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Insertion of terms satisfying the recurrence relations of Horadam sequence and Bifurcating Fibonacci sequences

Khushbu Das^{*}
Devbhadra Shah[†]

Abstract

In this article, we consider the problem of finding general formula for the terms introduced between two given positive integers ' a ' and ' b ' in such a way that the terms of newly formed finite sequence satisfy the recurrence relations of Horadam sequence and some bifurcating Fibonacci sequences.

Keywords: Generalized Fibonacci sequences, Horadam sequence, Bifurcating sequence, Inserted terms, Missing terms.

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1 Introduction

Consider any two fixed given positive integers a and b such that $a < b$. We insert k number of terms between a and b so that every term inserted between follows the given recurrence relation of any generalized Fibonacci sequence. Then we say x_n ($1 \leq n \leq k$) to be the *inserted term* (or *missing term*, as defined by some of the authors) in the finite (generalized Fibonacci-like) sequence $a, x_1, x_2, x_3, \dots, x_k, b$.

The problem of insertion of terms in arithmetic, harmonic, geometric sequence is considered to be elementary which occurs in the high school mathematics. However, not much work is done regarding the general formula for the inserted terms in a Fibonacci-like sequences. Howell [8] presented a proof for finding the n^{th} term of Fibonacci sequence using vectors and eigenvalues. But he said nothing for the Fibonacci-like sequences. Agnes, et al. [1] provided a formula for inclusion of three consecutive missing terms in Fibonacci-like sequence.

Horadam [7] defined a linear recurrence sequence of second order $F_n(p, q)$, acknowledged as *Horadam sequence*, by the recurrence relation $F_n^{(p,q)} = pF_{n-1}^{(p,q)} + qF_{n-2}^{(p,q)}$ with the initial conditions $F_0^{(p,q)} = a, F_1^{(p,q)} = c$, where a, c and p, q are arbitrary positive integers. First few terms of this sequence are shown in the Table 1.

n	$F_n^{(p,q)}$
0	a
1	c
2	$pc + qa$
3	$p^2c + pqa + qc$
4	$p^3c + p^2qa + 2pqc + q^2a$
5	$p^4c + p^3qa + 3p^2qc + 2pq^2a + q^2c$
6	$p^5c + p^4qa + 4p^3qc + 3p^2q^2a + 3pq^2c + q^3a$

Table 1

Insertion of terms satisfying the recurrence relations of Horadam sequence and Bifurcating Fibonacci sequences

The Binet-type explicit formula for the terms of this sequence is given by $F_k^{(p,q)} = \frac{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k}{\alpha - \beta}$, where $\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}$, $\beta = \frac{p - \sqrt{p^2 + 4q}}{2}$. Here we note that $\alpha - \beta = \sqrt{p^2 + 4q}$, $\alpha + \beta = p$ and $\alpha\beta = -q$.

Diwan, Shah [4,5] considered the sequence $\{F_n^{(p,q,r,s)}\}$ defined by the recurrence relation

$$F_n^{(p,q,r,s)} = p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,r,s)} + r^{\chi(n)} s^{1-\chi(n)} F_{n-2}^{(p,q,r,s)}, \quad (1.1)$$

where p, q, r, s are any fixed positive integers and $\chi(n) = \begin{cases} 1; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$.

They studied this sequence extensively for some explicit values of p, q, r, s . Verma and Bala, Bilgici, Yayenie [2,9,10] also studied this sequence for some specific values of p, q, r, s . It is easy to observe that this sequence is bifurcating sequence depending on the parity of n .

In this paper we derive the general formula which gives the value of any inserted term x_n ($1 \leq n \leq k$) for the sequence $\{F_n(p, q)\}$ and various bifurcating subsequence of $\{F_n^{(p,q,r,s)}\}$ by considering some fixed values of p, q, r, s . Throughout we assume that the positive integers a, b are the given first and last term respectively in the sequence to be considered.

2 Insertion of terms satisfying the recurrence relation of Horadam sequence

In this section, we find the formula for the inserted terms between the given fixed positive integers a, b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation of $F_n(p, q)$. Before finding the general formula, we find the formula for the first term x_1 in the finite sequence $a, x_1, x_2, x_3, \dots, x_k, b$. This value of x_1 will be further used to find the general formula for any x_n ($1 \leq n \leq k$).

In [7], Horadam obtained only the formula for the first missing term (inserted term) x_1 when k terms are inserted between given two positive

integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(p,q)}$. In fact, he proved that

$$x_1 = \frac{b - aqF_k^{(p,q)}}{F_{k+1}^{(p,q)}}. \quad (2.1)$$

In the following theorem, we obtain the general formula for any arbitrary term x_n ($1 \leq n \leq k$) when k terms $x_1, x_2, x_3, \dots, x_k$ are inserted between positive integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(p,q)}$.

Theorem 2.1. When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b in such a way that the finite sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfy the recurrence relation of $F_n^{(p,q)}$, then the general formula for any arbitrary term x_n ($1 \leq n \leq k$) is given by

$$x_n = \frac{F_n^{(p,q)}b(\alpha-\beta) - aqF_n^{(p,q)}\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} + aqF_{n-1}^{(p,q)}\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}. \quad (2.2)$$

Proof. We prove the result by the principle of mathematical induction. For we $k = 1$, we get $n = 1$ and

$$x_1 = \frac{F_1^{(p,q)}b(\alpha-\beta) - aqF_1^{(p,q)}\{(c-a\beta)\alpha - (c-a\alpha)\beta\} + aqF_0^{(p,q)}\{(c-a\beta)\alpha^2 - (c-a\alpha)\beta^2\}}{(c-a\beta)\alpha^2 - (c-a\alpha)\beta^2} = \frac{b-aqc}{cp+aq},$$

which is same as (2.1).

Next, we assume that (2.2) holds for some positive integer not exceeding n . Then both the following results hold:

$$x_{n-1} = \frac{F_{n-1}^{(p,q)}b(\alpha-\beta) - aqF_{n-1}^{(p,q)}\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} + aqF_{n-2}^{(p,q)}\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}$$

and

$$x_n = \frac{F_n^{(p,q)}b(\alpha-\beta) - aqF_n^{(p,q)}\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} + aqF_{n-1}^{(p,q)}\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}$$

Now consider

$$\begin{aligned} px_n + qx_{n-1} &= \frac{(pF_n^{(p,q)} + qF_{n-1}^{(p,q)})b(\alpha-\beta) - aq(pF_n^{(p,q)} + qF_{n-1}^{(p,q)})\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} \\ &\quad + aq(pF_{n-1}^{(p,q)} + qF_{n-2}^{(p,q)})\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}} \\ &= \frac{F_{n+1}^{(p,q)}b(\alpha-\beta) - aqF_{n+1}^{(p,q)}\{(c-a\beta)\alpha^k - (c-a\alpha)\beta^k\} \\ &\quad + aqF_n^{(p,q)}\{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}\}}{(c-a\beta)\alpha^{k+1} - (c-a\alpha)\beta^{k+1}}. \end{aligned}$$

It can be observed that the right side of this result simplifies to x_{n+1} . Thus, by (2.2) is true for every positive integer n .

3 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(p,q,1,1)}$

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (1.2).

If we let $p = q = r = s = 1$, the sequence $\{F_n^{(p,q,r,s)}\}$ is the sequence of usual Fibonacci numbers. If we define $F_0^{(p,q,r,s)} = 0, F_1^{(p,q,r,s)} = 1$, then first few terms of this sequence are shown in Table 2.

n	$F_n^{(p,q,r,s)}$
0	0
1	1
2	q
3	$pq + r$
4	$pq^2 + qr + qs$
5	$p^2q^2 + 2pqr + pqs + r^2$
6	$p^2q^3 + 2pq^2r + 2pq^2s + r^2q + sqr + qs^2$

Table 2

If we consider $r = s = 1$ in (1.1), we get the sequence $\{F_n^{(p,q,1,1)}\}$ whose terms are defined by the recurrence relation

$$F_n^{(p,q,1,1)} = p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)},$$

where $F_0^{(p,q,1,1)} = 0, F_1^{(p,q,1,1)} = 1$. This can be written in the form

$$F_n^{(p,q,1,1)} = \begin{cases} pF_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)}; & \text{when } n \text{ is odd} \\ qF_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)}; & \text{when } n \text{ is even} \end{cases} \quad (n \geq 2) \quad (3.1)$$

with the initial conditions $F_0^{(p,q,1,1)} = 0, F_1^{(p,q,1,1)} = 1$. First few terms of this sequence are shown in Table 3.

This sequence was studied in detail by Diwan, Shah [4] as well as Edson, Yayenie [6]. They derived the Binet-type explicit formula for the terms of this sequence as

$$F_k^{(p,q,1,1)} = \frac{q^{1-\chi(k)}}{(pq)^{\lfloor \frac{k}{2} \rfloor}} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right), \text{ where } \alpha = \frac{pq + \sqrt{p^2q^2 + 4pq}}{2}, \beta = \frac{pq - \sqrt{p^2q^2 + 4pq}}{2}.$$

Here we note that $\alpha - \beta = \sqrt{p^2q^2 + 4pq}$, $\alpha + \beta = pq$ and $\alpha\beta = -pq$.

n	$F_n^{(p,q,1,1)}$
0	0
1	1
2	q
3	$pq + 1$
4	$pq^2 + 2q$
5	$p^2q^2 + 3pq + 1$
6	$p^2q^3 + 4pq^2 + 3q$

Table 3

When we consider the sequence a, x_1, b , where x_1 is the only inserted term between a and b , then using $F_2^{(p,q,1,1)} = qF_1^{(p,q,1,1)} + F_0^{(p,q,1,1)}$, we get $b = qx_1 + a$. Thus $x_1 = \frac{b-a}{q}$.

When we consider the finite sequence a, x_1, x_2, b , so that it satisfies (3.1), we observe that $x_1 = \frac{x_2-a}{q}$ and $x_2 = \frac{b-x_1}{p}$. This gives $x_2 = \frac{bq+a}{pq+1}$ and $x_1 = \frac{b-ap}{pq+1}$.

Further, considering the sequence a, x_1, x_2, x_3, b , it is now easy to observe that $x_1 = \frac{x_2-a}{q}$, $x_2 = \frac{x_3-x_1}{p}$ and $x_3 = \frac{b-x_2}{q}$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-a(pq+1)}{pq^2+2q}$, $x_2 = \frac{bq+aq}{pq^2+2q}$, $x_3 = \frac{b-a+bpq}{pq^2+2q}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b . Using the similar approach as above, we obtain

$$x_1 = \frac{b-a(p^2q+2p)}{p^2q^2+3pq+1}, x_2 = \frac{bq+apq+a}{p^2q^2+3pq+1}, x_3 = \frac{b-ap+bpq}{p^2q^2+3pq+1} \text{ and } x_4 = \frac{a+2bq+bpq^2}{p^2q^2+3pq+1}.$$

We mention these results in the Table 4.

Number of inserted terms	Formula			
	x_1	x_2	x_3	x_4
1	$\frac{b-a}{q}$	---	---	---
2	$\frac{b-ap}{pq+1}$	$\frac{bq+a}{pq+1}$	---	---
3	$\frac{b-a(pq+1)}{pq^2+2q}$	$\frac{bq+aq}{pq^2+2q}$	$\frac{b-a+bpq}{pq^2+2q}$	---
4	$\frac{b-a(p^2q+2p)}{p^2q^2+3pq+1}$	$\frac{bq+apq+a}{p^2q^2+3pq+1}$	$\frac{b-ap+bpq}{p^2q^2+3pq+1}$	$\frac{a+2bq+bpq^2}{p^2q^2+3pq+1}$

Table 4

*Insertion of terms satisfying the recurrence relations of Horadam sequence
and Bifurcating Fibonacci sequences*

From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we can now write the general formula for x_1 as

$$x_1 = \frac{b - ap^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}. \quad (3.2)$$

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we obtain the general formula for n^{th} inserted term ($1 \leq n \leq k$) using the recurrence relation (3.1).

Theorem 3.1. When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(p,q,1,1)}$, the general formula for any arbitrary term x_n ($1 \leq n \leq k$) is given by

$$x_n = \frac{F_n^{(p,q,1,1)} b - a F_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since by (3.2), we have $x_1 = \frac{b - ap^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n . Then the following holds:

$$\begin{aligned} x_{n-1} &= \frac{F_{n-1}^{(p,q,1,1)} b - a F_{n-1}^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n-1)} q^{-\chi(n-1)} F_{n-2}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}} \\ x_{n-2} &= \frac{F_{n-2}^{(p,q,1,1)} b - a F_{n-2}^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} + ap^{\chi(n-2)} q^{-\chi(n-2)} F_{n-3}^{(p,q,1,1)} F_{k+1}^{(p,q,1,1)}}{F_{k+1}^{(p,q,1,1)}}. \end{aligned}$$

Now, $p^{\chi(n)} q^{1-\chi(n)} x_{n-1} + x_{n-2}$

$$\begin{aligned} &= \frac{\left(\begin{aligned} &\left(p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)} \right) b - a \left(p^{\chi(n)} q^{1-\chi(n)} F_{n-1}^{(p,q,1,1)} + F_{n-2}^{(p,q,1,1)} \right) \\ &\quad \times p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ &+ a \left(p^{\chi(n)} q^{1-\chi(n)} p^{\chi(n-1)} q^{-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n-2)} q^{-\chi(n-2)} F_{n-3}^{(p,q,1,1)} \right) F_{k+1}^{(p,q,1,1)} \end{aligned} \right)}{F_{k+1}^{(p,q,1,1)}} \\ &= \frac{\left(\begin{aligned} &F_n^{(p,q,1,1)} b - a F_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \\ &+ a \left(p^{\chi(n)} q^{1-\chi(n)} p^{\chi(n-1)} q^{-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \right) F_{k+1}^{(p,q,1,1)} \end{aligned} \right)}{F_{k+1}^{(p,q,1,1)}} \end{aligned}$$

$$= \frac{\left\{ F_n^{(p,q,1,1)} b - a F_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \right.}{F_{k+1}^{(p,q,1,1)}} \left. + a \left(p F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \right) F_{k+1}^{(p,q,1,1)} \right\}.$$

Now, when n is odd, we get

$$\begin{aligned} & p F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{1-\chi(n)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{-\chi(n)} \left(q F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)} \right) \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{\chi(n-1)} q^{1-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)} \right) \\ &= p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)}. \end{aligned}$$

Also, when n is even, we get

$$\begin{aligned} & p F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{1-\chi(n)} q^{\chi(n)} F_{n-2}^{(p,q,1,1)} + p^{\chi(n)} q^{-\chi(n)} F_{n-3}^{(p,q,1,1)} \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{1-2\chi(n)} q^{2\chi(n)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)} \right) \\ &= p^{\chi(n)} q^{-\chi(n)} \left(p^{\chi(n-1)} q^{1-\chi(n-1)} F_{n-2}^{(p,q,1,1)} + F_{n-3}^{(p,q,1,1)} \right) \\ &= p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)} \end{aligned}$$

Therefore, we have

$$x_n = p^{\chi(n)} q^{1-\chi(n)} x_{n-1} + x_{n-2} = \frac{\left\{ F_n^{(p,q,1,1)} b - a F_n^{(p,q,1,1)} p^{1-\chi(k)} q^{\chi(k)-1} F_k^{(p,q,1,1)} \right.}{F_{k+1}^{(p,q,1,1)}} \left. + a \left(p^{\chi(n)} q^{-\chi(n)} F_{n-1}^{(p,q,1,1)} \right) F_{k+1}^{(p,q,1,1)} \right\},$$

which proves the result for every positive integer n .

4 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(p,1,1,s)}$

If we consider $q = r = 1$ in (1.1), the sequence $\{F_n^{(p,1,1,s)}\}$ whose terms are defined by the recurrence relation

$$F_n^{(p,1,1,s)} = p^{\chi(n)} F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)} F_{n-2}^{(p,1,1,s)},$$

where $F_0^{(p,1,1,s)} = 0, F_1^{(p,1,1,s)} = 1$. This can be written in the form

$$F_n^{(p,1,1,s)} = \begin{cases} p F_{n-1}^{(p,1,1,s)} + F_{n-2}^{(p,1,1,s)} & \text{; when } n \text{ is odd} \\ F_{n-1}^{(p,1,1,s)} + s F_{n-2}^{(p,1,1,s)} & \text{; when } n \text{ is even} \end{cases} \quad (n \geq 2) \quad (4.1)$$

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with the initial conditions $F_0^{(p,1,1,s)} = 0, F_1^{(p,1,1,s)} = 1$. First few terms of this sequence are shown in Table 5.

n	$F_n^{(p,1,1,s)}$
0	0
1	1
2	1
3	$1 + p$
4	$1 + p + s$
5	$1 + 2p + ps + p^2$
6	$1 + s + 2p + 2ps + p^2 + s^2$

Table 5

This sequence was studied by Diwan, Shah [4]. They obtained Binet-type explicit formula for the terms of this sequence as

$$F_n^{(p,1,1,s)} = \frac{(\alpha-s)^{\chi(n)} \alpha^{\lfloor \frac{n}{2} \rfloor} - (\beta-s)^{\chi(n)} \beta^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta},$$

where $\alpha = \frac{(p+s+1) + \sqrt{(p+s+1)^2 - 4s}}{2}, \beta = \frac{(p+s+1) - \sqrt{(p+s+1)^2 - 4s}}{2}$. This gives $\alpha - \beta = \sqrt{(p+s+1)^2 - 4s}, \alpha + \beta = p + s + 1$ and $\alpha\beta = s$.

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (4.1).

When we consider the sequence a, x_1, b , by using $F_2^{(p,1,1,s)} = F_1^{(p,1,1,s)} + sF_0^{(p,1,1,s)}$, we get $b = x_1 + sa$. Thus $x_1 = b - sa$. This basic formula will be used to find the other terms.

When we consider the finite sequence as a, x_1, x_2, b , so that it satisfies (4.1), we observe that $x_1 = x_2 - s$ and $x_2 = \frac{b-x_1}{p}$. This gives $x_2 = \frac{b+sa}{1+p}$ and $x_1 = \frac{b-sap}{1+p}$.

Further, considering the sequence a, x_1, x_2, x_3, b , it is now easy to observe that $x_1 = x_2 - sa, x_2 = \frac{x_3-x_1}{p}$ and $x_3 = b - sx_2$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-as(p+s)}{1+p+s}, x_2 = \frac{b+sa}{1+p+s}, x_3 = \frac{b+bp-as^2}{1+p+s}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b . Using the similar approach as above, we

obtain $x_1 = \frac{b-sa(p^2+p+ps)}{1+2p+ps+p^2}$, $x_2 = \frac{b+sap+sa}{1+2p+ps+p^2}$, $x_3 = \frac{b+pb-s^2pa}{1+2p+ps+p^2}$ and $x_4 = \frac{b+pb+sb+s^2a}{1+2p+ps+p^2}$.

We mention these results in the Table 6.

Number of inserted terms	Formula			
	x_1	x_2	x_3	x_4
1	$b - sa$	---	---	---
2	$\frac{b - sap}{1 + p}$	$\frac{b + sa}{1 + p}$	---	---
3	$\frac{b - sa(p + s)}{1 + p + s}$	$\frac{b + sa}{1 + p + s}$	$\frac{b + bp - as^2}{1 + p + s}$	---
4	$\frac{b - sa(p^2 + p + ps)}{1 + 2p + ps + p^2}$	$\frac{b + sap + sa}{1 + 2p + ps + p^2}$	$\frac{b + pb - s^2pa}{1 + 2p + ps + p^2}$	$\frac{b + pb + sb + s^2a}{1 + 2p + ps + p^2}$

Table 6

From this Table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of k number of inserted terms and we can now write the general formula for x_1 as

$$x_1 = \frac{b-sa\{F_{k+1}^{(p,1,1,s)}-F_{k-1}^{(p,1,1,s)}\}}{F_{k+1}^{(p,1,1,s)}}. \quad (4.2)$$

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we obtain the general formula for n^{th} inserted term ($1 \leq n \leq k$) using the recurrence relation (4.1).

Theorem 4.1. When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(p,1,1,s)}$, the general formula for any arbitrary term x_n ($1 \leq n \leq k$) is given by

$$x_n = \frac{F_n^{(p,1,1,s)}b + saF_n^{(p,1,1,s)}F_{k-1}^{(p,1,1,s)} - saF_{n-2}^{(p,1,1,s)}F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since by (4.2), we have $x_1 = \frac{b-sa\{F_{k+1}^{(p,1,1,s)}-F_{k-1}^{(p,1,1,s)}\}}{F_{k+1}^{(p,1,1,s)}}$, which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n . Then the following holds:

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$$x_{n-1} = \frac{F_{n-1}^{(p,1,1,s)} b + s a F_{n-1}^{(p,1,1,s)} F_{k-1}^{(p,1,1,s)} - s a F_{n-3}^{(p,1,1,s)} F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}} \text{ and}$$

$$x_{n-2} = \frac{F_{n-2}^{(p,1,1,s)} b + s a F_{n-2}^{(p,1,1,s)} F_{k-1}^{(p,1,1,s)} - s a F_{n-4}^{(p,1,1,s)} F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}}.$$

Now, $p^{\chi(n)} x_{n-1} + s^{1-\chi(n)} x_{n-2}$

$$\begin{aligned} &= \frac{\left(\left(p^{\chi(n)} F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)} F_{n-2}^{(p,1,1,s)} \right) b + s a \left(p^{\chi(n)} F_{n-1}^{(p,1,1,s)} + s^{1-\chi(n)} F_{n-2}^{(p,1,1,s)} \right) F_{k-1}^{(p,1,1,s)} \right.}{F_{k+1}^{(p,1,1,s)}} \\ &\quad \left. - s a \left(p^{\chi(n)} F_{n-3}^{(p,1,1,s)} + s^{1-\chi(n)} F_{n-4}^{(p,1,1,s)} \right) F_{k+1}^{(p,1,1,s)} \right) \\ &= \frac{\left(F_n^{(p,1,1,s)} b + s a F_n^{(p,1,1,s)} F_{k-1}^{(p,1,1,s)} - s a \left(p^{\chi(n-2)} F_{n-3}^{(p,1,1,s)} + s^{1-\chi(n-2)} F_{n-4}^{(p,1,1,s)} \right) F_{k+1}^{(p,1,1,s)} \right)}{F_{k+1}^{(p,1,1,s)}} \\ &= \frac{F_n^{(p,1,1,s)} b + s a F_n^{(p,1,1,s)} F_{k-1}^{(p,1,1,s)} - s a F_{n-2}^{(p,1,1,s)} F_{k+1}^{(p,1,1,s)}}{F_{k+1}^{(p,1,1,s)}} = x_n, \text{ which proves the result for} \\ &\text{every positive integer } n. \end{aligned}$$

5 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(1,q,r,1)}$

If we consider $p = s = 1$ in (1.1), then we get the sequence $\{F_n^{(1,q,r,1)}\}$ whose terms are defined by the recurrence relation

$$F_n^{(1,q,r,1)} = q^{\chi(n)} F_{n-1}^{(1,q,r,1)} + r^{1-\chi(n)} F_{n-2}^{(1,q,r,1)},$$

where $F_0^{(1,q,r,1)} = 0, F_1^{(1,q,r,1)} = 1$. This can be written in the form

$$F_n^{(1,q,r,1)} = \begin{cases} F_{n-1}^{(1,q,r,1)} + r F_{n-2}^{(1,q,r,1)} ; & \text{when } n \text{ is odd} \\ q F_{n-1}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} ; & \text{when } n \text{ is even} \end{cases} \quad (n \geq 2) \quad (5.1)$$

with the initial conditions $F_0^{(1,q,r,1)} = 0, F_1^{(1,q,r,1)} = 1$. First few terms of this sequence are shown in Table 7.

Diwan, Shah [4] studied this sequence and obtained the Binet-type explicit formula for the terms of this sequence as

$$F_n^{(1,q,r,1)} = q^{1-\chi(n)} \left\{ \frac{(\alpha-1)^{\chi(n)} \alpha^{\lfloor \frac{n}{2} \rfloor} - (\beta-1)^{\chi(n)} \beta^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right\},$$

where $\alpha = \frac{(r+q+1) + \sqrt{(r+q+1)^2 - 4q}}{2}, \beta = \frac{(r+q+1) - \sqrt{(r+q+1)^2 - 4q}}{2}$ with $\alpha - \beta = \sqrt{(r+q+1)^2 - 4q}, \alpha + \beta = r+q+1, \alpha\beta = r$.

n	$F_n^{(1,q,r,1)}$
0	0
1	1
2	q
3	$r + q$
4	$q + rq + q^2$
5	$q + 2qr + r^2 + q^2$
6	$q + rq + 2rq^2 + 2q^2 + r^2q + q^3$

Table 7

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (5.1).

When we consider the sequence a, x_1, b , using $F_2^{(1,q,r,1)} = qF_1^{(1,q,r,1)} + F_0^{(1,q,r,1)}$, we get $b = qx_1 + a$. Thus $x_1 = \frac{b-a}{q}$.

When we consider the sequence as a, x_1, x_2, b , so that it satisfies (5.1), we observe that $x_1 = \frac{x_2-a}{q}$ and $x_2 = b - rx_1$. This gives $x_2 = \frac{qb+ra}{r+q}$ and $x_1 = \frac{b-a}{r+q}$.

Further, considering the sequence a, x_1, x_2, x_3, b , it is now easy to observe that $x_1 = \frac{x_2-a}{q}$, $x_2 = x_3 - rx_1$, $x_2 = x_3 - rx_1$ and $x_3 = \frac{b-x_2}{q}$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-a(q+1)}{q^2+q+rq}$, $x_2 = \frac{bq+rqa}{q^2+q+rq}$, $x_3 = \frac{bq+br-ra}{q^2+q+rq}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b . Using the similar approach as above, we obtain

$$x_1 = \frac{b-a(1+r+q)}{q^2+r^2+2rq+q}, x_2 = \frac{bq+ar^2+arq}{q^2+r^2+2rq+q}, x_3 = \frac{bq+rb-ar}{q^2+r^2+2rq+q}, x_4 = \frac{bq^2+rbq+qb+ar^2}{q^2+r^2+2rq+q}.$$

We mention these results in the Table 8. From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we can now write the general formula for x_1 as

$$x_1 = \frac{b-a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\}}{F_{k+1}^{(1,q,r,1)}}. \quad (5.2)$$

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Number of inserted term	Formula			
	x_1	x_2	x_3	x_4
1	$\frac{b-a}{q}$	---	---	---
2	$\frac{b-a}{r+q}$	$\frac{qb+ra}{r+q}$	---	---
3	$\frac{b-a(q+1)}{q^2+q+rq}$	$\frac{bq+rqa}{q^2+q+rq}$	$\frac{bq+br-ra}{q^2+q+rq}$	---
4	$\frac{b-a(1+r+q)}{q^2+r^2+2rq+q}$	$\frac{bq+ar^2+arq}{q^2+r^2+2rq+q}$	$\frac{bq+rb-ar}{q^2+r^2+2rq+q}$	$\frac{bq^2+rbq+qb+ar^2}{q^2+r^2+2rq+q}$

Table 8

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we obtain the general formula for n^{th} inserted term ($1 \leq n \leq k$) using the recurrence relation (5.1).

Theorem 5.1 When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(1,q,r,1)}$, the general formula for any arbitrary term x_n ($1 \leq n \leq k$) is given by

$$x_n = \frac{\left\{ F_n^{(1,q,r,1)} b - F_n^{(1,q,r,1)} a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right.}{F_{k+1}^{(1,q,r,1)}} + a \left\{ \left[\left(\frac{1-r}{q} \right) F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)} \right]^{1-\chi(n)} \left[\frac{1}{q} F_{n-1}^{(1,q,r,1)} \right]^{\chi(n)} \right\} F_{k+1}^{(1,q,r,1)} \right\}.$$

Proof. We use the principle of mathematical induction to prove the result.

By (5.2), we have $x_1 = \frac{b-a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\}}{F_{k+1}^{(1,q,r,1)}}$, which

proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n . Then the following holds:

$$x_{n-1} = \frac{\left\{ F_{n-1}^{(1,q,r,1)} b - F_{n-1}^{(1,q,r,1)} a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right.}{F_{k+1}^{(1,q,r,1)}} + a \left\{ \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right]^{1-\chi(n-1)} \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right]^{\chi(n-1)} \right\} F_{k+1}^{(1,q,r,1)} \right\}$$

$$x_{n-2} = \frac{\left\{ F_{n-2}^{(1,q,r,1)} b - F_{n-2}^{(1,q,r,1)} a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right.}{F_{k+1}^{(1,q,r,1)}} + a \left\{ \left[\left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right]^{1-\chi(n-2)} \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right]^{\chi(n-2)} \right\} F_{k+1}^{(1,q,r,1)} \right\}$$

$$\begin{aligned}
 & \text{Now, } q^{1-\chi(n)} x_{n-1} + r\chi(n) x_{n-2} \\
 & \left\{ \left(q^{1-\chi(n)} F_{n-1}^{(1,q,r,1)} + r\chi(n) F_{n-2}^{(1,q,r,1)} \right) b - a \left(q^{1-\chi(n)} F_{n-1}^{(1,q,r,1)} + r\chi(n) F_{n-2}^{(1,q,r,1)} \right) \right. \\
 & \quad \left. \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right. \\
 & \quad \left. + a \left\{ q^{1-\chi(n)} \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right]^{1-\chi(n-1)} \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right]^{\chi(n-1)} \right. \right. \\
 & \quad \left. \left. + r\chi(n) \left[\left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right]^{1-\chi(n-2)} \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right]^{\chi(n-2)} \right\} F_{k+1}^{(1,q,r,1)} \right\} \\
 & = \frac{F_{k+1}^{(1,q,r,1)}}{F_{k+1}^{(1,q,r,1)}} \\
 & \left\{ F_n^{(1,q,r,1)} b - a F_n^{(1,q,r,1)} \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right. \\
 & \quad \left. + a \left\{ q^{1-\chi(n)} \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right]^{\chi(n)} \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right]^{1-\chi(n)} \right. \right. \\
 & \quad \left. \left. + r\chi(n) \left[\left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right]^{1-\chi(n)} \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right]^{\chi(n)} \right\} F_{k+1}^{(1,q,r,1)} \right\} \\
 & = \frac{F_{k+1}^{(1,q,r,1)}}{F_{k+1}^{(1,q,r,1)}}
 \end{aligned}$$

We calculate the value of third term of numerator separately. Now, when n is odd, we get

$$\begin{aligned}
 & q^{1-\chi(n)} \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right]^{\chi(n)} \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right]^{1-\chi(n)} \\
 & + r\chi(n) \left[\left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right]^{1-\chi(n)} \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right]^{\chi(n)} \\
 & = q \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right] + \left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \\
 & = \frac{1}{q} \left[q F_{n-2}^{(1,q,r,1)} - r F_{n-4}^{(1,q,r,1)} + \left(q F_{n-3}^{(1,q,r,1)} + F_{n-4}^{(1,q,r,1)} \right) \right] \\
 & = \frac{1}{q} \left[q F_{n-2}^{(1,q,r,1)} - r F_{n-4}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right] \\
 & = \frac{1}{q} \left[(1+q) F_{n-2}^{(1,q,r,1)} - r F_{n-4}^{(1,q,r,1)} \right] \\
 & = \frac{1}{q} \left[(1-r) F_{n-2}^{(1,q,r,1)} + q F_{n-2}^{(1,q,r,1)} + r q F_{n-3}^{(1,q,r,1)} \right] \\
 & = \left(\frac{1-r}{q} \right) F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)} \\
 & = \left[\left(\frac{1-r}{q} \right) F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)} \right]^{1-\chi(n)}.
 \end{aligned}$$

Also, when n is even, we get

$$\begin{aligned}
 & q^{1-\chi(n)} \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right]^{\chi(n)} \left[\frac{1}{q} F_{n-2}^{(1,q,r,1)} \right]^{1-\chi(n)} \\
 & + r\chi(n) \left[\left(\frac{1-r}{q} \right) F_{n-4}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right]^{1-\chi(n)} \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right]^{\chi(n)} \\
 & = \left[\left(\frac{1-r}{q} \right) F_{n-3}^{(1,q,r,1)} + F_{n-2}^{(1,q,r,1)} \right] + r \left[\frac{1}{q} F_{n-3}^{(1,q,r,1)} \right] \\
 & = \frac{1}{q} \left[q F_{n-2}^{(1,q,r,1)} + F_{n-3}^{(1,q,r,1)} \right] = \frac{1}{q} \left[F_{n-1}^{(1,q,r,1)} \right] = \left[\frac{1}{q} F_{n-1}^{(1,q,r,1)} \right]^{\chi(n)}
 \end{aligned}$$

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Therefore, we have

$$x_n = q^{1-\chi(n)}x_{n-1} + r\chi(n)x_{n-2} \\ = \frac{\left\{ F_n^{(1,q,r,1)} b - F_n^{(1,q,r,1)} a \left\{ \left[\left(\frac{1-r}{q} \right) F_{k-1}^{(1,q,r,1)} + F_k^{(1,q,r,1)} \right]^{\chi(k)} \left[\frac{1}{q} F_k^{(1,q,r,1)} \right]^{1-\chi(k)} \right\} \right.}{F_{k+1}^{(1,q,r,1)}} + a \left\{ \left[\left(\frac{1-r}{q} \right) F_{n-2}^{(1,q,r,1)} + F_{n-1}^{(1,q,r,1)} \right]^{1-\chi(n)} \left[\frac{1}{q} F_{n-1}^{(1,q,r,1)} \right]^{\chi(n)} \right\} F_{k+1}^{(1,q,r,1)} \right\}, \quad \text{which}$$

proves the result for every positive integer n .

6 Insertion of terms satisfying the recurrence relation of bifurcating sequence $F_n^{(1,1,r,s)}$

If we consider $p = q = 1$ in (1.1), then we get the sequence $\{F_n^{(1,1,r,s)}\}$ whose terms are defined by the recurrence relation

$$F_n^{(1,1,r,s)} = F_{n-1}^{(1,1,r,s)} + r\chi(n)s^{1-\chi(n)}F_{n-2}^{(1,1,r,s)},$$

where $F_0^{(1,1,r,s)} = 0, F_1^{(1,1,r,s)} = 1$. This can be written in the form

$$F_n^{(1,1,r,s)} = \begin{cases} F_{n-1}^{(1,1,r,s)} + rF_{n-2}^{(1,1,r,s)} & ; \text{when } n \text{ is odd} \\ F_{n-1}^{(1,1,r,s)} + sF_{n-2}^{(1,1,r,s)} & ; \text{when } n \text{ is even} \end{cases} \quad (n \geq 2) \quad (6.1)$$

with the initial conditions $F_0^{(1,1,r,s)} = 0, F_1^{(1,1,r,s)} = 1$. First few terms of this sequence are shown in Table 9.

n	$F_n^{(1,1,r,s)}$
0	0
1	1
2	1
3	$1 + r$
4	$1 + r + s$
5	$1 + 2r + s + r^2$
6	$1 + rs + 2r + 2s + r^2 + s^2$

Table 9

Diwan, Shah [3] derived the Binet-type explicit formula for the terms of this sequence as

$$F_n^{(1,1,r,s)} = \frac{(\alpha-s)\chi^{(n)}\alpha^{\lfloor \frac{n}{2} \rfloor} - (\beta-s)\chi^{(n)}\beta^{\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta},$$

where $\alpha = \frac{(r+s+1)+\sqrt{(r+s+1)^2-4rs}}{2}$, $\beta = \frac{(r+s+1)-\sqrt{(r+s+1)^2-4rs}}{2}$ with $\alpha - \beta = \sqrt{(r+s+1)^2-4rs}$, $\alpha + \beta = r + s + 1$ and $\alpha\beta = rs$.

In this section, we find the formula for the inserted terms between the numbers a and b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation (6.1).

When we consider the sequence a, x_1, b , where x_1 is the only inserted term between a and b . Then using $F_2^{(1,1,r,s)} = F_1^{(1,1,r,s)} + sF_0^{(1,1,r,s)}$, we get $b = x_1 + sa$. Thus $x_1 = b - sa$.

When we consider the finite sequence as a, x_1, x_2, b , so that it satisfies (6.1), we observe that $x_1 = x_2 - sa$ and $x_2 = b - rx_1$. This gives $x_2 = \frac{b+rsa}{1+r}$. This also gives $x_1 = \frac{b-sa}{1+r}$.

Further, considering the sequence a, x_1, x_2, x_3, b , it is now easy to observe that $x_1 = x_2 - sa$, $x_2 = x_3 - rx_1$, $x_3 = b - sx_2$. Solving these three equations in three variables x_1, x_2, x_3 we get $x_1 = \frac{b-as(1+s)}{1+r+s}$, $x_2 = \frac{b-rsa}{1+r+s}$, $x_3 = \frac{b+br+ars^2}{1+r+s}$.

If we continue extending the above finite sequence one more time, then we get the sequence a, x_1, x_2, x_3, x_4, b . Using the similar approach as above, we obtain $x_1 = \frac{b-sa(1+r+s)}{1+2r+s+r^2}$, $x_2 = \frac{b+sra+sar^2}{1+2r+s+r^2}$, $x_3 = \frac{b+br-s^2ra}{1+2r+s+r^2}$ and $x_4 = \frac{b+br+bs+r^2s^2a}{1+2r+s+r^2}$.

We mention these results in the Table 10.

Number of inserted term	Formula			
	x_1	x_2	x_3	x_4
1	$b - sa$	---	---	---
2	$\frac{b - sa}{1 + r}$	$\frac{b + rsa}{1 + r}$	---	---
3	$\frac{b - sa(1 + s)}{1 + r + s}$	$\frac{b - rsa}{1 + r + s}$	$\frac{b + br + ars^2}{1 + r + s}$	---
4	$\frac{b - sa(1 + r + s)}{1 + 2r + s + r^2}$	$\frac{b + sra + sar^2}{1 + 2r + s + r^2}$	$\frac{b + br - s^2ra}{1 + 2r + s + r^2}$	$\frac{b + br + bs + r^2s^2a}{1 + 2r + s + r^2}$

Table 10

From this table, we observe that there is a similar pattern for the first inserted term in case of any number of inserted terms. We thus generalize it for

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the case of any number of inserted terms and when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we can now write the general formula for x_1 as

$$x_1 = \frac{b - sa \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\}}{F_{k+1}^{(1,1,r,s)}}, \quad (6.2)$$

Next, when the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given two positive integers a and b , we obtain the general formula for n^{th} inserted term ($1 \leq n \leq k$) using the recurrence relation (6.1).

Theorem 6.1 When the terms $x_1, x_2, x_3, \dots, x_k$ are inserted between given integers a and b , so that all x_i 's satisfy the recurrence relation of $F_n^{(1,1,r,s)}$, the general formula for any arbitrary term x_n ($1 \leq n \leq k$) is given by

$$x_n = \frac{\left\{ F_n^{(1,1,r,s)} b - F_n^{(1,1,r,s)} sa \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} \right\} + sa F_{k+1}^{(1,1,r,s)} \left(F_{n-1}^{(1,1,r,s)} \right)^{\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} s^{\lfloor \frac{n-2}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} \right)^{1-\chi(n)}}{F_{k+1}^{(1,1,r,s)}}.$$

Proof. We use the principle of mathematical induction to prove the result. Since

$$b - sa \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\}$$

by (6.2), we have $x_1 = \frac{\quad}{F_{k+1}^{(1,1,r,s)}}$,

which proves the result for $n = 1$. We next assume that it is true for all positive integers not exceeding n . Then the following holds:

$$x_{n-1} = \frac{\left\{ F_{n-1}^{(1,1,r,s)} b - F_{n-1}^{(1,1,r,s)} sa \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} \right\} + sa F_{k+1}^{(1,1,r,s)} \left(F_{n-2}^{(1,1,r,s)} \right)^{\chi(n-1)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{1-\chi(n-1)}}{F_{k+1}^{(1,1,r,s)}},$$

$$x_{n-2} = \frac{\left\{ F_{n-2}^{(1,1,r,s)} b - F_{n-2}^{(1,1,r,s)} sa \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} \right.}{F_{k+1}^{R(1,1,r,s)}} \left. + sa F_{k+1}^{(1,1,r,s)} \left(F_{n-3}^{(1,1,r,s)} \right)^{\chi(n-2)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{1-\chi(n-2)} \right\}.$$

$$\begin{aligned} & \text{Now, } x_{n-1} + r\chi(n) s^{1-\chi(n)} x_{n-2} \\ &= \frac{\left\{ \left(F_{n-1}^{(1,1,r,s)} + r\chi(n) s^{1-\chi(n)} F_{n-2}^{(1,1,r,s)} \right) b - \left(F_{n-1}^{(1,1,r,s)} + r\chi(n) s^{1-\chi(n)} F_{n-2}^{(1,1,r,s)} \right) sa \right.}{F_{k+1}^{(1,1,r,s)}} \left. \times \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} \right. \\ &= \frac{\left\{ +sa F_{k+1}^{R(1,1,r,s)} \left\{ \left(F_{n-2}^{(1,1,r,s)} \right)^{\chi(n-1)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{1-\chi(n-1)} \right. \right.}{F_{k+1}^{(1,1,r,s)}} \left. \left. + r\chi(n) s^{1-\chi(n)} \left(F_{n-3}^{(1,1,r,s)} \right)^{\chi(n-2)} \right. \right. \\ &= \frac{\left\{ \left(F_n^{(1,1,r,s)} b - F_n^{(1,1,r,s)} sa \times \left\{ \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} \right. \right.}{F_{k+1}^{(1,1,r,s)}} \left. \left. + sa F_{k+1}^{(1,1,r,s)} \left\{ \left(F_{n-2}^{(1,1,r,s)} \right)^{1-\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{\chi(n)} \right. \right. \right. \\ &= \frac{\left\{ +r\chi(n) s^{1-\chi(n)} \left(F_{n-3}^{(1,1,r,s)} \right)^{\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} \right)^{1-\chi(n)} \right\}}{F_{k+1}^{(1,1,r,s)}} \end{aligned}$$

We calculate the value of third term of numerator separately. Now, when n is odd, we get

$$\begin{aligned} & \left(F_{n-2}^{(1,1,r,s)} \right)^{1-\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{\chi(n)} \\ & + r\chi(n) s^{1-\chi(n)} \left(F_{n-3}^{(1,1,r,s)} \right)^{\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} \right)^{1-\chi(n)} \\ &= F_{n-2}^{(1,1,r,s)} + s \left(\sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} \right) \\ &= F_{n-2}^{(1,1,r,s)} + s \left(s^{\lfloor \frac{n-4}{2} \rfloor} F_0^{(1,1,r,s)} + s^{\lfloor \frac{n-6}{2} \rfloor} F_1^{(1,1,r,s)} + s^{\lfloor \frac{n-8}{2} \rfloor} F_2^{(1,1,r,s)} \right. \\ & \quad \left. + \dots + F_{n-4}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} \right) \end{aligned}$$

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$$\begin{aligned}
&= F_{n-2}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} F_0^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} F_1^{(1,1,r,s)} + s^{\lfloor \frac{n-6}{2} \rfloor} F_2^{(1,1,r,s)} \\
&+ \dots + s F_{n-4}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} \\
&= \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} s^{\lfloor \frac{n-2}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} \\
&= \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} s^{\lfloor \frac{n-2}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} \right)^{1-\chi(n)}.
\end{aligned}$$

Also, when n is even, we get

$$\begin{aligned}
&\left(F_{n-2}^{(1,1,r,s)} \right)^{1-\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right)^{\chi(n)} \\
&\quad + r \chi(n) s^{1-\chi(n)} \left(F_{n-3}^{(1,1,r,s)} \right)^{\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} \right)^{1-\chi(n)} \\
&= \sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} + r F_{n-3}^{(1,1,r,s)}.
\end{aligned}$$

To prove the main result, we need to prove that this value should be $F_{n-1}^{(1,1,r,s)}$. Clearly this result holds for $n = 3$, since $F_2^{(1,1,r,s)} = 1$. Assume that this result holds for all positive integers not exceeding n . Now,

$$\begin{aligned}
F_{n-2}^{(1,1,r,s)} &= \sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} + r F_{n-4}^{(1,1,r,s)} \text{ and} \\
F_{n-3}^{(1,1,r,s)} &= \sum_{m=0}^{\lfloor \frac{n-5}{2} \rfloor} s^{\lfloor \frac{n-5}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-5}{2} \rfloor} + r F_{n-5}^{(1,1,r,s)}. \\
\text{Now, } F_{n-2}^{(1,1,r,s)} + s F_{n-3}^{(1,1,r,s)} &= \sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} + r F_{n-4}^{(1,1,r,s)} \\
&+ s \left(\sum_{m=0}^{\lfloor \frac{n-5}{2} \rfloor} s^{\lfloor \frac{n-5}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-5}{2} \rfloor} + r F_{n-5}^{(1,1,r,s)} \right) \\
&= \sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} + r F_{n-4}^{(1,1,r,s)} \\
&+ \sum_{m=0}^{\lfloor \frac{n-5}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} + r s F_{n-5}^{(1,1,r,s)} \\
&= \left(\sum_{m=0}^{\lfloor \frac{n-4}{2} \rfloor} s^{\lfloor \frac{n-4}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-4}{2} \rfloor} + \sum_{m=0}^{\lfloor \frac{n-5}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} \right) \\
&+ r \left(F_{n-4}^{(1,1,r,s)} + s F_{n-5}^{(1,1,r,s)} \right) \\
&= \sum_{m=0}^{\lfloor \frac{n-3}{2} \rfloor} s^{\lfloor \frac{n-3}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-3}{2} \rfloor} + r F_{n-3}^{(1,1,r,s)} = F_{n-1}^{(1,1,r,s)} = \left(F_{n-1}^{(1,1,r,s)} \right)^{\chi(n)}.
\end{aligned}$$

Therefore, we have

$$x_n = x_{n-1} + r^{\chi(n)} s^{1-\chi(n)} x_{n-2} \\ = \frac{\left\{ F_n^{(1,1,r,s)} b - F_n^{(1,1,r,s)} s a \left(F_k^{(1,1,r,s)} \right)^{1-\chi(k)} \left(\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} s^{\lfloor \frac{k-1}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{k-1}{2} \rfloor} \right)^{\chi(k)} \right\} + s a F_{k+1}^{(1,1,r,s)} \left(F_{n-1}^{(1,1,r,s)} \right)^{\chi(n)} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} s^{\lfloor \frac{n-2}{2} \rfloor - m} F_{2m}^{(1,1,r,s)} + s^{\lfloor \frac{n-2}{2} \rfloor} \right)^{1-\chi(n)}}{F_{k+1}^{(1,1,r,s)}}, \text{ which}$$

proves the result for every positive integer n .

7 Conclusion

In this paper we derived the general formula which gives the value of any inserted term x_n ($1 \leq n \leq k$) between the given fixed positive integers a, b so that terms of the sequence $a, x_1, x_2, x_3, \dots, x_k, b$ satisfies the recurrence relation of $F_n(p, q)$ and various bifurcating subsequence of $\{F_n^{(p,q,r,s)}\}$ by considering some fixed values of p, q, r, s .

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*Insertion of terms satisfying the recurrence relations of Horadam sequence
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Topological indices of some anticancer drugs

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Abstract

Every sixth death in the world is due to cancer, making it the second leading cause of death. Around one-third of deaths from cancer are due to tobacco use, high body mass index, alcohol use, low fruit and vegetable intake, and lack of physical activity. Anticancer drugs are those which are used to cure malignant disease i.e., cancer. Topological indices are used to model physico-chemical properties and biological activities of chemical compounds. In this paper, we compute *M-polynomial* and *NM-polynomial* of anticancer drugs. Further, we retrieve some degree and neighborhood degree based topological indices of anticancer drugs from their respective *M-polynomial* and *NM-polynomial*. The theoretical results obtained in this article have promising aspects in designing novel drug for the treatment of cancer. This study may assist chemists and pharmaceutical industry workers forecast the features of anticancer drugs without experimenting.

Keywords: M-polynomial; NM-polynomial; degree-based topological indices; neighborhood degree-based topological indices; edge partitions; graph invariant.

2020 AMS subject classifications: 05C07, 05C31. ¹

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1 Introduction

Up to now, cancer remains a global and serious public health challenge. Cancer is the uncontrolled growth of abnormal cells in the body. Cancer develops when the body's normal control mechanism stops working. Old cells do not die and instead grow out of control, forming new, abnormal cells. It is estimated that there are more than 200 different types of cancer, generally named according to the tissue where the cancer was recognized for the first time. Any drug that is effective in the treatment of malignant, or cancerous, disease is called anticancer drug. There are several major classes of anticancer drugs; these include alkylating agents, antimetabolites, natural products, and hormones.

It is always interesting to find some properties of graphs which are invariant. Topological indices and polynomials are foremost among them. Over the last decade there are numerous research papers devoted to topological indices and polynomials. Several topological indices have been defined in the literature. For details of topological indices one can refer to [15, 19]. For different topological indices and their applications one can refer to [2, 3, 10–12]. Some topological indices are already computed in literature on various anticancer drugs to determine physico-chemical properties [9, 13, 24].

Molecular topological indices are important in mathematical chemistry, particularly in the study of quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR). It is an effective technique to eliminate expensive and time-consuming laboratory studies. Information on physicochemical qualities and biological activity of molecular graphs of compounds is required in pharmaceutical drug design. These features may be predicted using the topological index, a well-known method in chemical graph theory.

There are many graph polynomials refer to [5, 28]. The Hosoya polynomial is the most well-known polynomial which plays a vital role in determining distance-based topological indices such as Wiener index [27], hyper Wiener index [5] of graphs. There are various types of algebraic polynomials are found in the literature. Some examples for polynomials in the theory of chemical graphs are PI polynomial [1], Tutte polynomial [7], Theta polynomial [8], Schultz polynomial [17], Hub polynomial [25], Gourava and hyper-Gourava polynomial [4], M-polynomial [6], NM-polynomial [20] etc.

Throughout this paper, by a graph $G = (V, E)$ we mean a simple, undirected, finite graph of order n and size m . Let $V(G)$ and $E(G)$ denote the vertex set and an edge set, respectively. For undefined graph theoretic terminologies and notions, refer to [16, 18, 26].

The M -polynomial [6] is one among other algebraic polynomials which was introduced in 2015 and useful in determining many degree-based topological in-

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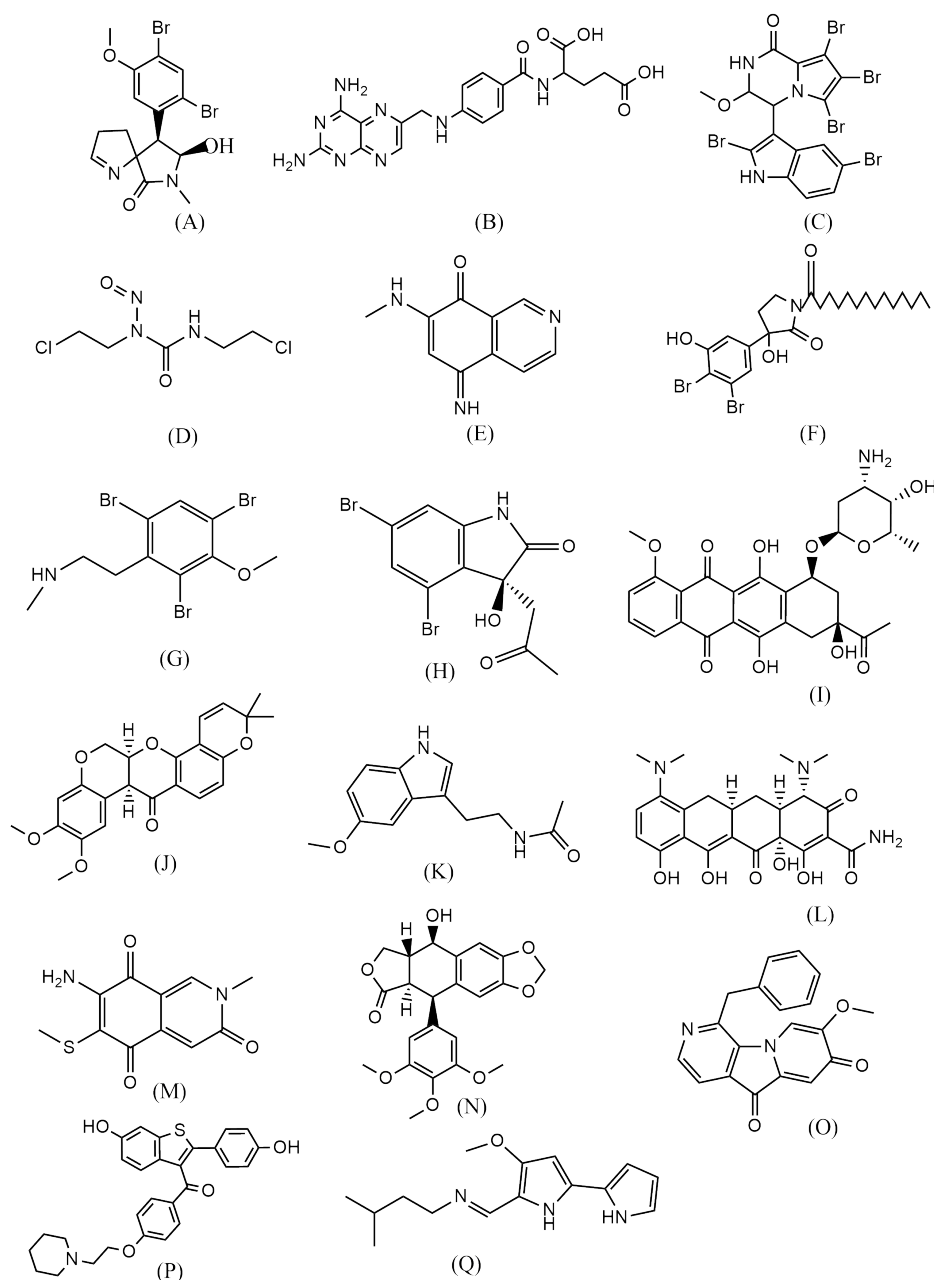


Figure 1: Molecular structure of anticancer drugs (A) Amathaspiramide E; (B) Aminopterin; (C) Aspidostomide E; (D) Carmustine; (E) caulibugulone E; (F) Convolutamide A; (G) Convolutamine F; (H) convolutamydine A; (I) Daunorubicin; (J) Deguelin; (K) Melatonin; (L) Minocycline; (M) Perfragilin A; (N) Podophyllotoxin; (O) Pterocellin; (P) Raloxifene; (Q) Tambjamine K.

dices (listed in Table 2) [15, 19]. Recently, the study of M -polynomial is reported in [21–23]. Let $\psi(u)$ denote the degree of vertex u of a graph G .

Definition 1.1. [6] If G is a graph, then M – polynomial of G is defined as

$$M(G; x, y) = \sum_{i \leq j} m_{ij}(G) x^i y^j, \quad (1)$$

where $m_{ij}, (i, j \geq 1)$, is the number [14] of edges uv in G such that $\psi(u) = i$ and $\psi(v) = j$.

Neighborhood M -polynomial [20] play the similar role for neighborhood degree sum-based indices. Let $\Psi(u)$ denote the degree sum of all vertices of G that are adjacent u . We call $\Psi(u)$ as neighborhood degree sum of u in G .

Definition 1.2. [20] If G is a graph, then NM – polynomial of G is defined as

$$NM(G; x, y) = \sum_{i \leq j} m_{ij}^*(G) x^i y^j, \quad (2)$$

where $m_{ij}^*(i, j \geq 1)$ is the number of edges uv in G such that $\Psi(u) = i$ and $\Psi(v) = j$.

D	$g(\psi(u), \psi(v))$	ND	$g(\Psi(u), \Psi(v))$
First Zagreb index $M_1(G)$	$\psi(u) + \psi(v)$	Third version Zagreb index $M_1'(G)$	$\Psi(u) + \Psi(v)$
Second Zagreb index $M_2(G)$	$\psi(u)\psi(v)$	Neighborhood second Zagreb index $M_2^*(G)$	$\Psi(u)\Psi(v)$
Forgotten topological index $F(G)$	$\psi(u)^2 + \psi(v)^2$	Neighborhood Forgotten topological index $F_N^*(G)$	$\Psi(u)^2 + \Psi(v)^2$
Second modified Zagreb index ${}^m M_2(G)$	$\frac{1}{\psi(u)\psi(v)}$	Neighborhood second modified Zagreb index ${}^{nm} M_2(G)$	$\frac{1}{\Psi(u)\Psi(v)}$
Randić index $R_\alpha(G)$	$(\psi(u)\psi(v))^\alpha$	Neighborhood Randić index $NR_\alpha(G)$	$(\Psi(u)\Psi(v))^\alpha$
Redefined third Zagreb index $ReZG_3(G)$	$(\psi(u) + \psi(v))\psi(u)\psi(v)$	Third NDe index $ND_3(G)$	$(\Psi(u) + \Psi(v))\Psi(u)\Psi(v)$
Symmetric division index $SDD(G)$	$\frac{\psi^2(u) + \psi^2(v)}{\psi(u)\psi(v)}$	Fifth NDe index $ND_5(G)$	$\frac{\Psi^2(u) + \Psi^2(v)}{\Psi(u)\Psi(v)}$
Harmonic index $H(G)$	$\frac{2}{\psi(u) + \psi(v)}$	Neighborhood Harmonic index $NH(G)$	$\frac{2}{\Psi(u) + \Psi(v)}$
Inverse sum indeg index $I(G)$	$\frac{\psi(u)\psi(v)}{(\psi(u) + \psi(v))}$	Neighborhood Inverse sum indeg index $NI(G)$	$\frac{\Psi(u)\Psi(v)}{(\Psi(u) + \Psi(v))}$
Augmented Zagreb index $A(G)$	$\left\{ \frac{\psi(u)\psi(v)}{(\psi(u) + \psi(v))} \right\}^3$	Sanskriti index $S(G)$	$\left\{ \frac{\Psi(u)\Psi(v)}{(\Psi(u) + \Psi(v))} \right\}^3$

Table 1: Description of some topological indices.

Degree based(D) and neighborhood degree sum-based (ND) topological indices defined on edge set $E(G)$ of a graph G can be expressed as

$$D(G) = \sum_{uv \in E(G)} g(\psi(u), \psi(v)), ND(G) = \sum_{uv \in E(G)} g(\Psi(u), \Psi(v)),$$

where the formulation of $g(\psi(u), \psi(v))$ and $g(\Psi(u), \Psi(v))$ are given in Table 1.

Topological indices of some anticancer drugs

Topological Index	Derivation from $M(G; x, y)$	Topological Index	Derivation from $NM(G; x, y)$
$M_1(G)$	$(D_x + D_y)(M(G; x, y)) _{x=y=1}$	$M'_1(G)$	$(D_x + D_y)(M(G; x, y)) _{x=y=1}$
$M_2(G)$	$(D_x D_y)(M(G; x, y)) _{x=y=1}$	$M_2^*(G)$	$(D_x D_y)(M(G; x, y)) _{x=y=1}$
$F(G)$	$(D_x^2 + D_y^2)(M(G; x, y)) _{x=y=1}$	$F_N^*(G)$	$(D_x^2 + D_y^2)(M(G; x, y)) _{x=y=1}$
${}^m M_2(G)$	$(I_x I_y)(M(G; x, y)) _{x=y=1}$	${}^{nm} M_2(G)$	$(I_x I_y)(M(G; x, y)) _{x=y=1}$
$R_\alpha(G)$	$(D_x^\alpha D_y^\alpha)(M(G; x, y)) _{x=y=1}$	$NR_\alpha(G)$	$(D_x^\alpha D_y^\alpha)(M(G; x, y)) _{x=y=1}$
$ReZG_3(G)$	$D_x D_y (D_x + D_y)(M(G; x, y)) _{x=y=1}$	$ND_3(G)$	$D_x D_y (D_x + D_y)(M(G; x, y)) _{x=y=1}$
$SDD(G)$	$(D_x I_y + D_y I_x)(M(G; x, y)) _{x=y=1}$	$ND_5(G)$	$(D_x I_y + D_y I_x)(M(G; x, y)) _{x=y=1}$
$H(G)$	$2I_x J(M(G; x, y)) _{x=1}$	$NH(G)$	$2I_x J(M(G; x, y)) _{x=1}$
$I(G)$	$I_x J D_x D_y (M(G; x, y)) _{x=1}$	$NI(G)$	$I_x J D_x D_y (M(G; x, y)) _{x=1}$
$A(G)$	$I_x^3 Q_{-2} J D_x^2 D_y^3 (M(G; x, y)) _{x=1}$	$S(G)$	$I_x^3 Q_{-2} J D_x^2 D_y^3 (M(G; x, y)) _{x=1}$

Table 2: Operations to derive degree-based topological indices from M -polynomial and NM -polynomial.

Where $D_x = x \frac{\partial f(x,y)}{\partial x}$, $D_y = y \frac{\partial f(x,y)}{\partial y}$, $I_x = \int_0^x \frac{f(t,y)}{t} dt$, $I_y = \int_0^y \frac{f(x,t)}{t} dt$, $J(f(x, y)) = f(x, x)$ and $Q_\alpha(f(x, y)) = x^\alpha f(x, y)$ are the operators.

Molecular topological indices are important in mathematical chemistry, particularly in the study of quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR). It is an effective technique to eliminate expensive and time-consuming laboratory studies. Information on physicochemical qualities and biological activity of molecular graphs of compounds is required in pharmaceutical drug design. These features may be predicted using the topological index, a well-known method in chemical graph theory. This motivates us to study M and NM polynomials from which we retrieve topological indices.

2 Main Results

In this section, we obtain the expression for M and NM -polynomials of molecular graphs of some anticancer drugs using edge partition technique, degree and neighborhood counting method.

Theorem 2.1. *Let G be the molecular graph of amathaspiramide E. Then,*

$$(i) \quad M(G; x, y) = xy^2 + 5xy^3 + 3x^2y^2 + 5x^2y^3 + 2x^2y^4 + 6x^3y^3 + 2x^3y^4,$$

$$(ii) \quad NM(G; x, y) = x^2y^4 + x^3y^5 + x^3y^6 + 2x^3y^7 + x^3y^8 + x^4y^4 + 2x^4y^6 \\ + x^4y^7 + x^5y^6 + x^5y^8 + x^6y^6 + 2x^6y^7 + x^6y^8 + 2x^6y^{10} \\ + x^7y^7 + x^7y^8 + x^7y^{10} + 2x^8y^{10} + x^{10}y^{10}.$$

Proof. Let G be the molecular graph of amathaspiramide E (Figure 1). It has 24 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j ,

i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1$, $m_{(1,3)} = 5$, $m_{(2,2)} = 3$, $m_{(2,3)} = 5$, $m_{(3,3)} = 6$, $m_{(3,4)} = 2$ and $m_{(2,4)} = 2$.

Thus, the M – polynomial of G is

$$\begin{aligned} M(G; x, y) &= \sum_{i \leq j} m_{(i,j)} x^i y^j \\ &= m_{(1,2)} x^1 y^2 + m_{(1,3)} x^1 y^3 + m_{(2,2)} x^2 y^2 + m_{(2,3)} x^2 y^3 \\ &\quad + m_{(2,4)} x^2 y^4 + m_{(3,3)} x^3 y^3 + m_{(3,4)} x^3 y^4 \\ &= xy^2 + 5xy^3 + 3x^2y^2 + 5x^2y^3 + 2x^2y^4 + 6x^3y^3 + 2x^3y^4. \end{aligned}$$

Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1$, $m_{(3,5)}^* = 1$, $m_{(3,6)}^* = 1$, $m_{(3,7)}^* = 2$, $m_{(3,8)}^* = 1$, $m_{(4,4)}^* = 1$, $m_{(2,4)}^* = 6$, $m_{(4,7)}^* = 1$, $m_{(5,6)}^* = 1$, $m_{(5,8)}^* = 1$, $m_{(6,6)}^* = 1$, $m_{(6,7)}^* = 2$, $m_{(6,8)}^* = 1$, $m_{(6,10)}^* = 2$, $m_{(7,7)}^* = 1$, $m_{(7,8)}^* = 1$, $m_{(7,10)}^* = 1$, $m_{(8,10)}^* = 2$ and $m_{(10,10)}^* = 1$.

Thus, the NM – polynomial of G is

$$\begin{aligned} NM(G; x, y) &= \sum_{i \leq j} m_{(i,j)}^* x^i y^j \\ &= m_{(2,4)}^* x^2 y^4 + m_{(3,5)}^* x^3 y^5 + m_{(3,6)}^* x^3 y^6 + m_{(3,7)}^* x^3 y^7 + m_{(3,8)}^* x^3 y^8 \\ &\quad + m_{(4,4)}^* x^4 y^4 + m_{(4,6)}^* x^4 y^6 + m_{(4,7)}^* x^4 y^7 + m_{(5,6)}^* x^5 y^6 + m_{(5,8)}^* x^5 y^8 \\ &\quad + m_{(6,6)}^* x^6 y^6 + m_{(6,7)}^* x^6 y^7 + m_{(6,8)}^* x^6 y^8 + m_{(6,10)}^* x^6 y^{10} \\ &\quad + m_{(7,7)}^* x^7 y^7 + m_{(7,8)}^* x^7 y^8 + m_{(7,10)}^* x^7 y^{10} + m_{(8,10)}^* x^8 y^{10} + m_{(10,10)}^* x^{10} y^{10} \\ &= x^2 y^4 + x^3 y^5 + x^3 y^6 + 2x^3 y^7 + x^3 y^8 + x^4 y^4 + 2x^4 y^6 + x^4 y^7 + x^5 y^6 + x^5 y^8 \\ &\quad + x^6 y^6 + 2x^6 y^7 + x^6 y^8 + 2x^6 y^{10} + x^7 y^7 + x^7 y^8 + x^7 y^{10} + 2x^8 y^{10} + x^{10} y^{10}. \end{aligned}$$

□

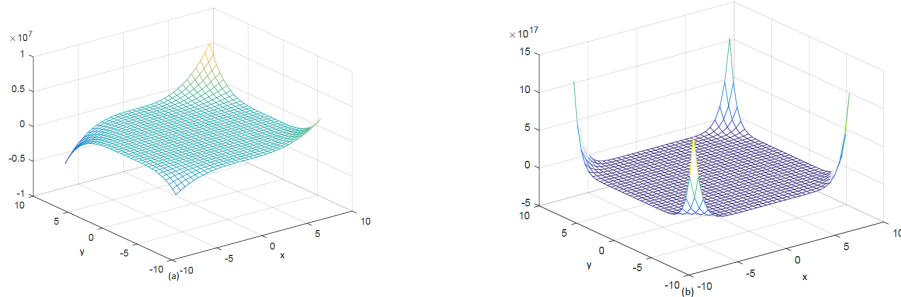


Figure 2: Plotting of (a) M-polynomial and (b) NM-polynomial of amathaspi-ramide E.

Corollary 2.1. *Let G be the molecular graph of amathaspiramide E. Then*

- (i) $M_1(G) = 122, M_1'(G) = 303,$
- (ii) $M_2(G) = 153, M_2^*(G) = 992,$
- (iii) $F(G) = 342, F_N^*(G) = 2141,$
- (iv) ${}^m M_2(G) = 4.833, {}^{nm} M_2(G) = 0.832,$
- (v) $R_\alpha(G) = 2^\alpha + 5(3)^\alpha + 3(4)^\alpha + 5(6)^\alpha + 2(8)^\alpha + 6(9)^\alpha + 2(12)^\alpha, NR_\alpha(G) = (8)^\alpha + (15)^\alpha + (18)^\alpha + 2(21)^\alpha + (24)^\alpha + (16)^\alpha + 2(24)^\alpha + (28)^\alpha + (30)^\alpha + (40)^\alpha + (36)^\alpha + 2(42)^\alpha + (48)^\alpha + 2(60)^\alpha + (49)^\alpha + (56)^\alpha + (70)^\alpha + 2(80)^\alpha + (100)^\alpha,$
- (vi) $ReZG_3(G) = 852, ND_3(G) = 14492,$
- (vii) $SDD(G) = 57.166, ND_5(G) = 53.65,$
- (viii) $H(G) = 9.904, NH(G) = 4.13,$
- (ix) $I(G) = 28.511, NI(G) = 72.36,$
- (x) $A(G) = 2572.2, S(G) = 852.94.$

Proof. Let $M(G) = f(x, y) = xy^2 + 5xy^3 + 3x^2y^2 + 5x^2y^3 + 2x^2y^4 + 6x^3y^3 + 2x^3y^4$. Then, we have

$$\begin{aligned}
 (D_x + D_y)(f(x, y)) &= 3xy^2 + 20xy^3 + 12x^2y^2 + 25x^2y^3 + 12x^2y^4 + 36x^3y^3 + 14x^3y^4, \\
 D_x D_y(f(x, y)) &= 2xy^2 + 15xy^3 + 12x^2y^2 + 30x^2y^3 + 16x^2y^4 + 54x^3y^3 + 24x^3y^4, \\
 (D_x^2 + D_y^2)(f(x, y)) &= 5xy^2 + 50xy^3 + 24x^2y^2 + 65x^2y^3 + 40x^2y^4 + 108x^3y^3 + 50x^3y^4, \\
 I_x I_y(f(x, y)) &= \frac{1}{2}xy^2 + \frac{5}{3}xy^3 + \frac{3}{4}x^2y^2 + \frac{5}{6}x^2y^3 + \frac{1}{4}x^2y^4 + \frac{2}{3}x^3y^3 + \frac{1}{6}x^3y^4, \\
 D_x^\alpha D_y^\alpha(f(x, y)) &= 2^\alpha xy^2 + 5(3)^\alpha xy^3 + 3(4)^\alpha x^2y^2 + 5(6)^\alpha x^2y^3 + 2(8)^\alpha x^2y^4 + 6(9)^\alpha x^3y^3 \\
 &\quad + 2(12)^\alpha x^3y^4, \\
 D_x D_y(D_x + D_y)(f(x, y)) &= 6xy^2 + 60xy^3 + 48x^2y^2 + 150x^2y^3 + 96x^2y^4 + 324x^3y^3 \\
 &\quad + 168x^3y^4, \\
 (D_x I_y + I_x D_y)(f(x, y)) &= \frac{5}{2}xy^2 + \frac{50}{3}xy^3 + 6x^2y^2 + \frac{65}{6}x^2y^3 + \frac{5}{x}x^2y^4 + 12x^3y^3 + \frac{25}{6}x^3y^4, \\
 I_x J(f(x, y)) &= \frac{2}{3}x^3 + 4x^4 + 2x^5 + \frac{8}{3}x^6 + \frac{4}{7}x^7, \\
 I_x J D_x D_y(f(x, y)) &= \frac{2}{3}x^3 + \frac{27}{4}x^4 + 6x^5 + \frac{35}{3}x^6 + \frac{24}{7}x^7, \\
 I_x^3 Q_{-2} J D_x^3 D_y^3(f(x, y)) &= 8x + \frac{327}{2}x^2 + 360x^3 + \frac{5398}{4}x^4 + \frac{3456}{5}x^5
 \end{aligned}$$

Using Table 2, we have

$$\begin{aligned}
 M_1(G) &= 3xy^2 + 20xy^3 + 12x^2y^2 + 25x^2y^3 + 12x^2y^4 + 36x^3y^3 + 14x^3y^4|_{x=y=1} = 122, \\
 M_2(G) &= 2xy^2 + 15xy^3 + 12x^2y^2 + 30x^2y^3 + 16x^2y^4 + 54x^3y^3 + 24x^3y^4|_{x=y=1} = 153,
 \end{aligned}$$

$$F(G) = 5xy^2 + 50xy^3 + 24x^2y^2 + 65x^2y^3 + 40x^2y^4 + 108x^3y^3 + 50x^3y^4|_{x=y=1} = 342,$$

$${}^mM_2(G) = \frac{1}{2}xy^2 + \frac{5}{3}xy^3 + \frac{3}{4}x^2y^2 + \frac{5}{6}x^2y^3 + \frac{1}{4}x^2y^4 + \frac{2}{3}x^3y^3 + \frac{1}{6}x^3y^4|_{x=y=1} = 4.833,$$

$$\begin{aligned} R_\alpha &= 2^\alpha xy^2 + 5(3)^\alpha xy^3 + 3(4)^\alpha x^2y^2 + 5(6)^\alpha x^2y^3 + 2(8)^\alpha x^2y^4 + 6(9)^\alpha x^3y^3 \\ &\quad + 2(12)^\alpha x^3y^4|_{x=y=1} = 2^\alpha + 5(3)^\alpha + 3(4)^\alpha + 5(6)^\alpha + 2(8)^\alpha + 6(9)^\alpha \\ &\quad + 2(12)^\alpha, \end{aligned}$$

$$\begin{aligned} ReZG_3(G) &= 6xy^2 + 60xy^3 + 48x^2y^2 + 150x^2y^3 + 96x^2y^4 + 324x^3y^3 \\ &\quad + 168x^3y^4|_{x=y=1} = 852, \end{aligned}$$

$$SDD(G) = \frac{5}{2}xy^2 + \frac{50}{3}xy^3 + 6x^2y^2 + \frac{65}{6}x^2y^3 + \frac{5}{x}y^4 + 12x^3y^3 + \frac{25}{6}x^3y^4|_{x=y=1} = 57.166,$$

$$H(G) = \frac{2}{3}x^3 + 4x^4 + 2x^5 + \frac{8}{3}x^6 + \frac{4}{7}x^7|_{x=1} = 9.904,$$

$$I(G) = \frac{2}{3}x^3 + \frac{27}{4}x^4 + 6x^5 + \frac{35}{3}x^6 + \frac{24}{7}x^7|_{x=1} = 28.511,$$

$$A(G) = 8x + \frac{327}{2}x^2 + 360x^3 + \frac{5398}{4}x^4 + \frac{3456}{5}x^5|_{x=1} = 2572.2.$$

For neighborhood degree sum-based indices, we consider $g(x, y) = NM(G) = x^2y^4 + x^3y^5 + x^3y^6 + 2x^3y^7 + x^3y^8 + x^4y^4 + 2x^4y^6 + x^4y^7 + x^5y^6 + x^5y^8 + x^6y^6 + 2x^6y^7 + x^6y^8 + 2x^6y^{10} + x^7y^7 + x^7y^8 + x^7y^{10} + x^8y^{10} + x^{10}y^{10}$. Then, applying the above operations and Table 2, we can easily obtain the neighborhood degree sum-based indices. This completes the proof. $\square$

Theorem 2.2. *Let G be the molecular graph of aminopterin. Then,*

$$(i) \quad M(G; x, y) = 7xy^3 + 5x^2y^2 + 18x^2y^3 + 4x^3y^3,$$

$$\begin{aligned} (ii) \quad NM(G; x, y) &= 3x^3y^4 + 2x^3y^5 + 2x^3y^6 + x^4y^5 + 2x^4y^6 + 5x^5y^5 \\ &\quad + 5x^5y^6 + 5x^5y^7 + 3x^6y^6 + 3x^6y^7 + 2x^6y^8 + x^7y^8. \end{aligned}$$

Proof. Let G be the molecular graph of aminopterin (Figure 1). It has 34 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,3)} = 7$, $m_{(2,2)} = 5$, $m_{(2,3)} = 18$ and $m_{(3,3)} = 4$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(3,4)}^* = 3$, $m_{(3,5)}^* = 2$, $m_{(3,6)}^* = 2$, $m_{(4,5)}^* = 1$, $m_{(4,6)}^* = 2$, $m_{(5,5)}^* = 5$, $m_{(5,6)}^* = 5$, $m_{(5,7)}^* = 5$, $m_{(6,6)}^* = 3$, $m_{(6,7)}^* = 3$, $m_{(6,8)}^* = 2$ and $m_{(7,8)}^* = 1$. The proof is similar as that of Theorem 2.1. $\square$

Topological indices of some anticancer drugs

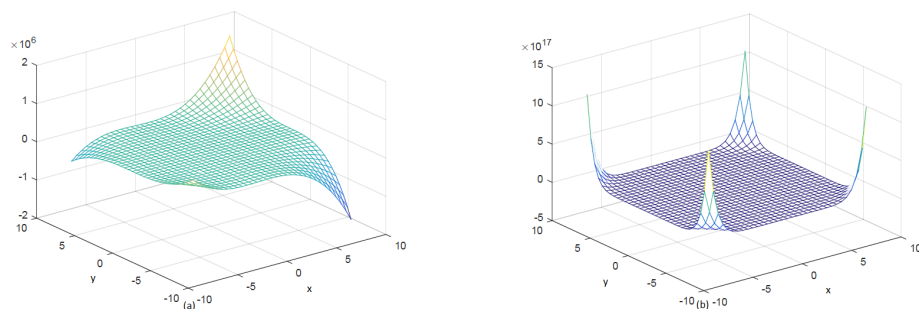


Figure 3: Plotting of (a) M-polynomial and (b) NM-polynomial of aminopterin.

Corollary 2.2. *Let G be the molecular graph of aminopterin. Then*

- (i) $M_1(G) = 162, M'_1(G) = 367,$
- (ii) $M_2(G) = 185, M_2^*(G) = 1006,$
- (iii) $F(G) = 416, F_N^*(G) = 2087,$
- (iv) ${}^mM_2(G) = 7.027, {}^{nm}M_2(G) = 1.35,$
- (v) $R_\alpha(G) = 7(3)^\alpha + 5(4)^\alpha + 18(6)^\alpha + 4(9)^\alpha, NR_\alpha(G) = 3(12)^\alpha + 2(15)^\alpha + 2(18)^\alpha + (20)^\alpha + 2(24)^\alpha + 5(25)^\alpha + 5(30)^\alpha + 5(35)^\alpha + 3(36)^\alpha + 3(42)^\alpha + 2(48)^\alpha + (56)^\alpha,$
- (vi) $ReZG_3(G) = 920, ND_3(G) = 11594,$
- (vii) $SDD(G) = 80.333, ND_5(G) = 71.16,$
- (viii) $H(G) = 7.266, NH(G) = 6.54,$
- (ix) $I(G) = 37.85, NI(G) = 89.91,$
- (x) $A(G) = 253.187, S(G) = 701.33.$

Theorem 2.3. *Let G be the molecular graph of aspidostomide E. Then,*

- (i) $M(G; x, y) = xy^2 + 6xy^3 + x^2y^2 + 9x^2y^3 + 12x^3y^3,$
- (ii) $NM(G; x, y) = x^2y^4 + x^3y^5 + 2x^3y^6 + 3x^3y^7 + x^4y^7 + 2x^5y^5 + x^5y^6 + x^5y^7 + 2x^6y^6 + 2x^6y^7 + x^6y^8 + 2x^6y^9 + 2x^7y^7 + x^7y^8 + 3x^7y^9 + x^8y^9 + 3x^9y^9.$

Proof. Let G be the molecular graph of aspidostomide E (Figure 1). It has 29 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1, m_{(1,3)} = 6, m_{(2,2)} = 1, m_{(2,3)} = 9$ and $m_{(3,3)} = 12$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1, m_{(3,5)}^* = 1, m_{(3,6)}^* = 2, m_{(3,7)}^* = 3, m_{(4,7)}^* = 1, m_{(5,5)}^* = 2, m_{(5,6)}^* = 1, m_{(5,7)}^* = 1, m_{(6,6)}^* = 2, m_{(6,7)}^* = 2, m_{(6,8)}^* = 1, m_{(6,9)}^* = 2, m_{(7,7)}^* = 1, m_{(7,8)}^* = 1, m_{(7,9)}^* = 3, m_{(8,9)}^* = 1$ and $m_{(9,9)}^* = 3$. Using the edge partition of G , the M and NM polynomials can be obtained easily. $\square$

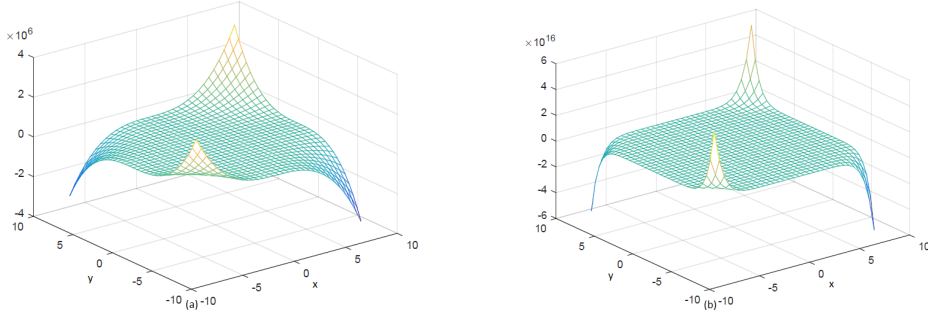


Figure 4: Plotting of (a) M-polynomial and (b) NM-polynomial of aspidostomide E.

Corollary 2.3. Let G be the molecular graph of aspidostomide E. Then

- (i) $M_1(G) = 148, M_1'(G) = 372,$
- (ii) $M_2(G) = 186, M_2^*(G) = 1235,$
- (iii) $F(G) = 406, F_N^*(G) = 2596,$
- (iv) ${}^m M_2(G) = 5.583, {}^{nm} M_2(G) = 0.94,$
- (v) $R_\alpha(G) = 2^\alpha + 6(3)^\alpha + 4^\alpha + 9(6)^\alpha + 12(9)^\alpha, NR_\alpha(G) = (8)^\alpha + (15)^\alpha + 2(18)^\alpha + 3(21)^\alpha + (28)^\alpha + 2(25)^\alpha + (30)^\alpha + (35)^\alpha + 2(36)^\alpha + 2(42)^\alpha + (48)^\alpha + 2(54)^\alpha + (56)^\alpha + 3(63)^\alpha + 2(49)^\alpha + (72)^\alpha + 3(81)^\alpha,$
- (vi) $ReZG_3(G) = 1012, ND_3(G) = 17762,$
- (vii) $SDD(G) = 68, ND_5(G) = 63.20,$
- (viii) $H(G) = 5.883, NH(G) = 4.85,$

$$(ix) \ I(G) = 34.966, \ NI(G) = 90.06,$$

$$(x) \ A(G) = 244.937, \ S(G) = 1070.91.$$

Theorem 2.4. *Let G be the molecular graph of carmustine. Then*

$$(i) \ M(G; x, y) = 3xy^2 + xy^3 + 3x^2y^2 + 3x^2y^3 + x^3y^3,$$

$$(ii) \ NM(G; x, y) = 2x^2y^3 + x^2y^4 + x^3y^4 + x^3y^5 + x^3y^6 + x^4y^5 + x^4y^7 \\ + x^5y^6 + x^5y^7 + x^6y^7.$$

Proof. Let G be the molecular graph of carmustine (Figure 1). It has 11 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 3$, $m_{(1,3)} = 1$, $m_{(2,2)} = 3$, $m_{(2,3)} = 3$ and $m_{(3,3)} = 1$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,3)}^* = 2$, $m_{(2,4)}^* = 1$, $m_{(3,4)}^* = 1$, $m_{(3,5)}^* = 1$, $m_{(3,6)}^* = 1$, $m_{(4,5)}^* = 1$, $m_{(4,7)}^* = 1$, $m_{(5,6)}^* = 1$, $m_{(5,7)}^* = 1$ and $m_{(6,7)}^* = 1$. The M and NM polynomials can be obtained by using the edge partition of G . $\square$

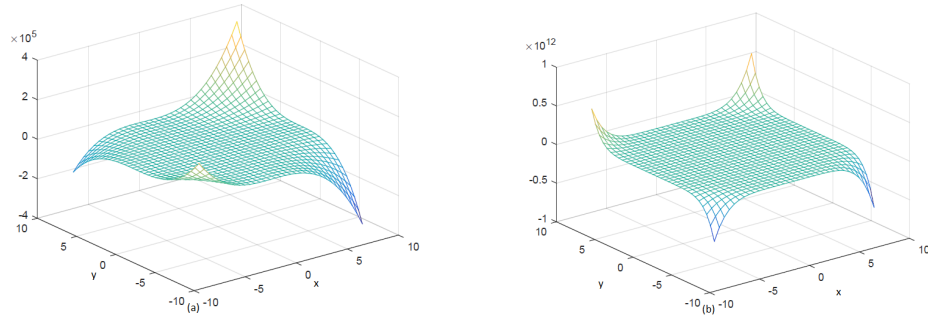


Figure 5: Plotting of (a) M-polynomial and (b) NM-polynomial of carmustine.

Corollary 2.4. *Let G be the molecular graph of carmustine. Then*

$$(i) \ M_1(G) = 46, \ M_1'(G) = 96,$$

$$(ii) \ M_2(G) = 48, \ M_2^*(G) = 220,$$

$$(iii) \ F(G) = 106, \ F_N^*(G) = 476,$$

$$(iv) \ {}^mM_2(G) = 3.194, \ {}^{nm}M_2(G) = 0.835,$$

- (v) $R_\alpha(G) = 3(2)^\alpha + (3)^\alpha + 3(4)^\alpha + 3(6)^\alpha + (9)^\alpha$, $NR_\alpha(G) = 2(6)^\alpha + (8)^\alpha + (12)^\alpha + (18)^\alpha + (15)^\alpha + (20)^\alpha + (28)^\alpha + (30)^\alpha + (35)^\alpha + (42)^\alpha$,
- (vi) $ReZG_3(G) = 222$, $ND_3(G) = 2258$,
- (vii) $SDD(G) = 25.33$, $ND_5(G) = 24.22$,
- (viii) $H(G) = 5.53$, $NH(G) = 2.79$,
- (ix) $I(G) = 10.85$, $NI(G) = 22.96$,
- (x) $A(G) = 12.86$, $S(G) = 131.74$.

Theorem 2.5. *Let G be the molecular graph of caulibugulone E. Then*

- (i) $M(G; x, y) = xy^2 + 2xy^3 + 3x^2y^2 + 5x^2y^3 + 4x^3y^3$,
- (ii) $NM(G; x, y) = x^2y^4 + x^3y^6 + x^3y^7 + x^4y^4 + 2x^4y^5 + x^4y^7 + 2x^5y^8 + x^6y^6 + x^6y^7 + x^7y^7 + x^7y^8 + x^8y^6 + x^8y^8$.

Proof. Let G be the molecular graph of caulibugulone E (Figure 1). It has 15 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1$, $m_{(1,3)} = 2$, $m_{(2,2)} = 3$, $m_{(2,3)} = 5$ and $m_{(3,3)} = 4$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1$, $m_{(3,6)}^* = 1$, $m_{(3,7)}^* = 1$, $m_{(4,4)}^* = 1$, $m_{(4,5)}^* = 2$, $m_{(4,7)}^* = 1$, $m_{(5,8)}^* = 2$, $m_{(6,6)}^* = 1$, $m_{(6,7)}^* = 1$, $m_{(6,8)}^* = 1$, $m_{(7,7)}^* = 1$, $m_{(7,8)}^* = 1$, $m_{(8,6)}^* = 1$ and $m_{(8,8)}^* = 1$. By making use of the edge partition of G , the M and NM polynomials can be computed. $\square$

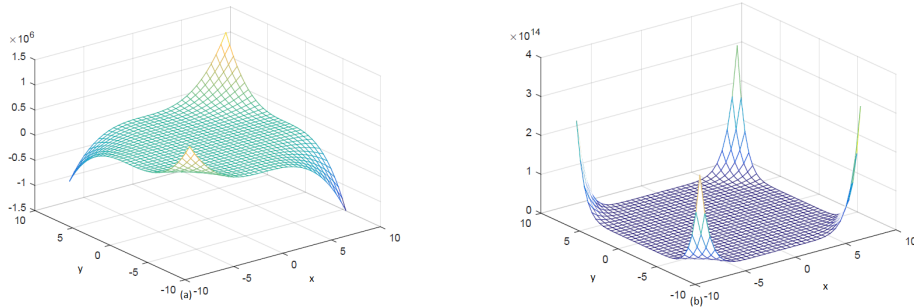


Figure 6: Plotting of (a) M-polynomial and (b) NM-polynomial of caulibugulone E.

Corollary 2.5. *Let G be the molecular graph of caulibugulone E. Then*

- (i) $M_1(G) = 72, M'_1(G) = 172,$
- (ii) $M_2(G) = 86, M_2^*(G) = 506,$
- (iii) $F(G) = 186, F_N^*(G) = 1076,$
- (iv) ${}^mM_2(G) = 3.194, {}^{nm}M_2(G) = 0.60,$
- (v) $R_\alpha(G) = (2)^\alpha + 2(3)^\alpha + 3(4)^\alpha + 5(6)^\alpha + 4(9)^\alpha, NR_\alpha(G) = (8)^\alpha + (18)^\alpha + (21)^\alpha + (16)^\alpha + 2(20)^\alpha + (28)^\alpha + 2(40)^\alpha + (36)^\alpha + (42)^\alpha + (48)^\alpha + (49)^\alpha + (56)^\alpha + (64)^\alpha,$
- (vi) $ReZG_3(G) = 444, ND_3(G) = 1072,$
- (vii) $SDD(G) = 34, ND_5(G) = 32.75,$
- (viii) $H(G) = 6.5, NH(G) = 2.80,$
- (ix) $I(G) = 17.16, NI(G) = 41.46,$
- (x) $A(G) = 26.28, S(G) = 384.26.$

Theorem 2.6. *Let G be the molecular graph of convolutamide A. Then*

- (i) $M(G; x, y) = xy^2 + 5xy^3 + xy^4 + 12x^2y^2 + 6x^2y^3 + x^2y^4 + 4x^3y^3 + 2x^3y^4,$
- (ii) $NM(G; x, y) = x^2y^3 + x^3y^4 + 3x^3y^6 + 2x^3y^7 + 9x^4y^4 + x^4y^5 + x^4y^9 + 2x^5y^6 + x^5y^8 + 2x^6y^6 + 2x^6y^7 + 3x^6y^8 + x^6y^9 + x^7y^8 + x^7y^9 + x^8y^9.$

Proof. Let G be the molecular graph of convolutamide A (Figure 1). It has 32 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1, m_{(1,3)} = 5, m_{(1,4)} = 1, m_{(2,2)} = 12, m_{(2,3)} = 6, m_{(2,4)} = 1, m_{(3,3)} = 4$ and $m_{(3,4)} = 2$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,3)}^* = 1, m_{(3,4)}^* = 1, m_{(3,6)}^* = 3, m_{(3,7)}^* = 2, m_{(4,4)}^* = 9, m_{(4,5)}^* = 1, m_{(4,9)}^* = 1, m_{(5,6)}^* = 2, m_{(5,8)}^* = 1, m_{(6,6)}^* = 2, m_{(6,7)}^* = 2, m_{(6,8)}^* = 3, m_{(6,9)}^* = 1, m_{(7,8)}^* = 1, m_{(7,9)}^* = 1$ and $m_{(8,9)}^* = 1$. Using the edge partition of G , the M and NM polynomials can be obtained easily. $\square$

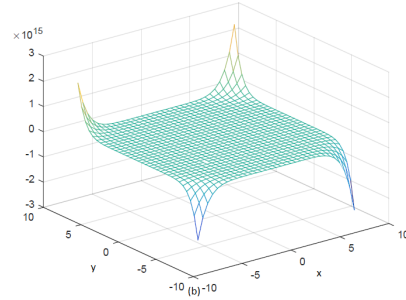


Figure 7: Plotting of (a) M-polynomial and (b) NM-polynomial of convolutamide A.

Corolary 2.6. Let G be the molecular graph of convolutamide A. Then

- (i) $M_1(G) = 150, M_1^*(G) = 343,$
- (ii) $M_2(G) = 173, M_2^*(G) = 959,$
- (iii) $F(G) = 388, F_N^*(G) = 2045,$
- (iv) ${}^m M_2(G) = 7.15, {}^{nm} M_2(G) = 1.47,$
- (v) $R_\alpha(G) = (2)^\alpha + 5(3)^\alpha + 13(4)^\alpha + 6(6)^\alpha + (8)^\alpha + 4(9)^\alpha + 2(12)^\alpha, NR_\alpha(G) = (6)^\alpha + (12)^\alpha + 3(18)^\alpha + 2(21)^\alpha + 9(16)^\alpha + (20)^\alpha + (36)^\alpha + 2(30)^\alpha + 2(36)^\alpha + 2(42)^\alpha + 3(48)^\alpha + (54)^\alpha + (56)^\alpha + (63)^\alpha + (72)^\alpha + (40)^\alpha,$
- (vi) $ReZG_3(G) = 890, ND_3(G) = 11854,$
- (vii) $SDD(G) = 75.07, ND_5(G) = 68.86,$
- (viii) $H(G) = 14.20, NH(G) = 6.47,$
- (ix) $I(G) = 35.17, NI(G) = 82.89,$
- (x) $A(G) = 51.23, S(G) = 701.26.$

Theorem 2.7. Let G be the molecular graph of convolutamine F. Then

- (i) $M(G; x, y) = 2xy^2 + 3xy^3 + 2x^2y^2 + 4x^2y^3 + 4x^3y^3,$
- (ii) $NM(G; x, y) = x^2y^3 + x^2y^4 + x^3y^4 + 2x^3y^6 + x^3y^7 + x^4y^5 + x^4y^8 + x^5y^8 + 2x^6y^6 + 2x^6y^8 + 2x^7y^8.$

Proof. Let G be the molecular graph of convolutamine F (Figure 1). It has 15 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in

$M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 2, m_{(1,3)} = 3, m_{(2,2)} = 2, m_{(2,3)} = 4$ and $m_{(3,3)} = 4$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,3)}^* = 1, m_{(2,4)}^* = 1, m_{(3,4)}^* = 1, m_{(3,6)}^* = 2, m_{(3,7)}^* = 1, m_{(4,5)}^* = 1, m_{(4,8)}^* = 1, m_{(5,8)}^* = 1, m_{(6,6)}^* = 2, m_{(6,8)}^* = 2$ and $m_{(7,8)}^* = 2$. The M and NM polynomials are obtained by using the edge partition of G . $\square$

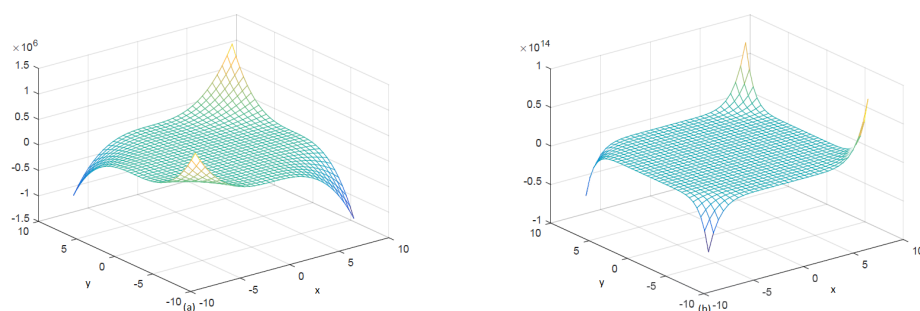


Figure 8: Plotting of (a) M-polynomial and (b) NM-polynomial of convolutamine F.

Corollary 2.7. Let G be the molecular graph of convolutamine F. Then

- (i) $M_1(G) = 70, M_1'(G) = 162,$
- (ii) $M_2(G) = 81, M_2^*(G) = 455,$
- (iii) $F(G) = 180, F_N^*(G) = 986,$
- (iv) ${}^mM_2(G) = 3.44, {}^{nm}M_2(G) = 0.77,$
- (v) $R_\alpha(G) = 2(2)^\alpha + 3(3)^\alpha + 2(4)^\alpha + 4(6)^\alpha + 4(9)^\alpha, NR_\alpha(G) = (6)^\alpha + (8)^\alpha + (12)^\alpha + 2(18)^\alpha + (21)^\alpha + (20)^\alpha + (32)^\alpha + (40)^\alpha + 2(36)^\alpha + 2(48)^\alpha + 2(56)^\alpha,$
- (vi) $ReZG_3(G) = 416, ND_3(G) = 5668,$
- (vii) $SDD(G) = 35.66, ND_5(G) = 33.48,$
- (viii) $H(G) = 6.76, NH(G) = 3.09,$
- (ix) $I(G) = 16.38, NI(G) = 38.63,$
- (x) $A(G) = 24.27, S(G) = 332.14.$

Theorem 2.8. Let G be the molecular graph of convolutamydine A. Then

- (i) $M(G; x, y) = 5xy^3 + xy^4 + 7x^2y^3 + x^2y^4 + 2x^3y^3 + 2x^3y^4,$
- (ii) $NM(G; x, y) = 2x^3y^4 + x^3y^5 + x^3y^6 + x^3y^7 + x^4y^7 + x^4y^9 + 2x^5y^6$
 $+ x^6y^6 + 3x^6y^7 + x^6y^{10} + 2x^7y^9 + x^7y^{10} + x^9y^{10}.$

Proof. Let G be the molecular graph of convolutamydine A (Figure 1). It has 18 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,3)} = 5$, $m_{(1,4)} = 1$, $m_{(2,3)} = 7$, $m_{(2,4)} = 1$, $m_{(3,3)} = 2$ and $m_{(3,4)} = 2$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(3,4)}^* = 2$, $m_{(3,5)}^* = 1$, $m_{(3,6)}^* = 1$, $m_{(3,7)}^* = 1$, $m_{(4,7)}^* = 1$, $m_{(4,9)}^* = 1$, $m_{(5,6)}^* = 2$, $m_{(6,6)}^* = 1$, $m_{(6,7)}^* = 3$, $m_{(6,10)}^* = 1$, $m_{(7,9)}^* = 2$, $m_{(9,10)}^* = 1$ and $m_{(7,10)}^* = 1$. The proof is similar as that of Theorem 2.1. $\square$

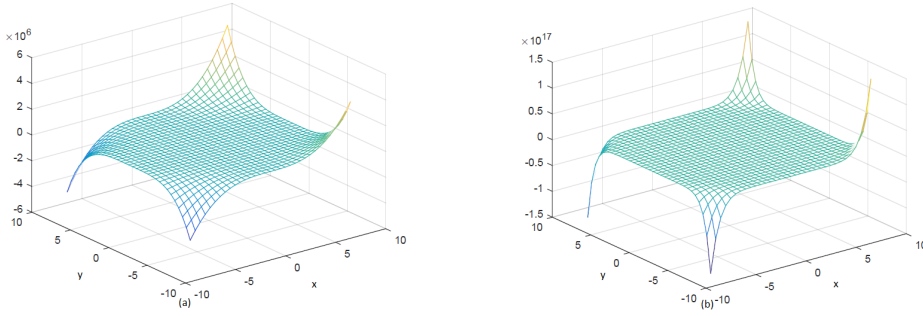


Figure 9: Plotting of (a) M-polynomial and (b) NM-polynomial of convolutamydine A.

Corolary 2.8. Let G be the molecular graph of convolutamydine A. Then

- (i) $M_1(G) = 82$, $M_1'(G) = 222$,
- (ii) $M_2(G) = 111$, $M_2^*(G) = 710$,
- (iii) $F(G) = 264$, $F_N^*(G) = 1524$,
- (iv) ${}^mM_2(G) = 3.59$, ${}^{nm}M_2(G) = 0.63$,
- (v) $R_\alpha(G) = 5(3)^\alpha + (4)^\alpha + (6)^\alpha + 2(9)^\alpha + 2(12)^\alpha$, $NR_\alpha(G) = 2(12)^\alpha + (15)^\alpha + (18)^\alpha + (21)^\alpha + (28)^\alpha + (36)^\alpha + 2(30)^\alpha + (36)^\alpha + 3(42)^\alpha + (60)^\alpha + 2(63)^\alpha + (70)^\alpha + (90)^\alpha$,
- (vi) $ReZG_3(G) = 614$, $ND_3(G) = 10042$,

$$(vii) \ SDD(G) = 42.58, \ ND_5(G) = 39.38,$$

$$(viii) \ H(G) = 4.27, \ NH(G) = 3.16,$$

$$(ix) \ I(G) = 20.71, \ NI(G) = 53.34,$$

$$(x) \ A(G) = 33.91, \ S(G) = 591.32.$$

Theorem 2.9. *Let G be the molecular graph of Daunorubicin. Then*

$$(i) \ M(G; x, y) = xy^2 + 9xy^3 + xy^4 + 2x^2y^2 + 10x^2y^3 + 2x^2y^4 + 16x^3y^3 + x^3y^4,$$

$$(ii) \ NM(G; x, y) = x^2y^4 + 4x^3y^6 + 5x^3y^7 + 2x^4y^5 + x^4y^7 + x^4y^8 + x^5y^7 + x^5y^8 + 5x^6y^6 + 3x^6y^7 + x^6y^8 + x^7y^7 + 5x^7y^8 + 8x^7y^9 + 2x^8y^9 + x^9y^9.$$

Proof. Let G be the molecular graph of Daunorubicin (Figure 1). It has 42 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1, m_{(1,3)} = 9, m_{(1,4)} = 1, m_{(2,2)} = 2, m_{(2,3)} = 10, m_{(2,4)} = 2, m_{(3,3)} = 16$ and $m_{(3,4)} = 1$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1, m_{(3,6)}^* = 4, m_{(3,7)}^* = 5, m_{(4,5)}^* = 2, m_{(4,7)}^* = 1, m_{(4,8)}^* = 1, m_{(5,7)}^* = 1, m_{(5,8)}^* = 1, m_{(6,6)}^* = 5, m_{(6,7)}^* = 3, m_{(6,8)}^* = 1, m_{(7,7)}^* = 1, m_{(7,8)}^* = 5, m_{(7,9)}^* = 8, m_{(8,9)}^* = 2$ and $m_{(9,9)}^* = 1$. Now by making use of the edge partition of G , the M and NM polynomials are obtained. $\square$

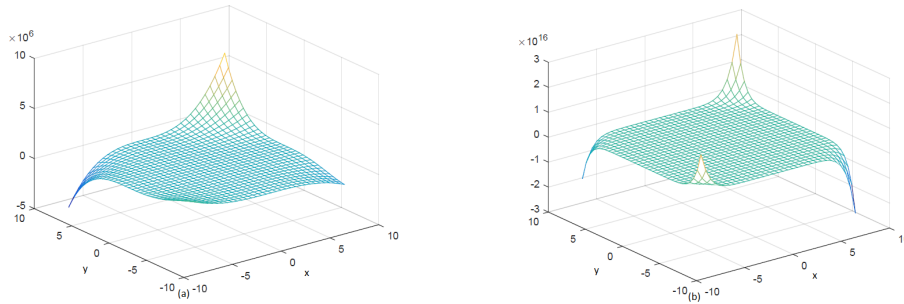


Figure 10: Plotting of (a) M-polynomial and (b) NM-polynomial of Daunorubicin.

Corollary 2.9. *Let G be the molecular graph of Daunorubicin. Then*

$$(i) \ M_1(G) = 217, \ M_1'(G) = 540,$$

- (ii) $M_2(G) = 273, M_2^*(G) = 1772,$
- (iii) $F(G) = 611, F_N^*(G) = 3750,$
- (iv) ${}^m M_2(G) = 8.02, {}^{nm} M_2(G) = 1.31,$
- (v) $R_\alpha(G) = (2)^\alpha + 9(3)^\alpha + (4)^\alpha + 2(4)^\alpha + 10(6)^\alpha + 2(8)^\alpha + 16(9)^\alpha + (12)^\alpha,$
 $NR_\alpha(G) = (8)^\alpha + 4(18)^\alpha + 5(21)^\alpha + 2(20)^\alpha + (28)^\alpha + (32)^\alpha + (35)^\alpha +$
 $(40)^\alpha + 5(36)^\alpha + 3(42)^\alpha + (48)^\alpha + (49)^\alpha + 5(56)^\alpha + 8(63)^\alpha + 2(72)^\alpha + (81)^\alpha,$
- (vi) $ReZG_3(G) = 1510, ND_3(G) = 25064,$
- (vii) $SDD(G) = 101.5, ND_5(G) = 92.35,$
- (viii) $H(G) = 16.85, NH(G) = 6.92,$
- (ix) $I(G) = 50.59, NI(G) = 130.24,$
- (x) $A(G) = 87.66, S(G) = 1502.98.$

Theorem 2.10. *Let G be the molecular graph of Deguelin. Then*

- (i) $M(G; x, y) = 2xy^2 + xy^3 + 2xy^4 + 3x^2y^2 + 14x^2y^3 + 2x^2y^4 + 9x^3y^3,$
- (ii) $NM(G; x, y) = 2x^2y^4 + x^3y^7 + 2x^4y^6 + 2x^4y^7 + 2x^5y^5 + x^5y^6$
 $+ 3x^5y^7 + 2x^5y^8 + x^6y^6 + 5x^6y^7 + 2x^6y^8 + 2x^7y^7$
 $+ 3x^7y^8 + 2x^7y^9 + 2x^8y^8 + x^8y^9.$

Proof. Let G be the molecular graph of Deguelin (Figure 1). It has 33 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 2, m_{(1,3)} = 1, m_{(1,4)} = 2, m_{(2,2)} = 3, m_{(2,3)} = 14, m_{(2,4)} = 2$ and $m_{(3,3)} = 9$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 2, m_{(3,7)}^* = 1, m_{(4,6)}^* = 2, m_{(4,7)}^* = 2, m_{(5,5)}^* = 2, m_{(5,6)}^* = 1, m_{(5,7)}^* = 3, m_{(5,8)}^* = 2, m_{(6,6)}^* = 1, m_{(6,7)}^* = 5, m_{(6,8)}^* = 2, m_{(7,7)}^* = 2, m_{(7,8)}^* = 3, m_{(7,9)}^* = 2, m_{(8,8)}^* = 2$ and $m_{(8,9)}^* = 1$. With the help of edge partition of G , the M and NM polynomials can be computed. $\square$

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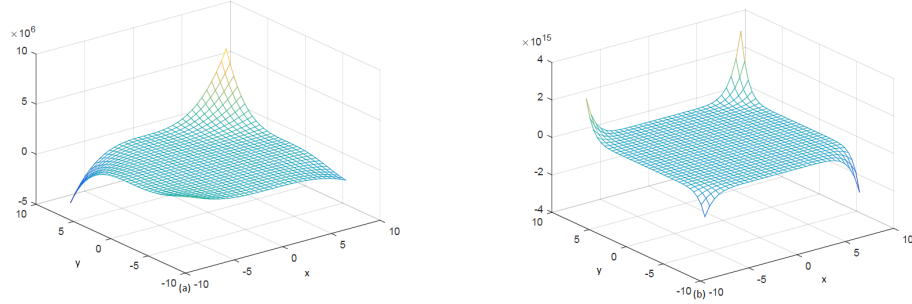


Figure 11: Plotting of (a) M-polynomial and (b) NM-polynomial of Deguelin.

Corollary 2.10. *Let G be the molecular graph of Deguelin. Then*

- (i) $M_1(G) = 158, M'_1(G) = 416,$
- (ii) $M_2(G) = 208, M_2^*(G) = 1340,$
- (iii) $F(G) = 462, F_N^*(G) = 2786,$
- (iv) ${}^m M_2(G) = 6.16, {}^{nm} M_2(G) = 1.06,$
- (v) $R_\alpha(G) = (2)^\alpha + (3)^\alpha + 2(4)^\alpha + 3(4)^\alpha + 14(6)^\alpha + 2(8)^\alpha + 9(9)^\alpha, NR_\alpha(G) = 2(8)^\alpha + (21)^\alpha + 2(24)^\alpha + 2(28)^\alpha + 2(25)^\alpha + (30)^\alpha + 3(35)^\alpha + 2(40)^\alpha + (36)^\alpha + 5(442)^\alpha + 2(48)^\alpha + 2(49)^\alpha + 3(56)^\alpha + 2(63)^\alpha + 2(64)^\alpha + (72)^\alpha,$
- (vi) $ReZG_3(G) = 1114, ND_3(G) = 18218,$
- (vii) $SDD(G) = 57.20, ND_5(G) = 70.04,$
- (viii) $H(G) = 13.4, NH(G) = 5.54,$
- (ix) $I(G) = 39.65, NI(G) = 101.61,$
- (x) $A(G) = 64.34, S(G) = 1103.$

Theorem 2.11. *Let G be the molecular graph of Melatonin. Then*

- (i) $M(G; x, y) = xy^2 + 2xy^3 + 4x^2y^2 + 9x^2y^3 + 2x^3y^3,$
- (ii) $NM(G; x, y) = x^2y^4 + 2x^3y^4 + 3x^4y^5 + x^4y^6 + 2x^5y^5 + x^5y^6 + 4x^5y^7 + x^6y^6 + x^6y^8 + 2x^7y^8.$

Proof. Let G be the molecular graph of Melatonin (Figure 1). It has 18 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1, m_{(1,3)} = 2, m_{(2,2)} = 4, m_{(2,3)} = 9$

and $m_{(3,3)} = 2$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1$, $m_{(3,4)}^* = 2$, $m_{(4,5)}^* = 3$, $m_{(4,6)}^* = 1$, $m_{(5,5)}^* = 2$, $m_{(5,6)}^* = 1$, $m_{(5,7)}^* = 4$, $m_{(6,6)}^* = 1$, $m_{(6,8)}^* = 1$ and $m_{(7,8)}^* = 2$. The proof is similar as that of Theorem 2.1. $\square$

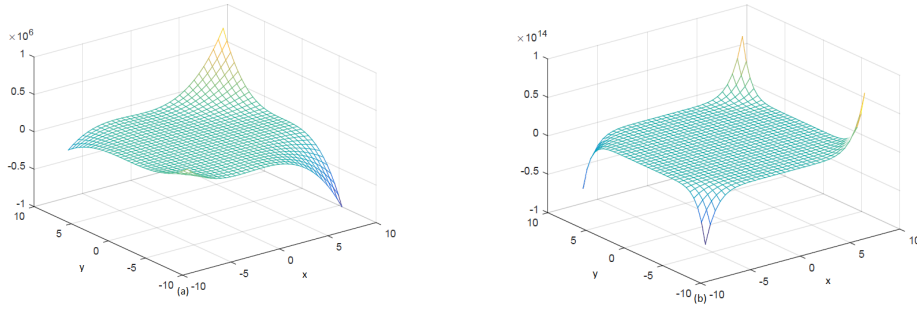


Figure 12: Plotting of (a) M-polynomial and (b) NM-polynomial of Melatonin.

Corollary 2.11. *Let G be the molecular graph of Melatonin. Then*

- (i) $M_1(G) = 84$, $M_1'(G) = 192$,
- (ii) $M_2(G) = 96$, $M_2^*(G) = 532$,
- (iii) $F(G) = 210$, $F_N^*(G) = 1100$,
- (iv) ${}^m M_2(G) = 3.88$, ${}^{nm} M_2(G) = 0.79$,
- (v) $R_\alpha(G) = (2)^\alpha + 2(3)^\alpha + 4(4)^\alpha + 9(6)^\alpha + 2(9)^\alpha$, $NR_\alpha(G) = (8)^\alpha + 2(12)^\alpha + 3(20)^\alpha + (24)^\alpha + 2(25)^\alpha + (30)^\alpha + 4(35)^\alpha + (36)^\alpha + (48)^\alpha + 2(56)^\alpha$,
- (vi) $ReZG_3(G) = 472$, $ND_3(G) = 6290$,
- (vii) $SDD(G) = 40.66$, $ND_5(G) = 37.59$,
- (viii) $H(G) = 7.93$, $NH(G) = 3.59$,
- (ix) $I(G) = 19.96$, $NI(G) = 47.11$,
- (x) $A(G) = 27.43$, $S(G) = 381.34$.

Theorem 2.12. *Let G be the molecular graph of Minocycline. Then*

- (i) $M(G; x, y) = 11xy^3 + xy^4 + x^2y^2 + 6x^2y^3 + 14x^3y^3 + 3x^3y^4$,

$$(ii) \quad NM(G; x, y) = 6x^3y^5 + x^3y^6 + 2x^3y^7 + 2x^3y^8 + x^4y^{10} + x^5y^5 + x^5y^6 \\ + 2x^5y^8 + 2x^5y^9 + 2x^6y^7 + x^6y^8 + 2x^6y^9 + 5x^7y^9 \\ + x^8y^8 + 3x^8y^9 + 2x^8y^{10} + x^9y^9 + x^9y^{10}.$$

Proof. Let G be the molecular graph of Minocycline (Figure 1). It has 36 edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,3)} = 11$, $m_{(1,4)} = 1$, $m_{(2,2)} = 1$, $m_{(2,3)} = 6$, $m_{(3,3)} = 14$ and $m_{(3,4)} = 3$. Let $M_{(i,j)}^*$ be the set of all edges with neighbourhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(3,5)}^* = 6$, $m_{(3,6)}^* = 1$, $m_{(3,7)}^* = 2$, $m_{(3,8)}^* = 2$, $m_{(4,10)}^* = 1$, $m_{(5,5)}^* = 1$, $m_{(5,6)}^* = 1$, $m_{(5,8)}^* = 2$, $m_{(5,9)}^* = 2$, $m_{(6,7)}^* = 2$, $m_{(6,8)}^* = 1$, $m_{(6,9)}^* = 2$, $m_{(7,9)}^* = 5$, $m_{(8,8)}^* = 1$, $m_{(8,9)}^* = 3$, $m_{(8,10)}^* = 2$, $m_{(9,9)}^* = 1$ and $m_{(9,10)}^* = 1$. With the help of edge partition of G , the M and NM polynomials can be computed. $\square$

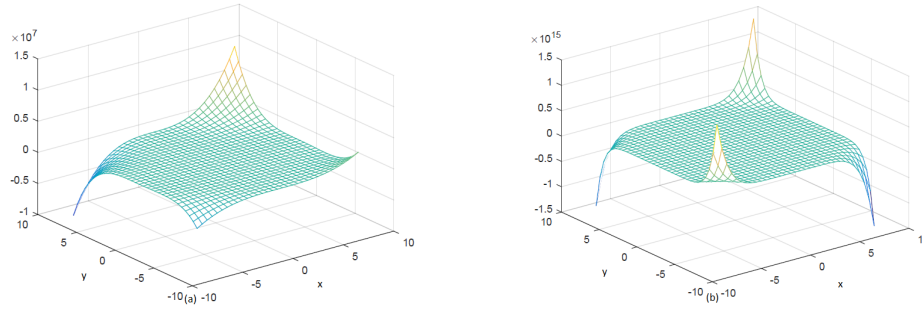


Figure 13: Plotting of (a) M-polynomial and (b) NM-polynomial of Minocycline.

Corollary 2.12. Let G be the molecular graph of Minocycline. Then, we have

- (i) $M_1(G) = 188$, $M_1'(G) = 478$,
- (ii) $M_2(G) = 239$, $M_2^*(G) = 1629$,
- (iii) $F(G) = 544$, $F_N^*(G) = 3516$,
- (iv) ${}^mM_2(G) = 6.97$, ${}^{nm}M_2(G) = 1.117$,
- (v) $R_\alpha(G) = 11(3)^\alpha + 2(4)^\alpha + 6(6)^\alpha + 14(9)^\alpha + 3(12)^\alpha$, $NR_\alpha(G) = 6(15)^\alpha + (18)^\alpha + 2(21)^\alpha + 2(24)^\alpha + (40)^\alpha + (25)^\alpha + (30)^\alpha + 2(40)^\alpha + 2(45)^\alpha + 2(42)^\alpha + (48)^\alpha + 2(54)^\alpha + 5(63)^\alpha + (64)^\alpha + 3(72)^\alpha + 2(80)^\alpha + (81)^\alpha + (90)^\alpha$,
- (vi) $ReZG_3(G) = 1356$, $ND_3(G) = 24438$,

(vii) $SDD(G) = 90.16$, $ND_5(G) = 80.73$,

(viii) $H(G) = 14.32$, $NH(G) = 5.86$,

(ix) $I(G) = 43.39$, $NI(G) = 114.08$,

(x) $A(G) = 78.87$, $S(G) = 1441.95$.

Theorem 2.13. *Let G be the molecular graph of Perfragilin A. Then*

(i) $M(G; x, y) = xy^2 + 5xy^3 + 5x^2y^3 + 7x^3y^3$,

(ii) $NM(G; x, y) = x^2y^4 + 2x^3y^6 + 3x^3y^7 + x^4y^8 + 3x^6y^6 + 2x^6y^8 + x^7y^7 + 4x^7y^8 + x^8y^8$.

Proof. Let G be the molecular graph of Perfragilin A (Figure 1). It has 18 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1$, $m_{(1,3)} = 5$, $m_{(2,3)} = 5$ and $m_{(3,3)} = 7$. Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1$, $m_{(3,6)}^* = 2$, $m_{(3,7)}^* = 3$, $m_{(4,8)}^* = 1$, $m_{(6,6)}^* = 3$, $m_{(6,8)}^* = 2$, $m_{(7,7)}^* = 1$, $m_{(7,8)}^* = 4$ and $m_{(8,8)}^* = 1$. Now by making use of the edge partition of G , the M and NM -polynomials are obtained. □

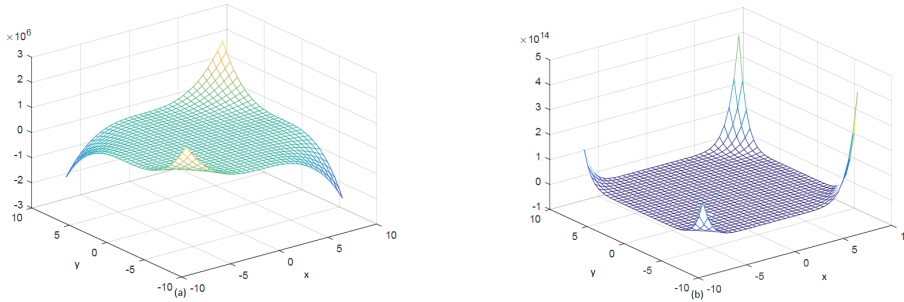


Figure 14: Plotting of (a) M-polynomial and (b) NM-polynomial of Perfragilin A.

Corolary 2.13. *Let G be the molecular graph of Perfragilin A. Then*

(i) $M_1(G) = 90$, $M_1'(G) = 220$,

(ii) $M_2(G) = 110$, $M_2^*(G) = 680$,

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$$(iii) \ F(G) = 246, F_N^*(G) = 1458,$$

$$(iv) \ {}^mM_2(G) = 3.77, {}^{nm}M_2(G) = 0.64,$$

$$(v) \ R_\alpha(G) = (2)^\alpha + 5(3)^\alpha + 5(6)^\alpha + 7(9)^\alpha, NR_\alpha(G) = (8)^\alpha + 2(18)^\alpha + 3(21)^\alpha + (32)^\alpha + 3(36)^\alpha + 2(48)^\alpha + (49)^\alpha + 4(56)^\alpha + (64)^\alpha,$$

$$(vi) \ ReZG_3(G) = 594, ND_3(G) = 9096,$$

$$(vii) \ SDD(G) = 42, ND_5(G) = 40.52,$$

$$(viii) \ H(G) = 7.5, NH(G) = 3.13,$$

$$(ix) \ I(G) = 20.91, NI(G) = 52.59,$$

$$(x) \ A(G) = 34.65, S(G) = 541.73.$$

Theorem 2.14. *Let G be the molecular graph of Podophyllotoxin. Then*

$$(i) \ M(G; x, y) = 3xy^2 + 2xy^3 + 3x^2y^2 + 15x^2y^3 + 11x^3y^3,$$

$$(ii) \ NM(G; x, y) = 3x^2y^4 + x^3y^6 + x^3y^7 + 2x^4y^5 + 2x^4y^7 + x^4y^8 + x^5y^5 \\ + x^5y^6 + 2x^5y^7 + x^5y^8 + 6x^6y^7 + 2x^6y^8 + x^6y^9 + x^7y^7 \\ + 4x^7y^8 + x^7y^9 + x^8y^8 + 2x^8y^9 + x^9y^9.$$

Proof. Let G be the molecular graph of Podophyllotoxin (Figure 1). It has 34 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 3, m_{(1,3)} = 2, m_{(2,2)} = 3, m_{(2,3)} = 15$ and $m_{(3,3)} = 11$. Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 3, m_{(3,6)}^* = 1, m_{(3,7)}^* = 1, m_{(4,5)}^* = 2, m_{(4,7)}^* = 2, m_{(4,8)}^* = 1, m_{(5,5)}^* = 1, m_{(5,6)}^* = 1, m_{(5,7)}^* = 2, m_{(5,8)}^* = 1, m_{(6,7)}^* = 6, m_{(6,8)}^* = 2, m_{(6,9)}^* = 1, m_{(7,7)}^* = 1, m_{(7,8)}^* = 4, m_{(7,9)}^* = 1, m_{(8,8)}^* = 1, m_{(8,9)}^* = 2$ and $m_{(9,9)}^* = 1$. Now by making use of the edge partition of G , the M and NM -polynomials can be computed. $\square$

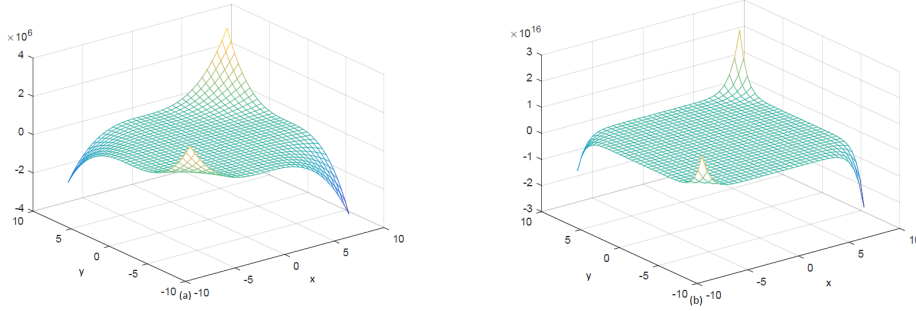


Figure 15: Plotting of (a) M-polynomial and (b) NM-polynomial of Podophyllotoxin.

Corolary 2.14. Let G be the molecular graph of Podophyllotoxin. Then

- (i) $M_1(G) = 170, M'_1(G) = 426,$
- (ii) $M_2(G) = 213, M_2^*(G) = 1383,$
- (iii) $F(G) = 452, F_N^*(G) = 2890,$
- (iv) ${}^m M_2(G) = 6.63, {}^{nm} M_2(G) = 1.2,$
- (v) $R_\alpha(G) = 3(2)^\alpha + 2(3)^\alpha + 3(4)^\alpha + 15(6)^\alpha + 11(9)^\alpha, NR_\alpha(G) = 3(8)^\alpha + (18)^\alpha + (21)^\alpha + 2(20)^\alpha + (25)^\alpha + 2(28)^\alpha + (32)^\alpha + 2(35)^\alpha + (30)^\alpha + (40)^\alpha + 6(42)^\alpha + 4(56)^\alpha + (49)^\alpha + (64)^\alpha + 2(48)^\alpha + 2(72)^\alpha + 2(72)^\alpha + (63)^\alpha + (81)^\alpha + (54)^\alpha,$
- (vi) $ReZG_3(G) = 1134, ND_3(G) = 19230,$
- (vii) $SDD(G) = 74.66, ND_5(G) = 73.13,$
- (viii) $H(G) = 14.16, NH(G) = 5.88,$
- (ix) $I(G) = 41, NI(G) = 103.62,$
- (x) $A(G) = 67.76, S(G) = 1161.65.$

Theorem 2.15. Let G be the molecular graph of Pterocellin. Then

- (i) $M(G; x, y) = xy^2 + 2xy^3 + 5x^2y^2 + 13x^2y^3 + 6x^3y^3,$
- (ii) $NM(G; x, y) = x^2y^4 + x^3y^6 + x^3y^7 + 2x^4y^4 + 2x^4y^5 + x^4y^7 + x^5y^5 + 3x^5y^6 + x^5y^8 + 4x^6y^6 + 2x^6y^7 + 3x^6y^8 + 2x^7y^8 + 3x^8y^8.$

Proof. Let G be the molecular graph of Pterocellin (Figure 1). It has 27 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1$, $m_{(1,3)} = 2$, $m_{(2,2)} = 5$, $m_{(2,3)} = 13$ and $m_{(3,3)} = 6$. Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1$, $m_{(3,6)}^* = 1$, $m_{(3,7)}^* = 1$, $m_{(4,4)}^* = 2$, $m_{(4,5)}^* = 2$, $m_{(4,7)}^* = 1$, $m_{(5,5)}^* = 1$, $m_{(5,6)}^* = 3$, $m_{(5,8)}^* = 1$, $m_{(6,6)}^* = 4$, $m_{(6,7)}^* = 2$, $m_{(6,8)}^* = 3$, $m_{(7,8)}^* = 2$, $m_{(8,8)}^* = 3$. With the help of edge partition of G , the M and NM -polynomials can be computed. $\square$

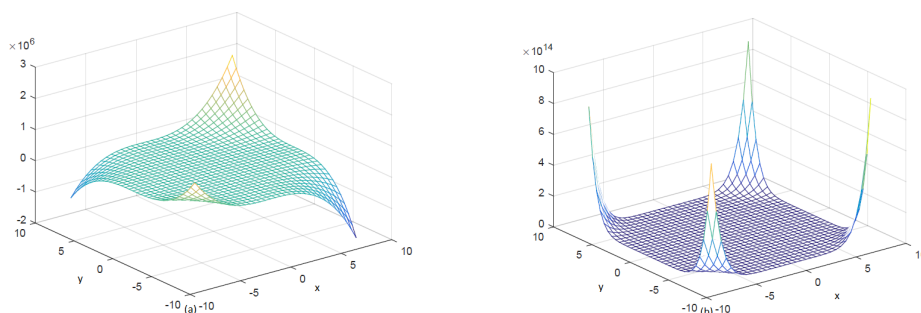


Figure 16: Plotting of (a) M-polynomial and (b) NM-polynomial of Pterocellin.

Corollary 2.15. Let G be the molecular graph of Pterocellin. Then

- (i) $M_1(G) = 132$, $M_1'(G) = 320$,
- (ii) $M_2(G) = 142$, $M_2^*(G) = 978$,
- (iii) $F(G) = 342$, $F_N^*(G) = 2024$,
- (iv) ${}^m M_2(G) = 5.25$, ${}^{nm} M_2(G) = 0.95$,
- (v) $R_\alpha(G) = (2)^\alpha + 2(3)^\alpha + 5(4)^\alpha + 13(6)^\alpha + 6(9)^\alpha$, $NR_\alpha(G) = (8)^\alpha + (18)^\alpha + (21)^\alpha + 2(16)^\alpha + 2(20)^\alpha + (28)^\alpha + (25)^\alpha + 3(30)^\alpha + 4(36)^\alpha + 2(42)^\alpha + 3(48)^\alpha + 2(56)^\alpha + 3(64)^\alpha + (40)^\alpha$,
- (vi) $ReZG_3(G) = 824$, $ND_3(G) = 12692$,
- (vii) $SDD(G) = 59.33$, $ND_5(G) = 56.84$,
- (viii) $H(G) = 11.36$, $NH(G) = 4.82$,
- (ix) $I(G) = 31.76$, $NI(G) = 78.39$,

(x) $A(G) = 48.84, S(G) = 772.10$.

Theorem 2.16. *Let G be the molecular graph of Raloxifene. Then*

(i) $M(G; x, y) = 3xy^3 + 11x^2y^2 + 18x^2y^3 + 6x^3y^3$,

(ii) $NM(G; x, y) = 2x^3y^5 + x^3y^7 + 2x^4y^4 + 4x^4y^5 + 8x^5y^5 + 7x^5y^6 + 4x^5y^7 + x^5y^8 + 2x^6y^7 + x^6y^8 + x^7y^7 + 2x^7y^8 + x^7y^9 + 2x^8y^9$.

Proof. Let G be the molecular graph of Raloxifene (Figure 1). It has 38 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,3)} = 3, m_{(2,2)} = 11, m_{(2,3)} = 18$ and $m_{(3,3)} = 6$. Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(3,5)}^* = 2, m_{(3,7)}^* = 1, m_{(4,4)}^* = 2, m_{(4,5)}^* = 4, m_{(5,5)}^* = 8, m_{(5,6)}^* = 7, m_{(5,7)}^* = 4, m_{(5,8)}^* = 1, m_{(6,7)}^* = 2, m_{(6,8)}^* = 1, m_{(7,7)}^* = 1, m_{(7,8)}^* = 2, m_{(7,9)}^* = 1, m_{(8,9)}^* = 2$. Now using the edge partition of G , the M and NM -polynomials can be evaluated. $\square$

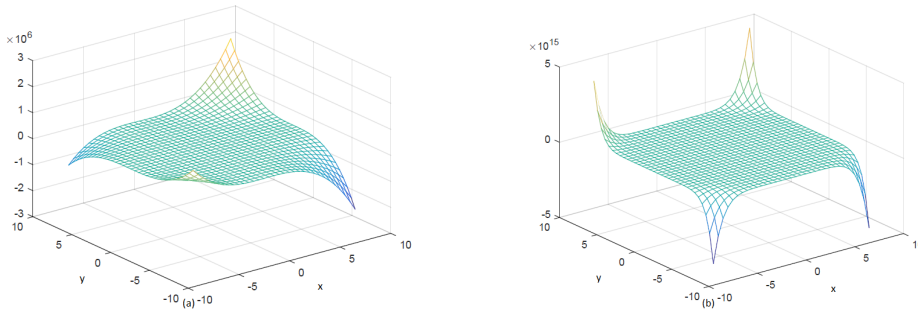


Figure 17: Plotting of (a) M-polynomial and (b) NM-polynomial of Raloxifene.

Corolary 2.16. *Let G be the molecular graph of Raloxifene. Then*

(i) $M_1(G) = 182, M_1'(G) = 430$,

(ii) $M_2(G) = 215, M_2^*(G) = 1253$,

(iii) $F(G) = 460, F_N^*(G) = 2580$,

(iv) ${}^mM_2(G) = 7.41, {}^{nm}M_2(G) = 1.36$,

(v) $R_\alpha(G) = 3(3)^\alpha + 11(4)^\alpha + 18(6)^\alpha + 6(9)^\alpha, NR_\alpha(G) = 2(15)^\alpha + (21)^\alpha + 2(16)^\alpha + 4(20)^\alpha + 8(25)^\alpha + 7(30)^\alpha + 4(35)^\alpha + (40)^\alpha + 2(42)^\alpha + (48)^\alpha + (49)^\alpha + 2(56)^\alpha + (63)^\alpha + 2(72)^\alpha$,

- (vi) $ReZG_3(G) = 1076, ND_3(G) = 15522,$
- (vii) $SDD(G) = 83, ND_5(G) = 78.66,$
- (viii) $H(G) = 16.2, NH(G) = 7.002,$
- (ix) $I(G) = 43.85, NI(G) = 105.83,$
- (x) $A(G) = 63.61, S(G) = 945.39.$

Theorem 2.17. *Let G be the molecular graph of Tambjamine K. Then*

- (i) $M(G; x, y) = xy^2 + 2xy^3 + 6x^2y^2 + 9x^2y^3 + 2x^3y^3,$
- (ii) $NM(G; x, y) = x^2y^4 + 2x^3y^4 + 2x^4y^4 + 5x^4y^5 + x^4y^7 + 3x^5y^7 + 4x^6y^7 + 2x^7y^7.$

Proof. Let G be the molecular graph of Tambjamine K (Figure 1). It has 20 number of edges. Let $M_{(i,j)}$ be the set of all edges with degree of end vertices i, j , i. e., $M_{(i,j)} = \{uv \in E(G) : \psi(u) = i, \psi(v) = j\}$. Let $m_{(i,j)}$ be the number of edges in $M_{(i,j)}$. From Figure 1, it is clear that $m_{(1,2)} = 1, m_{(1,3)} = 2, m_{(2,2)} = 5, m_{(2,3)} = 13$ and $m_{(3,3)} = 6$. Let $M_{(i,j)}^*$ be the set of all edges with neighborhood degree sum of end vertices i, j , i. e., $M_{(i,j)}^* = \{uv \in E(G) : \Psi(u) = i, \Psi(v) = j\}$. Let $m_{(i,j)}^*$ be the number of edges in $M_{(i,j)}^*$. From Figure 1, it is clear that $m_{(2,4)}^* = 1, m_{(3,4)}^* = 2, m_{(4,4)}^* = 2, m_{(4,5)}^* = 5, m_{(4,7)}^* = 1, m_{(5,7)}^* = 3, m_{(6,7)}^* = 4$ and $m_{(7,7)}^* = 2$. Using the edge partition of G , the M and NM -polynomials can be obtained easily. $\square$

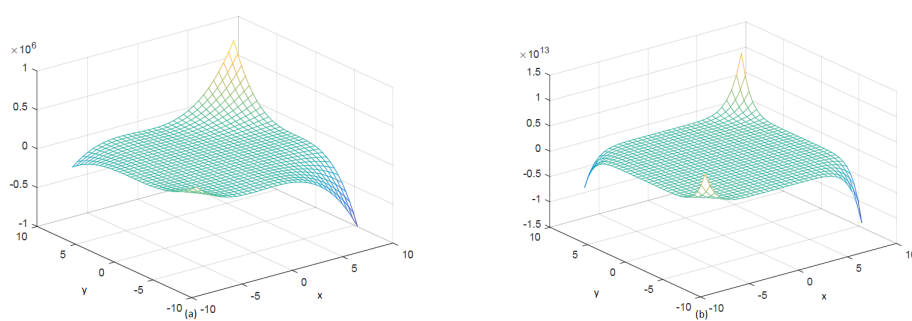


Figure 18: Plotting of (a) M-polynomial and (b) NM-polynomial of Tambjamine K.

Corollary 2.17. *Let G be the molecular graph of Tambjamine K. Then*

- (i) $M_1(G) = 92, M_1'(G) = 208,$

- (ii) $M_2(G) = 104, M_2^*(G) = 563,$
- (iii) $F(G) = 226, F_N^*(G) = 1162,$
- (iv) ${}^mM_2(G) = 4.38, {}^{nm}M_2(G) = 0.92,$
- (v) $R_\alpha(G) = (2)^\alpha + 2(3)^\alpha + 6(4)^\alpha + 9(6)^\alpha + 2(9)^\alpha, NR_\alpha(G) = (8)^\alpha + 2(12)^\alpha + 2(16)^\alpha + 5(20)^\alpha + (28)^\alpha + 3(35)^\alpha + 4(42)^\alpha + 2(49)^\alpha,$
- (vi) $ReZG_3(G) = 504, ND_3(G) = 6496,$
- (vii) $SDD(G) = 44.66, ND_5(G) = 41.67,$
- (viii) $H(G) = 8.93, NH(G) = 4.09,$
- (ix) $I(G) = 21.96, NI(G) = 51.09,$
- (x) $A(G) = 29.43, S(G) = 394.88.$



Figure 19: Plotting of R_α and NR_α for aminopterin, amathaspiramide E and carmustine.

M polynomial and NM polynomial give an extensive details on degree based and neighborhood degree based indices, respectively. We hope that a more in-depth analysis of the properties of M polynomial and NM polynomial will open up new general perspectives in the study of topological indices. To visualize the polynomials, their surface plots were performed by Matlab. The expressions of the M and NM polynomials for some anticancer drugs are shown in Figures 2 to 18, respectively. The graphs reveals that the polynomial displays different behaviors corresponding to different parameters. We can control the topology and therefore different properties and operations by tuning the M and NM polynomials via these parameters. Plotting of R_α and NR_α for aminopterin, amathaspiramide E and carmustine are shown in Figure 19. In the horizontal axis, $\alpha = 2, 4, 6, 8, 10$ are taken. In the vertical axis, the logarithmic values of the indices are considered to clearly represent the comparison.

3 Conclusion

In this paper, we have obtained M -polynomial and NM -polynomial of some anticancer drugs with graphical representations. The advantage of M -polynomial and NM -polynomial is from one expression we can obtain several degree-based and neighborhood degree topological indices. It is very challenging to bring all the degree-based and neighborhood degree based topological indices under M -polynomial and NM -polynomial, respectively. As considered topological indices are able to predict different physico-chemical properties. These results can be useful in designing new drug for the treatment of cancer. A research may be conducted for various chemical structures, and a conclusion can be drawn based on their topological indices range.

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A study on the number of edges of some families of graphs and generalized Mersenne numbers

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Abstract

The relationship between the Nandu sequence of the SM family of graphs and the generalized Mersenne numbers is demonstrated in this study. The sequences obtained from the peculiar number of edges of SM family of graphs are known as Nandu sequences. Nandu sequences are related to the two families of SM sum graphs and SM balancing graphs. The SM sum graphs are established from the inherent relationship between powers of 2 and natural numbers, whereas the SM balancing graphs are linked to the balanced ternary number system. In addition, some unusual prime numbers are discovered in this paper. These prime numbers best suit as an alternate for the Mersenne primes in the case of the public key cryptosystem.

Keywords: n^{th} SM balancing graphs, n^{th} SM sum graphs, Nandu Sequence, generalized Mersenne numbers, bipartite Kneser type-1 graphs.

2020 AMS subject classifications: 05C99, 40A30.¹

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1 Introduction

In computer science, number systems and related ideas are used, particularly in cryptography. The binary number system, which is used in binary computers, and the balanced ternary number system, which is used in ternary computers, are two number systems used in computers. The balanced ternary number system was used in the Russian SETUN computer. Graph theory is used to investigate the combinatorial structure of these two number systems. SM sum graphs and SM balancing graphs are the two categories of graphs based on these number systems that we will focus on here. These graphs are groups of graphs that have been structured in a specific order. SM family of graphs consists of SM sum graph, SM balancing graphs and its complement graphs. For large values of n , these graphs are all non-symmetric graphs [10] with bigger automorphism groups. The properties of these graphs lead to the discovery of some classical sequences. These sequences are called Nandu sequences or Ne-sequences. These Nandu sequences have a relation with Mersenne primes as well as generalized Mersenne numbers [8]. Furthermore, this is related to the low weight polynomial form of integers [4] which was used in elliptical curve cryptography. The Residue Number System (RNS) [2, 3] has an important role in modular multiplications in computer science. This RNS modular multiplication is used in the prime field based in elliptical curve cryptosystems too. Generalized Mersenne numbers are used in RNS modular multiplication for more efficiency. The newly defined sequence $\{\aleph_n\}$ in this paper are the particular cases of the generalized Mersenne numbers. At present, the modular multiplication requires a maximum of modulus 521. Eventually, the relationship between the Nandu sequence and the generalized Mersenne numbers are established in this work. The use of this relationship in elliptical curve cryptography is yet to be worked out. Also, a study on Mersenne primes in real quadratic fields was done by Sushma Palimar and B.R. Shankar [6]. The primality testing of large numbers in Maple was given in the work of S.Y. Yan [12]. A study on low weight polynomial form of integers for efficient modular multiplication was done by Jaewook Chung and M. Anwar Hasan [4]. A study on generalized double Fibonomial numbers was done by Mansi shah and Devbhadrha Shah [7].

In this paper, we established some properties of Nandu sequences. The Nandu sequence $\{Nt_{\sum_n}\}$ for $SM(\sum_n)$ satisfies the recurrence relation $Nt_{\sum_{n+1}} = 2Nt_{\sum_n} + 2^n + n - 1$, $Nt_{\sum_2} = 2$. Let $\aleph_n = \frac{Nt_{B_n} - Nt_{\sum_n}}{n}$, where Nt_{B_n} is the Nandu sequence of $SM(B_n)$. We derived a closed form of the generating function of the sequence $\aleph_n$ and is given by $G(x) = \sum_{r=0}^{n-2} 2^{n-2} \left(\frac{3}{2}\right)^r x^r$. The convergence of $\sum \frac{1}{\aleph_n}$ is then obtained.

Some preliminaries are given below.

2 Preliminary

In this section, we provide the basic definitions and some results from the related work mainly from [9, 10]. We begin with the definition of SM families of graphs. Let's look at how SM balancing graphs are defined [9]. Consider the set $T_n = \{3^m : m \text{ is an integer, } 0 \leq m \leq n-1\}$ for a fixed positive integer $n \geq 2$. Let $I = \{-1, 0, 1\}$. Let $x \leq \frac{1}{2}(3^n - 1)$ be any positive integer which is not a power of 3. Then x can be expressed as

$$x = \sum_{j=1}^n \alpha_j y_j, \quad (1)$$

where $\alpha_j \in I$, $y_j \in T_n$ and y_j 's are distinct. Each y_j such that $\alpha_j \neq 0$ is called a balancing component of x . Consider the simple digraph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_{\frac{1}{2}(3^n-1)}\}$ and adjacency of vertices is defined by: for any two distinct vertices v_x and v_{y_j} , $(v_x, v_{y_j}) \in E$ if (1) holds and $\alpha_j = -1$, and $(v_{y_j}, v_x) \in E$ if (1) holds and $\alpha_j = 1$. This digraph G is called the n^{th} SMD balancing graph, denoted by $SMD(B_n)$. Its underlying undirected graph is called the n^{th} SM balancing graph, denoted by $SM(B_n)$. Let us now look at the definition of SM sum graphs. If $p < 2^n$, is a positive integer which is not a power of 2, then $p = \sum_{i=1}^n x_i$, with $x_i = 0$ or 2^m , for some integer m , $0 \leq m \leq n-1$ and x_i 's are distinct. Here each $x_i \neq 0$ is called an additive component of p . For a fixed integer $n \geq 2$, the simple graph $SM(\sum_n)$, called n^{th} SM sum graph [9], is a graph with vertex set $\{v_1, v_2, \dots, v_{2^n-1}\}$ and adjacency of vertices defined by, v_i and v_j are adjacent if either i is an additive component of j or j is an additive component of i .

In $SM(\sum_n)$, degree of the vertex v_{2^n-1} is n and $\sum_{v \in V} \deg v = 2n(2^{n-1} - 1)$.

In $SM(B_n)$, the number of vertices is $\frac{1}{2}(3^n - 1)$ and $\sum_{v \in V} \deg v = 2n(3^{n-1} - 1)$.

Note: For a fixed integer $n \geq 2$, let $T_n = \{3^m : m \text{ is an integer, } 0 \leq m \leq n-1\}$, $N = \{1, 2, 3, \dots, t\}$, where $t = \frac{1}{2}(3^n - 1)$. Also, let $P_n = \{2^m : m \text{ is an integer, } 0 \leq m \leq n-1\}$, $M = \{1, 2, 3, \dots, 2^n - 1\}$. Then consider $P_n^c = M - P_n$, $T_n^c = N - T_n$ throughout this paper unless otherwise specified.

The Hamming weight of a string was defined as the number of 1's in the strings of 0 and 1. Here the number of additive components gives the Hamming weight of string (binary) representation of all numbers in P_n^c . The Hamming weight of string (binary) representation of numbers in P_n is always 1.

Bipartite Kneser type-1 graphs

Let $\mathcal{S}_n = \{1, 2, \dots, n\}$, for an integer $n > 1$. For any two integers $k \geq 1$ and $n \geq 2k + 1$, the bipartite Kneser graph [10] $H(n, k)$ has all the k -element subsets and all the $(n - k)$ -element subsets of $\mathcal{S}_n$ as vertices, and two vertices are adjacent if and only if one of them is a subset of the other. Here we define the bipartite Kneser type-1 graph as follows.

Definition 2.1. [10] Let $\mathcal{S}_n = \{1, 2, 3, \dots, n\}$ for a fixed integer $n > 1$. Let $\phi(\mathcal{S}_n)$ be the set of all non-empty subsets of $\mathcal{S}_n$. Let V_1 be the set of 1- element subsets of $\mathcal{S}_n$ and $V_2 = \phi(\mathcal{S}_n) - V_1$. Define a bipartite graph with adjacency of vertices as: a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$. This graph is called a bipartite Kneser type-1 graph.

This bipartite Kneser type-1 graph is isomorphic to the graph $SM(\sum_n)$ for each n . To study the structure of the bipartite Kneser type-1 graph, we can make use of $SM(\sum_n)$ graph. The automorphism groups of the bipartite Kneser type-1 graphs are isomorphic to the symmetric group S_n for each $n > 2$.

3 Nandu sequences of $SM(\sum_n)$ and $SM(B_n)$

The Nandu sequence or Ne-sequence $\{Nt_m\}_{m=1}^{n-1}$ of SM graphs are the sequence of numbers whose terms are the half of the sum of degrees of the vertices of $SM(\sum_n)$ or $SM(B_n)$ for all $n \geq 2$. Here we assume $n \geq 2$ for both the SM sum graph and SM balancing graphs unless otherwise specified. Kinkar Das and I Gutman [5] estimated the Wiener index by means of number of vertices, number of edges, and diameter.

3.1 Nandu sequence for the graph $SM(\sum_n)$

Definition 3.1. For the SM sum graph $SM(\sum_n)$, with vertex set $V = \{v_i : 1 \leq i \leq 2^n - 1\}$, the Nandu sequence $\{Nt_{\sum_n}\}$ is defined as a sequence with n^{th} term as $Nt_{\sum_n} = \frac{1}{2} \sum_{v \in V} \deg v$ and the sequence $\{DNt_{\sum_n}\}$ defined by $DNt_{\sum_n} = \sum_{v \in V} \deg v$ as double Nandu sequence.
i.e., $\{Nt_{\sum_n}\} = 2, 9, 28, 75, 186, \dots$

Theorem 3.1. Let $\{Nt_{\sum_n}\}$ be the Nandu sequence for the SM sum graph $SM(\sum_n)$, $n \geq 2$. Then $Nt_{\sum_{n+1}} = 2Nt_{\sum_n} + 2^n + n - 1$, $Nt_{\sum_2} = 2$.

Proof. Consider the graph $SM(\sum_n)$ with vertex set $V = \{v_i : 1 \leq i \leq 2^n - 1\}$.

The Nandu sequence is $\{Nt_{\sum_n}\}$ with $Nt_{\sum_n} = \frac{1}{2} \sum_{v \in V} \deg v$.

Then we have $Nt_{\sum_n} = n(2^{n-1} - 1)$.

$$\begin{aligned} Nt_{\sum_{n+1}} &= (n+1)(2^n - 1) \\ &= n(2^n - 1) + 2^n - 1 \\ &= 2n(2^{n-1} - 1) + 2^n - 1 + n \\ &= 2Nt_{\sum_n} + 2^n + n - 1. \end{aligned}$$

Hence proved. $\square$

Theorem 3.2. Let $Nt_{\sum_n}$ be a Nandu sequence of SM Sum graph. Then the following holds.

1. $Nt_{\sum_n}$ is a composite number for all $n > 2$ and is always divisible by n .
2. If $\frac{Nt_{\sum_n}}{n}$ is a prime, then $n - 1$ is a prime.
3. $\frac{1}{2n} \sum_{v \in V} \deg v = \frac{Nt_{\sum_n}}{n}$ is a Mersenne number.
4. $\frac{Nt_{\sum_n}}{n}$ gives the number of times each element of P_n is used to make numbers in P_n^c , the complement of P_n for a fixed n .

Proof. The proof is obvious from the definition of the sequence $Nt_{\sum_n}$. $\square$

Definition 3.2. Let V be the vertex set of $G = SM(\sum_n)$. Let Δ be the maximum degree of G and δ be the minimum degree of G . The vertex degree polynomial of G is given by $Deg(G, x) = \sum_{m=\delta}^{\Delta} \deg(G, m) \cdot x^m = \sum_{k=2}^n \binom{n}{k} x^k + n \cdot x^{2^{n-1}-1}$, where $\deg(G, m)$ is the number of vertices of degree m .

Let $G = SM(\sum_n)$ be an SM sum graph with vertex set V . The derivative of $Deg(G, x)$ at $x = 1$ is $DNt_{\sum_n}$, the $(n - 1)^{th}$ term of the double Ne-sequence of G .

Now let us see the summation of terms of Nandu Sequence of SM sum graph. Let $\{Nt_{\sum_n}\}$, where $Nt_{\sum_n} = \frac{1}{2} \sum_{v \in V} \deg v$, be the Nandu sequence for the SM sum graphs. Then its summation is given by $\sum_{r=1}^n Nt_{\sum_r} = n2^{n+1} - \frac{n(n+1)}{2} - n$.

Lemma 3.1. [9] If $G = SM(\sum_n)$, $P_n = \{2^m : m \text{ is an integer, } 0 \leq m \leq n-1\}$, then

$$d(v_i, v_j) = \begin{cases} 1 & \text{if } i \text{ is an additive component of } j \text{ or } j \text{ is an additive component of } i, \\ 2 & \text{if } i, j \in P_n \text{ or } i, j \notin P_n, i \text{ and } j \text{ have at least one common additive component,} \\ 3 & \text{if neither } i \text{ nor } j \text{ is an additive component but exactly one of them belongs to } P_n, \\ 4 & \text{if } i, j \notin P_n, i \text{ and } j \text{ have no common additive components.} \end{cases}$$

Proposition 3.1. [9] Let $G = SM(\sum_n)$ be an n^{th} SM sum graph. Let $d_r(v_i, v_j)$ denote the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Then

$$d_r(v_i, v_j) = \begin{cases} n \cdot (2^{n-1} - 1) & \text{if } r = 1, \\ \frac{n(n-1)}{2} + \left[\frac{(2^n - n - 2)(2^n - n - 1)}{2} - \delta_n \right] & \text{if } r = 2, \\ (n+1) \cdot 2^n - (n+2)2^{n-1} - n^2 & \text{if } r = 3, \\ \delta_n & \text{if } r = 4, \end{cases}$$

$$\text{where } \delta_n = \frac{1}{2} \sum_{r=2}^{n-2} \left[\binom{n}{r} \sum_{k=2}^{n-2} \binom{n-r}{k} \right].$$

Remark 3.1. For $n = 2$ or 3 , we get that $\delta_n = 0$. In these cases, the diameter of the graph $SM(\sum_n)$ is 2 or 3.

Theorem 3.3. Suppose $G = SM(\sum_n)$ $n \geq 2$ be an n^{th} SM sum graph. The $(n-1)^{\text{th}}$ term of the Nandu sequence is equal to $d_1(v_i, v_j)$, where $v_i v_j$ is an edge of G .

Proof. The proof follows from Proposition 3.1 . □

3.2 Nandu sequence for the graph $SM(B_n)$

We introduced two new sequences, called Nandu sequence and Double Nandu sequence for the SM balancing graphs also. Here we discuss some of their properties.

Proposition 3.2 ([9]). For the n^{th} SM Balancing graph $SM(B_n)$, let $d_r(v_i, v_j)$ be the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Let $t = \frac{1}{2}(3^n - 1)$. Then

$$d_r(v_i, v_j) = \begin{cases} n \cdot (3^{n-1} - 1) & \text{if } r = 1, \\ \frac{n(n-1)}{2} + \left[\frac{(t-n)(t-n-1)}{2} - \sigma_n \right] & \text{if } r = 2, \\ \frac{1}{2}(n \cdot 3^{n-1} + n - 2n^2) & \text{if } r = 3, \\ \sigma_n & \text{if } r = 4, \end{cases}$$

$$\text{where } \sigma_n = \frac{1}{2} \sum_{r=2}^{n-2} \left[\binom{n}{r} \sum_{k=2}^{n-2} \binom{n-r}{k} 2^{r+k-2} \right].$$

Definition 3.3. Consider the SM balancing graph $SM(B_n)$, $n \geq 2$, with vertex set $V = \{v_i : 1 \leq i \leq \frac{1}{2}(3^n - 1)\}$. The sequence $\{Nt_{B_n}\}$ with $Nt_{B_n} = \frac{1}{2} \sum_{v \in V} \deg v$ is called the Nandu sequence and the sequence $\{DNt_{B_n}\}$ defined by $DNt_{B_n} = \sum_{v \in V} \deg v$ is called the double Nandu sequence. i.e., $\{Nt_{B_n}\} = 4, 24, 104, 400, 1452, \dots$.

Definition 3.4. Let $G = SM(B_n)$ with vertex set V . Let Δ be the maximum degree of G and δ be the minimum degree of G . The vertex degree polynomial of G is given as

$$\begin{aligned} \text{Deg}(G, x) &= \sum_{m=\delta}^{\Delta} \deg(G, m) \cdot x^m \\ &= \sum_{k=2}^n 2^{k-1} \binom{n}{k} x^k + n \cdot x^{3^{n-1}-1}. \end{aligned}$$

Example 3.1. For $n = 5$, $\text{Deg}(G, x) = 16x^5 + 40x^4 + 40x^3 + 20x^2 + 5x^{80}$.

Theorem 3.4. Let $G = SM(B_n)$ be a SM balancing graph with vertex set V . The derivative of $\text{Deg}(G, x)$ at $x = 1$ is DNt_{B_n} , the $(n-1)^{\text{th}}$ term of the double Ne-sequence of G .

Proof. Let $\text{Deg}(G, x)'$ be the derivative of $\text{Deg}(G, x)$ w.r.to x .

$$\begin{aligned} \text{Deg}(G, x)' &= n(2x+1)^{n-1} - n + n \cdot (3^{n-1} - 1) \cdot x^{3^{n-1}-2} \\ \text{Hence, } \text{Deg}(G, 1)' &= n \cdot 3^{n-1} - n + n \cdot (3^{n-1} - 1) \\ &= 2n(3^{n-1} - 1) \\ &= DNt_{B_n}. \end{aligned}$$

□

Here we provide the summation for the Nandu sequence of SM balancing graphs.

Let $\{Nt_{B_n}\}$, where $Nt_{B_n} = \frac{1}{2} \sum_{v \in V} \deg v$, be the Nandu sequence for the SM balancing graphs. Then $\sum_{r=1}^n Nt_{B_r} = \frac{3}{4} [2n \cdot 3^n + 3^n - 1] - n - \frac{n(n+1)}{2}$.

Theorem 3.5. Let $SM(B_n)$ be the n^{th} SM balancing graph and Nt_{B_n} be the Nandu sequence. Then $Nt_{B_{n+1}} = 3Nt_{B_n} + 3^n + 2n - 1$, $Nt_{B_2} = 4$, for all $n \geq 2$.

Proof. Consider the graph $SM(B_n)$.

Then we have $Nt_{B_n} = n(3^{n-1} - 1)$, $Nt_{B_2} = 4$.

For all $n \geq 2$,

$$\begin{aligned} Nt_{B_{n+1}} &= (n+1)(3^n - 1) \\ &= n(3^n - 1) + 3^n - 1 \\ &= 3n(3^{n-1} - 1) + 3^n - 1 + n \\ &= 3Nt_{B_n} + 3^n + 2n - 1. \end{aligned}$$

Hence proved. $\square$

4 Relationship between Nandu sequence and generalized Mersenne numbers

Mersenne numbers are numbers of the form $M_n = 2^n - 1$. If a Mersenne number is prime, then n is a prime. But the converse is not true. Mersenne numbers are a particular case of a larger class of numbers, the generalized Mersenne numbers [11], $G_{a,n} = a^n - (a-1)^n$ characterised by their base a and exponent n . The idea of generalized Mersenne numbers was introduced by Solinas [8] in 1999 for the use in elliptic curve cryptography. The use of generalised Mersenne numbers in modular arithmetic to perform fast modular multiplications is well known.

Theorem 4.1. *Let Nt_{B_n} and $Nt_{\sum_n}$ be the terms of the Nandu sequence of SM balancing graph and SM sum graph respectively. Then $Nt_{B_n} - Nt_{\sum_n} = n(3^{n-1} - 2^{n-1})$.*

Proof. The proof follows from the definition of Nandu sequences of SM sum graph and SM balancing graphs. $\square$

Definition 4.1. *Let $SM(\sum_n)$ and $SM(B_n)$ be the SM sum graphs and SM balancing graphs respectively, $n \geq 3$. Then the sequence $\{\aleph_n\}$ is defined as $\aleph_n = \frac{Nt_{B_n} - Nt_{\sum_n}}{n}$.*

Theorem 4.2. *For $n \geq 3$ and when n is odd, $\aleph_n \equiv 0 \pmod{5}$*

Proof. We have $\aleph_n = \frac{Nt_{B_n} - Nt_{\sum_n}}{n} = 3^{n-1} - 2^{n-1}$

Since n is odd, then $n-1$ is even, say $n-1 = 2m$, for some integer m . Also, $3^{n-1} - 2^{n-1} = 3^{2m} - 2^{2m}$, which is a multiple of 5. Hence proved. $\square$

Theorem 4.3. $\sum \frac{1}{\aleph_n}$ converges.

Proof. Suppose $SM(\sum_n)$ and $SM(B_n)$ be the SM sum graphs and SM balancing graphs respectively. Then we have the sequence $\{\aleph_n\}$ as $\aleph_n = \frac{Nt_{B_n} - Nt_{\sum_n}}{n}$.

$$\begin{aligned} \text{Therefore } \aleph_n &= 3^{n-1} - 2^{n-1} \\ &= (3-2)(3^{n-2} + 3^{n-3} \cdot 2 + \dots + 2^{n-2}) \\ &\geq 2^{n-2} + 2^{n-2} + \dots + 2^{n-2} \\ &= (n-1) \cdot 2^{n-2} \end{aligned}$$

$$\text{But } \frac{1}{3^{n-1} - 2^{n-1}} \leq \frac{1}{(n-1) \cdot 2^{n-2}} = \frac{4}{(n-1) \cdot 2^n}.$$

Now consider the series $\sum \frac{1}{(n-1) \cdot 2^n}$. We have $2^n > n-1$, for $n \geq 3$.

To check the convergence of $\sum \frac{1}{(n-1) \cdot 2^n}$, it is enough to check the convergence of $\sum \frac{1}{(n-1)^2}$. But $\sum \frac{1}{(n-1)^2}$ is convergent. Therefore, by comparison test, $\sum \frac{1}{\aleph_n}$ converges. $\square$

Theorem 4.4. Let $SM(\sum_n)$ and $SM(B_n)$ be the SM sum graphs and SM balancing graphs. Then $\aleph_{n+1} = 2\aleph_n + 3^{n-1}$, for all $n \geq 2$, given $\aleph_2 = 1$.

Proof. Consider the graph $SM(\sum_n)$ and $SM(B_n)$.

Then we have $\aleph_n = \frac{Nt_{B_n} - Nt_{\sum_n}}{n}$.

$$\begin{aligned} \aleph_{n+1} &= 3^n - 2^n \\ &= 3 \cdot 3^{n-1} - 2 \cdot 2^{n-1} \\ &= 2(3^{n-1} - 2^{n-1}) + 3^{n-1} \\ &= 2\aleph_n + 3^{n-1}. \end{aligned}$$

Hence proved. $\square$

Lemma 4.1. A closed form of the generating function of the sequence $\aleph_n$ is given

$$\text{by } G(x) = \sum_{r=0}^{n-2} 2^{n-2} \left(\frac{3}{2}\right)^r x^r.$$

Proof. We have $\aleph_{n+1} = 2\aleph_n + 3^{n-1}$, for all $n \geq 2$, given $\aleph_2 = 1$.

When $n = 2$,

$$\begin{aligned} \aleph_3 &= 2\aleph_2 + 3 \\ \aleph_4 &= 2\aleph_3 + 3^2 = 2(2\aleph_2 + 3) + 3^2 = 2^2\aleph_2 + 2 \cdot 3 + 3^2 \\ \text{Similarly, } \aleph_5 &= 2\aleph_4 + 3^3 = 2^3\aleph_2 + 2^2 \cdot 3 + 2 \cdot 3^2 + 3^3 \\ \aleph_6 &= 2^4\aleph_2 + 2^3 \cdot 3 + 2^2 \cdot 3^2 + 2 \cdot 3^3 + 3^4. \end{aligned}$$

Continuing in this way we get,

$$\aleph_n = \sum_{r=0}^{n-2} 2^{n-2} \left(\frac{3}{2}\right)^r$$

Therefore, the required generating function is $G(x) = \sum_{r=0}^{n-2} 2^{n-2} \left(\frac{3}{2}\right)^r x^r$. $\square$

Theorem 4.5. Let $SM(\sum_n)$ and $SM(B_n)$ be the SM sum graphs and SM balancing graphs. Then $\aleph_{n+1} = G_{3,n}$.

Theorem 4.6. If $\aleph_n$ is prime, then n is even. But the converse is not true.

Proof. The proof follows from Theorem 4.2. $\square$

Definition 4.2. Let $SM(\sum_n)$ and $SM(B_n)$ be the SM sum graphs and SM balancing graphs. Then $\aleph_{n+1} = G_{3,n}$. For some values of n , $G_{3,n}$ is a prime. These prime numbers are called SM prime numbers.

These prime numbers can be used to replace the Mersenne primes in the new public key cryptosystem introduced by D. Aggarwal, et al [1]. In their work, they propose a new public-key cryptosystem whose security is based on the computational intractability of the problem: Given a Mersenne number $p = 2^n - 1$, where n is a prime, a positive integer h and an n -bit integer H , decide whether there exist n -bit integers F, G each of Hamming weight less than h such that $H = \frac{F}{G} \text{ modulo } p$.

Theorem 4.7. The series $\sum \frac{1}{Nt_{\sum_n}}$ and $\sum \frac{1}{Nt_{B_n}}$ converges.

Proof. We have for $n \geq 3$, $2^n - 1 > n$. So $\sum \frac{1}{n(2^n - 1)} \leq \frac{1}{n^2}$. As $\sum \frac{1}{n^2}$ is convergent, $\sum \frac{1}{n(2^n - 1)} = \sum \frac{1}{Nt_{\sum_n}}$ is convergent.

The same way $\sum \frac{1}{Nt_{B_n}}$ also converges. Hence proved. $\square$

We observed that $\aleph_n$ is always an odd number and is a prime when $n = 3, 4, 6, 18, 30$ and 32 so on. There exists $G_{3,n}$ primes. Currently, the largest modulus required for modular multiplication is 521. For $n = 6$, $\aleph_n = 665$.

Consider the function $A_{G_{3,n}}(x)$ as a function which gives the number of SM primes among the $G_{3,n}$ generalized Mersenne numbers which are less than or equal to the corresponding $\aleph_n$. We get that for $n = 2$, $A_{G_{3,n}}(x) = 1$, for $n = 3$, $A_{G_{3,n}}(x) = 2$, for $n = 4$, $A_{G_{3,n}}(x) = 3$ and for $n = 5$, $A_{G_{3,n}}(x) = 3$ etc.

In fact the sum of degrees of vertices $v_x, x \in T_n^c$ minus the sum of degrees of vertices $v_x, x \in P_n^c$ is the same as the sum of degrees of vertices $v_x, x \in T_n$ minus the sum of degrees of vertices $v_x, x \in P_n$. If these difference on either side is divided by n , then the quotient is equal to the $\aleph_n$.

Theorem 4.8. *If $p_1, p_2, p_3, \dots, p_n$ are some odd prime numbers, then $P = \prod_{i=1}^n p_i$ is having 1 as an additive component.*

Proof. Since $p_1, p_2, p_3, \dots, p_n$ are odd prime numbers, each is having 1 as an additive component. Then clearly $P = \prod_{i=1}^n p_i$ also has 1 as an additive component. This proves the theorem. $\square$

Corollary 4.1. *Let the number of odd prime numbers less than 2^n be denoted by $N(\beta)$. Then $N(\beta) \leq 2^{n-1} - 1$ for all $n \geq 3$.*

5 Conclusion

The Nandu sequences of the two SM families of graphs were examined, and a relation between these sequences and the generalized Mersenne numbers was established. As a consequence, we have a new sequence of integers called $G_{3,n}$, which is a type of generalized Mersenne number that may be employed in elliptical curve cryptography. These sequences are important in using a graph theory method to investigate the nature and structure of the two number systems - binary and balanced ternary. It has been noted that the $(n-1)^{th}$ term of the Nandu sequence of $SM(\sum_n)$ is identical to unordered pairs of vertices which are at distance one. The $\aleph_n$ functions can be studied further in relation to elliptic curve cryptography.

Conflict of Interest

The authors hereby declare that we have no potential conflict of interest.

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A Deep Learning Model for Classifying the Hate and Offensive Language in Social Media Text

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Abstract

Social media is a big data source for analyzing content. It requires a significant amount of computational cost. Recently introduced model for identifying and removing toxic content from Twitter, using an Information Retrieval (IR) model is working accurately but with small amount of data. Therefore, in this paper, a deep learning technique is used to process large-scale social media text data. First, it uses Natural Language Processing (NLP) based feature extraction to create four different sets of training samples i.e. TF-IDF-based features, POS Tagged Features, a reduced feature vector of POS and the combined vector of TF-IDF and POS tagged features. The deep Convolutional Neural Networks (CNN) is used to train the model and to classify hate and offensive language. The dataset has been obtained from Kaggle. The performance in terms of training accuracy, validation accuracy, training loss and validation loss has been measured with the time complexity. In addition, the class-wise Precision, Recall, F1-score and Mean accuracy have also been investigated. From experimental results, we found TF-IDF and POS-based combined features provide 82% and 83% of accuracy, respectively.

Keywords: Text mining, social media, semantic knowledge, sentiment analysis, deep learning, hate and offensive language.*

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1. Introduction

Information Retrieval (IR) is a technique used to locate precise information and supports different data formats, i.e. text, image, video, etc. Among these data formats, the text has a significant contribution. The text IR model uses text mining techniques. In IR techniques, text mining or data mining algorithms are employed to recover user query relevance information [1]. The IR model contains three key components: (1) User query (2) Query processing (3) Generation of outcomes[2]. However, the deficiency of these components can impact the performance of IR model such as lack of specific user query keywords, inappropriate keyword selection, lack of similar data, ranking of results etc. [3].

Our study is initiated with an IR model named SOIR (Semantic query Optimization-based Information Retrieval). This IR model incorporates query optimization and an FCM clustering technique. The results indicate the performance of the SOIR is better than previous models [4]. However, there are various applications of the IR systems. Beyond these applications, the IR model can also be applied for pattern recognition. In this context, the proposed work is extended in order to be used with the social media toxic content filtering. This paper is an extension of the SOIR model, which will be used for classifying the toxic contents from the social media posts. This model is a promising technique for handling various negative tweets from social media by using lexical as well as semantic pattern analysis.

In this paper, we are going to evaluate the SOIR and DGM based toxic content classification system on a large dataset. Additionally, we do an investigation of a deep learning technique for classifying the different NLP based text features. We organize the contents in the following manner:

1. Background: This section discusses the previously introduced SOIR and Directed Graph Model (DGM) for text IR and toxic content classification.
2. Proposed work: This Section provides the understanding of the proposed Deep learning model for classifying hate speech, offensive language and normal text.
3. Results analysis: This section contains the experimental analysis of the proposed model and the different parameters have been provided.
4. Conclusion: In this section, finally the work is concluded and future extension of work has been suggested.

2. Background

This section provides insight into SOIR model and previously developed DGM classification model.

A. SOIR Model

This model is aimed to improve the IR model for running time and relevancy. Due to the large amount of data, a significant amount of time is required to locate the precise information. In the documents database, a number of documents are present which do not belong to the similar category and content. This nature of database increases search

space. Therefore, some improvements have been made to enhance the user query representation and subjective data categorization. The query optimization involves a semantic model to recognize similar words to optimize the query.

First, we pre-process the data to improve the quality of data and exclusion of noise by removing stop words and special characters. Next, feature selection techniques are used to reduce data dimensions and speed up the search process. Thus, the TF-IDF is used [5], to compute the weight W for identifying important tokens. The fixed size of the feature i.e. 30 tokens is considered. Further, FCM clustering is being used to categorize feature vectors [6]. The clustering results are organized as a list of features:

$$F = \langle F_n, k_{1,2,...,n}, C \rangle$$

where, F is the feature set, F_n is the file name or index, $k_{1,2,...,n}$ is the list of keywords, and C is the class name or subject. The training feature vector F is stored in a database. The categorized features are helpful for efficient data retrieval. On the other hand, the user query is transformed into a vector Q as a set of keywords:

$$Q = \{q_1, q_2, \dots, q_k\}$$

In order to optimize the query, we initialize a set of queries by using synonyms. Additionally, a map is prepared that contains the keywords and the synonyms. In this algorithm, a single keyword is twisted multiple times to generate new queries using similar words. The different search query increases the chances of finding accurate data. The search process is developed on the basis of the k-NN (k-nearest neighbour) algorithm [7]. The distance between the query and data less than 0.25 is counted as the result. The developed SOIR model has been compared with the Cosine similarity-based and k-NN-based IR models. The precision, Recall and F-Score are calculated and visualized in figure 1.

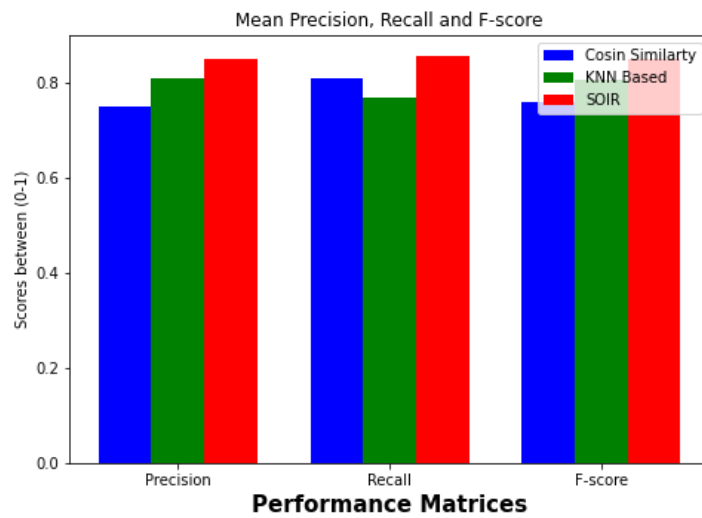


Figure 1. Mean Performance of SOIR Model

In this diagram, X-axis shows the measured matrix, and the Y-axis shows the precision, Recall and F-Score. According to the results, we found that the technique improves the precision, recall and f-score as we utilizing a large size of learning data.

B. SOIR for toxic content classification

In order to keep clean social media, we need an accurate model to identify malicious and toxic posts. We used an SOIR model for this task [4]. We summarized the negative emotions identified in table 1 below, which consists of Emotion classes and the relevant flow of emotions [8]. There are five main classes, and each class consists of its own subclasses. Thus, we need a multiclass classification system. So, we solve this problem by using the SOIR-based model.

Table 1. Emotion Classes

Emotions	Content
Distressed	Sad, Disappointed, Guilty, Missed
Surprised	Surprised
Fearful	Panic, Frightened, Shy
Angry	Angry
Disgusted	Dissatisfied, Annoyed, Doubtful, Hateful

In order to train the model, we have collected more than 3329 tweets and categorized them. Then we considered 2002 tweets for the experiment. The pre-processing has been applied for removing tags, special characters, and stop words. Further, part of speech tagging (POS) is used to understand the lexical structure [9]. The POS tags and sentiment classes are used to prepare a new dataset. On the other hand, tweets are also processed using the TF-IDF weights. The top 20 weighted tokens are picked and used. In the next process, we prepared thresholds using the tweets' POS tags. In the first step, we compute the mean feature of each feature group using:

$$M_F = \frac{1}{N} \sum_{i=1}^N F_i$$

Where M_F is the mean of the feature, N is the total number of samples in the groups. After measuring the mean value, we compute the distance from each point as the limit.

$$L_F = \frac{1}{N} \sum_{i=1}^N |F_i - M_F|$$

Thus the threshold of the feature can be defined as:

$$T_F^{Group} = M_F \pm L_F$$

Using this equation, we calculate the feature map FM for all the POS features. After that, we append features of the TF-IDF. Here the FCM clustering is used for creating the dictionary learning. We are just using the membership function of FCM for preparing the dictionary. The membership between data instance i and centroid j is measured using:

$$\mu_{i,j} = \frac{1}{\sum_{k=1}^c \left(\frac{d_{i,j}}{d_{i,k}} \right)^{\frac{2}{m-1}}}$$

According to this process, the emotion-labeled data is being used from each group. A tweet is tokenized and then inserted into the dictionary D with the TF-IDF weight. If a token exists in the dictionary, then we update the weight of the token and if the token is not available, then we simply insert the token. In order to compute the updated weight, we used:

$$NewWeight = 0.5 * Oldweight + 0.5 * \mu_{i,j}$$

where, $\mu_{i,j}$ is the membership between previous and new weight for the token.

The dictionaries are preserved in a database. Now we classify new tweets. For the classification, 50% of training samples and 50% of new tweets from Tweeter have been used. Then, we perform the POS tagging of test data. The tagged feature vector T_v is used to compare with the thresholds. For both pattern matching and decision making here the mean value M and limit L are used for computing upper threshold L_{max} and lower threshold L_{min} . The patterns between these two limits are used to compute the distance among queried tweets. POS tag feature and all the features threshold are used from feature map FM. Finally the higher matched value based class label is predicted as decision. After lexical pattern based decision making, we use the semantics for categorizing a tweet as final class label. Thus, the tweets are tokenized to regenerate it, as in SOIR. Let us have a tweet such that:

$$TW = \{k_1, k_2, \dots, k_n\}$$

After regeneration, we get:

$$TW_{m,n} = \begin{cases} k_{1,1}, k_{1,2}, \dots, k_{1,n} \\ k_{2,1}, k_{2,2}, \dots, k_{2,n} \\ \dots \\ k_{m,1}, k_{m,2}, \dots, k_{m,n} \end{cases}$$

After preparing the set of similar keywords, we train the model using keywords and relevant weights for all the classes. The trained model can be defined as:

$$D_{o,p} = \begin{cases} D_1 = \{[T_{1,1}, W_{1,1}], [T_{1,2}, W_{1,2}] \dots, [T_{1,p}, W_{1,p}]\} \\ D_2 = \{[T_{2,1}, W_{2,1}], [T_{2,2}, W_{2,2}] \dots, [T_{2,p}, W_{2,p}]\} \\ \dots \\ D_o = \{[T_{o,1}, W_{o,1}], [T_{o,2}, W_{o,2}] \dots, [T_{o,p}, W_{o,p}]\} \end{cases}$$

This is a function that generates a virtual directed graph using a weight matrix for decision-making. The matrix in the form of a graph model describes the association of a tweet with the given sentiment dictionary. An algorithm is developed to get the class label of the tweet, as given in table 2.

Table 2. DGM based Classification

Input: set of semantically similar twits $TW_{m,n}$, trained dictionary model $D_{o,p}$
Output: Class label C
Process:
1. <i>for</i>($i = 1; i < m; i++$) a. <i>for</i>($j = 1; j < n; j++$) i. <i>if</i> ($D_i.contains(TW_{i,j})$) 1. $W_i = W_{i-1} + W_{i,j}$ ii. <i>end if</i> b. <i>end for</i> c. $DW_i = \frac{W_i}{m}$ 2. <i>End for</i> 3. $C = getMaxVal(DW_i).classlabel$ 4. <i>Return</i> C

The above algorithm searches each word in the dictionary and the relevant weights are aggregated. A greater value of weights is used as the final sentiment label. In order to make final decision, we use a function as:

$$f(B, C) = \begin{cases} B = C & \text{then } C \\ B \neq C & \text{is } B_{i-1} == C \text{ then } C \end{cases}$$

Finally, if we find the Class label C belongs to shy, panic, sad and guilty then we label the tweet as nontoxic else C returns the toxic labels. This model is named as Directional Graph Model. Figure 2 below shows the precision, recall and f-score of different techniques. The results indicate the DGM shows better accuracy as compared to other models. However, the DGM performance is higher but the SOIR is consistent as compared to DGM.

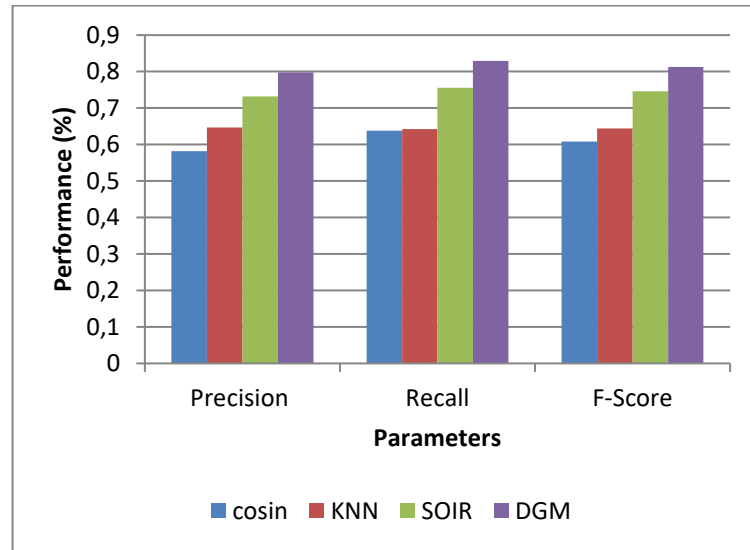


Figure 2. Performance summaries of classification models

Secondly, we measured the required time for training and reported it in figure 3. According to the results, the cosine-based and k-NN based techniques are winners. The main advantage of this model is that we can preserve the previously trained model for future use. Additionally, new data will also learn continuously. The model's performance replicates the efficient and accurate identification of toxic tweets.

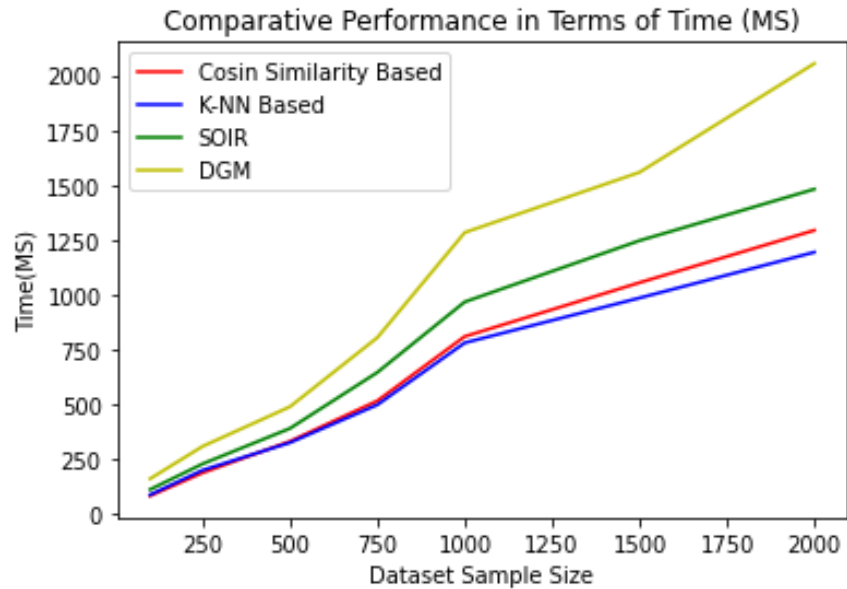


Figure 3. Time consumption

3. The proposed work

Emotion classification has become a common and classical domain of research and development but still there are many opportunities available for discovering something new. Initially, we started our work with the study of text mining and information retrieval. But due to new kinds of problems associated with text mining attracted us to classify social media text for finding toxic content. In order to design an effective method, we have implemented a DGM approach which is accurate and efficient for modelling negative emotions. But during the investigation, we found the following issues we tried to resolve in this study.

1. Social media content these days is full of offensive language even during formal communication, comments and others. Therefore, there is a need of distinguishing between the content as normal and toxic.
2. The previous experiments have been done with a small size of samples, but social media involve big data problems. Therefore, a previously defined model is reframed using the Deep learning technique to deal with a large amount of data.

In this context, we have obtained a dataset of Twitter from Kaggle. The dataset is hate speech and offensive language dataset. The dataset is distributed in three class labels hate speech, offensive and neither. The dataset involves 1430 instances as hate speech post of twitter, 19190 tweets as the offensive language and 4163 instances as the normal text. The dataset consists of 24783 instances of data. Among them, we make a split of 18587 instances for training and 6196 for testing. However, this dataset contains more data than the dataset on which we trained our previously designed models.

Therefore, initially, we have employed our DGM model for extracting features. During this, we found a significant drawback of this method, which we tried to demonstrate in figure 4. Figure 4 shows the time consumed during the experiment with only 50% of entire dataset for constructing the required features for DGM technique. However, the DGM based technique has accurate results but having huge time complexity for training with a large amount of data.

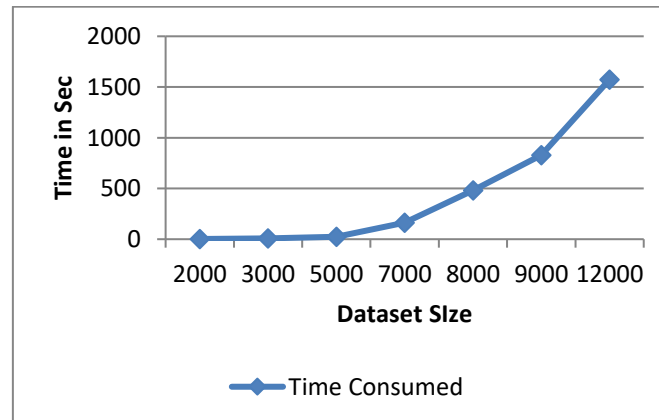


Figure 4. Time consumed with DGM

Therefore we drop the idea of utilizing the traditional classification technique for processing such a huge amount of data. The main reason behind the time consumption is computation of TF-IDF features and then computing the feature representation, which increases the time consumption significantly. Additionally, the measured features that need to train with the machine learning algorithms consume a significant amount of time for training. Therefore, the idea of working with the large amount of data with the classical and traditional feature construction method is becoming expensive. Finally, we proposed to use the Deep Learning technique for classifying the hate speech and offensive text to deal with the large amount of data. However, the deep learning models are mainly developed for image classification, but these techniques can also be used for other relevant classification and prediction tasks. The proposed model consists of the following steps to conduct the experiments.

Data preparation

In any machine learning model, data preparation is an essential step of data analysis. We have placed the obtained data from Kaggle into our Google drive and use the Google Colab to process data using the different techniques of text mining and machine learning.

	Unnamed: 0	count	hate_speech	offensive_language	neither	class	tweet
0	0	3	0	0	3	2	!!! RT @mayasolovely: As a woman you shouldn't...
1	1	3	0	3	0	1	!!!! RT @mleew17: boy dats cold...tyga dwn ba...
2	2	3	0	3	0	1	!!!!!! RT @UrKindOfBrand Dawg!!!! RT @80sbaby...
3	3	3	0	2	1	1	!!!!!!! RT @C_G_Anderson: @viva_based she lo...
4	4	6	0	6	0	1	!!!!!!!!!!!! RT @ShenikaRoberts: The shit you...

Figure 5. Dataset samples

In the first step, we load the dataset, the initial dataset samples are demonstrated in figure 5 with their relevant attributes. However, the dataset contains various attributes, but we are just utilizing the class labels and tweets here for data analysis.

Data pre-processing

In basic terminology, the data pre-processing is a technique by which we prepare the data to apply to the learning algorithm. Therefore in this context, we studied various NLP based text pre-processing techniques. From them, some essential techniques are applied for pre-processing of data. Here we applied the following pre-processing techniques:

- Punctuation Removal:** In this phase, we removed all the Punctuations from the input dataset instances by analyzing each character of the input strings.

- b. Tokenization: In this step, the aim is to divide the tweet strings into chunks of words.
- c. Stop-word Removal: In this step, the aim is to remove English stop words. The stop words are those words which frequently occur in the text but not essential for domain identification such as, 'this', 'that', 'is', 'am' so on. In this context, we apply the Natural Language Toolkit (NLTK) library for stop word removal.
- d. Stemming and Lemmatization: Both are generated form of the same words. The difference is that stem might not be an actual word and lemma is an actual language word. Stemming usage an algorithm to which makes word processing faster. On the other hand, in lemmatization, we used WordNet to produce lemma, which is slower than stemming.

Unnamed: 0	count	hate_speech	offensive_language	neither	class	tweet
0	0	3	0	0	3	2 [rt mayasolovely as a woman you shouldnt comp...
1	1	3	0	3	0	1 [rt mleew17 boy dats coldtyga dwn bad for cuf...
2	2	3	0	3	0	1 [rt urkindofbrand dawg rt 80sbaby4life you ev...
3	3	3	0	2	1	1 [rt cganderson vivabased she look like a tranni]
4	4	6	0	6	0	1 [rt shenikaroberts the shit you hear about me...

Figure 6. Pre-processed data

Features analysis

In this phase, we transform the tweets into a structured manner. So, we have used the following different ways to prepare four different set of feature vector.

- a. POS Tagging: In this phase, we utilized the NLTK based POS Tagger to process each string of the dataset. After that we prepared the vector for making it training and testing set. In this process we get 37 tags or features for training data. The transformed data is demonstrated in figure 7.

	NN	RB	IN	DT	PRP	VBP	VB	VBG	RP	PRP\$	MD	NNS	JJ	CD	VBZ	VBD	CC	TO	VBN	JJR	WP	WDT	WRB	RBS	PDT
0	6.0	2.0	3.0	3.0	2.0	1.0	2.0	1.0	2.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	7.0	0.0	2.0	1.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	2.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	5.0	1.0	1.0	1.0	3.0	1.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
3	3.0	0.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	3.0	0.0	2.0	2.0	4.0	1.0	3.0	0.0	0.0	0.0	2.0	0.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	1.0	1.0	0.0	0.0	0.0	0.0

Figure 7. Sample tagged data

- b. POS with reduced features: In this phase, the 37 features of NLP-POS tags are used for experimentation. In order to reduce the amount of features, we have used Principle Component Analysis (PCA) and reduced the features, finally we have getting only 20 features for applying the algorithms.

- c. Term Frequency – Inverted Document Frequency (TF-IDF): In this feature vector, we utilized TF-IDF Vectorizer to extract weighed keywords from the data. The dimensions of data are kept maximum 5000 keywords. The extracted features are demonstrated in figure 8.

	00sexilexi00	05professor	10	100	1000	10000	100046729	1008465039	100d	1017sqquad	1041204	10412041041204	1043359	11	12	1
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

5 rows × 5000 columns

Figure 8. extracted TF-IDF features

- d. Combined features: In this phase, we just combined TF-IDF features, and POS Tagged Features in a vertical manner to construct total of 5037 columns and 24783 rows of data.

Applying CNN

Here, we trained the high-dimensional data using Convolutional neural network (CNN). Therefore, we have developed a simple three-layer sequential model. First layer is configured with the total number of attributes in training features, and activation function 'ReLU' is used. The second hidden layer consists of 10 neurons and the activation function 'Sigmoid' has been used. The final layer contains three output neurons and 'Soft-max' activation function. The model takes very less amount of time to train the CNN model. The training time of the input training vectors has given in figure 9. The model will be trained in a few times even the combined features, which consist of 5037 columns and 24783 data. Therefore, we can use the large data for learning with deep learning techniques.

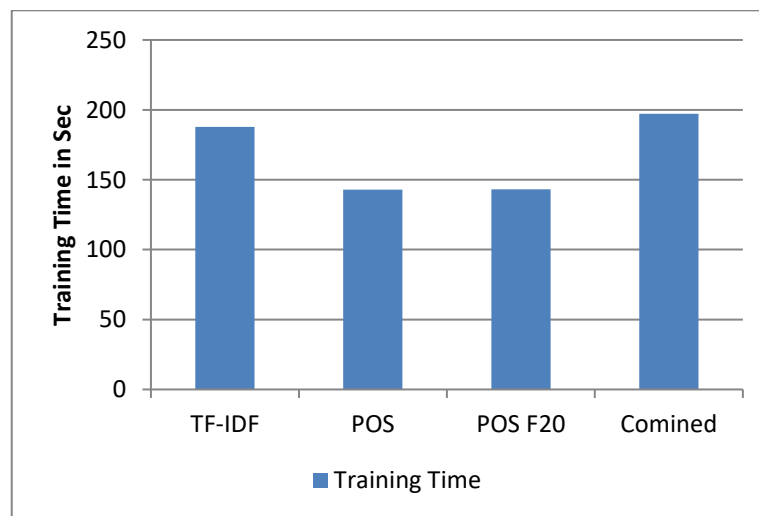


Figure 9. Training time

Performance analysis

In this phase, we evaluated our model for obtaining the performance matrix for learning as well as the validation of the model. Therefore, the training accuracy, validation accuracy, training loss and validation loss of the model have been measured. These performance parameters are discussed in next section. In addition, the class-wise performance report has also been prepared.

4. Results

This section details the conducted experiments and the obtained performance results. In deep learning, the accuracy metric is used to measure the 'algorithm's performance in an interpretable manner. The accuracy is determined in the form of a percentage. It measures how accurate our model's prediction is with the true data.

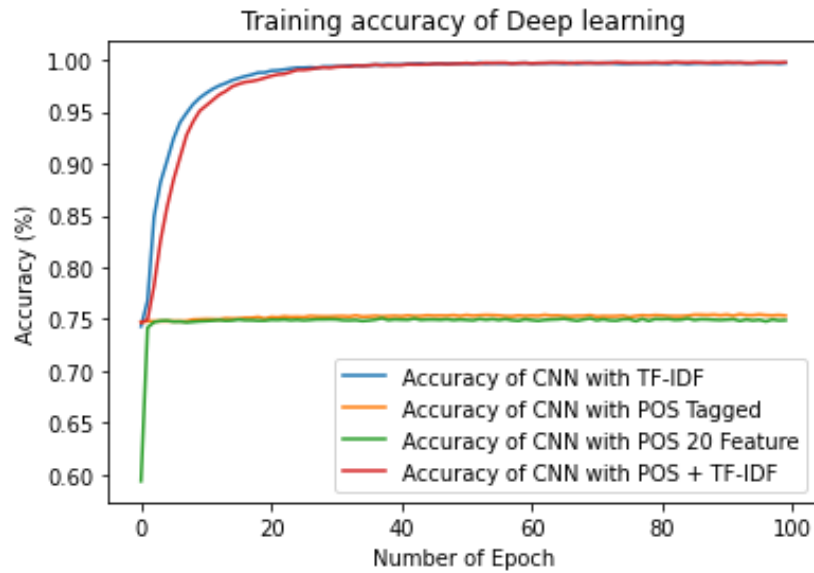


Figure 10. (A) Training Accuracy (%)

In this experiment, we have used four different set of feature vectors namely TF-IDF based weighted feature vector, second the POS tagged features, third we have reduce the size of POS tagged data for classification and finally the combined features of POS and TF-IDF has been employed with the CNN model. The performance of the model with the different sets of selected features in terms of accuracy (%) has been reported using figure 10(A). Additionally, the accuracy for validation or test dataset has been demonstrated in 10(B). According to the obtained consequences in terms of these two parameters, we found that the TF-IDF based and the combination of TF-IDF and POS

produces a higher performance than the other two algorithms. During training, we found the POS tagged data shows approximately 74.5 to 75.2% performance.

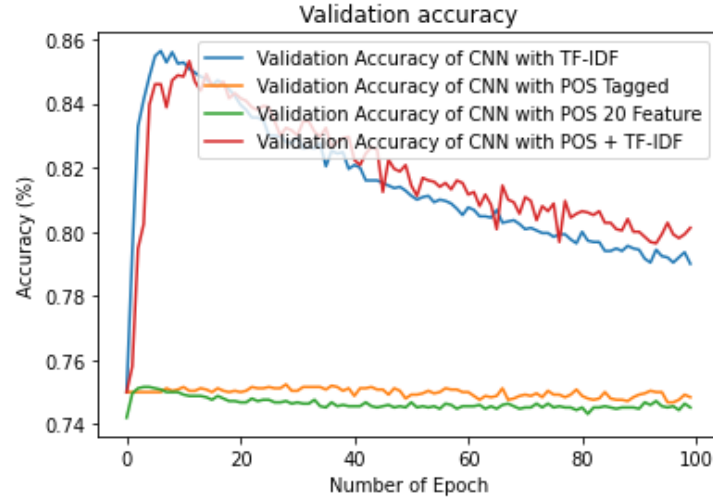


Figure 10. (B) Validation Accuracy (%)

On the other hand, when we used the reduced 20 features with the POS tagged data then we found a small improvement and we get an accuracy of 75.5%, which is higher than the POS tagged complete dataset, which has 36 features.

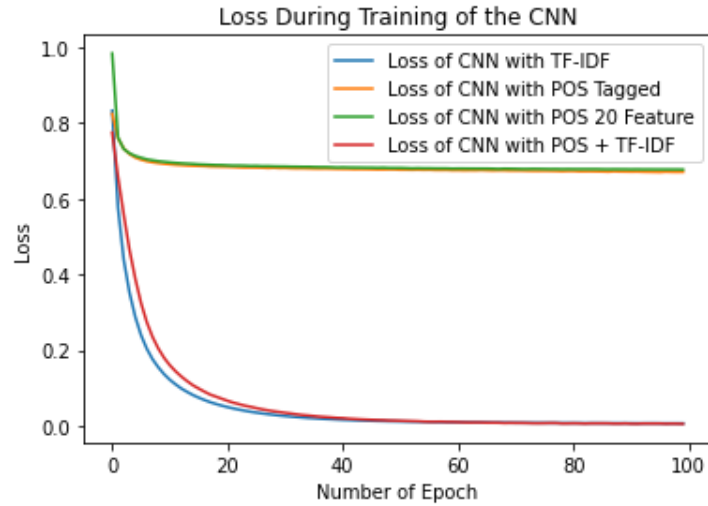


Figure 10. (C) Training Loss

On the other hand, when we train the model with the same data but TF-IDF features, model returns the total of 98.8% accurate results in consistent manner. And when we apply the combination of POS and TF-IDF features, we get improved results up to

99.8%. However, when working with the validation set, the performance of algorithm has dropped continuously. In figure 10(B), we can see the performance of all the feature based technique has been decreased significantly.

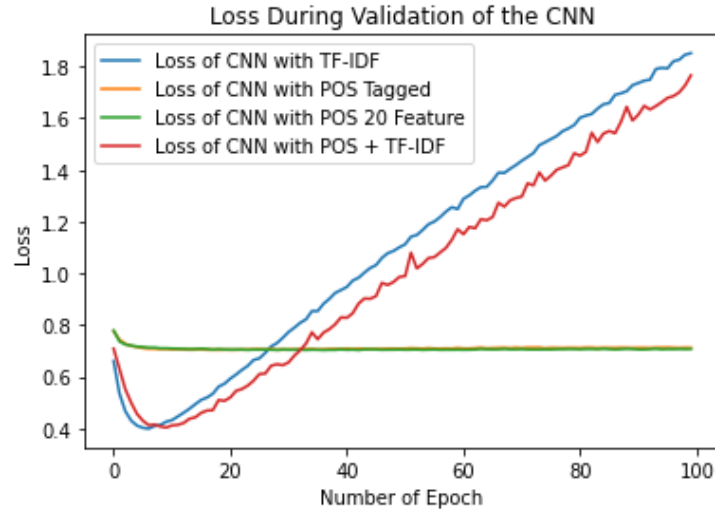


Figure 10. (D) Validation Loss

In addition, we have also computed the loss for training and validation. The loss function is used to optimize a machine learning algorithm. The loss is calculated on the training set and validation set and its interpretation is based on how well the model is doing in these two sets. It is the sum of errors made for each sample in training or test sets. Loss value implies how poorly a model behaves after each round of optimization. A loss is a number indicating how bad the 'model's prediction was on a sample. If the 'model's prediction is perfect, the loss is zero; otherwise, the loss is greater. The goal of a model is to find the set of weights and biases that have low loss. Higher loss is worse for any model. A loss is not a percentage. It is a sum of the errors made for each sample in training or validation.

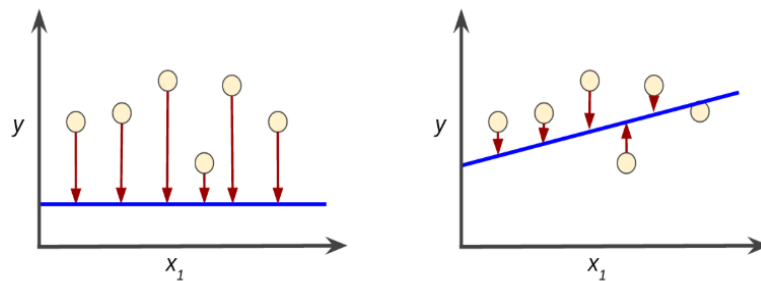


Figure 11. shows the loss measuring example

Figure 11 shows two graphs demonstrating the losses of two different models, the left graph has a high loss and the right graph has a low loss.

- The arrows represent a loss.
- The blue lines represent predictions.

The loss of the proposed deep learning model has demonstrated in figure 10(C) and 10(D). The figure 10(C) shows the training loss and 10(D) shows the validation loss of the model. In the diagram 10(C), we found that the loss of training reduced and become consistent. On the other hand, 10(D) demonstrates the loss during the validation of the model. The performance indicates when number of epoch has increases the loss of validation has also been increases in the case of TF-IDF based features learning, as well as also for the combined features, but the for combined features the loss function is less incline.

On the other hand for POS tagged data we can see the model produces similar or consistent performance in all the cases. Finally, for more simple understanding we have also measured the classification report.

Table 3. Performance summary of the models

	POS with 20 feature			POS Tagged			TF-IDF			Combined (TF-IDF + POS)		
	Precision	Recall	F1-score	Precision	Recall	F1-score	Precision	Recall	F1-score	Precision	Recall	F1-score
Class 0	0.0	0.0	0.0	0.0	0.0	0.0	0.25	0.31	0.28	0.23	0.29	0.26
Class 1	0.78	0.99	0.87	0.77	0.99	0.87	0.90	0.89	0.89	0.90	0.89	0.90
Class 2	0.51	0.06	0.51	0.42	0.05	0.09	0.70	0.70	0.70	0.74	0.72	0.73
Accuracy	0.77			0.77			0.82			0.83		
Marco avg	0.43	0.35	0.43	0.40	0.35	0.32	0.62	0.63	0.62	0.62	0.63	0.63
Weighted avg	0.69	0.77	0.69	0.67	0.77	0.68	0.83	0.82	0.83	0.84	0.83	0.83

The classification report for the implemented techniques has been given in table 3. The class-wise performance analysis is demonstrated in the table 3. That demonstrates the precision, recall and F1-score. Additionally, shows the accuracy, Marco avg, and weighted avg. according to these values we can say the combined and TF-IDF based features are providing higher accurate results. The accuracy has also visualized in figure 12. According to the given results of the implemented techniques, we found that the accuracy of POS based both the techniques i.e. entire features and top 20 features based approach demonstrate the similar accuracy of 77%. On the other hand the TF-IDF and combined features based technique provide the 82% and 83% of accuracy. Thus we can say the combined features based classification approach provide higher accurate results.

5. Conclusions

Social media is one of the user-centric and user-generated data source. The data may have a significant amount of knowledge, challenges and opportunity for diverse areas of

applications. However, there are a number of opportunities hidden in the social media data, but toxic intension based data contains can create the biggest issues in healthy social media surroundings. In this context, a DGM model is analysed, which is limited to working with a small amount of data. Therefore, in this paper, a deep learning Model is proposed for classifying hate speech, offensive language and the normal social media posts. The finding says the deep learning techniques are better for classifying large datasets.

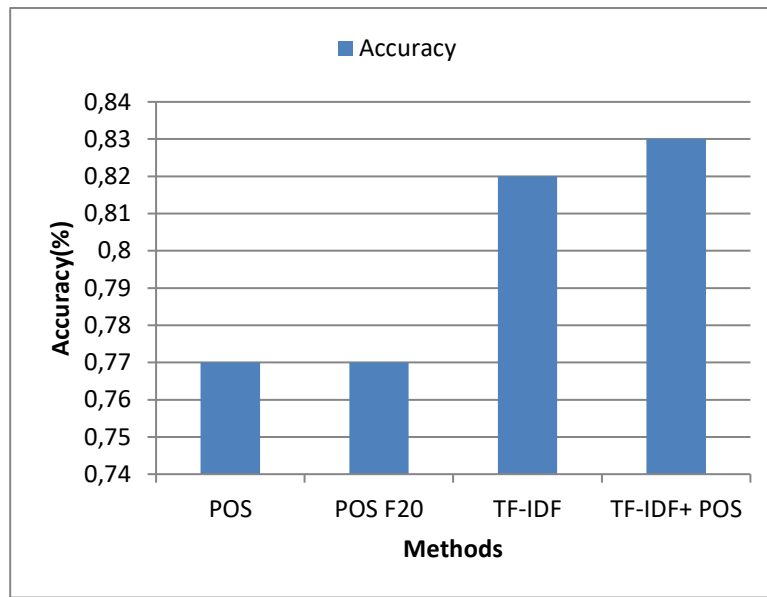


Figure 12. Comparison of Mean Accuracy

The DGM model utilizes the classical techniques for accurate classification of text but a limitation of the model motivates us to work with deep learning models. The proposed model includes the text feature analysis techniques and CNN algorithm for providing an accurate text classification technique. This model aims to differentiate between hate speech, offensive text and normal text by only considering text in social media post. During the experiments we conclude the following observations:

1. DGM is accurate but is limited to work with the less dimensional data on the other hand the deep learning is effective for learning with large scale data
2. Deep learning technique provides relevant or higher accuracy in less amount of time which is proved in the obtained performance.
3. In classification of offensive text and hate speech text, the key words are more essential then the sentence structures or sentence formation.
4. POS tagger is less effective for representing the text features to model difference between offensive and hate speech text.
5. The combination of keyword based features and POS tagged data can improve the performance significantly.

This study is helpful as a tutorial to demonstrate how the classical machine learning techniques are accurate but can be expensive to solve real-world problems. On the other hand, this study also tried to demonstrate how we can make use the deep learning techniques for processing the text data and get higher classification accuracy. In the near future, the following directions can be helpful for the extension of this study:

1. The deep learning techniques involve more complex data models, which will provide more accurate data analysis. Among them the LSTM has become much more popular. For extension, we can use them in future implementations.
2. The combination of two different kinds of features increases the dimension of the data to be trained. At the same time, in the selected TF-IDF features, various non-meaningful keywords are available. Therefore, the need is to design an enhanced semantic keyword selection technique for feature representation.

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A new class of almost continuity in topological spaces

Jagadeesh B.Toranagatti*

Abstract

In this paper, we apply the notion of $\delta g\beta$ -open sets due to Benchalli et al.[Benchalli et al., 2017] to present a new class of functions called almost $\delta g\beta$ -continuous functions along with its several properties, characterizations and mutual relationships.

Keywords: almost continuity, almost β -continuity, $\delta g\beta$ -continuity, almost $\delta g\beta$ -continuity.

2020 AMS subject classifications: 54A05, 54C08. ¹

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1 Introduction

The notion of continuity is an important concept in general topology as well as all branches of mathematics. Of course it's weak forms and strong forms are important, too. Singal and Singal[Singal et al., 1968], in 1968, defined almost continuous functions as a generalization of continuity.Noiri and Popa[Noiri and Popa, 1998], in 1998, defined almost β -continuous functions as a generalization of almost continuity.Recently, Benchalli et al.[Benchalli et al., 2017] introduced the notion of $\delta g\beta$ -continuous functions in topological spaces.

In this article, using the notion of $\delta g\beta$ -open sets given in [Benchalli et al., 2017], we introduce and study a new class of functions called almost $\delta g\beta$ -continuous functions. We investigate several properties of this class. The class of almost $\delta g\beta$ -continuity is a generalization of almost β -continuity and $\delta g\beta$ -continuity.

2 Preliminaries

Throughout this paper $(X, \tau), (Y, \sigma)$ and (Z, η) (or simply X, Y and Z) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise stated. Let M be a subset of X . The closure of M and the interior of M are denoted by $cl(M)$ and $int(M)$, respectively.

Definition 2.1. A set $M \subseteq X$ is called β -closed[Abd, 1983](=semi preclosed[Andrijević, 1986] (resp., pre-closed[Mashhour, 1982], regular closed[Stone, 1937], semi-closed [Levine, 1963] if $int(cl(int(M))) \subseteq M$ (resp., $cl(int(M)) \subseteq M, M = cl(int(M), int(cl(M))) \subseteq M$).

Definition 2.2. A set $M \subseteq X$ is called δ -closed [Velicko, 1968] if $M = cl_\delta(M)$ where $cl_\delta(M) = \{ p \in X : int(cl(N)) \cap M \neq \phi, N \in \tau \text{ and } p \in N \}$.

Definition 2.3. A set $M \subseteq X$ is called $g\beta$ -closed [Tahiliani, 2006](resp., $gspr$ -closed [Navalagi et al.] and $\delta g\beta$ -closed[Benchalli et al., 2017] if $\beta cl(M) \subseteq G$ whenever $M \subseteq G$ and G is open(resp. regular open and δ -open) in X .

The complements of the above mentioned closed sets are their respective open sets.

The class of $\delta g\beta$ -open (resp., $\delta g\beta$ -closed, open, closed, regular open, regular closed, preopen, semiopen and β -open) sets of (X, τ) containing a point $p \in X$ is denoted by $\delta G\beta O(X, p)$ (resp., $\delta G\beta C(X, p), O(X, p), C(X, p), RO(X, p), RC(X, p), PO(X, p), SO(X, p)$ and $\beta O(X, p)$).

Definition 2.4. A function $f:X \rightarrow Y$ is called almost continuous [Singal et al., 1968] (resp., almost β -continuous [Noiri and Popa, 1998] and almost gspr-continuous) if the inverse image of every regular open set G of Y is open (resp., β -open and gspr-open) in X .

Definition 2.5. [Benchalli et al., 2017] A function $f:X \rightarrow Y$ is called $\delta g\beta$ -continuous (resp., $\delta g\beta$ -irresolute) if the inverse image of every open (resp., $\delta g\beta$ -open) set G of Y is $\delta g\beta$ -open in X .

Definition 2.6. A function $f:X \rightarrow Y$ is called almost contra continuous [Baker, 2006] (resp. almost contra super-continuous [Ekici, 2004] and contr R -map [Ekici, 2006] if the inverse image of every regular closed set G of Y is open (resp. δ -open and regular open) in X .

Definition 2.7. A space X is said to be:

- (i) nearly compact [Singal and Mathur, 1969] if every regular open cover of X has a finite subcover,
- (ii) r - T_1 -space [Ekici, 2005] if for each pair of distinct points x and y of X , there exist regular open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$,
- (iii) r - T_2 -space [Ekici, 2005] if for each pair of distinct points x and y of X , there exist regular open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$,
- (iv) $\delta g\beta$ - T_1 space if for any pair of distinct points p and q , there exist $G, H \in \delta G\beta O(X)$ such that $p \in G$, $q \notin G$ and $q \in H$, $p \notin H$,
- (v) $\delta g\beta$ - T_2 space [BENCHALLI et al., 2017] if for each pair of distinct points x and y of X , there exist $G, H \in \delta G\beta O(X)$ such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$.

Definition 2.8. [Benchalli et al., 2017] A space X is said to be $T_{\delta g\beta}$ (resp., $\delta g\beta T_{\frac{1}{2}}$)-space if $\delta G\beta O(X) = O(X)$ (resp., $\delta G\beta O(X) = \beta O(X)$).

Definition 2.9. [Carnahan, 1972] A subset M of a space X is said to be N -closed relative to X if every cover of M by regular open sets of X has a finite subcover.

Theorem 2.1. [Benchalli et al., 2017] If A and B are $\delta g\beta$ -open subsets of a extremely disconnected and submaximal space X , then $A \cap B$ is $\delta g\beta$ -open in X .

Definition 2.10. [Jankovic, 1983] A space X is called locally indiscrete if $O(X) = RO(X)$.

Lemma 2.1. [Noiri, 1989]

Let (X, τ) be a space and let M be a subset of X .
 $M \in PO(X)$ if and only if $scl(M) = int(cl(M))$.

3 Almost $\delta g\beta$ -continuous functions

Definition 3.1. A function $f: X \rightarrow Y$ is said to be almost $\delta g\beta$ -continuous at $p \in X$ if for each $N \in \delta O(Y, f(p))$, there exists $M \in \delta G\beta O(X, p)$ such that $f(M) \subseteq N$. If f is almost $\delta g\beta$ -continuous at every point of X , then it is called almost $\delta g\beta$ -continuous.

Remark 3.1. We have the following implications

almost β -continuity $\longrightarrow$ almost $\delta g\beta$ -continuity $\longrightarrow$ almost $gspr$ -continuity.

$\uparrow$

$\delta g\beta$ -continuity.

None of these implications is reversible.

Example 3.1. Let $X = \{p, q, r, s\}$, $\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}\}$ and $\sigma = \{Y, \phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{p, q, r\}\}$. Define $f: (X, \tau) \rightarrow (X, \sigma)$ by $f(p) = f(r) = q$, $f(q) = p$ and $f(s) = r$. Clearly f is almost $\delta g\beta$ -continuous but for $\{q\} \in RO((X, \sigma))$, $f^{-1}(\{q\}) = \{p, r\} \notin G\beta O(X, \tau)$. Therefore f is not almost $g\beta$ -continuous. Define $g: (X, \tau) \rightarrow (X, \sigma)$ by $g(p) = p$, $g(q) = s$, $g(r) = r$ and $g(s) = q$. Then g is almost $\delta g\beta$ -continuous but for $\{p\} \in O(X, \sigma)$, $g^{-1}(\{p\}) = \{p\} \notin \delta G\beta O(X, \tau)$. Therefore g is not $\delta g\beta$ -continuous. Define $h: (X, \tau) \rightarrow (X, \sigma)$ by $h(p) = h(q) = q$, $h(r) = p$ and $h(s) = r$. Then h is almost $gspr$ -continuous but for $\{q\} \in RO(X, \sigma)$, $h^{-1}(\{q\}) = \{p, q\} \notin \delta G\beta O(X, \tau)$. Therefore h is not almost $\delta g\beta$ -continuous.

Theorem 3.1. If $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous and Y is locally indiscrete space, then f is $\delta g\beta$ -continuous.

Proof: It follows from the Definition 2.10

Theorem 3.2. Let X be a locally indiscrete space and $M \subseteq X$, then the following properties are equivalent:

- (i) M is $gspr$ -closed;
- (ii) M is $\delta g\beta$ -closed;
- (iii) M is $g\beta$ -closed.

As a consequence of above Theorem, we have the following result;

Theorem 3.3. Let X be a locally indiscrete space, then the following properties are equivalent:

- (i) $f: X \rightarrow Y$ is almost $gspr$ -continuous;
- (ii) $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous;
- (iii) $f: X \rightarrow Y$ is almost $g\beta$ -continuous.

Theorem 3.4. *Let X be a $\delta g\beta T_{\frac{1}{2}}$ -space. Then the following are equivalent:*

- (i) $f: X \rightarrow Y$ is almost β -continuous;
- (ii) $f: X \rightarrow Y$ is almost $g\beta$ -continuous;
- (iii) $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous.

Theorem 3.5. *Let X be a $T_{\delta g\beta}$ -space. Then the following are equivalent:*

- (i) $f: X \rightarrow Y$ is almost continuous;
- (ii) $f: X \rightarrow Y$ is almost β -continuous;
- (iii) $f: X \rightarrow Y$ is almost $g\beta$ -continuous;
- (iv) $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous;
- (v) $f: X \rightarrow Y$ is almost $gspr$ -continuous.

Lemma 3.1. [BENCHALLI et al., 2017] *For a subset M of a space X , the following are equivalent:*

- (i) M is δ -open and $\delta g\beta$ -closed;
- (ii) M is regular open;
- (iii) M is open and β -closed.

Theorem 3.6. *The following statements are equivalent for a $f: X \rightarrow Y$:*

- (i) f is almost contra super-continuous and almost $\delta g\beta$ -continuous;
- (ii) f is contra R -map;
- (iii) f is almost contra continuous and almost b -continuous.

Theorem 3.7. *The following statements are equivalent for a $f: X \rightarrow Y$:*

- (i) f is almost $\delta g\beta$ -continuous;
- (ii) For each point $p \in X$ and each $G \in \delta C(Y)$ with $f(p) \notin G$, there exists a $H \in \delta G\beta C(X)$ and $p \notin H$ such that $f^{-1}(G) \subseteq H$;
- (iii) For each point $p \in X$ and each $N \in RO(Y, f(p))$, there exists an $M \in \delta G\beta O(X, p)$ such that $f(M) \subseteq N$;
- (iv) For each point $p \in X$ and each $G \in RC(Y)$ with $f(p) \notin G$, there exists a $H \in \delta G\beta C(X)$ and $p \notin H$ such that $f^{-1}(G) \subseteq H$;

(v) for each $p \in X$ and each $N \in O(Y, f(p))$, there exists $M \in \delta G\beta O(X, p)$ such that $f(M) \subset \text{int}(cl(N))$;

(vi) for each $p \in X$ and each $N \in O(Y, f(p))$, there exists $M \in \delta G\beta O(X, p)$ such that $f(M) \subset scl(N)$.

Proof: (i) $\longleftrightarrow$ (ii) $\longrightarrow$ (iv) $\longleftrightarrow$ (iii) $\longleftrightarrow$ (v) $\longleftrightarrow$ (vi): obvious.

(iii) $\longrightarrow$ (i): Let $N \in \delta O(Y)$ such that $f(p) \in N$, then there exists $G \in RO(Y)$ such that $f(p) \in G \subseteq N$. By (iii), there exists an $M \in \delta G\beta O(X, p)$ such that $f(M) \subseteq G \subseteq N$.

Definition 3.2. A space X is said to be $\delta g\beta$ -additive if $\delta G\beta O(X)$ is closed under arbitrary union.

Theorem 3.8. Let X be a $\delta g\beta$ -additive space. Then $M \subseteq X$ is $\delta g\beta$ -closed (resp., $\delta g\beta$ -open) if and only if $\delta g\beta cl(M) = M$ (resp., $\delta g\beta int(M) = M$).

Theorem 3.9. The following statements are equivalent for a $f: X \rightarrow Y$ where X is $\delta g\beta$ -additive:

- (i) f is almost $\delta g\beta$ -continuous;
- (ii) $f(\delta g\beta cl(M)) \subseteq cl_\delta(f(M))$ for each $M \subseteq X$;
- (iii) $\delta g\beta cl(f^{-1}(N)) \subseteq f^{-1}(cl_\delta(N))$ for each $N \subseteq Y$;
- (iv) $f^{-1}(G) \in \delta G\beta C(X)$ for each δ -closed set G of Y ;
- (v) $f^{-1}(H) \in \delta G\beta O(X)$ for each δ -open set H of Y ;
- (vi) $f^{-1}(G) \in \delta G\beta C(X)$ for each regular closed set G of Y ;
- (vii) $f^{-1}(H) \in \delta G\beta O(X)$ for each regular open set H of Y .

Proof: (i) $\longrightarrow$ (ii) Let $N \in \delta C(Y)$ such that $f(M) \subseteq N$. Observe that $N = cl_\delta(N) = \bigcap \{F : N \subseteq F \text{ and } F \in RC(Y)\}$ and so $f^{-1}(N) = \bigcap \{f^{-1}(F) : N \subseteq F \text{ and } F \in RC(Y)\}$. By (i) and Definition 3.2, we have $f^{-1}(N) \in \delta G\beta C(X)$ and $M \subseteq f^{-1}(N)$. Hence $\delta g\beta cl(M) \subseteq f^{-1}(N)$, and it follows that $f(\delta g\beta cl(M)) \subseteq N$. Since this is true for any δ -closed set N containing $f(M)$, we have $f(\delta g\beta cl(M)) \subseteq cl_\delta(f(M))$.

(ii) $\longrightarrow$ (iii) Let $D \subseteq Y$, then $f^{-1}(D) \subseteq X$. By (ii), $f(\delta g\beta cl(f^{-1}(D))) \subseteq cl_\delta(f(f^{-1}(D))) \subseteq \delta g\beta cl(D)$. So that $\delta g\beta cl(f^{-1}(D)) \subseteq f^{-1}(cl_\delta(D))$.

(iii) $\longrightarrow$ (iv) Let G be a δ -closed subset of Y . Then by (iii), $\delta g\beta cl(f^{-1}(G)) \subseteq f^{-1}(cl_\delta(G)) = f^{-1}(G)$. In consequence, $\delta g\beta cl(f^{-1}(G)) = f^{-1}(G)$ and hence by Theorem 3.8, $f^{-1}(G) \in \delta G\beta C(X)$.

(iv) $\longrightarrow$ (v): Clear.

(v) $\longrightarrow$ (i): Let $N \in RO(Y)$. Then N is δ -open in Y . By (v), $f^{-1}(N) \in \delta G\beta O(X)$. Hence f is almost $\delta g\beta$ -continuous

Theorem 3.10. *The following statements are equivalent for a $f: X \rightarrow Y$ where X is $\delta g\beta$ -additive:*

- (i) f is almost $\delta g\beta$ -continuous;
- (ii) For every open subset K of Y , $f^{-1}(int(cl(K)) \in \delta G\beta O(X)$;
- (iii) For every closed subset M of Y , $f^{-1}(cl(int(M)) \in \delta G\beta C(X)$;
- (iv) For every β -open subset K of Y , $\delta g\beta cl(f^{-1}(K)) \subseteq f^{-1}(cl(K))$;
- (v) For every β -closed subset M of Y , $f^{-1}(int(M)) \subseteq \delta g\beta int(f^{-1}(M))$;
- (vi) For every semi-closed subset M of Y , $f^{-1}(int(M)) \subseteq \delta g\beta int(f^{-1}(M))$;
- (vii) For every semi-open subset K of Y , $\delta g\beta cl(f^{-1}(K)) \subseteq f^{-1}(cl(K))$;
- (viii) For every pre-open subset M of Y , $f^{-1}(M) \subseteq \delta g\beta int(f^{-1}(int(cl(M)))$.

Proof: (i) $\longleftrightarrow$ (ii): Let $K \subseteq Y$. Since $int(cl(N))$ is regular open in Y . Then by (i), $f^{-1}(int(cl(N))) \in \delta G\beta O(X)$. The converse is similar.

(i) $\longleftrightarrow$ (iii): It is similar to (i) $\longleftrightarrow$ (ii).

(i) $\longrightarrow$ (iv): Let $K \in \beta O(Y)$, then $cl(K)$ is regular closed in Y . So by (i), $f^{-1}(cl(K)) \in \delta G\beta C(X)$. Since $f^{-1}(N) \subseteq f^{-1}(cl(N))$, then $\delta g\beta cl(f^{-1}(N)) \subseteq f^{-1}(cl(N))$.

(iv) $\longrightarrow$ (v) and (vi) $\longrightarrow$ (vii): Obvious

(v) $\longrightarrow$ (vi): It follows from the fact that every semiclosed set is β -closed.

(vii) $\longrightarrow$ (i): It follows from the fact that every regular closed set is semi-open.

(i) $\longleftrightarrow$ (viii): Let $M \in PO(Y)$. Since $int(cl(N))$ is regular open in Y , then by (i), $f^{-1}(int(cl(N))) \in \delta G\beta O(X)$ and hence $f^{-1}(N) \subseteq f^{-1}(int(cl(N))) = \delta g\beta int(f^{-1}(int(cl(N))))$. Conversely, let $H \in RO(Y)$. Since H is preopen in Y , $f^{-1}(H) \subseteq \delta g\beta int(f^{-1}(int(Cl(N)))) = \delta g\beta int(f^{-1}(N))$, in consequence, $\delta g\beta int(f^{-1}(H)) = f^{-1}(H)$ and by Theorem 3.8, $f^{-1}(N) \in \delta G\beta O(X)$.

Theorem 3.11. *The following statements are equivalent for a $f: X \rightarrow Y$ where X is $\delta g\beta$ -additive:*

- (i) f is almost $\delta g\beta$ -continuous;
- (ii) For every e^* -open set K of Y , $f^{-1}(cl_\delta(K))$ is $\delta g\beta$ -closed in X ;
- (iii) For every δ -semiopen subset K of Y , $f^{-1}(cl_\delta(K))$ is $\delta g\beta$ -closed set in X ;
- (iv) For every δ -preopen subset K of Y , $f^{-1}(int(cl_\delta(K)))$ is $\delta g\beta$ -open set in X ;
- (v) For every open subset K of Y , $f^{-1}(int(cl_\delta(K)))$ is $\delta g\beta$ -open set in X ;

(vi) For every closed subset K of Y , $f^{-1}(cl(int_{\delta}(K)))$ is $\delta g\beta$ -closed set in X .

Proof: (i)→(ii): Let K be a e^* -open subset of Y . Then by Lemma 2.7 of [Ayhan and Özkoç, 2018], $cl_{\delta}(K) \in RC(Y)$. By (i), $f^{-1}(cl_{\delta}(K)) \in \delta G\beta C(X)$.

(ii)→(iii): Obvious since every δ -semiopen set is e^* -open.

(iii)→(iv): Let K be a δ -preopen subset of Y , then $int_{\delta}(Y \setminus K) \in \delta SO(Y)$. By (iii), $f^{-1}(cl_{\delta}(int_{\delta}(Y \setminus K))) \in \delta G\beta C(X)$ which implies $f^{-1}(int(cl_{\delta}(K))) \in \delta G\beta O(X)$.

(iv)→(v): Obvious since every open set is δ -preopen.

(v)→(vi): Clear

(vi)→(i): Let $K \in RO(Y)$. Then $K = int(cl_{\delta}(K))$ and hence $(Y \setminus K)$ is closed in X . By (vi), $f^{-1}(Y \setminus K) = X \setminus f^{-1}(int(cl_{\delta}(K))) = f^{-1}(cl(int_{\delta}(Y \setminus K))) \in \delta G\beta C(X)$. Thus $f^{-1}(K)$ is $\delta g\beta$ -open in X .

Theorem 3.12. The following are equivalent for a function $f: X \rightarrow Y$ where X is $\delta g\beta$ -additive:

- (i) f is almost $\delta g\beta$ -continuous;
- (ii) For every e^* -open subset G of Y , $f^{-1}(a-cl(G))$ is $\delta g\beta$ -closed set in X ;
- (iii) For every δ -semiopen subset G of Y , $f^{-1}(\delta-pcl(G))$ is $\delta g\beta$ -closed set in X ;
- (iv) For every δ -preopen subset G of Y , $f^{-1}(\delta-scl(G))$ is $\delta g\beta$ -open set in X .

Proof: Follows from the Lemma 3.1 of [Ayhan and Özkoç, 2018]

Theorem 3.13. If an injective function $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous and Y is $r-T_1$, then X is $\delta g\beta-T_1$.

Proof: Let (Y, σ) be $r-T_1$ and $p_1, p_2 \in X$ with $p_1 \neq p_2$. Then there exist regular open subsets G, H in Y such that $f(p_1) \in G$, $f(p_2) \notin G$, $f(p_1) \notin H$ and $f(p_2) \in H$. Since f is almost $\delta g\beta$ -continuous, $f^{-1}(G)$ and $f^{-1}(H) \in \delta G\beta O(X)$ such that $p_1 \in f^{-1}(G)$, $p_2 \notin f^{-1}(G)$, $p_1 \notin f^{-1}(H)$ and $p_2 \in f^{-1}(H)$. Hence X is $\delta g\beta-T_1$.

Theorem 3.14. If $f: X \rightarrow Y$ is an almost $\delta g\beta$ -continuous injective function and Y is $r-T_2$, then X is $\delta g\beta-T_2$.

Proof: Similar to the proof of Theorem 3.13

Theorem 3.15. If $f, g: X \rightarrow Y$ are almost $\delta g\beta$ -continuous where X is submaximal, extremely disconnected and $\delta g\beta$ -additive and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is $\delta g\beta$ -closed in X .

Proof: Let $D = \{x \in X : f(x) = g(x)\}$ and $x \notin (X \setminus D)$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \phi$, hence $int(cl(V)) \cap int(cl(W)) = \phi$. Since f and g are almost $\delta g\beta$ -continuous, there exist $G, H \in \delta G\beta O(X, x)$ such that $f(G) \subseteq int(cl(V))$ and

$g(H) \subseteq \text{int}(\text{cl}(W))$. Now, put $U = G \cap H$, then $U \in \delta GBO(X, x)$ and $f(U) \cap g(U) \subseteq \text{int}(\text{cl}(V)) \cap \text{int}(\text{cl}(W)) = \phi$. Therefore, we obtain $U \cap D = \phi$ and hence $x \notin \delta gbcl(D)$ then $D = \delta gbcl(D)$. Since X is δgb -additive, D is δgb -closed in X .

Definition 3.3. A space X is called $\delta g\beta$ -compact if every cover of X by $\delta g\beta$ -open sets has a finite subcover.

Definition 3.4. A subset M of a space X is said to be $\delta g\beta$ -compact relative to X if every cover of M by $\delta g\beta$ -open sets of X has a finite subcover.

Theorem 3.16. If $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous and K is $\delta g\beta$ -compact relative to X , then $f(K)$ is N -closed relative to Y .

Proof: Let $\{G_\alpha: \alpha \in \Omega\}$ be any cover of $f(K)$ by regular open sets of Y . Then $\{f^{-1}(G_\alpha): \alpha \in \Omega\}$ is a cover of K by $\delta g\beta$ -open sets of X . Hence there exists a finite subset Ω_o of Ω such that $K \subset \cup\{f^{-1}(G_\alpha): \alpha \in \Omega_o\}$. Therefore, we obtain $f(K) \subset \{G_\alpha: \alpha \in \Omega_o\}$. This shows that $f(K)$ is N -closed relative to Y .

Corollary 3.1. If a surjective function $f: X \rightarrow Y$ is almost $\delta g\beta$ -continuous and X is both $\delta g\beta$ -compact and $\delta g\beta$ -additive, then Y is nearly compact.

Lemma 3.2. Let X be a $\delta g\beta$ -compact, submaximal and extremely disconnected and $N \subset X$. Then N is $\delta g\beta$ -compact relative to X if N is $\delta g\beta$ -closed.

Proof: Let $\{B_\alpha: \alpha \in \Omega\}$ be a cover of N by $\delta g\beta$ -open sets of X . Note that $(X-N)$ is $\delta g\beta$ -open and that the set $(X-N) \cup \{B_\alpha: \alpha \in \Omega\}$ is a cover of X by $\delta g\beta$ -open sets. Since X is $\delta g\beta$ -compact, there exists a finite subset Ω_o of Ω such that the set $(X-N) \cup \{B_\alpha: \alpha \in \Omega_o\}$ is a cover of X by $\delta g\beta$ -open sets in X . Hence $\{B_\alpha: \alpha \in \Omega_o\}$ is a finite cover of N by $\delta g\beta$ -open sets in X .

Theorem 3.17. If the graph function $g: X \rightarrow X \times Y$ of $f: X \rightarrow Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$ is almost $\delta g\beta$ -continuous. Then f is almost $\delta g\beta$ -continuous.

Proof: Let $N \in RO(Y)$, then $X \times V \in RO(X \times Y)$. As g is almost $\delta g\beta$ -continuous, $f^{-1}(N) = g^{-1}(X \times N) \in \delta G\beta O(X)$.

Theorem 3.18. If the graph function $g: X \rightarrow X \times Y$ of $f: X \rightarrow Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$. If X is a submaximal and extremely disconnected space and $\delta g\beta$ -additive, then g is almost $\delta g\beta$ -continuous if and only if f is almost $\delta g\beta$ -continuous.

Proof: We only prove the sufficiency. Let $x \in X$ and $W \in RO(X \times Y)$. Then there exist regular open sets U_1 and V in X and Y , respectively such that $U_1 \times V \subset W$. If f is almost $\delta g\beta$ -continuous, then there exists a $\delta g\beta$ -open set U_2 in X such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = (U_1 \cap U_2)$. Then U is $\delta g\beta$ -open and $g(U) \subset U_1 \times V \subset W$. Thus g is almost $\delta g\beta$ -continuous.

Recall that for a $f: X \rightarrow Y$, the subset $G_f = \{(x, f(x)): x \in X\} \subset X \times Y$ is said to be graph of f .

Definition 3.5. A graph G_f of a function $f: X \rightarrow Y$ is said to be strongly $\delta g\beta$ -closed if for each $(p, q) \notin G_f$, there exist $V \in \delta G\beta O(X, p)$ and $W \in RO(Y, q)$ such that $(V \times W) \cap G_f = \phi$.

Lemma 3.3. For a graph G_f of a function $f: X \rightarrow Y$, the following properties are equivalent:

- (i) G_f is strongly $\delta g\beta$ -closed in $X \times Y$;
- (ii) For each $(p, q) \notin G_f$, there exist $U \in \delta G\beta O(X, p)$ and $V \in RO(Y, q)$ such that $f(U) \cap V = \phi$.

Theorem 3.19. Let $f: X \rightarrow Y$ have a strongly $\delta g\beta$ -closed graph G_f . If f is injective, then X is $\delta g\beta$ - T_1 .

Proof: Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ as f is injective. So that $(x_1, f(x_2)) \notin G_f$. Thus there exist $U \in \delta G\beta O(X, x_1)$ and $V \in RO(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Then $f(x_2) \notin f(U)$ implies $x_2 \notin U$ and it follows that X is $\delta g\beta$ - T_1 .

Theorem 3.20. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be any two functions.

- (i) If f is $\delta g\beta$ -continuous and g is almost continuous, then $(g \circ f)$ is almost $\delta g\beta$ -continuous.
- (ii) If f is $\delta g\beta$ -irresolute and g is almost $\delta g\beta$ -continuous, then $(g \circ f)$ is almost $\delta g\beta$ -continuous.
- (iii) If f is almost $\delta g\beta$ -continuous and g is R -map, then $(g \circ f)$ is almost $\delta g\beta$ -continuous.

Proof: (i) Let $N \in RO(Z)$. Then $g^{-1}(N)$ is open in Y since g is almost continuous. The $\delta g\beta$ -continuity of f implies $f^{-1}[g^{-1}(N)] = (g \circ f)^{-1}(N) \in \delta G\beta O(X)$. Hence $g \circ f$ is almost $\delta g\beta$ -continuous.

The proofs of (ii) and (iii) are similar to (i).

Definition 3.6. A function $f: X \rightarrow Y$ is said to be $\delta g\beta^*$ -continuous if for each $p \in X$ and each $N \in O(Y, f(p))$, there exists $M \in \delta G\beta O(X, p)$ such that $f(M) \subset cl(N)$.

Theorem 3.21. If $f: X \rightarrow Y$ is $\delta g\beta^*$ -continuous and K is $\delta g\beta$ -compact relative to X , then $f(K)$ is H -closed relative to Y .

Proof: Similar to the proof of Theorem 3.16

Theorem 3.22. If for each pair of distinct points p_1 and p_2 in a space X , there exists a function f of X into a Hausdorff space Y such that

- (i) $f(p_1) \neq f(p_2)$,

(ii) f is $\delta g\beta^*$ -continuous at p_1 and

(iii) almost $\delta g\beta$ -continuous at p_2 , then X is $\delta g\beta$ - T_2 .

Proof: As Y is Hausdorff, there exist disjoint open sets W_1 and W_2 of Y such that $f(p) \in W_1, f(q) \in W_2$. Hence $cl(W_1) \cap int(cl(W_2)) = \phi$. Since f is $\delta g\beta^*$ -continuous at p_1 , there exists $U_1 \in \delta G\beta O(X, p_1)$ such that $f(U_1) \subset cl(W_1)$. Since f is almost $\delta g\beta$ -continuous at p_2 , there exists $U_2 \in \delta G\beta O(X, p_2)$ such that $f(U_2) \subset int(cl(W_2))$. Therefore, we obtain $U_1 \cap U_2 = \phi$, X is $\delta g\beta$ - T_2 .

4 Conclusion

The notions of closed sets and continuous functions have been found to be useful in computer science and digital topology[[Khalimsky et al., 1990],[Kong et al., 1991]]. Professor El-Naschie[El Naschie, 2000] showed that the notion of fuzzy topology may be related to quantum physics in connection with string theory and ϵ^∞ theory. Therefore, the fuzzy topological version of the notions and results given in this paper will turn out to be useful in quantum physics.

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On integer cordial labeling of some families of graphs

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Lian Mathew[‡]

Abstract

An integer cordial labeling of a graph $G(p, q)$ is an injective map $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ or $[-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as p is even or odd, which induces an

edge labeling $f^* : E \rightarrow \{0, 1\}$ as $f^*(uv) = \begin{cases} 1, & f(u) + f(v) \geq 0 \\ 0, & \text{otherwise} \end{cases}$

such that the number of edges labelled with 1 and the number of edges labelled with 0 differ at most by 1. If a graph has integer cordial labeling, then it is called integer cordial graph. In this paper, we have proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph admits integer cordial labeling.

Keywords: Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph, Integer cordial labeling.

2020 AMS subject classifications: 05C78¹

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1 Introduction

Let G be a simple, finite, undirected graph. For terms not defined here, we refer to Harary [3]. A graph labeling is an assignment of integers to the vertices or edges, or both, under some conditions. It has wide applications in mathematics as well as in other fields such as circuit design, communication network addressing, data base management and so on. In this paper, we have proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jahangir graph, Jewel graph, Crown graph are integer cordial.

2 Preliminaries

Definition 2.1 [8] A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f . If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$.

Definition 2.2 [1] A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ = number of vertices having label i under f and $e_f(i)$ = number of edges having label i under f^* . A graph G is cordial if it admits cordial labeling. I. Cahit [1] introduced the concept of cordial labeling as a weaker version of graceful and harmonious graphs.

Definition 2.3 [7] An integer cordial labeling of a graph $G(p, q)$ is an injective map $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ or $[-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as p is even or odd, which induces an

edge labeling $f^* : E \rightarrow \{0, 1\}$ defined by $f^*(uv) = \begin{cases} 1, & f(u) + f(v) \geq 0 \\ 0, & \text{otherwise} \end{cases}$ such

that the number of edges labelled with 1 and the number of edges labelled with 0 differ at most by 1. If a graph has integer cordial labeling, then it is called integer cordial graph.

Definition 2.4 [4] A banana tree $B_{n,k}$ is a graph obtained by connecting one leaf of each of n copies of a k - star graph with a single root vertex. It has $nk + 1$ vertices and nk edges.

Definition 2.5 [5] $K_{1,n} * K_{1,m}$ is the graph obtained from $K_{1,n}$ by attaching root of a star $K_{1,m}$ at each pendant vertex of $K_{1,n}$.

Definition 2.6 [6] Olive tree T_k is a rooted tree consisting of k branches where the i th branch is a path of length i and it consists of $\frac{k(k+1)}{2} + 1$ vertices.

Definition 2.7 [5] The Jewel graph J_n is the graph with vertex set $V(J_n) = \{u, v, x, y, w_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uw_i, vw_i : 1 \leq i \leq n\}$.

Definition 2.8 [2] Jahangir graph $J_{n,m}$ for $m \geq 3$, is a graph on $nm + 1$ vertices, consisting of a cycle C_{nm} with one additional vertex which is adjacent to m ver-

tices of C_{nm} at a distance n to each other on C_{nm} .

Definition 2.9 [5] The crown $C_n \odot K_1$ is the graph obtained from a cycle by attaching a pendant edge to each vertex of the cycle.

In [7], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as Path P_n , Star graph $K_{1,n}$, Cycle C_n , Helm graph H_n , Closed helm graph CH_n are integer cordial. K_n is not integer cordial, $K_{n,n}$ is integer cordial iff n is even and $K_{n,n} \setminus M$ is integer cordial for any n , where M is perfect matching of $K_{n,n}$. In [8], Sarah et al. proved that the Sierpinski Sieve graph, the graph obtained by joining two friendship graphs by a path of arbitrary length, (n, k) –kite graph and Prism graph are integer cordial.

3 Main Results

Theorem 3.1. *Banana tree $B_{n,k}$ is integer cordial.*

Proof. **Case1:** When n is even (the total number of vertices is odd).

Let u denote the root vertex. Let $v_1, v_2, \dots, v_{\frac{nk}{2}}$ denote the vertices on $\frac{n}{2}$ leaves and $v_{\frac{nk}{2}+1}, v_{\frac{nk}{2}+2}, \dots, v_{nk}$ denote the vertices on the remaining $\frac{n}{2}$ leaves of $B_{n,k}$.

We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as follows:

$$\begin{aligned} f(u) &= 0 \\ f(v_i) &= -i; 1 \leq i \leq \frac{nk}{2} \\ f(v_i) &= i - \frac{nk}{2}; \frac{nk}{2} + 1 \leq i \leq nk \end{aligned}$$

Case 2: When n is odd and number of vertices is odd.

Let u denote the root vertex. Let $v_1, v_2, \dots, v_{\frac{(n-1)k}{2}}$ denote the vertices on $\lfloor \frac{n}{2} \rfloor$ leaves and $v_{\frac{(n-1)k}{2}+1}, v_{\frac{(n-1)k}{2}+2}, \dots, v_{(n-1)k}$ denote the vertices of remaining $\lfloor \frac{n}{2} \rfloor$ leaves. Let $u_1, u_2, \dots, u_k$ denote the k vertices of the another leaf such that u_1 is adjacent to u and u_k is adjacent to u_i where $1 \leq i \leq k$.

We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as follows:

$$\begin{aligned} f(u) &= 0 \\ f(v_i) &= -i; 1 \leq i \leq \frac{(n-1)k}{2} \\ f(v_i) &= i - \frac{(n-1)k}{2}; \frac{(n-1)k}{2} + 1 \leq i \leq k(n-1) \\ f(u_i) &= -(\frac{nk}{2} + 1 - i); 1 \leq i \leq \frac{k}{2} \\ f(u_i) &= \frac{k(n+1)}{2} + 1 - i; \frac{k}{2} + 1 \leq i \leq k \end{aligned}$$

Case 3: When n is odd and the number of vertices is even.

Let u denote the root vertex. Let $v_1, v_2, \dots, v_{\frac{(n-1)k}{2}}$ denote the vertices on $\lfloor \frac{n}{2} \rfloor$ leaves and $v_{\frac{(n-1)k}{2}+1}, v_{\frac{(n-1)k}{2}+2}, \dots, v_{k(n-1)}$ denote the vertices of another $\lfloor \frac{n}{2} \rfloor$ leaves where v_1 not adjacent to u . Let $u_1, u_2, \dots, u_k$ denote the k vertices of the remaining leaf such that u_1 is adjacent to u and u_k is adjacent to $u_i, 1 \leq i \leq k$.

We define $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ as follows:

$$\begin{aligned} f(u) &= 1 \\ f(v_i) &= -i; 1 \leq i \leq \frac{(n-1)k}{2} \\ f(v_i) &= i + 1 - \frac{(n-1)k}{2}; \frac{(n-1)k}{2} + 1 \leq i \leq k(n-1) \\ f(u_i) &= -(\lceil \frac{nk}{2} \rceil + 1 - i); 1 \leq i \leq \lceil \frac{k}{2} \rceil \\ f(u_i) &= \frac{k(n+1)}{2} + 2 - i; \lceil \frac{k}{2} \rceil + 1 \leq i \leq k \end{aligned}$$

Hence in all the possible cases, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore, Banana tree $B_{n,k}$ admits integer cordial labeling (See Figure 1).

□

Theorem 3.2. The graph $K_{1,n} * K_{1,m}$ is integer cordial.

Proof. **Case 1:** When n is even and m can be either odd or even.

Let $u_1, u_2, \dots, u_{\frac{n}{2}(1+m)}$ be the vertices of $\frac{n}{2}$ leaves and let $w_1, w_2, \dots, w_{\frac{n}{2}(1+m)}$ be the vertices of the other $\frac{n}{2}$ leaves and let u_0 be the center vertex.

We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as follows.

$$\begin{aligned} f(u_0) &= 0 \\ f(u_i) &= i, 1 \leq i \leq \frac{n}{2}(1+m) \\ f(w_i) &= -i, 1 \leq i \leq \frac{n}{2}(1+m) \end{aligned}$$

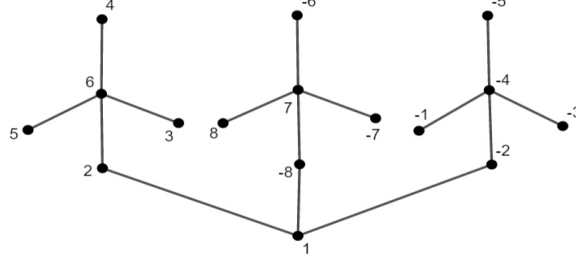


Figure 1: Integer cordial labeling of $B_{3,5}$

Case 2: When n is odd.

Let the center vertex be u_0 . Let $u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor(1+m)}$ be the vertices of $\lfloor \frac{n}{2} \rfloor$ leaves and let $w_1, w_2, \dots, w_{\lfloor \frac{n}{2} \rfloor(1+m)}$ be the vertices of the other $\lfloor \frac{n}{2} \rfloor$ leaves and let $v_1, v_2, \dots, v_{(m+1)}$ be the remaining vertices on the left out leaf, where v_1 is adjacent to u .

Case 2.1: When m is odd.

We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as follows:

$$f(u_0) = 0$$

$$f(u_i) = i, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor(1+m)$$

$$f(w_i) = -i, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor(1+m)$$

$$f(v_1) = -\left(\lfloor \frac{n}{2} \rfloor(2+m)\right)$$

$$f(v_i) = \lfloor \frac{n}{2} \rfloor(1+m+i), 2 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$$

$$f(v_{\lfloor \frac{m+1}{2} \rfloor+i}) = -\lfloor \frac{n}{2} \rfloor(1+m+i), \lceil \frac{m+1}{2} \rceil \leq i \leq m+1$$

Case 2.2: When m is even.

We define $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ as follows:

$$f(u_0) = 1$$

$$f(u_i) = i+1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor(1+m)$$

$$f(w_i) = -(i+1), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor(1+m)$$

$$f(v_1) = -1$$

$$f(v_i) = \lfloor \frac{n}{2} \rfloor(1+m+i), 2 \leq i \leq \frac{m+1}{2}$$

$$f(v_{\lfloor \frac{m+1}{2} \rfloor+i}) = -\left(\lfloor \frac{n}{2} \rfloor(1+m+i)\right), \frac{m+1}{2} \leq i \leq m+1$$

Here, for all possible cases, we have $|e_f(1) - e_f(0)| \leq 0$. Therefore $K_{1,n} * K_{1,m}$ is integer cordial (See Figure 2). $\square$

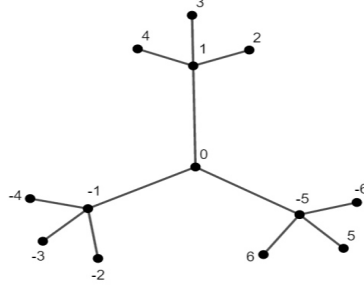


Figure 2: Integer cordial labeling of $K_{1,3} * K_{1,3}$

Theorem 3.3. *Olive tree T_k admits integer cordial labeling.*

Proof. Let U_i denote the $(k + 1 - i)^{th}$ branch and U denote the root vertex.

Case 1: When $\frac{k(k+1)}{2} + 1$ is an odd number.

An integer cordial labeling of T_k is obtained by assigning the positive integer from 1 to $\frac{k(k+1)}{4}$ to the vertices of the branches namely $U_2, U_3, U_6, U_7, U_{10}, U_{11}, U_{14}, \dots$, in any order and the negative integers from -1 to $\frac{-k(k+1)}{4}$ to the vertices of the branches namely $U_1, U_4, U_5, U_8, U_9, U_{12}, U_{13}, \dots$, in any order and let $U = 0$.

Case 2: When $\frac{k(k+1)}{2} + 1$ is an even number.

An integer cordial labeling of T_k is obtained by assigning the positive integers from 2 to $\frac{(k(k+1)+2)}{4}$ to the vertices of the branches namely $U_2, U_3, U_6, U_7, U_{10}, U_{11}, \dots$, in any order and the negative integers from -1 to $\frac{-(k(k+1)+2)}{4}$ to the vertices of the branches namely $U_1, U_4, U_5, U_8, U_9, U_{12}, U_{13}, \dots$, in any order and let $U = 1$.

Hence, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore, Olive tree admits integer cordial labeling (See Figure 3). $\square$

Theorem 3.4. *The Jewel graph J_n admits integer cordial labeling.*

Proof. Let $V(G_n) = \{u, v, x, y, w_i : 1 \leq i \leq n\}$ and $E(G) = \{ux, uy, xy, xv, yv, uw_i, vw_i : 1 \leq i \leq n\}$.

Case 1: When n is even.

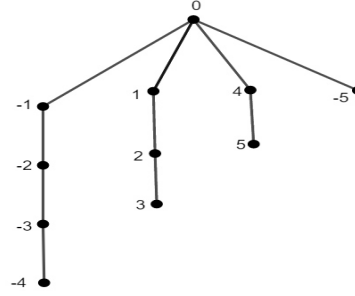


Figure 3: Integer cordial labeling of T_4

We define $f : V \rightarrow [-\frac{n}{2} \dots \frac{n}{2}]^*$ as follows:

$$\begin{aligned} f(u) &= 1 \\ f(v) &= -1 \\ f(x) &= 2 \\ f(y) &= -2 \\ f(w_i) &= i + 2, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ f(w_i) &= -(i - \lfloor \frac{n}{2} \rfloor + 2), \lceil \frac{n}{2} \rceil \leq i \leq n - 1 \end{aligned}$$

Case 2: When n is odd.

We define $f : V \rightarrow [-\lfloor \frac{n}{2} \rfloor \dots \lfloor \frac{n}{2} \rfloor]$ as follows:

$$\begin{aligned} f(u) &= 1 \\ f(v) &= -1 \\ f(x) &= 2 \\ f(y) &= -2 \\ f(w_i) &= i + 2, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ f(w_i) &= -(i - \lfloor \frac{n}{2} \rfloor + 2), \lceil \frac{n}{2} \rceil \leq i \leq n - 1 \\ f(w_n) &= 0 \end{aligned}$$

Here, for both the cases, we have $n + 3$ edges with label 1 and $n + 2$ edges with label 0. Hence in all possible cases, we have $|e_f(1) - e_f(0)| = 1$. Therefore, J_n is integer cordial (See Figure 4).

□

Theorem 3.5. *Jahangir graph $J_{n,m}$ is integer cordial except when $n = 1$.*

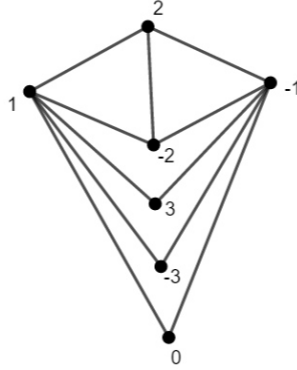


Figure 4: Integer cordial labeling of J_3

Proof. When $n = 1$, we have $J_{1,m}$ to be a complete graph with $m + 1$ vertices and hence not integer cordial.

Let u denote the central vertex adjacent to m vertices of C_{nm} and let $v_1, v_2, \dots, v_{nm}$ denote the vertices in the cycle C_{nm} .

Case 1: When the number of vertices $(nm + 1)$ is odd.

We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor \dots \lfloor \frac{p}{2} \rfloor]$ as follows:

$$\begin{aligned} f(u) &= 0 \\ f(v_i) &= i; 1 \leq i \leq \frac{nm}{2} \\ f(v_{(\frac{nm}{2}+i)}) &= -i; 1 \leq i \leq \frac{nm}{2} \end{aligned}$$

Case 2: When the number of vertices $(nm + 1)$ is even.

We define $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ as follows.

$$\begin{aligned} f(u) &= 1 \\ f(v_i) &= i + 1; 1 \leq i \leq \lfloor \frac{nm}{2} \rfloor \\ f(v_{\lfloor \frac{nm}{2} \rfloor + i}) &= -i; 1 \leq i \leq \lceil \frac{nm}{2} \rceil \end{aligned}$$

Hence in all possible cases, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore $J_{n,m}$ admits integer cordial labeling except when $n = 1$ (See Figure 5).

□

Theorem 3.6. *The Crown $C_n \odot K_1$ admits integer cordial labeling.*

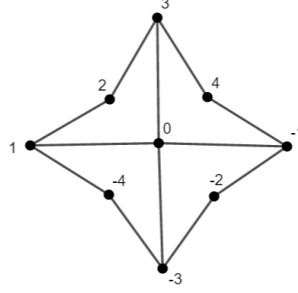


Figure 5: Integer cordial labeling of $J_{2,4}$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the inner cycle and let $u_1, u_2, \dots, u_n$ be the pendent vertices where u_i is adjacent to v_i .

We define $f : V \rightarrow [-\frac{p}{2} \dots \frac{p}{2}]^*$ as follows:

$$f(v_i) = -i$$

$$f(u_i) = i$$

Here, we have n edges with label 1 and n edges with label 0. Hence, $|e_f(1) - e_f(0)| = 0$. Therefore, the Crown $C_n \odot K_1$ is integer cordial (See Figure 6).

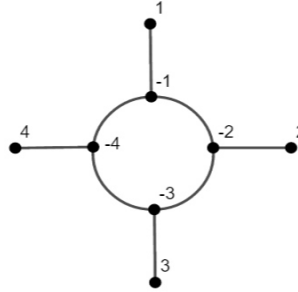


Figure 6: Integer cordial labeling of $C_4 \odot K_1$

□

4 Conclusion

In this paper, we proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph are integer cordial. Obtaining the integer cordial labeling of other class of graphs is still open. Further investigation can be done for all the networks.

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Arboricity and Span in M-Fuzzy Chromatic Index

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S. Suganthi†

Abstract

A fuzzy matching is a set of edges in which no two edges incident on a same membership value. In this paper, Arboricity and span of a fuzzy labeling graph are defined with suitable examples. Also, the relation between Arboricity and M -fuzzy Chromatic Index, Span and M -fuzzy Chromatic Index are discussed. Some results based on these concepts are stated and proved.

Keywords: M -fuzzy coloring; M -fuzzy chromatic index; Arboricity; span annuity

2010 AMS subject classification: 03E72, 05C72, 05C78‡

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1. Introduction

In 1736, Euler first invited the concept of graph theory. The theory of graphs is extremely useful for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science. Graph Theory is the main tool to constitute many real-world problems. Nowadays, graphs do not represent all systems properly due to the uncertainty or nebulousness of the framework of systems. This and many other circumstances stimulated to define fuzzy graphs.

Rosenfeld [8] first established the notion of fuzzy graphs in 1975. Crisp graph and fuzzy graph are both systematically the same, but when there is an uncertainty on points and or lines then fuzzy graph has a segregated significance. Since the world is brimming with uncertainty, so the fuzzy graph occurs in many real-life locations. Fuzzy graph theory is elaborated with a large number of branches. The fuzzy graph model is used to constitute the traffic network. Also it is used for job allocation, group structure and decision-making analysis etc. Seethalakshmi et al. [10] defined perfect fuzzy matching with some examples, and they derived that if G is a strong regular fuzzy graph with each vertex and is of degree at least two, then E is not a perfect fuzzy matching in G .

Ananthanarayanan et al. [1] discussed the coloring of fuzzy graphs using α -cut with appropriate examples. Anjaly Kishore et al.[2] defined the chromatic number of a fuzzy graph and they discussed some results based on these chromatic numbers. Arindam Dey et al.[3] explained the concept of complement fuzzy graph and their edge coloring. R. Jahir Hussain et al.[5] found the fuzzy chromatic number of some special fuzzy graphs S.Yahya Mohamed et al.[11-14] introduced the concept of fuzzy matching in fuzzy labeling graphs through some illustrations. In this paper, we introduced the new concept of Arboricity and Span of a fuzzy labeling graph G . Also, we discussed some properties based on Arboricity and Span of G .

2. Preliminaries

Definition 2.1

Let $G = (\alpha, \beta)$ be a fuzzy graph. The set of all subsets of β in which they received the same color is called *Color Class of G* . It is denoted by $CC(G)$.

Example 2.2

Consider the fuzzy graph G given in Figure 2.1
In Fig 2.1, the edge set is $\beta = \{e_1(0.2), e_2(0.5), e_3(0.35), e_4(0.14)\}$ and all edges receive different colors. So all edges belong to the color class of G .

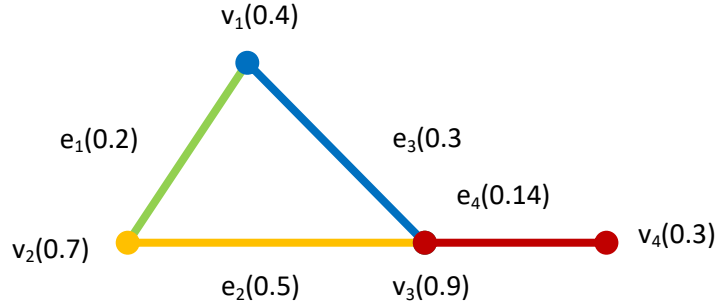


Figure 2.1: Color class of a fuzzy graph G

Definition 2.3

Let $G = (\alpha, \beta)$ be a fuzzy labeling graph. The number of elements in the color class is called as *coloring number* of G . It is denoted by $CN(G)$.

Definition 2.4

A family $\lambda = \{M_1, M_2, M_3, \dots, M_k\}$ of fuzzy matchings on a set β is called a M -fuzzy coloring of $G = (\alpha, \beta)$ if

- (i) $\vee \lambda = \beta$. It means single edge does not belong to two distinct color classes.
- (ii) $\lambda_i \wedge \lambda_j = 0$.
- (iii) For every edge (x, y) of G $\min \{\lambda_i(x), \lambda_j(y)\} = 0$ ($1 \leq i \leq k$).
(This means any one of the edges does not receive different color).

Example 2.5

Consider the fuzzy graph G given in figure 2.2,

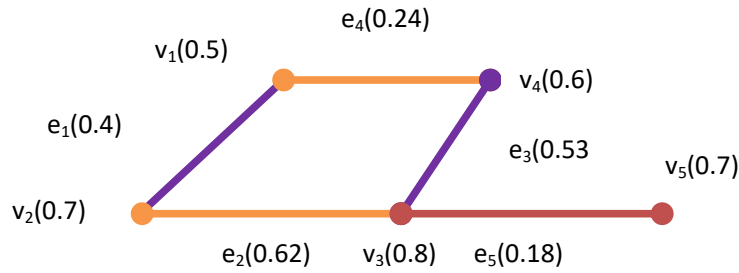


Figure 2.2: M -Fuzzy Coloring of G

In Fig 2.2, the fuzzy matchings are $M_1 = \{e_1(0.4), e_3(0.53)\}$ and $M_2 = \{e_4(0.24), e_2(0.62)\}$. Here $\lambda = \{M_1, M_2\}$ is M -fuzzy coloring of G .

Definition 2.6

Let $G = (\alpha, \beta)$ be a fuzzy labeling graph. The minimum number k for which there exists a M -fuzzy coloring is called *fuzzy matching chromatic index* of G . It is denoted by $\chi^{\text{fmci}}(G)$.

Example 2.7

Consider the fuzzy graph given in Figure 2.3

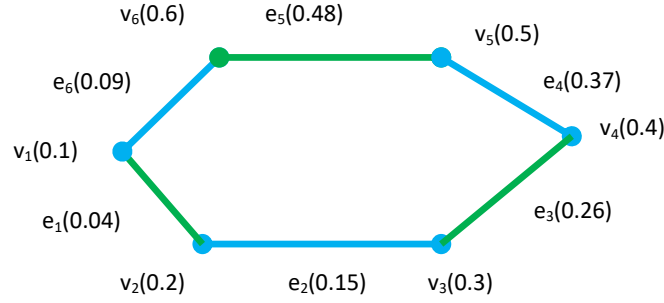


Figure 2.3: Fuzzy matching chromatic index of G

In Fig 2.3, the fuzzy matching of a given graph G are as follows $M_1 = \{e_1(0.04), e_3(0.26), e_5(0.48)\}$, $M_2 = \{e_2(0.15), e_4(0.37), e_6(0.09)\}$. Then the fuzzy matching chromatic index $\chi^{\text{fmci}}(G) = 2$ because $\lambda = \{M_1, M_2\}$ is M -fuzzy coloring of G .

3. Main Results

Definition 3.1

The minimum number of paths to cover all edges of a fuzzy graph G is called *Arboricity* of G . It is denoted by $A(G)$.

Example 3.2

Consider the following fuzzy graph G given in Figure 3.1

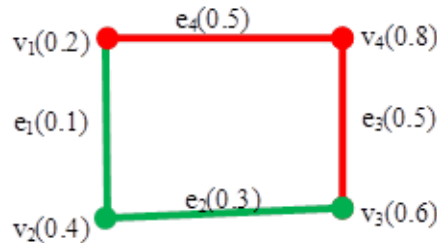


Figure 3.1: Arboricity of G

In fig 3.1 the path cover all edges of a given graph are $v_1e_1v_2e_2v_3$ and $v_3e_3v_4e_4v_1$. Here each colour represents one path and two distinct paths cover all edges of G . Hence, we obtain $A(G) = 2$.

Theorem 3.3

For any fuzzy labeling graph G with n vertices has Arboricity $\left\lceil \frac{n}{2} \right\rceil$.

Proof

Consider a fuzzy labeling graph $G = (\alpha, \beta)$ with n vertices. Now we construct a path to cover all edges of G . We obtain many paths to cover all edges of G . But we need minimum number of path to cover all edges. Then we construct the path with maximum number of edges in G . This path occurs with maximum number of edges $(n-1)$. Hence, we can find all paths with $(n-1)$ edges and which is equal to $\left\lceil \frac{n}{2} \right\rceil$. Therefore, G has Arboricity $\left\lceil \frac{n}{2} \right\rceil$.

Example 3.4

Consider the fuzzy graph given in Figure 3.2,

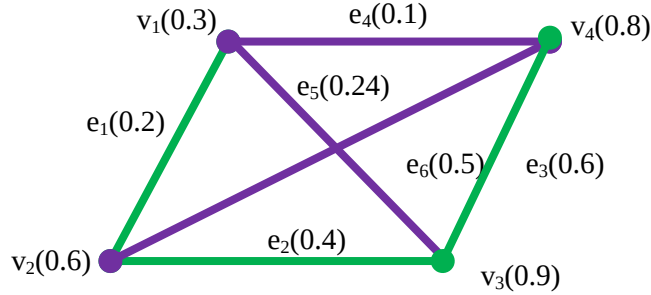


Figure 3.2: Fuzzy Labeling graph G

In fig 3.2 the path cover all edges of a given graph are $v_3e_5v_1e_4v_4e_6v_2$ and $v_1e_1v_2e_2v_3e_3v_4$. Here, each color represents one path and there are two paths exist to cover all edges of G . Hence, we get $A(G) = 2 = \left\lceil \frac{4}{2} \right\rceil$.

Theorem 3.5

For any fuzzy labeling graph G , $A(G) \leq \chi^{fmc_i}(G)$.

Proof

Consider a labeling fuzzy graph $G = (\alpha, \beta)$ and $M_1, M_2, M_3 \dots M_k$ be the distinct fuzzy matching in G . Here $M_1, M_2, M_3 \dots M_k$ contains the set of independent elements and each matching receive different colors. Now we find the path to cover all edges of G . Here we need more than one path to cover all edges of G . Already we have for

any fuzzy graph with n vertices has arboricity $\left\lceil \frac{n}{2} \right\rceil$. Here we require two matching or

more than two matching to cover all edges of G . Then the number of path to cover all edges is less than the number of matching in any color class of G . Hence, we obtain

$$A(G) \leq \chi^{fmc_i}(G).$$

Example 3.6

Consider the fuzzy graph given in Figure 3.3,

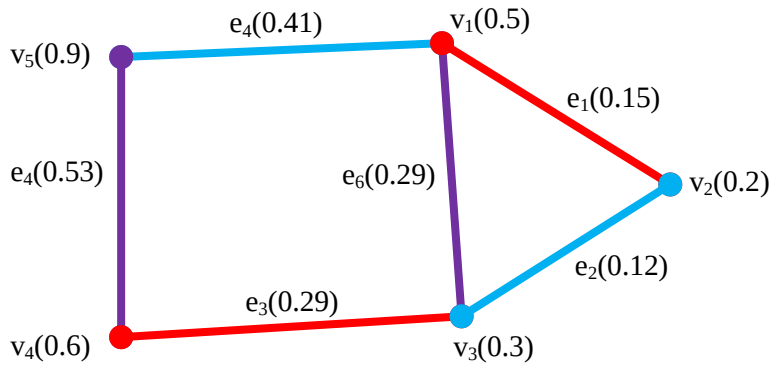


Figure 3.3: Fuzzy Graph G

In fig 3.3 the fuzzy matching of a given graph G is as follows

$$M_1 = \{e_4(0.53), e_6(0.29)\}, M_2 = \{e_1(0.15), e_3(0.29)\} \text{ and}$$

$M_3 = \{e_2(0.12), e_4(0.41)\}$. Also the color class $CC(G) = \{M_1, M_2, M_3\}$. Here $\chi^{fmc}(G) = 3$ and the paths to cover all edges of G are $v_5e_4v_1e_1v_2e_2v_3$ and $v_1e_6v_3e_3v_4e_4v_5$. Then $A(G) = 2$. Hence, we obtain $A(G) \leq \chi^{fmc}(G)$.

Definition 3.7

The minimum number of interchanging path to cover all vertices of a fuzzy labeling graph G is called *spanning arboricity* of G .

Example 3.8

Consider the following fuzzy graph G given in Figure 3.4

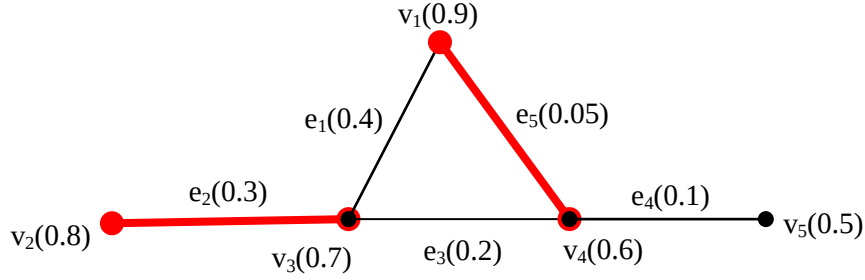


Figure 3.4: Spanning arboricity of G

In fig 3.4 one of the fuzzy matching of a given graph G is $M_1 = \{e_2(0.3), e_5(0.05)\}$. Then the interchanging path $v_2e_2v_3e_1v_1e_5v_4e_4v_5$ is to cover all points of G . Here spanning arboricity of G is one.

Theorem 3.9

A fuzzy labeling graph $G = (\alpha, \beta)$ with n vertices has spanning arboricity as unity.

Proof

Let us consider a fuzzy labeling graph $G = (\alpha, \beta)$ with n vertices. Then we identify the fuzzy matching of G and an interchanging path with maximum lines to cover all vertices of G . Now we can find an interchanging path to cover all vertices of G . An interchanging path is a path in which the edges are alternatively in M and $(\beta - M)$. But we can able to construct a fuzzy matching with maximum number of edges in the boundary of G . Then an interchanging path in the boundary path of any graph is only enough to cover all vertices of G

Example 3.10

Consider the following fuzzy graph given in Figure 3.5

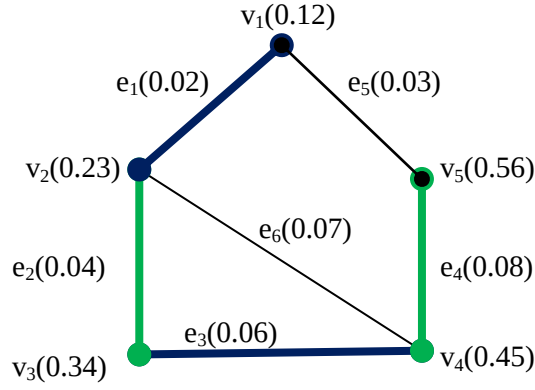


Figure 3.5: Spanning arboricity of G

In fig 3.5 the fuzzy matching of a given graph G is $M_1 = \{e_2(0.04), e_4(0.08)\}$ and $v_1e_1v_2e_2v_3e_3v_4e_4v_5$ is an interchanging path to cover all vertices of G . Hence, the spanning arboricity of G is one.

Definition 3.11

The number of edges in an interchanging path to cover all vertices of a fuzzy graph G is called *span of G* . It is denoted by $\mathcal{S}(G)$.

Example 3.12

Consider the following fuzzy graph given in Figure 3.6,

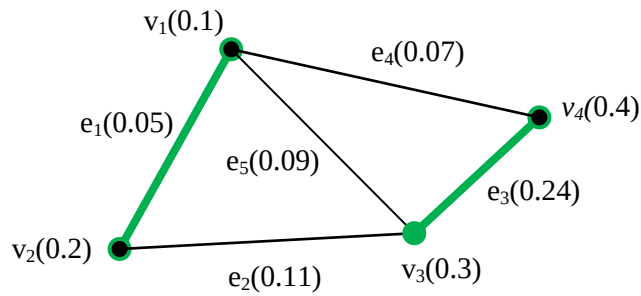


Figure 3.6 : Span of G

In fig 3.6 the fuzzy matching of a given graph G is $M_1 = \{e_1(0.05), e_3(0.24)\}$ and the number of edges in an interchanging path $v_1e_1v_2e_2v_3e_3v_4$ is 3. Hence, we have $\mathcal{S}(G) = 3$.

Theorem 3.13

For any fuzzy labeling graph $G = (\alpha, \beta)$ with n vertices, we have $\mathcal{S}(G) \geq \chi^{fmc}(G)$.

Proof

Let us consider a fuzzy labeling graph $G = (\alpha, \beta)$ with n vertices. Now we can form fuzzy matching of G and we find an interchanging path with maximum edges to cover all vertices of G . Then the interchanging path in the boundary of any graph is only enough to cover all vertices of G . to cover all vertices of G . But this path occurs with maximum number of edges $(n-1)$ because each fuzzy matching receives distinct colors. Therefore, span of G is $(n-1)$. Here the fuzzy matching chromatic index is less than $(n-1)$. and hence, we obtain $\mathcal{S}(G) \geq \chi^{fmc}(G)$.

Example 3.14

Consider the following fuzzy graph given in Figure 3.7,

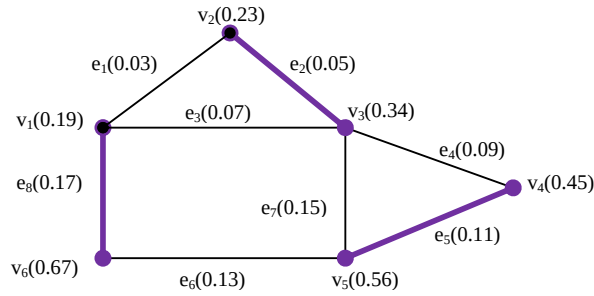


Figure 3.7: Span of G

In fig 3.7 the fuzzy matching is $M_1 = \{e_2(0.05), e_5(0.11), e_8(0.17)\}$ and $v_1e_1v_2e_2v_3e_4v_4e_5v_5e_6v_6$ is an interchanging path to cover all vertices of G . Then we have $\mathcal{S}(G) = 5 \geq \chi^{fmc}(G) = 2$.

4. Conclusion

Fuzzy set theory proposes to be a useful tool for handling vagueness and degrees of certainty, and for giving a consistent representation of linguistically formulated knowledge which allows the use of precise properties and algorithms. In this paper, we introduced the new concept of Arboricity and span of fuzzy labeling graph. Also we compared these values with M-fuzzy Chromatic Index with related examples. In future, we plan to extend our research work to operations like union, intersection and Cartesian product of two fuzzy labeling graphs.

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To some structural properties of ∞ -languages

Ivan Mezník¹

Abstract

Properties of catenation of sequences of finite (words) and infinite (ω -words) lengths are largely studied in formal language theory. These operations are derived from the mechanism how they are accepted or generated by the corresponding devices. Finite automata accept structures containing only words, ω -automata accept only ω -words. Structures containing both words and ω -words (∞ -words) are mostly generated by various types of ∞ -automata (∞ -machines). The aim of the paper is to investigate algebraic properties of operations on ∞ -words generated by IGk -automata, where k is to model the depth of memory. It has importance in many applications (shift registers, discrete systems with memory...). It is shown that resulting algebraic structures are of „pure“ groupoid or partial groupoid type.

Keywords: ∞ -words; ∞ -language; $\rho_{n,p,r}$ -catenation; closure of an ∞ -language; ρ -operation.

Mathematics Subject Classification: 68Q70, 08A55²

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1. Introduction

The notion of an ω -language was introduced by Nivat ([10]) as a free-monoid structure containing words finite („classical“ languages) and infinite (ω -languages) lengths. The theory of ω -languages has been intensively developed so far, mostly as a generalization of acceptance conditions of various types of automata ([1], [8], [9], [11], [12], [13], [14], [15] among others). Devices capable to accept (or generate) simultaneously words or finite or infinite (ω -words) lengths were described and investigated in [2], [3]. Such devices (k -machines, IGk -machines) also provide to implement the depth of memory and possess various applications (shift registers, modelling of phenomena working in a discrete time scale). They also make a lot of properties. In [3] lattice structures were described. The way how they generate words and ω -words motivates to study various types of catenation of words, which is the aim of the paper. The paper is organized as follows. In Section 2 we introduce basic concepts. In Sections 3 and 4 we examine properties of ρ -closure and ρ -operation, respectively.

2. Preliminaries

An *alphabet* is any finite set (including the empty set) and is denoted by Σ ; its elements are called *letters*. Let ω be the least infinite ordinal and n , $0 \leq n \leq \omega$ be an ordinal. For $\Sigma \neq \emptyset$ the set of all finite sequences of elements of Σ including the empty sequence λ is denoted by Σ^* , the set of all infinite sequences of elements of Σ by Σ^ω and the set $\Sigma^* \cup \Sigma^\omega$ by Σ^∞ . For $\Sigma = \emptyset$, by definition, $\Sigma^* = \Sigma^\omega = \Sigma^\infty = \emptyset$. The elements of Σ^* are *words*, the elements of Σ^ω are ω -*words*, the elements of Σ^∞ are ∞ -*words*. Instead of $(a_0, a_1, \dots, a_{n-1}) \in \Sigma^*$, $n \geq 1$ and $(a_0, a_1, \dots) \in \Sigma^\omega$ we write simply $a_0 a_1 \dots a_{n-1}$ and $a_0 a_1 \dots$. For $w \in \Sigma^\infty$, the *length* of w , denoted by $|w|$ is defined as follows: if $w = a_0 a_1 \dots a_{n-1} \in \Sigma^*$ then $|w| = n$, if $w = a_0 a_1 \dots \in \Sigma^\omega$ then $|w| = \omega$ and if $w = \lambda$ then $|w| = 0$. A subset of Σ^* and Σ^ω and Σ^∞ is referred to as a *language* and an ω -*language* and an ∞ -*language* (over Σ) respectively. For $L \subseteq \Sigma^\infty$ we define $m(L) = \inf\{|w|; w \in L\}$. Let $w \in \Sigma^\infty - \{\lambda\}$, $1 \leq m \leq n < 1 + |w|$; by $w(n)$ we denote the n -th letter of w , by $w[m, n]$ the word $w(m) \dots w(n)$ and if $|w| = \omega$ by $w[m, \omega]$ the word $w(m)w(m+1) \dots$. Instead of $w[1, n]$ we write only $w[n]$. In case $m > n$ we formally put $w[m, n] = \lambda$. The usual operation of *catenation* on Σ^* may be extended to Σ^∞ as a partial operation as follows. Let $w \in \Sigma^*$, $|w| = n$, $w' \in \Sigma^\infty$; if $w' \in \Sigma^*$, then ww' is defined by catenation on Σ^* and if $w' \in \Sigma^\omega$, then $ww' = w(1)w(2) \dots w(n)w'(1)w'(2) \dots$. For $w \in \Sigma^*$, $k \geq 1$, the symbol w^k denotes the result of k catenations of w and the symbol w^ω denotes the result of infinite number of catenations of w ; by definition, $w^0 = \lambda$.

3. Closure of an ∞ -language

In this section we define the notion of a ρ -catenation. Subsequently the concept of a ρ -closure is introduced and its closure characterization is derived.

2.1 Definition. Let n, p, r be positive integers and $u \in \Sigma^\infty, v \in \Sigma^\infty$. For u, v satisfying the property

$$u[p, p + n - 1] = v[r, r + n - 1]$$

we define the operation $\rho_{n,p,r}$ by

$$(2.1) \quad \rho_{n,p,r}(u, v) = u[p + n - 1]v[r + n, |v|]$$

called a $\rho_{n,p,r}$ -catenation or only ρ -catenation if n, p, r are given by the context.

Apparently, each $\rho_{n,p,r}$ -catenation is a partial operation in Σ^∞ . In other words, each $\rho_{n,p,r}$ -catenation defines a partial groupoid in Σ^∞ . In this manner (regarding the given n, p, r) the set of partial operations (groupoids) in Σ^∞ is given. Instead of $\rho_{n,p,r}(u, v)$ we write as customary $u\rho_{n,p,r}v$ or only $u\rho v$, if n, p, r are clear from the context. To simplify the text, by stating $u\rho_{n,p,r}v$ it is supposed that $(u, v) \in \text{Dom}(\rho_{n,p,r})$.

2.2 Lemma. Suppose $u \in \Sigma^\infty$ and $u\rho_{n,p,r}u$ for some n, p, r . Then it holds $u\rho_{n,p,r}u = u$.

Proof. It is an immediate consequence of Definition 2.1

Remarks. 1° Suppose $u \in \Sigma^\infty$ and $n, p, r \leq |u|$. Then there is obviously an infinite number of words $x \in \Sigma^\infty$ such that $u\rho_{n,p,r}x = u$ playing the role of the „identity“ element of $\rho_{n,p,r}$ -catenation.

2° Operation $\rho_{n,p,r}$ is in general not commutative. For example consider words u, v over $\Sigma = \{a, b\}$, $u = (ab)^3, v = a^3$. Applying the previous definition we get $u\rho_{1,3,1}v = aba^3, v\rho_{1,3,1}u = a^2(ab)^3, u\rho_{1,3,1}v \neq v\rho_{1,3,1}u$.

3° Operation $\rho_{n,p,r}$ is in general not associative. For example consider $\rho_{1,3,2}$ -catenation in $\{a, b\}^\infty$ and let $u = (ab)^4, v = a^5, w = a^7$. Construct $(u\rho_{1,3,2}v)\rho_{1,3,2}w, u\rho_{1,3,2}(v\rho_{1,3,2}w)$. Due to Definition 2.1 we get $(u\rho_{1,3,2}v)\rho_{1,3,2}w = aba^5$, whereas $u\rho_{1,3,2}(v\rho_{1,3,2}w) = aba^7$. Of course, some of catenations in the given expressions need not be defined.

2.3 Theorem Let $u \in \Sigma^\infty, v \in \Sigma^\infty$ and suppose $u\rho_{n,p,r}v, v\rho_{n,r,s}w$ for some $n, p, r, s \geq 1$. Then $u\rho_{n,p,s}w$ and there holds

$$(2.2) \quad u\rho_{n,p,s}w = u\rho_{n,p,r}(v\rho_{n,r,s}w).$$

Proof. Suppose that $u\rho_{n,p,r}v, v\rho_{n,r,s}w$ hold for given $n, p, r, s \geq 1$. From Definition 2.1 it follows that $u[p, p + n - 1] = v[r, r + n - 1]$ and $v[r, r + n - 1] = w[s, s + n - 1]$. Then evidently $u[p, p + n - 1] = w[s, s + n - 1]$, applying Definition 2.1 we get $u\rho_{n,p,s}w = u[p + n - 1]w[s + n, |w|]$ and the first part of the statement is verified.

Rewriting this expression we obtain

$$(2.3) \quad u\rho_{n,p,s}w = u[1] \dots u[p + n - 1]w[s + n]w[s + n + 1] \dots w[|w|].$$

Now we construct the right part of (2.2). By Definition 2.1 we have

$$v\rho_{n,r,s}w = v[1] \dots v[r] \dots v[r+n-1]w[s+n] \dots w[|w|],$$

where $v[r, r+n-1] = v[r] \dots v[r+n-1] = w[s] \dots w[s+n-1] = w[s, s+n-1]$ and

$$(2.4) \quad u\rho_{n,p,r}(v\rho_{n,r,s}w) = u[1] \dots u[p] \dots u[p+n-1]w[s+n] \dots w[|w|].$$

From (2.3) and (2.4) the statement (2.2) holds and the proof is completed.

2.4 Theorem Let $u, v \in \Sigma^\infty$ and suppose $u\rho_{n,p,r}v$ for fixed $n > 1, p \geq 1, r \geq 1$.

Then $u\rho_{m,p,r}v$ for any $m < n$ and it holds

$$(2.5) \quad u\rho_{n,p,r}v = u\rho_{m,p,r}v.$$

Proof. Let $u\rho_{n,p,r}v$ for the given n, p, r . From Definition 2.1 it follows that $u[p, p+n-1] = v[r, r+n-1]$. Since $m < n$, then apparently $u[p, p+m-1] = v[r, r+m-1]$ holds for any $m < n$ as well and thus $u\rho_{m,p,r}v$. Due to (2.1) $u\rho_{n,p,r}v = u[p+n-1]v[r+n, |v|]$. In a detailed version we have

$$(2.6) \quad u\rho_{n,p,r}v = u[1] \dots u[p+n-1]v[r+n] \dots v[|v|].$$

With a view to $m < n$, (2.6) may be rewritten as

$$(2.7) \quad u\rho_{n,p,r}v = u[1] \dots u[p+m-1]u[p+m] \dots u[p+n-1]v[r+n]v[r+n+1] \dots v[|v|].$$

Now, we construct $u\rho_{m,p,r}v$ for $m < n$. It holds $u[p, p+m-1] = v[r, r+m-1]$ and by (2.1)

$$(2.8) \quad u\rho_{m,p,r}v = u[p+m-1]v[r+m, |v|].$$

In detail

$$(2.9) \quad u\rho_{m,p,r}v = u[1] \dots u[p+m-1]v[r+m] \dots v[|v|].$$

With a view to $m < n$, (2.9) may be rewritten as

$$(2.10) \quad u\rho_{m,p,r}v = u[1] \dots u[p] \dots u[p+m-1]v[r+m] \dots v[r+n-1]v[r+n] \dots v[|v|].$$

Due to (2.7) and (2.10) $u[1] \dots u[p+m-1]$ and $v[r+n] \dots v[|v|]$ are common parts. It remains to verify that $v[r+m] \dots v[r+n-1] = u[p+m] \dots u[p+n-1]$. Using assumptions of Definition 2.1 we have $u[p, p+n-1] = v[r, r+n-1]$ and thus also $v[r+m] \dots v[r+n-1] = u[p+m] \dots u[p+n-1]$. Hence (2.7) and (2.10) are identical words and the proof is completed.

2.5 Definition. Let a $\rho_{n,p,r}$ -catenation be given. Define a relation $R_{n,p,r}$ on Σ^∞ by

$$(2.11) \quad R_{n,p,r} = \{u, v \in \Sigma^\infty; (u, v) \in \text{Dom}(\rho_{n,p,r})\} \subseteq \Sigma^\infty \times \Sigma^\infty.$$

2.6 Lemma. The relation $R_{n,p,r}$ is

- (i) reflexive,
- (ii) not symmetric,
- (iii) not antisymmetric,

(iv) not transitive.

Proof. (i) Reflexivity of $R_{n,p,r}$ follows immediately from Lemma 2.2. (ii) Consider $u = abbb, v = aabbb$. By Definition 2.1 it holds $u = abbb\rho_{2,3,3}aabbb = v$, whereas $v = aabbb\rho_{2,3,3}abbb = u$ does not hold, so the relation $R_{n,p,r}$ is not transitive. (iii) Put $u = abababab = (ab)^4, v = bababababa = (ba)^5$. By Definition 2.1 we have $u\rho_{1,2,5}v, v\rho_{1,2,5}u$, but $u \neq v$ and hence the relation $R_{n,p,r}$ is not antisymmetric. (iv) Let $u = abbb, v = bababa, w = bbbb$. From Definition it follows $u\rho_{1,1,2}v, v\rho_{1,1,2}w$, but $u\rho_{1,1,2}w$ does not hold and the relation $R_{n,p,r}$ is not transitive.

2.5 Definition. Let $L \subseteq \Sigma^\infty$ be an ∞ -language, $n \geq 1$ integer and $u, v \in \Sigma^\infty$. Put

$$C_n^\rho(u, v) = \bigcup_{p,r} u\rho_{n,p,r}v, C_n^\rho(L) = \bigcup_{u,v} C_n^\rho(u, v), C^\rho(L) = \bigcup_n C_n^\rho(L).$$

The set $C_n^\rho(L)$ is called the n -th ρ -closure of L and the set $C^\rho(L) = \bigcup_n C_n^\rho(L)$ the ρ -closure of L respectively.

2.6 Lemma. Let $L \subseteq \Sigma^\infty - \{\lambda\}$ be an ∞ -language. Then $L \subseteq C^\rho(L)$ holds true.

Proof. Suppose $w \in L$. Trivially $w(1) = w(1)$ and by Definition 2.1 it holds $w\rho_{1,1,1}w = w[1]w[2, |w|] = w$ and hence by Definition 2.5 $w \in C_1^\rho(L)$ and also $w \in C^\rho(L)$ and the statement holds true.

2.7 Theorem Let $L \subseteq (\Sigma^\infty - \{\lambda\})$ be an ∞ -language. Then for every $i \geq 1$ there holds

$$C_{i+1}^\rho(L) \subseteq C_i^\rho(L).$$

Proof. Let $w \in C_{i+1}^\rho(L)$. According to Definition 2.1 there exist $u, v \in \Sigma^\infty$ and $i, p, r \geq 1$ with the property $u[p, p+i] = v[r, r+i]$ for which $u\rho_{i+1,p,r}v = u[p+i]v[r+i+1, |v|] = w$ holds. Obviously if $u[p, p+i] = v[r, r+i]$ then also $u[p, p+i-1] = v[r, r+i-1]$ holds. By Definition 2.5 we get $u\rho_{i,p,r}v = u[p+i-1]v[r+i, |v|] = w' \in C_i^\rho(L)$. But apparently w, w' are identical words. Hence $w \in C_i^\rho(L)$ and the statement is valid.

As a consequence of Definition 2.5 and Theorem 2.7 the following Corollary 2.8 holds:

2.8 Corollary Let $L \subseteq (\Sigma^\infty - \{\lambda\})$ be an ∞ -language. Then $C^\rho(L) = C_1^\rho(L)$ holds true.

2.9 Example Let $L = \{ab, ba^k, a^\omega; k \geq 1\} \subseteq \{a, b\}^\infty$ be an ∞ -language. To find $C_n^\rho(L)$ and $C^\rho(L)$ applying Definition 2.5 we get the results as follows.

(i) $C_1^\rho(L)$: $C_1^\rho(ab, ab) = \{ab\}, C_1^\rho(ab, ba^k) = \{a^k, aba^k; k \geq 1\}, C_1^\rho(ba^k, ab) = \{b, ba^kb; k \geq 1\}, C_1^\rho(ba^k, ba^k) = \{ba^k; k \geq 1\}, C_1^\rho(ab, a^\omega) = \{a^\omega\}, C_1^\rho(a^\omega, ab) = \{a^kb; k \geq 1\}, C_1^\rho(a^\omega, a^\omega) = \{a^\omega\}, C_1^\rho(ba^k, a^\omega) = \{ba^\omega\}, C_1^\rho(a^\omega, ba^k) = \{a^k; k \geq 1\}$; therefore $C_1^\rho(L) = \{ab, a^k, aba^k, ba^kb, ba^k, a^\omega, a^kb, ba^\omega; k \geq 1\}$.

(ii) $C_2^\rho(L): C_2^\rho(ab, ab) = \{ab\}, C_2^\rho(ab, ba^k) = \emptyset, C_2^\rho(ba^k, ab) = \emptyset, C_2^\rho(a^\omega, a^\omega) = \{a^\omega\}, C_2^\rho(ba^k, a^\omega) = \{ba^\omega\}$ for $k \geq 2, C_2^\rho(a^\omega, ba^k) = \{a^k; k \geq 2\}, C_2^\rho(ba^k, ba^k) = \{ba^k; k \geq 1\}, C_2^\rho(ab, a^\omega) = \emptyset, C_2^\rho(a^\omega, ab) = \emptyset$; therefore $C_2^\rho(L) = \{ab, ba^k, a^\omega, ba^\omega, a^{k+1}; k \geq 1\}$.

(iii) $C_3^\rho(L): C_3^\rho(ab, ab) = C_3^\rho(ab, ba^k) = C_3^\rho(ba^k, ab) = C_3^\rho(ab, a^\omega) = C_3^\rho(a^\omega, ab) = \emptyset, C_3^\rho(ba^k, ba^k) = \{ba^k; k \geq 2\}, C_3^\rho(a^\omega, a^\omega) = \{a^\omega\}, C_3^\rho(ba^k, a^\omega) = \{ba^\omega\}$ for $k \geq 3, C_3^\rho(a^\omega, ba^k) = \{a^k; k \geq 3\}$; therefore $C_3^\rho(L) = \{ba^k, a^\omega, ba^\omega, a^{k+1}; k \geq 2\}$.

(iv) $C_n^\rho(L)$ for $n \geq 4: C_n^\rho(ab, ab) = C_n^\rho(ab, ba^k) = C_n^\rho(ba^k, ab) = C_n^\rho(ab, a^\omega) = C_n^\rho(a^\omega, ab) = \emptyset, C_n^\rho(ba^k, ba^k) = \{ba^k; k \geq n-1\}, C_n^\rho(a^\omega, a^\omega) = \{a^\omega\}, C_n^\rho(ba^k, a^\omega) = \{ba^\omega\}$ for $k \geq n-1, C_n^\rho(a^\omega, ba^k) = \{a^k; k \geq n-1\}$; therefore $C_n^\rho(L) = \{ba^k, a^\omega, ba^\omega, a^{k+1}; k \geq n-1\}$.

Conclusion: $C^\rho(L) = \{ab, a^k, aba^k, ba^k b, ba^k, a^\omega, a^k b, ba^\omega; k \geq 1\} = C_1^\rho(L)$.

2.10 Theorem. The set of ρ -closures is not closed under set union.

Proof. We state an counterexample. Consider $L_1 = \{ab\}, L_2 = \{a^\omega\}$ over $\{a, b\}^\infty$ and put $L = L_1 \cup L_2 = \{ab, a^\omega\}$. Applying Definition 2.1 and Corollary 2.8 we get $C^\rho(L_1) = C_1^\rho(L_1) = C^\rho(\{ab\}) = \{ab\}, C^\rho(L_2) = C_1^\rho(L_2) = C^\rho(\{a^\omega\}) = \{a^\omega\}$. Further, $C^\rho(L) = C^\rho(L_1 \cup L_2) = C^\rho(\{ab, a^\omega\}) = \{a^\omega, a^k b; k \geq 1\}$. Obviously $C^\rho(L_1 \cup L_2) \neq C^\rho(L_1) \cup C^\rho(L_2)$ and the statement is verified.

2.11 Theorem. Let $L_1, L_2 \subseteq \Sigma^\infty$. Then $C^\rho(L_1) \cup C^\rho(L_2) \subseteq C^\rho(L_1 \cup L_2)$.

Proof. With a view to Corollary 2.6 we may consider C_1^ρ instead of C^ρ . Let $w \in C_1^\rho(L_1) \cup C_1^\rho(L_2)$. According to Definition 2.3 then (a) there exist $u \in L_1, v \in L_1$ and positive integers p, r such that $u\rho_{1,p,r}v = w \in C_1^\rho(L_1)$ or (b) there exist $\bar{u} \in L_2, \bar{v} \in L_2$ and positive integers $\bar{p}, \bar{r}$ such that $\bar{u}\rho_{1,\bar{p},\bar{r}}\bar{v} = w \in C_1^\rho(L_2)$. Assuming (a), the statement there exist $u \in L_1 \cup L_2, v \in L_1 \cup L_2$ and positive integers p, r such that $u\rho_{1,p,r}v = w \in C_1^\rho(L_1 \cup L_2)$ is obviously also valid for an arbitrary set L_2 . Assuming (b), the statement there exist $\bar{u} \in L_2 \cup L_1, \bar{v} \in L_2 \cup L_1$ and positive integers $\bar{p}, \bar{r}$ such that $\bar{u}\rho_{1,\bar{p},\bar{r}}\bar{v} = w \in C_1^\rho(L_2 \cup L_1)$ is valid as well for an arbitrary set L_1 . Thus $w \in C_1^\rho(L_1 \cup L_2)$ and the proof is completed.

2.12 Theorem. The set of ρ -closures is not closed under set intersection.

Proof. We state an counterexample. Consider $L_1 = \{a^\omega, a^3, b\}, L_2 = \{a^3, ab\}$ over $\{a, b\}^\infty$ and put $L = L_1 \cap L_2 = \{a^3\}$. Applying Definition 2.1 and Corollary 2.8 we get $C^\rho(L_1) = C_1^\rho(L_1) = C_1^\rho(\{a^\omega, a^3, b\}) = \{a^\omega, b, a^k; k \geq 1\}, C^\rho(L_2) = C_1^\rho(L_2) = C_1^\rho(\{a^3, ab\}) = \{a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b\}, C^\rho(L_1) \cap C^\rho(L_2) = \{a, a^2, a^3, a^4, a^5, b\}$. Further, $C^\rho(L) = C^\rho(L_1 \cap L_2) = C^\rho(\{a^3\}) = \{a^k; 1 \leq k \leq 5\}$. Obviously $C^\rho(L_1 \cap L_2) \neq C^\rho(L_1) \cap C^\rho(L_2)$ and the statement is verified.

2.13 Theorem. Let $L_1, L_2 \subseteq \Sigma^\infty$. Then $C^\rho(L_1 \cap L_2) \subseteq C^\rho(L_1) \cap C^\rho(L_2)$.

Proof. With a view to Corollary 2.6 we may work with C_1^ρ instead of C^ρ . Let $w \in C_1^\rho(L_1 \cap L_2)$. According to Definition 2.3 there exist $u \in (L_1 \cap L_2), v \in (L_1 \cap L_2)$ and positive integers p, r such that $u\rho_{1,p,r}v = w \in C_1^\rho(L_1 \cap L_2)$. Since $u \in (L_1 \cap L_2), v \in (L_1 \cap L_2)$, then $w \in C_1^\rho(L_1)$ and also $w \in C_1^\rho(L_2)$. Thus $w \in C_1^\rho(L_1) \cap C_1^\rho(L_2)$ and the statement holds.

2.14 Example. Using the setting of the counterexample from the proof of Theorem 2.12, we have $C^\rho(L_1 \cap L_2) = C^\rho(\{a^3\}) = \{a^k; 1 \leq k \leq 5\} \subseteq C^\rho(L_1) \cap C^\rho(L_2) = \{a, a^2, a^3, a^4, a^5, b\}$ to illustrate Theorem 2.13. Further, we have $C^\rho(L_1) \cup C^\rho(L_2) = \{a^\omega, b, ab, a^2b, a^3b, a^k; k \geq 1\} \subseteq C^\rho(L_1 \cup L_2) = \{a^\omega, a^k, a^kb, b; k \geq 1\}$ to illustrate Theorem 2.11.

3. Operation ρ_n

3.1 Definition. Given $L_1, L_2 \subseteq \Sigma^\infty$ and $n \geq 1$, an operation ρ_n is defined as follows:
 $\rho_n(L_1, L_2) = \{xuy; \text{there exists } u \in \Sigma^n \text{ with } xu \in L_1 \text{ and } uy \in L_2\}$.

Clearly, for each n , ρ_n is the operation on 2^{Σ^∞} . In this manner the set of operations on 2^{Σ^∞} is given. Instead of $\rho_n(L_1, L_2)$ we also write $L_1\rho_nL_2$.

3.2 Theorem. (i) Let $L_1, L_2 \subseteq \Sigma^\omega$. Then for all $n \geq 1$ there holds $L_1\rho_nL_2 = \emptyset$.
(ii) Given $L_1, L_2 \subseteq \Sigma^*$ and let $L_1 \cup L_2$ be a finite set. Then for all $n > \max_{w \in L_1 \cup L_2} |w|$ here holds $L_1\rho_nL_2 = \emptyset$.

Proof. Both statements (i), (ii) follow immediately from Definition 3.1.

3.3 Example. (a) Let $L_1, L_2 \subseteq \{a, b\}^*$, $L_1 = \{(ab)^k; k \geq 1\}, L_2 = \{a^k, b^k; k \geq 1\}$. Applying Definition 3.1 we have $L_1\rho_1L_2 = \{ab^k, (ab)^k, (ab)^kb^m; k, m \geq 1\}$. Similarly, and with accordance to Theorem 3.2(ii) we get $L_1\rho_nL_2 = \emptyset$ and $L_2\rho_nL_1 = \emptyset$ for any $n \geq 2$. (b) Let $L_1, L_2, L_3 \subseteq \{a, b\}^\infty$, $L_1 = \{a^k, b; k \geq 1\}, L_2 = \{a^3, b^2\}, L_3 = \{a^\omega, ab\}$. Applying Definition 3.1 we have $L_1\rho_1L_2 = \{a^k, b^2; k \geq 3\}, L_2\rho_1L_3 = \{a^\omega, a^3b\}, (L_1\rho_1L_2)\rho_1L_3 = \{a^\omega, a^kb; k \geq 1\}, L_1\rho_1(L_2\rho_1L_3) = \{a^\omega, a^3b\}$.

3.4 Theorem. The operation ρ_n is generally

- (i) not commutative,
- (ii) not associative.

Proof. It follows immediately from the results of Example 3.3.

3.4 Remark. Theorem 3.4 justifies the conclusion that the set 2^{Σ^∞} with the operation ρ_n forms a „pure“ groupoid. Also nonexistence of an identity element may be simply verified.

4. Conclusion

In this paper we examined algebraic properties of operations on ∞ -words having direct relation to ∞ -languages generated by ∞ - automata. It may motivate to consider further types of operations, particularly modeling the depth of memory of such devices.

As a generalization a variant structure of ω -automata may be considered and the corresponding structures of their ω -languages studied.

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Abel fractional differential equations using Variation of parameters method

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Abstract

The Variation of Parameters Method (VPM) is utilized throughout the research to identify a numerical model for a nonlinear fractional Abel differential equation (FADE). The approach given here is used to solve the initial problem of fractional Abel differential equations. There is no conversion, quantization, disturbance, structural change, or precautionary concerns in the proposed method, although it is easy with numerical solutions. The measured values are graphed and tabulated to be compared with the numerical model.

Keywords: Abel fractional differential equations; Reimann-liouville fractional integral; Reimann-liouville fractional derivative; variation of parameters method.¹

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1 Introduction

Abel differential equations is well established, perturbed of implementations in structure and function in dynamics, linear systems with Stochastic, modeling technique, and linear algebra. Numerous research has been done mostly on methods of the Abel differential equations. we suggest several methods, including the iterative method of the Abel differential equation achieved from the variation iteration method. We consider the following nonlinear FADE:

$$D^\beta f(x) = P_1 f^3(x) + P_2 f^2(x) + P_3 f(x) + P_4, \quad 0 < \beta \leq 1, \quad 0 \leq x \leq R. \quad (1)$$

with the initial condition

$$f^k(0) = u_k, \quad k = 1, 2, 3, \dots, n-1. \quad (2)$$

where P_1, P_2, P_3 and P_4 are arbitrary number, D^β is the fractional derivative for order β and $f(x)$ is unknown function of the crisp variable x , k is an integer. However, assume IVPs (0.2) and each $x > 0$ has a unique fuzzy solution. The VPM calculation is a novel numeric plan created to examine and decipher the arrangement of first and second request dubious IVPs. Computer simulation, biophysics, synthetic science, applied mathematics, geophysics, material science, harmonies, and other domains rely heavily on linear and non-linear fractional equations. This is seeing fractional derivatives must store the record of the parameter under deliberation. The proposed approach targets constructing an answer of a VPM development just as limiting remaining blunder capacities for processing the obscure coefficients of VPM by applying a specific differential administrator without linearly or constraint on the structure. Again, we refer to see numerous qualities to show and reconsider some radical strategies for managing the various problems that occur in ordinary miracles.

2 Preliminaries

The definitions of significance and associated characteristics of the hypothesis are examined in this section.

Definition 2.1. *The fractional component of Riemann-Liouville, f the valued function of the fuzzy number β is considered to include $J_\alpha^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{f(\xi)}{(x-\xi)^{1-\beta}} d\xi$, $x > a$ where $\Gamma(\beta)$ is the famous Gamma characteristic.*

Definition 2.2. *The Riemann-Liouville fractional order derivative β of the crisp function f almost everywhere on I exists and can be represented by ${}_a^{RL}D^\beta f(x) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_0^x f(\xi)(x-\xi)^{m-\beta-1} d\xi$, where $m-1 \leq \beta < m \in \mathbb{Z}^+$.*

Definition 2.3. The modified Riemann-Liouville fractional order derivative β of the crisp function f almost everywhere on I exists and can be represented by ${}^{RL}_a D^\beta f(x) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_0^x (f(\xi) - f(0))(x - \xi)^{m-\beta-1} d\xi$, where $m - 1 \leq \beta < m$, and $m \geq 1$.

Remark 2.1. .

- (i) Riemann-Liouville derivative does not satisfy $D_\alpha^\alpha(1) = 0$ ($D_\alpha^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number
- (ii) All fractionals do not satisfy the known formula of the derivative of product of the two functions:

$$D_\alpha^\alpha(fg) = f(D_\alpha^\alpha(g)) + g(D_\alpha^\alpha(f))$$

- (ii) All fractionals do not satisfy the known formula of the derivative of quotient of the two functions:

$$D_\alpha^\alpha\left(\frac{f}{g}\right) = \frac{g(D_\alpha^\alpha(f)) - f(D_\alpha^\alpha(g))}{g^2}$$

3 Variation of parameters method

We consider the extensive expression to derive the main definition of the VPM

$$L(u) + N(u) + R(u) = f(x), \quad a \leq x \leq b \quad (3)$$

where L, N operators are linear and non-linear. R is a linear differential operator but L has the highest order than R , $f(x)$ is a source term in the given domain $[a, b]$. We have the following equation solution by using the VPM

$$u(x) = \sum_{l=0}^{k-1} \frac{p_{l+1} x^l}{l!} + \int_0^x f(x, \alpha) (-N(u)(\alpha) - R(u)(\alpha) + f(\alpha)) d\alpha, \quad (4)$$

where k represent the order of given differential equation and C_l where $l = 1, 2, 3$, are unknown. So

$$u(x) = \sum_{l=0}^{k-1} \frac{p_{l+1} x^l}{l!} \quad (5)$$

For homogeneous solution which is used by

$$L(u) = 0. \quad (6)$$

Another component obtained from definition (0.2) from VPM is

$$\int_0^x f(x, \alpha)(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \quad (7)$$

Where, $f(x, \alpha)$ is a Lagrange multiplier which that eliminates the incremental use of integers in the inverse problem and is dependent on the order of equations. The explore description is based on calculating the function of the variable $f(x, \alpha)$ from either a set of numbers.

$$f(x, \alpha) = \sum_{l=0}^{k-1} \frac{(-1)^{l-1} \alpha^{l-1} x^{k-1}}{(l-1)!(k-1)!} = \frac{(x-\alpha)^{k-1}}{(k-1)!}, \quad (8)$$

The sequence of the specific differential equations differs with k . We have always had the following criteria to explore:

$$\begin{aligned} k = 1, \quad f(x, \alpha) &= 1, \\ k = 2, \quad f(x, \alpha) &= (x - \alpha) \\ k = 3, \quad f(x, \alpha) &= \frac{x^2}{2!} + \frac{\alpha^2}{2!} - \alpha x \end{aligned} \quad (9)$$

As a result, we utilize its investigation to improve, the system to solve equations

$$u_{n+1} = u_0 + \int_0^x f(x, \alpha)(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \quad (10)$$

Using initial conditions, we can obtain the initial guess $u_0(x)$. We improve our estimate by using a specific value for the input parameter in each iteration. We are using Reimann-Liouville to solve the fractional Abel differential condition. When we combine VPM with a fractional integral for the arrangement process, the iterative plan for fractional equations is

$$u_{n+1} = u_0 + \frac{1}{\Gamma(\beta)} \int_0^x f(x, \alpha)^{x-\alpha} (-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \quad (11)$$

4 Numerical Examples

Example 4.1. We Consider the following fractional Abel problem,

$$D^\beta f(x) - 3f^3(x) + f(x) = 1, \quad 0 < \beta \leq 1, \quad x > 0, \quad (12)$$

with initial condition $f(0) = \frac{1}{3}$ can be found as follows:

$$f(x) = \frac{1}{\sqrt{6e^{2x+3}}}. \quad (13)$$

For the above problem, we create the iterative scheme shown below

$$f_{n+1} = f_0 + \frac{1}{\Gamma(\beta)} \int_0^x (x - \alpha)^{\beta-1} (3f^3(x) - f(x))d\alpha \quad (14)$$

x	Exact value	VPM	Absolute error
0.0	[0.3333]	[0.2916]	[0.0417]
0.1	[0.3111]	[0.3062]	[0.0049]
0.2	[0.2892]	[0.2730]	[0.0162]
0.3	[0.2679]	[0.2531]	[0.0148]
0.4	[0.2472]	[0.2374]	[0.0098]
0.5	[0.2275]	[0.2164]	[0.0111]
0.6	[0.2088]	[0.1943]	[0.0145]
0.7	[0.1912]	[0.1804]	[0.0108]
0.8	[0.1748]	[0.1692]	[0.0056]
0.9	[0.1595]	[0.1439]	[0.0156]
1.0	[0.1453]	[0.1379]	[0.0074]

Table 1: Value of $f(x)$, $\beta = 1$

x	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
0.0	0.3201	0.3197	0.2932
0.2	0.2817	0.2809	0.2782
0.4	0.2402	0.2397	0.2381
0.6	0.2007	0.1991	0.1962
0.8	0.1731	0.1706	0.1699
1.0	0.1497	0.1399	0.1388

Table 2: Different values of β

Taking initial condition $f(0) = \frac{1}{3}$, the following results for $\beta = 1$ are produced:

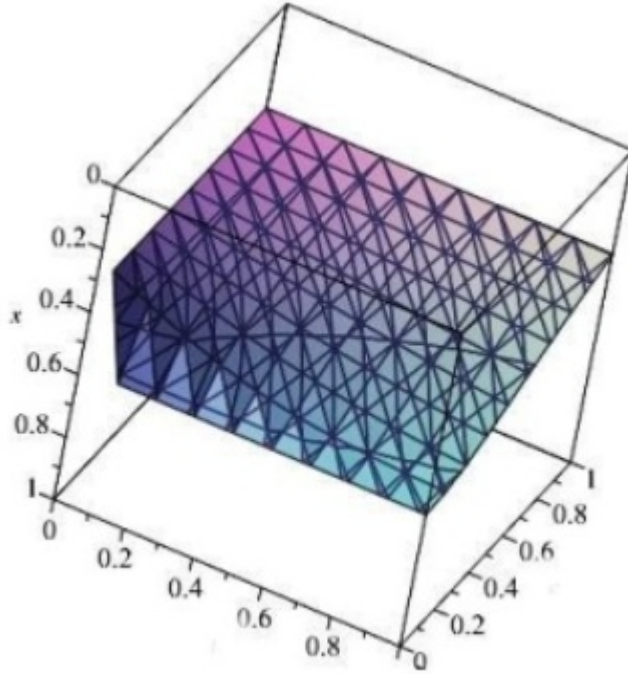
$$f_1(x) = \frac{1}{3} - \frac{2}{9}x$$

$$f_2(x) = \frac{7}{24} - \frac{4}{9}x + \frac{4}{81}x^3 - \frac{2}{243}x^4$$

$$f_3(x) = \frac{7}{24} - \frac{3049}{4608}x + \frac{5}{96}x^2 + \frac{2}{9}x^3 - \frac{133}{1728}x^4 - \frac{881}{3880}x^5 + \frac{13}{729}x^6 - \frac{2}{1701}x^7 \\ - \frac{13}{8748}x^8 + \frac{5}{13122}x^9 + \frac{8}{885735}x^{10} - \frac{32}{1948617}x^{11} + \frac{4}{1594323}x^{12} - \frac{8}{62178597}x^{13}.$$

Table 1 show a approximate solution and exact solution for $\beta = 1$. Table 2 shows different values of β . Fig 1 represents the exact and approximate solution.

Figure 1: Value of $f(x)$



Example 4.2. We Consider the following fractional Abel problem,

$$D^\beta f(x) + f^3(x) - f(x) = 1, \quad 0 < \beta \leq 1, \quad x > 0, \quad (15)$$

with initial condition $f(0) = \frac{1}{3}$ can be found as follows:

$$f(x) = \frac{e^t}{\sqrt{e^{2t}+8}}. \quad (16)$$

For the above problem, we create the iterative scheme shown below

$$f_{n+1} = f_0 + \frac{1}{\Gamma(\beta)} \int_0^x (x - \alpha)^{\beta-1} (f(x) - f^3(x)) d\alpha \quad (17)$$

Taking initial condition $f(0) = \frac{1}{3}$, the following results for $\beta = 1$ are produced:

$$f_1(x) = \frac{1}{3} + \frac{8}{27}x$$

$$f_2(x) = \frac{31}{96} + \frac{16}{27}x + \frac{8}{81}x^2 - \frac{64}{2187}x^3 - \frac{128}{19683}x^4$$

$$f_3(x) = \frac{31}{96} + \frac{780193}{884736}x + \frac{1045}{3456}x^2 - \frac{11201}{93312}x^3 - \frac{38591}{419904}x^4 - \frac{2657}{157464}x^5 + \frac{784}{177147}x^6 + \frac{3776}{1594323}x^7$$

$$- \frac{2240}{14348907}x^8 - \frac{100864}{1162261467}x^9 - \frac{851968}{52301766015}x^{10} - \frac{131072}{345191655699}x^{11} + \frac{262144}{847288609443}x^{12} + \frac{2097152}{99132767304831}x^{13}.$$

x	Exact value	VPM	Absolute error
0.0	[0.3333]	[0.3229]	[0.0104]
0.1	[0.3639]	[0.3508]	[0.0131]
0.2	[0.3964]	[0.3872]	[0.0092]
0.3	[0.4307]	[0.4139]	[0.0168]
0.4	[0.4665]	[0.4477]	[0.0188]
0.5	[0.5035]	[0.4972]	[0.0063]
0.6	[0.5415]	[0.5128]	[0.0287]
0.7	[0.5799]	[0.5524]	[0.0275]
0.8	[0.6183]	[0.6106]	[0.0077]
0.9	[0.6561]	[0.6392]	[0.0169]
1.0	[0.6929]	[0.6548]	[0.0381]

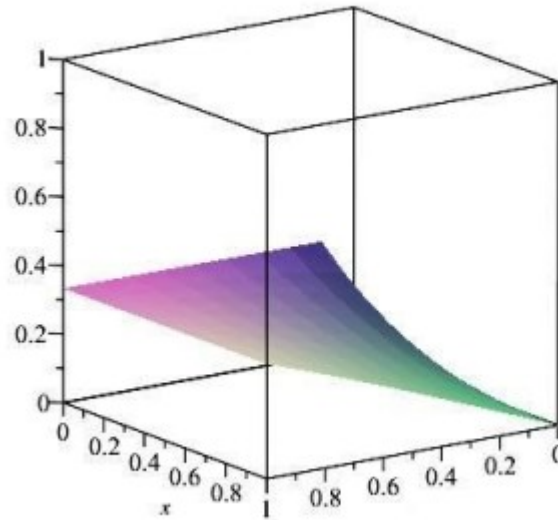
Table 3: Value of $f(x)$, $\beta = 1$

x	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
0.0	0.3301	0.3299	0.3271
0.2	0.3956	0.3907	0.3882
0.4	0.4641	0.4596	0.4481
0.6	0.5364	0.5209	0.5197
0.8	0.6180	0.6159	0.6132
1.0	0.6877	0.6792	0.6674

Table 4: Different values of β

Table 3 shows a approximate solution and exact solution for $\beta = 1$. Table 4 shows different values of β . Fig 2 represents the exact and approximate solution.

Figure 2: Value of $f(x)$



5 Conclusions

In this study, we examined the formulation to fractional Abel equation with the Variation of Parameters method. We demonstrate that VPM is a functional, efficient approach for the achievement of empirical and numerical testing for a wide variety of nonlinear fractional equations. We discovered that obtained estimate effects for different beta values interact simultaneously before the first-order derivative is exceeded.

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On the Roots and Stability of Vertex connectivity Polynomial

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Abstract

The introduction of the concept of vertex connectivity polynomial of graphs in [Priya and Anil Kumar, 2021b] triggered the need to study the nature of roots as well as the stability properties of the same for various graph classes. This paper mainly deals with results about the nature of roots, stability and schur stability of the vertex connectivity polynomial.

Keywords: Vertex connectivity polynomial, Vertex-connected vertices, Schur stability.

2020 AMS subject classifications: 05C31, 05C40.¹

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1 Introduction

The adjacency property of vertices in a graph is not sufficient to characterize a graph in terms of its connectedness as the deletion of the corresponding edge may or may not disconnect the graph. More generally, the connectivity property of vertices in a graph may not be preserved after the deletion of edges of the geodesics linking them. This motivated the authors to introduce the concept of “vertex-connected vertices”, which shares geodesics preserving the connectivity of a graph. The study of vertex-connected vertices in [Priya and Anil Kumar, 2021a] triggered the need to define vertex connectivity polynomial discussed in [Priya and Anil Kumar, 2021b] for simple finite connected graphs to explicitly reveal the number of vertex pairs that disconnect a graph.

A polynomial $p(z)$ is said to be *stable*, or a *Hurwitz polynomial*, if all its zeros lie in the open left half-plane [Fuhrmann, 2011] and is *schur stable* if all its zeros belong to the open unit disk. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems [Gajender and Sharma, 2014]. Thus the study of a graph polynomial is worthy only if it succeeds in predicting the behavior of some stable physical systems. This motivated the authors to study about the nature of roots and stability of the vertex connectivity polynomial of a graph. Throughout this paper, G denotes a finite simple connected graph with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively. All the graph theoretic terminology and notations used in the paper are as in [Harary, 1969].

2 Main Results

Definition 2.1. Let G be a graph of order n . Then, the vertex connectivity polynomial of G , denoted by $V[G; x]$, is defined in [Priya and Anil Kumar, 2021b] as:

$$V[G; x] = \sum_{i=1}^{\text{diam}(G)+1} |D_i| x^i,$$

where D_i is the set of all vertex pairs which disconnects G into i components.

Definition 2.2. Let $G = (V, E)$ be a graph and let $u, v \in V$. Then, the vertices u and v are said to be *vertex-connected* if the deletion of the edges of none of the geodesics connecting u and v disconnects G into multiple components. The graph G is *vertex-connected* if each of its vertex pairs are vertex-connected. [Priya and Anil Kumar, 2021a]

Definition 2.3. Let $G = (V, E)$ be a graph. Then, a pair of vertices $(u, v) \in V \times V$ at a distance $d > 0$ are said to be d -metrically connective if it disconnects the graph G into d components.

The graph G is metrically connective if every pair of its vertices are d -metrically connective for some $d > 0$. [Priya and Anil Kumar, 2021a]

Theorem 2.4. (Routh-Hurwitz Criteria [Fuhrmann, 2011]) Given a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where the coefficients a_i 's are real constants, the n Hurwitz matrices using the coefficients a_i of the above polynomial are defined as

$$H_1 = [a_1], H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \cdots,$$

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: $\det(H_j) > 0, j = 1, 2, \dots, n$.

Definition 2.5. Let G be a graph. Then, the roots of the vertex connectivity polynomial of G are called the vertex connectivity roots of G .

The following are some of the observations about the vertex connectivity roots of graphs.

- (i) $(0, \infty)$ is a zero-free interval of the vertex connectivity polynomial $V[G; x]$ for any graph G .
- (ii) If G is metrically connective, then it has at most $\text{diam}(G)$ vertex connectivity roots.
- (iii) If G is a tree, then it has $\text{diam}(G) + 1$ vertex connectivity roots counting multiplicities.

Theorem 2.6. Let G be a graph. Then, zero is a vertex connectivity root of G of multiplicity greater than one iff G is a tree.

Proof. Assume that zero is a vertex connectivity root of G of multiplicity $k > 1$. Then, $V[G; x] = x^k g(x)$, a polynomial of degree greater than or equal to 2. This means that every vertex pair of G disconnects the graph into at least two components so that none of its edges belongs to a cycle. Hence it follows that G is a tree. Conversely, if G is a tree, all its edges being cut edges, none of the vertex pairs are vertex-connected. That is, $V[G; x]$ is a polynomial free of linear coefficient.

This completes the proof. $\square$

Theorem 2.7. *Let G be a graph of order n . Then, zero is the only vertex connectivity root of G iff either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components.*

Proof. Assume that zero is the only vertex connectivity root of G . Then, $V[G; x] = rx^k$, where $r, k > 0$. But, $V[G; 1] = \sum_{i=1}^{\text{diam}(G)+1} |D_i| = \binom{n}{2}$ so that $r = \binom{n}{2}$. That is, $V[G; x] = \binom{n}{2}x^k$.

Case (i): If $k = 1$, then $V[G; x] = \binom{n}{2}x$ so that G is vertex-connected.

Case (ii): If $k > 1$, then all the $\binom{n}{2}$ vertex pairs disconnects the graph into k components. But if a vertex pair disconnects G into k components, then corresponding to that there exists a vertex pair disconnecting the graph into two components. Thus it follows that $V[G; x] = \binom{n}{2}x^2$.

Conversely, if either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components, then we get $V[G; x]$ as $\binom{n}{2}x$ and $\binom{n}{2}x^2$ respectively. In both the cases, it follows that zero is the only vertex connectivity root of G .

This completes the proof. $\square$

Corollary 2.8. *Let G be a non vertex-connected graph of order n . Then, zero is the only vertex connectivity root of G iff $G = P_2$.*

Proof. Let zero be the only vertex connectivity root of G . Since G is not vertex-connected, from theorem 2.7 it follows that G is a tree and $V[G; x] = \binom{n}{2}x^2$. This means that all the $\binom{n}{2}$ vertex pairs disconnects G into two components. Now if possible assume $n > 2$. Then, since every edge of G is a cut edge, we can find a vertex pair at 2 units distance which disconnects G into three components. Therefore $n = 2$ and $G = P_2$.

This completes the proof. $\square$

Theorem 2.9. *Let G be a graph and n_1, n_2 are respectively be the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G , where $n_1 > n_2$. If n_1 is prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Let $V[G; x] = xg(x)$, where $g(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + n_1$, $\sum_{i=1}^n a_i = n_2$, for some n . Clearly all the zeros of $g(x)$ are non-zero. Let β be a zero of $g(x)$ such that $|\beta| \leq 1$. Then,

$$g(\beta) = a_n \beta^n + a_{n-1} \beta^{n-1} + \dots a_1 \beta + n_1 = 0.$$

That is,

$$|n_1| = |a_n \beta^n + a_{n-1} \beta^{n-1} + \dots a_1 \beta| \leq |a_n| + |a_{n-1}| + \dots |a_1| = n_2,$$

a contradiction. Therefore, $|\beta| > 1$.

Now, if possible, assume that β is real. Then, $g(x)$ is reducible over the set of all real numbers. That is,

$$g(x) = (x - \beta)h(x),$$

where $h(x)$ is a nonconstant integer polynomial. Now,

$$n_1 = g(0) = -\beta h(0).$$

Since n_1 is prime and since $|\beta| > 1$, $|h(0)|$ must be 1. Let $\beta_1, \dots, \beta_{n-1}$ be the zeros of $h(x)$ and a be its leading coefficient. Then,

$$|\beta_1, \dots, \beta_{n-1}| = \frac{1}{|a|} \leq 1.$$

But $\beta_1, \dots, \beta_{n-1}$ are the zeros of $g(x)$ also so that $|\beta_i| > 1 \forall i = 1, \dots, n-1$, a contradiction. Therefore $g(x)$ has no real zeros and hence G lacks non-zero real vertex connectivity roots.

This completes the proof. □

Corollary 2.10. *Let G be a graph and n_1, n_2 are respectively be the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G , where $n_1 > n_2$. Then, all the non-zero vertex connectivity roots of G lie strictly outside the unit circle.*

Theorem 2.11. *Let G be a graph with $|D_i| = m \forall i \leq n$, where $2 \leq n \leq \text{diam}(G) + 1$. Then, all the non-zero vertex connectivity roots of G are distinct and are of unit modulus. In addition, if G is a tree of diameter p , where p is an odd prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Without loss of generality, let G be a graph which is not a tree. Since $|D_i| = m, \forall i = 1, 2, \dots, n$,

$$V[G, x] = m \sum_{i=1}^n x^i = mx[1 + x + \dots + x^{n-1}].$$

The zeros of the polynomial $1 + x + \dots + x^{n-1}$ are the n^{th} roots of unity except 1 and hence it follows that the vertex connectivity roots of G are distinct. Now, if G is a tree, then $V[G; x]$ is a polynomial of degree $diam(G) + 1$. That is,

$$V[G, x] = m \sum_{i=2}^{diam(G)+1} x^i = mx^2[1 + x + \dots + x^{diam(G)-1}].$$

Since $diam(G) = p$, the polynomial $1 + x + \dots + x^{p-1}$ is irreducible over the set of all rationals and hence it has no rational roots. But the only possible real $p - 1^{th}$ root of unity is -1 , which is rational. Therefore, it can be concluded that G is free of non-zero real vertex connectivity roots.

This completes the proof. $\square$

Corollary 2.12. *If n is odd, then all the non-zero vertex connectivity roots of C_n lies on the unit circle. In addition, if $\frac{n-1}{2}$ is even, then C_n has exactly two real vertex connectivity roots.*

Proof. This follows from the fact that $V[C_n; x] = \sum_{i=1}^{\frac{n-1}{2}} nx^i$ for odd n and -1 is a k^{th} root of unity if k is even. $\square$

Theorem 2.13. (i) *Let G_1 and G_2 be two disjoint graphs. If a is a vertex connectivity root of both G_1 and G_2 , then a is also a vertex connectivity root of $G_1 + G_2$.*

(ii) *Let G be a vertex-connected graph of order n and let G' be the graph obtained by adjoining one pendent vertex to m distinct vertices of $G, m \leq n$. Then, the vertex connectivity roots of G' are the roots of the polynomial $V[G; x] + nm x^2 + \binom{m}{2} x^3$.*

(iii) *Let G be metrically connective and let $d = diam(G)$. Then, G has exactly d vertex connectivity roots counting multiplicities.*

Proof. (i) This follows from the fact that $V[G_1 + G_2; x] = V[G_1; x] + V[G_2; x]$.

(ii) Since G is vertex-connected, the newly adjoined vertices along with every vertex of G disconnects G' into exactly two components and mutually disconnects G' into three components.

Therefore, $V[G'; x] = V[G; x] + nm x^2 + \binom{m}{2} x^3$.

(iii) This follows from the fact that the degree of $V[G; x]$ is d .

This completes the proof. $\square$

Theorem 2.14. *Let G be a graph of order n and diameter d . Then, any vertex connectivity root β of G either has nonpositive real part or satisfies*

$$|\beta| < \frac{1 + \sqrt{1 + 2n(n-1)}}{2}.$$

Proof. Let β be a non-zero vertex connectivity root of G . If $|\beta| \leq 1$, then the result holds trivially since $\frac{1 + \sqrt{1 + 2n(n-1)}}{2} > 1$.

Case(i) : Let G be a tree. Then, the degree of $V[G; x]$ is $d + 1$.

We have, $V[G; x] = x^2 g(x)$, where $g(x) = a_{d-1}x^{d-1} + \dots + a_1x + a_0$. Since $\sum_{i=0}^{d-1} a_i = \binom{n}{2}$, $|a_i| \leq \binom{n}{2} \forall i = 0, 1, \dots, d-1$. Now, if $|\beta| > 1$ and $Re(\beta) > 0$, then

$$\begin{aligned} \frac{|g(\beta)|}{|\beta^{d-1}|} &= |a_{d-1} + \frac{a_{d-2}}{\beta} + \dots + \frac{a_1}{\beta^{d-2}} + \frac{a_0}{\beta^{d-1}}| \\ &\geq |a_{d-1} + \frac{a_{d-2}}{\beta}| - \frac{n(n-1)}{2} [\frac{1}{|\beta^2|} + \dots + \frac{1}{|\beta^{d-1}|}] \\ &> Re(a_{d-1}) + a_{d-2} Re(\frac{1}{\beta}) - \frac{n(n-1)}{2} [\frac{1}{|\beta^2|} - \frac{1}{|\beta|}] \\ &\geq 1 - \frac{n(n-1)}{2} [\frac{1}{|\beta^2|} - \frac{1}{|\beta|}], \text{ since } a_{d-1}, a_{d-2} \geq 1 \text{ and } Re(\beta) > 0. \\ &= \frac{|\beta^2| - |\beta| - \frac{n(n-1)}{2}}{|\beta^2| - |\beta|} \\ &\geq 0, \end{aligned}$$

whenever $|\beta^2| - |\beta| - \frac{n(n-1)}{2} \geq 0$.

That is, if $|\beta| \geq \frac{1 + \sqrt{1 + 4 \frac{n(n-1)}{2}}}{2} = \frac{1 + \sqrt{1 + 2n(n-1)}}{2}$ and $Re(\beta) > 0$, then $\frac{|g(\beta)|}{|\beta^{d-1}|} > 0$ so that β cannot be a zero of $g(x)$.

Case(ii) Let G be a graph which is not a tree. Then at least one edge of G is a part of some cycle so that the linear coefficient of $V[G; x]$ is non-zero. Let $V[G; x] = xh(x)$, where $h(x) = a_{m-1}x^{m-1} + \dots + a_1x + a_0$. Here also $\sum_{i=0}^{m-1} a_i = \binom{n}{2}$, $|a_i| \leq \binom{n}{2} \forall i = 0, 1, \dots, m-1$, so that similar calculation as in case(i) yields that if $|\beta| \geq \frac{1 + \sqrt{1 + 2n(n-1)}}{2}$ and $Re(\beta) > 0$, then β cannot be a zero of $g(x)$.

Therefore, from the above two cases we get that every vertex connectivity root of the graph G either has nonpositive real part or has modulus strictly less than $\frac{1+\sqrt{1+2n(n-1)}}{2}$.

This completes the proof. $\square$

Theorem 2.15. *Let G be a tree with diameter $d \leq 2$. Then, all the non-zero vertex connectivity roots of G lie in the left half plane. In addition, if the leading coefficient of $V[G; x]$ dominates, then the polynomial is schur stable.*

Proof. Case(i): If $d = 1$, then $G = P_2$ so that $V[G; x] = x^2$. Thus, the result holds trivially.

Case(ii): If $d = 2$, then $V[G; x] = x^2(ax + b)$, a cubic polynomial where $a, b > 0$. Thus the only non-zero zero $x = \frac{-b}{a}$ of $V[G; x]$ lies in the left half plane. Now if the leading coefficient of $V[G; x]$ dominates, then $|\frac{b}{a}| < 1$ and thus all the vertex connectivity roots of G lies inside the unit circle. Hence $V[G; x]$ is schur stable. $\square$

Corollary 2.16. *$V[K_{1,n}; x]$ is schur stable iff $n > 3$.*

Theorem 2.17. *Every vertex connectivity root of the path graph P_n lie in the closed left half plane iff $n \leq 5$*

Proof. We have, $V[P_n; x] = \sum_{i=1}^{n-1} (n-i)x^{i+1}$. Thus the result holds trivially for $n = 2$ and $n = 3$.

Now for $n \geq 4$, let $V[P_n; x] = x^2g(x)$.

When $n = 4$, $g(x) = x^2 + 2x + 3$ so that

$$H_1 = [2], H_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

are having positive determinants.

If $n = 5$, then $g(x) = x^3 + 2x^2 + 3x + 4$ so that all the Hurwitz matrices

$$H_1 = [2], H_2 = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 5 & 4 \end{bmatrix}$$

have positive determinant.

For $n = 6$, $g(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$.

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

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so that $\det(H_3) < 0$.

For $n \geq 7$,

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 6 & 5 & 4 \end{bmatrix}$$

so that $\det(H_3) = 0$. Thus for $n \geq 6$, $g(x)$ is not stable so that all the vertex connectivity roots of P_n does not belong to the closed left half plane.

This completes the proof. □

A bistar graph $B_{m,m}$ is the union of two identical star graphs $K_{1,m}$ whose centres joined together to form a new edge.

Theorem 2.18. *The vertex connectivity roots of the bistar graph $B_{m,m}$ always lie in the closed left half plane and are all real iff $m \geq 7$. Also, for all $m \geq 2$, zero is the only vertex connectivity root of multiplicity > 1 .*

Proof. We have, $V[B_{m,m}; x] = x^2[m^2x^2 + m(m+1)x + (2m+1)]$. [Priya and Anil Kumar, 2021b]

The zeros of the polynomial $g(x) = m^2x^2 + m(m+1)x + (2m+1)$ are

$$\begin{aligned} x &= \frac{-m(m+1) \pm \sqrt{m^2(m+1)^2 - 4m^2(2m+1)}}{2m^2} \\ &= \frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}. \end{aligned}$$

Since $\sqrt{m^2 - 6m - 3} < \sqrt{m^2} < m+1$, all the zeros of $g(x)$ lie in the left half plane. The zeros of $g(x)$ are real iff $m^2 - 6m - 3 \geq 0$ iff $m \geq 3 + \sqrt{12}$ or $m \leq 3 - \sqrt{12}$. Since m is a positive integer and since $6 < 3 + \sqrt{12} < 7$, we get that the vertex connectivity roots of $B_{m,m}$ are all real iff $m \geq 7$. Now, since $m^2 - 6m - 3$ is never zero for any integer m , all the zeros of $g(x)$ are distinct.

This completes the proof. □

Corollary 2.19. *For $m \geq 3$, $V[B_{m,m}; x]$ is schur stable.*

Proof. In the proof of theorem 2.18, the zeros of $g(x)$ are $\frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}$. It can be observed that $|\frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}| < 1$ for $m \geq 3$ so that all the vertex connectivity roots of $B_{m,m}$ lie inside the unit circle.

This completes the proof. □

Theorem 2.20. *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a complex polynomial such that $|a_k| > |a_0| + |a_1| + \dots + |a_{k-1}| + |a_{k+1}| + \dots + |a_n|$ for some $0 \leq k \leq n$. Then exactly k zeros of P lie strictly inside the unit circle, and the other $n - k$ zeros of P lie strictly outside the unit circle. [Zhao, 1947]*

A helm H_n is obtained by adding pendent edges to every vertex on the graph obtained as the join of the cycle C_{n-1} and K_1 . A webgraph WB_n is obtained by joining the pendent vertices of H_n to form a cycle and then adding a single pendent edge to each vertex of the outer cycle. A butterfly graph is a double shell with same apex along with exactly two pendent edges at the apex.

Theorem 2.21. (i) *For $n > 3$, the helm H_n is not schur stable.*

(ii) *For $n > 3$, the web graph WB_n is not schur stable. But all the vertex connectivity roots of WB_n are real and belong to the closed left half plane.*

(iii) *For $n > 4$, the butterfly graph BF_n is not schur stable. But all the vertex connectivity roots of BF_n are real and belong to the closed left half plane.*

Proof. (i) From [Priya and Anil Kumar, 2021b], we get the vertex connectivity polynomial of H_n as $V[H_n; x] = \binom{n-1}{2} x^3 + n(n-1)x^2 + \binom{n}{2} x$. Here, the quadratic coefficient $n(n-1) > \binom{n-1}{2} + \binom{n}{2}$ so that from theorem 2.20 it follows that only two zeros of $V[H_n; x]$ lie inside the unit circle and one zero outside the unit circle. Hence H_n is not schur stable.

(ii) We have, $V[WB_n; x] = \binom{n-1}{2} x^3 + (2n-1)(n-1)x^2 + \binom{2n-1}{2} x$. That is, $V[WB_n; x] = \binom{n-1}{2} x [x^2 + \frac{2(2n-1)}{n-2} x + \frac{2(2n-1)}{n-2}]$. Since the discriminant of the quadratic polynomial $p(x) = x^2 + \frac{2(2n-1)}{n-2} x + \frac{2(2n-1)}{n-2}$ is $\frac{12(2n-1)}{(n-2)^2} > 0$, it follows that all the vertex connectivity roots of WB_n are real. The remaining results follows easily from the fact that the zeros of $p(x)$ are exactly $\frac{-2(2n-1) \pm \sqrt{12(2n-1)}}{2(n-2)}$.

(iii) From [Priya and Anil Kumar, 2021b], it follows that the vertex connectivity polynomial of BF_n is $V[BF_n; x] = x^3 + (2n-4)x^2 + \binom{n-2}{2} x$. For $n = 5, 6, 7$, the non-zero vertex connectivity roots of BF_n are exactly $-3 \pm \sqrt{6}$, $-4 \pm \sqrt{10}$ and $-5 \pm \sqrt{15}$ respectively so that in all those cases the graph is not schur stable. Now for $n > 7$, the linear coefficient $\binom{n-2}{2} > 1 + 2n - 4$ and hence by theorem 2.20, only one zero of $V[BF_n; x]$ lie inside the unit circle and the other two zeros lie outside the unit circle. Hence for $n > 4$, BF_n lacks schur stability. Now, since $2n - 4$ is greater than the magnitude of the square of the discriminant of the quadratic factor in $V[BF_n; x]$ and since the discriminant is positive, it follows that all the

non-zero vertex connectivity roots of BF_n belong to the left half plane.

This completes the proof. □

3 Conclusions

In this paper, the nature of roots and stability properties of some well known graphs were discussed in detail and inferences were made about its stability. More general properties regarding the graph stability can be inferred only after studying the same for graph operations.

4 Acknowledgements

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Some Edge Domination Parameters in Bipolar Hesitancy Fuzzy Graph

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Abstract

In this article, we establish edge domination in Bipolar Hesitancy Fuzzy Graph(BHFG). Various domination parameters such as inverse edge domination and total edge domination in BHFG are determined. Some theorems related to edge domination and examples are also discussed.

Keywords: Bipolar fuzzy graph; Hesitant fuzzy graph; Edge domination number; Total edge domination; Inverse edge domination;

2020 AMS subject classifications: 05C72, 05C69, 94D05.¹

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1 Introduction

The concept of fuzzy sets was first originated by L.A. Zadeh [Zadeh [1965]]. In 1973, Kaufmann established fuzzy graph using Zadeh's fuzzy relation. The domination concept in fuzzy graph was first established by A. Somasundaram and S. Somasundaram [Somasundaram and Somasundaram [1998]]. The edge domination in fuzzy graphs was initiated by S. Velammal and K.Thiagarajan [Vellamal and Thiagarajan [2012]]. The notion of Bipolar Fuzzy Graph(BFG) was established by M.Akram [Akram [2002]].The approach of domination in bipolar fuzzy graphs was proposed by M.G. Karunambigai, Palanivel and Akram [Karunambigai et al. [2013]].The book by Akram, Sarwar and Dudek entitled "*Graphs for the Analysis of Bipolar Fuzzy Information*" [Akram et al. [2021]] is a great tool for understanding the concepts of domination in BFGs. S. Ramya and S. Lavanya developed edge domination in bipolar fuzzy graphs [Ramya and Lavanya [2017]].The notion of hesitant fuzzy sets was first introduced by V.Torra [Torra [2010]] in the year 2010. Hesitancy fuzzy graph, a new approach to fuzzy graph theory was first established by T. Pathinathan,et.al [Pathinathan et al. [2015]].The idea of domination in hesitancy fuzzy graph was investigated by R. Sakthivel et.al,[Sakthivel et al. [2019]]. In the year 2021, K. Anantha Kanaga Jothi and K. Balasangu [Anantha Kanaga Jothi and Balasangu [2021]]defined the idea of irregular and totally irregular bipolar hesitancy fuzzy graphs and some of its properties.

2 Preliminaries

Definition 2.1 (Akram [2002]). *Let $\mathcal{X}$ be a non empty set. A bipolar fuzzy set B in $\mathcal{X}$ is an object having the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) | x \in \mathcal{X}\}$ where, $\mu_B^P : \mathcal{X} \rightarrow [0, 1]$ and $\mu_B^N : \mathcal{X} \rightarrow [-1, 0]$ are mappings.*

Definition 2.2 (Akram [2002]). *A Bipolar Fuzzy Graph (BFG) is of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where*

1. $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1^P : \mathcal{V} \rightarrow [0, 1]$ and $\mu_1^N : \mathcal{V} \rightarrow [-1, 0]$
2. $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ where $\mu_2^P : \mathcal{V} \times \mathcal{V} \rightarrow [0, 1]$ and $\mu_2^N : \mathcal{V} \times \mathcal{V} \rightarrow [-1, 0]$ such that

$$\mu_2^P(v_i, v_j) \leq \min(\mu_1^P(v_i), \mu_1^P(v_j))$$

and

$$\mu_2^N(v_i, v_j) \geq \max(\mu_1^N(v_i), \mu_1^N(v_j))$$

for all $(v_i, v_j) \in \mathcal{E}$.

Definition 2.3 (Akram [2002]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG is said to be strong then $\mu_2^P = \min(\mu_1^P(v_i), \mu_1^P(v_j))$ and $\mu_2^N = \max(\mu_1^N(v_i), \mu_1^N(v_j)) \forall v_i, v_j \in \mathcal{V}$.

Definition 2.4 (Akram [2002]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG is said to be complete then,

$$\mu_2^P(v_i, v_j) = \min(\mu_1^P(v_i), \mu_1^P(v_j))$$

$$\mu_2^N(v_i, v_j) = \max(\mu_1^N(v_i), \mu_1^N(v_j))$$

for all $v_i, v_j \in \mathcal{V}$.

Definition 2.5 (Karunambigai et al. [2013]). An arc (a, b) is said to be strong edge in a BFG, if

$$\mu_2^P(a, b) \geq (\mu_2^P)^\infty(a, b) \text{ and } \mu_2^N(a, b) \geq (\mu_2^N)^\infty(a, b)$$

whereas $(\mu_2^P)^\infty(a, b) = \max\{(\mu_2^P)^k(a, b) | k = 1, 2, \dots, n\}$
and $(\mu_2^N)^\infty(a, b) = \min\{(\mu_2^N)^k(a, b) | k = 1, 2, \dots, n\}$.

Definition 2.6 (Karunambigai et al. [2013]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG, then cardinality of $\mathcal{G}$ is defined as

$$|\mathcal{G}| = \sum_{v_i \in \mathcal{V}} \frac{(1 + \mu_1^P(v_i) + \mu_1^N(v_i))}{2} + \sum_{(v_i, v_j) \in \mathcal{E}} \frac{(1 + \mu_2^P(v_i, v_j) + \mu_2^N(v_i, v_j))}{2}$$

Definition 2.7 (Karunambigai et al. [2013]). The cardinality of $\mathcal{V}$, i.e., amount of nodes is termed as the order of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and is signified by $|\mathcal{V}|$ (or $O(\mathcal{G})$) and determined by

$$O(\mathcal{G}) = |\mathcal{V}| = \sum_{v_i \in \mathcal{V}} \frac{(1 + \mu_1^P(v_i) + \mu_1^N(v_i))}{2}$$

The no. of elements in a set of S , i.e., amount of edges is termed as size of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and signified as $|S|$ (or $S(\mathcal{G})$) and determined by

$$S(\mathcal{G}) = |S| = \sum_{(v_i, v_j) \in \mathcal{E}} \frac{(1 + \mu_2^P(v_i, v_j) + \mu_2^N(v_i, v_j))}{2}$$

for all $(v_i, v_j) \in \mathcal{E}$.

3 Bipolar hesitancy fuzzy graph

Definition 3.1 (Anantha Kanaga Jothi and Balasangu [2021]). *Let $\mathcal{X}$ be a non-empty set. A Bipolar hesitancy fuzzy set*

$$B = \{x, \mu_1^P(x), \mu_1^N(x), \gamma_1^P(x), \gamma_1^N(x), \beta_1^P(x), \beta_1^N(x) / x \in \mathcal{X}\}$$

where $\mu_1^P, \gamma_1^P, \beta_1^P : \mathcal{X} \rightarrow [0, 1]$ and $\mu_1^N, \gamma_1^N, \beta_1^N : \mathcal{X} \rightarrow [-1, 0]$ are mappings such that,

$$0 \leq \mu_1^P(x) + \gamma_1^P(x) + \beta_1^P(x) \leq 1$$

and

$$-1 \leq \mu_1^N(x) + \gamma_1^N(x) + \beta_1^N(x) \leq 0$$

.

Definition 3.2 (Anantha Kanaga Jothi and Balasangu [2021]). *Let $\mathcal{X}$ be a non empty set. Then we call mappings $\mu_2^P, \gamma_2^P, \beta_2^P : \mathcal{X} \times \mathcal{X} \rightarrow [0, 1]$, $\mu_2^N, \gamma_2^N, \beta_2^N : \mathcal{X} \times \mathcal{X} \rightarrow [-1, 0]$ are bipolar hesitancy fuzzy relation on $\mathcal{X}$ such that,*

$$\mu_2^P(x, y) \leq \mu_1^P(x) \wedge \mu_1^P(y); \mu_2^N(x, y) \geq \mu_1^N(x) \vee \mu_1^N(y); \gamma_2^P(x, y) \leq \gamma_1^P(x) \wedge \gamma_1^P(y); \gamma_2^N(x, y) \geq \gamma_1^N(x) \vee \gamma_1^N(y); \beta_2^P(x, y) \leq \beta_1^P(x) \wedge \beta_1^P(y); \beta_2^N(x, y) \geq \beta_1^N(x) \vee \beta_1^N(y).$$

Definition 3.3 (Anantha Kanaga Jothi and Balasangu [2021]). *A bipolar hesitancy fuzzy relation A on $\mathcal{X}$ is called symmetric relation if $\mu_2^P(x, y) = \mu_2^P(y, x)$, $\mu_2^N(x, y) = \mu_2^N(y, x)$, $\gamma_2^P(x, y) = \gamma_2^P(y, x)$, $\gamma_2^N(x, y) = \gamma_2^N(y, x)$, $\beta_2^P(x, y) = \beta_2^P(y, x)$, $\beta_2^N(x, y) = \beta_2^N(y, x)$ for all $(x, y) \in \mathcal{X}$*

Definition 3.4 (Pathinathan et al. [2015]). *A Hesitancy fuzzy graph is of the form $G = (V, E)$ where,*

$V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1, \gamma_1, \beta_1 : V \rightarrow [0, 1]$ denote the degree of membership, non-membership and hesitancy of the vertex $v_i \in V$ respectively and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$ for every $v_i \in V$ where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$ and

$E \subseteq V \times V$ where $\mu_2, \gamma_2, \beta_2 : V \times V \rightarrow [0, 1]$ denote the degree of membership, non-membership and hesitancy of the edge $(v_i, v_j) \in E$ respectively such that, $\mu_2(v_i, v_j) \leq \mu_1(v_i) \wedge \mu_1(v_j)$; $\gamma_2(v_i, v_j) \leq \gamma_1(v_i) \vee \gamma_1(v_j)$; $\beta_2(v_i, v_j) \leq \beta_1(v_i) \wedge \beta_1(v_j)$ and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$.

Definition 3.5 (Anantha Kanaga Jothi and Balasangu [2021]). *A Bipolar Hesitancy Fuzzy Graph (BHFG) is of the form $G = (V, E)$ where*

- (i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1^P, \gamma_1^P, \beta_1^P : V \rightarrow [0, 1]$ denote the degree of positive membership, positive non-membership and positive hesitancy of

Some Edge Domination Parameters in Bipolar Hesitancy Fuzzy Graph

the vertex $v_i \in V$ respectively, $\mu_1^N, \gamma_1^N, \beta_1^N : V \rightarrow [-1, 0]$ denote the degree of negative membership, negative non-membership and negative hesitancy of the vertex $v_i \in V$. For every $v_i \in V$,

$$\mu_1^P(v_i) + \gamma_1^P(v_i) + \beta_1^P(v_i) = 1 \text{ and } \mu_1^N(v_i) + \gamma_1^N(v_i) + \beta_1^N(v_i) = -1$$

$$\beta_1^P(v_i) = 1 - [\mu_1^P(v_i) + \gamma_1^P(v_i)] \text{ and } \beta_1^N(v_i) = -1 - [\mu_1^N(v_i) + \gamma_1^N(v_i)]$$

(ii) $E \subseteq V \times V$ where, $\mu_2^P, \gamma_2^P, \beta_2^P : V \times V \rightarrow [0, 1]$; $\mu_2^N, \gamma_2^N, \beta_2^N : V \times V \rightarrow [-1, 0]$ are mappings such that

$$\mu_2^P(v_i, v_j) \leq \mu_1^P(v_i) \wedge \mu_1^P(v_j)$$

$$\mu_2^N(v_i, v_j) \geq \mu_1^N(v_i) \vee \mu_1^N(v_j)$$

$$\gamma_2^P(v_i, v_j) \leq \gamma_1^P(v_i) \vee \gamma_1^P(v_j)$$

$$\gamma_2^N(v_i, v_j) \geq \gamma_1^N(v_i) \wedge \gamma_1^N(v_j)$$

$$\beta_2^P(v_i, v_j) \leq \beta_1^P(v_i) \wedge \beta_1^P(v_j)$$

$$\beta_2^N(v_i, v_j) \geq \beta_1^N(v_i) \vee \beta_1^N(v_j)$$

denote the degree of positive, negative membership, degree of positive, negative non membership and degree of positive, negative hesitancy of the edge $(v_i, v_j) \in E$ respectively and

$$0 \leq \mu_2^P(v_i, v_j) + \gamma_2^P(v_i, v_j) + \beta_2^P(v_i, v_j) \leq 1$$

,

$$-1 \leq \mu_2^N(v_i, v_j) + \gamma_2^N(v_i, v_j) + \beta_2^N(v_i, v_j) \leq 0$$

for every $(v_i, v_j) \in E$.

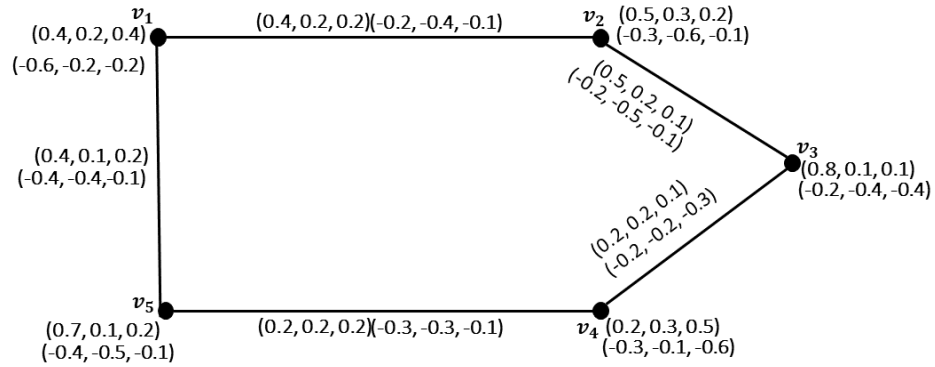


Figure 1: Bipolar Hesitancy Fuzzy Graph

Example 3.1. From Fig 1, for vertex v_1 ,

$$\mu_1^P(v_1) + \gamma_1^P(v_1) + \beta_1^P(v_1) = 0.4 + 0.2 + 0.4 = 1$$

$$\mu_1^N(v_1) + \gamma_1^N(v_1) + \beta_1^N(v_1) = -0.6 - 0.2 - 0.2 = -1.$$

$$\text{For edge } (v_1, v_2); \mu_2^P(v_1, v_2) + \gamma_2^P(v_1, v_2) + \beta_2^P(v_1, v_2) = 0.8 \leq 1$$

$$\mu_2^N(v_1, v_2) + \gamma_2^N(v_1, v_2) + \beta_2^N(v_1, v_2) = -0.7 \geq -1.$$

Definition 3.6. A Bipolar Hesitancy Fuzzy Graph $G = (V, E)$ is said to be complete when, $\mu_2^P(v_i, v_j) = \mu_1^P(v_i) \wedge \mu_1^P(v_j), \mu_2^N(v_i, v_j) = \mu_1^N(v_i) \vee \mu_1^N(v_j), \gamma_2^P(v_i, v_j) = \gamma_1^P(v_i) \vee \gamma_1^P(v_j), \gamma_2^N(v_i, v_j) = \gamma_1^N(v_i) \wedge \gamma_1^N(v_j), \beta_2^P(v_i, v_j) = \beta_1^P(v_i) \wedge \beta_1^P(v_j), \beta_2^N(v_i, v_j) = \beta_1^N(v_i) \vee \beta_1^N(v_j)$ for every $v_i, v_j \in V$.

Definition 3.7. A Bipolar Hesitancy Fuzzy Graph $G = (V, E)$ is said to be strong when, $\mu_2^P(v_i, v_j) = \mu_1^P(v_i) \wedge \mu_1^P(v_j), \mu_2^N(v_i, v_j) = \mu_1^N(v_i) \vee \mu_1^N(v_j), \gamma_2^P(v_i, v_j) = \gamma_1^P(v_i) \vee \gamma_1^P(v_j), \gamma_2^N(v_i, v_j) = \gamma_1^N(v_i) \wedge \gamma_1^N(v_j), \beta_2^P(v_i, v_j) = \beta_1^P(v_i) \wedge \beta_1^P(v_j), \beta_2^N(v_i, v_j) = \beta_1^N(v_i) \vee \beta_1^N(v_j)$ for every $(v_i, v_j) \in E$.

Definition 3.8. Let G be a Bipolar hesitancy fuzzy graph. The neighbourhood of a vertex x in G is defined by

$$N(x) = (N_\mu^P(x), N_\mu^N(x), N_\gamma^P(x), N_\gamma^N(x), N_\beta^P(x), N_\beta^N(x))$$

where

$$N_\mu^P(x) = \{y \in V / \mu_2^P(x, y) \leq \mu_1^P(x) \wedge \mu_1^P(y)\}; N_\mu^N(x) = \{y \in V / \mu_2^N(x, y) \geq \mu_1^N(x) \vee \mu_1^N(y)\}; N_\gamma^P(x) = \{y \in V / \gamma_2^P(x, y) \leq \gamma_1^P(x) \wedge \gamma_1^P(y)\}; N_\gamma^N(x) = \{y \in V / \gamma_2^N(x, y) \geq \gamma_1^N(x) \vee \gamma_1^N(y)\}; N_\beta^P(x) = \{y \in V / \beta_2^P(x, y) \leq \beta_1^P(x) \wedge \beta_1^P(y)\}; N_\beta^N(x) = \{y \in V / \beta_2^N(x, y) \geq \beta_1^N(x) \vee \beta_1^N(y)\}.$$

Definition 3.9. Let G be a Bipolar Hesitancy Fuzzy Graph. The neighborhood degree of a vertex x in G is defined by

$$\deg(x) = [\deg \mu^P(x), \deg \mu^N(x), \deg \gamma^P(x), \deg \gamma^N(x), \deg \beta^P(x), \deg \beta^N(x)]$$

$y \in V$, where

$$\deg \mu^P(x) = \sum_{y \in N(x)} \mu_1^P(y), \deg \mu^N(x) = \sum_{y \in N(x)} \mu_1^N(y), \deg \gamma^P(x) = \sum_{y \in N(x)} \gamma_1^P(y)$$

$$\deg \gamma^N(x) = \sum_{y \in N(x)} \gamma_1^N(y), \deg \beta^P(x) = \sum_{y \in N(x)} \beta_1^P(y), \deg \beta^N(x) = \sum_{y \in N(x)} \beta_1^N(y)$$

.

Definition 3.10. Let $G = (V, E)$ be a BHFG. The edge cardinality of G is given by, $|E| = r$

$$= \sum_{(u,v) \in E} \frac{3 + \mu_2^P(u, v) + \mu_2^N(u, v) + \gamma_2^P(u, v) + \gamma_2^N(u, v) + \beta_2^P(u, v) + \beta_2^N(u, v)}{3}$$

.

Definition 3.11. An Arc (u,v) is said to be strong edge in BHFG. then,
 $\mu_2^P(u,v) \geq (\mu_2^P)^\infty(u,v), \mu_2^N(u,v) \geq (\mu_2^N)^\infty(u,v), \gamma_2^P(u,v) \geq (\gamma_2^P)^\infty(u,v),$
 $\gamma_2^N(u,v) \geq (\gamma_2^N)^\infty(u,v), \beta_2^P(u,v) \geq (\beta_2^P)^\infty(u,v), \beta_2^N(u,v) \geq (\beta_2^N)^\infty(u,v)$
 whereas $(\mu_2^P)^\infty(u,v) = \max\{(\mu_2^P)^k(u,v) | k = 1, 2, \dots, n\};$
 $(\mu_2^N)^\infty(u,v) = \min\{(\mu_2^N)^k(u,v) | k = 1, 2, \dots, n\};$
 $(\gamma_2^P)^\infty(u,v) = \max\{(\gamma_2^P)^k(u,v) | k = 1, 2, \dots, n\};$
 $(\gamma_2^N)^\infty(u,v) = \min\{(\gamma_2^N)^k(u,v) | k = 1, 2, \dots, n\};$
 $(\beta_2^P)^\infty(u,v) = \max\{(\beta_2^P)^k(u,v) | k = 1, 2, \dots, n\};$
 $(\beta_2^N)^\infty(u,v) = \min\{(\beta_2^P)^k(u,v) | k = 1, 2, \dots, n\}.$

4 Edge domination in bipolar hesitancy fuzzy graph

Definition 4.1. Let $G = (V, E)$ be a Bipolar Hesitancy Fuzzy Graph. A set $S \subseteq E$ is said to be an edge dominating set of G if every edge not in S is incident to some edge in S .

Definition 4.2. An edge dominating set $S \subseteq E$ is said to be minimal if no proper subset of S is an edge dominating set.

Definition 4.3. The minimum cardinality out of all minimal dominating sets of BHFG G is said to be lower domination number of G and denoted as $d_{bh}(G)$.

Definition 4.4. The maximum cardinality out of all minimal dominating sets of BHFG G is said to be upper domination number of G and denoted as $D_{bh}(G)$.

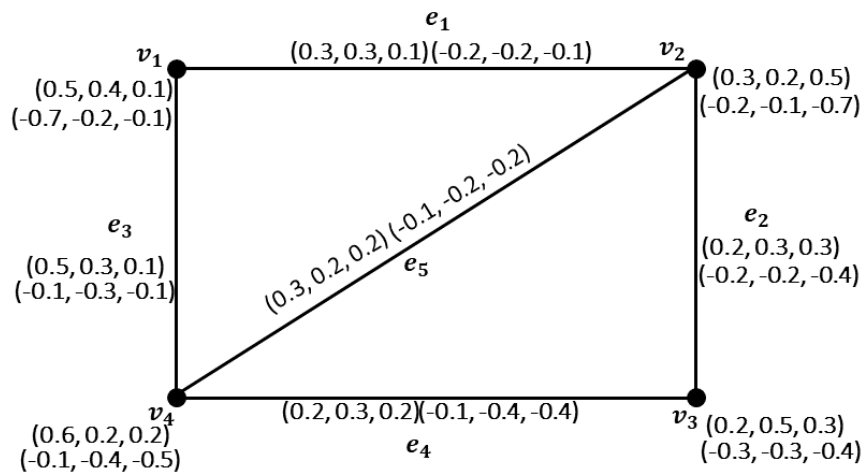


Figure 2: Edge domination in BHFG

Example 4.1. In the above figure 2, $\{e_1, e_2, e_4, e_5\}, \{e_2, e_3, e_5\}, \{e_1, e_3, e_4\}$ are edge dominating sets of G . $\{e_1, e_4\}, \{e_3, e_2\}, \{e_5\}$ are minimal edge dominating sets of G . Among all the minimal dominating sets, $\{e_5\}$ has minimum cardinality and edge domination number $\gamma_{bh}(G) = 1.06$.

Theorem 4.1. Let S be a minimal edge dominating set of a BHFG $G = (V, E)$. if for any edge $e \in S$, one of the following condition hold

- a) $N(e) \cap S = \phi$
- b) $\exists e' \in E - S$ such that $N(e) \cap S = \{e\}$.

Proof. Given $G = (V, E)$ is a BHFG and S is a minimal edge dominating set of G , Then for every edge $e \in S$, $S - \{e\}$ is not an edge dominating set and hence there exists an edge $e' \in E - S$ which is not adjacent to any element of $S - \{e\}$. Thus if $e' = e$ we get (a) and if $e' \neq e$ we get (b). $\square$

Definition 4.5. An edge e of a BHFG G is called an isolated edge if no effective edges is incident with the vertices of e and hence it doesn't dominate any other vertex in G .

Theorem 4.2. If $G = (V, E)$ is a BHFG without any isolated edges, then for every minimal edge dominating set S , prove that $E - S$ is also an edge dominating set.

Proof. Given $G = (V, E)$ a BHFG without any isolated edges. Let S be minimal edge dominating set of G , then there exists an edge $e' \in N((e))$. From theorem 5.4 we get $e' \in E - S$ which implies every edge in $E - S$ is adjacent to an edge in S . Hence $E - S$ is also an edge dominating set. $\square$

Corolary 4.1. For any graph G without isolated edges $\gamma_{bh}(G) \leq \frac{r}{3}$.

Definition 4.6. Let $G = (V, E)$ be a BHFG. Let S be a minimum edge set of G . If $E - S$ contains an edge dominating set S' of G , then S' is said to be inverse edge dominating set of G . The minimum cardinality out of all minmal inverse edge dominating sets is said to be inverse edge domination number and is denoted as $\gamma_{bh}^{-1}(G)$.

Proposition 4.1. For any graph G without isolated edges and vertices

$$\gamma_{bh}(G) \leq \gamma_{bh}^{-1}(G)$$

.

Proposition 4.2. If G is a graph without isolated edges and vertices and if number of vertices are greater than or equal to 3, then

$$\gamma_{bh}(G) + \gamma_{bh}^{-1}(G) \leq r$$

.

Definition 4.7. Let $G = (V, E)$ be BHFG without isolated edges. An edge dominating set S is called as total edge dominating set if $\langle S \rangle$ has no isolated edge. The minimum cardinality of all minimal total edge dominating sets is said to be total edge domination number of G and is denoted as γ_{tbh} .

A set $\mathcal{F} \subseteq E$ is said to be a total edge dominating set of G if for every edge in E is adjacent to at least one edge in $\mathcal{F}$.

Theorem 4.3. For any bipolar hesitancy fuzzy graph G , $\gamma_{bh}(G) \leq \gamma_{tbh}(G)$. $\square$

Theorem 4.4. For any bipolar fuzzy graph G with r edges then prove that $\gamma_{tbh} = r$ iff every edge of G has a unique neighbor.

Proof. Given a BHFG G with r edges. Let us consider every edge of G has a unique neighbor, then S is the only total edge dominating set of G which implies $\gamma_{tbh} = r$. Conversely, suppose $\gamma_{tbh} = r$ and if there exists an edge with neighbors s and t then $S - \{s\}$ gives a total edge dominating set of G . Thus $\gamma_{tbh} < r$ which is a contradiction. $\square$

5 Conclusions

We have established edge domination in Bipolar hesitancy fuzzy graph (BHFG). Along with various domination parameters such as inverse and total edge domination were also discussed. We have also given various examples and theorems supporting the main result. Our result can be extended to other domination parameters as well.

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D_4 -Magic Graphs

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Abstract

Consider the set $X = \{1, 2, 3, 4\}$ with 4 elements. A permutation of X is a function from X to itself that is both one one and on to. The permutations of X with the composition of functions as a binary operation is a nonabelian group, called the symmetric group S_4 . Now consider the collection of all permutations corresponding to the ways that two copies of a square with vertices 1, 2, 3 and 4 can be placed one covering the other with vertices on the top of vertices. This collection form a nonabelian subgroup of S_4 , called the dihedral group D_4 . In this paper, we introduce A -magic labelings of graphs, where A is a finite nonabelian group and investigate graphs that are D_4 -magic. This did not attract much attention in the literature.

Keywords: A -magic labeling; Dihedral group D_4 ; D_4 -magic.

2020 AMS subject classifications: 05C25, 05C78 .¹

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1 Introduction

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set whose elements are called vertices and $E(G)$ is a binary irreflexive and symmetric relation on $V(G)$ whose elements are called edges. For any abelian group A , written additively, any mapping $\ell : E(G) \rightarrow A \setminus \{0\}$ is called a labeling. Given a labeling on the edge set $E(G)$, one can introduce a vertex set labeling $\ell^+ : V(G) \rightarrow A$ as follows:

$$\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$$

A graph G is said to be A -magic if there is a labeling $\ell : E(G) \rightarrow A \setminus \{0\}$ such that for each vertex v , the sum of the labels of the edges incident with v are all equal to the same constant, that is, $\ell^+(v) = a$ for some fixed $a \in A$. The original concept of A -magic graph was introduced by Sedláček[1]. According to him, a graph G is A -magic if there exists an edge labeling on G such that (i) distinct edges have distinct non-negative labels; and (ii) the sum of the labels of the edges incident to a particular vertex is same for all vertices. When $A = \mathbb{Z}$, the $\mathbb{Z}$ -magic graphs are considered in Stanley[7]. Doob [5, 4] also considered A -magic graphs where A is an abelian group. Also he determined which wheels are $\mathbb{Z}$ -magic. Observe that several authors studied V_4 -magic graphs[8, 6]. It is natural to ask does there exist graphs which admits A -magic labeling, when A is nonabelian? In this paper, we address this question and investigate graphs that are D_4 -magic.

2 Main results

Let $G = (V(G), E(G))$ be a finite (p, q) graph and let $(A, *)$ be a finite non-abelian group with identity element 1. Let $f : E(G) \rightarrow N_q = \{1, 2, \dots, q\}$ and let $g : E(G) \rightarrow A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \rightarrow N_q \times A \setminus \{1\}$ by

$$\ell(e) := (f(e), g(e)), e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e'), g(e')) \text{ if and only if } f(e) \leq f(e').$$

Then obviously the relation $\leq$ is a partial order on the range of ℓ .

Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \dots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ . We define the product of elements of this chain as follows:

$$\prod_{i=1}^k (f(e_i), g(e_i)) := (((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \dots) * g(e_k).$$

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Let $u \in V$ and let $N^*(u)$ be the set of all edges incident with u . Note that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define,

$$\ell^*(u) = \prod_{i=1}^n (f(e_i), g(e_i)). \quad (1)$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A -magic. The map ℓ^* is called an A -magic labeling of G and the corresponding constant a is called the magic constant. For example, consider the cycle graph $C_4 = (uv, vw, wx, xu)$ and the permutation group D_4 . Note that the group D_4 is a non abelian group of order 8 and its elements are given by

$$\begin{aligned} \rho_0 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \mu_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \\ \rho_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, & \mu_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \\ \rho_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, & \delta_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \\ \rho_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, & \delta_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}. \end{aligned}$$

Define $f : E(G) \rightarrow N_4 = \{1, 2, 3, 4\}$ as $f(uv) = 1, f(wx) = 2, f(vw) = 3, f(xu) = 4$ and $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ as $g(uv) = g(wx) = \rho_1, g(vw) = g(xu) = \delta_1$. Thus

$$\ell^*(u) = (1, \rho_1)(4, \delta_1) = \rho_1 \delta_1 = \mu_2,$$

$\ell^*(v) = (1, \rho_1)(3, \delta_1) = \rho_1 \delta_1 = \mu_2$. Similarly, $\ell^*(w) = \mu_2$ and $\ell^*(x) = \mu_2$. Thus C_4 is D_4 -magic with magic constant μ_2 .

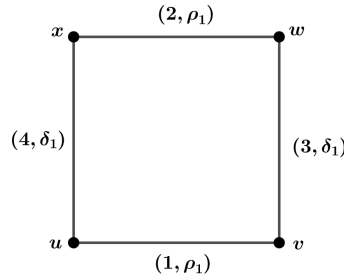


Figure 1: D_4 -magic labeling of C_4 .

In this paper, we will consider the symmetric group D_4 and investigate graphs that are D_4 -magic.

Theorem 2.1. *Let A be a non abelian group having an element of order 2 and let G be a graph. If either the degree of the vertices of G are all even or odd. Then G is A -magic.*

Proof. Let G be a (p, q) graph and A be a nonabelian group having an element of order 2. Let $a \in A$ is of order 2. Let $g : E(G) \rightarrow A \setminus \{1\}$ be the constant map $g(e) = a, \forall e \in E(G)$ and let f be any bijection from $E(G) \rightarrow N_q$. First assume that all the vertices of G are of even degree then $l^*(u) = 1, \forall u \in V(G)$. Similarly, if all the vertices of G are of odd degree then $l^*(u) = a, \forall u \in V(G)$. Hence the proof.

Corolary 2.1. *All Eulerian graphs are D_4 -magic.*

Theorem 2.2. *Any regular graph is D_4 -magic.*

Proof. Let $G = (V(G), E(G))$ be a regular graph with $|E(G)| = q$. Let $f : E(G) \rightarrow N_q$ be any bijection and g be any constant map from $E(G) \rightarrow D_4 \setminus \{\rho_0\}$. Obviously, f and g will determine a D_4 -magic labeling of G . This completes the proof of the theorem.

Corolary 2.2. *For any $n \geq 3$, the cycle graph C_n is D_4 -magic.*

Corolary 2.3. *For any $n \geq 2$, the complete graph K_n is D_4 -magic.*

Corolary 2.4. *The Peterson graph is D_4 -magic.*

Theorem 2.3. *The star graph $K_{1,n}, n \geq 2$ is D_4 -magic iff n is odd.*

Proof. Let $G = K_{1,n}$. Suppose that n is odd. Let $f : E(G) \rightarrow N_{n+1}$ be a bijection. Define $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$. Then clearly it is D_4 -magic with magic constant μ_1 .

Conversely, suppose $K_{1,n}$ is D_4 -magic with magic constant, say 'a'. So every pendent edge of $K_{1,n}$ should be mapped to a under g . Let u be the vertex of $K_{1,n}$ with degree n . Then

$$\ell^*(u) = \underbrace{aa \cdots a}_{n \text{ times}} = a.$$

This implies that $a^{n-1} = \rho_0$. If n is odd, the equation $a^{n-1} = \rho_0$ has five non trivial solutions in D_4 viz. $\mu_1, \mu_2, \delta_1, \delta_2$ and ρ_2 . On the other hand, if n is even there are no element in D_4 such that $a^{n-1} = \rho_0$. This completes the proof.

A bistar graph B_n is the graph obtained by connecting the apex vertices of two copies of star $K_{1,n}$ by a bridge.

Theorem 2.4. *The bistar graph $B_n, n > 1$ is D_4 -magic when $n \not\equiv 1 \pmod{4}$.*

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Proof. First, observe that there are $2n$ pendant edges and one bridge in B_n . Here we consider the following cases:

Case (i): n is even ($n \equiv 2 \pmod{4}$ or $n \equiv 0 \pmod{4}$).

If n is even, define $g : E(B_n) \rightarrow D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(B_n)$. Let f be any bijective map from $E(B_n) \rightarrow N_{2n+1}$. Then obviously, B_n is D_4 -magic with magic constant μ_1 .

Case (ii): $n \equiv 3 \pmod{4}$.

In this case we define $g : E(B_n) \rightarrow D_4 \setminus \{\rho_0\}$ by

$$g(e) = \begin{cases} \rho_1, & \text{if } e \text{ is a pendant edge,} \\ \rho_2, & \text{if } e \text{ is the bridge.} \end{cases}$$

Let f be any bijective map from $E(B_n)$ to N_{2n+1} . Then obviously B_n is D_4 -magic with the magic constant ρ_1 .

Case (iii): $n \equiv 1 \pmod{4}$.

Suppose that $n \equiv 1 \pmod{4}$. Let k_1 and k_2 be the apex vertices of the bistar graph. Assume that B_n is D_4 -magic with magic constant μ_1 . Therefore, $g(e) = \mu_1$ for all pendant edges e . Assume that $g(k_1 k_2) = a$, where $a \in D_4 \setminus \{\rho_0\}$. Without loss of generality assume that $f(k_1 k_2) > f(b)$, $\forall b \in E(G)$, where b denotes the pendant edge with one end point k_1 . Then

$$\ell^*(k_1) = \underbrace{\mu_1 \mu_1 \dots \mu_1}_{(n \text{ times})} a = \mu_1.$$

The above equation tells us that $a = \rho_0$, which is a contradiction. This contradiction shows that B_n is not D_4 -magic with magic constant μ_1 . In a similar manner, we can prove that B_n is not D_4 -magic with magic constants μ_2 , ρ_1 , ρ_2 , ρ_3 , δ_1 or δ_2 . Thus the bistar graph B_n is not D_4 -magic when $n \equiv 1 \pmod{4}$. This completes the proof of the theorem.

Theorem 2.5. *The complete bipartite graph $K_{m,n}$ is D_4 -magic, $m, n > 1$.*

Proof. Let $G = K_{m,n}$. Suppose $U = \{u_1, u_2, \dots, u_n\}$, and $V = \{v_1, v_2, \dots, v_m\}$ be the two partite sets of $K_{m,n}$. If m and n are both even or odd then the theorem is obvious by taking any constant map $g : E(G) \rightarrow \{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$.

Case (i): $n \equiv 0 \pmod{2}$ and $m \equiv 1 \pmod{4}$.

Let $U = \{u_1, u_2, \dots, u_{2l}\}$ and $V = \{v_1, v_2, \dots, v_{4r+1}\}$ where $n = 2l$, $m = 4r + 1$, and $l, r \in \mathbb{N}$. For $1 \leq i \leq n$ and $1 \leq j \leq m$ define

$$g(u_i v_{5k+1}) = \mu_1, \text{ where } k < m, k = 0, 1, 2, 3, \dots$$

$$g(u_i v_{5k+2}) = \mu_2, k < m, k = 0, 1, 2, 3, \dots$$

$$g(u_i v_j) = \rho_2, j \neq 5k + 1, 5k + 2 \text{ where } k = 0, 1, 2, \dots$$

$$f(u_i v_j) = (i - 1)m + j, 1 \leq i \leq n, 1 \leq j \leq m.$$

The maps f and g will determine a D_4 -magic labeling for $K_{m,n}$ with magic constant ρ_0 .

Case (ii): $n \equiv 0 \pmod{2}$ and $m \equiv 3 \pmod{4}$. Define g as follows:

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd } 1 \leq j < m, \\ \rho_3, & \text{if } i \text{ is even } 1 \leq j < m, \end{cases} \quad \text{and}$$

$$g(u_i v_m) = \rho_2, \quad \forall i, 1 \leq i \leq n,$$

and let f be any bijection from $E(G)$ to $\{1, 2, \dots, mn\}$. Then clearly f and g will determine a D_4 -magic labeling of $K_{m,n}$ with magic constant ρ_0 .

This completes the proof of the theorem.

3 Cycle Generated Graphs

In this section, we consider certain graphs which are constructed from cycles.

A wheel W_n of order $n + 1$, sometimes simply called an n wheel is a graph that contains a cycle of order n and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph join $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph.

Theorem 3.1. *If $n \geq 3$, the wheel W_n is D_4 -magic.*

Proof. Let the vertices of C_n be $u_1, u_2, \dots, u_n$ such that $u_i u_{i+1} \in E(C_n)$, $i = 1, 2, \dots, n$ and $u_{n+1} = u_1$. Denote the vertex of K_1 by k . Now we consider the following cases:

Case (i): n is odd.

If n is odd then every vertex of W_n is of odd degree. Thus we can take $g : E(W_n) \rightarrow D_4 \setminus \{\rho_0\}$ as any constant map from $E(W_n)$ to $\{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$. Since g is constant we can take f as any bijection from $E(W_n)$ to N_{2n} . Clearly this f and g will constitute a D_4 -magic labeling for W_n .

Case (ii): n is even.

Suppose n is even define $f : E(W_n) \rightarrow N_{2n}$ as

$$\begin{aligned} f(ku_i) &= i, \quad i = 1, 2, \dots, n, \\ f(u_i u_{i+1}) &= n + i, \quad 1 \leq i \leq n - 1, \\ f(u_1 u_n) &= 2n. \end{aligned}$$

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Now we can define $g : E(W_n) \rightarrow D_4 \setminus \{\rho_0\}$ by labeling each spokes by μ_1 and all the outer edges by μ_2 and ρ_2 alternatively. Then W_n becomes D_4 -magic with magic constant ρ_0 .

This completes the proof of the theorem.

The helm H_n is a graph obtained from a wheel W_n by attaching a pendant edge at each vertex of the n cycle.

Theorem 3.2. *The Helm graph H_n is D_4 -magic.*

Proof. Let $\{k, u_i, v_i : i = 1, 2, \dots, n\}$ be the vertex set of H_n , where k be the central vertex, $u_1, u_2, \dots, u_n$ are the vertices of the cycle, $v_1, v_2, \dots, v_n$ are the pendant vertices adjacent to $u_1, u_2, \dots, u_n$. The edge set of H_n is $E(H_n) = \{u_i u_{i+1}, k u_i, u_i v_i : i = 1, 2, \dots, n, u_{n+1} = u_1\}$. Now consider the following two cases:

Case(i): n is odd.

Suppose that n is odd. Define f and g as follows: Let $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ be defined as $g(k u_i) = \rho_2$, $1 \leq i \leq n$, $g(u_j u_{j+1}) = \rho_1$, $1 \leq j \leq n - 1$, $g(u_1 u_n) = \rho_1$, $g(u_k v_k) = \rho_2$, $1 \leq k \leq n$. Now let $f : E(G) \rightarrow N_{2n+1}$ be any bijection. Then clearly f and g will give a D_4 -magic labeling of H_n , where n is odd.

Case(ii): n is even.

Let f be defined as above and define $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ by

$$\begin{aligned} g(u_i v_i) &= \rho_2, \quad 1 \leq i \leq n, \quad g(v_1 v_n) = \rho_1 \\ g(k u_j) &= \begin{cases} \rho_2, & \text{if } 1 \leq j \leq n - 2, \\ \rho_1, & \text{if } j = n - 1, n. \end{cases} \\ g(u_k u_{k+1}) &= \begin{cases} \rho_1, & \text{if } 1 \leq k \leq n - 2, \\ \rho_2, & \text{if } k = n - 1. \end{cases} \end{aligned}$$

It follows that $l^*(u) = \rho_2, \forall u \in V(G)$. Hence H_n is D_4 -magic when n is even.

This completes the proof of the theorem.

The web graph $W(2, n)$ is a graph obtained joining the pendant points of a helm to form a cycle and adding a single pendant edge to each vertex of this outer graph.

Theorem 3.3. *The web graph $W(2, n), n \geq 3$ is D_4 -magic.*

Proof. Let $\{k, u_i, v_i, w_i : i = 1, 2, 3, \dots, n\}$ be the vertex set of $W(2, n)$, where k be the central vertex, $u_1, u_2, u_3, \dots, u_n$ are the vertices of inner cycle, $v_1, v_2, v_3, \dots, v_n$ are the vertices of outer cycle and $w_1, w_2, w_3, \dots, w_n$ are the pendant vertices adjacent to $v_1, v_2, v_3, \dots, v_n$ of $W(2, n)$. Let $E(W(2, n)) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i, v_i w_i : i = 1, 2, \dots, n \text{ and } u_{n+1} = u_1, v_{n+1} = v_1\}$. We define a D_4 -magic labeling for $W(2, n)$ with magic constant ρ_2 as follows:

Case (i): n is odd.

Let $f : E(G) \rightarrow N_{3n+1}$ be any bijection.

Define $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(ku_i) &= \rho_2 = g(u_i v_i) = g(v_i w_i), \quad 1 \leq i \leq n, \\ g(u_i u_{i+1}) &= \rho_1 = g(v_i v_{i+1}), \quad 1 \leq i \leq n-1, \\ g(u_1 u_n) &= \rho_1 = g(v_1 v_n). \end{aligned}$$

Case (ii): n is even.

Let $f : E(G) \rightarrow N_{3n+1}$ be any bijection.

Define $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(ku_i) &= \rho_2, \quad 1 \leq i < n-1, \quad g(ku_n) = g(ku_{n-1}) = \rho_1, \quad g(v_i v_{i+1}) = \\ \rho_1 &= g(u_i u_{i+1}) = \rho_1, \quad 1 \leq i \leq n-1, \quad g(v_i w_i) = \rho_2 = g(u_i v_i), \quad 1 \leq i \leq \\ n, & \quad g(v_1 v_n) = \rho_1, \quad g(u_1 u_n) = \rho_2. \end{aligned}$$

This completes the proof of the theorem.

A shell graph $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices. The vertex at which all chords are concurrent is called the apex. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all other vertices have degree 3.

Theorem 3.4. *Shell graphs $S_{n,n-3}$ are D_4 -magic.*

Proof. Let us denote the vertices of the shell graph $S_{n,n-3}$ by $u_1, u_2, \dots, u_n$ such that u_i is adjacent to u_{i+1} , where $i = 1, 2, \dots, n$ and $u_{n+1} = u_1$. Without loss of generality let the apex be u_1 . Now consider the following cases:

Case (i): n is even.

We will define the map $f : E(S_{n,n-3}) \rightarrow N_{2n-3}$ as

$$\begin{aligned} f(u_i u_{i+1}) &= i, \quad 1 \leq i \leq n-1, \\ f(u_n u_1) &= n, \\ f(u_1 u_j) &= n + (j-2), \quad 3 \leq j \leq n-1 \end{aligned}$$

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and we define $g : E(S_{n,n-3}) \rightarrow D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(u_1u_2) &= g(u_nu_1) = \rho_2, \\ g(u_1u_i) &= \mu_1, 3 \leq i \leq n-1, \\ g(u_iu_{i+1}) &= \mu_2, 2 \leq i \leq n-1. \end{aligned}$$

Clearly f and g define a D_4 -magic labeling with magic constant μ_1 .

Case (ii): n is odd.

Define f as

$$\begin{aligned} f(u_iu_{i+1}) &= i, 1 \leq i \leq n-1, \\ f(u_1u_n) &= n, \\ f(u_1u_j) &= n + (j-2), 3 \leq j \leq n-1 \end{aligned}$$

and define g as

$$\begin{aligned} g(u_1u_2) &= g(u_1u_n) = \rho_2, \\ g(u_1u_j) &= \mu_1, 3 \leq j \leq n-1, \\ g(u_iu_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \mu_2, & \text{if } i \text{ is odd, } 1 < i \leq n-1. \end{cases} \end{aligned}$$

Obviously the functions f and g define a D_4 -magic labeling of $S_{n,n-3}$ with magic constant ρ_0 .

This completes the proof of the theorem.

When k copies of C_n share a common edge it will form the n -gon book of k pages and is denoted by $B(n, k)$.

Theorem 3.5. *The graph n -gon book of k pages $B(n, k)$ is D_4 -magic.*

Proof. Let G be the graph $B(n, k)$. Denote the vertices of common edge by k_1 and k_n and the edges of i^{th} page other than k_1 and k_n by $u_{i2}, u_{i3}, \dots, u_{in-1}$ such that u_{i2} is adjacent to k_1 and u_{in-1} adjacent to k_n and u_{ij} adjacent to u_{ij+1} for all $2 \leq j < n-1$. Consider the following cases:

Case (i): k is even.

Define $g : E(G) \rightarrow D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(k_1k_n) &= \rho_2, \\ g(u_{1j}u_{1j+1}) &= \mu_1, 2 \leq j \leq n-2, \\ g(u_{1n-1}k_n) &= \mu_1 = g(k_1u_{12}), \\ g(u_{ij}u_{ij+1}) &= \mu_2, 2 \leq i \leq k, 2 \leq j \leq n-1, \\ g(k_1u_{l2}) &= g(u_{ln-1}) = \mu_2, 2 \leq l \leq k. \end{aligned}$$

Now define f as

$$\begin{aligned} f(k_1 k_n) &= 1, f(k_1 u_{12}) = 2, f(u_{1n-1} k_n) = n, \\ f(u_{1j} u_{1j+1}) &= j + 1, \forall 2 \leq j \leq n - 2, \\ f(k_1 u_{i2}) &= n + (i - 2)(n - 1) + 1, i \geq 2, \\ f(u_{ij} u_{ij+1}) &= n + (i - 2)(n - 1) + j, 2 \leq j \leq n - 2, 2 \leq i \leq k, \\ f(u_{i(n-1)} k_n) &= n + (i - 2)(n - 1) + (n - 1), 2 \leq i \leq k. \end{aligned}$$

The functions f and g determine a D_4 -magic labeling with magic constant ρ_0 .

Case (ii): k is odd.

Here define g as $g(e) = \rho_2, \forall e \in E(G)$ then g together with any bijection $f : E(G) \rightarrow N_{kn-1}$ will define a D_4 -magic labeling of $B(n, k)$ with magic constant ρ_0 .

This completes the proof of the theorem.

Note that, for any $n \geq 3$ the path graph of order n is not D_4 -magic.

4 Path Generated Graphs

In this section we will consider some graphs which are constructed from Paths. We start with the Splitting graph of Path.

A splitting graph $S(G)$ of a graph G is the graph obtained from G by adding to G a new vertex z' for each vertex z of G and joining z' to the neighbors of z in G .

Theorem 4.1. *Splitting graph of the path graph $P_n, n \geq 3$ is D_4 -magic.*

Proof. Let P_n be a path graph of order n , where $n \geq 3$. Let $u_1, u_2, \dots, u_n$ be the vertices of P_n , where $u_i u_{i+1} \in E(P_n), i = 1, 2, \dots, n - 1$. There are $2n$ vertices and $3n - 3$ edges in $S(P_n)$. Let u_{n+i} be the vertex corresponding to the i^{th} vertex in $S(P_n)$. Observe that there are two pendant edges in $S(P_n)$, one with end points u_2 and u_{n+1} and the other with end points u_{n-1} and u_{2n} .

Case (i): $n = 3$.

In this case, define $f : E(S(P_3)) \rightarrow N_6$ as

$$\begin{aligned} f(u_1 u_2) &= 1, f(u_2 u_3) = 3, f(u_3 u_5) = 2, f(u_1 u_5) = 4, f(u_2 u_4) = 5, \\ f(u_2 u_6) &= 6. \text{ Now define } g : E(G) \rightarrow D_4 \setminus \{\rho_0\} \text{ as} \\ g(u_1 u_2) &= g(u_3 u_5) = \rho_1, g(u_2 u_3) = g(u_1 u_5) = \delta_2, g(u_2 u_4) = g(u_2 u_6) = \mu_1. \end{aligned}$$

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Case (ii): $n > 3$.

In this case, define f and g as follows:

$$\begin{aligned} f(u_i u_{i+1}) &= i, \quad 1 \leq i \leq n-1, \\ f(u_i u_{n+(i-1)}) &= n + (i-2), \quad 2 \leq i \leq n, \\ f(u_i u_{n+(i+1)}) &= (2n-2) + i, \quad 1 \leq i \leq n-1 \text{ and} \\ g(u_1 u_2) &= \rho_2, \quad g(u_{n-1} u_n) = \mu_2, \\ g(u_2 u_{n+1}) &= g(u_{n-1} u_{2n}) = \mu_1, \\ g(u_i u_{i+1}) &= \mu_1, \quad 2 \leq i < n-1, \\ g(u_i u_{n+(i-1)}) &= \rho_2, \quad 3 \leq i \leq n, \\ g(u_i u_{n+(i+1)}) &= \mu_2, \quad 1 \leq i \leq n-2. \end{aligned}$$

In all the above cases, we can prove that the functions f and g defines a D_4 -magic labeling of $S(P_n)$ with magic constant μ_1 .

This completes the proof of the theorem.

The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if

- (i) They are adjacent edges of G or
- (ii) One is a vertex of G and the other is an edge incident with it.

Theorem 4.2. *Middle graph of the path graph P_n is D_4 -magic for $n \geq 3$.*

Proof. Let $M(P_n)$ be the middle graph of the path P_n . Denote the vertices of P_n by $u_1, u_2, \dots, u_n$ and edges by $e_1, e_2, \dots, e_{n-1}$ where e_i incident with u_i and u_{i+1} . There are $2n-1$ vertices and $3n-4$ edges in $M(P_n)$. Consider the following cases:

Case(i): $n = 3$.

Define $f : E(M(P_3)) \rightarrow N_{3n-4}$ as $f(e_1 u_1) = 1, f(e_1 u_2) = 2, f(e_1 e_2) = 3, f(e_2 u_2) = 4$ and $f(e_2 u_3) = 5$ and define $g : E(M(P_3)) \rightarrow D_4 \setminus \{\rho_0\}$ as $g(e_1 u_1) = \rho_2 = g(e_2 u_3), g(e_1 u_2) = \rho_1 = g(e_2 u_2), g(e_1 e_2) = \rho_3$. Then clearly the middle graph of the path P_3 is D_4 -magic with magic constant ρ_2 .

Case(ii): $n > 3$.

Define $f : E(M(P_n)) \rightarrow N_{3n-4}$ as follows:

For $1 \leq i \leq n-2, 1 \leq j \leq n, f(e_i u_j) = 2(i-1) + j$ and $f(e_i e_{i+1}) = 3i$.

Now define $g : E(M(P_n)) \rightarrow D_4 \setminus \{\rho_0\}$ by

$$\begin{aligned} g(e_1u_1) &= \rho_2 = g(e_{n-1}u_n), \\ g(e_1u_2) &= \mu_2, \quad g(e_2u_2) = \mu_1, \quad g(e_2e_3) = \rho_3 \\ g(e_2u_3) &= \rho_1 = g(e_3u_3), \\ g(e_ie_{i+1}) &= \mu_2, \text{ where } i \neq 2 \text{ and } 1 \leq i \leq n-1, \\ g(e_iu_j) &= \begin{cases} \mu_1, & \text{if } j = i+1 \text{ and } 3 \leq i < n-1, \\ \mu_2, & \text{if } i = j \text{ and } 4 \leq i \leq n-1. \end{cases} \end{aligned}$$

The above functions f and g will define a D_4 -magic labeling of $M(P_n)$ with magic constant ρ_2 .

This completes the proof of the theorem.

A triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3 .

Theorem 4.3. *The Triangular snake T_n is D_4 -magic.*

Proof. Note that every vertex of T_n has even degree. So the proof is indisputable from Theorem 2.1.

The alternate triangular snake $A(T_n)$ is obtained from the path $u_1, u_2, \dots, u_n$ by joining u_iu_{i+1} (alternatively) to a new vertex v_i .

Theorem 4.4. *The alternate triangular graph $A(T_n)$ is D_4 -magic.*

Proof. Let us denote the vertices of the path P_n be $u_1, u_2, \dots, u_n$ and the vertex which join u_i and u_{i+1} be denoted by v_i . Now consider the following cases:

Case (i): n is even and triangle starts from u_1 .

Suppose that n is even and the triangle starts from the first vertex u_i , then there are $n + \frac{n}{2}$ vertices and $2n - 1$ edges.

Suppose $n = 2$ then $A(T_n)$ is C_3 itself. So there is nothing to prove.

Suppose $n = 4$ then take f be any bijection from $E(A(T_n))$ to N_7 and define $g : E(A(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ by

$$\begin{aligned} g(u_1v_1) &= g(u_4v_3) = \rho_1, \quad g(u_2u_3) = \rho_2, \quad g(u_2v_1) = g(u_3v_3) = g(u_1u_2) = \\ g(u_3u_4) &= \rho_3. \end{aligned}$$

Then $A(T_4)$ becomes D_4 -magic with magic constant ρ_0 .

Suppose $n > 4$, then let $f : E(A(T_n)) \rightarrow N_{2n-1}$ be any bijection and define

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$g : E(A(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(u_1u_2) &= g(u_{n-1}u_n) = g(u_2v_1) = g(u_{n-1}v_{n-1}) = \rho_3, \\ g(u_1v_1) &= g(u_{n-1}v_{n-1}) = \rho_1. \text{ For } 2 \leq i < n-1, \text{ define} \\ g(u_iu_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \mu_2, & \text{if } i \text{ is odd.} \end{cases} \\ g(u_{2k+1}v_{2k+1}) &= g(u_{2k+2}v_{2k+1}) = \mu_1, k = 1, 2, \dots, \frac{n-4}{2} \end{aligned}$$

Obviously the functions f and g will constitute a D_4 -magic labeling for $A(T_n)$ with $l^*(u) = \rho_0, \forall u \in V(A(T_n))$.

Case (ii): n is even and the triangle starts from the second vertex u_2 .

We can define a magic labeling for $A(T_n)$, where n is even and cycle starts from u_2 as follows:

Let f be any bijection as above and define g as

$$\begin{aligned} g(u_iu_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_3, & \text{if } i \text{ is even,} \end{cases} \quad 1 \leq i \leq n-1, \\ g(u_{2k}v_{2k}) &= g(u_{2k+1}v_{2k}) = \rho_1, \quad k = 1, 2, 3, \dots, \frac{n-2}{2}. \end{aligned}$$

Clearly l^* is a constant map, i.e., $l^*(u) = \rho_2, \forall u \in V(A(T_n))$.

Case (iii): n is odd and the triangle starts from the first vertex.

Suppose $n = 3$ and the triangle starts from the first vertex u_1 .

Let $f : E(A(T_n)) \rightarrow N_4$ be any bijection.

Now define $g : E(A(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ by

$g(u_1u_2) = g(u_2v_1) = \delta_2, g(u_1v_1) = \delta_1, g(u_2u_3) = \rho_2$. Using these maps we can show that the graph is D_4 -magic with magic constant ρ_2 .

When n is odd and $n > 3$, there are $n + \frac{(n-1)}{2}$ vertices and $2(n-1)$ edges in $A(T_n)$. Suppose that $n > 3$, n is odd and the triangle of $A(T_n)$ starts from the first vertex u_1 . Here we take f as any bijection and $g : E(A(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ be defined as follows:

$$\begin{aligned} g(u_1u_2) &= g(u_2v_1) = \delta_2, \quad g(u_1v_1) = \delta_1, \\ g(u_iu_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \rho_3, & \text{if } i \text{ is odd,} \end{cases} \quad 1 < i < n. \\ g(u_iv_i) &= g(u_{i+1}v_i) = \rho_1, \quad 1 < i < n \text{ and } i \text{ is odd.} \end{aligned}$$

Then clearly $l^*(u) = \rho_2, \forall v \in V(A(T_n))$

Case (iv): n is odd and triangle starts from the second vertex.

$A(T_n)$ with n odd and the triangle starts from the first vertex is just the mirror image of the $A(T_n)$ in Case (iii). So we can define f and g similarly as in Case(iii) and obtain a D_4 -magic labeling for $A(T_n)$ with magic constant ρ_2 .

This completes the proof of the theorem.

A double triangular snake $D(T_n)$ consists of two triangular snakes that have a common path P_n .

Theorem 4.5. *The double triangular graph is D_4 -magic.*

Proof. Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n and let $v_1, v_2, \dots, v_{n-1}, w_1, w_2, \dots, w_{n-1}$ be the remaining vertices of $D(T_n)$ such that the vertex v_i is adjacent to u_i and u_{i+1} , where $1 \leq i < n$. Similarly the vertex w_i is adjacent to u_i and u_{i+1} . Without loss of generality let $v_1, v_2, \dots, v_{n-1}$ and $w_1, w_2, \dots, w_{n-1}$ be the vertices of upper triangles and lower triangles respectively.

Now we define a D_4 -magic labeling for $D(T_n)$ as follows:

Let $f : E(D(T_n)) \rightarrow N_{5(n-1)}$ be any bijection and let $g : E(D(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ be defined by $g(u_i u_{i+1}) = \rho_2$, $g(u_i v_i) = g(u_i w_i) = \rho_1$, and $g(u_{i+1} v_i) = g(u_{i+1} w_i) = \rho_3$ where $1 \leq i < n$. Thus we can see that $l^*(u) = \rho_0$, $\forall u \in V(D(T_n))$. This completes the proof.

5 Conclusions

In this paper, we introduced the concept of A -magic labeling of graphs, where A is a nonabelian group. Furthermore, we characterised graphs which are D_4 magic.

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Vague Positive Implicative and Associative W - Implicative Ideals of Lattice Wajsberg Algebras

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Abstract

In the present paper, we introduce the notions of vague positive implicative, and vague associative W -implicative ideals of lattice Wajsberg algebra. Further, we investigate some relevant properties. Moreover, we obtain some relationship between the vague associative W -implicative ideal, and the vague W -implicative ideal.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; W -implicative ideal; Vague set; Vague W -implicative ideal; Vague positive implicative W -implicative ideal; Vague associative W -implicative ideal.

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1. Introduction

The concept of Wajsberg algebra was first proposed by Mordchaj Wajsberg[10] in 1935, and analysed by Font, Rodriguez and Torrens[4] in 1984. Zadeh introduced the notion of fuzzy set in 1965. Fuzzy logic has been applied to many fields, from control theory to artificial intelligence. In 1993, the idea of vague set was introduced by Gau and Buehrer[5]. Vague set as an extension of fuzzy sets, the idea of vague set is that the membership of every element can be divided into two aspects including supporting and opposing. It is the new extension not only provides a significant addition to existing theories for handling uncertainties, but it leads to potential areas of further field research and pertinent applications. The authors [9], introduced the notions of vague W -implicative ideals, vague implicative W -implicative ideals of lattice Wajsberg algebra, and investigated some properties.

In this paper, we introduce the definitions of the vague positive implicative W -implicative ideal, and the vague associative W -implicative ideal of lattice Wajsberg algebra. Also, we discuss the relationship between the vague positive implicative W -implicative ideal, and the vague W -implicative ideal.

2. Preliminaries

In this section, we recall some basic definitions and results that are helpful in developing our main results.

Definition 2.1[4] Let $(\mathbb{W}, \rightarrow, *, 1)$ be an algebra with a binary operation “ $\rightarrow$ ” and a quasi complement “ $*$ ”. Then it is called Wajsberg algebra, if the following axioms are satisfied for all $x, y, z \in \mathbb{W}$,

- i $1 \rightarrow x = x$
- ii $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$
- iii $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
- iv $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$.

Proposition 2.2[4] A Wajsberg algebra $(\mathbb{W}, \rightarrow, *, 1)$ is satisfied the following axioms for all $x, y, z \in \mathbb{W}$,

- i $x \rightarrow x = 1$
- ii If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$
- iii $x \rightarrow 1 = 1$
- iv $(x \rightarrow (y \rightarrow x)) = 1$
- v If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$
- vi $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$
- vii $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
- viii $x \rightarrow 0 = x \rightarrow 1^* = x^*$
- ix $(x^*)^* = x$

$$x \quad (x^* \rightarrow y^*) = y \rightarrow x.$$

Definition 2.3[4] A Wajsberg algebra $(\mathbb{W}, \rightarrow, *, 1)$ is called a lattice Wajsberg algebra, if the following conditions are satisfied for all $x, y \in \mathbb{W}$,

- i The partial ordering " $\leq$ " on a Wajsberg algebra $\mathbb{W}$, such that $x \leq y$ if and only if $x \rightarrow y = 1$
- ii $(x \vee y) = (x \rightarrow y) \rightarrow y$
- iii $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$.

Thus, $(\mathbb{W}, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[4] Let $(\mathbb{W}, \rightarrow, *, 1)$ be lattice Wajsberg algebra. Then the following axioms hold for all $x, y, z \in \mathbb{W}$,

- i If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$
- ii $x \leq y \rightarrow z$ if and only if $y \leq x \rightarrow z$
- iii $(x \vee y)^* = (x^* \wedge y^*)$
- iv $(x \wedge y)^* = (x^* \vee y^*)$
- v $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
- vi $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
- vii $(x \rightarrow y) \vee (y \rightarrow x) = 1$
- viii $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$
- ix $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$
- x $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$
- xi $(x \wedge y) \rightarrow z = (x \rightarrow y) \wedge (x \rightarrow z)$.

Definition 2.5[6] Let $(\mathbb{W}, \rightarrow, *, 1)$ be a lattice Wajsberg algebra. Then it is called lattice H -Wajsberg algebra, if it satisfied $(x \vee y) \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in \mathbb{W}$.

Note. In a lattice H -Wajsberg algebra $\mathbb{W}$, the following hold:

- i $x \rightarrow (x \rightarrow y) = (x \rightarrow y)$
- ii $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$ for all $x, y, z \in \mathbb{W}$.

Definition 2.6[6] Let L be a lattice. An ideal I of L is a non-empty subset of L is called a lattice ideal, if the following axioms are satisfied for all $x, y \in I$,

- i $x \in I$ and $y \leq x$ imply $y \in I$

- ii $x, y \in I$ implies $x \vee y \in I$.

Definition 2.7[6] Let $(\mathbb{W}, \rightarrow, *, 1)$ be a lattice Wajsberg algebra. Let I be a non-empty subset of $\mathbb{W}$. Then I is called a W -implicative ideal of $\mathbb{W}$, if the following axioms are satisfied for all $x, y \in \mathbb{W}$,

- i $0 \in I$
 ii $(x \rightarrow y)^* \in I$ and $y \in I$ imply $x \in I$.

Definition 2.8[5] A vague set A in the universal of discourse $\mathbb{W}$ is characterized by two membership functions given by:

- i A truth membership function $t_A: X \rightarrow [0,1]$ and
 ii A false membership function $f_A: X \rightarrow [0,1]$.

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the “evidence for x ”, and $f_A(x)$ is a lower bound on the negation of x derived from the “evidence against x ” and $t_A(x) + f_A(x) \leq 1$. Thus the grade of membership of x in the vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. The vague set A is written as $A = \{\langle x, [t_A(x), f_A(x)] \rangle / x \in \mathbb{W}\}$.

Where the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in the vague set A and denoted by $V_A(x)$.

Definition 2.9[5] A vague set A of a universe X with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in \mathbb{W}$, is called the zero vague set of $\mathbb{W}$.

Definition 2.10[5] A vague set A of a universe X with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in \mathbb{W}$ is called the one vague set of $\mathbb{W}$.

Definition 2.11[5] Let A be a vague set of a universe X with the truth membership function t_A and the false membership function f_A . For any $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α, β) – cut of a vague set A is a crisp subset $A_{(\alpha, \beta)}$ of the set X given by $A_{(\alpha, \beta)} = \{x \in \mathbb{W} / V_A(x) \geq [\alpha, \beta]\}$.

Definition 2.12[5] The α -cut, A_α of the vague set is the (α, α) -cut of A and hence given by $A_\alpha = \{x \in \mathbb{W} / t_A(x) \geq \alpha\}$.

Definition 2.13[4] Let $I = [0,1]$ denote the family of all closed subintervals of $[0,1]$. If $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$ are two elements of $I[0,1]$, we call $I_1 \geq I_2$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. We define the term $rmax$ to mean the maximum of two intervals as $rmax[I_1, I_2] = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$.

Similarly, we can define the term $rmin$ of any two intervals.

Definition 2.14[5] The intersection of two vague sets A and B with respective truth membership functions and the false membership functions t_A, t_B, f_A and f_B is a vague set $C = A \cap B$, whose truth membership function and false membership functions are related to those of A and B by

$$t_C = \min\{t_A, t_B\}, 1 - f_C = \min\{1 - f_A, 1 - f_B\} = 1 - \max\{f_A, f_B\}.$$

Definition 2.15[9] Let A be a vague set of lattice Wajsberg algebra $\mathbb{W}$. Then A is called a vague WI -ideal of $\mathbb{W}$, if the following axioms are satisfied for all $x, y \in \mathbb{W}$,

- i $V_A(0) \geq V_A(x)$,
- ii $V_A(x) \geq rmin\{V_A(x \rightarrow y)^*, V_A(y)\}$.

3.1. Vague positive implicative W-implicative ideal

In this section, we introduce the definition of vague positive implicative W-implicative ideal of lattice Wajsberg algebra, and investigate some related properties.

Definition 3.1.1 A vague set A of lattice Wajsberg algebra $\mathbb{W}$ is called a vague positive implicative W-implicative ideal of $\mathbb{W}$, if for all $x, y, z \in \mathbb{W}$,

- i $V_A(0) \geq V_A(x)$
- ii $V_A(y) \geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\}$.

Example 3.1.2 Consider a set $\mathbb{W} = \{0, a, b, c, 1\}$ with partial ordering as in Figure 3.1.1. Defining a binary operation ' $\rightarrow$ ' and a quasi complement ' $*$ ' on $\mathbb{W}$ as given in tables 3.1.1 and 3.1.2.

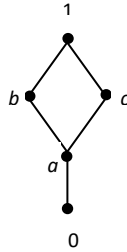


Figure: 3.1.1
Lattice diagram

x	x^*
0	1
a	b
b	a
c	c
1	0

Table: 3.1.1
Complement

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	a	a	c	1	1
b	c	b	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Table: 3.1.2
Implication

Define " $\vee$ " and " $\wedge$ " operations on $\mathbb{W}$ as follows:

$$(x \vee y) = (x \rightarrow y) \rightarrow y$$

$$(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^* \text{ for all } x, y \in \mathbb{W}.$$

Then, $(\mathbb{W}, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra.

Let A be a vague set of $\mathbb{W}$ defined by

$$A = \{\langle 0, [0.6, 0.3] \rangle, \langle a, [0.5, 0.2] \rangle, \langle b, [0.6, 0.2] \rangle, \langle c, [0.7, 0.2] \rangle, \langle 1, [0.7, 0.3] \rangle\}$$

Then, A is is vague positive implicative W-implicative ideal of $\mathbb{W}$.

Theorem 3.1.3 Every vague positive implicative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ is a vague W-implicative ideal of $\mathbb{W}$.

Proof: Let A be a vague positive implicative W-implicative ideal of $\mathbb{W}$.

Then from (ii) of definition 3.1.1,

$$\text{we have } V_A(y) \geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\} \text{ for all } x, y, z \in \mathbb{W}. \quad (3.1.1)$$

Taking $x = y, y = x$, and $z = x$ in (3.1.1), we get

$$\begin{aligned} V_A(x) &\geq rmin\{V_A(((x \rightarrow (x \rightarrow x)^*)^* \rightarrow y)^*), V_A(y)\} \\ &= rmin\{V_A(((x \rightarrow 1)^* \rightarrow)^*), V_A(y)\} \\ &= rmin\{V_A(((x \rightarrow 0)^* \rightarrow)^*), V_A(y)\} \\ &= rmin\{V_A((x \rightarrow y))^*, V_A(y)\} \end{aligned}$$

Thus, $V_A(x) \geq rmin\{V_A((x \rightarrow y))^*, V_A(y)\}$, and $V_A(0) \geq V_A(x)$. ■

Note. The converse of the above proposition may not be true.

Proposition 3.1.4 Let V_A be a vague implicative W -implicative ideal of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague positive implicative W -implicative ideal of $\mathbb{W}$ if and only if $V_A(x) \geq V_A(((x \rightarrow (y \rightarrow x)^*)^* \rightarrow 0)^*)^*$ for all $x, y \in \mathbb{W}$.

Proof: Let V_A be a vague positive implicative W -implicative ideal of $\mathbb{W}$, then from (ii) of definition 3.1.1 we have

$$V_A(y) \geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\} \text{ for all } x, y, z \in \mathbb{W}. \quad (3.1.2)$$

Substituting $x = 0, y = x$ and $z = y$ in (3.1.2) we get

$$\begin{aligned} V_A(x) &\geq rmin\{V_A(((x \rightarrow (y \rightarrow x)^*)^* \rightarrow 0)^*)^*, V_A(0)\} \\ &= rmin\{V_A((x \rightarrow (y \rightarrow x)^*)), V_A(0)\} \\ &= V_A((x \rightarrow (y \rightarrow x)^*)^*) \end{aligned}$$

Conversely, suppose V_A is a vague W -implicative ideal and it satisfies the inequality,

$$V_A(x) \geq V_A((x \rightarrow (y \rightarrow x)^*)^*) \text{ for all } x, y, z \in \mathbb{W} \quad (3.1.3)$$

Put $x = y$ in (3.1.3) then, we have

$$\begin{aligned} V_A(y) &\geq V_A((y \rightarrow (z \rightarrow y)^*)^*) \\ &\geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\} \end{aligned}$$

Thus, we have $V_A(y) \geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\}$, and

$$V_A(0) \geq V_A(x) \quad [\text{From (i) of definition 3.1.1}]$$

Hence, V_A is a vague positive implicative W -implicative ideal of $\mathbb{W}$. ■

Proposition 3.1.5 If V_A is a vague positive implicative W -implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ then, $I = \{x \in A/V_A(x) = V_A(0)\}$ is a positive implicative W -implicative ideal of $\mathbb{W}$.

Proof: Let V_A be a vague positive implicative W-implicative ideal of $\mathbb{W}$ and $I = \{x \in A/V_A(x) = V_A(0)\}$.

Obviously, $0 \in A$.

Let $((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^* \in I, x \in I$ for all $x, y, z \in \mathbb{W}$

Then, we have $V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*) = V_A(0)$ and $V_A(x) = V_A(0)$ (3.1.4)

Since V_A is a vague positive implicative W-implicative ideal, we have

$$\begin{aligned} V_A(y) &\geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\} && [\text{From (ii) of definition 3.1.1}] \\ &= V_A(0) && [\text{From 3.1.4}] \end{aligned}$$

and $V_A(0) \geq V_A(y)$ [From (i) of definition 3.1.1]

Then, we get $V_A(y) = V_A(0)$

Thus, $y \in I$ it follows that I is a positive implicative W-implicative ideal of $\mathbb{W}$. ■

Theorem 3.1.6 Let V_A be a vague subset of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$ if and only if $V_A(\alpha, \beta) \neq \emptyset$; $\alpha, \beta \in [0, 1]$.

Proof: Let V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$ and $\alpha, \beta \in [0, 1]$ such that $V_A(\alpha, \beta) \neq \emptyset$. Clearly $0 \in V_A(\alpha, \beta)$.

Let $((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^* \in V_A(\alpha, \beta)$ and $x \in V_A(\alpha, \beta)$ for all $x, y, z \in \mathbb{W}$

Then, we have $V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*) \geq [\alpha, \beta], V_A(x) \geq [\alpha, \beta]$.

It follows that, $V_A(y) \geq rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\} \geq [\alpha, \beta]$.

Thus, $y \in V_A[\alpha, \beta]$.

Hence, $[\alpha, \beta]$ is a positive implicative W-implicative ideal of $\mathbb{W}$.

Conversely, if $V_A(\alpha, \beta) \neq \emptyset$ is a positive implicative W-implicative ideal of $\mathbb{W}$, where $\alpha, \beta \in [0, 1]$. For any $x \in \mathbb{W}$ and $x \in V_A(A)$, it follows that $V_A(A)(x)$ is a positive implicative W-implicative ideal of $\mathbb{W}$.

Thus, $0 \in V_A(A)(x)$. That is, $V_A(0) \geq V_A(x)$ for all $x, y, z \in \mathbb{W}$.

Let $[\alpha, \beta] = rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\}$, it follows that $V_A(\alpha, \beta)$ is a positive implicative W-implicative ideal and $((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^* \in V_A[\alpha, \beta]$ and $x \in V_A[\alpha, \beta]$.

This implies that $y \in V_A[\alpha, \beta]$.

So, $V_A(y) \geq [\alpha, \beta] = rmin\{V_A(((y \rightarrow (z \rightarrow y)^*)^* \rightarrow x)^*), V_A(x)\}$

Thus, V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$. ■

Corollary 3.1.7 A vague subset V_A of lattice Wajsberg algebra $\mathbb{W}$ is a vague positive implicative W -implicative ideal of $\mathbb{W}$ if and only if V_α is a positive implicative W -implicative ideal of $\mathbb{W}$, when $V_\alpha \neq \emptyset, \alpha \in [0,1]$.

Proposition 3.1.8 Let M and N be implicative W -implicative ideals of lattice Wajsberg algebra $\mathbb{W}$, such that $M \subseteq N$. If V_A is a vague positive implicative W -implicative ideal of M . Then so on N .

Proof: Let M and N be implicative W -implicative ideals of lattice Wajsberg algebra $\mathbb{W}$. Let V_A be a vague positive implicative W -implicative ideal of M .

Since $M \subseteq N$, $V_M(x) \leq V_N(x)$ for all $x \in \mathbb{W}$.

Then, clearly $M_\alpha \leq N_\alpha$ for every $\alpha \in [0,1]$.

If V_M is a vague positive implicative W -implicative ideal of $\mathbb{W}$.

Hence, we get M_α is a positive implicative W -implicative ideal of $\mathbb{W}$.

[From corollary 3.1.7]

Then, N_α is a positive implicative W -implicative ideal of $\mathbb{W}$. [From Proposition 2.10]

Thus, V_N is a positive implicative W -implicative ideal.

Hence V_A is a vague positive implicative W -implicative ideal of N . ■

3.2. Vague Associative W -implicative ideal

In this section, we introduce an notation of vague associative W -implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ and examine its properties.

Definition 3.2.1 A vague subset V_A of lattice Wajsberg algebra $\mathbb{W}$ is said to be a vague associative W -implicative ideal of $\mathbb{W}$ with respect to x , where x is a fixed element of $\mathbb{W}$, if it satisfies,

i $V_A(0) \geq V_A(y)$

ii $V_A(z) \geq rmin\{V_A((z \rightarrow y)^* \rightarrow x)^*, V_A((y \rightarrow x)^*)\}$ for all $x, y, z \in \mathbb{W}$

Note. A vague associative W -implicative ideal with respect to all $x \neq 1$ is called a vague associative W -implicative ideal. Vague associative W -implicative ideal with respect to 1 is constant.

Example 3.2.2 Consider a set $A = \{0, f, g, 1\}$ with partial ordering as in Figure 3.2.1. Define ‘ $*$ ’ and ‘ $\rightarrow$ ’ on $\mathbb{W}$ as given in tables 3.2.1 and 3.2.2.

$\rightarrow$	0	f	g	1
0	1	1	1	1

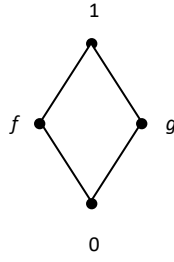


Figure 3.2.1
Lattice diagram

x	x^*
0	1
f	g
g	f
1	0

Table 3.2.1
Complement

f	f	1	1	1
g	f	g	1	1
1	0	f	g	1

Table 3.2.2
Implication

Here, $\mathbb{W}$ is a lattice Wajsberg algebra.

A vague subset V_A of $\mathbb{W}$ is defined by,

$$A = \{\langle 0, [0.7, 0.2] \rangle, \langle f, [0.7, 0.2] \rangle, \langle g, [0.5, 0.3] \rangle, \langle 1, [0.5, 0.3] \rangle\}.$$

Then, V_A is a vague associative W-implicative ideal of $\mathbb{W}$.

Proposition 3.2.3 If V_A is a vague associative W-implicative ideal of $\mathbb{W}$ with respect to x then $V_A(0) = V_A(y)$.

Proof: Let V_A be a vague associative W-implicative ideal of $\mathbb{W}$ with respect to x if $x = (0, 1)$. Then it is trivial.

If x is neither 0 nor 1. Then,

$$V_A(x) \geq \text{rmin}\{V_A((x \rightarrow 0)^* \rightarrow x)^*, V_A((y \rightarrow x)^*)\} \text{ [From (ii) of definition 3.2.1]}$$

Thus, $V_A(x) = V_A(0)$. ■

Proposition 3.2.4 Every vague associative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ with respect to 0 is a vague W-implicative ideal of $\mathbb{W}$.

Proof: Let V_A be a vague associative W-implicative ideal of $\mathbb{W}$ with respect to 0.

Then, we have $V_A(x) \geq \text{rmin}\{V_A((x \rightarrow y)^* \rightarrow 0)^*, V_A((y \rightarrow 0)^*)\}$ for all $x, y \in \mathbb{W}$

[From (ii) of definition 3.2.1]

$$= \text{rmin}\{V_A((x \rightarrow y)^*), V_A(y)\}$$

Thus, V_A is a vague W-implicative ideal of $\mathbb{W}$. ■

Theorem 3.2.5 Let V_A be a vague W-implicative ideal of lattice Wajsberg algebra of $\mathbb{W}$. V_A is a vague associative W-implicative ideal of $\mathbb{W}$ if and only if it satisfies, $V_A((z \rightarrow (y \rightarrow x)^*)^* \geq V_A((z \rightarrow y)^* \rightarrow x)^*$ for all $x, y, z \in \mathbb{W}$.

Proof: Let V_A be a vague W-implicative ideal of $\mathbb{W}$ satisfying

$$V_A((z \rightarrow (y \rightarrow x)^*)^*) \geq V_A((z \rightarrow y)^* \rightarrow x)^* \text{ for all } x, y, z \in \mathbb{W}$$

Then, $V_A(z) \geq \text{rmin}\{V_A((z \rightarrow (y \rightarrow x)^*)^*), V_A((y \rightarrow x)^*)\}$

$$= \text{rmin}\{V_A(((z \rightarrow y)^* \rightarrow x)^*), V_A((y \rightarrow x)^*)\}$$

Thus, V_A is a vague associative W -implicative ideal of $\mathbb{W}$.

Conversely, if V_A be a vague associative W -implicative ideal of $\mathbb{W}$.

Then, $V_A((z \rightarrow (y \rightarrow x)^*)^*) \geq \text{rmin}\{V_A(((z \rightarrow (y \rightarrow x)^*)^* \rightarrow (z \rightarrow y)^*)^* \rightarrow x), V_A((z \rightarrow (y \rightarrow x)^*)^*)\}$

Let us consider, $((z \rightarrow (y \rightarrow x)^*)^* \rightarrow (z \rightarrow y)^*)^* \rightarrow x)$

$$\begin{aligned} &= x^* \rightarrow ((z \rightarrow (y \rightarrow x)^*)^* \rightarrow (z \rightarrow y)^*) \\ &= x^* \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow (y \rightarrow x)^*)) \\ &= (x \rightarrow y) \rightarrow (x^* \rightarrow ((y \rightarrow x) \rightarrow z^*)) \\ &= (z \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow (x^* \rightarrow z^*)) \\ &= (z \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow y)) \\ &= 1 \end{aligned}$$

It follows that $V_A((z \rightarrow (y \rightarrow x)^*)^*) \geq \text{rmin}\{V_A(0), V_A(((z \rightarrow y)^* \rightarrow x)^*)\}$

$$= V_A(((z \rightarrow y)^* \rightarrow x)^*)$$

Thus, $V_A((z \rightarrow (y \rightarrow x)^*)^*) \geq V_A(((z \rightarrow y)^* \rightarrow x)^*)$. ■

Theorem 3.2.6 Let V_A be a vague W -implicative ideal of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague associative W -implicative ideal of $\mathbb{W}$ if and only if $V_A(z) \geq V_A(((z \rightarrow x)^* \rightarrow x)^*)$ for all $x, y, z \in \mathbb{W}$.

Proof: Let V_A be a vague associative W -implicative ideal of $\mathbb{W}$.

Then, $V_A(z) \geq \text{rmin}\{V_A((z \rightarrow y)^* \rightarrow x)^*, V_A((y \rightarrow x)^*)\}$ for all $x, y, z \in \mathbb{W}$.

[From (ii) of definition 3.2.1]

Taking $y = x$ we get,

$$\begin{aligned} V_A(z) &\geq \text{rmin}\{V_A((z \rightarrow x)^* \rightarrow x)^*, V_A((x \rightarrow x)^*)\} \\ &= \text{rmin}\{V_A((z \rightarrow y)^* \rightarrow x)^*, V_A(0)\} \\ &= V_A(((z \rightarrow y)^* \rightarrow x)^*) \end{aligned}$$

Conversely, if V_A is a vague W -implicative ideal and satisfies

$$V_A(z) \geq V_A(((z \rightarrow y)^* \rightarrow x)^*) \text{ for all } x, y, z \in \mathbb{W}$$

Clearly, $((z \rightarrow x)^*) \rightarrow (y \rightarrow x)^*)^* \rightarrow (z \rightarrow y)^*)^* = 0$

and $((z \rightarrow y)^* \rightarrow (z \rightarrow x)^*)^* \leq (x \rightarrow y)^*$

It follows that, $((z \rightarrow (y \rightarrow x)^*)^* \rightarrow x)^* \rightarrow ((z \rightarrow y)^* \rightarrow x)^*)^* = 0$

$$\begin{aligned} V_A((z \rightarrow (y \rightarrow x)^*)^*) &\geq V_A(((z \rightarrow y \rightarrow x)^* \rightarrow x)^* \rightarrow x)^*) \\ &\geq \text{rmin}\{V_A(((z \rightarrow (y \rightarrow x)^* \rightarrow x)^*), V_A((z \rightarrow y)^* \rightarrow x)^*)\} \\ V_A(((z \rightarrow y)^* \rightarrow x)^*) &= \text{rmin}\{V_A(0), V_A(((z \rightarrow y)^* \rightarrow x)^*)\} \\ &= V_A(((z \rightarrow y)^* \rightarrow x)^*) \end{aligned}$$

From the proposition 3.2.3, V_A is a vague associative W-implicative ideal of $\mathbb{W}$. ■

4. Conclusions

In this paper, we have introduced the notions of vague positive implicative W-implicative ideal and vague associative W-implicative ideal of lattice Wajsberg algebras. Further, we have discussed the relationship between vague positive implicative W-implicative ideal, and vague W-implicative ideal. Moreover, we have given some of the characterization of the vague associative W-implicative ideal.

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Common Neighbor Polynomial of Some Special Trees

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Abstract

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E . Let (u, v) denotes an unordered vertex pair of distinct vertices of G and let $N(u)$ denote the open neighborhood of the vertex u in G . The i -common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n - 2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common neighbor polynomial of G . In this paper, we study the common neighbor polynomial of some special type of trees such as complete m -ary tree, caterpillar tree, star like tree and fire cracker graph.

Keywords: common neighbor, common neighbor polynomial.

2020 AMS subject classifications: 05C31, 05C39.¹

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1 Introduction

Vertex similarity of nodes is a well studied concept in graph theory as it is highly significant in various fields such as in the study of molecular structure of chemical graphs, measuring consensus rate of different individuals/organizations in network analysis etc. The number of common neighbors shared by two nodes in a network system can be treated as a measure of consensus among the corresponding individuals. This motivates the authors to define the i -common neighbor set and the common neighbor polynomial of a graph.

Let $G(V, E)$ be a graph (simple graph) of order n with vertex set V and edge set E . Let (u, v) denotes an unordered vertex pair of distinct vertices of G and let $N(u)$ denote the open neighborhood of the vertex u in G . The i -common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n - 2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common neighbor polynomial of G . This polynomial was introduced by the present authors in (3).

Trees are commonly used to represent hierarchical data structures involved in network file system, possibility spaces, algorithmic routing etc. It is easy to dissect a tree data structure to access the information about a particular part of a huge data. Due to this reason, many researches were conducted to explore the properties of trees. In this paper, we derive the common neighbor polynomial of some special trees like rooted trees, caterpillars etc.

2 Main Results

A rooted tree is a tree in which one of the vertices is distinguished as the root. According to the distance of other vertices from the root vertex, there is a hierarchy on the vertices of a rooted tree. The distance of a vertex v from the root is called the depth or level of the vertex. The height of a rooted tree is the greatest depth of a vertex of the tree. Considering a path from the root to a vertex w , if a vertex v immediately precedes w , then v is called the parent of w and w is called the child of v . Vertices having same parent are called siblings. An m -ary tree ($m \geq 2$) is a rooted tree in which every vertex has m or fewer number of children. A complete m -ary tree is an m -ary tree in which every internal vertices has exactly m children and all leaves are of same distance from the root.

Theorem 1. *Let T be a tree on n vertices. Let v be a vertex of T with degree k . If T' is a tree obtained from T by attaching p pendent edges at the vertex v , we have the following:*

$$N[T'; x] = N[T; x] + \frac{p}{2}(2k + p - 1)x + p(n - k).$$

Proof. Let $\{v_1, v_2, \dots, v_k\}$ be the neighbors of v in T . When we attach a pendent edge vw to v , n number of new pairs of vertices are introduced in which the pairs (v_i, w) where $i \in \{1, 2, \dots, k\}$ have one common neighbor v and the remaining $n - k$ new pairs have no common neighbors. There will be no change in the number of common neighbors of pairs of vertices of T by the introduction of the pendent edge vw . Hence the common neighbor polynomial becomes $N[T; x] + kx + (n - k)$. Repeating the process p times, after attaching the p -th pendent edge to v , the common neighbor polynomial of resulting graph becomes,

$$\begin{aligned} N[T'; x] &= N[T; x] + kx + (n - k) + (k + 1)x + (n - k) + \dots \\ &\quad + (k + p - 1)x + (n - k) \\ &= N[T; x] + [k + (k + 1) + (k + 2) + \dots + (k + p - 1)]x + p(n - k) \\ &= N[T; x] + \frac{p}{2}(2k + p - 1)x + p(n - k). \end{aligned}$$

This completes the proof. $\square$

Theorem 2. Let T be a complete m -ary tree with p levels where the root vertex is considered to be in the 0-th level. Then we have the following:

$$\begin{aligned} N[T; x] &= \frac{m^2(m^{p-1} - 1)}{m - 1}x + \binom{m}{2} \frac{m^p - 1}{m - 1}x + \frac{m[m^{2p-2} - m^p + m - 1]}{m - 1} + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} \left[\frac{m^{2i+3} - m^{p+i+1}}{1 - m} \right] + \frac{m^2[m^{2p} - m^{p+1} - m^p + m]}{2(m^2 - 1)}. \end{aligned}$$

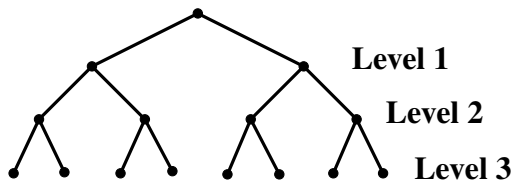


Figure 1: Complete binary tree with 3 levels of vertices

Proof. Let (u, v) be any pair of vertices of T . Here we consider 4 different cases according to the levels in which the vertices u and v lie in T .

Case(i) For $i \in \{0, 1, 2, \dots, p-2\}$ let u be a vertex in the i -th level and v a vertex in the $(i + 1)^{th}$ or $(i + 2)^{th}$ level.

If v is an $(i + 1)^{th}$ level vertex, the vertex pair (u, v) have no common neighbors and there are $m^i m^{i+1}$ such pairs of vertices in T . If v is in the $(i + 2)^{th}$

level, then there are $m^i m^{i+2}$ pairs of vertices (u, v) in which $m^i[m^{i+2} - m^2]$ pairs of vertices have no common neighbors and $m^i m^2$ pairs have exactly one common neighbor.

Case(ii) Let u be a vertex in the $(p - 1)$ -th level and v a vertex in the p -th level. In this case the pairs of vertices (u, v) have no common neighbors and there are $m^{p-1} m^p$ such pairs of vertices.

Case(iii) For $i \in \{0, 1, 2, \dots, p - 3\}$ let u be a vertex in the i -th level and v a vertex in the j -th level where $j = i + 3, i + 4, \dots, p$. All the pairs of vertices under this case have no common neighbors and there are $m^i[m^{i+3} + m^{i+4} + \dots + m^p]$ such pairs of vertices.

Case(iv) For $i \in \{1, 2, \dots, p\}$ let u and v be vertices of same level. In this case $\frac{1}{2}m^i[m^i - m]$ distinct pairs of vertices which are not siblings have no common neighbors and $\binom{m}{2}m^{i-1}$ pairs of vertices which are siblings have exactly one common neighbor.

From the above cases, it follows that

$$\begin{aligned} N[T; x] &= \sum_{i=0}^{p-2} m^i m^2 x + \binom{m}{2} \sum_{i=1}^p m^{i-1} x + \sum_{i=0}^{p-2} m^i [m^{i+1} + m^{i+2} - m^2] + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} m^i [m^{i+3} + m^{i+4} + \dots + m^p] + \sum_{i=1}^p \frac{1}{2} m^i (m^i - m). \\ &= \frac{m^2(m^{p-1} - 1)}{m - 1} x + \binom{m}{2} \frac{m^p - 1}{m - 1} x + \frac{m[m^{2p-2} - m^p + m - 1]}{m - 1} + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} \left[\frac{m^{p+i+1} - m^{2i+3}}{m - 1} \right] + \frac{m^2[m^{2p} - m^{p+1} - m^p + m]}{2(m^2 - 1)}. \end{aligned}$$

This completes the proof. $\square$

A derivative of a graph G is a graph obtained from G by deleting all the pendent vertices of G . A caterpillar is a tree whose derivative is a path graph (5). Consequently, a caterpillar $P_n(m_1, m_2, \dots, m_n)$ is obtained by attaching m_i pendent edges to the vertex v_i of a path P_n where $i \in \{1, 2, \dots, n\}$.

Theorem 3. *The common neighbor polynomial of a caterpillar tree $P_n(m_1, m_2, \dots, m_n)$ is given by the following:*

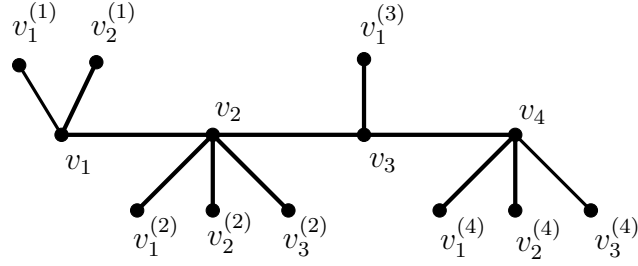


Figure 2: The caterpillar $P_4(2, 3, 1, 3)$

$$\begin{aligned}
 N[P_n(m_1, m_2, \dots, m_n); x] &= N[P_n; x] + \sum_{j=1}^n N[K_{1, m_j}; x] + \sum_{\substack{l, k \in \{1, 2, \dots, n\} \\ l \neq k}} m_l m_k \\
 &+ \left[m_1 + m_n + 2 \sum_{j=2}^{n-1} m_j \right] x + (n-2)(m_1 + m_n) + (n-3) \sum_{j=2}^{n-1} m_j.
 \end{aligned}$$

Proof. Let $P_n(m_1, m_2, \dots, m_n)$ be a caterpillar and let $v_1, v_2, \dots, v_n$ be the vertices of its derived graph which is a path. Also let $v_1^{(j)}, v_2^{(j)}, \dots, v_{m_j}^{(j)}$ be the pendent vertices of the caterpillar tree attached to the vertex v_j where $j \in \{1, 2, \dots, n\}$. Let (u, v) be any pair of vertices of the caterpillar. We consider the following cases to build up its common neighbor polynomial.

Case(i) Let $u, v \in \{v_1, v_2, \dots, v_n\}$. Here the number of pairs of vertices (u, v) with i common neighbors equals $|N(P_n, i)|$. So the pairs of vertices under this case contribute the term $N[P_n; x]$ to the common neighbor polynomial of the caterpillar.

Case(ii) Let $u, v \in \{v_j, v_1^{(j)}, v_2^{(j)}, \dots, v_{m_j}^{(j)}\}$. Here the vertices of the set under consideration spans a star graph K_{1, m_j} and hence the number of pairs of vertices (u, v) with i common neighbors equals $|N(K_{1, m_j}, i)|$.

Case(iii) Let $u \in \{v_1^{(l)}, v_2^{(l)}, \dots, v_{m_l}^{(l)}\}$ and $v \in \{v_1^{(k)}, v_2^{(k)}, \dots, v_{m_k}^{(k)}\}$ where $l \neq k$ and $l, k \in \{1, 2, \dots, n\}$.

Here u and v are the pendent vertices attached to the vertices v_l and v_k respectively where $l \neq k$. No pair of vertices under this case have common neighbors and there are $\sum_{\substack{l, k \\ l \neq k}} m_l m_k$ such pairs.

Case(iv) Let $u \in \{v_1, v_n\}$, a pendent vertex of the derived graph P_n and let v be any vertex attached to the vertices of P_n such that uv is not an edge of the caterpillar.

In this case pairs of vertices of the form $(v_1, v_l^{(2)})$ and $(v_n, v_k^{(n-1)})$ where $l \in \{1, 2, \dots, m_2\}$ and $k \in \{1, 2, \dots, m_{n-1}\}$ have exactly one common neighbor each and there are $m_2 + m_{n-1}$ such pairs of vertices. There remains $m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j$ pairs of vertices under this case which have no common neighbors.

Case(v) Let $u \in \{v_2, v_3, \dots, v_{n-1}\}$ and let v be any vertex selected in a way same as in Case(iv).

For $i \in \{2, 3, \dots, n-1\}$, the pairs of vertices of the form $(v_i, v_l^{(i-1)})$ where $l \in \{1, 2, \dots, m_{i-1}\}$ have exactly one common neighbor v_{i-1} and pairs of vertices of the form $(v_i, v_k^{(i+1)})$ where $k \in \{1, 2, \dots, m_{i+1}\}$ have exactly one common neighbor v_{i+1} . There are $m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j$ such pairs of vertices.

The remaining pairs of vertices under this case have no common neighbors and there are $(n-3)(m_1 + m_n) + (n-4)(m_2 + m_{n-1}) + (n-5) \sum_{j=3}^{n-2} m_j$ such pairs of vertices.

From the above cases, it follows that

$$\begin{aligned} N[P_n(m_1, m_2, \dots, m_n); x] &= N[P_n; x] + \sum_{j=1}^n N[K_{1, m_j}; x] + \sum_{\substack{l, k \\ l \neq k}} m_l m_k \\ &\quad + [m_2 + m_{n-1}]x + m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j \\ &\quad + \left[m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j \right] x + (n-3)(m_1 + m_n) \\ &\quad + (n-4)(m_2 + m_{n-1}) + (n-5) \sum_{j=3}^{n-2} m_j. \end{aligned}$$

Now the result follows after some rearrangement of the terms. $\square$

Corollary 4. For a caterpillar $P_n(m, m, \dots, m)$ where m pendent edges are attached to each vertex of the path P_n , we have

$$\begin{aligned} N[P_n(m, m, \dots, m); x] &= N[P_n; x] + nN[K_{1, m}; x] + \binom{n}{2} m^2 + 2m(n-1)x \\ &\quad + m(n-1)(n-2). \end{aligned}$$

A star like tree(2) $S(n_1, n_2, \dots, n_k)$ is a graph having only one vertex w of degree greater than 2 such that deletion of w results in a disjoint union of the path

graphs $P_{n_1}, P_{n_2}, \dots, P_{n_k}$. The star like tree graphs are used to represent proteins which will have generally 20 branches where each branch indicates the presence of one of the 20 natural amino acids. For example, the strand A of human insulin has 21 amino acids of 11 kinds which can be modelled by a star like tree graph with 11 branches as shown in Figure 3.

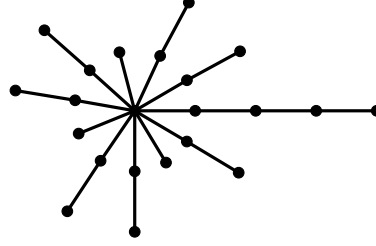


Figure 3: Strand A of human insulin

Theorem 5. *The common neighbor polynomial of a star like tree $S(n_1, n_2, \dots, n_k)$ with $N + 1$ vertices is given by*

$$N[S(n_1, n_2, \dots, n_k); x] = \sum_{r=1}^k N[P_{n_r+1}; x] + \binom{k}{2}x + \binom{N}{2} - \sum_{r=1}^k \binom{n_r}{2} - \binom{k}{2}$$

where $N = n_1 + n_2 + \dots + n_k$.

Proof. Let $S(n_1, n_2, \dots, n_k)$ be a star like tree with a vertex w such that $S(n_1, n_2, \dots, n_k) - w = P_{n_1} \cup P_{n_2} \cup \dots \cup P_{n_k}$. Any pair of vertices $(u, v) \in P_{n_r} \cup \{w\}$ where $r \in \{1, 2, \dots, k\}$, has as many common neighbors in $S(n_1, n_2, \dots, n_k)$ as it has in P_{n_r+1} .

Let (u, v) be a pair of vertices in $S(n_1, n_2, \dots, n_k)$ such that $u \in P_{n_r}$ and $v \in P_{n_s}$ where $r \neq s, r, s \in \{1, 2, \dots, k\}$. Then the vertex pair (u, v) has a single common neighbor w if both u and v are adjacent to w and there are no common neighbors otherwise. Hence there are $\binom{k}{2}$ pairs of vertices with one common neighbor and $\binom{N}{2} - \sum_{r=1}^k \binom{n_r}{2} - \binom{k}{2}$ pairs of vertices with no common neighbors. Hence the result follows. $\square$

Corollary 6. *The common neighbor polynomial of the graphical representation of Strand A of human insulin is given by*

$$N[G; x] = 65x + 166.$$

Proof. The proof follows from the fact that Strand A of human insulin can be graphically represented as a star like tree graph $S(1, 2, 4, 2, 2, 1, 2, 2, 1, 2, 2)$. $\square$

The (n, k) firecracker graph(1) is obtained by identifying each vertex of a path P_n with one of the pendent vertices of the star graph $K_{1,k}$. In particular, the $(n, 2)$ firecracker graph is known as the centipede graph.

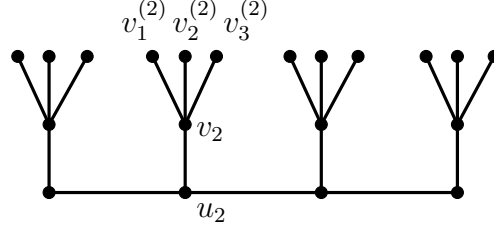


Figure 4: The $(4, 4)$ - firecracker graph

Lemma 7. (see (3)) For a complete bipartite graph $K_{m,n}$, we have

$$N[K_{m,n}; x] = \binom{m}{2} x^n + \binom{n}{2} x^m + mn.$$

Lemma 8. (see (3)) For a path P_n with $n \geq 2$ vertices, we have

$$N[P_n; x] = (n-2)x + \binom{n-1}{2} + 1.$$

Theorem 9. The common neighbor polynomial of (n, k) firecracker graph G is given by the following:

$$N[G; x] = \left[n \binom{k}{2} + 3n - 4 \right] x + nk + \binom{n-1}{2} + 1 + \binom{n}{2} k^2 + (n-1)(n-2) + n(n-1)(k-1).$$

Proof. For $j \in \{1, 2, \dots, n\}$, let $v_1^j, v_2^j, \dots, v_k^j$ be the pendent vertices of the j^{th} star where the vertex v_k^j is identified with the vertex u_j of the path P_n . Let v_j be the center vertex of the j^{th} star attached to the vertex u_j of P_n . Let (u, v) be any pair of vertices of G . Here we consider 5 cases:

Case(i) Let $u, v \in \{v_j, v_1^j, v_2^j, \dots, v_k^j\}$ where $j \in \{1, 2, \dots, n\}$.

In this case, the pair (u, v) has as many common neighbors in G as in $K_{1,k}$.

Case(ii) Let $u, v \in \{u_1, u_2, \dots, u_n\}$.

Here the vertex pair (u, v) has as many common neighbors in G as in P_n .

Case(iii) Let $u = v_j$ and $v \in \{u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}$ where $j \in \{1, 2, \dots, n\}$

In this case, for each j , the pairs (v_j, u_{j-1}) and (v_j, u_{j+1}) has exactly one

common neighbor. Also the pairs (v_1, u_2) and (v_n, u_{n-1}) have one common neighbor each. Hence there are $2(n-1)$ pairs of vertices (u, v) with one common neighbor and all other $(n-1)(n-2)$ pairs of vertices have no common neighbors.

Case(iv) Let $u \in \{v_r, v_1^r, v_2^r, \dots, v_{k-1}^r\}$ and $v \in \{v_s, v_1^s, v_2^s, \dots, v_{k-1}^s\}$ where $r, s \in \{1, 2, \dots, n\}$.

In this case, there are $\binom{n}{2}k^2$ pairs of vertices which have no common neighbors.

Case(v) Let $u \in \{v_1^j, v_2^j, \dots, v_{k-1}^j\}$ and $v \in \{u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}$ where $j \in \{1, 2, \dots, n\}$,

In this case, there are $n(n-1)(k-1)$ pairs of vertices which have no common neighbors.

Using lemmas 7 and 8, it follows that,

$$\begin{aligned}
 N[G; x] &= nN[K_{1,k}; x] + N[P_n; x] + 2(n-1)x + (n-1)(n-2) + \binom{n}{2}k^2 \\
 &\quad + n(n-1)(k-1) \\
 &= n\left[\binom{k}{2}x + k\right] + (n-2)x + \binom{n-1}{2} + 1 + 2(n-1)x + \binom{n}{2}k^2 \\
 &\quad + (n-1)(n-2) + n(n-1)(k-1) \\
 &= \left[n\binom{k}{2} + 3n - 4\right]x + nk + \binom{n-1}{2} + 1 + \binom{n}{2}k^2 \\
 &\quad + (n-1)(n-2) + n(n-1)(k-1).
 \end{aligned}$$

This completes the proof. □

Corollary 10. *The common neighbor polynomial of the centipede graph G is given by*

$$N[G; x] = 4(n-1)x + 2n + \frac{3(n-1)(3n-2)}{2} + 1.$$

Proof. The proof follows from the fact that the centipede graph is a special case of (n, k) - firecracker graph when $k = 2$. □

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An efficient block-based image compression and quality-wise decompression algorithm

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Abstract

In this paper, we propose a block-based lossy image compression algorithm that makes use of spatial redundancies of neighboring pixels in image data. Compression is achieved by replacing a block of pixels with their statistical mean. The algorithm helps in decompressing the image at different quality levels. Quality matrices constructed from the quantization table of the JPEG baseline algorithm are used to achieve different qualities of the reconstructed data. Experimental results show that the proposed method outperforms existing polynomial-based algorithms both in computation time and complexity.

Keywords: lossy compression; JPEG compression; polynomial-based compression.

2020 AMS subject classifications: 97M10 ¹

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1 Introduction

Image compression is a process that reduces the size of image data files while keeping necessary information. We can classify image compression schemes into four groups according to the process element as pixel-based, block-based, subband-based, and region-based. Image compression has been a hot topic of research for many years and several image compression standards have been developed ([Sonal, 2007], [Memon and Sayood, 1995]). Among these, the block-based compression scheme JPEG that uses discrete cosine transform (DCT) and the subband-based scheme JPEG 2000 has got much attention and popularity ([Wallace, 1992], [Rabbani, 2002]). Since these two methods involve transforms such as DCT and wavelet transform, their computational complexity is very high. Researches are still going on in developing simple and fast compression algorithms that can show better performance than the existing one. As an alternative to transform-based techniques, polynomial-based compression and statistical approach in compression are also developed([Shukla et al., 2005], [Ameer, 2009], [Sajikumar and Anilkumar, 2017], [Sajikumar et al., 2021]). Even though many algorithms have been reported in this field, research is still needed to cope with the continuous demand for efficient transmission or storage of image data.

If the information retained after decompression is 100%, the compression method is called lossless otherwise it is lossy. If we take a pixel in an image at random there is a good chance that its neighbours will have the same intensity or very similar intensity. Typically hence, image compression is based on the fact that the neighbouring pixels are highly correlated ([Salomon, 2007], [Sayood, 2012]). Most image compression methods exploit this feature to obtain efficient compression. Lossless compression can be achieved with the techniques like Run Length Encoding (RLE), Huffman coding, Arithmetic coding, etc.([Gallager, 1978], [Jain, 1989], [Taubman and Marcellin, 2012], [Witten et al., 1987]). Lossy techniques include transform coding methods such DCT/JPEG, JPEG2000, etc. ([Pennebaker and Mitchell, 1992], [Gonzalez and Woods, 2008], Goyal [2001]). Polynomial-based compression is another type of lossy compression method ([Sadeh, 1996], [Eden et al., 1986]). S. Sajikumar and A. K. Anilkumar [Sajikumar and Anilkumar, 2017] introduced a compression scheme using Chebyshev polynomials. Lossy compression techniques tested for their performance based on three commonly used measures, the Root Mean Square Error (RMSE), Peak Signal to Noise Ratio (PSNR) and the Compression Ratio (CR). The RMSE between original image $f(x, y)$ and reconstructed image $\hat{f}(x, y)$ of size $M \times N$ is defined by [Joshi, 2018]:

$$RMSE = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - \hat{f}(x, y)]^2 \right]^{\frac{1}{2}} \quad (1)$$

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For an 8-bit gray level image,

$$PSNR = 10 \log_{10} \left(\frac{255^2}{MSE} \right) (dB) \quad (2)$$

$$CR = \frac{\text{compressed image size}}{\text{uncompressed image size}} \% \quad (3)$$

In digital image compression, the basic data redundancies are due to coding redundancy, inter-pixel redundancy, and psycho-visual redundancy. Statistical approaches in image compression are the compression techniques that try to decorrelate this inter-pixel redundancy. Dimensionality reduction is another aspect of image compression. The Principal Component Analysis (PCA) is the significant one in this area ([Du and Fowler, 2007], [Sonal, 2007]). The PCA approach is implemented via the Statistical approach and the Neural Network approach [Dony and Haykin, 1995]. To reduce the storage space required, we can make use of statistical measures of central dispersion such as mean and variance of pixel values in an image [Sajikumar et al., 2021]. This paper presents a simple and efficient lossy image compression method using the mean value of a block of pixels. The input image is partitioned into non-overlapping blocks and the mean pixel value for each block is used to represent the entire block of pixels followed by a quality gradation at the decompression stage. We compare the proposed method with polynomial-based compression techniques such as plane fitting model and Chebyshev polynomial surface fit method ([Ameer and Basir, 2006], [Sajikumar and Anilkumar, 2017]). In comparison, it is found that the proposed method outperforms these algorithms.

2 Proposed method

Divide the input image matrix into non-overlapping blocks of size $n \times n$. Subtract 128 from each pixel in the image matrix to change the gray levels from $[0, 255]$ to values centered about zero. Thus the modified range becomes $[-128, 127]$. Compute the mean value of each block of pixels and store this mean for each block as the reconstruction parameter. At the reconstruction stage, replace all pixels in each block by the respective mean values. That is, n^2 pixel values in each block are replaced by a single parameter and hence high compression can be achieved as the block size increases.

To reduce the loss of information at the decompression stage, a quality matrix of dimension $n \times n$ is introduced. This matrix allows us to decompress output images at different quality levels. The quality determination process outputs images at different bit-rates of lower to a higher order. We have adopted this matrix from

the JPEG's baseline compression algorithm where it is used as the quantization matrix ([Wallace, 1992], [Ahumada Jr and Peterson, 1992], [Watson, 1993]).

$$Q_{50} = \begin{pmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{pmatrix} \quad (4)$$

If N is the visual quality level of the decompressed image, then we can obtain different quality matrices Q_N using the following equation [Khedr and Abdelrazek, 2016]:

$$Q_N = \begin{cases} \left(\frac{100 - N}{50} \right) Q_{50}, & N > 50 \\ \left(\frac{50}{N} \right) Q_{50}, & N < 50 \end{cases} \quad (5)$$

We have considered submatrices of size $n \times n$ with elements taken in order from the top left corner of the matrix Q_{50} . The quality matrix Q_{50} for different block sizes 2×2 , 3×3 , 4×4 are:

$$\begin{pmatrix} 16 & 11 \\ 12 & 12 \end{pmatrix}, \begin{pmatrix} 16 & 11 & 10 \\ 12 & 12 & 14 \\ 14 & 13 & 16 \end{pmatrix}, \begin{pmatrix} 16 & 11 & 10 & 16 \\ 12 & 12 & 14 & 19 \\ 14 & 13 & 16 & 24 \\ 14 & 17 & 22 & 29 \end{pmatrix}$$

3 Experimental results

Test images of size 256×256 with gray levels in the range $[0, 256]$ are considered. Experimental results with block sizes 2×2 , 3×3 , 4×4 are given in Tables 1-3 and Figures 1-5. Compression qualities are analyzed at different levels 5, 10, 50, 90, and 95. Decompressed images at these levels are given in Figures 2-5.

In 2×2 blocks, four gray values are replaced by the mean and hence save 75% storage space with CR 25%. At this CR, all the test images show reasonable reconstructed image quality with low RMSE and exhibit superior performance to polynomial-based compression schemes. With the 25% CR and quality index 95, Rice image shows PSNR value 31.5585 (dB), Lena 27.5988 (dB), and Cameraman

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25.4852(dB). These results are promising in comparison with the polynomial-based algorithms. Detailed comparison results with different block sizes are given in the next section. For 3×3 blocks, CR is 11.11% and it is 6.25% for 4×4 blocks. As the block size increases reconstruction quality becomes poor with a marginal increase in RMSE.

Test Image	CR %	Performance	Quality Index				
			95	90	50	10	5
Rice	25	PSNR	31.5585	31.5340	30.4099	21.0875	16.0389
		RMSE	6.7393	6.7583	7.6921	22.4491	40.2340
Lena	25	PSNR	27.5988	27.5823	27.0974	21.7713	16.3263
		RMSE	10.6317	10.6518	11.2634	220.7958	38.9249
Cameraman	25	PSNR	25.4852	25.4779	25.1629	20.6225	17.0641
		RMSE	13.5604	13.5720	14.0733	23.7365	35.7547

Table 1: Compression performance at different reconstruction qualities in the case of 2×2 blocks.

Test Image	CR %	Performance	Quality Index				
			95	90	50	10	5
Rice	11.11	PSNR	26.0410	26.0311	25.6843	20.7885	15.2735
		RMSE	12.7202	12.7346	13.2534	23.2871	43.9405
Lena	11.11	PSNR	23.7492	23.7424	23.5305	19.9055	15.5763
		RMSE	16.5607	16.5738	16.9831	25.77901	42.4353
Cameraman	11.11	PSNR	22.1119	22.1079	21.9610	19.4491	16.6085
		RMSE	19.9961	20.0053	20.3466	21.1698	37.6804

Table 2: Compression performance at different reconstruction qualities in the case of 3×3 blocks.

Test Image	CR %	Performance	Quality Index				
			95	90	50	10	5
Rice	6.25	PSNR	25.3334	25.3145	24.8320	18.4780	15.4024
		RMSE	13.7997	13.8298	14.6197	30.3836	43.2934
Lena	6.25	PSNR	23.3583	23.3464	23.0116	18.6289	14.3644
		RMSE	17.3231	17.3468	18.0285	29.8604	48.7890
Cameraman	6.25	PSNR	21.7695	21.7629	21.5635	17.9073	14.8582
		RMSE	20.8001	20.8160	21.2994	32.4471	46.0924

Table 3: Compression performance at different reconstruction qualities in the case of 4×4 blocks.

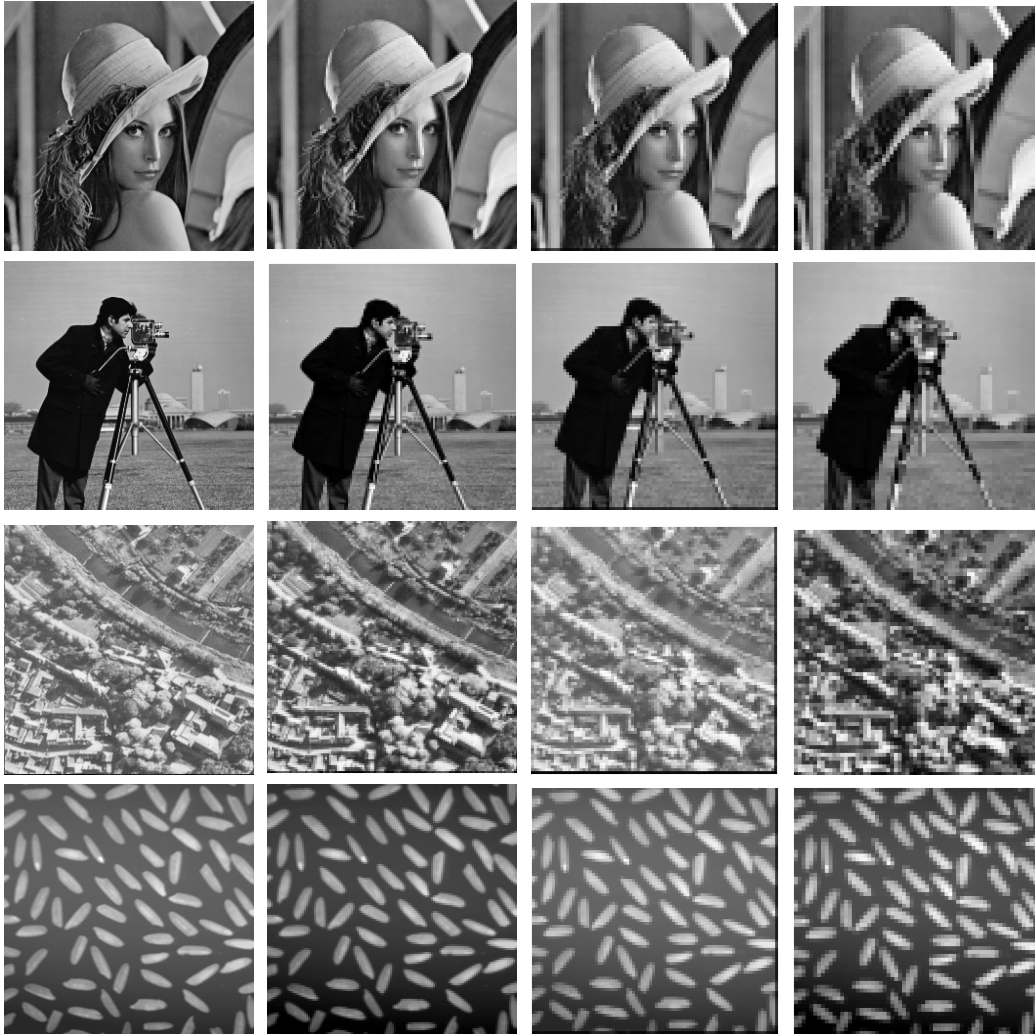


Figure 1: **The first column:** orinal images; **the second column:** reconstructed images using 2×2 blocks at quality index 95; **the third column:** reconstructed images using 3×3 blocks at quality index 95; **the fourth column:** reconstructed images using 4×4 blocks at quality index 95.

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Figure 2: **Top left corner:** orinal image; **left to right:** reconstructed Lena image using 2×2 blocks at quality indices 5, 10, 50, 90, 95.

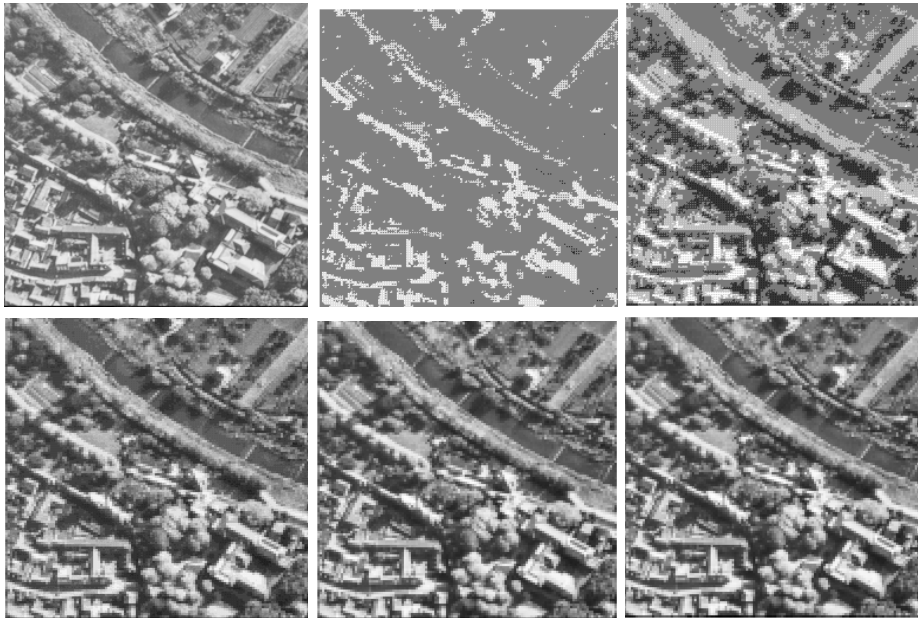


Figure 3: **Top left corner:** orinal image; **left to right:** reconstructed Aerial image using 2×2 blocks at quality indices 5, 10, 50, 90, 95.

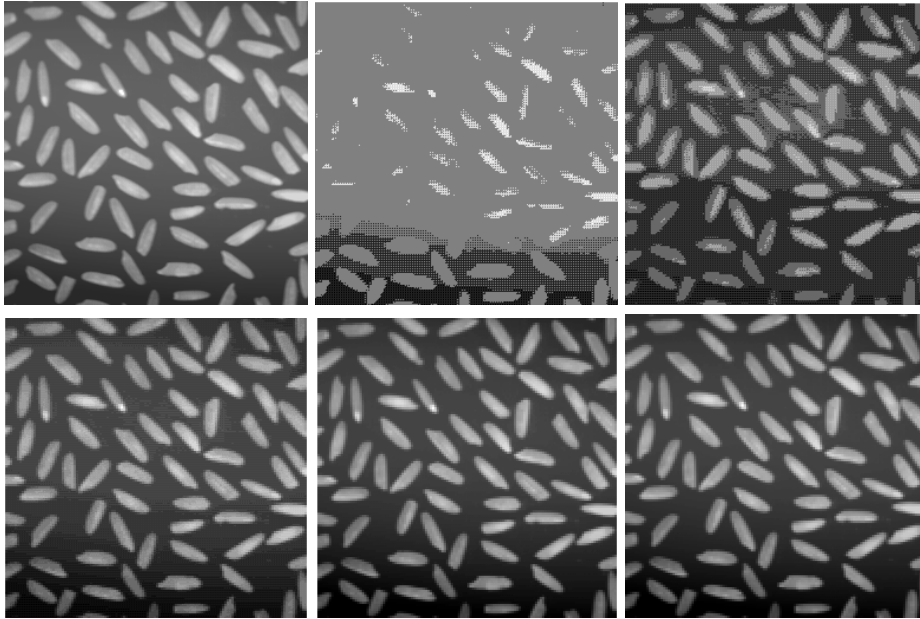


Figure 4: **Top left corner:** orinal image; **left to right:** reconstructed Rice image using 2×2 blocks at quality indices 5, 10, 50, 90, 95.



Figure 5: **Top left corner:** orinal image; **left to right:** reconstructed Cameraman image using 2×2 blocks at quality indices 5, 10, 50, 90, 95.

4 Comparison with Polynomial models

Experimental results are compared with the plane fitting model proposed by S. Ameer and O. Basir [Ameer and Basir, 2006] and the Chebyshev polynomial surface fit model proposed by S. Sajikumar and A. K. Anilkumar [Sajikumar and Anilkumar, 2017]. Both these methods are block-based algorithms and the proposed method outperforms these two for any block size. Comparison results for 2×2 blocks and 4×4 blocks are given in Tables 4-5. The plane fitting model and Chebyshev polynomial model have CR's 75% with 2×2 blocks and 18.75% with 4×4 blocks respectively. But the proposed method has a CR of only 25% with 2×2 blocks and it decreases as the block size increases. Even at 25% CR, the proposed method can give an improved result.

Test Image	CR %	Performance	Plane Model	Chebyshev Poly. fit	Proposed Method
Rice	25	PSNR	28.9417	28.9417	31.5585
		RMSE	9.1104	9.1104	6.7393
Lena	25	PSNR	26.1790	26.1790	27.5988
		RMSE	12.5299	12.5299	10.6317
Cameraman	25	PSNR	23.8796	23.8796	25.4852
		RMSE	16.3095	16.3095	13.5607

Table 4: Performance comparison with Polynomial fitting model and Chebyshev polynomial surface fit model in the case of 2×2 blocks.

Test Image	CR %	Performance	Plane Model	Chebyshev Poly. fit	Proposed Method
Rice	6.25	PSNR	24.4904	24.9876	25.3334
		RMSE	15.1987	14.3527	13.7997
Lena	6.25	PSNR	23.2617	23.2985	23.3583
		RMSE	17.5214	17.4356	17.3231
Cameraman	6.25	PSNR	21.3521	21.3374	21.7695
		RMSE	21.8174	21.8632	20.8001

Table 5: Performance comparison with Polynomial fitting model and Chebyshev polynomial surface fit model in the case of 4×4 blocks.

5 Conclusions

This paper presents a simple and efficient lossy image compression algorithm based on mean values of non-overlapping blocks of pixels. This mean value is taken as the parameter for reconstruction. A method for obtaining decompressed images at desired quality is also implemented. Using the quality matrix for different block sizes, the end-user or application has a choice for getting decompressed images according to the use. The proposed method outperforms existing polynomial-based methods in its speed of execution and computational complexity.

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Comparative study between local and global optimization for the Heston model

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Abstract

The objective of this study is to estimate the calibration parameters of the Heston stochastic volatility model by the two optimization methods: local and global, then to compare their performances and finally to recommend one of the two methods. To predict the prices of EUR/USD currency options, we use the Heston stochastic volatility model. We will first present the model and the two optimization methods: local and global, then we will estimate the calibration parameters using the two optimization methods with MATLAB software, then compare the two and recommend the most efficient method. Results have shown that the local optimization provides excellent calibration parameters with a reduced computational time compared to the global optimization. Therefore, we can clearly recommend it for the Heston model.

Keywords: Heston Model; Local calibration; Global calibration.

2010 AMS subject classification: 46N10, 91B60, 91B70, 91G20, 91G30

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1. Introduction

Calibration is the process of adjusting the parameters of a model by integrating the uncertainty of the parameters and/or the model to obtain a representation of the modeled system that satisfies a predefined criterion. The model used is the one proposed by Heston in 1993, which is a stochastic volatility model. The success of Heston's model lies in the calibration of its parameters and in the analytical form of its characteristic function. In this study, first, we will give the explicit form of call currency option prices based on the Heston model as well as the calibration parameters that must be computed. Then we will present the two optimization methods: local and global as well as the criterion according to which we look for its parameters. finally, we calculate the parameters of the model for each method and we do an analysis of the performances.

This work will help the trader who seeks to estimate the prices of currency options on the EUR/USD parity using the Heston model, to choose the optimization method that provides a more accurate calibration of the model.

2. Theoretical principles of Heston model

The present value of the expected payoff from a European currency option is given by:

$$E[S_t/S] = S_0 e^{(r_d - r_f)t} \quad (1)$$

Under the equivalent martingale measure, Heston model is given by: [1]:

$$\begin{aligned} dS_t &= (r_d - r_f)S_t dt + \sqrt{V_t}S_t d\bar{B}_t \\ dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t}d\bar{Z}_t \end{aligned} \quad (2)$$

Where $d\bar{B}_t d\bar{Z}_t = \rho dt$

r_d and r_f are the domestic and foreign interest rates respectively. θ is the long-term mean of variance. κ is the rate of mean reversion. σ is the coefficient of volatility. S_t and V_t are the price and volatility of the process. $\bar{B}_t$ and $\bar{Z}_t$ are correlated Wiener process, and the correlation coefficients is ρ .

according to feller the following condition will be satisfied (feller condition): $2\kappa\theta > \sigma^2$ [3]. we assume that the rates r_d and r_f are constant. According to Heston and Nandi [8] currency call option at time t, given a strike price K, that expires at time T is given by:

$$C(S, v, t) = S e^{-r_f \tau} P_1 - K e^{-r_d \tau} P_2 \quad (3)$$

where $\tau = T - t$, P_1 and P_2 can be defined via the Fourier Inversion Transformation [2]:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\psi \ln K} f_j(x, v, \tau, \psi)}{i\psi} \right] d\psi \quad (4)$$

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Where $j=1,2$. And the characteristic function for the logarithm of the exchange rate, $x = \ln(S_t)$ is given by [10]: $f_j(x, v, \tau, \psi) = e^{C(\tau, \phi) + D(\tau, \phi)v_t + i\phi x}$ Where:

$$\begin{aligned} C(\tau, \phi) &= (r_d - r_f)\phi i \tau + \frac{a}{\sigma^2} [(b_j - \rho\sigma\phi i + d)\tau - 2\ln(\frac{1 - ge^{d\tau}}{1 - g})] \\ D(\tau, \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2} (\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}) \\ g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \\ d &= \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)} \\ u_1 &= \frac{1}{2}; u_2 = -\frac{1}{2}; a = \kappa\theta; b_1 = \kappa\lambda - \rho\sigma; b_2 = \kappa + \lambda \end{aligned}$$

3. Materials and methods

3.1 Calibration methods

The parameters to be estimated are: $v, \kappa, \sigma, \theta$ and ρ . In this study, we use inverse problem method, which consists of collecting market data and then using it to search for the model parameters while minimizing the difference between the Heston model prices and the real market prices. In other words, we look for the parameters that minimize the squared error between the market prices of European currency options and those of the model. We assume that for a call option, there are several realizations of the parameters of the Heston model. Then we can write:

$$\text{Min } S(\Omega) = \min(\Omega) \sum_{i=1}^N \left(C_i^H(K_i, T_i) - C_i^M(K_i, T_i) \right)^2 \quad (5)$$

Ω is a vector of realizations $C_i^H(K_i, T_i)$, and $C_i^M(K_i, T_i)$ are prices estimated by the model and the market prices of the option i respectively, with strike K_i and maturity T_i . N is the number of options used for calibration. It is then a nonlinear optimization problem under the constraint: $2\kappa\theta - \sigma^2 > 0$ [5]. Since it is a nonlinear function then there are several local extrema depending on the initially estimated value, and even with a judicious choice of the initial value, convergence to the global optimum is not certain. Therefore, we will try both local and global optima.

3.1.1 Local optimization

If there are multiple local minima of the objective function, we cannot know whether the minimum that is found is the optimal one. In other words, we cannot know if it is a local or global minimum and there is no easy way to measure the distance to the global solution. For this, we will define an acceptance threshold, if the minimum found is acceptable, we

can at least ensure that any solution found is consistent with our criteria and if no solution was found, we start the process again with another initial value [6].

In this work, we will consider the following set of acceptable solutions:

$$\sum_{i=1}^N |C_i^H(K_i, T_i) - C_i^M(K_i, T_i)| \leq \frac{1}{2} \sum_{i=1}^N |bid_i - ask_i| \quad (6)$$

bid and ask are the market prices for buying and selling respectively. for local optimization, we use the lsqnonlin (least-squares non-linear) function in MATLAB [12], which uses a reflective trust-region minimization algorithm (see Yuan (1999) [9]. In addition, we will also define lower and upper bounds for the optimal parameters. These bounds allow us to reject solutions that are not economically acceptable. we will consider the following bounds [7]:

- We consider the $[0,1]$ interval for both for θ and V_0 .
- ρ takes values from -1 to 1.
- For σ an interval of 0 to 5 will be considered in the calibration.
- To have a mean reversion, the parameter κ must take positive values. the maximum values of κ will be set dynamically during calibration as a consequence of the non-negativity constraint.
- To this we add the feller condition which allows the model to avoid negative or zero values $2\kappa\theta - \sigma^2 > 0$, [3].

3.1.2 Global optimization

Global optimization allows to search for the global optimum using stochastic methods, even if a local minimum is found the search process does not stop. However, stochastic methods also have some drawbacks. The mathematical properties of these algorithms are less easy to handle than those of local algorithms. Moreover, despite their name, their convergence to the global minimum is not guaranteed. Indeed, the output sequence being stochastically determined, the algorithm may decide to stop prematurely and, in some cases, the solution obtained may be inferior to that of a local search. In sum, even if global optimization is theoretically more efficient, when working with functions of unknown form, it is not easy to decide which calibration procedure will be the most powerful [6]. In order to compare the results of the global optimization with the local one, we use the Simulated Annealing framework. This algorithm proceeds by iteration taking into account the previous information but also introducing randomization. Initially, the algorithm starts with a high tolerance for random shocks, and different regions are examined during the first phase. As a result, even if a minimum is found, the algorithm continues to look for improved solutions. As time progresses, the algorithm reduces its tolerance until it eventually settles on the best optimum reached.

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In this work, we use the MATLAB function `asamin` which was developed by Professor Shinichi Sakata and involves the Adaptive Simulated Annealing (ASA) algorithm, with dynamic adjustment for random shock tolerance [4]. Goel and Stander (2009) showed that the ASA framework provides good results among a range of different global optimizers. For comparison purposes, we will use the same parameter bounds as outlined above.

4. Results and discussion

For the calibration of the parameters of the Heston model, we use the EUR/USD parity as an underlying asset. To price the currency option, Eur Libor (US Libor) data is used to determine the domestic (foreign) interest rate and the following market data had to be taken from the forex market. This data includes: EUR/USD Exchange rate, Strike price, Bid and Ask [11]

The currency call option prices have been obtained from the Bloomberg database. This data will be used to calibrate the model. The time period is from May 28, 2020 to May 7, 2021 and we use 5 maturities and for each maturity we consider 5 strike prices (25 currency options in all). The domestic and foreign risk-free interest rates are also provided by Bloomberg and the available maturities match those of the options. We use In-The-Money (ITM) currency call options sorted by $\frac{S_t}{K}$.

4.1 Local calibration results

We have developed MATLAB scripts to obtain the following parameters:

ν	0
θ	0.0041
σ	0.3163
ρ	0.9925
κ	12.4467

using these parameters to estimate the following currency option prices:

[0.02332; 0.0206; 0.01800; 0.0155; 0.0132; 0.0506; 0.0456; 0.0408; 0.0363; 0.0321; 0.0501; 0.0463; 0.0427; 0.0393; 0.0361; 0.0556; 0.0521; 0.0487; 0.0455; 0.0424; 0.0560; 0.0527; 0.0496; 0.0466; 0.0437]

By taking the difference between these estimated prices and the market price (the price considered is the median price i.e., $Mid\ price = \frac{bid+ask}{2}$) and comparing this difference with the difference between the market bid and ask prices, we find that all the options considered meet the selection criteria chosen.

The table above shows that the calibrated Heston model gives an accurate match for all traded options. All currency options have a predicted value within the observed bid-ask spread. Moreover, the average deviation of the model from the average market price is 0.00068153, which is less than the average deviation of the bid-ask spreads (0.003978) when evaluated against our acceptance criterion. The time elapsed for the local calibration is 2.377027 seconds.

4.2 Global calibration results

We have developed MATLAB scripts to obtain the following parameters:

v	0
θ	0.0051
σ	0.2898
ρ	1
κ	12.5445
κ	12.4467

using these parameters to estimate the following currency option prices:

[0.0234; 0.0208; 0.0183; 0.0161; 0.0139; 0.0509; 0.0463; 0.0419; 0.0378; 0.0340; 0.0524; 0.0488; 0.0453; 0.0421; 0.0390; 0.0587; 0.0553; 0.0521; 0.0490; 0.0460; 0.0597; 0.0565; 0.0534; 0.0505; 0.0477]

By calculating the difference between these estimated prices and the market price (the price considered is the median price i.e., $Mid\ price = \frac{bid+ask}{2}$) by comparing this difference with the difference between the market bid and ask prices, we find that 21 options out of the 25 options considered meet the selection criterion.

As can be noticed, the parameters found with the ASA algorithm are somewhat different from those found in the local optimization case. Moreover, in the global calibration, 21 out of 25 satisfied the chosen selection criterion and the average deviation from the average market price is 0.00252278. Therefore, the quality of the ASA solution is less accurate than the results obtained by MATLAB's lsqnonlin. In addition, the main drawback of ASA is its significantly higher computation time (590.013684 seconds in ASA versus 6.5 seconds in MATLAB's lsqnonlin). Based on these analyses, we can confirm that MATLAB's lsqnonlin provides excellent calibration results (the average distance to the average market price is 0.00068153 versus 0.00252278 for ASA and all currency call option prices are within the observed bid-ask spreads), and also uses lower computation times.

However, these results depend on the complexity of the objective function; it is possible that this function is not complicated enough to take advantage of the ASA algorithm. In particular, since we generally do not know whether the objective function can have

multiple local minima, a prudent choice would be to use both calibration approaches. The drawback is that a global search does not necessarily improve the results delivered by a local search. However, progress in computing capacity and numerical methods is reducing the time required for global calibration. In our calibration, the execution time of ASA was less than 10 minutes, which for many practical applications is worth testing better solutions.

5. Conclusion

In this study, we used local and global optimization to look for the calibration parameters of the Heston stochastic volatility model used to estimate the prices of EUR/USD currency options. Our results showed that local optimization provides excellent calibration parameters with reduced computational time compared to global optimization. Therefore, we can clearly recommend the local optimization for the Heston model.

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Mixed Picture Fuzzy Graph

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Abstract

A new form of picture fuzzy graph has been identified and extended here as Mixed Picture Fuzzy Graph (MPFG). The picture fuzzy set is formed from the fuzzy set and the intuitionistic fuzzy set. It is helpful when there are multiple options, such as yes, no, rejection and abstain. MPFG, which is dependent on the picture fuzzy relation, is defined in this paper. The properties of various types of degrees, order and size of MPFG are examined. Also some types of MPFG such as regular, strong, complete and complement of MPFG are introduced and their properties were analysed. As an application part, the concept of MPFG has been applied in instagram and the result has been discussed here.

Keywords: picture fuzzy graph, MPFG, degree, size & order of MPFG, regular, strong, complete MPFG and complement of MPFG

2020 AMS subject classifications: 05C72 ¹

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1 Introduction

Many decision-making problems in unpredictable environments have been modelled using fuzzy graphs. A variety of generalisations of fuzzy graphs have really been implemented to deal with the uncertainty of complex real-life circumstances. Zadeh's(19) fuzzy set theory played a significant role in decision making in unpredictable environments. Rosenfeld(14), developed the basic conception of fuzzy graph 10 years after Zadeh's seminal article on fuzzy sets. As compared to the graph, the fuzzy graph seems to be a beneficial tool for modelling those problems because it is more efficient, flexible and compatible with any real-world problem. Mordeson & Nair(8) introduced the idea of a complement fuzzy graph, in which Sunitha & Kumar(17) expanded the concept.

The principle of Atanassov's(2) Intuitionistic Fuzzy Set (IFS) allocates a membership and non-membership degree individually, with the sum of the two degrees not exceeding the value one. Shannon and Atanassov proposed a description for intuitionistic fuzzy relations and intuitionistic fuzzy graphs, as well as a list of properties in (16). Different operations on intuitionistic fuzzy graphs were defined by Parvathi et al.(10; 11). Nagoor Gani and Shajitha Begum(9) has characterised about degree, order and size of intuitionistic fuzzy graphs.

The Picture Fuzzy Set (PFS) is a new idea that deals with uncertainties and is a direct continuation of the IFS. It can simulate uncertainty in circumstances including multiple types of answers: yes, abstention, rejection, and no. It is shown about one of the most fundamental concepts of degree of neutrality goes absent from the IFS principle. Cuong & Kreinovich(6) proposed PFS, a direct extension of fuzzy set and IFS that integrates the principle of positive, negative, and neutral membership degree of an element. Cuong(4) investigated some PFS properties and proposed distance measures between them. Phong and Co-authors(13) investigated some picture fuzzy relation compositions. Then, Cuong and Hai(5) extended some fuzzy logic operators for PFSs, including such conjunctions, complements, and disjunctions. Peng & Dai(12) proposed and implemented an algorithmic solution for PFS in a decision-making problem. New concepts of PFG with application was published by Cen Zuo et al.,(3). L. T. Koczy et al.(7) analyzed the study of social networks and Wi-Fi networks using the concept of picture fuzzy graphs. Wei Xiao, Arindam Dey, and Le Hoang Son(18) spoke about their research on regular picture fuzzy graphs and how they can be used in communication networks. Sankar Das and Ganesh Ghorai(15) investigated the creation of a road map based on a multigraph using picture fuzzy information. And Abdelkadir Muzey Mohammed(1) explained about mixed graph representation.

2 Basic Definitions

We stepped over some fundamental definitions in this section that are related to our main concept.

Definition 2.1. (3) Let $G_{pf}^* = (V, \mathcal{E})$ be a graph. A pair $G_{pf} = (A, B)$ is called a *picture fuzzy graph* on G^* where $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy set on V and $B = (\mu_B, \eta_B, \nu_B)$ is a picture fuzzy set on $\mathcal{E} \subseteq V \times V$ such that for each arc $vu \in \mathcal{E}$.

$$\left. \begin{aligned} \mu_B(v, u) &\leq \min(\mu_A(v), \mu_A(u)) \\ \eta_B(v, u) &\leq \min(\eta_A(v), \eta_A(u)) \\ \nu_B(v, u) &\geq \max(\nu_A(v), \nu_A(u)) \end{aligned} \right\} \quad (1)$$

denotes the degree of positive membership, neutral membership & membership membership of the edge $(v, u) \in \mathcal{E}$.

Definition 2.2. (1) A *mixed graph* $G_m = (V, \mathcal{E}, A)$ is a graph consists from set of vertices V , set of undirected edges $\mathcal{E}$ & set of directed edges(or arcs) A .

2.1 Notations

The following mathematical symbols were used throughout the paper:

G_{pf} —picture fuzzy graph

G_m —mixed graph

G_{mpf} —mixed picture fuzzy graph

$\mu_A(v), \eta_A(v), \nu_A(v)$ —positive, neutral & negative membership of a vertex v in G_{mpf}

$\mu_B(v, u), \eta_B(v, u), \nu_B(v, u)$ —positive, neutral & negative membership of an edge vu in G_{mpf}

$\mu_{\vec{B}}(v, u), \eta_{\vec{B}}(v, u), \nu_{\vec{B}}(v, u)$ —positive, neutral & negative membership of an arc vu in G_{mpf}

$d(v_i)$ —degree of a vertex v_i in G_{mpf}

$\delta(G_{mpf})$ —minimum degree of a G_{mpf}

$\Delta(G_{mpf})$ —maximum degree of a G_{mpf}

$(G_{mpf})^c$ —complement of a G_{mpf}

$(G_{mpf}^c)^c$ —complement of complement G_{mpf}

$O(G_{mpf})$ —order of a G_{mpf}

$S(G_{mpf})$ —size of a G_{mpf}

$\mathbb{S}_P$ —strength of a path P

H' —subgraph of G_{mpf}

$CD_{mpf}(v_i, v_j)$ —circle-distance between v_i and v_j of G_{mpf}

$C(S)$ —centrality of a squad

3 Mixed Picture Fuzzy Graph[MPFG]

The popularity of social media sites and networks are growing every day. Positive, neutral & negative membership of a vertex can be classified as good, neutral and bad activities in PFG. The situation now is why we should have to switch from PFG to MPFG? MPFG is the combination of both directed and undirected edges. Many real-life situations take the shape of MPFG. Further a real life problem has been identified and resolved using this MPFG.

Definition 3.1. Let $G_{mpf}^* = (V, \mathcal{E}, \vec{\mathcal{E}})$ be a graph. An ordered triple $G_{mpf} = (A, B, \vec{B})$ is called *mixed picture fuzzy graph* on G_{mpf}^* , where $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy set on V , $B = (\mu_B, \eta_B, \nu_B)$ is a picture fuzzy relation on the undirected edge $\mathcal{E} \subseteq V \times V$ and $\vec{B} = (\mu_{\vec{B}}, \eta_{\vec{B}}, \nu_{\vec{B}})$ is a picture fuzzy relation on the directed edge $\vec{\mathcal{E}} \subseteq V \times V$, which satisfies,

$$\left. \begin{aligned} \mu_B(v, u) &\leq \min(\mu_A(v), \mu_A(u)) \\ \eta_B(v, u) &\leq \min(\eta_A(v), \eta_A(u)) \\ \nu_B(v, u) &\geq \max(\nu_A(v), \nu_A(u)) \end{aligned} \right\} \forall (v, u) \in \mathcal{E} \quad \&$$

$$\left. \begin{aligned} \mu_{\vec{B}}(v, u) &\leq \min(\mu_A(v), \mu_A(u)) \\ \eta_{\vec{B}}(v, u) &\leq \min(\eta_A(v), \eta_A(u)) \\ \nu_{\vec{B}}(v, u) &\geq \max(\nu_A(v), \nu_A(u)) \end{aligned} \right\} \forall (v, u) \in \vec{\mathcal{E}} \quad (2)$$

Also $\vec{B}$ must not have a symmetric relation.

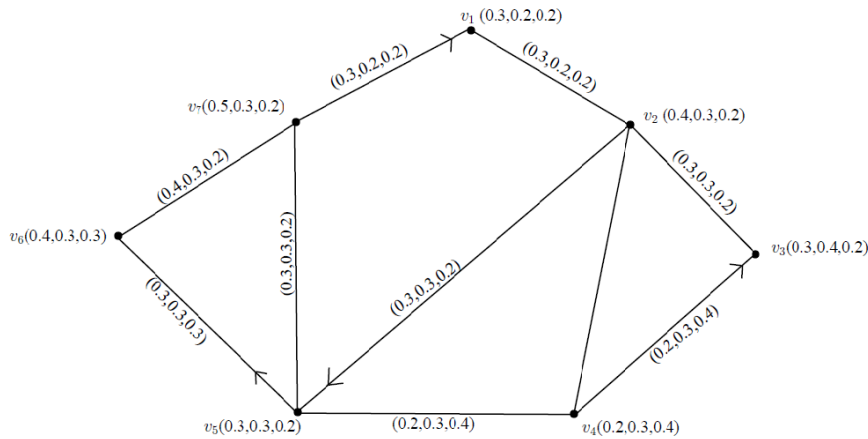


Figure 1: Mixed Picture Fuzzy Graph

Definition 3.2. Consider a graph $H' = (V', \mathcal{E}', \vec{\mathcal{E}}')$ is Mixed Picture Fuzzy Sub-graph (MPFSG) of MPFG if $V' \subseteq V, \mathcal{E}' \subseteq \mathcal{E}$ and $\vec{\mathcal{E}}' \subseteq \vec{\mathcal{E}}$ if, $\mu'_A(v) \leq \mu_A(v), \eta'_A(v) \leq \eta_A(v), \nu'_A(v) \geq \nu_A(v), \mu'_B(v, u) \leq \mu_B(v, u), \eta'_B(v, u) \leq \eta_B(v, u), \nu'_B(v, u) \geq \nu_B(v, u), \mu'_{\vec{B}}(v, u) \leq \mu_{\vec{B}}(v, u), \eta'_{\vec{B}}(v, u) \leq \eta_{\vec{B}}(v, u), \nu'_{\vec{B}}(v, u) \geq \nu_{\vec{B}}(v, u)$.

Theorem 3.1. A MPFG is a expansion of IFG.

Proof. The statement becomes trivial by assuming the neutral membership/abstain is equal to zero. Hence MPFG can reduce to IFG.

Similarly, the statement “A MPFG is a generalization of PFG” is also true. $\square$

Theorem 3.2. If $V = \{v_1, v_2, \dots, v_n\}$ is vertex set of MPFG, $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$. Then total number of edges denoted by $|\mathcal{E}_{mpf}|$ in MPFG G_{mpf} is given by,

$$|\mathcal{E}_{mpf}| = 1/2 \left[\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg_{in}(v) \right] \quad \text{or}$$

$$|\mathcal{E}_{mpf}| = 1/2 \left[\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg_{out}(v) \right]$$

Proof. Let $G_u = (V, \mathcal{E})$ be undirected subgraph of G_{mpf} and $G_d = (V, \vec{\mathcal{E}})$ with directed edges which are disjoint MPFSGs of MPFG $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$ such that

$$\mathcal{E}_{mpf} = \mathcal{E} \cup \vec{\mathcal{E}}$$

Handshaking theorem and Elementary counting principle, which states that

$$|\mathcal{E}| = \frac{1}{2} \sum_{v \in V} \deg(v) \quad \text{and} \quad |\vec{\mathcal{E}}| = \sum_{v \in V} \deg_{in}(v) = \sum_{v \in V} \deg_{out}(v) \quad (3)$$

$$|\mathcal{E}_{mpf}| = |\mathcal{E} \cup \vec{\mathcal{E}}| = |\vec{\mathcal{E}}| + |\mathcal{E}| - |\mathcal{E} \cap \vec{\mathcal{E}}| \quad (4)$$

since, $G_u = (V, \mathcal{E})$ and $G_d = (V, \vec{\mathcal{E}})$ are disjoint MPFSGs,

$$|\mathcal{E} \cap \vec{\mathcal{E}}| = 0$$

then (4) is reduced to

$$|\mathcal{E}_{mpf}| = |\vec{\mathcal{E}}| + |\mathcal{E}| \quad (5)$$

substituting (3) in (5), we get

$$|\mathcal{E}_{mpf}| = 1/2 \left[\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg_{in}(v) \right] \quad \text{or}$$

$$|\mathcal{E}_{mpf}| = 1/2 \left[\sum_{v \in V} \deg(v) + \sum_{v \in V} \deg_{out}(v) \right]$$

$$[\because \sum_{v \in V} \deg_{in}(v) = \sum_{v \in V} \deg_{out}(v)]$$

Hence proved $\square$

Definition 3.3. The *degree of a vertex* in a MPFG denoted as,

$$d(v_i) = (d_\mu(v_i), d_\eta(v_i), d_\nu(v_i))$$

where,

$$\left. \begin{aligned} d_\mu(v_i) &= \sum_{v \neq u} \mu_B(v, u) + \frac{1}{2} \left[\sum_{v \neq u} \mu_{\vec{B}_{in}}(v, u) + \sum_{v \neq u} \mu_{\vec{B}_{out}}(v, u) \right] \\ d_\eta(v_i) &= \sum_{v \neq u} \eta_B(v, u) + \frac{1}{2} \left[\sum_{v \neq u} \eta_{\vec{B}_{in}}(v, u) + \sum_{v \neq u} \eta_{\vec{B}_{out}}(v, u) \right] \\ d_\nu(v_i) &= \sum_{v \neq u} \nu_B(v, u) + \frac{1}{2} \left[\sum_{v \neq u} \nu_{\vec{B}_{in}}(v, u) + \sum_{v \neq u} \nu_{\vec{B}_{out}}(v, u) \right] \end{aligned} \right\} \quad (6)$$

From figure 1, we get, $d(v_1)=(0.45,0.3,0.3)$, $d(v_2)=(0.95,0.95,0.9)$, $d(v_3)=(0.4,0.45,0.4)$, $d(v_4)=(0.5,0.75,1.0)$, $d(v_5)=(0.8,0.9,0.85)$, $d(v_6)=(0.55,0.45,0.35)$, $d(v_7)=(0.45,0.3,0.3)$, $\delta(G_{mpf})=(0.4,0.3,0.3)$ and $\Delta(G_{mpf})=(0.95,0.95,1.0)$

Definition 3.4. Consider $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$ be a MPFG. The *neighbourhood of a vertex* is represented as,

$$Nh(v) = (Nh_\mu(v), Nh_\eta(v), Nh_\nu(v))$$

where,

$$\left. \begin{aligned} Nh_\mu(v) &= \{u \in V / \mu_B(v, u) = \min(\mu_A(v), \mu_A(u)), \mu_{\vec{B}}(v, u) = \min(\mu_A(v), \mu_A(u))\} \\ Nh_\eta(v) &= \{u \in V / \eta_B(v, u) = \min(\eta_A(v), \eta_A(u)), \eta_{\vec{B}}(v, u) = \min(\eta_A(v), \eta_A(u))\} \\ Nh_\nu(v) &= \{u \in V / \nu_B(v, u) = \max(\nu_A(v), \nu_A(u)), \nu_{\vec{B}}(v, u) = \max(\nu_A(v), \nu_A(u))\} \end{aligned} \right\} \quad (7)$$

and $Nh[v] = Nh(v) \cup \{v\}$ represents closed neighbourhood of a vertex.

Definition 3.5. The *neighbourhood degree of a vertex* is represented as,

$$d_{Nh}(v) = (d_{Nh_\mu}(v), d_{Nh_\eta}(v), d_{Nh_\nu}(v))$$

where,

$$\left. \begin{aligned} d_{Nh_\mu}(v) &= \sum_{u \in Nh(v)} \mu_A(u) \\ d_{Nh_\eta}(v) &= \sum_{u \in Nh(v)} \eta_A(u) \\ d_{Nh_\nu}(v) &= \sum_{u \in Nh(v)} \nu_A(u) \end{aligned} \right\} \quad (8)$$

Mixed Picture Fuzzy Graph

Note: If a vertex is an isolated vertex then $Nh(v) = \emptyset$

Definition 3.6. The *closed neighbourhood degree* of a vertex is denoted as,

$$d_{Nh}[v] = (d_{Nh_\mu}[v], d_{Nh_\eta}[v], d_{Nh_\nu}[v])$$

where,

$$\left. \begin{aligned} d_{Nh_\mu}[v] &= \sum_{u \in Nh(v)} \mu_A(u) + \mu_A(v), \\ d_{Nh_\eta}[v] &= \sum_{u \in Nh(v)} \eta_A(u) + \eta_A(v) \\ d_{Nh_\nu}[v] &= \sum_{u \in Nh(v)} \nu_A(u) + \nu_A(v) \end{aligned} \right\} \quad (9)$$

Definition 3.7. A *path* in $G_{mpf} = (A, B, \vec{B})$ is a distinct vertices sequence $v_0, v_1, v_2, \dots, v_k$ one of the succeeding responses are satisfied with both directed & undirected edges,

$$\begin{aligned} &\mu_B(v_{i-1}, v_i), \eta_B(v_{i-1}, v_i) > 0 \text{ and } \nu_B(v_{i-1}, v_i) = 0 \\ &\mu_B(v_{i-1}, v_i), \eta_B(v_{i-1}, v_i) = 0 \text{ and } \nu_B(v_{i-1}, v_i) > 0 \\ &\mu_B(v_{i-1}, v_i), \eta_B(v_{i-1}, v_i), \nu_B(v_{i-1}, v_i) > 0 \\ &\mu_{\vec{B}}(v_{i-1}, v_i), \eta_{\vec{B}}(v_{i-1}, v_i) > 0 \text{ and } \nu_{\vec{B}}(v_{i-1}, v_i) = 0 \\ &\mu_{\vec{B}}(v_{i-1}, v_i), \eta_{\vec{B}}(v_{i-1}, v_i) = 0 \text{ and } \nu_{\vec{B}}(v_{i-1}, v_i) > 0 \\ &\mu_{\vec{B}}(v_{i-1}, v_i), \eta_{\vec{B}}(v_{i-1}, v_i), \nu_{\vec{B}}(v_{i-1}, v_i) > 0 \end{aligned} \quad i = 1, 2, \dots, k.$$

Where k denotes the *length of the path*.

Definition 3.8. A MPFG $G_{mpf} = (A, B, \vec{B})$ seems to be *connected*, if each set of vertices possesses atleast 1 mixed picture fuzzy path connecting them, else it is said to be disconnected.

Definition 3.9. If there is a path $P = v_n, v_1, \dots, v_n$ for $n \geq 3$ then it's a *cycle*.

Definition 3.10. The *complement* of a $G_{mpf} = (A, B, \vec{B})$ is a $G_{mpf}^c = (A^c, B^c, \vec{B}^c)$ iff it follows,

$\mu_A^c = \mu_A, \eta_A^c = \eta_A, \nu_A^c = \nu_A$ and

$$\left. \begin{aligned} \mu_B^c(v, u) &= \min(\mu_A(v), \mu_A(u)) - \mu_B(v, u) \\ \eta_B^c(v, u) &= \min(\eta_A(v), \eta_A(u)) - \eta_B(v, u) \\ \nu_B^c(v, u) &= \max(\nu_A(v), \nu_A(u)) - \nu_B(v, u) \\ \mu_{\vec{B}}^c(v, u) &= \min(\mu_A(v), \mu_A(u)) - \mu_{\vec{B}}(v, u) \\ \eta_{\vec{B}}^c(v, u) &= \min(\eta_A(v), \eta_A(u)) - \eta_{\vec{B}}(v, u) \\ \nu_{\vec{B}}^c(v, u) &= \max(\nu_A(v), \nu_A(u)) - \nu_{\vec{B}}(v, u) \end{aligned} \right\} \quad (10)$$

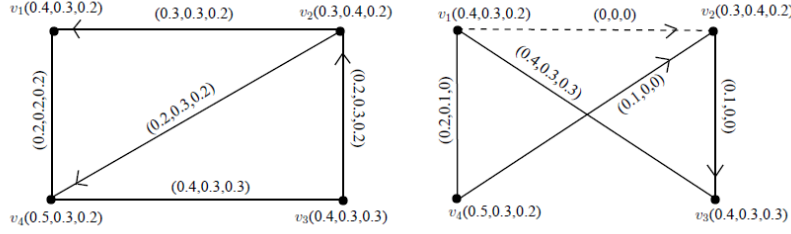


Figure 2: Complement of Mixed Picture Fuzzy Graph

Theorem 3.3. If G_{mpf}^c be a complement of MPFG, then $(G_{mpf}^c)^c = G$

Note: A MPFG is self-complementary if $(G_{mpf}^c)^c = G$

Definition 3.11. The order of a G_{mpf} is represented by,

$$O(G_{mpf}) = (O_\mu(G_{mpf}), O_\eta(G_{mpf}), O_\nu(G_{mpf}))$$

where,

$$\left. \begin{aligned} O_\mu(G_{mpf}) &= \sum_{u \in V} \mu_A(v) \\ O_\eta(G_{mpf}) &= \sum_{u \in V} \eta_A(v) \\ O_\nu(G_{mpf}) &= \sum_{u \in V} \nu_A(v) \end{aligned} \right\} \quad (11)$$

Here $O_\mu(G_{mpf})$, $O_\eta(G_{mpf})$ & $O_\nu(G_{mpf})$ are the order of positive, neutral & negative membership degree respectively.

Definition 3.12. Let $G_{mpf} = (A, B, \vec{B})$ is MPFG. The size of a G_{mpf} is represented by,

$$\mathcal{S}(G_{mpf}) = (\mathcal{S}_\mu(G_{mpf}), \mathcal{S}_\eta(G_{mpf}), \mathcal{S}_\nu(G_{mpf}))$$

where,

$$\left. \begin{aligned} \mathcal{S}_\mu(G_{mpf}) &= \sum_{v,u \in V} \mu_B(v, u) + \sum_{v,u \in V} \mu_{\vec{B}}(v, u) \\ \mathcal{S}_\eta(G_{mpf}) &= \sum_{v,u \in V} \eta_B(v, u) + \sum_{v,u \in V} \eta_{\vec{B}}(v, u) \\ \mathcal{S}_\nu(G_{mpf}) &= \sum_{v,u \in V} \nu_B(v, u) + \sum_{v,u \in V} \nu_{\vec{B}}(v, u), \forall j \neq i. \end{aligned} \right\} \quad (12)$$

Here $\mathcal{S}_\mu(G_{mpf})$, $\mathcal{S}_\eta(G_{mpf})$ and $\mathcal{S}_\nu(G_{mpf})$ are the size of positive, neutral & negative membership respectively.

Definition 3.13. For a path P,

$$\left. \begin{aligned} \mathbb{S}_\mu &= \min_{v,u \in V} \{\mu_B(v, u)\} + \min_{v,u \in V} \{\mu_{\vec{B}}(v, u)\} \\ \mathbb{S}_\eta &= \min_{v,u \in V} \{\eta_B(v, u)\} + \min_{v,u \in V} \{\eta_{\vec{B}}(v, u)\} \\ \mathbb{S}_\nu &= \max_{v,u \in V} \{\nu_B(v, u)\} + \max_{v,u \in V} \{\nu_{\vec{B}}(v, u)\} \end{aligned} \right\} \quad (13)$$

The *strength of a path* $\mathbb{S}_P = (\mathbb{S}_\mu, \mathbb{S}_\eta, \mathbb{S}_\nu)$.

Definition 3.14. A $G_{mpf} = (A, B, \vec{B})$ is said to be *strong MPFG* if,

$$\left. \begin{aligned} \mu_B(v, u) &= \min(\mu_A(v), \mu_A(u)) \\ \eta_B(v, u) &= \min(\eta_A(v), \eta_A(u)) \\ \nu_B(v, u) &= \max(\nu_A(v), \nu_A(u)), \forall (v, u) \in \mathcal{E} \text{ \& } \\ \mu_{\vec{B}}(v, u) &= \min(\mu_A(v), \mu_A(u)) \\ \eta_{\vec{B}}(v, u) &= \min(\eta_A(v), \eta_A(u)) \\ \nu_{\vec{B}}(v, u) &= \max(\nu_A(v), \nu_A(u)), \forall (u, v) \in \vec{\mathcal{E}}. \end{aligned} \right\} \quad (14)$$

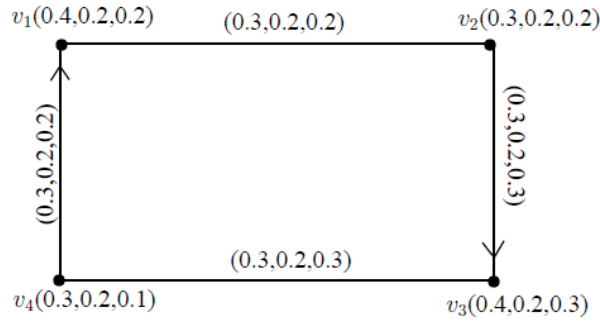


Figure 3: Strong Mixed Picture Fuzzy Graph

Note: $(G_{mpf}^c)^c = G_{mpf}$ iff G is strong MPFG

Definition 3.15. A MPFG $G_{mpf} = (A, B, \vec{B})$ is said to be *complete MPFG* if,

$$\left. \begin{aligned} \mu_B(v, u) &= \min(\mu_A(v), \mu_A(u)) \\ \eta_B(v, u) &= \min(\eta_A(v), \eta_A(u)) \\ \nu_B(v, u) &= \max(\nu_A(v), \nu_A(u)) \text{ and } \\ \mu_{\vec{B}}(v, u) &= \min(\mu_A(v), \mu_A(u)) \\ \eta_{\vec{B}}(v, u) &= \min(\eta_A(v), \eta_A(u)) \\ \nu_{\vec{B}}(v, u) &= \max(\nu_A(v), \nu_A(u)), \forall v, u \in V \end{aligned} \right\} \quad (15)$$

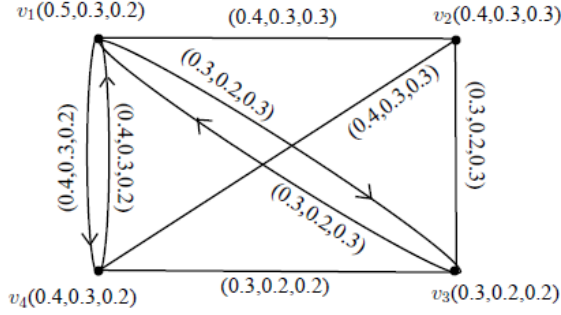


Figure 4: Complete Mixed Picture Fuzzy Graph

Note: Every complete MPFG becomes a strong MPFG. But the contrary, does not have to be true.

Theorem 3.4. *The order of a complete MPFG is equal to the closed neighbourhood degree of every vertex (i.e), $O_\mu(G_{mpf}) = \{d_{N_\mu}[v]|v \in V\}$, $O_\eta(G_{mpf}) = \{d_{N_\eta}[v]|v \in V\}$, $O_\nu(G_{mpf}) = \{d_{N_\nu}[v]|v \in V\}$.*

Proof. Consider $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$ be a complete MPFG. The μ , η and ν -order of G_{mpf} , is the sum of the positive, neutral and negative membership value of each vertex respectively.

We know that, if G_{mpf} is a complete MPFG, then the closed neighbourhood μ , η and ν -degree of every vertex is the sum of the positive membership, neutral membership & negative membership values of the vertices respectively. Therefore, $O_\mu(G_{mpf}) = \{d_{N_\mu}[v]|v \in V\}$, $O_\eta(G_{mpf}) = \{d_{N_\eta}[v]|v \in V\}$, $O_\nu(G_{mpf}) = \{d_{N_\nu}[v]|v \in V\}$. Hence the result. $\square$

Definition 3.16. A MPFG $G_{mpf} = (A, B, \vec{B})$ is defined as *regular MPFG* if,

$$\left. \begin{aligned} \mu_B(v, u) &= \min(\mu_A(v), \mu_A(u)) \text{ and } \sum_{u \neq v} \mu_B(u, v) = \text{constant}, \\ \eta_B(v, u) &= \min(\eta_A(v), \eta_A(u)) \text{ and } \sum_{u \neq v} \eta_B(u, v) = \text{constant}, \\ \nu_B(v, u) &= \max(\nu_A(v), \nu_A(u)) \text{ and } \sum_{u \neq v} \nu_B(u, v) = \text{constant}, \\ \mu_{\vec{B}}(v, u) &= \min(\mu_A(v), \mu_A(u)) \text{ and } \sum_{u \neq v} \mu_{\vec{B}}(u, v) = \text{constant}, \\ \eta_{\vec{B}}(v, u) &= \min(\eta_A(v), \eta_A(u)) \text{ and } \sum_{u \neq v} \eta_{\vec{B}}(u, v) = \text{constant}, \\ \nu_{\vec{B}}(v, u) &= \max(\nu_A(v), \nu_A(u)) \text{ and } \sum_{u \neq v} \nu_{\vec{B}}(u, v) = \text{constant}. \end{aligned} \right\} \quad (16)$$

Mixed Picture Fuzzy Graph

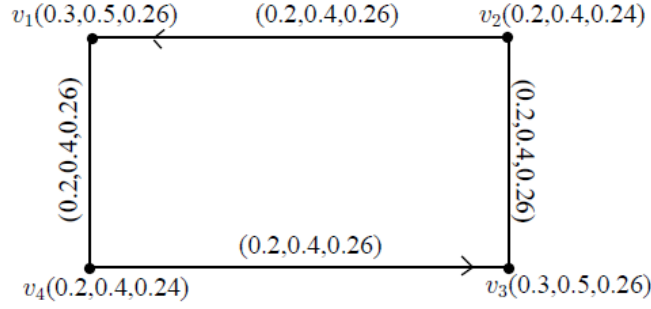


Figure 5: Regular Mixed Picture Fuzzy Graph

Theorem 3.5. Every complete MPFG is a regular MPFG.

Proof. Consider $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$ be a MPFG. From the definition of complete MPFG we have, $\mu_B(v, u) = \min(\mu_A(v), \mu_A(u))$, $\eta_B(v, u) = \min(\eta_A(v), \eta_A(u))$, $\nu_B(v, u) = \max(\nu_A(v), \nu_A(u))$ and $\mu_{\vec{B}}(v, u) = \min(\mu_A(v), \mu_A(u))$, $\eta_{\vec{B}}(v, u) = \min(\eta_A(v), \eta_A(u))$, $\nu_{\vec{B}}(v, u) = \max(\nu_A(v), \nu_A(u)) \forall v, u \in V$.

Then, the closed neighbourhood μ , η and ν -degree of every vertex is the sum of the positive membership, neutral membership & negative membership values of the vertices and itself respectively. As a result, the closed neighbourhood μ -degree, closed neighbourhood η -degree, & closed neighbourhood ν -degree were the same for all vertices. Therefore, min. closed neighbourhood degree is equal to max. closed neighbourhood degree. Hence G_{mpf} is a regular MPFG. $\square$

Definition 3.17. Let $G_{mpf} = (A, B, \vec{B})$ be a MPFG. If two vertices v & u are linked by a length of a path k in G_{mpf} is $P : v_0, v_1, v_2, \dots, v_{n-1}, v_n$ then $\mu_B(v, u)$, $\eta_B(v, u)$, $\nu_B(v, u)$ and $\mu_{\vec{B}}(v, u)$, $\eta_{\vec{B}}(v, u)$, $\nu_{\vec{B}}(v, u)$ are described as follows

$$\begin{aligned}\mu_B^k(v, u) &= \min\{\mu_B(v, v_1), \mu_B(v_1, v_2), \dots, \mu_B(v_{k-1}, u)\} \\ \eta_B^k(v, u) &= \min\{\eta_B(v, v_1), \eta_B(v_1, v_2), \dots, \eta_B(v_{k-1}, u)\} \\ \nu_B^k(v, u) &= \max\{\nu_B(v, v_1), \nu_B(v_1, v_2), \dots, \nu_B(v_{k-1}, u)\} \\ \mu_{\vec{B}}^k(v, u) &= \min\{\mu_{\vec{B}}(v, v_1), \mu_{\vec{B}}(v_1, v_2), \dots, \mu_{\vec{B}}(v_{k-1}, u)\} \\ \eta_{\vec{B}}^k(v, u) &= \min\{\eta_{\vec{B}}(v, v_1), \eta_{\vec{B}}(v_1, v_2), \dots, \eta_{\vec{B}}(v_{k-1}, u)\} \\ \nu_{\vec{B}}^k(v, u) &= \max\{\nu_{\vec{B}}(v, v_1), \nu_{\vec{B}}(v_1, v_2), \dots, \nu_{\vec{B}}(v_{k-1}, u)\}\end{aligned}$$

Let $\mu^\infty(v, u)$, $\eta^\infty(v, u)$, $\nu^\infty(v, u)$ is *Strength of connectedness* between the

two nodes v & u of MPFG.

$$\begin{aligned}\mu_B^\infty(v, u) &= \sup\{\mu_B^k(v, u)/k = 1, 2, \dots\} \\ \eta_B^\infty(v, u) &= \sup\{\eta_B^k(v, u)/k = 1, 2, \dots\} \\ \nu_B^\infty(v, u) &= \inf\{\nu_B^k(v, u)/k = 1, 2, \dots\} \\ \mu_{\vec{B}}^\infty(v, u) &= \sup\{\mu_{\vec{B}}^k(v, u)/k = 1, 2, \dots\} \\ \eta_{\vec{B}}^\infty(v, u) &= \sup\{\eta_{\vec{B}}^k(v, u)/k = 1, 2, \dots\} \\ \nu_{\vec{B}}^\infty(v, u) &= \inf\{\nu_{\vec{B}}^k(v, u)/k = 1, 2, \dots\}\end{aligned}$$

here $\inf$ has been used to determine the minimum membership value and $\sup$ is used to determine the maximum membership value.

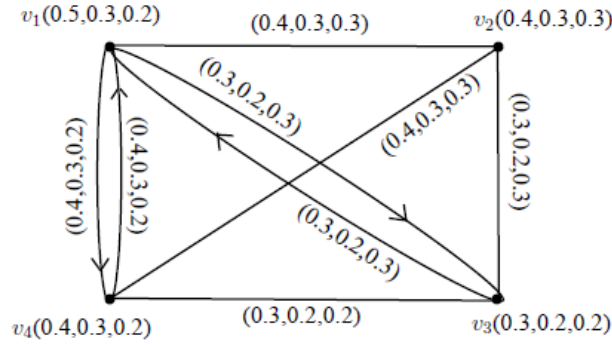


Figure 6: Strength of connectedness

Consider a conneted MPFG as shown in the figure 6

The possible paths between v_1 to v_4 are

- $P_1 : v_1 - v_4$ along with the value of membership $(0.4, 0.3, 0.2)$
- $P_2 : v_1 - v_2 - v_4$ along with the value of membership $(0.4, 0.3, 0.3)$
- $P_3 : v_1 - v_3 - v_4$ along with the value of membership $(0.3, 0.2, 0.3)$
- $P_4 : v_1 - v_2 - v_3 - v_4$ along with the value of membership $(0.3, 0.2, 0.3)$
- $P_5 : v_1 - v_2 - v_3 - v_4$ along with the value of membership $(0.3, 0.2, 0.3)$
- $P_6 : v_1 - v_3 - v_2 - v_4$ along with the value of membership $(0.3, 0.2, 0.3)$

We've arrived to this conclusion through routine calculations,

$$\mu^\infty(v_1, v_4) = \sup\{0.4, 0.4, 0.3, 0.3, 0.3, 0.3\} = 0.4$$

$$\eta^\infty(v_1, v_4) = \sup\{0.3, 0.3, 0.2, 0.2, 0.2, 0.2\} = 0.3$$

$$\nu^\infty(v_1, v_4) = \inf\{0.2, 0.3, 0.3, 0.3, 0.3, 0.3\} = 0.2$$

The *strength of connectedness* between 2 vertices v_1 & v_4 of a MPFG is $(0.4, 0.3, 0.2)$

Mixed Picture Fuzzy Graph

Definition 3.18. Consider $G_{mpf} = (V, \mathcal{E}, \vec{\mathcal{E}})$ be a MPFG & v, u be any two distinct vertices. In G_{mpf} , eliminating an edge or arc (v, u) decreases the strength between some pair of vertices and is described to as a bridge.

Definition 3.19. Let $G'_{mpf} = (A_1, B_1, \vec{B}_1)$ and $G''_{mpf} = (A_2, B_2, \vec{B}_2)$ be two MPFGs. A *homomorphism* $h : G'_{mpf} \rightarrow G''_{mpf}$ is a mapping function h from V_1 to V_2 if:

- $\mu_{A_1}(v_1) \leq \mu_{A_2}(h(v_1))$
 $\eta_{A_1}(v_1) \leq \eta_{A_2}(h(v_1))$
 $\nu_{A_1}(v_1) \geq \nu_{A_2}(h(v_1))$
- $\mu_{B_1}(v_1, u_1) \leq \mu_{B_2}(h(v_1), h(v_2))$
 $\eta_{B_1}(v_1, u_1) \leq \eta_{B_2}(h(v_1), h(v_2))$
 $\nu_{B_1}(v_1, u_1) \geq \nu_{B_2}(h(v_1), h(v_2)), \forall v_1 \in V_1 \quad \& \quad v_1, u_1 \in E_1$
- $\mu_{\vec{B}_1}(v_1, u_1) \leq \mu_{\vec{B}_2}(h(v_1), h(v_2))$
 $\eta_{\vec{B}_1}(v_1, u_1) \leq \eta_{\vec{B}_2}(h(v_1), h(v_2))$
 $\nu_{\vec{B}_1}(v_1, u_1) \geq \nu_{\vec{B}_2}(h(v_1), h(v_2)), \forall v_1 \in V_1 \quad \& \quad v_1, u_1 \in \vec{E}_1$

Definition 3.20. Let $G'_{mpf} = (A_1, B_1, \vec{B}_1)$ and $G''_{mpf} = (A_2, B_2, \vec{B}_2)$ be two MPFGs. An *isomorphism* $h : G'_{mpf} \rightarrow G''_{mpf}$ is a bijective mapping function h from V_1 to V_2 if:

- $\mu_{A_1}(v_1) = \mu_{A_2}(h(v_1))$
 $\eta_{A_1}(v_1) = \eta_{A_2}(h(v_1))$
 $\nu_{A_1}(v_1) = \nu_{A_2}(h(v_1))$
- $\mu_{B_1}(v_1, u_1) = \mu_{B_2}(h(v_1), h(v_2))$
 $\eta_{B_1}(v_1, u_1) = \eta_{B_2}(h(v_1), h(v_2))$
 $\nu_{B_1}(v_1, u_1) = \nu_{B_2}(h(v_1), h(v_2)), \forall v_1 \in V_1 \quad \& \quad v_1, u_1 \in E_1$
- $\mu_{\vec{B}_1}(v_1, u_1) = \mu_{\vec{B}_2}(h(v_1), h(v_2))$
 $\eta_{\vec{B}_1}(v_1, u_1) = \eta_{\vec{B}_2}(h(v_1), h(v_2))$
 $\nu_{\vec{B}_1}(v_1, u_1) = \nu_{\vec{B}_2}(h(v_1), h(v_2)), \forall v_1 \in V_1 \quad \& \quad v_1, u_1 \in \vec{E}_1$

Theorem 3.6. *Isomorphism of MPFG is an equivalence relation.*

Proof. For show that MPFG isomorphism is an equivalence relation, we must first prove that it is reflexive, symmetric, and transitive.

Reflexive: consider $\theta : G_{mpf} \rightarrow G_{mpf}$ is a mapping, therefore θ is an identity function. Hence it is reflexive.

Symmetric: In isomorphic MPFG G_{mpf} & H_{mpf} , there exist a 1-1 correspondence $\theta : G_{mpf} \rightarrow H_{mpf}$ which sustains adjacency. From θ is 1-1 correspondence from G_{mpf} to H_{mpf} , here 1-1 correspondence θ^{-1} from H_{mpf} to G_{mpf} which sustains adjacency. Hence isomorphism of MPFG is symmetric.

Transitive: If G_{mpf} is isomorphic to H_{mpf} and H_{mpf} is isomorphic to K_{mpf} , then there are 1-1 correspondences between θ & ϕ from G_{mpf} to H_{mpf} & H_{mpf} to K_{mpf} respectively, which sustains adjacency. It follows $\phi \circ \theta$ is a 1-1 correspondence between from G_{mpf} to K_{mpf} which sustains adjacency. Hence it is transitive.

Therefore, isomorphism of MPFG is an equivalence relation. $\square$

Definition 3.21. Let $G_{mpf} = (A, B, \vec{B})$ be a MPFG. A vertex v of G_{mpf} is said to be *busy vertex* if $\mu_A(v) \leq d_\mu(v), \eta_A(v) \leq d_\eta(v), \nu_A(v) \geq d_\nu(v)$. otherwise, it is called *free vertex*.

Definition 3.22. Let $G_{mpf} = (A, B, \vec{B})$ be a MPFG. Then an edge (v, u) is defined as an *effective edge* iff $\mu_B(v, u) = \min(\mu_A(v), \mu_A(u)), \eta_B(v, u) = \min(\eta_A(v), \eta_A(u)), \nu_B(v, u) = \max(\nu_A(v), \nu_A(u)), \mu_{\vec{B}}(v, u) = \min(\mu_A(v), \mu_A(u)), \eta_{\vec{B}}(v, u) = \min(\eta_A(v), \eta_A(u)), \nu_{\vec{B}}(v, u) = \max(\nu_A(v), \nu_A(u))$.

Note: When all edges in a graph are effective, the graph is complete.

4 Application of MPFG in instagram

Social media has grown gaining popularity in latest years of its user-friendliness. Social media services such as Whatsapp, Facebook, Twitter, and Instagram allow people to communicate across long distances. To put it another way, social media has made the entire globe available at the touch of a button. Social media sites are also valuable resources for public awareness creation, as they rapidly distribute information about natural disasters and terrorist/criminal attacks to a mass audience.

Social network is a collection of vertices and edges. Persons, groups, countries, associations, locations, business and other entities are represented by vertices, while edges define the relationship between vertices. We commonly use a classical graph to describe a social network, with vertices representing persons and edges representing relationships/flows between vertices. Several manuscripts

have been shared on social media platforms. However, a classical graph cannot accurately model a social network. Since all vertices in a classical graph are extremely significant. As a result, in today's social networks, every social units (personal or organisational) are given equal weight. In fact, however, not all social units are equal in importance. In a classical graph, all edges (relationships) have the same weight. For example, a person may be well-versed in certain practises. On the other hand, they have no experience of certain activities, and he has a very little knowledge of others. We can easily represent these three kinds (positive, neutral and negative) of vertex and edge membership degrees with a picture fuzzy set, which has three membership values for each element.

In Instagram, we can classify three activities namely good, neutral and bad activities which is represented in PFG as positive, neutral & negative membership values of a vertex. Similarly, edge membership value can be used to describe the strength of relationship between two vertices. Since social media has such a vast number of clients, it also contains mutual and single-sided relationships; it is not restricted to directed or undirected relationships. As a result, we have introduced a mixed picture fuzzy graph which includes both directed and undirected edges. It provides a more accurate result than previous methods. For example, in Instagram an undirected edge exists when two friends have a mutual relationship. Similarly, if a friend-1 follows friend-2 but friend-2 doesn't then there occurs directed edge.

The vertex effect on a social media platform is identified via centrality, which is one of the most significant concepts in social networking. The degree of centrality determines how closely a social squad is linked to other social squads. It essentially provides the social squad's/person's participation in the social network. A vertex's centrality seems more central than that of other vertex's. The centre people are muchis closer to the others and has access to more information. It should be noticed that a person's information is shared by a friend of a friend. However, friends of friends communicate less information than direct friends. As a result, the importance of the relationship gradually decreases as it passes from one member to the next along a connected path.

In MPFG, suppose a friend-1 directly connected to a friend-2, then we say v_1 is circle distance-1(CD-1) friend of v_2 . The set of all CD-1 friends of v represented as $cd_1(v)$. i.e., $cd_1(v) = \{v_i \in V; v_i \text{ is a CD-1 friends of } v\}$. Correspondingly, suppose there is a shortest path between v_1 & v_2 with m edges, then v_1 is a CD- m friend of v_2 . That is, $cd_m(v) = \{v_i \in V; v_i \text{ is a CD-}m \text{ friends of } v\}$. Now, consider $cd'_m(v) = cd_m(v) - cd'_{m-1}(v)$, where $m = 2, 3, \dots$ and $cd'_m(v) = cd_m(v)$.

CD-1 friends are obviously more significant than CD-2 friends, and CD-2 friends are more significant than CD-3 friends, and so on. The linguistic term "more significant" could be denoted by weights(w_m). Let $0 \leq w_m \leq 1$ have been the weights that gradually decreases, when the CD between the friends increases. Then $w_1 \geq w_2 \geq \dots \geq w_m \geq \dots$

Let $u_1(=v_i), u_2, u_3, \dots, u_m(=v_j)$ are the vertices upon the path between v_i and v_j . We have to derive MPFCD $CD_{mpf}(v_i, v_j)$ between v_i and v_j with this path as

$$CD_{mpf}(v_i, v_j) = \sum_{n=1}^{m-1} \mu(u_n, u_{n+1}) + \sum_{n=1}^{m-1} \eta(u_n, u_{n+1}) + \sum_{n=1}^{m-1} \nu(u_n, u_{n+1})$$

There could be several paths connecting two vertices in a networks. Let us assume these paths of equal length whose MPFCD (CD_{mpf}) seems to be the maximum in G_{mpf} . Suppose these are n edgeds in this path of maximum MPFCD, we designate CD_{mpf}^n i.e, $CD_{mpf}^n(v_i, v_j)$ represents the MPFCD between the vertices v_i & v_j in MPFG with the particular path having accurately n edges. We stated that social squad S with atmost CD - p friends.

The centrality $C(S)$ of a social squad is defined as follows:

$$C(S) = \sum_{u_1 \in cd_1^1(v)} w_1 CD_{mpf}^1(V, u_1) + \sum_{u_2 \in cd_2^1(v)} w_2 CD_{mpf}^2(V, u_2) + \dots + \sum_{u_p \in cd_p^1(v)} w_p CD_{mpf}^p(V, u_p) \quad (17)$$

Close friends are valued more than the next closest friends, while the significance of the furthest friend gradually decreases. The significance is established by including the weight w_i , which stands for CD - i friend, $i = 1, 2, 3, \dots$. For example, MPFG of 7 people after 7 days is shown in figure 7. Also the link membership values are shown in same figure. In the definition of centrality of a social unit, p can be taken as fixed for a social network. Here we assumed that $p = 3$ and measure the centrality of social squad. Here, we take $w_1 = 1$ and $w_{i+1} = 1/2w_i$, $i = 1, 2, \dots$.

4.1 Centrality of v_1

Here $cd_1(v_1) = \{v_2, v_4, v_3\} = cd_1'(v_1)$, $cd_2(v_1) = \{v_3, v_5, v_7, v_2\}$, $cd_2'(v_1) = cd_2(v_1) - cd_1'(v_1) = \{v_5, v_7\}$, $cd_3(v_1) = \{v_6, v_7, v_3\}$, $cd_3'(v_1) = cd_3(v_1) - cd_2'(v_1) = \{v_6, v_3\}$.

Mixed Picture Fuzzy Graph

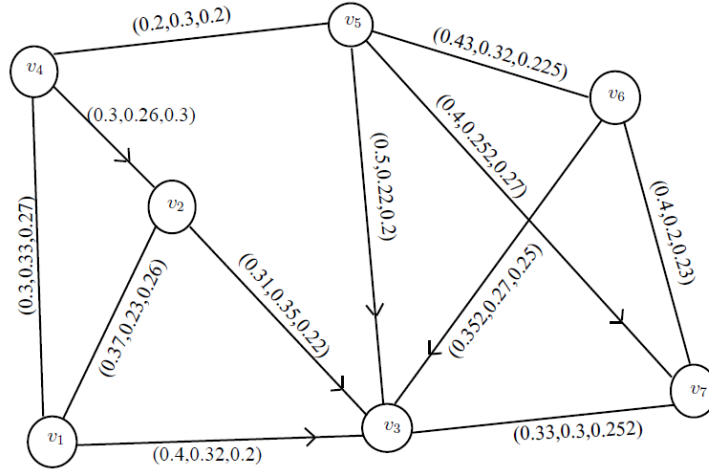


Figure 7: Mixed picture fuzzy network

Now,

$$\begin{aligned}
 \sum_{u_1 \in (cd_1)'(v_1)} (CD_{mpf})^1(v_1, u_1) &= \{\text{membership values of}(v_1, v_2)\} + \{\text{membership values of}(v_1, v_4)\} + \{\text{membership values of}(v_1, v_3)\} \\
 &= (\mu(v_1, v_2), \eta(v_1, v_2), \nu(v_1, v_2)) + ((\mu(v_1, v_4), \eta(v_1, v_4), \nu(v_1, v_4)) + (\mu(v_1, v_3), \eta(v_1, v_3), \nu(v_1, v_3))) \\
 &= (0.37, 0.23, 0.26) + (0.3, 0.33, 0.27) + (0.4, 0.32, 0.2) \\
 &= (1.07, 0.88, 0.73).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{u_2 \in (cd_2)'(v_1)} (CD_{mpf})^2(v_1, u_2) &= (CD_{mpf})^2(v_1, v_5) + (CD_{mpf})^2(v_1, v_7) \\
 &= \{\text{membership values of}\{(v_1, v_5) + (v_1, v_5)\}\} + \{\text{membership values of}\{(v_1, v_3) + (v_3, v_7)\}\} \\
 &= \{(0.3, 0.33, 0.27) + (0.2, 0.3, 0.2)\} + \{(0.4, 0.32, 0.2) + (0.33, 0.3, 0.252)\} \\
 &= (1.23, 0.25, 0.922).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{u_3 \in (cd_3)'(v_1)} (CD_{mpf})^3(v_1, u_3) &= (CD_{mpf})^3(v_1, v_6) + (CD_{mpf})^3(v_1, v_3) \\
 &= \{\text{membership values of}\{(v_1, v_3) + (v_3, v_7) \\
 &\quad + (v_7, v_6)\}\} + \{\text{membership values of}\{(v_1, v_4) \\
 &\quad + (v_4, v_5) + (v_5, v_3)\}\} \\
 &= \{(0.4, 0.32, 0.2) + (0.33, 0.3, 0.252) \\
 &\quad + (0.4, 0.2, 0.23)\} + \{(0.3, 0.33, 0.27) \\
 &\quad + (0.2, 0.3, 0.2) + (0.52, 0.27, 0.25)\} \\
 &= (2.15, 1.72, 1.402).
 \end{aligned}$$

The centrality of v_1 is

$$\begin{aligned}
 C(v_1) &= \sum_{u_1 \in cd'_1(v_1)} w_1 CD_{mpf}^1(v_1, u_1) + \sum_{u_2 \in cd'_2(v_1)} 0.5 \times CD_{mpf}^2(v_1, u_2) \\
 &\quad + \sum_{u_3 \in cd'_3(v_1)} 0.25 \times CD_{mpf}^3(v_1, u_3) \\
 &= (2.2225, 1.935, 1.5415)
 \end{aligned}$$

Similarly, we can calculate centralities of other vertices.

$$C(v_2) = (1.8125, 1.6125, 1.363), C(v_3) = (0.985, 0.755, 0.66975),$$

$$C(v_4) = (1.715, 1.781, 1.4775), C(v_5) = (2.818, 2.317, 1.91225),$$

$$C(v_6) = (2.8005, 2.1905, 1.824), C(v_7) = (3.442, 2.33, 2.042).$$

4.1.1 Discussion

Suppose there are more than one paths between two vertices, we have to choose the shortest distance path to calculate the centrality. From the results, we have centrality of v_3 is comparatively less than other vertices. Because v_3 has less number of mutual friends. So degree of centrality depends on mutual friends and friends of circle distance-i.

Social networks are built on the backs of millions of users and massive amounts of data. We used a simple numerical example of a MPFG to describe a small social network problem in this study. The smaller examples are really useful in understanding the benefits of our suggested model.

5 Conclusion

The prime goal of this paper is to just introduce the terms and concepts of a MPFG and examined the various types of MPFG. Initially, we present a definition of an MPFG built from a picture fuzzy graph in this paper. Few types of degrees were discussed with its properties. We discuss about regular, strong, complete and complement of MPFG are some of the different forms of MPFG. The isomorphic property has also been analysed in MPFG. When comparing to picture fuzzy graph models, the MPFG can boost effectiveness, reliability, flexibility and comparability in modelling complex real-world scenarios. A model has been developed to represent a social network problem using MPFG. The concept of a MPFG can be used to a database system, a computer network, a traffic signal system, a social network, a transportation network and image processing among other things.

6 Acknowledgements

Future work is to develop this concept in the field of transversals of MPFG.

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Common Fixed Point for a Contractive Mapping via (α, β, ψ) -Admissibility in b -metric space

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Abstract

In this paper, we establish the concept of a common fixed point theorem for new type of generalized contractive mappings. Furthermore, we employ our main result to shows a common fixed point theorem for a pair of self-mappings (R, S) in b -metric space via (α, β, ψ) -admissibility type contractive condition. An example is also given to verify the main result.

Keywords: α -admissible mapping, common fixed point, b -metric spaces.

2020 AMS subject classifications: 47H10, 54H25, 54M20.

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1 Introduction

In the last fifty years, fixed point theories lie in finding and proving the uniqueness of solutions for many questions of Applied Sciences such as Physics, Chemistry, Economics, and Engineering. In 1922, Stefan Banach [S.Banach [1922]] proved a fixed point theorem for contractive mappings in complete metric spaces. In 1969, Nadler [Nadler [1969]] introduced the concept of multi-value function. Later, Czerwik [Czerwik [1993]] and Bakhtin [Bakhtin [1989]] initiate the concept of b-metrics metric space. Khan [Khan et al. [1984]] introduced the altering distance mapping to formulate a new contractive condition in fixed point theory in order to extend the Banach fixed point theorem to new forms. For some extension to the Banach contraction theorem. Recently, Abodayeh et al. [Abodayeh et al. [2017]] introduced a new notion, named almost perfect function, to formulate new contractive conditions to modify and extend some fixed point theorems known in the literature. Now, we mention the notions of altering distance function and almost perfect function.

2 Preliminaries

Definition 2.1 (Khan et al. [1984]). *A self-function ψ on $\mathbb{R}^+ \cup \{0\}$ is called an altering distance function if ψ satisfies the following conditions:*

- (1) $\psi(s) = 0 \iff s = 0$.
- (2) ψ is a nondecreasing and continuous function.

Definition 2.2 (Abodayeh et al. [2017]). *A nondecreasing self-function ψ on $\mathbb{R}^+ \cup \{0\}$ is called an almost perfect function if ψ satisfies the following conditions:*

- (1) $\psi(s) = 0 \iff s = 0$.
- (2) *If for all sequence (s_n) in $\mathbb{R}^+ \cup \{0\}$ with $\psi(s_n) \rightarrow 0$ it holds $s_n \rightarrow 0$.*

Definition 2.3 (Samet et al. [2012]). *Let R be a self-mapping on X and $\alpha : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be a function. Then, R is called α -admissible if for all $v, w \in X$ with $\alpha(v, w) \geq 1$ it holds $\alpha(Rv, Rw) \geq 1$.*

The definition of triangular α -admissibility for a single mapping

Definition 2.4 (Karupinar et al. [2013]). *Let R be a self-mapping on X and $\alpha : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$. Then, we call R triangular α -admissible if*

- (1) R is α -admissible; and

Common fixed point for a contractive mapping via (α, β, ψ) -admissibility in b -metric space

- (2) For all $v, w, u \in X$ with $\alpha(v, w) \geq 1$ and $\alpha(w, u) \geq 1$ it holds $\alpha(v, u) \geq 1$.

Definition 2.5 (Abdeljawad [2013]). Let R and S be two self mappings on X and $\alpha : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be a function. Then, the pair (R, S) is called α -admissible if $z, w \in X$ and $\alpha(z, w) \geq 1$ imply $\alpha(Rz, Sw) \geq 1$ and $\alpha(Sz, Rw) \geq 1$.

Definition 2.6 (Hussain et al. [2014]). Let D_b be a metric on a set X and $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be functions. Then, X is called α, β -complete if and only if $\{x_n\}$ is a Cauchy sequence in X and $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply (x_n) converges to some $x \in X$.

Definition 2.7 (Hussain et al. [2014]). Let D_b be a metric on a set X and $\alpha, \eta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be functions. A self-mapping S on X is called α, β -continuous if $\{x_n\}$ is a sequence in X , $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$.

Definition 2.8 (Mehemet and Kiziltunc [2013]). Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $D_b : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ is called a b -metric provide that, for all $x, y, z \in X$,

- (1) $D_b(x, y) = 0$ if and only if $x = y$ (non-negative axiom)
- (2) $D_b(x, y) = D_b(y, x)$ (symmetric axiom)
- (3) $D_b(x, z) \leq s[D_b(x, y) + D_b(y, z)]$. (s -Triangular inequality).

A pair (X, D_b) is called a b -metric space.

Definition 2.9. Let R, S , be two self-mappings on the set X and $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be functions. We say that (R, S) is a pair of (α, β) -admissibility if $z, w \in X$ and $\alpha(z, w) \geq \beta(z, w)$ imply $\alpha(Rz, Sw) \geq \beta(Rz, Sw)$ and $\alpha(Sz, Rw) \geq \beta(Sz, Rw)$.

Example 2.1. Define self-mappings R and S on a set of real numbers by $Ru = u^2$ and

$$Su = \begin{cases} -u^2, & \text{if } u < 0; \\ u^2, & \text{if } u \geq 0. \end{cases}$$

Additionally, define $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ via $\alpha(u, v) = e^{u+v}$ and $\beta(u, v) = e^u$. Then, (R, S) is a pair of (α, β) -admissibility.

3 Main Results

Definition 3.1. Let ψ be a nondecreasing function on $\mathbb{R}^+ \cup \{0\}$. We call ψ a perfect control function if the following conditions hold:

- (i) $\psi(t) = 0 \iff t = 0$.
- (ii) If (t_n) is a sequence in $\mathbb{R}^+ \cup \{0\}$ and $\psi(t_n) \rightarrow 0$ as $n \rightarrow +\infty$ implies $t_n \rightarrow 0$ as $n \rightarrow +\infty$.
- (iii) $\psi(u + v) \leq \psi(u) + \psi(v)$ for all $u, v \in \mathbb{R}^+ \cup \{0\}$.
- (iv) $\psi^n(\lambda x) = \lambda^n \psi(x)$.

Definition 3.2. Let (X, D_b) be a b -metric space with constant $s \geq 1$. Let R, S be two self-mappings on X , ψ be a perfect self-mapping on $\mathbb{R}^+ \cup \{0\}$, $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be functions. We say that the pair (R, S) is an (α, β, ψ) -Admissibility type contraction if there exists $\lambda \in [0, 1)$ such that $z, w \in X$ and $\alpha(z, w) \geq \beta(z, w)$ imply

$$\begin{aligned} \psi(D_b(Rz, Sw)) &\leq \lambda\psi(D_b(z, w)) + \lambda\psi(D_b(z, Rz)) + \lambda\psi(D_b(w, Sw)) \\ &\quad + \lambda\psi(D_b(w, Rz)) + \lambda\psi(D_b(z, Sw)) \end{aligned} \quad (1)$$

and

$$\begin{aligned} \psi(D_b(Sz, Rw)) &\leq \lambda\psi(D_b(z, w)) + \lambda\psi(D_b(z, Sz)) + \lambda\psi(D_b(w, Rw)) \\ &\quad + \lambda\psi(D_b(w, Sz)) + \lambda\psi(D_b(z, Rw)) \end{aligned} \quad (2)$$

Theorem 3.1. Let (X, D_b) be a b -metric space with constant $s \geq 1$. Let $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ be function and (R, S) be a self-mappings on X . Assume following conditions:

- (i) (X, D_b) is an α, β -complete b -metric space.
- (ii) R and S are α, β -continuous.
- (iii) (R, S) is pair of (α, β) -admissibility.
- (iv) If v, w, z are in X , with $\alpha(v, w) \geq \beta(v, w)$ and $\alpha(w, z) \geq \beta(w, z)$, then $\alpha(v, z) \geq \beta(v, z)$.
- (v) There exists $x_0 \in X$ such that $\alpha(Rx_0, SRx_0) \geq \beta(Rx_0, SRx_0)$ and $\alpha(SRx_0, Rx_0) \geq \beta(SRx_0, Rx_0)$.

Then R and S have a common fixed point.

Common fixed point for a contractive mapping via (α, β, ψ) -admissibility in b -metric space

Proof. In view of condition (v) we start with $x_0 \in X$ in such away that $\alpha(Rx_0, SRx_0) \geq \beta(Rx_0, SRx_0)$ and $\alpha(SRx_0, Rx_0) \geq \beta(SRx_0, Rx_0)$. Now, let $x_1 = Rx_0$ and $x_2 = Sx_1$. Then $\alpha(x_0, x_1) \geq \beta(x_0, x_1)$ and $\alpha(x_1, x_0) \geq \beta(x_1, x_0)$. In view of condition (iii), we have

$$\alpha(x_1, x_2) = \alpha(Rx_0, Sx_1) \geq \beta(Rx_0, Sx_1) = \beta(x_1, x_2)$$

and

$$\alpha(x_2, x_1) = \alpha(Sx_1, Rx_0) \geq \beta(Sx_1, Rx_0) = \beta(x_2, x_1)$$

Again we put $x_3 = Sx_2$. Then condition (iii) implies that

$$\alpha(x_2, x_3) = \alpha(Sx_1, Rx_2) \geq \beta(Sx_1, Rx_2) = \beta(x_2, x_3)$$

and

$$\alpha(x_3, x_2) = \alpha(Rx_2, Sx_1) \geq \beta(Rx_2, Sx_1) = \beta(x_3, x_2)$$

Putting $x_4 = Sx_3$ and referring to condition (iii) we conclude

$$\alpha(x_3, x_4) = \alpha(Rx_2, Sx_3) \geq \beta(Rx_2, Sx_3) = \beta(x_3, x_4)$$

and

$$\alpha(x_4, x_3) = \alpha(Sx_3, Rx_2) \geq \beta(Sx_3, Rx_2) = \beta(x_4, x_3)$$

Continuing in the same manner, we contract a sequence (x_n) in X with $x_{2n+1} = Rx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ such that

$$\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}) \quad \forall n \in \mathbb{N}$$

and

$$\alpha(x_{n+1}, x_n) \geq \beta(x_{n+1}, x_n) \quad \forall n \in \mathbb{N}$$

From condition (iv) we see that

$$\alpha(x_n, x_m) \geq \beta(x_n, x_m) \quad \forall n, m \in \mathbb{N}$$

If there exists $q \in \mathbb{N}$ such that $x_{2q} = x_{2q+1}$, then $x_{2q} = Rx_{2q}$ and hence R has a fixed point. From contractive condition (I), we have

$$\begin{aligned} & \psi(D_b(x_{2q+1}, x_{2q+2})) = \psi(D_b(Rx_{2q}, Sx_{2q+1})) \\ & \leq \psi \left(\begin{aligned} & \lambda(D_b(x_{2q}, x_{2q+1})) + \lambda(D_b(x_{2q}, Rx_{2q})) + \lambda(D_b(x_{2q+1}, Sx_{2q+1})) \\ & + \lambda(D_b(x_{2q+1}, Rx_{2q})) + \lambda(D_b(x_{2q}, Sx_{2q+1})) \end{aligned} \right) \\ & \leq \psi \left(\begin{aligned} & 2\lambda(D_b(x_{2q}, x_{2q+1})) + \lambda(D_b(x_{2q+1}, x_{2q+2})) \\ & + \lambda s[(D_b(x_{2q}, x_{2q+1}) + D_b(x_{2q+1}, x_{2q+2}))] \end{aligned} \right) \\ & \leq \psi \left(\begin{aligned} & \lambda(2+s)(D_b(x_{2q}, x_{2q+1})) \\ & + \lambda(1+s)(D_b(x_{2q+1}, x_{2q+2})) \end{aligned} \right) \end{aligned}$$

$$\leq \psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)} (D_b(x_{2q}, x_{2q+1})) \right) \quad (3)$$

The last inequality is correct only if $\psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)} (D_b(x_{2q}, x_{2q+1})) \right) = 0$. The properties of ψ and D_b imply that $x_{2q+1} = x_{2q+2}$. Hence, $x_{2q} = Rx_{2q} = Sx_{2q}$. Thus, R and S have a common fixed point of R and S . If there exists $q \in \mathbb{N}$ such that $x_{2q+1} = x_{2q+2}$ then $x_{2q+1} = Tx_{2q+1}$ and hence S has a fixed point.

From contractive condition (2), we have

$$\begin{aligned} \psi(D_b(x_{2q+2}, x_{2q+3})) &= \psi(D_b(Sx_{2q+1}, Rx_{2q+2})) \\ &\leq \psi \left(\begin{aligned} &\lambda(D_b(x_{2q+1}, x_{2q+2})) + \lambda(D_b(x_{2q+1}, Sx_{2q+1})) + \lambda(D_b(x_{2q+2}, Rx_{2q+2})) \\ &+ \lambda(D_b(x_{2q+2}, Sx_{2q+1})) + \lambda(D_b(x_{2q+1}, Rx_{2q+2})) \end{aligned} \right) \\ &\leq \psi \left(\begin{aligned} &2\lambda(D_b(x_{2q+1}, x_{2q+2})) + \lambda(D_b(x_{2q+2}, x_{2q+3})) \\ &+ \lambda s[(D_b(x_{2q+1}, x_{2q+2}) + D_b(x_{2q+2}, x_{2q+3}))] \end{aligned} \right) \\ &\leq \psi \left(\begin{aligned} &\lambda(2+s)(D_b(x_{2q+1}, x_{2q+2})) \\ &+ \lambda(1+s)(D_b(x_{2q+2}, x_{2q+3})) \end{aligned} \right) \\ &\leq \psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)} (D_b(x_{2q+1}, x_{2q+2})) \right) \end{aligned} \quad (4)$$

The last inequality is correct only if $\psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)} (D_b(x_{2q+1}, x_{2q+2})) \right) = 0$. The properties of ψ and D_b imply that $x_{2q+2} = x_{2q+3}$. Hence, $x_{2q+1} = Rx_{2q+1} = Sx_{2q+1}$. Thus, R and S have a common fixed point of R and S .

Now, assume that $x_n \neq x_{n+1} \forall n \in \mathbb{N}$. For $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} \psi(D_b(x_{2n+1}, x_{2n+2})) &= \psi(D_b(Rx_{2n}, Sx_{2n+1})) \\ &\leq \psi \left(\begin{aligned} &\lambda(D_b(x_{2n}, x_{2n+1})) + \lambda(D_b(x_{2n}, Rx_{2n})) + \lambda(D_b(x_{2n+1}, Sx_{2n+1})) \\ &+ \lambda(D_b(x_{2n+1}, Rx_{2n})) + \lambda(D_b(x_{2n}, Sx_{2n+1})) \end{aligned} \right) \\ &\leq \psi \left(\begin{aligned} &2\lambda(D_b(x_{2n}, x_{2n+1})) + \lambda(D_b(x_{2n+1}, x_{2n+2})) \\ &+ \lambda s[(D_b(x_{2n}, x_{2n+1}) + D_b(x_{2n+1}, x_{2n+2}))] \end{aligned} \right) \\ &\leq \psi \left(\begin{aligned} &\lambda(2+s)(D_b(x_{2n}, x_{2n+1})) \\ &+ \lambda(1+s)(D_b(x_{2n+1}, x_{2n+2})) \end{aligned} \right) \\ &\leq \psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)} (D_b(x_{2n}, x_{2n+1})) \right) \end{aligned} \quad (5)$$

Let $[\delta = \frac{\lambda(2+s)}{1-\lambda(1+s)}]$. Hence

$$\psi(D_b(x_{2n+1}, x_{2n+2})) \leq \psi(\delta(D_b(x_{2n}, x_{2n+1})))$$

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Using argument similar to the above, we may show that

$$\begin{aligned}
 \psi(D_b(x_{2n}, x_{2n+1})) &= \psi(D_b(Sx_{2n-1}, Rx_{2n})) \\
 &\leq \psi \left(\begin{aligned} &\lambda(D_b(x_{2n-1}, x_{2n})) + \lambda(D_b(x_{2n-1}, Sx_{2n-1})) + \lambda(D_b(x_{2n}, Rx_{2n})) \\ &+ \lambda(D_b(x_{2n}, Sx_{2n-1})) + \lambda(D_b(x_{2n-1}, Rx_{2n})) \end{aligned} \right) \\
 &\leq \psi \left(\begin{aligned} &2\lambda(D_b(x_{2n-1}, x_{2n})) + \lambda(D_b(x_{2n}, x_{2n+1})) \\ &+ \lambda s[(D_b(x_{2n-1}, x_{2n}) + D_b(x_{2n}, x_{2n+1}))] \end{aligned} \right) \\
 &\leq \psi \left(\begin{aligned} &\lambda(2+s)(D_b(x_{2n-1}, x_{2n})) \\ &+ \lambda(1+s)(D_b(x_{2n}, x_{2n+1})) \end{aligned} \right) \\
 &\leq \psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2n-1}, x_{2n})) \right) \\
 &\leq \psi \left((\delta(D_b(x_{2n-1}, x_{2n}))) \right) \tag{6}
 \end{aligned}$$

Combining equation (5) and (6) together, we reach

$$\begin{aligned}
 \psi(D_b(x_n, x_{n+1})) &= \psi(D_b(Sx_{n-1}, Rx_n)) \\
 &\leq \psi \left(\begin{aligned} &\lambda(D_b(x_{n-1}, x_n)) + \lambda(D_b(x_{n-1}, Sx_{n-1})) + \lambda(D_b(x_n, Rx_n)) \\ &+ \lambda(D_b(x_n, Sx_{n-1})) + \lambda(D_b(x_{n-1}, Rx_n)) \end{aligned} \right) \\
 &\leq \psi \left(\begin{aligned} &2\lambda(D_b(x_{n-1}, x_n)) + \lambda(D_b(x_n, x_{n+1})) \\ &+ \lambda s[(D_b(x_{n-1}, x_n) + D_b(x_n, x_{n+1}))] \end{aligned} \right) \\
 &\leq \psi \left(\begin{aligned} &\lambda(2+s)(D_b(x_{n-1}, x_n)) \\ &+ \lambda(1+s)(D_b(x_n, x_{n+1})) \end{aligned} \right) \\
 &\leq \psi \left(\frac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{n-1}, x_n)) \right) \\
 &\leq \psi \left((\delta(D_b(x_{n-1}, x_n))) \right) \tag{7}
 \end{aligned}$$

By recurring equation (7) n -times, we deduce

$$\begin{aligned}
 \psi(D_b(x_n, x_{n+1})) &\leq \psi(\delta(D_b(x_{n-1}, x_n))) \\
 &\leq \delta\psi(D_b(x_{n-2}, x_{n-1})) \\
 &\leq \delta(\delta\psi(D_b(x_{n-2}, x_{n-1}))) \\
 &= \delta^2\psi(D_b(x_{n-2}, x_{n-1})) \\
 &\vdots
 \end{aligned}$$

$$\leq \delta^n \psi(D_b(x_0, x_1)). \quad (8)$$

On allowing $n \rightarrow \infty$ in equation (8), we get

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_{n+1})) = 0 \quad (9)$$

The properties of ψ implies that

$$\lim_{n \rightarrow +\infty} D_b(x_n, x_{n+1}) = 0 \quad (10)$$

We intend to prove that (x_n) is Cauchy sequence in X , take $n, m \in \mathbb{N}$ with $m > n$. We divide the proof into four cases:

Case 1: n is an odd integer and m is an even integer. Therefore, there exists $t \in \mathbb{N}$ and an odd integer h such that $n = 2t + 1$ and $m = 2t + 1 + h$. Since $\alpha(x_n, x_m) \geq \beta(x_n, x_m)$, we have

$$\begin{aligned} \psi(D_b(x_n, x_m)) &= \psi(D_b(x_{2t+1}, x_{2t+1+h})) \\ &= \psi(D_b(Rx_{2t}, Sx_{(2t+h)})) \\ &\leq \psi \left(\lambda(D_b(x_{2t}, x_{2t+h})) + \lambda(D_b(x_{2t}, Rx_{2t})) + \lambda(D_b(x_{2t+h}, Sx_{2t+h})) \right. \\ &\quad \left. + \lambda(D_b(x_{2t+h}, Rx_{2t})) + \lambda(D_b(x_{2t}, Sx_{2t+h})) \right) \\ &= \psi \left(\lambda(D_b(x_{2t}, x_{2t+h})) + \lambda(D_b(x_{2t}, x_{2t+1})) + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) \right. \\ &\quad \left. + \lambda(D_b(x_{2t+h}, x_{2t+1})) + \lambda(D_b(x_{2t}, x_{2t+1+h})) \right) \\ &\leq \psi \left(\lambda \sum_{i=2t}^{2t+h-1} (D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) \right. \\ &\quad \left. + \lambda \sum_{i=2t+1}^{2t+h-1} (D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1+h})) \right) \\ &\leq \psi \left(\lambda(2+s) \sum_{i=2t}^{\infty} (D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) \right. \\ &\quad \left. + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \right) \end{aligned}$$

Where, $k = \lambda(2+s)$

$$\begin{aligned} &\leq \psi \left(k \sum_{i=2t}^{\infty} (D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) \right. \\ &\quad \left. + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \right) \\ &\leq \psi \left(\frac{k^{2t+1}}{1-k} (D_b(x_0, x_1)) + \lambda(D_b(x_{2t}, x_{2t+1})) \right. \\ &\quad \left. + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \right) \end{aligned}$$

By permitting $n, m \rightarrow \infty$ in above inequalities and considering equation (9)

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_m)) = 0$$

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The properties of ψ implies that

$$\lim_{n \rightarrow +\infty} D_b(x_n, x_m) = 0 \quad (11)$$

Case 2: n and m are both even integers. Applying the triangular inequality of the b -metric d_b , we have

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + D_b(x_{n+1}, x_m)], \text{ for } m \geq n$$

Letting $n \rightarrow \infty$ and in view of equation (10) and (11), we get

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_m)) = 0.$$

Case 3: n is an even integer and m is an odd integer. Applying the triangular inequality of the b -metric D_b , we have

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + D_b(x_{n+1}, x_m)]$$

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + s[D_b(x_{n+1}, x_{m-1}) + D_b(x_{m-1}, x_m)]], \text{ for } m \geq n$$

On permitting $m, n \rightarrow \infty$ and considering equation (10) and (11), we get

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_m)) = 0.$$

Case 4: n and m are both odd integers. Applying the triangular inequality of the b -metric D_b , we have

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{m-1}) + D_b(x_{m-1}, x_m)], \text{ for } m \geq n$$

On permitting $n \rightarrow \infty$ and in view of equation (10) and (11), we get

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_m)) = 0.$$

Combining all cases with each other, we conclude that

$$\lim_{n \rightarrow +\infty} \psi(D_b(x_n, x_m)) = 0.$$

Thus, we conclude that (x_n) is a Cauchy sequence in X . The α, β -completeness of the b -metric space (X, D_b) ensures that there is $x \in X$ such that $x_n \rightarrow x$. Using the α, β -continuity of the mappings R and S , we deduce that $x_{2n+1} = Rx_{2n} \rightarrow Rx$ and $x_{2n+2} = Sx_{2n+1} \rightarrow Sx$. by uniqueness of limit, we obtain $Rx = Sx = x$. Thus, x is a fixed point of R . $\square$

Example 3.1. Define $D_b : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ by $D_b(z, w) = |z - w|$ and let R, S be two self-mappings on $\mathbb{R}_0^+$ define by $Rz = \frac{z}{2}$ and $Sw = \frac{w}{4}$. In addition, define the function $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ by $\psi^n \lambda x = \lambda^n \psi(x)$, where $\psi(x) = \frac{x}{1+x}$. Furthermore, the functions $\alpha, \beta : X \times X \rightarrow \mathbb{R}_0^+$ define by

$$\alpha(p, q) = \begin{cases} e^{p+q}, & \text{if } p, q \in [0, 1]; \\ 0, & \text{if } p > 1 \text{ or } q > 1. \end{cases} \quad \text{and } \beta(p, q) = \begin{cases} e^p, & \text{if } p, q \in [0, 1]; \\ 1, & \text{if } p > 1 \text{ or } q > 1. \end{cases}.$$

Then:

1. ψ is a perfect control function.
2. There exists $u_0 \in X$ such that

$$\alpha(Ru_0, Ru_1) \geq \beta(Ru_0, Ru_1) \text{ and } \alpha(Ru_1, Ru_0) \geq \beta(Ru_1, Ru_0).$$
3. (R, S) is a pair of (α, β) -admissibility.
4. R and S are α, β -continuous.
5. (X, D_b) is an α, β -complete b -metric space.
6. (R, S) is an (α, β, ψ) -contraction.

Proof. It is an easy matter to see equations (1) to (3). To prove (4), let (u_n) be any sequence in $\mathbb{R}_0^+$ whenever $u_n \rightarrow u \in \mathbb{R}_0^+$ and $\alpha(u_n, u_{n+1}) \geq \beta(u_n, u_{n+1}) \forall n \in \mathbb{N}$.

Case 1: If $u_n = u$ for all n , where $u_n \in [0, 1] \forall n \in \mathbb{N}$. We conclude that $Ru_n \rightarrow Ru$ as $n \rightarrow \infty$.

Case 2: If $u_n \neq u$, for all n , we notice that $u = 0$. Hence, $u_n \rightarrow 0$ in $([0, 1], |\cdot|)$. Therefore, $|\frac{u}{2}, 0| \rightarrow 0 = Ru$ in $(\mathbb{R}_0^+, D_b)$; that is R is α, β -continuous.

To prove (5), let (u_n) be a Cauchy sequence in $(\mathbb{R}_0^+, D_b)$ such that

$$\alpha(u_n, u_{n+1}) \geq \beta(u_n, u_{n+1}).$$

Then, $u_n \in [0, 1] \forall n \in \mathbb{N}$. If there exists $u \in [0, 1]$ such that $u_n = u$ for all n , then, $u_n \rightarrow u$ as $n \rightarrow +\infty$. Now, suppose the elements of (u_n) are distinct. Give $\epsilon > 0$, since (u_n) is a Cauchy sequence in $(\mathbb{R}_0^+, D_b)$, then there exists $n_0 \in \mathbb{N}$ such that $|u_n, u_m| < \epsilon \forall m > n \geq n_0$. Therefore, $|u_n, 0| < \epsilon \forall n \geq n_0$. So, $u_n \rightarrow 0$ in $(\mathbb{R}_0^+, D_b)$. Thus, $(\mathbb{R}_0^+, D_b)$ is an α, β -complete b -metric space.

To prove (6), let $z, w \in X$ be such that $\alpha(z, w) \geq \beta(z, w)$. Then, $z, w \in [0, 1]$.

Common fixed point for a contractive mapping via (α, β, ψ) -admissibility in b -metric space

So,

$$\begin{aligned}
 \psi(D_b(Rz, Sw)) &= \psi(D_b(\frac{z}{2}, \frac{w}{4})) \\
 &= \psi(|\frac{z}{2}, \frac{w}{4}|) \\
 &= \frac{|\frac{z}{2}, \frac{w}{4}|}{1+|\frac{z}{2}, \frac{w}{4}|} \\
 &= \frac{\frac{1}{2}|z, \frac{w}{2}|}{1+\frac{1}{2}|z, \frac{w}{2}|} \\
 &= \frac{|z, \frac{w}{2}|}{2+|z, \frac{w}{2}|} \\
 &\leq \frac{1}{6}(\frac{|z, w|}{1+|z, w|}) \\
 &\leq \frac{1}{6}\psi(D_b(z, w))
 \end{aligned}$$

$$\begin{aligned}
 \psi(D_b(Rz, Sw)) &\leq \frac{1}{6}\psi(D_b(z, w)) + \frac{1}{6}\psi(D_b(z, Rz)) + \frac{1}{6}\psi(D_b(w, Sw)) + \\
 &\quad \frac{1}{6}\psi(D_b(w, Rz)) + \frac{1}{6}\psi(D_b(z, Sw)), \quad (\because \lambda = \frac{1}{6})
 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
 \psi(D_b(Sz, Rw)) &\leq \frac{1}{6}\psi(D_b(z, w)) + \frac{1}{6}\psi(D_b(z, Sz)) + \frac{1}{6}\psi(D_b(w, Rw)) + \\
 &\quad \frac{1}{6}\psi(D_b(w, Sz)) + \frac{1}{6}\psi(D_b(z, Rw)), \quad (\because \lambda = \frac{1}{6})
 \end{aligned}$$

Hence, R and S satisfy definition 3.2. Therefore, R and S satisfy all the condition of theorem. Therefore, R and S have a common fixed point.

4 Conclusions

In Theorem 3.1 we have formulated a new contractive conditions to modify and extend some common fixed point theorem for a pair of self-mappings (R, S) in b -metric space via (α, β, ψ) -admissibility type. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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The Adomian Decomposition Method for Standard Power Options

S. J. Ghevariya*

Abstract

Black-Scholes model derived by Black and Scholes is worldwide used mathematical model for valuing option price. This model brings a new quantitative approach for researcher to finding theoretical values of options. They derived a model of European options for plain vanilla payoffs. Black-Scholes model derived from Black-Scholes differential equation, which is parabolic in nature. In this paper, a well-known, accurate, simple, semi-analytical method, Adomian decomposition method (ADM) is used for Black-Scholes differential equation for standard power payoffs. This model is the generalization of plain vanilla payoffs. Further, it can be seen that the cumulative distribution function of standard normal random variable is used in the closed form formulas of standard power options, while our formulas do not involve any term regarding random variable. In fact our formulas are impressive, fruitful and very close to the closed form formulas. Numerical results shows that our approach gives very accurate results.

Keywords: Black-Scholes theory; Adomian Decomposition Method; Standard Power Options.

2020 AMS subject classifications: Primary: 91G20. Secondary: 91B25, 35Q91.¹

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1 Introduction

In finance, risk is one of the serious factor for all traders in financial markets. Financial derivatives are used to manage risk significantly. In fact, financial derivatives minimized risk by transferring it to the other traders or financial institutes which are linked with the asset price. Financial derivatives are the contracts between two parties or financial institutes. Options, Futures, Forwards are examples of financial derivatives. Option is one of the important derivatives for not only investors or traders but also for economic point of view. Options help to diversify the portfolios of investors. Option is a contract to buy or sell the underlying asset at pre-determined price and pre-determined date. The pre-determined price is known as the striking price and pre-determined date is known as expiry date of option. Two types of options are available due to its trading strategies. One is European and the other is American. European option is exercised at the expiry date only, while American option can be exercised before expiry date. Further, option can be classified in two types, namely call and put. Call (put) option gives holder the right to buy (sell) the underlying asset at striking price and expiry date. Acquiring such type of freedom one has to sacrifices by paying some amount at the beginning of the contract. This amount is known as option price or premium. Nowadays, option pricing become one of the crucial problem in financial world. The economists Black and Scholes obtained formula of option pricing for plain vanilla payoffs in 1973, which is known as Black-Scholes model. The amounts $\max\{S - K, 0\}$ and $\max\{K - S, 0\}$ are known as plain vanilla payoffs, where S is the value of asset price at expiry date of option and K is the striking price of option. Due to widely acceptance of this model, Black and Scholes awarded by a Nobel prize in 1997. After that, many researcher worked with different payoffs [Dedania and Ghevariya, 2013a,b, Ghevariya and Thakkar, 2019, R. J. Haber, P. J. Schonbucher and Wilmott, 1999, Haug, 2007, P. Wilmott, S. Howison and Dewynne, 1993]. Black-Scholes model derived from Black-Scholes differential equation using boundary conditions. This equation is a second order linear partial differential equation of parabolic in nature. The solution of Black-Scholes equation become intrinsic goal of many researchers. So different methods have been used to solve Black-Scholes equation to get approximate or closed form solutions with various payoffs like binomial method (See P. Wilmott, S. Howison and Dewynne [2002]), projected differential transform method (PDTM) [S. O. Edeki, O. O. Ugberbor and Owoloko, 2015, S. O. Edeki, R. M. Jena, O. P. Ogundile and Chakraverty, 2021, Ghevariya, 2020, 2022b], homotopy perturbation method (HPM) [Ghevariya, 2022a], Mellin transform [Fadugba and Nwozo, 2016, Ghevariya, 2018, 2019, Panini and Srivastav, 2004], etc. In this paper, we presented a model by solving Black-Scholes differential equation for standard power payoffs, $\max\{S^p - K\}$ and $\max\{K - S^p\}$, where $p \in (0, \infty)$ using Adomian de-

composition method (ADM). This model is the generalization of model for plain vanilla payoffs by taking $p = 1$. Section-2 deals with the introduction of ADM. In section-3, the derivation of Black-Scholes model for standard power options using ADM has been discussed. Section-4 deals with comparisons of option prices of closed form formulas and using ADM. Discussions related to derived models have been given in section-5. Conclusion is given in the last section.

2 Adomian Decomposition Method (ADM)

The ADM is a simple, semi-analytical and accurate method used for solving linear as well as nonlinear differential equations. The ADM was introduced by George Adomian (See Adomian [1986, 1994]). In recent years, this method is very popular in applied mathematics, in particular, area related to initial and boundary value problems. The approach of this method is to decompose nonlinear differential operator into series of polynomials known as Adomian polynomials. The advantage of this method is to provide an efficient and accurate solution in terms of series which converges to the exact solution in most of the cases. The convergence of series has been discussed in [Cherruault and Adomian, 1993, Cherruault, 1989]. In this section, we discuss basic idea about ADM. For that, consider general form of partial differential equation

$$\mathcal{L}_y[u(x, y)] + \mathcal{R}[u(x, y)] + \mathcal{N}[u(x, y)] = g(x, y), \quad (1)$$

where $\mathcal{L}_y$ is an invertible operator which is differentiable with respect to y , $\mathcal{R}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator and g is any function. Let $u(x, 0) = f(x)$ be initial condition satisfying by Equation (1). Note that Equation (1) can be written as

$$u(x, y) = f(x) + \mathcal{L}_y^{-1}[g(x, y)] - \mathcal{L}_y^{-1}[\mathcal{R}[u(x, y)]] - \mathcal{L}_y^{-1}[\mathcal{N}[u(x, y)]], \quad (2)$$

where $\mathcal{L}_y^{-1} = \int_0^y (\cdot) dy$. The ADM defines the solution of Equation (1) in terms of series given by

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y). \quad (3)$$

Further, ADM assumes the nonlinear term, $\mathcal{N}$ can be represented by

$$\mathcal{N}[u(x, y)] = \sum_{n=0}^{\infty} A_n(u_0, \dots, u_n), \quad (4)$$

where A_n 's are known as Adomian polynomials. Adomian polynomials can be defined by the general formula

$$A_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\rho^n} \left[\mathcal{N} \left(\sum_{l=0}^n \rho^l u_l \right) \right]_{\rho=0}.$$

In particular, A_0, A_1, A_2 determined as

$$\begin{aligned} A_0 &= \mathcal{N}[u_0(x, y)] \\ A_1 &= u_1 \frac{d}{du_0} \mathcal{N}[u_0(x, y)] \\ A_2 &= u_2 \frac{d}{du_0} \mathcal{N}[u_0(x, y)] + \frac{u_1^2}{2!} \frac{d^2}{du_0^2} \mathcal{N}[u_0(x, y)]. \end{aligned}$$

Substituting values of u and $\mathcal{N}$ from Equations (3) and (4) into Equation (2), we get

$$\sum_{n=0}^{\infty} u_n(x, y) = f(x) + \mathcal{L}_y^{-1}[g(x, y)] - \mathcal{L}_y^{-1} \mathcal{R} \left[\sum_{n=0}^{\infty} u_n(x, y) \right] - \mathcal{L}_y^{-1} \left[\sum_{n=0}^{\infty} A_n(u_0, \dots, u_n) \right]. \quad (5)$$

From Equation (5), we obtain the following recurrence relation given by

$$\begin{aligned} u_0(x, y) &= f(x) + \mathcal{L}_y^{-1}[g(x, y)] \quad \text{and} \\ u_{l+1}(x, y) &= \mathcal{L}_y^{-1} \mathcal{R}[u_l(x, y)] - \mathcal{L}_y^{-1}[A_l(u_0, \dots, u_n)], \quad l = 0, 1, 2, \dots \end{aligned}$$

Hence, the solution of Equation (1) can be written as

$$u(x, y) = \lim_{n \rightarrow \infty} \sum_{l=0}^n u_l(x, y).$$

3 Adomian Decomposition Method for Standard Power Options

In this section, we derive Black-Scholes model for the standard power payoff functions in the money. That means for call (put) option, the striking price is less (greater) than the asset price.

Theorem 3.1. *The solution of Black-Scholes equation using ADM of European call option for the standard power payoff function, $\max\{S^p - K, 0\}$ ($p \in (0, \infty)$) is*

$$C_1(S, t) = S^p e^{(p-1)(r+\frac{1}{2}p\sigma^2)(T-t)} - K e^{-r(T-t)}, \quad S \geq K. \quad (6)$$

Proof. The Black-Scholes equation (see P. Wilmott, S. Howison and Dewynne [2002, P.76]) is given by

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + rS \frac{\partial C_1}{\partial S} - rC_1 = 0 \quad (7)$$

with $C_1(S, t) \rightarrow S$ if $S \rightarrow \infty$, $C_1(0, t) = 0$ and $C_1(S, T) = \max\{S^p - K, 0\}$, where C_1 denotes price of call option, S is the asset price at time t , K is the striking price, T is an expiry date, σ is the volatility and r is the rate of interest which is constant during expiry date of option. Take $S = K^{\frac{1}{p}} e^x$, $T - t = \frac{2\tau}{\sigma^2}$, $C_1(S, t) = Ku(x, \tau)$, the Equation (7) reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k - 1) \frac{\partial u}{\partial x} - ku, \quad (8)$$

where $k = \frac{2r}{\sigma^2}$. Now $C_1(S, T) = \max\{S^p - K, 0\}$ gives

$$u(x, 0) = \max\{e^{px} - 1, 0\}. \quad (9)$$

Now, we find solution of Equation (8) with initial condition given in Equation (9) using ADM. Note that the Equation (8) can be written as

$$\mathcal{L}_\tau[u(x, \tau)] - \mathcal{R}[u(x, \tau)] = 0, \quad (10)$$

where $\mathcal{L}_\tau = \frac{\partial}{\partial \tau}$ and $\mathcal{R} = \frac{\partial^2}{\partial x^2} + (k - 1) \frac{\partial}{\partial x} - k$. Taking $\mathcal{L}_\tau^{-1} = \int_0^\tau (\cdot) d\tau$ on both sides of Equation (10), we get

$$u(x, \tau) = \max\{e^{px} - 1, 0\} + \mathcal{L}_\tau^{-1} \mathcal{R}[u(x, \tau)]. \quad (11)$$

Consider solution of Equation (10) as

$$u(x, \tau) = \sum_{n=0}^{\infty} u_n(x, \tau). \quad (12)$$

From Equations (11) and (12), we obtain

$$\sum_{n=0}^{\infty} u_n(x, \tau) = \max\{e^{px} - 1, 0\} + \mathcal{L}_\tau^{-1} \mathcal{R} \left[\sum_{n=0}^{\infty} u_n(x, \tau) \right]. \quad (13)$$

Recurrence relation can be obtained from Equation (13) as

$$\begin{aligned} u_0(x, \tau) &= \max\{e^{px} - 1, 0\} \quad \text{and} \\ u_{n+1}(x, \tau) &= \mathcal{L}_\tau^{-1} \mathcal{R}[u_n(x, \tau)], \quad n = 0, 1, 2, 3, \dots \end{aligned} \quad (14)$$

Taking $n = 0, 1, 2, \dots$ in Equation (14), we get

$$\begin{aligned} u_1(x, \tau) &= [(p-1)(p+k) + k]\tau e^{px} - k\tau \max\{e^{px} - 1, 0\} \\ u_2(x, \tau) &= \left[[(p-1)(p+k)]^2 - k^2 \right] \frac{\tau^2}{2!} e^{px} + \frac{(k\tau)^2}{2!} \max\{e^{px} - 1, 0\} \\ &\vdots \\ u_n(x, \tau) &= \left[[(p-1)(p+k)]^n - (-1)^n k^n \right] \frac{\tau^n}{n!} e^{px} \\ &\quad + (-1)^n \frac{(k\tau)^n}{n!} \max\{e^{px} - 1, 0\} \end{aligned}$$

Hence, solution of Equation (10) can be written as

$$\begin{aligned} u(x, \tau) &= \sum_{n=0}^{\infty} u_n(x, \tau) \\ &= \sum_{n=0}^{\infty} \left[[(p-1)(p+k)]^n - (-1)^n k^n \right] \frac{\tau^n}{n!} e^{px} \\ &\quad + (-1)^n \frac{(k\tau)^n}{n!} \max\{e^{px} - 1, 0\} \\ &= \begin{cases} e^{px + [(p-1)(p+k)]\tau} - e^{-k\tau}, & x \geq 0 \\ 0, & x < 0. \end{cases} \end{aligned}$$

But we have $x = \ln\left(\frac{S}{K^{\frac{1}{p}}}\right)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, $k = \frac{2r}{\sigma^2}$ and $C_1(S, t) = Ku(x, \tau)$. Thus the above Equation reduces to

$$C_1(S, t) = S^p e^{(p-1)(r + \frac{1}{2}p\sigma^2)(T-t)} - Ke^{-r(T-t)}, \quad S \geq K.$$

□

Theorem 3.2. *The solution of Black-Scholes equation using ADM of European put option for the standard power payoff function, $\max\{K - S^p, 0\}$ ($p \in (0, \infty)$) is*

$$P_1(S, t) = Ke^{-r(T-t)} - S^p e^{(p-1)(r + \frac{1}{2}p\sigma^2)(T-t)}, \quad S \leq K. \quad (15)$$

Proof. The Black-Scholes equation (see P. Wilmott, S. Howison and Dewynne [2002, P.76]) is given by

$$\frac{\partial P_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_1}{\partial S^2} + rS \frac{\partial P_1}{\partial S} - rP_1 = 0 \quad (16)$$

with $P_1(S, t) \rightarrow 0$ if $S \rightarrow \infty$, $P_1(0, t) = e^{-r(T-t)}$ and $P_1(S, T) = \max\{K - S^p, 0\}$, where P_1 denotes price of put option, K is the striking price, S is the

asset price at time t , K is the striking price, T is an expiry date, σ is the volatility and r is the rate of interest which is constant during expiry date of option. Take $S = K^{\frac{1}{p}} e^x$, $T - t = \frac{2\tau}{\sigma^2}$, $P_1(S, t) = K u(x, \tau)$, the Equation (16) reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k - 1) \frac{\partial u}{\partial x} - k u, \quad (17)$$

where $k = \frac{2r}{\sigma^2}$. Now $P_1(S, T) = \max\{K - S^p, 0\}$ gives

$$u(x, 0) = \max\{1 - e^{px}, 0\}. \quad (18)$$

Now, we find solution of Equation (17) with initial condition given in Equation (18) using ADM. Note that the Equation (17) can be written as

$$\mathcal{L}_\tau[u(x, \tau)] - \mathcal{R}[u(x, \tau)] = 0, \quad (19)$$

where $\mathcal{L}_\tau = \frac{\partial}{\partial \tau}$ and $\mathcal{R} = \frac{\partial^2}{\partial x^2} + (k - 1) \frac{\partial}{\partial x} - k$. Taking $\mathcal{L}_\tau^{-1} = \int_0^\tau (\cdot) d\tau$ on both sides of Equation (19), we get

$$u(x, \tau) = \max\{1 - e^{px}, 0\} + \mathcal{L}_\tau^{-1} \mathcal{R}[u(x, \tau)]. \quad (20)$$

Consider solution of Equation (19) as

$$u(x, \tau) = \sum_{n=0}^{\infty} u_n(x, \tau). \quad (21)$$

From Equations (20) and (21), we obtain

$$\sum_{n=0}^{\infty} u_n(x, \tau) = \max\{1 - e^{px}, 0\} + \mathcal{L}_\tau^{-1} \mathcal{R}\left[\sum_{n=0}^{\infty} u_n(x, \tau)\right]. \quad (22)$$

Recurrence relation can be obtained from Equation (22) as

$$\begin{aligned} u_0(x, \tau) &= \max\{1 - e^{px}, 0\} \quad \text{and} \\ u_{n+1}(x, \tau) &= \mathcal{L}_\tau^{-1} \mathcal{R}[u_n(x, \tau)], \quad n = 0, 1, 2, 3, \dots \end{aligned} \quad (23)$$

Taking $n = 0, 1, 2, \dots$ in Equation (23), we get

$$\begin{aligned} u_1(x, \tau) &= -k\tau \max\{1 - e^{px}, 0\} - [k + (p - 1)(p + k)]\tau e^{px} \\ u_2(x, \tau) &= \frac{(k\tau)^2}{2!} \max\{1 - e^{px}, 0\} + \left[k^2 - (p - 1)(p + k) \right]^2 \frac{\tau^2}{2!} e^{px} \\ &\vdots \\ u_n(x, \tau) &= (-1)^n \frac{(k\tau)^n}{n!} \max\{1 - e^{px}, 0\} \\ &\quad + (-1)^n \left[k^n - (-1)^n [(p - 1)(p + k)]^n \right] \frac{\tau^n}{n!} e^{px} \end{aligned}$$

Hence, solution of Equation (19) can be written as

$$\begin{aligned}
 u(x, \tau) &= \sum_{n=0}^{\infty} u_n(x, \tau) \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{(k\tau)^n}{n!} \max\{1 - e^{px}, 0\} \\
 &\quad + (-1)^n \left[k^n - (-1)^n [(p-1)(p+k)]^n \right] \frac{\tau^n}{n!} e^{px} \\
 &= \begin{cases} e^{-k\tau} - e^{px + [(p-1)(p+k)]\tau}, & x \leq 0 \\ 0, & x > 0. \end{cases}
 \end{aligned}$$

But we have $x = \ln\left(\frac{S}{K^{1/p}}\right)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, $k = \frac{2r}{\sigma^2}$ and $P_1(S, t) = Ku(x, \tau)$. Thus the above Equation reduces to

$$P_1(S, t) = Ke^{-r(T-t)} - S^p e^{(p-1)(r + \frac{1}{2}p\sigma^2)(T-t)}, \quad S \leq K.$$

□

The closed form formulas of standard power options are stated in the next result. It can be seen that these formulas consist the distribution function of standard normal random variable.

Theorem 3.3. *Haug [2007, P.119] European option pricing formulas for standard power payoff functions are*

$$C_2(S, t) = S^p e^{(p-1)(r + \frac{1}{2}p\sigma^2)(T-t)} \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \quad \text{and} \quad (24)$$

$$P_2(S, t) = Ke^{-r(T-t)} \Phi(-d_2) - S^p e^{(p-1)(r + \frac{1}{2}p\sigma^2)(T-t)} \Phi(-d_1), \quad (25)$$

where $d_1 = \frac{\ln\left(\frac{S}{K^{1/p}}\right) + (r + (p - \frac{1}{2})\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - p\sigma\sqrt{T-t}$ and $\Phi(\cdot)$ is cdf of standard normal random variable.

4 Comparisons

In this section, we analyze prices of call and put options from ADM with the closed form solutions. For that, we consider the expiry date, $T = 1$, the risk-free interest rate, $r = 0.08$ and the volatility, $\sigma = 0.3$. For Figure 1, we consider initial value of asset price, $S = 100$ and the present time, $t = 0$. Figure 1(a) represents comparisons of call option values from Equations (6) and (24) for striking price, $K = 90$, while Figure 1(b) represents comparisons of put option values from Equations (15) and (25) for $K = 110$ against different values of p . Moreover,

Figure 1 shows that Equations (6) and (15) are valid for $p \geq 1.0$ and $p \leq 1.0$, respectively. This can be justified due to the term containing p^{th} power of asset price in the formulas. Also, it can be seen that formulas of call options from Equations (6) and (24) are coincide for $p > 0.9$, while formulas for put options from Equations (15) and (25) are coincide for $p < 0.9$. Further, Figures 2 & 3 define the absolute and relative errors of call and put option values given in Figure 1. Also, Figure 4 represents call and put option values against p and time to expiration. The singularity near $p = 0.9$ can be observed in all the Figures. In Figure 3, it can be seen that the relative error of call values is not more than 0.5%, when $p \geq 1.1$, while error for put option is not more than 0.035%, when $p \leq 0.85$. Moreover, by approaching expiry date of options, Figure 4, represents absolute and relative errors of call and put options for larger and smaller values of p , respectively.

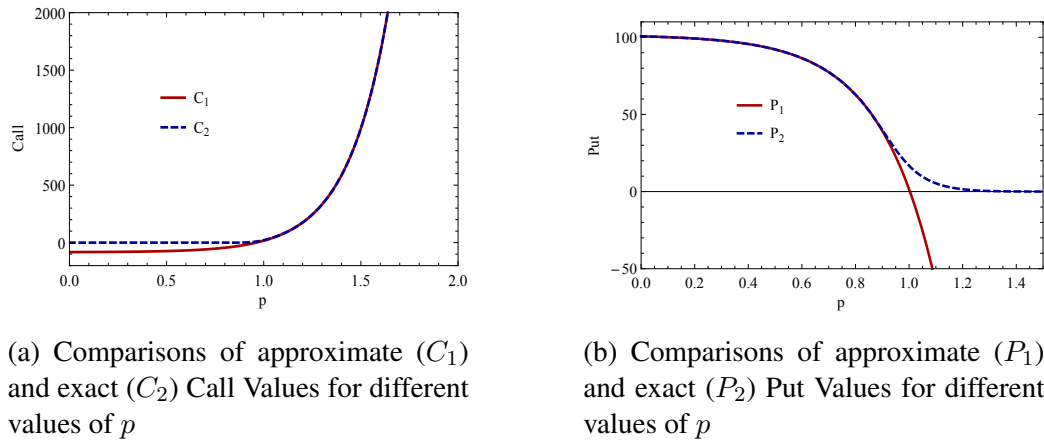


Figure 1

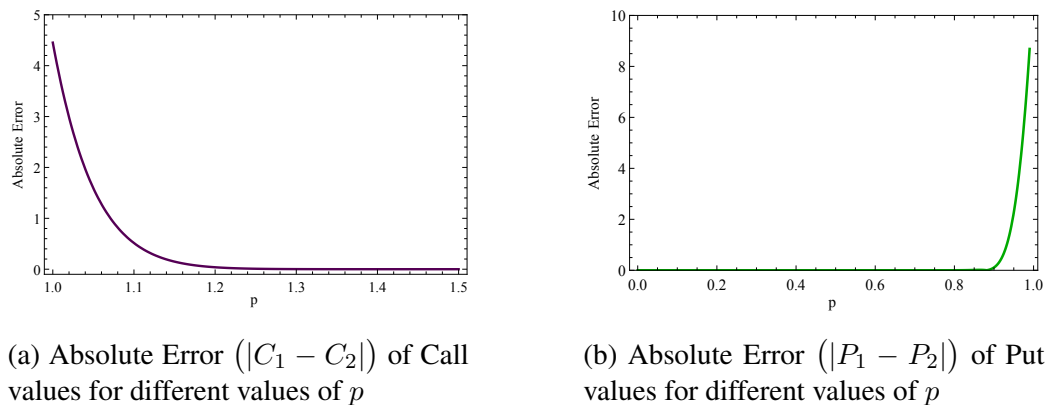
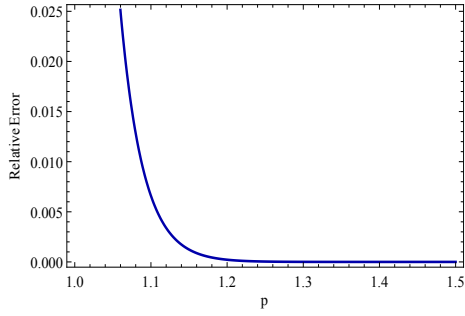
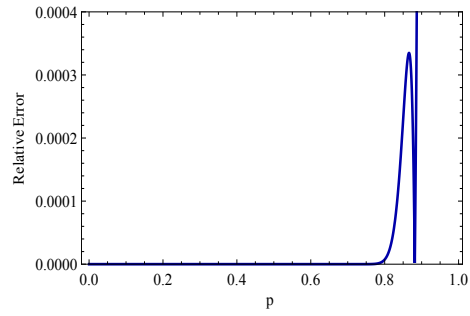


Figure 2

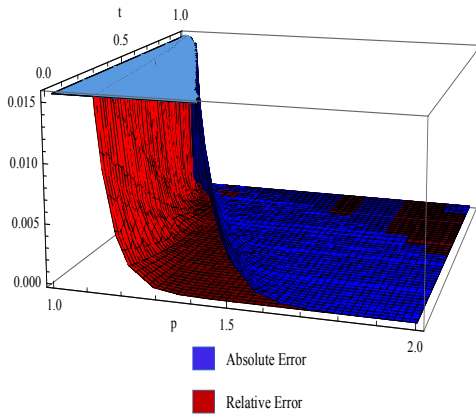


(a) Relative Error $(|C_1 - C_2|/C_2)$ of Call values for different values of p

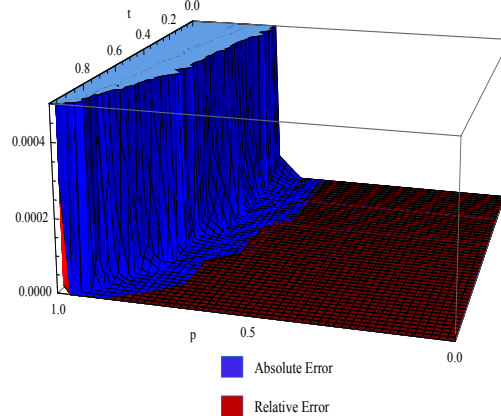


(b) Relative Error $(|P_1 - P_2|/P_2)$ of Put values for different values of p

Figure 3



(a) Absolute and relative errors of Call values for different values of p against time to expiration



(b) Absolute and relative errors of Put values for different values of p against time to expiration

Figure 4

5 Discussion

Any Black-Scholes model rooted with the Nobel prize celebrated model for plain vanilla payoffs has an importance in finance. Hence, the techniques used to derive either exact or approximate values for Black-Scholes model for various payoffs have been studied. One of the simple and powerful technique to get approximation is ADM. The model derived in this paper is the generalization of

plain vanilla payoffs by considering $p = 1$. Further, this model for call and put options involve well known functions namely exponential and polynomial while the exact formulas involve terms regarding random variable which is quite difficult to handle. Numerical comparisons shows that the model derived in this paper is quite accurate in the deleted neighborhood of $p = 0.9$.

6 Conclusion

This paper deals with the derivation of Black-Scholes models of standard power options using ADM. It can be observed that the ADM method is quite simple, straightforward and powerful technique for solving Black-Scholes differential equation. On the other hand, the limitations of this methods are that this method applied to nonlinear equations does not seems to be fast enough to get accurate solution and the convergence of the series involved in the method needs to be checked. Moreover, it can be seen that the closed form formulas of standard power options given by Equations (24) & (25) involve the cdf of standard normal random variable, while the formulas derived using ADM given by Equations (6) & (15) do not involve any term regarding random variable. Hence, these formulas are very simple to handle and also they are very close to the exact solutions. Further, it has been verified by numerically that for larger values of p , our formula for call option fit with the exact solution, while for smaller values of p , put option fit with the exact solution. It might be interesting to derive Black-Scholes model having payoffs $\max\{p(S) - K, 0\}$ and $\max\{K - p(S)\}$ for call and put options, respectively, where $p(x) = \sum_{k=0}^n a_k x^k$ ($a_i \in [0, \infty)$, $1 \leq i \leq n$, $n \in \mathbb{N}$). Obviously, this model will be the generalization of the model derived in this paper.

Conflict of interest: Author has no conflict of interest.

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On the approximation of conjugate of functions belonging to the generalized Lipschitz class by Euler-matrix product summability method of conjugate series of Fourier series

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Abstract

In this paper, a new theorem on the approximation of conjugate of functions belonging to the generalized Lipschitz class by Euler-Matrix product summability method of conjugate series of Fourier series has been obtained. Sometimes a series is not summable by any individual summability method. But it becomes summable by taking product summability means of given series. So working in this direction we have used Euler-Matrix product summability method. Since $Lip\alpha$ $Lip(\alpha, p)$ classes are the particular cases of generalized Lipschitz class. Therefore, many of the known results may become particular cases of our result. On the bases of above facts we can say that our result may be useful for the coming researchers in future.

Keywords: generalized Lipschitz class, conjugate series of Fourier series, product summability method, Euler mean, matrix mean.

2022 AMS subject classifications: 42B05, 42B08.¹

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1 Introduction

The degree of approximation of functions belonging to various classes by using Cesáro, Nörlund and generalized Nörlund summability methods has been obtained by a number of researchers like Chandra [1], Holland [3], Lal et al ([5],[6]) and Kushwaha [4], Qureshi [7]. Later on Tiwary et al [9] has discussed the degree of approximation of functions by using $(E, q)A$ product summability means of Fourier series. No work seems to have been done so far to find the degree of approximation of conjugate of functions belonging to generalized Lipschitz class by using Euler-Matrix product summability means. Now, in this paper, we are presenting a new theorem on the degree of approximation of conjugate functions belonging to the generalized Lipschitz class by Euler-Matrix product summability method. This new result may become the generalization of many of the known results.

2 Definitions

In this section, we have given following definitions:

Definition 2.1. A function $f \in Lip\alpha$ if

$$|f(x+t) - f(x-t)| = O(|t|^\alpha) \text{ for } 0 \leq \alpha < 1.$$

Definition 2.2. A function $f \in Lip(\alpha, p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p dx \right)^{1/p} = O(|t|^\alpha), 0 \leq \alpha < 1, p \geq 1.$$

Given a positive increasing function $\xi(t)$ and integer $p \geq 1$,

$f \in Lip(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p dx \right)^{1/p} = O(\xi(t)).$$

If $\xi(t) = t^\alpha$, then $Lip(\xi(t), p)$ coincides to $Lip(\alpha, p)$.

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Definition 2.3. L_∞ -norm of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\|f\|_\infty = \sup \{|f(x)| : f : \mathbb{R} \rightarrow \mathbb{R}\}.$$

L_p -norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \right)^{1/p}, p \geq 1.$$

The degree of approximation of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by a trigonometric polynomial t_n (Zygmund [11]) is defined by

$$\|t_n - f\|_\infty = \sup \{|t_n - f| : x \in \mathbb{R}\} \text{ or } \|t_n - f\|_p = \min \|t_n - f\|.$$

Let f be 2π periodic and integrable over $(-\pi, \pi)$ in Lebesgue sense and $f \in Lip(\xi(t), p)$. Let its Fourier series be given by

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x). \end{aligned} \tag{1}$$

The conjugate series of the Fourier series (1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = - \sum_{n=1}^{\infty} B_n(x). \tag{2}$$

If f is Lebesgue integrable, then

$$\bar{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot(t/2) dt = -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_0^\pi \psi(t) \cot(t/2) dt$$

exists for almost all x (Zygmund [11]).

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying Töeplitz (p. 131) condition of regularity i.e. $a_{n,k} \rightarrow 1$ as $n \rightarrow \infty$, $a_{n,k} = 0$ for $k > n$ and $\sum_{k=0}^n |a_{n,k}| \leq M$ a finite constant. Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose k^{th} partial sums is $s_k = \sum_{n=0}^k u_n$. The sequence to sequence transformation $t_n = \sum_{k=0}^n a_{n,k} s_k$ defines the sequence $\{t_n\}$ of lower triangular matrix summability means of sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to summable to sum s by lower triangular matrix method if $\lim_{n \rightarrow \infty} t_n$ exists and is equal to s (Zygmund; p.74) and we write $t_n \rightarrow s(T)$, as $n \rightarrow \infty$.

The (E, q) means of $\{s_n\}$ is defined by -

$$W_n = \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} s_k.$$

The (E, q) transform of Matrix transform A of s_n is defined by

$$\begin{aligned} \bar{\eta}_n &= \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \bar{t}_k \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \bar{s}_\nu \right\}. \end{aligned}$$

If $\bar{\eta}_n \rightarrow \infty$ as $n \rightarrow \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E, q)A$ -summable to sum s .

We use the following notations :

(i) $\psi(x, t) = f(x+t) - f(x-t).$

(ii) $\bar{M}_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} \right\}.$

3 Lemmas:-

For the proof of our theorem, we have required following lemmas:

Lemma 3.1. $\overline{M}_n(t) = O\left(\frac{1}{t}\right)$ for $0 \leq t \leq (n+1)^{-1}$.

Proof. For $0 \leq t \leq (n+1)^{-1}$, we have

$$\begin{aligned} |\overline{M}_n(t)| &= \frac{1}{2\pi(1+q)^n} \left| \left[\sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} \right\} \right] \right| \\ &\leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \frac{|\cos(\nu+1/2)t|}{(t/\pi)} \right\} \right| \\ &\leq \frac{1}{2\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \right\} \right| \\ &= O\left(\frac{1}{t}\right). \end{aligned}$$

Lemma 3.2. $\overline{M}_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$ for $(n+1)^{-1} \leq t \leq \pi$.

Proof. For $(n+1)^{-1} \leq t \leq \pi$, we have by Jordan's Lemma, $\sin(t/2) \geq (t/\pi)$, then

$$\begin{aligned} |\overline{M}_n(t)| &= \frac{1}{2\pi(1+q)^n} \left| \left[\sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} \right\} \right] \right| \\ &\leq \frac{1}{2\pi(1+q)^n} \left| \left[\sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \frac{\cos(\nu+1/2)t}{(t/\pi)} \right\} \right] \right| \\ &= \frac{1}{2t(1+q)^n} \left| \left[\sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{\nu,k} \cos(\nu+1/2)t \right\} \right] \right| \\ &= \frac{1}{2t(1+q)^n} \left| \left[\sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ O\left(\frac{a_{k,k-\tau-1}}{t}\right) + A_{k,\tau} \right\} \right] \right| \\ &= O\left(\frac{A_{n,\tau}}{t}\right). \end{aligned}$$

4 Theorem

In this section, we have proved the theorem:

Theorem 4.1. *Let f be a 2π -periodic, Lebesgue integrable function belonging to $Lip(\xi(t), p)$ class and $T = (a_{m,n})$ be an infinite lower triangular matrix. Then the degree of approximation of conjugate function by (E, q) A-summability means of its conjugate series of Fourier series is given by*

$$\|\bar{\eta} - \bar{f}(x)\|_p = O\left((n+1)^{1/p} \xi\left(\frac{1}{n+1}\right)\right),$$

provided $\xi(t)$ satisfies following conditions:

$$\left\{ \int_0^{1/(n+1)} \left(\frac{t|\psi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left(\frac{1}{n+1}\right), \quad (3)$$

and

$$\left\{ \int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)} \right)^q dt \right\}^{1/q} = O((n+1)^\delta), \quad (4)$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0$, $p^{-1} + q^{-1} = 1$ such that $1 \leq p < \infty$, conditions (3) and (4) holds uniformly in x .

Proof. The k^{th} partial sums of conjugate series of Fourier series (2) is given by

$$\bar{s}_k(f; x) = -\frac{1}{2\pi} \int_0^{\pi} \cot(t/2) \psi(t) dt + \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(k+1/2)t}{\sin(t/2)} \psi(t) dt.$$

$$\bar{s}_k(f; x) = -\frac{1}{2\pi} \int_0^{\pi} \cot(t/2) \psi(t) dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(k+1/2)t}{\sin(t/2)} \psi(t) dt.$$

Therefore making (A-transform) of $\bar{s}_k(f; x)$, we get

$$\bar{t}_n - \bar{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sum_{k=0}^n a_{n,k} \frac{\cos(\nu+1/2)t}{\sin(t/2)} dt.$$

Now, making (E, q) A -transform of $\bar{s}_k(f; x)$, we get

$$\begin{aligned}
 \bar{\eta}_n - \bar{f}(x) &= \frac{1}{2\pi(1+q)^n} \\
 &\times \int_0^\pi \psi(t) \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^k a_{n,k} \frac{\cos(\nu + 1/2)t}{\sin(t/2)} \right\} dt \\
 &= \int_0^\pi \psi(t) \bar{M}_n(t) dt \\
 &= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^\pi \right) \psi(t) \bar{M}_n(t) dt \\
 &= I_1 + I_2.
 \end{aligned} \tag{5}$$

Clearly,

$$|\psi(x+t) - \psi(x-t)| \leq |f(u+x+t) - f(x+t)| + |f(u-x-t) - f(x-t)|.$$

Now, let

$$\begin{aligned}
 \psi(x, t) &= \psi(x+t) - \psi(x-t) \\
 \psi_1(u, x, t) &= f(u+x+t) - f(x+t) \\
 \psi_2(u, x, t) &= f(u-x-t) - f(x-t)
 \end{aligned}$$

Hence, by Minkowski's inequality

$$\begin{aligned}
 \left\{ \int_0^{2\pi} |\psi(x, t)|^p dx \right\}^{1/p} &\leq \left\{ \int_0^{2\pi} |\psi_1(u, x, t)|^p dx \right\}^{1/p} \\
 &+ \left\{ \int_0^{2\pi} |\psi_2(u, x, t)|^p dx \right\}^{1/p} \\
 &= O(\xi(t)).
 \end{aligned} \tag{6}$$

Then

$$f \in Lip(\xi(t), p) \Rightarrow \psi \in Lip(\xi(t), p).$$

Using the Hölder's inequality, $\psi(t) \in Lip(\xi(t), p)$, condition (3), $\sin t \geq (2\pi/t)$, Lemma 1, and second mean value theorem for integrals, we have

$$\begin{aligned} |I_1| &\leq \left[\int_0^{1/(n+1)} \left(\frac{t\psi(t)}{\xi(t)} \right)^p dt \right]^{1/p} \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t} \right)^q dt \right]^{1/q} \\ &= O((n+1)^{-1}) \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t} \right)^q dt \right]^{1/q} \\ &= O((n+1)^{-1}) \left[\int_0^{1/(n+1)} \left(\frac{\xi(t)}{t^2} \right)^q dt \right]^{1/q} \\ &= O\left((n+1)^{1/p} \xi\left(\frac{1}{n+1}\right)\right), p^{-1} + q^{-1} = 1. \end{aligned} \quad (7)$$

Using the Hölder's inequality, Lemma 2, $|\sin t| \leq 1$, $\sin t \geq (2\pi/t)$, and condition (4), we have

$$\begin{aligned} |I_2| &\leq \left[\int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}\psi(t)}{\xi(t)} \right)^p dt \right]^{1/p} \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t^{-\delta}} \right)^q dt \right]^{1/q} \\ &\leq O((n+1)^\delta) \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)|\overline{M}_n(t)|}{t^{-\delta}} \right)^q dt \right]^{1/q} \\ &\leq O((n+1)^\delta) \left[\int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta}} O\left(\frac{A_{n,\tau}}{t}\right) \right)^q dt \right]^{1/q} \end{aligned}$$

$$\begin{aligned}
 &= O((n+1)^\delta) \left[\int_{1/(n+1)}^{\pi} \left(\left(\frac{\xi(t)}{t^{1-\delta}} A_{n,\tau} \right) \right)^q dt \right]^{1/q} \\
 &= O((n+1)^\delta) \left[\int_{1/\pi}^{(n+1)} \left(\frac{\xi(1/y)}{y^{\delta-1}} A_{n,[y]} \right)^q \frac{dy}{y^2} \right]^{1/q} \\
 &= O\left((n+1)^\delta \xi\left(\frac{1}{n+1} \right) \right) \left[\int_{1/\pi}^{(n+1)} \frac{dy}{y^{\delta q - q + 2}} \right]^{1/q} \\
 &= \left\{ (n+1)^\delta \xi\left(\frac{1}{n+1} \right) (O(n+1)^{-q(\delta-1)-1})^{1/q} \right\} \\
 &= O\left((n+1)^{1/p} \xi\left(\frac{1}{n+1} \right) \right), p^{-1} + q^{-1} = 1. \tag{8}
 \end{aligned}$$

Collecting equations from (5) to (8), we get

$$\|\bar{\eta} - \bar{f}(x)\|_p = O\left((n+1)^{1/p} \xi\left(\frac{1}{n+1} \right) \right), 1 \leq p < \infty.$$

5 Corollaries:-

Corollary 5.1. If $\xi(t) = t^\alpha$ then the degree of approximation of a function $\bar{f}(x)$, conjugate of $f \in Lip(\alpha, p)$, $\frac{1}{p} < \alpha < 1$ by $(E, q)A$ means is given by

$$\|\bar{\eta} - \bar{f}(x)\|_p = O\left((n+1)^{-\alpha+1/p} \right), 1 \leq p < \infty.$$

Corollary 5.2. If $p \rightarrow \infty$ in case 1, then for $0 < \alpha < 1$, the degree of approximation of a function $\bar{f}(x)$, conjugate of $f \in Lip\alpha$ by $(E, q)A$ means is given by

$$\|\bar{\eta} - \bar{f}(x)\|_p = O\left((n+1)^{-\alpha} \right).$$

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The AHPSort II to evaluate the High-level instruction performances

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Abstract

This paper proposes a model for ranking Italian high schools based on several performance outputs. To analyze the performance of Italian public High Schools we consider the students' school performance and their academic achievements; also the school characteristics may influence the performance evaluation of high schools, although the importance of these aspects is certainly less than the results achieved by students. Data are from Eduscopio and ScuolaInChiaro portals and refers to the 2019/20 school year. We analyze a sample of 263 high schools (HS) in all Italian Regions. For each school we consider nine outputs related to students' school and academic performance, and school characteristics. We assess the performance of high schools using a multi-criteria approach. Our analysis involves a high number of schools, so we apply the AHPSort II method which in addition to defining the ranking of schools also defines their classification. Our results show that scientific lyceums are all in the first class regardless of the geographic area.

Keywords: school ranking; academic performance; students' achievements; AHPSort II.

2020 AMS subject classifications: 90 Operations research, mathematical programming. ¹

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1 Introduction

This paper focuses on the evaluation of Italian high schools' performance. In Italy, Eduscopio (Giovanni Agnelli Foundation) and ScuolaInChiaro (Ministry of Education) represent important sources of information for such an evaluation; they provide annually, for each school, data on students' school careers, their academic achievements and school characteristics. In particular, Eduscopio provides students' and their families with a ranking of high school in the area of residence based on university performance of school leavers; ScuolaInChiaro makes available to the community all the information relating to Italian schools of all levels, in an organic and structured form.

This paper aims to provide the ranking of Italian high schools, taking into account the school and academic careers of students as well as the characteristics of the school. The multi-criteria approach may be a useful tool to assess the performance of high schools. By considering the above datasets, [Mancini and Marcarelli, 2019] derived the ranking among the typologies of schools; more recently, [Mancini and Marcarelli, 2022] provide a ranking among the school types both at a national level and within each geographic area. Furthermore, applying the AHP method and comparing the results with those obtained by a further MCDM method, PROMETHEE, they found significant differences between HS according to criteria related to school and academic performance both within and between geographic areas. Many studies have dealt with the application of multi-criteria methods in the field of education [Giannoulis and Ishizaka, 2010, Goztepe, 2020, Mancini and Marcarelli, 2022, Stamenkovic et al., 2016].

By taking into account some performance indicators used by Eduscopio and ScuolaInChiaro and according to the approach proposed by [Mancini and Marcarelli, 2022, 2019], this study analyzes nine performance outputs for a sample of 263 Italian high schools. However, unlike [Mancini and Marcarelli, 2022, 2019] in which the schools were grouped into 6 different types of schools and 3 geographic areas, we provide the ranking among all the schools. Due to the characteristics of the problem (e.g., independence among the elements, the high number of alternatives) and the output required, among multi-criteria methods proposed in the literature, this paper focuses on the Analytic Hierarchy Process Sort II (AHPSort II) method. The available data allows us to avoid some disadvantages of the AHP. There is no inconsistency in the judgment matrices because entries of matrices are ratios between performance indices. Furthermore, the AHPSort II allows to analyze a sorting decision problem through a feasible interaction with the decision makers precisely because it foresees a limited number of interactions. The goal of this paper is to obtain a ranking of high schools in Italy (we consider 251 schools in our study) by sorting them into ordered classes.

Finally, in order to verify the impact (role) of the geographical and/or the school

typology factors on the performance of a school, we compare our results with those obtained by [Mancini and Marcarelli, 2022, 2019].

The paper is organized as follows: Section 2 reports a literature overview on the topic; Section 3 defines the methodology; Section 4 reports the case study and the main results; Sections 5 and 6 provide a discussion and some concluding remarks, respectively.

2 Literature overview

The literature on school ranking is vast. Past studies have mainly focused on the school quality and its student achievements [Eide and Showalter, 1998], some other on school's contribution to student academic performance [Jamelske, 2009, Kelly and Downey, 2010] or on the question of “school accountability” affecting the school choice [Burgess et al., 2013, Hart and Figlio, 2015, Nunes et al., 2018]. Many factors may influence students' achievements, such as their socioeconomic status, family background, geographical area of residence and the type of school attended [Agasisti and Murtinu, 2012, Lauer, 2003] as far as school and class size, students' features and school management and resources [Giambona and Porcu, 2018, Masci et al., 2018]. As regards the impact of secondary school on academic performance, recently, Aina et al. [2011] and Aina et al. [2021] have demonstrated that differences in university students' achievements across high schools cannot be limited to the first-year and have to consider the geographic differences. In recent study, several authors used multi-criteria methods for analysed school ranking and their performance. Bana e Costa and Oliveira [2012] use the Measuring Attractiveness by a Categorical Based Evaluation Technique (MACBETH) which allows a multi-criteria evaluation of the different scientific areas within high schools in order to offer an accurate evaluation for each range of activities proposed by the school. Blasco-Blasco et al. [2021] analyze the performance of high schools with a particular focus on student achievement indicators. The authors analyze the student support policies of the schools with the use of Technique for Others Preference by Similarity to Ideal Solution (TOPSIS). Yendra et al. [2018] also use TOPSIS integrated with AHP for the analysis of the quality of the training offer of the high school and higher level. Recently, Mancini and Marcarelli [2019] analyzed the performance of Italian high schools, and derived a ranking among different typologies of schools based on the students' academic achievements, their school performance and the school characteristics. Then, Mancini and Marcarelli [2022] made an in-deep analysis taking into account the geographic areas. Using AHP and Promethee methods they derive a ranking among the school typologies both at a national level and within each geographic area.

3 Methodology

AHPSort II is a Multi-Criteria Decision Analysis (MCDA) method for solving sorting problems. This method allows you to use several criteria and a very large number of alternatives [Ishizaka et al., 2012]. Furthermore, it is a method that provides for a limited interaction with the decision maker [Fattoruso et al., 2022]. The procedure can be repeated or easily automated. The AHPSort II method is described below. We consider a set of alternatives $A=(a_1, \dots, a_i)$ evaluated respect a set of criteria $G=(g_1, \dots, g_j)$; therefore, $g_j(a_i)$ represents the evaluation of alternative a_i on criterion g_j . Moreover, we consider a set of classes $C=(C_1, \dots, C_o)$; the construction of the classes requires the definition by the Decision Maker (DM) of the limiting profiles lp_{ij} or of the central profiles cp_{ij} for each considered g_j criterion [Ishizaka et al., 2020]. The lp_{ij} are defined by the DM when he is able to clearly separate the classes from each other, therefore they represent thresholds that separate the classes from each other; alternatively, when this is not possible, the DM opts to define the cp_{ij} which represent the centroids of each class for all the considered criteria. The AHPSort II allows you to analyze a large number of alternatives by using representative profiles $rp_{sj} = (rp_{1j}, \dots, rp_{sj})$. The rp are points homogeneously distributed in the observed data for each g_j criterion. Once all the elements that constitute the decision problem have been defined, AHPSort II foresees the evaluation of local priorities: w_j for g_j ; p_{sj} for rp_{sj} ; and, p_{oj} alternatively for lp_{ij} or cp_{ij} . The local priorities are determined using Pairwise Comparison Matrices (PCMs) with the eigenvalue method [Ishizaka et al., 2020]. The determination of the local priority of the alternatives p_{ij} is instead defined through the following linear interpolation formula:

$$p_{ij} = p_{sj} + \frac{p_{s+1j} - p_{sj}}{rp_{s+1j} - rp_{sj}} \cdot (g_j(a_i) - rp_{sj}). \quad (1)$$

The global priority of the alternative a_i is defined as follow:

$$p_i = \sum_{j=1}^J p_{ij} \cdot w_j. \quad (2)$$

while the global priority p_k of lp_{ij} or cp_{ij} is defined as:

$$p_k = \sum_{j=1}^J p_{oj} \cdot w_j \quad (3)$$

The sorting of a_i to a C class takes place considering the final global priority; therefore, considering the proximity of p_i to p_k .

4 Case study

4.1 Data

Our case study concerns the evaluation of the performance of schools in Italy. In particular, our reference sample is composed of 56 scientific lyceums (SL), 38 classical high schools (CL), 39 linguistic high schools (LL), 26 high schools of human sciences (HSL), 43 commercial technical high schools (CTHS), 49 High School Technological Technician (TTHS). Figure 1 shows the percentage of schools considered in this study divided by typology.

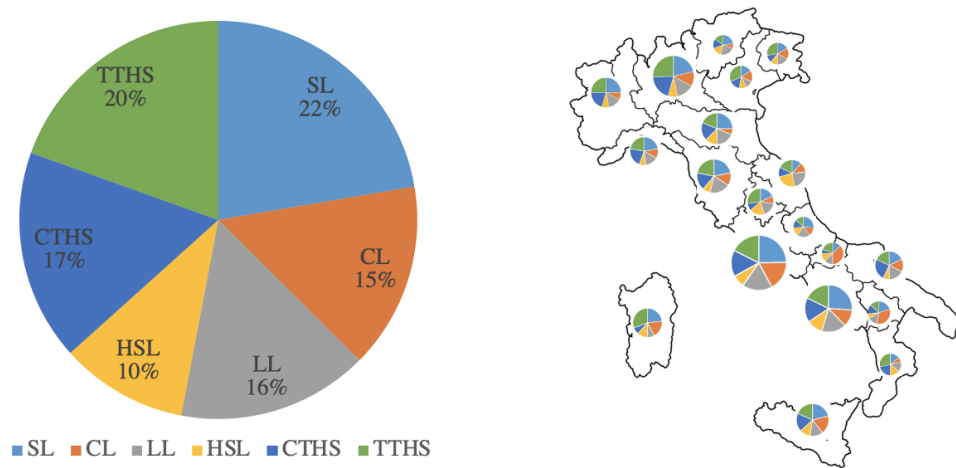


Figure 1: Typologies of high schools in Italy

Each school is evaluated against criteria that determine the performance of the school among these are: Maturity score (g_1) defined as the weighted average score between the high school graduation score of enrolled students e non-enrolled students; INVALSI test score (g_2) defined as the average of each student's math, reading and foreign language test scores; Percentage of graduates in good standing (without failures) (g_3); Students enrolled in the academic year (g_4). In addition to the school performance criteria, schools are evaluated for the academic performance of their students; this type of criteria include: the percentage of students who pass the first year (g_5); percentage of academic credits achieved at the end of the first year (g_6); average exam score (g_7). Finally, the evaluation of schools also takes into account criteria that consider the characteristics of each school, including the average number of students per class (g_8) and the percentage of teachers employed part-time (g_9). The data was collected by the Eduscopio and ScuolaInChiaro portals for the 2019-2020 academic year.

4.2 Results

In our paper we consider a set of alternatives $A = (a_1, \dots, a_{251})$ evaluated respect the criteria set $G = (g_1, \dots, g_9)$; the evaluation table of $g_j(a_i)$ is reported in Appendix A.

The decision-makers involved in the construction of our study are school managers of the different typology of schools considered which from here on we will generically call DMs.

We report in Table 1 the weights w_j of criteria defined with the eigenvalue method.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
w_j	0,10	0,15	0,10	0,10	0,15	0,15	0,15	0,05	0,05

Table 1: Criteria weights w_j

For each g_j criterion considered, we have defined three priority classes C_1 , C_2 and C_3 to which we have associated for simplicity the labels of LOW (C_1), MEDIUM (C_2), and HIGH (C_3). Classes define the performance of high schools. Therefore in the High (C_3) the schools with the best performances will be sorted, in the Low class (C_1) those with the worst performances. In Table 2, we report the central profiles cp_{ij} defined by the DMs, for each class C and each criterion g_j .

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
Low (C_1)	65,79	1	17,4	0,33	0,2	21,26	20	10	1
Medium (C_2)	75,71	4	53,8	0,64	0,55	54,4	24,44	19	26,14
High (C_3)	80	6	80	0,8	0,8	80	28	26	50

Table 2: Central profiles cp_{ij} for each criterion g_j

As suggested by Abastante et al. [2019], we have built the reference profiles $rp_{sj} = (rp_{1j}, \dots, rp_{6j})$. We report in Table 3 the rp_{sj} for each criterion g_j .

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
rp_{1j}	0	0	0	0	0	0	0	0	0
rp_{2j}	17,12	1,4	18,04	0,192	0,18	17,50	4	5,6	10,25
rp_{3j}	34,25	2,8	36,08	0,384	0,36	35,01	8	11,2	20,51
rp_{4j}	51,38	4,2	54,12	0,576	0,54	52,52	12	16,8	30,77
rp_{5j}	68,50	5,6	72,16	0,768	0,72	70,03	16	22,4	41,03
rp_{6j}	85,63	7	90,2	0,96	0,9	87,54	20	28	51,29

Table 3: Reference profiles rp_{sj} for each criterion g_j

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Moreover, we have defined in Table 4 the local priorities p_{sj} and in Table 5 the local priorities p_{oj} .

	p_{s1}	p_{s2}	p_{s3}	p_{s4}	p_{s5}	p_{s6}	p_{s7}	p_{s8}	p_{s9}
rp_{1j}	0,020	0,032	0,030	0,025	0,023	0,022	0,027	0,023	0,030
rp_{2j}	0,031	0,045	0,043	0,038	0,034	0,034	0,038	0,034	0,042
rp_{3j}	0,041	0,058	0,054	0,067	0,063	0,059	0,051	0,062	0,055
rp_{4j}	0,069	0,090	0,092	0,088	0,089	0,087	0,065	0,085	0,090
rp_{5j}	0,104	0,197	0,187	0,155	0,147	0,156	0,086	0,150	0,154
rp_{6j}	0,280	0,234	0,259	0,267	0,297	0,298	0,128	0,300	0,261

Table 4: Local priorities p_{sj}

	p_{o1}	p_{o2}	p_{o3}	p_{o4}	p_{o5}	p_{o6}	p_{o7}	p_{o8}	p_{o9}
C_1	0,084	0,036	0,033	0,046	0,045	0,042	0,128	0,042	0,032
C_2	0,174	0,091	0,083	0,108	0,104	0,102	0,193	0,103	0,084
C_3	0,196	0,217	0,218	0,205	0,198	0,198	0,282	0,200	0,252

Table 5: Local priorities p_{oj}

After, define the local priority p_{ij} with the use of (1), we calculate the global priority p_i . In Figure 2, in Appendix, we report the ranking of the high school.

Finally, we obtained the classification of the high schools in terms of performance; the results are shown in Table 6.

	North	Center	South	Italy
High	27	26	23	76
Medium	57	52	60	169
Low	2	3	1	6

Table 6: Schools performance sorted in geographical areas

In Figure 3, in Appendix, it's shown how the different typologies of high schools have been sorted into the different performance classes.

5 Discussion

As shown by the results obtained, it emerges that the schools that obtain the best performance throughout Italy are the SLs and the CLs with the best positions for those in the south follow. A portion of CL is positioned in medium-sized performances mainly in northern and central Italy. In the medium classes the LL,

HSL, CTHS and TTHS converge in order. A few CL from the north, south and center are included in the performance of the lower class. SL have the best performance regardless of the geographical area. Moving from North to South the first class is almost exclusively composed by SL, in the North, by a great number of SL and a few CL, in the Center, and fifty-fifty by SL and CL, in the South. Comparing the results with those obtained by Mancini and Marcarelli [2019] and Mancini and Marcarelli [2022], an inversion of ordering emerges in the first positions between SL and CL. The other positions are confirmed. It should be noted that in the works of Mancini and Marcarelli [2019] and Mancini and Marcarelli [2022] the initial data are represented by the average performance values by type of school for each criterion considered. In this paper, however, the performances are evaluated for individual schools. In this sense, the difference in the overall results may be due to the influence of anomalous values in the calculation of the average values by type of school. For a more detailed comparison, we also checked the ranking of schools for each criterion. Figure 4, in Appendix, shows in particular the ranking of the SL and CL for each criterion, geographical area and class considered. As can be seen in the Figure 4, SLs obtain higher performances for all the criteria except for the g_3 criterion for the CLs of northern Italy.

6 Conclusions

This paper investigates the performance of Italian high schools in order to derive a ranking considering the typology and the geographic area. Using AHPSort II, we obtain a classification of Italian high schools into different categories. The results show that the ranking among the types of schools does not vary moving from North to South: scientific lyceums are all in the first class regardless the geographic area. However, the limit of this work is the lack of a model validation. The model may assist students in selecting the type of school to attend; the information makes it possible to make an appropriate choice according to their academic perspectives.

Our future works will address a comparative analysis to test the model proposed: ELECTRE TRI [Corrente et al., 2016] may provide a classification of high schools into different categories such as ‘over-performing schools’, ‘average-performing schools’ and ‘weak-performing schools’; then we may compare results with those obtained by our model. Furthermore, if the decision makers are not sure about the correct level of reference profiles, it could be interesting to perform a sensitivity analysis with several limiting profiles to test the robustness of the process. Finally, when applied to small territorial districts, our model may be a useful tool to help public administrations distribute additional financial resources to public schools based on their performance rank.

Declarations

Conflict of interest . The authors declare that they have no conflict of interest.

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Appendix A

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
SL	a_1	North	80,20	6	61,9	0,94	0,88	80,2	27,33	27	9,7
SL	a_2	North	78,61	3	67,9	0,90	0,80	74,6	26,62	21	18,7
SL	a_3	North	76,47	4	59,2	0,90	0,81	69,1	25,89	23	16,3
SL	a_4	North	74,87	4	59,2	0,92	0,74	48,2	24,91	24	27,2
SL	a_5	North	72,85	4	38,0	0,83	0,68	52,4	24,07	24	25,2
SL	a_6	North	77,94	7	65,6	0,94	0,89	87,2	28,62	26	16,3
SL	a_7	North	80,68	7	59,3	0,93	0,87	85,6	28,62	23	7,8
SL	a_8	North	77,29	6	62,8	0,95	0,89	82,0	27,57	26	11,7
SL	a_9	North	75,02	4	59,1	0,93	0,84	74,2	26,20	22	14,5
SL	a_{10}	North	75,70	4	40,7	0,88	0,76	67,1	25,02	21	17,3
SL	a_{11}	North	72,22	4	44,0	0,91	0,80	66,5	24,73	22	23,7
SL	a_{12}	North	70,66	4	46,9	0,78	0,61	46,3	25,53	21	27,7
SL	a_{13}	North	78,54	3	68,2	0,92	0,86	75,8	27,09	22	24,6
SL	a_{14}	North	76,10	5	67,6	0,87	0,79	67,6	26,55	21	5,8
SL	a_{15}	North	78,91	7	69,1	0,94	0,90	87,5	28,81	22	12,2
SL	a_{16}	North	76,66	4	52,1	0,92	0,84	75,8	26,71	24	11,7
SL	a_{17}	North	68,76	4	37,7	0,86	0,72	55,7	23,56	22	39,0
SL	a_{18}	North	79,52	5	74,6	0,93	0,90	87,2	28,89	24	22,3
SL	a_{19}	North	75,18	5	57,4	0,91	0,86	76,1	27,51	19	38,5
SL	a_{20}	North	75,54	4	46,3	0,92	0,84	62,6	25,04	23	11,8
SL	a_{21}	Center	81,37	5	70,3	0,84	0,80	77,2	27,61	22	14,3
SL	a_{22}	Center	78,38	6	62,9	0,93	0,86	72,1	26,81	23	19,8
SL	a_{23}	Center	78,03	4	60,0	0,90	0,83	70,7	26,59	21	20,6
SL	a_{24}	Center	80,23	6	76,6	0,93	0,87	76,7	27,92	25	6,2
SL	a_{25}	Center	81,64	5	72,6	0,92	0,85	71,5	27,57	20	17,9
SL	a_{26}	Center	81,01	6	56,3	0,90	0,84	82,7	28,47	25	7,1
SL	a_{27}	Center	77,34	5	56,6	0,93	0,86	78,1	27,59	22	7,5
SL	a_{28}	Center	77,44	3	65,4	0,90	0,81	76,6	27,51	24	10,6
SL	a_{29}	Center	77,35	5	60,2	0,93	0,86	77,4	27,10	24	16,7
SL	a_{30}	Center	77,78	2	62,4	0,91	0,82	71,3	26,56	23	15,6
SL	a_{31}	Center	78,45	6	65,8	0,91	0,83	73,1	26,27	23	23,0
SL	a_{32}	Center	76,15	6	66,1	0,90	0,81	73,5	26,21	22	12,9
SL	a_{33}	Center	77,36	5	57,1	0,91	0,85	73,0	26,01	23	7,6
SL	a_{34}	Center	77,11	2	49,9	0,90	0,76	71,1	26,15	24	20,5
SL	a_{35}	Center	73,94	5	53,1	0,84	0,71	56,1	24,99	24	20,5
SL	a_{36}	Center	77,02	3	59,9	0,89	0,76	64,3	25,18	22	9,7
SL	a_{37}	Center	75,24	3	53,2	0,80	0,68	55,0	25,33	22	18,2
SL	a_{38}	Center	75,8	4	65,1	0,88	0,74	59,9	25,23	23	18,1
SL	a_{39}	South	81,32	4	76,9	0,90	0,81	66,7	25,87	22	9,6
SL	a_{40}	South	82,01	6	74,6	0,93	0,85	68,4	26,09	22	4,9

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T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
SL	a_{41}	South	77,97	5	66,1	0,94	0,84	74,9	28,12	18	16,7
SL	a_{42}	South	81,79	5	57,1	0,93	0,87	80,0	27,29	24	6,8
SL	a_{43}	South	79,05	2	62,4	0,94	0,87	76,3	26,67	23	16,5
SL	a_{44}	South	78,65	6	61,1	0,90	0,75	61,6	25,44	21	3,4
SL	a_{45}	South	81,41	5	55,7	0,78	0,68	65,8	25,42	18	5,9
SL	a_{46}	South	79,16	6	59,2	0,89	0,79	68,3	25,95	22	10,0
SL	a_{47}	South	79,86	2	52,6	0,87	0,77	67,9	25,94	21	5,2
SL	a_{48}	South	74,06	3	20,6	0,80	0,60	43,3	22,65	14	21,2
SL	a_{49}	South	81,07	4	84,6	0,93	0,85	74,9	26,15	27	2,7
SL	a_{50}	South	79,60	2	61,7	0,96	0,87	68,9	25,57	23	10,0
SL	a_{51}	South	82,32	5	77,9	0,93	0,85	74,1	25,89	21	4,3
SL	a_{52}	South	78,27	5	62,3	0,91	0,85	76,5	26,57	24	4,3
SL	a_{53}	South	77,21	4	50,5	0,83	0,79	69,4	25,18	21	6,4
SL	a_{54}	South	72,66	1	39,8	0,80	0,66	53,3	23,32	21	0,0
SL	a_{55}	South	78,94	4	51,3	0,91	0,84	74,5	26,88	20	9,9
SL	a_{56}	South	74,99	4	50,3	0,84	0,72	58,5	25,38	21	11,9
CL	a_{57}	North	78,96	6	66,9	0,95	0,86	75,0	28,31	22	16,9
CL	a_{58}	North	79,47	5	67,3	0,86	0,77	67,7	26,48	24	30,8
CL	a_{59}	North	79,26	6	61,9	0,93	0,88	80,2	27,89	24	14,0
CL	a_{60}	North	79,90	6	68,2	0,94	0,88	78,5	28,41	22	10,1
CL	a_{61}	North	74,10	5	55,5	0,91	0,81	64,8	26,41	23	34,4
CL	a_{62}	North	65,79	2	73,3	0,86	0,73	55,1	26,08	19	25,0
CL	a_{63}	North	75,87	3	74,6	0,88	0,80	66,8	26,88	22	24,6
CL	a_{64}	North	80,86	4	65,3	0,87	0,79	68,9	27,76	21	14,7
CL	a_{65}	North	80,55	5	72,2	0,95	0,88	72,7	27,73	21	29,6
CL	a_{66}	North	78,82	5	68,8	0,92	0,84	64,6	26,63	22	21,2
CL	a_{67}	North	79,53	6	72,7	0,93	0,87	71,3	28,29	23	7,7
CL	a_{68}	Center	81,30	6	60,1	0,92	0,84	71,4	28,27	21	24,6
CL	a_{69}	Center	79,34	5	69,4	0,91	0,82	60,3	26,98	22	14,3
CL	a_{70}	Center	83,12	7	75,9	0,92	0,86	67,7	26,80	24	7,0
CL	a_{71}	Center	84,57	6	88,5	0,95	0,86	65,8	27,62	23	31,9
CL	a_{72}	Center	81,25	6	59,4	0,92	0,85	76,0	28,86	24	13,3
CL	a_{73}	Center	78,85	6	62,6	0,90	0,85	73,1	28,23	24	7,0
CL	a_{74}	Center	82,45	6	73,9	0,93	0,86	78,5	27,54	23	8,6
CL	a_{75}	Center	82,77	3	62,6	0,95	0,82	69,3	27,28	23	8,6
CL	a_{76}	Center	82,4	4	64,4	0,87	0,77	68,5	27,19	23	13,1
CL	a_{77}	Center	81,29	2	72,4	0,93	0,83	69,5	27,07	23	15,0
CL	a_{78}	Center	81,41	2	78,9	0,79	0,74	56,7	25,80	23	15,5
CL	a_{79}	Center	80,06	2	66,5	0,90	0,77	55,2	25,09	23	20,8
CL	a_{80}	Center	75,70	3	67,5	0,86	0,71	50,7	24,62	23	15,8

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
CL	a_{81}	Center	81,39	4	83,9	0,95	0,90	66,2	27,06	23	5,4
CL	a_{82}	South	83,77	5	90,2	0,95	0,88	68,0	26,30	22	8,5
CL	a_{83}	South	83,78	5	72,2	0,94	0,87	80,0	27,77	25	9,9
CL	a_{84}	South	81,21	6	78,3	0,95	0,89	77,3	27,41	22	6,8
CL	a_{85}	South	79,67	6	57,1	0,88	0,74	68,7	26,36	21	3,4
CL	a_{86}	South	74,90	3	73,0	0,83	0,68	55,9	25,83	19	12,4
CL	a_{87}	South	79,90	3	80,5	0,76	0,62	42,3	25,69	23	26,4
CL	a_{88}	South	82,55	6	83,4	0,91	0,83	67,4	26,12	23	0,0
CL	a_{89}	South	85,2	4	80,0	0,96	0,82	60,5	26,23	14	3,7
CL	a_{90}	South	84,13	5	86,7	0,93	0,84	63,0	24,70	20	3,2
CL	a_{91}	South	80,54	5	71,3	0,88	0,80	69,4	26,63	22	3,4
CL	a_{92}	South	79,32	5	63,30	0,75	0,62	53,0	24,78	17	23,0
CL	a_{93}	South	81,20	3	67,5	0,88	0,79	64,4	26,71	19	2,7
CL	a_{94}	South	77,03	3	45,6	0,84	0,73	46,8	23,38	19	14,4
LL	a_{95}	North	80,33	5	50,8	0,86	0,77	66,3	26,31	23	0,0
LL	a_{96}	North	77,98	4	55,4	0,78	0,72	58,6	25,08	19	11,1
LL	a_{97}	North	73,42	6	63,7	0,7	0,56	41,0	24,09	22	37,2
LL	a_{98}	North	75,23	4	59,5	0,83	0,76	69,9	26,06	23	10,8
LL	a_{99}	North	76,91	7	75,4	0,78	0,74	64,0	25,46	23	11,6
LL	a_{100}	North	73,4	4	40,6	0,75	0,65	40,6	24,12	21	27,7
LL	a_{101}	North	78,8	3	53,5	0,78	0,63	48,2	24,04	21	33,8
LL	a_{102}	North	78,78	4	67,6	0,76	0,70	65,0	24,15	24	20,8
LL	a_{103}	North	79,1	4	64,8	0,8	0,71	61,4	26,39	21	14,7
LL	a_{104}	North	80,05	4	70,3	0,73	0,61	51,3	24,05	24	7,4
LL	a_{105}	North	82,81	4	77,9	0,78	0,74	73,6	26,19	24	21,1
LL	a_{106}	North	78,27	3	54,4	0,75	0,64	50,8	24,96	23	17,2
LL	a_{107}	Center	81,49	5	66,4	0,79	0,71	64,9	26,87	22	14,3
LL	a_{108}	Center	78,25	5	44,3	0,74	0,66	54,9	25,73	22	12,5
LL	a_{109}	Center	75,69	3	35,3	0,63	0,50	43,5	25,54	20	25,8
LL	a_{110}	Center	80,3	4	63,5	0,72	0,64	59,0	25,22	23	4,5
LL	a_{111}	Center	78,86	5	60,0	0,74	0,65	55,2	24,93	20	17,9
LL	a_{112}	Center	80,25	3	56,0	0,79	0,71	62,2	26,00	23	15,8
LL	a_{113}	Center	76,92	6	52,5	0,72	0,66	64,6	25,49	23	14,66
LL	a_{114}	Center	76,42	4	56,1	0,73	0,65	67,9	24,78	21	13,01
LL	a_{115}	Center	82,52	4	66,7	0,8	0,62	53,3	25,33	24	20,95
LL	a_{116}	Center	76,91	4	70,8	0,76	0,63	52,7	24,63	23	16,13
LL	a_{117}	Center	78,09	6	63,0	0,67	0,56	54,2	25,27	24	26,35
LL	a_{118}	Center	75,51	4	47,3	0,71	0,60	44,9	23,91	22	28,24
LL	a_{119}	Center	73,76	3	55,2	0,7	0,55	37,1	24,55	21	23,36
LL	a_{120}	Center	74,28	3	41,8	0,64	0,51	41,4	23,01	20	26,32

The AHPSort II to evaluate the High-level instruction performances

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
LL	a_{121}	Center	79,18	4	69,4	0,72	0,64	53,9	24,88	23	5,41
LL	a_{122}	South	80,26	5	67,8	0,77	0,75	74,8	25,95	22	9,17
LL	a_{123}	South	79,24	4	74,5	0,86	0,78	64,3	26,10	21	13,73
LL	a_{124}	South	80,25	3	39,6	0,72	0,67	67,6	25,53	22	7,95
LL	a_{125}	South	78,5	4	54,6	0,8	0,66	51,3	25,82	28	3,97
LL	a_{126}	South	79,12	3	60,0	0,7	0,58	56,8	24,71	22	13,43
LL	a_{127}	South	72,71	4	30,0	0,52	0,41	37,8	24,12	20	2,70
LL	a_{128}	South	81,31	5	40,0	0,56	0,44	40,5	23,43	18	5,94
LL	a_{129}	South	76,84	4	54,7	0,49	0,35	43,0	25,21	16	4,23
LL	a_{130}	South	78,93	5	62,9	0,83	0,72	53,4	24,27	20	6,67
LL	a_{131}	South	85,63	6	71,3	0,76	0,72	61,1	24,97	20	7,29
LL	a_{132}	South	78,32	3	67,3	0,52	0,49	69,8	24,12	21	19,72
LL	a_{133}	South	74,38	4	41,3	0,56	0,47	40,9	23,39	18	29,89
HSL	a_{134}	North	78,25	5	63,9	0,76	0,65	55,8	24,11	19	18,62
HSL	a_{135}	North	76,15	4	65,0	0,76	0,59	39,8	23,62	23	27,2
HSL	a_{136}	North	75,85	4	46,7	0,75	0,65	62,2	25,74	23	24,81
HSL	a_{137}	North	75,31	4	68,0	0,84	0,64	37,4	23,04	21	28,57
HSL	a_{138}	North	72,69	4	59,5	0,61	0,52	53,3	24,46	24	20,8
HSL	a_{139}	North	73,27	3	46,2	0,71	0,56	46,7	24,60	19	26,94
HSL	a_{140}	North	76,43	4	51,7	0,73	0,69	59,6	24,48	23	11,8
HSL	a_{141}	North	73,01	2	47,0	0,76	0,65	46,5	24,43	22	36,80
HSL	a_{142}	Center	74,40	5	58,8	0,77	0,71	54,7	24,65	22	14,29
HSL	a_{143}	Center	77,24	4	59,5	0,77	0,67	50,9	24,37	23	4,5
HSL	a_{144}	Center	79,55	6	65,7	0,77	0,65	45,1	24,67	23	31,9
HSL	a_{145}	Center	75,58	4	64,2	0,8	0,68	52,2	24,70	23	16,13
HSL	a_{146}	Center	78,21	4	42,8	0,67	0,58	50,3	23,55	23	13,1
HSL	a_{147}	Center	73,24	4	56,6	0,69	0,59	52,1	23,40	22	28,24
HSL	a_{148}	Center	75,46	3	61,8	0,55	0,43	45,2	23,12	23	21,43
HSL	a_{149}	South	76,32	6	81,6	0,64	0,57	43,0	23,88	19	16,43
HSL	a_{150}	South	75,13	4	70,3	0,83	0,66	34,3	24,25	23	5,41
HSL	a_{151}	South	73,61	3	56,3	0,55	0,47	54,4	25,53	22	13,43
HSL	a_{152}	South	77,02	4	61,8	0,80	0,65	44,9	23,58	21	13,73
HSL	a_{153}	South	78,46	5	55,1	0,49	0,35	51,6	22,87	18	5,94
HSL	a_{154}	South	73,78	3	34,3	0,44	0,33	33,7	23,59	18	13,6
HSL	a_{155}	South	77,18	4	66,9	0,52	0,44	53,9	24,67	20	4,82
HSL	a_{156}	South	78,76	3	65,9	0,80	0,70	56,1	24,42	22	0,00
HSL	a_{157}	South	80,11	6	70,0	0,69	0,61	54,7	23,84	20	7,29
HSL	a_{158}	South	74,91	4	52,9	0,67	0,54	47,7	24,02	20	16,45
HSL	a_{159}	South	78,07	4	32,4	0,57	0,43	22,1	21,22	18	29,89
CTHS	a_{160}	North	72,22	5	42,4	0,51	0,41	43,6	24,14	19	18,62

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
CTHS	a_{161}	North	73,57	5	38,9	0,56	0,44	42,1	24,18	18	27,41
CTHS	a_{162}	North	73,75	3	32,1	0,50	0,42	44,1	22,47	19	23,97
CTHS	a_{163}	North	75,78	6	54,4	0,48	0,37	32,0	23,52	21	13,50
CTHS	a_{164}	North	74,27	3	27,2	0,42	0,36	61,0	24,47	21	22,92
CTHS	a_{165}	North	74,13	4	34,7	0,47	0,42	57,0	24,33	21	28,57
CTHS	a_{166}	North	74,98	4	37,0	0,45	0,34	52,0	23,41	21	27,71
CTHS	a_{167}	North	73,82	4	53,5	0,51	0,37	41,4	24,28	19	26,39
CTHS	a_{168}	North	71,55	2	35,0	0,52	0,40	35,2	22,30	21	22,22
CTHS	a_{169}	North	70,75	4	36,6	0,38	0,27	23,7	22,98	25	20,69
CTHS	a_{170}	North	76,60	2	60,1	0,39	0,33	47,9	24,13	20	12,96
CTHS	a_{171}	North	74,22	3	34,5	0,47	0,38	47,5	24,33	18	0,00
CTHS	a_{172}	North	74,99	4	58,3	0,61	0,50	54,5	23,94	24	7,41
CTHS	a_{173}	North	72,10	3	35,1	0,47	0,27	24,1	23,29	17	50,29
CTHS	a_{174}	North	74,51	5	55,1	0,55	0,45	58,5	24,02	21	38,5
CTHS	a_{175}	North	74,55	4	49,6	0,64	0,54	46,8	23,47	22	26,00
CTHS	a_{176}	Center	75,49	3	42,9	0,63	0,53	51,3	25,73	22	37,14
CTHS	a_{177}	Center	74,04	4	44,8	0,56	0,42	34,5	23,27	21	25,81
CTHS	a_{178}	Center	74,71	5	49,6	0,54	0,49	60,7	24,57	20	24,24
CTHS	a_{179}	Center	78,75	5	52,6	0,64	0,61	56,4	25,03	20	17,86
CTHS	a_{180}	Center	73,78	3	47,0	0,36	0,30	57,3	24,67	21	24,66
CTHS	a_{181}	Center	75,37	4	32,6	0,45	0,36	52,5	24,72	21	20,39
CTHS	a_{182}	Center	75,27	2	49,4	0,42	0,28	47,2	24,57	19	9,93
CTHS	a_{183}	Center	72,25	4	57,4	0,50	0,30	34,3	24,37	15	25,30
CTHS	a_{184}	Center	73,75	2	57,7	0,43	0,38	53,0	21,91	22	13,16
CTHS	a_{185}	Center	75,93	5	37,9	0,35	0,22	32,0	22,22	20	27,97
CTHS	a_{186}	Center	71,22	3	58,3	0,36	0,23	30,2	21,72	21	34,52
CTHS	a_{187}	Center	71,99	5	53,8	0,53	0,40	32,6	20,95	26	20,55
CTHS	a_{188}	Center	80,18	4	75,8	0,67	0,50	35,1	24,75	19	17,47
CTHS	a_{189}	South	73,81	3	61,5	0,47	0,42	53,8	24,17	20	11,82
CTHS	a_{190}	South	76,48	2	43,2	0,43	0,34	50,7	24,35	20	15,38
CTHS	a_{191}	South	74,61	4	27,2	0,43	0,33	44,5	23,95	20	16,67
CTHS	a_{192}	South	73,44	3	36,1	0,40	0,31	42,0	22,98	20	21,58
CTHS	a_{193}	South	75,22	3	50,4	0,38	0,29	36,8	23,24	22	10,58
CTHS	a_{194}	South	76,45	4	38,4	0,36	0,26	25,7	23,32	17	9,70
CTHS	a_{195}	South	69,12	2	33,8	0,38	0,26	31,0	20,00	15	0,00
CTHS	a_{196}	South	73,76	5	59,5	0,48	0,39	52,8	24,27	22	6,40
CTHS	a_{197}	South	76,97	4	49,9	0,53	0,43	52,7	23,90	20	9,40
CTHS	a_{198}	South	73,45	5	49,9	0,58	0,48	49,6	22,36	16	6,67
CTHS	a_{199}	South	76,37	5	62,0	0,49	0,42	47,4	22,60	18	17,20
CTHS	a_{200}	South	74,49	5	36,0	0,56	0,50	60,5	22,89	16	0,00

The AHPSort II to evaluate the High-level instruction performances

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
CTHS	a_{201}	South	73,75	4	21,8	0,35	0,29	44,4	22,77	17	5,00
CTHS	a_{202}	South	74,72	4	31,3	0,49	0,37	36,1	23,14	17	4,39
TTHS	a_{203}	South	71,06	3	31,6	0,52	0,37	50,6	25,04	19	23,97
TTHS	a_{204}	North	71,6	4	32,0	0,45	0,36	55,0	24,15	23	27,23
TTHS	a_{205}	North	73,32	4	39,9	0,70	0,58	49,4	24,52	20	11,11
TTHS	a_{206}	North	73,78	4	37,1	0,36	0,27	35,7	22,47	22	29,81
TTHS	a_{207}	North	74,79	6	39,4	0,37	0,26	43,7	22,99	21	22,40
TTHS	a_{208}	North	74,99	4	36,5	0,77	0,67	68,8	25,97	21	27,69
TTHS	a_{209}	North	69,93	5	38,8	0,37	0,27	46,2	26,39	21	3,64
TTHS	a_{210}	North	67,97	4	30,3	0,41	0,35	58,4	24,64	19	28,28
TTHS	a_{211}	North	72,2	4	43,9	0,50	0,37	41,9	24,59	20	35,00
TTHS	a_{212}	North	70,51	5	26,8	0,48	0,35	51,3	23,42	21	17,65
TTHS	a_{213}	North	69,62	4	34,7	0,35	0,24	39,5	23,62	20	24,81
TTHS	a_{214}	North	70,55	4	52,4	0,36	0,20	30,4	25,26	19	26,39
TTHS	a_{215}	North	73,92	4	28,3	0,40	0,28	39,6	23,93	21	15,61
TTHS	a_{216}	North	76,79	6	48,4	0,65	0,57	64,6	26,16	23	23,26
TTHS	a_{217}	North	74,27	3	48,6	0,33	0,25	46,7	24,91	18	36,46
TTHS	a_{218}	North	79,65	6	73,2	0,37	0,24	42,4	24,86	10	40,38
TTHS	a_{219}	North	72,53	4	32,1	0,61	0,49	53,0	25,41	23	32,82
TTHS	a_{220}	North	73,00	3	55,6	0,45	0,32	39,3	25,17	18	23,40
TTHS	a_{221}	North	73,55	4	43,0	0,51	0,43	56,6	26,87	22	26,00
TTHS	a_{222}	North	74,49	4	39,5	0,44	0,37	60,1	25,41	22	29,72
TTHS	a_{223}	Center	71,38	4	30,2	0,43	0,37	62,1	25,65	21	21,71
TTHS	a_{224}	Center	73,81	3	33,8	0,49	0,38	47,4	24,36	21	25,84
TTHS	a_{225}	Center	73,48	6	39,4	0,44	0,30	39,9	23,52	20	10,29
TTHS	a_{226}	Center	77,89	3	41,9	0,68	0,59	53,5	24,47	19	20,00
TTHS	a_{227}	Center	75,76	4	54,8	0,58	0,47	52,7	25,36	18	28,42
TTHS	a_{228}	Center	71,09	3	46,0	0,33	0,23	48,9	26,39	19	8,85
TTHS	a_{229}	Center	72,35	4	45,8	0,50	0,38	52,4	25,59	16	19,53
TTHS	a_{230}	Center	73,03	3	28,3	0,45	0,33	54,7	24,67	22	24,66
TTHS	a_{231}	Center	70,88	3	47,9	0,53	0,36	33,7	24,48	20	36,36
TTHS	a_{232}	Center	72,64	3	30,7	0,41	0,29	38,4	23,84	20	20,35
TTHS	a_{233}	Center	71,8	5	46,1	0,50	0,32	40,4	23,50	20	11,67
TTHS	a_{234}	Center	68,81	5	31,8	0,33	0,21	36,9	23,11	21	18,60
TTHS	a_{235}	Center	73,21	5	50,5	0,42	0,24	26,1	22,66	19	25,81
TTHS	a_{236}	Center	70,51	3	23,8	0,37	0,31	49,5	23,32	18	8,48
TTHS	a_{237}	South	71,06	3	70,8	0,35	0,32	65,0	25,87	20	11,82
TTHS	a_{238}	South	79,55	3	46,7	0,51	0,44	70,8	24,43	20	21,58
TTHS	a_{239}	South	77,18	3	42,3	0,40	0,24	39,9	25,48	17	21,70
TTHS	a_{240}	South	77,84	5	48,4	0,36	0,23	43,3	24,63	19	14,86

T	S	Area	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9
TTHS	a_{241}	South	76,68	4	26,0	0,34	0,26	48,7	24,03	22	16,17
TTHS	a_{242}	South	74,62	3	31,7	0,63	0,47	48,5	23,42	16	6,06
TTHS	a_{243}	South	74,07	4	35,9	0,39	0,26	37,3	23,39	20	6,09
TTHS	a_{244}	South	80,27	3	24,5	0,47	0,27	41,5	21,07	18	21,24
TTHS	a_{245}	South	74,97	3	35,2	0,38	0,29	50,5	24,48	19	9,65
TTHS	a_{246}	South	72,50	4	40,8	0,48	0,39	54,3	24,08	17	0,73
TTHS	a_{247}	South	76,93	4	48,2	0,53	0,40	48,2	23,44	18	8,33
TTHS	a_{248}	South	71,41	3	25,8	0,34	0,29	49,8	26,01	19	9,33
TTHS	a_{249}	South	72,75	4	17,4	0,50	0,42	43,0	22,34	17	5,00
TTHS	a_{250}	South	75,08	5	24,7	0,36	0,30	58,6	23,12	16	27,91
TTHS	a_{251}	South	76,04	4	31,7	0,52	0,26	21,3	22,60	17	29,89

Table 7: Evaluation table of $g_j(a_i)$.
Legend: T=Type; S=School; Area=Geo-position.

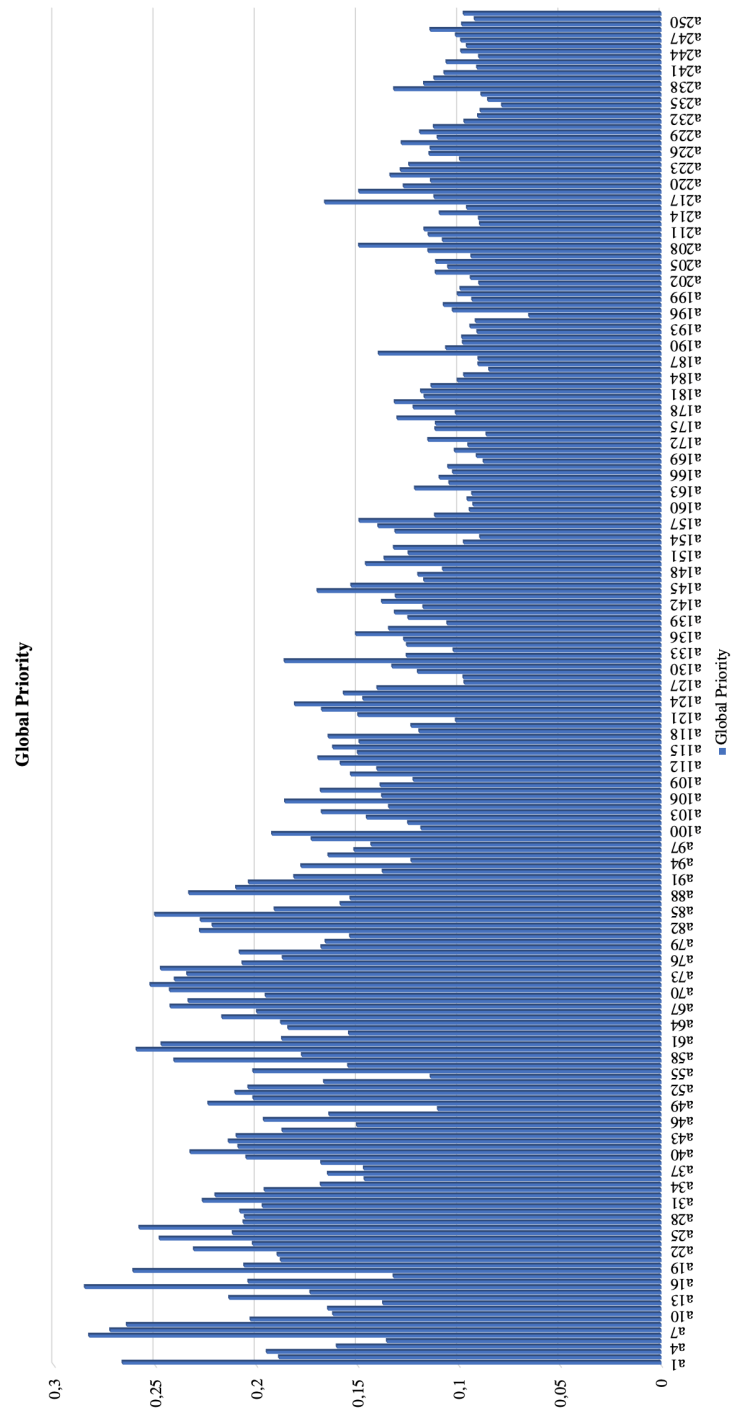


Figure 2: Ranking of school defined by p_i

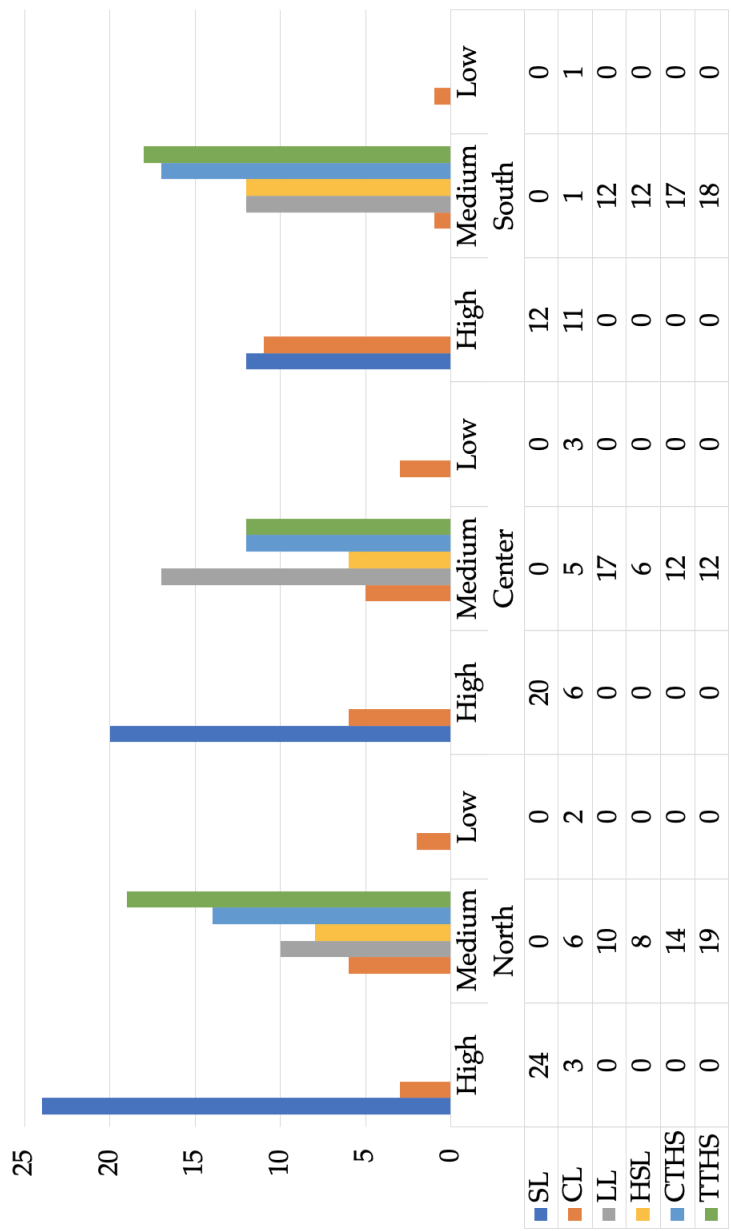


Figure 3: Performance of School sorted for typology and geographical areas

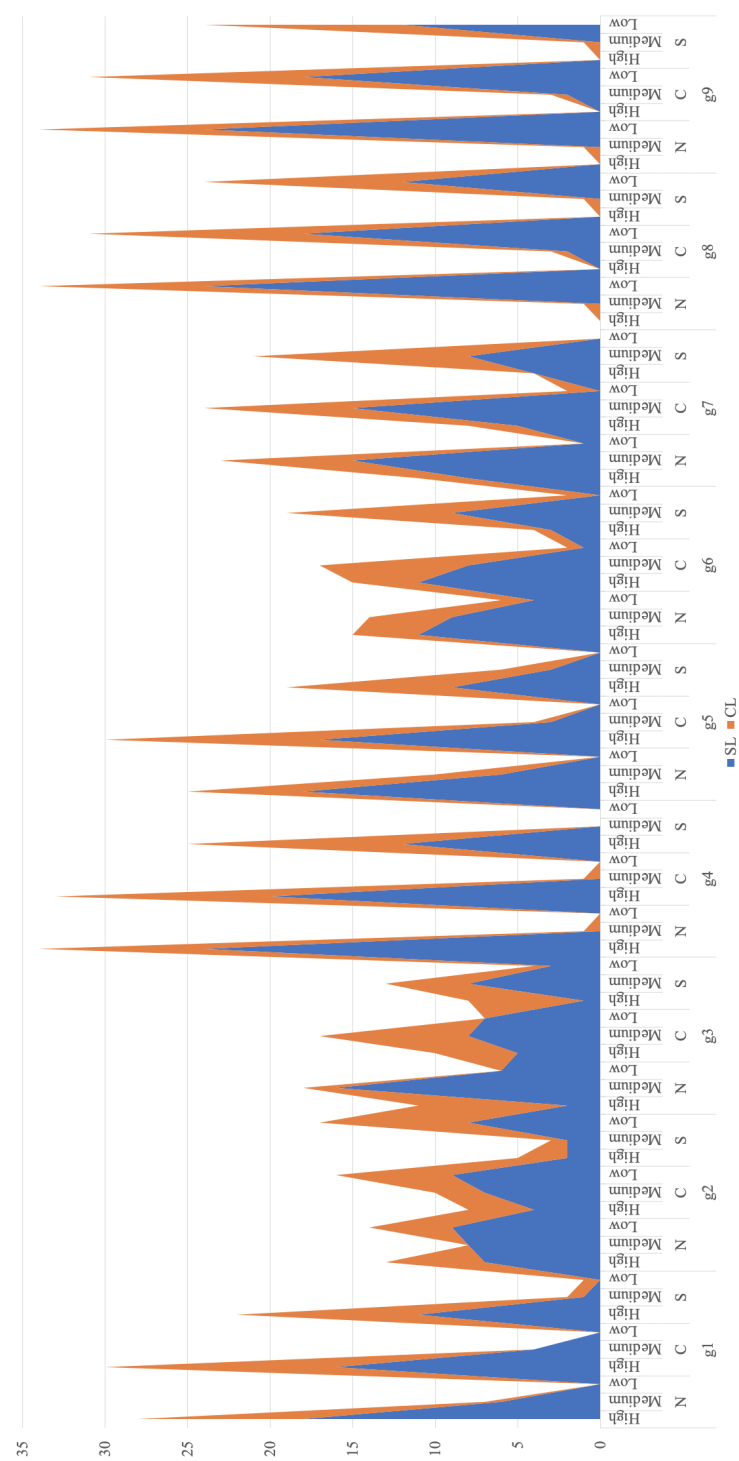


Figure 4: Performance of SLs and CLs sorted for criteria and geographical areas

Closed Isometric Linear Transformations of Complex Spacetime endowed with Euclidean, or Lorentz, or generally Isotropic Metric

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Elias Vossos[†]
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Abstract

This paper is the first in a series of documents showing that *Newtonian physics* and *Einsteinian relativity theory* can be unified, by using a Generalized Real Boost (GRB), which expresses both the *Galilean Transformation* (GT) and the *Lorentz Boost*. Here, it is proved that the Closed Linear Transformations (CLTs) in Spacetime (ST) correlating frames having parallel spatial axes, are expressed via a 4x4 matrix Λ_I , which contains complex Cartesian Coordinates (CCs) of the velocity of one Observer / Frame (O/F) wrt another. In the case of generalized Special Relativity (SR), the inertial Os/Fs are related via isotropic ST endowed with constant real metric, which yields the constant characteristic parameter ω_I that is contained in the CLT and GRB of the specific SR. If ω_I is imaginary number, the ST can only be described by using complex CCs and there exists real Universal Speed (c_I). The specific value $\omega_I = \pm i$ gives the *Lorentzian-Einsteinian versions* of CLT and GRB in ST endowed with metric: $-g_{100}\eta$ and $c_I = c$, where i ; c ; g_{100} ; η are the imaginary unit; speed of light in vacuum; time-coefficient of metric; *Lorentz metric*, respectively. If ω_I is real number, the corresponding ST can be described by using real CCs, but does not exist c_I . The specific value $\omega_I = 0$ gives GT with infinite c_I . GT is also the reduction of the CLT and GRB, if one O/F has small velocity wrt another. The results may be applied to any ST

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endowed with isotropic metric, whose elements (four-vectors) have spatial part (vector) that is element of the ordinary *Euclidean space*.

Keywords: 5th Euclidean postulate; complex space; electromagnetic tensor; Euclidean metric; Euclidean space; Galilean Transformation, general relativity; isometry; linear transformation; Lorentz boost, Lorentz metric, Lorentz transformation, Minkowski spacetime, Newtonian physics, spacetime; special relativity; universal speed.

2010 AMS subject classification: 15A04; 83A05.[§]

Abbreviations-Annotations

CCs: Cartesian Coordinates
CILToCST: Closed Isometric Linear Transformation of Complex Spacetime
CLT: Closed Linear Transformation
 c_1 : Universal Speed
 E^3 : three-dimensional Euclidean Space
 E^4 : Euclidean Spacetime
ERT: Einsteinian Relativity Theory
ESR: Einsteinian Special Relativity
GR: General Relativity
GRB: Generalized Real Boost
GSR: Generalized Special Relativity
GT: Galilean Transformation
IO: Inertial Observer
LT: Linear Transformation
LB: Lorentz Boost
 M^4 : Minkowski Spacetime
NPs: Newtonian Physics
O/F: Observer / Frame
QMs: Quantum Mechanics
RT: Relativity Theory
SR: Special Relativity
ST: Spacetime (four-dimensional Space)
TPs: Theory of Physics
 U : Invariant Speed
wrt: with respect to

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1 Introduction

Linear transformations (LTs) are very important in Relativity Theory (RT) and Quantum Mechanics (QMs) [1]. Moreover, there exist many different approaches of RT, which emerge the corresponding QMs. For instance, *Galilean Transformation* (GT) endowed with the corresponding metric of Spacetime (ST) produces *Newtonian Physics* (NPs), which gives the classic QMs (*Schrödinger Equation*). Thus, many low-velocity phenomena, like the atomic spectra (without fine structure) were explained. On the other hand, *Lorentz Transformation* (endowed with the *Lorentz metric* of ST) produces *Einsteinian Special Relativity* (ESR), which gives relativistic QMs (*Klein-Gordon Equation*). Thus, many high-velocity phenomena and the fine structure of atomic spectra were explained [2].

In this paper, we prove that there exist two types of Closed Isometric Linear Transformation of Complex Spacetime (CILToCST) with common solution the GT. These can apply not only to Special Relativity (SR), but also to General Relativity (GR), because they are reached without adopting one specific metric of spacetime. In addition, any complex *Cartesian Coordinates* (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses [3] (pp. 5-6).

SR relates the frames of Inertial Observers (IOs), via LTs of linear spacetime. ESR uses real spacetime (*Minkowski spacetime*) (M^4) endowed with *Lorentz Metric* (η) and the frames of two IOs with parallel spatial axes are always related via *Lorentz Boost* (LB). But is known that LB is not Closed Linear Transformation (CLT). In contrast, Lorentz Transformation (combination of *spatial Euclidean Rotation* with LB) is CLT (e.g. see [4], p. 41, eq. 1.104). Thus, if three Observers / Frames (Os/Fs): $Oxyz$, $O'x'y'z'$ and $O''x''y''z''$ are related, where the axes of $O'x'y'z'$ are parallel not only to the corresponding axes of $Oxyz$, but also to the corresponding axes of $O''x''y''z''$, then the axes of $Oxyz$ and $O''x''y''z''$ are not parallel (Figure 1). Thus, the transitive attribute in parallelism (which is equivalent to the 5th *Euclidean postulate*) is cancelled, when more than two Os/Fs are related. This consideration leads to successful results, such as *Thomas Precession*, which explains the fine structure of atomic spectra. But this happens only if we take successive observers O , O' and O'' with Thomas' order [5]. The reversed order of this sequence yields a result with 200% relative error.

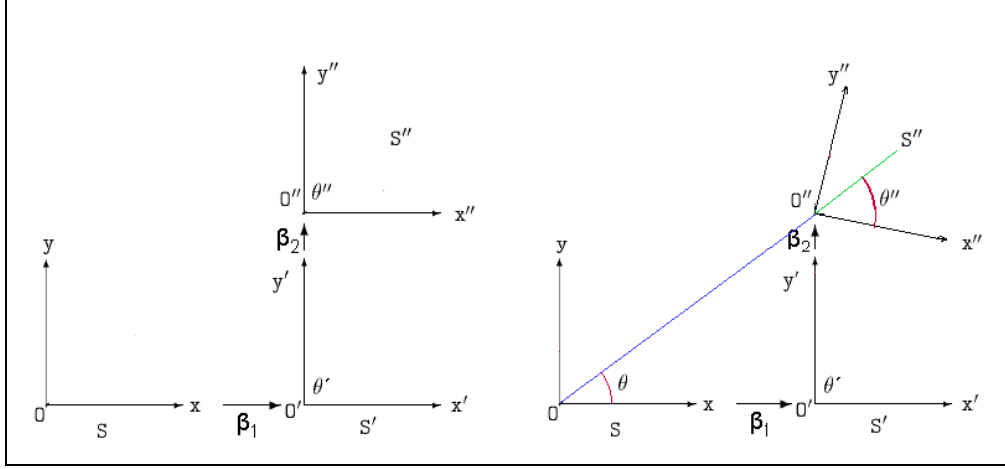


Figure 1: Correlation of three successive observers (frames), by using Lorentz Boost. The frame $O'x'y'z'$ has parallel axes to the corresponding of frame $Oxyz$, moving with velocity $(\beta_1 c, 0, 0)$ wrt $Oxyz$. The frame $O''x''y''z''$ has parallel axes to the corresponding of frame $O'x'y'z'$, moving with velocity $(0, \beta_2 c, 0)$ wrt $O'x'y'z'$. The correlation of the observers, by using Lorentz Boost, cancels the absolute character of parallelism. Thus, the axes of frame $O''x''y''z''$ are not parallel to the corresponding of frame $Oxyz$ (Thomas Rotation).

In this paper, we prove that there exists CLT, which relates Os/Fs with parallel spatial axes (in case of IOs, or observers that have the same acceleration). Thus, the transitive attribute in parallelism is valid in complex three-dimensional Euclidean Space (E^3) and the axes rotation that happens in real space, when more than two observers are related, is the equivalent phenomenon of the corresponding Generalized Real Boost (GRB) [3] (pp. 5-6). The CLT is divided into *two cases*: *one*, where time depends on the position where the event happens, which can have real Invariant Speed (U) and *another*, where time is independent from the position and has $U=\infty$. Moreover, the demand that the CLT is isometric, gives the CILToCST. If *the metric of ST is independent from the position* of the event in ST, we have the *case of SR* and the CILToCST may be applied globally, *relating IOs*. Thus, infinite number of SR-theories is produced (each one of which with the corresponding metric of ST), keeping the ESR-formalism. In the case that *the metric of ST depends on the position* of the event in ST, we have the *case of GR* and the CILToCST may be applied locally, *relating Os/Fs with the same acceleration*. Thus, infinite number of GR-theories is produced (each one of which with the corresponding metric of ST of IOs), all of them keeping Einsteinian GR-formalism. Of course, zero acceleration leads to the corresponding SR. Finally, we present the *improper isometric LT* in ST endowed with *Euclidean*, or *Lorentz*, or generally any *isotropic metric*.

2 The Matrix of Closed Linear Transformation of Complex Spacetime

Initially, we determine the matrix A of active interpretation of the CLT of complex ST endowed with any metric.

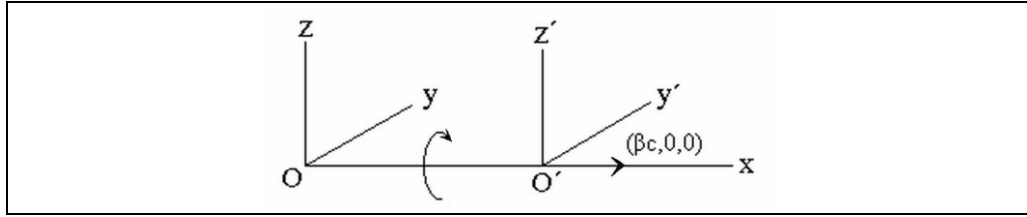


Figure 2: Two frames $Oxyz$ and $O'x'y'z'$ initially coincide. The second is moving with velocity $(\beta c, 0, 0)$ wrt to $Oxyz$.

2.1 Motion in the x-Direction

We consider one unmoved O/F $Oxyz$, measuring real spacetime and another O/F $O'x'y'z'$ with parallel spatial axes, moving to the right, along x-axis with velocity $v = |\vec{\beta}|c = \beta c$ wrt O/F $Oxyz$ (Figure 2), where $c=299,792,458 \text{ m s}^{-1}$ is the speed of light in vacuum and the frames initially coincide. Supposing the next linear transformation

$$cdt' = bcdt + adx + kdy + vdz \quad (1)$$

$$dx' = gcdt + fdx + \delta dy + \theta dz \quad (2)$$

$$dy' = g_1cdt + f_1dx + hdy + \lambda dz \quad (3)$$

$$dz' = g_2cdt + f_2dx + \zeta dy + \mu dz, \quad (4)$$

we determine the coefficients with the following condition: the space has isotropy. Rotating the coordinates system about the x-axis, by one negative right angle (Figure 1), we correspond the new axes to the initial axes: $t \rightarrow t'$, $x \rightarrow x'$, $y \rightarrow -z$, $z \rightarrow y$ and $y' \rightarrow -z'$, $z' \rightarrow y'$. Thus, from (1), we have

$$cdt' = bcdt + adx - kdz + vdy. \quad (5)$$

(1) compared to (5), gives $k=v=0$. Besides, from (2) we have

$$dx' = gcdt + fdx - \delta dz + \theta dy. \quad (6)$$

(2) compared to (6), gives $\delta=\theta=0$. Besides, from (3) we obtain

$$-dz' = g_1cdt + f_1dx - hdy + \lambda dy. \quad (7)$$

(4) compared to (7), gives $g_2=-g_1$, $f_2=-f_1$, $\zeta=-\lambda$ and $\mu=h$. Besides, from (4), we have

$$dy' = g_2cdt + f_2dx - \zeta dz + \mu dy. \quad (8)$$

(3) compared to (8), gives $g_2=g_1$, $f_2=f_1$, $\zeta=-\lambda$ and $\mu=h$. So, $k=v=\delta=\theta=g_1=g_2=f_1=f_2=0$, $\zeta=-\lambda$, $\mu=h$ and the transformation becomes

$$cdt' = bcdt + adx \quad (9)$$

$$dx' = gcdt + fdx \quad (10)$$

$$dy' = hdy + \lambda dz \quad (11)$$

$$dz' = -\lambda dy + hdz. \quad (12)$$

Using matrices we have the *active interpretation* of the LT [4] (p. 6):

$$\begin{bmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{bmatrix} = \begin{bmatrix} b & a & 0 & 0 \\ g & f & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \cdot \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix}, \quad (13)$$

or equivalently,

$$dX' = A_{1(x)} dX, \quad (14)$$

where the base and the coordinates are

$$[\vec{e}_\mu] = [\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3]; \quad dX = \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} \quad (15)$$

respectively. Besides, the velocities are related in the following way:

$$v'_x = \frac{gc + fv_x}{bc + av_x} c; \quad v'_y = \frac{hv_y + \lambda v_z}{bc + av_x} c; \quad v'_z = \frac{-\lambda v_y + hv_z}{bc + av_x} c. \quad (16)$$

2.2 General Linear Transformation (Motion in a random direction)

We then consider one unmoved O/F Oxyz and another O/F O'x'y'z' with parallel spatial axes, moving with velocity (v_x, v_y, v_z) wrt Oxyz, where they initially coincide (Figure 3). We rotate Oxyz, in order to parallelize the unitary vector $\hat{x}$ to the velocity vector $\vec{v}$ of the moving O'x'y'z'. This is sequentially achieved as following (Figure 4). We firstly rotate the coordinate system Oxyz about z-axis, through an angle θ : $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k})$. We then rotate the coordinate system $O(\hat{i}, \hat{j}, \hat{k})$ about $\hat{j}$, by an angle ω : $O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$. Thus, we have the transformation

$$\begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} = \begin{bmatrix} \cos \omega \cos \theta & \cos \omega \sin \theta & \sin \omega \\ -\sin \theta & \cos \theta & 0 \\ -\sin \omega \cos \theta & -\sin \omega \sin \theta & \cos \omega \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad (17)$$

where

$$\sin \theta = \frac{v_y}{\sqrt{v_x^2 + v_y^2}}; \quad \cos \theta = \frac{v_x}{\sqrt{v_x^2 + v_y^2}}; \quad \sin \omega = \frac{v_z}{|\vec{v}|}; \quad \cos \omega = \frac{\sqrt{v_x^2 + v_y^2}}{|\vec{v}|}. \quad (18)$$

As a result, the 3x3 matrix of (17) becomes

*Closed Isometric Linear Transformations of Complex Spacetime endowed with
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$$R = \begin{bmatrix} \frac{\beta_x}{|\vec{\beta}|} & \frac{\beta_y}{|\vec{\beta}|} & \frac{\beta_z}{|\vec{\beta}|} \\ -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & 0 \\ -\frac{\beta_x \beta_z}{|\vec{\beta}| \sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_y \beta_z}{|\vec{\beta}| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\sqrt{\beta_x^2 + \beta_y^2}}{|\vec{\beta}|} \end{bmatrix} \quad (19)$$

and we define

$$\tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}. \quad (20)$$

The unit means that time is not affected by the spatial rotation. Moreover, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence of successive transformations:

$$\begin{aligned} O(\hat{x}, \hat{y}, \hat{z}) &\rightarrow O(\hat{i}', \hat{j}', \hat{k}'); \quad O(\hat{i}', \hat{j}', \hat{k}') \rightarrow O'(\hat{i}', \hat{j}', \hat{k}'); \\ O'(\hat{i}', \hat{j}', \hat{k}') &\rightarrow O'(\hat{x}, \hat{y}, \hat{z}). \end{aligned}$$

The above simple transformations have active interpretations:

$$X_R = \tilde{R}X \quad ; \quad X'_R = \Lambda_{1(x)}X_R \quad ; \quad X' = \tilde{R}^T X'_R, \quad (21)$$

respectively, where $\tilde{R}^T$ is the transpose matrix of $\tilde{R}$. Thus, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$dX' = \tilde{R}^T \Lambda_{1(x)} \tilde{R} dX = \Lambda_{(\beta)} dX. \quad (22)$$

So, we calculate

$$\Lambda_{(\beta)} = \begin{bmatrix} b & \frac{a}{|\vec{\beta}|} \beta_x & \frac{a}{|\vec{\beta}|} \beta_y & \frac{a}{|\vec{\beta}|} \beta_z \\ \frac{g}{|\vec{\beta}|} \beta_x & (f-h) \frac{\beta_x^2}{|\vec{\beta}|^2} + h & (f-h) \frac{\beta_x \beta_y}{|\vec{\beta}|^2} + \frac{\beta_z \lambda}{|\vec{\beta}|} & (f-h) \frac{\beta_x \beta_z}{|\vec{\beta}|^2} - \frac{\beta_y \lambda}{|\vec{\beta}|} \\ \frac{g}{|\vec{\beta}|} \beta_y & (f-h) \frac{\beta_x \beta_y}{|\vec{\beta}|^2} - \frac{\beta_z \lambda}{|\vec{\beta}|} & (f-h) \frac{\beta_y^2}{|\vec{\beta}|^2} + h & (f-h) \frac{\beta_y \beta_z}{|\vec{\beta}|^2} + \frac{\beta_x \lambda}{|\vec{\beta}|} \\ \frac{g}{|\vec{\beta}|} \beta_z & (f-h) \frac{\beta_x \beta_z}{|\vec{\beta}|^2} + \frac{\beta_y \lambda}{|\vec{\beta}|} & (f-h) \frac{\beta_y \beta_z}{|\vec{\beta}|^2} - \frac{\beta_x \lambda}{|\vec{\beta}|} & (f-h) \frac{\beta_z^2}{|\vec{\beta}|^2} + h \end{bmatrix}. \quad (23)$$

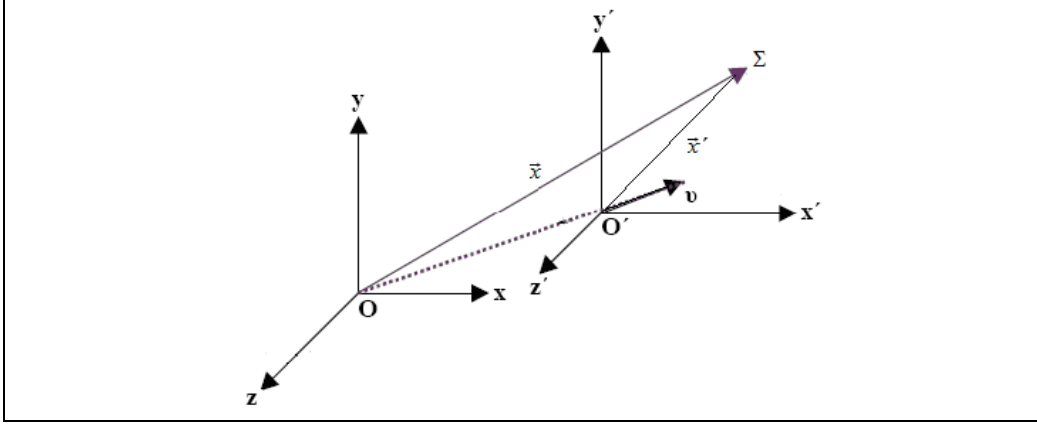


Figure 3: Two frames $Oxyz$ and $O'x'y'z'$, which initially coincide. The second is moving with random velocity (v_x, v_y, v_z) wrt to $Oxyz$.

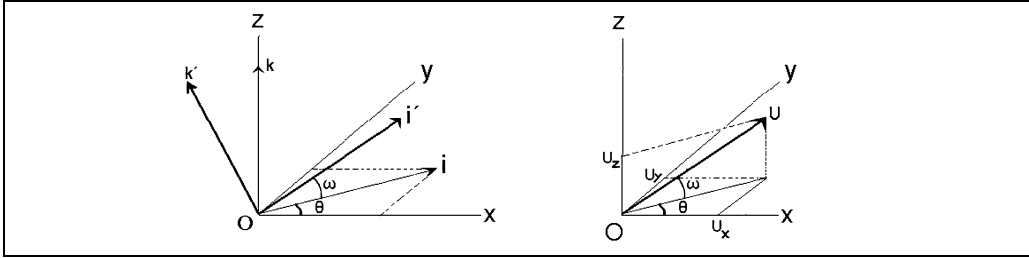


Figure 4: Rotation of the initial frame $Oxyz$, in order to achieve parallelization of vector $\hat{x}$ to the velocity vector $\vec{v}$ of the moving observer $O'x'y'z'$ [$O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$].

2.3 Solution of the proper Closed Linear Transformation of Complex Spacetime (Correlation of two perpendicular moving Observers / Frames)

We consider one unmoved O/F $Oxyz$, another O/F $O'x'y'z'$ with parallel spatial axes, moving to the right, along x -axis with velocity $(\beta c, 0, 0)$ wrt $Oxyz$ and also a third O/F $O''x''y''z''$ with parallel spatial axes, moving upward, along y -axis with velocity $(0, \beta c, 0)$ wrt $Oxyz$ (Figure 5). All of them initially coincide and also $\beta > 0$, because

$$\beta = |\vec{\beta}|. \quad (24)$$

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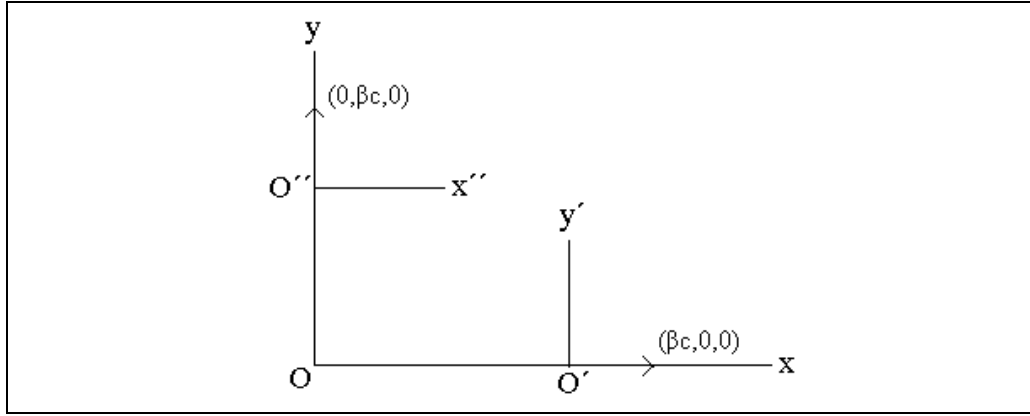


Figure 5: Two frames $O'x'y'z'$ and $O''x''y''z''$ moving with corresponding velocities $(\beta c, 0, 0)$ and $(0, \beta c, 0)$ wrt $Oxyz$.

The transformation $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence: $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{x}, \hat{y}, \hat{z})$; $O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$. The above simple transformations have active interpretations, respectively:

$$X = \Lambda_{1(x)}^{-1} X' ; \quad X'' = \Lambda_{2(y)} X .$$

Thus, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$X'' = \Lambda_{2(y)} \Lambda_{1(x)}^{-1} X' = \Pi X' .$$

According to equation (23), it is

$$\Lambda_{1(x)} = \begin{bmatrix} b & a & 0 & 0 \\ g & f & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \quad (25)$$

and

$$\Lambda_{2(y)} = \begin{bmatrix} b & 0 & a & 0 \\ 0 & h & 0 & -\lambda \\ g & 0 & f & 0 \\ 0 & \lambda & 0 & h \end{bmatrix} . \quad (26)$$

Thus, we have

$$\Pi = \Lambda_{2(y)} \Lambda_{1(x)}^{-1} = \Lambda_{2(y)} \cdot \begin{bmatrix} \frac{f}{bf-ag} & \frac{-a}{bf-ag} & 0 & 0 \\ \frac{-g}{bf-ag} & \frac{b}{bf-ag} & 0 & 0 \\ 0 & 0 & \frac{h}{h^2+\lambda^2} & \frac{-\lambda}{h^2+\lambda^2} \\ 0 & 0 & \frac{\lambda}{h^2+\lambda^2} & \frac{h}{h^2+\lambda^2} \end{bmatrix}. \quad (27)$$

or equivalently,

$$\Pi = \begin{bmatrix} \frac{bf}{bf-ag} & \frac{-ab}{bf-ag} & \frac{ah}{h^2+\lambda^2} & \frac{-a\lambda}{h^2+\lambda^2} \\ \frac{-gh}{bf-ag} & \frac{bh}{bf-ag} & \frac{-\lambda^2}{h^2+\lambda^2} & \frac{-h\lambda}{h^2+\lambda^2} \\ \frac{gf}{bf-ag} & \frac{-ag}{bf-ag} & \frac{fh}{h^2+\lambda^2} & \frac{-f\lambda}{h^2+\lambda^2} \\ \frac{-g\lambda}{bf-ag} & \frac{b\lambda}{bf-ag} & \frac{h\lambda}{h^2+\lambda^2} & \frac{h^2}{h^2+\lambda^2} \end{bmatrix}. \quad (28)$$

Now, we calculate the velocity factor $\vec{\beta}'_4$ of observer $O'x'y'z'$ wrt $O'x'y'z'$. Equation (16) can be applied, because O/F $O'x'y'z'$ is moving in the x-direction and observer O'' can be considered as the observed body. So, it is

$$\beta'_{4x} = \frac{g}{b}, \beta'_{4y} = \frac{h\beta}{b}, \beta'_{4z} = -\frac{\lambda\beta}{b} \quad (29)$$

and we obtain

$$|\vec{\beta}'_4| = \frac{\sqrt{g^2 + (h^2 + \lambda^2)\beta^2}}{b} = \frac{|\vec{\beta}|}{b} \sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2 + \lambda^2}. \quad (30)$$

Replacing the above to (23), yields $\Lambda_{(\beta_4)} = \Lambda_4$. The condition that the transformation is closed, gives

$$\Pi = \Lambda_4. \quad (31)$$

Comparing the matrices, element by element, we shall calculate the parameters α , f and g . The transformation must be reduced to GT, if one IO has small velocity wrt another IO. So, it must be $b, g, f, h \neq 0$. We have two cases: (i) $\lambda=0$ and (ii) $\lambda \neq 0$.

2.3.1 The case of proper Closed Linear Transformations of Complex Spacetime with $\lambda=0$ (time independent from the position, i.e. $a=0$).

When $\lambda=0$, we compare matrices Π and Λ_4 element by element and we also take into account (29). Thus, we have: $h_4=1$ (from element Π_{33}) and $\lambda_4=0$ (from element Π_{13}). We then obtain $f_4=1$ (from element Π_{12}). So,

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$$h_4=f_4=1 \ ; \ \lambda_4=0. \quad (32)$$

From elements Π_{10} and Π_{20} , we get

$$\frac{g_4 g}{|\vec{\beta}| \sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2}} = -\frac{gh}{bf - ag} \ ; \ \frac{g_4 h}{\sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2}} = \frac{gf}{bf - ag} \quad (33)$$

respectively. Thus,

$$g = -\frac{|\vec{\beta}| h^2}{f}. \quad (34)$$

From elements Π_{01} and Π_{02} , we have

$$\frac{a_4 g}{|\vec{\beta}| \sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2}} = -\frac{ab}{bf - ag} \ ; \ \frac{a_4 h}{\sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2}} = \frac{a}{h} \quad (35)$$

respectively. So,

$$\frac{g}{|\vec{\beta}| h} = -\frac{bh}{bf - ag}. \quad (36)$$

Replacing (34) to the above, implies

$$\alpha=0, \quad (37)$$

for the *CLT*, or $g=0$ for the *non-closed LT* (because (34) gives $h=0$ and matrix (25) cannot be identical). Thus, element Π_{11} gives

$$f=h \quad (38)$$

and (34) becomes

$$g = -|\vec{\beta}| h. \quad (39)$$

Finally, (23) yields the general matrix of *CLT*:

$$\Lambda_{(\beta)} = \begin{bmatrix} b & 0 & 0 & 0 \\ -h\beta_x & h & 0 & 0 \\ -h\beta_y & 0 & h & 0 \\ -h\beta_z & 0 & 0 & h \end{bmatrix} = \begin{bmatrix} b & \mathbf{O}^T \\ -h\beta & h\mathbf{I}_3 \end{bmatrix}; \ \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}; \ \mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (40)$$

and the typical matrix *CLT* (along x-axis):

$$\Lambda_{(x)} = \begin{bmatrix} b & 0 & 0 & 0 \\ -h\beta & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{bmatrix}, \quad (41)$$

where $b=b_{(\beta)}$ and $h=h_{(\beta)}$.

Next, we calculate the corresponding CILToCST. The representation of the non-degenerate inner product in basis $[\vec{e}_\mu] = [\vec{e}_0 \ \vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] = [ct, \hat{x}, \hat{y}, \hat{z}]$ is the real *matrix of metric*

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix}. \quad (42)$$

In this paper, we consider $g_{00} < 0$ [signature of spacetime: (-+++), or (----)]. The fundamental equation of isometry - *Killing's equation* in a linear space - (see e.g. [4], p.10, eq.1.15) is

$$g' = A^T g A. \quad (43)$$

The element by element comparison of the above matrices gives

$$g_{11} = g_{22} = g_{33} = g_{ii} = g'_{ii} = 0, \quad g'_{00} = b^2 g_{00}. \quad (44)$$

The isometry of spacetime [see e.g. [4], (p. 240)] is

$$dS'^2 = dS^2, \quad (45)$$

or equivalently,

$$g'_{00} c^2 dt'^2 + dx'_i g'_{ij} dx'^j = g_{00} c^2 dt^2 + dx_i g_{ij} dx^j, \quad (46)$$

which combined with (44) and (40) gives

$$b=1 \text{ for the } CLT, \quad (47)$$

or $b = -1, \pm i$ for the *non-closed LT* (because matrix (25) cannot be identical). So, since $b=1$, *CLT* keeps *time invariant*. The *Einstein's summation convention* [4] (p. 3) was used in (46) and will be used in the equations that follow. Besides, (44ii) becomes

$$g'_{00} = g_{00}. \quad (48)$$

Thus, for any O/F the *metric of the ST* in accordance with the complex LT is

$$g_\Gamma = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (49)$$

We observe that $\det g_\Gamma = 0$. So, this *spacetime is degenerate* [6] (p. 174). In order to calculate function h , we consider the unmoved O/F Oxyz, another O/F O'x'y'z' moving to the right, along the x-axis with velocity $(\beta c, 0, 0)$ wrt Oxyz and a third O/F O''x''y''z'' moving to the left, along the x-axis with velocity $(-\beta c, 0, 0)$ wrt Oxyz. Thus, $X' = \Lambda_{(x)(\beta)} X$ and $X'' = \Lambda_{(x)(-\beta)} X$ give

$$X'' = \Lambda_{(x)(-\beta)} \Lambda_{(x)(\beta)}^{-1} X'. \quad (50)$$

Also, the *typical transformation of velocities* (16) becomes

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$$v'_x = h(-\beta c + v_x), \quad v'_y = h v_y, \quad v'_z = h v_z. \quad (51)$$

Thus, the calculation of the velocity factor of observer O'' wrt O/F $O'x'y'z'$ gives

$$\beta'_{3x} = -2h\beta, \quad \beta'_{3y} = 0, \quad \beta'_{3z} = 0. \quad (52)$$

As the transformation is closed, we have

$$\Lambda_{(x)(-\beta)} \Lambda_{(x)(\beta)}^{-1} = \Lambda_{(x)(\beta_3)}, \quad (53)$$

or equivalently,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta h & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta & \frac{1}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{h} & 0 \\ 0 & 0 & 0 & \frac{1}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2h_3 h \beta & h_3 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_3 \end{bmatrix}, \quad (54)$$

from which it derives that $h_3 = h_{(\beta_3)} = 1$ for any value of β . As h depends only on the norm of velocity factor β , the only solution is $h=1$. Hence, there derives the GT, which is expressed by the general matrix

$$\Lambda_{\Gamma(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta_x & 1 & 0 & 0 \\ -\beta_y & 0 & 1 & 0 \\ -\beta_z & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & O^T \\ -\beta & I_3 \end{bmatrix}; \quad \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}, \quad (55)$$

and typical matrix along x -axis

$$\Lambda_{\Gamma(x)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (56)$$

which produces NPs with *invariant time* and *infinite universal speed*. As unmoved O/F $Oxyz$ measures *real velocity*, the transformation matrix (Λ_Γ) contains only real numbers. So, the spacetime is limited to the *real domain* R^4 . Moreover, this ST (*Galilean spacetime*) endowed with the *Galilean metric* (49), is degenerate.

2.3.2 The case of proper Closed Linear Transformations of Complex Spacetime with $\lambda \neq 0$ (time dependent on the position).

When $\lambda \neq 0$, we compare matrices Π and Λ_4 element by element and we also take into account (29). We start with $(\Lambda_4)_{21} + (\Lambda_4)_{12} = \Pi_{21} + \Pi_{12}$ and we obtain

$$\frac{2gh(f_4 - h_4)}{\left| \vec{\beta} \right| \left(\frac{g^2}{\left| \vec{\beta} \right|^2} + h^2 + \lambda^2 \right)} = -\frac{ag}{bf - ag} - \frac{\lambda^2}{h^2 + \lambda^2}. \quad (57)$$

We then use $(\mathcal{A}_4)_{32} + (\mathcal{A}_4)_{23} = \Pi_{32} + \Pi_{23}$, which gives

$$\frac{2h(f_4 - h_4)}{\frac{g^2}{\left| \vec{\beta} \right|^2} + h^2 + \lambda^2} = \frac{f - h}{h^2 + \lambda^2}. \quad (58)$$

The combination of (57) with the above equation implies

$$\frac{g(f - h)}{\left| \vec{\beta} \right| (h^2 + \lambda^2)} = -\frac{ag}{bf - ag} - \frac{\lambda^2}{h^2 + \lambda^2}. \quad (59)$$

Also, $(\mathcal{A}_4)_{31} + (\mathcal{A}_4)_{13} = \Pi_{31} + \Pi_{13}$ gives

$$\frac{-2g(f_4 - h_4)}{\left| \vec{\beta} \right| \left(\frac{g^2}{\left| \vec{\beta} \right|^2} + h^2 + \lambda^2 \right)} = \frac{b}{bf - ag} - \frac{h}{h^2 + \lambda^2}. \quad (60)$$

The combination of the above equation with (58) also gives

$$\frac{-g(f - h)}{\left| \vec{\beta} \right| (h^2 + \lambda^2)} = \frac{bh}{bf - ag} - \frac{h^2}{h^2 + \lambda^2}. \quad (61)$$

We then add (59) and (61) and get

$$f = h ; f_4 = h_4. \quad (62)$$

Moreover, from $(\mathcal{A}_4)_{11} = \Pi_{11}$ and $(\mathcal{A}_4)_{00} = \Pi_{00}$, we have

$$h_4 = \frac{bh}{bf - ag} = \frac{bf}{bf - ag} = b_4, \quad (63)$$

which combined with (62) gives

$$f = h = b. \quad (64)$$

We then use $(\mathcal{A}_4)_{22} = \Pi_{22}$, and we obtain

$$h_4 = \frac{fh}{h^2 + \lambda^2}. \quad (65)$$

The combination of the above equation with (63) gives

$$\frac{b}{bf - ag} = \frac{f}{h^2 + \lambda^2}. \quad (66)$$

Furthermore, $(\mathcal{A}_4)_{01} = \Pi_{01}$ and $(\mathcal{A}_4)_{02} = \Pi_{02}$ give respectively:

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$$\frac{ga_4}{|\vec{\beta}| \sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2 + \lambda^2}} = -\frac{ab}{bf - ag}; \quad (67)$$

$$\frac{a_4}{\sqrt{\frac{g^2}{|\vec{\beta}|^2} + h^2 + \lambda^2}} = \frac{a}{h^2 + \lambda^2}. \quad (68)$$

The substitution of (68) to (67) gives

$$\frac{g}{|\vec{\beta}|(h^2 + \lambda^2)} = -\frac{b}{bf - ag}. \quad (69)$$

Moreover, the combination of the above equation with (66) gives

$$g = -|\vec{\beta}|h. \quad (70)$$

Finally, (66) combined with (64) and (70) yields

$$\alpha = \frac{\lambda^2}{h|\vec{\beta}|}. \quad (71)$$

The replacement of (64), (70), (71) and

$$\lambda = \omega|\vec{\beta}|b, \quad (72)$$

makes the general matrix (23) equivalent to

$$A = \begin{bmatrix} b & \frac{\lambda^2}{b|\vec{\beta}|^2}\beta_x & \frac{\lambda^2}{b|\vec{\beta}|^2}\beta_y & \frac{\lambda^2}{b|\vec{\beta}|^2}\beta_z \\ -b\beta_x & b & \frac{\beta_z\lambda}{|\vec{\beta}|} & -\frac{\beta_y\lambda}{|\vec{\beta}|} \\ -b\beta_y & -\frac{\beta_z\lambda}{|\vec{\beta}|} & b & \frac{\beta_x\lambda}{|\vec{\beta}|} \\ -b\beta_z & \frac{\beta_y\lambda}{|\vec{\beta}|} & -\frac{\beta_x\lambda}{|\vec{\beta}|} & b \end{bmatrix} = b \begin{bmatrix} 1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z \\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix}. \quad (73)$$

We also define

$$\beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}; \quad \delta = \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = \begin{bmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \end{bmatrix}; \quad A_{(\beta)} = \begin{bmatrix} 0 & \beta_z & -\beta_y \\ -\beta_z & 0 & \beta_x \\ \beta_y & -\beta_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta^3 & -\beta^2 \\ -\beta^3 & 0 & \beta^1 \\ \beta^2 & -\beta^1 & 0 \end{bmatrix}. \quad (74)$$

It is noted that the antisymmetric matrix $A_{(\beta)}$ is related to the *cross product (external product)* [7] (p. 1048), because

$$A_{(\beta)} \delta = [-\vec{\beta} \times \vec{\delta}] = [\vec{\delta} \times \vec{\beta}]. \quad (75)$$

Thus, the four-vectors of two observers are related, by using the *general Matrix*:

$$\Lambda_{(\omega, \vec{\beta})} = b \begin{bmatrix} 1 & \omega^2 \beta_x & \omega^2 \beta_y & \omega^2 \beta_z \\ -\beta_x & 1 & \omega \beta_z & -\omega \beta_y \\ -\beta_y & -\omega \beta_z & 1 & \omega \beta_x \\ -\beta_z & \omega \beta_y & -\omega \beta_x & 1 \end{bmatrix} = b \begin{bmatrix} 1 & \omega^2 \beta^T \\ -\beta & I_3 + \omega A_{(\beta)} \end{bmatrix}. \quad (76)$$

Besides, the *typical Matrix* along x-axis is

$$\Lambda_{(x)(\omega, \beta)} = b \begin{bmatrix} 1 & \omega^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega \beta \\ 0 & 0 & -\omega \beta & 1 \end{bmatrix}. \quad (77)$$

So, the *proper Closed Linear Transformation of Complex Spacetime* (22) is

$$\begin{bmatrix} cd t' \\ dx' \\ dy' \\ dz' \end{bmatrix} = b \begin{bmatrix} 1 & \omega^2 \beta_x & \omega^2 \beta_y & \omega^2 \beta_z \\ -\beta_x & 1 & \omega \beta_z & -\omega \beta_y \\ -\beta_y & -\omega \beta_z & 1 & \omega \beta_x \\ -\beta_z & \omega \beta_y & -\omega \beta_x & 1 \end{bmatrix} \begin{bmatrix} cd t \\ dx \\ dy \\ dz \end{bmatrix}. \quad (78)$$

The pure mathematical approach is simply obtained by replacing $ct \rightarrow x^0$. Thus,

$$\begin{bmatrix} dx'^0 \\ dx'^1 \\ dx'^2 \\ dx'^3 \end{bmatrix} = b \begin{bmatrix} 1 & \omega^2 \beta^1 & \omega^2 \beta^2 & \omega^2 \beta^3 \\ -\beta^1 & 1 & \omega \beta^3 & -\omega \beta^2 \\ -\beta^2 & -\omega \beta^3 & 1 & \omega \beta^1 \\ -\beta^3 & \omega \beta^2 & -\omega \beta^1 & 1 \end{bmatrix} \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}. \quad (79)$$

Below, we calculate the corresponding CILToCST. For simplicity reasons, when we write i (the imaginary unit), we mean $\pm i$:

$$i \rightarrow \pm i \quad ; \quad -i \rightarrow \mp i. \quad (80)$$

The combination of the fundamental equation of isometry (43) (the *Killing's equation* in a linear space) with the above, gives

$$g_{11} = g_{22} = g_{33} = g_{ii}; \quad g_{00} = \frac{g_{ii}}{\omega^2}; \quad g'_{00} = b^2(1 + \omega^2 \beta^2)g_{00} = b^2(1 + \omega^2 |\vec{\beta}|^2) \frac{g_{ii}}{\omega^2}; \quad (81)$$

$$g'_{ii} = b^2(1 + \omega^2 |\vec{\beta}|^2)g_{ii}. \quad (82)$$

So, for any O/F, the metric of spacetime in accordance with the CILToCST is isotropic:

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} \quad (83)$$

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and also

$$\omega^2 = \frac{g_{ii}}{g_{00}}. \quad (84)$$

Thus, ω^2 is a real number. So, ω is a real or an imaginary number, which only depends on the metric of spacetime. Besides, the metrics of the spacetime of two observers (frames) Oxyz and O'x'y'z', are related using the formulas

$$g' = b^2(1 + \omega^2 |\vec{\beta}|^2)g; \quad (85)$$

$$g'_{ii} = b^2(1 + \omega^2 |\vec{\beta}|^2)g_{ii}. \quad (86)$$

So,

$$b^2 = \frac{g'_{ii}}{g_{ii}} \frac{1}{1 + \omega^2 |\vec{\beta}|^2}. \quad (87)$$

Using the well-known Lorentz γ -factor function

$$\gamma_{(\vec{\delta})} = \frac{1}{\sqrt{1 - \vec{\delta}^T \vec{\delta}}} = \frac{1}{\sqrt{1 - \vec{\delta} \cdot \vec{\delta}}} = \frac{1}{\sqrt{1 - |\vec{\delta}|^2}} = \gamma_{(\vec{\delta})}, \quad (88)$$

equation (87) may be written as

$$b^2 = \frac{g'_{ii}}{g_{ii}} \gamma_{(i\omega\vec{\beta})}^2. \quad (89)$$

Besides, the isometry of spacetime (45) combined with (89) and (78) gives

$$g'_{ii} = g_{ii}. \quad (90)$$

Thus, (89) gives $b^2 = \gamma_{(i\omega\vec{\beta})}^2$ and we obtain

$$b = \gamma_{(i\omega\vec{\beta})} > 0. \quad (91)$$

Moreover, (81iii) gives

$$g'_{00} = g_{00}. \quad (92)$$

This means that the metric of ST must be affected in the same way for any O/F, in the case of CILToCST. Equivalently, the observers that are related must be IOs or must have the *same acceleration*. Thus, the metric of spacetime in accordance with the CILToCST is

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} = g_{ii} \begin{bmatrix} \frac{1}{\omega^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{bmatrix}. \quad (93)$$

In case of SR (the frames are moved with constant velocity / the observes are IOs), equation (84) becomes

$$\omega_1^2 = \frac{g_{1ii}}{g_{100}}. \quad (94)$$

So, the time and space metric's coefficients are independent from the position and they are combined to produce ω_1 , which is *the characteristic parameter of SR*. The *continuity of the metric of spacetime at the point $\lambda=\omega=0$* and the matrices (49) and (93) gives

$$\lim_{\omega_1 \rightarrow 0} g_{1ii} = \lim_{\omega \rightarrow 0} g_{ii} = 0. \quad (95)$$

We also observe that *if the metric's coefficients have the same sign* [signature of spacetime: (----)], then the characteristic parameter ω_1 is *a real number*, in contrast with the case that *the coefficients of metric have different signs* [signature of spacetime: (-+++)], where the characteristic parameter ω_1 is *an imaginary number*.

The representation of the non-degenerate inner product in basis $[\vec{e}_\mu] = [\vec{e}_0 \ \vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] = [ct, \hat{x}, \hat{y}, \hat{z}]$ for IOs is the matrix

$$g_1 = \begin{bmatrix} g_{100} & 0 & 0 & 0 \\ 0 & g_{1ii} & 0 & 0 \\ 0 & 0 & g_{1ii} & 0 \\ 0 & 0 & 0 & g_{1ii} \end{bmatrix} = g_{1ii} \begin{bmatrix} \frac{1}{\omega_1^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{100} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega_1^2 & 0 & 0 \\ 0 & 0 & -\omega_1^2 & 0 \\ 0 & 0 & 0 & -\omega_1^2 \end{bmatrix}. \quad (96)$$

Generally, equation (78) gives the *active transformation* of O/F Oxyz to O/F O'x'y'z' (if they are accelerated with the same acceleration):

$$\begin{bmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{bmatrix} = \gamma_{(i\omega\vec{\beta})} \begin{bmatrix} 1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z \\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix}. \quad (97)$$

The replacement $ct \rightarrow x^0$ gives the pure mathematical approach. Thus,

$$\begin{bmatrix} dx'^0 \\ dx'^1 \\ dx'^2 \\ dx'^3 \end{bmatrix} = \gamma_{(i\omega\vec{\beta})} \begin{bmatrix} 1 & \omega^2\beta^1 & \omega^2\beta^2 & \omega^2\beta^3 \\ -\beta^1 & 1 & \omega\beta^3 & -\omega\beta^2 \\ -\beta^2 & -\omega\beta^3 & 1 & \omega\beta^1 \\ -\beta^3 & \omega\beta^2 & -\omega\beta^1 & 1 \end{bmatrix} \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}. \quad (98)$$

Using vectors, the above transformation becomes

$$cdt' = \gamma_{(i\omega\vec{\beta})} (cdt + \omega^2 \vec{\beta} \cdot d\vec{x}) ; d\vec{x}' = \gamma_{(i\omega\vec{\beta})} [(d\vec{x} - \vec{\beta} cdt) - \omega \vec{\beta} \times d\vec{x}]. \quad (99)$$

Moreover, the *general* and *typical matrices* of CILToCST are, respectively:

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$$\Lambda_{(\omega, \beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta_x & \omega^2 \beta_y & \omega^2 \beta_z \\ -\beta_x & 1 & \omega \beta_z & -\omega \beta_y \\ -\beta_y & -\omega \beta_z & 1 & \omega \beta_x \\ -\beta_z & \omega \beta_y & -\omega \beta_x & 1 \end{bmatrix} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta^T \\ -\beta & I_3 + \omega A_{(\beta)} \end{bmatrix}; \quad (100)$$

$$\beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}; \quad A_{(\beta)} = \begin{bmatrix} 0 & \beta_z & -\beta_y \\ -\beta_z & 0 & \beta_x \\ \beta_y & -\beta_x & 0 \end{bmatrix}; \quad \Lambda_{(x)(\omega, \beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega \beta \\ 0 & 0 & -\omega \beta & 1 \end{bmatrix}. \quad (101)$$

The above matrices Λ have the following properties:

$$\Lambda_{(\omega, 0)} = I_4; \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad (102)$$

$$\Lambda_{(\omega, \beta)}^{-1} = \Lambda_{(\omega, -\beta)}; \quad (103)$$

$$\det \Lambda_{(\omega, \beta)} = 1. \quad (104)$$

In case of SR, the matrices form a new *group* (which corresponds to *Lorentz group*) with elements

$$d = (\Lambda_{(\omega_1, \beta)}, B) \quad ; \quad B = \begin{bmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{bmatrix} = \begin{bmatrix} ct_0 \\ b_x \\ b_y \\ b_z \end{bmatrix}, \quad (105)$$

and operation

$$d_1 * d_2 = (\Lambda_{(\omega_1, \beta_2)} \Lambda_{(\omega_1, \beta_1)}, \Lambda_{(\omega_1, \beta_2)} B_1 + B_2), \quad (106)$$

where:

b_1^μ is the μ -coordinate which is measured in $O'x'y'z'$, when all the coordinates x^ν , for $\nu=0, 1, 2, 3$, in $Oxyz$ are equal to zero and

b_2^μ is the μ -coordinate which is measured in $O''x'y'z''$, when all the coordinates x'^ν , for $\nu=0, 1, 2, 3$, in $O'xyz$ are equal to zero. The above operation expresses the successive transformations:

$$O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z}); \quad O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z}). \quad (107)$$

These have active interpretations:

$$X' = \Lambda_{(\omega_1, \beta_1)} X + B_1; \quad X'' = \Lambda_{(\omega_1, \beta_2)} X' + B_2. \quad (108)$$

respectively. Thus, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$X'' = \Lambda_{(\omega_1, \beta_2)} \Lambda_{(\omega_1, \beta_1)} X + \Lambda_{(\omega_1, \beta_2)} B_1 + B_2. \quad (109)$$

As ω^2 is a real number, we observe that we always have *real time*. Besides, the *norm of the position four-vector* for Os/Fs with the same acceleration / the same metric of ST, is the corresponding *invariant quantity*

$$dS^2 = dX^T g dX = g_{00} c^2 dt^2 + g_{ii} d\bar{x}^2 = g_{ii} \left(\frac{1}{\omega^2} c^2 dt^2 + d\bar{x}^2 \right) = -g_{00} (-c^2 dt^2 - \omega^2 d\bar{x}^2). \quad (110)$$

In the case of SR, the above equation becomes

$$dS^2 = dX^T g dX = g_{100} c^2 dt^2 + g_{1ii} d\bar{x}^2 = g_{1ii} \left[\frac{1}{\omega_1^2} c^2 dt^2 + d\bar{x}^2 \right] = -g_{00} (-c^2 dt^2 - \omega_1^2 d\bar{x}^2). \quad (111)$$

If ω is a real number [the coefficients of metric of time and space have the same sign: signature of spacetime: (----)], then

$$\omega = \pm \sqrt{\frac{g_{ii}}{g_{00}}} = s \quad (112)$$

with $s \in \mathbb{R}$. Thus, the transformation matrix (A) contains only real numbers and the ST is limited to the *real domain* $\mathbb{R}^4$. Finally, the four-vectors of two Os/Fs have the same metric

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} = g_{ii} \begin{bmatrix} \frac{1}{s^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -s^2 & 0 & 0 \\ 0 & 0 & -s^2 & 0 \\ 0 & 0 & 0 & -s^2 \end{bmatrix} \quad (113)$$

and their CCs are related via the matrix:

$$\Lambda_{(s,\beta)} = \gamma_{(i\xi\beta)} \begin{bmatrix} 1 & s^2\beta_x & s^2\beta_y & s^2\beta_z \\ -\beta_x & 1 & s\beta_z & -s\beta_y \\ -\beta_y & -s\beta_z & 1 & s\beta_x \\ -\beta_z & s\beta_y & -s\beta_x & 1 \end{bmatrix} = \gamma_{(is\beta)} \begin{bmatrix} 1 & s^2\beta^T \\ -\beta & I_3 + s A_{(\beta)} \end{bmatrix}. \quad (114)$$

The typical matrix along x-axis is

$$\Lambda_{(x)(s,\beta)} = \gamma_{(i\xi\beta)} \begin{bmatrix} 1 & s^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & s\beta \\ 0 & 0 & -s\beta & 1 \end{bmatrix}. \quad (115)$$

If ω is an imaginary number [the coefficients of metric of time and space have different sign: signature of spacetime: (-+++)], then

$$\omega = i \sqrt{\frac{g_{ii}}{-g_{00}}} = \xi i ; \quad \xi = \sqrt{\frac{g_{ii}}{-g_{00}}} \quad (116)$$

with $\xi \in \mathbb{R}_+$. Thus, the transformation matrix (A) contains complex numbers and the spacetime is represented by the *complex domain* $\mathbb{R} \times \mathbb{C}^3$. Finally, the four-vectors of two Os/Fs have the same metric

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$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} = g_{ii} \begin{bmatrix} -\frac{1}{\xi^2} & 0 & 0 & 0 \\ \xi^2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & \xi^2 \end{bmatrix} \quad (117)$$

and their CCs are related via the matrix:

$$\Lambda_{(\xi i, \beta)} = \gamma_{(\xi \beta)} \begin{bmatrix} 1 & -\xi^2 \beta_x & -\xi^2 \beta_y & -\xi^2 \beta_z \\ -\beta_x & 1 & i \xi \beta_z & -i \xi \beta_y \\ -\beta_y & -i \xi \beta_z & 1 & i \xi \beta_x \\ -\beta_z & i \xi \beta_y & -i \xi \beta_x & 1 \end{bmatrix} = \gamma_{(\xi \beta)} \begin{bmatrix} 1 & -\xi^2 \beta^T \\ -\beta & I_3 + i \xi A_{(\beta)} \end{bmatrix}. \quad (118)$$

Besides, the typical Matrix along x-axis is

$$\Lambda_{(x)(\xi i, \beta)} = \gamma_{(\xi \beta)} \begin{bmatrix} 1 & -\xi^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i \xi \beta \\ 0 & 0 & -i \xi \beta & 1 \end{bmatrix}. \quad (119)$$

The substitution of (64), (70), (71) and (72) to (16) gives the *velocities typical transformation of CILToCST*:

$$\nu'_x = \frac{\nu_x - \beta c}{c + \omega^2 \beta \nu_x} c ; \quad \nu'_y = \frac{\nu_y + \omega \beta \nu_z}{c + \omega^2 \beta \nu_x} c ; \quad \nu'_z = \frac{\nu_z - \omega \beta \nu_y}{c + \omega^2 \beta \nu_x} c. \quad (120)$$

For the purpose of finding a possible Invariant Speed (U) for *Os/Fs with the same ω (or equivalently the same acceleration)*, we assume that a particle is moving to the right with velocity $\vec{\nu} = U \vec{e}_1$; $U > 0$. So, we have

$$\nu'_x = \frac{U - \beta c}{c + \omega^2 \beta U} c ; \quad \nu'_y = 0 ; \quad \nu'_z = 0. \quad (121)$$

According to the *Euclidean metric in the ordinary space E^3* , the norm of U is

$$U^2 = \left(\frac{U - \beta c}{c + \omega^2 \beta U} c \right)^2 + 0 + 0 = \frac{U^2 - 2\beta c U + \beta^2 c^2}{c^2 + \omega^4 \beta^2 U^2 + 2\omega^2 \beta c U} c^2, \quad (122)$$

which may be written as

$$(\omega^4 U^4 - c^4) \beta^2 + 2U(\omega^2 c U^2 + c^3) \beta = 0. \quad (123)$$

So, we obtain

$$U^2 = -\frac{c^2}{\omega^2}, \quad (124)$$

or equivalently,

$$\omega^2 = -\frac{c^2}{U^2}. \quad (125)$$

Since *norm* $U > 0$, ω is an *imaginary number* ($\omega = \xi i, \xi \in \mathbb{R}_+$) independent from the velocity (i.e. depends on the acceleration or equivalently the gravitation). Thus, we have

$$U = \frac{1}{\xi} c. \quad (126)$$

So, it is

$$\gamma_{(i\omega\beta)} = \frac{1}{\sqrt{1 + \omega^2 \beta^T \beta}} = \frac{1}{\sqrt{1 + \omega^2 \vec{\beta} \cdot \vec{\beta}}} = \frac{1}{\sqrt{1 - \xi^2 |\vec{\beta}|^2}} = \gamma_{(\xi\vec{\beta})} = \frac{1}{\sqrt{1 - \left(\frac{\vec{u}}{U}\right)^2}} = \gamma_{\left(\frac{\vec{u}}{U}\right)} \quad (127)$$

and (110) can also be written as

$$dS^2 = dX^T g dX = g_{ii} [-U^2 dt^2 + d\vec{x}^2] = -g_{00} \left[-c^2 dt^2 + \left(\frac{c}{U}\right)^2 d\vec{x}^2 \right]. \quad (128)$$

In the case of *spacetime endowed with constant metric* (or equivalently IOs), equation (126) becomes

$$c_1 = \frac{1}{\xi_1} c. \quad (129)$$

and we obtain the *Universal Speed* (c_1) of the *specific SR*. Besides, we have

$$\gamma_{(i\omega_1\vec{\beta})} = \frac{1}{\sqrt{1 + \omega_1^2 \beta^T \beta}} = \frac{1}{\sqrt{1 + \omega_1^2 |\vec{\beta}|^2}} = \frac{1}{\sqrt{1 - \xi_1^2 |\vec{\beta}|^2}} = \gamma_{(\xi_1\vec{\beta})} = \frac{1}{\sqrt{1 - \left(\frac{\vec{u}}{c_1}\right)^2}} = \gamma_{\left(\frac{\vec{u}}{c_1}\right)} \quad (130)$$

and

$$dS^2 = dX^T g dX = g_{iii} [-c_1^2 dt^2 + d\vec{x}^2] = -g_{00} \left[-c^2 dt^2 + \left(\frac{c}{c_1}\right)^2 d\vec{x}^2 \right]. \quad (131)$$

Now, let us find the corresponding *Euclidean CILToCST*. We initially define

$$dX_\omega = \begin{bmatrix} dX_\omega^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} c dt \\ dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}; \quad dX_\omega^0 = \frac{1}{\omega} dx^0, \quad (132)$$

where x^0 ; X_ω^0 are the zeroth-coordinates, by using the bases

$$[\vec{e}_\mu] = [\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3]; \quad [\vec{E}_\mu] = [\vec{E}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3] \quad (133)$$

of ST endowed with metric (93) and *Euclidean metric*, respectively. Thus,

$$\vec{e}_0 \cdot \vec{e}_0 = g_{00}; \quad \vec{E}_0 \cdot \vec{E}_0 = 1, \quad (134)$$

where dot “.” is *Euclidean inner product* [4] (p. 7). So, we understand that

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$$\vec{E}_0 = \frac{i}{\sqrt{-g_{00}}} \vec{e}_0. \quad (135)$$

Then, the CILToCST (97) can be written as

$$\begin{bmatrix} \frac{1}{\omega} \text{cd } t' \\ \text{d } x' \\ \text{d } y' \\ \text{d } z' \end{bmatrix} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega\beta_x & \omega\beta_y & \omega\beta_z \\ -\omega\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\omega\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\omega\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\omega} \text{cd } t \\ \text{d } x \\ \text{d } y \\ \text{d } z \end{bmatrix}, \quad (136)$$

or equivalently,

$$\text{d}X'_{\omega} = \tilde{R}_{(\omega\beta)} \text{d}X_{\omega}, \quad (137)$$

where

$$\tilde{R}_{(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega\beta_x & \omega\beta_y & \omega\beta_z \\ -\omega\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\omega\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\omega\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix}; \quad \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}, \quad (138)$$

or equivalently,

$$\tilde{R}_{(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega\beta^T \\ -\omega\beta & \text{I}_3 + \omega \text{A}_{(\beta)} \end{bmatrix} = \gamma_{(i\omega\beta)} \left\{ \text{I}_4 + \omega \begin{bmatrix} 0 & \beta^T \\ -\beta & \text{A}_{(\beta)} \end{bmatrix} \right\}. \quad (139)$$

The above matrix $\tilde{R}$ is a *rotation matrix* with the following properties:

$$\tilde{R}_{(\omega 0)} = \text{I}_4; \quad \text{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (140)$$

$$\tilde{R}_{(\omega\beta)}^{-1} = \tilde{R}_{(\omega\beta)}^T = \tilde{R}_{(-\omega\beta)}, \quad (141)$$

$$\det \tilde{R}_{(\omega\beta)} = 1. \quad (142)$$

The corresponding typical matrix along the x-axis in *Euclidean spacetime* (E^4) is

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega\beta & 0 & 0 \\ -\omega\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix}. \quad (143)$$

Now, using vectors the CILToCST of equation (136) becomes

$$\frac{\text{cd } t'}{\omega} = \gamma_{(i\omega\vec{\beta})} \left(\frac{\text{cd } t}{\omega} + \omega\vec{\beta} \cdot \text{d } \vec{x} \right); \quad \text{d } \vec{x}' = \gamma_{(i\omega\vec{\beta})} \left[\left(\text{d } \vec{x} - \omega\vec{\beta} \frac{\text{cd } t}{\omega} \right) - \omega\vec{\beta} \times \text{d } \vec{x} \right], \quad (144)$$

or equivalently,

$$dX'_\omega{}^0 = \gamma_{(i\omega\vec{\beta})} (dX_\omega{}^0 + \omega\vec{\beta} \cdot d\vec{x}) ; d\vec{x}' = \gamma_{(i\omega\vec{\beta})} \left[(d\vec{x} - \omega\vec{\beta} dX_\omega{}^0) - \omega\vec{\beta} \times d\vec{x} \right]. \quad (145)$$

Thus, we have the *Euclidean metric* of the position four-vector X_ω in E^4 , and the corresponding invariant quantity is

$$dS_\omega^2 = dX_\omega{}^r g_E dX_\omega = \frac{1}{\omega^2} c^2 dt^2 + d\vec{x}^2 = (dX_\omega{}^0)^2 + d\vec{x}^2 = \frac{1}{g_{ii}} dS^2, \quad (146)$$

according to (110ii).

Also, we observe that β -factor can be written as

$$\beta^i = \frac{dx^i}{dx^0} = \frac{dx^i}{\omega dX_\omega{}^0} = \frac{B^i}{\omega} ; B^i = \frac{dx^i}{dX_\omega{}^0}, \quad (147)$$

by using (132ii). The quantity B^i is called *B-factor* and it can substitute the β -factor, in E^4 . Then, equations (136-139) are rewritten:

$$\begin{bmatrix} dX'_\omega{}^0 \\ dx'^1 \\ dx'^2 \\ dx'^3 \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} 1 & B^1 & B^2 & B^3 \\ -B^1 & 1 & B^3 & -B^2 \\ -B^2 & -B^3 & 1 & B^1 \\ -B^3 & B^2 & -B^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} dX_\omega{}^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}, \quad (148)$$

$$dX'_\omega = \tilde{R}_{(B)} dX_\omega, \quad (149)$$

$$\tilde{R}_{(B)} = \gamma_{(iB)} \begin{bmatrix} 1 & B^1 & B^2 & B^3 \\ -B^1 & 1 & B^3 & -B^2 \\ -B^2 & -B^3 & 1 & B^1 \\ -B^3 & B^2 & -B^1 & 1 \end{bmatrix} ; B = \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix}, \quad (150)$$

$$\tilde{R}_{(B)} = \gamma_{(iB)} \begin{bmatrix} 1 & B^T \\ -B & I_3 + A_{(B)} \end{bmatrix} = \gamma_{(iB)} \left\{ I_4 + \begin{bmatrix} 0 & B^T \\ -B & A_{(\beta)} \end{bmatrix} \right\}. \quad (151)$$

The above matrix $\tilde{R}$ is a *rotation matrix* having the following properties:

$$\tilde{R}_{(0)} = I_4 ; \quad (152)$$

$$\tilde{R}_{(B)}^{-1} = \tilde{R}_{(B)}^T = \tilde{R}_{(-B)} ; \quad (153)$$

$$\det \tilde{R}_{(B)} = 1. \quad (154)$$

Besides, the corresponding typical matrix along the x-axis in E^4 is

$$\tilde{R}_{(x)(B)} = \gamma_{(iB)} \begin{bmatrix} 1 & B & 0 & 0 \\ -B & 1 & 0 & 0 \\ 0 & 0 & 1 & B \\ 0 & 0 & -B & 1 \end{bmatrix}. \quad (155)$$

Note that the above transformation can be limited in the real spacetime (R^4), because the corresponding *Lorentz γ -factor* is positive for any real *B-factor*.

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We observe that the above results could be obtained from the initial equations of E^4 : (136-146), when $\omega \rightarrow 1$ and $ct \rightarrow X_\omega$. We also observe that $\tilde{R}$ reminds us of the *contravariant electromagnetic tensor* [3] (p. 14), [4] (p. 414):

$$F_{(E, B_m)} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & c B_m^3 & -c B_m^2 \\ -E^2 & -c B_m^3 & 0 & c B_m^1 \\ -E^3 & c B_m^2 & -c B_m^1 & 0 \end{bmatrix}, \quad (156)$$

where E and B_m are the *intensity of electric field* and *induction of magnetic field*, respectively [4] (p. 396). Actually, they are correlated via the formula

$$\tilde{R}_{(B)} = F_{(E, B_m)} + \gamma_{(iB)} I_4 ; \quad E^j = \gamma_{(iB)} B^j ; \quad B_m^j = \frac{\gamma_{(iB)} B^j}{c}. \quad (157)$$

Thus, it is

$$E^j = c B_m^j ; \quad E^j E_j = c^2 (B_m^j B_{mj}) \quad (158)$$

where (158ii) is the same as the *electromagnetic waves in vacuum*, while (158i) means that the vectors of the induction of magnetic field and intensity of electric field are parallel. This reveals a hidden correlation between the *spacetime* and *electromagnetism (Maxwell equations)*.

Moreover, for any *constant value of ω_1* (or more precisely for any *constant metric*, i.e. *constant values of g_{100} and g_{iii}*), we have a specific CILToCST which correlates IOs and the corresponding *SR-theory*.

Furthermore, the limit $s \rightarrow s_I \rightarrow 0$ in the equations (113-115) and their combination with (95) gives GT of complex spacetime with *infinite universal speed*. In the same way, the limit $\zeta \rightarrow \zeta_I \rightarrow 0$ in the equations (117-119) and their combination with (95), gives again GT. Thus, the result when $\lambda=0$ (GT) is embedded to the case when $\lambda \neq 0$, if we take the corresponding limit to zero ($\lambda \rightarrow 0$, or equivalently, $\omega \rightarrow 0$).

Besides, if one O/F has small velocity wrt another, the CILToCST (even been complex) is reduced to GT.

The replacement $\zeta \rightarrow \zeta_I = 1$ to the equations (118) and (119), produces the *Lorentzian-Einsteinian* version of CILToCST (Λ_B) [7] (pp.1047-1048), which is expressed via the general matrix

$$\Lambda_{B(\beta)} = \gamma \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix} \quad (159)$$

and the typical matrix along the x-axis

$$\Lambda_{B(x)(\beta)} = \gamma \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix}. \quad (160)$$

From (96), we take the corresponding metric of complex spacetime

$$g_B = g_{1ii} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = g_{1ii} \eta, \quad (161)$$

which for $g_{1ii}=1$ becomes the *Lorentz metric*. Thus, we have the SR-theory with *universal speed* being the *speed of light in vacuum* ($c_1=c$) [7,8]. This theory gives results that are exactly the same as ESR, when only two Os/Fs are related. But the results are different, when more than two Os/Fs are related. Besides, it calculates the fine structure peaks of atomic hydrogen's spectrum [8] (p. 4) more accurately than ESR. The explicit form of forward *Lorentzian-Einsteinian* CILToCST is

$$ct' = \gamma(ct - \beta_x x - \beta_y y - \beta_z z) \quad (162)$$

$$x' = \gamma(-\beta_x ct + x + i\beta_z y - i\beta_y z) \quad (163)$$

$$y' = \gamma(-\beta_y ct - i\beta_z x + y + i\beta_x z) \quad (164)$$

$$z' = \gamma(-\beta_z ct + i\beta_y x - i\beta_x y + z) \quad (165)$$

The explicit form of reverse *Lorentzian-Einsteinian* CILToCST is

$$ct = \gamma(ct' + \beta_x x' + \beta_y y' + \beta_z z') \quad (166)$$

$$x = \gamma(\beta_x ct' + x' - i\beta_z y' + i\beta_y z') \quad (167)$$

$$y = \gamma(\beta_y ct' + i\beta_z x' + y' - i\beta_x z') \quad (168)$$

$$z = \gamma(\beta_z ct' - i\beta_y x' + i\beta_x y' + z') \quad (169)$$

When the metric of ST depends on the position of the event in spacetime (GR), the transformation is applied locally, not globally (correlating Os/Fs with the same acceleration / gravitation). A metric is in accordance with the CILToCST, if only the limit of vanishing acceleration leads to the corresponding SR. Thus, the usage of (84) and (94) leads to

$$\lim_{\vec{a} \rightarrow 0} g_{ii} = g_{1ii}; \quad (170)$$

$$\lim_{\vec{a} \rightarrow 0} g_{00} = \lim_{\vec{a} \rightarrow 0} \frac{g_{ii}}{\omega^2} = \frac{g_{1ii}}{\omega_1^2} = g_{100}. \quad (171)$$

3 Proper Time – Special and General Relativity

Let P be a particle moving with velocity $\vec{v}_p$ wrt observer O ($\vec{v}_p'$ wrt observer O') in spacetime. The generalized definition of proper time (τ) is

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$$d\tau^2 = \frac{dS^2}{g_{00} c^2}. \quad (172)$$

Using (84) and (110), we have

$$d\tau^2 = \frac{g_{ii}}{g_{00} c^2} \left(\frac{1}{\omega^2} c^2 dt^2 + d\bar{x}^2 \right) = \frac{\omega^2}{c^2} \left(\frac{1}{\omega^2} c^2 dt^2 + d\bar{x}^2 \right), \quad (173)$$

or equivalently,

$$d\tau^2 = dt^2 + \frac{\omega^2}{c^2} d\bar{x}^2 = dt^2 \left(1 + \frac{\omega^2}{c^2} \bar{v}_p^2 \right). \quad (174)$$

Thus, the relation between the time and the proper time is

$$\frac{dt^2}{d\tau^2} = \gamma_{(i\omega\beta_p)}^2. \quad (175)$$

For $\omega=s$ with $s \in \mathbb{R}$, there does not exist real Invariant Speed (U) and the γ -factor is always positive. So,

$$\frac{dt}{d\tau} = \gamma_{(is\beta_p)}. \quad (176)$$

When $\omega=\zeta i$ with $\zeta \in \mathbb{R}$, there exists a real U . If the speed of particle is less than the invariant speed ($|\bar{v}_p| < U$), then the γ -factor is positive again. Thus,

$$\frac{dt}{d\tau} = \gamma_{(\xi\beta_p)}. \quad (177)$$

For GT with $s \rightarrow \zeta \rightarrow 0$ (the limit of degenerate spacetime), $\gamma_{(is\beta_p)} = \gamma_{(\xi\beta_p)} = 1$. So, $d\tau = dt' = dt$ (time is invariant) as we know in NPs.

In the case of ST with constant metric (IOs), equation (175) becomes

$$\frac{dt^2}{d\tau^2} = \gamma_{(i\omega_1\beta_p)}^2. \quad (178)$$

Thus, the *Lorentzian-Einsteinian* version of CILToCST with $\omega_1=i$ ($\zeta_1=1$), gives $U=c_1=c$. If the speed of a particle is less than the *speed of light in vacuum* ($|\bar{v}_p| < c$), then γ -factor is positive again. Thus,

$$\frac{dt}{d\tau} = \gamma_{(\beta_p)}. \quad (179)$$

and we have the same result as the ESR.

In any case, using proper time, we can define four-velocity, four-momentum etc, building the whole structure of Generalized SR and GR.

4 The Results of Closed Linear Transformation of Complex Spacetime - Discussion

In this section, we present the typical matrix $\Lambda_{(x)(\beta)}$, the general matrix $\Lambda_{(\beta)}$, the covariant matrix of spacetime metric g , the invariant speed U and the domain of the coordinates C^4 that corresponds to the transformation of a contravariant infinitesimal four-vector in spacetime: $dX' = \Lambda dX$.

$$\begin{array}{c}
 (\Lambda_{(x)(\beta)}, \Lambda_{(\beta)}, g, U, C^4) \\
 | \\
 \lambda = \omega |\vec{\beta}| b = ? \\
 \begin{array}{cc}
 \lambda \neq 0 & \lambda = 0
 \end{array} \\
 \hline
 \Lambda_{(x)(\beta)} = b_{(\beta)} \begin{bmatrix} 1 & \omega^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega \beta \\ 0 & 0 & -\omega \beta & 1 \end{bmatrix}, \quad \Lambda_{(x)(\beta)} = \begin{bmatrix} b_{(\beta)} & 0 & 0 & 0 \\ -h_{(\beta)} \beta & h_{(\beta)} & 0 & 0 \\ 0 & 0 & h_{(\beta)} & 0 \\ 0 & 0 & 0 & h_{(\beta)} \end{bmatrix}, \\
 \Lambda_{(\beta)} = b_{(\beta)} \begin{bmatrix} 1 & \omega^2 \beta^T \\ -\beta & I_3 + \omega A_{(\beta)} \end{bmatrix}, U \in \{R, I\}. \quad \Lambda_{(\beta)} = \begin{bmatrix} b_{(\beta)} & O^T \\ -h_{(\beta)} \beta & h_{(\beta)} I_3 \end{bmatrix}, U = +\infty. \\
 \begin{array}{cc}
 | \text{isometry} & | \text{isometry} \\
 \hline
 \Lambda_{(x)(\omega, \beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega \beta \\ 0 & 0 & -\omega \beta & 1 \end{bmatrix}, \quad \Lambda_{\Gamma(x)(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 \Lambda_{(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta^T \\ -\beta & I_3 + \omega A_{(\beta)} \end{bmatrix}, \quad \Lambda_{\Gamma(\beta)} = \begin{bmatrix} 1 & O^T \\ -\beta & I_3 \end{bmatrix}, \\
 g = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{bmatrix}, \quad g_{\Gamma} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, U = +\infty, R^4. \\
 | \\
 \omega = ?
 \end{array}
 \end{array}$$

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$\omega = \xi i \in I$	$\omega = s \in R$
$\Lambda_{(x)(i\xi, \beta)} = \gamma_{(\xi\beta)} \begin{bmatrix} 1 & -\xi^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\xi\beta \\ 0 & 0 & -i\xi\beta & 1 \end{bmatrix},$ $\Lambda_{(i\xi, \beta)} = \gamma_{(\xi\beta)} \begin{bmatrix} 1 & -\xi^2 \beta^T \\ -\beta & I_3 + i\xi A_{(\beta)} \end{bmatrix},$ $g = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & \xi^2 \end{bmatrix},$ $U = \frac{1}{\xi} c \in R_+, \quad X \in RC^3.$	$\Lambda_{(x)(\xi, \beta)} = \gamma_{(is\beta)} \begin{bmatrix} 1 & s^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & s\beta \\ 0 & 0 & -s\beta & 1 \end{bmatrix},$ $\Lambda_{(\xi, \beta)} = \gamma_{(is\beta)} \begin{bmatrix} 1 & s^2 \beta^T \\ -\beta & I_3 + s A_{(\beta)} \end{bmatrix},$ $g = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -s^2 & 0 & 0 \\ 0 & 0 & -s^2 & 0 \\ 0 & 0 & 0 & -s^2 \end{bmatrix},$ $U \text{ does not exist, } X \in R^4.$

The results may be applied to any complex or real isotropic space of dimension four (spacetime), endowed with the corresponding metric, whose elements (four-vectors) have spatial part (vector) with *Euclidean metric*. We simply put

$$ct \rightarrow x^0 ; x \rightarrow x^1 ; y \rightarrow x^2 ; z \rightarrow x^3. \quad (180)$$

So, the β -factor is written as

$$\beta^i = \frac{dx^i}{dx^0}. \quad (181)$$

New spaces are produced from the initial space: (i) with derivation of the initial four-vector wrt an invariant quantity, such as ‘proper time’, or (ii) with multiplication of the initial four-vector with invariant quantity, such as ‘mass’ (see e.g. [4] p. 109). Moreover, there exist applications beyond physics as biometry, econometrics etc, producing suitable vectors and four-vectors.

The specific value $\omega_I = i$ ($\xi_I = 1$) gives the *Lorentzian-Einsteinian version of CILToCST* endowed with *Lorentz metric* (for $g_{lii} = 1$), which produces the *Lorentzian Complex Relativity Theory*, which

- (i) is using the *Lorentzian-Einsteinian version of CILToCST* instead of *Lorentz transformation* that is used by ESR,
- (ii) is using complex Cartesian Coordinates creating a Generalized Euclidean Geometry,
- (iii) creates the *new group of CILToCST* with elements complex matrices, instead of the *Lorentz group* of ESR,
- (iv) can produce the *Lorenz Boost* of ESR [3] (p. 6),

- (v) is successfully applied to mechanics and electromagnetism,
- (vi) maintains the Classical Physical Laws and the formalism of ERT,
- (vii) is in accordance with Quantum Mechanics,
- (viii) gives results that are exactly the same as ERT, if only two observers are related,
- (ix) gives different results than ERT, when more than two observers are related, and
- (x) calculates with better accuracy the fine structure peaks of spectrum of atomic Hydrogen than ESR [8] (p. 4).

Finally, we can consider that the value of ω_I depends on the *cosmic time* (t_c). Thus, *Lorentz metric* is valid, only for ‘events’ near to (nowadays, Earth). This could be an explanation for the problem of *dark matter* and *dark energy*.

5 Improper isometric Linear Transformations in Spacetime endowed with Euclidean, or Lorentz, or generally Isotropic metric

In the derivation of proper closed isometric LT (ICLTtoCST) ($\downarrow\uparrow$), we have chosen positive b [the *lower sign* ($\downarrow$) in (91)] and via (64) f is positive ($\uparrow$), too. So, there have remained the following three (3) improper non-closed isometric LTs (which do not contain the *identity transformation*) [see also the *Lorentzian-Einsteinian version* [7] (pp. 1049-1050)]:

(i) *Space inversion non-closed isometric Linear Transformation* ($\downarrow\downarrow$) in *isotropic ST* and E^4 with corresponding matrices ($\det\Lambda = \det\tilde{R} = -1$):

$$\Lambda_{(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta^T \\ \beta & -I_3 - \omega A_{(\beta)} \end{bmatrix}; \quad (182)$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega \beta^T \\ \omega \beta & -I_3 - \omega A_{(\beta)} \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} 1 & B^T \\ B & -I_3 - A_{(B)} \end{bmatrix} = \tilde{R}_{(B)}. \quad (183)$$

The respective *typical transformations* along the x -axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega^2 \beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega \beta \\ 0 & 0 & \omega \beta & -1 \end{bmatrix}; \quad (184)$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} 1 & \omega \beta & 0 & 0 \\ \omega \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega \beta \\ 0 & 0 & \omega \beta & -1 \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} 1 & B & 0 & 0 \\ B & -1 & 0 & 0 \\ 0 & 0 & -1 & -B \\ 0 & 0 & B & -1 \end{bmatrix} = \tilde{R}_{(x)(B)}. \quad (185)$$

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(ii) *Time inversion non-closed isometric Linear Transformation* ($\uparrow\uparrow$) in isotropic ST and E^4 with corresponding matrices ($\det\Lambda = \det\tilde{R} = -1$):

$$\Lambda_{B(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega^2 \beta^T \\ -\beta & I_3 + \omega A_{(\beta)} \end{bmatrix}; \quad (186)$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega\beta^T \\ -\omega\beta & I_3 + \omega A_{(\beta)} \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} -1 & -B^T \\ -B & I_3 + A_{(B)} \end{bmatrix} = \tilde{R}_{(B)}. \quad (187)$$

The respective *typical transformations* along the x -axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega^2 \beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix}; \quad (188)$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega\beta & 0 & 0 \\ -\omega\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} -1 & -B & 0 & 0 \\ -B & 1 & 0 & 0 \\ 0 & 0 & 1 & B \\ 0 & 0 & -B & 1 \end{bmatrix} = \tilde{R}_{(x)(B)}. \quad (189)$$

(iii) *Spacetime inversion non-closed isometric Linear Transformation* ($\uparrow\downarrow$) in isotropic ST and E^4 with corresponding matrices ($\det\Lambda = \det\tilde{R} = 1$):

$$\Lambda_{(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega^2 \beta^T \\ \beta & -I_3 - \omega A_{(\beta)} \end{bmatrix}; \quad (190)$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega\beta^T \\ \omega\beta & -I_3 - \omega A_{(\beta)} \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} -1 & -B^T \\ B & -I_3 - A_{(B)} \end{bmatrix} = \tilde{R}_{(B)}. \quad (191)$$

The respective *typical transformations* along the x -axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega^2 \beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix}; \quad (192)$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(i\omega\beta)} \begin{bmatrix} -1 & -\omega\beta & 0 & 0 \\ \omega\beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix} = \gamma_{(iB)} \begin{bmatrix} -1 & -B & 0 & 0 \\ B & -1 & 0 & 0 \\ 0 & 0 & -1 & -B \\ 0 & 0 & B & -1 \end{bmatrix} = \tilde{R}_{(x)(B)}. \quad (193)$$

These matrices are exactly the opposite of the corresponding *proper ICLToCST*.

The above can be compared to the case of *Lorentz Boost* [4] (pp. 30-31), [7] (pp. 1050-1052), where we have:

(a) *Space inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \tilde{R}_L = -1$):

$$\Lambda_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)} \beta^T \\ -\gamma_{(\beta)} \beta & -I_3 - \frac{\gamma_{(\beta)} - 1}{\beta^T \beta} \beta \beta^T \end{bmatrix}; \quad (194)$$

$$\tilde{R}_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -i \gamma_{(\beta)} \beta^T \\ -i \gamma_{(\beta)} \beta & -I_3 - \frac{\gamma_{(\beta)} - 1}{\beta^T \beta} \beta \beta^T \end{bmatrix} = \begin{bmatrix} \gamma_{(iB)} & -\gamma_{(iB)} B^T \\ -\gamma_{(iB)} B & -I_3 - \frac{\gamma_{(iB)} - 1}{B^T B} B B^T \end{bmatrix} = \tilde{R}_{L(B)}. \quad (195)$$

The respective *typical transformations* along the x -axis, have

$$\Lambda_{L(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)} \beta & 0 & 0 \\ -\gamma_{(\beta)} \beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \quad (196)$$

$$\tilde{R}_{L(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -i \gamma_{(\beta)} \beta & 0 & 0 \\ -i \gamma_{(\beta)} \beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \gamma_{(iB)} & -\gamma_{(iB)} B & 0 & 0 \\ -\gamma_{(iB)} B & -\gamma_{(iB)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \tilde{R}_{L(x)(B)}. \quad (197)$$

(b) *Time inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \tilde{R}_L = 1$):

$$\Lambda_{L(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)} \beta^T \\ \gamma_{(\beta)} \beta & I_3 - \frac{\gamma_{(\beta)} + 1}{\beta^T \beta} \beta \beta^T \end{bmatrix}; \quad (198)$$

$$\tilde{R}_{L(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -i \gamma_{(\beta)} \beta^T \\ i \gamma_{(\beta)} \beta & I_3 - \frac{\gamma_{(\beta)} + 1}{\beta^T \beta} \beta \beta^T \end{bmatrix} = \begin{bmatrix} -\gamma_{(iB)} & -\gamma_{(iB)} B^T \\ \gamma_{(iB)} B & I_3 - \frac{\gamma_{(iB)} + 1}{B^T B} B B^T \end{bmatrix} = \tilde{R}_{L(B)}. \quad (199)$$

The respective *typical transformations* along the x -axis, have

$$\Lambda_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)} \beta & 0 & 0 \\ \gamma_{(\beta)} \beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad (200)$$

$$\tilde{R}_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -i \gamma_{(\beta)} \beta & 0 & 0 \\ i \gamma_{(\beta)} \beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma_{(iB)} & -\gamma_{(iB)} B & 0 & 0 \\ \gamma_{(iB)} B & -\gamma_{(iB)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tilde{R}_{L(x)(B)}. \quad (201)$$

(c) *Spacetime inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \tilde{R}_L = -1$):

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$$\Lambda_{L(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ \gamma_{(\beta)}\beta & -I_3 + \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}; \quad (202)$$

$$\tilde{R}_{L(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & i\gamma_{(\beta)}\beta^T \\ i\gamma_{(\beta)}\beta & -I_3 + \frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix} = \begin{bmatrix} -\gamma_{(iB)} & i\gamma_{(iB)}B^T \\ i\gamma_{(iB)}B & -I_3 + \frac{\gamma_{(iB)}+1}{B^TB}BB^T \end{bmatrix} = \tilde{R}_{L(B)}. \quad (203)$$

The respective *typical transformations* along the x-axis, have

$$\Lambda_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \quad (204)$$

$$\tilde{R}_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & i\gamma_{(\beta)}\beta & 0 & 0 \\ i\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\gamma_{(iB)} & \gamma_{(iB)}B & 0 & 0 \\ \gamma_{(iB)}B & \gamma_{(iB)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \tilde{R}_{L(x)(B)}. \quad (205)$$

5 Conclusions

In a 3D complex ‘space’ endowed with *Euclidean metric*, we consider one frame Oxyz, where a ‘position’ vector has real *Cartesian Coordinates*. Another real independent variable (‘time’) and the aforementioned coordinates produce a real four-vector. There exist *two cases* of closed linear ‘spacetime’ transformation of this real four-vector: *one* with the ‘time’ depending on the position, where the ‘event’ happens and *another* with the ‘time’ being independent from the position. The *first case* can have real Invariant ‘Speed’ (*U*), in contrast to the *second case*, which has only infinite *U*.

Moreover, for transformation having *isometry*, the *first case* transformation matrix is totally calculated and contains a parameter ω , with $\omega^2 = g_{ii}/g_{00}$ (the ratio of coefficients of ‘spacetime’ metric) in addition to the ‘velocity’ of the frame O’x’y’z’ wrt Oxyz. The *second case* is turned to *Galilean Transformation* (GT). The assumption that $\omega \rightarrow 0$ in the *first case* yields GT. So, in isometry, the *second case* is *embedded to the first case* transformation. Besides the *first case* is divided to *two types*: *one type*, where ‘time’ and ‘space’ have ‘spacetime’ metric coefficients with different signs [signature of spacetime: (-+++)], which leads to complex 3D ‘space’ with real *U*. The *second type*, where ‘time’ and ‘space’ have metric coefficients with the same sign [signature of spacetime: (----)], leads to real 3D ‘space’ without *U*. *Time remains real*, in both cases.

If the *metric is independent from the ‘position’* of the ‘event’ in ‘spacetime’, it is $\omega = \omega_I = \text{constant}$ and we have the case of ‘Special Relativity’

(‘SR’) and the transformation can be applied globally, relating ‘Inertial Observers / Frames’ (‘IOs/Fs’). Thus, infinite number of ‘SRs’ are produced (each one with the corresponding metric), all of them keeping Einsteinian SR-formalism. In the case that *metric depends on the ‘position’* of the ‘event’ in ‘spacetime’, we have the ‘General Relativity’ (‘GR’) and the transformation may be applied locally, relating ‘accelerated observers / frames’. Thus, infinite number of ‘GRs’ are produced (each one with the corresponding metric of IOs’ spacetime), all of them keeping Einsteinian GR-formalism. Of course, vanishing ‘acceleration’ leads to the corresponding ‘SR’.

This new modeling of study allows studying *Einsteinian Relativity Theory*, *Newtonian Physics* (NPs), or any other Theory of Physics that is in accordance with closed Linear Spacetime Transformations simultaneously. This is achieved, because the coefficients of spacetime metric are contained in the transformation matrix. Besides, NPs is obtained, not only by the low velocity limit, but also by the zero limit of the space coefficient of spacetime metric ($g_{ii} \rightarrow 0$), or equivalently $\omega \rightarrow 0$. Finally, we can consider that the value of ω_i depends on the *cosmic time* (t_c) and *Lorentz metric* is valid only for ‘events’ near to the (nowadays, Earth). This could be used for the explanation of *dark matter* and *dark energy*.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Some characteristics of picture fuzzy subgroups via cut set of picture fuzzy set

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Abstract

Given any picture fuzzy set Q of a universe Y , the set $C_{r,s,t}(Q)$ called the (r, s, t) -cut set of Q was studied. In this paper, some characteristics of picture fuzzy subgroup of a group are obtained via the cut sets of picture fuzzy set.

Keywords: fuzzy set, picture fuzzy set, picture fuzzy subgroup, cut set

2020 AMS subject classifications: 20N25, 08A72, 03E72.¹

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1 Introduction

Zadeh [18] introduced the notion of fuzzy sets (FSs). This theory has been generalised by many researchers. Atanassov [1] initiated the concept of intuitionistic fuzzy sets (IFSs). For more on intuitionistic fuzzy set, also see [2, 3, 4]. The application of these fuzzy sets have taken different directions such as optimization and decision-making [15] control theory [13] and many others. One particular application is the work of Cuong and Kreinovich [5] who put forward the notion of picture fuzzy sets (PFSs) as a generalisation of fuzzy sets and intuitionistic fuzzy sets. Picture fuzzy set has been extensively studied and applied (see [6, 7, 8, 9, 10, 11, 12, 14, 17] for details).

Rosenfeld [16] generalised fuzzy sets to fuzzy groups (FGs). The idea of cut set of picture fuzzy sets was initiated by Dutta and Ganju [12] and they obtained some of its properties. Dogra and Pal [11] corrected the definition of cut set that was given by Dutta and Ganju [12] and they initiated the notion of picture fuzzy subgroups (PFSGs) of a group.

In this paper, motivated by the work of Dogra and Pal [11], investigation of the characteristics of picture fuzzy subgroups of a group via the cut sets of picture fuzzy sets was done. The organisation of the paper is as follows: Section 2 gives the basic definitions and preliminary ideas of PFSs; Section 3 investigates some properties of picture fuzzy subgroups of a group via cut sets of picture fuzzy sets.

2 Preliminaries

Definition 2.1. [18] Let Y be a nonempty set. A FS Q of Y is an object of the form

$$Q = \{\langle y, \sigma_Q(y) \rangle | y \in Y\}$$

with a membership function

$$\sigma_Q : Y \longrightarrow [0, 1]$$

where the function $\sigma_Q(y)$ denotes the degree of membership of $y \in Q$.

Definition 2.2. [1] Let a nonempty set Y be fixed. An IFS Q of Y is an object of the form

$$Q = \{\langle y, \sigma_Q(y), \tau_Q(y) \rangle | y \in Y\}$$

where the functions

$$\sigma_Q : Y \rightarrow [0, 1] \text{ and } \tau_Q : Y \rightarrow [0, 1]$$

are called the membership and non-membership degrees of $y \in Q$, respectively, and for every $y \in Y$,

$$0 \leq \sigma_Q(y) + \tau_Q(y) \leq 1.$$

Definition 2.3. [5] A picture fuzzy set Q of Y is defined as

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \gamma_Q(y)) | y \in Y\},$$

where the functions

$$\sigma_Q : Y \rightarrow [0, 1], \tau_Q : Y \rightarrow [0, 1] \text{ and } \gamma_Q : Y \rightarrow [0, 1]$$

are called the positive, neutral and negative membership degrees of $y \in Q$, respectively, and $\sigma_Q, \tau_Q, \gamma_Q$ satisfy

$$0 \leq \sigma_Q(y) + \tau_Q(y) + \gamma_Q(y) \leq 1, \forall y \in Y.$$

For each $y \in Y$, $S_Q(y) = 1 - (\sigma_Q(y) + \tau_Q(y) + \gamma_Q(y))$ is called the refusal membership degree of $y \in Q$.

Definition 2.4. [5] Let Q and R be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:

(i) $Q \subseteq R$ if and only if for all $y \in Y$, $\sigma_Q(y) \leq \sigma_R(y)$, $\tau_Q(y) \leq \tau_R(y)$ and $\gamma_Q(y) \geq \gamma_R(y)$.

(ii) $Q = R$ if and only if $Q \subseteq R$ and $R \subseteq Q$.

(iii) $Q \cup R = \{(y, \sigma_Q(y) \vee \sigma_R(y), \tau_Q(y) \vee \tau_R(y), \gamma_Q(y) \wedge \gamma_R(y)) | y \in Y\}$.

(iv) $Q \cap R = \{(y, \sigma_Q(y) \wedge \sigma_R(y), \tau_Q(y) \wedge \tau_R(y), \gamma_Q(y) \vee \gamma_R(y)) | y \in Y\}$.

Definition 2.5. [11] Let $(G, *)$ be a crisp group and $Q = \{(y, \sigma_Q(y), \tau_Q(y), \eta_Q(y)) | y \in G\}$ be a PFS in G . Then, Q is called picture fuzzy subgroup of G (PFSG) if

(i) $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b)$, $\tau_Q(a * b) \geq \tau_Q(a) \wedge \tau_Q(b)$, $\eta_Q(a * b) \leq \eta_Q(a) \vee \eta_Q(b)$

(ii) $\sigma_Q(a^{-1}) \geq \sigma_Q(a)$, $\tau_Q(a^{-1}) \geq \tau_Q(a)$, $\eta_Q(a^{-1}) \leq \eta_Q(a)$ for all $a, b \in G$.

Notice that a^{-1} is the inverse of $a \in G$,

or equivalently, Q is a PFSG of G if and only if

$\sigma_Q(a * b^{-1}) \geq \sigma_Q(a) \wedge \sigma_Q(b)$, $\tau_Q(a * b^{-1}) \geq \tau_Q(a) \wedge \tau_Q(b)$, $\eta_Q(a * b^{-1}) \leq \eta_Q(a) \vee \eta_Q(b)$.

Definition 2.6. [11] Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G . Then, for $a \in G$ the picture fuzzy left coset of $Q \in G$ is the PFS $aQ = (\sigma_{aQ}, \tau_{aQ}, \eta_{aQ})$ defined by

$$\sigma_{aQ}(u) = \sigma_Q(a^{-1} * u), \tau_{aQ}(u) = \tau_Q(a^{-1} * u) \text{ and } \eta_{aQ}(u) = \eta_Q(a^{-1} * u)$$

for all $u \in G$.

Definition 2.7. [11] Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G . Then, for $a \in G$ the picture fuzzy right coset of $Q \in G$ is the PFS $Qa = (\sigma_{Qa}, \tau_{Qa}, \eta_{Qa})$ defined by

$$\sigma_{Qa}(u) = \sigma_Q(u * a^{-1}), \tau_{Qa}(y) = \tau_Q(u * a^{-1}) \text{ and } \eta_{Qa}(u) = \eta_Q(u * a^{-1})$$

for all $u \in G$.

Definition 2.8. [11] Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G . Then, Q is called a picture fuzzy normal subgroup (PFNSG) of G if

$$\sigma_{Qa}(y) = \sigma_{aQ}(y), \tau_{Qa}(y) = \tau_{aQ}(y), \eta_{Qa}(y) = \eta_{aQ}(y)$$

for all $a, y \in G$.

Remark 2.1. Dogra and Pal established that PFSG of G is normal if and only if
(i) $\sigma_Q(u^{-1} * a * u) = \sigma_Q(a)$
(ii) $\tau_Q(u^{-1} * a * u) = \tau_Q(a)$
(ii) $\eta_Q(u^{-1} * a * u) = \eta_Q(a)$. For all $a \in Q$ and $u \in G$.

Cut set of picture fuzzy sets was introduced by Dutta and Ganju ? but the definition did not capture the cut set very well. Thus, Dogra and Pal ? corrected it in their paper.

Definition 2.9. [11] Let $Q = \{(x, \sigma_Q, \tau_Q, \gamma_Q) | a \in Y\}$ be PFS over the universe Y . Then, (r, s, t) -cut set of Q is the crisp set in Q , denoted by $C_{r,s,t}(Q)$ and is defined by

$$C_{r,s,t}(Q) = \{a \in Y | \sigma_Q(a) \geq r, \tau_Q(a) \geq s, \gamma_Q(a) \leq t\}$$

$r, s, t \in [0, 1]$ with the condition $0 \leq r + s + t \leq 1$.

Theorem 2.1. [11] Let $(G, *)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G . Then, Q is a PFSG if and only if $C_{r,s,t}(Q)$ is a crisp subgroup of G .

Theorem 2.2. [12] If Q and R are two PFSs of a universe Y , then the following results hold

$$(i) \quad C_{r,s,t}(Q) \subseteq C_{u,v,w}(Q) \text{ if } r \geq u, s \leq v, t \leq w.$$

$$(ii) \quad C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,1-r-t,t}(Q).$$

$$(iii) \quad Q \subseteq R \text{ implies } C_{r,s,t}(Q) \subseteq C_{r,s,t}(R).$$

$$(iv) \quad C_{r,s,t}(Q \cap R) = C_{r,s,t}(Q) \cap C_{r,s,t}(R).$$

$$(v) \quad C_{r,s,t}(Q \cup R) \supseteq C_{r,s,t}(Q) \cup C_{r,s,t}(R).$$

$$(vi) \quad C_{r,s,t}(\cap Q_i) = \cap C_{r,s,t}(Q_i).$$

$$(vii) \quad C_{1,0,0}(Q) = Y.$$

3 Main Results

Remark 3.1. *It is important to note that [12] misquoted the result of [5]. Hence, the results built on this foundation cannot be or at all correct. The counter example in Example (3.1) shows that the claims in Theorem 2.2 (i), (ii) and (vii) are wrong. The correct version of Theorem (2.2) is given in Theorem (3.1).*

Example 3.1. *Let*

$$Y = \{y_1, y_2, y_3, y_4\},$$

$$Q = \{(y_1, 0, 0.1, 0.8), (y_2, 0.2, 0.5, 0.3), (y_3, 0.4, 0.2, 0.1), (y_4, 0.5, 0.3, 0.2)\},$$

and

$$R = \{(y_1, 0.1, 0.4, 0.5), (y_2, 0.3, 0.6, 0.1), (y_3, 0.5, 0.3, 0), (y_4, 0.5, 0.4, 0.1)\}$$

be PFSs, taking $r = 0.1$, $s = 0.3$, $t = 0.5$. Thus, the $(0.1, 0.3, 0.5)$ -cut set of Q are

$$C_{0.1,0.3,0.5}(Q) = \{y_2, y_4\}$$

and

$$C_{0.1,0.3,0.5}(R) = \{y_1, y_2, y_3, y_4\}.$$

Hence,

$$C_{1-s-t,s,t}(Q) = C_{0.2,0.3,0.5}(Q) = \{y_2, y_4\},$$

$$C_{r,s,t}(Q) = C_{0.1,0.3,0.5}(Q) = \{y_2, y_4\},$$

$$C_{0.1,0.2,0.3}(Q) = \{y_3\}$$

and

$$C_{r,1-r-t,t}(Q) = C_{0.1,0.4,0.5}(Q) = \{y_2\}.$$

Thus,

$$C_{1-s-t,s,t}(R) \subseteq C_{r,s,t}(R) \subsetneq C_{r,1-r-t,t}(R), \text{ which means (i) and (ii) fail.}$$

Note that the left side of the inclusion which holds obey the condition (i) of our Theorem 3.1

Furthermore,

$$C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(Q) \subsetneq C_{r,1-r-t,t}(Q), \text{ which means (i) and (ii) fail.}$$

Note that the left side of the inclusion which holds obey the condition (i) of our Theorem 3.1

In addition,

$$C_{0.1,0.2,0.3}(Q) = \{y_3\} \subsetneq \{y_2, y_4\} = C_{0.1,0.3,0.5}(Q) \text{ which means (i) and (ii) fail.}$$

$$\text{Also, } C_{1,0,0}(Q) = \emptyset \neq Y, \text{ which means (vii) fails.}$$

Theorem 3.1. Let Q and R be two PFSs of a universe Y . Then, the following assertions hold:

$$(i) \quad C_{r,s,t}(Q) \subseteq C_{u,v,w}(Q) \text{ if } r \geq u, s \geq v, t \leq w.$$

$$(ii) \quad C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q),$$

$$(iii) \quad C_{1-s-t,1-r-t,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q),$$

$$(iv) \quad C_{r,1-r-t,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q),$$

$$(v) \quad C_{0,0,1}(Q) = Y.$$

Proof. (i) Let $x \in C_{r,s,t}(Q)$. Using Definition 2.9, $\sigma_Q(x) \geq r \geq u$, $\tau_Q(x) \geq s \geq v$, $\gamma_Q(x) \leq t \leq w$. Thus, $x \in C_{u,v,w}(Q)$ and, as such, $C_{r,s,t}(Q) \subseteq C_{u,v,w}(Q)$.

(ii) Since $r + s + t \leq 1$ implies that $1 - s - t \geq r$, $s \geq s$ and $t \leq t$, and $r \geq r$, $s \geq s$ and $t \leq 1 - r - s$, by Theorem 3.1 (i), the result holds.

(iii) Since $r + s + t \leq 1$ implies that $1 - s - t \geq r$, $1 - r - t \geq s$ and $t \leq t$, and $r \geq r$, $s \geq s$ and $t \leq 1 - s - r$, by Theorem 3.1 (i), the result holds.

(iv) Since $r + s + t \leq 1$ implies that $r \geq r$, $1 - r - t \geq s$ and $t \leq t$, and $r \geq r$, $s \geq s$ and $t \leq 1 - s - r$, by Theorem 3.1 (i), the result holds.

(v) Since $\forall y \in Y$, $\sigma(y) \geq 0$, $\tau(y) \geq 0$, $\gamma(y) \leq 1$, $C_{0,0,1}(Q) = Y$,

□

Proposition 3.1. *If Q is PFSG of G , then $C_{r,s,t}(Q)$ is a subgroup of G , where $\sigma_Q(e) \geq r$, $\tau_Q(e) \geq s$, and $\eta_Q(e) \leq t$ and e is the identity element of G .*

Proof. Clearly $C_{r,s,t}(Q) \neq \emptyset$ as $e \in C_{r,s,t}(Q)$. Let $a, b \in C_{r,s,t}(Q)$ be any two elements. Then,

$$\sigma_Q(a) \geq r, \tau_Q(a) \geq s, \eta_Q(a) \leq t \text{ and } \sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t$$

$$\sigma_Q(a) \wedge \sigma_Q(b) \geq r, \tau_Q(a) \wedge \tau_Q(b) \geq s \text{ and } \eta_Q(a) \vee \eta_Q(b) \leq t$$

Since Q is a PFSG of G ,

$$\sigma_Q(a * b^{-1}) \geq \sigma_Q(a) \wedge \sigma_Q(b) \geq r, \tau_Q(a * b^{-1}) \geq \tau_Q(a) \wedge \tau_Q(b) \geq s$$

$$\text{and } \eta_Q(a * b^{-1}) \leq \eta_Q(a) \vee \eta_Q(b) \leq t.$$

Hence, $a * b^{-1} \in C_{r,s,t}(Q)$ and $C_{r,s,t}(Q)$ is a subgroup of G . $\square$

Proposition 3.2. *If Q is PFNSG of G , then $C_{r,s,t}(Q)$ is a normal subgroup of G , where $\sigma_Q(e) \geq r$, $\tau_Q(e) \geq s$, and $\eta_Q(e) \leq t$ and e is the identity element of G .*

Proof. Let $a \in C_{r,s,t}(Q)$ and $u \in G$ be any element. Then,

$$\sigma_Q(a) \geq r, \tau_Q(a) \geq s, \eta_Q(a) \leq t.$$

Also, since Q is a PFNSG of G ,

$$\sigma_Q(u^{-1} * a * u) = \sigma_Q(a) \geq r, \tau_Q(u^{-1} * a * u) = \tau_Q(a) \geq s, \text{ and } \eta_Q(u^{-1} * a * u) = \eta_Q(a) \leq t$$

$\forall a \in Q$ and $u \in G$. Therefore, $u^{-1} * a * u \in C_{r,s,t}(Q)$, so $a * u \in u * C_{r,s,t}(Q)$ which implies $C_{r,s,t}(Q) * u \subseteq u * C_{r,s,t}(Q)$. Also, $u * a * u^{-1} \in C_{r,s,t}(Q)$, so $u * a \in C_{r,s,t}(Q) * u$ which implies $u * C_{r,s,t}(Q) \subseteq C_{r,s,t}(Q) * u$. Hence, $u * C_{r,s,t}(Q) = C_{r,s,t}(Q) * u$. Thus, $C_{r,s,t}(Q)$ is a normal subgroup of G . $\square$

Proposition 3.3. *Given two PFSGs Q and R of a group $(G, *)$. Then, $Q \cap R$ is a PFSG of G .*

This proposition has been proved by Dogra and Pal [11] by using PFSG definition. But, in this paper, a rather simpler approach of cut set of PFS is used to prove it.

Proof. By Theorem 2.1, $Q \cap R$ is a PFSG of G if and only if $C_{r,s,t}(Q \cap R)$ is a crisp subgroup of G . Since $C_{r,s,t}(Q \cap R) = C_{r,s,t}(Q) \cap C_{r,s,t}(R)$ (Theorem 2.2, iv) and both $C_{r,s,t}(Q)$ and $C_{r,s,t}(R)$ are subgroups of G and intersection of two subgroups of a group is its subgroup, $C_{r,s,t}(Q \cap R)$ is a subgroup of G . Therefore, $Q \cap R$ is a PFSG of G . $\square$

Notice that the union of two PFSGs of G need not be a PFSG of G [11].

Proposition 3.4. *Let Q be a PFSG of G and a be any fixed element of G . Then,*

$$(i.) \quad a * C_{r,s,t}(Q) = C_{r,s,t}(a * Q).$$

$$(ii.) \quad C_{r,s,t}(Q) * a = C_{r,s,t}(Q * a), \quad \forall r, s, t \in [0, 1] \text{ with } r + s + t \leq 1.$$

Proof. (i.)

$$C_{r,s,t}(a * Q) = \{u \in G \mid \sigma_{a*Q}(u) \geq r, \tau_{a*Q}(u) \geq s, \eta_{a*Q}(u) \leq t\}$$

with the condition $0 \leq r + s + t \leq 1$.

Also

$$\begin{aligned} a * C_{r,s,t}(Q) &= a * \{b \in G \mid \sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t\} \\ &= \{a * b \in G \mid \sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t\} \end{aligned}$$

Set $a * b = u$, so that $b = a^{-1} * u$.

Therefore,

$$\begin{aligned} a * C_{r,s,t}(Q) &= \{u \in G \mid \sigma_Q(a^{-1} * u) \geq r, \tau_Q(a^{-1} * u) \geq s, \eta_Q(a^{-1} * u) \leq t\} \\ &= \{u \in G \mid \sigma_{a*Q}(u) \geq r, \tau_{a*Q}(u) \geq s, \eta_{a*Q}(u) \leq t\} \\ &= C_{r,s,t}(a * Q) \end{aligned}$$

for all $r, s, t \in [0, 1]$ with $r + s + t \leq 1$.

(ii.)

$$C_{r,s,t}(Q * a) = \{u \in G \mid \sigma_{Q*a}(u) \geq r, \tau_{Q*a}(u) \geq s, \eta_{Q*a}(u) \leq t\}$$

with the condition $0 \leq r + s + t \leq 1$.

Also

$$\begin{aligned} C_{r,s,t}(Q * a) &= \{b \in G \mid \sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t\} * a \\ &= \{b * a \in G \mid \sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t\} \end{aligned}$$

Set $b * a = u$, so that $b = u * a^{-1}$.

Therefore,

$$\begin{aligned} C_{r,s,t}(Q) * a &= \{u \in G \mid \sigma_Q(u * a^{-1}) \geq r, \tau_Q(u * a^{-1}) \geq s, \eta_Q(u * a^{-1}) \leq t\} \\ &= \{u \in G \mid \sigma_{Q*a}(u) \geq r, \tau_{Q*a}(u) \geq s, \eta_{Q*a}(u) \leq t\} \\ &= C_{r,s,t}(Q * a) \end{aligned}$$

for all $r, s, t \in [0, 1]$ with $r + s + t \leq 1$.

□

Proposition 3.5. *Let Q be a PFSG of G . Let $a, b \in G$ such that*

$$\sigma_Q(a) \wedge \sigma_Q(b) = r, \tau_Q(a) \wedge \tau_Q(b) = s, \eta_Q(a) \vee \eta_Q(b) = t.$$

Then,

$$(i.) \quad a * Q = b * Q \Leftrightarrow a^{-1} * b \in C_{r,s,t}(Q)$$

$$(ii.) \quad Q * a = Q * b \Leftrightarrow a * b^{-1} \in C_{r,s,t}(Q)$$

$$\text{Proof.} \quad (i.) \quad a * Q = b * Q \Leftrightarrow C_{r,s,t}(a * Q) = C_{r,s,t}(b * Q)$$

$$\Leftrightarrow a * C_{r,s,t}(Q) = b * C_{r,s,t}(Q) \quad [\text{by Proposition 3.4}]$$

$$\Leftrightarrow a^{-1} * b \in C_{r,s,t}(Q) \quad [\text{since } C_{r,s,t}(Q) \text{ is a subgroup of } G \text{ and } a, b \in C_{r,s,t}(Q)].$$

$$(ii.) \quad Q * a = Q * b \Leftrightarrow C_{r,s,t}(Q * a) = C_{r,s,t}(Q * b)$$

$$\Leftrightarrow C_{r,s,t}(Q) * a = C_{r,s,t}(Q) * b \quad [\text{by Proposition 3.4}]$$

$$\Leftrightarrow a * b^{-1} \in C_{r,s,t}(Q) \quad [\text{since } C_{r,s,t}(Q) \text{ is a subgroup of } G \text{ and } a, b \in C_{r,s,t}(Q)].$$

□

Corolary 3.1. *If $C_{r,s,t}(Q)$ is PFNSG of G , then Proposition 3.4 (i) and (ii) coincide.*

Proof.

$$C_{r,s,t}(a * Q) = a * C_{r,s,t}(Q) = C_{r,s,t}(Q) * a = C_{r,s,t}(Q * a).$$

□

4 Conclusion

In this paper, some of the existing results in fuzzy group and fuzzy cut groups of a fuzzy group have been extended to picture fuzzy group and cut subgroup of a picture fuzzy subgroup. In further research, it will be of interest to see the relationship between a picture fuzzy set and bipolar fuzzy set and establish which one is a generalisation of the other.

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