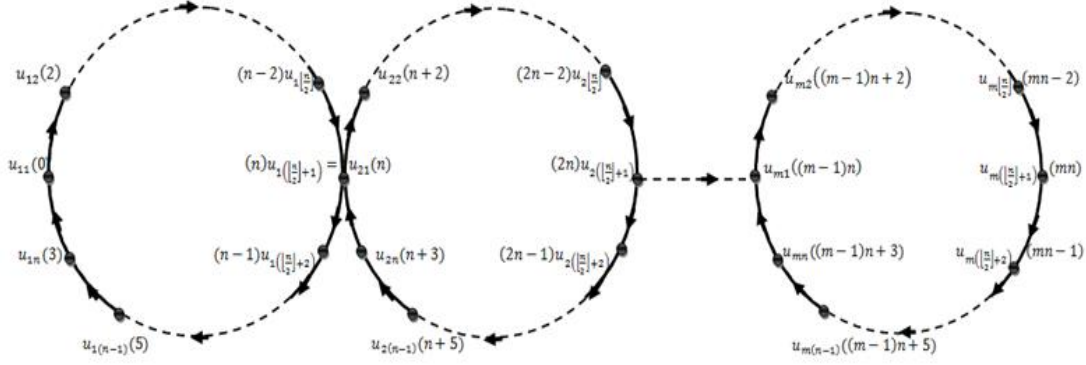


Figure 3.3: The labeling of the Di-cyclic Snake $m\vec{C}_n$ when n is even

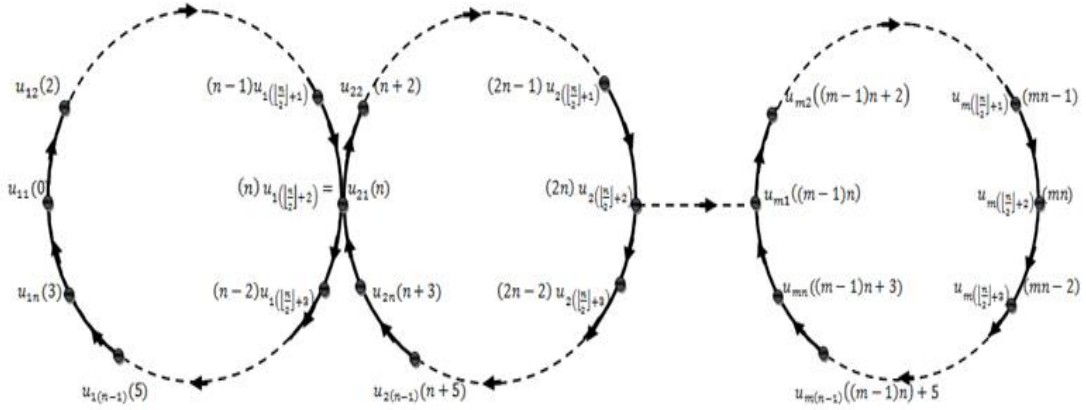


Figure 3.4: The labeling of the Di-cyclic Snake $m\vec{C}_n$ when n is odd

Now to find $f^*(\overrightarrow{u_{ij}u_{i(j+1)}})$

For $i = 1$ to m , $j = 1$ to $\left\lfloor \frac{n}{2} \right\rfloor$

$$\begin{aligned} f^*(\overrightarrow{u_{ij}u_{i(j+1)}}) &= \left\lfloor \frac{f(u_{ij}) + f(u_{i(j+1)})}{2} \right\rfloor \\ &= \left\lfloor \frac{[(i-1)n + 2(j-1)] + [(i-1)n + 2((j+1)-1)]}{2} \right\rfloor \\ &= \left\lfloor \frac{2(i-1)n + 4j - 2}{2} \right\rfloor = (i-1)n + 2j - 1 = ni + 2j - n - 1. \end{aligned}$$

Therefore, $f^*(\overrightarrow{u_{ij}u_{i(j+1)}}) = ni + 2j - n - 1$ for $i = 1$ to m , $j = 1$ to $\left\lfloor \frac{n}{2} \right\rfloor$ (3.5.1)

$$\begin{aligned}
 f^*\left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}}\right) &= \left\lfloor \frac{f(u_{i(\lfloor \frac{n}{2} \rfloor + 1)}) + f(u_{i(\lfloor \frac{n}{2} \rfloor + 2)})}{2} \right\rfloor \\
 &= \left\lfloor \frac{[(i-1)n+2(\lfloor \frac{n}{2} \rfloor + 1 - 1)] + [(i-1)n+2(n - (\lfloor \frac{n}{2} \rfloor + 2 - 1)) + 1]}{2} \right\rfloor \\
 &= \left\lfloor \frac{2(i-1)n+2n-1}{2} \right\rfloor = \left\lfloor \frac{2(i-1+1)n-1}{2} \right\rfloor = \left\lfloor \frac{2in-1}{2} \right\rfloor = \frac{2in}{2} = ni
 \end{aligned}$$

Therefore, $f^*\left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}}\right) = ni$ for $i = 1$ to m and $n \geq 3$ (3.5.2)

For $i = 1$ to m , $j = \lfloor \frac{n}{2} \rfloor + 2$ to $n - 1$

$$\begin{aligned}
 f^*\left(\overrightarrow{u_{ij} u_{i(j+1)}}\right) &= \left\lfloor \frac{f(u_{ij}) + f(u_{i(j+1)})}{2} \right\rfloor \\
 &= \left\lfloor \frac{[(i-1)n+2(n-(j-1))+1] + [(i-1)n+2(n-((j+1)-1))+1]}{2} \right\rfloor \\
 &= \left\lfloor \frac{2(i-1)n+4n-4j+4}{2} \right\rfloor \\
 &= (i+1)n - 2j + 2 = ni - 2j + n + 2 \\
 f^*\left(\overrightarrow{u_{ij} u_{i(j+1)}}\right) &= ni - 2j + n + 2 \text{ for } i = 1 \text{ to } m, j = \lfloor \frac{n}{2} \rfloor + 2 \text{ to } n - 1
 \end{aligned} \tag{3.5.3}$$

$$\begin{aligned}
 f^*\left(\overrightarrow{u_{in} u_{i1}}\right) &= \left\lfloor \frac{f(u_{in}) + f(u_{i1})}{2} \right\rfloor \\
 &= \left\lfloor \frac{[(i-1)n+2(n-(n-1))+1] + [(i-1)n+2(1-1)]}{2} \right\rfloor \\
 &= \left\lfloor \frac{2(i-1)n+2(n-n+1)+1}{2} \right\rfloor = \left\lfloor \frac{2(i-1)n+3}{2} \right\rfloor = \frac{2(i-1)n+4}{2} \\
 &= (i-1)n + 2 = ni - n + 2.
 \end{aligned}$$

Therefore, $f^*\left(\overrightarrow{u_{in} u_{i1}}\right) = ni - n + 2$ for $i = 1$ to m and $n \geq 3$ (3.5.4)

Next to find $f^*(u_{ij})$

$$\begin{aligned}
 f^*(u_{11}) &= |f^*\left(\overrightarrow{u_{11} u_{12}}\right) - f^*\left(\overrightarrow{u_{1n} u_{11}}\right)| \\
 &= |[n(1) + 2(1) - n - 1] - [n(1) - n + 2]| \quad [\text{by (3.5.1) \& (3.5.4)}] \\
 &= |1 - 2| = |-1| < 2
 \end{aligned}$$

Therefore, $f^*(u_{11}) < 2$ (3.5.5)

For $i = 1$ to m , $j = 2$ to $\lfloor \frac{n}{2} \rfloor$

$$\begin{aligned}
 f^*(u_{ij}) &= |f^*\left(\overrightarrow{u_{ij} u_{i(j+1)}}\right) - f^*\left(\overrightarrow{u_{i(j-1)} u_{ij}}\right)| \\
 &= |[ni + 2j - n - 1] - [ni + 2(j-1) - n - 1]| \quad [\text{by (3.5.1)}] \\
 &= |-1 + 3| = |2| = 2
 \end{aligned}$$

Therefore, $f^*(u_{ij}) = 2$ for $i = 1$ to m , $j = 2$ to $\lfloor \frac{n}{2} \rfloor$ (3.5.6)

For $i = 1$ to $m - 1$, two cases for $f^*(u_{ij})$ when $j = \lfloor \frac{n}{2} \rfloor + 1$ & $\lfloor \frac{n}{2} \rfloor + 2$

Case (i): n is odd.

$$\begin{aligned}
 f^*\left(u_{i(\lfloor \frac{n}{2} \rfloor + 1)}\right) &= \left| f^*\left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}}\right) - f^*\left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor)} u_{i(\lfloor \frac{n}{2} \rfloor + 1)}}\right) \right| \\
 &= \left| ni - \left(ni + 2\left\lfloor \frac{n}{2} \right\rfloor - n - 1 \right) \right| \quad [\text{by (3.5.2) \& (3.5.1)}]
 \end{aligned}$$

$$\begin{aligned}
 &= \left| n - 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 \right| \\
 &= \left| n - 2 \left(\frac{n-1}{2} \right) + 1 \right| = |2| = 2 \\
 f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 1)} \right) &= 2 \text{ when } n \text{ is odd}
 \end{aligned} \tag{3.5.7}$$

$$\begin{aligned}
 f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 2)} \right) &= \left| \left(f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 2)} u_{i(\lfloor \frac{n}{2} \rfloor + 3)}} \right) + f^* \left(\overrightarrow{u_{(i+1)1} u_{(i+1)2}} \right) \right) \right. \\
 &\quad \left. - \left(f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}} \right) + f^* \left(\overrightarrow{u_{(i+1)n} u_{(i+1)1}} \right) \right) \right| \\
 &= \left| \left[(ni - 2 \left(\left\lfloor \frac{n}{2} \right\rfloor + 2 \right) + n + 2) + n(i + 1) + 2 - n - 1 \right] - [ni + n(i + 1) - n + 2] \right| \\
 &\quad \text{[by (3.5.3), (3.5.1), (3.5.2) \& (3.5.4)]} \\
 &= \left| \left[\left(ni - 2 \left(\left\lfloor \frac{n}{2} \right\rfloor \right) - 4 + n + 2 \right) + (ni + n + 2 - n - 1) \right] - [ni + ni + 2] \right| \\
 &= \left| \left[2ni - 2 \left(\frac{n-1}{2} \right) + n - 1 \right] - [2ni + 2] \right| \\
 &= |-2| \leq 2
 \end{aligned}$$

$$f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 2)} \right) \leq 2 \text{ when } n \text{ is odd.} \tag{3.5.8}$$

Case (ii): n is even.

$$\begin{aligned}
 f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 1)} \right) &= \left| \left[f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}} \right) + f^* \left(\overrightarrow{u_{(i+1)1} u_{(i+1)2}} \right) \right] \right. \\
 &\quad \left. - \left[f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor)} u_{i(\lfloor \frac{n}{2} \rfloor + 1)}} \right) + f^* \left(\overrightarrow{u_{(i+1)n} u_{(i+1)1}} \right) \right] \right| \\
 &= \left| [ni + (n(i + 1) + 2(1) - n - 1)] - [(ni + 2 \left\lfloor \frac{n}{2} \right\rfloor - n - 1) + (n(i + 1) - n + 2)] \right| \\
 &\quad \text{[by (3.5.2), (3.5.1) \& (3.5.4)]} \\
 &= \left| [ni + ni + n + 2 - n - 1] - [ni + 2 \left(\frac{n}{2} \right) - n - 1 + ni + 2] \right| \\
 &= |0| < 2
 \end{aligned}$$

$$f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 1)} \right) < 2 \text{ when } n \text{ is even} \tag{3.5.9}$$

$$\begin{aligned}
 f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 2)} \right) &= \left| f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 2)} u_{i(\lfloor \frac{n}{2} \rfloor + 3)}} \right) - f^* \left(\overrightarrow{u_{i(\lfloor \frac{n}{2} \rfloor + 1)} u_{i(\lfloor \frac{n}{2} \rfloor + 2)}} \right) \right| \\
 &= \left| (ni - 2 \left(\left\lfloor \frac{n}{2} \right\rfloor + 2 \right) + n + 2) - ni \right| \quad \text{[by (3) \& (4)]} \\
 &= \left| n - 2 \left\lfloor \frac{n}{2} \right\rfloor - 4 + 2 \right| \\
 &= \left| n - 2 \left(\frac{n}{2} \right) - 2 \right| \\
 &= |-2| \leq 2
 \end{aligned}$$

$$f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 2)} \right) \leq 2 \text{ when } n \text{ is even} \tag{3.5.10}$$

$$\text{Cases (i) and (ii) imply } f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 1)} \right) \leq 2 \text{ for } i = 1 \text{ to } m - 1 \tag{3.5.11}$$

$$\text{and } f^* \left(u_{i(\lfloor \frac{n}{2} \rfloor + 2)} \right) \leq 2 \text{ for } i = 1 \text{ to } m - 1 \tag{3.5.12}$$

Now to find $f^*(u_{i(\lfloor \frac{n}{2} \rfloor + 1)})$ and $f^*(u_{i(\lfloor \frac{n}{2} \rfloor + 2)})$ for $i = m$

$$\begin{aligned} f^*(u_{m(\lfloor \frac{n}{2} \rfloor + 1)}) &= \left| f^*\left(\overrightarrow{u_{m(\lfloor \frac{n}{2} \rfloor + 1)} u_{m(\lfloor \frac{n}{2} \rfloor + 2)}}\right) - f^*\left(\overrightarrow{u_{m\lfloor \frac{n}{2} \rfloor} u_{m(\lfloor \frac{n}{2} \rfloor + 1)}}\right) \right| \\ &= \left| nm - \left(nm + 2 \left\lfloor \frac{n}{2} \right\rfloor - n - 1 \right) \right| \quad [\text{by (2) \& (1)}] \\ &= \left| n - 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 \right| \\ &= \begin{cases} \left| n - 2 \left(\frac{n-1}{2} \right) + 1 \right| & \text{if } n \text{ is odd} \\ \left| n - 2 \left(\frac{n}{2} \right) + 1 \right| & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} |2| & \text{if } n \text{ is odd} \\ |1| & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\text{Therefore, } f^*(u_{m(\lfloor \frac{n}{2} \rfloor + 1)}) \leq 2 \quad (3.5.13)$$

$$\begin{aligned} f^*(u_{m(\lfloor \frac{n}{2} \rfloor + 2)}) &= \left| f^*\left(\overrightarrow{u_{m(\lfloor \frac{n}{2} \rfloor + 2)} u_{m(\lfloor \frac{n}{2} \rfloor + 3)}}\right) - f^*\left(\overrightarrow{u_{m(\lfloor \frac{n}{2} \rfloor + 1)} u_{m(\lfloor \frac{n}{2} \rfloor + 2)}}\right) \right| \\ &= \left| (nm - 2 \left(\left\lfloor \frac{n}{2} \right\rfloor + 2 \right) + n + 2) - nm \right| \quad [\text{by (3.5.3) \& (3.5.2)}] \\ &= \left| n - 2 \left\lfloor \frac{n}{2} \right\rfloor - 2 \right| \\ &= \begin{cases} \left| n - 2 \left(\frac{n-1}{2} \right) - 2 \right| & \text{if } n \text{ is odd} \\ \left| n - 2 \left(\frac{n}{2} \right) - 2 \right| & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} |-1| & \text{if } n \text{ is odd} \\ |-2| & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\text{Therefore, } f^*(u_{m(\lfloor \frac{n}{2} \rfloor + 2)}) \leq 2 \quad (3.5.14)$$

For $i = 1$ to m , $j = \left\lfloor \frac{n}{2} \right\rfloor + 3$ to $n - 1$

$$\begin{aligned} f^*(u_{ij}) &= |f^*(\overrightarrow{u_{ij} u_{i(j+1)}}) - f^*(\overrightarrow{u_{i(j-1)} u_{ij}})| \\ &= |[ni - 2j + n + 2] - [ni - 2(j - 1) + n + 2]| \quad [\text{by (3)}] \\ &= |-2| \end{aligned}$$

$$\text{Therefore, } f^*(u_{ij}) \leq 2 \text{ for } i = 1 \text{ to } m, j = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ to } n - 1 \quad (3.5.15)$$

For $i = 1$ to m and $j = n$

$$\begin{aligned} f^*(u_{in}) &= |f^*(\overrightarrow{u_{in} u_{i1}}) - f^*(\overrightarrow{u_{i(n-1)} u_{in}})| \\ &= |[ni - n + 2] - [ni - 2(n - 1) + n + 2]| \quad [\text{by (4)}] \\ &= |ni - n + 2 - ni + 2n - 2 - n - 2| = |-2| \end{aligned}$$

$$\text{Therefore, } f^*(u_{in}) \leq 2 \quad (3.5.16)$$

Equations (3.5.5), (3.5.6) and (3.5.11) to (3.5.16) imply $f^*(u_{ij}) \leq 2$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Thus, Di-Cyclic snake $m\overrightarrow{C_n}$ is a near mean digraph for $m \geq 1$ and $n \geq 3$.

Theorem 3.6. Alternating Di-Cyclic snakes are non near mean digraphs.

Proof: In an alternating Di-Cyclic snake, either $d^+(u_{21}) = 0$ and $d^-(u_{21}) = 4$ (or) $d^+(u_{21}) = 4$ and $d^-(u_{21}) = 0$.

Therefore, Corresponding to every $f: V \rightarrow \{0, 1, 2, \dots, q\}$, $\sum_{w \in V} f^*(\overrightarrow{u_{21}w}) = 0$ and $\sum_{w \in V} f^*(\overrightarrow{wu_{21}})$ is a sum of at least three positive integers (or) $\sum_{w \in V} f^*(\overrightarrow{wu_{21}}) = 0$ and $\sum_{w \in V} f^*(\overrightarrow{u_{21}w})$ is a sum of at least three positive integers.

Therefore, $f^*(u_{21}) > 2$.

Therefore, No function $f: V \rightarrow \{0, 1, 2, \dots, q\}$ is a near mean labeling.

Thus, an alternating di-cyclic snake is a non near mean digraph.

4. Conclusions

In this article, different dicyclic snakes are introduced. Also, existence of near mean labeling is verified to dicyclic snakes and its generalisation. Most of the labeling are proved only for graphs. In this way, we develop the concept of labeling into digraphs

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Radio Mean Labeling of Digraphs

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Abstract

Let D be a strong digraph and let $\vec{d}(u, v)$ denote the distance between any two vertices in D . A radio mean labeling is a one-to-one mapping f from $V(D)$ to N satisfying the condition $\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + \text{diam}(D)$ for every $u, v \in V(D)$. The span of a labeling f is the maximum integer that f maps to a vertex of D . The radio mean number of D , $rmn(D)$ is the lowest span taken over all radio mean labelings of the graph D . In this paper, we analyze radio mean labeling for some newly defined digraphs.

Keywords: Radio Mean, Radio Mean Number, Radio Mean Labeling, Digraphs.

AMS Subject Classification: 05C78[‡].

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1. Introduction

The graph labeling problem is one of the recent developing area in graph theory. Alex Rosa first introduced this problem in 1967[10]. Radio labeling is motivated by the channel assignment problem introduced by W. K. Hale in 1980[4]. In 2001, Gary Chartrand defined the concept of radio labeling of G [2]. Liu and Zhu first determined the radio number in 2005[5]. Ponraj et al.[8] introduced the notion of radio mean labeling of graphs and investigated radio mean number of some graphs [9].

In this paper, we introduce a new definition for radio mean labeling of digraphs and also we study radio mean number of some newly defined digraphs.

Radio Labeling is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing, cell biology etc.,

The following results are used in the subsequent section.

Definition 1.1. Let D be a strong digraph and let $\vec{d}(u, v)$ denote the distance between any two vertices in D . A radio mean labeling is a one-to-one mapping f from $V(D)$ to N satisfying the condition $\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + \text{diam}(D)$ for every $u, v \in V(D)$. The span of a labeling f is the maximum integer that f maps to a vertex of D . The radio mean number of D , $rmn(D)$ is the lowest span taken over all radio mean labelings of the graph D .

Definition 1.2. Consider a globe. Let u, v be the vertices of degree n . Orient the edges of all but one $u - v$ path of length two in same direction. Orient the left out $u - v$ path in opposite direction. It is strongly connected and is called a Diglobe. It is denoted as $\overrightarrow{Gl(n)}$.

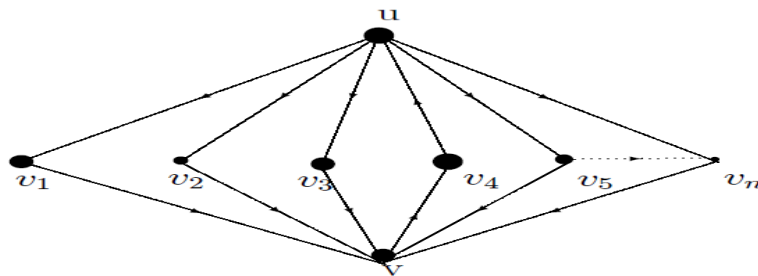


Figure 1.1. Diglobe

Definition 1.3. If the edges of all $u - v$ paths are in a single common direction it is not a strong digraph. We name it as sole diglobe ($\overrightarrow{SGL(n)}$).

Remark 1.4. If there are atleast two paths are oriented in opposite directions, the diglobe becomes strongly connected.

Remark 1.5. Orient the edges of the globe in such a way that the two edges in each $u - v$ path of length 2 get opposite directions. The resulting digraph is called alternate diglobe and is denoted as $\overrightarrow{AGl(n)}$. $\overrightarrow{AGl(n)}$ is not a strong digraph.

Definition 1.6. Consider a book with n pages sharing a common edge. The common edge is called the spine or base of the book. Orient all the edges except the spine in the one single direction and the spine in opposite direction. The resulting digraph is strongly connected and is called as directed book. The directed book is denoted as $\overrightarrow{B(m, n)}$.

Remark 1.7. The ditriangular book with n -pages is defined as n copies of cycle C_3 sharing a common edge in a directed book. The ditriangular book is denoted as $\overrightarrow{B(3, n)}$.

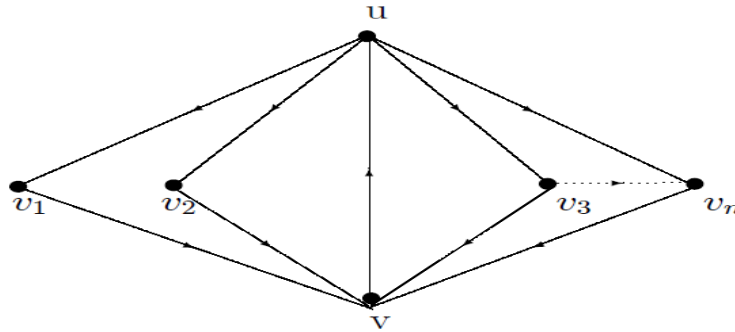


Figure 1.2. Ditriangular Book

Remark 1.8. The diquadrilateral book with n -pages is defined as n copies of cycle C_4 sharing a common edge in a directed book. The diquadrilateral book is denoted as $\overrightarrow{B(4, n)}$.

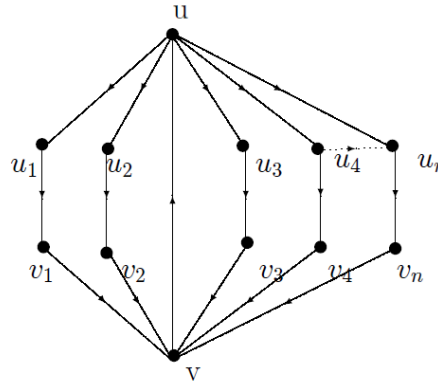


Figure 1.3. Diquadrilateral Book

2. Main Results

Theorem 2.1. The radio mean number of diglobe $(\overrightarrow{Gl(n)})$ is less than or equal to $n + 3$ for $3 \leq n \leq 4$.

Proof. Let D be a diglobe.

Let $V(D) = \{v_1, v_2, v_3, \dots, v_n, u, v\}$ and

$$A(D) = \{\overrightarrow{uv_i}/1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{\overrightarrow{uv_i}/\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{v_{\lfloor \frac{n}{2} \rfloor + 1}u}\} \cup \{\overrightarrow{v_i v}/1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{\overrightarrow{v_i v}/\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{vv_{\lfloor \frac{n}{2} \rfloor + 1}}\}$$

The diameter of diglobe is 4.

Define $f: V(D) \rightarrow N$ by

$$f(v_i) = i + 1, \quad 1 \leq i \leq n$$

$$f(u) = n + 2$$

$$f(v) = n + 3$$

Claim. f is a valid radio mean labeling.

Since the diameter is 4, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 5$ for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$ values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i) . Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 2 + i + 1}{2} \right\rceil \geq 5$$

Case b. Consider the pairs (v_i, v) . Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil \frac{f(v_i) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i + 1 + n + 3}{2} \right\rceil \geq 5$$

Case c. Consider the pairs (u, v) and $\vec{d}(u, v) = 2$

$$\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{n + 2 + n + 3}{2} \right\rceil > 5$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 2$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 2 + \left\lceil \frac{i + 1 + i + 2}{2} \right\rceil > 5$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Further, f attains its maximum corresponding to v and therefore $rmn(\overrightarrow{Gl(n)}) \leq n + 3$ for $3 \leq n \leq 4$.

Theorem 2.2. The radio mean number of diglobe $(\overrightarrow{Gl(n)})$ is $n + 2$ for $n > 4$.

Proof. Let D be a diglobe.

Let $V(D) = \{v_1, v_2, v_3, \dots, v_n, u, v\}$ and

$$A(D) = \{\overrightarrow{uv_i}/1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{\overrightarrow{uv_i}/\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\} \cup \{\overrightarrow{v_{\lfloor \frac{n}{2} \rfloor + 1}u}\} \cup \{\overrightarrow{v_i v}/1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{\overrightarrow{v_i v}/\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\}$$

$$\leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{ \overrightarrow{v_i v} / \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \} \cup \{ \overrightarrow{v v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}} \}$$

The diameter of diglobe is 4.

Define $f: V(D) \rightarrow N$ by

$$f(v_i) = i, \quad 1 \leq i \leq n$$

$$f(u) = n + 1$$

$$f(v) = n + 2$$

Claim. f is a valid radio mean labeling.

Since the diameter is 4, to prove f is a radio mean labeling, it is enough to prove that,

$$\vec{d}(x, y) + \left\lfloor \frac{f(x) + f(y)}{2} \right\rfloor \geq 5 \dots \dots \dots (1) \text{ for every pair of vertices } (x, y) \text{ where } x \neq y.$$

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$ values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i) . Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lfloor \frac{f(u) + f(v_i)}{2} \right\rfloor \geq 1 + \left\lfloor \frac{n + 1 + i}{2} \right\rfloor \geq 5$$

Case b. Consider the pairs (v_i, v) . Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lfloor \frac{f(v_i) + f(v)}{2} \right\rfloor \geq 1 + \left\lfloor \frac{i + n + 2}{2} \right\rfloor \geq 5$$

Case c. Consider the pairs (u, v) and $\vec{d}(u, v) = 2$

$$\vec{d}(u, v) + \left\lfloor \frac{f(u) + f(v)}{2} \right\rfloor \geq 2 + \left\lfloor \frac{n + 1 + n + 2}{2} \right\rfloor > 5$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 2$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lfloor \frac{f(v_i) + f(v_{i+1})}{2} \right\rfloor \geq 2 + \left\lfloor \frac{i + i + 1}{2} \right\rfloor > 5$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Further, f attains its maximum corresponding to v and is $n + 2$ for $n > 4$.

Since D contains only $n + 2$ vertices, $n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of D .

Hence $rmn(D) = n + 2$ for $n > 4$.

Theorem 2.3. The radio mean number of ditriangular book $(\overrightarrow{B(3, n)})$ is less than or equal to $n + 3$ for $n = 2$.

Proof. Let D be a ditriangular book.

Let $V(D) = \{v_1, v_2, v_3, \dots, v_n, u, v\}$ and

$$A(D) = \{ \overrightarrow{uv_i} / 1 \leq i \leq n \} \cup \{ \overrightarrow{v_i v} / 1 \leq i \leq n \} \cup \{ \overrightarrow{vu} \}$$

The diameter of ditriangular book is 3.

Define $f: V(D) \rightarrow N$ by

$$f(v_i) = i + 1, \quad 1 \leq i \leq n$$

$$f(u) = n + 2$$

$$f(v) = n + 3$$

Claim. f is a valid radio mean labeling.

Since the diameter is 3, to prove f is a radio mean labeling, it is enough to prove that, $\vec{d}(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 4 \dots \dots \dots (1)$ for every pair of vertices (x, y) where $x \neq y$.

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$ values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i) . Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 2 + i + 1}{2} \right\rceil \geq 4$$

Case b. Consider the pairs (v_i, v) . Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil \frac{f(v_i) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i + 1 + n + 3}{2} \right\rceil > 4$$

Case c. Consider the pairs (u, v) and $\vec{d}(u, v) = 1$

$$\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 2 + n + 3}{2} \right\rceil > 4$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 3$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 3 + \left\lceil \frac{i + 1 + i + 2}{2} \right\rceil > 4$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Further, f attains its maximum corresponding to v and therefore $rmn(\overrightarrow{B(3, n)}) \leq n + 3$ for $n = 2$.

Theorem 2.4. The radio mean number of ditriangular book $\overrightarrow{B(3, n)}$ is $n + 2$ for $n > 2$.

Proof. Let D be a ditriangular book.

Let $V(D) = \{v_1, v_2, v_3, \dots, v_n, u, v\}$ and

$A(D) = \{\overrightarrow{uv_i} / 1 \leq i \leq n\} \cup \{\overrightarrow{v_i v} / 1 \leq i \leq n\} \cup \{\overrightarrow{vu}\}$

The diameter of ditriangular book is 3.

Define $f: V(D) \rightarrow N$ by

$$f(v_i) = i, \quad 1 \leq i \leq n$$

$$f(u) = n + 1$$

$$f(v) = n + 2$$

Claim. f is a valid radio mean labeling.

Since the diameter is 3, to prove f is a radio mean labeling, it is enough to prove that,

$$\vec{d}(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 4 \dots \dots \dots (1)$$

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$ values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, v_i) . Here, $\vec{d}(u, v_i) \geq 1$

$$\vec{d}(u, v_i) + \left\lceil \frac{f(u) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{n + 1 + i}{2} \right\rceil \geq 4$$

Case b. Consider the pairs (v_i, v) . Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil \frac{f(v_i) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{i + n + 2}{2} \right\rceil \geq 4$$

Case c. Consider the pairs (u, v) and $\vec{d}(u, v) = 1$

$$\vec{d}(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{n+1 + n+2}{2} \right\rceil > 4$$

Case d. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 3$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 3 + \left\lceil \frac{i+i+1}{2} \right\rceil > 4$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Further, f attains its maximum corresponding to v and is $n+2$ for $n > 2$.

Since D contains only $n+2$ vertices, $n+2$ is the minimum of the maximum integer that could be assigned to the vertices of D .

Hence $rmn(D) = n+2$ for $n > 2$.

Theorem 2.5. The radio mean number of diquadrilateral book $(\overrightarrow{B(4, n)})$ is $2n+2$ for $n > 3$.

Proof. Let D be a diquadrilateral book.

Let $V(D) = \{v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_n, u, v\}$ and

$A(D) = \{\overrightarrow{uu_i} / 1 \leq i \leq n\} \cup \{\overrightarrow{u_i v_i} / 1 \leq i \leq n\} \cup \{\overrightarrow{v_i v} / 1 \leq i \leq n\} \cup \{\overrightarrow{vu}\}$

The diameter of diquadrilateral book is 5.

Define $f: V(D) \rightarrow N$ by

$$f(u_i) = i, \quad 1 \leq i \leq n$$

$$f(v_i) = 2n - i + 1, \quad 1 \leq i \leq n$$

$$f(u) = 2n + 1$$

$$f(v) = 2n + 2$$

Claim. f is a valid radio mean labeling.

Since the diameter is 5, to prove f is a radio mean labeling, it is enough to prove that,

$$\vec{d}(x, y) + \left\lceil \frac{f(x) + f(y)}{2} \right\rceil \geq 6 \dots \dots \dots (1) \text{ for every pair of vertices } (x, y) \text{ where } x \neq y.$$

Equivalently, it is enough to prove (1) for pair of vertices with minimum f values and minimum $\vec{d}(x, y)$ values. Hence, the proof involves the following cases

Case a. Consider the pairs (u, u_i) . Here, $\vec{d}(u, u_i) \geq 1$

$$\vec{d}(u, u_i) + \left\lceil \frac{f(u) + f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n+1 + i}{2} \right\rceil \geq 6$$

Case b. Consider the pairs (v_i, v) . Here, $\vec{d}(v_i, v) \geq 1$

$$\vec{d}(v_i, v) + \left\lceil \frac{f(v_i) + f(v)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n-i+1 + 2n+2}{2} \right\rceil > 6$$

Case c. Consider the pairs (u_i, v_i) and $\vec{d}(u_i, v_i) = 1$

$$\vec{d}(u_i, v_i) + \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{i + 2n - i + 1}{2} \right\rceil > 6$$

Case d. Consider the pairs (u_i, u_{i+1}) and $\vec{d}(u_i, u_{i+1}) = 4$, then,

$$\vec{d}(u_i, u_{i+1}) + \left\lceil \frac{f(u_i) + f(u_{i+1})}{2} \right\rceil \geq 4 + \left\lceil \frac{i + i + 1}{2} \right\rceil \geq 6$$

Case e. Consider the pairs (v_i, v_{i+1}) and $\vec{d}(v_i, v_{i+1}) = 4$, then,

$$\vec{d}(v_i, v_{i+1}) + \left\lceil \frac{f(v_i) + f(v_{i+1})}{2} \right\rceil \geq 4 + \left\lceil \frac{2n - i + 1 + 2n - i}{2} \right\rceil > 6$$

Hence, by all the above cases, the radio mean condition is satisfied by f .

Further, f attains its maximum corresponding to v and is $2n + 2$ for $n > 3$.

Since D contains only $2n + 2$ vertices, $2n + 2$ is the minimum of the maximum integer that could be assigned to the vertices of D .

Hence, $rmn(D) = 2n + 2$ for $n > 3$.

3. Conclusions

In this papers we compute some types of newly defined digraphs. In future we will find that digraphs (Radio Labeling) is used for X-ray, crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing, cell biology etc.,

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Changing and Unchanging strong efficient edge domination number of some standard graphs when a vertex is removed or an edge is added

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Abstract

Let $G = (V, E)$ be a simple graph. A subset S of $E(G)$ is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$ ($|N_w[e] \cap S| = 1$ for all $e \in E(G)$) where $N_s(e) = \{f / f \in E(G), f \text{ is adjacent to } e \text{ \& } \deg f \geq \deg e\}$ ($N_w(e) = \{f / f \in E(G), f \text{ is adjacent to } e \text{ \& } \deg f \leq \deg e\}$) and $N_s[e] = N_s(e) \cup \{e\}$ ($N_w[e] = N_w(e) \cup \{e\}$). The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong efficient edge domination number of G and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$). When a vertex is removed or an edge is added to the graph, the strong efficient edge domination number may or may not be changed. In this paper the change or unchanged of the strong efficient edge domination number of some standard graphs are determined, when a vertex is removed or an edge is added.

Keywords: Domination, edge domination, strong edge domination, efficient edge domination, strong efficient edge domination.

AMS subject classification: 05C69³

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1. Introduction

It is meant by the graph that it is a finite, undirected graph without loops and multiple edges. The concept of domination in graphs was introduced by Ore. Two volumes on domination have been published by T. W. Haynes, S. T. Hedetniemi and P. J. Slater [4, 5]. Let G be a graph with vertex set V and edge set E . A subset D of $V(G)$ is called a strong dominating set of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D . Similarly, a set D is a subset of $V(G)$ is called a weak dominating set of G if every vertex in $V - D$ is weakly dominated by at least one vertex in D . The strong (weak) domination number $\gamma_s(G)$ ($\gamma_w(G)$) respectively of G is the minimum cardinality of a strong (weak) dominating set of G . A subset D of $V(G)$ is called an efficient dominating set of G if for every vertex $u \in V(G)$, $|N[u] \cap D| = 1$ [1, 2]. Edge dominating sets were also studied by Mitchell and Hedetniemi [6, 7]. A subset F of edges in a graph $G = (V, E)$ is called an edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F . The edge domination number $\gamma'(G)$ of a graph G is the smallest cardinality among all minimum edge dominating sets of G . The degree of an edge uv is defined to be $\deg u + \deg v - 2$. An edge uv is called an isolated edge if $\deg uv = 0$. A subset F of E is called an efficient edge dominating set if every edge in E is either in F or dominated by exactly one edge in F . The cardinality of minimum efficient edge dominating set is called the edge domination number of G . Motivated by these definitions; strong efficient edge domination in graphs is defined as follows. A subset S of $E(G)$ is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$ [$|N_w[e] \cap S| = 1$ for all $e \in E(G)$] where $N_s(e) = \{f/f \in E(G) \& \deg f \geq \deg e\}$ ($N_w(e) = \{f/f \in E(G) \& \deg f \leq \deg e\}$) and $N_s[e] = N_s(e) \cup \{e\}$ ($N_w[e] = N_w(e) \cup \{e\}$). The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called as a strong efficient edge domination number of G (weak efficient edge domination number of G) and also denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$).

Definition 1.1. Let $G = (V, E)$ be a simple graph. Let $E(G) = \{e_1, e_2, e_3, e_4, \dots, e_n\}$. An edge e_i is said to be the full degree edge if and only if $\deg e_i = n-1$.

Observation 1.2. $\gamma'_{se}(G) = 1$ if and only if G has a full degree edge.

Observation 1.3. $\gamma'_{se}(K_{1,n}) = 1, n \geq 1$ and $\gamma'_{se}(D_{r,s}) = 1, r, s \geq 1$

Theorem 1.4. For any path P_m , $\gamma'_{se}(P_m) = \begin{cases} n, & \text{if } m = 3n + 1, n \geq 1 \\ n + 1, & \text{if } m = 3n, n \geq 2 \\ n + 1, & \text{if } m = 3n + 2, n \geq 1 \end{cases}$

Theorem 1.5. $\gamma'_{se}(C_{3n}) = n, \forall n \in \mathbb{N}$

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Theorem 1.6. Let W_m be a wheel graph. Then W_m has a strong efficient edge dominating set if and only if $m = 3n$, $n \geq 2$ and $\gamma'_{se}(W_{3n}) = n$, $n \geq 2$.

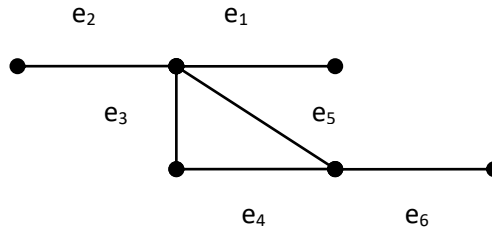
2. Main Results

Definition 2.1. $(E'_{se})^0(G) = \{e \in E(G) / (\gamma'_{se})(G + e) = (\gamma'_{se})(G)\}$

$(E'_{se})^+(G) = \{e \in E(G) / (\gamma'_{se})(G + e) > (\gamma'_{se})(G)\}$,

$(E'_{se})^-(G) = \{e \in E(G) / (\gamma'_{se})(G + e) < (\gamma'_{se})(G)\}$

Example 2.2. Consider the following graph



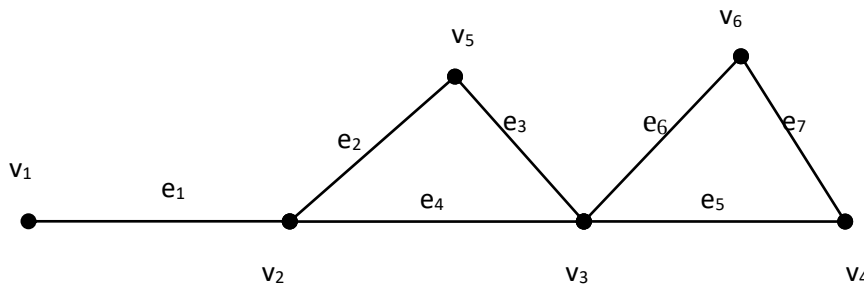
Since e_5 is the full degree edge of G , $\gamma'_{se}(G) = 1$. $\{e_3, e_6\}$ is the unique strong efficient edge dominating set of $G - e_5$. Therefore $(\gamma'_{se})(G - e_5) = 2 > (\gamma'_{se})(G)$ and e_5 is the full degree edge of $G - e_3$ and $\gamma'_{se}(G - e_3) = 1 = \gamma'_{se}(G)$. Hence $\gamma'_{se}(G) = (\gamma'_{se})(G - e_3)$.

Definition 2.3. $(V'_{se})^0(G) = \{v \in V(G) / \gamma'_{se}(G - v) = \gamma'_{se}(G)\}$

$(V'_{se})^+(G) = \{v \in V(G) / \gamma'_{se}(G - v) > \gamma'_{se}(G)\}$,

$(V'_{se})^-(G) = \{v \in V(G) / \gamma'_{se}(G - v) < \gamma'_{se}(G)\}$

Example 2.4. Consider the following graph



$S = \{e_4, e_7\}$ is the strong efficient edge dominating set of G and $\gamma'_{se}(G) = 2$.
 $\gamma'_{se}(G - v_i) = \gamma'_{se}(G)$, $i = 1, 3$ and $\gamma'_{se}(G - v_i) < \gamma'_{se}(G)$, $i \neq 1, 3$

Theorem 2.5. Let $G = P_{3n}$, $n \geq 1$. Then $(V_{se}')^+(G) = \phi$

Proof: Case (1): Let $G = P_{3n}$, $n \geq 1$. Let v be the end vertex of G . Then $G - v = P_{3n-1}$. $\gamma_{se}'(P_{3n-1}) = \gamma_{se}'(P_{3(n-1)+2}) = n-1+1 = n$ and $\gamma_{se}'(G) = n+1$. Therefore $\gamma_{se}'(G-v) < \gamma_{se}'(G)$. Hence $v_i \notin (V_{se}')^+(G)$.

Case (2): Let $v = v_{3k}$, $1 \leq k \leq n-1$. Thus $G - v = P_{3k-1} \cup P_{3n-3k}$ and $\gamma_{se}'(P_{3k-1}) = k$, $\gamma_{se}'(P_{3n-3k}) = \gamma_{se}'(P_{3(n-k)}) = n-k+1$. Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k-1}) + \gamma_{se}'(P_{3n-3k}) = k + n - k + 1 = n+1 = \gamma_{se}'(G)$. Hence $v \in (V_{se}')^0(G)$.

Case (3): Let $v = v_{3k+1}$, $1 \leq k \leq n-1$. Thus $G - v = P_{3k} \cup P_{3n-3k-1}$ and $\gamma_{se}'(P_{3k}) = k+1$, $\gamma_{se}'(P_{3n-3k-1}) = \gamma_{se}'(P_{3(n-k-1)+2}) = n-k$. Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k}) + \gamma_{se}'(P_{3n-3k-1}) = k+1 + n - k = n+1 = \gamma_{se}'(G)$. Hence $v \in (V_{se}')^0(G)$.

Case (4): Let $v = v_{3k+2}$, $1 \leq k \leq n-2$. Thus $G - v = P_{3k+1} \cup P_{3n-3k-2}$ and $\gamma_{se}'(P_{3k+1}) = k$, $\gamma_{se}'(P_{3n-3k-2}) = \gamma_{se}'(P_{3(n-k-1)+1}) = n-k-1$. Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k+1}) + \gamma_{se}'(P_{3n-3k-2}) = k + n - k - 1 = n-1 < \gamma_{se}'(G)$. Hence $v \in (V_{se}')^-(G)$.

Case (5): when $v = v_2$ or v_{3n-1} . Thus $G-v = P_{3n-2} \cup P_1$ having no strong efficient dominating set. From the above given the cases it is identified that, $(V_{se}')^+(G) = \phi$

Theorem 2.6. Let $G = P_{3n+1}$, $n \geq 1$. Then $(V_{se}')^-(G) = \phi$

Proof: Case (1): Let $G = P_{3n+1}$, $n \geq 1$. Let v be the end vertex of G . Then $G - v = P_{3n}$. $\gamma_{se}'(G-v) = n+1$ but $\gamma_{se}'(G) = n$. Therefore $\gamma_{se}'(G-v) > \gamma_{se}'(G)$. Hence $v \in (V_{se}')^+(G)$

Case (2): Let $v = v_{3k}$, $1 \leq k \leq n-1$. Thus $G - v = P_{3k-1} \cup P_{3n+1-3k}$. Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k-1}) + \gamma_{se}'(P_{3n+1-3k}) = \gamma_{se}'(P_{3(k-1)+2}) + \gamma_{se}'(P_{3(n-k)+1}) = k + n - k = n = \gamma_{se}'(G)$. Hence $v \in (V_{se}')^0(G)$.

Case (3): Let $v = v_{3k+1}$, $1 \leq k \leq n-1$. Thus $G - v = P_{3k} \cup P_{3n-3k}$. Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k}) + \gamma_{se}'(P_{3(n-k)}) = k+1 + n - k + 1 = n+2 > n = \gamma_{se}'(G)$. Hence $v \in (V_{se}')^+(G)$.

Case (4): Let $v = v_{3k+2}$, $1 \leq k \leq n-2$. Thus $G - v = P_{3k+1} \cup P_{3n-3k-1} = P_{3k+1} \cup P_{3(n-k-1)+2}$ and, Therefore $\gamma_{se}'(G-v) = \gamma_{se}'(P_{3k+1}) + \gamma_{se}'(P_{3(n-k-1)+2}) = k + n - k - 1 + 1 = n = \gamma_{se}'(G)$. Hence $v \in (V_{se}')^0(G)$.

Case (5): when $v = v_2$ or v_{3n} . $G-v = P_{3n-1} \cup P_1$ which has no strong efficient dominating set. From the above all the cases, $(V_{se}')^-(G) = \phi$

Theorem 2.7. Let $G = P_{3n+2}$, $n \geq 1$. Then $(V_{se}')^-(G) = \phi$

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Proof: Case (1): Let $G = P_{3n+2}$, $n \geq 1$. Let v be the end vertex of G . Then $G - v = P_{3n+1}$. $\gamma'_{se}(G - v) = \gamma'_{se}(P_{3n+1}) = n$ but $\gamma'_{se}(G) = n + 1$. Therefore $\gamma'_{se}(G - v) < \gamma'_{se}(G)$. Hence $v \notin (V'_{se})^-(G)$

Case (2): Let $v = v_{3k}$, $1 \leq k \leq n - 1$. Thus $G - v = P_{3k-1} \cup P_{3n+2-3k}$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(P_{3k-1}) + \gamma'_{se}(P_{3n+2-3k}) = \gamma'_{se}(P_{3(k-1)+2}) + \gamma'_{se}(P_{3(n-k)+2}) = k + n - k + 1 = n + 1 = \gamma'_{se}(G)$. Hence $v \in (V'_{se})^0(G)$.

Case (3): Let $v = v_{3k+1}$, $1 \leq k \leq n - 1$. Thus $G - v = P_{3k} \cup P_{3n-3k+1}$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(P_{3k}) + \gamma'_{se}(P_{3(n-k)+1}) = k + 1 + n - k = n + 1 = \gamma'_{se}(G)$. Hence $v \in (V'_{se})^0(G)$.

Case (4): Let $v = v_{3k+2}$, $1 \leq k \leq n - 2$. Thus $G - v = P_{3k+2} \cup P_{3n-3k-1} = P_{3k+2} \cup P_{3(n-k-1)+2}$ and, Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(P_{3k+2}) + \gamma'_{se}(P_{3(n-k-1)+2}) = k + 2 + n - k - 1 = n + 1 = \gamma'_{se}(G)$. Hence $v \in (V'_{se})^0(G)$.

Case (5): when $v = v_2$ or v_{3n+1} . $G - v = P_{3n-2} \cup P_1$ which has no strong efficient dominating set. From the above all the cases, $(V'_{se})^-(G) = \emptyset$

Theorem 2.8. Let $G = C_{3n}$, $n \geq 1$. Then $(V'_{se})^0(G) = V(G)$.

Proof:

Let $G = C_{3n}$, $n \geq 1$. Let $v \in V(G)$. Then $\gamma'_{se}(G) = n$, $G - v = P_{3n-1}$ and $\gamma'_{se}(P_{3n-1}) = \gamma'_{se}(P_{3(n-1)+2}) = n$ Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(G)$. Hence $(V'_{se})^0(G) = V(G)$.

Theorem 2.9. Let $G = K_{1,n}$, $n \geq 2$. Then $(V'_{se})^0(G) = V(G)$.

Proof:

Let $G = K_{1,n}$, $n \geq 2$. Let $v \in V(G)$. Then $\gamma'_{se}(G) = 1$, $G - v = K_{1, n-1}$ and $\gamma'_{se}(K_{1, n-1}) = 1$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(G)$. Hence $(V'_{se})^0(G) = V(G)$.

Theorem 2.10. Let $G = D_{r,s}$, $r, s \geq 1$. Then $|(V'_{se})^0(G)| = r + s - 2$.

Proof:

Let $G = D_{r,s}$, $r, s \geq 1$. Let $V(G) = \{u, v, u_i, v_j / 1 \leq i \leq r, 1 \leq j \leq s\}$, $E(G) = \{uv, uu_i, vv_j / 1 \leq i \leq r, 1 \leq j \leq s\}$. Let $v = u_i$ or v_j . Then $\gamma'_{se}(G) = 1$, $G - v = D_{r-1, s} = D_{r, s-1}$ and $\gamma'_{se}(D_{r-1, s}) = \gamma'_{se}(D_{r, s-1}) = 1$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(G)$. Hence $(V'_{se})^0(G) = V(G) - \{u, v\}$. Therefore $|(V'_{se})^0(G)| = r + s - 2$

Theorem 2.11. Let $G = W_{3n}$, $n \geq 2$. Then $v \in (V'_{se})^0(G)$ if $v \in (K_1)$ and $v \in (V'_{se})^-(G)$ if $v \in V(C_{3n-1})$

Proof:

Let $G = W_{3n}, n \geq 2$. Let $V(G) = \{v, v_i / 1 \leq i \leq 3n\}$,
 $E(G) = \{vv_j, v_i v_{i+1}, v_{3n} v_1 / 1 \leq j \leq 3n, 1 \leq i \leq 3n-1\}$

Case (1): $G - v = C_{3n}$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(C_{3n}) = n$. Hence $v \in (V'_{se})^0(G)$.

Case (2): Let $v = v_i, 1 \leq i \leq 3n$. $G - v = F_{3n-1}$. Therefore $\gamma'_{se}(G - v) = \gamma'_{se}(F_{3n-1}) = \gamma'_{se}(F_{3(n-1)+2}) = n$. but $\gamma'_{se}(G) = 2n$. Hence $\gamma'_{se}(G - v) < \gamma'_{se}(G)$.
 Therefore $v \in (V'_{se})^-(G)$

Theorem 2.12.

Let $G = P_{3n}, n \geq 2$. Let $e = uv$ be any edge incident with any vertex of G and $G' = G + e$.
 Then $\gamma'_{se}(G + e) = \gamma'_{se}(G) - 1$ if e is incident with u_1 or $u_{3i}, 1 \leq i \leq 3n-2$ and $\gamma'_{se}(G + e)$
 has no strong efficient edge dominating set if e is incident with $u_2, u_{3n-1},$
 $u_{3i+2}, 1 \leq i \leq 3n-2$

Proof:

Let $G = P_{3n}, n \geq 2$. $V(G) = \{u_i / 1 \leq i \leq 3n\}$, $E(G) = \{e_i = u_i u_{i+1} / 1 \leq i \leq 3n-1\}$. Let $e = uv$ be the new edge incident with any vertex of G and $G' = G + e$.

Case 1: Let e be an end edge of G' . Then $G' = P_{3n+1}$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(P_{3n+1}) = n$
 but $\gamma'_{se}(G) = n + 1$. Therefore $\gamma'_{se}(G + e) < \gamma'_{se}(G)$. Hence $e \in (E'_{se})^-(G)$.

Case 2: Let the edge e be incident with the vertex u_2 . Let S be a strong efficient edge dominating set of G' . Suppose $n \geq 2$. Among all the edges, the edge e_2 have maximum degree. It must belong to S . It strongly efficiently dominates e, e_3, e_1 . Also the edges $e_5, e_8, e_{11}, \dots, e_{3n-4}$ belong to S . If the edge e_{3n-2} belongs to S , then $|N_S[e_{3n-3}] \cap S| = |\{e_{3n-4}, e_{3n-2}\}| = 2 > 1$, a contradiction. Hence G' has no strong efficient edge dominating set. The proof is similar if the edge e is added at the vertex u_{3n-1} .

Case 3: Let the edge e be incident with the vertex u_3 . e_2 and e_3 are the only maximum degree edges. Hence any strong efficient edge dominating set contains either e_2 or e_3 . Then $S_1 = \{e_1, e_3, e_6, \dots, e_{3n-3}, e_{3n-1}\}$, $S_2 = \{e_2, e_4, e_7, \dots, e_{3n-2}\}$ are the strong efficient edge dominating sets of G' . Therefore $|S_1| = n+1, |S_2| = n$. Hence $\gamma'_{se}(G') = n < \gamma'_{se}(G)$.

Therefore $e \in (E'_{se})^-(G)$. The proof is similar if the edge e is incident with the vertex $u_{3i}, 2 \leq i \leq n-1$

Case 4: Let the edge e be incident with the vertex u_4 . Then $S = \{e_2, e_4, e_7, \dots, e_{3n-2}\}$ is the unique strong efficient edge dominating set of G' and $\gamma'_{se}(G') = n$. Therefore $\gamma'_{se}(G + e) < \gamma'_{se}(G)$. Hence $e \in (E'_{se})^-(G)$. The proof is similar if the edge e is incident with the vertex $u_{3i+1}, 2 \leq i \leq n-1$

Case 5: Let the edge e be incident with the vertex u_5 . Let S be a strong efficient edge dominating set of G' . The edge e_4 & e_5 are the only maximum degree edges. If the edge e_4 belongs to S then no edge in S to strongly efficiently dominate e_2 . If the edge e_5 belongs to S then $e_2, e_8, e_{11}, \dots, e_{3n-4}$ belongs to S and there is no edge in S to strongly

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efficiently dominate e_{3n-2} . Hence strong efficient edge dominating set does not exists. Proof is similar if the edge e is incident with the vertex u_{3i+2} , $2 \leq i \leq n-2$. From all the

above cases, $(E'_{se})^0(G) = \phi$

Remark: Let $n = 1$, $G' = K_{1,3}$. $\gamma'_{se}(G') = 1 = \gamma'_{se}(G)$. Therefore $e \in (E'_{se})^0(G)$

Theorem 2.13.

Let $G = P_{3n+1}$, $n \geq 1$. Let $e = uv$ be any edge incident with any vertex of G and $G' = G + e$. Then $\gamma'_{se}(G + e) = \gamma'_{se}(G)$ if e is incident with u_2 or u_{3i} , $1 \leq i \leq n-1$, u_{3j+2} , $1 \leq j \leq n-2$ and $\gamma'_{se}(G + e) = \gamma'_{se}(G) + 1$ if e is incident with u_1 , u_{3n} , u_{3i+1} , $1 \leq i \leq n-1$.

Proof:

Let $G = P_{3n+1}$, $n \geq 1$. $V(G) = \{u_i / 1 \leq i \leq 3n+1\}$, $E(G) = \{e_i = u_i u_{i+1} / 1 \leq i \leq 3n\}$. Let $e = uv$ be the any edge incident with any vertex of G and $G' = G + e$.

Case 1: Let e be an end edge of G' . Then $G' = P_{3n+2}$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(P_{3n+2}) = n+1$ but $\gamma'_{se}(G) = n$. Therefore $\gamma'_{se}(G + e) > \gamma'_{se}(G)$. Hence $e \in (E'_{se})^-(G)$. Therefore $\gamma'_{se}(G + e) = \gamma'_{se}(G) + 1$

Case 2: Let the edge e be incident with the vertex u_2 . Suppose $n \geq 1$. $S = \{e_2, e_5, e_8, \dots, e_{3n-1}\}$ is the unique strong efficient edge dominating sets of G' and $|S| = n$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(G) = n$. Hence $e \in (E'_{se})^0(G)$. The proof is similar if the edge e is incident with the vertex u_{3n} .

Case 3: Let the edge e be incident with the vertex u_3 . e_2 and e_3 are the only maximum degree edges. Hence any strong efficient edge dominating set contains either e_2 or e_3 . If e_2 belongs to S then $S = \{e_2, e_5, e_8, \dots, e_{3n-3}, e_{3n-1}\}$ is the unique strong efficient edge dominating sets of G' and $|S| = n$. Hence $\gamma'_{se}(G') = n = \gamma'_{se}(G)$. Therefore $(E'_{se})^+(G) = \phi$. If the edge e_3 belongs to S then there is no edge in S to strongly efficiently dominate e_3 . The proof is similar if the edge e is incident with the vertex u_{3i} , $2 \leq i \leq n-1$.

Case 4: Let the edge e be incident with the vertex u_4 . Then $S_1 = \{e_1, e_3, e_5, \dots, e_{3n-1}\}$, $S_2 = \{e_2, e_4, e_7, \dots, e_{3n-2}, e_{3n}\}$ are the strong efficient edge dominating sets of G' and $|S_1| = |S_2| = n+1$. Therefore $\gamma'_{se}(G') = n+1$ but $\gamma'_{se}(G) = n$. Therefore $\gamma'_{se}(G + e) > \gamma'_{se}(G)$. Hence $e \in (E'_{se})^+(G)$. The proof is similar if the edge e is incident with the vertex u_{3i+1} , $2 \leq i \leq n-1$

Case 5: Let the edge e be incident with the vertex u_5 . Let S be a strong efficient edge dominating set of G' . The edge e_4 & e_5 are the only maximum degree edges. If the edge e_4 belongs to S then no edge in S to strongly efficiently dominate e_2 . If the edge e_5 belongs to S then $e_2, e_8, e_{11}, \dots, e_{3n-4}$ belongs to S and there is no edge in S to strongly efficiently dominate e_{3n-2} . Hence strong efficient edge dominating set does not exists.

Proof is similar if the edge e is incident with the vertex u_{3i+2} , $2 \leq i \leq n-2$. From all the above cases, $(E'_{se})^0(G) = \phi$

Theorem 2.14.

Let $G = P_{3n+2}$ $n \geq 1$. Let $e = uv$ be any edge incident with any vertex of G and $G' = G+e$. Then $\gamma'_{se}(G+e) = \gamma'_{se}(G)$ if e is incident with all $u_i / 1 \leq i \leq 3n+2$ except u_1, u_{3n} and $\gamma'_{se}(G+e) = \gamma'_{se}(G) + 1$ if e is incident with u_1, u_{3n} .

Proof:

Let $G = P_{3n+2}$, $n \geq 1$. $V(G) = \{u_i / 1 \leq i \leq 3n+2\}$, $E(G) = \{e_i = u_i u_{i+1} / 1 \leq i \leq 3n+1\}$. Let $e = uv$ be the any edge incident with any vertex of G and $G' = G+e$.

Case 1: Let e be an end edge of G' . Then $G' = P_{3n+3} = P_{3(n+1)}$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(P_{3(n+1)}) = n+2$ but $\gamma'_{se}(G) = n+1$. Therefore $\gamma'_{se}(G+e) > \gamma'_{se}(G)$. Hence $e \in (E'_{se})^+(G)$. Therefore $\gamma'_{se}(G+e) = \gamma'_{se}(G) + 1$

Case 2: Let the edge e be incident with the vertex u_2 . Suppose $n \geq 1$. $S = \{e_2, e_5, e_8, \dots, e_{3n-1}, e_{3n+1}\}$ is the unique strong efficient edge dominating sets of G' and $|S| = n+1$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(G) = n+1$. Hence $e \in (E'_{se})^0(G)$. The proof is similar if the edge e is incident with the vertex u_{3n} .

Case 3: Let the edge e be incident with the vertex u_3 . e_2 and e_3 are the only maximum degree edges. Hence any strong efficient edge dominating set contains either e_2 or e_3 . If e_2 belongs to S then $S = \{e_2, e_5, e_8, \dots, e_{3n-3}, e_{3n-1}, e_{3n+1}\}$ is the unique strong efficient edge dominating set of G' and $|S| = n+1$. Hence $\gamma'_{se}(G') = n+1 = \gamma'_{se}(G)$. Therefore $e \in (E'_{se})^0(G)$. If e_3 belongs to S then $S = \{e_1, e_3, e_6, \dots, e_{3n}\}$ is the unique strong efficient edge dominating set of G' and $|S| = n+1$. Hence $\gamma'_{se}(G') = n+1 = \gamma'_{se}(G)$. Therefore $e \in (E'_{se})^0(G)$. The proof is similar if the edge e is incident with the vertex u_{3i} , $2 \leq i \leq n-1$.

Case 4: Let the edge e be incident with the vertex u_4 . The edge e_3 & e_4 are the only maximum degree edges. Hence any strong efficient edge dominating set S contains either e_3 or e_4 . If the edge e_3 belongs to S then $S = \{e_1, e_3, e_6, \dots, e_{3n}\}$ is the unique strong efficient edge dominating set of G' and $|S| = n+1$. Hence $\gamma'_{se}(G') = n+1 = \gamma'_{se}(G)$. Therefore $e \in (E'_{se})^0(G)$. If the edge e_4 belongs to S then there is no edge in S to strongly efficiently dominate e_{3n} . Hence strong efficient edge dominating set does not exist. The proof is similar if the edge e is incident with the vertex u_{3i+1} , $2 \leq i \leq n-1$

Case 5: Let the edge e be incident with the vertex u_5 . The edge e_4 & e_5 are the only maximum degree edges. Hence any strong efficient edge dominating set S contains either e_4 or e_5 . If the edge e_4 belongs to S then there is no edge in S to strongly efficiently dominate e_2 . If the edge e_5 belongs to S then $\{e_2, e_5, e_8, \dots, e_{3n-3}, e_{3n-1}, e_{3n+1}\}$ is the unique strong efficient edge dominating set of G' and $|S| = n+1$. Hence

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$\gamma'_{se}(G') = n + 1 = \gamma'_{se}(G)$. Therefore $e \in (E'_{se})^0(G)$. Proof is similar if the edge e is incident with the vertex u_{3i+2} , $2 \leq i \leq n - 2$.

Theorem 2.15.

Let $G = C_{3n}$, $n \geq 1$. Let $e = uv$ be any edge incident with any vertex of G and $G' = G + e$. Then $\gamma'_{se}(G) = \gamma'_{se}(G + e)$.

Proof: Let $G = C_{3n}$, $n \geq 1$. Let $e = uv$ be the new edge. $V(G') = \{u_i, u / 1 \leq i \leq 3n\}$, $E(G') = \{e_i = u_i u_{i+1} / 1 \leq i \leq 3n - 1, e_{3n} = u_{3n} u_1, e = u_i v\}$ and $G' = G + e$. Let the edge e be incident with the vertex u_1 . e_1 and e_{3n} are the maximum degree edges and they are adjacent. Any strong efficient dominating set contains e_1 or e_{3n} . Then $S_1 = \{e_1, e_4, e_7, \dots, e_{3n-2}\}$, $S_2 = \{e_3, e_6, e_9, \dots, e_{3n-3}, e_{3n}\}$ are the strong efficient edge dominating sets of G' and $|S_1| = |S_2| = n$, $n \geq 1$. $\gamma'_{se}(G') = n$, $n \geq 1$. No other strong efficient edge dominating set exists without e_1 and e_{3n} . Therefore $\gamma'_{se}(G) = \gamma'_{se}(G + e) = n$, $n \geq 1$. The proof is similar if the edge e is with any u_i , $2 \leq i \leq 3n$.

Theorem 2.16.

Let $G = K_{1, n}$, $n \geq 1$. Let e be any edge incident with any vertex of G and $G' = G + e$. Then $\gamma'_{se}(G) = \gamma'_{se}(G + e)$.

Proof: Let $G = K_{1, n}$, $n \geq 1$. Let $V(G) = \{u_i, u / 1 \leq i \leq n\}$, $E(G) = \{uu_i / 1 \leq i \leq n\}$. $V(G') = \{u_i, u, v / 1 \leq i \leq n\}$.

Case 1: Let e be the new edge incident with u . Then $G' = K_{1, n+1}$. Therefore $\gamma'_{se}(G') = \gamma'_{se}(G) = 1$. **Case 2:** Let $u_i v$ be the new edge incident with u_i . Then $\{uu_i\}$ is the unique strong efficient edge dominating set of G' and $\gamma'_{se}(G') = 1$. Therefore $\gamma'_{se}(G) = \gamma'_{se}(G + e) = 1$.

Theorem 2.17.

Let $G = D_{r, s}$, $r, s \geq 1$. Let $e = xy$ be any edge incident with any vertex of G and $G' =$

$G + e$. Then $\gamma'_{se}(G') = \begin{cases} \gamma'_{se}(G), & \text{if } e = wu \text{ or } wv \\ \gamma'_{se}(G) + 1, & \text{if } e = u_i w \text{ or } v_j w \end{cases}$.

Proof: $G = D_{r, s}$, $r, s \geq 1$. Let $e = xy$ be the new edge. $V(G) = \{u_i, v_j, u, v / 1 \leq i \leq r, 1 \leq j \leq s\}$, $V(G') = \{u_i, v_j, u, v, x, y / 1 \leq i \leq r, 1 \leq j \leq s\}$, $E(G') = \{e_i = uu_i, f = uv, f_j = vv_j, e = xy / 1 \leq i \leq r, 1 \leq j \leq s\}$. Then $G' = G + e$.

Case 1: If the edge e is incident with either the vertex u or the vertex v . Then $G' = D_{r+1, s}$ or $G' = D_{r, s+1}$ and $\gamma'_{se}(G') = \gamma'_{se}(G) = 1$.

Case 2: Let the edge e be incident with the vertex u_i , $1 \leq i \leq r$ or v_j , $1 \leq j \leq s$. Then $S = \{uv, u_iw\}$ or $S = \{uv, v_jw\}$ is the strong efficient edge dominating set of G' and $|S|=2$.
 $\gamma'_{se}(G') = 2$. $\gamma'_{se}(G) = \gamma'_{se}(G + e) = 2$.

3. Conclusions

In this paper, the change or unchanged of the strong efficient edge domination number of some standard graphs are determined, when a vertex is removed or an edge is added.

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Application of Analytic Hierarchy Process in Engineering Education

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Abstract

Analytic Hierarchy Process (AHP) provides a mathematical technique to formulate a problem as a hierarchical structure and believes in an amalgamation of quantitative and qualitative criteria. It is this uniqueness of AHP that makes it one of the important inclusive systems, considered to make decisions with multiple criteria. This paper focuses on conducting Analytic Hierarchy Process, based on the data collected from several Engineering colleges in the state of Telangana. This paper aims to understand the reasons for removing the staple Engineering streams such as Mechanical engineering, Production engineering, Electronics and Instrumentation engineering and introducing new and contemporary streams such as Artificial Intelligence and Data Science, Artificial Intelligence and Machine Learning and Internet of Things. The World Economic Forum's latest "Future of Jobs" report highlights the impact of 'double disruption' of Automation, followed by COVID-19. The report indicates that while 85 million jobs will be displaced, 47% of core skills will change by 2025. The topic thus is of immense value since it looks closely at the paradigm shift mentioned above and its further consequences. The result of the present study would be helpful to indicate the exact rankings of the programming and non-programming branches in the engineering field and thus would be instrumental in gauging learners' inclination towards studying specific branches. This paper aims to analyze the growing demand of programming branches over traditional, non-programming branches.

Keywords: Analytic Hierarchy Process, Pair-wise comparison, Priority vector.

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1. Introduction

The process of choosing the optimal alternative among all potential options is classified as decision-making, however in practice, attaining an optimized result can be difficult because decision-makers are frequently challenged with diverse decision-making problems [1]. Multi-criteria decision-making (MCDM) is one of the most significant fields of decision theory, and it is used to find the optimum solution out of all the possibilities [4]. MCDM has been improved by the development of several approaches, including: Analytic Hierarchy Process (AHP) (Saaty 1980)[8]; Techniques for determining superiority and inferiority [10]; Simos' technique of ranking [6]; multi-attribute utility theory (MAUT) [3]; elimination and selection in accordance with reality [7]; For enrichment evaluations, a preference ranking organization method is used [2]; and selecting based on benefits [9]. These MCDM techniques are commonly used to help solve real-world decision-making difficulties. Saaty's (1980) AHP is a popular MCDM method that has gotten a lot of attention in the industry, including education and management. Earlier many researchers have done a good amount of work on decision making [5].

Students with technological capabilities will have excellent work possibilities in the post-COVID global economic landscape. According to the World Economic Forum, 92 per cent of firms are speeding up their digitalization efforts, with more than 90 per cent using technology such as artificial intelligence, big data analytics, and cloud computing. Students who graduate with specialized technical skills will have access to desirable job possibilities in fields such as IT, cloud services, and healthcare.

It is observed that, many Engineering colleges have dropped the non-programming branches such as Mechanical Engineering, Production Engineering, Electronics & Instrumentation Engineering etc. and these are being replaced by Computer Science Engineering-Emerging Technology courses such as Artificial Intelligence & Data Science, Artificial Intelligence & Machine Learning, Internet of Things & Cyber Security etc. The authors of the paper are motivated by this scenario to present a paper on this topic. This paper tries to prove that, Emerging Technology courses are providing more employability skills required by the students in the current scenario. With this goal, variables were selected and study was done to observe the results.

There are not too many discussions to understand this issue with the application of AHP method. In this context, the approach towards understanding the change in terms of choices of courses among engineering students, using AHP technique is not only a novel idea but also presents a vivid picture to understand the paradigm shift.

2. The Physical Importance of all Criteria

Scope of Employment (C₁):

The Emerging Technology courses have a wide scope of employability in many sectors like telecommunication, transportation, corporate sector, medical etc. These graduates progress more quickly in terms of developing the necessary abilities to obtain and

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manage a job, resulting in better chances. However, while having such abilities increases these graduates' chances of finding work, it does not guarantee it [3]. Subject-specific knowledge and abilities are essential for work. Other talents, undoubtedly, play an important role in the lives of students, assisting them in obtaining not just a job, but also a profession of their choice. Acquiring the right skills not only ensures sustenance but also promotes growth in the employment sector. Subject-related skills are those that students gain throughout their degree program, whereas transferable skills are those that people learn on the job and may be used to other occupations inside or outside of their organization, allowing them to go further in their careers [5]. Because information technology has such a strong influence on today's global marketplaces, this study focuses mostly on IT workers. IT firms are constantly developing and adopting new technology.

Employers also seek talented candidates that can adapt to today's fast-paced work environment and are eager to learn new skills. While working to improve employability, the International Labor Organization emphasized fast globalization, new working environments, and technological advancement. This will necessitate investing in their workforce's skill development and training.

The Indian economy has been moving toward digitalization since 2015, necessitating a highly qualified and capable workforce. Through numerous open online learning platforms such as MOOCS, SWAYAM, and others, where at least 350 online courses enable students to virtually attend courses taught by the best faculty, many jobs will be created in the future.

Right Employability Skills (C₂):

Computer Science and Technology is more of an applied computer science than a completely theoretical discipline. It focuses on addressing user needs in an organizational and societal environment by performing the following tasks: Selection, creation, application, integration, and administration of computing technologies.

As a result, computer graduates must have particular abilities and knowledge in order to obtain relevant positions at work; they must be able to anticipate changes in the field of technology and express the value of the same to an individual or organization. Every year, the software sector employs roughly 10 million people. In the \$124–130 billion market, it contributes 67 percent. For the fiscal year 2016-17, it grew at a rate of 12- 14 percent.

Because the software business is growing, it generates strong cash flows and a high return on equity. Nonetheless, rather than adding new staff, numerous IT organizations are deploying automation to improve digitalization. Employability skills are a continuous practice that can be included into the curriculum as well as the unpredictable work environment of an organization. This emphasizes the need of having the appropriate employment skills.

Higher pay package (C₃):

Computer Science Engineering is a field that is in high demand in the industry. As a result, obtaining a professional degree or certification in Computer Science will make one a valued asset to a company, especially in the IT area. Furthermore, the pay packages in these companies are fairly high.

Wide career options and career stability (C₄):

According to data from popular study destinations,

- Computer and Information Technology (IT) occupations are expected to expand by 12% in the United States by 2028, according to the Bureau of Labor Statistics.
- By 2023, the number of Software and Application Programmers in Australia is predicted to increase from 121,300 to 146,800.
- Immigration.ca reports that “Qualified Software Engineers are being hired by Canadian employers as quickly as they become available.”

Freedom to work from anywhere (C₅):

Computer science and information technology have an impact on everything from scientific research to health development, transportation, banking, and communications, among other things. Microwave ovens, refrigerators, and door locks are now all connected to our Wi-Fi networks.

Applications in all the Fields (C₆):

Artificial intelligence applications have advanced tremendously in recent years. Its applications can be found in virtually every industry.

Artificial Intelligence Applications in E-Commerce

- Personalized Purchasing
- Assistants with Artificial Intelligence
- Preventing Fraud

Applications of Artificial Intelligence in Education

- Educators will benefit from automated administrative tasks
- Creating Intelligent Content
- Virtual Assistants
- Personalized Education

Applications of Artificial Intelligence in Lifestyle

- Vehicles that drive themselves
- Filters for Spam
- Recognition of faces
- System of Recommendation

Navigation, robotics, human resources, healthcare, agriculture, gaming, automobiles, social media, marketing, chat bots, finance, and many more fields have been transformed by AI applications.

3. Significance of each Alternative

Artificial Intelligence & Data Science (AI&DS) (A₁):

The Artificial Intelligence and Data Science Program teaches students how to do intelligent data analysis, which is a critical component in many real-world applications. Data science has evolved as one of the most high-growth, dynamic, and rewarding occupations in technology during the last 10 years.

This course intends to teach not only essential technologies like artificial intelligence, data mining, and data modeling, but also advanced topics like machine learning and big data analytics. Students will gain cross-disciplinary skills in fields such as statistics, computer science, machine learning, and logic, as well as data scientists, and will have

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career opportunities in healthcare, business, e-commerce, social networking companies, climatology, biotechnology, genetics, and other important fields by the end of this course. Students will learn statistical, mathematical reasoning, machine learning, knowledge discovery, and visualization abilities as part of this program.

Artificial Intelligence & Machine Learning (AI&ML) (A₂):

The student under B.Tech Artificial Intelligence and Machine Learning is required to write the code of the said machine. This code in essence works as a guiding instruction for the machine, where it can perform tasks with less human intervention.

Internet of Things/Cyber Security (IOT/CS)(A₃):

B.Tech CSE (IoT& Cyber Security including Block chain Technology), undergraduate program familiarizes students with the functional and operational aspects of IoT, Cyber Security and Block chain Technology. Cyber security is a specialist topic of Information Technology (IT) that is considered a sub-discipline of Computer Science.

Students will gain the information and abilities needed to safeguard computer operating systems, networks, and data from cyber-attacks in Cyber Security courses. Because of the rising occurrence of cybercrime, cyber security as a profession has evolved over time. Any industry that transacts online or handles sensitive data requires a Cyber Security expert to protect its data from such criminals. Because cyberspace is a global platform that anybody may access from anywhere in the world, the scope of cyber security is equally distributed.

Cyber security is a lucrative and rapidly expanding subject that focuses on safeguarding businesses from digital threats and keeping their data and networks secure. Experts in cyber security identify flaws, offer software and hardware programs to limit risks, and create rules and processes to ensure security. The demand for qualified cyber security specialists is expected to expand as more firms transfer their activities online and cyber-attacks become more common, particularly in healthcare and financial institutions. Information security analysts, for example, are expected to expand by 40% between 2020 and 2038, according to the Bureau of Labor Statistics.

Mechanical Engineering (Mech) (A₄):

Students in this program will learn how to become Mechanical Engineers. The goal of this curriculum is to prepare students to use mechanical engineering principles in the design, manufacture, and maintenance of mechanical systems.

Production Engineering (PE) (A₅):

Production engineering is a branch of engineering that is closely related to mechanical engineering. Production engineers are educated to increase the efficiency and effectiveness of manufacturing and service industries. Manufacturing technology, which is a branch of mechanical engineering, is combined with management science in production engineering. A production engineer works in a variety of industries, dealing with engineering methods and management difficulties relating to manufacturing.

Electronics and Instrumentation Engineering (A₆):

Electronics and Instrumentation Engineering is a program combining motor skills and academic skills which carve out a career in various fields of electronics, measurement, and complex process understanding.

4. Methodology

The AHP technique is broken down into the following steps.

- I) Choosing criteria and structuring a decision-making problem
- II) Prioritization of criteria using pair wise comparison
- III) On each criterion, compare options in pairs
- IV) Calculating a relative score for each option

4.1 Prioritization Methods:

There are a few methods for determining alternate priorities, like as i) Geometric Mean Method, ii) Additive Normalization Method, iii) Stochastic Vector Method are available to find the priorities of alternatives. The Geometric Mean Method (GMM) is employed in this paper.

4.1.1 Geometric Mean Method (GMM):

The weights for the criteria or alternatives are determined using this procedure. The alternate pair-wise comparison matrix is shown in Table 1. Here $K_1, K_2, \dots, K_n$ represents the alternatives which are to be ranked and $k_{11}, k_{12}, \dots, k_{nn}$ represents expert opinions. The Geometric Mean Method, which is used to calculate the priority weight vectors, is described below.

	K_1	K_2		K_n
K_1	k_{11}	k_{12}		k_{1n}
K_2	k_{21}	k_{22}		k_{2n}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
K_n	k_{n1}	k_{n2}		k_{nn}

Table 1: Pair-wise comparisons

Obtain the geometric row means of each row as

$$\text{Priority vector } k_1 = (k_{11} \times k_{12} \times \dots \times k_{1n})^{\frac{1}{n}}$$

$$\text{Priority vector } k_2 = (k_{21} \times k_{22} \times \dots \times k_{2n})^{\frac{1}{n}}$$

$$\text{Priority vector } k_n = (k_{n1} \times k_{n2} \times \dots \times k_{nn})^{\frac{1}{n}}$$

The normalized vector of $(k_1, k_2, \dots, k_n)$ becomes the solution vector.

Table 2 describes AHP Measurement scale about the importance of Saaty's crisp numbers.

Intensity of importance	Definition	Explanation
1	Same importance	Two elements contribute same to the property
3	Moderate importance of one over another	Experience and judgment some favor one over the other
5	Essential or high	Experience and judgment highly favor

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	importance	one over another
7	Very strong importance	An element is highly favored and its dominance is demonstrated in practice
9	Extreme importance	One of the most possible orders of affirmation is evidence favoring one element over another.
2,4,6,8	Lying between two adjacent judgments	Comprise is needed between two judgments
Reciprocals	Whenever activity i compared to j is assigned one of the above numbers, the activity j compared to i is assigned its reciprocal	
Rational	Ratios occurring from forcing consistency of judgments	

Table 2: AHP Measurement Scale

Table 3 lists the number of courses offered in selected engineering colleges.

S.No	Name of the college	TS EAMCET CODE	Number of courses	Percentage
1.	MuffakhamJah College of Engineering & Technology	MJCT	10	15
2.	CVR College of Engineering	CVRH	10	15
3.	Geetanjali College of Engineering and Technology	GCTC	9	13
4.	Guru Nanak Institute of Technology	GNIT	8	12
5.	Guru Nanak Institutions Technical Campus	GURU	9	13
6.	Methodist College of Engineering and Technology	METH	6	8
7.	Nalla Narasimha Reddy Educational Society Group of Institutions	NNRG	8	12
8.	Lords Institute of Engineering & Technology	LRDS	8	12

Table 3: Number of Courses from selected Engineering Colleges

Table 4 exhibits Program-wise intake in three consecutive academic years 2019-2020, 2020-2021 and 2021-2022.

MJCT											
	CSE	INF	ECE	CIV	MEC H	PE	EEE	EIE	AI& DS	AI& ML	IOT/ CS
2019-2020	120	120	120	120	120	60	60	60	0	0	---
2020-2021	120	120	120	120	120	0	60	60	60	0	---
2021-2022	120	120	120	120	120	0	60	0	60	60	---
CVRH											
2019-2020	300	240	240	120	120	---	120	60	0	0	0
2020-2021	300	240	240	60	60	---	60	60	60	60	60

2021-2022	300	240	120	60	60	---	60	60	120	120	60
GCTC											
2019-2020	240	60	240	120	120	---	120	---	0	0	0
2020-2021	240	60	240	60	60	---	60	---	60	60	120
2021-2022	240	60	240	60	60	---	60	---	60	180	120
GNIT											
2019-2020	180	60	120	120	120	---	60	---	---	0	0
2020-2021	180	60	120	120	120	---	60	---	---	60	60
2021-2022	180	60	120	120	120	---	60	---	---	60	60
GURU											
2019-2020	300	60	300	180	300	---	120	---	0	0	0
2020-2021	300	60	300	180	300	---	120	---	60	60	60
2021-2022	300	60	300	180	180	---	120	---	60	60	60
METH											
2019-2020	120	---	120	120	120	---	60	---	0	---	---
2020-2021	120	---	120	120	60	---	60	---	60	---	---
2021-2022	120	---	120	60	60	---	60	---	120	---	---
NNRG											
2019-2020	180	0	180	60	120	---	60	---	0	0	---
2020-2021	180	60	180	60	60	---	60	---	0	0	---
2021-2022	180	60	180	30	30	---	30	---	60	60	---
LRDS											
2019-2020	180	120	120	180	180	---	60	---	0	0	---
2020-2021	180	180	120	180	180	---	60	---	60	120	---
2021-2022	180	180	120	180	120	---	60	---	60	180	---

Table 4: Program-wise intake in three consecutive academic years

5. Weight Vectors of each Criteria

To find the weight vectors of each criterion Geometric Mean Method (GMM) has been applied. The data furnished below is fetched from inputs collected from the major stake holders like students, parents, alumni and employers. Table 5 to Table 10 shows the measurement of the weight vectors of Criteria 1 to 6. Table 11 displays the weight vectors of all Criteria and Table 12 presents average weights of each Alternative with respect to all Criteria.

	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1/2	2	3	5	7	2.1720
AI&ML	2	1	3	4	5	7	3.0717
IOT/CS	1/2	1/3	1	2	3	5	1.3076
Mech	1/3	1/4	1/2	1	2	3	0.7937
PE	1/5	1/5	1/3	1/2	1	2	0.4869
EIE	1/7	1/7	1/5	1/3	1/2	1	0.2965

Table 5: Weights of C₁ (Scope of Employment)

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	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1	2	4	3	5	2.2209
AI&ML	1	1	2	3	3	4	2.0396
IOT/CS	1/2	1/2	1	2	2	2	1.1224
Mech	1/4	1/3	1/2	1	1	2	0.6609
PE	1/3	1/3	1/2	1	1	2	0.6933
EIE	1/5	1/4	1/2	1/2	1/2	1	0.4291

Table 6: Weights of C₂ (Right Employability Skills)

	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1/2	3	3	3	6	2.0800
AI&ML	2	1	2	4	5	7	2.8709
IOT/CS	1/3	1/2	1	3	3	5	1.3990
Mech	1/3	1/4	1/3	1	1	2	0.6177
PE	1/3	1/5	1/3	1	1	1/2	0.4723
EIE	1/6	1/7	1/5	1/2	2	1	0.4101

Table 7: Weights of C₃ (Higher pay package)

	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1/2	2	4	4	7	2.1955
AI&ML	2	1	3	5	5	8	3.2598
IOT/CS	1/2	1/3	1	2	2	4	1.1775
Mech	1/4	1/5	1/2	1	1	3	0.6493
PE	1/4	1/5	1/2	1	1	1/3	0.4502
EIE	1/7	1/8	1/4	1/3	3	1	0.4057

Table 8: Weights of C₄ (Wide career options and career stability)

	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1	1	5	5	7	2.3650
AI&ML	1	1	1	5	5	7	2.3650
IOT/CS	1	1	1	5	5	7	2.3650
Mech	1/5	1/5	1/5	1	1	1	0.4472
PE	1/5	1/5	1/5	1	1	1	0.4472
EIE	1/7	1/7	1/7	1	1	1	0.3779

Table 9: Weights of C₅ (Freedom to work from anywhere)

	AI&DS	AI&ML	IOT/CS	Mech	PE	EIE	Weight Vector
AI&DS	1	1	1	5	5	7	2.3650
AI&ML	1	1	1	7	7	5	2.5014

IOT/CS	1	1	1	6	6	4	2.2894
Mech	1/5	1/7	1/6	1	1	1	0.4101
PE	1/5	1/7	1/6	1	1	1	0.4101
EIE	1/7	1/5	1/4	1	1	1	0.4388

Table 10: Weights of C_6 (Applications in all the fields)

	C_1	C_2	C_3	C_4	C_5	C_6	Priority vector
C_1	1	2	2	1	1/2	1/3	0.9346
C_2	1/2	1	1/2	1/2	1/3	1/4	0.4673
C_3	1/2	2	1	1/2	1/3	1/4	0.5887
C_4	1	2	2	1	1/3	1/3	0.8735
C_5	2	3	3	3	1	1/2	1.7320
C_6	3	4	4	3	2	1	2.5697

Table 11: Weight Vectors of all Criteria

	C_1	C_2	C_3	C_4	C_5	C_6	Average weight
A_1	2.1720	2.2209	2.0800	2.1955	2.3650	2.3650	2.2307
A_2	3.0717	2.0396	2.8709	3.2598	2.3650	2.5014	2.6506
A_3	1.3076	1.1224	1.3990	1.1775	2.3650	2.2894	1.5351
A_4	0.7937	0.6609	0.6177	0.6493	0.4472	0.4101	0.5813
A_5	0.4869	0.6933	0.4723	0.4502	0.4472	0.4101	0.4859
A_6	0.2965	0.4291	0.4101	0.4057	0.3779	0.4388	0.3898

Table 12: Average weights of each Alternative with respect to all Criteria

Following are the 3-D clustered column charts representing the weights of Alternatives with respect to Criteria. Figure 1 to 6 will show the graphical representation of all the criteria i.e. criteria 1 to criteria 6.

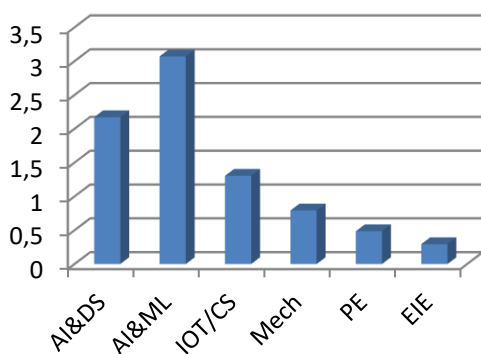


Figure 1: Weights of alternatives with respect to C_1

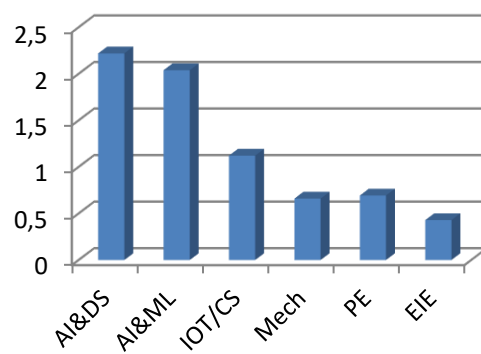


Figure 2: Weights of alternatives with respect to C_2

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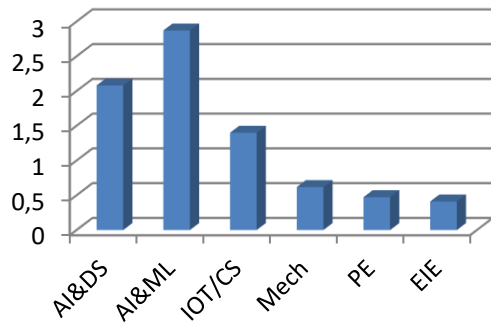


Figure 3: Weights of alternatives with respect to C_3

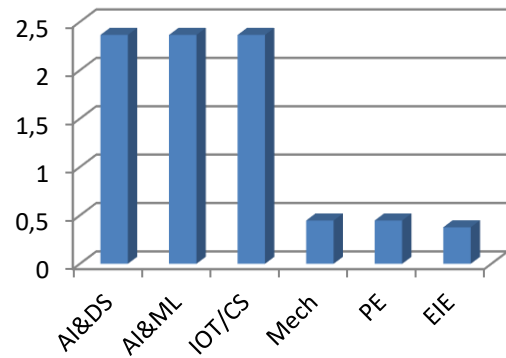


Figure 5: Weights of alternatives with respect to C_5

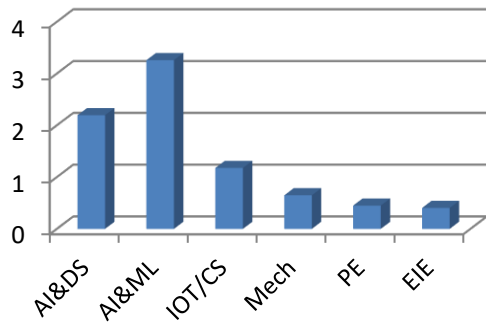


Figure 4: Weights of alternatives with respect to C_4

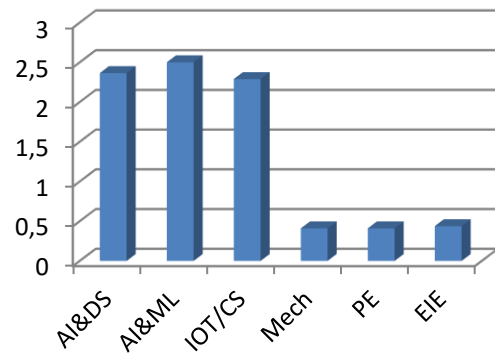


Figure 6: Weights of alternatives with respect to C_6

Table 13 indicates the ranking and weights of alternatives which is further graphically presented through bar chart(Figure 7).

S.No	Alternatives	Weights	Ranks
1	AI&DS	2.2307	2
2	AI&ML	2.6506	1
3	IOT/CS	1.5351	3
4	Mech	0.5813	4
5	PE	0.4859	5
6	EIE	0.3898	6

Table 13: Ranking and Weights of Alternatives

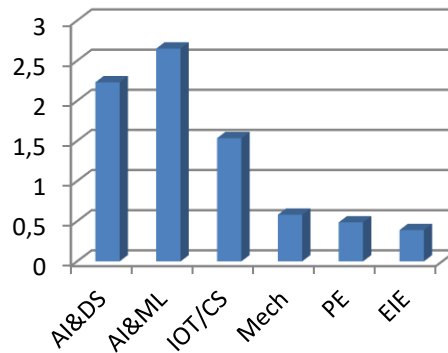


Figure 7: Ranks of Alternatives

6. Conclusions

According to the ranking of the Alternative weights, Artificial Intelligence & Machine Learning branch is the most important branch in the field of engineering education, followed by Artificial Intelligence & Data Science, Internet of Things, Mechanical Engineering, Production Engineering and finally Electronics and Instrumentation Engineering. The authors are of the opinion that the technological disruptions appear to be more powerful in the post-Covid scenario. This is one of the potential reasons behind the current change in the employment scenario, affecting the equilibrium of the workforce. The stakeholders as well as the government policy makers must be aware of the quick transition in the educational field and thus make judicious plans where the boom of a particular field does not create a vacuum or dearth of skilled labourers in another field especially in the core, non-programming branches of engineering education.

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Strong forms of b-continuous multifunctions

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Abstract

In this paper we have introduced strong forms of b-continuous multifunctions namely $b^\#$ -multicontinuity and $^*b$ -multicontinuity and studied their properties and characterizations. Also investigate the relationship with other type of functions with suitable examples.

Keywords: b-open, multi-function, $u.b^\#$ -c, $u.^*b$ -c

2010 AMS subject classification: 54C05, 54C08, 54C60^{**}.

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1. Introduction

Recently topologists concentrate their research in several types of continuous multi functions. A weak form of b-continuous multifunctions was studied in [4]. The variations of multi continuity were discussed in [5]. The weak and strong forms of continuity of multi functions were introduced in [6]. Certain properties of topological spaces preserved under multivalued continuous mappings were investigated in [7]. Certain strong forms of mixed continuous multi functions were characterized in [8] and the upper and lower β -continuous multi functions were studied in [11]. The notions of $b^\#$ -continuity and $*b$ -continuity were respectively discussed and studied in [9] and [3].

In this paper we have introduced strong forms of b-continuous multifunctions namely $b^\#$ -multicontinuity and $*b$ -multicontinuity and also studied their properties and characterizations with suitable examples.

2. Preliminaries

Throughout this paper it is assumed that X and Y are non-empty sets and τ and σ are topologies on X and Y respectively and τ' and σ' denote the collections of closed sets in X and Y respectively. The notation $P: X \Rightarrow Y$ is used for a multivalued function. For the notations in multifunction theory, the reader may consult (Thangavelu, Premakumari, 2015). We use the following abbreviations and notations.

“continuous” = “c”, “upper continuous” = “u.c” and “lower continuous” = “l.c”. Further

$V \in (\sigma, x, P(x), \subseteq) \Rightarrow V \in \sigma, x \in X \text{ and } P(x) \subseteq V.$

$U \in [\tau, x, P, V, \subseteq] \Rightarrow U \in \tau, x \in U \text{ and } P(U) \subseteq V.$

$V \in (\sigma, x, P(x), \emptyset) \Rightarrow V \in \sigma, x \in X \text{ and } P(x) \cap V \neq \emptyset.$

$U \in [\tau, x, P, V, \emptyset] \Rightarrow U \in \tau, x \in U \text{ and } P(u) \cap V \neq \emptyset \forall u \in U.$

$\chi \in \{b^\#, *b\}.$

Definition 2.1. The set A is called β (resp. b , $*b$)-open[1] (resp.[2], resp.[3]) if $A \subseteq Cl(Int(Cl(A)))$ (resp. $Cl(Int(A)) \cup Int(Cl(A))$, $Cl(Int(A)) \cap Int(Cl(A))$ and $b^\#$ -open [9,10] if $A = Cl(Int(A)) \cup Int(Cl(A))$. The complements of β (resp. b , $*b$, $b^\#$)-open sets are β (resp. b , $*b$, $b^\#$)-closed sets.

Lemma 2.2. The set B is

- (i) χ -open $\Rightarrow$ b-open
- (ii) open $\Rightarrow$ b-open
- (iii) b-open $\Rightarrow$ β -open

Definition 2.3. The multifunction P is u.c [5,6,7] if $\forall V \in (\sigma, x, P(x), \subseteq), \exists U \in [\tau, x, P, V, \subseteq]$ and is l.c if $\forall V \in (\sigma, x, P(x), \emptyset), \exists U \in [\tau, x, P, V, \emptyset].$

Analogously $u.b$ -c [4] and $u.\beta$ -c [11] may be defined by replacing “ τ ” in $[\tau, x, P, V, \subseteq]$ respectively by “ $bO(X, \tau)$ ” and “ $\beta O(X, \tau)$ ”. Also $l.b$ -c [4] and $l.\beta$ -c [11] may be defined by replacing “ τ ” in $[\tau, x, P, V, \emptyset]$ by “ $bO(X, \tau)$ ” and “ $\beta O(X, \tau)$ ” respectively.

Definition 2.4. The multifunction P is c if P is $u.c$ and $l.c$. The notions b -c and β -c can be similarly defined.

3. χ -multi continuity where $\chi \in \{^*b, b^\#\}$

Definition 3.1. The multivalued function P is $u.b^\#$ -c (resp. $u.^*b$ -c) if $P^+(V)$ is $b^\#$ -open (resp. $^*b$ -open) $\forall V \in \sigma$.

Proposition 3.2. Consider the following statements.

- (i) P is $u.\chi$ -c.
- (ii) $P^-(B)$ is χ -closed $\forall B \in \sigma'$.
- (iii) $P^-(Cl(B))$ is χ -closed $\forall B \subseteq Y$.
- (iv) $P^+(Int(B))$ is χ -open $\forall B \subseteq Y$.

The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) always hold.

Proof: Suppose (i) holds. Let $B \in \sigma'$ that implies, $P^+(Y \setminus B)$ is χ -open so that $X \setminus P^-(B) = P^+(Y \setminus B)$ is χ -open that further shows that $P^-(B)$ is χ -closed. This proves (i) $\Rightarrow$ (ii).

Now we assume (ii). Let $V \in \sigma$ that implies by (ii), $P^-(Y \setminus V)$ is χ -closed so that $X \setminus P^+(V)$ is χ -closed that further shows that $P^+(V)$ is χ -open. This proves (ii) $\Rightarrow$ (i). Other implications follow easily.

Proposition 3.3. If P is $u.\chi$ -c then $\forall V \in (\sigma, x, P(x), \subseteq), \exists U \in [\chi O(X, \tau), x, P, V, \subseteq]$.

Proof: Let P be $u.\chi$ -c and $V \in (\sigma, x, P(x), \subseteq)$. Since $P(x) \subseteq V$, $x \in P^+(V)$. Since $P^+(V)$ is χ -open $\exists$ a χ -open set U with $x \in U \subseteq P^+(V)$. Clearly $U \in [\chi O(X, \tau), x, P, V, \subseteq]$.

Proposition 3.4. P is $u.\chi$ -c $\Rightarrow$ it is $u.b$ -c and $u.\beta$ -c.

Definition 3.5. The multifunction P is $l.b^\#$ -c (resp. $l.^*b$ -c) if $P^-(V)$ is $b^\#$ - (resp. $^*b$)-open $\forall V \in \sigma$.

Proposition 3.6. Consider the following statements.

- (i) P is $l.\chi$ -c.
- (ii) $P^+(B)$ is χ -closed $\forall B \in \sigma'$.
- (iii) $P^+(Cl(B))$ is χ -closed $\forall B \subseteq Y$.
- (iv) $P^-(Int(B))$ is χ -open $\forall B \subseteq Y$.

The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) always hold.

Proof: Suppose (i) holds. Let $B \in \sigma'$ that implies $P^-(Y \setminus B)$ is χ -open so that $X \setminus P^+(B)$ is χ -open that further shows that $P^+(B)$ is χ -closed. This proves (i) $\Rightarrow$ (ii).

Now we assume (ii). Let $V \in \sigma$ that implies by (ii), $P^+(Y \setminus V)$ is χ -closed so that $X \setminus P^-(V)$ is χ -closed that further shows that $P^-(V)$ is χ -open. This proves (ii) $\Rightarrow$ (i). The rest follows easily.

Proposition 3.7. If P is $l.\chi$ -c then $\forall V \in (\sigma, x, F(x), \emptyset), \exists U \in [\chi O(X, \tau), x, P, V, \emptyset]$.

Proof: Analogous to Proposition 3.3.

Proposition 3.8. P is $l.\chi$ -c $\Rightarrow$ it is $l.b$ -c and $l.\beta$ -c

Definition 3.9. P is $b^\#$ -c (resp. $*b$ -c) if it is $u.b^\#$ -c (resp. $u.*b$ -c) and $l.b^\#$ -c (resp. $l.*b$ -c).

The next proposition follows from previous definition, Proposition 3.2 and Proposition 3.6.

Proposition 3.10. Consider the following statements.

- (i) P is χ -continuous.
 - (ii) $P^+(V)$ and $P^-(V)$ are χ -open $\forall V \in \sigma$.
 - (iii) $P^+(B)$ and $P^-(B)$ are χ -closed $\forall B \in \sigma'$.
 - (iv) $P^+(Int(B))$ and $P^-(Int(B))$ are χ -open $\forall B \subseteq Y$.
 - (v) $P^+(Cl(B))$ and $P^-(Cl(B))$ are χ -closed $\forall B \subseteq Y$.
- The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) always hold.
The following diagrams always hold.

Diagram 3.11. Let $t=u$ or l .

- (i) $t.b^\#$ -c $\Rightarrow t.b$ -c $\Leftarrow t.*b$ -c.
- (ii) $t.c \Rightarrow t.b$ -c $\Rightarrow t.\beta$ -c.

Examples 3.12. In this section some examples are given to illustrate certain results in the third section.

Let $X = \{p, q, r, s\}$, $Y = \{1, 2, 3\}$, $\sigma = \{\emptyset, \{1\}, Y\}$,
 $\tau = \{\emptyset, \{r\}, \{q\}, \{q, r\}, \{p, q\}, \{p, q, r\}, \{q, r, s\}, X\}$.

(i) $F_1(p) = \{1, 2\}$, $F_1(q) = \{1, 3\}$, $F_1(r) = \{1\}$ and $F_1(s) = \{1\}$ then $F_1^+(\{1\}) = \{r, s\}$ is $b^\#$ -open so that F_1 is $u.b^\#$ -c.

(ii) If $F_2(p) = \{1, 2\}$, $F_2(q) = \{1\}$, $F_2(r) = \{1, 3\}$ and $F_2(s) = \{3\}$ then $F_2^+(\{1\}) = \{r\}$ is $*b$ -open that implies F_2 is $u.*b$ -c.

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(iii) If $F_3(p) = \{1\}$, $F_3(q) = \{1\}$, $F_3(r) = \{1, 2\}$ and $F_3(s) = \{1\}$ then $F_3^+(\{1\}) = \{p, q, s\}$ is b-open and β -open and hence F_3 is u.b-c and u. β -c. However F_3 is not u. χ -c as $F_3^+(\{1\}) = \{p, q, s\}$ is not χ -open.

(iv) If $F_4(p) = \{2\}$, $F_4(q) = \{3\}$, $F_4(r) = \{1, 2\}$ and $F_4(s) = \{1, 3\}$ then $F_4^-(\{1\}) = \{r, s\}$ is $b^\#$ -open that implies F_4 is $l.b^\#$ -c.

(v) If $F_5(p) = \{1, 2\}$, $F_5(q) = \{1, 3\}$, $F_5(r) = \{2\}$ and $F_5(s) = \{3\}$ then $F_5^-(\{1\}) = \{p, q\}$ is $\ast b$ -open and hence F_5 is $l.\ast b$ -c.

(vi) If $F_6(p) = \{2\}$, $F_6(q) = \{1, 2\}$, $F_6(r) = \{3\}$ and $F_6(s) = \{1, 3\}$ then $F_6^-(\{1\}) = \{q, s\}$ is b-open and β -open so that F_6 is $l.b$ -c and $l.\beta$ -c. However F_6 is not $l.\chi$ -c as $F_6^-(\{1\}) = \{q, s\}$ is not χ -open.

(vii) If $F_7(p) = Y$, $F_7(q) = \{1, 3\}$, $F_7(r) = \{1\}$ and $F_7(s) = \{1\}$ then $F_7^+(\{1\}) = \{r, s\}$ and $F_7^-(\{1\}) = X$ are $b^\#$ -open we see that F_7 is u. $b^\#$ -c and $l.b^\#$ -c and hence $b^\#$ -c.

(viii) If $G_1(p) = \{2\}$, $G_1(q) = \{1\}$, $G_1(r) = \{1, 2\}$ and $G_1(s) = \{3\}$ then $G_1^+(\{1\}) = \{q\}$ and $G_1^-(\{1\}) = \{q, r\}$ are $\ast b$ -open so that G_1 is u. $\ast b$ -c and $l.\ast b$ -c and hence $\ast b$ -c.

(ix) If $G_2(p) = \{1, 3\}$, $G_2(q) = \{1\}$, $G_2(r) = \{2\}$ and $G_2(s) = \{1\}$ then $G_2^+(\{1\}) = \{q, s\}$ and $G_2^-(\{1\}) = \{p, q, s\}$ are b-open and β -open we see that G_2 is b-c and β -c. However G_2 is not χ -c as $G_2^+(\{1\}) = \{q, s\}$ and $G_2^-(\{1\}) = \{p, q, s\}$ are not χ -open.

(x) If $G_3(p) = \{2, 3\}$, $G_3(q) = \{1\}$, $G_3(r) = \{1\}$ and $G_3(s) = \{1\}$ then $G_3(\{1\}) = \{q, r, s\}$ is open that implies G_3 is u.c. However G_3 is not u. $b^\#$ -c as $G_3^+(\{1\}) = \{q, r, s\}$ is not $b^\#$ -open. If $G_4(p) = \{2, 3\}$, $G_4(q) = \{1, 2\}$, $G_4(r) = \{1\}$ and $G_4(s) = \{1\}$ then $G_4^+(\{1\}) = \{r, s\}$ is $b^\#$ -open so that G_4 is u. $b^\#$ -c. However G_4 is not u.c as $G_4^+(\{1\}) = \{r, s\}$ is not open.

(xi) If $G_5(p) = \{2, 3\}$, $G_5(q) = \{1, 2\}$, $G_5(r) = \{1, 3\}$ and $G_5(s) = \{1\}$ then $G_5^-(\{1\}) = \{q, r, s\}$ is open we see that G_5 is $l.c$. However G_5 is not $l.b^\#$ -c as $G_5^+(\{1\}) = \{q, r, s\}$ is not $b^\#$ -open. If $G_6(p) = \{3\}$, $G_6(q) = \{2\}$, $G_6(r) = Y$ and $G_6(s) = \{1, 3\}$ then $G_6^-(\{1\}) = \{r, s\}$ is $b^\#$ -open we see that G_6 is $l.b^\#$ -c. However G_6 is not $l.c$ as $G_6^-(\{1\}) = \{r, s\}$ is not open.

(xii) If $F(p) = \{2, 3\}$, $F(q) = \{1\}$, $F(r) = \{1, 3\}$ and $F(s) = \{1, 2\}$ then $F^+(\{1\}) = \{q\}$ and $F^-(\{1\}) = \{q, r, s\}$ are open we see that F is u.c and $l.c$ so that it is c. However F is neither u. $b^\#$ -c nor $l.b^\#$ -c as $F^+(\{1\}) = \{q\}$ and $F^-(\{1\}) = \{q, r, s\}$ are not $b^\#$ -open. If $G(p) = \{3\}$, $G(q) = \{2, 3\}$, $G(r) = \{1\}$ and $G(s) = \{1\}$ then $G^+(\{1\}) = \{r, s\} = G^-(\{1\})$ is $b^\#$ -open we see that G is u. $b^\#$ -c and $l.b^\#$ -c so that it is $b^\#$ -c. However G is not c as $G^+(\{1\}) = \{r, s\} = G^-(\{1\})$ is not open.

4. Conclusions

The concepts of strong forms of b -continuous multifunctions namely $b^\#$ -multicontinuous and $*b$ -multicontinuous functions are suitable for future extension research.

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Forcing vertex square free detour number of a graph

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Abstract

Let G be a connected graph and S a square free detour basis of G . A subset $T \subseteq S$ is called a forcing subset for S if S is the unique square free detour basis of G containing T . A forcing subset for S of minimum order is a minimum forcing subset of G . The forcing square free detour number of G is $fdn_{\square_{fu}}(G) = \min\{fdn_{\square_{fu}}(S_u)\}$, where the minimum is taken over all square free detour bases S in G . In this paper, we introduce the forcing vertex square free detour sets. The general properties satisfied by these forcing subsets are discussed and the forcing square free detour number for a certain class of standard graphs are determined. We show that the two parameters $dn_{\square_{fu}}(G)$ and $fdn_{\square_{fu}}(G)$ satisfy the relationship $0 \leq fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$. Also, we prove the existence of a graph G with $fdn_{\square_{fu}}(G) = \alpha$ and $dn_{\square_{fu}}(G) = \beta$, where $0 \leq \alpha \leq \beta$ and $\beta \geq 2$ for some vertex u in G .

Keywords: forcing square free detour number; forcing vertex square free detour set; forcing vertex square free detour number.

2020 AMS subject classifications: 05C12, 05C69¹

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1 Introduction

In a simple connected graph $G = (V, E)$ of order at least two, a longest xy path is called an $x - y$ detour path for any two vertices x and y in G . A set S of vertices of G is a detour set of G if each vertex v of G lies on an $x - y$ detour path for some elements x and y in S . The minimum cardinality of a detour set of G is the detour number of G and is denoted by $dn(G)$. A subset T of a minimum detour set S of G is a forcing detour subset for S if S is the unique detour basis containing T . A forcing detour subset for S of minimum cardinality is a minimum forcing detour subset of S . The forcing detour number $fdn(S)$ of S , is the minimum cardinality of a forcing detour subset for S . The forcing detour number $fdn(G)$ of G , is $\min\{fdn(S)\}$, where the minimum is taken over all detour bases S in G . This notion of forcing detour number was initiated by Chartrand et al. [2003]. From then onwards there have been many relevant studies about the forcing detour concept in the past few decades. The connected detour parameter was defined by Santhakumaran and Athisayanathan [2009]. Also, Santhakumaran and Athisayanathan [2010] focussed on edge of a graph and introduced forcing edge detour number and the forcing weak edge detour number. Further results on connected detour number was studied by Ali and Ali [2019]. Titus and Balakrishnan [2016] concentrated on a vertex of a graph and compiled the findings on forcing vertex detour monophonic number arising from the chordless path. Ramalingam et al. [2016] extended this concept to triangle free and extracted forcing results. The square free detour concept was introduced by Rani and Pacifica [2022]. Though there are several novel forcing parameters, this paper aims at filling some significant research gaps and brings forth the results on the forcing vertex square free detour number of a graph. We determine the bounds for it and compare the relationship with the vertex square free detour number of a graph. These ideas have interesting application in channel assignment problem and in radio technologies. Moreover, this concept can be applied in security based community design. For the basic terminologies we refer to Chartrand et al. [1993].

2 Preliminaries

Theorem 2.1. *Let u be any vertex of a connected graph G .*

- (i) *Every end-vertex of G other than the vertex u (whether u is an end-vertex or not) belongs to every u -square free detour set.*
- (ii) *No cut-vertex of G belongs to any $dn_{\square_{fu}}$ -set.*

Theorem 2.2. *For any vertex u in a non-trivial connected graph G of order n , $1 \leq dn_{\square_{fu}}(G) \leq n - 1$.*

Theorem 2.3. *Let G be a connected graph.*

- (i) *If G is the complete graph K_n , then $dn_{\square_{fu}}(G) = 1$ for every vertex u in K_n .*
- (ii) *If G is the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$) with partitions X and Y , then*

$$dn_{\square_{fu}}(G) = \begin{cases} m-1 & \text{if } u \in X \\ n-1 & \text{if } u \in Y \end{cases}$$

- (iii) *If G is the cycle C_n , then $dn_{\square_{fu}}(G) = 1$ for every vertex u in C_n .*
- (iv) *For a wheel W_n ($n \geq 4$),*

$$dn_{\square_{fu}}(W_n) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil & \text{if } u \in K_1, n \geq 4 \\ 2 & \text{if } u \in C_{n-1}, n \geq 6 \\ 1 & \text{if } u \in C_{n-1}, 4 \leq n \leq 5 \end{cases}$$

3 Forcing vertex square free detour number of a graph

Even though every connected graph contains a vertex square free detour set, some connected graph may contain several vertex square free detour sets. For each minimum vertex square free detour set S_u in a connected graph there is always some subset T of S_u that uniquely determines S_u as the minimum vertex square free detour set containing T such forcing subsets are considered in this section.

Definition 3.1. *Let u be any vertex of a connected graph G and S_u be a $dn_{\square_{fu}}$ -set of G . A subset $F_u \subseteq S_u$ is called a u -forcing subset for S_u if S_u is the unique $dn_{\square_{fu}}$ -set consisting of F_u . The u -forcing subset for S_u of minimum cardinality is a minimum u -forcing subset of S_u . The forcing u -square free detour number of S_u , denoted by $fdn_{\square_{fu}}(S_u)$ is the order of a minimum u -forcing subset for S_u . The forcing u -square free detour number of G is $fdn_{\square_{fu}}(G) = \min fdn_{\square_{fu}}(S_u)$ where the minimum is considered over all $dn_{\square_{fu}}$ -sets S_u in G .*

Example 3.1. *For the graph G shown in Figure 1, the only $dn_{\square_{fu}}$ -sets are $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_4\}$ and $\{v_3, v_5\}$. Hence $fdn_{\square_{fu}}(G) = 2$. Also $dn_{\square_{fu}}(G) = 2$ and $fdn_{\square_{fu}}(G) = 1$ for $u = v_3$ and v_4 in G . Moreover v_5 and v_4 are the unique vertex square free detour sets for vertices v_2 and v_5 respectively and so $fdn_{\square_{fu}}(G) = 0$ for $u = v_2, v_5$.*

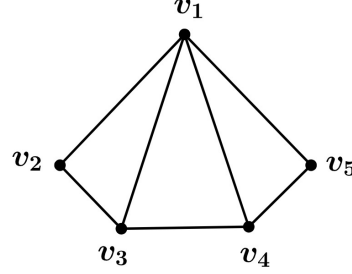


Figure 1: G

The following theorem follows from the definitions of vertex square free detour number and forcing vertex square free detour number of a graph G .

Theorem 3.1. *For any vertex u in a connected graph G , $0 \leq fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$.*

Proof. Let u be any vertex of G . From the definition of $fdn_{\square_{fu}}(G)$, we find that $fdn_{\square_{fu}}(G) \geq 0$, consider a $dn_{\square_{fu}}$ -set S_u in G . We have $fdn_{\square_{fu}}(G) = \min\{fdn_{\square_{fu}}(S_u) : S_u \text{ is a } dn_{\square_{fu}}\text{-set in } G\}$ and so $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$. Hence $0 \leq fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G)$. □

Now, we characterize the graph G for which the bounds in Theorem 3.1 are reached and the graph for which $fdn_{\square_{fu}}(G) = 1$.

Theorem 3.2. *Let u be any vertex of a connected graph G . Then*

- (i) $fdn_{\square_{fu}}(G) = 0$ if and only if G has a unique $dn_{\square_{fu}}$ -set,
- (ii) $fdn_{\square_{fu}}(G) = 1$ if and only if G has at least two $dn_{\square_{fu}}$ -sets one of which is a unique $dn_{\square_{fu}}$ -set containing any one of its elements,
- (iii) $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G)$ if and only if no $dn_{\square_{fu}}$ -set of G is the unique $dn_{\square_{fu}}$ -set containing any of its proper subsets.

Proof. (i) Let $fdn_{\square_{fu}}(G) = 0$. Then, $fdn_{\square_{fu}}(S_u) = 0$ where S_u is any $dn_{\square_{fu}}$ -set by definition and so the empty set is the minimum u -forcing subset for S_u . Since the empty set ϕ is a subset of every set we have S_u is the unique minimum u -forcing subset of G . The converse of this Theorem is obvious.

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- (ii) Let $fdn_{\square_{fu}}(G) = 1$. Then by (i), G has at least two $fdn_{\square_{fu}}$ -sets. Also, since $fdn_{\square_{fu}}(G) = 1$, there is a singleton subset F of a $dn_{\square_{fu}}$ -set S_u of G such that F is not a subset of any other $dn_{\square_{fu}}$ -set of G . Thus S_u is the unique $dn_{\square_{fu}}$ -set containing one of its elements. The converse is obvious.
- (iii) Let $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G)$. Then $fdn_{\square_{fu}}(S_u) = dn_{\square_{fu}}(G)$ for every $dn_{\square_{fu}}$ -set S_u in G . By Theorem 2.1, $dn_{\square_{fu}}(G) \geq 1$ and so $fdn_{\square_{fu}}(G) = 1$. Also by (i), G has at least two $dn_{\square_{fu}}$ -sets and hence the empty set ϕ is not a u -forcing subset of any $dn_{\square_{fu}}$ -set S_u of G is unique $dn_{\square_{fu}}$ -set which consists of its proper subsets.

□

Theorem 3.3. *Let $G = (V, E)$ be a connected graph and let S_u^* be the set of all u -square free detour vertices of G . Then $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G) - |S_u^*|$.*

Proof. Let S_u be any square free detour basis of G . Then $dn_{\square_{fu}}(S_u) = |S_u|$, $S_u^* \subseteq S_u$ and S_u is the unique square free detour basis containing $S_u - S_u^*$. Thus $fdn_{\square_{fu}}(G) \leq |S_u - S_u^*| = |S_u| - |S_u^*| = dn_{\square_{fu}}(G) - |S_u^*|$. □

Remark 3.1. *The bound in Theorem 3.3 is sharp. For the graph G given in Figure 2, $fdn_{\square_{fv_1}}(G) = 0$, $|S_{v_1}^*| = 2$ and $dn_{\square_{fv_1}}(G) = 2$. Also, the inequality in Theorem 3.3, can be strict. For the graph G given in Figure 1, $fdn_{\square_{fv_3}}(G) = 1$, $|S_{v_3}^*| = 0$ and $dn_{\square_{fv_3}}(G) = 2$. Thus $fdn_{\square_{fv_3}}(G) < dn_{\square_{fv_3}}(G) - |S_{v_3}^*|$.*

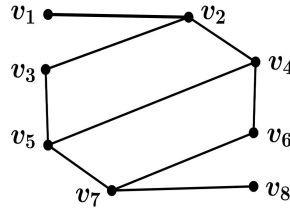


Figure 2: G

Theorem 3.4. *Let G be a connected graph and let $\mathfrak{S}$ be the set of relative complements of the minimum u -forcing subsets in their respective minimum u -square free detour sets in G . Then $\cap_{F \in \mathfrak{S}} F$ is the set of all u -square free detour vertices of G .*

Proof. Let S_u^* be the set of all u -square free detour vertices of G . We claim that $S_u^* \subseteq \cap_{F \in \mathfrak{S}} F$. Let $x \in S_u^*$. Then x is a u -square free detour vertex of G so that x belongs to every u -square free detour set S_u of G . Let $T \subseteq S_u$ be any minimum u -forcing subset for any u -square free detour basis S_u of G . We claim that $x \notin T$.

If $x \in T$, then $T' = T - \{x\}$ is a proper subset of S_u is the unique u -square free detour containing T' so that T' is a u -forcing subset for S_u with $|T'| < |T|$, which is a contradiction to T is a minimum u -forcing subset for S_u . Thus $x \notin T$ and so $x \in F$, where F is the relative complement of T in S_u . Hence $x \in \cap_{F \in \mathfrak{S}} F$ so that $S_u^* \subseteq \cap_{F \in \mathfrak{S}} F$.

Conversely, let $x \in \cap_{F \in \mathfrak{S}} F$. Then x belongs to the relative complement of T in S_u for every T for every S_u such that $T \subset S_u$, where T is a minimum u -forcing subset for S_u . Since F is the relative complement of T in S_u , $F \subset S_u$ and so $x \in S_u$ for every S_u so that x is a u -square free detour vertex of G . Thus $x \in S_u^*$ and so $\cap_{F \in \mathfrak{S}} F \subset S_u^*$. Hence $S_u^* \cap_{F \in \mathfrak{S}} F$. □

Theorem 3.5. *Let u be any vertex of a connected graph G and let S_u be any $dn_{\square_{fu}}$ -set of G . Then*

- (i) *No u -square free detour vertex belongs to any minimum u -forcing subset of S_u .*
- (ii) *No cut-vertex of G belongs to any minimum u -forcing subset of S_u .*

Proof. (i) The proof follows from the first part of Theorem 3.4.

- (ii) Since any minimum u -forcing subset of S_u is a subset of S_u , the proof follows from Theorem 2.1(ii). □

Corollary 3.1. *Let u be any vertex of a connected graph G . If G contains l end-vertices, then $fdn_{\square_{fu}}(G) \leq dn_{\square_{fu}}(G) - l + 1$.*

Proof. This follows from Theorems 2.1(i) and 3.5(i). □

Remark 3.2. *The bound in Corollary 3.1 is sharp. For a tree T with l end-vertices, $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G) - l + 1$ for any end-vertex u in T .*

Theorem 3.6. *Let G be any connected graph of order n . Then*

- (i) *If G is a tree with l end-vertices, then $fdn_{\square_{fu}}(G) = 0$ for every vertex l in G .*
- (ii) *If G is the complete graph K_n , then*

$$fdn_{\square_{fu}}(G) = \begin{cases} 0 & \text{if } n = 4 \\ n & \text{otherwise} \end{cases}$$

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(iii) If G is the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$), with partitions X and Y , $dn_{\square_{fu}}(K_{m,n}) = 0$ for every vertex u in $K_{m,n}$.

(iv) If G is the cycle C_n , then

$$fdn_{\square_{fu}}(G) = \begin{cases} 0 & \text{if } n = 4 \\ 1 & \text{otherwise} \end{cases}$$

(v) If G is the wheel W_n , then

$$fdn_{\square_{fu}}(W_n) = \begin{cases} 0 & \text{if } n = 5, u \in K_1 \\ 1 & \text{if } n = 4, u \in W_n \text{ and } n \geq 6, u \in C_{n-1} \\ \left\lceil \frac{n-1}{3} \right\rceil & \text{if } n \geq 6, u \in K_1. \end{cases}$$

Proof. (i) By the fact that $dn_{\square_{fu}}(G) = l - 1$ or $dn_{\square_{fu}}(G) = l$ when u is an end-vertex or not an end-vertex. Since the set of all end-vertices of a tree is the unique $dn_{\square_{fu}}$ -set, the result follows from Theorem 3.2(i) that $fdn_{\square_{fu}}(G) = 0$.

(ii) By Theorem 2.3(i) for the complete graph K_4 , S_u consists of the antipodal vertex of u . Since the set of antipodal vertex is unique for K_4 , the result follows from Theorem 3.2(i) that $fdn_{\square_{fu}}(G) = 0$. For K_n ($n \neq 4$) it follows from Theorem 2.3(i), that S_u consists of exactly one vertex of $V - \{u\}$. Thus there exist $n - 1$ distinct vertices other than u in K_n . Then the result follows from Theorem 3.2(ii) that $fdn_{\square_{fu}}(G) = 1$.

(iii) By Theorem 2.3(ii), for $K_{m,n}$ ($2 \leq m \leq n$) with partitions X, Y with $|X| = m$ and $|Y| = n$, we have $dn_{\square_{fu}}(G) = m - 1$ or $dn_{\square_{fu}}(G) = n - 1$ according to whether the vertex u lies in X or Y . Since the $dn_{\square_{fu}}$ -set S_u is unique in both the cases, the result follows from Theorem 3.2(i) that $fdn_{\square_{fu}}(G) = 0$.

(iv) By Theorem 2.3(iii) for C_4 , $dn_{\square_{fu}}$ -set S_u consists of the antipodal vertex of u . Thus we observe that S_u is unique and so $fdn_{\square_{fu}}(G) = 0$ by Theorem 3.2(i). By Theorem 2.3(iii), for an even cycle C_n ($n \neq 4$), S_u consists of exactly one vertex which is antipodal or adjacent to u . Also, for an odd cycle C_n ($n \geq 3$) $dn_{\square_{fu}}$ -set S_u contains exactly one vertex which is adjacent to u . Since there exist two adjacent vertices for an odd cycle and in addition an antipodal vertex for an even cycle we have $fdn_{\square_{fu}}(G) = 1$.

(v) By Theorem 2.3(iv), for $W_n = K_1 + C_{n-1}$ ($n \geq 5$) a $dn_{\square_{fu}}$ -set S_u consists of $\left\lceil \frac{n-1}{3} \right\rceil$ vertices of the rim C_{n-1} of W_n , where u is a central vertex of W_n that is in K_1 called as hub. Thus there exist three different $dn_{\square_{fu}}$ -sets with $\left\lceil \frac{n-1}{3} \right\rceil$ vertices. Therefore, from Theorem 3.2(ii) the result follows that $fdn_{\square_{fu}}(G) = \left\lceil \frac{n-1}{3} \right\rceil$ for the central vertex of W_n ($n \geq 5$).

When u is a vertex on C_{n-1} of W_n by Theorem 2.3(iv), S_u contains exactly one adjacent or antipodal vertex on C_{n-1} with the hub of the wheel, according as $n-1$ is odd or $n-1$ is even. Thus there are two adjacent vertices for a vertex on an odd cycle and an antipodal vertex besides two adjacent vertices for a vertex on even cycle. Hence the result follows from Theorem 3.2(ii) that $fdn_{\square_{fu}}(G) = 1$ for $u \in C_{n-1}$ ($n \geq 6$).

By Theorem 2.3(iv) for W_4 , S_u consists of exactly one adjacent vertex for every $u \in W_4$. Thus there are $n-1$ such $dn_{\square_{fu}}$ -sets, for the vertices of W_4 are adjacent to each other. Hence by Theorem 3.2(ii), we have $fdn_{\square_{fu}}(G) = 1$.

Also, for W_5 , S_u contains two antipodal vertices of C_4 where u is the central vertex of W_5 . Since there exist two different S_u with distinct pair of antipodal vertices of C_4 , from Theorem 3.2(iii), the result follows that $fdn_{\square_{fu}}(G) = dn_{\square_{fu}}(G) = 2$. Furthermore, when u is a vertex on the rim of W_5 , S_u consists of the antipodal vertex of u . Since there is only one antipodal vertex for any vertex u on C_4 of W_5 , we have unique S_u for all vertices on the rim of W_5 . Hence the result follows from Theorem 3.2(i) that $fdn_{\square_{fu}}(G) = 0$. $\square$

Theorem 3.7. For every pair α, β of positive integers with $0 \leq \alpha \leq \beta$ and $\beta \geq 2$, there exists a connected graph G with $fdn_{\square_{fu}}(G) = \alpha$ and $dn_{\square_{fu}}(G) = \beta$ for some vertex u in G .

Proof. We consider two cases.

Case 1: Let $\alpha = 0$. Let G be any tree with $\beta + 1$ end-vertices. Then for an end-vertex u in G , $fdn_{\square_{fu}}(G) = 0$ by Theorems 3.2(i) and 3.5(i).

Case 2: Let $\alpha \geq 1$. For each i ($1 \leq i \leq \alpha$), let D_6^i be a Dutch Windmill graph consisting of i copies of C_6 : $u, p_i, q_i, r_i, s_i, t_i, u$. Let H be a graph obtained by adding α new vertices $r'_1, r'_2, \dots, r'_\alpha$ and joining each r'_i ($1 \leq i \leq \alpha$) to both the vertices q_i and s_i of D_6^i ($1 \leq i \leq \alpha$).

Forcing vertex square free detour number of a graph

Let $K_{1,\beta-\alpha}$ be the star with common vertex w_o and let $W = \{w_1, w_2, \dots, w_{\beta-\alpha}\}$ be the set of all end-vertices of $K_{1,\beta-\alpha}$. Let G be the required graph produced by identifying the vertex u of D_6^i with the common vertex w_o of $K_{1,\beta-\alpha}$ as pictured in Figure 3.

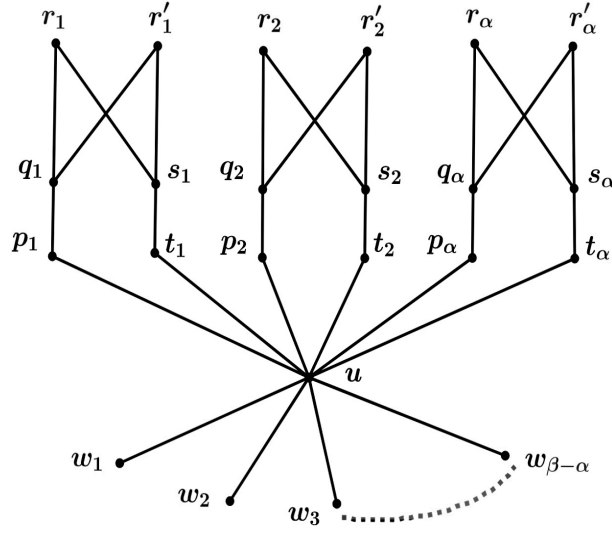


Figure 3: G

First we show that $dn_{\square_{fu}}(G) = \beta$ for some vertex u in G . By the fact that every $dn_{\square_{fu}}$ -set of G contains W and exactly one vertex from each C_6 of $D_6^i (1 \leq i \leq \alpha)$. Then $dn_{\square_{fu}}(G)(\beta - \alpha) + \alpha$ and so $dn_{\square_{fu}}(G) = \beta$. Let $S_u = W \cup \{x_1, x_2, \dots, x_\alpha\}$, where $x_j = p_j$ or t_j of $C_6^i (1 \leq i \leq \alpha)$. Clearly S_u is a $dn_{\square_{fu}}$ -set of G and so $(G) \leq |S_u| = (\beta - \alpha) + \alpha = \beta$. Thus $dn_{\square_{fu}}(G) = \beta$.

Next we show that $fdn_{\square_{fu}}(G) = \alpha$. Since $dn_{\square_{fu}}(G) = \beta$, we find that every $dn_{\square_{fu}}$ -set of G consists of W and exactly one vertex from each C_6 of $D_6^i (1 \leq i \leq \alpha)$. Let $F \subseteq S_u$ be any minimum u -forcing subset of S_u . Then By Theorem 3.5(ii), $F \subseteq S_u - W$ and so $|F| \leq \alpha$. If $|F| < \alpha$ then there is a vertex y_j of $C_6^i (1 \leq i \leq \alpha)$ distinct from $x_j (1 \leq i, j \leq \alpha)$. Then $S'_u = (S_u - x_j) \cup y_j$ is also a minimum u -forcing subset containing F such that $x_j \notin F$. Thus S_u is not a unique u -square free detour set which consists of F and so F is not a minimum u -forcing subset of S_u . Thus $fdn_{\square_{fu}}(G) = \alpha$.

□

4 Conclusion

In this paper, we computed the forcing vertex square free detour number of some standard graphs. We discussed the characteristics of the forcing vertex square free detour sets. Also, the relationship between the vertex square free detour number and the forcing vertex square free detour number has been exhibited. In future, this concept can be extended to edge related parameter. To derive similar results in the context of some other variants of detour number is the open area of research.

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Solving Fully Fuzzy Linear Systems with Triangular and Pentagonal Fuzzy Numbers

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Abstract

In this paper algorithms are proposed to solve fully fuzzy linear system (FFLS) using triangular fuzzy number and pentagonal fuzzy number by converting the system into block system of linear equations known as associated fuzzy system. The method is illustrated by numerical examples.

Keywords: Fully fuzzy linear systems, Triangular fuzzy number, Pentagonal fuzzy number, fuzzy systems.

AMS Subject Classification: 47H10[‡]

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1 Introduction

In many real-life applications, fuzzy sets (Zadeh 1965) play an important role instead of crisp approach. Many applications have made use of the fuzzy concept including computer applications, communication such as. solving fuzzy linear system of equations (FLS) is one field of applied mathematics that shows in various applications when the coefficients of system can be taken by fuzzy numbers. The solutions of FLS were first introduced by Friedman et al. (1998) by proposing a method for solving $n \times n$ system which includes crisp coefficients matrix with an arbitrary fuzzy number vector as right-hand side. A fuzzy linear system where all the elements of the system are fuzzy numbers is called a fully fuzzy linear system (FFLS). In this paper we use the multiplication operation on matrices to solve fully fuzzy linear system. In section 2, we define some of definitions and basics of triangular and pentagonal fuzzy numbers. In section 3, an algorithm to solve triangular fuzzy system of equations and pentagonal fuzzy system of equations is discussed in section 4. In section 5, numerical examples are solved. In section 6, conclusion is given.

2 Preliminaries

Definition 2.1. The characteristic function μ_A of a crisp set A of X assigns a value either 0 or 1 to each member in X . This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e., $\mu_{\tilde{A}} : X \rightarrow [0,1]$. The assigned value indicates the membership grade of the element in the set A . The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ is called fuzzy set.

Definition 2.2. A triangular fuzzy number $\tilde{A}$ is fuzzy number whose membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 + \frac{x-a}{\epsilon}, & \text{if } a - \epsilon \leq x \leq a \\ 1 - \frac{x-a}{\epsilon}, & \text{if } a \leq x \leq a + \delta \\ 0, & \text{otherwise} \end{cases}$$

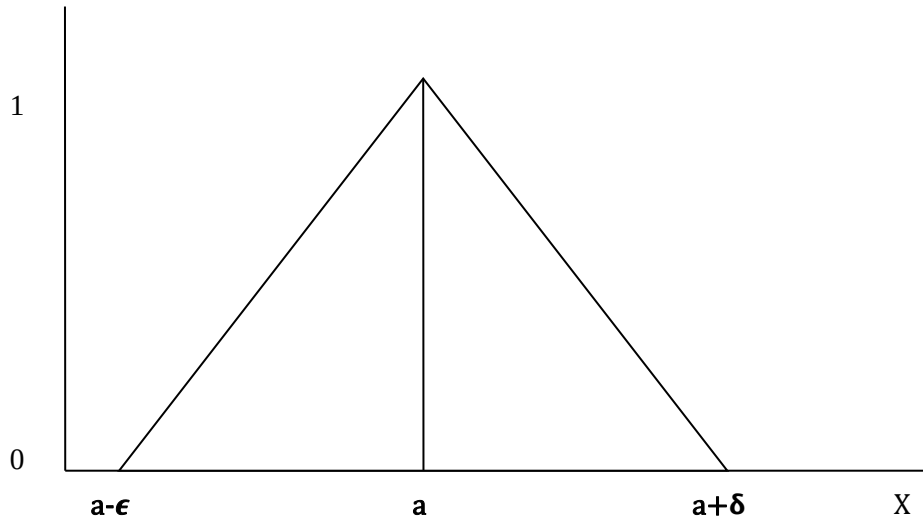


Figure 1.1 Triangular Fuzzy Number

and simply denoted by $\tilde{A} = (a, \epsilon, \delta)$

Definition 2.3. A fuzzy number $\tilde{A}$ is positive $\mu_{\tilde{A}}(x) = 0$ for every $x \leq 0$.

Definition 2.4. Let $\tilde{A} = (a, \epsilon_1, \delta_1)$ and $\tilde{B} = (b, \epsilon_2, \delta_2)$ are two triangular fuzzy numbers. Then addition of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\begin{aligned}\tilde{A} \oplus \tilde{B} &= (a, \epsilon_1, \delta_1) \oplus (b, \epsilon_2, \delta_2) \\ &= (a + b, \epsilon_1 + \epsilon_2, \delta_1 + \delta_2)\end{aligned}$$

Definition 2.5. Let $\tilde{A} = (a, \epsilon_1, \delta_1)$ and $\tilde{B} = (b, \epsilon_2, \delta_2)$ are two triangular fuzzy numbers. Then product of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \otimes \tilde{B} = (a, \epsilon_1, \delta_1) \otimes (b, \epsilon_2, \delta_2) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$$

$$\text{Where, } \mathcal{A} = ab, \mathcal{B} = a\epsilon_2 + b\epsilon_1 - \epsilon_1\epsilon_2, \mathcal{C} = a\delta_2 + b\delta_1 - \delta_1\delta_2$$

Definition 2.6. A fuzzy number $\tilde{A} = (a, b, c, d, e)$ is said to be a pentagonal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x < a - \varepsilon_2 \\ \frac{1}{2} \left(\frac{x - a + \varepsilon_1}{\varepsilon_1 - \varepsilon_2} \right), & \text{if } a - \varepsilon_1 \leq x \leq a - \varepsilon_2 \\ 1 + \frac{1}{2} \left(\frac{x - a}{\varepsilon_2} \right), & \text{if } a - \varepsilon_2 \leq x \leq a \\ 1, & \text{if } x = a \\ 1 - \frac{1}{2} \left(\frac{x - a}{\delta_1} \right), & \text{if } a \leq x \leq a + \delta_1 \\ \frac{1}{2} \left(\frac{x - a + \delta_2}{\varepsilon_1 - \delta_2} \right), & \text{if } a + \delta_1 \leq x \leq a + \delta_2 \\ 0, & \text{if } x > a + \delta_2 \end{cases}$$

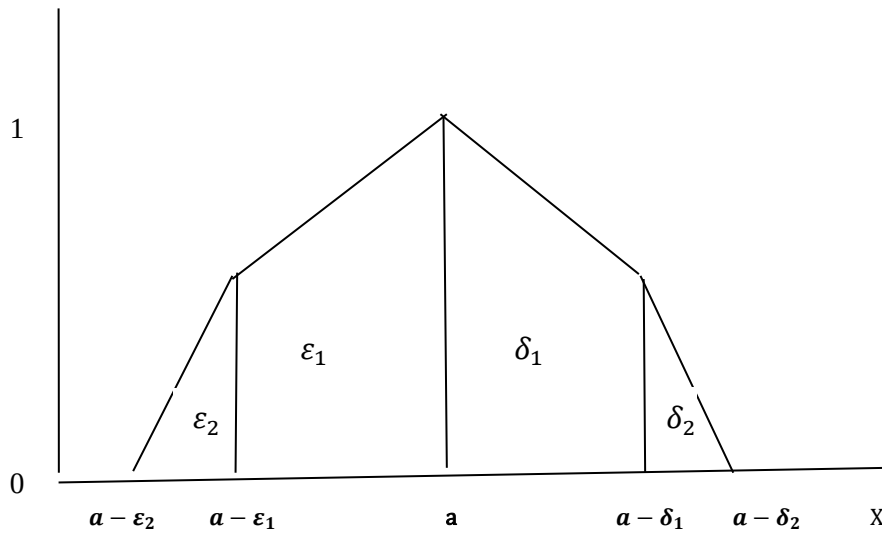


Figure 2.2 Pentagonal Fuzzy Number

Definition 2.7. Let $\tilde{A} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2)$ are two pentagonal fuzzy numbers. Then addition of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\begin{aligned} \tilde{A} \oplus \tilde{B} &= (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1) \oplus (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2) \\ &= (a + b, \varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2, \gamma_1 + \gamma_2, \theta_1 + \theta_2) \end{aligned}$$

Definition 2.8. Let $\tilde{A} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2)$ are two pentagonal fuzzy numbers. Then product of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \otimes \tilde{B} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1) \otimes (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2) = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E})$$

where $\mathcal{A} = ab$, $\mathcal{B} = a\varepsilon_2 + b\varepsilon_1 - \varepsilon_1\varepsilon_2$, $\mathcal{C} = a\delta_2 + b\delta_1 - \delta_1\delta_2$
 $\mathcal{D} = a\gamma_2 + b\gamma_1 - \gamma_1\gamma_2$, $\mathcal{E} = a\theta_2 + b\theta_1 - \theta_1\theta_2$

3 Triangular fuzzy linear systems

Consider the fully fuzzy triangular linear system $AX = B$. where $A = (a_{ij})$ is the coefficient matrix, $X = (x_1, x_2, x_3, \dots, x_j)$ is the unknown variable, $B = (b_1, b_2, b_3, \dots, b_j)$ is the constant vector. Let $a_{ij} = (a_{ij}, \varepsilon_{ij}, \delta_{ij})$, $b_j = (b_j, g_j, h_j)$ and $x_j = (x_j, y_j, z_j)$ be triangular fuzzy numbers by $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

Then $AX = B$ ----- (1)
 is rewritten in the form

$$(a_{ij}, \varepsilon_{ij}, \delta_{ij}) \otimes (x_j, y_j, z_j) = (b_j, g_j, h_j)$$

$$(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum x_j\varepsilon_{ij} - \sum y_j\varepsilon_{ij}, \sum a_{ij}z_j + \sum x_j\delta_{ij} - \sum z_j\delta_{ij}) = (b_j, g_j, h_j) \text{----} (2)$$

$$(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum \varepsilon_{ij}(x_j - y_j), \sum a_{ij}z_j + \sum \delta_{ij}(x_j - z_j)) = (b_j, g_j, h_j) \text{-----} (3)$$

Comparing on both sides, the system is rewritten as

$$Sx = b \text{-----} (4)$$

$$Sy + T(x - y) = g \text{-----} (5)$$

$$Sz + Q(x - z) = h \text{-----} (6)$$

Where, $S = \sum a_{ij}$, $T = \sum \varepsilon_{ij}$, $Q = \sum \delta_{ij}$

Equation (4) to (6) can be rewritten as a $3n \times 3n$ block system linear equation. This system is known as the associated system of fuzzy system and is written in matrix representation as follows,

$$\begin{pmatrix} S & 0 & 0 \\ T & (S - T) & 0 \\ Q & 0 & (S - Q) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ g \\ h \end{pmatrix} \text{-----} (7)$$

The solution of the above matrix is

$$\left. \begin{aligned} x &= S^{-1}b \\ y &= (T^{-1}S - I)^{-1}(T^{-1}h - Ix) \\ z &= (Q^{-1}S - I)^{-1}(Q^{-1}h - Ix) \end{aligned} \right\} \text{-----} (8)$$

4 Pentagonal fuzzy linear systems

Consider the fully fuzzy pentagonal linear system $AX = B$. where $A = (a_{ij})$ is the coefficient matrix, $X = (x_1, x_2, x_3, \dots, x_j)$ is the unknown variable, $B = (b_1, b_2, b_3, \dots, b_j)$ is the constant vector. $a_{ij} = (a_{ij}, \varepsilon_{ij}, \delta_{ij}, \gamma_{ij}, \theta_{ij})$, $b_j = (b_j, g_j, h_j, k_j, l_j)$ and

$x_j = (x_j, y_j, z_j, w_j, v_j)$ are pentagonal fuzzy numbers by $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

Then $AX = B$ ----- (9)

is rewritten in the form

$$(a_{ij}, \varepsilon_{ij}, \delta_{ij}, \gamma_{ij}, \theta_{ij}) \otimes (b_j, g_j, h_j, k_j, l_j) = (b_j, g_j, h_j, k_j, l_j) \text{----- (10)}$$

$$(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum x_j\varepsilon_{ij} - \sum y_j\varepsilon_{ij}, \sum a_{ij}z_j + \sum x_j\delta_{ij} - \sum z_j\delta_{ij}, \sum a_{ij}w_j + \sum x_j\gamma_{ij} - \sum w_j\gamma_{ij}, \sum a_{ij}v_j + \sum x_j\theta_{ij} - \sum v_j\theta_{ij}) = (b_j, g_j, h_j, k_j, l_j) \text{----- (11)}$$

$$(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum \varepsilon_{ij}(x_j - y_j), \sum a_{ij}z_j + \sum \delta_{ij}(x_j - z_j), \sum a_{ij}w_j + \sum \gamma_{ij}(x_j - w_j), \sum a_{ij}v_j + \sum \theta_{ij}(x_j - v_j)) = (b_j, g_j, h_j, k_j, l_j) \text{----- (12)}$$

Comparing on both sides, the system is rewritten as

$$\left. \begin{aligned} Sx &= b \\ Sy + T(x - y) &= g \\ Sz + Q(x - z) &= h \\ Sw + R(x - w) &= k \\ Sv + H(x - v) &= l \end{aligned} \right\} \text{----- (13)}$$

where,

$$\left. \begin{aligned} S &= \sum a_{ij} \\ T &= \sum \varepsilon_{ij} \\ Q &= \sum \delta_{ij} \\ R &= \sum \gamma_{ij} \\ H &= \sum \theta_{ij} \end{aligned} \right\} \text{----- (14)}$$

Equation (13) can be rewritten as a $5n \times 5n$ block system linear equation. This system is known as the associated system of fuzzy system and is written in matrix representation as follows,

$$\begin{pmatrix} S & 0 & 0 & 0 & 0 \\ T & (S - T) & 0 & 0 & 0 \\ Q & 0 & (S - Q) & 0 & 0 \\ R & 0 & 0 & (S - R) & 0 \\ H & 0 & 0 & 0 & (S - H) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} b \\ g \\ h \\ k \\ l \end{pmatrix} \text{----- (15)}$$

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 \\ I & (T^{-1}S - I) & 0 & 0 & 0 \\ I & 0 & (Q^{-1}S - I) & 0 & 0 \\ I & 0 & 0 & (R^{-1}S - I) & 0 \\ I & 0 & 0 & 0 & (H^{-1}S - I) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} S^{-1}b \\ T^{-1}h \\ Q^{-1}g \\ R^{-1}k \\ H^{-1}l \end{pmatrix} \text{----- (16)}$$

The solution of the above matrix is

$$\left. \begin{aligned} x &= S^{-1}by = (T^{-1}S - I)^{-1}(T^{-1}h - Ix) \\ z &= (Q^{-1}S - I)^{-1}(Q^{-1}h - Ix) \\ w &= (R^{-1}S - I)^{-1}(R^{-1}K - Ix) \\ v &= (H^{-1}S - I)^{-1}(H^{-1}l - Ix) \end{aligned} \right\} \text{----- (17)}$$

5 Numerical Examples

1. Consider the following triangular fully fuzzy linear system

$$\begin{bmatrix} (3,1,2) & (5,2,3) \\ (4,2,1) & (6,1,2) \end{bmatrix} \begin{bmatrix} (x_1, x_2, x_3) \\ (y_1, y_2, y_3) \end{bmatrix} = \begin{bmatrix} (11,15,12) \\ (14,18,20) \end{bmatrix}$$

Let us find the 2 x 2 coefficient matrices defined in Section 3.

$$S = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}, T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

The corresponding right-hand side vectors are

$$b = \begin{bmatrix} 11 \\ 14 \end{bmatrix}, g = \begin{bmatrix} 15 \\ 18 \end{bmatrix}, h = \begin{bmatrix} 12 \\ 20 \end{bmatrix}.$$

The solution of the matrix is presented in section 3.

$$x = S^{-1}b.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$y = (T^{-1}S - I)^{-1}(T^{-1}h - Ix).$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

$$z = (Q^{-1}s - I)^{-1}(Q^{-1}h - Ix).$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \end{bmatrix}.$$

2. Consider the following pentagonal fully fuzzy linear system

$$\begin{bmatrix} (3,1,1,2,2) & (1,2,2,3,3) \\ (1,3,3,4,4) & (4,2,2,5,5) \end{bmatrix} \begin{bmatrix} (x_1, x_2, x_3) \\ (y_1, y_2, y_3) \end{bmatrix} = \begin{bmatrix} (22,10,10,12,12) \\ (24,12,12,14,14) \end{bmatrix}$$

As per first step let us find the 2 x 2 matrices defined section 4.

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}, T = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix},$$

$$R = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, H = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

The corresponding right-hand side vectors:

$$b = \begin{bmatrix} 22 \\ 24 \end{bmatrix}, g = \begin{bmatrix} 10 \\ 12 \end{bmatrix}, h = \begin{bmatrix} 10 \\ 12 \end{bmatrix}, k = \begin{bmatrix} 12 \\ 14 \end{bmatrix}, l = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

The solution presented in section 4.

$$x = S^{-1}b.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5.82 \\ 4.55 \end{bmatrix}.$$

$$y = (T^{-1}S - I)^{-1}(T^{-1}h - Ix).$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -15.34 \\ 2.27 \end{bmatrix}.$$

$$z = (Q^{-1}s - I)^{-1}(Q^{-1}h - Ix).$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 22.45 \\ 5.32 \end{bmatrix}.$$

$$w = (R^{-1}s - I)^{-1}(R^{-1}k - Ix).$$

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} 10.25 \\ 7.67 \end{bmatrix}. \\ v &= (H^{-1}s - I)^{-1}(H^{-1}l - Ix). \\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 14.45 \\ -2.38 \end{bmatrix}. \end{aligned}$$

6 Conclusion

In this paper we consider fully fuzzy triangular linear system of equations and fully fuzzy pentagonal system of equations, the solution of FFLS is obtained by constructing associated system of equations by multiplication operation on matrices. This method yields solution faster than other existing ones.

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Pell Even Sum Cordial Labeling of Graphs

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Abstract

Let $G = (V, E)$ be a simple graph and let P_i be Pell numbers. For a bijection $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{|V|-1}\}$, assign the label 1 for the edge $e = uv$ if $f(u) + f(v)$ is even and label 0 otherwise. Then f is said to be a Pell even sum cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with 0 and 1 respectively. If any graph admits Pell even sum cordial labeling, it is called Pell even sum cordial graph. In this study, we show that star, comb, bistar, jewel, crown, bipartite graph $K_{m,m}$, flower graph, helm, wheel, triangular book, $K_2 + mK_1$ are Pell even sum cordial.

Keywords: Cordial Labeling, Pell Numbers, Pell Even Sum Cordial Labeling.

2010 AMS Subject Classification: 05C78[‡].

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected simple graph. We refer to Harary [5] for graph theory concepts and refer Gallian [4] for the literature on graph labeling. The Pell numbers are defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ where $P_0 = 0$ and $P_1 = 1$. The Pell number sequence is given as 0, 1, 2, 5, 12, ... Cahit [2] is credited for inventing cordial labeling. Shiama [11] defined Pell labeling and shown that path, cycle, star, double star, coconut tree, bistar and $B_{m,n,k}$ are Pell graphs. Muthu Ramakrishnan and Sutha [8] proposed Pell graceful labeling as an extension of Fibonacci graceful labeling and demonstrated that cycle, path, olive tree, comb graph are Pell graceful graphs whereas complete graph and wheel graphs are not Pell graceful. Indira et al. [6] suggested some algorithms for the existence of Pell labeling in quadrilateral snake, extended duplicate graph of quadrilateral graph. Sriram et al. [12] investigated the Pell labeling for the joins of square of a path. Muthu Ramakrishnan and Sutha [7] also demonstrated that bistar, subdivision of bistar, caterpillar graphs, Jelly fish graph, star graph, coconut tree are Pell graceful. Sharon Philomena and Thirusangu [10] shown that $\langle K_{1,n} : 2 \rangle$ is a Pell graph. Avudainayaki and Selvam [1] shown that the extended duplicate graph of arrow graph and splitting graph of path admits harmonious and Pell labeling. Celine Mary et al. [3] demonstrated through an algorithm that inflation of alternate triangular snake graph of odd length in which the alternate block starts from the second vertex is a Pell graph. Muthu Ramakrishnan and Sutha [9] proposed the Pell square graceful labeling and proved that subdivision of the edges of a path P_n in $P_n \odot K_1$, $\langle S_n : m \rangle$, Olive tree, twig graph are Pell square graceful. Inspired by the concepts discussed above, the *Pell even sum cordial labeling* is being introduced here. It is defined as follows.

Definition 1.1 Let $G = (V, E)$ be a simple graph and let P_i be Pell numbers. For a bijection $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{|V|-1}\}$, assign the label 1 for the edge $e = uv$ if $f(u) + f(v)$ is even and label 0 otherwise. Then f is said to be a Pell even sum cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with 0 and 1 respectively. If any graph admits Pell even sum cordial labeling, it is called Pell even sum cordial graph.

Definition 1.2 The Comb $P_n \odot K_1$ is the graph created by adding a pendent edge to each vertex of a path.

Definition 1.3 The bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of $K_{1,n}$.

Definition 1.4 The jewel graph J_n is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$.

Definition 1.5 The crown $C_n \odot K_1$ is the graph obtained from a cycle by attaching a pendent edge to each vertex of the cycle.

Definition 1.6 A complete bipartite is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph.

Definition 1.7 The helm is the graph obtained from a wheel graph by adjoining a pendent edge at each node of the cycle.

Definition 1.8 The Flower Fl_n is the graph obtained from a helm by attaching each pendent vertex to the apex of the helm.

Definition 1.9 The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in G_1, v \in G_2\}$.

Definition 1.10 One edge union of cycles of same length is called a book. The common edge is called the base of the book. If we consider t copies of cycles of length $n \geq 3$, then the book is denoted by B_n^t . If $n = 3, 4, 5$ or 6 , the book B is called book with triangular, rectangular, pentagonal or hexagonal pages respectively.

Definition 1.11 The Wheel W_n can be defined as the graph join $K_1 + C_{n-1}$.

Definition 1.12 $K_{1,k}$ is a tree with one internal node and k leaves.

2. Pell Even Sum Cordial Graphs

In this section, we prove that star, comb, bistar, jewel, crown, bipartite graph $K_{n,n}$, flower graph, helm, wheel, triangular book, $K_2 + mK_1$ are Pell even sum cordial.

Theorem 2.1. For $n \geq 2$, the comb is Pell even sum cordial graph

Proof. Recall that comb $G = (V, E)$ is a graph obtained from the path with vertices $u_1, u_2, \dots, u_n$ by joining a vertex v_i to each u_i where $1 \leq i \leq n$. Actually $|V(G)| = 2n$ and $|E(G)| = 2n - 1$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n-1}\}$ defined by $f(u_i) = P_{2i-1}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-2}$ for $1 \leq i \leq n$. The induced edge labels are given $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n - 1$ and $f^*(u_i v_i) = 0$ for $1 \leq i \leq n$. From this, $e_f(0) = n, e_f(1) = n - 1$ and so $|e_f(0) - e_f(1)| \leq 1$.

Therefore, for $n \geq 2$, the comb is a Pell even sum cordial graph.

Theorem 2.2. For $n \geq 2$, the bistar $B_{m,m}$ is a Pell even sum cordial.

Proof. Consider the bistar $G = B_{m,m}$. Then $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq m\}, E(G) = \{uv, uu_i, vv_i : 1 \leq i \leq m\}, |V(G)| = 2m + 2$ and $|E(G)| = 2m + 1$. Consider

$f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2m+1}\}$ defined by $f(u) = P_0, f(v) = P_1, f(u_i) = P_{i+1}$ for $1 \leq i \leq m-1$ and $f(v_i) = P_{m+i}$ for $1 \leq i \leq m$. The induced edge labels are given by $f^*(uv) = 0$.

For $1 \leq i \leq m-1, f^*(uu_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}; \\ 0 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$

For $1 \leq i \leq m, f^*(vv_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}; \\ 0 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$

When m is odd, $e_f(1) = m+1, e_f(0) = m$ and on the otherhand, when m is even, $e_f(1) = m, e_f(0) = m+1$. Therefore, for $n \geq 2$, bistar $B_{m,m}$ is Pell even sum cordial.

Theorem 2.3. For $n \geq 1$, the jewel graph J_n is Pell even sum cordial.

Proof. Consider the jewel graph $G = J_n$. Here $V(G) = \{u, v, x, y, u_i: 1 \leq i \leq n\}, E(G) = \{ux, uy, xy, xv, yv, uu_i, vv_i: 1 \leq i \leq n\}, |V(G)| = n+4$ and $|E(G)| = 2n+5$.

Let $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{n+3}\}$ be defined by $f(u) = P_0, f(v) = P_1, f(x) = P_2$ and $f(u_i) = P_{i+3}$ for $1 \leq i \leq n$. Then the induced edge labels are given by, $f^*(uv) = 1, f^*(uy) = 1, f^*(xy) = 0, f^*(vx) = 0$ and $f^*(vy) = 1$

For $1 \leq i \leq n,$

$f^*(uu_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{2}; \\ 0 & \text{if } i \equiv 1 \pmod{2}. \end{cases}$

$f^*(vv_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}; \\ 0 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$

From the above, $e_f(0) = n+3, e_f(1) = n+2$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for $n \geq 1$, the jewel graph J_n is Pell even sum cordial.

Theorem 2.4 For $n \geq 3$, the crown $C_n \odot K_1$ is Pell even sum cordial.

Proof. Consider the crown $G = C_n \odot K_1$. Here $V(G) = \{u_i, v_i: 1 \leq i \leq n\}$ and $E(G) = \{u_i u_{i+1}: 1 \leq i \leq n-1; u_n u_1, u_i v_i: 1 \leq i \leq n\}, |V(G)| = 2n = |E(G)|$.

Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n-1}\}$ defined by $f(u_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-1}$ for $1 \leq i \leq n$. Then the induced edge labels are given by $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n, f^*(u_n u_1) = 1$ and $f^*(u_i v_i) = 0$ for $1 \leq i \leq n$.

From the above, $e_f(0) = n+3, e_f(1) = n+2$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for $n \geq 1$, the crown $C_n \odot K_1$ is Pell even sum cordial.

Theorem 2.5 For $n \geq 2$, the complete bipartite graph $K_{n,n}$ is Pell even sum cordial.

Proof. Let $G = K_{n,n}$. Let the partitions of vertex set be $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. Hence $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and $E(G) = \{u_i v_i: 1 \leq i \leq n\}$. Then $|V(G)| = 2n, |E(G)| = n^2$.

Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n-1}\}$ defined by $f(u_i) = P_{i-1}$ for $1 \leq i \leq n$ and $f(v_i) = P_{(n-1)+i}$ for $1 \leq i \leq n$. The induced edge labels are given by,

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For $1 \leq i \leq n$,

$$f^*(u_i v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 0 \pmod{2}; \\ 0 & \text{otherwise.} \end{cases}$$

When n is odd, $e_f(0) = \frac{n^2+1}{2}$, $e_f(1) = \frac{n^2-1}{2}$ and when, n is even, $e_f(0) = e_f(1) = \frac{n^2}{2}$. Therefore, for $n \geq 2$, the complete bipartite graph $K_{n,n}$ is Pell even sum cordial.

Theorem 2.6 For $n \geq 3$, the flower graph Fl_n is Pell even sum cordial.

Proof. Let $G = Fl_n$. Then $V(G) = \{v, u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_i u_i, v u_i : 1 \leq i \leq n; v_n v_1; v_i v_{i+1} : 1 \leq i \leq n-1\}$. Then, $|V(G)| = 2n+1$, $|E(G)| = 4n$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n}\}$ defined by $f(v_n) = P_{2n}$ for $1 \leq i \leq n$ and $f(v_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(u_i) = P_{2i-1}$ for $1 \leq i \leq n$. The induced edge labels are given by $f^*(v_i v_{i+1}) = 1$, $f^*(v_i u_i) = 0$, $f^*(v u_i) = 0$, $f^*(v v_i) = 1$ and $f^*(v_n v_1) = 1$ for $1 \leq i \leq n$.

From the above, $e_f(0) = e_f(1) = 2n$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, the flower graph Fl_n is Pell even sum cordial for $n \geq 3$.

Theorem 2.7 For $n \geq 3$ and n is even, the Helm H_n is Pell even sum cordial.

Proof. Let $G = H_n$. Then $V(G) = \{v, u_i, v_i : 1 \leq i \leq n\}$, $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1; u_i v_i : 1 \leq i \leq n; v u_i : 1 \leq i \leq n-1; u_1 u_n\}$, $|V(G)| = 2n+1$, $|E(G)| = 3n$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n}\}$ defined by $f(u_i) = P_i$ for $1 \leq i \leq n-1$, $f(v_i) = P_{n+i}$ for $1 \leq i \leq n$ and $f(v) = 0$. Then the induced edge labels are given by, For $1 \leq i \leq n-1$,

$$f^*(u_i u_{i+1}) = 0, f^*(v_i u_i) = 1, f^*(u_n u_1) = 0 \text{ and } f^*(v u_i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod{2}; \\ 1 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$$

From the above, $e_f(0) = e_f(1) = \frac{3n}{2}$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for an even integer ≥ 3 , the Helm H_n is Pell even sum cordial.

Theorem 2.8 For $n \geq 4$, the wheel graph W_n is Pell even sum cordial.

Proof. Let $G = W_n$. Then $V(G) = \{u_0, u_1, u_2, \dots, u_n\}$ and $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1; u_0 v_i : 1 \leq i \leq n-1\}$, $|V(G)| = n+1$ and $|E(G)| = 2n-2$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{2n}\}$ defined by $f(u_i) = P_{2i-2}$ for $1 \leq i \leq n$ and $f(u_0) = P_1$. Then the induced edge labels are given by $f^*(u_i u_{i+1}) = 1$ for $1 \leq i \leq n-1$, $f^*(u_n u_1) = 1$ and $f^*(u_0 u_i) = 0$ for $1 \leq i \leq n$. From this, $e_f(0) = e_f(1) = n-1$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for $n \geq 4$, the wheel graph W_n is Pell even sum cordial.

Theorem 2.9 For $n \geq 2$, the star graph $K_{1,n}$ is Pell even sum cordial.

Proof. Let $G = K_{1,n}$. Then $V(G) = \{v, u_1, u_2, \dots, u_n\}$ and $E(G) = \{v v_i : 1 \leq i \leq n\}$, $|V(G)| = n+1$ and $|E(G)| = n$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_n\}$ defined by $f(v) = P_0$, $f(u_i) = P_i$ for $1 \leq i \leq n-1$. The induced edge labels are given by, for $1 \leq i \leq n$, $f^*(v u_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{2}; \\ 0 & \text{if } i \equiv 1 \pmod{2}. \end{cases}$

When n is odd, $e_f(0) = \frac{n+1}{2}, e_f(1) = \frac{n-1}{2}$ whereas when n is even, $e_f(0) = \frac{n}{2} = e_f(1)$. Therefore, for $n \geq 2$, the star graph $K_{1,n}$ is Pell even sum cordial.

Theorem 2.10 For $n \geq 3$, the triangular book TB_n is Pell even sum cordial.

Proof. Let $G = TB_n$. Then $V(G) = \{v_1, v_2, \dots, v_{n-1}, v_n\}$, $E(G) = \{v_1v_i, v_nv_i, v_nv_1 : 2 \leq i \leq n-1\}$, $|V(G)| = n$ and $|E(G)| = 2n-3$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{n-1}\}$ defined by $f(v_1) = P_0, f(v_n) = P_1$ and $f(v_i) = P_{i+1}$ for $1 \leq i \leq n-2$. Then the induced edge labels are given by $f^*(v_1v_n) = 0$.

For $1 \leq i \leq n-2$,

$$f^*(v_1v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{2}; \\ 0 & \text{if } i \equiv 1 \pmod{2}. \end{cases}$$

$$f^*(v_nv_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}; \\ 0 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$$

From the above, $e_f(0) = n-1, e_f(1) = n-2$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for $n \geq 3$, the triangular book TB_n is Pell even sum cordial.

Theorem 2.11. For $m \geq 3$, the graph $K_2 + mK_1$ is Pell even sum cordial.

Proof. Let $G = K_2 + mK_1$. Then $V(G) = \{u_1, u_2, v_1, v_2, \dots, v_m\}$, $E(G) = \{u_1u_2, u_1v_i, u_2v_i : 1 \leq i \leq m\}$, $|V(G)| = m+2$ and $|E(G)| = 2m+1$. Consider $f: V(G) \rightarrow \{P_0, P_1, \dots, P_{m+1}\}$ defined by $f(u_1) = P_0, f(u_2) = P_1$ and $f(v_i) = P_{i+1}$ for $1 \leq i \leq m$. Then the induced edge labels are given by $f^*(u_1u_2) = 0$.

For $1 \leq i \leq m$,

$$f^*(u_1v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}; \\ 0 & \text{if } i \equiv 0 \pmod{2}. \end{cases}$$

$$f^*(u_2v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{2}; \\ 0 & \text{if } i \equiv 1 \pmod{2}. \end{cases}$$

From the above, $e_f(0) = m+1, e_f(1) = m$ and so $|e_f(0) - e_f(1)| \leq 1$. Therefore, for $m \geq 3$, the graph $K_2 + mK_1$ is Pell even sum cordial.

3. Conclusions

This article introduces a novel graph labeling technique called Pell even sum cordial labeling and it is demonstrated that several typical graphs are Pell even sum cordial. More Pell even sum graphs will be provided in the following article.

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