

ISSN 1592-7415

eISSN 2282-8214

Volume n. 41 – 2021

RATIO MATHEMATICA

Journal of Mathematics, Statistics, and Applications

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RATIO MATHEMATICA
Journal of Mathematics, Statistics, and Applications

ISSN 1592-7415 - eISSN 2282-8214

Ratio Mathematica is an International, double peer-reviewed, open access journal, published every six months (June-December).

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Publisher: APAV - Accademia Piceno-Aprutina dei Velati in Teramo

Tax Code 92036140678, **Vat Id. Number** 02184450688

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On the stability of a multiplicative type sum form functional equation

Surbhi Madan*
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Dhiraj Kumar Singh^{‡§}

Abstract

In this paper we intend to discuss the stability of a sum form functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n k(p_i) \sum_{j=1}^m q_j^\beta$$

where f, k are real valued mappings each having the domain I ; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ are fixed integers and β is a fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$ and $1^\beta := 1$.

Keywords: Additive mapping; bounded mapping; functional equation; stability of a sum form functional equation.

2020 AMS subject classifications: 39B52, 39B82. ¹

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¹Received on November 23, 2021. Accepted on December 27, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.690. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers; $\mathbb{R}$ denotes the set of real numbers and I denotes the closed unit interval $[0, 1]$. For $n \in \mathbb{N}$,

$$\Gamma_n = \left\{ (p_1, \dots, p_n); p_i \geq 0, i = 1, \dots, n; \sum_{i=1}^n p_i = 1 \right\}$$

denotes the set of all finite n -component complete discrete probability distributions.

A mapping $a : I \rightarrow \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. Further, a mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}, y \in \mathbb{R}$. An interesting relation between these two additive mappings was established by Daróczy and Losonczi (2). They proved that additive mapping $a : I \rightarrow \mathbb{R}$ can be uniquely extended to the set of real numbers.

The stability of functional equations has intrigued mathematicians for more than eight decades now. The problem primarily aims to study the functional inequality corresponding to the given functional equation. Thereafter, it focuses on examining the proximity of their solutions. It needs to be remarked that without knowing the general solution of an equation we can not discuss its stability.

The seminal problem entered the corpus of sum form functional equations with the paper of Maksa (9). One of the interesting aspect explored in this field is to obtain *general solution* and discuss the *stability* of those sum form functional equations which are useful in characterizing entropies. Working in this direction, Nath and Singh (15; 16; 18); Singh and Grover (25) have recently addressed few sum form functional equations. These equations were useful in characterizing the Shannon entropies (19) defined as:

$$H_n(p_1, \dots, p_n) = - \sum_{i=1}^n p_i \log_2 p_i \quad (\text{with } 0 \log_2 0 := 0) \quad (1.1)$$

where $H_n : \Gamma_n \rightarrow \mathbb{R}, n \in \mathbb{N}$ and $(p_1, \dots, p_n) \in \Gamma_n$.

With the aim to further delve in the area of entropies, Havrda and Charvát (4) generalized the Shannon entropies (1.1) by introducing entropies of degree β defined as:

$$H_n^\beta(p_1, \dots, p_n) = (1 - 2^{1-\beta})^{-1} \left[1 - \sum_{i=1}^n p_i^\beta \right] \quad (1.2)$$

where $H_n^\beta : \Gamma_n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$; $(p_1, \dots, p_n) \in \Gamma_n$ and β is a fixed positive real power different from 1, such that $0^\beta := 0$ and $1^\beta := 1$.

Losonczi and Maksa (8) were first to address the sum form functional equation that characterized the entropies (1.2) by considering the multiplicative type functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) \sum_{j=1}^m f(q_j) \quad (1.3)$$

where $f : I \rightarrow \mathbb{R}$; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$ and $n \geq 3, m \geq 3$ are fixed integers. Thereafter, Nath and Singh have analysed Pexiderized forms of (1.3), containing two and three unknown mappings for $n \geq 3, m \geq 3$ being fixed integers in (11) and (12).

It needs to be highlighted that many research papers in reference to the sum form functional equations characterizing several entropies have been written. In brief, these papers reflected upon: some generalizations; Pexiderized forms; importance and applications. Some significant contributions are: Nath and Singh (10; 17); Singh and Dass (20); Singh and Grover (21; 22; 23). The primary focus of these authors had been to obtain the general solution (or solutions) of the sum form functional equations for fixed integers $n \geq 3, m \geq 3$ or $n \geq 3, m \geq 2$. The stability problem for some of these sum form functional equations has been discussed by the authors but for most of them it remains unaddressed.

One of the interesting equation which motivated us is the functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) \sum_{j=1}^m q_j^\beta \quad (1.4)$$

where $f : I \rightarrow \mathbb{R}$; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$ and β is a fixed positive real power different from 1 satisfying $0^\beta := 0$ and $1^\beta := 1$. In the recent past, Nath and Singh (13) have studied the equation (1.4) and obtained its general solutions for $n \geq 3, m \geq 3$ being fixed integers. The authors have further explained the relation of these solutions with entropies (1.2). The stability of (1.4) is established by Singh and Grover (24) for $n \geq 3, m \geq 3$ being fixed integers. Indeed, with the goal of getting a deeper insight of the equation (1.4), Nath and Singh (14); Garg, Grover and Singh (3) have recently studied a Pexiderized form (1.4), that is

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n k(p_i) \sum_{j=1}^m q_j^\beta \quad (1.5)$$

where $f : I \rightarrow \mathbb{R}$, $k : I \rightarrow \mathbb{R}$; $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$ and β is a fixed positive real power different from 1 satisfying $0^\beta := 0$ and $1^\beta := 1$. In (14)

and (3), the authors have obtained the general solutions of (1.5) for $n \geq 3, m \geq 3$ and $n \geq 3, m \geq 2$. The authors have further reflected upon the significance of general solutions in the light of entropies of degree β and diversity index. So far the stability problem remains unaddressed. The objective of this paper is to discuss the stability of functional equation (1.5).

For the problem of stability concerning functional equations, we refer to the survey paper of Hyers and Rassias (5) and Hyers, Isac and Rassias (6). The problem of stability of the functional equation (1.5) is given along the following lines: *Let $n \geq 3, m \geq 3$ be fixed integers; $0 \leq \varepsilon \in \mathbb{R}$ be fixed. Find all the mappings $f : I \rightarrow \mathbb{R}, k : I \rightarrow \mathbb{R}$ satisfying the functional inequality*

$$\left| \sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) - \sum_{i=1}^n k(p_i) \sum_{j=1}^m q_j^\beta \right| \leq \varepsilon \quad (1.6)$$

for all $(p_1, \dots, p_n) \in \Gamma_n, (q_1, \dots, q_m) \in \Gamma_m$.

Below we will provide some known results.

Lemma 1.1 ((8)). *Suppose a mapping $\phi : I \rightarrow \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^n \phi(p_i) = c_1$ for all $(p_1, \dots, p_n) \in \Gamma_n, n \geq 3$ a fixed integer and c_1 a real constant. Then there exists an additive mapping $a : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c_1}{n}$ for all $p \in I$.*

Lemma 1.2 ((9)). *Let $0 \leq \varepsilon \in \mathbb{R}, n \geq 3$ be fixed integer and $\psi : I \rightarrow \mathbb{R}$ be a mapping which satisfies the functional inequality $\left| \sum_{i=1}^n \psi(p_i) \right| \leq \varepsilon$ for all $(p_1, \dots, p_n) \in \Gamma_n$. Then there exist an additive mapping $A_1 : \mathbb{R} \rightarrow \mathbb{R}$ and a mapping $B_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that $|B_1(p)| \leq 18\varepsilon$ for all $p \in \mathbb{R}, B(0) = 0$ and $\psi(p) - \psi(0) = A_1(p) + B_1(p)$ for all $p \in I$.*

Lemma 1.3 ((7)). *Let $A_2 : \mathbb{R} \rightarrow \mathbb{R}$ be an additive mapping, $M : I \rightarrow \mathbb{R}$ a multiplicative mapping, $B_2 : \mathbb{R} \rightarrow \mathbb{R}$ a bounded mapping and $c_2 \in \mathbb{R}$. If $A_2(p) = M(p) + c_2$ for all $p \in I$, then $A_2(p) = dp, p \in \mathbb{R}$ for some $d \in \mathbb{R}$ and $M(p) = 0$ or $M(p) = p, p \in I$. Also if $A_2(p) = M(p) + B_2(p)$ for all $p \in I$, then $A_2(p) = dp, p \in \mathbb{R}$ for some $d \in \mathbb{R}$ and $M(p) = 0$ or $M(p) = p^\alpha, p \in I$ for some $0 \leq \alpha \in \mathbb{R}$.*

Lemma 1.4 ((26)). *If f is a solution of the functional equation $f(x+y) = f(x) + f(y)$ which is bounded over an interval $[a, b]$, then it is of the form $f(x) = c_3 x$ for some real number c_3 .*

Lemma 1.5 ((3)). *Let $n \geq 3, m \geq 2$ be fixed integers; β be fixed positive real power different from 1 satisfying the conventions $0^\beta := 0, 1^\beta := 1$ and $f : I \rightarrow \mathbb{R}$,*

$k : I \rightarrow \mathbb{R}$. The pair (f, k) satisfies (1.5) if and only if there exist the additive mappings $a_1, a_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$\left. \begin{array}{l} \text{(i)} \quad f(p) = cp^\beta + a_1(p) - \frac{1}{nm}a_1(1), \\ \text{(ii)} \quad k(p) = cp^\beta + a_2(p) - \frac{1}{n}a_2(1). \end{array} \right\}$$

2 The stability of the functional equation (1.5)

In this section our primary aim is to find the solutions of inequality (1.6). Thereafter, we need to observe: *What is the difference between these solutions and the solutions (given by Lemma 1.5) of equation (1.5)?* In the sense of Hyers, Isac and Rassias (6), if the difference is only a bounded mapping, we would say that functional equation (1.5) is stable. Following this we establish the the main result as follows:

Theorem 2.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers; β be fixed positive real power different from 1 satisfying the conventions $0^\beta := 0$ and $1^\beta := 1$; ε be a nonnegative real constant and let $f : I \rightarrow \mathbb{R}$, $k : I \rightarrow \mathbb{R}$ be real valued mappings. Suppose the pair (f, k) satisfies (1.6), then there exist the additive mappings $a_1, a_2 : \mathbb{R} \rightarrow \mathbb{R}$, the bounded mappings $b_1, b_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $0 \neq \bar{c}, c \in \mathbb{R}$ such that*

$$\left. \begin{array}{l} \text{(i)} \quad f(p) - f(0) = cp^\beta + a_1(p) + b_1(p), \\ \text{(ii)} \quad k(p) - k(0) = cp^\beta + a_2(p) + b_2(p) \end{array} \right\} \quad (2.1)$$

with

$$\left. \begin{array}{l} \text{(i)} \quad |b_1(p)| \leq 1296\varepsilon\bar{c}(2m+1), \quad b_1(0) = 0, \\ \text{(ii)} \quad |b_2(p)| \leq 1296\varepsilon\bar{c}(2m+1) + 18\varepsilon, \quad b_2(0) = 0. \end{array} \right\} \quad (2.2)$$

Proof. Let us put $q_1 = 1, q_2 = \dots = q_m = 0$ in (1.6). We get

$$\left| \sum_{i=1}^n \left[f(p_i) + (m-1)f(0) - k(p_i) \right] \right| \leq \varepsilon$$

for all $(p_1, \dots, p_n) \in \Gamma_n$. By Lemma 1.2, there exists an additive mapping $A_1 : \mathbb{R} \rightarrow \mathbb{R}$ and a mapping $B_1^* : \mathbb{R} \rightarrow \mathbb{R}$ with $|B_1^*(p)| \leq 18\varepsilon$, $B_1^*(0) = 0$, such that

$$f(p) - k(p) - f(0) + k(0) = A_1(p) + B_1^*(p)$$

for all $p \in I$. From this, we obtain the expression

$$k(p) = f(p) - A_1(p) - B_1(p) \quad (2.3)$$

where $B_1 : \mathbb{R} \rightarrow \mathbb{R}$, defined as $B_1(x) = B_1^*(x) + f(0) - k(0)$ is a bounded mapping. With the aid of (2.3), inequality (1.6) can be written as

$$\left| \sum_{i=1}^n \left[\sum_{j=1}^m f(p_i q_j) - [f(p_i) - A_1(1)p_i - B_1(p_i)] \sum_{j=1}^m q_j^\beta \right] \right| \leq \varepsilon$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. By Lemma 1.2, there exists a mapping $A_2 : \mathbb{R} \times \Gamma_m \rightarrow \mathbb{R}$, additive in the first variable and a mapping $B_2 : \mathbb{R} \times \Gamma_m \rightarrow \mathbb{R}$, bounded in the first variable by 18ε with $B_2(0; q_1, \dots, q_m) = 0$, such that

$$\begin{aligned} \sum_{j=1}^m f(p q_j) - [f(p) - A_1(1)p - B_1(p) - f(0) + B_1(0)] \sum_{j=1}^m q_j^\beta - m f(0) \\ = A_2(p; q_1, \dots, q_m) + B_2(p; q_1, \dots, q_m). \end{aligned} \quad (2.4)$$

Let $x \in I$ and $(r_1, \dots, r_m) \in \Gamma_m$ be an arbitrary probability distribution. Now, replacing p successively by $x r_t$, $t = 1, \dots, m$ in (2.4); summing the resulting m equations so obtained and using the additivity of the mapping $A_2 : \mathbb{R} \times \Gamma_m \rightarrow \mathbb{R}$ in the first variable, we have

$$\begin{aligned} \sum_{t=1}^m \sum_{j=1}^m f(x r_t q_j) - \left[\sum_{t=1}^m f(x r_t) - A_1(1)x - \sum_{t=1}^m B_1(x r_t) - m f(0) + m B_1(0) \right] \sum_{j=1}^m q_j^\beta \\ - m^2 f(0) = A_2(x; q_1, \dots, q_m) + \sum_{t=1}^m B_2(x r_t; q_1, \dots, q_m) \end{aligned} \quad (2.5)$$

for all $x \in I$, $(q_1, \dots, q_m) \in \Gamma_m$ and $(r_1, \dots, r_m) \in \Gamma_m$. Now for $p = x$ and $q_1 = r_1, \dots, q_m = r_m$, functional equation (2.4) gives

$$\begin{aligned} \sum_{t=1}^m f(x r_t) = [f(x) - A_1(1)x - B_1(x) - f(0) + B_1(0)] \sum_{t=1}^m r_t^\beta + m f(0) \\ + A_2(x; r_1, \dots, r_m) + B_2(x; r_1, \dots, r_m). \end{aligned} \quad (2.6)$$

From (2.5) and (2.6), we get

$$\begin{aligned} \sum_{t=1}^m \sum_{j=1}^m f(x r_t q_j) - [f(x) - A_1(1)x - B_1(x) - f(0) + B_1(0)] \sum_{t=1}^m r_t^\beta \sum_{j=1}^m q_j^\beta \\ - m^2 f(0) = \left[A_2(x; r_1, \dots, r_m) + B_2(x; r_1, \dots, r_m) - A_1(1)x - \sum_{t=1}^m B_1(x r_t) \right. \\ \left. + m B_1(0) \right] \sum_{j=1}^m q_j^\beta + A_2(x; q_1, \dots, q_m) + \sum_{t=1}^m B_2(x r_t; q_1, \dots, q_m). \end{aligned} \quad (2.7)$$

On the stability of a multiplicative ...

We see that, the left hand side of (2.7) is commutative in r_t and q_j , $t = 1, \dots, m$; $j = 1, \dots, m$ (Ac zel (1)). So, the commutativity on the right hand side implies

$$\begin{aligned} & A_2(x; r_1, \dots, r_m) \left[1 - \sum_{j=1}^m q_j^\beta \right] - A_2(x; q_1, \dots, q_m) \left[1 - \sum_{t=1}^m r_t^\beta \right] \\ &= \sum_{t=1}^m B_2(xr_t; q_1, \dots, q_m) - \sum_{j=1}^m B_2(xq_j; r_1, \dots, r_m) \\ &+ \left[B_2(x; r_1, \dots, r_m) - A_1(1)x - \sum_{t=1}^m B_1(xr_t) + mB_1(0) \right] \sum_{j=1}^m q_j^\beta \\ &- \left[B_2(x; q_1, \dots, q_m) - A_1(1)x - \sum_{j=1}^m B_1(xq_j) + mB_1(0) \right] \sum_{t=1}^m r_t^\beta. \end{aligned}$$

For fixed $(q_1, \dots, q_m) \in \Gamma_m$ and $(r_1, \dots, r_m) \in \Gamma_m$, the right hand side of the above equation is bounded on I while the left hand side is additive in $x \in I$. So, by Lemma 1.4, it follows that

$$\begin{aligned} & [A_2(x; r_1, \dots, r_m) - xA_2(1; r_1, \dots, r_m)] \left[1 - \sum_{j=1}^m q_j^\beta \right] \\ &= [A_2(x; q_1, \dots, q_m) - xA_2(1; q_1, \dots, q_m)] \left[1 - \sum_{t=1}^m r_t^\beta \right]. \end{aligned} \quad (2.8)$$

Now, we assert that for $m \geq 3$ and fixed positive real power $\beta \neq 1$, $1 - \sum_{t=1}^m r_t^\beta$ does not vanish identically on Γ_m . To the contrary, suppose $1 - \sum_{t=1}^m r_t^\beta$ vanishes identically on Γ_m . Then, $1 = \sum_{t=1}^m r_t^\beta$ for all $(r_1, \dots, r_m) \in \Gamma_m$. By Lemma 1.1, there exists an additive mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ such that $r^\beta = A(r)$ with $A(1) = 1$. By Lemma 1.3, $r^\beta = 0$ or $r^\beta = r$ for all $r \in I$. This gives a contradiction as for the former case, our supposition $1 = \sum_{t=1}^m r_t^\beta$ is contradicted while for the latter case, our assumption $\beta \neq 1$ is contradicted. This proves our assertion and so there exists a probability distribution $(r_1^*, \dots, r_m^*) \in \Gamma_m$ such that $1 - \sum_{t=1}^m r_t^{*\beta} \neq 0$. Therefore with the substitution $r_t = r_t^*$, $t = 1, \dots, m$, functional equation (2.8) reduces to

$$A_2(x; q_1, \dots, q_m) = A_0(x) \left[1 - \sum_{j=1}^m q_j^\beta \right] + x A_2(1; q_1, \dots, q_m) \quad (2.9)$$

where $A_0 : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$A_0(x) = \left[1 - \sum_{t=1}^m r_t^{*\beta} \right]^{-1} [A_2(x; r_1^*, \dots, r_m^*) - x A_2(1; r_1^*, \dots, r_m^*)]$$

is an additive mapping with $A_0(1) = 0$. Further from (2.4), we have

$$\begin{aligned} A_2(1; q_1, \dots, q_m) &= \sum_{j=1}^m f(q_j) - [f(1) - A_1(1) - B_1(1) - f(0) + B_1(0)] \sum_{j=1}^m q_j^\beta \\ &\quad - m f(0) - B_2(1; q_1, \dots, q_m). \end{aligned} \quad (2.10)$$

From (2.4), (2.9), (2.10) with $A_0(1) = 0$, we gather that

$$\begin{aligned} &\sum_{j=1}^m [f(pq_j) - f(0) - A_0(pq_j) - (f(1) - f(0))pq_j] - [f(p) - f(0) - A_0(p) \\ &\quad - p(f(1) - f(0))] \sum_{j=1}^m q_j^\beta - p \sum_{j=1}^m [f(q_j) - f(0) - A_0(q_j) - (f(1) - f(0))q_j] \\ &= [B_1(0) - B_1(p) + pB_1(1) - pB_1(0)] \sum_{j=1}^m q_j^\beta - pB_2(1; q_1, \dots, q_m) \end{aligned} \quad (2.11)$$

for all $p \in I$ and $(q_1, \dots, q_m) \in \Gamma_m$.

Now, define a mapping $\overline{F} : I \rightarrow \mathbb{R}$ as

$$\overline{F}(x) = f(x) - f(0) - A_0(x) - (f(1) - f(0))x \quad (2.12)$$

for all $x \in I$. With the aid of (2.12), equation (2.11) can be written as

$$\begin{aligned} &\sum_{j=1}^m \overline{F}(pq_j) - \overline{F}(p) \sum_{j=1}^m q_j^\beta - p \sum_{j=1}^m \overline{F}(q_j) \\ &= [B_1(0) - B_1(p) + pB_1(1) - pB_1(0)] \sum_{j=1}^m q_j^\beta - pB_2(1; q_1, \dots, q_m). \end{aligned} \quad (2.13)$$

It clearly follows from (2.12), that $\overline{F}(0) = 0$. Also we observe, the right hand side of (2.13) is bounded by $18\varepsilon(2m+1)$, consequently by Lemma 1.2, along with $\overline{F}(0) = 0$, there exists a mapping $A_3 : I \times \mathbb{R} \rightarrow \mathbb{R}$, additive in the second variable and a mapping $B_3 : I \times \mathbb{R} \rightarrow \mathbb{R}$, bounded in the second variable by $324\varepsilon(2m+1)$ with $B_3(p; 0) = 0$, such that

$$\overline{F}(pq) - q^\beta \overline{F}(p) - p \overline{F}(q) = A_3(p; q) + B_3(p; q). \quad (2.14)$$

On the stability of a multiplicative ...

Define a mapping $K : I \times I \rightarrow \mathbb{R}$ as

$$K(p; q) = \overline{F}(pq) - q^\beta \overline{F}(p) - p \overline{F}(q) \quad (2.15)$$

for all $p \in I, q \in I$. With the help of (2.15), it can be verified easily that

$$\begin{aligned} \overline{F}(pqr) - pq \overline{F}(r) - q^\beta r^\beta \overline{F}(p) - pr^\beta \overline{F}(q) &= K(pq; r) + r^\beta K(p; q) \\ &= K(p; qr) + pK(q; r) \end{aligned} \quad (2.16)$$

for all $p \in I, q \in I$ and $r \in I$. From (2.14), (2.15) and (2.16), it follows that

$$\begin{aligned} A_3(p; qr) + pA_3(q; r) - A_3(pq; r) \\ = B_3(pq; r) + r^\beta A_3(p; q) + r^\beta B_3(p; q) - B_3(p; qr) - pB_3(q; r). \end{aligned} \quad (2.17)$$

Apparently, the left hand side of (2.17) is additive in $r \in I$, while its right hand side is bounded on I . Consequently by Lemma 1.4, its left hand side must be linear therefore, we get

$$A_3(p; qr) + pA_3(q; r) - A_3(pq; r) = r [A_3(p; q) + pA_3(q; 1) - A_3(pq; 1)]. \quad (2.18)$$

Now, for the substitution $r = 1$, equation (2.17) gives

$$pA_3(q; 1) - A_3(pq; 1) = B_3(pq; 1) - pB_3(q; 1). \quad (2.19)$$

From (2.17), (2.18) and (2.19), we obtain

$$\begin{aligned} (r^\beta - r) A_3(p; q) &= rB_3(pq; 1) - rpB_3(q; 1) - B_3(pq; r) \\ &\quad - r^\beta B_3(p; q) + B_3(p; qr) + pB_3(q; r). \end{aligned} \quad (2.20)$$

Since for fixed positive real number $\beta \neq 1$, we have ' $r^\beta - r$ ' does not vanish identically on I , there exists some $r^* \in I$, such that

$$r^{*\beta} - r^* \neq 0. \quad (2.21)$$

Using this in (2.20), it follows that

$$\begin{aligned} A_3(p; q) &= (r^{*\beta} - r^*)^{-1} \left\{ r^* B_3(pq; 1) - r^* p B_3(q; 1) - B_3(pq; r^*) \right. \\ &\quad \left. - r^{*\beta} B_3(p; q) + B_3(p; qr^*) + p B_3(q; r^*) \right\} \end{aligned}$$

for all $p \in I, q \in I$. This yield that additive mapping $A_3(p; q)$ is bounded in the second variable also. Hence by Lemma 1.4, it must be linear therein. So,

$$A_3(p; q) = qA_3(p; 1) \quad (2.22)$$

for all $p \in I, q \in I$. From (2.19), with the substitution $q = 1$ it follows that

$$A_3(p; 1) = pA_3(1; 1) - B_3(p; 1) + pB_3(1; 1) \quad (2.23)$$

for all $p \in I$. From (2.22) and (2.23), it can be concluded that mapping $A_3(p; q)$ is bounded. Moreover, from (2.14) and (2.23) with $\bar{F}(1) = 0$ we obtain, $A_3(p; 1) = -B_3(p; 1)$. Consequently we get $|A_3(p; q)| \leq 324\varepsilon(2m + 1)$. Hence, using this in (2.14), it follows that

$$|\bar{F}(pq) - q^\beta \bar{F}(p) - p \bar{F}(q)| \leq 648\varepsilon(2m + 1) \quad (2.24)$$

for all $p \in I, q \in I$. Now on interchanging the places of p and q in the functional inequality (2.24), we have

$$|\bar{F}(pq) - p^\beta \bar{F}(q) - q \bar{F}(p)| \leq 648\varepsilon(2m + 1). \quad (2.25)$$

Applying triangle inequality to functional inequalities (2.24) and (2.25), we obtain

$$|(q^\beta - q) \bar{F}(p) - (p^\beta - p) \bar{F}(q)| \leq 1296\varepsilon(2m + 1) \quad (2.26)$$

where $p \in I, q \in I$ and β is a fixed positive real power different from 1. With the aid of (2.21) we get, $q^{*\beta} - q^* \neq 0$ for some $q^* \in I$. On taking $\bar{c} := (q^{*\beta} - q^*)^{-1} \in \mathbb{R}$; $c := \bar{F}(q^*)(q^{*\beta} - q^*)^{-1} \in \mathbb{R}$ in (2.26), it follows that there exist a mapping $b_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that (2.2)(i) holds for all $p \in \mathbb{R}$ and

$$\bar{F}(p) = cp^\beta - cp + b_1(p) \quad (2.27)$$

for all $p \in I$. Thus, the solution (2.1)(i) follows from (2.12) and (2.27) by defining additive mapping $a_1 : \mathbb{R} \rightarrow \mathbb{R}$ as $a_1(x) = A_0(x) + (f(1) - f(0))x - cx$. Further, the solution (2.1)(ii) with (2.2)(ii) follows from (2.1)(i), (2.2)(i) and (2.3) by defining additive mapping $a_2 : \mathbb{R} \rightarrow \mathbb{R}$ as $a_2(x) = a_1(x) - A_1(x)$ and bounded mapping $b_2 : \mathbb{R} \rightarrow \mathbb{R}$ as $b_2(x) = b_1(x) - B_1^*(x)$. This completes the proof. $\square$

3 Acknowledgements

The third author is grateful for the support from the SERB-MATRICES scheme (MTR/2020/000508) of the Department of Science and Technology, Government of India.

References

- [1] Aczel 1966 J. Aczel. *Lectures on Functional Equations and Their Applications*, Academic Press, New York and London, 1966.

- [2] Z. Daróczy and L. Losonczi. Über die Erweiterung der auf einer Punktmenge additiven Funktionen, *Publicationes Mathematicae (Debrecen)*, **14** (1967), 239 – 245.
- [3] P. Garg, S. Grover, and D.K. Singh. On a functional equation related to diversity, To appear in *Montes Taurus Journal of Pure and Applied Mathematics*.
- [4] J. Havrda and F. Charvát. Quantification method of classification processes. Concept of structural α -entropy, *Kybernetika*, **3** (1967), 30 – 35.
- [5] D.H. Hyers and T.M. Rassias. Approximate homomorphisms, *Aequationes Mathematicae*, **44(2-3)** (1992), 125 – 153.
- [6] D.H. Hyers, G. Isac, and Th.M. Rassias. *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel-Berlin, 1998.
- [7] I. Kocsis. On the stability of a sum form functional equation of multiplicative type, *Acta Acad. Paed. Agriensis, Sectio Mathematicae*, **28** (2001), 43 – 53.
- [8] L. Losonzi and Gy. Maksa. On some functional equations of the information theory, *Acta Mathematica Academiae Scientiarum Hungarica*, **39(1-3)** (1982), 73 – 82.
- [9] Gy. Maksa. On the stability of a sum form equation, *Results in Mathematics*, **26(3-4)** (1994), 342 – 347.
- [10] P. Nath and D.K. Singh. On a sum form functional equation and its role in information theory, *Proc. of National Conference on 'Information Technology: Setting Trends In Modern Era' and 8th Annual Conference of Indian Society of Information Theory and Applications*, (March 18–20, 2006), 88 – 94.
- [11] P. Nath and D.K. Singh. On a multiplicative type sum form functional equation and its role in information theory, *Applications of Mathematics*, **51(5)** (2006), 495 – 516.
- [12] P. Nath and D.K. Singh. Some sum form functional equations containing at most two unknown mappings, *Proc. of 9th National Conference of Indian Society of Information Theory and Applications (ISITA) on "Information Technology and Operational Research Applications"*, (December 8–9, 2007) (2008), 54 – 71.
- [13] P. Nath and D.K. Singh. On a sum form functional equation related to entropies of type (α, β) , *Asian-European Journal of Mathematics*, **6(2)** (2013), 13 pages.

- [14] P. Nath and D.K. Singh. On a sum form functional equation related to various nonadditive entropies in information theory, *Tamsui Oxford Journal of Information and Mathematical Sciences*, **30** (2014), 23 – 43.
- [15] P. Nath and D.K. Singh. Some functional equations and their corresponding sum forms, *Sarajevo Journal of Mathematics*, **12(24)** (2016), 89 – 106.
- [16] P. Nath and D.K. Singh. On a sum form functional equation containing five unknown mappings, *Aequationes Mathematicae*, **90** (2016), 1087 – 1101.
- [17] P. Nath and D.K. Singh. On an equation related to nonadditive entropies in information theory, *Mathematics for Applications*, **6(1)** (2017), 31 – 41.
- [18] P. Nath and D.K. Singh. On the stability of a functional equation, *Palestine Journal of Mathematics*, **6(2)** (2017), 573 – 579.
- [19] C.E. Shannon. A mathematical theory of communication, *Bell System Technical Journal*, **27** (1948), 379 – 423; 623 – 656.
- [20] D.K. Singh and P. Dass. On a functional equation related to some entropies in information theory, *Journal of Discrete Mathematical Sciences and Cryptography*, **21(3)** (2018), 713 – 726.
- [21] D.K. Singh and S. Grover. On the stability and general solution of a sum form functional equation emerging from information theory, *Bulletin of Mathematical Analysis and Applications*, **12(3)** (2020), 1 – 18.
- [22] D.K. Singh and S. Grover. On the stability of a nonmultiplicative type sum form functional equation, *Italian Journal of Pure and Applied Mathematics*, **45** (2021), 360 – 374.
- [23] D.K. Singh and S. Grover. On a sum form functional equation emerging from statistics and its applications, *Results in Nonlinear Analysis*, **4(2)** (2021), 65 – 76.
- [24] D.K. Singh and S. Grover. On the stability of a sum form functional equation related to entropies of type (α, β) , *Journal of Nonlinear Sciences and Applications*, **14(3)** (2021), 168 – 180.
- [25] D.K. Singh and S. Grover. On the stability of a sum form functional equation related to nonadditive entropies, *Journal of Mathematics and Computer Science*, **23(4)** (2021), 328 – 340.
- [26] G.S. Young. The linear functional equation, *The American Mathematical Monthly*, **65(1)** (1958), 37 – 38.

Encryption using semigroup action

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Abstract

An enciphering transformation is a function f that converts any plaintext message into a ciphertext message and deciphering transformation is a function f^{-1} , which reverse the process. Such a set-up is called a cryptosystem. In this paper, we extend a generalization of the original Diffie-Hellman key exchange and ElGamal cryptosystem in $(\mathbb{Z}/p\mathbb{Z})^*$ by constructing a semigroup action on a finite dimensional vector space T over F_2 .

Keywords: semigroup action; enciphering; plaintext; ciphertext; cryptosystem

2010 AMS subject classifications: 94A60, 08A70, 08A62. ¹

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¹Received on August 23, 2021. Accepted on December 14, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.649. ISSN: 1592-7415. eISSN: 2282-8214. ©Anooja et al. This paper is published under the CC-BY licence agreement.

1 Introduction

Recently there has been a lot of on-going research work to find more secure and efficient public key cryptosystems based on algebraic structures such as non-abelian groups, linear groups, semigroups and power series rings (see Anshel et al. [1999], Baumslag et al. [2006], Maze et al. [2007], Shpilrain and Zapata [2006]), and where the security is based on hard algorithmic problems from combinatorial group theory. The hard problems from combinatorial group theory include the conjugacy search problem, the decomposition search problem and the subgroup membership search problem. Most common public key cryptosystems and public key exchange protocols presently in use, such as the RSA algorithm, Diffie-Hellman, and elliptic curve methods are number theory based and hence depend on the structure of abelian groups.

The idea of using semigroups as platforms for public key cryptosystems has appeared in several papers. Yamamura [1998] has considered a group action of $Sl_2(\mathbb{Z})$. Blackburn and Galbraith [1999] have analyzed the system of Yamamura and they have shown that it is insecure. Maze et al. [2007] showed that the discrete logarithm problem over a group can be considered as a special case of an action by a semigroup on a set. They showed that every semigroup action by an abelian semigroup on a set gives rise to a Diffie-Hellman key exchange. By taking the action of the semigroup on itself, a semigroup can then be used as a platform for a public key cryptosystem. Kropholler et al. [2010] studied the potential of the semigroup $\langle a, b ; a^p = b^r, a^q = b^s \rangle$ as platforms for the Diffie-Hellman key exchange protocol. Special instances of semigroup actions appears in Anshel et al. [1999], Shpilrain and Ushakov [2005], Ko et al. [2000] and Slavin [2007]. In this paper, we try to extend a generalization of the original Diffie-Hellman key exchange and ElGamal cryptosystem in $(\mathbb{Z}/p\mathbb{Z})^*$ by constructing a semigroup from a (p, q) -graph G and defining a semigroup action on a finite vector space of dimension q over the field F_2 .

2 Notations and Basic Results

Most of the notations, definitions and results we mentioned here are standard and can be found in Menezes et al. [1996], Koblitz [1998], Lyndon and Schupp [1977], Maze et al. [2007] and Diffie and Hellman [1976].

Most of the public key cryptosystems and public key exchange protocols currently in use, like the Diffie and Hellman [1976] key exchange protocol, the ElGamal [1985] public key cryptosystem, the Digital Signature Algorithm (DSA) and the ElGamal's signature scheme, use the discrete logarithm problem as the basis of their security.

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The discrete logarithm problem can be defined as follows.

Problem 2.1. (*Discrete Logarithm Problem*) Let G be a group and $a, b \in G$. Find an integer $n \in \mathbb{N}$ such that $a^n = b$.

Problem (2.1) has a solution if and only if $b \in \langle a \rangle$, the cyclic group generated by a . If $b \in \langle a \rangle$ then there is a unique integer n satisfying $1 \leq n \leq \text{ord}(a)$ such that $a^n = b$. This unique integer is called the discrete logarithm of b with base a and denote it by $\log_a b$. Discrete Logarithm Problem plays important role in the Diffie-Hellman key agreement and the ElGamal public key cryptosystem, the digital signature algorithm and ElGamal's signature scheme. Currently the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ of integers modulo p where p is a prime is widely used as the platform group.

Protocol 2.1. (*Diffie-Hellman Key Exchange Protocol*) Let G be a group.

1. Alice and Bob publicly agree on an element $g \in G$.
2. Alice chooses $n \in \mathbb{N}$ and computes g^n . Alice's private key is n , her public key is g^n .
3. Bob chooses $m \in \mathbb{N}$ and computes g^m . Bob's private key is m , his public key is g^m .
4. Their common secret key is then g^{mn} .

The ElGamal public key cryptosystem works as follows:

Alice chooses $n \in \mathbb{N}$, $a, b \in G$ where $b = a^n$. The private key of Alice is (a, b, n) , the public key is (a, b) . Bob chooses a random integer $r \in \mathbb{N}$ and he applies the encryption function

$$\begin{aligned} \varphi : G &\rightarrow G \times G \\ m &\rightarrow (c_1, c_2) = (a^r, mb^r) \end{aligned}$$

Alice computes m from the ciphertext (c_1, c_2) by $m = c_2(c_1^n)^{-1}$.

3 Construction of a semigroup from a (p, q) - graph

Let G be a finite (p, q) -graph and H be a subgraph of G . Let x_H denote a vector corresponding to H such that $x_H = (x_1, x_2, \dots, x_q)$ where

$$x_i = \begin{cases} 1 & \text{if } e_i \text{ is in } H \\ 0 & \text{otherwise} \end{cases}$$

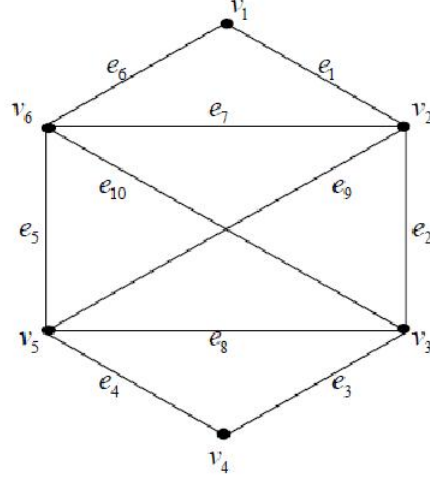


Figure 1: Graph G

Let S be a set of such vectors. Then S is a semigroup under the operation defined by

$$\begin{aligned} x_H y_K &= (x_1, x_2, \dots, x_q)(y_1, y_2, \dots, y_q) \\ &= (x_1 y_1, x_2 y_2, \dots, x_q y_q) \end{aligned}$$

We shall illustrate this with the following example.

Example 3.1. Consider the graph G with $p = 6$ and $q = 10$ given in Figure 1. Let us consider seven subgraphs of G , which are displayed in Figure 2. Let $x_{H_1}, x_{H_2}, \dots, x_{H_7}$ be the vectors corresponding to the subgraphs $H_1, H_2, \dots, H_7$ respectively. Let $S = \{x_{H_1}, x_{H_2}, \dots, x_{H_7}\}$. Then

$$\begin{aligned} x_{H_1} &= (1, 0, 1, 0, 0, 1, 0, 0, 1, 1) \\ x_{H_2} &= (1, 1, 0, 0, 0, 1, 1, 0, 0, 0) \\ x_{H_3} &= (1, 1, 1, 0, 0, 0, 0, 0, 1, 0) \\ x_{H_4} &= (1, 0, 0, 0, 0, 1, 0, 0, 0, 0) \\ x_{H_5} &= (1, 0, 1, 0, 0, 0, 0, 0, 1, 0) \\ x_{H_6} &= (1, 1, 0, 0, 0, 0, 0, 0, 0, 0) \\ x_{H_7} &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

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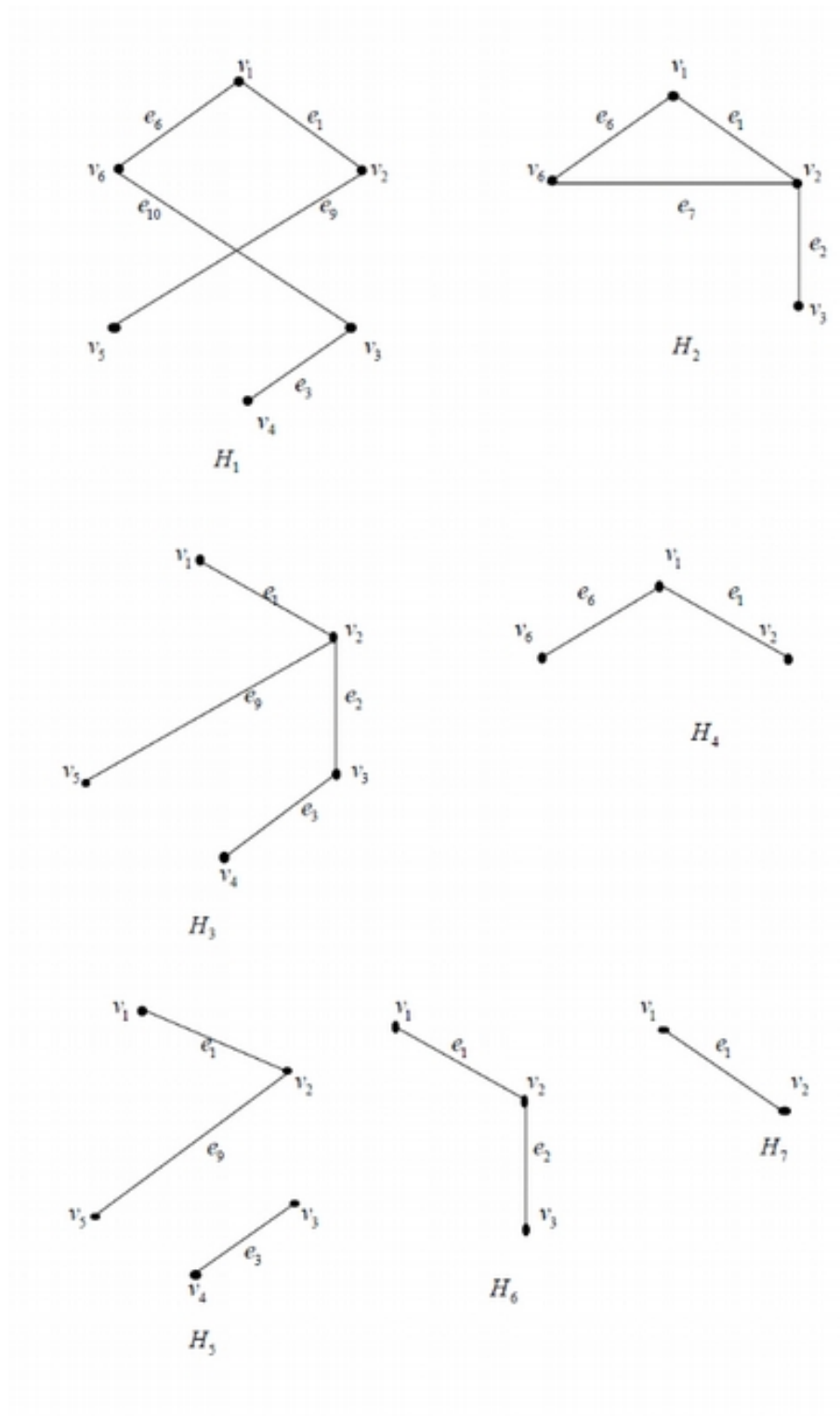


Figure 2: Subgraphs of the graph G

Now,

$$\begin{aligned}
 x_{H_1}x_{H_2} &= x_{H_4}, & x_{H_1}x_{H_3} &= x_{H_5}, & x_{H_1}x_{H_4} &= x_{H_4}, & x_{H_1}x_{H_5} &= x_{H_5}, \\
 x_{H_1}x_{H_6} &= x_{H_7}, & x_{H_1}x_{H_7} &= x_{H_7}, & x_{H_2}x_{H_3} &= x_{H_6}, & x_{H_2}x_{H_4} &= x_{H_4}, \\
 x_{H_2}x_{H_5} &= x_{H_7}, & x_{H_2}x_{H_6} &= x_{H_6}, & x_{H_2}x_{H_7} &= x_{H_7}, & x_{H_3}x_{H_4} &= x_{H_7}, \\
 x_{H_3}x_{H_5} &= x_{H_5}, & x_{H_3}x_{H_6} &= x_{H_6}, & x_{H_3}x_{H_7} &= x_{H_7}, & x_{H_4}x_{H_5} &= x_{H_7}, \\
 x_{H_4}x_{H_6} &= x_{H_7}, & x_{H_4}x_{H_7} &= x_{H_7}, & x_{H_5}x_{H_6} &= x_{H_7}, & x_{H_5}x_{H_7} &= x_{H_7}, \\
 x_{H_6}x_{H_7} &= x_{H_7}
 \end{aligned}$$

Also,

$$x_{H_i}(x_{H_j}x_{H_k}) = (x_{H_i}x_{H_j})x_{H_k}, \quad i, j, k = 1, 2, \dots, 7.$$

Hence S is a semigroup.

4 Key Exchange using S-action

Let T be a q dimensional vector space over F_2 . Define the left action of S on T , $\varphi : S \times T \rightarrow T$ such that $\varphi(x, t) = xt$. We call this action as an S -action on the vector space T . The right action is similarly defined.

Let G be a (p, q) -graph, S an abelian semigroup associated with the graph G , T be a q dimensional vector space over F_2 , and an S -action on T as defined above.

Diffie-Hellman key exchange using S -action is as follows:

1. Alice and Bob agree on an element $t \in T$.
2. Alice chooses $x \in S$ and computes xt . Alice's private key is x , her public key is xt .
3. Bob chooses $y \in S$ and computes yt . Bob's private key is y , his public key is yt .
4. Their common secret key is then $x(yt) = (xy)t = (yx)t = y(xt)$.

Example 4.1. Consider the semigroup S in the example 3.1. Let T be a 10 dimensional vector space over F_2 . Suppose Alice and Bob want to agree on a key. Suppose they choose $t = (0, 1, 1, 0, 1, 0, 1, 1, 0, 0) \in T$. Then Alice chooses $x_{H_1} = (1, 0, 1, 0, 0, 1, 0, 0, 1, 1) \in S$ and computes $x_{H_1}t = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$. Then send it to Bob. Similarly, Bob chooses $x_{H_6} = (1, 1, 0, 0, 0, 1, 0, 0, 1, 1) \in S$ and computes $x_{H_6}t = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$. Then send it to Alice. Their common key is $x_{H_1}(x_{H_6}t) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

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S-action Problem

Let G be a (p, q) -graph and S be a semigroup associated with the graph G , acting on a q dimensional vector space T over F_2 .

Given elements $t \in T$ and $y \in S$, find $x \in S$ such that $xt = y$.

4.1 Diffie-Hellman Problem using S-action

Let G be a (p, q) -graph, S be a semigroup associated with the graph G , T be a q dimensional vector space over F_2 and φ be an S -action on T .

Given $r, s, t \in T$ with $s = xr$ and $t = yr$ for some $x, y \in S$, find $(xy)r \in T$.

5 Cryptosystem using S-action

Let G be a (p, q) -graph, S be a semigroup associated with the graph G , T be a q dimensional vector space over F_2 , T is an additive abelian group and an action on T as defined above.

ElGamal cryptosystem using S -action is as follows:

1. Alice chooses elements $t \in T$ and $x \in S$. Alice's public key is (t, xt) .
2. Bob chooses a random element $y \in S$ and encrypts a message m using the encryption function $(m, y) \mapsto (yt, (y(xt)) + m) = (c_1, c_2)$.
3. Alice can decrypt the message using

$$\begin{aligned} m &= (y(xt))^{-1} + (y(xt)) + m \\ &= (xc_1)^{-1} + c_2 \end{aligned}$$

Note: Message m is also represented as vectors. Each letter in the message represents a vector $(x_1, x_2, \dots, x_q)$, $q \geq 26$ such that

$$x_i = \begin{cases} 1 & \text{if the corresponding letter is in } i^{th} \text{ position of the alphabet} \\ 0 & \text{otherwise} \end{cases}$$

Example 5.1. Let G be any (p, q) -graph with $q = 26$. Let T be a 26 dimensional vector space over F_2 , T be an additive abelian group and S be the semigroup associated with the graph G . The action of S on T is as defined earlier. Suppose Alice wants to receive a message.

1. Alice chooses $t = (0, 1, 1, 0, 1, 0, 1, 1, 0, \dots, 0) \in T$. Then chooses $x = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots, 0) \in S$ corresponding to one subgraph H_1 of G and compute $xt = (0, 0, 1, 0, 1, 0, 1, 0, 0, \dots, 0)$. Her public key is (t, tx) .
2. Bob wishes to send a message $m = \text{MEET ME TOMORROW}$ to Alice. He send it letter by letter. So, first he wants to send the letter $M = m_1(m) = (0, 0, \dots, 0, 0, 1, 0, 0, \dots, 0, 0)$.

For, he chooses $y = (0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, \dots, 0, 0) \in S$ that is a vector corresponding to one subgraph H_2 of G and compute

$$yt = (0, 0, 1, 0, 1, 0, 0, 1, 0, 0, \dots, 0, 0) = c_1$$

$$y(xt) = (0, 0, 1, 0, 1, 0, 0, \dots, 0, 0)$$

and

$$y(xt) = (0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, \dots, 0, 0) = c_2$$

Then he sends (c_1, c_2) to Alice.

3. After receiving this, Alice decrypt the message by computing $(xc_1)^{-1} + c_2$.

$$xc_1 = (0, 0, 1, 0, 1, 0, 0, \dots, 0, 0)$$

$$(xc_1)^{-1} = (0, 0, 1, 0, 1, 0, 0, \dots, 0, 0)$$

$$(xc_1)^{-1} + c_2 = (0, 0, 0, \dots, 0, 0, 1, 0, \dots, 0, 0)$$

$$= m_1(m) = M$$

Similarly, they transfer each letter in the message.

References

- Iris Anshel, Michael Anshel, and Dorian Goldfeld. An algebraic method for public key cryptography. *Mathematical Research Letters*, 6(3–4):287–291, 1999.
- Gilbert Baumslag, Benjamin Fine, and Xiaowei Xu. Cryptosystems using linear groups. *Applicable Algebra in Engineering, Communication and Computing*, 17(3):205–217, 2006.
- Simon R Blackburn and Steven Galbraith. Cryptanalysis of two cryptosystems based on group actions. In *International Conference on the Theory and Application of Cryptology and Information Security*, volume 3531, pages 52–61. Springer, 1999.

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- Whitfield Diffie and Martin Hellman. New directions in cryptography. *IEEE transactions on Information Theory*, 22(6):644–654, 1976.
- Taher ElGamal. A public key cryptosystem and a signature scheme based on discrete logarithms. *IEEE transactions on information theory*, 31(4):469–472, 1985.
- Ki Hyoung Ko, Sang Jin Lee, Jung Hee Cheon, Jae Woo Han, Ju-sung Kang, and Choonsik Park. New public-key cryptosystem using braid groups. In *Annual International Cryptology Conference*, pages 166–183. Springer, 2000.
- Neal Koblitz. *Algebraic methods of cryptography*. Berlin Heidelberg New York: Springer, 1998.
- PH Kropholler, SJ Pride, WAM Othman, KB Wong, and PC Wong. Properties of certain semigroups and their potential as platforms for cryptosystems. In *Semigroup Forum*, volume 81, pages 172–186. Springer, 2010.
- Roger C Lyndon and Paul E Schupp. *Combinatorial group theory*. Berlin Heidelberg New York: Springer, 1977.
- Gérard Maze, Chris Monico, and Joachim Rosenthal. Public key cryptography based on semigroup actions. *Advances of Mathematics of Communications*, 1(4):489–507, 2007.
- Alfred J Menezes, Paul C Van Oorschot, and Scott A Vanstone. *Handbook of Applied Cryptography. Discrete Mathematics and Its Applications*. CRC press, New York, 1996.
- Vladimir Shpilrain and Alexander Ushakov. Thompson’s group and public key cryptography. In *International Conference on Applied Cryptography and Network Security*, pages 151–163. Springer, 2005.
- Vladimir Shpilrain and Gabriel Zapata. Combinatorial group theory and public key cryptography. *Applicable Algebra in Engineering, Communication and Computing*, 17(3):291–302, 2006.
- Keith R Slavin. Public key cryptography using matrices. 2007. US Patent 10260818, <http://www.patentstorm.us/patents/7184551-fulltext.html>.
- Akihiro Yamamura. Public-key cryptosystems using the modular group. In *International Workshop on Public Key Cryptography*, pages 203–216. Springer, 1998.

A Result on b-metric space using α -compatible mappings

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Abstract

The objective of this paper is to generate a common fixed point theorem in b-metric space using α -compatible and α -continuous mappings. This result generalizes the Theorem proved by J.R. Roshan and others. Further our findings are supported by discussing some valid examples.

Keywords: b-metric space; fixed point; α -fixed point; α -compatible; α -continuous mappings.

2010 AMS subject classification: 54H25, 47H10.

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^{*}Received on September 26, 2021. Accepted on December 5, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.666. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

Fixed point theory plays an important role in mathematics and it is fast growing in the fields of analysis because of its applications in mathematics and allied subjects. Several authors [1, 2, 3, 4] established many results in fixed point theory using various weaker conditions. In the recent past, b-metric space was emerged as one of the generalizations of metric space. During this period Czerwik [5] introduced the concept of b -metric space. In recent years, number of well-known fixed point theorems have been established in b-metric space such as [6, 7, 8, 9]. The concept of alpha compatible and alpha continuous mappings were introduced in metric space [10] and some results were established in the recent past under certain weaker conditions. J.R. Roshan, N. Shobkolaei, S.Sedghi and M.Abbas [11] proved a common fixed point theorem using compatible and continuous mappings in b-metric space. In this paper, we use the concept of α -compatible and α -continuous maps and generate a fixed point theorem in b-metric space.

2.2. Preliminaries

Definition 2.1. A function $d : X \times X \rightarrow R^+$ where X is a nonempty set and $m \geq 1$ is a b-metric space if and only if for each $\phi, \varphi, \gamma \in X$

- (i) $d(\phi, \varphi) = 0 \Leftrightarrow \phi = \varphi$
- (ii) $d(\phi, \varphi) = d(\varphi, \phi)$
- (iii) $d(\phi, \gamma) \leq m[d(\phi, \varphi) + d(\varphi, \gamma)]$.

Definition 2.2. Two self maps M, N of a b-metric space X is said to be compatible if $d(MN\eta_k, NM\eta_k) = 0$ whenever $\{\eta_k\}$ sequence in X such that $M\eta_k = N\eta_k = \gamma$ for some $\gamma \in X$ as $k \rightarrow \infty$.

Definition 2.3. A point $\mu \in X$ is said to be an α -fixed point of map $M : X \rightarrow X$ if $(\alpha \circ M)\mu = \mu$.

Remark 2.4. A fixed point is not necessarily α -fixed point and α -fixed point is not necessarily a fixed point. If $\alpha=1$, the identity map then they coincide.

Example 2.5. Let $M, \alpha : R \rightarrow R$ be defined by $M(u) = u^2 - 1$ and $\alpha(u) = u^3 + 1$.

Then $(\alpha \circ M)(0) = \alpha(-1) = 0$ and $(\alpha \circ M)(1) = \alpha(0) = 1$.

Therefore 0 and 1 are α -fixed points but not fixed points.

Example 2.6. Let $M, \alpha : R \rightarrow R$ be defined by $M(u) = \frac{u^2}{3}$ and $\alpha(u) = u^3$.

Here $(\alpha \circ M)(0) = \alpha(0) = 0$ and $M(0) = 0$.

Therefore 0 is α -fixed point of M which is also fixed point of M .

Definition 2.7. A pair of self maps M and N of a b-metric space X is called α -commuting if $((\alpha \circ M) \circ (\alpha \circ N))\mu = ((\alpha \circ N) \circ (\alpha \circ M))\mu$ for all $\mu \in X$.

The preceding example show the relation between commuting and α -commuting mappings.

Example 2.8. Let $M, N, \alpha : R \rightarrow R$ be defined by $M(u) = u^4$, $N(u) = \sqrt{u}$ and $\alpha(u) = 3u$ for all $u \in R$.

Then $MN(u) = M(\sqrt{u}) = (\sqrt{u})^4 = u^2$ and $NM(u) = N(u^4) = u^2$.

Therefore $MN(u) = NM(u)$.

Hence M and N are commuting mappings.

Also for $u \in R$, $(\alpha \circ M)(u) = \alpha(u^4) = 3u^4$, $(\alpha \circ N)(u) = \alpha(\sqrt{u}) = 3\sqrt{u}$.

$(\alpha \circ M) \circ (\alpha \circ N)(u) = (\alpha \circ M)(3\sqrt{u}) = \alpha(3^4 u^2) = 3^5 u^2$

and $(\alpha \circ N) \circ (\alpha \circ M)(u) = (\alpha \circ N)(3u^4) = \alpha(\sqrt{3u^4}) = 3\sqrt{3}u^2$.

Therefore $(\alpha \circ N) \circ (\alpha \circ M)(u) \neq (\alpha \circ M) \circ (\alpha \circ N)(u)$.

Hence M and N are not α -commuting mappings.

Example 2.9. Suppose $M, N, \alpha : R - \{0\} \rightarrow R - \{0\}$ given by

$M(u) = u^4$, $N(u) = u^5$ and $\alpha(u) = \frac{1}{u}$ for all $u \in R$.

Then $MN(u) = M(u^5) = u^{20}$ and $NM(u) = N(u^4) = u^{20}$.

Therefore $MN(u) = NM(u)$.

Hence M and N are commuting mappings.

Also for $u \in R - \{0\}$, $(\alpha \circ M)(u) = \alpha(u^4) = \frac{1}{u^4}$, $(\alpha \circ N)(u) = \alpha(u^5) = \frac{1}{u^5}$

$(\alpha \circ M) \circ (\alpha \circ N)(u) = (\alpha \circ M)\left(\frac{1}{u^5}\right) = \alpha\left(T\left(\frac{1}{u^5}\right)\right) = \alpha\left(\frac{1}{u^{20}}\right) = u^{20}$.

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$$\text{and } (\alpha o N) o (\alpha o M)(u) = (\alpha o N) \left(\frac{1}{u^4} \right) = \alpha \left(S \left(\frac{1}{u^4} \right) \right) = \alpha \left(\frac{1}{u^{20}} \right) = u^{20}.$$

Therefore $(\alpha o M) o (\alpha o N)(u) = (\alpha o N) o (\alpha o M)(u)$

Hence M and N are α -commuting mappings.

Example 2.10. Let M, N, $\alpha: R \rightarrow R$ be defined by $M(u) = u^4$, $N(u) = 4u$

and $\alpha(u) = \frac{u}{4}$ for all $u \in R$.

$$MN(u) = M(4u) = (4u)^4 = 4^4 u^4, \quad NM(u) = N(u^4) = 4u^4.$$

Therefore $MN(u) \neq NM(u)$.

Hence M and N are not commuting mappings.

$$\text{Also for } u \in R, (\alpha o M)(u) = \alpha(u^4) = \frac{u^4}{4}, \quad (\alpha o N)(u) = \alpha(4u) = u$$

$$(\alpha o M) o (\alpha o N)(u) = (\alpha o M)(u) = \alpha(u^4) = \frac{u^4}{4}$$

$$(\alpha o N) o (\alpha o M)(u) = (\alpha o N) \left(\frac{u^4}{4} \right) = \alpha(u^4) = \frac{u^4}{4}.$$

Therefore, $(\alpha o M) o (\alpha o N)(u) = (\alpha o N) o (\alpha o M)(u)$.

Hence M and N are α -commuting mappings.

Example 2.11. Suppose M, N, $\alpha: R - \{0\} \rightarrow R - \{0\}$ given by

$$M(u) = u^3, \quad N(u) = 3u^2 \quad \text{and} \quad \alpha(u) = \frac{1}{u^2} \quad \text{for all } u \in R.$$

$$MN(u) = M(3u^2) = 3^3 u^6, \quad NM(u) = N(u^3) = 3u^6.$$

Therefore $MN(u) \neq NM(u)$.

Hence M and N are not commuting mappings.

$$\text{Also for } u \in R - \{0\}, (\alpha o M)(u) = \alpha(u^3) = \frac{1}{u^6}, \quad (\alpha o N)(u) = \alpha(3u^2) = \frac{1}{3^2 u^4}$$

$$(\alpha o M) o (\alpha o N)(u) = (\alpha o M) \left(\frac{1}{9u^4} \right) = \alpha \left(M \left(\frac{1}{9u^4} \right) \right) = \alpha \left(\frac{1}{9^3 u^{12}} \right) = 9^6 u^{24}.$$

$$(\alpha o N) o (\alpha o M)(u) = (\alpha o N) \left(\frac{1}{u^6} \right) = \alpha \left(S \left(\frac{1}{u^6} \right) \right) = \alpha \left(3 \left(\frac{1}{u^{12}} \right) \right) = \frac{u^{24}}{9}.$$

Therefore, $(\alpha o M) o (\alpha o N)(u) \neq (\alpha o N) o (\alpha o M)(u)$.

Hence, M and N are not α -commuting mappings.

Definition 2.12. A pair of self maps of a b-metric space (X, d) is called weakly α -commuting mappings if $(\alpha o M)$ and $(\alpha o N)$ are weakly commuting maps. i.e $d(((\alpha o M) o (\alpha o N))(u), ((\alpha o N) o (\alpha o M))(u)) \leq d((\alpha o M)(u), (\alpha o N)(u))$ for all $u \in X$.

Definition 2.13. The self maps M, N of a b-metric space X are called α -compatible maps if $(\alpha o M)$ and $(\alpha o N)$ are compatible if whenever $\{u_n\}$ is a Sequence in X such that $d((\alpha o M)(u_n), (\alpha o N)(u_n)) \rightarrow \mu \in X$, then $d(((\alpha o M) o (\alpha o N))(u_n), ((\alpha o N) o (\alpha o M))(u_n)) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.14. Two self maps M and N are called weakly α -compatible if $(\alpha o M)$ and $(\alpha o N)$ are weakly compatible, i.e $(\alpha o M)$ and $(\alpha o N)$ commute at their coincidence points.

Remark 2.15. It may be observed that α -commuting maps are weakly α -commuting maps, weakly α -commuting maps are α -compatible maps and α -compatible mappings are weakly α -compatible maps. But converse is not true in each case. These facts are presented in the following example.

Example 2.16. Let M, N, $\alpha : R - \{0\} \rightarrow R - \{0\}$ given by $M(u) = u^5$, $N(u) = u^4$ and $\alpha(u) = \frac{1}{u}$ for all $u \in R$.

Here $(\alpha o M)(u) = \alpha(u^5) = \frac{1}{u^5}$, $(\alpha o N)(u) = \alpha(u^4) = \frac{1}{u^4}$

$$(\alpha o M) o (\alpha o N)(u) = (\alpha o M)\left(\frac{1}{u^4}\right) = \alpha\left(\frac{1}{u^{20}}\right) = u^{20}.$$

$$\text{and } (\alpha o N) o (\alpha o M)(u) = (\alpha o N)\left(\frac{1}{u^5}\right) = \alpha\left(\frac{1}{u^{20}}\right) = u^{20}.$$

Therefore, $(\alpha o M) o (\alpha o N)(u) = (\alpha o N) o (\alpha o M)(u)$

Hence, N and M are α -commuting mappings.

Also for $u \in R - \{0\}$

$$d(((\alpha o M) o (\alpha o N))(u), ((\alpha o N) o (\alpha o M))(u)) = |u^{20} - u^{20}|^2 = 0$$

$$d((\alpha o M)u, (\alpha o N)u) = \left| \frac{1}{u^5} - \frac{1}{u^4} \right|^2 = \left| \frac{1-u}{u^5} \right|^2$$

$$d(((\alpha o M) o (\alpha o N))(u), ((\alpha o N) o (\alpha o M))(u)) = 0$$

$$\leq \left| \frac{1-u}{u^5} \right|^2 = d((\alpha o M)u, (\alpha o N)u).$$

Therefore M and N are weakly α -commuting maps.

$$\text{Now } (\alpha o M)(u_n) = \frac{1}{u_n^5}, (\alpha o N)(u_n) = \frac{1}{u_n^4}$$

$$d((\alpha o M)(u_n), (\alpha o N)(u_n)) = \left| \frac{1}{u_n^5} - \frac{1}{u_n^4} \right|^2 \rightarrow 0 \text{ as } u_n \rightarrow 1.$$

Hence M and N are α -compatible mappings.

Definition 2.16. The self mapping M of a b-metric space (X,d) is said to be α -continuous if $(\alpha o M)$ is continuous. In other words for every $\varepsilon \geq 0$, for all $\delta > 0$ such that $d(x, y) \leq \delta \Rightarrow d((\alpha o M)u, (\alpha o M)v) \leq \varepsilon$.

The following theorem was proved in [11].

Theorem 2.17[10]. Let f, g, S and T be four self mappings defined on a complete b-metric space(X, d) with the following conditions:

$$(C1) \ f(X) \subseteq T(X) \text{ and } g(X) \subseteq S(X)$$

$$(C2)$$

$$d(fu, gv) \leq \frac{q}{k^4} \max \left\{ d(Su, Tv), d(fu, SDu), d(gv, Tv), \frac{1}{2}(d(Su, gv) + d(fu, Tv)) \right\}$$

holds for every $u, v \in X$ with $0 < q < 1$.

$$(C3) \text{ The self mappings T and S are both continuous}$$

$$(C4) \text{ two pairs } (f, S) \text{ and } (g, T) \text{ are compatible.}$$

Then the above four maps will be having a unique fixed point which is common.

Now we prove the generalization of **Theorem (2.17)** in the preceding Theorem under some modified conditions.

To do so, we'll need to recall the following lemmas.

Lemma 2.18[10]. Let (X, d) be a b -metric space with $k \geq 1$ and two sequences $\{u_n\}$ and $\{v_n\}$ are b -convergent to u and v respectively. Then we have

$$\frac{1}{k^2} d(u, v) \leq \liminf_{n \rightarrow \infty} d(u_n, v_n) \leq \limsup_{n \rightarrow \infty} d(u_n, v_n) \leq k^2 d(u, v).$$

Lemma 2.19[9]. Let M and N be α -compatible mappings from a b -metric space (X, d) into itself such that $\lim_{j \rightarrow \infty} (\alpha \circ M)u_n = \lim_{j \rightarrow \infty} (\alpha \circ N)u_n = \mu$, for some $\mu \in X$. Then $\lim_{j \rightarrow \infty} ((\alpha \circ M) \circ (\alpha \circ N))u_n = (\alpha \circ M)\mu = \mu$, if M is α -continuous.

3. Main Result

Theorem 3.1. Let M, N, P and Q be four self maps and α as defined on a b -metric space (X, d) which is complete with the given conditions:

(b1) $(\alpha \circ M)(X) \subseteq (\alpha \circ Q)(X)$ and $(\alpha \circ N)(X) \subseteq (\alpha \circ P)(X)$

$$(b2) \quad d((\alpha \circ M)u, (\alpha \circ N)v) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d((\alpha \circ P)u, (\alpha \circ Q)v), \\ d((\alpha \circ M)u, (\alpha \circ P)u), \\ d((\alpha \circ N)v, (\alpha \circ Q)v), \\ \frac{1}{2} \left(d((\alpha \circ P)u, (\alpha \circ N)v) \right. \\ \left. + d((\alpha \circ M)u, (\alpha \circ Q)v) \right) \end{array} \right\},$$

holds for every $u, v \in X$ with $q \in (0, 1)$.

(b3) The mappings Q and P are α -continuous

(b4) the pair of maps (M, P) and (N, Q) are α -compatible.

Then the above four maps will be having a unique fixed point which is common.

Proof:

Using the condition (b1) for the point $u_0 \in X \exists u_1 \in X$ such that $(\alpha \circ M)u_0 = (\alpha \circ Q)u_1$. For this point u_1 we can select a point $u_2 \in X$ such that $(\alpha \circ N)u_1 = (\alpha \circ P)u_2$ and so on. Continuing this process it is possible to construct a sequence $\{v_j\}$ such that $v_{2j} = (\alpha \circ M)u_{2j} = (\alpha \circ Q)u_{2j+1}$ and $v_{2j+1} = (\alpha \circ N)u_{2j+1} = (\alpha \circ P)u_{2j+2} \forall j \geq 0$.

We now demonstrate that $\{v_j\}$ is a cauchy sequence.

$$\text{Take } d(v_{2j}, v_{2j+1}) = d((\alpha \circ M)u_{2j}, (\alpha \circ N)u_{2j+1})$$

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$$\leq \frac{q}{k^4} \max\{d((\alpha o P)u_{2j}, (\alpha o Q)u_{2j+1}), d((\alpha o M)u_{2j}, (\alpha o P)u_{2j}), d((\alpha o N)u_{2j+1}, (\alpha o Q)u_{2j+1}), \\ \frac{1}{2}(d((\alpha o P)u_{2j}, (\alpha o N)u_{2j+1}) + d((\alpha o M)u_{2j}, (\alpha o Q)u_{2j+1}))\}$$

$$= \frac{q}{k^4} \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j-1}), d(v_{2j+1}, v_{2j}), \\ \frac{1}{2}(d(v_{2j-1}, v_{2j+1}) + d(v_{2j}, v_{2j}))\}. \\ = \frac{q}{k^4} \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1}), \frac{d(v_{2j-1}, v_{2j+1})}{2}\} \\ \leq \frac{q}{k^4} \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1}), \frac{k}{2}(d(v_{2j-1}, v_{2j}) + d(v_{2j}, v_{2j+1}))\}.$$

If $d(v_{2j}, v_{2j+1}) > d(v_{2j-1}, v_{2j})$ for some j , then the above inequality gives

$$d(v_{2j}, v_{2j+1}) \leq \frac{q}{k^3} d(v_{2j}, v_{2j+1})$$

a contradiction.

Hence $d(v_{2j}, v_{2j+1}) \leq d(v_{2j-1}, v_{2j})$ for all $j \in N$.

Now the above inequality gives

$$d(v_{2j}, v_{2j+1}) \leq \frac{q}{k^3} d(v_{2j-1}, v_{2j}) \text{-----(1)}$$

$$\text{Similarly } d(v_{2j-1}, v_{2j}) \leq \frac{q}{k^3} d(v_{2j-2}, v_{2j-1}) \text{-----(2)}$$

From (1) and (2) we have

$$d(v_j, v_{j-1}) \leq \lambda d(v_{j-1}, v_{j-2}), \text{ where } \lambda = \frac{q}{k^3} < 1 \text{ and } j \geq 2.$$

Hence for all $j \geq 2$, we obtain

$$d(v_j, v_{j-1}) \leq \lambda d(v_{j-1}, v_{j-2}) \leq \dots \leq \lambda^{j-1} d(v_1, v_0) \text{-----(3)}$$

So for all $j > l$, we have

$$d(y_j, y_l) \leq k d(y_l, y_{l+1}) + k^2 d(y_{l+1}, y_{l+2}) + \dots + k^{j-l-1} d(y_{j-1}, y_j).$$

Now from (3), we have

$$d(v_j, v_l) \leq (k\lambda^l + k^2\lambda^{l+1} + \dots + k^{j-l-1}\lambda^{j-1})d(v_1, v_0)$$

$$\leq k\lambda^l (1 + k\lambda + k^2\lambda^2 + \dots)d(v_1, v_0)$$

$$\leq \frac{k\lambda^l}{1 - k\lambda} d(v_1, v_0).$$

Taking limits as $l, j \rightarrow \infty$, we have $d(v_j, v_l) \rightarrow 0$ as $k\lambda$ is less than one.

Therefore $\{v_j\}$ is a cauchy sequence in X and by completeness of X , it converges to some point μ in X such that

$$\lim_{j \rightarrow \infty} (\alpha o M)u_{2j} = \lim_{j \rightarrow \infty} (\alpha o Q)u_{2j+1} = \lim_{j \rightarrow \infty} (\alpha o N)u_{2j+1} = \lim_{j \rightarrow \infty} (\alpha o P)u_{2j+2} = \mu.$$

Since P is α -continuous, therefore $\lim_{j \rightarrow \infty} (\alpha o P)o(\alpha o P)u_{2j+2} = (\alpha o P)\mu$ and

$$\lim_{j \rightarrow \infty} (\alpha o P)o(\alpha o M)u_{2j} = (\alpha o P)\mu.$$

By (b4) we have (M, P) is α -compatible,

$$\lim_{j \rightarrow \infty} d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o P)o(\alpha o M)u_{2j}) = 0$$

so by Lemma (2.19), we have $\lim_{j \rightarrow \infty} (\alpha o M)o(\alpha o P)u_{2j} = (\alpha o P)\mu$.

Now putting $u = (\alpha o P)u_{2j}$ and $v = u_{2j+1}$ in (b2), we get

take sup limit as $j \rightarrow \infty$ on both the sides and by Lemma (2.18), we get

$$d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o N)u_{2j+1}) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d((\alpha o P)o(\alpha o P)u_{2j}, (\alpha o Q)u_{2j+1}), \\ d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o P)o(\alpha o P)u_{2j}), \\ d((\alpha o N)u_{2j+1}, (\alpha o Q)u_{2j+1}), \\ \frac{1}{2} \left(d((\alpha o P)o(\alpha o P)u_{2j}, (\alpha o N)u_{2j+1}) \right. \\ \left. + d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o Q)u_{2j+1}) \right) \end{array} \right\},$$

$$\begin{aligned} \frac{d((\alpha o P)\mu, \mu)}{k^2} &\leq \lim_{j \rightarrow \infty} d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o N)u_{2j+1}) \\ &\leq \frac{q}{k^4} \max \left\{ \begin{array}{l} \limsup_{j \rightarrow \infty} d((\alpha o P)o(\alpha o P)u_{2j}, (\alpha o Q)u_{2j+1}), \\ \limsup_{j \rightarrow \infty} d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o P)o(\alpha o P)u_{2j}), \\ \limsup_{j \rightarrow \infty} d((\alpha o N)u_{2j+1}, (\alpha o Q)u_{2j+1}), \\ \frac{1}{2} \left(\limsup_{j \rightarrow \infty} d((\alpha o P)o(\alpha o P)u_{2j}, (\alpha o N)u_{2j+1}) \right. \\ \left. + \limsup_{j \rightarrow \infty} d((\alpha o M)o(\alpha o P)u_{2j}, (\alpha o Q)u_{2j+1}) \right) \end{array} \right\}, \\ &\leq \frac{q}{k^4} \max \left\{ \begin{array}{l} k^2 d((\alpha o P)\mu, \mu), d((\alpha o P)\mu, (\alpha o P)\mu), d(\mu, \mu), \\ \frac{1}{2} (d((\alpha o P)\mu, \mu) + d((\alpha o P)\mu, \mu)) \end{array} \right\} \\ &\leq \frac{q}{k^4} \max \left\{ \begin{array}{l} k^2 d((\alpha o P)\mu, \mu), 0, 0, \\ d((\alpha o P)\mu, \mu) \end{array} \right\} \\ &\leq \frac{q}{k^4} \max \left\{ \begin{array}{l} k^2 d((\alpha o P)\mu, \mu), 0, 0, \\ d((\alpha o P)\mu, \mu) \end{array} \right\} \\ &= \frac{q}{k^4} k^2 d((\alpha o P)\mu, \mu) \end{aligned}$$

$$= \frac{q}{k^2} d((\alpha o P)\mu, \mu).$$

Therefore $d((\alpha o P)\mu, \mu) \leq qd((\alpha o P)\mu, \mu)$.

As $0 < q < 1$, so $(\alpha o P)\mu = \mu$.

Since Q is α -continuous, therefore $\lim_{j \rightarrow \infty} (\alpha o T)o(\alpha o T)u_{2j+2} = (\alpha o T)\mu$

and $\lim_{j \rightarrow \infty} (\alpha o Q)o(\alpha o N)u_{2j} = (\alpha o Q)\mu$.

Since the pair (N, Q) is α -compatible, we have

$$\lim_{j \rightarrow \infty} d((\alpha o N)o(\alpha o Q)u_{2j}, (\alpha o Q)o(\alpha o N)u_{2j}) = 0.$$

So by Lemma (2.18) we have $\lim_{j \rightarrow \infty} (\alpha o N)o(\alpha o Q)u_{2j} = (\alpha o Q)\mu$.

Now putting $u = u_{2j}$ and $v = (\alpha o Q)u_{2j+1}$ in (b2), we get

$$d((\alpha o M)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d((\alpha o P)u_{2j}, (\alpha o Q)o(\alpha o Q)u_{2j+1}), \\ d((\alpha o M)u_{2j}, (\alpha o P)u), \\ d((\alpha o N)o(\alpha o Q)u_{2j+1}, (\alpha o Q)o(\alpha o Q)u_{2j+1}), \\ \frac{1}{2} \left(d((\alpha o P)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \right. \\ \left. + d((\alpha o M)u_{2j}, (\alpha o Q)o(\alpha o Q)u_{2j+1}) \right) \end{array} \right\}$$

take sup limit as $j \rightarrow \infty$ on both the sides and by Lemma (2.18), we get

$$d(\mu, (\alpha o Q)\mu) \leq d((\alpha o M)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} k^2 d(\mu, (\alpha o Q)\mu), 0, 0, \\ \frac{k^2}{2} \left(d(\mu, (\alpha o Q)\mu) \right. \\ \left. + d(\mu, (\alpha o Q)\mu) \right) \end{array} \right\}$$

$$d(\mu, (\alpha o Q)\mu) \leq \frac{q}{k^2} d(\mu, (\alpha o Q)\mu)$$

which implies that $\mu = (\alpha o Q)\mu$.

Therefore $(\alpha o P)\mu = (\alpha o Q)\mu = \mu$. -----(4)

Again putting $u = \mu$ and $v = u_{2j+1}$ in (b2)

$$d((\alpha o M)\mu, (\alpha o N)u_{2j+1}) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d((\alpha o P)\mu, (\alpha o Q)u_{2j+1}), \\ d((\alpha o M)\mu, (\alpha o P)\mu), \\ d((\alpha o N)u_{2j+1}, (\alpha o Q)u_{2j+1}), \\ \frac{1}{2} \left(d((\alpha o P)\mu, (\alpha o N)u_{2j+1}) \right. \\ \left. + d((\alpha o M)\mu, (\alpha o Q)u_{2j+1}) \right) \end{array} \right\},$$

take sup limit as $j \rightarrow \infty$ on both the sides and by Lemma (2.18) we get

$$d((\alpha o M)\mu, \mu) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} k^2 d((\alpha o P)\mu, \mu), k^2 d((\alpha o M)\mu, (\alpha o P)\mu), \\ k^2 d(\mu, \mu), \\ \frac{k^2}{2} \left(d((\alpha o P)\mu, \mu) + d((\alpha o M)\mu, \mu) \right) \end{array} \right\},$$

$$= \frac{q}{k^2} d((\alpha o M)\mu, \mu).$$

This implies that $d((\alpha o M)\mu, \mu) = 0$.

That gives $(\alpha o M)\mu = \mu$ as $0 < q < 1$.

Again putting $u = \mu$ and $v = \mu$ in (b2), we get

$$d((\alpha o M)\mu, (\alpha o N)\mu) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d((\alpha o P)\mu, (\alpha o Q)\mu), \\ d((\alpha o M)\mu, (\alpha o P)\mu), \\ d((\alpha o N)\mu, (\alpha o Q)\mu), \\ \frac{1}{2} \left(d((\alpha o P)\mu, (\alpha o N)\mu) + d((\alpha o M)\mu, (\alpha o Q)\mu) \right) \end{array} \right\},$$

$$d(\mu, (\alpha o N)\mu) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} d(\mu, \mu), d(\mu, \mu), d((\alpha o N)\mu, \mu), \\ \frac{1}{2} (d(\mu, (\alpha o N)\mu) + d(\mu, \mu)) \end{array} \right\},$$

$$d(\mu, (\alpha o N)\mu) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} 0, 0, d((\alpha o N)\mu, \mu), \\ \frac{1}{2} (d(\mu, (\alpha o N)\mu) + 0) \end{array} \right\},$$

$$d(\mu, (\alpha o N)\mu) \leq \frac{q}{k^4} \max \left\{ \begin{array}{l} 0, 0, d((\alpha o N)\mu, \mu), \\ \frac{1}{2} (d(\mu, (\alpha o N)\mu) + 0) \end{array} \right\},$$

$$= \frac{q}{k^2} d((\alpha o N)\mu, \mu)$$

$$\leq q d((\alpha o N)\mu, \mu)$$

which implies that $d(\mu, (\alpha o N)\mu) = 0$

$$\mu = (\alpha o N)\mu.$$

Therefore $(\alpha o M)\mu = (\alpha o N)\mu = \mu$.------(5)

Hence from (4) and (5) we obtain

$$(\alpha o P)\mu = (\alpha o Q)\mu = (\alpha o M)\mu = (\alpha o N)\mu = \mu.$$

Therefore μ is a common α -fixed point of M, N, P and Q.

Uniqueness:

Assume that η ($\eta \neq \mu$) is another common fixed point of the four mappings M, N, P, and Q.

Put $u = \eta$ and $v = \mu$ in (b2)

$$\begin{aligned}
 d((\alpha o M)\eta, (\alpha o N)\mu) &\leq \frac{q}{k^4} \max \left\{ \begin{aligned} &d((\alpha o P)\eta, (\alpha o Q)\mu), \\ &d((\alpha o M)\eta, (\alpha o P)\eta), \\ &d((\alpha o N)\mu, (\alpha o Q)\mu), \\ &\frac{1}{2} \left(d((\alpha o P)\eta, (\alpha o N)\mu) \right. \\ &\quad \left. + d((\alpha o M)\eta, (\alpha o Q)\mu) \right) \end{aligned} \right\} \\
 &\leq \frac{q}{k^4} \max \left\{ \begin{aligned} &k^2 d(\eta, \mu), k^2 d(\eta, \eta), k^2 d(\mu, \mu), \\ &\frac{k^2}{2} (d(\eta, \mu) + d(\eta, \mu)) \end{aligned} \right\} \\
 &= \frac{q}{k^4} \max \{k^2 d(\eta, \mu), 0, 0, k^2 d(\eta, \mu)\} \\
 \frac{d(\eta, \mu)}{k^2} &= \frac{q}{k^4} k^2 d(\eta, \mu) \\
 d(\eta, \mu) &\leq q d(\eta, \mu).
 \end{aligned}$$

As $0 < q < 1$, so $\eta = \mu$.

Hence the four maps M, N, P and Q will be having a unique common fixed point.

Now we give an illustration to support our result.

Example 3.2: Suppose $X = [0, 1]$ is a b-metric space $d(u, v) = |u - v|^2$ where $u, v \in X$.

Define the four self maps M, N, P, Q and α as follows

$$\alpha(u) = u \quad ; \quad M(u) = N(u) = \begin{cases} \frac{u+2}{6}, & 0 \leq u \leq \frac{1}{2} \\ \frac{1}{2}, & u = \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{2} < u \leq 1 \end{cases} ;$$

$$Q(u) = P(u) = \begin{cases} \frac{4\alpha + 3}{6}, & 0 \leq u \leq \frac{1}{2} \\ 1 - u, & \frac{1}{2} < u \leq 1 \end{cases} ;$$

$$M(X) = N(X) = (0.16, 0.33] \cup \left\{ \frac{1}{2} \right\}, \quad P(X) = Q(X) = \left[0, \frac{1}{2} \right],$$

clearly the condition (b1) is satisfied.

Take a sequence as $u_j = \frac{1}{2} - \frac{1}{j}$ for $j \geq 0$.

$$\text{Now } \lim_{j \rightarrow \infty} Mu_j = \lim_{j \rightarrow \infty} M\left(\frac{1}{2} - \frac{1}{j}\right) = \lim_{j \rightarrow \infty} \frac{\left(\frac{1}{2} - \frac{1}{j} + 2\right)}{6} = \frac{5}{6} - \frac{1}{6n} = \frac{5}{6}$$

$$\text{and } \lim_{j \rightarrow \infty} Pu_j = \lim_{j \rightarrow \infty} Q\left(\frac{1}{2} - \frac{1}{j}\right) = \lim_{j \rightarrow \infty} \frac{\left(4\left(\frac{1}{2} - \frac{1}{j}\right) + 3\right)}{6} = \frac{5}{6}$$

that is $\exists$ a sequence $\{u_j\}$ in X such that $\lim_{j \rightarrow \infty} Mu_j = \lim_{j \rightarrow \infty} Pu_j = \frac{5}{6}$.

Similarly $\lim_{j \rightarrow \infty} Nu_j = \lim_{j \rightarrow \infty} Qu_j = \frac{5}{6}$. Also

$$MPu_j = MP\left(\frac{1}{2} - \frac{1}{j}\right) = M\left(\frac{4\left(\frac{1}{2} - \frac{1}{j}\right) + 3}{6}\right) = M\left(\frac{5}{6} - \frac{2}{3j}\right) = \frac{1}{2} \text{ as } k \rightarrow \infty \text{ and}$$

$$PMu_j = PM\left(\frac{1}{2} - \frac{1}{j}\right) = P\left(\frac{\left(\frac{1}{2} - \frac{1}{j}\right) + 2}{6}\right) = P\left(\frac{5}{6} - \frac{1}{6n}\right) = 1 - \frac{5}{6} + \frac{1}{6n} = \frac{1}{6} \text{ as } j \rightarrow \infty.$$

A Result on b-metric space using α -compatible mappings

$$\text{So that } \lim_{j \rightarrow \infty} d(MPu_j, PMu_j) = d\left(\frac{1}{2}, \frac{5}{6}\right) = \left|\frac{1}{2} - \frac{5}{6}\right|^2 = \frac{4}{36} \neq 0.$$

$$\text{Similarly } \lim_{j \rightarrow \infty} d(NQu_j, QNu_j) \neq 0.$$

Showing that the pairs (M, P) and (N, Q) are not compatible mappings.

$$\text{Again } (\alpha \circ M)(u_j) = (\alpha \circ M)\left(\frac{1}{2} - \frac{1}{j}\right) = \alpha\left(\frac{\left(\frac{1}{2} - \frac{1}{j}\right) + 2}{6}\right) = \alpha\left(\frac{5}{6} - \frac{1}{6j}\right) = \frac{5}{6} \text{ as } j \rightarrow \infty$$

$$\text{and also } (\alpha \circ P)(u_j) = (\alpha \circ P)\left(\frac{1}{2} - \frac{1}{j}\right) = \alpha\left(\frac{4\left(\frac{1}{2} - \frac{1}{j}\right) + 3}{6}\right) = \alpha\left(\frac{5}{6} - \frac{2}{3j}\right) = \frac{5}{6} \text{ as } j \rightarrow \infty.$$

$$(\alpha \circ M) \circ (\alpha \circ P)u_j = (\alpha \circ M)\left(\frac{5}{6} - \frac{2}{3j}\right) = \alpha\left[M\left(\frac{5}{6} - \frac{2}{3j}\right)\right] = \alpha\left(\frac{1}{6}\right) = \frac{1}{6} \text{ as } j \rightarrow \infty \text{ and}$$

$$(\alpha \circ P) \circ (\alpha \circ M)u_j = (\alpha \circ P)\left(\frac{5}{6} - \frac{1}{6j}\right) = \alpha\left[1 - \frac{5}{6} + \frac{1}{6n}\right] = \alpha\left(\frac{1}{6} + \frac{1}{6n}\right) = \frac{1}{6} \text{ as } j \rightarrow \infty.$$

$$\text{So that } \lim_{j \rightarrow \infty} d((\alpha \circ M) \circ (\alpha \circ P)u_j, (\alpha \circ P) \circ (\alpha \circ M)u_j) = d\left(\frac{1}{6}, \frac{1}{6}\right) = \left|\frac{1}{6} - \frac{1}{6}\right|^2 = 0.$$

$$\text{Similarly } \lim_{j \rightarrow \infty} d((\alpha \circ Q) \circ (\alpha \circ N)u_j, (\alpha \circ N) \circ (\alpha \circ Q)u_j) = 0.$$

Showing that the pairs $(\alpha \circ M, \alpha \circ P)$ and $(\alpha \circ N, \alpha \circ Q)$ are α -compatible mappings.

Now we fulfill the requirement that the mappings M, N, P and Q satisfy the condition (b2).

We have $(\alpha o M)(u) = 0$, $(\alpha o N)(v) = \frac{v}{16}$

$$d((\alpha o M)u, (\alpha o N)v) = d\left(0, \frac{v}{16}\right) = \left|\frac{v}{16}\right|^2 = \frac{v^2}{256}.$$

Also $(\alpha o P)(u) = u$ and $(\alpha o Q)(v) = \frac{v}{4}$,

$$d((\alpha o P)u, (\alpha o Q)v) = d\left(u, \frac{v}{4}\right) = \left|u - \frac{v}{4}\right|^2,$$

$$d((\alpha o M)u, (\alpha o P)u) = d(0, u) = u^2,$$

$$d((\alpha o g)v, (\alpha o T)v) = d\left(\frac{v}{16}, \frac{v}{4}\right) = \left|\frac{v}{16} - \frac{v}{4}\right|^2 = \frac{9v^2}{256},$$

$$d((\alpha o S)u, (\alpha o g)v) = d\left(u, \frac{v}{16}\right) = \left|u - \frac{v}{16}\right|^2,$$

$$d((\alpha o M)u, (\alpha o Q)v) = d\left(0, \frac{v}{4}\right) = \frac{v^2}{16}.$$

Substituting all these in the inequality (b2), we obtain

$$\frac{v^2}{256} \leq \frac{q}{k^4} \max \left\{ k^2 \left(u - \frac{v}{4} \right)^2, k^2 u^2, k^2 \frac{9v^2}{256}, \frac{k^2}{2} \left(\left(u - \frac{v}{16} \right)^2 + \frac{v^2}{16} \right) \right\}$$

If we choose $u = 0.5, v = 0.9$ and $k = 2$ we obtain

$$0.00316 \leq \frac{q}{16} \max \{0.3024, 1, k^2 0.112, 0.4948\}$$

$$0.00316 \leq \frac{q}{16} \quad (1)$$

$$0.00316 \leq \frac{q}{16} (1) \Rightarrow q = 0.05 \in (0, 1).$$

Hence the condition (b2) is satisfied.

4. Conclusion

This work is focused to generate the existence of common fixed point theorem proved by J.R.Roshan and others mentioned in Theorem (2.17) by employing

some weaker conditions α -compatible and α -continuous mappings instead of compatible and continuous mappings. At the end of the theorem our result is justified with a suitable example.

References

- [1] V.Srinivas and K.Mallaiah. A Result on Multiplicative Metric Space. Journal of Mathematical and computational science, 10(5),1384-1394, 2020.
- [2] V.Nagaraju,Bathini Raju and P.Thirupathi. Common fixed point theorem for four self maps satisfying common limit point property. Journal of Mathematical and Computational Science, 10(4), 1228-1238, 2020.
- [3] B.Vijayabaskar Reddy and V.Srinivas. Fixed point results on multiplicative semi metric space. Journal of Scientific Research, 12(3), 341-348, 2020.
- [4] V.Srinivas,T.Thirupathi and K.Mallaiah. A Fixed Point Theorem Using E.A Property on Multiplicative Metric Space. Journal of Mathematical and Computational Science, 10 (5), 1788-1800, 2020.
- [5] Czerwik,S.Nonlinear set valued contraction mappings in b-metric spaces. Atti.Sem.Mat.Fis.Uni.Sem.Mat.Fis.Univ.Modena, 46(2), 263-276, 1998.
- [6] Boriceanu, Monica, Marius Bota and Adrian Petruşel. Multivalued fractals in b-metric spaces. Central European Journal of Mathematics, 8(2), 367-377, 2010.
- [7] T.Thirupathi and V. Srinivas. Some outcomes on b-metric space. Journal of Mathematical and Computational Science, 10(6), 3012-3025, 2020.
- [8] A. Aghajani, M.Abbas and J. Roshan. Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Mathematica Slovaca ,64 (4), 941-960, 2014.
- [9] T.Thirupathi and V.Srinivas. Certain results in b-metric space using subcompatible, faintly compatible mappings. Journal of Mathematical and Computational Science, 11(6), 8382-8399, 2021.

- [10] Shivram Sharma and Praveen Kumar Sharma. On common α -fixed point theorems. Journal of Mathematical and Computational Science, 11(1), 87-108, 2021.
- [11] J.R.Roshan, N.Sobkolaei, S.Sedghi and M.Abbas. Common fixed point of four maps in b-metric Spaces. Hacettepe Journal of Mathematics and Statistics, 43(4), 613-624, 2014.

On decomposition of multistars into multistars

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Abstract

The multistar $S^{w_1, \dots, w_n}$ is the multigraph whose underlying graph is an n -star and the multiplicities of its n edges are $w_1, \dots, w_n$. Let G and H be two multigraphs. An H -decomposition of G is a set D of H -subgraphs of G , such that the sum of $\omega(e)$ over all graphs in D which include an edge e , equals the multiplicity of e in G , for all edges e in G . In this paper, we fully characterize $S^{1,2,3}$, $K_{1,m}$ and S^{m^l} decomposable multistars, where m^l is m repeated l times.

Keywords: decomposition; multigraph; multistar

2020 AMS subject classifications: 05C51; 05C70; 05C75 ¹

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¹Received on October 24, 2021. Accepted on November 24, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.681. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

If G and H are two simple graphs with out isolated vertices, then G is H -decomposable or H divides G if there exists a partition of the edge set of G into disjoint isomorphic copies of H .

The above definition can be extended to multigraphs also. Let G and H be two multigraphs. Then the corresponding H -decomposition problem is to decide for a fixed H and an input G , whether such a partition exists. We can formally define the concepts about multigraphs and the multigraph decomposition problems as follows.

Definition 1.1. A multigraph (V, E, w) consists of a simple underlying graph (V, E) and a multiplicity function $w : E \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

The multigraph on an underlying graph G with constant multiplicity λ is denoted by $\lambda.G$ and this is different from λG , denoting λ disjoint copies of G . When referring to a simple graph G as a multigraph we mean $1.G$. An isomorphism between multigraphs is an isomorphism between their underlying simple graphs which preserves edge multiplicity.

Definition 1.2. A subgraph H of a multigraph G is a multigraph H whose underlying graph is a subgraph of that of G and its multiplicity function is dominated by the multiplicity function of G , i.e. the multiplicity of an edge in H does not exceed its multiplicity in G .

Definition 1.3. An H -subgraph of G is a subgraph of a multigraph G , isomorphic to a multigraph H .

Definition 1.4. Let G and H be two multigraphs. An H -decomposition of G is a set D of H -subgraphs of G , such that the sum of $\omega(e)$ over all graphs in D which include an edge e , equals the multiplicity of e in G , for all edges e in G .

Definition 1.5. The multistar $S^{w_1, w_2, \dots, w_n}$ is the multigraph whose underlying graph is an n -star and the multiplicities of its n edges are $w_1, w_2, \dots, w_n$.

There are considerable number of papers dealing with an H -decomposition of G and some them are provided in the reference [Shyu, 2013, Lin and Shyu, 1996, Lin, 2010, Lee and Lin, 2005, Lee et al., 2005, Bryant et al., 2001, Bialostocki and Roditty, 1982]. Priesler and Tarsi [Priesler and Tarsi, 2004] showed that, for any multistar H (except a few cases), H -decomposition is NP -complete. Priesler and Tarsi [Priesler and Tarsi, 2005] fully characterized $S^{1,2}$ -decomposable multistars in the following theorem.

Theorem 1.1. [Priesler and Tarsi, 2005] *The multistar $S^{w_1, w_2, \dots, w_n}$, $n \geq 2$ is $S^{1,2}$ -decomposable if and only if*

1. $\sum_{i=1}^n w_i \equiv 0 \pmod{3}$
2. *The number of odd multiplicities among the w_i is at most $\frac{1}{3}(\sum_{i=1}^n w_i)$*
3. *The largest among the w_i is at most twice the sum of all the others.*

In this paper we fully characterize those multistars which are $S^{1,2,3}$ -decomposable, $K_{1,m}$ -decomposable, S^{2^m} -decomposable and S^{m^l} -decomposable where m^l denotes m repeated l times.

2 Main Results

2.1 $S^{1,2,3}$ decomposability of S^{w_1, w_2, w_3}

Theorem 2.1. *Let $w_1 \geq w_2 \geq w_3 \geq 2$ be positive integers and $n = \frac{w_1 + w_2 + w_3}{6}$. Then S^{w_1, w_2, w_3} is $S^{1,2,3}$ -decomposable if*

1. $w_1 + w_2 + w_3 \equiv 0 \pmod{6}$
2. $2 \leq w_1 - n \leq 2n, 2 \leq w_2 - n \leq 2n$
3. $5w_2 \leq w_1 + 7w_3 - 12$
4. $w_1 \leq w_2 + w_3 - 6$ if m and $l - (\frac{m-1}{2})$ are odd where $m = w_1 - n, l = w_2 - n$.

Proof. Consider the equations

$$\begin{aligned} 3x_1 + 2x_2 + 1(n - (x_1 + x_2)) &= w_1 \\ 3y_1 + 2y_2 + 1(n - (y_1 + y_2)) &= w_2 \\ 3(n - (x_1 + y_1)) + 2(n - (x_2 + y_2)) + 1(n - (2n - (x_1 + x_2 + y_1 + y_2))) &= w_3 \end{aligned}$$

Let the above three equations be called as (A). Firstly we claim that under the given conditions (1),(2),(3) and (4) we can find non negative integers x_1, x_2, y_1, y_2 satisfying equations (A) such that $n - (x_1 + x_2) \geq 0, n - (y_1 + y_2) \geq 0, n - (x_1 + y_1) \geq 0, n - (x_2 + y_2) \geq 0, n - [2n - (x_1 + x_2 + y_1 + y_2)] \geq 0$. Let these five inequalities be called as (B). The equations in (A) can be simplified as

$$2x_1 + x_2 = w_1 - n \tag{2.1.1}$$

$$2y_1 + y_2 = w_2 - n \tag{2.1.2}$$

$$2x_1 + 2y_1 + x_2 + y_2 = 4n - w_3 \tag{2.1.3}$$

Since $n = \frac{w_1+w_2+w_3}{6}$, $w_1 - n + w_2 - n = 4n - w_3$. Thus to prove our claim we have to solve the equations (2.1.1), (2.1.2) such that all the inequalities in (B) are satisfied. Observing (2.1.1) and (2.1.2), it is clear that they have a positive integral solution such that all the inequalities in (B) are satisfied if and only if $w_1 - n \leq 2n$ and $w_2 - n \leq 2n$. By condition (2), these inequalities holds. Let $m = w_1 - n$, $l = w_2 - n$.

Case 1: $w_2 - n \leq n$

Here we have to solve the equations $2x_1 + x_2 = m$, $2y_1 + y_2 = l$.

Subcase 1.1: m is even

Take $x_1 = \frac{m}{2}$, $x_2 = 0$, $y_1 = 0$, $y_2 = l$. Thus $x_1 + x_2 = \frac{m}{2} \leq n$, $y_1 + y_2 = l \leq n$, $x_l + y_l = \frac{m}{2} \leq n$, $x_2 + y_2 = l \leq n$. So the only inequality in (B), which has to be verified is $n - (2n - (x_l + x_2 + y_1 + y_2)) \geq 0$. Since $n - (x_l + x_2) \geq 0$ and $n - (y_1 + y_2) \geq 0$, $x_l + x_2 + y_1 + y_2 \leq 2n$. Thus $n - (2n - (x_l + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow x_l + x_2 + y_1 + y_2 \geq n \Leftrightarrow \frac{m}{2} + l \geq n \Leftrightarrow \frac{w_1-n}{2} + w_2 - n \geq n \Leftrightarrow w_1 + 2w_2 \geq 5n \Leftrightarrow w_1 + 2w_2 \geq 5(\frac{w_1+w_2+w_3}{6}) \Leftrightarrow 5w_3 \leq w_1 + 7w_2$, which is always true, since $w_1 \geq w_2 \geq w_3$. Thus in this subcase all the inequalities in (B) are satisfied.

Subcase 1.2: m is odd

Take $x_1 = \frac{m-1}{2}$, $x_2 = 1$, $y_1 = 1$, $y_2 = l - 2$. Here $x_1 + y_1 = \frac{m-1}{2} + 1 \leq n$ [since m is odd and $m \leq 2n$]. $x_2 + y_2 = 1 + l - 2 = l - 1 \leq n$ [since in this subcase $l = w_2 - n \leq n$], $y_1 + y_2 = l - 1 \leq n$ and $x_1 + x_2 = \frac{m-1}{2} + 1 \leq n$. As in the above subcase $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow x_1 + x_2 + y_1 + y_2 \geq n \Leftrightarrow m - 1 + 2l \geq 2n \Leftrightarrow w_1 + 2w_2 - 1 \geq 5n \Leftrightarrow 5w_3 + 6 \leq w_1 + 7w_2$. Since $w_3 \geq 2$, $w_1 > 2 + n$, $w_2 \geq 2 + n$, we get $w_1 \geq 3$, $w_2 \geq 3$, $w_3 \geq 2$. Also $w_1 \geq w_2 \geq w_3$. Thus $7w_2 \geq 5w_3 + 2w_3 \geq 5w_3 + 4$. Thus $w_1 + 7w_2 \geq 5w_3 + 4 + w_1 \geq 5w_3 + 7 > 5w_3 + 6$. Hence $x_1 + x_2 + y_1 + y_2 \geq n$ and thus $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0$. Thus in this subcase also all the conditions in (B) are satisfied.

Case 2: $w_2 - n > n$

Here also we have to solve the equations $2x_1 + x_2 = m$, $2y_1 + y_2 = l$.

Subcase 2.1: m is even and $l - \frac{m}{2}$ is even

Take $x_1 = \frac{m}{2}$, $x_2 = 0$, $y_1 = \frac{l-\frac{m}{2}}{2}$, $y_2 = \frac{m}{2}$. Here $n - (x_l + x_2) = n - \frac{m}{2} \geq 0$, since $\frac{m}{2} \leq n$. $n - (y_1 + y_2) = n - (\frac{2l-m}{4} + \frac{m}{2}) = n - \frac{2l+m}{4}$. Thus $n - (y_1 + y_2) \geq 0 \Leftrightarrow \frac{2l+m}{4} \leq n \Leftrightarrow 2(w_2 - n) + w_1 - n \leq 4n \Leftrightarrow 5w_2 \leq w_1 + 7w_3$, which is true by the given condition (3). Similarly $n - (x_1 + y_1) \geq 0$ and $n - (x_2 + y_2) \geq 0$. As in the above case, $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow x_1 + x_2 + y_1 + y_2 \geq n \Leftrightarrow \frac{m}{2} + \frac{2l-m}{4} + \frac{m}{2} \geq n \Leftrightarrow 3m + 2l \geq 4n \Leftrightarrow 3w_1 + 2w_2 \geq 9n \Leftrightarrow 3w_3 \leq 3w_1 + w_2$, which is always true since $w_1 \geq w_2 \geq w_3$.

Subcase 2.2: m is even and $l - \frac{m}{2}$ is odd

Here take $x_1 = \frac{m}{2}$, $x_2 = 0$, $y_1 = \frac{l-\frac{m}{2}+1}{2}$, $y_2 = \frac{m}{2} - 1$. We can easily verify that $n - (x_l + x_2) \geq 0 \Leftrightarrow 5w_2 \leq w_1 + 7w_3 - 12$, which is true by condition (3).

Similarly $n - (y_1 + y_2) \geq 0$. Also it easily follows that $n - (x_1 + y_1) \geq 0$ and $n - (x_2 + y_2) \geq 0$. $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow 3w_1 + w_2 \geq 3w_3 + 4$. But $w_1 \geq w_2 \geq w_3 \geq 2$ and by condition (2), $w_1 \geq n + 2$ and $n = \frac{w_1 + w_2 + w_3}{6}$. Thus $w_1 \geq 4$. Also $3w_1 + w_2 = w_1 + 2w_1 + w_2 \geq w_1 + 3w_3 \geq 3w_3 + 4$ (since $w_1 \geq 4$). Thus $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0$ and hence all the inequalities in (B) are satisfied.

Subcase 2.3: m is odd and $l - \frac{m-1}{2}$ is even

Take $x_1 = \frac{m-1}{2}$, $x_2 = 1$, $y_1 = \frac{l - \frac{m-1}{2}}{2}$, $y_2 = \frac{m-1}{2}$. $n - (x_1 + x_2) = n - (\frac{m-1}{2} + 1) \geq 0 \Leftrightarrow 1 + \frac{m-1}{2} \leq n$. This is true since $m \leq 2n$ and m is odd. Similarly $n - (x_2 + y_2) \geq 0$. Also $n - (x_1 + y_1) \geq 0 \Leftrightarrow 5w_2 \leq w_1 + 7w_3 + 6$, which is true by condition (3). Similarly $n - (y_1 + y_2) \geq 0$. Also $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow 3w_1 + w_2 + 2 \geq 3w_3$, which is always true since $w_1 \geq w_2 \geq w_3$. Thus all the inequalities in (B) are satisfied.

Subcase 2.4: m is odd and $l - \frac{m-1}{2}$ is odd

Take $x_1 = \frac{m-1}{2}$, $x_2 = 1$, $y_1 = \frac{l - \frac{m+1}{2}}{2}$, $y_2 = \frac{m+1}{2}$. $n - (x_1 + x_2) = n - \frac{m-1}{2} + 1 = n - \frac{m+1}{2}$. Since m is odd and $m \leq 2n$, $\frac{m+1}{2} \leq n$. So $n - (x_1 + x_2) \geq 0$. $n - (x_2 + y_2) = n - (\frac{m+1}{2} + 1) \geq 0 \Leftrightarrow w_1 \leq w_2 + w_3 - 6$, which is true by condition (4). Also we can verify that $n - (x_1 + y_1) \geq 0$ and $n - (y_1 + y_2) \geq 0$ by condition (3). Similarly $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow 3w_1 + w_2 + 6 \geq 3w_3$, which is always true. Hence all the conditions in (B) are satisfied in this subcase also. Hence our claim is proved in both cases. Thus using equations (A), we can properly partition w_1 into x_1 copies of 3, x_2 copies of 2 and $n - (x_1 + x_2)$ copies of 1's. w_2 can be partitioned into y_1 copies of 3, y_2 copies of 2 and $n - (y_1 + y_2)$ copies of 1. w_3 can be partitioned into $n - (x_1 + y_1)$ copies of 3, $n - (x_2 + y_2)$ copies of 2 and $n - (2n - (x_1 + x_2 + y_1 + y_2))$ copies of 1. Using these partitions of w_1, w_2, w_3 , we can decompose S^{w_1, w_2, w_3} into copies of $S^{1, 2, 3}$. $\square$

2.2 $K_{1,m}$ decomposability of $S^{w_1, w_2, \dots, w_n}$

Theorem 2.2. *The multistar $S^{w_1, w_2, \dots, w_n}$, $w_1 \geq w_2 \geq \dots \geq w_n$, is $K_{1,m}$ -decomposable ($n \geq m$) if and only if*

1. $\sum_{i=1}^n w_i \equiv 0 \pmod{m}$
2. For each $k = 1, 2, \dots, m-1$, $\sum_{i=1}^k w_i \leq \frac{k}{m-k}(w_{k+1} + \dots + w_n)$

Proof. Suppose the multistar the multistar $S^{w_1, w_2, \dots, w_n}$, $w_1 \geq w_2 \geq \dots \geq w_n$, is $K_{1,m}$ -decomposable ($n \geq m$). Then clearly $\sum_{i=1}^n w_i \equiv 0 \pmod{m}$.

To prove (2), assume the contrary. Suppose that $\sum_{i=1}^k w_i > \frac{k}{m-k}(w_{k+1} + \dots + w_n)$

w_n), for some k with $1 \leq k \leq m-1$. This implies

$$\begin{aligned} (m-k) \sum_{i=1}^k w_i > k(w_{k+1} + \cdots + w_n) &\Rightarrow m \sum_{i=1}^k w_i > k \sum_{i=1}^n w_i \\ &\Rightarrow \sum_{i=1}^k w_i > \frac{k}{m} \left(\sum_{i=1}^n w_i \right). \end{aligned}$$

This is not possible, since $\frac{1}{m}(\sum_{i=1}^n w_i)$ is the number of copies of $K_{1,m}$ to which $S^{w_1, w_2, \dots, w_n}$ can be decomposed. Each copy of $K_{1,m}$ can contribute at most k to $\sum_{i=1}^k w_i$. Thus $\sum_{i=1}^k w_i \leq \frac{k}{m}(\sum_{i=1}^n w_i)$.

We prove sufficiency by induction on $w = \sum_{i=1}^n w_i$. For $w = m$, the multistar is $K_{1,m}$ itself. If $w \geq 2m$, one copy of $K_{1,m}$ is deleted from $S^{w_1, w_2, \dots, w_n}$ by subtracting m number of 1's from the largest m multiplicities. The multistar obtained after this process still satisfies conditions 1 and 2. Hence by induction the proof follows. $\square$

2.3 S^{2^m} decomposability of $S^{w_1, w_2, \dots, w_n}$

Theorem 2.3. *The multistar $S^{w_1, w_2, \dots, w_n}$, $w_1 \geq w_2 \geq \cdots \geq w_n$, is S^{2^m} -decomposable ($n \geq m$) if and only if*

1. $\sum_{i=1}^n w_i \equiv 0 \pmod{2m}$
2. For $1 \leq i \leq n$, $w_i \equiv 0 \pmod{2}$
3. For each $k = 1, 2, \dots, m-1$, $\sum_{i=1}^k w_i \leq \frac{k}{m-k}(w_{k+1} + \cdots + w_n)$

Proof. Assume that the multistar $S^{w_1, w_2, \dots, w_n}$ is S^{2^m} -decomposable. Then as in the above theorems conditions 1 and 3 hold. Since $S^{w_1, w_2, \dots, w_n}$ is S^{2^m} -decomposable, clearly $w_i \equiv 0 \pmod{2}$.

We prove sufficiency by induction on $w = \sum_{i=1}^n w_i$. For $w = 2m$, the multistar is S^{2^m} itself. If $w \geq 4m$, delete one copy of S^{2^m} from $S^{w_1, w_2, \dots, w_n}$ by subtracting m number of 2's from the largest m multiplicities. The multistar obtained after this deletion still satisfies all the three conditions. Hence by induction the proof follows. $\square$

The above two theorems can be generalized to characterize S^{m^l} -decomposable multistars.

2.4 S^{m^l} decomposability of $S^{w_1, w_2, \dots, w_n}$

Theorem 2.4. *The multistar $S^{w_1, w_2, \dots, w_n}$, $w_1 \geq w_2 \geq \cdots \geq w_n$, is S^{m^l} -decomposable ($n \geq l$) if and only if*

On decomposition of multistars into multistars

1. $\sum_{i=1}^n w_i \equiv 0 \pmod{lm}$
2. For $1 \leq i \leq n$, $w_i \equiv 0 \pmod{m}$
3. For each $k = 1, 2, \dots, l-1$, $\sum_{i=1}^k w_i \leq \frac{k}{l-k}(w_{k+1} + \dots + w_n)$

Proof. Assume that the multistar $S^{w_1, w_2, \dots, w_n}$ is S^{m^m} -decomposable. Then conditions 1, 2 and 3 follows as in the above theorem.

The sufficiency can similarly be proved using induction by deleting one copy of S^{m^m} from $S^{w_1, w_2, \dots, w_n}$ by subtracting l number of m 's from the largest l multiplicities. $\square$

3 Conclusions

In this paper we have characterized those multistars which are $S^{1,2,3}$ -decomposable, $K_{1,m}$ -decomposable, S^{2^m} -decomposable and S^{m^l} -decomposable.

4 Acknowledgement

We are indebted to Prof.M.I.Jinnah, formerly University of Kerala, for his guidance and valuable suggestions during the preparation of this manuscript.

References

- A Bialostocki and Y Roditty. 3k 2-decomposition of a graph. *Acta Mathematica Academiae Scientiarum Hungarica*, 40(3-4):201–208, 1982.
- Darryn E Bryant, Saad El-Zanati, Charles Vanden Eynden, and Dean G Hoffman. Star decompositions of cubes. *Graphs and Combinatorics*, 17(1):55–59, 2001.
- Hung-Chih Lee and Chiang Lin. Balanced star decompositions of regular multigraphs and λ -fold complete bipartite graphs. *Discrete mathematics*, 301(2-3):195–206, 2005.
- Hung Chih Lee, Jeng Jong Lin, Chiang Lin, and Tay Woei Shyu. Multistar decomposition of complete multigraphs. *Ars Combinatoria*, 74:49–63, 2005.
- Chiang Lin and Tay-Woei Shyu. A necessary and sufficient condition for the star decomposition of complete graphs. *Journal of Graph Theory*, 23(4):361–364, 1996.
- Jenq-Jong Lin. Decomposition of balanced complete bipartite multigraphs into multistars. *Discrete mathematics*, 310(5):1059–1065, 2010.

- Miri Priesler and Michael Tarsi. On some multigraph decomposition problems and their computational complexity. *Discrete Mathematics*, 281(1-3):247–254, 2004.
- Miri Priesler and Michael Tarsi. Multigraph decomposition into stars and into multistars. *Discrete mathematics*, 296(2-3):235–244, 2005.
- Tay-Woei Shyu. Decomposition of complete bipartite graphs into paths and stars with same number of edges. *Discrete Mathematics*, 313(7):865–871, 2013.

An image compression method based on Ramanujan Sums and measures of central dispersion

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Abstract

This paper introduces a simple lossy image compression method based on Ramanujan Sums $c_q(n)$ and the statistical measures of numerical data such as mean and standard deviation. The Ramanujan Sum $c_q(n)$ has been used in digital signal processing for a variety of applications nowadays. Some of them include the recently developed image kernels for edge detection, extraction of periodicity from signals, etc. The presented compression algorithm is an extension of the edge detection algorithm using an integer image kernel based on Ramanujan Sums. We propose a block-based compression algorithm that detects edges in the images using this image kernel and then compresses the image by storing kernel operation values, the mean and standard deviation for each block instead of pixel values. The proposed method has the advantage of low computational complexity and shows its ability in fast reconstruction and high compression that can be achieved for different block sizes.

Keywords: Ramanujan Sum $c_q(n)$; lossy image compression; mean; standard deviation.

2020 AMS subject classifications: 11Z05; 97M10. ¹

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¹Received on October 28, 2021. Accepted on December 20, 2021. Published on December 31, 2021. doi: doi:10.23755/rm.v41i0.683. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

In the rapid popularization of the internet and social media, the role of a digital image is indispensable. As we have to transmit a lot of information over communication networks, bandwidth reduction is necessary. To achieve this, audio and video signals need to be compressed. Image or video signals in compressed form are convenient for editing, storing, utilizing, and transmitting. Image compression techniques can be classified into two categories. If the information retained after decompression is 100%, it is called lossless compression otherwise lossy. If we take a pixel in an image at random there is a good chance that its neighbours will have the same intensity or very similar intensity. Typically hence, image compression is based on the fact that the neighbouring pixels are highly correlated ([Salomon, 2007], [Sayood, 2012]). Most image compression methods exploit this feature to obtain efficient compression.

Lossless compression can be achieved with the techniques like Run Length Encoding (RLE), Huffman coding, Arithmetic coding etc.([Gallager, 1978], [Jain, 1989], [Taubman and Marcellin, 2012], [Witten et al., 1987]). Lossy techniques include transform coding methods such as Discrete Cosine Transform (DCT), JPEG, JPEG2000 etc.([Pennebaker and Mitchell, 1992], [Gonzalez and Woods, 2008], [Goyal, 2001]). Polynomial-based compression is another lossy compression method ([Sadeh, 1996],[Eden et al., 1986]). Sajikumar S et al., [Sajikumar and Anilkumar, 2017] introduced a compression scheme using Chebyshev polynomials. The proposed compression algorithm differs from the standard compression algorithms in its low computational complexity and fast reconstruction.

Lossy compression techniques are tested for their performance based on three commonly used measures, the Root Mean Square Error (RMSE), Peak Signal to Noise Ratio (PSNR) and the Compression Ratio (CR). The RMSE between original image $f(x, y)$ and reconstructed image $\hat{f}(x, y)$ of size $M \times N$ is defined by [Joshi, 2018]:

$$\text{RMSE} = \sqrt{\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - \hat{f}(x, y)]^2} \quad (1)$$

For an 8-bit gray level image,

$$\text{PSNR} = 10 \log_{10} \left(\frac{255^2}{\text{MSE}} \right) (dB) \quad (2)$$

$$\text{CR} = \frac{\text{compressed image size}}{\text{uncompressed image size}} \% \quad (3)$$

An image kernel is a matrix used to obtain effects like blurring, sharpening, outlining, etc. Computer vision applications of image kernel mainly include feature extraction and edge detection. A geometric perspective of kernel methods

can be seen in [Lampert, 2009]. Zhang et al., studies various non-local kernel regression for image and video restoration tasks [Zhang et al., 2010]. Odone et al., describes methods for building kernels from binary strings for image matching [Odone et al., 2005]. In 2016, Krishnaprasad P et al., [Krishnaprasad and Ramanujan, 2016] presented an image kernel based on Ramanujan Sums to detect edges. For each 3×3 block of pixels in the image, they multiplied each pixel by the corresponding entry of the 3×3 kernel matrix constructed from $c_3(n)$ and then takes the sum. This sum is considered as a new pixel in the image.

Ramanujan Sums $c_q(n)$ is a family of trigonometric sums defined by Srinivasa Ramanujan in 1918 [Ramanujan, 1918]. In the last fifteen years, Ramanujan Sums have aroused some interest in signal processing. Cohen initially introduced Ramanujan Sums to the signal processing community ([Cohen, 1955], [Cohen, 1958]). In 1950, he observed that the DFT coefficients of even symmetric signals can be computed by integer-valued weighting coefficients. Later it was proved that these integer-valued coefficients are nothing but the well-known Ramanujan Sums.

The rest of the paper is organized as follows. A brief description of the Ramanujan Sum is given in section 2. Image kernel construction and the proposed compression algorithm are given in sections 3 and 4 respectively. Results and discussion are included in section 5 and section 6 concludes the paper.

2 Review of Ramanujan Sums

The Ramanujan Sum $c_q(n)$ has been used by mathematicians to derive many important infinite series expansions for arithmetic functions in number theory [Apostol, 1976]. Interestingly, this sum has many properties which are attractive from the point of view of digital signal processing. Srinivasa Ramanujan defined the q^{th} Ramanujan Sum by

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q W_q^{kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^q W_q^{-kn} \quad (4)$$

where $W_q = e^{-i2\pi/q}$, $i = \sqrt{-1}$ and (k, q) denotes the gcd of k and q . Here the sum runs over those k satisfying $(k, q) = 1$ means that we are considering all the integers which are coprime to q in the summation.

For example, if $q = 8$ then $k \in \{1, 3, 5, 7\}$ so that

$$c_8(n) = e^{i2n\pi/8} + e^{i6n\pi/8} + e^{i10n\pi/8} + e^{i14n\pi/8}$$

In number theory, the number of integers less than or equal to q and coprime to q is called the Euler's totient function $\phi(q)$ Apostol [1976]. Since 1, 3, 5, 7 are coprime to 8, $\phi(8) = 4$.

So the sum given in equation (4) has precisely $\phi(q)$ terms and it is clear that $c_q(0) = \phi(q)$. Also

$$c_q(n+q) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{i2n\pi k/q} \cdot e^{i2\pi k} = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{i2n\pi k/q} = c_q(n)$$

That is $c_q(n)$ is periodic with period q .

If $(k, q) = 1$, we have $(q-k, q) = 1$. Therefore,

$$(W_q^k)^* = W_q^{-k} = W_q^{-(q-k)} = W_q^k$$

where $*$ is the complex conjugate. This implies that the summation (4) is real valued and it can also be written as :

$$c_q(n+q) = \sum_{\substack{k=1 \\ (k,q)=1}}^q \cos \frac{2n\pi k}{q} \quad (5)$$

From (5), $c_q(n) = c_q(-n)$ shows that $c_q(n)$ is symmetric. Thus $c_q(n)$ is a real, symmetric, and periodic sequence in n .

For $0 \leq n \leq q-1$, first few Ramanujan sequences are

$$\begin{aligned} c_1(n) &= 1 \\ c_2(n) &= 1, -1 \\ c_3(n) &= 2, -1, -1 \\ c_4(n) &= 2, 0, -2, 0 \\ c_5(n) &= 4, -1, -1, -1, -1 \end{aligned}$$

Note that $c_q(n)$ is integer-valued and further properties can be seen in [Vaidyanathan, 2014].

3 Image kernel

Krishnaprasad et al., [Krishnaprasad and Ramanujan, 2016], has introduced a kernel matrix M_q of size $q \times q$ constructed from $c_q(n)$ by considering the circular

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shifts of the q elements

$c_q(0), c_q(1), \dots, c_q(q-1)$ in each row. The first row of the kernel matrix contains the q elements in the order $c_q(0), c_q(1), \dots, c_q(q-1)$

$$\text{where } c_q(r) = \sum_{\substack{k=1 \\ (k,q)=1}}^q e^{i2\pi kr/q} \quad \text{for } 0 \leq r \leq q-1.$$

The second-row elements are $c_q(q-1), c_q(0), c_q(1), \dots, c_q(q-2)$ and so on. Thus

$$M_q = \begin{bmatrix} c_q(0) & c_q(1) & c_q(2) & \dots & c_q(q-1) \\ c_q(q-1) & c_q(0) & c_q(1) & \dots & c_q(q-2) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ c_q(1) & c_q(2) & c_q(3) & \dots & c_q(0) \end{bmatrix}$$

4 Proposed method

Partition the input image into non-overlapping blocks of size $q \times q$. Test images of size 256×256 with 8-bit gray levels between 0 and 255 are considered. Multiply each pixel in the $q \times q$ block with the corresponding elements of the kernel M_q and take their sum. This sum is stored for edge detection. Represent the entire block of pixel values with this sum obtained. After the edge detection process, we move on to the compression part. In this step, we are computing the mean and standard deviation of each block of pixels to obtain the texture at decompression. Two different ways of compressing an image with the statistical measures of pixel values are proposed.

A. Method 1

For each $q \times q$ block, a block value is computed by adding the kernel multiplication sum which is obtained at the edge detection stage, the mean and standard deviation of each block of pixels. Represent the entire block with this sum at the reconstruction stage. By varying the block size we can compress the image at different compression levels.

B. Method 2

Instead of taking the sum of three quantities to represent each block, consider the kernel sum and the mean value only. Thus we need to store only two values in place of q^2 pixels and hence high compression is achieved.

Experimental results with the test images are given in Tables 1-3 and Figures 1-2. In both methods, no quantization or postprocessing is done at the reconstruction step.

Algorithm

Step 1 Load an input gray image.

Step 2 Partition the image matrix into non-overlapping blocks of size $q \times q$.

Step 3 For each block compute the elementwise product sum with the kernel matrix and the pixel values of the $q \times q$ block. Also find the mean and standard deviation of the q^2 pixels. Store these values for the reconstruction of each block.

Step 4 Replace the q^2 gray values by the elementwise product sum computed in *Step 3* to detect edges.

Step 5 Replace the q^2 gray values by the sum of the three quantities stored in *Step 3* to compress the image.

OR

Replace the q^2 gray values by the sum of the elementwise product sum and the mean to achieve high compression.

5 Results and Discussion

Edge detection results for the test images of different block sizes 2×2 , 3×3 , 4×4 are given in Figure 1. Edge detection with 2×2 blocks shows better results as compared to others.

In the first method, we replace q^2 pixel values with three quantities. Hence in 2×2 , 3×3 , and 4×4 blocks compression ratios are 75%, 33.33%, and 18.75% respectively. But in the second method, we need to store only two quantities instead of q^2 values in each block. Thus the compression ratios are 50% for 2×2 blocks, 22.22% for 3×3 blocks, and 12.5% for 4×4 blocks. From Figure 2, and Tables 1-3 we can conclude that method 2 shows better CR with reasonable reconstructed image quality measures PSNR and RMSE. The test image Rice achieves an appreciable PSNR 30.3117(dB) with RMSE 7.7796 in the case of 2×2 blocks. As the quality measures PSNR decreases and RMSE increases with an increase of the block size, the algorithm works better with smaller blocks.

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Experimental results shows that PSNR values are greater than 22(dB) and RMSE is less than 20 for different test images when we apply the second method of compression. In practical applications this is an acceptable range at the CR 50%. Also, as the kernel operation doesn't involve usual convolution product at the edge detection stage the number of additions and multiplications required is reduced and hence saves a lot of computation time.

Test Image	Bolck Size	Block Value	PSNR(dB)	RMSE
Lena	2×2	method 1	22.8412	18.3857
	2×2	method 2	24.8995	14.5066
Cameraman	2×2	method 1	20.7047	23.5127
	2×2	method 2	22.6443	18.8073
Aerial	2×2	method 1	19.9407	25.6747
	2×2	method 2	22.3203	19.5221
Rice	2×2	method 1	27.7290	10.4734
	2×2	method 2	30.3117	7.7796

Table 1: Compression quality measures with 2×2 blocks using methods 1& 2

Test Image	Bolck Size	Block Value	PSNR(dB)	RMSE
Lena	3×3	method 1	16.9428	36.2579
	3×3	method 2	18.0869	31.7830
Cameraman	3×3	method 1	14.9823	45.4389
	3×3	method 2	15.8706	41.0294
Aerial	3×3	method 1	14.3263	49.0033
	3×3	method 2	15.1876	44.3774
Rice	3×3	method 1	21.5605	21.3067
	3×3	method 2	23.6949	16.7801

Table 2: Compression quality measures with 3×3 blocks using methods 1& 2

Test Image	Bolck Size	Block Value	PSNR(dB)	RMSE
Lena	4×4	method 1	12.5169	60.3522
	4×4	method 2	12.8137	58.3253
Cameraman	4×4	method 1	10.5412	75.7666
	4×4	method 2	10.9338	72.4182
Aerial	4×4	method 1	8.7710	92.8943
	4×4	method 2	9.0363	90.0997
Rice	4×4	method 1	16.8793	36.5237
	4×4	method 2	17.5469	33.8217

Table 3: Compression quality measures with 4×4 blocks using methods 1& 2

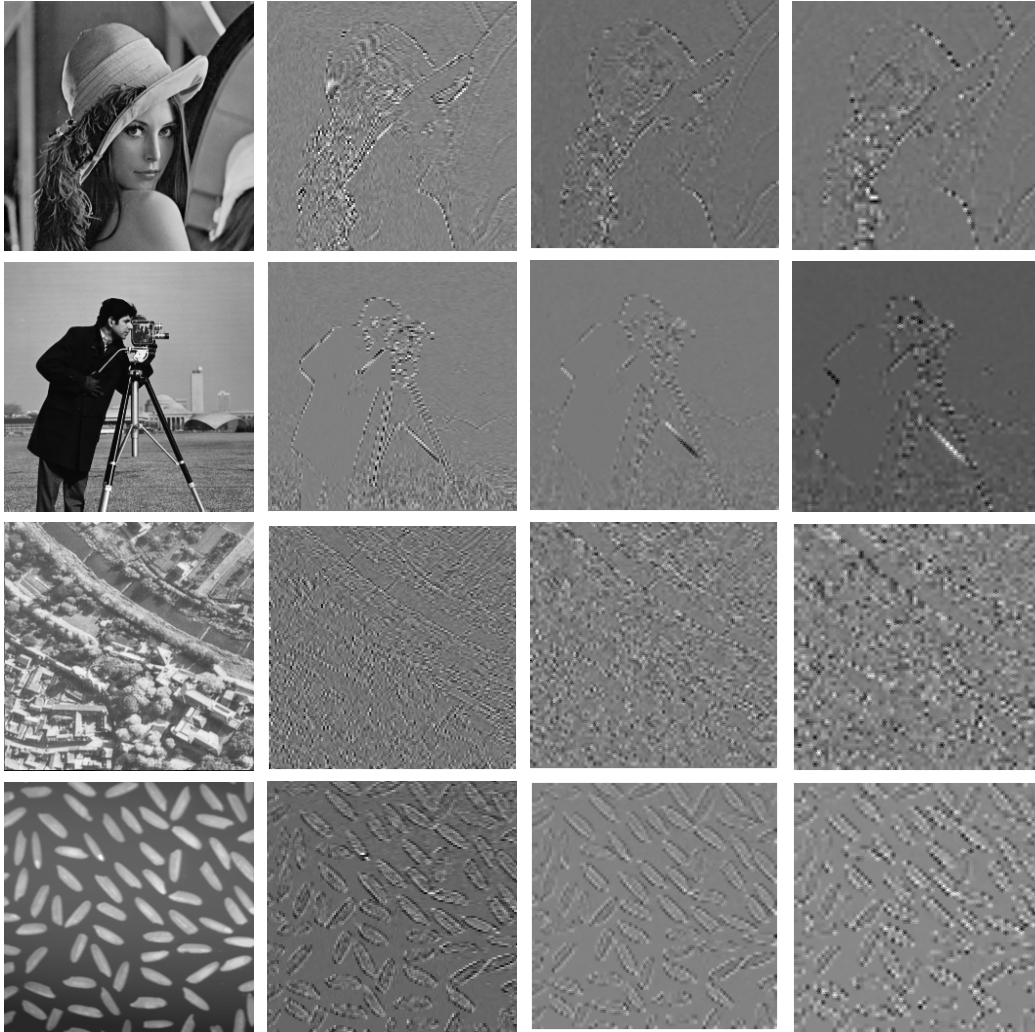


Figure 1: **The first column:** orinal images; **the second column:** edges by M_2 ; **the third column:** edges by M_3 ; **the fourth column:** edges by M_4 .

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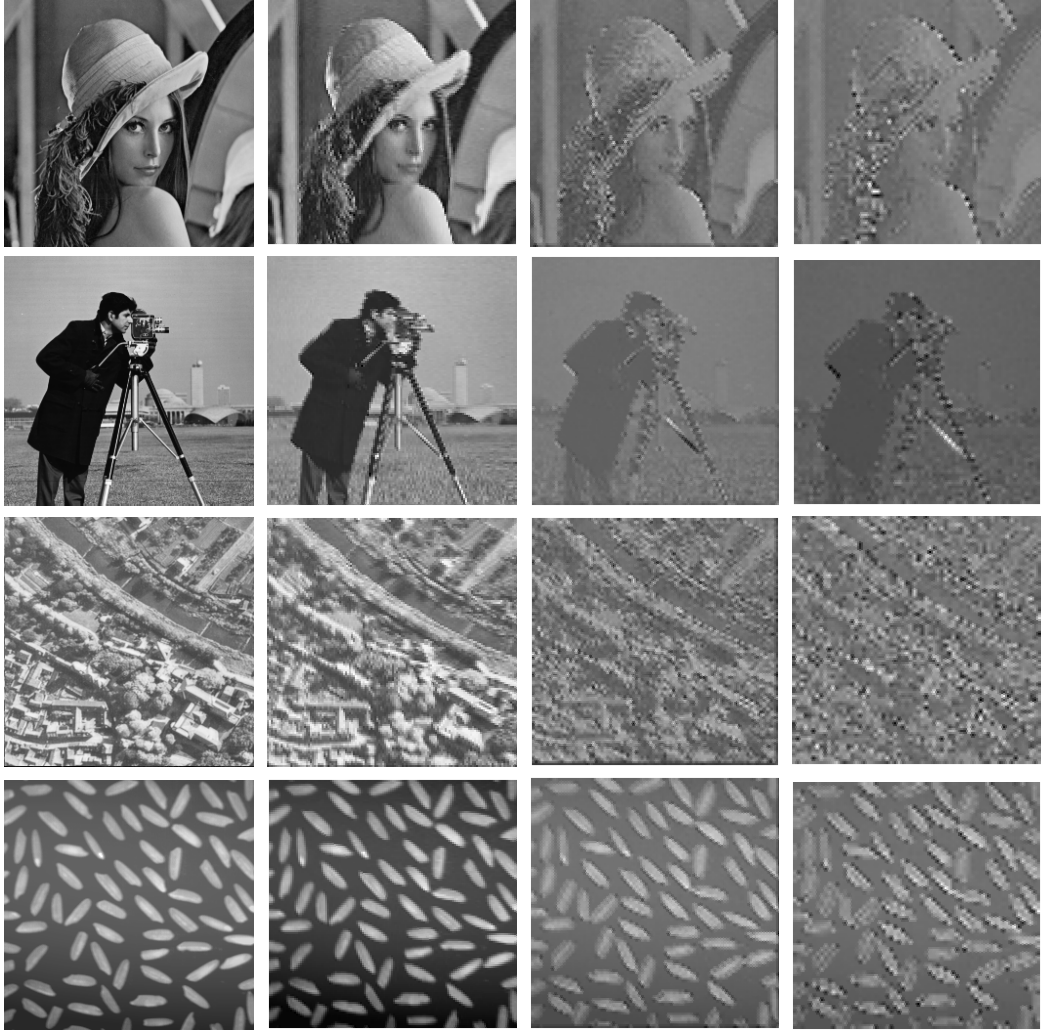


Figure 2: **The first column:** orinal images; **the second column:** reconstructed image using method 2 at the CR 50%; **the third column:** reconstructed image using method 2 at the CR 22.22%; **the fourth column:** reconstructed image using method 2 at the CR 12.5%.

6 Conclusions

In this paper, we presented an edge detection and compression algorithm based on Ramanujan Sums and measures of central tendency and dispersion such as mean and standard deviation. The edge detection algorithm using kernels constructed from Ramanujan Sums has been extended to a compression algorithm. Here the compression is achieved by replacing each block of pixels with a single

value obtained by adding edges with texture. The advantage of this method is its low computational complexity and fast reconstruction.

References

- Tom M Apostol. Analytic number theory. 1976.
- Eckford Cohen. A class of arithmetical functions. *Proceedings of the National Academy of Sciences of the United States of America*, 41(11):939, 1955.
- Eckford Cohen. Representations of even functions (mod r), arithmetical identities. *Duke Mathematical Journal*, 25(3):401–421, 1958.
- Murray Eden, Michael Unser, and Riccardo Leonardi. Polynomial representation of pictures. *Signal Processing*, 10(4):385–393, 1986.
- Robert Gallager. Variations on a theme by huffman. *IEEE Transactions on Information Theory*, 24(6):668–674, 1978.
- Rafael C Gonzalez and Richard E Woods. Digital image processing. *Nueva Jersey*, 2008.
- Vivek K Goyal. Theoretical foundations of transform coding. *IEEE Signal Processing Magazine*, 18(5):9–21, 2001.
- Anil K Jain. *Fundamentals of digital image processing*. Prentice-Hall, Inc., 1989.
- Madhuri A Joshi. *Digital image processing: An algorithmic approach*. PHI Learning Pvt. Ltd., 2018.
- P Krishnaprasad and Ajeesh Ramanujan. Ramanujan sums based image kernels for computer vision. In *2016 International Conference on Electrical, Electronics, and Optimization Techniques (ICEEOT)*, pages 835–839. IEEE, 2016.
- Christoph H Lampert. *Kernel methods in computer vision*. Now Publishers Inc, 2009.
- Francesca Odone, Annalisa Barla, and Alessandro Verri. Building kernels from binary strings for image matching. *IEEE Transactions on Image Processing*, 14(2):169–180, 2005.
- William B Pennebaker and Joan L Mitchell. *JPEG: Still image data compression standard*. Springer Science & Business Media, 1992.

An image compression method based on Ramanujan Sums and measures of central dispersion

- Srinivasa Ramanujan. On certain trigonometrical sums and their applications in the theory of numbers. *Trans. Cambridge Philos. Soc*, 22(13):259–276, 1918.
- I Sadeh. Polynomial approximation of images. *Computers & Mathematics with Applications*, 32(5):99–115, 1996.
- S Sajikumar and AK Anilkumar. Image compression using chebyshev polynomial surface fit. *Int. J. Pure Appl. Math. Sci*, 10:15–27, 2017.
- David Salomon. *A concise introduction to data compression*. Springer Science & Business Media, 2007.
- Khalid Sayood. *Introduction to data compression*. Newnes, 2012.
- David Taubman and Michael Marcellin. *JPEG2000 Image Compression Fundamentals, Standards and Practice: Image Compression Fundamentals, Standards and Practice*, volume 642. Springer Science & Business Media, 2012.
- PP Vaidyanathan. Ramanujan sums in the context of signal processing—part i: Fundamentals. *IEEE transactions on signal processing*, 62(16):4145–4157, 2014.
- Ian H Witten, Radford M Neal, and John G Cleary. Arithmetic coding for data compression. *Communications of the ACM*, 30(6):520–540, 1987.
- Haichao Zhang, Jianchao Yang, Yanning Zhang, and Thomas S Huang. Non-local kernel regression for image and video restoration. In *European Conference on Computer Vision*, pages 566–579. Springer, 2010.

Estimation of fuzzy metric spaces from metric spaces

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Abstract

In this paper we estimate the new way for analyzing the fuzzy metric spaces from metric spaces using fuzzy fixed point theorem and vice versa. We derive some definitions and theorems for analyzing the metric spaces with new structure of fuzzy metric space using fixed point theorems. Also, we have given new examples for fuzzy metric spaces using fixed point theorem.

Keywords: metric space, fuzzy logic, fixed point theorem.

2020 AMS subject classifications: 54E35, 03B52, 47H10

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Received on October 10, 2021. Accepted on December 15, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.670. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

In earlier of 1906, Maurice Freechet provided the concept of metric spaces. In 1922, Banach developed a reliable result called Banach contraction principle on the basis of fixed point theory [15]. Later in 1965, Zadeh [10] defined the way of fuzzy set in metric spaces. This fuzzy concept used in various field of engineering, science and technology. This fuzzy concept is used to analyze the complex state as to easy by simple condition and conversion. Then several mathematicians gave their various concept towards the concept of fuzzy metric space Erceg [4] [5], Diamond and Kolden [7], George & Veeramani [6], Gregori and Romaguera [11]. The analysis of fuzzy metric spaces was introduced in various way and its topologies developed by many researchers. On the line of this, we frame a new structure to analyze the fuzzy metric spaces from metric spaces using fuzzy fixed point theorem [13] [14]. In this paper, we discuss and give various definition and theorem to estimate the fuzzy metric space from metric space and converse also using fixed point theorem[1][9][12][16].

2. Prefatory

In this part, we look back on some basic concepts and results in both metric and fuzzy based metric spaces.

Definition 2.1: [2] A metric space is given by a set X and a distance function $\bar{d} : X \times X \rightarrow \mathbb{R}$ defined on X such that $a, b, c \in X$

$$(i) \bar{d}(a, b) \geq 0, \bar{d}(a, b) = 0 \leftrightarrow a = b$$

$$(ii) \bar{d}(a, b) = \bar{d}(b, a)$$

$$(iii) \bar{d}(a, c) \leq \bar{d}(a, b) + \bar{d}(b, c)$$

Definition 2.2: [10] A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 2.3: [8] A binary operation $* : [0,1]^2 \rightarrow [0,1]$ is called a continuous triangular norm (called t- norm) if it satisfies the following conditions:

$$(i) \quad * \text{ is associative and commutative,}$$

$$(ii) \quad * \text{ is continuous,}$$

$$(iii) \quad p * 1 = p \text{ for all } p, q, r \in [0,1],$$

$$(iv) \quad p * q \leq r * t \text{ whenever } p \leq r \text{ and } q \leq t \text{ for all } p, q, r, t \in [0,1]$$

Examples of t-norm are $p * q = pq$, $p * q = \min\{p, q\}$ and $p * q = \max\{p, q\}$.

Definition 2.4: [3, 6] The 3-tuple $(X, F_{\bar{d}}, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and $F_{\bar{d}}$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for all $a, b, c \in X$, each t and $u > 0$

- (i) $F_{\bar{d}}(a, b, t) > 0$
- (ii) $F_{\bar{d}}(a, b, t) = 0$ if and only if $a = b$,
- (iii) $F_{\bar{d}}(a, b, t) * F_{\bar{d}}(b, a, t)$,
- (iv) $F_{\bar{d}}(a, b, t) * F_{\bar{d}}(b, c, u) \leq F_{\bar{d}}(a, c, t + u)$,
- (v) $F_{\bar{d}}(a, b, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.

Then is $F_{\bar{d}}$ called a fuzzy metric on X . Then $F_{\bar{d}}(a, b, t)$ denotes the degree of nearness between a and b with respect to t .

Lemma 2.5: $(X, F_{\bar{d}}, *)$ is non-decreasing for all $a, b \in X$.

3. Main Results

The main aim of this paper is to estimate the fuzzy metric spaces from any ordinary metric spaces and converse also and justify the Banach fixed point.

Theorem 3.1: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively, so the following diagram

$$\begin{array}{ccc} X \times X \times \mathcal{R}^+ & \xrightarrow{F_{\bar{d}}} & I \\ \text{dpr} \downarrow & & \uparrow \beta \\ X \times X & \xrightarrow{\bar{d}} & \mathcal{R}^+ \end{array}$$

Figure (1): commutative diagram

is commutative. Where, $\bar{d}_{xy}: (a, b, t) \rightarrow (ta, tb)$, $\bar{d}(ta, tb) \rightarrow ty$ for some metric $\bar{d}(a, b) = y > 0$ and $\beta: (ty) \rightarrow 1 - \sin(ty) =: \hat{t} \in I$. Further more $F_{\bar{d}} = \beta \circ \bar{d} \circ \bar{d}_{xy}$.

Proof: According to the below equation it is very easy to check that β is continuous.

$\sin^{-1}(ty)$ in $\mathcal{R}^+$ is continuous $\Rightarrow \beta$ is continuous.

Now we justify that $\bar{d} \circ \bar{d}_{xy} = \beta^{-1} \circ F_{\bar{d}}$. For $(a, b, c) \in X \times X \times \mathcal{R}^+$, we have,

$$\bar{d} \circ \bar{d}_{xy}(a, b, t) = \bar{d}(ta, tab) = ty := q > 0.$$

On the other hand,

$$\beta^{-1} \circ F_{\bar{d}}(a, b, t) = \beta^{-1}(\hat{t}) = \beta^{-1}(1 - \sin ty) = \sin^{-1}\{1 - (1 - \sin ty)\} = ty$$

Therefore, the above diagram (Figure 1) is commutative.

Lemma 3.2:

Let $t_1, t_2 \in \mathcal{R}^+$, if $t_1 \leq t_2$, then $\beta(t_1) \geq \beta(t_2)$ and $\beta(t_1 + t_2) \geq \max \{ \beta(t_1), \beta(t_2) \}$.

Proof:

If $t_1 \leq t_2$, this $\Rightarrow \sin t_1 \leq \sin t_2 \Rightarrow -\sin t_1 \geq -\sin t_2 \Rightarrow 1 - \sin t_1 \geq 1 - \sin t_2$

Hence, $\beta(t_1) \geq \beta(t_2)$. Now, the second declaration is clear.

Theorem 3.3: Let $(X, \bar{d})$ be the metric space and $(X, F_{\bar{d}}, *)$ is a fuzzy metric space with $p * q = \max \{p, q\}$ for all $p, q \in I$. Then for all $a, b, c \in X, t, q, y \in \mathcal{R}^+$, we have $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$ is a fuzzy metric space.

Proof:

We verify the fuzzy metric space conditions (i), (ii), (iii) from the above definition (2.4),

For (i) $\beta \circ \bar{d} \circ \bar{d}_{xy}(a, b, t) = \beta \circ \bar{d}(ta, tb) = \beta \{ \bar{d}(ta, tb) \} = \hat{t} > 0$.

For (ii) $\beta \circ \bar{d} \circ \bar{d}_{xy}(a, a, t) = \beta \circ \bar{d}(ta, ta) = \beta \{ \bar{d}(ta, ta) \} = \beta(0) = 1$

For (iii) $\beta \circ \bar{d} \circ \bar{d}_{xy}(a, b, t) = \beta \circ \bar{d}(ta, tb) = \beta \{ \bar{d}(ta, tb) \} = \beta(ty) = \hat{t}$.

Another side,

$$\beta \circ \bar{d} \circ \bar{d}_{xy}(b, a, t) = \beta \circ \bar{d}(tb, ta) = \beta \{ \bar{d}(tb, ta) \} = \beta(ty) = \hat{t} > 0$$

Therefore, symmetric condition satisfied.

Next, we check the condition (iv)

$$\begin{aligned} \beta \circ \bar{d} \circ \bar{d}_{xy}(a, c, t + u) &= \beta \circ \bar{d} \{ (t + u)a, (t + u)c \} \\ &= \beta \{ \bar{d}(t + u)a, (t + u)c \} \\ &= \beta \{ \bar{d}(t + u)r \} \\ &= \beta(tr + ur) \end{aligned}$$

Using the above lemma, $\beta(tr + ur) \geq \max \{ \beta(tr), \beta(ur) \}$
 $= \beta(tr) * \beta(ur)$

$$\begin{aligned} &= \beta \{ \bar{d}(ta, tb) \} * \beta \{ \bar{d}(ua, uc) \} \\ &= \beta \circ \bar{d} \circ \bar{d}_{xy}(a, b, t) * \beta \circ \bar{d} \circ \bar{d}_{xy}(b, a, u) \end{aligned}$$

For (v) is trivially true. Therefore, $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$ is a fuzzy metric space.

Comments:

On the other hand we reach the metric space from the fuzzy metric space basis on the above commutative diagram (Figure 1). So, if $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$ is a fuzzy metric space, then the associative metric is $(X, \beta^{-1} \circ F_{\bar{d}} \circ \bar{d}_{xy}^{-1})$.

Definition 3.4: Let $(X, \bar{d})$ be a metric space on X , and $\{a_n\}$ be a sequence in X the n is $\{a_n\}$ called converge Sequence to some fixed $a \in X$ if $\varepsilon > 0, N \in \mathbb{N}$, $\bar{d}(a_n, a) < \varepsilon$ for all $n > N$

We represent $a_n \rightarrow a$ if $\{a_n\}$ converge to a ; and $\{a_n\}$ is called a Cauchy

sequence. $\bar{d}(a_n, a_m) < \epsilon$ for all $n, m > N$

Definition 3.5: Let $(X, \bar{d})$ and $(X, F_{\bar{d}}, *)$ are metric and fuzzy metric space on X , respectively. And $\{a_n\}$ is a sequence in X then the following is equivalent.

- (i) $\{a_n\}$ is convergent in the metric space $(X, \bar{d})$
- (ii) $\bar{d}(a_n, a) < \epsilon$ for all $n > N$
- (iii) $\{a_n\}$ is convergent in the fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$
- (iv) For any $0 < \epsilon < 1$ and $t > 0$ there exists $n > N$ such that $\beta \circ \bar{d} \circ \bar{d}_{xy}(a_n, a, t) > 1 - \epsilon$

Definition 3.6: A metric space $(X, \bar{d})$ is complete if every Cauchy sequence in X is convergent.

Definition 3.7: A fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$ is complete if and only if $(X, \bar{d})$ is complete.

In the below mentioned theorem we prove that if any self map has fixed point theorems in the metric space, then we induced fuzzy metric space for the same point and vice versa. We point to Mihet & Shen et al. [5] for fixed point theorems in the fuzzy metric spaces.

Theorem 3.8: Let $(X, \bar{d})$ be a complete metric space on X , suppose the mapping $S: X \rightarrow X$ satisfy the contractive condition, thus $\bar{d}(Sa, Sb) < k \bar{d}(a, b)$, for all $a, b \in X, k \in [0, 1)$ is a constant. If T has a unique fixed point in X with respect to the metric $(X, \bar{d})$, then T has a unique fixed point with respect to the induced fuzzy metric $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$.

Proof: Suppose that T has a unique fixed point in X with respect to the metric space $(X, \bar{d})$. So, we have $\bar{d}(Sa, a) = 0$ for some x .

$$\begin{aligned} \text{Therefore} \quad F_{\bar{d}}(Sa, a, t) &= \beta \circ \bar{d} \circ \bar{d}_{xy}(Sa, a, t) \\ &= \beta \circ \{s(Sa), s(Sb)\} \\ &= \beta(ty) = \beta(0) = 1 \end{aligned}$$

This represents that $Sa = a$ with respect to the fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$

If there is another fixed point $b \in X$ then,

$$\begin{aligned} F_{\bar{d}}(a, b, t) &= \beta \circ \bar{d} \circ \bar{d}_{xy}(Sa, Sb, t) \\ &= \beta \circ \{s(Sa), s(Sb)\} \\ &= \beta(ty) = \beta(0) = 1 \end{aligned}$$

and therefore, $a = b$.

Theorem 3.9: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively and commutative.

Where, $\bar{d}_{xy}: (a, b, t) \rightarrow (ta, tb), \bar{d}(ta, tb) \rightarrow ty$ for some metric $\bar{d}(a, b) = y > 0$ and $\beta: (ty) \rightarrow 1 - \cos(ty) =: \hat{t} \in I$. Further more $F_{\bar{d}} = \beta \circ \bar{d} \circ \bar{d}_{xy}$.

Proof for this theorem is similar to the proof of theorem 3.1.

Example 3.10: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively.
 $\bar{d}_{xy}: (a, b, t) \rightarrow (ta, tb), \bar{d}(ta, tb) \rightarrow ty$ for some metric $\bar{d}(a, b) = y > 0$ and
 $\beta: (ty) \rightarrow 1 - \sin^2(ty) =: \hat{t} \in I$. Furthermore $F_{\bar{d}} = \beta \circ \bar{d} \circ \bar{d}_{xy}$. And we prove
it is commutative.

Solution: According to the below equation it is very easy to check that β is
continuous.

$$\sin^2(ty) \text{ in } \mathcal{R}^+ \text{ is continuous} \Rightarrow \beta \text{ is continuous.}$$

Now we justify that $\bar{d} \circ \bar{d}_{xy} = \beta^{-1} \circ F_{\bar{d}}$. For $(a, b, c) \in X \times X \times \mathcal{R}^+$, we have,
 $\bar{d} \circ \bar{d}_{xy}(a, b, t) = \bar{d}(ta, tab) = ty := q > 0$.

On the other hand,

$$\begin{aligned} \beta^{-1} \circ F_{\bar{d}}(a, b, t) &= \beta^{-1}(\hat{t}) \\ &= \beta^{-1}(1 - \sin^2(ty)) = \sin^{-1}\{\sqrt{1 - (1 - \sin^2(ty))}\} = ty \end{aligned}$$

Therefore, it is commutative.

4. Conclusion

In this article we design a new structure to estimate the fuzzy metric space with the help of metric space using fuzzy fixed point theorems and converse also. We discussed the various definitions and provided the proof for the same definitions with new structure using fuzzy fixed point theorems. We have also given some examples which satisfies the condition of our new structure basis on fixed point theorem. This paper makes away to analysis the applications of fuzzy metric space in various field of engineering.

References

- [1] Saleh Omran, H.S. Al-Saadi (2017), *Some notes on metric and fuzzy metric spaces*, International Journal of Advanced and Applied Sciences, 4(5); 41-43.
- [2] Fréchet MM (1906). *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo (1884-1940), 22(1); 1-72.
- [3] George A and Veeramani P (1994), *On some results of fuzzy metric spaces*, Fuzzy Sets and Systems, 64(3); 395-399.
- [4] Erceg MA (1979), *Metric spaces in fuzzy set theory*, Journal of Mathematical Analysis and Applications, 69(1); 205-230.

- [5] Mihet D (2004), A *Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets and Systems, 144(3); 431-439.
- [6] Valentin Gregori, Samuel Morillas, Almanzor Sapena (2011), *Examples of fuzzy metrics and applications*, Fuzzy Sets and Systems 170; 95-111.
- [7] Diamond P and Kloden P (1999), *Metric space so fuzzy sets*, Fuzzy Sets and Systems, 35(2):241-249.
- [8] Schweizer B and Skala A (1960), *Statistical metric spaces*, The Pacific Journal of Mathematics, 10 (1); 314-334.
- [9] Dhamodharan D, Nihal Tas and Krishnakumar R (2019), *Common Fixed-Point Theorems Satisfying Implicit Relations on 2-cone Banach Space with an Application*, Mathematical Sciences and Applications E-Notes, 7 (1); 9-18.
- [10] Zadeh LA (1965), *Fuzzy sets*, Information and Control, 8(3); 338-353.
- [11] Gregori V and Romaguera S (2000), *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, 115(3):485-489.
- [12] Shen Y, Qiu D, and Chen W (2012), *Fixed point theorems in fuzzy metric spaces*, Applied Mathematics Letters, 25(2); 138-141.
- [13] George A and Veeramani P (1997), *On Some Results of Analysis for Fuzzy Metric Spaces*, Fuzzy Sets and Systems (90) 365-368.
- [14] Karmosil O, Michalek J (1975), *Fuzzy metric and Statistical Metric Spaces*, Kybernetika 11, 326-334.
- [15] Banach S (1922), *sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae, Vol.3, pp.133-181
- [16] Krishnakumar R, Damodharan D (2016), *Fixed point theorems in normal cone metric spaces*, International Journal of Math.Sci & Engg.appls.10, 213-224.

Blocks within the period of Lucas sequence

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Abstract

In this paper, we consider the periodic nature of the sequence of Lucas numbers L_n defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$; for all $n \geq 2$; with initial condition $L_0 = 2$ and $L_1 = 1$. For any modulo $m > 1$, we introduce the ‘blocks’ within this sequence by observing the distribution of residues within a single period of Lucas sequence. We show that length of any one period of the Lucas sequence contains either 1, 2 or 4 blocks.

Keywords: Fibonacci sequence, Lucas sequence, Periodicity of Lucas sequence

2010 AMS subject classification[‡]: 11B37, 11B39, 11B50

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[‡] Received on September 21, 2021. Accepted on December 5, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.664. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

The Fibonacci sequence and the Lucas sequence are well-known sequences among all the integer sequences. The Fibonacci sequence $\{F_n\}$ satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with the initial conditions $F_0 = 1$ and $F_1 = 1$. Lucas sequence $\{L_n\}$ is considered as the ‘twin sequence’ of Fibonacci sequence which satisfies the similar recursive relation $L_n = L_{n-1} + L_{n-2}$, with the initial conditions $L_0 = 2$ and $L_1 = 1$.

On the other hand, some researchers have conducted important research on the period of these two recursive sequences [1, 3, 4, 5, 6]. Wall [1] defines the length of period of the Fibonacci sequence by reducing it through the modulo any positive integer $m > 1$. Kramer and Hoggatt Jr. [3] also defined the length of the period of the Lucas sequence obtained by reducing the sequence through modulo any positive integer $m > 1$.

In this paper, we take deep insight in to the periodic nature of Lucas sequence and introduce the concept of ‘blocks’ by observing the distribution of residues within a single period of Lucas sequence when considered modulo any positive integer $m > 1$.

We denote the sequence of least non-negative residues of the terms of $\{L_n\}$ taken modulo m ($m \geq 2$) by $L(\text{mod } m)$. If we examine the sequence of final digits of $\{L_n\}$, then we notice an interesting pattern that the sequence $L(\text{mod } 10) = \{2, 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, \dots\}$ repeats after the 12 terms again and again all the way. For any modulo m , it is easy to observe that the sequence $\{L_n\}$ is always periodic and it repeats from starting values 0 and 1. By $k_L = k_L(m)$, we mean the lengths of the period of $\{L_n\}$ modulo any positive integer m . These leads us to the following easy consequences.

Lemma 1.1. (a) $L_{k_L(m)-1} \equiv -1(\text{mod } m)$ (b) $L_{k_L(m)} \equiv 2(\text{mod } m)$
 (c) $L_{k_L(m)+1} \equiv 1(\text{mod } m)$ (d) $L_{k_L(m)+2} \equiv 3(\text{mod } m)$
 (e) $L_{k_L(m)+nr} \equiv L_n(\text{mod } m)$, for all $r \in \mathbb{Z}$.

The following is an important result which speaks about the divisibility property of $k_L(m)$.

Fact 1.2. For any $m > 1$, since $L(\text{mod } m)$ is always periodic, we conclude that if $L_n \equiv 2(\text{mod } m)$ and $L_{n+1} \equiv 1(\text{mod } m)$, then $k_L(m) \mid n$.

2. Blocks within $L(\text{mod } m)$

In this article, we restrict our attention to the behavior of the blocks within the residues for a given modulus and consequently some interesting relationships will be derived.

Definition 2.1. By $\alpha(m)$ we mean the smallest positive value of the index n of Lucas numbers such that $L_n \equiv 2L_{n+1}(\text{mod } m)$, when $n > 1$. We call $\alpha(m)$ the *restricted period* of $L(\text{mod } m)$.

Equivalently, $\alpha(m)$ is the position of the first repeated term in the sequence $L(\text{mod } m)$. Thus, $L_{\alpha(m)} \equiv 2L_{\alpha(m)+1}$ when considered $(\text{mod } m)$.

As an illustration, if we consider $L(\text{mod } 3) = \{2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1 \dots\}$, then it is apparent that $L_4 \equiv 1(\text{mod } 3)$ and $L_5 \equiv 2(\text{mod } 3)$. Thus $L_4 \equiv 2L_5(\text{mod } 3)$, which gives $\alpha(3) = 4$.

We call the finite sequence $L_0, L_1, \dots, L_{\alpha(m)-1}$ to be the *first block* occurring in $L(\text{mod } m)$. It may happen that $\alpha(m) = k_L(m)$. In such case, we call $L(\text{mod } m)$ to be without restricted period.

For $L(\text{mod } 4) = \{2, 1, 3, 0, 3, 3, 2, 1, 3 \dots\}$, clearly $\alpha(4) = k_L(4) = 6$, and thus $L(\text{mod } 4)$ has no restricted period.

Definition 2.2. By $s(m)$ we mean the second positive residue ' t ', which appears after the first block in $L(\text{mod } m)$.

This clearly means that $2s(m) \equiv L_{\alpha(m)}(\text{mod } m)$; $s(m) = L_{\alpha(m)+1}$. Using the definition of L_n , we now conclude that $L_{\alpha(m)+2} = 3s(m)$, $L_{\alpha(m)+3} = 4s(m)$, $L_{\alpha(m)+4} = 7s(m)$, ... Also the first block ends with $m - s(m)$. Thus,

$$(L_{\alpha(m)}, L_{\alpha(m)+1}, L_{\alpha(m)+2}, L_{\alpha(m)+3}, \dots) = s(m)(2, 1, 3, 4, 7, \dots)(\text{mod } m).$$

This implies that the successive terms in $L(\text{mod } m)$ after the first block are the multiples of $s(m)$. We therefore call $s(m)$ to be a *multiplier*.

Again, in the sequence $L(\text{mod } m)$, the blocks are of the form $2, 1, \dots, m - s(m), 2s(m), s(m), \dots, m - x, x, x, \dots$; where $2, 1, \dots, m - s(m)$ is the first block, $2s(m), s(m), \dots, m - x$ is the second block, and so on. The occurrence of $3 - 2m, m - 1$ in $L(\text{mod } m)$ will indicate that the end of the period has been reached and there after repetition begins, since the next two terms will be $2, 1$. Here we note that each block contains the same (that is $\alpha(m)$) number of terms and the subscripts are in arithmetic progression. Thus, $L_{\alpha(m)u} \equiv$

$2L_{\alpha(m)u+1}(\text{mod } m)$, for each positive integer u . Since $L_{k_L(m)} \equiv 2L_{k_L(m)+1}(\text{mod } m)$, we conclude that $\alpha(m)u = k_L(m)$, where u is a positive integer, which implies that $\alpha(m) \mid k_L(m)$. Later in the paper we show that the value of u is either 1 or 2 or 4.

Definition 2.3. By $\beta(m)$ we mean the order of $s(m)$, when considered modulo m . That is, $s(m)^{\beta(m)} \equiv 1 (\text{mod } m)$ and if $n < \beta(m)$ then $s(m)^n \not\equiv 1 (\text{mod } m)$.

To illustrate above definitions, we consider the following three examples:

- (1) For $L(\text{mod } 4) = \{2, 1, 3, 0, 3, 3, 2, 1, \dots\}$, clearly $k_L(4) = 6$. Also, the restricted period $\alpha(4)$ is 6 and multiplier $s(4)$ is 1. Thus, the order of $s(4) = 1$ is 1 and hence $\beta(4) = 1$.
- (2) If we consider $L(\text{mod } 6) = \{2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, \dots\}$, then clearly $k_L(6)$ is 24, $\alpha(6)$ is 12 and $s(m)$ is 5. Since $5^2 \equiv 1 (\text{mod } 6)$, we get $\beta(11) = 2$.
- (3) If we consider $L(\text{mod } 13) = 2, 1, 3, 4, 7, 11, 18, 3, 8, 11, 6, 4, 10, 1, 11, 12, 10, 9, 6, 2, 8, 10, 5, 2, 7, 9, 3, 12, 2, 1, 3, 4, \dots$, then $k_L(13) = 28$, $\alpha(13) = 7$ and $s(m) = 8$. Since $8^4 \equiv 1 (\text{mod } 13)$, we have $\beta(13) = 4$.

The following theorem ties together the three functions $k_L(m)$, $\alpha(m)$ and $\beta(m)$.

Theorem 2.4. $k_L(m) = \alpha(m) \times \beta(m)$.

Proof: We first divide the single period of $L(\text{mod } m)$ into smaller finite subsequences, say $R_0, R_1, R_2, \dots, R_n, \dots$ as shown below:

$$\begin{array}{c} \overbrace{2, 1, \dots, 3s_1 - m, m - s_1}^{R_0}, \overbrace{2s_1, s_1, \dots, 3s_2 - m, m - s_2}^{R_1}, \overbrace{2s_2, s_2, \dots, 3s_3 - m, m - s_3}^{R_2}, \dots \\ \overbrace{\dots, 2s_n, s_n, \dots, 3 - m, m - 1}^{R_n}, \overbrace{2, 1, \dots, 3s_1 - m, m - s_1}^{R_{n+1}}, \dots, \end{array} \quad (2.1)$$

where $s_1 = s(m)$.

Obviously each finite subsequence ' R_i ' has $\alpha(m)$ terms and it contains exactly one block. Hence every subsequence $R_i (i \geq 1)$ is a multiple of ' R_0 '. Therefore, we have the following congruences modulo m :

$R_1 = s_1 R_0$; $R_2 = s_2 R_0$; $R_3 = s_3 R_0$; $\dots$; $R_{n-1} = s_{n-1} R_0$; $R_n = s_n R_0$. Since the first term of R_1 is $m - s_2$ and that of R_0 is $m - s_1$ and we also have $R_1 = s_1 R_0$, we get $m - s_2 = s_1(m - s_1)$. If we consider the modulo m , we get $s_2 = s_1 \times s_1$. According to similar arguments, when considering the modulo

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m , we have $s_3 = s_2 \times s_1, s_4 = s_3 \times s_1, s_5 = s_4 \times s_1, \dots, s_n = s_{n-1} \times s_1$. Therefore, we have

$$\begin{aligned} s_n &= s_{n-1} \times s_1 \\ &= (s_{n-2} \times s_1) \times s_1 \\ &= (s_{n-3} \times s_1) \times s_1 \times s_1 \\ &\quad \vdots \\ &= (s_{n-(n-1)} \times s_1) \times \overbrace{s_1 \times s_1 \times \dots \times s_1}^{(n-2) \text{ times}} \end{aligned}$$

Thus, $s_n = s_1^n$.

Now since the order of s_1 is $\beta(m)$, we can write single period of $L(mod\ m)$ as follows:

$$2, 1, 3, 4, 7, \dots, 3s_1 - m, m - s_1, s_1, \dots, 3s_1^2 - m, m - s_1^2, s_1^2, \dots, 3s_1^3 - m, m - s_1^3, \dots, 3 - m, m - 1, s_1^{\beta(m)-1}, \dots, 2, 1.$$

Therefore, $\beta(m)$ can be interpreted differently as the number of blocks in a single period of $L(mod\ m)$. It now follows easily that $k_L(m) = \alpha(m) \times \beta(m)$.

Following are some interesting consequences which follows from these results.

Corollary 2.5. $L_{n \times \alpha(m) + r} \equiv (s(m))^n \times L_r(mod\ m)$.

Proof: From the previous theorem, we have $R_n \equiv s_n R_0(mod\ m)$ and $s_n \equiv s_1^n(mod\ m)$. Thus, we have

$$R_n \equiv s_1^n R_0(mod\ m). \quad (2.2)$$

This shows that the r^{th} term of R_n is equal to s_1^n times the r^{th} term of R_0 , when considering the modulo m . Also, from the definition of $s(m)$, an immediate conclusion that would be drawn is $s_1 = 2L_{\alpha(m)}$ when considering modulo m . Therefore, from lemma 1.1 and above arguments, we can say that $L_{n \times \alpha(m) + r} \equiv (L_{\alpha(m)})^n \times L_r(mod\ m)$. This finally gives $L_{n \times \alpha(m) + r} \equiv (s(m))^n \times L_r(mod\ m)$.

Corollary 2.6. $gcd(m, s_i) = 1$; for all $i \geq 1$.

Proof: From the definition, when considered modulo m we have $s_n = s_1^n$. Therefore, we write $s_i^{\beta(m)} \equiv (s_1^i)^{\beta(m)} \equiv (s_1^{\beta(m)})^i(mod\ m)$. Thus, since

$s_1^{\beta(m)} \equiv 1(\text{mod } m)$, we have $(s_1^{\beta(m)})^i \equiv 1(\text{mod } m)$. This gives $s_i^{\beta(m)} \equiv 1(\text{mod } m)$. Now suppose $\gcd(m, s_i) = d$. Then, $d \mid m$ and $d \mid s_i$, which gives $d \mid s_i^{\beta(m)}$. Also, $m \mid (s_i^{\beta(m)} - 1)$. Using both together, we have $d \mid (s_i^{\beta(m)} - 1)$. This gives, $d = 1$. Thus, $\gcd(m, s_i) = 1$.

Corollary 2.7. $s_n^r \equiv s_{n \times r}(\text{mod } m)$.

Proof: From the definition of $s(m)$, we have $s_n \equiv s_1^n(\text{mod } m)$. Then we can write $s_n^r \equiv (s_1^n)^r \equiv s_1^{n \times r} \equiv s_{n \times r}(\text{mod } m)$. It now follows that $s_n^r \equiv s_{n \times r}(\text{mod } m)$.

The following theorem doesn't seem to give us an immediate idea about $L(\text{mod } m)$, but some good results follow. The evidence comes from Robinson [2] but admits that Morgan Wood knew the result in the early 1930's.

Theorem 2.8. $k_L(m) = \gcd(2, \beta(m)) \times \text{lcm}[2, \alpha(m)]$, for $m > 2$.

Proof: Koshy [5] proved that $L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n$. Taking $n = \alpha(m)$, we get

$$L_{\alpha(m)}^2 - L_{\alpha(m)-1}L_{\alpha(m)+1} = 5(-1)^{\alpha(m)}. \quad (2.3)$$

Now, $L_{\alpha(m)} \equiv 2s(m)(\text{mod } m)$, $L_{\alpha(m)-1} \equiv -s(m)(\text{mod } m)$ and $L_{\alpha(m)+1} \equiv s(m)(\text{mod } m)$. Therefore, by (2.2) we have

$$(2s(m))^2 - (-s(m))(s(m)) \equiv 5(-1)^{\alpha(m)}(\text{mod } m).$$

This gives

$$5(s(m))^2 \equiv 5(-1)^{\alpha(m)}(\text{mod } m). \quad (2.4)$$

Thus $(s(m))^2$ and $(-1)^{\alpha(m)}$ has same order modulo m . But the order of -1 is 2 and the order of $s(m)$ is $\beta(m)$ modulo m . Thus,

$$\frac{\beta(m)}{\gcd(2, \beta(m))} = \frac{2}{\gcd(2, \alpha(m))}.$$

Thus,

$$k_L(m) = \alpha(m)\beta(m) = \alpha(m) \frac{2 \gcd(2, \beta(m))}{\gcd(2, \alpha(m))} = \gcd(2, \beta(m)) \times \text{lcm}[2, \alpha(m)],$$

as required.

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Finally, we calculate the possible values of $\beta(m)$.

Theorem 2.9. $\beta(m) = 1$ or 2 or 4 ; for any $m \geq 2$.

Proof: By above theorem we have

$$\mu(m) = \gcd(2, \beta(m)) \times \text{lcm}[2, \alpha(m)], = (1 \text{ or } 2) \times (\alpha(m) \text{ or } 2\alpha(m)).$$

Therefore, $\mu(m) = \alpha(m)$ or $2\alpha(m)$ or $4\alpha(m)$. Thus, we have $\beta(m) = 1$ or 2 or 4 ; for any $m \geq 2$.

We conclude the paper by noting the following obvious result which is a direct consequence of theorem 2.4 and theorem 2.9.

Corollary 2.10. $k(m) = \alpha(m)$ or $2\alpha(m)$ or $4\alpha(m)$.

3. Conclusion

In this article we had introduced the ‘blocks’ within the period of the Lucas sequence and shown that length of any one period of the Lucas sequence contains either 1, 2 or 4 blocks.

References

- [1] D. D. Wall. Fibonacci series modulo m . The American Mathematical Monthly, 67, 525 – 532, 1960.
- [2] D. W. Robinson. The Fibonacci matrix modulo m . The Fibonacci Quarterly, 1, 29 – 36, 1963.
- [3] J. Kramer, V. E. Jr. Hoggatt. Special cases of Fibonacci Periodicity. The Fibonacci Quarterly, 1:5, 519 – 522, 1972.
- [4] K. Thomas. Fibonacci and Lucas Numbers with Applications. John Wiley and Sons, Inc., New York, 2001.

- [5] R. Marc. Properties of the Fibonacci sequence under various moduli. Master's Thesis, Wake Forest University, 1996.
- [6] R. P. Patel, D. V. Shah. Periodicity of generalized Lucas numbers and the length of its period under modulo 2^e . The Mathematics Today, 33, 67 – 74, 2017.

On extended quasi-MV algebras

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Abstract

In this paper, we introduce a new algebraic structure called extended quasi-MV algebras, which are generalizations of quasi-MV algebras. The notions of ideals, ideal congruences and filters in Equasi-MV algebras were introduced and their mutual relationships were investigated. There is a bijection between the set of all ideals and the set of all ideal congruences on an Equasi-MV algebra.

Keywords: Equasi-MV algebras; Quasi-MV algebras; Idempotent elements; Ideal congruences; Filters

2020 AMS subject classifications: 03G27 ¹

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¹Received on October 15, 2021. Accepted on December 18, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.680. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

MV-algebras were introduced by Chang Chang [1958] as an algebraic counterpart of infinite valued logic. There are many papers on MV-algebras. Also, many algebraic structures are defined, which extend the notion of MV-algebras. Quantum computation logics M. L. Dalla Chiara and Leporini [2005] received more attention in recent years, which are new forms of quantum logics G. Cattaneo and Leporini [2004]. These logics determine the meaning of a sentence with a mixture of quregisters M. L. Dalla Chiara and Greechie [2013]. Corresponding to quantum computational, Ledda, Konig, Paoli and Giuntini introduced the notion of quasi-MV algebras in A. Ledda and Giuntini [2006], which are generalizations of MV-algebras. The element 0 in a quasi-MV algebra is not necessarily a neutral element of the operation $\oplus$. Since then, many authors continued to study quasi-MV algebras. For example, Ledda etc. studied some properties of quasi-MV algebras and $\sqrt{I}$ quasi-MV algebras F. Bou and Freytes [2008], F. Paoli and Freytes [2009]; Chen introduced pseudo-quasi-MV algebras which are non-commutative generalizations of quasi-MV algebras Liu and Chen [2016].

EMV-algebras (extended MV-algebras) Dvurečenskij and Zahiri [2019] are also generalizations of MV-algebras. An EMV-algebra does not necessarily have a top element. Dvurečenskij and Zahiri gave some properties of EMV-algebras. The notions of ideals, congruences and filters in EMV-algebras were also introduced and the relationships between them were investigated. One of the main results is that every EMV-algebra can be embedded into an EMV-algebra with a top element. Liu presented EBL-algebras in Liu [2020], which extended the notion of BL-algebras. The author gave some properties of EBL-algebras. Also, the concepts of ideals, congruences and filters were introduced and the relationships between them were studied.

Inspired by Dvurečenskij and Zahiri [2019], we shall give the definition of Equasi-MV algebras. In these algebras, 0 is not necessarily the neutral element and the complement element of 0 does not necessarily exist. The structure of this paper is as follows. In Sect.2, we give some definitions and results of quasi-MV algebras. In Sect.3, we introduce Equasi-MV algebras and present some examples of Equasi-MV algebras. In Sect.4, we define ideals and ideal congruences in Equasi-MV algebras. And we study the relationships between them. In Sect.5, we introduce the notions of filters and prime ideals. Moreover, every Equasi-MV algebra has at least one maximal ideal.

2 Preliminaries

In this section, we will give some notions and results on quasi-MV algebras, which will be used in the following.

A quasi-MV algebra $\mathbf{A}$. Ledda and Giuntini [2006] is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ satisfying the following conditions:

$$\text{QMV1) } x \oplus (y \oplus z) = (x \oplus z) \oplus y;$$

$$\text{QMV2) } x'' = x;$$

$$\text{QMV3) } x \oplus 1 = 1;$$

$$\text{QMV4) } (x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x;$$

$$\text{QMV5) } (x \oplus 0)' = x' \oplus 0;$$

$$\text{QMV6) } (x \oplus y) \oplus 0 = x \oplus y;$$

$$\text{QMV7) } 0' = 1.$$

In any quasi-MV algebra $\mathbf{A}$, we can define the following operations:

$$x \otimes y = (x' \oplus y')'; \quad x \uplus y = x \oplus (x' \otimes y); \quad x \mathbin{\mathbb{M}} y = x \otimes (x' \oplus y).$$

It is obvious that $x \uplus y = (x \uplus y) \oplus 0$ and $x \mathbin{\mathbb{M}} y = (x \mathbin{\mathbb{M}} y) \oplus 0$. Moreover, we can also define an binary relation $\leq$ on A as follows: $x \leq y$ iff $x \mathbin{\mathbb{M}} y = x \oplus 0$. The relation $\leq$ is a preordering of A , but not a partial ordering.

Lemma 2.1. [A. Ledda and Giuntini, 2006, Lemma 8] *Let $\mathbf{A}$ be a quasi-MV algebra. For all $x, y, z \in A$, the following statements are equivalent.*

- (i) $x \leq y$;
- (ii) $x' \oplus y = 1$;
- (iii) $x \uplus y = y \oplus 0$.

In the following, we give some properties of quasi-MV algebras, including a few properties of preordering $\leq$ and the operations $\mathbin{\mathbb{M}}$ and $\uplus$.

Lemma 2.2. [A. Ledda and Giuntini, 2006, Lemma 11] *Let $\mathbf{A}$ be a quasi-MV algebra. For all $x, y, z, w \in A$:*

- (i) $x \oplus 0 \leq y \oplus 0, y \oplus 0 \leq x \oplus 0$ imply $x \oplus 0 = y \oplus 0$;
- (vi) $x \leq x \oplus 0$ and $x \oplus 0 \leq x$;
- (ii) $x \leq y$ and $z \leq w$ imply $x \oplus z \leq y \oplus w$;
- (vii) $x \otimes y \leq z$ iff $x \leq y' \oplus z$;
- (iii) $x \leq y$ and $z \leq w$ imply $x \otimes z \leq y \otimes w$;
- (viii) if $x \leq y$, then $y' \leq x'$;
- (iv) $x \leq y$ and $z \leq w$ imply $x \mathbin{\mathbb{M}} z \leq y \mathbin{\mathbb{M}} w$;
- (ix) $0 \leq x, x \leq 1$.
- (v) $x \leq y$ and $z \leq w$ imply $x \uplus z \leq y \uplus w$;

Lemma 2.3. [A. Ledda and Giuntini, 2006, Lemma 12] *Let $\mathbf{A}$ be a quasi-MV*

algebra. For all $x, y, z \in A$:

- (i) $x \sqcap y = y \sqcap x$; (vii) $x \otimes (y \sqcup z) = (x \otimes y) \sqcup (x \otimes z)$;
- (ii) $x \sqcup y = y \sqcup x$; (viii) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$;
- (iii) $x \sqcap y \leq x, y$ and $x, y \leq x \sqcup y$; (ix) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$;
- (iv) if $x \leq y, z$, then $x \leq y \sqcap z$; (x) $x \leq x \sqcap x$ and $x \sqcap x \leq x$;
- (v) if $x, y \leq z$, then $x \sqcup y \leq z$; (xi) $(x \sqcap y)' = x' \sqcup y'$ and $(x \sqcup y)' = x' \sqcap y'$.
- (vi) $x \oplus (y \sqcap z) = (x \oplus y) \sqcap (x \oplus z)$;

The following lemma gives the distributivity between $\sqcap$ and $\sqcup$ on quasi-MV algebras.

Lemma 2.4. *Let A be a quasi-MV algebra. For all $x, y, z \in A$,*

- (i) $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$;
- (ii) $(x \sqcap y) \sqcup z = (x \sqcup z) \sqcap (y \sqcup z)$;
- (iii) $x \sqcap (y \oplus z) \leq (x \sqcap y) \oplus (x \sqcap z)$;
- (iv) $(x \sqcup y) \otimes (x \sqcup z) \leq x \sqcup (y \otimes z)$.

Proof. (i) For any $x, y \in A$, we have $x, y \leq x \sqcup y$ and so $x \sqcap z, y \sqcap z \leq (x \sqcup y) \sqcap z$ by Lemma 2.2 (iv). It follows from Lemma 2.3 (v) that $(x \sqcap z) \sqcup (y \sqcap z) \leq (x \sqcup y) \sqcap z$. Conversely, we have

$$\begin{aligned}
 (x \sqcup y) \sqcap z &= (x \sqcup y) \otimes ((x \sqcup y)' \oplus z) \\
 &= (x \sqcup y) \otimes ((x' \oplus z) \sqcap (y' \oplus z)) \text{ (Lemma 2.3 (xi) and (vi))} \\
 &\leq (x \otimes (x' \oplus z)) \sqcup (y \otimes (y' \oplus z)) \text{ (Lemma 2.3 (vii) and (iii))} \\
 &= (x \sqcap z) \sqcup (y \sqcap z).
 \end{aligned}$$

Then $((x \sqcap z) \sqcup (y \sqcap z)) \oplus 0 \leq ((x \sqcup y) \sqcap z) \oplus 0$ and $((x \sqcup y) \sqcap z) \oplus 0 \leq ((x \sqcap z) \sqcup (y \sqcap z)) \oplus 0$. Note that $((x \sqcap z) \sqcup (y \sqcap z)) \oplus 0 = (x \sqcap z) \sqcup (y \sqcap z)$ and $((x \sqcup y) \sqcap z) \oplus 0 = (x \sqcup y) \sqcap z$. It follows that $(x \sqcap z) \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$ by Lemma 2.2 (i).

Similarly, we can prove (ii).

(iii) For any $x, y, z \in A$, since $x \leq x \oplus 0 \leq x \oplus y$, we have

$$\begin{aligned}
 (x \sqcap y) \oplus (x \sqcap z) &= ((x \sqcap y) \oplus x) \sqcap ((x \sqcap y) \oplus z) \text{ (Lemma 2.3 (vi))} \\
 &= (x \oplus x) \sqcap (y \oplus x) \sqcap (x \oplus z) \sqcap (y \oplus z) \\
 &\geq x \sqcap x \sqcap x \sqcap (y \oplus z) \\
 &= (x \oplus 0) \sqcap x \sqcap (y \oplus z) \text{ (Lemma 2.3 (x))} \\
 &= (x \oplus 0) \sqcap (y \oplus z).
 \end{aligned}$$

Note that $(x \oplus 0) \sqcap (y \oplus z) = x \sqcap (y \oplus z)$. It follows that $x \sqcap (y \oplus z) \leq (x \sqcap y) \oplus (x \sqcap z)$.

(iv) For any $x, y, z \in A$, it follows from $(x \otimes y)' \oplus y = x' \oplus y' \oplus y = 1$ that

$x \otimes y \leq y$. Then we have

$$\begin{aligned}
 (x \uplus y) \otimes (x \uplus z) &= ((x \uplus y) \otimes x) \uplus ((x \uplus y) \otimes z) \text{ (Lemma 2.3 (vii))} \\
 &= (x \otimes x) \uplus (y \otimes x) \uplus (x \otimes z) \uplus (y \otimes z) \\
 &\leq x \uplus x \uplus x \uplus (y \otimes z) \\
 &= (x \oplus 0) \uplus x \uplus (y \otimes z) \text{ (Lemma 2.3 (x))} \\
 &= (x \oplus 0) \uplus (y \otimes z).
 \end{aligned}$$

Note that $(x \oplus 0) \uplus (y \otimes z) = x \uplus (y \otimes z)$. It follows that $(x \uplus y) \otimes (x \uplus z) \leq x \uplus (y \otimes z)$. $\square$

Let $\mathbf{A}$ be a quasi-MV algebra and $a \in A$. If $a \oplus a = a$, we call a to be idempotent. We use $\mathcal{I}(\mathbf{A})$ to denote the set of all idempotent elements of A . For $a \in A$, we call a regular if $a \oplus 0 = a$. We denote the set of all regular elements of A by $\mathcal{R}(\mathbf{A})$.

Lemma 2.5. *Let $\mathbf{A}$ be a quasi-MV algebra. For any $x \in A$, $a \in \mathcal{I}(\mathbf{A})$, we have*

- (i) $x \oplus a = x \uplus a$;
- (ii) $x \otimes a = x \mathbin{\frown} a$.

Proof. (i) For any $x \in A$ and $a \in \mathcal{I}(\mathbf{A})$, we have $x, a \leq x \oplus a$. Then $x \uplus a \leq x \oplus a$ by Lemma 2.3 (v). Conversely,

$$\begin{aligned}
 (x \oplus a) \otimes (x \uplus a)' &= (x \oplus a) \otimes (x' \mathbin{\frown} a') \text{ (Lemma 2.3 (xi))} \\
 &\leq ((x \oplus a) \otimes x') \mathbin{\frown} ((x \oplus a) \otimes a') \text{ (Lemma 2.2(iii) and 2.3(iv))} \\
 &= (a \mathbin{\frown} x') \mathbin{\frown} (x \mathbin{\frown} a') \\
 &= (a \mathbin{\frown} a') \mathbin{\frown} (x \mathbin{\frown} x') \\
 &= 0 \mathbin{\frown} (x \mathbin{\frown} x') = 0.
 \end{aligned}$$

This means that $(x \oplus a)' \oplus (x \uplus a) = 1$. It follows that $x \oplus a \leq x \uplus a$.

(ii) By (i), we have $x' \oplus a' = x' \uplus a'$, that is $(x' \oplus a')' = (x' \uplus a')' = x \mathbin{\frown} a$. It follows that $x \mathbin{\frown} a = x \otimes a$. $\square$

The application of the above lemma will be reflected in the following proof process.

Example 2.1. [A. Ledda and Giuntini, 2006, Example 3] *The Diamond is the 4-element quasi-MV algebra, where the operations $\oplus$ and $'$ are defined as following tables:*

$\oplus$	0	a	b	1	$'$	
0	0	b	b	1	0	1
a	b	1	1	1	a	a
b	b	1	1	1	b	b
1	1	1	1	1	1	0

Remark that $a \oplus a = 1$, but $a \mathbin{\frown} a = (a' \oplus (a' \oplus a'))' = (a \oplus (a \oplus a'))' = b \neq 1$.

3 Equasi-MV algebras

In the section, we shall define the notion of extended quasi-MV algebras, which are generalizations of quasi-MV algebras. Some basic properties of these algebras are presented.

Definition 3.1. A extended quasi-MV algebra (abbreviated as Equasi-MV algebra) is an algebra $\mathbf{A} = \langle A, \oplus, 0 \rangle$, if the following conditions are satisfied:

EQMV1) $\langle A, \oplus, 0 \rangle$ is a commutative preordered semigroup and $(x \oplus y) \oplus 0 = x \oplus y$ for all $x, y \in A$;

EQMV2) for each $x \in A$, there is $b \in \mathcal{I}(\mathbf{A})$ such that $x \leq b$, and the element $\lambda_b(x) = \min\{z \in [0, b] : z \oplus x = b\}$ exists in A for all $x \in [0, b]$ such that $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$ is a quasi-MV algebra.

Note that for any $x, y \in A$, there exist $a, b \in \mathcal{I}(\mathbf{A})$ such that $x \leq a$ and $y \leq b$. Then there exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leq c$. In fact, take $c = a \oplus b$. It is obvious that $a, b \leq a \oplus b$ and $a \oplus b \in \mathcal{I}(\mathbf{A})$. Therefore, an Equasi-MV algebra has enough idempotent elements. That is, for all $x \in A$, there is $a \in \mathcal{I}(\mathbf{A})$ such that $x \leq a$.

Let $\mathbf{A}$ be an Equasi-MV algebra. For all $n \in \mathbb{N}$ and $x \in A$, we define

$$0.x = 0, 1.x = x, \dots, (n+1).x = n.x \oplus x.$$

An Equasi-MV algebra $\langle A, \oplus, 0 \rangle$ is called a proper Equasi-MV algebra if 0 has no complement element.

Example 3.1. If $\langle A, \oplus, ', 0, 1 \rangle$ is a quasi-MV algebra, then $\langle A, \oplus, 0 \rangle$ is an Equasi-MV algebra. Also, if $\langle A, \vee, \wedge, \oplus, 0 \rangle$ is an EMV-algebra, it is obvious that $\langle A, \oplus, 0 \rangle$ is an Equasi-MV algebra.

Example 3.2. Let $\langle A, \oplus, ', 0, 1 \rangle$ be a quasi-MV algebra and $\langle B, \vee, \wedge, \oplus, 0 \rangle$ be an EMV-algebra. We define that the operation on the algebra $A \times B$ is point by point. That is, for any $\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \in A \times B$,

$$\langle x_1, x_2 \rangle \oplus \langle y_1, y_2 \rangle = \langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle.$$

And the least element of $A \times B$ is $0 = \langle 0, 0 \rangle$. For any $x \in B$, there exists $b \in \mathcal{I}(\mathbf{B})$ such that $x \leq b$. Then for any $\langle x_1, x_2 \rangle \in A \times B$, there exists $\langle 1, b \rangle \in \mathcal{I}(\mathbf{A}) \times \mathcal{I}(\mathbf{B})$. It suffices to show that $\langle [0, 0], \langle 1, b \rangle, \oplus, \lambda_{\langle 1, b \rangle}, \langle 0, 0 \rangle, \langle 1, b \rangle \rangle$ is a quasi-MV algebra. We define $\lambda_{\langle 1, b \rangle}(\langle x_1, x_2 \rangle) = \langle (x_1)', \lambda_b(x_2) \rangle$, for all $\langle x_1, x_2 \rangle \in A \times B$. As a result, $A \times B$ is an Equasi-MV algebra.

Example 3.3. Let e be a smallest idempotent of an Equasi-MV algebra $\mathbf{A}$. Then an Equasi-MV algebra is the algebra $\mathbf{S} = \langle A \times A, \oplus^S, 0^S \rangle$, where:

(i) $0^S = \langle 0, \frac{e}{2} \rangle$;

(ii) $x^S \oplus^S y^S = \langle x_1 \oplus y_1, \frac{e}{2} \rangle$, for all $x^S = \langle x_1, x_2 \rangle$ and $y^S = \langle y_1, y_2 \rangle$.

For any $a \in \mathcal{I}(\mathbf{A})$, we define $a^S = \langle a, \frac{e}{2} \rangle$. Then $a^S = a^S \oplus^S a^S \in \mathcal{I}(\mathbf{S})$. Now we show that $\langle [0^S, a^S], \oplus^S, \lambda_{a^S}, 0^S, a^S \rangle$ is a quasi-MV algebra, where $\lambda_{a^S}(x^S) =$

$\langle \lambda_a(x_1), x_2 \rangle$ and $a \in \mathcal{I}(\mathbf{A})$. It is easy to show that $\lambda_{a^S}(x^S)$ is the least element such that $x^S \oplus z^S = a^S$ for all $x^S \in [0^S, a^S]$.

It is clear that $\lambda_{a^S}\lambda_{a^S}(x^S) = \lambda_{a^S}\langle \lambda_a(x_1), x_2 \rangle = \langle x_1, x_2 \rangle = x^S$. And $\lambda_{a^S}(x^S \oplus 0^S) = \lambda_{a^S}\langle x_1 \oplus 0, \frac{e}{2} \rangle = \langle \lambda_a(x_1) \oplus 0, \frac{e}{2} \rangle$, $\lambda_{a^S}(x^S) \oplus 0^S = \langle \lambda_a(x_1), x_2 \rangle \oplus 0^S = \langle \lambda_a(x_1) \oplus 0, \frac{e}{2} \rangle$. What's more, $\lambda_{a^S}(0^S) = \langle \lambda_a(0), \frac{e}{2} \rangle = \langle a, \frac{e}{2} \rangle = a^S$.

Example 3.4. Let $\langle A, \vee, \wedge, 0 \rangle$ be a generalized Boolean algebra Conrad and Darnel [1997]. For any $x, y \in [0, b]$, where $\oplus = \vee$ and $\lambda_b(x)$ is the unique relative complement of x in $[0, b]$. Then $\langle A, \oplus, 0 \rangle$ is an EMV-algebra by Example 3.2 (2) in Dvurečenskij and Zahiri [2019]. Hence, $\langle A, \oplus, 0 \rangle$ is an Equasi-MV algebra.

Example 3.5. Let $\langle A, \oplus, ', 0, 1 \rangle$ be a quasi-MV algebra and $\langle B, \vee, \wedge, 0 \rangle$ be a generalized Boolean algebra. It is easy to show that $A \times B$ is an Equasi-MV algebra.

Proof. The operation $\oplus$ on $A \times B$ is defined pointwise. For all $\langle x, y \rangle \in A \times B$, there exist $a \in \mathcal{I}(\mathbf{A})$ and $b \in \mathcal{I}(\mathbf{B})$ such that $\langle x, y \rangle \leq \langle a, b \rangle$ and $\langle [0, 0], \langle a, b \rangle, \oplus, \lambda_{\langle a, b \rangle}, \langle 0, 0 \rangle, \langle a, b \rangle \rangle$ is a quasi-MV algebra.

Let's give a specific description of the above example. Let the Diamond (Example 2.6) be the 4-element quasi-MV algebra $\mathbf{A}$ and $\mathbf{M} = \langle M, \vee, \wedge, 0 \rangle$ be the generalized Boolean algebra Conrad and Darnel [1997], where M is the set of components of any positive element $\mathbb{N}^+$ and the least element $0 := \emptyset$. That is, $M = \{N : N \subseteq \mathbb{N}^+\}$. Then every element N in M is idempotent. It is easily shown that $A \times M$ with the pointwise operation is an Equasi-MV algebra. $\square$

Example 3.6. Let $\mathbf{S} = \langle [0, 1] \times [0, 1], \oplus, ', 0, 1 \rangle$ be a standard quasi-MV algebra A. Ledda and Giuntini [2006, Example 5]. Let $\mathbf{A} = \mathbf{S} \oplus \mathbf{S} \oplus \mathbf{S} \oplus \dots$. Then $\mathbf{A}$ is an Equasi-MV algebra.

Proof. Obviously, $\langle A, \oplus, 0 \rangle$ is a commutative preordered semigroup and $(x \oplus y) \oplus 0 = x \oplus y$ for all $x, y \in A$. For any $x, y \in \mathbf{A}$. Suppose $x = (x_i), y = (y_i)$. If $x_i \neq 0$ or $y_i \neq 0$, there exists $u_i \in \mathcal{I}(\mathbf{A})$ such that $x_i, y_i \leq u_i$ for all $i \geq 1$. If $x_i = y_i = 0$, take $u_i = 0$. We have an idempotent $u = (u_i) \in A$ such that $x, y \leq u$ and $\langle [0, u], \oplus, \lambda_u, 0, u \rangle$ is a quasi-MV algebra. $\square$

Remark 3.1. Let $\mathbf{A}$ be an Equasi-MV algebra. For all $x, y \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \in [0, b]$. In the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$, we denote

$$x \uplus_b y = \lambda_b(\lambda_b(x) \oplus y) \oplus y, \quad x \mathbin{\text{\tiny \(\mathbb{M}\)}}_b y = \lambda_b(\lambda_b(x) \oplus \lambda_b(\lambda_b(x) \oplus y)).$$

Proposition 3.1. Let $\mathbf{A}$ be an Equasi-MV algebra and $a, b \in \mathcal{I}(\mathbf{A})$ such that $a \leq b$. For each $x \in [0, a]$, we have

- (i) $\lambda_b(a)$ is an idempotent, and $\lambda_a(a) = 0$;
- (ii) $\lambda_a(x) \oplus 0 = \lambda_b(x) \mathbin{\text{\tiny \(\mathbb{M}\)}} a$;
- (iii) $\lambda_b(x) \oplus 0 = \lambda_a(x) \oplus \lambda_b(a)$;
- (iv) $\lambda_a(x) \leq \lambda_b(x)$.

Proof. Since $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$ is a quasi-MV algebra and $a \in \mathcal{I}(\mathbf{A})$, by Lemma 2.5 (i) we get that $x \oplus a = x \uplus a$ for all $x \in [0, b]$.

(i) Since $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$ is a quasi-MV algebra, $\lambda_b(a)$ is also an idempotent element by Lemma 26 in A. Ledda and Giuntini [2006]. It is obvious $\lambda_a(a) = 0$ in the quasi-MV algebra $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$.

(ii) For all $x \in [0, a]$, we have

$$\begin{aligned} (\lambda_b(x) \mathbin{\frown} a) \oplus (x \oplus 0) &= (\lambda_b(x) \oplus (x \oplus 0)) \mathbin{\frown} (a \oplus (x \oplus 0)) \text{ (Lemma 2.3 (vi))} \\ &= b \mathbin{\frown} a = a. \end{aligned}$$

It follows that $\lambda_a(x) \oplus 0 = \lambda_a(x \oplus 0) \leq \lambda_b(x) \mathbin{\frown} a$ in the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$. Conversely, since $b = a \oplus \lambda_b(a) = x \oplus (\lambda_a(x) \oplus \lambda_b(a))$, we get $\lambda_b(x) \leq \lambda_a(x) \oplus \lambda_b(a)$. Since $\lambda_b(a)$ is an idempotent, by Lemma 2.5 (i) we have $\lambda_a(x) \oplus \lambda_b(a) = \lambda_a(x) \uplus \lambda_b(a)$. Hence, $\lambda_b(x) \leq \lambda_a(x) \uplus \lambda_b(a)$. Thus

$$\begin{aligned} \lambda_b(x) \mathbin{\frown} a &\leq (\lambda_a(x) \uplus \lambda_b(a)) \mathbin{\frown} a \text{ (Lemma 2.2 (iv))} \\ &= \lambda_a(x) \oplus 0 \text{ (Lemma 2.4 (i)).} \end{aligned}$$

Summary of the above results, we get that $\lambda_a(x) \oplus 0 = \lambda_b(x) \mathbin{\frown} a$.

(iii) By (ii) we have

$$\begin{aligned} \lambda_a(x) \oplus \lambda_b(a) &= (\lambda_a(x) \oplus 0) \oplus \lambda_b(a) \\ &= (\lambda_b(x) \mathbin{\frown} a) \oplus \lambda_b(a) \\ &= \lambda_b(x) \uplus \lambda_b(a) \text{ (Lemma 2.3 (vi) and Lemma 2.5 (i)).} \end{aligned}$$

It follows from $x \leq a$ that $\lambda_b(a) \leq \lambda_b(x)$. Then $\lambda_b(x) \uplus \lambda_b(a) = \lambda_b(x) \oplus 0$. Therefore, $\lambda_b(x) \oplus 0 = \lambda_a(x) \oplus \lambda_b(a)$.

(iv) It follows from (ii) or (iii). $\square$

The following statement shows that $\uplus_a$ and $\mathbin{\frown}_a$ on $[0, a]$ are coincide with $\uplus$ and $\mathbin{\frown}$ on A , respectively.

Proposition 3.2. *Let $\mathbf{A}$ be an Equasi-MV algebra. For all $x, y \in A$, there exist $a, b \in \mathcal{I}(\mathbf{A})$ such that $x, y \in [0, a]$ and $x, y \in [0, b]$. Then we have*

- (i) $x \mathbin{\frown}_a y = x \mathbin{\frown}_b y$;
- (ii) $x \uplus_a y = x \uplus_b y$.

Proof. (i) By Definition 3.1, for all $a, b \in \mathcal{I}(\mathbf{A})$, there exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leq c$. Then we have

$$\begin{aligned} x \uplus_c y &= x \oplus \lambda_c(x \oplus \lambda_c(y) \oplus 0) \\ &= x \oplus \lambda_c(x \oplus \lambda_a(y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))} \\ &= x \oplus (\lambda_c(x \oplus \lambda_a(y)) \otimes_c a) \text{ (the definition of } \otimes_c) \\ &= x \oplus (\lambda_c(x \oplus \lambda_a(y)) \mathbin{\frown} a) \text{ (Lemma 2.5 (ii))} \\ &= x \oplus ((\lambda_a(x \oplus \lambda_a(y)) \uplus \lambda_c(a)) \mathbin{\frown} a) \text{ (Proposition 3.1(iii), Lemma 2.5(i))} \\ &= x \oplus (\lambda_a(x \oplus \lambda_a(y)) \mathbin{\frown} a) \text{ (Lemma 2.4 (i))} \\ &= x \oplus (\lambda_a(x \oplus \lambda_a(y)) \oplus 0) = x \uplus_a y. \end{aligned}$$

Similarly, we can show that $x \uplus_c y = x \uplus_b y$. Hence, $x \uplus_a y = x \uplus_b y$.

(ii) We also have

$$\begin{aligned}
 x \mathbin{\mathbb{M}}_c y &= \lambda_c(\lambda_c(x) \oplus \lambda_c(\lambda_c(x) \oplus y)) \\
 &= \lambda_c(\lambda_c(x) \oplus \lambda_c(\lambda_a(x) \oplus y \oplus \lambda_c(a))) \text{ (Proposition 3.1 (iii))} \\
 &= \lambda_c(\lambda_c(x) \oplus (\lambda_c(\lambda_a(x) \oplus y) \otimes_c a)) \text{ (definition of } \otimes_c) \\
 &= \lambda_c(\lambda_c(x) \oplus ((\lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \otimes_c a)) \text{ (Proposition 3.1 (iii))} \\
 &= \lambda_c(\lambda_c(x) \oplus ((\lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \mathbin{\mathbb{M}} a)) \text{ (Lemma 2.5 (i))} \\
 &= \lambda_c(\lambda_c(x) \oplus (\lambda_a(\lambda_a(x) \oplus y) \mathbin{\mathbb{M}} a)) \text{ (Lemma 2.4 (i))} \\
 &= \lambda_c(\lambda_c(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \\
 &= \lambda_c(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))} \\
 &= \lambda_c(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \otimes_c a \text{ (definition of } \otimes_c) \\
 &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \oplus \lambda_c(a)) \otimes_c a \text{ (Proposition 3.1 (iii))} \\
 &= \lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \mathbin{\mathbb{M}} a \\
 &= x \mathbin{\mathbb{M}}_a y.
 \end{aligned}$$

Similarly, we can show that $x \mathbin{\mathbb{M}}_c y = x \mathbin{\mathbb{M}}_b y$ and so $x \mathbin{\mathbb{M}}_a y = x \mathbin{\mathbb{M}}_b y$. $\square$

Definition 3.2. Let $\mathbf{A}$ be an Equasi-MV algebra and $x, y \in [0, a]$ where $a \in \mathcal{I}(\mathbf{A})$. A preordering $\leq_a$ on the quasi-MV algebra $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$ defined as follows:

$$x \leq_a y \iff x \mathbin{\mathbb{M}}_a y = x \oplus 0.$$

By Proposition 3.2, for any $x, y \leq a, b$, where $a, b \in \mathcal{I}(\mathbf{A})$, we have $x \leq_a y \iff x \leq y \iff x \leq_b y$. Then we can also define a preordering $\leq$ on A by $x \leq y \iff x \mathbin{\mathbb{M}} y = x \oplus 0$, where $x \mathbin{\mathbb{M}} y = x \mathbin{\mathbb{M}}_a y$.

Lemma 3.1. Let $\mathbf{A}$ be an Equasi-MV algebra. For all $x, y \in A$, the operation $\otimes$: $A \times A \rightarrow A$ defined by $x \otimes y = \lambda_a(\lambda_a(x) \oplus \lambda_a(y))$, where $a \in \mathcal{I}(\mathbf{A})$ and $x, y \leq a$. Then

(i) the well-defined binary operation $\otimes$ on A is not determined by the choice of a and is also order preserving and associative.

(ii) if $x, y \in A$, $x \leq y$, then $y \otimes \lambda_a(x) = y \otimes \lambda_b(x)$ and $y \oplus 0 = x \oplus (y \otimes \lambda_a(x))$ for all $a, b \in \mathcal{I}(\mathbf{A})$ and $x, y \leq a, b$.

(iii) if $x, y \in [0, a]$ and $a \in \mathcal{I}(\mathbf{A})$, then $x \otimes \lambda_a(y) = x \otimes \lambda_a(x \mathbin{\mathbb{M}} y)$ and $x \oplus 0 = (x \mathbin{\mathbb{M}} y) \oplus (x \otimes \lambda_a(y))$.

(iv) an element $a \in A$ is idempotent iff $a \otimes a = a$.

Proof. (i) Let $x, y \in A$ and $a, b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq a, b$. We claim that $\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) = \lambda_b(\lambda_b(x) \oplus \lambda_b(y))$. Indeed, there exists an element $c \in \mathcal{I}(\mathbf{A})$

such that $a, b \leq c$. Then

$$\begin{aligned}
 \lambda_c(\lambda_c(x) \oplus \lambda_c(y)) &= \lambda_c(\lambda_a(x) \oplus \lambda_c(a) \oplus \lambda_a(y) \oplus \lambda_c(a)) \text{ (Proposition 3.1 (iii))} \\
 &= \lambda_c(\lambda_a(x) \oplus \lambda_a(y)) \otimes_c \lambda_c(\lambda_c(a)) \text{ (Proposition 3.1 (i))} \\
 &= \lambda_c(\lambda_a(x) \oplus \lambda_a(y)) \mathbin{\small{\frown}} a \text{ (Lemma 2.5 (ii))} \\
 &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \oplus \lambda_c(a)) \mathbin{\small{\frown}} a \text{ (Lemma 3.1 (iii))} \\
 &= (\lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \mathbin{\small{\cup}} \lambda_c(a)) \mathbin{\small{\frown}} a \text{ (Lemma 2.5 (i))} \\
 &= \lambda_a(\lambda_a(x) \oplus \lambda_a(y)) \mathbin{\small{\frown}} a \\
 &= \lambda_a(\lambda_a(x) \oplus \lambda_a(y)).
 \end{aligned}$$

Similarly, we have $\lambda_c(\lambda_c(x) \oplus \lambda_c(y)) = \lambda_b(\lambda_b(x) \oplus \lambda_b(y))$.

Let $x, y, z \in A$. There exists $c \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leq c$. It follows from the definition of $\otimes$ that $x \otimes y, y \otimes z \in [0, c]$. Then

$$\begin{aligned}
 (x \otimes y) \otimes z &= \lambda_c(\lambda_c(x \otimes y) \oplus \lambda_c(z)) \\
 &= \lambda_c((\lambda_c(x) \oplus \lambda_c(y)) \oplus \lambda_c(z)) \\
 &= \lambda_c(\lambda_c(x) \oplus (\lambda_c(y) \oplus \lambda_c(z))) \\
 &= \lambda_c(\lambda_c(x) \oplus \lambda_c(y \otimes z)) = x \otimes (y \otimes z).
 \end{aligned}$$

This proves that $\otimes$ is associative. It is easy to prove that $\otimes$ is order preserving.

(ii) Let $x \leq y$ and $x, y \leq a, b$, where $a, b \in \mathcal{I}(\mathbf{A})$. There exists $c \in \mathcal{I}(\mathbf{A})$ such that $a, b \leq c$. By Proposition 3.1, we have

$$\begin{aligned}
 y \otimes \lambda_a(x) &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x))) \\
 &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x)) \oplus 0) \\
 &= \lambda_c(\lambda_c(y) \oplus \lambda_c(\lambda_a(x) \oplus 0)) \\
 &= y \otimes (\lambda_a(x) \oplus 0).
 \end{aligned}$$

Then

$$\begin{aligned}
 y \otimes \lambda_c(x) &= y \otimes (\lambda_c(x) \oplus 0) \\
 &= y \otimes (\lambda_a(x) \oplus \lambda_c(a)) \\
 &= y \otimes (\lambda_a(x) \mathbin{\small{\cup}} \lambda_c(a)) \text{ (Lemma 2.5 (i))} \\
 &= (y \otimes \lambda_a(x)) \mathbin{\small{\cup}} (y \otimes \lambda_c(a)) \text{ (Lemma 2.3 (vii))}.
 \end{aligned}$$

Since $\lambda_c(a) \leq \lambda_c(y)$, we have $y \otimes \lambda_c(a) \leq y \otimes \lambda_c(y) = 0$, where $y \leq a \leq c$. This implies $y \otimes \lambda_c(x) = y \otimes \lambda_a(x)$. Similarly, we have $y \otimes \lambda_c(x) = y \otimes \lambda_b(x)$. It follows that $y \otimes \lambda_a(x) = y \otimes \lambda_b(x)$.

In the quasi-MV algebra $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$, we have

$$x \oplus (y \otimes \lambda_a(x)) = x \oplus \lambda_a(\lambda_a(y) \oplus x) = x \mathbin{\small{\cup}} y = y \oplus 0.$$

(iii) Let $x, y \leq a$ and $a \in \mathcal{I}(\mathbf{A})$. We have

$$\begin{aligned}
 x \otimes \lambda_a(x \mathbin{\small{\frown}} y) &= x \otimes (\lambda_a(x) \mathbin{\small{\cup}} \lambda_a(y)) \\
 &= (x \otimes \lambda_a(x)) \mathbin{\small{\cup}} (x \otimes \lambda_a(y)) \text{ (Lemma 2.3 (vii))} \\
 &= x \otimes \lambda_a(y).
 \end{aligned}$$

$$\begin{aligned}
 (x \mathbin{\&}\! y) \oplus (x \otimes \lambda_a(y)) &= (x \mathbin{\&}\! y) \oplus (x \otimes \lambda_a(x \mathbin{\&}\! y)) \\
 &= (x \mathbin{\&}\! y) \oplus \lambda_a(\lambda_a(x) \oplus (x \mathbin{\&}\! y)) \\
 &= x \oplus \lambda_a(x \oplus \lambda_a(x \mathbin{\&}\! y)) \text{ (QMV 4)} \\
 &= x \oplus \lambda_a(x \oplus \lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus y)) \\
 &= x \oplus 0.
 \end{aligned}$$

(iv) $\implies$: Suppose $a, b \in \mathcal{I}(\mathbf{A})$ with $a \leq b$. We have $\lambda_b(a) \oplus \lambda_b(a) = \lambda_b(a)$ by Proposition 3.1 (i). In the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$, we have $a \otimes a = \lambda_b(\lambda_b(a) \oplus \lambda_b(a)) = \lambda_b(\lambda_b(a)) = a$.

$\impliedby$: For each $a \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $a \leq b$. Suppose $a \otimes a = a$. We have $\lambda_b(\lambda_b(a) \oplus \lambda_b(a)) = a$. Then $\lambda_b(\lambda_b(\lambda_b(a) \oplus \lambda_b(a))) = \lambda_b(a)$. It follows from $\lambda_b(a) \oplus \lambda_b(a) = \lambda_b(a)$ that $\lambda_b(a) \in \mathcal{I}(\mathbf{A})$. By Proposition 3.1 (i), we have $\lambda_b(\lambda_b(a)) \oplus \lambda_b(\lambda_b(a)) = \lambda_b(\lambda_b(a))$. That is $a \oplus a = a$. It implies $a \in \mathcal{I}(\mathbf{A})$. $\square$

Theorem 3.1. *Let $\mathbf{A}$ be an Equasi-MV algebra. Then $\langle \mathcal{R}(\mathbf{A}), \mathbb{U}_R, \mathbin{\&}\!_R, \oplus_R, 0_R \rangle$ is an EMV-subalgebra of $\mathbf{A}$.*

Proof. It is obvious that $\mathcal{R}(\mathbf{A})$ is closed under the operations $\mathbb{U}_R, \mathbin{\&}\!_R, \oplus_R, 0_R$. For all $x, y \in \mathcal{R}(\mathbf{A})$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq a$. Then $[0, a] \cap \mathcal{R}(\mathbf{A})$ is an MV-algebra of $[0, a]$ by Lemma 15 in A. Ledda and Giuntini [2006]. This means that $\mathcal{R}(\mathbf{A})$ is an EMV-subalgebra of $\mathbf{A}$. $\square$

4 Ideals and congruences

In this section, we give the notions of ideals and ideal congruences of Equasi-MV algebras. We also give an equivalent definition of ideals. Moreover, there is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences.

Definition 4.1. *Let $\mathbf{A}$ be an Equasi-MV algebra. An equivalence relation θ on A is called a congruence, if the following conditions hold:*

- (i) θ is compatible with $\oplus$;
- (ii) for all $b \in \mathcal{I}(\mathbf{A})$, $\theta \cap ([0, b] \times [0, b])$ is a congruence on the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$.

The set of all congruences on A represented by $\text{Con}(A)$.

Definition 4.2. *Let $\mathbf{A}_1, \mathbf{A}_2$ be two Equasi-MV algebras. We call a map $f : A_1 \longrightarrow A_2$ to be an Equasi-MV homomorphism, if it satisfies the following statements:*

- (i) $f(x \oplus y) = f(x) \oplus f(y)$ and $f(0) = 0$, for all $x, y \in A_1$;
- (ii) for all $x, y \in [0, a]$ and $a \in \mathcal{I}(\mathbf{A}_1)$, $f(\lambda_a(x)) = \lambda_{f(a)}(f(x))$.

Example 4.1. Let $f : A_1 \rightarrow A_2$ be an Equasi-MV homomorphism. We can define $\theta = \{(x, y) \in A_1 \times A_1 : f(x) = f(y)\}$, then θ is a congruence.

Let $\mathbf{A}$ be an Equasi-MV algebra and θ be a congruence on $\mathbf{A}$. We denote

$$A/\theta = \{x/\theta : x \in A\}, \text{ where } x/\theta = \{y \in A : \langle x, y \rangle \in \theta\}.$$

We define operations $\cap, \cup, \oplus$ on A/θ as follows: for any $x, y \in A$,

$$x/\theta \cap y/\theta = (x \cap y)/\theta, x/\theta \cup y/\theta = (x \cup y)/\theta, x/\theta \oplus y/\theta = (x \oplus y)/\theta.$$

Suppose $x/\theta \leq y/\theta$. Then $(x \cap y)/\theta \geq x/\theta$. For all $z \in A$, we have

$$\begin{aligned} x/\theta \oplus z/\theta &= (x \oplus z)/\theta \\ &\leq ((x \cap y) \oplus z)/\theta \\ &\leq (y \oplus z)/\theta \\ &= y/\theta \oplus z/\theta. \end{aligned}$$

This proves that $\langle A/\theta, \oplus, 0/\theta \rangle$ is a commutative preordered semigroup and $(x/\theta \oplus y/\theta) \oplus 0/\theta = x/\theta \oplus y/\theta$.

For all $x \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x \leq a$. It is easily shown that a/θ is an idempotent element and $x/\theta \leq a/\theta$. Since $\mathbf{A}$ is an Equasi-MV algebra, we have that $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$ is a quasi-MV algebra. And let $\theta_a = \theta \cap ([0, a] \times [0, a])$ be an ideal congruence on $\langle [0, a], \oplus, \lambda_a, 0, a \rangle$. For any $x/\theta_a \in [0/\theta_a, a/\theta_a]$, we define $\lambda_{a/\theta_a}(x/\theta_a) = \lambda_a(x)/\theta_a$. Then $[0/\theta_a, a/\theta_a]$ is a quasi-MV algebra.

Now we show that $\langle [0/\theta, a/\theta], \oplus, \lambda_{a/\theta}, 0/\theta, a/\theta \rangle$ is a quasi-MV algebra. For all $x/\theta \in [0/\theta, a/\theta]$, there exists $y/\theta \in [0/\theta, a/\theta]$ such that $x/\theta \oplus y/\theta = a/\theta$. It follows that $\langle x \oplus y, a \rangle \in \theta$. And since $x, y \leq a$, we have $\langle x \oplus y, a \rangle \in \theta_a$. That is, $x/\theta_a \oplus y/\theta_a = a/\theta_a$. Thus $y/\theta_a \geq \lambda_a(x)/\theta_a$ and so $y/\theta \geq \lambda_a(x)/\theta$. This implies that $\lambda_{a/\theta}(x/\theta)$ exists and equals to $\lambda_a(x)/\theta$. It can be easily shown that $\langle [0/\theta, a/\theta], \oplus, \lambda_{a/\theta}, 0/\theta, a/\theta \rangle$ is a quasi-MV algebra. Thus, $\langle A/\theta, \oplus, 0/\theta \rangle$ is an Equasi-MV algebra.

And the map $\pi : \langle A, \oplus, 0 \rangle \longrightarrow \langle A/\theta, \oplus, 0/\theta \rangle$ defined by $x \longmapsto x/\theta$ is an Equasi-MV homomorphism from A onto A/θ .

Definition 4.3. Let $\mathbf{A}$ be an Equasi-MV algebra and I be a nonempty subset of A . We call I to be an ideal of A if the following conditions hold:

- (I1) $0 \in I$;
- (I2) for all $x, y \in I$, then $x \oplus y \in I$;
- (I3) $x \in I$ and $y \leq x$ imply $y \in I$.

If I is an ideal of A and $x \in A$, we have $x \in I$ iff $x \oplus 0 \in I$ by (I3).

Definition 4.4. Let $\mathbf{A}$ be an Equasi-MV algebra and I be a nonempty subset of A . If the following statements hold, I is a weak ideal of A :

- (W1) $0 \in I$;
- (W2) for all $x, y \in I$, then $x \oplus y \in I$;
- (W3) $x \in I$ and $y \in A$ imply $x \otimes y \in I$.

Lemma 4.1. *Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. Then I is a weak ideal.*

Proof. Let I be an ideal of A and $x \in I$. If $y \in A$ with $y \leq x$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq b$. Then we have

$$\begin{aligned} (x \otimes y) \mathbin{\mathbb{M}} x &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y)) \mathbin{\mathbb{M}} x \\ &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus x)) \\ &= \lambda_b(\lambda_b(x) \oplus \lambda_b(y) \oplus \lambda_b(b)) = x \otimes y. \end{aligned}$$

It follows that $x \otimes y \leq x$. Thus $x \otimes y \in I$ and so I is a weak ideal of A . $\square$

The converse of Lemma 4.1 is not true. For example, $\{0\}$ is a weak ideal, but not an ideal.

Proposition 4.1. *Let I be a nonempty subset of an Equasi-MV algebra $\mathbf{A}$ and $0 \in I$. Then I is an ideal iff for all $x, y \in A$, $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leq a$, $\lambda_a(x) \otimes y \in I$ and $x \in I$ implies $y \in I$.*

Proof. $\implies$: Let I be an ideal of A . For all $x, y \in A$ and $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leq a$, if $\lambda_a(x) \otimes y \in I$ and $x \in I$, we have $(\lambda_a(x) \otimes y) \oplus x \in I$. Since

$$\begin{aligned} \lambda_a(y) \oplus ((\lambda_a(x) \otimes y) \oplus x) &= \lambda_a(y) \oplus (\lambda_a(x \oplus \lambda_a(y)) \oplus x) \\ &= \lambda_a(y) \oplus (\lambda_a(\lambda_a(x) \oplus y) \oplus y) \text{ (QMV4)} \\ &= \lambda_a(y) \oplus y \oplus \lambda_a(\lambda_a(x) \oplus y) \\ &= a, \end{aligned}$$

we have $y \leq (\lambda_a(x) \otimes y) \oplus x \in I$ and $y \in I$.

$\impliedby$: For any $x, y \in I$ and $a \in \mathcal{I}(\mathbf{A})$ with $x \leq y$ and $x, y \leq a$, we have $\lambda_a(x) \otimes y = 0 \in I$. Hence, $y \in I$ is obtained from propositional conditions. And then

$$\begin{aligned} \lambda_a(x) \otimes (x \oplus y) &= \lambda_a(x \oplus \lambda_a(x \oplus y)) \\ &= \lambda_a(x) \mathbin{\mathbb{M}} y \\ &\leq y \in I. \end{aligned}$$

Then $\lambda_a(x) \otimes (x \oplus y) \in I$. It follows from $x \in I$ that $x \oplus y \in I$. $\square$

Definition 4.5. *Let $\mathbf{A}$ be an Equasi-MV algebra. We define a binary relation $\preccurlyeq$ as follows: for all $x, y \in A$,*

$$x \preccurlyeq y \text{ iff } x \mathbin{\mathbb{M}} y = x.$$

The binary relation $\preccurlyeq$ satisfies antisymmetry and transitivity, but when x is a regular element, it satisfies reflexivity.

Lemma 4.2. *Let $\mathbf{A}$ be an Equasi-MV algebra and $x, y \in A$. Then*

$$x \preccurlyeq y \text{ iff } x \leq y \text{ and } x \in \mathcal{R}(\mathbf{A}).$$

Proof. If $x \preceq y$, we have $x \sqcap y = x$ and $x \sqcap y = (x \sqcap y) \oplus 0 = x \oplus 0$. It follows that $x \leq y$ and $x \oplus 0 = x$. Thus $x \in \mathcal{R}(\mathbf{A})$. Conversely, if $x \leq y$ and $x \in \mathcal{R}(\mathbf{A})$, we have $x \sqcap y = x \oplus 0 = x$ and so $x \preceq y$. $\square$

Lemma 4.3. *Let $\mathbf{A}$ be an Equasi-MV algebra and $J \subseteq A$. Then the following statements are equivalent:*

- (i) J is a weak ideal of A ;
- (ii) (1) if $x, y \in J$, then $x \oplus y \in J$; (2) if $x \in J, y \preceq x$, then $y \in J$.

Proof. (i) $\implies$ (ii): Suppose $x \in J$ and $y \preceq x$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x \leq b$. Then $x \otimes (\lambda_b(x) \oplus y) = x \sqcap y \in J$. Since $y \preceq x$, we have $x \sqcap y = y \in J$.

(ii) $\implies$ (i): For any $x \in J, y \in A$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq b$. Since $x \otimes y \leq x$ and $x \otimes y \in \mathcal{R}(\mathbf{A})$ by Lemma 4.2, we have $x \otimes y \preceq x$. Therefore, $x \otimes y \in J$. $\square$

Let $\mathbf{A}$ be an Equasi-MV algebra and H be a subset of A . The ideal generated by H is the smallest ideal of $\mathbf{A}$ containing H , denoted by $\langle H \rangle$.

Lemma 4.4. *Let $\mathbf{A}$ be an Equasi-MV algebra and $H \subseteq A$, then*

- (i) $\langle H \rangle = \{x \in A : \text{there exist } h_1, \dots, h_n \in H, n \in \mathbb{N} \text{ such that } x \leq h_1 \oplus \dots \oplus h_n\}$;
- (ii) $\langle 0 \rangle$ is the smallest ideal of $\mathbf{A}$;
- (iii) If I is an ideal of A and $x \in A$, we have $\langle I \cup \{x\} \rangle = \{z \in A : z \leq a \oplus n.x \text{ for some } a \in I \text{ and } n \in \mathbb{N}\}$.

Proof. (i) We write $M = \{x \in A : \text{there exist } h_1, \dots, h_n \in H, n \in \mathbb{N} \text{ such that } x \leq h_1 \oplus \dots \oplus h_n\}$. Then M is an ideal of A . Now we show that M is the smallest ideal of $\mathbf{A}$ containing H . Suppose M' is an ideal of $\mathbf{A}$ containing H . For any $x \in M$, there exist $h_1, \dots, h_n \in H$ such that $x \leq h_1 \oplus \dots \oplus h_n$. As $H \subseteq M'$, we get $x \in M'$ and so $M \subseteq M'$.

(ii) By (i) we obvious get the result. $\square$

Definition 4.6. *An ideal I of an Equasi-MV algebra $\mathbf{A}$ is maximal if for all $x \in A \setminus I$, $\langle I \cup \{x\} \rangle = A$.*

Definition 4.7. *Let $\mathbf{A}$ be an Equasi-MV algebra and θ be a congruence on $\mathbf{A}$. θ is an ideal congruence if for all $x, y \in A$, $(x \oplus 0)\theta(y \oplus 0) \Rightarrow x\theta y$.*

Example 4.2. *Let $\mathbf{A}$ be an Equasi-MV algebra and $x, y \in A$. A binary relation χ defined as follows: $x\chi y$ iff $x \leq y$ and $y \leq x$.*

It is easy to show that χ is compatible with $\oplus$. We now show that for all $b \in \mathcal{I}(\mathbf{A})$, $\chi \cap ([0, b] \times [0, b])$ is congruence on the quasi-MV algebra $\langle [0, b], \oplus, \lambda_b, 0, b \rangle$. Suppose $\langle x, y \rangle \in \chi \cap ([0, b] \times [0, b])$. It follows from $\langle x, y \rangle \in \chi$ that $x \leq y$ and $y \leq x$. Hence, $\lambda_b(y) \leq \lambda_b(x)$ and $\lambda_b(x) \leq \lambda_b(y)$. Therefore, $\langle \lambda_b(x), \lambda_b(y) \rangle \in \chi \cap ([0, b] \times [0, b])$. That is, χ is a congruence on A . As a result, χ is an ideal congruence.

Definition 4.8. Let $\mathbf{A}$ be an Equasi-MV algebra, I be an ideal of $\mathbf{A}$ and θ be an ideal congruence on $\mathbf{A}$. We define two relations $f(J)$ on $A \times A$ and $g(\theta)$ on A as follows:

$$\begin{aligned} \langle x, y \rangle &\in f(J) \text{ iff there exists } b \in \mathcal{I}(\mathbf{A}) \text{ such that } x \otimes \lambda_b(y), y \otimes \lambda_b(x) \in J; \\ g(\theta) &= 0/\theta = \{x \in A : x\theta 0\}. \end{aligned}$$

Theorem 4.1. Let $\mathbf{A}$ be an Equasi-MV algebra, J be an ideal of $\mathbf{A}$ and θ be an ideal congruence on $\mathbf{A}$.

- (i) $f(J)$ is an ideal congruence on $\mathbf{A}$;
- (ii) $g(\theta)$ is an ideal of $\mathbf{A}$;
- (iii) $J = g(f(J))$;
- (iv) $\theta = f(g(\theta))$.

Proof. (i) Obviously, $f(J)$ is a congruence on A . Now we show that $f(J)$ is an ideal congruence. Let $\langle x \oplus 0, y \oplus 0 \rangle \in f(J)$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq b$. Then $\lambda_b(x \oplus 0) \otimes (y \oplus 0), \lambda_b(y \oplus 0) \otimes (x \oplus 0) \in J$. It follows that $\lambda_b(x) \otimes y = \lambda_b(x \oplus 0) \otimes (y \oplus 0) \in J$. Similarly, $\lambda_b(y) \otimes x \in J$. Thus, $\langle x, y \rangle \in f(J)$. Therefore, $f(J)$ is an ideal congruence on $\mathbf{A}$.

(ii) Suppose $\langle x, 0 \rangle \in \theta$ and $y \leq x$. We have $\langle \lambda_b(x), b \rangle \in \theta$. That implies $\langle \lambda_b(x) \oplus y, b \rangle \in \theta$ and so $\langle x \otimes (\lambda_b(x) \oplus y), x \otimes b \rangle \in \theta$. That is, $\langle x \sqcap y, x \oplus 0 \rangle \in \theta$. It follows from $y \leq x$ that $x \sqcap y = y \oplus 0$. Thus, $\langle y \oplus 0, x \oplus 0 \rangle \in \theta$. Since θ is an ideal congruence on $\mathbf{A}$, we have $\langle y, x \rangle \in \theta$. This together with $\langle 0, x \rangle \in \theta$ implies that $\langle y, 0 \rangle \in \theta$ and so $y \in g(\theta)$. Therefore, $g(\theta)$ is an ideal of $\mathbf{A}$.

(iii) It is easily seen that $g(f(J)) = \{x \in A : x \oplus 0 \in J\}$. For all $x \in A$, we have $x \in J$ iff $x \oplus 0 \in J$. Thus $g(f(J)) = \{x \in A : x \in J\}$.

(iv) For any $x, y \in A$, if $\langle x, y \rangle \in f(g(\theta))$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq b$, $\langle \lambda_b(x) \otimes y, 0 \rangle \in \theta$ and $\langle \lambda_b(y) \otimes x, 0 \rangle \in \theta$. Then $\langle (\lambda_b(x) \otimes y) \oplus x, 0 \oplus x \rangle \in \theta$. By $(\lambda_b(x) \otimes y) \oplus x = x \sqcup y$, we get $\langle x \sqcup y, 0 \oplus x \rangle \in \theta$. Similarly, we have $\langle x \sqcup y, 0 \oplus y \rangle \in \theta$. Thus, $\langle 0 \oplus x, 0 \oplus y \rangle \in \theta$. Since θ is an ideal congruence on $\mathbf{A}$, we have $\langle x, y \rangle \in \theta$. Therefore, $f(g(\theta)) \subseteq \theta$.

Conversely, if $\langle x, y \rangle \in \theta$, there exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq b$ and so $\langle y \otimes \lambda_b(x), x \otimes \lambda_b(x) \rangle \in \theta$. This together with $x \otimes \lambda_b(x) = 0$ implies $\langle y \otimes \lambda_b(x), 0 \rangle \in \theta$. Similarly, $\langle x \otimes \lambda_b(y), 0 \rangle \in \theta$. Thus, $\langle x, y \rangle \in f(g(\theta))$. Therefore, $\theta \subseteq f(g(\theta))$. $\square$

Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. The relation θ_I is defined as follows: for all $x, y \in A$,
 $(x, y) \in \theta_I \iff \exists b \in \mathcal{I}(\mathbf{A}) \text{ with } x, y \leq b \text{ such that } \lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$.

Proposition 4.2. Let $\mathbf{A}$ be an Equasi-MV algebra. If I is an ideal of A , the relation θ_I is an ideal congruence on A .

Proof. Let I be an ideal of A . Suppose $\langle x, y \rangle, \langle y, z \rangle \in \theta_I$. We have $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$ and $\lambda_b(\lambda_b(y) \oplus z), \lambda_b(\lambda_b(z) \oplus y) \in I$ where $b \in \mathcal{I}(\mathbf{A})$ such

that $x, y, z \leq b$. Since I is an ideal of A , we have $\lambda_b(\lambda_b(x) \oplus y) \oplus \lambda_b(\lambda_b(y) \oplus z) \in I$ and $\lambda_b(\lambda_b(y) \oplus x) \oplus \lambda_b(\lambda_b(z) \oplus y) \in I$. And $(\lambda_b(x) \oplus z) \oplus (\lambda_b(\lambda_b(x) \oplus y) \oplus \lambda_b(\lambda_b(y) \oplus z)) = b$. It follows that $\lambda_b(\lambda_b(x) \oplus z) \in I$. Similarly, $\lambda_b(\lambda_b(z) \oplus x) \in I$. Then $\langle x, z \rangle \in \theta_I$. The reflexivity and symmetry is clear.

It is easy to prove that θ_I is compatible with $\oplus$. For all $u \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leq u$. Now, we show that $\theta_{I_u} = \theta_I \cap ([0, u] \times [0, u])$ is a congruence on the quasi-MV algebra $\langle [0, u], \oplus, \lambda_u, 0, u \rangle$. Suppose $\langle x, y \rangle \in \theta_{I_u}$, we have $\lambda_u(\lambda_u(x) \oplus y), \lambda_u(\lambda_u(y) \oplus x) \in I \cap ([0, u] \times [0, u])$. Then

$$\begin{aligned} & (\lambda_u(x \oplus z) \oplus (y \oplus z)) \oplus \lambda_u(\lambda_u(x) \oplus y) \\ &= \lambda_u(x \oplus z) \oplus x \oplus z \oplus \lambda_u(\lambda_u(y) \oplus x) \\ &= \lambda_u(\lambda_u(x) \oplus \lambda_u(z)) \oplus \lambda_u(z) \oplus z \oplus \lambda_u(\lambda_u(y) \oplus x) \\ &= u. \end{aligned}$$

It follows that $\lambda_u(\lambda_u(x \oplus z) \oplus (y \oplus z)) \leq \lambda_u(\lambda_u(x) \oplus y) \in \theta_I$. Then $\lambda_u(\lambda_u(x \oplus z) \oplus (y \oplus z)) \in \theta_I$. Similarly, $\lambda_u(\lambda_u(y \oplus z) \oplus (x \oplus z)) \in \theta_I$. Thus, $\langle x \oplus z, y \oplus z \rangle \in \theta_{I_u}$. And $\langle \lambda_u(x), \lambda_u(z) \rangle \in \theta_{I_u}$ is obvious. Therefore, θ_I is a congruence on A .

For each $\langle x \oplus 0, y \oplus 0 \rangle \in \theta_I$, we have $\lambda_b(\lambda_b(x \oplus 0) \oplus (y \oplus 0)), \lambda_b(\lambda_b(y \oplus 0) \oplus (x \oplus 0)) \in I$. That is, $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I$. Thus $\langle x, y \rangle \in \theta_I$. Therefore, θ_I is an ideal congruence. $\square$

Theorem 4.2. *Let $\mathbf{A}$ be an Equasi-MV algebra. There is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences.*

Proof. Let I be an ideal of A and θ_I be an ideal congruence induced by I . Now we show that $I = 0/\theta_I$. Since $0 \in I$, we have $\langle x, 0 \rangle \in \theta_I$, for all $x \in I$. It follows that $x \in 0/\theta_I$. Conversely, suppose $x \in 0/\theta_I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x \leq a$. By Proposition 4.1, since $\lambda_a(x) \otimes 0 \in I$ and $0 \in I$, we have $x \in I$. Hence, $I = 0/\theta_I$.

Let θ be an ideal congruence on A . Let $I = 0/\theta$. Suppose $\langle x, y \rangle \in \theta_I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq a$ and $\lambda_b(\lambda_b(x) \oplus y), \lambda_b(\lambda_b(y) \oplus x) \in I = 0/\theta$. That is, $\langle \lambda_b(\lambda_b(x) \oplus y), 0 \rangle \in \theta$ and $\langle \lambda_b(\lambda_b(y) \oplus x), 0 \rangle \in \theta$. Hence, $\langle \lambda_b(\lambda_b(x) \oplus y) \oplus y, 0 \oplus y \rangle \in \theta$ and $\langle \lambda_b(\lambda_b(y) \oplus x) \oplus x, 0 \oplus x \rangle \in \theta$. Since $\lambda_b(\lambda_b(x) \oplus y) \oplus y = \lambda_b(\lambda_b(y) \oplus x) \oplus x$, we have $\langle x \oplus 0, y \oplus 0 \rangle \in \theta$. And since θ is an ideal congruence on A , we have $\langle x, y \rangle \in \theta$.

Conversely, let $\langle x, y \rangle \in \theta$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq a$. Then $\langle \lambda_a(x), \lambda_a(y) \rangle \in \theta$ and $\langle \lambda_a(x) \otimes y, \lambda_a(y) \otimes x \rangle \in \theta$. Since $\lambda_a(y) \otimes y = 0$, we have $\lambda_a(x) \otimes y \in 0/\theta$. Similarly, $\lambda_a(y) \otimes x \in 0/\theta$. That is, $\langle x, y \rangle \in \theta_I$. Therefore, $\theta = \theta_I$. $\square$

Theorem 4.3. *Let $\mathbf{A}$ be an Equasi-MV algebra. Then $f(I) \circ f(J) = f(J) \circ f(I)$ is valid, where I and J are ideals of $\mathbf{A}$.*

Proof. Suppose $f(I), f(J) \in \text{Con}I(\mathbf{A})$ and $\langle x, y \rangle \in f(I) \circ f(J)$ for $x, y \in A$. So there exists $z \in A$ such that $\langle x, z \rangle \in f(I)$ and $\langle z, y \rangle \in f(J)$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $x, y, z \leq b$. Let p be a ternary term defined as follows:

$$p_b(x, y, z) = (x \otimes (\lambda_b(y) \oplus (y \mathbin{\frown} z))) \uplus (z \otimes (\lambda_b(y) \oplus (y \mathbin{\frown} x))).$$

Then

$$(x \otimes (\lambda_b(z) \oplus (z \mathbin{\frown} y))) \uplus (y \otimes (\lambda_b(z) \oplus (z \mathbin{\frown} x))) f(I) p_b(z, z, y) = y \oplus 0$$

and

$$(x \otimes (\lambda_b(z) \oplus (z \mathbin{\frown} y))) \uplus (y \otimes (\lambda_b(z) \oplus (z \mathbin{\frown} x))) f(J) p_b(x, y, y) = x \oplus 0.$$

Let

$$(x \otimes (\lambda_b(z) \oplus (z \mathbin{\frown} y))) \uplus (y \otimes (\lambda_b(z) \oplus (y \mathbin{\frown} x))) = t,$$

where $t \leq b \in \mathcal{I}(\mathbf{A})$. It follows from $\langle t, y \oplus 0 \rangle \in f(I)$ and $\langle t, x \oplus 0 \rangle \in f(J)$ that

$$(y \oplus 0) \otimes \lambda_b(t), \lambda_b(y \oplus 0) \otimes t \in I;$$

$$(x \oplus 0) \otimes \lambda_b(t), \lambda_b(x \oplus 0) \otimes t \in J.$$

Now, $y \otimes \lambda_b(t) \leq (y \oplus 0) \otimes \lambda_b(t) \in I$, $x \otimes \lambda_b(t) \leq (x \oplus 0) \otimes \lambda_b(t) \in J$. Similarly, $\lambda_b(y) \otimes t \leq \lambda_b(y \oplus 0) \otimes t \in I$, $\lambda_b(x) \otimes t \leq \lambda_b(x \oplus 0) \otimes t \in J$. Thus, $\langle t, y \rangle \in f(I)$ and $\langle t, x \rangle \in f(J)$. That is, $\langle x, y \rangle \in f(J) \circ f(I)$. $\square$

Lemma 4.5. *If $\mathbf{A}$ is an Equasi-MV algebra, the lattice $\text{Con}I(\mathbf{A})$ of ideal congruences on $\mathbf{A}$ is a sublattice of $\text{Con}(\mathbf{A})$.*

Proof. Let I, J be two ideals of $\mathbf{A}$. It is easy to prove that $f(I \cap J) = f(I) \cap f(J)$. Now we show that $f(I \vee J) = f(I) \vee f(J)$.

Since $g(f(I \vee J)) = I \vee J$ and $g(f(I)) \vee g(f(J)) = I \vee J$, we claim that $g(f(I) \vee f(J)) = g(f(I)) \vee g(f(J))$. Let $x \in g(f(I)) \vee g(f(J))$ such that $x \leq y \oplus z$ where $y \in g(f(I))$ and $z \in g(f(J))$. Then we get $\langle y, 0 \rangle \in f(I)$, $\langle z, 0 \rangle \in f(J)$ and $\langle y, z \rangle \in f(I) \circ f(J) = f(I) \vee f(J)$. It follows that $\langle z \oplus 0, 0 \rangle \in f(J)$, $\langle y \oplus z, z \oplus 0 \rangle \in f(I)$ and $\langle y \oplus z, 0 \rangle \in f(I) \circ f(J) = f(I) \vee f(J)$. And then $x \leq y \oplus z \in g(f(I) \vee f(J))$. Therefore, $g(f(I)) \vee g(f(J)) \subseteq g(f(I) \vee f(J))$.

Conversely, for any $x \in g(f(I) \vee f(J))$, we have $\langle x, 0 \rangle \in f(I) \vee f(J) = f(I) \circ f(J)$. Then there exist $z \in A$ and $b \in \mathcal{I}(\mathbf{A})$ such that $\langle x, z \rangle \in f(I)$ and $\langle z, 0 \rangle \in f(J)$. And $\langle x \otimes \lambda_b(z), 0 \rangle \in f(I)$, $\langle z, 0 \rangle \in f(J)$. Then $x \leq (x \otimes \lambda_b(z)) \oplus z$. Since $x \otimes \lambda_b(z) \in g(f(I))$ and $z \in g(f(J))$, we have $x \in g(f(I)) \vee g(f(J))$. Thus, $g(f(I) \vee f(J)) \subseteq g(f(I)) \vee g(f(J))$. $\square$

Theorem 4.4. *$\text{Con}I(\mathbf{A})$ is distributive.*

Proof. By Theorem 4.2, we only need to prove that the lattice of ideals on A is distributive. Suppose I, J, K are ideals on A and $x \in I \cap (J \vee K)$. Then $x \in I$ and $x \leq y \oplus z$, for some $y \in J, z \in K$. Hence, $x \leq (x \mathbin{\frown} y) \oplus (x \mathbin{\frown} z)$. It follows from $x \mathbin{\frown} y \in I \cap J, x \mathbin{\frown} z \in I \cap K$ that $x \in (I \cap J) \vee (I \cap K)$. $\square$

5 Filters and prime ideals

In this section, we introduce the notions of filters and prime ideals of Equasi-MV algebras. Moreover, we study some properties of them. We prove that every Equasi-MV algebra has at least one maximal ideal. Also, we get prime theorem on Equasi-MV algebras.

Definition 5.1. Let $\langle A, \oplus, 0 \rangle$ be an Equasi-MV algebra and F be a nonempty subset of A . F is called a filter if the following conditions are satisfied:

- (i) for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$;
- (ii) for all $x, y \in F$, then $x \otimes y \in F$.

Definition 5.2. We call a filter F is proper if $F \neq A$. A proper filter F is maximal, if for all $x \in A \setminus F$, $\langle F \cup \{x\} \rangle = A$.

Let A be an Equasi-MV algebra. For $x \in A$ and $n \in \mathbb{N}$, we define

$$x^1 = x, \dots, x^n = x^{n-1} \otimes x, n \geq 2.$$

Proposition 5.1. Let A be an Equasi-MV algebra and F be a filter of A . Then I_F is an ideal of A , where

$$I_F := \{\lambda_a(x) : x \in F, \exists a \in \mathcal{I}(A), x \leq a\}.$$

Proof. For all $x \in A$, we have

$$x \in I_F \iff \exists a \in \mathcal{I}(A) \text{ s.t. } x \leq a, \lambda_a(x) \in F.$$

It is obvious that $0 \in I_F$. Suppose $x, y \in I_F$. There exist $a, b \in \mathcal{I}(A)$ such that $x \leq a$ and $y \leq b$. It follows $\lambda_a(x), \lambda_b(y) \in F$. Let $c \in \mathcal{I}(A)$ such that $a, b \leq c$. Then $\lambda_c(x), \lambda_c(y) \in F$ by Proposition 3.1 (iv). That implies $\lambda_c(x) \otimes \lambda_c(y) \in F$. Since $\lambda_c(x), \lambda_c(y) \leq c$ and $\lambda_c(x) \otimes \lambda_c(y) = \lambda_c(x \oplus y)$, we have $x \oplus y \in I_F$.

Suppose $x, y \in A$ with $x \in I_F$ and $y \leq x$. There exists $a \in \mathcal{I}(A)$ such that $x \leq a$ and $\lambda_a(x) \in F$. Since $x, y \in [0, a]$ and $y \leq x$, we have $\lambda_a(x) \leq \lambda_a(y)$. It implies $\lambda_a(y) \in F$ and $y \in I_F$. $\square$

In the following, we give an equivalent condition of maximal filters.

Proposition 5.2. Let A be an Equasi-MV algebra and F be a proper filter of A .

- (i) For all $x \in A$, $\langle F \cup \{x\} \rangle = \{z \in A : z \geq y \otimes x^n, \exists n \in \mathbb{N}, y \in F\}$;
- (ii) F is a maximal filter iff for all $x \notin F$, there exist $n \in \mathbb{N}$ and $b \in \mathcal{I}(A)$ with $x \leq b$ such that $\lambda_b(x^n) \in F$.

Proof. (i) It is obvious.

(ii) Let F be a maximal filter and $x \notin F$. We have $0 \in \langle F \cup \{x\} \rangle$ by (i) and so there exist $n \in \mathbb{N}$ and $y \in F$ such that $0 = y \otimes x^n$. There exists $b \in \mathcal{I}(A)$ such that $x, y \leq b$. Then $b = \lambda_b(y \otimes x^n) = \lambda_b(y) \oplus \lambda_b(x^n)$, it follows that $y \leq \lambda_b(x^n)$ and $\lambda_b(x^n) \in F$. Conversely, for any $x \in A \setminus F$, there exist $n \in \mathbb{N}$, $b \in \mathcal{I}(A)$ such that $\lambda_b(x^n) \in F$. Then $0 = \lambda_b(x^n) \otimes x^n$ and $0 \in \langle F \cup \{x\} \rangle$. It follows that $\langle F \cup \{x\} \rangle = A$ and F is a maximal filter. $\square$

Lemma 5.1. *Let F be a proper filter of an Equasi-MV algebra $\mathbf{A}$.*

- (i) *If $a \in F \cap \mathcal{I}(\mathbf{A})$, we have $a \notin I_F$.*
- (ii) *If $a \in F \cap \mathcal{I}(\mathbf{A})$, then for all $b \in \mathcal{I}(\mathbf{A})$ with $a < b$, we have $\lambda_b(a) \in I_F$.*
- (iii) *If F is a maximal filter of A , then for all $a \in \mathcal{I}(\mathbf{A})$, $a \notin I_F$ implies $a \in F$.*
- (iv) *If J is a maximal ideal of A , then*

$$\forall a \in \mathcal{I}(\mathbf{A}) \setminus J \implies \lambda_b(a) \in J, \text{ where } b \in \mathcal{I}(\mathbf{A}) \text{ and } a < b. \quad (*)$$
- (v) *If J is an ideal of A satisfying $(*)$, then F_J is a filter of A , where*

$$F_J := \{\lambda_a(x) : x \in J, a \in \mathcal{I}(\mathbf{A}) \setminus J, x < a\}.$$

Proof. (i) Suppose $a \in F \cap \mathcal{I}(\mathbf{A})$ and $a \in I_F$. There exists $b \in \mathcal{I}(\mathbf{A})$ such that $a \leq b$ and $\lambda_b(a) \in F$. It follows from $\lambda_b(a), a \in F$ that $0 = a \otimes \lambda_b(a) \in F$, which is a contradiction.

(ii) It is obvious.

(iii) Let $a \in \mathcal{I}(\mathbf{A})$ and $a \notin I_F$. For all $b \in \mathcal{I}(\mathbf{A})$ with $a \leq b$, we have $\lambda_b(a) \notin F$ by Proposition 5.1. Suppose $a \notin F$. Since F is a maximal filter, we have $\langle F \cup \{a\} \rangle = A$. By Proposition 5.2, there exist $n \in \mathbb{N}$ and $x \in F$ such that $0 = x \otimes a^n$. We have $u \in \mathcal{I}(\mathbf{A})$ such that $x, a \leq u$ and $0 = x \otimes a^n = x \otimes_u a^n$. Since $a \in \mathcal{I}(\mathbf{A})$, we get $a^n = a$ and so $u = \lambda_u(x) \oplus \lambda_u(a)$. It follows that $x \leq \lambda_u(a)$ and $\lambda_u(a) \in F$, which is a contradiction.

(iv) Suppose $a \in \mathcal{I}(\mathbf{A})$ and $a \notin J$. For any $b \in \mathcal{I}(\mathbf{A})$ and $a < b$, we have $\lambda_b(a) \in \langle J \cup \{a\} \rangle = A$. By Lemma 4.4, there exist $n \in \mathbb{N}$ and $x \in J$ such that $\lambda_b(a) \leq x \oplus n.a$. Since $a, \lambda_b(a) \in [0, b]$, we have

$$\begin{aligned} \lambda_b(a) &= \lambda_b(a) \oplus 0 \\ &= \lambda_b(a) \mathbin{\small{\frown}} (x \oplus n.a) \\ &\leq (\lambda_b(a) \mathbin{\small{\frown}} x) \oplus (\lambda_b(a) \mathbin{\small{\frown}} n.a) \text{ (Lemma 2.4 (iii))} \\ &= \lambda_b(a) \mathbin{\small{\frown}} x. \end{aligned}$$

It follows $\lambda_b(a) \leq x \in J$ and so $\lambda_b(a) \in J$.

(v) Suppose $x, y \in A$ with $x \leq y$ and $x \in F_J$. There exists $a \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x < a$ and $\lambda_a(x) \in J$. Let $b \in \mathcal{I}(\mathbf{A})$ and $a, y \leq b$. We have $\lambda_b(y) \leq \lambda_b(x) \leq \lambda_a(x) \oplus \lambda_b(a)$. By (iv), we have $\lambda_b(a) \in J$ and $\lambda_a(x) \oplus \lambda_b(a) \in J$. That implies $\lambda_b(y) \in J$ and $y \in F_J$.

Let $x, y \in F_J$. There exist $a, b \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x \leq a, y \leq b$ and $\lambda_a(x), \lambda_b(y) \in J$. Let $c \in \mathcal{I}(\mathbf{A})$ and $a, b \leq c$. We have $\lambda_c(a), \lambda_c(b) \in J$ by (iv) and $\lambda_c(x) \leq \lambda_c(x) \oplus 0 = \lambda_a(x) \oplus \lambda_c(a) \in J, \lambda_c(y) \leq \lambda_c(y) \oplus 0 = \lambda_b(y) \oplus \lambda_c(b) \in J$ by Proposition 3.1. It follows that $\lambda_c(x), \lambda_c(y) \in J$ and $\lambda_c(x) \oplus \lambda_c(y) \in J$. Thus $\lambda_c(\lambda_c(x) \oplus \lambda_c(y)) \in F_J$. That is, $x \otimes y = x \otimes_c y \in F_J$. $\square$

Definition 5.3. *Let $\mathbf{A}$ be an Equasi-MV algebra and I be an ideal of $\mathbf{A}$. We call I to be prime if for all $x, y \in A$, $x \mathbin{\small{\frown}} y \in I$ implies that $x \in I$ or $y \in I$.*

Proposition 5.3. *Let I be an ideal of an Equasi-MV algebra $\mathbf{A}$. Then I is prime iff*

for any $x, y \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ with $x, y \leq a$ such that $\lambda_a(\lambda_a(x) \oplus y) \in I$ or $\lambda_a(\lambda_a(y) \oplus x) \in I$.

Proof. $\Leftarrow$: Let $\pi: A \rightarrow A/I$ be the canonical projection and θ be an ideal congruence. If $x \mathbin{\frown} y \in I$, we have $(x \mathbin{\frown} y)/\theta = x/\theta \mathbin{\frown} y/\theta \in \pi(I)$. Let $x/\theta = [i]$ or $y/\theta = [j]$, where $i, j \in I$. There exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y, i, j \leq a$, $\lambda_a(x) \otimes i \in I$, $\lambda_a(i) \otimes x \in I$ or $\lambda_a(y) \otimes j \in I$, $\lambda_a(j) \otimes y \in I$. It follows from Proposition 4.1 that $x \in I$ or $y \in I$.

$\Rightarrow$: For any $x, y \in A$, there exists $a \in \mathcal{I}(\mathbf{A})$ such that $x, y \leq a$. We have

$$\begin{aligned} & (\lambda_a(x) \oplus y) \uplus (\lambda_a(y) \oplus x) \\ &= \lambda_a(x) \oplus y \oplus \lambda_a(\lambda_a(x) \oplus y \oplus \lambda_a(\lambda_a(y) \oplus x)) \\ &= \lambda_a(x) \oplus \lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(y) \oplus x)) \oplus \lambda_a(\lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(y) \oplus x)) \oplus \lambda_a(y)) \\ &= \lambda_a(y) \oplus x \oplus \lambda_a(\lambda_a(y) \oplus x \oplus x) \oplus \lambda_a(\lambda_a(\lambda_a(x) \oplus \lambda_a(\lambda_a(y) \oplus x)) \oplus \lambda_a(y)) \\ &= \lambda_a(x) \oplus \lambda_a(\lambda_a(y) \oplus x) \oplus \lambda_a((\lambda_a(x) \oplus \lambda_a(\lambda_a(y) \oplus x)) \oplus y) \oplus x \oplus \lambda_a(\lambda_a(y) \oplus x \oplus x) \\ &= a. \end{aligned}$$

It follows $\lambda_a((\lambda_a(x) \oplus y) \uplus (\lambda_a(y) \oplus x)) = 0 \in I$. That is, $\lambda_a(\lambda_a(x) \oplus y) \mathbin{\frown} \lambda_a(\lambda_a(y) \oplus x) = 0 \in I$. Therefore, $\lambda_a(\lambda_a(x) \oplus y) \in I$ or $\lambda_a(\lambda_a(y) \oplus x) \in I$. $\square$

Example 5.1. Let $A \times M$ be an Equasi-MV algebra mentioned in Example 3.6. It can be easily proved that $P = \{0, b\}$ is a prime ideal of a quasi-MV algebra A . Now we show that $P \times M$ is a prime ideal of an Equasi-MV algebra $A \times M$. Obviously, $\langle 0, 0 \rangle \in P \times M$ and $\langle 0, M \rangle \oplus \langle b, M \rangle = \langle b, M \rangle \in P \times M$. And for any $\langle x, M \rangle \leq \langle b, M \rangle$, we have $\langle x, M \rangle \in P \times M$. Then $P \times M$ is an ideal of $A \times M$. For any $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in A \times M$, suppose $\langle x_1, y_1 \rangle \mathbin{\frown} \langle x_2, y_2 \rangle = \langle x_1 \mathbin{\frown} x_2, y_1 \wedge y_2 \rangle \in P \times M$, we have $x_1 \in P$ or $x_2 \in P$. That is, $\langle x_1, y_1 \rangle \in P \times M$ or $\langle x_2, y_2 \rangle \in P \times M$.

Let $\mathbf{A}$ be a proper Equasi-MV algebra and $a \in \mathcal{I}(\mathbf{A}) \setminus \{0\}$. We define

$$\uparrow a = \{x \in A : x > a\}.$$

Then $\uparrow a$ is a filter of A . Moreover, $\uparrow a$ is a proper filter of A .

Proposition 5.4. Let F be a maximal filter of an Equasi-MV algebra $\mathbf{A}$. Then

$$I_F = \{\lambda_a(x) : x \in F, \exists a \in \mathcal{I}(\mathbf{A}), x \leq a\}$$

is a maximal ideal of A .

Proof. We know that I_F is an ideal of A by Proposition 5.1. As $F \neq \emptyset$, we have $a \in \mathcal{I}(\mathbf{A}) \cap F$ and so $a \notin I_F$ by Lemma 5.1 (i).

Let J be an ideal of A and $I_F \subseteq J$. Suppose $a \notin J$ and $a \in \mathcal{I}(\mathbf{A})$, we have $a \notin I_F$ and so $a \in F$ by Lemma 5.1 (iii). Then for any $b \in \mathcal{I}(\mathbf{A})$ with $a \leq b$, we have $\lambda_b(a) \in I_F \subseteq J$. Hence, J satisfies condition $(*)$ in Lemma 5.1 (iv). It follows from Lemma 5.1 (iv) that F_J is a filter of A .

Suppose $x \in F$ and $w \in \mathcal{I}(\mathbf{A}) \setminus J$. There exists $u \in \mathcal{I}(\mathbf{A})$ such that $x, w \leq u$. Since J is a proper ideal, we have $u \notin J$. It follows from the definition of I_F that $\lambda_u(x) \in I_F \subseteq J$ and then $x \in F_J$. That implies $F \subseteq F_J$.

Since F is a maximal filter, we have $F_J = F$ or $F_J = A$. If $F_J = A$, then there exist $x \in J$ and $a \in \mathcal{I}(\mathbf{A})$ such that $x < a$ and $\lambda_a(x) = 0$, which is a contradiction. Thus $F_J = F$. By Lemma 5.1 (v), for all $x \in J$, there exists $a \in \mathcal{I}(\mathbf{A}) \setminus J$ such that $x < a$ and $\lambda_a(x) \in F_J = F$. Hence, we have $x \in I_F$. That is, $J \subseteq I_F$. Thus $J = I_F$. This proves that I_F is a maximal ideal of A . $\square$

Theorem 5.1. *Let $\mathbf{A}$ be a proper Equasi-MV algebra. Then $\mathbf{A}$ has at least one maximal ideal.*

Proof. Suppose $0 \neq a \in A$. Note that $\uparrow a$ is a filter and $\{0\} \neq \uparrow a$. By Zorn's lemma, we know that the set of all filters that does not contain 0 has a maximal element, which is a maximal filter of A , denoted by F . It follows from Proposition 5.4 that I_F is a maximal ideal. $\square$

The following statement gives the prime theorem on Equasi-MV algebras.

Theorem 5.2. *Let I be a proper ideal of an Equasi-MV algebra $\mathbf{A}$ and $a \in A \setminus I$. Then there exists a maximal ideal P which contains I and $a \in A \setminus P$. Moreover, P is prime.*

Proof. Let $M = \{J : I \subseteq J, a \notin J\}$ where I, J are ideals of A . By Zorn's lemma, M has a top element P . It follows from $I \in M$ that $M \neq \emptyset$. We claim that P is prime. Suppose $x \mathbin{\&}\! y \in P$ and $x, y \notin P$. We have $a \in \langle P \cup \{x\} \rangle$ and $a \in \langle P \cup \{y\} \rangle$. Then there exist $n \in \mathbb{N}$ and $u, v \in P$ such that $a \leq u \oplus n.x$ and $a \leq v \oplus n.y$. It follows that

$$a \leq (u \oplus n.x) \mathbin{\&}\! (v \oplus n.y) \leq (u \oplus v \oplus n.x) \mathbin{\&}\! (u \oplus v \oplus n.y).$$

By Lemma 2.4 (iii), we have

$$a \leq (u \oplus v \oplus n.x) \mathbin{\&}\! (u \oplus v \oplus n.y) = (u \oplus v) \oplus (n.x \mathbin{\&}\! n.y) \leq (u \oplus v) \oplus n^2.(x \mathbin{\&}\! y) \in P.$$

It follows that $a \in P$, which is a contradiction. Thus, we have $x \in P$ or $y \in P$. $\square$

6 Conclusion

In this paper, we introduce the notion of Equasi-MV algebras, which are generalizations of quasi-MV algebras. We study some basic properties of Equasi-MV algebras, such as ideals, ideal congruences and filters and investigate their mutual relationships. We show that there is a one-to-one correspondence between the set of all ideals and the set of all ideal congruences on an Equasi-MV algebra. And we also studied some results on maximal ideals and prime ideals.

There are many topics that deserve further study. For example, (1) can any Equasi-MV algebra be embedded into an Equasi-MV algebra with a top element?

(2) Does any simple Equasi-MV algebra have a top element? (3) The author introduced ME-algebras and studied the categorical equivalence between equality algebras and abelian lattice-ordered groups in Liu [2019]. We will study the relationships between monadic Equasi-MV algebras and monadic equality algebras.

References

- F. Paoli A. Ledda, M. König and R. Giuntini. MV-algebras and quantum computation. *Studia Logica*, 82(2):245–270, 2006.
- C. C. Chang. Algebraic analysis of many valued logics. *Transactions of the American Mathematical society*, 88(2):467–490, 1958.
- P. F. Conrad and M. R. Darnel. Generalized Boolean algebras in lattice-ordered groups. *Order*, 14(4):295–319, 1997.
- A. Dvurečenskij and O. Zahiri. On EMV-algebras. *Fuzzy Sets and Systems*, 373: 116–148, 2019.
- A. Ledda F. Bou, F. Paoli and H. Freytes. On some properties of quasi-MV algebras and $\sqrt{\prime}$ quasi-MV algebras. Part II. *Soft Computing*, 12(4):341–352, 2008.
- R. Giuntini F. Paoli, A. Ledda and H. Freytes. On some properties of quasi-MV algebras and $\sqrt{\prime}$ quasi-MV algebras. *Reports Math. Log.*, 44:31–63, 2009.
- R. Giuntini G. Cattaneo, M. L. Dalla Chiara and R. Leporini. An unsharp logic from quantum computation. *International Journal of Theoretical Physics*, 43 (7):1803–1817, 2004.
- H. Liu. On categorical equivalences of equality algebras and monadic equality algebras. *Logic Journal of the IGPL*, 27(3):267–280, 2019.
- H. Liu. EBL-algebras. *Soft Computing*, 24(19):14333–14343, 2020.
- J. Liu and W. Chen. A non-commutative generalization of quasi-MV algebras. In *2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 122–126. IEEE, 2016.
- R. Giuntini M. L. Dalla Chiara and R. Greechie. *Reasoning in quantum theory: sharp and unsharp quantum logics*, volume 22. Springer Science & Business Media, 2013.
- R. Giuntini M. L. Dalla Chiara and R. Leporini. Logics from quantum computation. *International Journal of Quantum Information*, 3(02):293–337, 2005.

Some common fixed point theorems in complex valued fuzzy metric spaces

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Abstract

In this paper, we aim to prove some common fixed point theorems for pairs of any mappings, for pairs of occasionally weakly compatible mappings satisfying some conditions in complex valued fuzzy metric spaces.

Keywords: Complex fuzzy set, complex valued continuous t -norm, complex valued fuzzy metric spaces, occasionally weakly compatible mappings, common fixed point.

2020 AMS subject classifications: 47H10, 54H25. ¹

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¹Received on September 1, 2021. Accepted on November 30, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.645 . ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [Zadeh, 1965] in 1965. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Since then it has become a vigorous area of research in engineering, medical science, social science, graph theory, metric space theory, complex analysis etc. Deng [Deng, 1982], Erceg [Erceg, 1979], Kaleva and Seikkala [Kaleva and Seikkala, 1984], Kramosil and Michalek [Kramosil and Michalek, 1975] have introduced the concepts of fuzzy metric spaces in different ways. George and Veermani [George and Veeramani, 1994] modified the notion of fuzzy metric spaces with the help of continuous t -norms. The concepts of compatible, weakly compatible, occasionally weakly compatible mappings in fuzzy metric space are present in the paper by C. T. Aage and J. N. Salunke [Aage and Salunke, 2009]. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions in fuzzy metric spaces.

The concept of fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley (see [Buckley, 1987], [Buckley, 1989], [Buckley, 1991], [Buckley, 1992]). Motivated by the work of Buckley some authors continued research in fuzzy complex numbers. In this series Ramot *et al.* [D. Ramot and Kandel, 2002] extended fuzzy sets to complex fuzzy sets as a generalization. According to Ramot *et al.* [D. Ramot and Kandel, 2002], the complex fuzzy set is characterized by a membership function, whose range is not limited to $[0, 1]$ but extended to the unit circle in the complex plane.

Azam *et al.* [A. Azam and Khan, 2011] introduced the notion of complex valued metric space which is a generalization of classical metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces. Additionally it offers numerous research activities in mathematical analysis.

Later, D. Singh *et al.* [D. Singh and Kumam, 2016] defined the notion of complex valued fuzzy metric spaces with the help of complex valued continuous t -norm and also defined the notion of convergent sequence, cauchy sequence in complex valued fuzzy metric spaces.

This paper presents some common fixed point theorems for pairs of any mappings and pairs of occasionally weakly compatible mappings satisfying some conditions in the complex valued fuzzy metric spaces. We also provide some examples which support the main results here.

2 Definitions and Notations

Here and in the following, let $\mathbb{R}, \mathbb{R}_0^+, \mathbb{C}$ and $\mathbb{N}$ be the sets of real numbers, non negative real numbers, complex numbers and positive integers, respectively.

Definition 2.1. [J. Choi and Islam, 2017] Define a partial order relation $\preceq$ on $\mathbb{C}$ as follows: For $z_1, z_2 \in \mathbb{C}$,

$$z_1 \preceq z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2) \text{ and } \Im(z_1) \leq \Im(z_2).$$

Thus $z_1 \preceq z_2$ if any one of the following statements holds:

- (o_1) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$
- (o_2) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$
- (o_3) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$
- (o_4) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$,

where $\Re(z)$ and $\Im(z)$ indicates respectively the real and imaginary parts of the complex number z .

We write $z_1 \prec z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$, i.e., any one of (o_2), (o_3) and (o_4) is satisfied and we write $z_1 \prec z_2$ if only (o_4) is satisfied. Considering (o_1)-(o_4), the following properties for the partial order $\preceq$ on $\mathbb{C}$ hold true:

- (i) $0 \preceq z_1 \preceq z_2 \implies |z_1| \leq |z_2|$
- (ii) $z_1 \preceq z_2$ and $z_2 \preceq z_3 \implies z_1 \preceq z_3$
- (iii) $z_1 \preceq z_2$ and $\lambda > 0$ ($\lambda \in \mathbb{R}$) $\implies \lambda z_1 \preceq \lambda z_2$.

Note: $z_1 \preceq z_2$ and $z_2 \preceq z_1$ have the same meaning.

Definition 2.2. [J. Choi and Islam, 2017] The max function for the partial order $\preceq$ on $\mathbb{C}$ is defined as follows:

- (i) $\max\{u, v\} = v \iff u \preceq v$
- (ii) $u \preceq \max\{u, v\} \implies u \preceq v$
- (iii) $u \preceq \max\{v, w\} \implies u \preceq v$ or $u \preceq w$.

For any $0 \preceq u, 0 \preceq v$, we can easily prove that $|\max\{u, v\}| = \max\{|u|, |v|\}$, where $|\cdot|$ denotes the usual modulus of complex number.

Definition 2.3. [J. Choi and Islam, 2017] The min function for the partial order $\preceq$ on $\mathbb{C}$ is defined as follows:

- (i) $\min\{u, v\} = u \iff u \preceq v$
- (ii) $\min\{u, v\} \preceq v \implies u \preceq v$
- (iii) $\min\{u, w\} \preceq v \implies u \preceq v$ or $w \preceq v$.

For any $0 \preceq u, 0 \preceq v$, we can easily prove that $|\min\{u, v\}| = \min\{|u|, |v|\}$.

Definition 2.4. [Zadeh, 1965] A fuzzy set on a non empty set X is just a function $\mu : X \rightarrow [0, 1]$.

Definition 2.5. [D. Ramot and Kandel, 2002] A complex fuzzy set S , defined on a universe discourse U , is characterized by a membership function $\mu_s(x)$ that assigns every element $x \in U$, a complex valued grade of membership in S . The values $\mu_s(x)$ lie within the unit circle in the complex plane and are thus of the form $\mu_s(x) = r_s(x)e^{iw_s(x)}$, ($i = \sqrt{-1}$), where $r_s(x)$ and $w_s(x)$ both real valued, with $r_s(x) \in [0, 1]$. The complex fuzzy set S may be represented as the set of ordered pairs given by $S = \{(x, \mu_s(x)) : x \in U\}$.

Definition 2.6. [D. Singh and Kumam, 2016] A binary operation $*$: $r_se^{i\theta} \times r_se^{i\theta} \rightarrow r_se^{i\theta}$, where $r_s \in [0, 1]$ and a fix $\theta \in [0, \frac{\pi}{2}]$, is called complex valued continuous t -norm if it satisfies the followings:

- (t1) $*$ is associative and commutative,
- (t2) $*$ is continuous,
- (t3) $a * e^{i\theta} = a, \forall a \in r_se^{i\theta}$,
- (t4) $a * b \preceq c * d$ whenever $a \preceq c$ and $b \preceq d, \forall a, b, c, d \in r_se^{i\theta}$.

Example 2.1. [D. Singh and Kumam, 2016] The followings are examples for complex valued continuous t -norm: (i) $a * b = \min(a, b), \forall a, b \in r_se^{i\theta}$ and a fix $\theta \in [0, \frac{\pi}{2}]$ (ii) $a * b = \max(a + b - e^{i\theta}, 0), \forall a, b \in r_se^{i\theta}$ and a fix $\theta \in [0, \frac{\pi}{2}]$.

Definition 2.7. [D. Singh and Kumam, 2016, Demir, 2021] The triplet $(X, M, *)$ is said to be complex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a complex valued continuous t -norm and $M : X \times X \times (0, \infty) \rightarrow r_se^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:

- (cf1) $0 \prec M(x, y, t)$,
- (cf2) $M(x, y, t) = e^{i\theta}, \forall t \in (0, \infty)$ if and only if $x = y$,

- (cf3) $M(x, y, t) = M(y, x, t)$,
- (cf4) $M(x, y, t + s) \succeq M(x, z, t) * M(z, y, s)$,
- (cf5) $M(x, y, t) : (0, \infty) \rightarrow r_se^{i\theta}$ is continuous,

for all $x, y, z \in X ; t, s \in (0, \infty); r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. The pair $(M, *)$ is called a complex valued fuzzy metric and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t . It is noted that if we take $\theta = 0$, then complex valued fuzzy metric simply goes to real valued fuzzy metric.

Note: It is clear that $r_se^{i\theta} \preceq e^{i\theta}$ and consequently, $M(x, y, t) \preceq e^{i\theta}$ for all $x, y \in X, t \in (0, \infty), r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$.

Example 2.2. [D. Singh and Kumam, 2016] Let (X, d) be a real valued metric space. Let $a * b = \min\{a, b\}$, for all $a, b \in r_se^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$.

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For each $t > 0$, $x, y \in X$, we define

$$M(x, y, t) = e^{i\theta} \frac{kt^n}{kt^n + md(x, y)},$$

where $k, m, n \in \mathbb{N}$. Then $(X, M, *)$ is a complex valued fuzzy metric space. By choosing $k = m = n = 1$, we get

$$M(x, y, t) = e^{i\theta} \frac{t}{t + d(x, y)}.$$

This complex valued fuzzy metric space induced by a metric d is referred to as a standard complex valued fuzzy metric space.

Example 2.3. [D. Singh and Kumam, 2016] Let $X = \mathbb{R}$. Let $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$, $x, y \in X$, we define

$$M(x, y, t) = e^{i\theta} e^{-\frac{|x-y|}{t}}.$$

Then $(X, M, *)$ is a complex valued fuzzy metric space.

Definition 2.8. [D. Singh and Kumam, 2016] Let $(X, M, *)$ be a complex valued fuzzy metric space. We define an open ball $B(x, r, t)$ with centre $x \in X$ and radius $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}$, $t > 0$ as: $B(x, r, t) = \{y \in X : M(x, y, t) \succ e^{i\theta} - r\}$, where $\theta \in [0, \frac{\pi}{2}]$. A point $x \in X$ is said to be interior point of a set $A \subset X$, whenever there exists $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}$ such that $B(x, r, t) = \{y \in X : M(x, y, t) \succ e^{i\theta} - r\} \subset A$, where $\theta \in [0, \frac{\pi}{2}]$. A subset A of X is called open if every element of A is an interior point of A . If we define $\tau = \{A \subset X : x \in A \text{ if and only if there exists } t > 0 \text{ and } r \in \mathbb{C}, 0 \prec r \prec e^{i\theta}, \theta \in [0, \frac{\pi}{2}] \text{ such that } B(x, r, t) \subset A\}$. Then one can easily check that τ is a topology on X .

Definition 2.9. [D. Singh and Kumam, 2016] Let $(X, M, *)$ be complex valued fuzzy metric space and τ be the topology induced by complex valued fuzzy metric. Let $\{x_n\}$ be any sequence in X . The sequence $\{x_n\}$ is said to converges to $x \in X$ if and only if for any $t > 0$, $M(x_n, x, t) \rightarrow e^{i\theta}$ as $n \rightarrow \infty$ or $|M(x_n, x, t)| \rightarrow 1$ as $n \rightarrow \infty$. A sequence $\{x_n\}$ in a complex valued fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for any $t > 0$, $M(x_m, x_n, t) \rightarrow e^{i\theta}$ as $m, n \rightarrow \infty$ or $|M(x_m, x_n, t)| \rightarrow 1$ as $m, n \rightarrow \infty$. A complex valued fuzzy metric space in which every cauchy sequece is convergent is called complex valued complete fuzzy metric space.

For example (see [D. Singh and Kumam, 2016]), let $X = \mathbb{R}$ and $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define $M(x, y, t) = \frac{te^{i\theta}}{t + |x-y|}$. Then $(X, M, *)$ is complex valued complete fuzzy metric space.

Definition 2.10. Let $(X, M, *)$ be a complex valued fuzzy metric space and $S, T : X \rightarrow X$ be two mappings. A point $x \in X$ is said to be a coincidence point of S and T if and only if $Sx = Tx$. We shall call $w = Sx = Tx$ a point of coincidence of S and T . Moreover, if $Sx = Tx = x$, then the point $x \in X$ is called common fixed point of S and T .

By study of the paper by C. T. Aage and J. N. Salunke (see [Aage and Salunke, 2009]), we find the definitions of compatible, weakly compatible, occasionally weakly compatible mappings in fuzzy metric space. In the similar way, we can define the same definitions in complex valued fuzzy metric space as follows:

Definition 2.11. Let $(X, M, *)$ be a complex valued fuzzy metric space and $S, T : X \rightarrow X$ be two self mappings. The self maps S and T on X are said to be commuting if $STx = TSx$, for all $x \in X$. The self maps S and T are said to be compatible if

$$\lim_{n \rightarrow \infty} M(STx_n, TSx_n, t) = e^{i\theta}, t > 0 \text{ or } \lim_{n \rightarrow \infty} |M(STx_n, TSx_n, t)| = 1, t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = x$, for some $x \in X$.

Definition 2.12. Let $(X, M, *)$ be a complex valued fuzzy metric space and $S, T : X \rightarrow X$ be two mappings. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, that is, they commute at their coincidence points.

Definition 2.13. Let $(X, M, *)$ be a complex valued fuzzy metric space and $S, T : X \rightarrow X$ be two mappings. The self-maps S and T are said to be occasionally weakly compatible if and only if there is a coincidence point x (say) in X of S and T at which S and T commute, i.e., $STx = TSx$.

It is needed to mention that every weakly compatible map is occasionally weakly compatible but converse is not always true. The supporting example is given below:

Example 2.4. Let $\mathbb{R}$ be the set of real numbers with standard complex valued fuzzy metric space. Define $S, T : \mathbb{R} \rightarrow \mathbb{R}$ by $Sx = x^2 + x$ and $Tx = 2x$ for all $x \in \mathbb{R}$. Then $Sx = Tx$ for $x = 0, 1$ but $ST0 = TS0$ and $ST1 \neq TS1$. Therefore S and T are occasionally weakly compatible self maps but not weakly compatible.

3 Lemmas

Here we set lemmas which will be required in the sequel.

Lemma 3.1. [Jungck and Rhoades, 2006] *Let X be any set and S, T be occasionally weakly compatible self maps of X . If S and T have a unique point of coincidence $w = Sx = Tx$, then w is the unique common fixed point of S and T .*

D. Singh *et al.* [D. Singh and Kumam, 2016] state and prove the following lemma for complex valued complete fuzzy metric space, but it is also true without the restriction of completeness. So we state the lemma without the restriction of completeness.

Lemma 3.2. *Let $(X, M, *)$ be a complex valued fuzzy metric space such that*

$$\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}, \text{ for all } x, y \in X.$$

If $M(x, y, kt) \succsim M(x, y, t)$, for some $0 < k < 1$, for all $x, y \in X, t \in (0, \infty)$, then $x = y$.

Lemma 3.3. [D. Singh and Kumam, 2016] *Let $\{x_n\}$ be a sequence in a complex valued fuzzy metric space $(X, M, *)$ with*

$$\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}, \text{ for all } x, y \in X.$$

If there exists $k \in (0, 1)$ such that $M(x_{n+1}, x_{n+2}, kt) \succsim M(x_n, x_{n+1}, t)$, for all $t > 0$ and $n = 0, 1, 2, 3, \dots$. Then $\{x_n\}$ is a Cauchy sequence in X .

We can easily prove the following lemma.

Lemma 3.4. *Let $(X, M, *)$ be a complex valued fuzzy metric space. Let $\{x_n\}$ be a sequence in X converging to $x \in X$. Then any subsequence of $\{x_n\}$ converges to the same point $x \in X$.*

4 Main Results

Here we present some fixed point theorems on complex valued fuzzy metric spaces.

Theorem 4.1. *Let $(X, M, *)$ be a complex valued complete fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X, t > 0$ and let S and T be self-mappings on X . If there exists $k \in (0, 1)$ such that*

$$M(Sx, Ty, kt) \succsim M(x, y, t), \tag{1}$$

for all $x, y \in X$ and for all $t > 0$, then S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point and we define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}; n = 0, 1, 2, 3, \dots$$

Now, for $k \in (0, 1)$ and for all $t > 0$, we have by the condition (1)

$$\begin{aligned} M(x_{2n+1}, x_{2n+2}, kt) &= M(Sx_{2n}, Tx_{2n+1}, kt) \\ &\succsim M(x_{2n}, x_{2n+1}, t). \end{aligned}$$

and

$$\begin{aligned} M(x_{2n}, x_{2n+1}, kt) &= M(Sx_{2n-1}, Tx_{2n}, kt) \\ &\succsim M(x_{2n-1}, x_{2n}, t). \end{aligned}$$

In general, we have $M(x_{n+1}, x_{n+2}, kt) \succsim M(x_n, x_{n+1}, t)$, for all $t > 0$ and $k \in (0, 1)$; $n = 0, 1, 2, 3, \dots$. By Lemma 3.3, $\{x_n\}$ is a cauchy sequence in X . Since X is complete, then there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Obviously $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ in X , by Lemma 3.4, they are also converge to the same point $u \in X$, i.e., $x_{2n} \rightarrow u$ and $x_{2n+1} \rightarrow u$ as $n \rightarrow \infty$. Now, we claim that u is the common fixed point of S and T .

For this, using condition (1), we have

$$\begin{aligned} M(Su, u, kt) &= M(Su, u, \frac{kt}{2} + \frac{kt}{2}) \\ &\succsim M(Su, x_{2n+2}, \frac{kt}{2}) * M(x_{2n+2}, u, \frac{kt}{2}) \\ &= M(Su, Tx_{2n+1}, \frac{kt}{2}) * M(x_{2n+2}, u, \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{t}{2}) * M(x_{2n+2}, u, \frac{kt}{2}). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get, $M(Su, u, kt) \succsim e^{i\theta} * e^{i\theta} = e^{i\theta}$. So $Su = u$. Again,

$$\begin{aligned} M(u, Tu, kt) &= M(u, Tu, \frac{kt}{2} + \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{kt}{2}) * M(x_{2n+1}, Tu, \frac{kt}{2}) \\ &= M(u, x_{2n+1}, \frac{kt}{2}) * M(Sx_{2n}, Tu, \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{kt}{2}) * M(x_{2n}, u, \frac{t}{2}). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get $M(u, Tu, kt) \succsim e^{i\theta} * e^{i\theta} = e^{i\theta}$. Therefore $Tu = u$.

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Thus $Su = Tu = u$ and therefore u is a common fixed point of S and T .

For uniqueness, let z be another fixed point of S and T .

Now, using condition (1),

$$\begin{aligned} M(u, z, kt) &= M(Su, Tz, kt). \\ &\succsim M(u, z, t). \end{aligned}$$

By Lemma 3.2, $u = z$. This completes the theorem. $\square$

If we consider $S = T$ in Theorem 4.1, we get the following corollary.

Corollary 4.1. *Let $(X, M, *)$ be a complex valued complete fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$, $t > 0$ and let S be self-mapping on X . If there exists $k \in (0, 1)$ such that*

$$M(Sx, Sy, kt) \succsim M(x, y, t),$$

for all $x, y \in X$ and for all $t > 0$, then S has a unique fixed point in X .

A supporting example to Corollary 4.1 is given below.

Example 4.1. *Let $X = \mathbb{R}$ and $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define*

$$M(x, y, t) = \frac{te^{i\theta}}{t + |x - y|}.$$

Then certainly $(X, M, *)$ is complex valued complete fuzzy metric space with

$$\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}.$$

We define the self map S on X by

$$Sx = \frac{x+1}{2}, \text{ for all } x \in X.$$

Now, for any $t > 0$ and for $k = \frac{1}{2}$,

$$\begin{aligned} M(Sx, Sy, \frac{t}{2}) &= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2} + |Sx - Sy|}. \\ &= \frac{te^{i\theta}}{t + 2 \left| \frac{x+1}{2} - \frac{y+1}{2} \right|} = \frac{te^{i\theta}}{t + |x - y|} = M(x, y, t). \end{aligned}$$

Thus all the conditions of Corollary 4.1 are satisfied and $x = 1$ is the unique fixed point of S .

Theorem 4.2. Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$ and let A, B, S and T be self-mappings on X . Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \succsim \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}, \quad (2)$$

for all $x, y \in X$ and for all $t > 0$, then A, B, S and T have a unique common fixed point in X .

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.

Now, by the given condition (2), we get

$$\begin{aligned} M(Ax, By, kt) &\succsim \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\} \\ &= \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ &\quad M(Ax, By, t), M(By, Ax, t)\} \\ &= \min\{M(Ax, By, t), e^{i\theta}, e^{i\theta}, M(Ax, By, t), M(By, Ax, t)\} \\ &= M(Ax, By, t). \end{aligned}$$

In view of Lemma 3.2, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \quad (3)$$

Suppose that the pair $\{A, S\}$ have an another coincidence point $z \in X$, i.e., $Az = Sz$.

Now,

$$\begin{aligned} M(Az, By, kt) &\succsim \min\{M(Sz, Ty, t), M(Sz, Az, t), M(By, Ty, t), \\ &\quad M(Az, Ty, t), M(By, Sz, t)\} \\ &= \min\{M(Az, By, t), M(Az, Az, t), M(By, By, t), \\ &\quad M(Az, By, t), M(By, Az, t)\} \\ &= \min\{M(Az, By, t), e^{i\theta}, e^{i\theta}, M(Az, By, t), M(By, Az, t)\} \\ &= M(Az, By, t). \end{aligned}$$

Again, in view of Lemma 3.2, we have $Az = By$. Therefore

$$Az = Sz = By = Ty. \quad (4)$$

From (3) and (4), $Ax = Az$ and therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, w is the unique common fixed point of $\{A, S\}$. Similarly, we can show that the pair $\{B, T\}$ have also a unique common fixed point. Suppose this is $u \in X$.

Now,

$$\begin{aligned}
 M(w, u, kt) &= M(Aw, Bu, kt) \\
 &\succsim \min\{M(Sw, Tu, t), M(Sw, Aw, t), M(Bu, Tu, t), \\
 &\quad M(Aw, Tu, t), M(Bu, Sw, t)\} \\
 &= \min\{M(w, u, t), M(w, w, t), M(u, u, t), \\
 &\quad M(w, u, t), M(u, w, t)\} \\
 &= \min\{M(w, u, t), e^{i\theta}, e^{i\theta}, M(w, u, t), M(u, w, t)\} \\
 &= M(w, u, t).
 \end{aligned}$$

Therefore using Lemma 3.2, we have $w = u$ and consequently, w is common fixed point of A, B, S and T .

To show, this common fixed point is unique, let v is an another common fixed point of A, B, S and T .

Now,

$$\begin{aligned}
 M(w, v, kt) &= M(Aw, Bv, kt) \\
 &\succsim \min\{M(Sw, Tv, t), M(Sw, Aw, t), M(Bv, Tv, t), \\
 &\quad M(Aw, Tv, t), M(Bv, Sw, t)\} \\
 &= \min\{M(w, v, t), M(w, w, t), M(v, v, t), \\
 &\quad M(w, v, t), M(v, w, t)\} \\
 &= \min\{M(w, v, t), e^{i\theta}, e^{i\theta}, M(w, v, t), M(v, w, t)\} \\
 &= M(w, v, t).
 \end{aligned}$$

By Lemma 3.2, we have $w = v$. Hence A, B, S and T have a unique common fixed point. $\square$

In the following an supporting example to Theorem 4.2 is given.

Example 4.2. Let $X = \mathbb{R}$. Consider the metric $d(x, y) = |x| + |y|$, for all $x \neq y$ and $d(x, y) = 0$, for $x = y$ on X . Let $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define

$$M(x, y, t) = \frac{te^{i\theta}}{t + d(x, y)}.$$

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Then certainly $(X, M, *)$ is complex valued fuzzy metric space with

$$\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}.$$

Now, we define the self maps A, B, S and T on X by

$$Ax = \frac{x}{3}, Bx = \frac{x}{4}, Sx = x \text{ and } Tx = \frac{x}{2}, \text{ for all } x \in X.$$

Set $k = \frac{1}{2}$. Now, for $x \neq y$,

$$\begin{aligned} M(Ax, By, \frac{t}{2}) &= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2} + |Ax| + |By|} \\ &= \frac{te^{i\theta}}{t + 2\frac{|x|}{3} + 2\frac{|y|}{4}} \\ &\succsim \frac{te^{i\theta}}{t + 2\frac{|x|}{2} + 2\frac{|y|}{4}} \\ &= \frac{te^{i\theta}}{t + |x| + \frac{|y|}{2}} \\ &= \frac{te^{i\theta}}{t + |Sx| + |Ty|} = M(Sx, Ty, t). \end{aligned}$$

For the case $x = y$,

$$M(Ax, By, \frac{t}{2}) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2} + 0} = e^{i\theta} = M(Sx, Ty, t).$$

Therefore, for any $x, y \in X$,

$$\begin{aligned} M(Ax, By, \frac{t}{2}) &\succsim M(Sx, Ty, t) \\ &= \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}. \end{aligned}$$

Therefore, the maps A, B, S and T satisfies the condition (2) of Theorem 4.2 for $k = \frac{1}{2}$. Also the pairs $\{A, S\}$ and $\{B, T\}$ are obviously occasionally weakly compatible. Thus all the conditions of Theorem 4.2 are satisfied and $x = 0$ is the unique common fixed point of A, B, S and T in X .

Setting $A = B$ and $S = T$ in Theorem 4.2, we get the following corollary.

Corollary 4.2. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$ and let A, S be self-mappings on X . Let the pair $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$M(Ax, Ay, kt) \succeq \min\{M(Ax, Sy, t), M(Ay, Sx, t), M(Sx, Sy, t), \\ M(Sx, Ax, t), M(Sy, Ay, t)\},$$

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X .

Theorem 4.3. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$ and let A, B, S and T be self-mappings on X . Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$M(Ax, By, kt) \succeq g(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ M(Ax, Ty, t), M(By, Sx, t)\}),$$

for all $x, y \in X$ and for all $t > 0$, where $g : r_s e^{i\theta} \rightarrow r_s e^{i\theta}$ with $g(x) \succeq x$ for all $x \in r_s e^{i\theta}$, where $r_s \in (0, 1)$ and $\theta \in [0, \frac{\pi}{2}]$. Then A, B, S and T have a unique common fixed point in X .

Proof. The proof follows from Theorem 4.2. □

Theorem 4.4. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$, and let A, B, S and T be self-mappings on X . Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$M(Ax, By, kt) \succeq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t), \quad (5)$$

for all $x, y \in X$ and for all $t > 0$. Then A, B, S and T have a unique common fixed point in X .

Proof. The pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.

Now, the condition (5) gives

$$\begin{aligned} M(Ax, By, kt) &\succeq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(Ax, Ty, t) \\ &= M(Ax, By, t) * M(Ax, Ax, t) * M(By, By, t) \\ &\quad * M(Ax, By, t) \\ &= M(Ax, By, t) * e^{i\theta} * e^{i\theta} * M(Ax, By, t) \\ &= M(Ax, By, t). \end{aligned}$$

In view of Lemma 3.2, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \quad (6)$$

Suppose that the pair $\{A, S\}$ have an another coincidence point z (say) $\in X$, i.e., $Az = Sz$.

Now,

$$\begin{aligned} M(Az, By, kt) &\succsim M(Sz, Ty, t) * M(Az, Sz, t) * M(By, Ty, t) \\ &\quad * M(Az, Ty, t) \\ &= M(Az, By, t) * M(Az, Az, t) * M(By, By, t) \\ &\quad * M(Az, By, t) \\ &= M(Az, By, t) * e^{i\theta} * e^{i\theta} * M(Az, By, t) \\ &= M(Az, By, t). \end{aligned}$$

Using Lemma 3.2, $Az = By$ and consequently

$$Az = Sz = By = Ty. \quad (7)$$

From (6) and (7) we get, $Ax = Az$ and therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, w is the unique common fixed point of $\{A, S\}$. Similarly, we can show that there is a unique common fixed point u (say) $\in X$ of $\{B, T\}$.

Now,

$$\begin{aligned} M(w, u, kt) &= M(Aw, Bu, kt) \\ &\succsim M(Sw, Tu, t) * M(Aw, Sw, t) * M(Bu, Tu, t) \\ &\quad * M(Aw, Tu, t) \\ &= M(Aw, Bu, t) * M(Aw, Aw, t) * M(Bu, Bu, t) \\ &\quad * M(Aw, Bu, t) \\ &= M(Aw, Bu, t) * e^{i\theta} * e^{i\theta} * M(Aw, Bu, t) \\ &= M(Aw, Bu, t) \\ &= M(w, u, t). \end{aligned}$$

By Lemma 3.2, we have $w = u$ and consequently w is common fixed point of A, B, S and T . For uniqueness, let v is an another common fixed point of A, B, S and T .

Therefore

$$\begin{aligned}
 M(w, v, kt) &= M(Aw, Bv, kt) \\
 &\succeq M(Sw, Tv, t) * M(Aw, Sw, t) * M(Bv, Tv, t) \\
 &\quad * M(Aw, Tv, t) \\
 &= M(w, v, t) * M(w, w, t) * M(v, v, t) * \\
 &\quad M(w, v, t) \\
 &= M(w, v, t) * e^{i\theta} * e^{i\theta} * M(w, v, t). \\
 &= M(w, v, t)
 \end{aligned}$$

and in view of Lemma 3.2, we have $w = v$. Hence A, B, S and T have a unique common fixed point. $\square$

A supporting example to Corollary 4.4 is given below.

Example 4.3. Let $X = \mathbb{R}$. Define the metric $d(x, y) = |x| + |y|$, for all $x \neq y$ and $d(x, y) = 0$, for $x = y$ on X . Let $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define

$$M(x, y, t) = e^{i\theta} e^{-\frac{d(x, y)}{t}}.$$

Then certainly $(X, M, *)$ is complex valued fuzzy metric space with

$$\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}.$$

Now, we define the self maps A, B, S and T on X by

$$Ax = \frac{x}{5}, Bx = \frac{x}{15}, Sx = x \text{ and } Tx = \frac{x}{4}, \text{ for all } x \in X.$$

Let $k = \frac{1}{3}$. Now for $x \neq y$,

$$\begin{aligned}
 M(Ax, By, \frac{t}{3}) &= \frac{\frac{t}{3} e^{i\theta}}{\frac{t}{3} + |Ax| + |By|} \\
 &= \frac{te^{i\theta}}{t + 3\frac{|x|}{5} + 3\frac{|y|}{15}} \\
 &\succeq \frac{te^{i\theta}}{t + |x| + \frac{|y|}{4}} \\
 &= \frac{te^{i\theta}}{t + |Sx| + |Ty|} = M(Sx, Ty, t).
 \end{aligned}$$

For $x = y$,

$$M(Ax, By, \frac{t}{3}) = \frac{\frac{t}{3}e^{i\theta}}{\frac{t}{3} + 0} = e^{i\theta} = M(Sx, Ty, t).$$

Thus for any $x, y \in X$,

$$\begin{aligned} M(Ax, By, \frac{t}{3}) &\succeq M(Sx, Ty, t) \\ &= \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}. \end{aligned}$$

Therefore the maps A, B, S and T satisfies the condition (5) of Theorem 4.4 for $k = \frac{1}{3}$. Also the pairs $\{A, S\}$ and $\{B, T\}$ are obviously occasionally weakly compatible. Thus all the conditions of Theorem 4.3 are satisfied and $x = 0$ is the unique common fixed point of A, B, S and T in X .

Setting $A = B$ and $S = T$ in Theorem 4.4, we get the following corollary.

Corollary 4.3. Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$, and let A, S be self-mappings on X . Let the pair $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, Ay, kt) \succeq M(Sx, Sy, t) * M(Ax, Sx, t) * M(Ay, Sy, t) * M(Ax, Sy, t),$$

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X .

Theorem 4.5. Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$. Let the pair $\{A, S\}$ be occasionally weakly compatible self maps on X . If there exists $k \in (0, 1)$ such that

$$\begin{aligned} M(Sx, Sy, kt) &\succeq pM(Ax, Ay, t) + \\ &\quad q \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\}, \end{aligned} \quad (8)$$

for all $x, y \in X$ and for all $t > 0$, where $p, q > 0$ are reals with $p + q \geq 1$. Then A and S have a unique common fixed point in X .

Proof. The pair $\{A, S\}$ are occasionally weakly compatible, then there exists $x \in X$ such that $Ax = Sx$. Suppose that there exists another $y \in X$ for which $Ay = Sy$.

From the condition (8),

$$\begin{aligned}
 M(Sx, Sy, kt) &\succeq pM(Ax, Ay, t) + \\
 &\quad q \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\} \\
 &= pM(Sx, Sy, t) + \\
 &\quad q \min\{M(Sx, Sy, t), M(Sx, Sx, t), M(Sy, Sy, t)\} \\
 &= pM(Sx, Sy, t) + q \min\{M(Sx, Sy, t), e^{i\theta}, e^{i\theta}\} \\
 &= pM(Sx, Sy, t) + qM(Sx, Sy, t) \\
 &= (p + q)M(Sx, Sy, t).
 \end{aligned}$$

Since $p + q \geq 1$, $M(Sx, Sy, kt) \succeq M(Sx, Sy, t)$.

In view of Lemma 3.2, we have $Sx = Sy$ and consequently $Ax = Ay$. Therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, A and S have a unique common fixed point in X . $\square$

The following corollary is the direct consequence of Theorem 4.5.

Corollary 4.4. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t \rightarrow \infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$. Let A be self map on X . If there exists $k \in (0, 1)$ such that*

$$M(Ax, Ay, kt) \succeq \alpha M(Ax, Ay, t),$$

for all $x, y \in X$ and for all $t > 0$, where $\alpha \geq 1$ real. Then A has a unique fixed point in X .

5 Conclusions

In the Theorem 4.1 we have shown a sufficient condition for existence of a unique common fixed point of pair of any mappings in complex valued fuzzy metric spaces. Similar results are found in the remaining theorems 4.2, 4.3, 4.4 and 4.5 but for pairs of occasionally weakly compatible mappings.

References

- F. Brain A. Azam and M. Khan. Common fixed point theorems in complex valued metric spaces. *Numer. Funct. Anal. Optim.*, 32(3):243–253, 2011.
- C. T. Aage and J. N. Salunke. Common fixed point theorems in fuzzymetric spaces. *International Journal of Pure and Applied Mathematics*, 56(2):155–164, 2009.

- J. J. Buckley. Fuzzy complex numbers. *Proceedings of ISFK, Guangzhou, China*, pages 597–700, 1987.
- J. J. Buckley. Fuzzy complex numbers. *Fuzzy sets and systems*, 33(3):333–345, 1989.
- J. J. Buckley. Fuzzy complex analysis i: Differentiation. *Fuzzy sets and systems*, 41:269–284, 1991.
- J. J. Buckley. Fuzzy complex analysis ii: Integration. *Fuzzy sets and systems*, 49: 171–179, 1992.
- M. Friedman D. Ramot, R. Milo and A. Kandel. Complex fuzzy sets. *IEEE Transactions of Fuzzy systems*, 10(2):171–186, 2002.
- M. Imdad D. Singh, V. Joshi and P. Kumam. A novel framework of complex fuzzy metric spaces and fixed point theorems. *Journal of Intelligent and Fuzzy Systems*, 30(6):3227–3238, 2016.
- I. Demir. Fixed point theorems in complex valued fuzzy bapplication to integral equations. *Miskolc Mathematical Notes*, 22(1):153–171, 2021.
- Z.K. Deng. Fuzzy pseudo metric spaces. *J. Math. Anal. Appl.*, 86:74–95, 1982.
- M.A. Erce. A metric space in fuzzy set theory. *J. Math. Anal. Appl.*, 69:205–230, 1979.
- A. George and P. Veeramani. On some results in fuzzy metric spaces. *Fuzzy Sets and Systems*, 64:395–399, 1994.
- T. Biswas J. Choi, S.K. Datta and M.N. Islam. Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam Mathematical J.*, 39(1):115–126, 2017.
- G. Jungck and B.E. Rhoades. Fixed point theorems for occasionally weakly compatible mappings. *Fixed Point Theory*, 7(2):287–296, 2006.
- O. Kaleva and S. Seikkala. On fuzzy metric spaces. *Fuzzy Sets and Systems*, 12: 215–229, 1984.
- I. Kramosil and J. Michalek. Fuzzy metric and statistical metric spaces. *Kybernetika*, 11:336–344, 1975.
- L.A. Zadeh. Fuzzy sets. *Information and control*, 8:338–353, 1965.

A unified shape model for sunspot number cycles

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Abstract

We proposed a model which can unify many of the shape models existing in the literature and show that the shape of the sunspot number cycle can be described as a product of a polynomial and a negative exponential function. The proposed model has certain free parameters, which need to be estimated from the observed sunspot number data. Since all the models reviewed in this paper are a product of a polynomial and a negative exponential along with a number of parameters, we have seen that all these models can be derived from a modified generalized Gamma probability density function by transforming certain parameters and fixing certain parameters. In this paper, we estimate the parameters of the model from the revised monthly averaged sunspot numbers available in the SIDC website. Finally, a preliminary level prediction has also been attempted to forecast the characteristics of sunspot number cycle 25.

Keywords: sunspot numbers; gamma distribution; free parameter; etc.

2010 AMS subject classification: 70F15; 97M10.[§]

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§ Received on xxx, xxx. Accepted on xxx, xxx. Published on xxx, xxx. doi: xxx. ISSN: xxx. eISSN: doi. 10.23755/rm.v41i0.671.. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Solar activity is the key factor which drives the space weather. Refined modelling and accurate prediction of the solar activity intensity has been an important activity of space faring agencies across the world due to the impact of solar activity on satellites as well as on weather (Haigh, 2007, Hathaway et.al., 2004). Solar flux causes the upper atmosphere density variation and in turn it affects directly the lifetime of the Earth-orbiting satellites especially in the low Earth orbit. Solar activity intensity has been measured as the number of dark spots, called sunspot numbers, appears in the visible solar disc through direct observation since 1749 onwards. Irrespective of the measurement interval (daily, monthly, and yearly), a definite pattern is existing in the sunspot number time series.

Accurate predictions of the intensity of solar activity are increasingly important as we become more reliant upon satellites in low-Earth orbits, which provide crucial contribution in communication, national defence and Earth mapping. Also, such satellites provide an abundance of scientific data. During higher solar activity period, the increased ultraviolet emission from Sun heats up the Earth's upper atmosphere and this causes the atmosphere to expand and results in the increased drag on low Earth orbits satellites, thereby leading to early decay into the Earth's atmosphere. Therefore, better predictions of solar activity are essential to help mission planning and design of satellites (Vallado et.al., 2014).

Sunspot number cycle time series is one of the longest time series which was studied by many experts. First of all, this time series is non stationary, cyclic and highly nonlinear in the time domain. The more interesting and difficult part to deal with is the high dispersion between consecutive observations (Withbroe,1989) which in fact makes the prediction of sunspot numbers tedious.

Many attempts to model and predict the future behaviour of the solar activity are well documented in the literature. Depending on the nature of the prediction methods, we can classify the methodology in to five classes as: 1) Curve fitting 2) Precursor 3) Spectral 4) Neural Networks and 5) Climatology. The first attempt using the curve fitting methodology was by the McNish-Lincoln curve fitting (Hathaway, D. H, 2015). Subsequently, several authors have studied the highly nonlinear behaviour of sunspot number cycle and proposed various models to handle the studies related to the prediction.

Many mathematical functions have been appeared in the literature as model of the shape of the sunspot number cycle. Due to the exponential rise and decay of sunspot number cycle, a model involving exponential function was proposed by Nordemann (1992). The bell shape and the asymmetry along the peak amplitude of most of the sunspot number cycle were explored and there by a suitable mathematical function was introduced by Hathaway et.al. (1994). Few

statistical probability distribution functions were also proposed by various authors (Sabarinath et.al.,2008, Du et.al., 2011, Li et.al., 2017, Sabarinath et.al., 2020) for modeling the shape of sunspot number cycles. De Meyer (1981) proposed a model using periodic functions. As far as prediction of a future sunspot number cycle is concerned, statistical averaged models are used as an initial estimate of the future sunspot number cycle.

2. Existing Models

Several authors developed different mathematical functions to describe the shape of the sunspot number cycle. In particular Stewart and Panofsky (SP) (1938) proposed a function for the shape of a sunspot number cycle with the form,

$$R(t) = a(t - t_0)^b e^{-c(t-t_0)} \quad (1)$$

where a, b, c and t_0 are parameters that vary from cycle to cycle and t is the independent variable represents time. The important thing to be noticed is that, this model resembles a power law for the rising phase of a sunspot number cycle and an exponential for the declining phases of a cycle.

Nordemann (N) (1992) proposed another fit. He used the solution of the differential equation $\frac{dN}{dt} = KN$, in analogy with the nuclear decay process. The declining phase (maximum to minimum) of a sunspot number cycle is represented by

$$N = N_0 e^{Kt}, \quad K < 0 \quad (2)$$

and the solution of $\frac{dN}{dt} = A + KN$, is used to represent the first phase or ascent phase (minimum to maximum) of a sunspot number cycle. Thus, the model for the ascent phase is:

$$N = \frac{A}{K} (1 - e^{Kt}), \quad K < 0 \quad (3)$$

where N represents sunspot numbers, K is the decay constant and A is a production parameter.

Hathaway et.al. (H) (1994) established a model with four parameters along with a measure for the goodness of fit. The functional form is:

$$f(t) = \frac{a(t - t_0)^3}{e^{\left[\frac{(t-t_0)^2}{b^2}\right]} - c} \quad (4)$$

where, a represents the amplitude; b represents the time in months and t_0 denotes the starting time; c gives the asymmetry of the sunspot number cycle. This function is derived from Stewart and Panofsky model, but requires two

parameters for each sunspot number cycle. Where $c = 0.71$, and b is a dependent parameter given by,

$$b = 27.12 + \frac{25.15}{[a \times 10^3]^{\frac{1}{4}}} \quad (5)$$

Sabarinath et.al. (S) (2008) used a binary mixture of a modified Laplace Distribution. Laplace Distribution is a function f of two parameters m and s is given by,

$$f(x) = \frac{1}{2s} e^{\frac{-|x-m|}{s}} \quad (6)$$

where, m is the location parameter and s is the scale parameter. They modified this form and used the binary mixture of this distribution and reducing the number of parameters into six (later reduced to two floating parameters) to fit the predominant double peaks during the high solar activity regime of a sunspot number cycle. Modified final model is:

$$f(t) = \frac{A_1}{33.2} \exp\left(\frac{-|t - t_0 - 41.7|}{16.6}\right) + \frac{A_2}{46} \exp\left(\frac{-|t - t_0 - 67.3|}{23}\right) \quad (7)$$

Volobuev (V) (2009) found a function similar to that used by Stewart and Panofsky to fit the shape of sunspot number cycle that requires only one parameter for each cycle. The empirical model used is:

$$R = \left(\frac{t - t_0}{T_s}\right)^2 e^{-\left(\frac{t-t_0}{T_d}\right)^2} \quad (8)$$

We can see that this model is also similar to that of Stewart and Panofsky (1938) by putting $b = 2$ and modifying the growth multiplier and decay multiplier properly by introducing the new parameters T_s and T_d .

Du (2011) suggested modified Gaussian function with four parameters viz. peak size A , peak timing t_m , width B , and asymmetry α , in the form:

$$R(t) = A \exp\left(\frac{-(t - t_m)^2}{2B^2[1 + \alpha(t - t_m)]^2}\right) \quad (9)$$

Li et al (L) (2017) used a binary mixture of Gaussian function (suggested by Du),

$$f(x) = A_1 \exp\left(-\frac{(x - m_1)^2}{s_1}\right) + A_2 \exp\left(-\frac{(x - m_2)^2}{s_2}\right) \quad (10)$$

this model has six parameters. Peak sizes are denoted by A_1, A_2 , s_1 and s_2 represents gradients and peak time is represented by m_1 and m_2 .

Sabarinath et. al. (SB) (2020) fit the full sunspot number cycle perfectly with modified Maxwell Boltzmann probability distribution function. The final modified model is:

$$f(x; \alpha; A) = \frac{A}{\alpha^3} \sqrt{\frac{2}{\pi}} x^2 e^{-\frac{x^2}{2\alpha^2}} \quad (11)$$

where A is the area parameter, $\alpha > 0$.

All these models discussed so far has a common form that is these models are a product of a polynomial and a negative exponential function. Since the solar activity like process can be modelled by a bell-shaped curve viz., Gamma family of probability density function, we propose that the sunspot number cycles can be effectively modelled as a generalized Gamma distribution with certain free parameters. These free parameters can be estimated in the maximum likelihood sense from the sunspot data. Also, all these models discussed so far can be derived from this generalized Gamma distribution model as special cases. In the next section we brief on the derivation of the proposed model.

3. Methodology

New model-Generalized Gamma distribution

The Gamma distribution is often used to describe variables bounded on one side. A version of this distribution is obtained by adding a third parameter and gets the generalized Gamma distribution (Walck, 2001).

Probability density function is,

$$f(x; p, q, r) = \frac{p^r}{\Gamma(q)} (px)^{qr-1} e^{-(px)^r} \quad (12)$$

where p (a scale parameter) and q are the real positive parameters and a third parameter r has been added ($r = 1$ for the ordinary Gamma distribution) to generalize the Gamma distribution. This new parameter takes any real value but normally we consider the case where $c > 0$

Put the following substitutions,

$$\left. \begin{aligned} c &= \frac{p^r}{\Gamma(q)} p^{qr-1} \\ \alpha &= qr \\ K &= p^r \\ \delta &= r \end{aligned} \right\} \quad (13)$$

Then, Equation (12) becomes

$$f(x; c, k, \alpha, \delta) = cx^{\alpha-1} e^{-kx^\delta} \quad (14)$$

where k is the scale parameter, α is the shape parameter and δ is the location parameter. Depending on the values of the parameters we can arrive all the

existing models. Table1 list the existing models and their derivation into Equation (14).

Once we fix a model, the next step is to evaluate the best estimates of the model parameters in statistical parameter estimation sense. Here our measurement data is the monthly averaged sunspot numbers. The model we indent to fit over this data is given in Equation (14). We estimate the parameters by least square method. Using simple random search technique, we estimate the parameters. The mathematical algorithms used are given in detail in the next part.

Data

In the present study, we use the monthly averaged sunspot numbers, and these sunspot numbers for all the 24 sunspot number cycles were used. On July 1st, 2015, the sunspot number series has been replaced by a new improved version called version 2.0 data, that includes several corrections of past in homogeneities in the sunspot number time series.

In the new version 2.0 data, the conventional 0.6 Zürich scale factor, has been replaced by a factor of 1/0.6. This scale change, when combined with the recalibration, leads to a net increase of about 45% (correction variable with time) of the most recent part of the series, after 1947. This data can be obtained from https://www.bis.sidc.be/silso/DATA/SN_m_tot_V2.0.txt.

Estimation techniques

The function in which parameters to be estimated is,

$$f(x; c, k, \alpha, \delta) = cx^{\alpha-1}e^{-kx^\delta} \quad (15)$$

The maximum likelihood estimates of the parameters α , δ and k are considered to be the best unbiased, consistent and sufficient estimate of the parameters (Sorenson, 1980). Practically, the least square estimate is considered to be the maximum likelihood estimate. The simple mathematical procedure to estimate the parameters is to minimize the sum of squared error function J .

$$J = \sum_r e_r^2 \quad (16)$$

where e_r is the error. The minimum of J can be found by differentiating J with respect to the parameters α , δ and k .

In the present study, if we consider without loss of generality, a sunspot number cycle having a length of 132 months (≈ 11 year), and if we assume $\{s_n; n = 1, 2, \dots, 132\}$ as the realised sunspot number values, then the function J can be written as,

$$J = \sum_{i=1}^{132} [s_n - f(x_n, \alpha, \delta, k)]^2 \quad (17)$$

where, $x_n = 1, 2, \dots, 132$ represents the months for each $n = 1, 2, \dots, 132$. Then our objective is to compute and solve α , δ and k from

$$\frac{\partial J}{\partial \alpha} = 0 \quad (18)$$

$$\frac{\partial J}{\partial \delta} = 0 \quad (19)$$

$$\frac{\partial J}{\partial k} = 0 \quad (20)$$

Analytically solving the Equations (18) to (20) for α , δ and k is cumbersome. Hence, we go with numerical procedures for estimating the parameters. Monte Carlo based simple random search-based procedure is considered here to estimate the parameters. This procedure is described below as an algorithm.

Step-1. Start with a search region α , δ and k . Let S_α , S_δ and S_k are the bounded search regions of α , δ and k respectively. Our objective is to find an $\alpha_0 \in S_\alpha$, $\delta_0 \in S_\delta$ and $k_0 \in S_k$ such that,

$$J_{\alpha_0, \delta_0, k_0} = \sum_{i=1}^{132} [s_n - f(x_n, \alpha_0, \delta_0, k_0)]^2 \quad (21)$$

is minimum. That is,

$$J_{\alpha_0, \delta_0, k_0} \leq J_{\alpha, \delta, k} \quad (22)$$

for any $\alpha \in S_\alpha$, $\delta \in S_\delta$ and $k \in S_k$

Step-2. Start with a random initial value of α in S_α , δ in S_δ and k in S_k .

Compute J and in each iteration keep the minimum value of J , α , δ and k . After a very large number of iterations take the value of α , δ and k corresponds to the global minimum value of J .

4. Results

Estimates for the parameters

Table 2 shows typical converged values of the four model parameters c , α , k and δ . If we do a Monte Carlo based estimation of these parameters, due to the initial random number variation the optimum value differ numerically due to different starting points. But the variation is insignificant. This was proved in many Monte Carlo based optimizations (Ji et.al., 2006). Hence without loss of generality we consider a typical Monte Carlo run and a converged value of the parameters for further analysis. Table 2 gives one such value. The range of

search or feasible region was found by trial-and-error method. The range considered for the simulation is given below.

$$0.0001 \leq c \leq 0.0030, 3 \leq \alpha \leq 7, 0 \leq k \leq 0.9, \text{ and } 0 \leq \delta \leq 3.1.$$

Model Name and Year in which it is proposed	Functional form	Parameters in Equation (14)				Generalized form
		c	α	δ	k	
SP (1938)	$a(t - t_0)^b \exp(-c(t - t_0))$	free	free	1	free	$ax^\alpha \exp(-cx)$
N (1992)	$N = N_0 \exp(kt)$ $\frac{A}{K}(1 - \exp(-kt))$	free	1	1	free	$a \exp(-cx)$
H (1994)	$\frac{a(t - t_0)^3}{\exp\left(\frac{(t - t_0)^2}{b^2}\right) - c}$	free	4	2	free	$ax^3 \exp(-kx^2)$
S (2008)	$\frac{A_1}{33.2} \exp\left(\frac{- t - 41.7 }{16.6}\right)$ $+\frac{A_2}{46} \exp\left(\frac{- t - 67.3 }{23}\right)$	free	1	1	fixed	$c_1 \exp(-k_1 x) + c_2 \exp(-k_2 x)$
V (2009)	$\frac{(t - t_0)^2}{T_s^2} \exp\left(\frac{-(t - t_0)^2}{T_d^2}\right)$	free	3	2	free	$cx^2 \exp(-kx^2)$
Du (2011)	$A \exp\left(-\frac{(t - t_m)^2}{2B^2(1 + \alpha(t - t_m)^2)}\right)$	free	1	2	free	$c \exp(-kx^2)$
L (2017)	$A_1 \exp\left(\frac{-(t - m_1)^2}{s_1}\right) + A_2 \exp\left(\frac{-(t - m_2)^2}{s_2}\right)$	free	1	2	free	$c_1 \exp(-k_1 x^2) + c_2 \exp(-k_2 x^2)$
SB (2020)	$\frac{A}{\alpha^3} \sqrt{2/\pi} t^2 \exp\left(\frac{-t^2}{2\alpha^2}\right)$	free	3	2	free	$cx^2 \exp(-kx^2)$

Table 1. Different models, their parameters and its values.

Figure 1, 2, and 3 shows the model fit of the model on sunspot number cycles 13, 23 and 24, respectively.

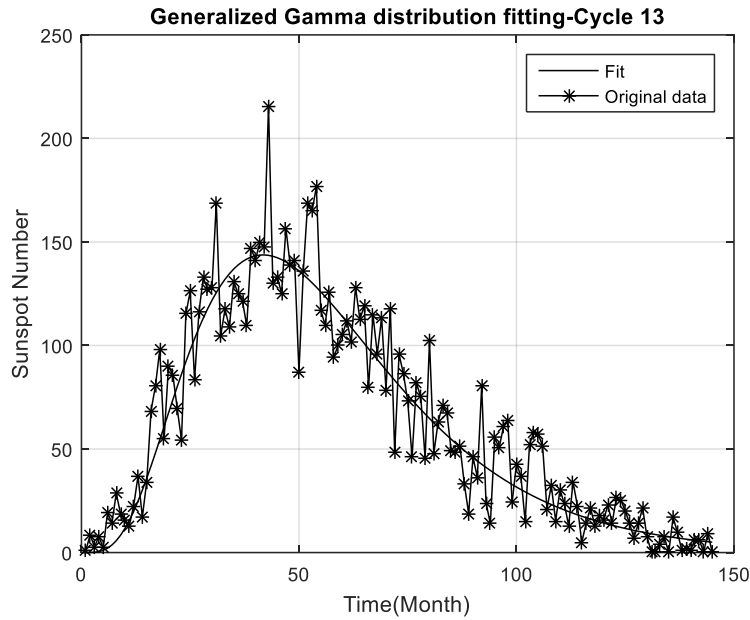


Figure 1. Generalized Gamma distribution fit on the monthly averaged sunspot number cycle 13.

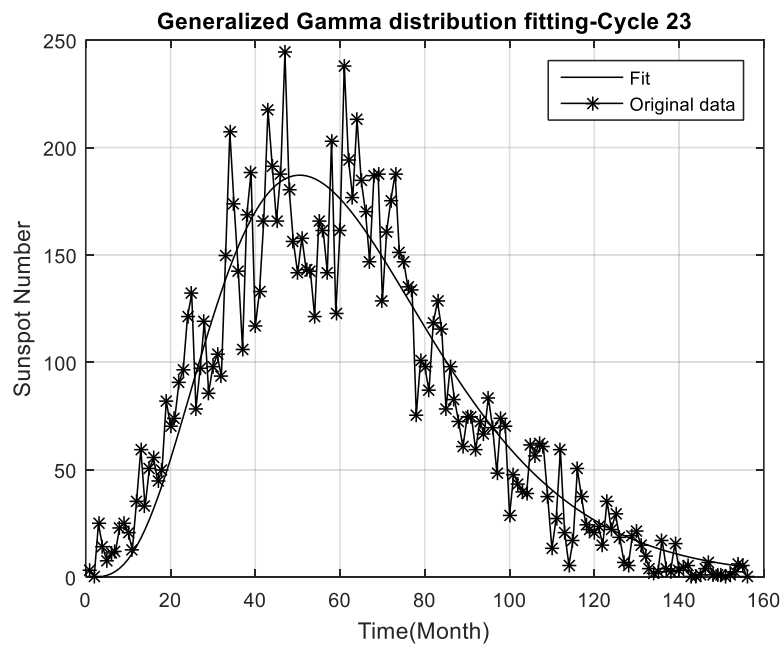


Figure 2. Generalized Gamma distribution fit on the monthly averaged sunspot number cycle 23.

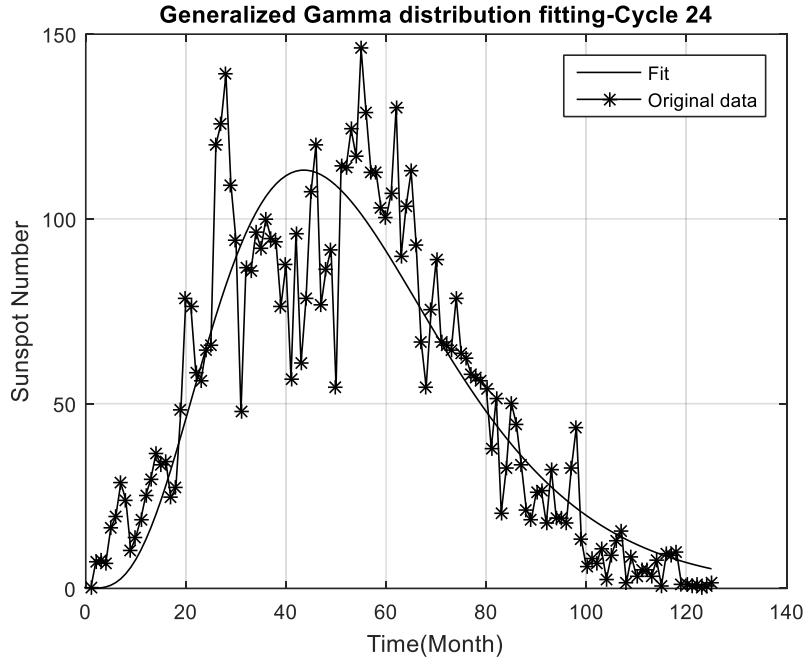


Figure 3. Generalized Gamma distribution fit on the monthly averaged sunspot number cycle 24.

Analysis of the parameters

The parameters are estimated on each of the sunspot number cycle independently. Figures 4 and 5 show these estimated model parameters for sunspot number cycle 1 to 24. The parameters are estimated in the maximum likelihood (ML) sense using the random search method. In table 2 Column number 2 to 5 gives the ML estimate of the parameters c, α, k and δ . Column 6 gives the coefficient of determination R^2 value. From these values one can see that the goodness of fit of the model is fair and on all modern sunspot number cycles are of high degree, since the coefficient of determination is greater than 0.8. The important thing to be noted is that, since the model is derived from the Gamma distribution probability density function as given in Equation (12), there must be a correlation among the model parameters. As evident through the set of Equations (13) the correlation between the model parameters is given in Table 3. α has a very high correlation with k and δ . Figure 6 and 7 shows this correlation along with the linear regression model derived out of this correlation. of course, α and k has a positive correlation and between α and δ a negative correlation. The corresponding linear regression fits are given in Equation (23) and (24).

$$k = 0.29\alpha - 1.3 \quad (23)$$

and

$$\delta = -0.32\alpha + 2.7 \quad (24)$$

Now, substitute Equation (23) and (24) in Equation (14), we can reduce the proposed model into a two-parameter model. Again, if we re-estimate the two parameters in a maximum likelihood sense, we can again obtain a correlation between the parameters, there by the model reduce to a one parameter model as proposed in the shape of the sunspot number cycle-a one parameter fit by Volobuev (2009). In fact, the reduction of model parameters into a single parameter does not add any predictive power in the characterisation of a sunspot number cycle via the prediction of the peak amplitude, location of the sunspot maximum, cycle length etc. Hence, an indicative parameter for these characters of a sunspot number cycle is essential in a sunspot model. So minimum two parameters, a location parameter and a scale parameter must be there in a sunspot model. If there is a shape parameter, it can characterize the degree of asymmetry present in a cycle.

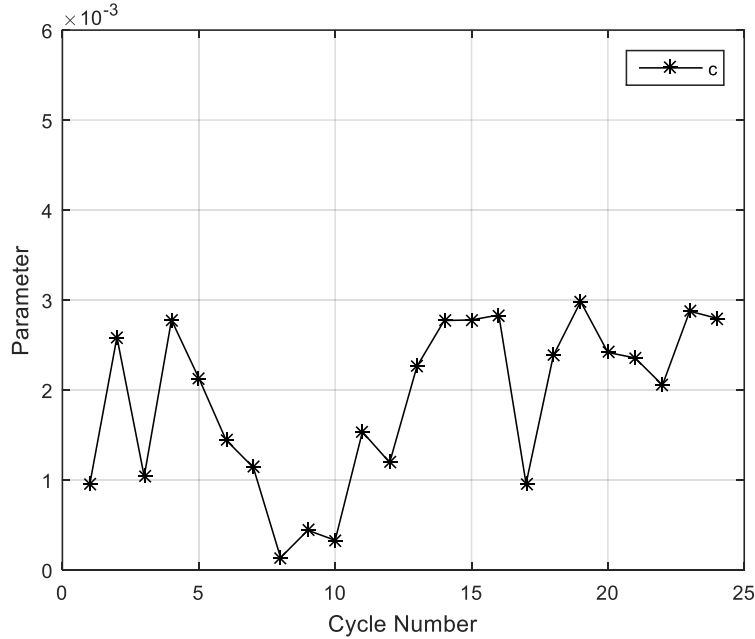


Figure 4. Variation of the ML estimate of the parameter c over different sunspot number-cycles

Coefficient of determination (R^2) is considered as one of the measures of goodness of fit for a regression fit. For each cycle the coefficient of determination for the best fit are computed and is given in Table 2, column 6 and the pictorial version is given in Figure 8.

Sunspot number cycle Number	c	α	k	δ	Coefficient of determination
1	0.00095	4.62004	0.03367	1.10253	0.7
2	0.00259	5.86033	0.52436	0.69427	0.7
3	0.00104	6.79747	0.80867	0.65381	0.9
4	0.00278	5.83371	0.56733	0.66284	0.9
5	0.00213	4.21489	0.01729	1.22798	0.8
6	0.00144	3.97518	0.00191	1.62772	0.6
7	0.00114	4.23167	0.00323	1.53445	0.7
8	0.00014	6.71511	0.47050	0.72610	0.8
9	0.00044	5.30111	0.10092	0.93291	0.8
10	0.00033	6.01638	0.37419	0.72687	0.9
11	0.00153	5.29585	0.14993	0.89630	0.9
12	0.00119	4.53186	0.01235	1.33943	0.8
13	0.00226	6.11178	0.74136	0.63797	0.9
14	0.00277	5.06902	0.26096	0.77246	0.7
15	0.00278	4.34677	0.00650	1.48307	0.8
16	0.00283	4.81611	0.08682	0.98842	0.8
17	0.00096	4.98223	0.04514	1.10391	0.9
18	0.00238	4.96288	0.06138	1.07036	0.9
19	0.00298	5.12209	0.08539	1.02239	0.9
20	0.00242	4.74031	0.08620	0.95019	0.9
21	0.00236	5.06018	0.08613	1.00536	0.9
22	0.00205	5.00316	0.06184	1.07074	0.9
23	0.00288	4.78601	0.07280	1.00605	0.9
24	0.00280	4.84851	0.09596	0.98271	0.8

Table 2. The estimated model parameters for sunspot number cycles 1 to 24.

It may be seen that the R^2 value for most of the cycles are greater than 0.8. Especially, for modern cycles (cycle 15 to 24) the R^2 values are greater than 0.8. This shows that the generalized Gamma model is best for modelling the shape of a sunspot number cycle.

	c	α	k	δ
c	1	-0.27	-0.09	-0.02
α		1	0.89	-0.86
k			1	-0.78
δ				1

Table 3. The correlation among the estimated model parameters.

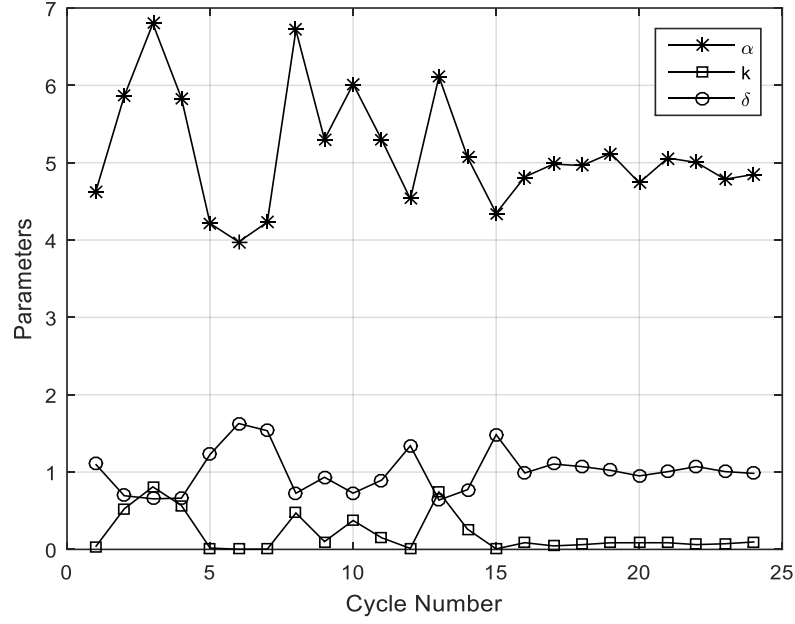


Figure 5. Variation of the ML estimate of the parameter α , k , and δ over different sunspot number cycles.

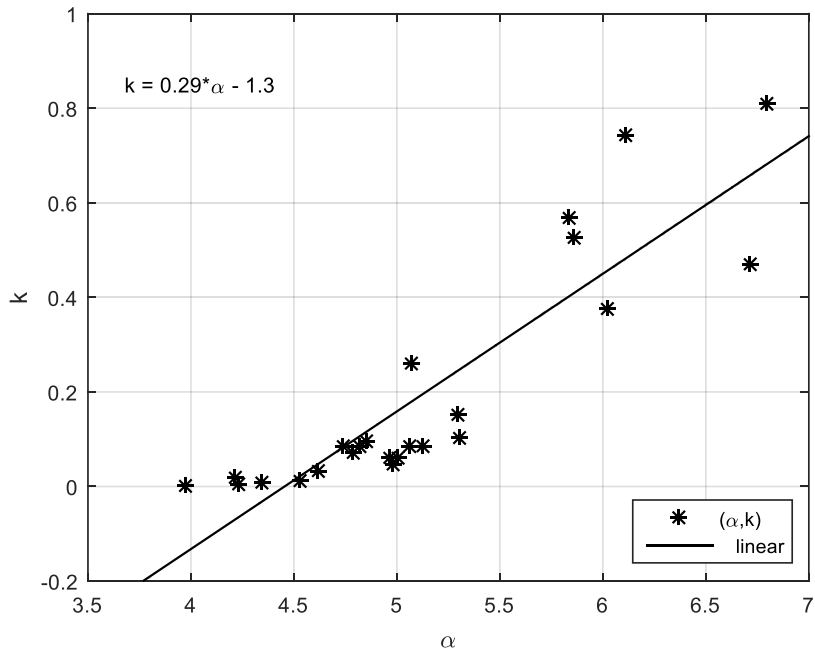


Figure 6. Linear correlation between the model parameters α and k .

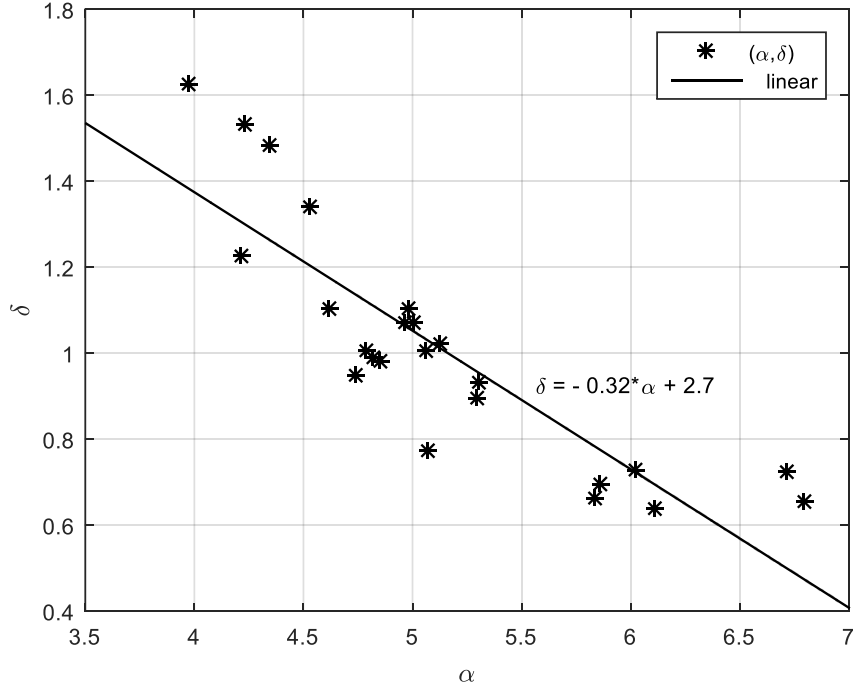


Figure 7. Linear correlation between the model parameters α and δ .

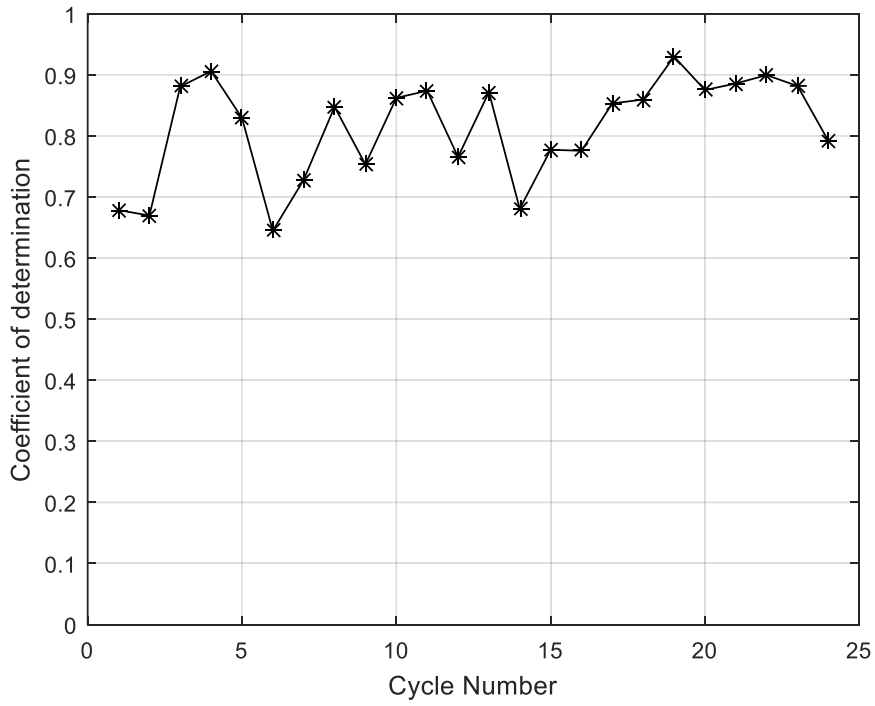


Figure 8. Coefficient of determination (R^2) of the ML fit on different cycles.

Prediction of sunspot number cycle 25

As an attempt has been made to predict the shape of the sunspot number cycle 25. Before attempting the predict cycle 25 from the proposed model, a direct comparison of the model of all the sunspot number cycles from 1 to 24 has been done. Figure 9 shows this model comparison. It may be noted that, the length of the cycle is not being considered here. Without loss of generality one can assume the length as 132 months. The variation in amplitude and the location of the peak amplitude varies between cycles. The range of variation of peak amplitude is from 70 to 300 units of sunspot numbers. Since the range of variation of the peak amplitude is quite large, the problem of prediction of cycle 25 is cumbersome from the previous cycle's model parameters alone. Hence, we make two kinds of predictions which are statistically more probable forecasts. They are (1) Since the parameters α , k and δ has lesser variation if we consider cycle 16-24, we fix these parameters as their average over cycle 16 to 24 and re-estimated the other parameter c alone for these cycles. Then for cycle 25, the parameter is c taken as the average of the re-estimated values of c for cycle 16 to 24. This value is 0.002. The remaining parameters are, $\alpha = 4.9246$, $k = 0.0757$, $\delta = 1.02$. The prediction of cycle 25 in this direction is plotted in Figure 10.

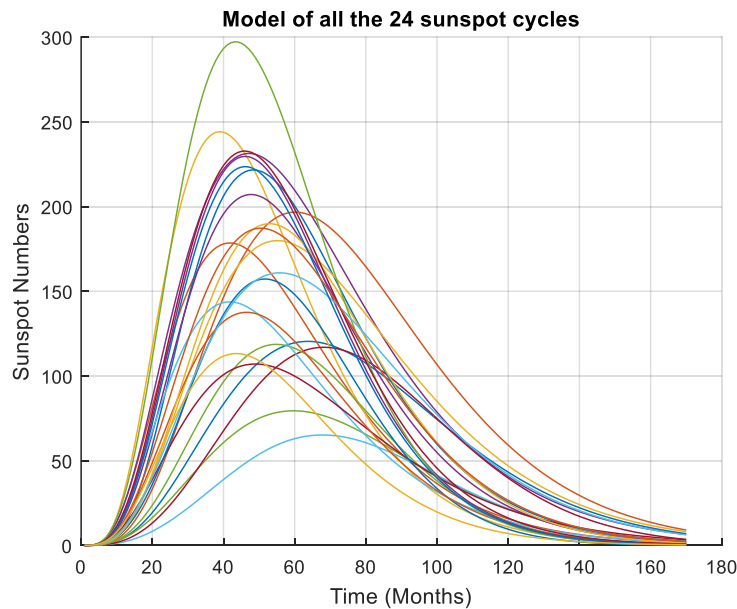


Figure 9. Model of all the 24 Sunspot number cycles.

This prediction shows the peak amplitude as 157 units occurring at 47 months from the beginning of sunspot number cycle 25. The second methods are (2) keeping the parameters α , k , and δ similar to cycle 24 and keeping the

parameter c as taken in method (1). In this way the peak amplitude of cycle 25 is 81 units of sunspot numbers occurring at 44 months from the beginning of the cycle 25. These two predictions can be considered as a band, inside which the actual cycle 25 may occur.

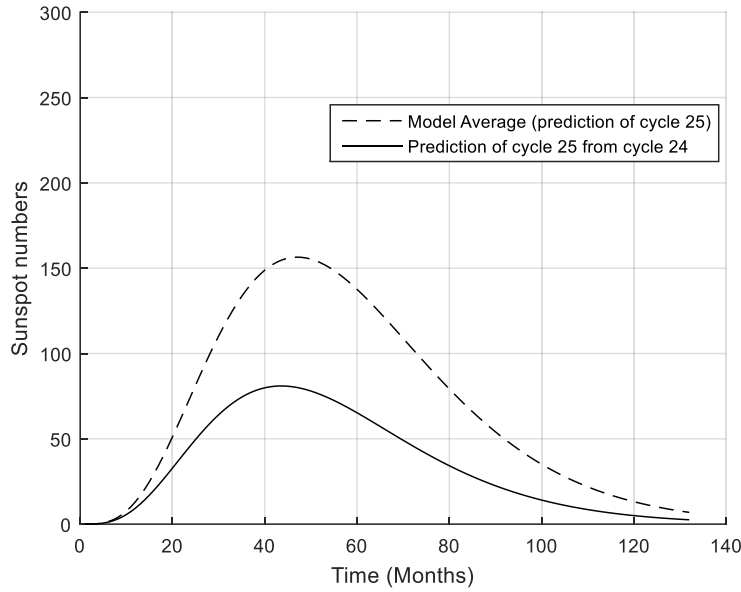


Figure 10. Prediction of Sunspot number cycle 25
Maximum peak occurring at 44 months with a sunspot number value of 81 units and 47 months with a sunspot value of 157 units.

5 Conclusion

Proposed a model which can unify many of the shape models existing in the literature. Also, it is shown that the shape model of sunspot number cycle can be described by a product of a polynomial and a negative exponential function. Since all the models reviewed in this paper are a product of a polynomial and a negative exponential, we proposed that all these models can be derived from the generalized Gamma probability density function by giving suitable parameter values. In this paper, we derived the existing models from the proposed generalized Gamma model and estimated the parameters of the proposed model from the revised version-2 monthly averaged sunspot numbers available in the SIDC's website. Prediction of sunspot number cycle 25 shows that the peak amplitude of cycle 25 can vary between 81 to 157 units of sunspot numbers and this peak amplitude may occur between 44 to 47 months from the beginning of cycle 25. In actual date, this shows cycle 25 may peak during August 2023 to November 2023.

References

- [1] Boslaugh, S. Watters, P.A. Statistics in a nutshell. Sebastopol, CA: O'Reilly Media. 2008.
- [2] De Meyer, F. Mathematical modelling of the sunspot cycle. *Solar Physics*, 70(2): 259-272, 1981.
- [3] Du, Z. The shape of solar cycle described by a modified Gaussian function. *Solar Physics*, 273(1): 231-253, 2011.
- [4] Hathaway, D. H. Wilson, R. M. Reichmann, E. J. The shape of the sunspot cycle. *Solar Physics*, 151(1): 177-190, 21, 1994.
- [5] Haigh, J.D. The Sun and the Earth's climate. *Living reviews in solar physics*. 4(1): 1-64, 2007.
- [6] Hathaway, D. H. Wilson, R. M. What the sunspot record tells us about space climate. *Solar Physics*, 224(1): 5-19, 2004.
- [7] Hathaway, D. H. The solar cycle. *Living reviews in solar physics*, 12(1): 4, 2015.
- [8] Ji, M., Klinowski, J. Taboo evolutionary programming: a new method of global optimization. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 462: 3613–3627, 2006.
- [9] Li, F. Y. Xiang, N. B. Kong, D. F. Xie, J. L. The shape of solar cycles described by a simplified binary mixture of Gaussian functions. *The Astrophysical Journal*, 834(2): 192, 2017.
- [10] Nordemann, D. J. R. Sunspot number time series: exponential fitting and solar behavior. *Solar physics*, 141(1): 199-202, 1992.
- [11] Sabarinath, A. Anilkumar, A.K. Modeling of Sunspot numbers by a modified binary mixture of Laplace distribution functions. *Solar Physics*, 250(1): 183-197, 2008.
- [12] Sabarinath, A. Beena, G. P. Anilkumar, A. K. Modelling the shape of sunspot cycle using a modified Maxwell-Boltzmann probability distribution function. *Ratio Mathematica*, 39: 33-54, 2020.
- [13] Sorenson, H.W. Parameter Estimation-Principles and Problems. Marcel Dekker, New York. 1980.
- [14] Stewart, J. Q. Panofsky, H. A. A. The Mathematical characteristics of sunspot variations. *The Astrophysical Journal*, 88: 385, 1938.
- [15] Vallado, D. A. Finkleman, D. A critical assessment of satellite drag and atmospheric density modeling. *Acta Astronautica*, 95: 141-165, 2014.

- [16] Volobuev, D. M. The shape of the sunspot cycle: A one-parameter fit. *Solar physics*, 258(2): 319-330, 2019.
- [17] Withbroe, G.L. Solar activity cycle-History and predictions. *Journal of Spacecraft and Rockets*, 26(6): 394-402. 22. 1989.
- [18] Walck, C. Hand-book on statistical distributions for experimentalists. Internal Report SUF-PFY/96-01, Particle Physics Group, University of Stockholm. 2001.

A characterization of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces

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Abstract

The idea of intuitionistic fuzzy metric space introduced by Park (2004). In this paper, we introduce the notion of strong intuitionistic fuzzy diameter zero for a family of subsets based on the intuitionistic fuzzy diameter for a subset of A . Then we introduce nested sequence of subsets having strong intuitionistic fuzzy diameter zero using their intuitionistic fuzzy diameter.

Keywords: strong fuzzy diameter; intuitionistic fuzzy metric space; strong completeness.

2010 AMS subject classification: 05C72, 54E50, 03F55.[‡]

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[‡] Received on September 18, 2021. Accepted on December 1, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.661. ISSN: 1592-7415. eISSN: 2282-8214. © The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh [17] in 1965. Kramosil and Michalek [6] introduced the fuzzy metric spaces (FM-spaces) by generalizing the concept of probabilistic metric spaces to fuzzy situation. George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] with a view to obtain a Hausdorff topology on fuzzy metric spaces which have very important applications in quantum particle particularly in connection with both string and E-infinity theory. In 2004, Park [8] defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms.

Several researchers have shown interest in the intuitionistic fuzzy metric space successfully applied in many fields, it can be found in [5, 10, 11, 13, 14, 15, 16]. Theory of fuzzy sets have been widely used and developed in different fields of sciences, including mathematical programing, theory of modeling, theory of optimal control, theory of neural network, engineering and medical sciences, coloured image processing, etc.

In this paper, the concept of characterization of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces are introduced and also discuss some properties of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces.

2 Preliminaries

Definition 2.1[17] Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu_A(x)) | x \in X\}$.

Definition 2.2[4] The 3-tuple $(A, M, *)$ is said to be a fuzzy metric space if A be a non-empty set and $*$ be a continuous t-norm. A fuzzy set $A^2 \times (0, \infty)$ is called a fuzzy metric on A if $a, b, c \in A$ and $s, t > 0$, the following condition holds:

1. $M(a, b, t) = 0$
2. $M(a, b, t) = 1$ if and only if $a = b$

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3. $M(a, b, t) = M(b, a, t)$
4. $M(a, b, t + s) \geq M(a, b, t) * M(a, b, s)$
5. $M(a, b, \bullet): (0, \infty) \rightarrow [0, 1]$ is left continuous

The function $M(a, b, t)$ denote the degree of nearness between a and b with respect to t respectively.

Definition 2.3[1, 2] Let a set E be fixed. An intuitionistic fuzzy set A in E is an object of the following $A = \{(x, \mu_A(x), \nu_A(x)), x \in E\}$ where the functions $\mu_A(x): E \rightarrow [0, 1]$ and $\nu_A(x): E \rightarrow [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E: 0 \leq \mu_A(x) + \nu_A(x) \leq 1$, when $\nu_A(x) = 1 - \mu_A(x)$ for all $x \in E$ is an ordinary fuzzy set. In addition, for each IFS A in E , if $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. Then $\mu_A(x)$ is called the degree of indeterminacy of x to A or called the degree of hesitancy of x to A . It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in E$.

Definition 2.4 [7] A 5-tuple $(A, M, N, *, \circ)$ is said to be an intuitionistic fuzzy metric space if A is an arbitrary set, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm and, M, N are fuzzy sets on $A^2 \times [0, \infty)$ satisfying the conditions:

1. $M(a, b, t) + N(a, b, t) \leq 1$, for all $a, b \in A$ and $t > 0$
2. $M(a, b, 0) = 0$, for all $a, b \in A$
3. $M(a, b, t) = 1$, for all $a, b \in A$ and $t > 0$ if and only if $a = b$
4. $M(a, b, t) = M(b, a, t)$, for all $a, b \in A$ and $t > 0$
5. $M(a, b, t) * M(b, c, s) \leq M(a, c, t + s)$, for all $a, b, c \in A$ and $s, t > 0$
6. $M(a, b, \bullet): [0, \infty) \rightarrow [0, \infty]$ is left continuous for all $a, b \in A$
7. $\lim_{t \rightarrow \infty} M(a, b, t) = 1$, for all $a, b \in A$ and $t > 0$
8. $N(a, b, 0) = 1$, for all $a, b \in A$
9. $N(a, b, t) = 0$, for all $a, b \in A$ and $t > 0$ if and only if $a = b$
10. $N(a, b, t) = N(b, a, t)$, for all $a, b \in A$ and $t > 0$
11. $N(a, b, t) \circ N(b, c, s) \geq N(a, c, t + s)$, for all $a, b, c \in A$ and $s, t > 0$
12. $N(a, b, \bullet): [0, \infty) \rightarrow [0, 1]$ is right continuous for all $a, b \in A$
13. $\lim_{t \rightarrow \infty} N(a, b, t) = 0$, for all $a, b \in A$.

The functions $M(a, b, t)$ and $N(a, b, t)$ denote the degree of nearness and the degree of non-nearness between a and b w.r.t t respectively.

Definition 2.5 [9] The **fuzzy diameter** of a non-empty set B of a fuzzy metric space A , with respect to t , is the function $\varphi_B: (0, +\infty) \rightarrow [0, 1]$ given by $\varphi_B(t) = \inf\{M(a, b, t): a, b \in B\}$ for each $t \in R^+$.

Definition 2.6 [9] A collection of sets $\{B_i\}_{i \in I}$ is said to have fuzzy diameter zero if given $r \in (0, 1)$ and $t \in R^+$ there exists $i \in I$ such that $M(a, b, t) > 1 - r$ for all $a, b \in B_i$.

3 Strong fuzzy diameter zero in intuitionistic fuzzy metric spaces

Definition 3.1 The fuzzy diameter of a non-empty set B of a intuitionistic fuzzy metric space $(A, M, N, *, \circ)$, with respect to t , is the function $\varphi_B: (0, +\infty) \rightarrow [0, 1]$ given by $\varphi_B(t) = \inf\{M(a, b, t): a, b \in B\}$ and $\psi_B: (0, +\infty) \rightarrow [0, 1]$ given by $\psi_B(t) = \sup\{N(a, b, t): a, b \in B\}$ for each $t \in R^+$

Definition 3.2 A collection of sets $\{B_i\}_{i \in I}$ of a intuitionistic fuzzy metric space $(A, M, N, *, \circ)$ is said to have fuzzy diameter zero if given $r \in (0, 1)$ and $t \in R^+$ there exists $i \in I$ such that $M(a, b, t) > 1 - r$ and $N(a, b, t) < r$ for all $a, b \in B_i$.

Theorem 3.3 Let $\{B_n\}_{n \in \mathbb{N}}$ be a nested sequence of sets of the intuitionistic fuzzy metric space $(A, M, N, *, \circ)$. Then the following statements are equivalent:

- (i) $\{B_n\}_{n \in \mathbb{N}}$ has fuzzy diameter zero.
- (ii) $\lim_{n \rightarrow \infty} \varphi_{B_n}(t) = 1$, $\lim_{n \rightarrow \infty} \psi_{B_n}(t) = 0$ for all $t \in R^+$.

Proof:

(i)→(ii):

Let $t \in R^+$. Given $r \in (0, 1)$ exists $n_{r,t} \in \mathbb{N}$ such that $M(a, b, t) > 1 - r$, $N(a, b, t) < r$ for each $a, b \in B_n$ with $n \geq n_{r,t}$.

Then, $\varphi_{B_n}(t) = \inf\{M(a, b, t): a, b \in B_n\} \geq 1 - r$ and $\psi_{B_n}(t) = \sup\{N(a, b, t): a, b \in B_n\} \leq r$ for all $n \geq n_{r,t}$.

Hence,

$$\lim_{n \rightarrow \infty} \varphi_{B_n}(t) = 1 \text{ and } \lim_{n \rightarrow \infty} \psi_{B_n}(t) = 0, \text{ since } r \text{ is arbitrary in } (0, 1).$$

(ii)→(i):

Suppose $\lim_{n \rightarrow \infty} \varphi_{B_n}(t) = 1$ and $\lim_{n \rightarrow \infty} \psi_{B_n}(t) = 0$, for all $t \in R^+$.

Let $t \in R^+$ and let $r \in (0, 1)$.

We can find $n_{r,t} \in \mathbb{N}$ such that $\varphi_{B_n}(t) > 1 - r$ and $\psi_{B_n}(t) < r$ for all $n \geq n_{r,t}$. Thus, $M(a, b, t) > 1 - r$ and $N(a, b, t) < r$ for each $a, b \in B_n$ with $n \geq n_{r,t}$ i.e., $\{B_n\}_{n \in \mathbb{N}}$ has fuzzy diameter zero.

A characterization of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces

Definition 3.4 A family of non-empty sets $\{B_i\}_{i \in I}$ of a intuitionistic fuzzy metric space $(A, M, N, *, \circ)$ has strong fuzzy diameter zero if for $r \in (0, 1)$ there exists $i \in I$ such that $M(a, b, t) > 1 - r$ and $N(a, b, t) < r$ for each $a, b \in B_i$ and all $t \in R^+$.

Theorem 3.5 Let $(A, M, N, *, \circ)$ be an intuitionistic fuzzy metric space and let $\{B_n\}_{n \in \mathbb{N}}$ be a nested sequence of sets of A . Then the following statements are equivalent.

- (i) $\{B_n\}_{n \in \mathbb{N}}$ has strong fuzzy diameter zero.
- (ii) $\lim_{n \rightarrow \infty} \varphi_{B_n}(t_n) = 1$, $\lim_{n \rightarrow \infty} \psi_{B_n}(t_n) = 0$ for every decreasing and increasing sequence of positive real numbers $\{t_n\}_{n \in \mathbb{N}}$ that converges and diverges respectively.

Proof:

(i) $\rightarrow$ (ii): Let $\{t_n\}_{n \in \mathbb{N}}$ be a decreasing increasing sequence of positive real numbers that converges and diverges respectively. Given $r \in (0, 1)$, we can find $n_r \in \mathbb{N}$ such that $M(a, b, t) > 1 - r$ and $N(a, b, t) < r$ for each $a, b \in B_{n_r}$ with $n \geq n_r$ and all $t \in R^+$. In particular, $M(a, b, t_n) > 1 - r$ and $N(a, b, t_n) < r$ for all $a, b \in B_{n_r}$ with $n \geq n_r$, i.e., $\varphi_{B_{n_r}}(t_n) \geq 1 - r$, $\psi_{B_{n_r}}(t_n) \leq r$ for all $n \geq n_r$, i.e., $\lim_{n \rightarrow \infty} \varphi_{B_{n_r}}(t_n) = 1, \lim_{n \rightarrow \infty} \psi_{B_{n_r}}(t_n) = 0$.

(ii) $\rightarrow$ (i): Suppose that $\{B_n\}_{n \in \mathbb{N}}$ has not strong fuzzy diameter zero. Let $r \in (0, 1)$ such that $I = \{n \in \mathbb{N}: M(a, b, t) \leq 1 - r, N(a, b, t) \geq r \text{ for some } a, b \in B_n \text{ and some } t \in R^+\}$, is infinite. Take $n_1 = \min I$. Then, there exist $a_{n_1}, b_{n_1} \in B_{n_1}$ such that $M(a_{n_1}, b_{n_1}, t_{n_1}) \leq 1 - r, N(a_{n_1}, b_{n_1}, t_{n_1}) \geq r$ with $0 < t_{n_1} < 1$.

Take $n_2 > n_1$, with $n_2 \in \mathbb{N}$, such that $M(a_{n_1}, b_{n_1}, t_{n_1}) \leq 1 - r, N(a_{n_1}, b_{n_1}, t_{n_1}) \geq r$ for some $a_{n_2}, b_{n_2} \in B_{n_2}$ and $0 < t_{n_2} < \min\{t_{n_1}, \frac{1}{2}\}$. In this way, we construct, by induction, a sequence $\{t_{n_i}\}_{i \in \mathbb{N}}$ such that $M(a_{n_i}, b_{n_i}, t_{n_i}) \leq 1 - r, N(a_{n_i}, b_{n_i}, t_{n_i}) \geq r$ for some $a_{n_i}, b_{n_i} \in B_{n_i}$, $n_i \in \mathbb{N}$ with $n_i > n_{i-1}$ and $0 < t_{n_i} < \min\{t_{n_{i-1}}, \frac{1}{i}\}$.

Then,

$$\varphi_{B_{n_i}}(t_{n_i}) = \inf\{M(a, b, t_{n_i}): a, b \in B_{n_i}\} \leq 1 - r,$$

$$\psi_{B_{n_i}}(t_{n_i}) = \sup\{N(a, b, t_{n_i}): a, b \in B_{n_i}\} \geq r \text{ for all } i \in \mathbb{N}.$$

Hence $\{\varphi_{B_{n_i}}(t_{n_i})\}_{i \in \mathbb{N}}$, $\{\psi_{B_{n_i}}(t_{n_i})\}_{i \in \mathbb{N}}$ does not converge and diverge

respectively. Now, $\{t_{n_i}\}_{i \in \mathbb{N}}$ is a subsequence of the decreasing and increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ that converges and diverges respectively, given by

$$t_n = \begin{cases} t_{n_1} & n \leq n_1 \\ t_{n_{i+1}} & n_i \leq n \leq n_{i+1} \end{cases}$$

and the sequence $\{\varphi_{B_n}(t_n)\}_{n \in \mathbb{N}}$, $\{\psi_{B_n}(t_n)\}_{n \in \mathbb{N}}$ does not converge and diverge respectively. Thus, we get the contradiction.

Theorem 3.6 Let $\{B_n\}_{n \in \mathbb{N}}$ be a nested sequence of sets with fuzzy diameter zero in a intuitionistic fuzzy metric space $(A, M, N, *, \circ)$. $\{B_n\}_{n \in \mathbb{N}}$ has strong fuzzy diameter zero if and only if $\{B_n\}$ is a singleton set after a certain stage.

Proof:

Suppose $\{B_n\}_{n \in \mathbb{N}}$ is not eventually constant. Put $p_n = \sup\{d(a, b) : a, b \in B_n\}$, $q_n = \inf\{d(a, b) : a, b \in B_n\}$ and take $t_n = p_n$ and $t_n = q_n$ for all $n \in \mathbb{N}$. Then, $\{t_n\}_{n \in \mathbb{N}}$ is a decreasing and increasing sequence of positive real numbers converges and diverges respectively.

$$\begin{aligned} \text{Then, } \lim_{n \rightarrow \infty} \varphi_{B_n}(t) &= \lim_{n \rightarrow \infty} \inf\{M_d(a, b, t_n) : a, b \in B_n\} \\ &= \lim_{n \rightarrow \infty} \frac{t_n}{t_n + \text{diam}(B_n)} \\ &= \lim_{n \rightarrow \infty} \frac{p_n}{p_n + p_n} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_{B_n}(t) &= \lim_{n \rightarrow \infty} \sup\{N_d(a, b, t_n) : a, b \in B_n\} \\ &= \lim_{n \rightarrow \infty} \frac{t_n}{t_n + \text{diam}(B_n)} \\ &= \lim_{n \rightarrow \infty} \frac{q_n}{q_n + q_n} = \frac{1}{2} \end{aligned}$$

Hence $\{B_n\}_{n \in \mathbb{N}}$ has not strong fuzzy diameter zero.

Theorem 3.7 Let $(A, M, N, *, \circ)$ be a intuitionistic fuzzy metric space. If $\{B_n\}_{n \in \mathbb{N}}$ is a nested sequence of sets of A which has strong fuzzy diameter zero then $\{B_n\}_{n \in \mathbb{N}}$ has strong fuzzy diameter zero.

Proof:

First, we prove that $\varphi_{\bar{B}}(t) = \varphi_B(t)$, $\psi_{\bar{B}}(t) = \psi_B(t)$ for every subset B of A . Indeed, take $a, b \in B$. Then, we can find two sequences $\{a_n\}_{n \in \mathbb{N}}$ and

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$\{b_n\}_{n \in \mathbb{N}}$ in B that converge to a and b , respectively. Let $t \in R^+$ and an arbitrary $\varepsilon \in (0, 1)$. We have that

$$\begin{aligned} M(a, b, t + 2\varepsilon) &\geq M(a, b_n, \varepsilon) * M(a_n, b_n, t) * M(b_n, b, \varepsilon) \\ &\geq M(a, a_n, \varepsilon) * \varphi_B(t) * M(b_n, b, \varepsilon), \\ N(a, b, t + 2\varepsilon) &\leq N(a, b_n, \varepsilon) \circ N(a_n, b_n, t) \circ N(b_n, b, \varepsilon) \\ &\leq N(a, a_n, \varepsilon) \circ \psi_B(t) \circ N(b_n, b, \varepsilon) \end{aligned}$$

and taking limit on the inequality when n tends to ∞ , we obtain

$$\begin{aligned} M(a, b, t + 2\varepsilon) &\geq 1 * \varphi_B(t) * 1 = \varphi_B(t), \\ N(a, b, t + 2\varepsilon) &\leq 1 \circ \psi_B(t) \circ 1 = \psi_B(t). \end{aligned}$$

Since ε is arbitrary, due to the continuity of $M(a, b, t), N(a, b, t)$ we obtain $M(a, b, t) \geq \varphi_B(t), N(a, b, t) \leq \psi_B(t)$ and then $\varphi_{\bar{B}}(t) \geq \varphi_B(t), \psi_{\bar{B}}(t) \geq \psi_B(t)$. On the other hand, we have $\varphi_{\bar{B}}(t) \leq \varphi_B(t), \psi_{\bar{B}}(t) \leq \psi_B(t)$ and hence $\varphi_{\bar{B}}(t) = \varphi_B(t), \psi_{\bar{B}}(t) = \psi_B(t)$. Let $\{t_n\}_{n \in \mathbb{N}}$ be a decreasing and increasing sequence of positive real numbers convergent and divergent respectively. By theorem 3.5, we have that $\lim_{n \rightarrow \infty} \varphi_{B_n}(t_n) = 1, \lim_{n \rightarrow \infty} \psi_{B_n}(t_n) = 0$.

Then, by our last argument, we have that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi_{B_n}(t_n) &= \lim_{n \rightarrow \infty} \varphi_{\bar{B}_n}(t_n) = 1, \\ \lim_{n \rightarrow \infty} \psi_{B_n}(t_n) &= \lim_{n \rightarrow \infty} \psi_{\bar{B}_n}(t_n) = 0, \end{aligned}$$

and consequently, by theorem 3.5, $\{B_n\}_{n \in \mathbb{N}}$ has strong fuzzy diameter zero.

4 Conclusion

Intuitionistic fuzzy set theory plays a vital role in uncertain situations in all aspects. In this paper, the characterizations of strong fuzzy diameter zero in intuitionistic fuzzy metric spaces are discussed and proved that the nested sequences having the strong fuzzy diameter zero in intuitionistic fuzzy metric space. We have also provided that nested sequences of subsets has strong fuzzy diameter zero if and only if singleton set after a certain stage in a intuitionistic fuzzy metric spaces.

References

- [1] K.T.Atanassov. More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33(1), 37-45. 1989.
- [2] K.T.Atanassov. Intuitionistic Fuzzy sets. *Fuzzy sets and system*, 20, 87-96. 1986.
- [3] M.A.Erceg. Metric spaces in fuzzy set theory. *Journal of Mathematical Analysis and Applications*. 69(1), 205–230. 1979.
- [4] A.George, P.Veeramani. On some results in fuzzy metric spaces, *Fuzzy sets and Systems*. 1994.
- [5] V.Gregori, S.Romaguera, P.Veeramani. A note on intuitionistic fuzzy metric spaces. *Chaos, Solitons and Fractals*. 28, 902-905. 2006.
- [6] Valentine Gregori and Juan Jose Minana, Bernardino Roig, Almanzor Sapena. A Characterization of strong completeness in fuzzy metric spaces. *Mathematics*, 8(861). doi:10.3390/math8060861. 2020.
- [7] Kramosil, Michalek. Fuzzy metrics and statistical metric spaces. *Kybernetika*, 11, 326-334. 1975.
- [8] J. H. Park, Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals*, 22 (5), 1039-1046. 2004.
- [9] R. Roopkumar, R. Vembu. Some remarks on metrics induced by a fuzzy metric. *arXiv preprint arXiv:1802.03031*. 2018.
- [10] S.Yahya Mohamed, E.Naargees Begum. A Study on Intuitionistic L-Fuzzy Metric Spaces. *Annals of Pure and Applied Mathematics* ,15(1), 67-75. 2017.
- [11] S.Yahya Mohamed, E.Naargees Begum. A Study on Properties of Connectedness in Intuitionistic L-Fuzzy Special Topological Spaces. *International Journal of Emerging Technologies and Innovative Research*, 5(7), 533-538. 2018.
- [12] S.Yahya Mohamed, E. Naargees Begum. A Study on Fuzzy Fixed Points and Coupled Fuzzy Fixed Points in Hausdorff L-Fuzzy Metric Spaces. *Journal of Computer and Mathematical Sciences*, 9(9), 1187-1200. 2018.
- [13] S.Yahya Mohamed, E.Naargees Begum. A Study on Fixed Points and Coupled Fuzzy Fixed Points in Hausdorff Intuitionistic L-Fuzzy Metric Spaces. *International Journal of Advent Technology*, 6, 12719-2725. 2018.

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- [14] S.Yahya Mohamed, A.Mohamed Ali. Fixed point theorems in intuitionistic fuzzy graph metric space. *Annals of Pure and Applied Mathematics*, 118(6), 67-74. 2017.
- [15] S.Yahya Mohamed, A.Mohamed Ali. Intuitionistic fuzzy graph metric space. *International Journal of Pure and Applied Mathematics*, 118(6), 67-74. 2018.
- [16] S.Yahya Mohamed, E. Naargees Begum. A study on concepts of Balls in a intuitionistic fuzzy D-metric spaces. *Advances in Mathematics: Scientific Journal*, 9(3), 1019-1025. 2020.
- [17] L.A. Zadeh, Fuzzy sets. *Information and control*, 8(3), 338-356. 1965.

Study of feedback queueing system with unreliable waiting server under multiple differentiated vacation policy

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Abstract

This manuscript analyses a queueing system with Bernoulli schedule feedback of customers, unreliable waiting server under differentiated vacations. The unsatisfied customer may again join the queue with probability α , following the Bernoulli schedule. The stationary solution is obtained for the model with aid of the Probability Generating function technique. Some important system performance measures are derived and the graphical behavior of these measures with some parameters is analysed. Finally, to obtain the optimal value of service rate for the model, cost optimization is performed through the quadratic fit approach.

Keywords: Feedback; differentiated vacations; optimization; Bernoulli schedule; unreliable waiting server.

2010 AMS subject classification: 60K25, 60K30.

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¹Received on June 22, 2021. Accepted on December 19, 2021. Published on December 31, 2021.
doi: 10.23755/rm.v41i0.619ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Queueing system with server vacations and feedback is a powerful tool and is successfully applied in many real-life congestion problems such as traffic systems, operating systems, communication systems, manufacturing and production lines. Queueing systems with feedback assume that unsatisfied customers may join the queue again to repeat their request before leaving the system. This concept was first introduced by Takacs [16]. Disney and Konig [4] analysed Bernoulli feedback queueing system. Sharma and Kumar [14] studied the markovian feedback queueing model with the impatient behaviour of customers. Later on, Kalidass and Kasthuri [6] did the pioneering work on M/G/1 queueing model with immediate feedback. Sunder et al. [15] analysed the feedback queueing model with different service stations and vacation policies under reneging.

The vacation queueing model is another remarkable and unavoidable feature due to its widespread applications in real-life situations. The server can go on vacations for many reasons like insufficient workload, server breakdown and some secondary tasks assigned to him, etc. The assumption of the service station being a hundred percent reliable is not a feasible one. The server may suffer a breakdown at any instant of time. Ke, J.C. [7, 8, 9] analysed different queueing models with an unreliable server. Later on, Ke et al. [10] performed pioneer work on finite buffer M/M/C queueing system with server breakdown. Kim et al. [11] analysed M/G/1 queueing model with a working breakdown. Levy and Yechiali [12] first analysed vacation queueing systems. Later on Doshi [3], Tian and Zhang [17] performed a comprehensive survey on vacation queueing models. Researchers analysed queues with different vacation strategies like single vacations, multiple vacations and working vacations. Servi and Finn [13] first introduced the concept of working vacation in which the server remains engaged in some auxiliary works and thereby provides service at a relatively slower rate. Banik et al. [2] analysed multiple working vacation queueing systems. A new class of differentiated vacations is introduced by Ibe and Isijola [5]. Zhang and Zhou [20] studied queueing model with m kinds of differentiated working vacations. The transient solution of a differentiated vacation queueing system was carried out by Vijyashree and Janani [19]. Vyshna Unni and Rose Mary [18] considered multi-server queues with differentiated vacations. Amar Aissani et al. [1] studied differentiated vacation queues under general service times.

In this paper, a queue with waiting and unreliable servers under differentiated vacation policy and Bernoulli feedback of customers with different arrival rates in various states is considered.

2. Mathematical description of model

The queueing model is analysed under the following assumptions:

1. The customers arrive in accordance with the Poisson process. The customers are served on an FCFS basis by the server, the service time is assumed to be exponentially distributed with a mean of $1/\mu$.
2. On getting the service, the customers may decide to leave the queue with probability β or unsatisfied customers may re-join the queue for another service with complementary probability $1-\beta$ ($=\alpha$).
3. When the system gets empty on serving all the customers, the server keeps on waiting for customers for a random period exponentially distributed with a mean $1/w$ and leaves for vacation only if none arrives in that duration. On returning from type I vacation, if it finds customers waiting for the service, it resumes to active state otherwise it goes for type II vacation. In the same way, on returning from vacation II, it switches to Vacation I or resumes to active state depending on whether the system is empty or customers are waiting in the system respectively. This process continues in the same manner. The period of both the vacations are exponentially distributed but with different means $1/\theta_1$ and $1/\theta_2$ respectively.
4. The server breaks down at any point of time in the active service state. The breakdowns occur according to Poisson distribution with parameter γ . The server is immediately sent for repair in such a situation. The repair time is also assumed to follow an exponential distribution with a mean of $1/\delta$. The customers have to wait for their service until the server gets repaired.
5. The arrival rates of customers in two types of vacations, downstate of server and active state are taken to be $\lambda_1, \lambda_2, \lambda_0, \lambda_3$ respectively.

3. Steady state equations

Denoting the number of customers in the system at any time t by $C(t)$ and the server state at time t by $S(t)$, we observe that $\{C(t), S(t)\}$ is a continuous Markov chain.

The different possible server states are

$$S(t) = \begin{cases} 0, \text{server in vacation I} \\ 1, \text{server in vacation II} \\ 2, \text{server in active state} \\ 3, \text{server under repair} \end{cases}$$

Denoting the probability of n customers in the i^{th} state of server by p_{ni} , the steady-state equations governing the proposed quasi birth-death model using the Markov process are

$$(\lambda_2 + \theta_2)p_{01} = \theta_1 p_{00} \quad (1)$$

$$(\lambda_2 + \theta_2)p_{n1} = \lambda_2 p_{n-1,1}, \quad n \geq 1 \quad (2)$$

$$(\lambda_3 + w)p_{02} = \mu\beta p_{12} \quad (3)$$

$$(\lambda_3 + \mu\beta + \gamma)p_{n2} = \lambda_3 p_{n-1,2} + \mu\beta p_{n+1,2} + \delta p_{n3} + \theta_1 p_{n0} + \theta_2 p_{n1}, \quad n \geq 1 \quad (4)$$

$$(\lambda_0 + \delta)p_{13} = \gamma p_{12} \quad (5)$$

$$(\lambda_0 + \delta)p_{n3} = \gamma p_{n2} + \lambda_0 p_{n-1,3}, \quad n \geq 2 \quad (6)$$

$$(\lambda_1 + \theta_1)p_{00} = \theta_2 p_{01} + w p_{02} \quad (7)$$

$$(\lambda_1 + \theta_1)p_{n0} = \lambda_1 p_{n-1,0}, \quad n \geq 1 \quad (8)$$

Defining probability generating functions as

$$H_i(z) = \sum_{n=0}^{\infty} p_{ni} z^n, \quad \text{for } i = 0, 1, 2. \quad (9)$$

$$H_3(z) = \sum_{n=1}^{\infty} p_{n3} z^n \quad (10)$$

Multiplying system of equations (1) and (2) by appropriate powers of z and summing over n from 1 to ∞ .

$$H_1(z) = \frac{\theta_1}{(\lambda_2 + \theta_2 - \lambda_2 z)} p_{00} \quad (11)$$

Similarly, equations (7) and (8) yield

$$H_0(z) = \frac{(\lambda_1 + \theta_1)}{(\lambda_1 + \theta_1 - \lambda_1 z)} p_{00} \quad (12)$$

On similar steps from equations (5) and (6) we have

$$(\lambda_0 + \delta - \lambda_0 z) H_3(z) = \gamma H_2(z) - \gamma p_{02}$$

After some rearrangement of terms, we obtain

$$H_3(z) = \frac{\gamma(H_2(z) - p_{02})}{\lambda_0(1 - z) + \delta} \quad (13)$$

Where,

$$p_{02} = \frac{1}{w} \left(\lambda_1 + \theta_1 - \frac{\theta_1 \theta_2}{\lambda_2 + \theta_2} \right) p_{00}, \quad \text{Using (7)} \quad (14)$$

Multiplying equations (3) and (4) by an appropriate power of z and taking summation over n , together with the use of equations (9) and (10) we obtain

$$\begin{aligned} \left(\lambda_3 + \mu\beta + \gamma - \lambda_3 z - \frac{\mu\beta}{z} \right) H_2(z) \\ = \theta_1 H_0(z) + \theta_2 H_1(z) + \delta H_3(z) - (\lambda_1 + 2\theta_1) p_{00} \\ + \left(\mu\beta + \gamma - \frac{\mu\beta}{z} \right) p_{02} \end{aligned} \quad (15)$$

Considering equations (13) and (15) simultaneously, we obtain

$$H_2(z) = \frac{\theta_1 H_0(z) + \theta_2 H_1(z) - (\lambda_1 + 2\theta_1) p_{00} + g_2(z) p_{02}}{g_1(z)} \quad (16)$$

Where,

$$g_1(z) = \lambda_3(1 - z) + \mu\beta \left(1 - \frac{1}{z} \right) - \frac{\gamma\delta}{\lambda_0(1 - z) + \delta} + \gamma \quad (17)$$

$$g_2(z) = \mu\beta \left(1 - \frac{1}{z} \right) - \frac{\gamma\delta}{\lambda_0(1 - z) + \delta} + \gamma \quad (18)$$

Taking limits $z \rightarrow 1$ in equation (11) and (12) the closed-form expressions for P.G.F's are

$$H_0(1) = \frac{(\lambda_1 + \theta_1)}{\theta_1} p_{00} \quad (19)$$

$$H_1(1) = \frac{\theta_1}{\theta_2} p_{00} \quad (20)$$

Differentiating equation (11) and (12) and taking limits $z \rightarrow 1$

$$H'_0(1) = \frac{\lambda_1(\lambda_1 + \theta_1)}{\theta_1^2} p_{00} \quad (21)$$

$$H'_1(1) = \frac{\lambda_2\theta_1}{\theta_2^2} p_{00} \quad (22)$$

$$H''_0(1) = \frac{2\lambda_1^2(\lambda_1 + \theta_1)}{\theta_1^3} p_{00} \quad (23)$$

$$H''_1(1) = \frac{2\lambda_2^2\theta_1}{\theta_2^3} p_{00} \quad (24)$$

Similarly, taking limits in equation (16) and using the L-Hospital rule

$$H_2(1) = \frac{\left(\frac{\lambda_1(\lambda_1 + \theta_1)}{\theta_1} + \frac{\theta_1\lambda_2}{\theta_2}\right) p_{00} + \left(\mu\beta - \frac{\lambda_0\gamma}{\delta}\right) p_{02}}{\beta\mu - \lambda_3 - \frac{\lambda_0\gamma}{\delta}} \quad (25)$$

On taking limit in equation (13),

$$H_3(1) = \frac{\gamma(H_2(1) - p_{02})}{\delta} \quad (26)$$

Now, differentiating $H_2(z)$, taking limits $z \rightarrow 1$ and applying L' Hospital rule twice,

$$H'_2(1) = \frac{(\theta_1 H''_0(1) + \theta_2 H''_1(1) + g''_2(1) p_{02}) g'_1(1)}{2(g'_1(1))^2} - \frac{(\theta_1 H'_0(1) + \theta_2 H'_1(1) + g'_2(1) p_{02}) g''_1(1)}{2(g'_1(1))^2} \quad (27)$$

Where,

$$g'_1(1) = \mu\beta - \frac{\lambda_0\gamma}{\delta} - \lambda_3$$

$$g'_2(1) = \mu\beta - \frac{\lambda_0\gamma}{\delta}$$

$$g_1''(1) = g_2''(1) = -2 \left(\mu\beta + \gamma \left(\frac{\lambda_0}{\delta} \right)^2 \right)$$

Similarly, after differentiating equation (11) and taking limits $z \rightarrow 1$, we get

$$H_3'(1) = \frac{\gamma(\delta H_2'(1) + \lambda_0(H_2(1) - p_{02}))}{\delta^2} \quad (28)$$

The closed-form expressions for all the P.G.F's are implicitly expressed in terms of only one probability p_{00} .

To obtain p_{00} we use the following normalization condition.

$$\sum_{i=0}^3 H_i(1) = 1 \quad (29)$$

After some rearrangements of terms, we get:

$$p_{00} \left(\frac{\lambda_1 + \theta_1}{\theta_1} + \frac{\theta_1}{\theta_2} + \left(\frac{\gamma + \delta}{\delta} \right) (A + BC) - \frac{\gamma C}{\delta} \right) = 1 \quad (30)$$

Where,

$$\begin{aligned} A &= \frac{\lambda_1 \theta_2 (\lambda_1 + \theta_1) + \lambda_2 \theta_1^2}{\theta_1 \theta_2 \left(\mu\beta - \lambda_3 - \frac{\lambda_0 \gamma}{\delta} \right)} \\ B &= \frac{\mu\beta\delta - \lambda_0 \gamma}{\delta \left(\mu\beta - \lambda_3 - \frac{\lambda_0 \gamma}{\delta} \right)} \\ C &= \frac{\lambda_1 \lambda_2 + \lambda_1 \theta_2 + \lambda_2 \theta_1}{w(\lambda_2 + \theta_2)} \\ p_{00} &= \left(\frac{\lambda_1 + \theta_1}{\theta_1} + \frac{\theta_1}{\theta_2} + \left(\frac{\gamma + \delta}{\delta} \right) (A + BC) - \frac{\gamma C}{\delta} \right)^{-1} \end{aligned} \quad (31)$$

4. System performance measures

Expected system length = Mean number of customers in the system

$$\begin{aligned} &= EL_s \\ &= \sum_{i=0}^3 \sum_{n=1}^{\infty} n p_{ni} \end{aligned}$$

Expected system length in breakdown state = Mean number of customers in the system in down-state of the server (repair state)
= EL[B]

$$= \sum_{n=1}^{\infty} n p_{n3}$$

$$\text{Probability of server being on vacations } P_v = \sum_{n=1}^{\infty} \sum_{i=0}^1 p_{ni}$$

$$\text{Probability of server in the active (normal) state } P_w = \sum_{n=1}^{\infty} p_{n2}$$

$$\text{Probability of server being under repair } P_r = \sum_{n=1}^{\infty} p_{n3}$$

5. Numerical results

In this section, the sensitivity of different performance measures of the queueing model towards the system parameters is analysed and the observed numerical results are graphically represented with aid of MATLAB software. For the purpose, the parameters are fixed as $\lambda_3 = 2.4$, $\lambda_1 = 2$, $\lambda_0 = 1.2$, $\mu = 7$, $w = 0.3$, $\gamma = 0.7$, $\lambda_2 = 2.2$, $\delta = 0.6$, $\beta = 0.8$, $\theta_1 = 0.6$, $\theta_2 = 0.8$, unless they are changed in graphs as shown.

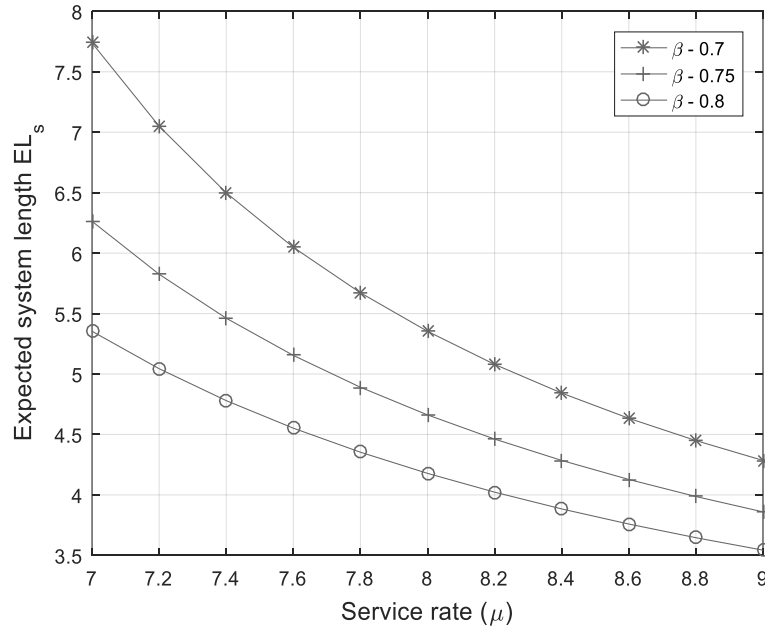


Figure 1: Effect of service rate μ on mean system length

Figure 1 shows variation in mean (expected) system length with μ . The mean system length decreases as μ increases. This is due to a decrease in mean service time with increasing μ . On increasing the probability of leaving the system β by unsatisfied customers on service completion, this system length goes on decreasing further, the reason being that with increasing β , the feedback probability decreases and hence the length of the system.

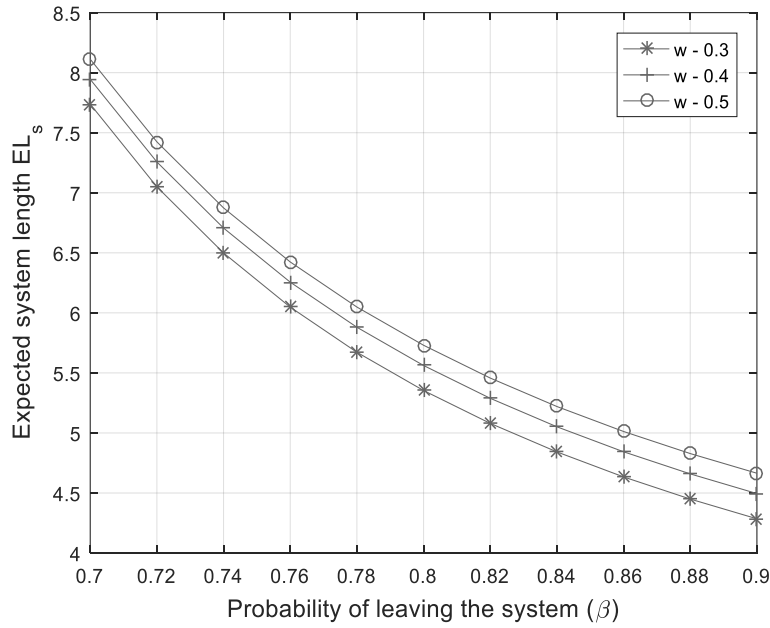


Figure 2: Variation in mean system length versus leaving probability β

Figure 2 illustrates how the expected system length varies with change in leaving probability of customers for different values of w . As w increases, the mean waiting time of the server in normal state decreases and this results in a corresponding increase in mean system length.

Figure 3 reveals the effect of variation in leaving probability β on the probability of the server being on vacation. As we observe from the figure, with an increase in leaving probability, the probability of the server being on vacations increases. This increase is more obvious with increasing values of repair rate δ ; this is due to the reason that as δ increases, the mean repair time decreases, thereby increasing the probability of servers being on vacation.

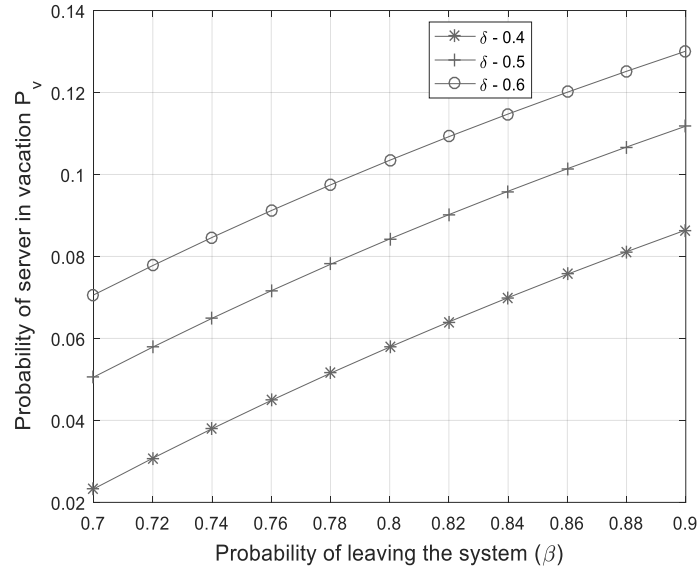


Figure 3: Variation in the probability of server in vacations versus β for different repair rates δ

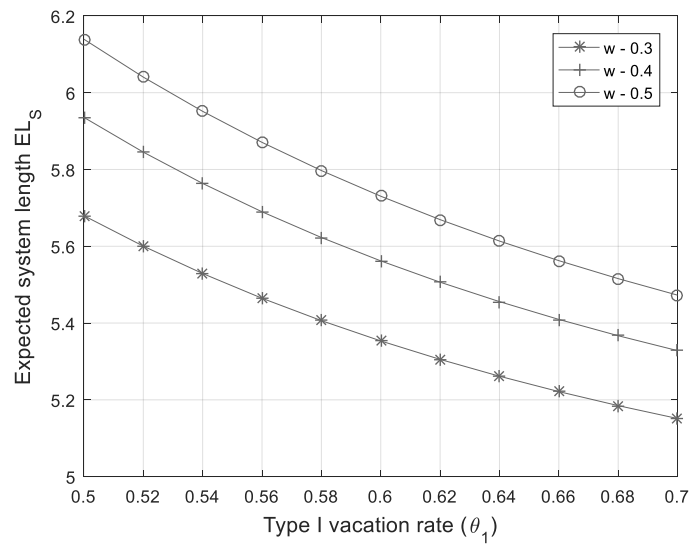


Figure 4: Effect of rate of type I vacation on expected system length

The effect of change in the rate of type I vacation on expected system length is depicted in figure 4. The expected system length decreases as θ_1 increases. This is due to an increase in the duration of type I vacation with decreasing θ_1 . The Expected system length further increases as the parameter of waiting for the server w increases. The reason is that with an increase in w , the waiting time of the server in normal state decreases.

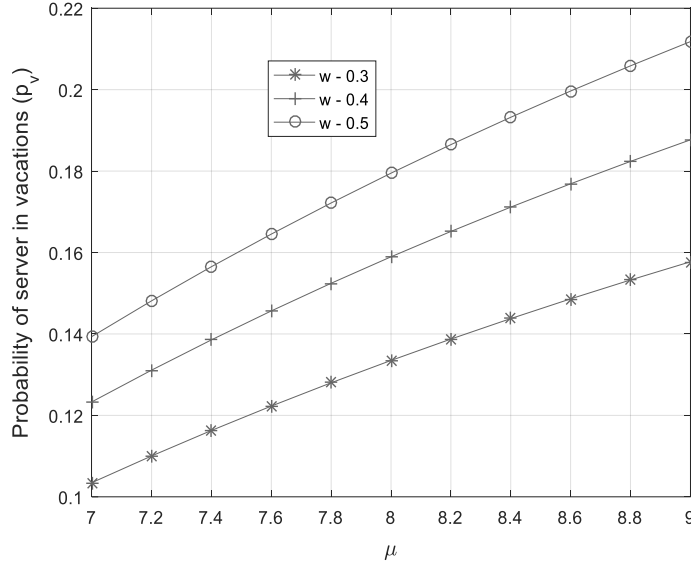


Figure 5: Variation in the probability of server in vacations versus service rate μ

The variation in the probability of the server being on vacation with service rate μ is shown in figure 5. As μ increases the mean service time decreases. This results in faster service thereby increasing the probability of the server being on vacation. This increase is more obvious with increasing the waiting time parameter w . This is because of the corresponding decrease in the mean waiting time of the server in the active (normal) state.

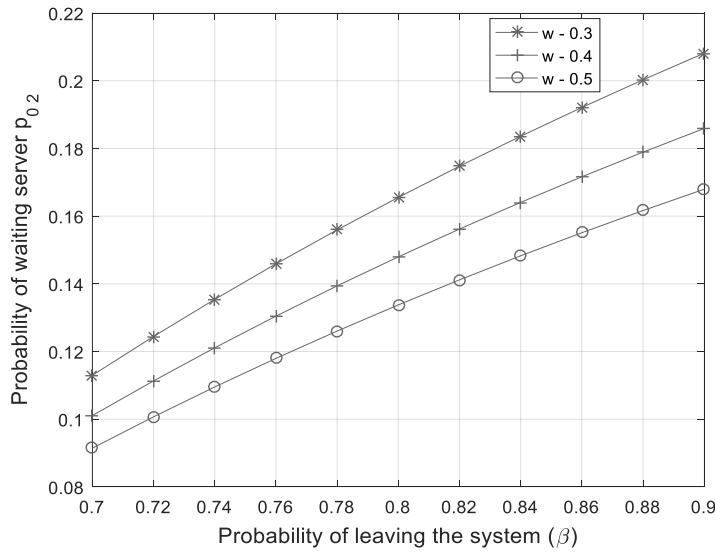


Figure 6: Probability of waiting server versus leaving probability β

Figure 6 represents the effect of leaving probability β of the customer after being served on the probability of waiting server p_{02} . The probability of waiting server increases with an increase in the probability of leaving the system β . As β increases, the probability of waiting for server p_{02} decreases. This is due to a decrease in mean waiting time with an increase in β .

6. Cost optimization

In this section, the operating cost function is optimized relative to μ . Here we define some cost elements as

C_{L_S} = Cost per unit time for the customer present in the system.

C_μ = Cost per unit time for service in the active state.

C_{θ_1} = Cost per unit time in the period of type I vacation.

C_{θ_2} = Cost per unit time in the period of type II vacation.

C_γ = Cost per unit time in the breakdown state

C_δ = Cost per unit time for repair

The cost function per unit time is defined as

$$F(\mu) = EL_S C_{L_S} + \mu C_\mu + \theta_1 C_{\theta_1} + \theta_2 C_{\theta_2} + \gamma C_\gamma + \delta C_\delta$$

Fix $C_{L_S} = 14$, $C_\mu = 20$, $C_{\theta_1} = 10$, $C_{\theta_2} = 8$, $C_\gamma = 7$, $C_\delta = 10$ in parabolic method to find the optimal cost $F(x)$ and corresponding value of x . This method starts by generating the quadratic function through calculated points in every iteration. The point at which $F(x)$ is optimum in three-point pattern $\{x_1, x_2, x_3\}$ is given by

$$x_L = \frac{0.5(F(x_1)(x_2^2 - x_3^2) + F(x_2)(x_3^2 - x_1^2) + F(x_3)(x_1^2 - x_2^2))}{F(x_1)(x_2 - x_3) + F(x_2)(x_3 - x_1) + F(x_3)(x_1 - x_2)}$$

This value obtained improves the current three-point pattern by replacing one of the three points. Optimum value up to the desired degree of accuracy is obtained by recursively using the process.

Table 1 shows that optimum value $F(\mu) = 310.19155$ with the permissible error of 10^{-4} for $\mu = 7.36987$. This value verifies the results of Figure 7.

S.No	μ_1	μ_2	μ_3	$F(\mu_1)$	$F(\mu_2)$	$F(\mu_3)$	μ_L
1	7.0	7.50000	8.00000	311.66873	310.34146	313.16952	7.40971
2	7.0	7.40971	7.50000	311.66873	310.20608	310.34146	7.38091
3	7.0	7.38091	7.40971	311.66873	310.19268	310.20608	7.37334
4	7.0	7.37334	7.38091	311.66873	310.19166	310.19268	7.37086
5	7.0	7.37086	7.37334	311.66873	310.19155	310.19166	7.37016
6	7.0	7.37016	7.37086	311.66873	310.19155	310.19155	7.36994
7	7.0	7.36994	7.37016	311.66873	310.19154	310.19155	7.36987

Table 1: Optimal service rate for operating cost by quadratic fit approach

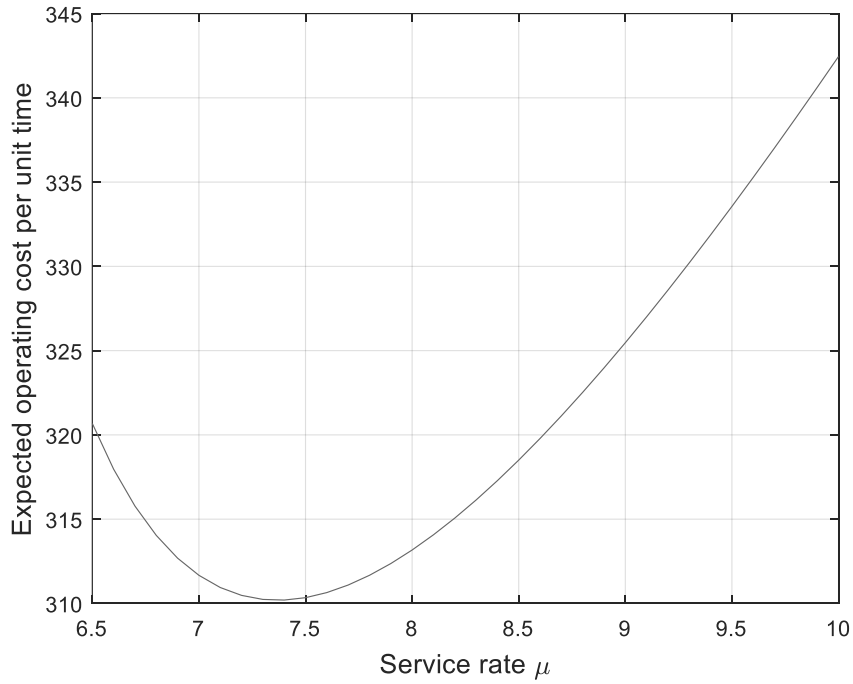


Figure 7: Expected operating cost per unit time versus service rate μ

7. Conclusion

The queueing model under differentiated vacations with an unreliable waiting server and Bernoulli schedule feedback of customers is considered. The impact of state-dependent arrival of customers is studied on the queueing model in steady-state. The sensitivity of some important system measures towards feedback probability, waiting parameter of server, service rate, and duration of vacations is illustrated graphically. Cost is also optimized for the model using the parabolic method. The model can be extended to bulk arrival, general service times for future research.

Conflicts of interests

The authors declare that there is no conflict of interest.

References

- [1] Aissani, A., Braham, K. A. and Taleb, S. (2020). On a Queueing System with Differentiated Vacations and General Service Time. In: 2020 International Conference on Advanced Computing and Applications (ACOMP), 7-14.
- [2] Banik, A. D., Gupta, U.C. and Pathak, S.S. (2007). On the GI/M/1/N queue with multiple working vacations—analytic analysis and computation. *Applied Mathematical Modelling*, 31(9), 1701-1710. Available at: doi:10.1016/j.apm.2006.05.010
- [3] Doshi, B. T. (1986). Queueing systems with vacations, a survey. *Queueing Systems*, 1, 29-66. Available at: <http://dx.doi.org/10.1007/BF01149327>
- [4] Disney, R. L. and Konig, D. (1985). Queueing Networks: A Survey of Their Random Processes. *SIAM Review*, 27(3), 335-403. Available at: <https://www.jstor.org/stable/2031581>
- [5] Ibe, O. C. and Isijola, O. A. (2014). M/M/1 multiple vacation queueing systems with differentiated vacations. *Modelling and Simulation in Engineering*, 2014. Available at: <https://doi.org/10.1155/2014/158247>

- [6] Kalidass, K. and Kasthuri, R. (2013). A two phase service M/G/1 queue with a finite number of immediate Bernoulli feedbacks. *Opsearch*, 51(2), 201-218. Available at: DOI:10.1007/s12597-013-0136-3
- [7] Ke, J.C. (2004). Bi-level control for batch arrival queues with an early startup and un-reliable server. *Applied Mathematical Modelling*, 28(5), 469–485. Available at: DOI:10.1016/j.apm.2003.10.007
- [8] Ke, J.C. (2005). Modified T vacation policy for an M/G/1 queueing system with an unreliable server and startup. *Mathematical and Computer Modelling*, 41(11-12), 1267–1277. Available at: DOI:10.1016/j.mcm.2004.08.009
- [9] Ke, J.C. (2006). On M/G/1 system under NT policies with breakdowns, startup and closedown. *Applied Mathematical Modelling*, 30(1), 49-66. Available at: <https://doi.org/10.1016/j.apm.2005.03.022>
- [10] Ke, J.C., Lin, C.H., Yang, J. Y. & Zhang, Z. G. (2009). Optimal (d, c) vacation policy for a finite buffer M/M/c queue with unreliable servers and repairs. *Applied Mathematical Modelling*, 33(10), 3949–3962. Available at: DOI:10.1016/j.apm.2009.01.008
- [11] Kim, B. K. and Lee, D. H. (2014). The M/G/1 queue with disasters and working breakdowns. *Applied Mathematical Modelling*, 38(5-6), 1788-1798. Available at: <https://doi.org/10.1016/j.apm.2013.09.016>
- [12] Levy, Y. and Yechiali, U. (1975). Utilization of idle time in an M/G/1 queueing system. *Management Science*, 22(2), 202-211. Available at: <http://www.jstor.org/stable/2629709>
- [13] Servi, L. D. and Finn, S. G. (2002). M/M/1 queues with working vacations (M/M/1/WV). *Performance Evaluation*, 50(1), 41-52. Available at: [https://doi.org/10.1016/S0166-5316\(02\)00057-3](https://doi.org/10.1016/S0166-5316(02)00057-3)
- [14] Sharma, S. K. and Kumar, R. (2012). A Markovian feedback queue with retention of reneged customers and balking. *Advanced Modelling and Optimization*, 14(3), 681-688.
- [15] Sundar Rajan B. Ganesan, V. & Rita, S. (2017). Feedback Queue with Services in Different Stations under Reneging and Vacation Policies. *International Journal of Applied Engineering Research* 12(22), 11965-11969.

- [16] Takacs, L. (1963). A single server queue with feedback. Bell System Technical Journal, 42(2), 505-519. Available at: <https://doi.org/10.1002/j.1538-7305.1963.tb00510.x>
- [17] Tian, N. and Zhang, Z. G.(2006). Vacation Queueing Models: Theory and Applications. Springer, New York, NY, USA, 2006.
- [18] Unni, V., Mary, K.J.R. (2019). Queueing systems with multiple servers under differentiated working vacations. International journal of research in advent technology, 7(4), 392-396. Available at: DOI:10.32622/ijrat.742019141
- [19] Vijayashree, K.V. and Janani, B. (2017). Transient analysis of an M/M/1 queueing system subject to differentiated vacations. Quality Technology and Quantitative Management. 15(6), 730-748. Available at: <https://doi.org/10.1080/16843703.2017.1335492>
- [20] Zhang, H. and Zhou, G. (2017). M/M/1 queue with m kinds of differentiated working vacations. Journal of Applied Mathematics and Computing, 54, 213-227. Available at: DOI10.1007/s12190-016-1005-z

Perfect edge domination in vague graphs

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Abstract

In this paper, we introduce the notions of perfect edge domination set and perfect edge domination number in vague graphs. Also, we introduce the definitions of connected perfect edge domination set and connected perfect edge dominating number. Moreover, we investigate some related properties of these concepts with comprehensive results and illustrations.

Keywords: Vague graphs; Edge dominating set; Perfect edge domination set; Perfect edge domination number; Connected perfect edge domination set; Connected perfect edge domination number.

2010 AMS subject classification[§]: 03B60, 06B10, 06B20.

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[§] Received on June 25, 2021. Accepted on December 19, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.588. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Edge domination sets in graphs are a phenomenon in research that has gained prominence due to the extreme scope of offering research solutions with varied dimensions. With the aid of perfect edge dominance and successful edge dominance in graphs, Chin Lung Lua, Ming-Tat Koa and Chuan Yi Tangb [3] investigated perfect edge dominance in graphs. R. S. Chitra and N. Prabhavathi [2] performed an analysis of perfect edges, perfect edge covering, and perfect edge vertex domination sets in graphs. The concept of ideal domination sets was proposed by S. Revathi, P.J. Jayalakshmi, and C.V. R. Harinarayanan [10]. The notion of connected edge domination in fuzzy graphs was suggested by C.Y. Ponnappan, S. Basheer Ahamed, and P. Surulinathan [7]. P. Karpagam and V. Revathi [6] introduced the idea of connected edge perfect domination in fuzzy graphs with the idea of the principle of connected edge dominance in fuzzy graphs. S. Revathi, C.V.R. Harinarayanan, and R. Muthuraj [9] introduced an advanced idea of perfect domination in intuitionistic fuzzy graphs. W. L. Gau and D. J. Buehrer [4] proposed the concept of a vague set. R.A. Borzooei and H. Rashmanlou [1] introduced the notion of domination in vague graphs, and obtained strong domination numbers with applications. The definition of dominating sets in vague graphs was efficiently used in vague graphs by Yahya Talebi and Hossein Rashmanlou [11]. Recently, the authors [5] explored the concepts of edge dominance, independent edge domination in vague graphs, and obtained its related properties.

In this paper, we present the notion of a perfect edge domination set and the perfect edge domination number of the vague graphs. Further, we introduce the connected perfect edge domination set and connected perfect edge dominating number. Also, we obtained some relevant properties.

2. Preliminaries

In this section, we will show some basic definitions and properties that are helpful in developing our main results.

Definition 2.1[4] A vague set P in the universe of discourse X is characterized by two membership functions given by

- i. A truth membership function $t_P: X \rightarrow [0, 1]$,
- ii. A false membership function $f_P: X \rightarrow [0, 1]$.

Where $t_P(x)$ is lower bound of the grade of membership of x derived from the ‘evidence for x ’, and $f_P(x)$ is a lower bound of the negation of x derived from the ‘evidence against x ’ and $t_P(x) + f_P(x) \leq 1$. Thus the grade

of membership of x in the vague set P is bounded by a subinterval $[t_P(x), 1 - f_P(x)]$ of $[0, 1]$. The vague set P is written as $P = \{(x, [t_P(x), f_S(x)]) / x \in X\}$, where the interval $[t_P(x), 1 - f_S(x)]$ is called the value of x in the vague set P .

Definition 2.2[1] A vague graph is of the form $G = (P, Q)$, where

- i. A sequence of distinct vertices $P = \{v_1, v_2, \dots, v_n\}$, such that $t_P : P \rightarrow [0, 1]$ and $f_P : P \rightarrow [0, 1]$ are truth and false membership functions, respectively such that $0 \leq t_P(x) + f_P(x) \leq 1$ for all $x \in P$.
- ii. A vague relation of the vague subsets $X \times Y$ is an expression R , defined by $R = \{(x, y), t_R(x, y), f_R(x, y)\} / x \in X, y \in Y\}$, where $t_R : X \times Y \rightarrow [0, 1]$ and $f_R : X \times Y \rightarrow [0, 1]$, which satisfies the condition $0 \leq t_R(x, y) + f_R(x, y) \leq 1$ for all $(x, y) \in X \times Y$.

Definition 2.3[1] Let $G = (P, Q)$ be a vague graph, where $P = (t_P, f_P)$ is a vague set of vertex and $Q = (t_Q, f_Q)$ is a set edge in G , then for every $uv \in Q$, such that $t_Q(uv) \leq \min\{t_Q(u), t_Q(v)\}$ and $f_Q(uv) \leq \max\{f_Q(u), f_Q(v)\}$.

Definition 2.4 [1] Let $G = (P, Q)$ be a vague graph. If cardinality of the arcs $t_Q(v_i v_j) = \min\{t_P(v_i), f_P(v_j)\}$ and $f_Q(v_i v_j) = \max\{t_P(v_i), f_P(v_j)\}$ for all $v_i v_j \in$, then G is called a complete vague graph.

Definition 2.5[10] An arc of a vague graph $G = (P, Q)$ is said to be a strong edge in G . If $t_Q(uv) \geq (t_Q)^\infty(uv)$ and $f_Q(uv) \leq (f_Q)^\infty(uv)$.

- Where
- i. $(t_Q)^\infty(uv) = \sup\{(t_Q)^r(uv) : r = 1, 2, \dots, n\}$
 - ii. $(f_Q)^\infty(uv) = \inf\{(f_Q)^r(uv) : r = 1, 2, \dots, n\}$.

Definition 2.6[10] Let e_i be an edge in a vague graph $G = (P, Q)$. Then the neighborhood of e_j is representing by $N(e_i) = \{e_j \in Q / (u, v) \text{ is a strong arc}\}$.

Definition 2.7[7] Let $G = (P, Q)$ be a vague graph and $e_i, e_j \in Q$. If a strong arc e_i is adjacent to e_j . Then, we say that e_i dominates e_j . It is denoted by D .

Definition 2.8[1] An edge e_j in a vague graph $G = (P, Q)$ is called a neighbor of $e_i \in D$ with respect to D , if $N(e_j) \cap D = \{e_i\}$.

Definition 2.9[1] Let $G = (P, Q)$ be a vague graph. If neighborhood degree is defined by

- i. The minimum neighborhood degree of G is $\delta(G) = \min\{d_N(e), e \in Q\}$
- ii. The maximum neighborhood degree of G is $\Delta(G) = \max\{d_N(e), e \in Q\}$.

Definition 2.10[4] Two vertices v_i and v_j in a vague graph $G = (P, Q)$ are called a strong neighborhood G . If either one of the conditions are hold,

- i. $t_Q(v_i v_j) > 0, f_Q(v_i v_j) > 0$
- ii. $t_Q(v_i v_j) = 0, f_Q(v_i v_j) > 0$
- iii. $t_Q(v_i v_j) > 0, f_Q(v_i v_j) = 0$.

Definition 2.11[10] Let $G = (P, Q)$ be a vague graph. Then number of edge (the cardinality of Q) is called the order size of a vague graph and is denoted by

$$O(S) = \sum_{v_i v_j \in Q} \left(\frac{1+t_Q(v_i v_j)-f_Q(v_i v_j)}{2} \right) \text{ for all } v_i v_j \in Q.$$

Definition 2.12[4] Two edges in a vague graph $G = (P, Q)$ is called an independent if there is no any strong arcs between them.

Definition 2.13[4] Let $G = (P, Q)$ be a vague graph. If the sub graph is induced by D has an isolated edge.

Definition 2.14[4] A edge in vague graph $G = (P, Q)$ is an isolated edge, if it is not adjacent to any strong arc in G .

3. Main Results

In this section, we introduce perfect edge domination set, perfect edge domination number, connected perfect edge domination set and connected perfect edge dominating number of vague graphs, and obtain some properties with illustrations.

Definition 3.1 Let $G = (P, Q)$ be a vague graph and D be an edge dominating set in G . If for every edge of $Q(G) - D$ is adjacent to exactly one edge in D , then D is called a perfect edge domination set in G .

Example 3.2 Let $G = (P, Q)$ be a vague graph as shown in the figure 3.1. Consider the edge set $Q = \{e_1, e_2, e_3, e_4, e_5\}$. We have e_1, e_4 and e_5 are strong arcs in G .

Here, $\{e_5\}$ and $\{e_1, e_4\}$ are edge dominating sets in G . Now, $D = \{e_5\}$ is a perfect edge domination in G . Since, $\{e_5\}$ is dominates the all other neighbor edges in G .

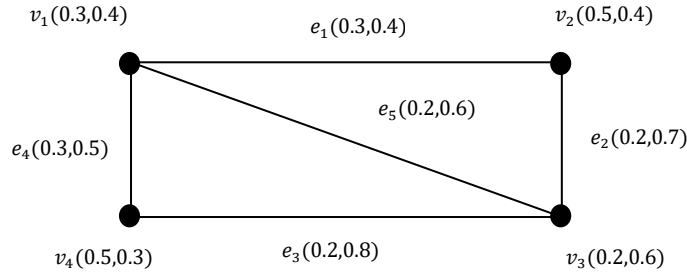


Figure 3.1: Perfect edge domination set

Definition 3.3 Let $G = (P, Q)$ be a vague graph. If it has a minimum perfect edge dominating set of vague graph G . Then it is called a perfect edge dominating number of G , and it is denoted by $\gamma_p(D)$ in G .

Example 3.4 Let $G = (P, Q)$ be a vague graph as shown in the figure 3.2. From the edge set $Q = \{e_1, e_2, e_3, e_4\}$, we see that $\{e_1\}$ and $\{e_4\}$ are strong arc G . Then $\{e_1\}$ and $\{e_4\}$ are a perfect edge dominating sets of G . Then minimum perfect edge dominating number is $\gamma_p(D) = 0.70$.

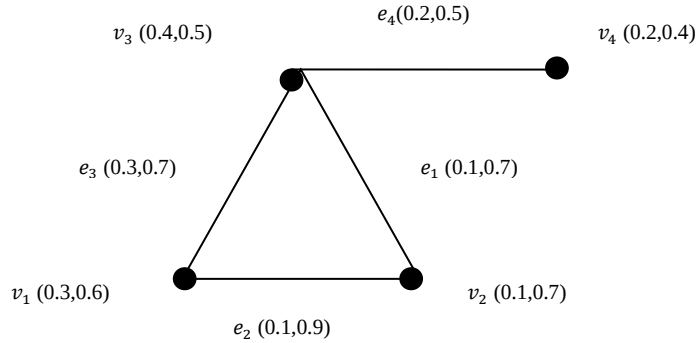


Figure 3.2: Minimum perfect edge dominating number.

Proposition 3.5 Let $G = (P, Q)$ be a vague graph with at least one isolated edge, then perfect edge dominating set does not exist.

Proof: Let D be a minimal perfect edge dominating set and e_i be the path of D , since G has at least one isolated edge. Incase $Q(G) - D$ is a perfect edge dominating set of vague graph, it has e_j as neighborhood of perfect edge

domination set D . There exist a complement path of vague graph is denoted by $N(e_j)$.

From the definition 2.6, we have $|N(e_j) \cap D|=1$, but in this vague graph $|N(e_j) \cap D| \neq 1$.

This is a contradiction. Therefore, $Q(G) - D$ not a perfect dominating set. ■

Proposition 3.6 Let $G = (P, Q)$ be a complete vague graph. Then, every edge in D is a perfect edge dominating set.

Proof: Let $G = (P, Q)$ be a complete vague graph and let $e_i \in D$ be a edge dominating set in G . Then every edges in the graph is $t_Q(v_i v_j) = \min\{t_P(v_i), f_P(v_j)\}$ and $f_Q(v_i v_j) = \max\{t_P(v_i), f_P(v_j)\}$ for all $v_i v_j \in Q$.

Therefore, every path in G is a strong arc and complement edges are $e_j \in Q(G) - D$ adjacent to exactly one edge in D is said to be a perfect edge domination set in vague graph. Here, D is a perfect edge dominating set of vague graph G , and then every complete vague graph G is a perfect edge dominating set. ■

Example 3.7 Let $G = (P, Q)$ be a complete vague graph as shown in the figure 3.3. From the edge set $Q = \{e_1, e_2, e_3, e_4, e_5\}$ is an edge dominating set. Then, we have e_1, e_4 and e_5 which are strong arcs of the vague graph.

Here $\{e_5\}$ and $\{e_1, e_4\}$ are edge dominating set of G , for $D = \{e_5\}$ is a perfect edge domination in vague graph.

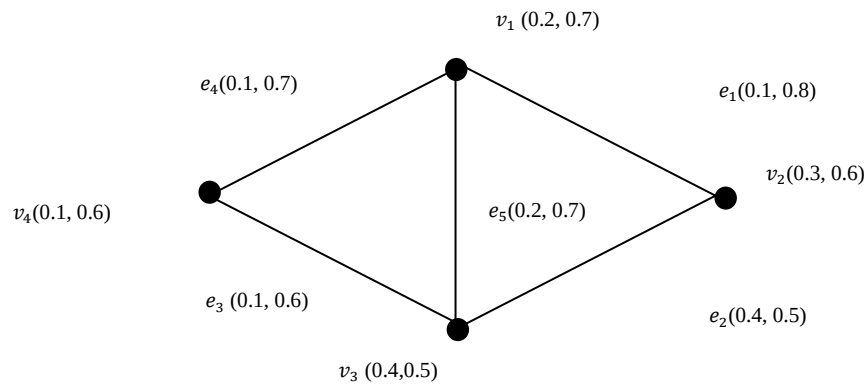


Figure 3.3: Perfect edge domination set

Proposition 3.8 Let $G = (P, Q)$ be a vague graph, then D is a minimal perfect edge dominating set. If for each $Q(G) - D$ is not a perfect edge dominating set.

Proof: Let G be a vague graph and has a minimal perfect edge domination set D . From the definition 3.1, if D is minimum, the arcs must be strong. Suppose e_i and e_j are any two edges adjacent in G , but $e_i \in D$ is a minimum perfect edge domination set of a vague graph. Then e_j may or may not have any strong in this graph. Thus, each edge in e_j has no strong neighbor of edge in $Q(G) - D$. Hence, $Q(G) - D$ is not perfect edge domination set of vague graph G . ■

Definition 3.9 Let $G = (P, Q)$ be a vague graph and S is an edge dominating set of G is connected perfect edge domination set with $\langle S \rangle$ is connected.

Example 3.10 Let $G = (P, Q)$ be a vague graph as shown in the figure3.4. From the edge set to $Q = \{e_1, e_2, e_3, e_4, e_5\}$, which are strong arcs of G . Here $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are edge dominating sets of G . Then $D = \{e_1, e_4\}$ is connected perfect edge domination in G . Since, $\{e_1, e_4\}$ is a dominating set in G .

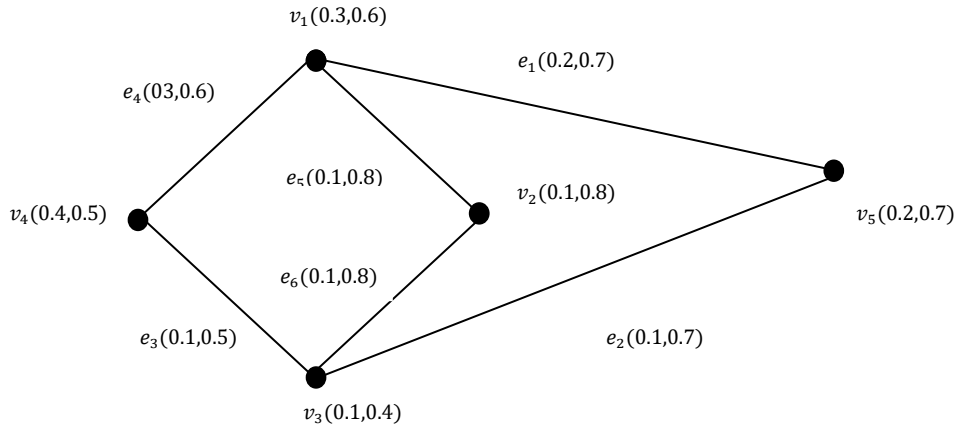


Figure 3.4: Connected perfect edge domination

Connected strong perfect edge dominating set $Q_{cs} = \{e_1, e_2\}$ is connected, then $Q - Q_{cs} = \{e_3, e_4, e_5, e_6\}$ is connected

$$N(e_3) = \{e_2, e_4, e_6\} \cap \{e_1, e_2\} = e_2.$$

$$N(e_5) = \{e_1, e_4, e_5\} \cap \{e_1, e_2\} = e_1.$$

$$N(e_6) = \{e_1, e_5, e_3\} \cap \{e_1, e_2\} = e_2.$$

$$N(e_3) = \{e_1, e_5, e_3\} \cap \{e_1, e_2\} = e_1.$$

Definition 3.11 Let $G = (P, Q)$ be a vague graph. Then the smallest cardinality number of the arc in any edge connected perfect edge dominating set of G is called connected perfect edge domination number, which is denoted as γ_{cp} .

Proposition 3.12 Let $G = (P, Q)$ be a vague graph and $\langle S \rangle$ be a minimal connected edge perfect dominating set of G , then $Q(G) - \langle S \rangle$ is also a connected edge perfect dominating set of G .

Proof: Let $\{e_1, e_2, e_3, \dots, e_n\}$ be an edge set of vague graph G and M be a minimal connected edge dominating set of $Q(G)$. If $Q(G) - M$ is not a connected perfect edge dominating set. Then from, the definition 3.1, we have D has minimum set of arcs that should be strong. if e_i and e_j are any two edges adjacent in G , but $e_i \in D$ is a minimum perfect edge domination set of a vague graph, then e_j is may (or) may not have any strong arc in this graph, Thus, every edge in e_j has no strong neighbor of edge in $Q(G) - D$. Hence $Q(G) - D$ is not a perfect edge domination set of vague graph G . ■

Example 3.13 Let $G = (P, Q)$ be a vague graph as shown in the figure 3.5. From the edge set $Q = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, we see that e_1, e_2 and e_5 are strong arcs in the graph G . Then $\{e_1, e_5\}$ are a connected perfect edge dominating set of G . Thus, $\{e_1, e_5\}$ is minimum connected perfect edge dominating number is $\gamma_{cp}(D) = 0.50$.

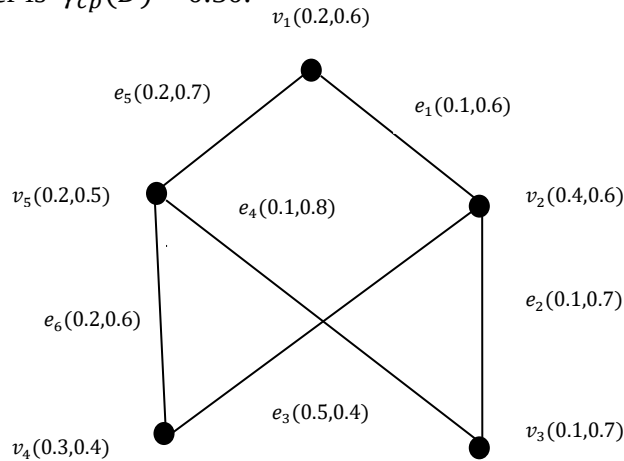


Figure 3.5: Minimum perfect edge dominating number

Remark 3.14 Let $G = (P, Q)$ be a vague graph and $\langle S \rangle$ be a minimal connected edge perfect dominating set of G , then $Q(G) - \langle S \rangle$ is a minimal connected edge perfect dominating set of G .

Proposition 3.15 Let $G = (P, Q)$ be a vague graph and D is a connected perfect edge domination set and it is without isolated edges, then $\frac{r}{\Delta_c(G)+1} \geq \gamma_{CP}(G)$.

Proof: Let G be a vague graph and D be a connected perfect edge domination set. From the definition 3.9 of maximum neighbourhood and minimum neighbourhood of connected perfect edge domination set, then

$$|D|\Delta_c(G) \leq \sum_{e \in D} d_Q(e) = \sum_{e \in D} |N(e)|$$

$$\leq \left| \bigcup_{e \in D} N(e) \right|$$

$$\leq |Q - D|$$

$$\leq r - |D|$$

$$|D|\Delta_c(G) + |D| \leq r$$

$$\text{Therefore, } \frac{r}{\Delta_c(G)+1} \geq \gamma_{CP}(G). \blacksquare$$

Proposition 3.16 Every connected perfect edge dominating set of a vague graph $G = (P, Q)$ is not independent.

Proof: Let G be a vague graph and D be a minimal connected perfect edge dominating set, then path of graphs have strong arcs. Clearly, every edges in the graph are satisfied by the condition of $t_Q(uv) \geq (t_Q)^\infty(uv)$ and $f_Q(uv) \leq (f_Q)^\infty(uv)$.

Suppose, we assume that a vague graph D contains an isolated edge e_i in G . Since G is a connected perfect edge dominating set, e_i is a strong neighborhood of at least one edge $D - \{e_j\}$. Thus, $D - \{e_j\}$ is an edge dominating set which contradicts in the minimum edge domination set D . Hence a vague graph is not independent. $\blacksquare$

Proposition 3.17 Let $G = (P, Q)$ be a vague graph and S_c be a minimal connected perfect edge dominating set of G .

Proof: Let S_c be a minimal connected perfect edge dominating set of vague graph $G = (P, Q)$. If e_j is not dominated by edge e_i in S_c and the induced vague graph $\langle S_c \rangle$ is disconnected. We know that from the definition of

connected perfect dominating set, if every edge $e_j \in Q - S_c$ is perfect dominated by exactly one edge e_i in S_c and the induced vague sub graph $\langle S_c \rangle$ is connected. So, we have every edge $e_j \in Q - S_c$ is dominated by some edge e_i in S_c which is a contradiction. Therefore, S_c is connected dominating set. ■

4. Conclusions

In this study, we have introduced the notions of perfect edge domination set and perfect edge domination number of vague graphs. Furthermore, we have investigated some related properties with suitable examples. Moreover, we have introduced the notion of connected perfect domination set and connected perfect edge domination number of the vague graph. Finally, we have obtained some properties.

References

- [1] R. A. Borzooei and H. Rashmanlou, Domination in Vague Graphs and its Applications, Journal of Intelligent & Fuzzy Systems, 29, 1933 -1940, 2015.
- [2] R. S .Chitra and N. Prabhavathi, A Study of Perfect Edge Covering and Perfect Edge Vertex Domination Sets In Graphs, International Journal of Research and Analytical Reviews, Volume 7, Number 1,408 - 415, 2020.
- [3] Chin Lung Lu, Ming-Tat Ko, and Chuan Yi Tang, Perfect Edge Domination and Efficient Edge Domination, Discrete Applied Mathematics, 119, 227–250, 2002.
- [4] W. L. Gau and D. J. Buehrer, Vague Sets, IEEE Transactions on Systems, Man and Cybernetics 23(2),(1993), 610-614.
- [5] M. Kaliraja, P. Kanibose and A. Ibrahim, Edge Domination in Vague Graph, International Journal of Advanced Science and Technology, Volume 29, Number 1, 1474 – 1480,2020.
- [6] P. Karpagam and V. Revathi, Connected Edge Perfect Domination in Fuzzy Graphs, International Journal of Innovative Research in Science, Engineering and Technology, Volume 7, Number 2, 1836-1840, 2018.

- [7] C. Y. Ponnappan, S. Basheer Ahamed and P. Surulinathan, Connected Edge Domination in Fuzzy Graphs, International Journal of Mathematics And its Applications, Volume 3, Number 2, 133 -137, 2015.
- [8] N. Ramakrishna, Vague Graphs, International Journal of Computational Cognition, Number 7, 51-58, 2009.
- [9] S. Revathi, C. V. R. Harinarayanan and R. Muthuraj, Connected Total Perfect Dominating Set in Fuzzy Graph, International Journal of Computational and Applied Mathematics, Volume 12, Number 1, 84 - 98, 2017.
- [10] S. Revathi, P. J. Jayalakshmi and C. V. R. Harinarayanan, Perfect Dominating Sets in Fuzzy Graphs, IOSR Journal of Mathematics, Volume 8, Number 3, 43-47, 2013.
- [11] Yahya Talebiy and Hossein Rashmanlouz, Application of Dominating Sets in Vague Graphs. Applied Mathematics E-Notes, 17, 251- 267, 2017.
- [12] L. A. Zadeh, Fuzzy Sets, Information and control, 8, 338 - 353, 1965.

Tikhonov type regularization for unbounded operators

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Abstract

In this paper, we introduce a Tikhonov type regularization method for an ill-posed operator equation $Tx = y$, where T is a closed densely defined unbounded operator on a Hilbert space H .

Keywords: densely defined operator, closed operator, Tikhonov type regularization.

2020 AMS subject classifications: 47A10, 47A52 ¹

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¹Received on September 20, 2021. Accepted on December 10, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.663. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Most of the problems arise in the field of science and engineering can be modelled as an operator equation

$$Tx = y \quad (1)$$

where $T : X \rightarrow Y$ is a bounded linear map from a normed linear space X to a normed linear space Y . In most of the cases (1) is Ill-posed. Certain regularization procedures are known for solving ill-posed operator equation (1). For example Tikhonov regularization, Mollifier method, Ritz method [5, 3]. In this paper we introduce a Tikhonov type regularization method for solving an ill-posed operator equation (1), where T is a closed densely defined operator on a Hilbert space H and we study the order of convergence.

2 Preliminaries

Let $L(H)$, $C(H)$ and $B(H)$ denote the space of all linear, closed linear and bounded linear operators on a Hilbert space H respectively. For $T \in L(H)$, the domain, range of T are denoted by $D(T)$, $N(T)$ respectively. An operator $T \in L(H)$ is said to be densely defined if $\overline{D(T)} = H$. For example let $T : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \dots, x_n, \dots) = (x_1, 2x_2, 3x_3, \dots, nx_n, \dots)$$

with domain

$$D(T) = \{(x_1, x_2, x_3, \dots, x_n, \dots) \in H : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\}.$$

Then T is closed and unbounded. Since $c_{00} \subseteq D(T)$ and c_{00} is dense in $l^2(\mathbb{N})$, $D(T)$ is dense in $l^2(\mathbb{N})$.

Proposition 2.1. *Let $T \in C(H)$ be a densely defined operator. Then there exist a unique operator $T^* \in C(H)$ such that*

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x \in D(T), \forall y \in D(T^*).$$

Proof. Let $D(T^*) = \{y \in H : \langle Tx, y \rangle \text{ is continuous for every } x \in D(T)\}$. For $y \in D(T)$, define $f : D(T) \rightarrow \mathbb{C}$ by $f(x) = \langle Tx, y \rangle \forall x \in D(T)$. Extend f to $f_0 : H \rightarrow \mathbb{C}$ by $f_0(x) = \lim_{n \rightarrow \infty} \langle Tx_n, y \rangle$ where (x_n) is a sequence in $D(T)$ such that $x_n \rightarrow x$.

Next we prove that f_0 is well defined. For, let (x_n) and (y_n) be two sequences in $D(T)$ converges to x . Since T is closed, $T(x_n - y_n) \rightarrow 0$. If $\langle Tx_n, y \rangle \rightarrow \langle x, y \rangle$, then

$$\begin{aligned} |\langle Ty_n, y \rangle - \langle x, y \rangle| &= |\langle Ty_n - Tx_n + Tx_n - x, y \rangle| \\ &\leq \|T(y_n - x_n)\| \|y\| + |\langle Tx_n - x, y \rangle| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Hence f_0 is well defined.

Since f_0 is a bounded linear functional on the Hilbert space H , by Riesz representation theorem there exist a unique $y^* \in H$ such that $f_0(x) = \langle x, y^* \rangle$. Thus $\langle Tx, y \rangle = \langle x, y^* \rangle \forall x \in D(T)$. Define $T^* : D(T^*) \rightarrow H$ by $T^*y = y^*$. Then T^* is well-defined. Also $\langle Tx, y \rangle = \langle x, T^*y \rangle \forall x \in D(T), \forall y \in D(T^*)$. $\square$

Consider an ill-posed operator equation

$$Tx = y \tag{2}$$

where T is a closed densely defined operator on H .

Definition 2.1. [7]

Let $T \in C(H)$ be densely defined. Then there exist a unique densely defined operator $T^\dagger \in C(H)$ with domain $D(T^\dagger) = R(T) \oplus R(T)^\perp$ satisfies the following properties

- (i) $TT^\dagger y = P_{\overline{R(T)}} y$ for all $y \in D(T^\dagger)$,
- (ii) $T^\dagger Tx = Q_{N(T)^\perp} x$ for all $x \in D(T)$.
- (iii) $N(T^\dagger) = R(T)^\perp$.

where P and Q are the orthogonal projection on to $\overline{R(T)}$ and $N(T^\perp)$ respectively. The operator $T^\dagger$ is called the Moore-Penrose inverse of T .

For $y \in D(T^\dagger)$, let $S_y = \{x \in D(T) : \|Tx - y\| \leq \|Tu - y\| \forall u \in D(T)\}$. Then $u \in S_y$ is called least square solution of the operator equation (2). Note that $\|T^\dagger y\| \leq \|x\| \forall x \in S_y$, is called least square solution of minimal norm and is denoted by $\hat{x}$ [7].

If $R(T)$ is not closed, then $T^\dagger$ is not continuous. Now we introduce a Tikhonov type regularization procedure for finding an approximate solution for $T^\dagger y$.

3 Tikhonov type regularization

In this section we introduce a Tikhonov type regularization procedure for solving (2).

Lemma 3.1. *Let $T \in C(H)$ be densely defined and $\alpha > 0$. Then $T^*T + \alpha I$ and $TT^* + \alpha I$ are bijective closed densely defined operators on H . Also $(TT^* + \alpha I)^{-1}$ and $(T^*T + \alpha I)^{-1}$ are bounded, self adjoint operators on H .*

Proof. Let $T \in C(H)$ and $\alpha > 0$. By proposition 2.1, we have $T^* \in C(H)$. Hence, $(TT^* + \alpha I)$ and $(T^*T + \alpha I)$ are closed densely defined operators on H . Since $\langle (TT^* + \alpha I)x, x \rangle = \langle T^*x, T^*x \rangle + \alpha \langle x, x \rangle \geq 0, \forall x \in D(T^*)$, we have $(TT^* + \alpha I)$ is a positive operator. Similarly $(T^*T + \alpha I)$ is also a positive operator. Since $T^*T + \alpha I$ is positive,

$$\begin{aligned} \|(T^*T + \alpha I)x\| \|x\| &\geq \langle (T^*T + \alpha I)x, x \rangle \\ &= \langle T^*Tx, x \rangle + \alpha \|x\|^2 \\ &\geq \alpha \|x\|^2 \quad \forall x \in H \end{aligned}$$

Thus

$$\|(T^*T + \alpha I)x\| \geq \alpha \|x\| \quad \forall x \in H \quad (3)$$

Since $T^*T + \alpha I$ is bounded below, it is one-one and its inverse from the range is continuous. Also $R(T^*T + \alpha I)$ is closed. Since $T^*T + \alpha I$ is also self adjoint, $R(T^*T + \alpha I) = N(T^*T + \alpha I)^\perp = H$. Hence $T^*T + \alpha I$ is onto. Therefore $(T^*T + \alpha I)^{-1} \in B(H)$. Similarly $(TT^* + \alpha I)^{-1} \in B(H)$. From (3), $\|(T^*T + \alpha I)^{-1}\| \leq \frac{1}{\alpha}$. $\square$

Theorem 3.1. *Let $T \in C(H)$ be densely defined. Then $T^*(TT^* + \alpha I)^{-1}$ and $T(T^*T + \alpha I)^{-1}$ are bounded operators on H . Also $\|T^*(TT^* + \alpha I)^{-1}\| \leq \frac{1}{\sqrt{\alpha}}$ and $\|T(T^*T + \alpha I)^{-1}\| \leq \frac{1}{\sqrt{\alpha}}$.*

Proof. We have $(T^*T + \alpha I)^{-1}T^*T = I - \alpha(T^*T + \alpha I)^{-1}$

Since $\langle (T^*T + \alpha I)^{-1}x, x \rangle \geq 0 \quad \forall x \in H$,

$$\begin{aligned} \langle (T^*T + \alpha I)^{-1}T^*Tx, x \rangle &= \langle I - \alpha(T^*T + \alpha I)^{-1}x, x \rangle \\ &= \langle x, x \rangle - \alpha \langle (T^*T + \alpha I)^{-1}x, x \rangle \leq \langle x, x \rangle. \end{aligned}$$

Since $(T^*T + \alpha I)^{-1}T^*T$ self adjoint, $\|(T^*T + \alpha I)^{-1}T^*T\| \leq 1$.

Let $x \in H$.

$$\begin{aligned} \|T^*(TT^* + \alpha I)^{-1}x\|^2 &= \langle T^*(TT^* + \alpha I)^{-1}x, T^*(TT^* + \alpha I)^{-1}x \rangle \\ &= \langle TT^*(TT^* + \alpha I)^{-1}x, (TT^* + \alpha I)^{-1}x \rangle \\ &= \langle (TT^* + \alpha I)^{-1}TT^*x, (TT^* + \alpha I)^{-1}x \rangle \\ &\leq \|(TT^* + \alpha I)^{-1}TT^*x\| \|(TT^* + \alpha I)^{-1}x\| \\ &\leq \frac{1}{\alpha} \|x\|^2 \end{aligned}$$

we have $\|T^*(TT^* + \alpha I)^{-1}x\|^2 \leq \frac{1}{\alpha}\|x\|^2 \forall x \in H$.

Thus $\|T^*(TT^* + \alpha I)^{-1}\| \leq \frac{1}{\sqrt{\alpha}}$. Hence $T^*(TT^* + \alpha I)^{-1}$ is bounded. Similarly $T(T^*T + \alpha I)^{-1}$ is bounded. □

Lemma 3.2. [7]

Let $T \in C(H)$ be densely defined. Then

- (i) $(TT^* + I)^{-1}T \subseteq T(T^*T + I)^{-1}$
- (ii) $(T^*T + I)^{-1}T^* \subseteq T^*(TT^* + I)^{-1}$

Remark 3.1. *From Theorem 3.1, we have $T^*(TT^* + \alpha I)^{-1}$ and $T(T^*T + \alpha I)^{-1}$ are bounded. Therefore from Lemma 3.2, we have $(TT^* + \alpha I)^{-1}T$ and $(T^*T + \alpha I)^{-1}T^*$ are bounded.*

Lemma 3.3. *Let $T \in C(H)$ be densely defined. For every $x \in D(T) \cap N(T)^\perp$ $\|\alpha(T^*T + \alpha I)^{-1}x\| \rightarrow 0$, as $\alpha \rightarrow 0$.*

Proof. Let $T_\alpha = \alpha(T^*T + \alpha I)^{-1}$, $\alpha > 0$.

From (3.1) we have $\|(T^*T + \alpha I)^{-1}\| \leq \frac{1}{\alpha}$. Hence $\|T_\alpha\| \leq 1$ for every $\alpha > 0$. Let $u \in R(T^*T)$ then there exist $v \in D(T^*T)$ such that $T^*Tv = u$.

$$\begin{aligned} \|T_\alpha u\| &= \|T_\alpha T^*Tv\| \\ &= \alpha\|(T^*T + \alpha I)^{-1}T^*Tv\| \\ &\leq \alpha\|(T^*T + \alpha I)^{-1}T^*T\|\|v\| \\ &\leq \alpha\|v\| \end{aligned}$$

Hence $\|T_\alpha u\| \leq \alpha\|v\| \forall u \in R(T^*T)$.

Thus for every $u \in R(T^*T)$, $\|\alpha(T^*T + \alpha I)^{-1}u\| \rightarrow 0$ as $\alpha \rightarrow 0$. Since $\overline{R(T^*T)} = N(T)^\perp$, $\|\alpha(T^*T + \alpha I)^{-1}x\| \rightarrow 0, \forall x \in D(T) \cap N(T)^\perp$. □

Theorem 3.2. *Let $T \in C(H)$ be densely defined and $R_\alpha = (T^*T + \alpha I)^{-1}T^*$. Then $\{R_\alpha\}_{\alpha>0}$ is a regularization family for (2).*

Proof. Let $y \in D(T^*)$. Then $(T^*T + \alpha I)\hat{x} = T^*y + \alpha\hat{x}$.

Hence $\hat{x} = (T^*T + \alpha I)^{-1}(T^*y + \alpha\hat{x})$. Thus

$$\begin{aligned} T^\dagger y - R_\alpha y &= \hat{x} - (T^*T + \alpha I)^{-1}T^*y \\ &= (T^*T + \alpha I)^{-1}(T^*y + \alpha\hat{x}) - (T^*T + \alpha I)^{-1}T^*y \\ &= (T^*T + \alpha I)^{-1}\alpha\hat{x} \end{aligned}$$

Hence $\|T^\dagger y - R_\alpha y\| = \alpha \|(T^*T + \alpha I)^{-1} \hat{x}\|$.

Since $\hat{x} \in D(T) \cap N(T)^\perp$, by Lemma 3.3, $\|T^\dagger y - R_\alpha y\| \rightarrow 0$ as $\alpha \rightarrow 0$.

Thus $\{R_\alpha\}_{\alpha>0}$ is a regularization family for (2). □

4 Order estimate

In this section we find an error estimate for the regularization family $R_\alpha = (T^*T + \alpha I)^{-1}T^*$, where T is a closed densely defined operator. We use the following lemmas.

Lemma 4.1. [7]

For $T \in C(H)$ we have the following

- (i) If $\mu \in \mathbb{C}$ and $\lambda \in \sigma(T)$ then $\lambda + \mu \in \sigma(T + \mu I)$
- (ii) If $\alpha \in \mathbb{C}$ and $\lambda \in \sigma(T)$ then $\alpha\lambda \in \sigma(\alpha T)$
- (iii) $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$

Lemma 4.2. [7]

Let $T \in L(H)$ be a positive operator. Then the following results hold.

- (i) $T^\dagger$ is positive.
- (ii) $\sigma(T) = \sigma_a(T)$
- (iii) $0 \notin \sigma(I + T)$ that is $(I + T)^{-1} \in B(H)$
- (iv) If $0 \notin \sigma(T)$ then $0 \neq \lambda \in \sigma(T)$ if and only if $\frac{1}{\lambda} \in \sigma(T^{-1})$

Theorem 4.1. Suppose $T \in C(H)$ is densely defined positive operator. Then for every $\alpha > 0$

$$\sigma\left((T + \alpha I)^{-2}T\right) = \left\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\right\}$$

Proof. Since T is positive, $T + \alpha I$ is bijective.

Also $(T + \alpha I)^{-2}T = (T + \alpha I)^{-1} - \alpha(T + \alpha I)^{-2}$.

From Lemmas 4.1, 4.2 for $\alpha, \lambda > 0$, we have

$\lambda \in \sigma(T)$ if and only if $(\lambda + \alpha)^{-1} \in \sigma\left((T + \alpha I)^{-1}\right)$.

Hence

$$\begin{aligned} \sigma\left((T + \alpha I)^{-2}T\right) &= \left\{\mu - \alpha\mu^2 : \mu \in \sigma\left((T + \alpha I)^{-1}\right)\right\} \\ &= \left\{\frac{1}{\lambda + \alpha} - \frac{\alpha}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\right\} \\ &= \left\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\right\}. \end{aligned}$$

□

Corolary 4.1. Let $T \in C(H)$ be densely defined and $\alpha > 0$.

$$\text{Then } \|(T^*T + \alpha I)^{-1}T^*\| = \sup \left\{ \frac{\sqrt{\lambda}}{\lambda + \alpha} : \lambda \in \sigma(T^*T) \right\} \leq \frac{1}{2\sqrt{\alpha}}$$

Proof. We have $R_\alpha = (T^*T + \alpha I)^{-1}T^*$. Hence $R_\alpha^*R_\alpha = T(T^*T + \alpha I)^{-2}T^*$.

From Lemma 2.2 in [2], we have $R_\alpha^*R_\alpha = (TT^* + \alpha I)^{-2}TT^*$.

Since $R_\alpha^*R_\alpha$ is self adjoint and bounded, $\|R_\alpha\|^2 = \|R_\alpha^*R_\alpha\|$

$$\begin{aligned} &= \sup \left\{ |k| : k \in \sigma(R_\alpha^*R_\alpha) \right\} \\ &= \sup \left\{ \frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(TT^*) \right\} \end{aligned}$$

$$\|R_\alpha\| = \sup \left\{ \frac{\sqrt{\lambda}}{\lambda + \alpha} : \lambda \in \sigma(TT^*) \right\}.$$

Since $2\sqrt{\alpha\lambda}(\lambda + \alpha)^{-1} \leq 1$ for $\lambda, \alpha > 0$, we have $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$. □

Now we find an order estimate for R_α .

Corolary 4.2. Let $T \in C(H)$ is densely defined and $R_\alpha = (T^*T + \alpha I)^{-1}T^*$. For every $\alpha > 0$ and $\delta > 0$, let $y^\delta \in H$ be such that $\|y - y^\delta\| \leq \delta$. Then

$$\|R_\alpha y - R_\alpha y^\delta\| \leq \frac{\delta}{2\sqrt{\alpha}}.$$

Proof. For $\|y - y^\delta\| \leq \delta$,

$$\begin{aligned} \|R_\alpha y - R_\alpha y^\delta\| &\leq \|R_\alpha\| \|y - y^\delta\| \\ &\leq \frac{1}{2\sqrt{\alpha}} \|y - y^\delta\| \\ &\leq \frac{\delta}{2\sqrt{\alpha}} \end{aligned}$$

□

Theorem 4.2. Let $T \in C(H)$ is densely defined and $R_\alpha = (T^*T + \alpha I)^{-1}T^*$. Then

$$\|\hat{x} - R_\alpha y^\delta\| \leq \|\hat{x} - R_\alpha y\| + \frac{\delta}{2\sqrt{\alpha}}. \text{ If } \alpha = \alpha(\delta) \text{ is chosen such that } \alpha(\delta) \rightarrow 0$$

and $\frac{\delta}{\sqrt{\alpha}(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$, then $\|\hat{x} - R_{\alpha(\delta)}^\delta\| \rightarrow 0$ as $\delta \rightarrow 0$.

$$\begin{aligned} \text{Proof. } \|\hat{x} - R_\alpha y^\delta\| &\leq \|\hat{x} - R_\alpha y\| + \|R_\alpha y - R_\alpha y^\delta\| \\ &\leq \|\hat{x} - R_\alpha y\| + \frac{\delta}{2\sqrt{\alpha}} \end{aligned}$$

by Theorem 3.6, $\|\hat{x} - R_\alpha y\| \rightarrow 0$ as $\alpha \rightarrow 0$.

□

References

- [1] P Mathe B Hofmann and H. R.V Weizsacker. *Regularization in Hilbert space under Unbounded operators and general source conditions*. IOP Publishing Inverse Problems, 2009.
- [2] C. W Groetsch. Spectral methods for linear inverse problems with unbounded operators. *Journal of approximation theory*, 70:16–28, 1992.
- [3] S. H Kulkarni and M. T Nair. A characterization of closed range operators. *Indian J.Pure Appl. Math.*, 31:353–361, 2000.
- [4] S. H Kulkarni and G Ramesh. Projection method for inversion of unbounded operators. *Indian J.Pure Appl. Math*, 39:185–202, 2008.
- [5] A. K Louis and P Maass. A mollifier method for linear operator equations of the first kind. *inverse problem*, 1990.
- [6] A. G Rahim. Ill-posed problems with unbounded operators. *Journal of mathematical analysis and application*, 325:490–495, 2007.
- [7] M. T Nair S. H Kulkarni and G Ramesh. Some properties of unbounded operators with closed range. *Proc.Indian Acadamy of science*, 118:613–625, 2008.

The sequence of trifurcating Fibonacci numbers

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Abstract

One of the remarkable generalizations of Fibonacci sequence is a k -Fibonacci sequence and subsequently generalized into the ‘Bifurcating Fibonacci sequence’. In this paper, further generalization as the sequence of ‘trifurcating Fibonacci numbers’ is studied and Binet-like formula for these numbers is obtained. The analogous of Cassini’s identity, Catalan’s identity, d’Ocagne’s identity and some fundamental identities for the terms of this sequence has also been investigated.

Keywords: Fibonacci sequence, bifurcating Fibonacci sequence, generalization of Fibonacci sequence, Binet formula, identities related with the Fibonacci sequence.

2010 AMS subject classification[‡]: 11B39, 11B83, 11B99.

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[‡] 1Received on September 30, 2021. Accepted on December 10, 2021. Published on December 31, 2021.doi: 10.23755/rm.v41i0.668. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY license agreement.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n \geq 0}$ is a sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... , where each term is the sum of two preceding terms. The corresponding recurrence relation is $F_n = F_{n-1} + F_{n-2}$; $n \geq 2$. Various generalizations of this sequence have appeared in recent years [5, 7, 8]. Related to this work (i) some work alters the first two terms of the sequence from 0,1 to any arbitrary integers a, b while maintaining the recurrence relation (ii) some more work preserves the first two terms of the sequence but alters the recurrence relation (iii) even the combined approach of altering the initial terms as well as recurrence relation was considered by several authors. For further details about this sequence, one can refer Koshy [6], Patel, Shah [7], Singh, Sikhwal, Bhatnagar [8] and related papers available in the literature.

One interesting generalization depending on exactly one real parameter k is the sequence of k -Fibonacci numbers $\{F_{k,n}\}$ which is defined using a linear recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$; $n \geq 2$ where $F_{k,0} = 0$ and $F_{k,1} = 1$. For $k = 1$, we get the standard Fibonacci sequence and for $k = 2$, we get the sequence of Pell numbers. This sequence was studied by Arvadia and Shah [1]. Edson and Yayenie [4] generalized this sequence to a sequence which depends on two real parameters a, b . They defined the bifurcating sequence $\{F_n^{(a,b)}\}_{n \geq 0}$ by the recurrence relation

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}; & \text{if } n \text{ is even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}; & \text{if } n \text{ is odd} \end{cases}; n \geq 2$$

where $F_0^{(a,b)} = 0, F_1^{(a,b)} = 1$.

Diwan and Shah [2, 3], Yayenie [10], Verma and Bala [9] studied this sequence extensively and obtained significant results. It is easy to observe that (i) by considering $a = b = 1$, we get standard Fibonacci sequence (ii) by considering $a = b = 2$, we get the sequence of Pell numbers and (iii) by considering $a = b = k$, we get the sequence of k -Fibonacci numbers. In this paper, we further generalize this sequence to a sequence of trifurcating Fibonacci numbers, which depends on the three real parameters a, b, c .

Definition: For any three nonzero positive integers a, b and c , the trifurcating Fibonacci sequence $\{F_n^{(a,b,c)}\}_{n \geq 0}$ is defined recursively by $F_0^{(a,b,c)} = 0, F_1^{(a,b,c)} = 1$ and the recurrence relation

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$$F_n^{(a,b,c)} = \begin{cases} aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; & \text{if } n \equiv 0(\text{mod } 3) \\ bF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; & \text{if } n \equiv 1(\text{mod } 3). \\ cF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}; & \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

To avoid cumbersome notation, we denote $F_n^{(a,b,c)}$ by P_n . Few terms of this trifurcating generalized Fibonacci sequence are shown in the Table 1. In this paper we obtain various interesting results for this sequence.

n	P_n
0	0
1	1
2	c
3	$ac + 1$
4	$abc + b + c$
5	$abc^2 + bc + c^2 + ac + 1$
6	$a^2bc^2 + 2abc + ac^2 + a^2c + a + b + c$
7	$a^2b^2c^2 + 2ab^2c + 2abc^2 + a^2bc + ab + 2bc + b^2 + c^2 + ac + 1$

Table 1

2. Fundamental identities for the trifurcating sequence $\{P_n\}_{n \geq 0}$

In this section, we derive some interesting identities for the terms of the sequence $\{P_n\}_{n \geq 0}$. We first show that any two consecutive terms of $\{P_n\}_{n \geq 0}$ are always relatively prime.

Theorem 2.1. $\gcd(P_n, P_{n-1}) = 1$; for all $n = 1, 2, \dots$.

Proof. We prove the result by considering three cases when $n = 3k, 3k + 1$ or $3k + 2$. We present the proof only for the case $n = 3k$ and other cases follows accordingly. Now Euclidean algorithm leads to the following system of equations:

$$\begin{aligned} P_{3k} &= aP_{3k-1} + P_{3k-2} \\ P_{3k-1} &= cP_{3k-2} + P_{3k-3} \\ P_{3k-2} &= bP_{3k-3} + P_{3k-4} \\ &\vdots \end{aligned}$$

$$\begin{aligned}P_4 &= bP_3 + P_2 \\P_3 &= aP_2 + P_1 \\P_2 &= cP_1 + 0\end{aligned}$$

It now easily follows from Euclidean algorithm that $\gcd(P_n, P_{n-1}) = P_1 = 1$.

We now prove certain summation formulae for the terms of $\{P_n\}_{n \geq 0}$.

Lemma 2.2.

- a) $P_{3n+2} = (bc + 1)(P_3 + P_6 + \dots + P_{3n}) + (P_2 + P_5 + \dots + P_{3n-2}) + c$
- b) $P_{3n+1} = (ab + 1)(P_2 + P_5 + P_8 + \dots + P_{3n-1})$
 $\quad + (b - 1)(P_1 + P_4 + \dots + P_{3n-2}) + 1$
- c) $P_{3n} = (ac + 1)(P_1 + P_4 + \dots + P_{3n-2}) + (a - 1)(P_0 + P_3 + \dots + P_{3n-3})$.

Proof. Since, $P_{3n+2} = cP_{3n+1} + P_{3n}$ we get

$$\begin{aligned}P_2 &= cP_1 + P_0 \\P_5 &= cP_4 + P_3 \\P_8 &= cP_7 + P_6 \\&\vdots \\P_{3n-1} &= cP_{3n-2} + P_{3n-3} \\P_{3n+2} &= cP_{3n+1} + P_{3n}.\end{aligned}$$

Adding all these equations we get

$$P_2 + P_5 + P_8 + \dots + P_{3n+2} = c(P_1 + P_4 + \dots + P_{3n+1}) + (P_0 + P_3 + \dots + P_{3n}) \quad (2.1)$$

Again, $P_{3n} = aP_{3n-1} + P_{3n-2}$ gives

$$\begin{aligned}P_3 &= aP_2 + P_1 \\P_6 &= aP_5 + P_4 \\P_9 &= aP_8 + P_7 \\&\vdots \\P_{3n-3} &= aP_{3n-4} + P_{3n-5} \\P_{3n} &= aP_{3n-1} + P_{3n-2}\end{aligned}$$

Adding these equations, we get

$$\begin{aligned}P_3 + P_6 + P_9 + \dots + P_{3n} &= a(P_2 + P_5 + P_8 + \dots + P_{3n-1}) \\&\quad + (P_1 + P_4 + \dots + P_{3n-2})\end{aligned} \quad (2.2)$$

Also, since $P_{3n+1} = bP_{3n} + P_{3n-1}$, we have

$$\begin{aligned}P_4 &= bP_3 + P_2 \\P_7 &= bP_6 + P_5 \\P_{10} &= bP_9 + P_8 \\&\vdots\end{aligned}$$

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$$\begin{aligned} P_{3n-2} &= bP_{3n-3} + P_{3n-4} \\ P_{3n+1} &= bP_{3n} + P_{3n-1} \end{aligned}$$

Adding all these equations we get

$$\begin{aligned} P_4 + P_7 + \cdots + P_{3n-2} + P_{3n+1} \\ = b(P_3 + P_6 + \cdots + P_{3n}) + (P_2 + P_5 + \cdots + P_{3n-1}) \end{aligned} \quad (2.3)$$

Using (2.3) in (2.1) we get

$$\begin{aligned} P_2 + P_5 + P_8 + \cdots + P_{3n+2} \\ = c(1 + b(P_3 + P_6 + \cdots + P_{3n}) + (P_2 + P_5 + \cdots + P_{3n-1})) \\ + (P_0 + P_3 + \cdots + P_{3n}) \\ = c + (bc + 1)(P_0 + P_3 + P_6 + \cdots + P_{3n}) + c(P_2 + P_5 + \cdots + P_{3n-1}) \end{aligned}$$

This finally gives

$$\begin{aligned} P_{3n+2} &= (bc + 1)(P_0 + P_3 + P_6 + \cdots + P_{3n}) \\ &+ (c - 1)(P_2 + P_5 + \cdots + P_{3n-1}) + c \end{aligned} \quad (2.4)$$

Other results can be proved accordingly by considering the pair of equations (2.1), (2.2) and further (2.1), (2.3) together.

Lemma 2.3.

$$\begin{aligned} \text{a) } \sum_{i=0}^n P_{3i} &= \frac{aP_{3n}(b-1)(c-1) - (P_{3n+1}-1)(ac+1)(c-1) + (P_{3n+2}-c)(ab+1)(ac+1)}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)} \\ \text{b) } \sum_{i=0}^n P_{3i+1} &= \frac{(P_{3n+1}-1)(a-1)(c-1) - (P_{3n+2}-c)(ab+1)(a-1) + aP_{3n}(ab+1)(bc+1)}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)} \\ \text{c) } \sum_{i=0}^n P_{3i+2} &= \frac{(P_{3n+2}-c)(a-1)(b-1) - aP_{3n}(bc+1)(b-1) + (P_{3n+1}-1)(ac+1)(bc+1)}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)} \end{aligned}$$

Proof. We only prove result (a) here and other two results can be proved in a similar way. Using (b) and (c) of lemma 2.2, we get

$$\begin{aligned} aP_{3n} &= \frac{(ac+1)}{(b-1)} \{ (P_{3n+1}-1) - (ab+1)(P_2 + P_5 + P_8 + \cdots + P_{3n-1}) \} \\ &+ (a-1)(P_0 + P_3 + \cdots + P_{3n}) \end{aligned}$$

$$\begin{aligned} \text{Then } aP_{3n} - \frac{(ac+1)}{(b-1)} (P_{3n+1}-1) - (a-1)(P_0 + P_3 + \cdots + P_{3n}) \\ = - \frac{(ac+1)(ab+1)}{(b-1)} (P_2 + P_5 + P_8 + \cdots + P_{3n-1}) \end{aligned}$$

Using (2.4) we get

$$\begin{aligned} aP_{3n} - \frac{(ac+1)}{(b-1)} (P_{3n+1}-1) - (a-1)(P_0 + P_3 + \cdots + P_{3n}) \\ = - \frac{(ab+1)(ac+1)}{(b-1)(c-1)} ((P_{3n+2}-c) - (bc+1)(P_0 + P_3 + P_6 + \cdots + P_{3n})) \end{aligned}$$

$$\begin{aligned} \text{Then, } ((ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1))(P_0 + P_3 + \cdots + P_{3n}) \\ = a(b-1)(c-1)P_{3n} - (ac+1)(c-1)(P_{3n+1}-1) \end{aligned}$$

$$\begin{aligned}
 & +(ab+1)(ac+1)(P_{3n+2}-c). \\
 \text{Hence, } P_0 + P_3 + \dots + P_{3n} & = \frac{aP_{3n}(b-1)(c-1) - (P_{3n+1}-1)(ac+1)(c-1) + (P_{3n+2}-c)(ab+1)(ac+1)}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}.
 \end{aligned}$$

We now obtain the sum of first k trifurcating Fibonacci numbers.

Theorem 2.4.

$$\begin{aligned}
 & \left\{ (b-1)(c-1) - \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor (bc+1)(b-1) \right\} \left\lfloor \frac{4-\chi(k)}{3} \right\rfloor aP_{3\left\lfloor \frac{k}{3} \right\rfloor - 3} \\
 & + \left\{ \left\lfloor \frac{1+\chi(k)}{3} \right\rfloor (b-1)(c-1) + (ab+1)(bc+1) - \left\lfloor \frac{2+\chi(k)}{3} \right\rfloor (bc+1)(b-1) \right\} aP_{3\left\lfloor \frac{k}{3} \right\rfloor} \\
 & + \left\{ \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor [(a-1)(c-1) + (ac+1)(bc+1)] - (ac+1)(c-1) \right\} \left\lfloor \frac{4-\chi(k)}{3} \right\rfloor \left(P_{3\left\lfloor \frac{k}{3} \right\rfloor - 2} - 1 \right) \\
 & + \left\lfloor \frac{2+\chi(k)}{3} \right\rfloor \left\{ (a-1)(c-1) + (ac+1)(bc+1) - \left\lfloor \frac{1+\chi(k)}{3} \right\rfloor (ac+1)(c-1) \right\} \\
 & + \left\lfloor \frac{4-\chi(k)}{3} \right\rfloor \left\{ \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor (a-1)(b-1) + (ab+1)(ac+1) \right\} \left(P_{3\left\lfloor \frac{k}{3} \right\rfloor - 1} - c \right) \\
 & + \left\{ \left\lfloor \frac{2+\chi(k)}{3} \right\rfloor (a-1)(b-1) + \left\lfloor \frac{1+\chi(k)}{3} \right\rfloor (ab+1)(ac+1) - (ab+1)(a-1) \right\} \left(P_{3\left\lfloor \frac{k}{3} \right\rfloor + 2} - c \right) \\
 \sum_{n=1}^k P_n & = \frac{\quad}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}.
 \end{aligned}$$

Proof. We first obtain the value of $\sum_{n=1}^k P_n$ in three cases when k is of the form $3m-2$, $3m-1$ and $3m$ and then combine the results to obtain a single result.

$$\begin{aligned}
 & \text{For } k = 3m-2, \text{ using the above lemma we get} \\
 \sum_{n=1}^k P_n & = (P_3 + \dots + P_{3m-3}) + (P_1 + P_4 + \dots + P_{3m-2}) \\
 & \quad + (P_2 + P_5 + \dots + P_{3m-4}) \\
 & = \frac{\begin{cases} aP_{3m-3}(b-1)(c-1) - (P_{3m-2}-1)(ac+1)(c-1) + (P_{3m-1}-c)(ab+1)(ac+1) + \\ (P_{3m-2}-1)(a-1)(c-1) - (P_{3m+2}-c)(ab+1)(a-1) + aP_{3m}(ab+1)(bc+1) + \\ (P_{3m-1}-c)(a-1)(b-1) - aP_{3m-3}(bc+1)(b-1) + (P_{3m-2}-1)(ac+1)(bc+1) \end{cases}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}
 \end{aligned}$$

On simplification, we get,

$$\sum_{n=1}^k P_n = \frac{\begin{cases} aP_{3m-3}\{(b-1)(c-1) - (bc+1)(b-1)\} + (P_{3m-2}-1)\{(a-1)(c-1) + \\ (ac+1)(bc+1) - (ac+1)(c-1)\} + (P_{3m-1}-c)\{(a-1)(b-1) + (ab+1)(ac+1)\} \\ - (P_{3m+2}-c)(ab+1)(a-1) + aP_{3m}(ab+1)(bc+1) \end{cases}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}$$

Next, for the case $k = 3m-1$, we get

$$\begin{aligned}
 \sum_{n=1}^k P_n & = (P_3 + \dots + P_{3m-3}) + (P_1 + P_4 + \dots + P_{3m-2}) \\
 & \quad + (P_2 + P_5 + \dots + P_{3m-1})
 \end{aligned}$$

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$$= \frac{\begin{cases} aP_{3m-3}(b-1)(c-1) - (P_{3m-2}-1)(ac+1)(c-1) + (P_{3m-1}-c)(ab+1)(ac+1) + \\ (P_{3m+1}-1)(a-1)(c-1) - (P_{3m+2}-c)(ab+1)(a-1) + aP_{3m}(ab+1)(bc+1) + \\ (P_{3m+2}-c)(a-1)(b-1) - aP_{3m}(bc+1)(b-1) + (P_{3m+1}-1)(ac+1)(bc+1) \end{cases}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}$$

Simplifying this we get

$$\sum_{n=1}^k P_n = \frac{aP_{3m-3}(b-1)(c-1) + (P_{3m-1}-c)(ab+1)(ac+1) - (P_{3m-2}-1)(ac+1)(c-1) + \{(ac+1)(bc+1) + (a-1)(c-1)\}(P_{3m+1}-1) + (P_{3m+2}-c)\{(a-1)(b-1) - (ab+1)(a-1)\} + aP_{3m}\{(ab+1)(bc+1) - (bc+1)(b-1)\}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}$$

Finally, for $k = 3m$, we get

$$\begin{aligned} \sum_{n=1}^k P_n &= (P_3 + \dots + P_{3m}) + (P_1 + P_4 + \dots + P_{3m-2}) \\ &\quad + (P_2 + P_5 + \dots + P_{3m-1}) \\ &= \frac{\begin{cases} aP_{3m}(b-1)(c-1) - (P_{3m+1}-1)(ac+1)(c-1) + (P_{3m+2}-c)(ab+1)(ac+1) + \\ (P_{3m+1}-1)(a-1)(c-1) - (P_{3m+2}-c)(ab+1)(a-1) + aP_{3m}(ab+1)(bc+1) + \\ (P_{3m+2}-c)(a-1)(b-1) - aP_{3m}(bc+1)(b-1) + (P_{3m+1}-1)(ac+1)(bc+1) \end{cases}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)} \end{aligned}$$

This on simplification gives

$$\sum_{n=1}^k P_n = \frac{aP_{3m}\{(b-1)(c-1) + (ab+1)(bc+1) - (bc+1)(b-1)\} + (P_{3m+1}-1)\{(a-1)(c-1) + (ac+1)(bc+1) - (ac+1)(c-1)\} + (P_{3m+2}-c)\{(a-1)(b-1) + (ab+1)(ac+1) - (ab+1)(a-1)\}}{(ab+1)(ac+1)(bc+1) + (a-1)(b-1)(c-1)}$$

Combining all these three results, we finally get the required result.

The following are the interesting identities related with the summation of trifurcating Fibonacci numbers as well as its squares.

Theorem 2.5. $\sum_{k=0}^n a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k = P_n + P_{n+1} - 1.$

Proof. Using the definition of trifurcating Fibonacci numbers, we have

$$P_{3n} = aP_{3n-1} + P_{3n-2}; P_{3n+1} = bP_{3n} + P_{3n-1}; P_{3n+2} = cP_{3n+1} + P_{3n}$$

This can be written as

$$aP_{3n-1} = P_{3n} - P_{3n-2}; bP_{3n} = P_{3n+1} - P_{3n-1}; cP_{3n+1} = P_{3n+2} - P_{3n}$$

Thus, we have the following system of equations:

$$cP_1 = P_2 - P_0$$

$$aP_2 = P_3 - P_1$$

$$bP_3 = P_4 - P_2$$

$\vdots$

$$aP_n = P_{n+1} - P_{n-1}; \text{ if } n \equiv 0(\text{mod } 3)$$

$$bP_n = P_{n+1} - P_{n-1}; \text{ if } n \equiv 1(\text{mod } 3)$$

$$P_n = P_{n+1} - P_{n-1}; \text{ if } n \equiv 2(\text{mod } 3)$$

Adding all the above equations and using the fact that $P_0 = 0$ and $P_1 = 1$, we get $\sum_{k=0}^n a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k = P_n + P_{n+1} - 1.$

Theorem 2.6. $\sum_{k=0}^n a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k^2 = P_n P_{n+1}$.

Proof. We prove this result only for the case $n \equiv 0 \pmod{3}$ and remaining cases can be proved accordingly. We let $n = 3m$ and apply induction on m .

For $m = 1$, we have

$$\sum_{k=0}^3 a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k^2 = bP_0^2 + cP_1^2 + aP_2^2 + bP_3^2$$

Since $P_0 = 0, P_1 = 1, P_2 = c$ and $P_3 = (1 + ac)$, we get

$$\sum_{k=0}^3 a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor = c(1 + ac) + b(1 + ac)^2 = P_3 P_4$$

We next assume that the result holds for some positive integer $m = l > 1$.

That is let $\sum_{k=0}^{3l} a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k^2 = P_{3l} P_{3l+1}$ holds.

$$\begin{aligned} \text{Now, } \sum_{k=0}^{3(l+1)} a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k^2 \\ = \sum_{k=0}^{3l} a \left\lfloor 1 - \frac{\chi(k+1)}{3} \right\rfloor b \left\lfloor 1 - \frac{\chi(k)}{3} \right\rfloor c \left\lfloor 1 - \frac{\chi(k+2)}{3} \right\rfloor P_k^2 + cP_{3l+1}^2 + aP_{3l+2}^2 + bP_{3l+3}^2 \\ = P_{3l}P_{3l+1} + cP_{3l+1}^2 + aP_{3l+2}^2 + bP_{3l+3}^2 \\ = P_{3l+1}P_{3l+2} + aP_{3l+2}^2 + bP_{3l+3}^2 \\ = P_{3l+2}P_{3l+3} + bP_{3l+3}^2 = P_{3(l+1)}P_{3(l+1)+1} \end{aligned}$$

Thus, by induction, the result to be proved holds for every positive integer n .

3. Binet-like formula for the trifurcating Fibonacci sequence:

Generating function is used to solve the linear homogeneous recurrence relations. In this section, the generating function for the trifurcating Fibonacci sequence is derived and it is used to obtain Binet-like formula for these numbers. We first prove a result which will be needed to obtain the generating function of P_n .

Lemma 3.1. $P_{n+3} - (abc + a + b + c)P_n + P_{n-3} = 0$.

Proof. We prove the result by considering the three cases when $n = 3k, 3k + 1$ or $3k + 2$. We present the proof only for the case $n = 3k$ and other cases follows accordingly.

Using the definition of P_n , we get

$$P_{3k+3} = aP_{3k+2} + P_{3k+1} = a(cP_{3k+1} + P_{3k}) + bP_{3k} + P_{3k-1}$$

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$$= acP_{3k+1} + (a+b)P_{3k} + cP_{3k-2} + P_{3k-3}.$$

Now, by definition we have $P_{3k+1} = bP_{3k} + P_{3k-1}$. Multiplying this by ac we get $acP_{3k+1} = abcP_{3k} + acP_{3k-1} = abcP_{3k} + c(P_{3k} - P_{3k-2})$. Substituting this value in above equation we get

$$P_{3k+3} = (abc + a + b + c)P_{3k} - P_{3k-3}$$

$$\text{Hence, } P_{3k+3} - (abc + a + b + c)P_{3k} + P_{3k-3} = 0.$$

Lemma 3.2. The generating function of the subsequence $\{P_m\}_{m \geq 0}$ of $\{P_n\}_{n=0}^{\infty}$ is

$$\begin{aligned} \text{(i)} \quad f(x) &= \frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}; \text{ when } m \equiv 2(mod 3) \\ \text{(ii)} \quad g(x) &= \frac{x-ax^4}{(1-(abc+a+b+c)x^3-x^6)}; \text{ when } m \equiv 1(mod 3) \\ \text{(iii)} \quad h(x) &= \frac{(ac+1)x^3}{(1-(abc+a+b+c)x^3-x^6)}; \text{ when } m \equiv 0(mod 3). \end{aligned}$$

Proof. We present the proof only for the case $m \equiv 2(mod 3)$. The other cases can be proved accordingly. We first let $f(x) = \sum_{n=1}^{\infty} P_{3n-1}x^{3n-1} = P_2x^2 + P_5x^5 + P_8x^8 + \dots$. Then

$$\begin{aligned} (abc + a + b + c)x^3 f(x) &= (abc + a + b + c)P_2x^5 + (abc + a + b + c)P_5x^8 \\ &\quad + (abc + a + b + c)P_8x^{11} \dots \end{aligned}$$

Using lemma 3.1, we get

$$\begin{aligned} (1 - (abc + a + b + c)x^3 - x^6)f(x) &= cx^2 + x^5 - \sum_{m=2}^{\infty} (p_{3k+2} - (abc + a + b + c)p_{3k-1} - p_{3k-4}) = cx^2 + x^5. \end{aligned}$$

$$\text{Thus, } f(x) = \frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}, \text{ as required.}$$

The following result gives the generating function for P_n .

Theorem 3.3. The generating function for the trifurcating Fibonacci sequence $\{P_n\}$ is $F(x) = \frac{x(1+cx+x^2+acx^2-ax^3+x^4)}{1-(abc+a+b+c)x^3-x^6}$.

Proof. We begin with the formal power series representation of the generating function for $\{P_n\}$. Let

$$F(x) = P_0 + P_1x + P_2x^2 + \dots + P_kx^k + \dots = \sum_{m=0}^{\infty} P_mx^m$$

$$\begin{aligned} \text{Then, } cxF(x) &= cP_0x + cP_1x^2 + cP_2x^3 + \dots + cP_kx^{k+1} + \dots \\ &= \sum_{m=0}^{\infty} cP_mx^{m+1} = \sum_{m=1}^{\infty} cP_{m-1}x^m. \end{aligned}$$

$$\text{Also, } x^2F(x) = P_0x^2 + P_1x^3 + P_2x^4 + \dots + P_kx^{k+2} + \dots = \sum_{m=0}^{\infty} P_mx^{m+2}$$

Since $P_{3k+2} = cP_{3k+1} + P_{3k}$, we get

$$(1 - cx - x^2)F(x) = x + \sum_{n=1}^{\infty} (P_{3n} - cP_{3n-1} - P_{3n-2})x^{3n}$$

Since $P_{3k} = aP_{3k-1} + P_{3k-2}$ and $P_{3k+1} = bP_{3k} + P_{3k-1}$ we get
 $(1 - cx - x^2)F(x) = x + (a - c) \sum_{n=1}^{\infty} P_{3n-1} x^{3n} + (b - c) \sum_{n=1}^{\infty} P_{3n} x^{3n+1}$.
 For convenience, we let $f(x) = \sum_{n=1}^{\infty} P_{3n-1} x^{3n}$ and $g(x) = \sum_{n=1}^{\infty} P_{3n} x^{3n}$
 Using lemma 3.2 (a) and 3.2 (b) we get

$$(1 - cx - x^2)F(x) = x + (a - c) \frac{(ac+1)x^3}{(1-(abc+a+b+c)x^3-x^6)} + (b - c) \frac{cx^2+x^5}{(1-(abc+a+b+c)x^3-x^6)}$$

On simplification, we get the required result.

We now obtain the Binet-like formula for the sequence of trifurcating Fibonacci numbers.

Theorem 3.4. The terms of the trifurcating Fibonacci sequence $\{P_n\}$ are given

by $P_n = \frac{\gamma(n)\alpha^{\lfloor \frac{n}{3} \rfloor} - \delta(n)\beta^{\lfloor \frac{n}{3} \rfloor}}{\alpha - \beta}$
 where $\gamma(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\lfloor \frac{\chi(n)+1}{3} \rfloor} \alpha + (-1)^{\chi(n)} a^{\lfloor \frac{4-\chi(n)}{3} \rfloor} c^{\lfloor 1-\frac{\chi(n)}{3} \rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right)$
 and $\delta(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\lfloor \frac{\chi(n)+1}{3} \rfloor} \beta + (-1)^{\chi(n)} a^{\lfloor \frac{4-\chi(n)}{3} \rfloor} c^{\lfloor 1-\frac{\chi(n)}{3} \rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right)$
 with $\alpha = \frac{u+\sqrt{u^2+4}}{2}, \beta = \frac{u-\sqrt{u^2+4}}{2}, u = a + b + c + abc$ and

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ 2 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof. From the generating function of $\{P_n\}$, we have

$$F(x) = -\frac{x(1+cx+(1+ac)x^2-ax^3+x^4)}{(x^3+\alpha)(x^3+\beta)}$$

This can be rewritten as

$$F(x) = -\frac{1}{\alpha - \beta} \left[\frac{(1+ac)\alpha - (a\alpha+1)x + (\alpha-c)x^2}{(x^3+\alpha)} - \frac{(1+ac)\beta - (a\beta+1)x + (\beta-c)x^2}{(x^3+\beta)} \right]$$

Using McLaurin series expansion, we get

$$F(x) = -\frac{1}{\alpha - \beta} \left[\sum_{n=0}^{\infty} \frac{(-1)^n(1+ac)\alpha}{\alpha^{n+1}} x^{3n} - \sum_{n=0}^{\infty} \frac{(-1)^n(a\alpha+1)}{\alpha^{n+1}} x^{3n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha-c)}{\alpha^{n+1}} x^{3n+2} - \sum_{n=0}^{\infty} \frac{(-1)^n(1+ac)\beta}{\beta^{n+1}} x^{3n} + \sum_{n=0}^{\infty} \frac{(-1)^n(a\beta+1)}{\beta^{n+1}} x^{3n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n(\beta-c)}{\beta^{n+1}} x^{3n+2} \right]$$

The sequence of trifurcating Fibonacci numbers

$$= -\frac{1}{\alpha-\beta} \left[\sum_{n=0}^{\infty} (-1)^n (1+ac) \left(\frac{\beta^n - \alpha^n}{(\alpha\beta)^n} \right) x^{3n} \right. \\ \left. - \sum_{n=0}^{\infty} (-1)^n \left(\frac{(a\alpha+1)\beta^{n+1} - (a\beta+1)\alpha^{n+1}}{(\alpha\beta)^{n+1}} \right) x^{3n+1} \right. \\ \left. + \sum_{n=0}^{\infty} (-1)^n \left(\frac{(\alpha-c)\beta^{n+1} - (\beta-c)\alpha^{n+1}}{(\alpha\beta)^{n+1}} \right) x^{3n+2} \right]$$

Now, if α, β are the roots of $1 - (a+b+c+abc)x - x^2 = 0$ then

$$\alpha = \frac{u+\sqrt{u^2+4}}{2}, \beta = \frac{u-\sqrt{u^2+4}}{2}.$$

If we let $u = a+b+c+abc$, then it is easy to observe that $\alpha\beta = -1, \alpha + \beta = u$ and $\alpha - \beta = \sqrt{u^2+4}$.

Then

$$F(x) = \left[\sum_{n=0}^{\infty} (1+ac) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) x^{3n} - \sum_{n=0}^{\infty} \left(\frac{(a\beta+1)\alpha^{n+1} - (a\alpha+1)\beta^{n+1}}{\alpha - \beta} \right) x^{3n+1} \right. \\ \left. + \sum_{n=0}^{\infty} \left(\frac{(\beta-c)\alpha^{n+1} - (\alpha-c)\beta^{n+1}}{\alpha - \beta} \right) x^{3n+2} \right]$$

Thus,

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha - \beta} \left(\begin{aligned} & \left(\left(\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor + (-1)^{\chi(n)} a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor} \right) \alpha^{\left\lfloor \frac{n}{3} \right\rfloor} \right. \\ & \quad \left. + \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} \alpha \right) \\ & - \left(\left(\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor + (-1)^{\chi(n)} a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor} \right) \beta^{\left\lfloor \frac{n}{3} \right\rfloor} \right. \\ & \quad \left. + \left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} \beta \right) \end{aligned} \right) x^n$$

For convenience if we write $\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ 2 & \text{if } n \equiv 2 \pmod{3} \end{cases}$ and

$$\gamma(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} \alpha + (-1)^{\chi(n)} a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right),$$

$$\delta(n) = \left(\left\lfloor \frac{\chi(n)+2}{3} \right\rfloor c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} \beta + (-1)^{\chi(n)} a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor} + \left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor \right) \text{ then}$$

$$F(x) \text{ can be written as } F(x) = \sum_{n=0}^{\infty} \frac{\gamma(n)\alpha^{\left\lfloor \frac{n}{3} \right\rfloor} - \delta(n)\beta^{\left\lfloor \frac{n}{3} \right\rfloor}}{\alpha - \beta} x^n$$

This gives $P_n = \frac{\gamma(n)\alpha^{\left\lfloor \frac{n}{3} \right\rfloor} - \delta(n)\beta^{\left\lfloor \frac{n}{3} \right\rfloor}}{\alpha - \beta}$, as desired.

The following results are the easy consequence from this theorem.

Corollary 3.5. (i) $P_{3n} = (1+ac) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$

$$(ii) P_{3n+1} = \left(\frac{(a\beta+1)\alpha^{n+1} - (a\alpha+1)\beta^{n+1}}{\alpha-\beta} \right)$$

$$(iii) P_{3n+2} = \left(\frac{(\beta-c)\alpha^{n+1} - (\alpha-c)\beta^{n+1}}{\alpha-\beta} \right).$$

4. Some more identities relating trifurcating Fibonacci numbers:

In this section, we use the above Binet-like formula to derive some interesting properties for the terms of trifurcating Fibonacci sequence. If we let $w = \gamma(n)\delta(n)$, then we observe that

$$w = \left[1 - \frac{\chi(n)}{3} \right]^2 + 2(-1)^{\chi(n)} \left[1 - \frac{\chi(n)}{3} \right] a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor}$$

$$+ \left[1 - \frac{\chi(n)}{3} \right] \left[\frac{\chi(n)+2}{3} \right] c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} (\alpha + \beta) + a^{2\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{2\left\lfloor 1 - \frac{\chi(n)}{3} \right\rfloor}$$

$$+ (-1)^{\chi(n)} \left[\frac{\chi(n)+2}{3} \right] a^{\left\lfloor \frac{4-\chi(n)}{3} \right\rfloor} c^{\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} + \left[1 - \frac{\chi(n)}{3} \right] (\alpha + \beta) + \left[\frac{\chi(n)+2}{3} \right]^2 c^{2\left\lfloor \frac{\chi(n)+1}{3} \right\rfloor} \alpha \beta$$

This on simplification gives the value of $w = \gamma(n)\delta(n)$ as

$$w = \begin{cases} (1+ac)^2 & ; \text{if } n \equiv 0(\text{mod } 3) \\ (1+ac)(1+b) & ; \text{if } n \equiv 1(\text{mod } 3) \\ (1+ac)(1+bc) & ; \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

$$\text{This can be further written as } w = \begin{cases} P_3^2 & ; \text{if } n \equiv 0(\text{mod } 3) \\ (1+b)P_3 & ; \text{if } n \equiv 1(\text{mod } 3) \\ (1+bc)P_3 & ; \text{if } n \equiv 2(\text{mod } 3) \end{cases}$$

We first obtain an identity for the terms of $\{P_n\}$, which is analogous to that of Catalan's identity for Fibonacci numbers.

Theorem 4.1. For any two nonnegative integers k and $r \left(\leq \frac{k}{3} \right)$, we have

$$P_{k-3r}P_{k+3r} - P_k^2 = (-1)^{l-r+1} w \left(\frac{P_{3r}}{P_3} \right)^2.$$

Proof. If $k \equiv 0(\text{mod } 3)$ then taking $k = 3l$ and using corollary 3.5, we get

$$P_{3l-3r}P_{3l+3r} - P_{3l}^2$$

$$= (1+ac) \left(\frac{\alpha^{l-r} - \beta^{l-r}}{\alpha-\beta} \right) (1+ac) \left(\frac{\alpha^{l+r} - \beta^{l+r}}{\alpha-\beta} \right) - (1+ac)^2 \left(\frac{\alpha^l - \beta^l}{\alpha-\beta} \right)^2$$

$$= \frac{(1+ac)^2}{(\alpha-\beta)^2} \{ (\alpha^{l-r} - \beta^{l-r})(\alpha^{l+r} - \beta^{l+r}) - (\alpha^l - \beta^l)^2 \}$$

$$= \frac{(1+ac)^2}{(\alpha-\beta)^2} \left\{ \alpha^{2l} - (-1)^l \left(\frac{\beta}{\alpha} \right)^r - (-1)^l \left(\frac{\alpha}{\beta} \right)^r + \beta^{2l} - (\alpha^{2l} - 2(-1)^l + \beta^{2l}) \right\}$$

$$\begin{aligned}
 &= \frac{(1+ac)^2}{(\alpha-\beta)^2} (-1)^{l-r+1} \{\alpha^{2r} - 2(-1)^r + \beta^{2r}\} \\
 &= \frac{(1+ac)^2}{(\alpha-\beta)^2} (-1)^{l-r+1} (\alpha^r - \beta^r)^2 = (-1)^{l-r+1} \left(\frac{(1+ac)(\alpha^r - \beta^r)}{(\alpha-\beta)} \right)^2
 \end{aligned}$$

$$\text{Thus, } P_{3l-3r}P_{3l+3r} - P_{3l}^2 = (-1)^{l-r+1}P_{3r}^2$$

Next, if we let $k \equiv 1 \pmod{3}$ then by considering $k = 3l + 1$, we get

$$\begin{aligned}
 &P_{3l-3r+1}P_{3l+3r+1} - P_{3l+1}^2 \\
 &= \left(\frac{(a\beta+1)\alpha^{l-r+1} - (a\alpha+1)\beta^{l-r+1}}{\alpha-\beta} \right) \left(\frac{(a\beta+1)\alpha^{l+r+1} - (a\alpha+1)\beta^{l+r+1}}{\alpha-\beta} \right) \\
 &\quad - \left(\frac{(a\beta+1)\alpha^{l+1} - (a\alpha+1)\beta^{l+1}}{\alpha-\beta} \right)^2 \\
 &= \frac{1}{(\alpha-\beta)^2} \left[\begin{aligned} &a^2(\alpha^{l-r} - \beta^{l-r})(\alpha^{l+r} - \beta^{l+r}) \\ &-a(\alpha^{l-r} - \beta^{l-r})(\alpha^{l+r+1} - \beta^{l+r+1}) \\ &-a(\alpha^{l-r+1} - \beta^{l-r+1})(\alpha^{l+r} - \beta^{l+r}) \\ &+(\alpha^{l-r+1} - \beta^{l-r+1})(\alpha^{l+r+1} - \beta^{l+r+1}) \\ &- \{a^2(\alpha^l - \beta^l)^2 - 2a(\alpha^l - \beta^l)(\alpha^{l+1} - \beta^{l+1}) + (\alpha^{l+1} - \beta^{l+1})^2\} \end{aligned} \right] \\
 &= \frac{1}{(\alpha-\beta)^2} \left[\begin{aligned} &a^2(-1)^{l-r+1}[\beta^{2r} + \alpha^{2r} + 2(-1)^r] \\ &-(-1)^{l-r+1}[\beta^{2r} + \alpha^{2r} + 2(-1)^r] \\ &-a(-1)^{l+1} \left[\left(\frac{\beta}{\alpha} \right)^r (\alpha + \beta) + \left(\frac{\alpha}{\beta} \right)^r (\alpha + \beta) + 2(\alpha + \beta) \right] \end{aligned} \right] \\
 &= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2} \{a^2(\alpha^r - \beta^r)^2 - au[\beta^{2r} + \alpha^{2r} + 2(-1)^r] - (\alpha^r - \beta^r)^2\} \\
 &= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2} (\alpha^r - \beta^r)^2 \{a^2 - au - 1\} = \frac{(-1)^{l-r+1}}{(1+ac)^2} \{a^2 - au - 1\} P_{3r}^2
 \end{aligned}$$

$$\text{Thus, } P_{3l-3r+1}P_{3l+3r+1} - P_{3l+1}^2 = (-1)^{l-r+1}P_{3r}^2P_3^{-2}.$$

Finally, if $k \equiv 2 \pmod{3}$ then by considering $k = 3l + 2$, we get

$$\begin{aligned}
 &P_{3l-3r+2}P_{3l+3r+2} - P_{3l+2}^2 \\
 &= \left(\frac{(\beta-c)\alpha^{l-r+1} - (\alpha-c)\beta^{l-r+1}}{\alpha-\beta} \right) \left(\frac{(\beta-c)\alpha^{l+r+1} - (\alpha-c)\beta^{l+r+1}}{\alpha-\beta} \right) - \left(\frac{(\beta-c)\alpha^{l+1} - (\alpha-c)\beta^{l+1}}{\alpha-\beta} \right)^2 \\
 &= \frac{1}{(\alpha-\beta)^2} \left[\begin{aligned} &(\alpha^{l-r} - \beta^{l-r})(\alpha^{l+r} - \beta^{l+r}) + c(\alpha^{l-r} - \beta^{l-r})(\alpha^{l+r+1} - \beta^{l+r+1}) \\ &+c(\alpha^{l-r+1} - \beta^{l-r+1})(\alpha^{l+r} - \beta^{l+r}) \\ &+c^2(\alpha^{l-r+1} - \beta^{l-r+1})(\alpha^{l+r+1} - \beta^{l+r+1}) \\ &- (\alpha^{l-r} - \beta^{l-r})^2 - (\alpha^l - \beta^l)(\alpha^{l+1} - \beta^{l+1}) - c^2(\alpha^{l+1} - \beta^{l+1})^2 \\ &(-1)^{l-r+1}[\beta^{2r} + \alpha^{2r} + 2(-1)^r] \\ &-c^2(-1)^{l-r+1}[\beta^{2r} + \alpha^{2r} + 2(-1)^r] \end{aligned} \right] \\
 &= \frac{1}{(\alpha-\beta)^2} \left[\begin{aligned} &-c(-1)^{l+1} \left[\left(\frac{\beta}{\alpha} \right)^r (\alpha + \beta) + \left(\frac{\alpha}{\beta} \right)^r (\alpha + \beta) - 2(\alpha + \beta) \right] \end{aligned} \right] \\
 &= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2} \{(\alpha^r - \beta^r)^2 - cu[\beta^{2r} + \alpha^{2r} + 2(-1)^r] - c^2(\alpha^r - \beta^r)^2\}
 \end{aligned}$$

$$= \frac{(-1)^{l-r+1}}{(\alpha-\beta)^2} (\alpha^r - \beta^r)^2 \{1 + cu - c^2\} = \frac{(-1)^{l-r+1}}{(1+ac)^2} \{1 + cu - c^2\} P_{3r}^2$$

Thus, $P_{3l-3r+2}P_{3l+3r+2} - P_{3l+2}^2 = (-1)^{l-r+1} w P_{3r}^2 P_3^{-2}$

Hence, in general we write $P_{k-3r}P_{k+3r} - P_k^2 = (-1)^{l-r+1} w P_{3r}^2 P_3^{-2}$.

The following identity is analogous to the Cassini's identity for Fibonacci numbers which follows easily from the above theorem.

Corollary 4.2. $P_{k-3}P_{k+3} - P_k^2 = (-1)^n w$ for any integer $k \geq 3$.

The following identity is similar to d'Ocagne's identity of Fibonacci numbers.

Theorem 4.3. $P_m P_{n+3} - P_{m+3} P_n = (-1)^n P_{m-n} \left(\frac{w}{P_3}\right)$ where m, n are nonnegative integers such that $m \geq n$ and $m \equiv n \pmod{3}$.

Proof. Since $m \equiv n \pmod{3}$, we first let both m, n to be of the form $3j, 3k$ respectively, for positive integers j and $k \leq j$. Then

$$\begin{aligned} P_{3j}P_{3k+3} - P_{3j+3}P_{3k} &= (1+ac) \left(\frac{\alpha^j - \beta^j}{\alpha - \beta}\right) (1+ac) \left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right) \\ &\quad - (1+ac) \left(\frac{\alpha^{j+1} - \beta^{j+1}}{\alpha - \beta}\right) (1+ac) \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right) \\ &= \frac{(1+ac)^2}{(\alpha - \beta)^2} \{ \alpha^j \beta^k (\alpha - \beta) - \alpha^k \beta^j (\alpha - \beta) \} \\ &= (1+ac)^2 (-1)^k \left(\frac{\alpha^{j-k} - \beta^{j-k}}{\alpha - \beta}\right) = (-1)^{\lfloor \frac{n-1}{3} \rfloor} w P_{m-n} P_3^{-1} \end{aligned}$$

If m, n are of the form $3j + 1$ and $3k + 1$ respectively, then for positive integers j and $k \leq j$, we have

$$\begin{aligned} &P_{3j+1}P_{3k+3+1} - P_{3j+3+1}P_{3k+1} \\ &= \left[\left(\frac{(a\beta+1)\alpha^{j+1} - (a\alpha+1)\beta^{j+1}}{\alpha - \beta}\right) \left(\frac{(a\beta+1)\alpha^{k+2} - (a\alpha+1)\beta^{k+2}}{\alpha - \beta}\right) \right. \\ &\quad \left. - \left(\frac{(a\beta+1)\alpha^{j+2} - (a\alpha+1)\beta^{j+2}}{\alpha - \beta}\right) \left(\frac{(a\beta+1)\alpha^{k+1} - (a\alpha+1)\beta^{k+1}}{\alpha - \beta}\right) \right] \\ &= \frac{1}{(\alpha - \beta)^2} \left[\begin{aligned} &a^2 (\alpha^j \beta^k (\alpha - \beta) - \alpha^k \beta^j (\alpha - \beta)) \\ &- a (\alpha^j \beta^k (\alpha^2 - \beta^2) - \alpha^k \beta^j (\alpha^2 - \beta^2)) \\ &- (\alpha^j \beta^k (\alpha - \beta) - \alpha^k \beta^j (\alpha - \beta)) \end{aligned} \right] \\ &= \frac{1}{(\alpha - \beta)} [(a^2 - au - 1) \alpha^k \beta^k (\alpha^{j-k} - \beta^{j-k})] \\ &= (-1)^k (a^2 - au - 1) \left(\frac{\alpha^{j-k} - \beta^{j-k}}{\alpha - \beta}\right) = (-1)^{\lfloor \frac{n-1}{3} \rfloor} w P_{m-n} P_3^{-1} \end{aligned}$$

Finally, if m, n are of the form $3j + 2$ and $3k + 2$ respectively, then for positive integers j and $k \leq j$, we have

$$\begin{aligned}
 P_{3j+2}P_{3k+3+2} - P_{3j+3+2}P_{3k+2} &= \left(\frac{(\beta-c)\alpha^{j+1}-(\alpha-c)\beta^{j+1}}{\alpha-\beta} \right) \left(\frac{(\beta-c)\alpha^{k+2}-(\alpha-c)\beta^{k+2}}{\alpha-\beta} \right) \\
 &\quad - \left(\frac{(\beta-c)\alpha^{j+2}-(\alpha-c)\beta^{j+2}}{\alpha-\beta} \right) \left(\frac{(\beta-c)\alpha^{k+1}-(\alpha-c)\beta^{k+1}}{\alpha-\beta} \right) \\
 &= \frac{1}{(\alpha-\beta)^2} \left[\alpha^j \beta^k (\alpha - \beta) - \alpha^k \beta^j (\alpha - \beta) - c^2 (\alpha^j \beta^k (\alpha - \beta) - \alpha^k \beta^j (\alpha - \beta)) \right. \\
 &\quad \left. + c (\alpha^j \beta^k (\alpha^2 - \beta^2) - \alpha^k \beta^j (\alpha^2 - \beta^2)) \right] \\
 &= \frac{1}{(\alpha-\beta)} [(1 + cu - c^2) \alpha^k \beta^k (\alpha^{j-k} - \beta^{j-k})] \\
 &= (-1)^k (a^2 - au - 1) \left(\frac{\alpha^{j-k} - \beta^{j-k}}{\alpha - \beta} \right) = (-1)^{\lfloor \frac{n-1}{3} \rfloor} w P_{m-n} P_3^{-1}
 \end{aligned}$$

Combining all the above cases, we finally get

$$P_m P_{n+3} - P_{m+3} P_n = (-1)^{\lfloor \frac{n-1}{3} \rfloor} w P_{m-n} P_3^{-1}.$$

We use above Binet-like formula to prove the following identity which combines four consecutive P_n 's.

Theorem 4.4.

$$\begin{aligned}
 a \lfloor 1 - \frac{\chi(k+2)}{3} \rfloor b \lfloor 1 - \frac{\chi(k+1)}{3} \rfloor c \lfloor 1 - \frac{\chi(k)}{3} \rfloor P_{k+1}^2 + a \lfloor 1 - \frac{\chi(k)}{3} \rfloor b \lfloor 1 - \frac{\chi(k+2)}{3} \rfloor c \lfloor 1 - \frac{\chi(k+1)}{3} \rfloor P_{k+2}^2 \\
 = P_{k+2} P_{k+3} - P_k P_{k+1}.
 \end{aligned}$$

Proof. We prove the result only for the case $k = 3n$ and the remaining cases $k = 3n + 1$ and $k = 3n + 2$ can be handled accordingly.

$$\begin{aligned}
 \text{Now, } c P_{3n+1}^2 + a P_{3n+2}^2 \\
 = c \left\{ \left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta} \right)^2 \right\} + a \left\{ \left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta} \right)^2 \right\} \\
 = \frac{1}{(\alpha-\beta)^2} \left[\begin{aligned} &a^2 c (\alpha^{2n} - 2(-1)^n + \beta^{2n}) \\ &- 2ac (\alpha^{2n+1} - (-1)^n \alpha - (-1)^n \beta + \beta^{2n+1}) \\ &+ c (\alpha^{2n+2} - 2(-1)^{n+1} + \beta^{2n+2}) \\ &+ 2ac (\alpha^{2n+1} - (-1)^n \beta + (-1)^n \alpha + \beta^{2n+1}) \\ &+ c^2 (\alpha^{2n+2} - 2(-1)^{n+1} + \beta^{2n+2}) + a (\alpha^{2n} - 2(-1)^n + \beta^{2n}) \end{aligned} \right] \\
 = \frac{(1+ac)}{(\alpha-\beta)^2} \{ a (\alpha^n - \beta^n)^2 + c (\alpha^{n+1} - \beta^{n+1})^2 \}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } P_{3n+2} P_{3n+3} - P_{3n} P_{3n+1} \\
 = \left(\frac{(\beta-c)\alpha^{n+1}-(\alpha-c)\beta^{n+1}}{\alpha-\beta} \right) (1 + ac) \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) \\
 - (1 + ac) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \left(\frac{(a\beta+1)\alpha^{n+1}-(a\alpha+1)\beta^{n+1}}{\alpha-\beta} \right)
 \end{aligned}$$

$$\begin{aligned} &= \frac{(1+ac)}{(\alpha-\beta)^2} \left[(\alpha^n - \beta^n)(\alpha^{n+1} - \beta^{n+1}) + c(\alpha^{n+1} - \beta^{n+1})^2 \right] \\ &= \frac{(1+ac)}{(\alpha-\beta)^2} \{ a(\alpha^n - \beta^n)^2 + c(\alpha^{n+1} - \beta^{n+1})^2 \} \end{aligned}$$

This proves the required result.

4. Conclusions

In this paper, we considered the sequence of ‘trifurcating Fibonacci numbers’ and obtained its Binet-like formula. We also obtained the analogous of Cassini’s identity, Catalan’s identity, d’Ocagne’s identity and some fundamental identities for the terms of this sequence.

References

- [1] Arvadia M. P., Shah D. V. Left k-Fibonacci sequence and related identities. Journal Club for Applied Sciences. 2 (1), 20 – 26, July 2015.
- [2] Diwan D. M., Shah D. V. Explicit and recursive formulae for the class of generalized Fibonacci sequence. International Journal of Advanced Research in Engineering, Science and Management. 1 (10), 1 – 6, July 2015.
- [3] Diwan D. M., Shah D. V. Extended Binet’s formula for the class of generalized Fibonacci sequences. VNSGU Journal of Science and Technology. 4 (1), 205 – 210, 2015.
- [4] Edson M., Yayenie O. A new generalization of Fibonacci sequence and extended Binet’s formula. INTEGERS, Electron. J. Comb. Number Theor., 9, 639 – 654, 2009.
- [5] Gupta V. K., Panwar Y. K., O. Sikhwal. Generalized Fibonacci sequences. Theoretical Mathematics & Applications. 2 (2), 115 – 124, 2012.
- [6] Koshy Thomas. Fibonacci and Lucas Numbers with applications. John Wiley and Sons, Inc., N. York., 2001.
- [7] Patel Vandana R., Shah Devbhadra V. Generalized Fibonacci sequence and its properties. Int. Journal of Physics and Mathematical Sciences. 4 (2), 118 – 124, 2014.
- [8] Singh B., Sikhwal O., Bhatnagar S. Fibonacci-like sequence and its properties. Int. J. Contemp-Math. Sciences. 5 (18), 857 – 868, 2010.
- [9] Verma, Ankur Bala. On Properties of generalized Bi-variate Bi-Periodic Fibonacci polynomials. International journal of Advanced science and Technology. 29 (3), 8065 – 8072, 2020.
- [10] Yayenie O. A note on generalized Fibonacci sequence. Applied Mathematics and computation. 217, 5603 – 5611, 2011.

On Cesàro's Means of First Order of Wavelet Packet Series

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Abstract

Wavelet packets have the capability of partitioning the higher-frequency octaves to yield better frequency localisation. Ahmad and Kumar [2000] have obtained the pointwise convergence of the wavelet packet series. But till now no work seems to have been done to obtain Cesàro summability of order 1 of wavelet packet series. In an attempt to make an advanced study in this direction, a novel theory on $(C, 1)$, Cesàro summability of order 1 of wavelet packet series is obtained in this study.

Keywords: Multiresolution analysis, $(C, 1)$ summability, wavelet packets, periodic wavelet packets, wavelet packet expansions.

2020 AMS subject classifications: 40A30, 42C15. ¹

1 Introduction

Several researchers, including S. E. Kelly and Raphael [1994a], S. E. Kelly and Raphael [1994b], Kumar and Lal [2013], Meyer [1992], Walter [1992], Walter [1995], Wickerhauser [1994], have investigated the problem of wavelet packet series convergence and demonstrated that wavelet packets are a basic yet effective wavelet and multiresolution analysis extension. Wavelet packet functions are a collection of functions that can be used to create other functions. Wavelet packet

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¹Received on November 18, 2021. Accepted on December 12, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.687. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

functions are still time-localized, but they have more versatility in describing diverse types of signals than wavelets. Wavelet packets, in particular, are better at encoding signals with periodic behaviour. Wavelet packets can partition higher-frequency octaves, resulting in more accurate frequency localization.

Working in slight different directions, Ahmad and Kumar [2000] have obtained the pointwise convergence of the wavelet packet series. But till now no work seems to have been done to obtain Cesàro summability of order 1 of wavelet packet series. It is important to note that Cesàro summability is a strong tool to obtain the convergence than that of ordinary convergence. This work establishes a new theory on Cesàro summability of order 1 of wavelet packet series in an attempt to make a more advanced study in this field.

2 Definitions and Preliminaries

Let $L^2(\mathbb{R})$ be the space of measurable and square integrable functions over set of real numbers $\mathbb{R}$. If a function $\phi \in L^2(\mathbb{R})$ generates nested sequences of closed subspaces, it is said to produce an MRA (multiresolution analysis), $Q_i = \overline{\text{span}}\{\phi_{i,j} : i, j \in \mathbb{Z}\}$, where $\phi_{i,j}(t) = 2^{i/2}\phi(2^i t - j)$ and $\mathbb{Z}$ is the set of integers, satisfying the following conditions

- (i) $\dots \subset Q_{-2} \subset Q_{-1} \subset Q_0 \subset Q_1 \subset Q_2 \subset \dots$, i.e. $Q_i \subset Q_{i+1}$, $i \in \mathbb{Z}$;
- (ii) $\overline{(\cup_{i \in \mathbb{Z}} Q_i)} = L^2(\mathbb{R})$;
- (iii) $\cap_{i \in \mathbb{Z}} Q_i = \{0\}$;
- (iv) $\lambda(t) \in Q_i \Leftrightarrow \lambda(2t) \in Q_{i+1}$, $i \in \mathbb{Z}$

such that $\phi_{0,j}$ form a Riesz basis of $\{Q_0\}$. A function ϕ which generates a multiresolution analysis, is called a scaling function. Wavelet packets can be constructed with the help of multiresolution analysis. We know that if $\mathbb{H}$ is a Hilbert space with ONB (orthonormal basis) $\{\epsilon_j\}_{j \in \mathbb{Z}}$ then,

$$\lambda_{2k} = \sqrt{2} \sum_{j \in \mathbb{Z}} \alpha_{2k-j} \epsilon_j, \quad \lambda_{2k+1} = \sqrt{2} \sum_{j \in \mathbb{Z}} \beta_{2k-j} \epsilon_j,$$

where $\{\alpha_k\}_{k \in \mathbb{Z}}$ and $\{\beta_k\}_{k \in \mathbb{Z}}$ are in $l^2(\mathbb{Z})$, are orthonormal bases of two orthogonal closed subspaces $\mathbb{H}_1$ and $\mathbb{H}_0$ respectively, such that $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_0$.

Using the foregoing decomposition strategy, we now build the fundamental wavelet packets connected with the scaling function $\phi \in L^2(\mathbb{R})$ which is already defined in multiresolution analysis.

On Cesàro's means of first order of wavelet packet series

Let $\{\xi_k, k = 0, 1, 2, \dots\}$ denote a wavelet packet family that corresponds to the scaling function ϕ which is orthonormal. Consider $\xi_0 = \phi$. Recursively, the wavelet packets $\xi_k, k = 0, 1, 2, \dots$, are defined by

$$\begin{cases} \xi_{2k}(t) = \sqrt{2} \sum_{j \in \mathbb{Z}} h_j \xi_k(2t - j) \\ \xi_{2k+1}(t) = \sqrt{2} \sum_{j \in \mathbb{Z}} g_j \xi_k(2t - j). \end{cases} \quad (1)$$

As a result, the $\{\xi_k\}$ family is a generalisation of the orthonormal wavelet $\xi_1 = \psi$, often known as the mother wavelet. For the Hilbert space $L^2(\mathbb{R})$, the set $\{\xi_k(t-j) : k = 0, 1, 2, \dots, j \in \mathbb{Z}\}$ form an ONB.

Consider the family of subspaces of $L^2(\mathbb{R})$ as

$$P_i^k = \text{span}\{2^i \xi_k(2^i t - j) : j \in \mathbb{Z}\}, i \in \mathbb{Z}, \quad (2)$$

formed by the family of wavelet packets $\{\xi_k\}$ for each $k = 0, 1, 2, \dots$

Observe that $P_i^0 = Q_i$ and $P_i^1 = W_i$, where $\{Q_i\}$ is the multiresolution analysis of $L^2(\mathbb{R})$ produced by $\xi_0 = \phi$ and $\{W_i\}$ is the sequence of orthogonal complimentary subspaces generated by the wavelet $\xi_1 = \psi$. The orthogonal decomposition $Q_{i+1} = Q_i \oplus W_i, i \in \mathbb{Z}$ can then be expressed as

$$P_{i+1}^0 = P_i^0 \oplus P_i^1. \quad (3)$$

As follows, this orthogonal decomposition can be extended from $k = 0$ to any $k = 1, 2, 3, \dots$ in the form of

$$P_{i+1}^k = P_i^{2k} \oplus P_i^{2k+1}, i \in \mathbb{Z}. \quad (4)$$

Now we'll state a result that will be employed in the theorem's proof.

The decomposition trick (4) produces

$$\begin{aligned} W_i &= P_i^1 = P_{i-1}^2 \oplus P_{i-1}^3 \\ &= P_{i-2}^4 \oplus P_{i-2}^5 \oplus P_{i-2}^6 \oplus P_{i-2}^7 \\ &\vdots \\ &= P_{i-j}^{2^j} \oplus P_{i-j}^{2^j+1} \oplus \dots \oplus P_{i-j}^{2^{j+1}-1} \\ &\vdots \\ &= P_0^{2^i} \oplus P_0^{2^i+1} \oplus \dots \oplus P_0^{2^{i+1}-1}, \end{aligned} \quad (5)$$

for each $i = 1, 2, \dots$, where (2) declares P_i^k . Furthermore, the family $\left\{2^{\frac{i-j}{2}} \xi_r(2^{i-j} t - l) : l \in \mathbb{Z}\right\}$ is an ONB of P_{i-j}^r , where $r = 2^j + \mu$ for each $\mu = 0, 1, 2, \dots, 2^j - 1, j = 1, 2, \dots, i$;

and $\iota = 1, 2, \dots$. All of the elements of this base, however, have the same basic shape:

$$\xi_{\iota,k,j}(t) = 2^{\iota/2} \xi_k(2^\iota t - j). \quad (6)$$

Let $\lambda \in L^2(\mathbb{R})$, then the function λ can be approximated by a wavelet packet series as follows:

$$\lambda(t) \sim \sum_{\iota \in \mathbb{Z}} \sum_{k=2^r}^{2^{r+1}-1} \sum_{j \in \mathbb{Z}} C_{l,k,j} \xi_{l,k,j}(t), \quad (7)$$

where $l = \iota - r$, $r = 0$ if $\iota < 0$ and $r = 0, 1, 2, \dots, \iota$ if $\iota \geq 0$; and the coefficients $C_{l,k,j}$ defined by

$$C_{l,k,j} = \langle \lambda, \xi_{l,k,j} \rangle, \quad (8)$$

are called the wavelet packet coefficients.

Wavelet packets are a scalable time signal analysis method that combines the advantages of windowed Harmonic and wavelet processing. Wavelet bundles, which are periodic as well, offer a fascinating supplement to Fourier series.

Using the periodization techniques for period 1 on the basis functions, an MRA for $L^2(\mathbb{R})$ can be transformed into an MRA for $L^2(0, 1)$. Let $\{\xi_k : k \in \mathbb{Z}\}$ denote the family of wavelet packets presented previously which is nonstationary in nature. Define general periodic wavelet packets $\xi_{k,\iota,j}^{per}$ by

$$\xi_{k,\iota,j}^{per} = \sum_{l \in \mathbb{Z}} 2^{\iota/2} \xi_k(2^\iota(t + l) - j)$$

for $0 \leq j < 2^\iota$ and $k, \iota = 1, 2, 3, \dots$.

With ξ_k^{per} , We now define an operator $S_\nu \lambda$ as follows:

$$(S_\nu \lambda)(t) = \sum_{k=2^r}^{2^{r+1}-1} \sum_{j=0}^{\nu} \langle \lambda, \xi_{l,k,j}^{per} \rangle \xi_{l,k,j}^{per}(t). \quad (9)$$

Let $s_k = \sum_{\nu=0}^k a_\nu$ be the k^{th} partial sum of an infinite series $\sum_{k=0}^{\infty} a_k$.

If $\sigma_k = \frac{1}{k+1} \sum_{\nu=0}^k s_\nu \rightarrow s$ as $k \rightarrow \infty$ then the series $\sum_{k=0}^{\infty} a_k$ is called summable to s by $(C, 1)$ i.e. Cesàro means of order 1 (Titchmarsh [1939]).

Let $D_\mu (\mu = 1, 2, 3, \dots)$ be the collection of constant dyadic step functions on the intervals $[j2^{-\mu}, (j+1)2^{-\mu})$; $0 < j \leq 2^\mu$. Let $D = \cup_{\mu=1}^{\infty} D_\mu$. Let $\mathbb{B}$ be a Banach space and σ_ζ be a bounded linear functional on $\mathbb{B}$ which must be generated by any function $\zeta \in D$ as

$$\sigma_\zeta \lambda = \int_0^1 \lambda \zeta \text{ for } \lambda \in \mathbb{B}.$$

On Cesàro's means of first order of wavelet packet series

We have

$$|\sigma_\zeta \lambda| \leq \|\zeta\|_\infty \|\lambda\|_{\mathbb{B}}.$$

Now if we take $\mathbb{B} = L^q$ and define

$$\|\zeta\|_r = \|\sigma_\zeta\| = \sup_{\|\lambda\|_q \leq 1} \int_0^1 \lambda \zeta \quad \text{for any } \zeta \in D. \quad (10)$$

Then clearly

$$\left| \int_0^1 \lambda \zeta \right| \leq \|\lambda\|_q \|\zeta\|_r, \lambda \in L^q, \zeta \in D. \quad (11)$$

Let us write

$$\begin{aligned} \Pi_i \lambda(t) &= \sum_{\mu=0}^{2^i-1} \left(\frac{1}{\mu+1} \sum_{\nu=0}^{\mu} (S_\nu \lambda)(t) \right) \delta_{[\mu 2^{-i}, (\mu+1) 2^{-i})} \\ &= \sum_{\mu=0}^{2^i-1} \sigma_\mu \lambda(t) \delta_{[\mu 2^{-i}, (\mu+1) 2^{-i})} \end{aligned}$$

and

$$A_i = \sum_{\mu=0}^{2^i-1} C_{l,k,j}^{per} \delta_{[\mu 2^{-i}, (\mu+1) 2^{-i})},$$

where (9) defines $S_\nu \lambda$ and δ_I is the characteristic function on $I \subset \mathbb{R}$.

We're going to define an operator now

$$\begin{aligned} T_i(t, x) &= 2^{-i} \sum_{j=0}^{2^i-1} C_{l,o,j}^{per} \phi_{i,j}^{per}(t) \overline{\phi_{i,j}^{per}(x)} \\ &= 2^{-i} \sum_{k=2^r}^{2^{r+1}-1} \sum_{\mu < i} \sum_{j=0}^{2^i-1} \xi_{l,k,j}^{per}(t) \overline{\xi_{l,k,j}^{per}(x)}, \end{aligned}$$

where $l = \mu - r$, $r = 0$ if $\mu < 0$ and $r = 0, 1, 2, \dots, \mu$ if $0 \leq \mu < i$.

In this paper, an estimate for the Cesàro summability of wavelet packet series has been determined in the following form:

Theorem 2.1. *Let λ be 1-periodic continuous function. Then*

$$\left\| \left(2^{-i} \sum_{\mu=0}^{2^i-1} \left| \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} S_\nu \lambda \right|^r \right)^{1/r} \right\|_\infty \leq C \|\lambda\|_\infty \quad (12)$$

if and only if

$$\|T_i\|_1 \leq C \|A_i\|_q, \quad (13)$$

where $C > 0$, a constant and $1 < r < \infty$.

Furthermore,

$$\lim_{i \rightarrow \infty} \|\Pi_i \lambda(t) - \lambda(t)\|_r = 0$$

uniformly in $[0, 1]$.

Proof. By equation 12 we have

$$\begin{aligned} \left(2^{-i} \sum_{\mu=0}^{2^i-1} \left| \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} S_{\nu} \lambda \right|^r \right)^{\frac{1}{r}} &= \|\Pi_i \lambda(t)\|_r = \sup_{\|A_i\|_q \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} C_{l,k,j}^{per} \sigma_{\mu} \lambda(t) \\ &= \sup_{\|A_i\|_q \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} S_{\nu} \lambda(t) \\ &= \sup_{\|A_i\|_q \leq 1} \int_0^1 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} K_{\nu}(t, x) \lambda(x) dx \\ &\leq \|\lambda\|_{\infty} \sup_{\|A_i\|_q \leq 1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} \|T_{\nu}(t, x)\|_1 \\ &\leq \|\lambda\|_{\infty} \sup_{\|A_i\|_q \leq 1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} (C \|A_i\|_q), \quad \text{by (13)} \\ &= \|\lambda\|_{\infty} \sup_{\|A_i\|_q \leq 1} C \|A_i\|_q \leq C \|\lambda\|_{\infty}, \end{aligned}$$

where

$$K_i(t, x) = \sum_{j=0}^{2^i-1} \phi_{i,j}^{per}(t) \overline{\phi_{i,j}^{per}(x)} = \sum_{k=2^r}^{2^{r+1}-1} \sum_{\mu < i} \sum_{j=0}^{2^i-1} \xi_{l,k,j}^{per}(t) \overline{\xi_{l,k,j}^{per}(x)}.$$

If, on the other hand, (12) is true, we have

$$\begin{aligned}
\|T_i(t, x)\|_1 &= \sup_{\|\lambda\|_\infty \leq 1} \int_0^1 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} T_\nu(0, x) \lambda(x) dx \\
&= \sup_{\|\lambda\|_\infty \leq 1} \int_0^1 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} (2^{-\nu} \sum_{k=2^r}^{2^{r+1}-1} \sum_{j=0}^{2^\nu-1} \xi_{l,k,j}^{per}(0) \overline{\xi_{l,k,j}^{per}(x)}) \lambda(x) dx \\
&= \sup_{\|\lambda\|_\infty \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} (2^{-\nu} \sum_{k=2^r}^{2^{r+1}-1} \sum_{j=0}^{2^\nu-1} \xi_{l,k,j}^{per}(0)) \int_0^1 \lambda(x) \overline{\xi_{l,k,j}^{per}(x)} dx \\
&= \sup_{\|\lambda\|_\infty \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} (2^{-\nu} \sum_{k=2^r}^{2^{r+1}-1} \sum_{j=0}^{2^\nu-1} \langle \lambda, \xi_{l,k,j}^{per} \rangle \xi_{l,k,j}^{per}(0)) \\
&= \sup_{\|\lambda\|_\infty \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} C_{l,k,j}^{per} (S_\nu \lambda)(0) \\
&= \sup_{\|\lambda\|_\infty \leq 1} 2^{-i} \sum_{\mu=0}^{2^i-1} C_{l,k,j}^{per} (\sigma_\mu \lambda)(0) \\
&= \sup_{\|\lambda\|_\infty \leq 1} \int_0^1 \Pi_i \lambda(0) A_i \\
&\leq \sup_{\|\lambda\|_\infty \leq 1} \|A_i\|_q \|\Pi_i \lambda(0)\|_r \\
&\leq \|A_i\|_q \sup_{\|\lambda\|_\infty \leq 1} \left\| \left(2^{-i} \sum_{\mu=0}^{2^i-1} \left| \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} (S_\nu \lambda)(0) \right|^r \right)^{1/r} \right\|_\infty \\
&= \|A_i\|_q \sup_{\|\lambda\|_\infty \leq 1} \left\| \left(2^{-i} \sum_{\mu=0}^{2^i-1} |(\sigma_\mu \lambda)(0)|^r \right)^{1/r} \right\|_\infty, \quad \text{by (12)} \\
&\leq \|A_i\|_q \sup_{\|\lambda\|_\infty \leq 1} C \|\lambda\|_\infty \\
&\leq C \|A_i\|_q.
\end{aligned}$$

Now

$$\begin{aligned}
\Pi_i \lambda(t) - \lambda(t) &= \sum_{\mu=0}^M ((\sigma_\mu \lambda)(t) - \lambda(t)) \delta_{[\mu 2^{-l}, (\mu+1) 2^{-l})} \\
&= \sum_{\mu=0}^M \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} ((S_\nu \lambda)(t) - \lambda(t)) \delta_{[\mu 2^{-l}, (\mu+1) 2^{-l})}
\end{aligned}$$

for any $l \geq M \geq 2^l$. As a result,

$$\begin{aligned} \|\Pi_l \lambda(t) - \lambda(t)\| &\leq \sum_{\mu=0}^M \left\| \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} ((S_{\nu} \lambda)(t) - \lambda(t)) \right\|_{\infty} \|\delta_{[0, 2^{-l})}\|_r \\ &\leq \sum_{\mu=0}^M \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} \|S_{\nu} \lambda - \lambda\|_{\infty} \|\delta_{[0, 2^{-l})}\|_r, \end{aligned}$$

since the limit of the characteristic function of $[0, 2^{-l})$ in all L^r -space ($1 < r < \infty$) is 0 and thus the ultimate result is followed.

The theorem's proof is now complete. $\square$

3 Conclusions

The estimate for the Cesàro summability of order 1 of wavelet packet series has been determined in the form of

$$\lim_{i \rightarrow \infty} \|\Pi_i \lambda(t) - \lambda(t)\|_r = 0$$

uniformly in $[0, 1]$.

Acknowledgements. One of the authors, Shyam Lal, is grateful to DST-CIMS for supporting his work.

Manoj Kumar is thankful to the Director, Institute of Engineering and Technology, Lucknow for promoting this research activity.

References

- K. Ahmad and R. Kumar. Pointwise convergence of wavelet packet series. *Atti Sem. Fis. Univ. Modena*, 48:107–120, 2000.
- M. Kumar and S. Lal. On generalized carleson operators of periodic wavelet packet expansions. *The Scientific world Journal*, 2013:1–10, 2013.
- Y. Meyer. *Wavelets and Operators*. Cambridge University Press, Cambridge, 1992.
- M. A. Kon S. E. Kelly and L. A. Raphael. Pointwise convergence of wavelet expansions. *Bull. Amer. Math. Soc.*, 30:87–94, 1994a.
- M. A. Kon S. E. Kelly and L. A. Raphael. Local convergence for wavelet expansions. *J. Funct. Anal.*, 126:102–138, 1994b.

On Cesàro's means of first order of wavelet packet series

- E. C. Titchmarsh. *The Theory of functions*. Oxford University Press, 1939.
- G. G. Walter. Approximation of delta function by wavelets. *J. Approx. Theory*, 71:329–343, 1992.
- G. G. Walter. Pointwise convergence for wavelet expansions. *J. Approx. Theory*, 80:108–118, 1995.
- M. V. Wickerhauser. *Adapted Wavelet Analysis from Theory to Software*. A.K. Peters, Ltd., Wellesley, 1994.

On the solutions of Pellian equation

$$U^2 - DV^2 = k^2N$$

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Abstract

In this paper we consider a class of Pell's equation $U^2 - DV^2 = k^2N$, where D, N are positive integers, D is non-square and k is any integer. When (u, v) satisfy this equation, we define $\frac{u+v\sqrt{D}}{k}$ to be its *solution*. We first introduce the *class of solutions* of this equation and call $\frac{u+v\sqrt{D}}{k}$ to be the *fundamental solution* of the class, if v is the smallest positive value which occurs in the solutions of that class. We first derive the necessary and sufficient condition for any two solutions of proposed equation to belong to the same class and the bounds for the values of u, v occurring in the fundamental solution. We also derive an explicit formula which gives all the solutions of this equation. We further present some interesting recurrence relations connecting the values of u, v . Finally, we obtain the results for total number of positive solutions of proposed equation not exceeding any given positive real number Z .

Keywords: Pell's equation, Solutions of Pell's equation, Recurrence relations.

2010 AMS subject classification[‡]: 11D09, 11D45.

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[‡]Received on July 15, 2021. Accepted on December 25, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.628. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

In number theory, Pell's equation falls in the category of Diophantine equations and it is considered to be one of the oldest Diophantine equations. Specifically, the term Pell's equation is used to refer any Diophantine equation of the form

$$u^2 - Dv^2 = N, \quad (1.1)$$

where D and N are fixed non-zero integers and we wish to find integers u and v that satisfy the equation. Throughout, we consider the integer D to be positive and non-square. This condition is helpful because only it leaves open the possibility of infinitely many solutions in positive integers u, v .

Pellian equation

$$x^2 - Dy^2 = 1, \quad (1.2)$$

where D is fixed positive non-square integer, attracted attention of early mathematicians. It is known that (1.2) always has infinitude of solutions. We refer to Stolt [11, 12] who nicely studied the equation $x^2 - Dy^2 = \pm 4N$. In this paper we use the notions proposed by Stolt. Many authors considered some specific Pell equations and discussed about their solutions. For completeness we recall that there are many papers which considered different types of Pell's equation. For extensive resources on Pell's equation, one can refer [1] – [14].

In the present paper we consider the Pellian equation

$$U^2 - DV^2 = k^2N, \quad (1.3)$$

where D, N are positive integers, D is non-square and k is any integer. Throughout we assume that (1.3) is solvable. If (u, v) satisfy this equation, we define $\frac{u+v\sqrt{D}}{k}$ to be the *solution* of (1.3). If $\frac{u+v\sqrt{D}}{k}$ is any solution of (1.3), $x + y\sqrt{D}$ is any solution of (1.2) and $x_1 + y_1\sqrt{D}$ is its smallest positive solution, then it is easy to observe that $\frac{u+v\sqrt{D}}{k} \times (x + y\sqrt{D})$ is also a solution of (1.3).

This solution is said to be *associated* with the solution $\frac{u+v\sqrt{D}}{k}$. Here we note that if two positive solutions $\frac{u_\alpha+v_\alpha\sqrt{D}}{k}$ and $\frac{u_\beta+v_\beta\sqrt{D}}{k}$ are associated then there exists an integer t such that

$$\frac{u_\beta+v_\beta\sqrt{D}}{k} = \frac{u_\alpha+v_\alpha\sqrt{D}}{k} \times (x_1 + y_1\sqrt{D})^t; t = 0, \pm 1, \pm 2, \dots \quad (1.4)$$

The set of all solutions associated with other forms a *class of solutions* of (1.3). Among all the solutions $\frac{u+v\sqrt{D}}{k}$ in any given class K we now choose a solution $\frac{u_1+v_1\sqrt{D}}{k}$ in the following way: Let v_1 be the least non-negative value of v which occurs in K . Then in this case, u_1 is also uniquely determined. The solution $\frac{u_1+v_1\sqrt{D}}{k}$ defined in this way is said to be *fundamental solution* of

the class. It can be observed that $\frac{u_1+v_1\sqrt{D}}{k}$ is a fundamental solution of class K if $\frac{u_1+v_1\sqrt{D}}{k} \times (x_1 - y_1\sqrt{D})$ is not a positive solution of (1.3). Thus if $\frac{u_\alpha+v_\alpha\sqrt{D}}{k}$ is any fixed fundamental solution then all the positive solutions given by (1.4) are associated with each other. Moreover, we observe that if (1.3) is solvable, then it has only finite number of classes of solutions.

2. The equation $U^2 - DV^2 = k^2N$

It is easy to observe that if $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are any two integer solutions of (1.3) such that $u \equiv u' \pmod{N}$ and $v \equiv v' \pmod{N}$, then $\left(\frac{uu'-Dvv'}{kN}, \frac{uv'-u'v}{kN}\right)$ is the positive solution of the equation $R^2 - DS^2 = k^2$. We first present the necessary and sufficient condition for any two solutions of (1.3) to be associated with each other.

Theorem 2.1. If $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are any two solutions of (1.3) then the necessary and sufficient condition for these two solutions to be associated with each other is that $\frac{uv'-u'v}{kN}$ is an integer.

Proof. Since $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are solutions of (1.3), we get $u^2 - Dv^2 = k^2N$ and $u'^2 - Dv'^2 = k^2N$. Multiplying these two equations we get $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{uv'-u'v}{kN}\right)^2 = k^2$. We first show that the solutions $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ of the Pellian equation (1.3) are associated with each other if and only if both $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ are integers.

If two numbers $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ are integers, then $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{uv'-u'v}{kN}\right)^2 = k^2$. Thus $\frac{\frac{uu'-Dvv'}{kN} + \frac{uv'-u'v}{kN}\sqrt{D}}{k}$ is a solution of Pellian equation $R^2 - DS^2 = k^2$. Now

$$\frac{\frac{u+v\sqrt{D}}{k}}{\frac{u'+v'\sqrt{D}}{k}} = \frac{u+v\sqrt{D}}{u'+v'\sqrt{D}} = (u + v\sqrt{D}) \times \frac{u'-v'\sqrt{D}}{\pm k^2N} = \frac{uu'-Dvv'}{k^2N} - \frac{uv'-u'v}{k^2N}\sqrt{D}.$$

Clearly this is a solution of (1.2). Thus, both the solutions $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ of (1.3) are associated.

Conversely let two solutions $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ of (1.3) be associated with each other. Then clearly, they should lie in the same solution class of (1.3). Then

On the solutions of Pellian equation $U^2 - DV^2 = k^2N$

we have $\frac{\frac{u+v\sqrt{D}}{k}}{\frac{u'+v'\sqrt{D}}{k}} = \frac{uu'-Dvv'}{k^2N} - \frac{uv'-u'v}{k^2N}\sqrt{D}$, which should be some solution of (1.2). In this case we have $u^2 - Dv^2 = k^2N$ and $u'^2 - Dv'^2 = k^2N$. This gives $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{uv'-u'v}{kN}\right)^2 = k^2$. Thus,

$$k \times \left(\frac{uu'-Dvv'}{k^2N}, \frac{uv'-u'v}{k^2N}\right) = \left(\frac{uu'-Dvv'}{kN}, \frac{uv'-u'v}{kN}\right)$$

is an integer solution of $R^2 - DS^2 = k^2$. Hence $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ are integers.

Thus, it is now sufficient to show that $\frac{uu'-Dvv'}{kN}$ is an integer when $\frac{uv'-u'v}{kN}$ is an integer; and $\frac{uv'-u'v}{kN}$ is not an integer when $\frac{uu'-Dvv'}{kN}$ is not an integer. Since $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are solutions of (1.3), we have $u^2 - Dv^2 = \pm k^2N$, $u'^2 - Dv'^2 = \pm k^2N$. Multiplying these two equations we get

$$(uu' - vv'D)^2 - D(uv' - vu')^2 = k^2(kN)^2. \quad (2.1)$$

It is obvious from (2.1) that $uu' - Dvv'$ is divisible by kN when $uv' - u'v$ is divisible by kN . That is, $\frac{uu'-Dvv'}{kN}$ is an integer when $\frac{uv'-u'v}{kN}$ is an integer. Conversely, if $\frac{uu'-Dvv'}{kN}$ is not an integer, then there exists an integer d such that $d \mid kN$ but $d \nmid (uu' - Dvv')$. Now from (2.1) it is seen that if $d \mid (uv' - u'v)$, then $d \mid (uu' - vv'D)$ too, which is contrary to the assumption. Thus $d \nmid (uv' - u'v)$, that is $\frac{uv'-u'v}{kN}$ is not an integer, as required. This proves the required result.

We now derive the bounds for the values of u, v occurring in the fundamental solutions.

Theorem 2.2. If $\frac{u+v\sqrt{D}}{k}$ is the smallest fundamental solution of the class K of the Pellian equation (1.3) and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of (1.2), then $0 < |u| \leq k\sqrt{\frac{(x_1+1)N}{2}}$ and $0 \leq v \leq ky_1\sqrt{\frac{N}{2(x_1+1)}}$.

Proof. If both the inequalities to be proved are true for a class K , then they are also true for the conjugate class $\bar{K}$. Thus, we can assume that u is positive.

$$\begin{aligned} \text{Note that } \frac{ux_1 - Dvy_1}{k} &= \frac{ux_1}{k} - (y_1\sqrt{D})\left(\frac{v\sqrt{D}}{k}\right) = \frac{ux_1}{k} - \sqrt{x_1^2 - 1}\sqrt{\frac{u^2 - k^2N}{k^2}} \\ &= \frac{ux_1}{k^2} - \sqrt{(x_1^2 - 1)\left(\frac{u^2}{k^2} - N\right)} > 0. \end{aligned}$$

Now consider the solution $\left(\frac{u+v\sqrt{D}}{k}\right)(x_1 - y_1\sqrt{D}) = \frac{ux_1 - Dvy_1 + (x_1v - y_1u)\sqrt{D}}{k}$ of (1.3) which belongs to the same class as $\frac{u+v\sqrt{D}}{k}$. But $\frac{u+v\sqrt{D}}{k}$ is the fundamental solution of the class and by above $\frac{ux_1 - Dvy_1}{k}$ is positive. Thus, we must have $\frac{ux_1 - Dvy_1}{k} \geq \frac{u}{k}$, as $\frac{u}{k}$ occurs in fundamental solution of (1.3). From this inequality it now follows that $ux_1 - Dvy_1 \geq u$. This gives $u(x_1 - 1) \geq Dvy_1$, which gives $u^2(x_1 - 1)^2 \geq D^2v^2y_1^2 = (u^2 - k^2N)(x_1^2 - 1)$. This can be written as $u^2 \frac{x_1 - 1}{x_1 + 1} \geq u^2 - k^2N$, which eventually gives $u^2 \leq \frac{(x_1 + 1)k^2N}{2}$. This proves the first of required inequality.

Again by (1.3) we have $Dv^2 = u^2 - k^2N$. Using above inequality, we get $Dv^2 \leq \frac{(x_1 + 1)k^2N}{2} - k^2N = \frac{(x_1 - 1)k^2N}{2}$. This gives

$$v \leq k \sqrt{\frac{(x_1 - 1)N}{2D}} = k \sqrt{\frac{(x_1^2 - 1)N}{2D(x_1 + 1)}}.$$

Since $x_1 + y_1\sqrt{D}$ is the solution of (1.2), we thus get $0 \leq v \leq ky_1 \sqrt{\frac{N}{2(x_1 + 1)}}$, as required.

We now present an illustration to demonstrate the results of this theorem.

Illustration. Consider the Pellian equation $U^2 - 7V^2 = 18$. Then clearly, $D = 7, k = 3$ and $N = 2$. If we consider the equation $x^2 - 7y^2 = 1$, then it is easy to see that $8 + 3\sqrt{7}$ is its fundamental solution. This gives $x_1 = 8, y_1 = 3$. Now if $\frac{u+v\sqrt{7}}{3}$ is the smallest fundamental solution of equation $U^2 - 7V^2 = 18$, then by above theorem we should have

$$0 < |u| \leq 3 \times \sqrt{\frac{9 \times 2}{2}} \text{ and } 0 \leq v \leq 3 \times 3 \sqrt{\frac{2}{2 \times 9}}.$$

This gives $0 < |u| \leq 9$ and $0 \leq v \leq 3$. These are indeed true as $\frac{5+\sqrt{7}}{3}$ is the smallest fundamental solution of the equation $U^2 - 7V^2 = 18$.

We now prove a very important result which produces all the fundamental solutions of (1.3).

Theorem 2.3. If $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1.1) and $\frac{r_j + s_j\sqrt{D}}{k}$ runs through all the fundamental solutions of $R^2 - DS^2 = k^2$, then all the fundamental solutions of $U^2 - DV^2 = k^2N$ are covered by

$$\frac{A_{ij} + B_{ij}\sqrt{D}}{k} = \left(\frac{r_j + s_j\sqrt{D}}{k}\right)(u_i \pm v_i\sqrt{D}). \quad (2.2)$$

Proof. If we multiply the surd conjugate of (2.2) with (2.2) then we get $\frac{A_{ij}^2 - DB_{ij}^2}{k^2} = \left(\frac{r_j^2 - Ds_j^2}{k^2}\right)(u_i^2 - Dv_i^2)$. Since $u_i^2 - Dv_i^2 = N$ and $r_j^2 - Ds_j^2 = k^2$, we get $A_{ij}^2 - DB_{ij}^2 = k^2N$. This shows that $\frac{A_{ij} + B_{ij}\sqrt{D}}{k}$ defined by (2.2) are the solutions of (1.3).

We next show that the solutions $\frac{A_{ij} + B_{ij}\sqrt{D}}{k}$ defined by (2.2) covers all the fundamental solutions of (1.3). On the contrary assume that there exists some positive solution, say $\frac{X+Y\sqrt{D}}{k}$ of (1.3) which is not covered by (2.2). Then this solution will lie between any two successive solutions of (1.3) of some fixed class generated by $\frac{r_j + s_j\sqrt{D}}{k}$. This means for some fixed j , we have

$$\left(\frac{r_j + s_j\sqrt{D}}{k}\right)(u_i \pm v_i\sqrt{D}) \leq \frac{X+Y\sqrt{D}}{k} < \left(\frac{r_j + s_j\sqrt{D}}{k}\right)(u_{i+1} \pm v_{i+1}\sqrt{D}).$$

Then $u_i \pm v_i\sqrt{D} \leq \left(\frac{X+Y\sqrt{D}}{k}\right)\left(\frac{k}{r_j + s_j\sqrt{D}}\right) < u_{i+1} \pm v_{i+1}\sqrt{D}$. This gives

$$u_i \pm v_i\sqrt{D} \leq \left(\frac{X+Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right) < u_{i+1} \pm v_{i+1}\sqrt{D}.$$

We denote

$$\epsilon + \delta\sqrt{D} = \left(\frac{X+Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right). \quad (2.3)$$

Then

$$u_i \pm v_i\sqrt{D} \leq \epsilon + \delta\sqrt{D} < u_{i+1} \pm v_{i+1}\sqrt{D}. \quad (2.4)$$

To prove the required result, it is sufficient to prove that

(i) $\epsilon + \delta\sqrt{D}$ is a solution of (1.1), and

(ii) $\epsilon > 0$ and $\delta > 0$ or $\delta < 0$ depending on the sign of $u_i \pm v_i\sqrt{D}$.

This will produce one positive solution of (1.1) between two consecutive fundamental solutions of (1.1) for any fixed class j , which is a contradiction.

To prove (i), we multiply surd conjugate of (2.3) with (2.3). This gives $\epsilon^2 - D\delta^2 = \left(\frac{X^2 - DY^2}{k^2}\right)\left(\frac{r_j^2 - Ds_j^2}{k^2}\right)$. Since $X^2 - DY^2 = k^2N$ and $r_j^2 - Ds_j^2 = k^2$, we get $\epsilon^2 - D\delta^2 = N$, which proves the first part.

Next, we show that ϵ and δ defined by (2.3) are positive. Now $r_j^2 - Ds_j^2 = k^2$ implies $\left(\frac{r_j + s_j\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right) = 1$ and $\frac{X+Y\sqrt{D}}{k} > 1$. Since $0 < \frac{r_j + s_j\sqrt{D}}{k} < \infty$, clearly $0 < \frac{r_j - s_j\sqrt{D}}{k} < \infty$. Since $\epsilon + \delta\sqrt{D} = \left(\frac{X+Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right)$, we get $0 < \epsilon + \delta\sqrt{D} < \infty$. Also, since $\epsilon^2 - D\delta^2 = N$, we get $0 < \frac{N}{\epsilon - \delta\sqrt{D}} < \infty$, that is $0 < \frac{\epsilon - \delta\sqrt{D}}{N} < \infty$. This gives $0 < \epsilon - \delta\sqrt{D} < \infty$. Adding this result with $0 < \epsilon + \delta\sqrt{D} < \infty$ proves that $\epsilon > 0$.

We further observe by (2.4) that

$$1 < u_i \pm v_i\sqrt{D} \leq \epsilon + \delta\sqrt{D} < u_{i+1} \pm v_{i+1}\sqrt{D} < \infty.$$

By considering the ‘+’ sign, we get $0 < \frac{1}{\epsilon + \delta\sqrt{D}} \leq \frac{1}{u_i + v_i\sqrt{D}} < 1$.

Then $0 < \frac{\epsilon - \delta\sqrt{D}}{N} \leq \frac{u_i - v_i\sqrt{D}}{N}$, which gives $0 < \epsilon - \delta\sqrt{D} \leq u_i - v_i\sqrt{D}$.

Subsequently we get

$$2\delta\sqrt{D} = (\epsilon + \delta\sqrt{D}) - (\epsilon - \delta\sqrt{D}) \geq (u_i + v_i\sqrt{D}) - (u_i - v_i\sqrt{D}) = 2v_i\sqrt{D}.$$

This gives $\delta \geq v_i > 0$.

Also, if we select ‘-’ sign in $u_i \pm v_i\sqrt{D}$, then we have

$$\epsilon + \delta\sqrt{D} < u_{i+1} - v_{i+1}\sqrt{D} < \infty,$$

which implies that $0 < \frac{1}{u_{i+1} - v_{i+1}\sqrt{D}} < \frac{1}{\epsilon + \delta\sqrt{D}}$. Then $0 < \frac{u_{i+1} + v_{i+1}\sqrt{D}}{N} < \frac{\epsilon - \delta\sqrt{D}}{N}$.

This gives $\epsilon - \delta\sqrt{D} > u_{i+1} + v_{i+1}\sqrt{D}$. Thus, we get

$$2\delta\sqrt{D} = (\epsilon + \delta\sqrt{D}) - (\epsilon - \delta\sqrt{D}) < (u_{i+1} - v_{i+1}\sqrt{D}) - (u_{i+1} + v_{i+1}\sqrt{D}) \\ = -2v_{i+1}\sqrt{D}.$$

This gives $\delta < -v_{i+1} < 0$. Hence $\delta > 0$ or $\delta < 0$ depends on the sign of $u_i \pm v_i\sqrt{D}$, which proves (ii). Hence all the fundamental solutions $\frac{A_{ij} + B_{ij}\sqrt{D}}{k}$ of (1.3) are covered by (2.2).

We illustrate this by an example which justifies the meaning of “... covered by (2.2)”.

Illustration. As earlier we once again consider the equation $U^2 - 7V^2 = 18$. Then we have $D = 7, k = 3$ and $N = 2$. Next, we consider the Pellian equation $R^2 - 7S^2 = 9$ and if $\frac{r_j + s_j\sqrt{7}}{3}$ runs through all of its fundamental solutions, then it can be observed that $\frac{r_1 + s_1\sqrt{7}}{3} = \frac{4 + \sqrt{7}}{3}$, $\frac{r_2 + s_2\sqrt{7}}{3} = \frac{11 + 4\sqrt{7}}{3}$, $\frac{r_3 + s_3\sqrt{7}}{3} = \frac{24 + 9\sqrt{7}}{3}$. Also $u_i + v_i\sqrt{7}$ runs through all the fundamental solutions of $u^2 - 7v^2 = 2$, then we have $u_1 \pm v_1\sqrt{7} = 3 \pm \sqrt{7}$. Then above theorem claims that all the fundamental solutions of $U^2 - 7V^2 = 18$ are covered by

$$\frac{A_{1j} + B_{1j}\sqrt{2}}{3} = \left(\frac{r_j + s_j\sqrt{7}}{3} \right) (3 \pm \sqrt{7}); j = 1, 2, 3.$$

Thus, $\frac{A_{11} + B_{11}\sqrt{2}}{3} = \left(\frac{4 + \sqrt{7}}{3} \right) (3 \pm \sqrt{7}) = \frac{19 + 7\sqrt{7}}{3}, \frac{5 - \sqrt{7}}{3};$

$\frac{A_{12} + B_{12}\sqrt{2}}{3} = \left(\frac{11 + 4\sqrt{7}}{3} \right) (3 \pm \sqrt{7}) = \frac{61 + 23\sqrt{7}}{3}, \frac{5 + \sqrt{7}}{3}$ and

$\frac{A_{13} + B_{13}\sqrt{2}}{3} = \left(\frac{24 + 9\sqrt{7}}{3} \right) (3 \pm \sqrt{7}) = \frac{135 + 51\sqrt{7}}{3}, \frac{9 + 3\sqrt{7}}{3}.$

On the solutions of Pellian equation $U^2 - DV^2 = k^2N$

It can be observed that $\frac{5+\sqrt{7}}{3}, \frac{9+3\sqrt{7}}{3}$ and $\frac{19+7\sqrt{7}}{3}$ are the only three fundamental solutions of the equation $U^2 - 7V^2 = 18$. Thus $\frac{A_{1j}+B_{1j}\sqrt{2}}{3}; j = 1,2,3$ covers all the fundamental solutions of $U^2 - 7V^2 = 18$.

Here the smallest fundamental solution of $U^2 - 7V^2 = 18$ is $\frac{5+\sqrt{7}}{3}$ and $x_1 + y_1\sqrt{D} = 8 + 3\sqrt{7}$ is the fundamental solution of $x^2 - 7y^2 = 1$. Then every fundamental solution of $U^2 - 7V^2 = 18$ should be smaller than $\frac{5+\sqrt{7}}{3} \times (8 + 3\sqrt{7}) = \frac{61+23\sqrt{7}}{3}$. This is indeed true as the only fundamental solutions of $U^2 - 7V^2 = 18$ are $\frac{A_{11}+B_{11}\sqrt{2}}{3} = \frac{19+7\sqrt{7}}{3}, \frac{A_{12}+B_{12}\sqrt{2}}{3} = \frac{5+\sqrt{7}}{3}$ and $\frac{A_{13}+B_{13}\sqrt{2}}{3} = \frac{9+3\sqrt{7}}{3}$.

We next derive an explicit formula which produces all the positive solutions of (1.3).

Theorem 2.4. If $x_1 + y_1\sqrt{D}$ is the smallest positive solution of (1.2) and $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ defined by (2.2) runs through all the fundamentals solutions of (1.3), then all the integer solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ of $U^2 - DV^2 = k^2N$ are given by

$$\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} = \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \right) (x_1 + y_1\sqrt{D})^n; n \geq 0. \quad (2.5)$$

Proof. If we consider the surd conjugate of (2.5) and multiply with (2.5), we get $\frac{u_{ij,n}^2 - Dv_{ij,n}^2}{k^2} = \left(\frac{A_{ij}^2 - DB_{ij}^2}{k^2} \right) (x_1^2 - Dy_1^2)^n$. Since $A_{ij}^2 - DB_{ij}^2 = k^2N$ and $x_1^2 - Dy_1^2 = 1$, we get $u_{ij,n}^2 - Dv_{ij,n}^2 = k^2N$. Thus $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ defined by (2.5) are the solutions of (1.3).

We next show that the solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ defined by (2.5) gives all the positive solutions of (1.3). On the contrary assume that there exists some positive solution, say $\frac{X+Y\sqrt{D}}{k}$ of (1.3) which is not covered by (2.5). Then this solution will lie between any two successive solutions of (1.3) of some classes generated by $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ (for a fixed i, j). This means for some fixed i, j and for some fixed m , we have

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \right) (x_1 + y_1\sqrt{D})^m \leq \frac{X+Y\sqrt{D}}{k} < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \right) (x_1 + y_1\sqrt{D})^{m+1}.$$

$$\text{Then } \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \right) \leq \left(\frac{X+Y\sqrt{D}}{k} \right) (x_1 - y_1\sqrt{D})^m < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \right) (x_1 + y_1\sqrt{D}).$$

We denote

$$\frac{\epsilon + \delta\sqrt{D}}{k} = \left(\frac{X+Y\sqrt{D}}{k}\right) (x_1 - y_1\sqrt{D})^m. \quad (2.6)$$

Thus,

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) \leq \frac{\epsilon + \delta\sqrt{D}}{k} < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) (x_1 + y_1\sqrt{D}). \quad (2.7)$$

To prove the required result, it is sufficient to prove that

(i) $\frac{\epsilon + \delta\sqrt{D}}{k}$ is a solution of (1.3)

(ii) $\epsilon > 0, \delta > 0$.

(iii) $\frac{\epsilon + \delta\sqrt{D}}{k}$ is the solution of (1.3) smaller than all its fundamental solutions for the fixed classes i, j . This will contradict the fact that $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through every fundamental solution of (1.3) for some fixed value of i, j .

To prove (i), we take surd conjugate of (2.6) and multiply it with (2.6). This gives $\frac{\epsilon^2 - D\delta^2}{k^2} = \left(\frac{X^2 - DY^2}{k^2}\right) (x_1^2 - Dy_1^2)^m$. Since $X^2 - DY^2 = k^2N$ and $x_1^2 - Dy_1^2 = 1$, we get $\epsilon^2 - D\delta^2 = k^2N$, as required.

Again, since $1 < (x_1 + y_1\sqrt{D})^m < \infty$ and $\frac{X+Y\sqrt{D}}{k} > 1$, we have $0 < (x_1 - y_1\sqrt{D})^m < 1$. Hence, we get $0 < \frac{\epsilon + \delta\sqrt{D}}{k} < \infty$, as $\frac{\epsilon + \delta\sqrt{D}}{k} = \left(\frac{X+Y\sqrt{D}}{k}\right) (x_1 - y_1\sqrt{D})^m$. This gives $0 < \frac{\epsilon - \delta\sqrt{D}}{kN} < \infty$, that is $0 < \frac{\epsilon - \delta\sqrt{D}}{k} < \infty$, as $\epsilon^2 - D\delta^2 = k^2N$. Adding this result with $0 < \frac{\epsilon + \delta\sqrt{D}}{k} < \infty$ proves that $\epsilon > 0$.

We further observe by (2.7) that $1 < \frac{A_{ij}+B_{ij}\sqrt{D}}{k} \leq \frac{\epsilon + \delta\sqrt{D}}{k} < \infty$. Taking reciprocal, we get $0 < \frac{k}{\epsilon + \delta\sqrt{D}} \leq \frac{k}{A_{ij}+B_{ij}\sqrt{D}} < 1$. Then $0 < \frac{\epsilon - \delta\sqrt{D}}{kN} \leq \frac{A_{ij}-B_{ij}\sqrt{D}}{kN}$. This gives $0 < \frac{\epsilon - \delta\sqrt{D}}{k} \leq \frac{A_{ij}-B_{ij}\sqrt{D}}{k}$. Subsequently we get

$$\frac{2\delta\sqrt{D}}{k} = \left(\frac{\epsilon + \delta\sqrt{D}}{k}\right) - \left(\frac{\epsilon - \delta\sqrt{D}}{k}\right) \geq \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) - \left(\frac{A_{ij}-B_{ij}\sqrt{D}}{k}\right) = \frac{2B_{ij}\sqrt{D}}{k}.$$

This now gives $\delta \geq B_{ij} > 0$. Hence both $\epsilon > 0, \delta > 0$, which proves (ii).

Finally, we prove that $\frac{\epsilon + \delta\sqrt{D}}{k}$ is the smallest solution of (1.3) for the fixed classes i, j . On the contrary suppose $\frac{\epsilon + \delta\sqrt{D}}{k}$ is positive solution of (1.3) but not the smallest solution of any fixed classes i, j . In this case $\frac{\epsilon + \delta\sqrt{D}}{k} (x_1 - y_1\sqrt{D})$ will be the positive solution of (1.3). Then by (2.7) we have

$$\frac{A_{ij}+B_{ij}\sqrt{D}}{k} \leq \frac{\epsilon + \delta\sqrt{D}}{k} < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) (x_1 + y_1\sqrt{D}),$$

which gives

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) (x_1 - y_1\sqrt{D}) \leq \frac{\epsilon + \delta\sqrt{D}}{k} (x_1 - y_1\sqrt{D}) < \frac{A_{ij}+B_{ij}\sqrt{D}}{k}.$$

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Here $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through every fundamental solution of (1.3). Thus, we can now say that $\frac{\epsilon+\delta\sqrt{D}}{k}(x_1 - y_1\sqrt{D})$ is a positive solution of (1.3) smaller than all the fundamental solutions $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ of (1.3), which is a contradiction. This contradiction finally assures that there cannot exist any solution of (1.3) which is not covered by (2.5), as required.

We illustrate this theorem by the following example:

Illustration. Once again we consider the Pellian equation $U^2 - 7V^2 = 18$ whose all the fundamental solutions are given by $\frac{A_{11}+B_{11}\sqrt{2}}{3} = \frac{19+7\sqrt{7}}{3}$; $\frac{A_{12}+B_{12}\sqrt{2}}{3} = \frac{5+\sqrt{7}}{3}$ and $\frac{A_{13}+B_{13}\sqrt{2}}{3} = \frac{9+3\sqrt{7}}{3}$ and $x_1 + y_1\sqrt{D} = 8 + 3\sqrt{7}$ is the fundamental solution of $x^2 - 7y^2 = 1$. Then the above theorem asserts that all the integer solutions $\frac{u_{1j,n}+v_{1j,n}\sqrt{7}}{3}$ of Pellian equation $U^2 - 7V^2 = 18$ are covered by $\frac{u_{1j,n}+v_{1j,n}\sqrt{7}}{3} = \left(\frac{A_{1j}+B_{1j}\sqrt{7}}{3}\right)(x_1 + y_1\sqrt{D})^n$; $j = 1, 2, 3$; $n \geq 0$. This gives $\frac{u_{11,n}+v_{11,n}\sqrt{7}}{3} = \left(\frac{19+7\sqrt{7}}{3}\right)(8 + 3\sqrt{7})^n$; $\frac{u_{12,n}+v_{12,n}\sqrt{7}}{3} = \left(\frac{5+\sqrt{7}}{3}\right)(8 + 3\sqrt{7})^n$ and $\frac{u_{13,n}+v_{13,n}\sqrt{7}}{3} = \left(\frac{9+3\sqrt{7}}{3}\right)(8 + 3\sqrt{7})^n$.

We now prove some recurrence relations for the values of $u_{ij,n}$ and $v_{ij,n}$. We assume that $x_1 + y_1\sqrt{D}$ is the fundamental solution of (1.2), $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1.1), $\frac{r_j+s_j\sqrt{D}}{k}$ runs through all the fundamental solutions of $R^2 - DS^2 = k^2$, $r \geq 1$ is a fixed integer and $n \geq 1$.

Theorem 2.5. a) $u_{ij,n+r} = \frac{x_ru_{ij,n}+Dy_rv_{ij,n}}{k}$, $v_{ij,n+r} = \frac{y_ru_{ij,n}+x_rv_{ij,n}}{k}$.
 b) $u_{ij,n+r} = \frac{x_1u_{ij,n+r-1}+Dy_1v_{ij,n+r-1}}{k}$, $v_{ij,n+r} = \frac{y_1u_{ij,n+r-1}+x_1v_{ij,n+r-1}}{k}$.
 c) $u_{ij,n+2r} = \frac{2kx_ru_{ij,n+r}-u_{ij,n}}{k^2}$, $v_{ij,n+2r} = \frac{2kx_rv_{ij,n+r}-v_{ij,n}}{k^2}$.

Proof. (a) By (2.5), we have

$$\begin{aligned} \frac{u_{ij,n+r}+v_{ij,n+r}\sqrt{D}}{k} &= \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)(x_1 + y_1\sqrt{D})^{n+r} = \left(\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}\right)(x_r + y_r\sqrt{D}) \\ &= \frac{(x_ru_{ij,n}+Dy_rv_{ij,n})+(y_ru_{ij,n}+x_rv_{ij,n})\sqrt{D}}{k}. \end{aligned}$$

$$\text{Hence } u_{ij,n+r} = \frac{x_ru_{ij,n}+Dy_rv_{ij,n}}{k}, v_{ij,n+r} = \frac{y_ru_{ij,n}+x_rv_{ij,n}}{k}.$$

(b) To prove the second result, we first consider $r = 1$ and replace n by $n - 1$ in the above result. We thus get

$$\begin{aligned}
 u_{ij,n} &= \frac{x_1 u_{ij,n-1} + D y_1 v_{ij,n-1}}{k}, v_{ij,n} = \frac{y_1 u_{ij,n-1} + x_1 v_{ij,n-1}}{k}. \\
 \text{Then } u_{ij,n+r} &= \frac{x_r u_{ij,n} + D y_r v_{ij,n}}{k} \\
 &= \frac{1}{k} \left\{ x_r \left(\frac{x_1 u_{ij,n-1} + D y_1 v_{ij,n-1}}{k} \right) + D y_r \left(\frac{y_1 u_{ij,n-1} + x_1 v_{ij,n-1}}{k} \right) \right\} \\
 &= \frac{1}{k} \left\{ x_1 \left(\frac{x_r u_{ij,n-1} + D y_r v_{ij,n-1}}{k} \right) + D y_1 \left(\frac{x_r v_{ij,n-1} + y_r u_{ij,n-1}}{k} \right) \right\} \\
 &= \frac{x_1 u_{ij,n+r-1} + D y_1 v_{ij,n+r-1}}{k}
 \end{aligned}$$

Value of $v_{ij,n+r}$ can be obtained accordingly.

(c) To prove the final part, we replace n by $n + r$ in the first result and using that in (a) above, we obtain

$$\begin{aligned}
 u_{ij,n+2r} &= \frac{1}{k} \left\{ x_r u_{ij,n+r} + D y_r \left(\frac{y_r u_{ij,n} + x_r v_{ij,n}}{k} \right) \right\} \\
 &= \frac{1}{k} \left\{ x_r u_{ij,n+r} + \frac{D y_r^2 u_{ij,n}}{k} + x_r \left(\frac{D y_r v_{ij,n}}{k} \right) \right\} \\
 &= \frac{1}{k} \left\{ x_r u_{ij,n+r} + \frac{D y_r^2 u_{ij,n}}{k} + x_r \left(u_{ij,n+r} - \frac{x_r u_{ij,n}}{k} \right) \right\} \\
 &= \frac{1}{k} \left\{ 2 x_r u_{ij,n+r} - \frac{u_{ij,n}}{k} (x_r^2 - D y_r^2) \right\}
 \end{aligned}$$

Since $x_r + y_r \sqrt{D}$ is a solution of (1.2), we have $x_r^2 - D y_r^2 = 1$. Thus, we obtain

$$u_{ij,n+2r} = \frac{2 k x_r u_{ij,n+r} - u_{ij,n}}{k^2}.$$

Value of $v_{ij,n+2r}$ can also be obtained accordingly.

We further derive some more interesting properties related with the value of $u_{ij,n}$ and $v_{ij,n}$. The following interesting recursive formula connects three $u_{ij,n}$'s as well as $v_{ij,n}$'s when the suffixes are in arithmetic progression.

Corollary 2.6. (a) $u_{ij,n} u_{ij,n+2r} - u_{ij,n+r}^2 = k^2 D N y_r^2$.

(b) $v_{ij,n} v_{ij,n+2r} - v_{ij,n+r}^2 = -k^2 N y_r^2$.

Proof. By (2.5) we have $\frac{u_{ij,n} + v_{ij,n} \sqrt{D}}{k} = \left(\frac{A_{ij} + B_{ij} \sqrt{D}}{k} \right) (x_1 + y_1 \sqrt{D})^n$.

Taking its surd conjugate, we get

$$\frac{u_{ij,n} - v_{ij,n} \sqrt{D}}{k} = \left(\frac{A_{ij} - B_{ij} \sqrt{D}}{k} \right) (x_1 - y_1 \sqrt{D})^n.$$

For convenience we write $\gamma = x_1 + y_1 \sqrt{D}$, $\bar{\gamma} = x_1 - y_1 \sqrt{D}$ and $\mu_{ij} = \frac{A_{ij} + B_{ij} \sqrt{D}}{k}$, $\bar{\mu}_{ij} = \frac{A_{ij} - B_{ij} \sqrt{D}}{k}$. Then we have $\frac{u_{ij,n} + v_{ij,n} \sqrt{D}}{k} = \mu_{ij} \gamma^n$ and $\frac{u_{ij,n} - v_{ij,n} \sqrt{D}}{k} = \bar{\mu}_{ij} \bar{\gamma}^n$.

Adding and subtracting these two relations, we get

$$u_{ij,n} = \frac{k}{2} \{ \mu_{ij} \gamma^n + \bar{\mu}_{ij} \bar{\gamma}^n \} \text{ and } v_{ij,n} = \frac{k}{2\sqrt{D}} \{ \mu_{ij} \gamma^n - \bar{\mu}_{ij} \bar{\gamma}^n \}.$$

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It can be easily observed that $\gamma\bar{\gamma} = x_1^2 - Dy_1^2 = 1$ and $\mu_{ij}\bar{\mu}_{ij} = \frac{A_{ij}^2 - DB_{ij}^2}{k^2} = N$.

Then

$$\begin{aligned} & u_{ij,n}u_{ij,n+2r} - u_{ij,n+r}^2 \\ &= \frac{k^2}{4} \left\{ (\mu_{ij}\gamma^n + \bar{\mu}_{ij}\bar{\gamma}^n)(\mu_{ij}\gamma^{n+2r} + \bar{\mu}_{ij}\bar{\gamma}^{(n+2r)}) \right. \\ & \quad \left. - (\mu_{ij}\gamma^{n+r} + \bar{\mu}_{ij}\bar{\gamma}^{(n+r)})^2 \right\} \\ &= \frac{k^2}{4} \left\{ \mu_{ij}\bar{\mu}_{ij}(\gamma^{n+2r}\bar{\gamma}^n + \gamma^n\bar{\gamma}^{(n+2r)}) \right. \\ & \quad \left. - 2\mu_{ij}\bar{\mu}_{ij}\gamma^{n+r}\bar{\gamma}^{(n+r)} \right\} = \frac{k^2}{4} \{N(\gamma^{2r} + \bar{\gamma}^{2r}) - 2N\} \\ &= \frac{k^2N}{4} (\gamma^r - \bar{\gamma}^r)^2. \end{aligned}$$

Since $x_r + y_r\sqrt{D} = (x_1 + y_1\sqrt{D})^r = \gamma^r$ and $x_r - y_r\sqrt{D} = (x_1 - y_1\sqrt{D})^r = \bar{\gamma}^r$, we get $u_{ij,n}u_{ij,n+2r} - u_{ij,n+r}^2 = k^2DN\gamma_r^2$.

Second result can be proved accordingly.

Following result gives some more recurrence relations in the form of a determinant.

Theorem 2.7. a) $\begin{vmatrix} u_{ij,n} & u_{ij,n+r} \\ v_{ij,n} & v_{ij,n+r} \end{vmatrix} = ky_rN$

b) $\begin{vmatrix} u_{ij,n+r-1} & u_{ij,n+r} \\ v_{ij,n+r-1} & v_{ij,n+r} \end{vmatrix} = ky_1N$

c) $\begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r} \end{vmatrix} = -2kNy_r(x_r - 1).$

Proof. We only prove (c), since first two results follow easily through theorem

2.5. Now

$$\begin{aligned} & \begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r} \end{vmatrix} \\ &= \begin{vmatrix} u_{ij,n} & u_{ij,n+r} \\ v_{ij,n} & v_{ij,n+r} \end{vmatrix} - \begin{vmatrix} u_{ij,n-r} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n+r} \end{vmatrix} + \begin{vmatrix} u_{ij,n-r} & u_{ij,n} \\ v_{ij,n-r} & v_{ij,n} \end{vmatrix} \\ &= ky_rN - ky_{2r}N + ky_rN = kN(2y_r - y_{2r}). \end{aligned}$$

Now $(x_1 + y_1\sqrt{D})^{2r} = x_{2r} + y_{2r}\sqrt{D} = (x_r + y_r\sqrt{D})^2$. This gives $y_{2r} = 2x_ry_r$. Thus

$$\begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r} \end{vmatrix} = kN(2y_r - 2x_ry_r) = -2kNy_r(x_r - 1).$$

3. Number of solutions up to a desired limit

In this final section, we define and obtain the values of the sums $R(Z) = \sum_{\substack{\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} \leq Z \\ U^2-DV^2=k^2N}} 1$, $S(Z) = \sum_{U^2-DV^2=k^2N} u_{ij,n} \leq Z$ and $T(Z) = \sum_{U^2-DV^2=k^2N} v_{ij,n} \leq Z$ 1, the

total number of positive solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$, $u_{ij,n}$ and $v_{ij,n}$ respectively of (1.3) that do not exceed any given large positive real number Z . For convenience, we denote $\delta = \frac{1}{\log \gamma}$, $\gamma = x_1 + y_1\sqrt{D}$ and let $\mathcal{A}_{ij} = \frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through all the fundamental solutions of (1.3) for any fixed class i . We also assume that (1.1) has β fundamental solutions and the equation $R^2 - DS^2 = k^2$ has η fundamental solutions.

Thus, throughout we have $1 \leq i \leq \beta$ and $1 \leq j \leq \eta$. We first obtain the value of $R(Z)$ which gives the number of all the solutions of (1.3) not exceeding any fixed given positive real number Z .

Theorem 3.1. $R(Z) = \delta \left\{ \beta\eta \log(Z) - \log\left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij}\right) \right\} + C$, where C is the effective constant such that $0 \leq C < \beta\eta$.

Proof. To find the value of $R(Z) = \sum_{\substack{\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} \leq Z \\ U^2-DV^2=k^2N}} 1$, we first find the number

of positive solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ of (1.3) that do not exceed Z for some fixed class $i = \alpha$ ($1 \leq j \leq \eta$). Since γ and $\mathcal{A}_{ij}$ are solutions of (1.2) and (1.3) respectively, (2.5) can be written as $\frac{u_{\alpha j,n}+v_{\alpha j,n}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n$. Now, for any given Z , it is clear that for some fixed class $i = \alpha$, there exists some n such that $\frac{u_{\alpha j,n}+v_{\alpha j,n}\sqrt{D}}{k} \leq Z < \frac{u_{\alpha j,n+1}+v_{\alpha j,n+1}\sqrt{D}}{k}$. Then we get $\mathcal{A}_{\alpha j}\gamma^n \leq Z < \mathcal{A}_{\alpha j}\gamma^{n+1}$.

This implies $n < \frac{\log Z - \log \mathcal{A}_{\alpha j}}{\log \gamma} < n + 1$. Since n is an integer, we get $n = \left\lfloor \frac{\log Z - \log \mathcal{A}_{\alpha j}}{\log \gamma} \right\rfloor$, where $[x]$ denotes the integer part of x .

Now since $[x] = x - \{x\}$, where $\{x\}$ is the fractional part of x and as $0 \leq \{x\} < 1$, we have

$$R(Z) = \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \left\lfloor \frac{\log Z - \log \mathcal{A}_{ij}}{\log \gamma} \right\rfloor = \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \left(\frac{\log Z - \log \mathcal{A}_{ij}}{\log \gamma} + c' \right),$$

where $0 \leq c' < 1$. Thus $R(Z) = \frac{1}{\log \gamma} \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} (\log Z - \log \mathcal{A}_{ij}) + \beta\eta c'$.

If we write $C = \beta\eta c'$, then we get $0 \leq C < \beta\eta$ and

$$R(Z) = \delta \left\{ \beta\eta \log(Z) - \log\left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij}\right) \right\} + C.$$

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We next find the value of $S(Z)$.

Theorem 3.2. $S(Z) = \delta \left\{ \beta \eta \log(2Z/k) - \log \left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij} \right) \right\} + C$, where C is the effective constant such that $-\beta\eta \leq C < \beta\eta$.

Proof. To find the value of $S(Z)$, we first find the number of positive solutions of (1.3) where the values $u_{ij,n}$ of U do not exceed Z for some fixed class $i = \alpha$ ($1 \leq j \leq \eta$). Now (2.5) can be written as $\frac{u_{\alpha j,n} + v_{\alpha j,n}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n$. Then for any given Z , it is clear that for some fixed class $i = \alpha$, there exists some n such that $u_{\alpha j,n} \leq Z < u_{\alpha j,n+1}$. Since $\mathcal{A}_{ij} = \frac{A_{ij} + B_{ij}\sqrt{D}}{k}$, we write $\overline{\mathcal{A}_{ij}} = \frac{A_{ij} - B_{ij}\sqrt{D}}{k}$. We also have $\gamma^{-1} = x_1 - y_1\sqrt{D}$. Since γ and $\mathcal{A}_{ij}, \overline{\mathcal{A}_{ij}}$ are the solutions of (1.2) and (1.3) respectively, we have $\gamma\gamma^{-1} = 1$ and $\mathcal{A}_{\alpha j}\overline{\mathcal{A}_{\alpha j}} = N$. Then (2.5) can be written as

$$\frac{u_{\alpha j,n} + v_{\alpha j,n}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n. \quad (3.1)$$

Now taking surd-conjugate of (3.1) we get $\frac{u_{\alpha j,n} - v_{\alpha j,n}\sqrt{D}}{k} = \overline{\mathcal{A}_{\alpha j}}\gamma^{-n}$. Adding this with (3.1) we now have

$$u_{\alpha j,n} = \frac{k}{2} \left\{ \mathcal{A}_{\alpha j}\gamma^n + \overline{\mathcal{A}_{\alpha j}}\gamma^{-n} \right\} = \frac{k}{2} \left\{ \mathcal{A}_{\alpha j}\gamma^n + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} \right\}.$$

Since $u_{\alpha j,n} \leq Z < u_{\alpha j,n+1}$, for some n , we get

$$\mathcal{A}_{\alpha j}\gamma^n + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} \leq \frac{2Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+1} + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n-1}.$$

Also, since $\frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} > 0$, $\overline{\mathcal{A}_{\alpha j}} = \frac{N}{\mathcal{A}_{\alpha j}} < \mathcal{A}_{\alpha j}$ and $\gamma^{-1} < \gamma$, we have $\mathcal{A}_{\alpha j}\gamma^n < \frac{2Z}{k} < 2\mathcal{A}_{\alpha j}\gamma^{n+1}$. Now $\gamma = x_1 + y_1\sqrt{D} > 2$. Then we have $\mathcal{A}_{\alpha j}\gamma^n < \frac{2Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+2}$. This implies $n < \frac{\log 2Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} < n + 2$. Since

n is an integer, we get $n \leq \left\lfloor \frac{\log 2Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} \right\rfloor \leq n + 1$. This implies

$$n = \left\lfloor \frac{\log 2Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} \right\rfloor \text{ or } n = \left\lfloor \frac{\log 2Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} \right\rfloor - 1.$$

Thus $S(Z) = \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \left\lfloor \frac{\log 2Z - \log \mathcal{A}_{ij} - \log k}{\log \gamma} \right\rfloor + c$, where $c = 0$ or -1 .

Then $S(Z) = \frac{1}{\log \gamma} \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} (\log 2Z - \log \mathcal{A}_{ij} - \log k + c + c')$, where $0 \leq c' < 1$. Now considering $c + c' = C'$, we have $-1 \leq C' < 1$. We can now write $S(Z) = \frac{\beta\eta}{\log \gamma} \log(2Z/k) - \frac{1}{\log \gamma} \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \log \mathcal{A}_{ij} + \beta\eta C'$.

If we write $C = \beta\eta C'$, then we get $-\beta\eta \leq C < \beta\eta$ and

$$S(Z) = \delta \left\{ \beta\eta \log(2Z/k) - \log \left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij} \right) \right\} + C.$$

Finally, we find the value of $T(Z)$.

Theorem 3.3. $T(Z) = \delta \left\{ \beta \eta \log(2\sqrt{D}Z/k) - \log\left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij}\right) \right\} + C$, where C is the effective constant such that $0 \leq C < 2\beta\eta$.

Proof. To find the value of $T(Z)$, we first find the number of positive solutions of (1.3) where the values $v_{ij,n}$ of V do not exceed Z for some fixed class $i = \alpha$ ($1 \leq j \leq \eta$). Now by (2.5) since $\frac{u_{\alpha j,n} + v_{\alpha j,n}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n$ and $\frac{u_{\alpha j,n} - v_{\alpha j,n}\sqrt{D}}{k} = \overline{\mathcal{A}_{\alpha j}}\gamma^{-n}$, on subtraction, we get

$$v_{\alpha j,n} = \frac{k}{2\sqrt{D}} \left\{ \mathcal{A}_{\alpha j}\gamma^n - \overline{\mathcal{A}_{\alpha j}}\gamma^{-n} \right\} = \frac{k}{2\sqrt{D}} \left\{ \mathcal{A}_{\alpha j}\gamma^n - \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} \right\}.$$

Since $v_{\alpha j,n} \leq Z < v_{\alpha j,n+1}$, for some n , we have

$$\mathcal{A}_{\alpha j}\gamma^n - \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} \leq \frac{2\sqrt{D}Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+1} - \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n-1}.$$

Thus, we have $\mathcal{A}_{\alpha j}\gamma^{n-1} \leq \frac{2\sqrt{D}Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+1}$. This implies

$$n - 1 < \frac{\log 2\sqrt{D}Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} < n + 1.$$

Since n is an integer, we have

$$n = \left\lfloor \frac{\log 2\sqrt{D}Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} \right\rfloor \text{ or } n = \left\lfloor \frac{\log 2\sqrt{D}Z - \log \mathcal{A}_{\alpha j} - \log k}{\log \gamma} \right\rfloor + 1.$$

Thus $T(Z) = \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \left\lfloor \frac{\log 2\sqrt{D}Z - \log \mathcal{A}_{ij} - \log k}{\log \gamma} \right\rfloor + c$, where $c = 0$ or 1 . Then

$$T(Z) = \frac{1}{\log \gamma} \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} (\log 2\sqrt{D}Z - \log \mathcal{A}_{ij} - \log k + c + c'),$$

where $0 \leq c' < 1$. Considering $c + c' = C'$, we have $0 \leq C' < 2$. Thus, we

now write $T(Z) = \frac{\beta\eta}{\log \gamma} \log(2\sqrt{D}Z/k) - \frac{1}{\log \gamma} \sum_{i=1}^{\beta} \sum_{j=1}^{\eta} \log \mathcal{A}_{ij} + \beta\eta C'$.

If we write $C = \beta\eta C'$, then we get $0 \leq C < 2\beta\eta$ and

$$T(Z) = \delta \left\{ \beta \eta \log(2\sqrt{D}Z/k) - \log\left(\prod_{i=1}^{\beta} \prod_{j=1}^{\eta} \mathcal{A}_{ij}\right) \right\} + C.$$

The Following interesting conclusions are now an easy consequence from these theorems.

Corollary 3.4. $T(Z) - S(Z) \approx \delta\beta\eta \log \sqrt{D}$.

Corollary 3.5. If the solutions $\frac{u_{ij,n} + v_{ij,n}\sqrt{D}}{k}$ of $U^2 - DV^2 = k^2N$ are considered as lattice points within the square $[0, Z] \times [0, Z]$, then density of these lattice points is zero.

This follows from the fact that $\lim_{n \rightarrow \infty} \frac{\log Z}{Z} = 0$.

4. Conclusions

In this paper, we derived the necessary and sufficient condition for any two solutions of $U^2 - DV^2 = k^2N$ to belong to the same class and the bounds for the values of u, v occurring in the fundamental solution. We also derived an explicit formula which gives all its positive solutions. We further obtained some interesting recurrence relations connecting the values of u, v . Finally, we obtained the results for total number of its positive solutions not exceeding any given positive real number Z .

References

- [1] Andreescu Titu, Andrica Dorin, Cucurezeanu Ion. An Introduction to Diophantine Equations. Birkhäuser Boston Inc, Secaucus, United States. 2010.
- [2] Burton D. M. Elementary Number Theory. Tata Mc Graw–Hill Pub. Co. Ltd., New Delhi. 2007.
- [3] Kaplan P., Williams K. S. Pell's Equation $x^2 - Dy^2 = -1, -4$ and continued fractions. Journal of Number Theory. 23, 169 – 182, 1986.
- [4] LeVeque William J. Topics in Number Theory, Vol. I. Addition-Wesley Pub. Co., Inc. 137 – 158, 1956.
- [5] Madni Bilkis M., Shah Devbhadra V. Alternate proofs for the infinite number of solutions of Pell's equation. International Journal of Engineering, Science and Mathematics. 7 (4), 255 – 259, April 2018.
- [6] Matthews K. The Diophantine Equation $x^2 - Dy^2 = N, D > 0$. Expositiones Mathematicae. 18, 323 – 331, 2000.
- [7] Mollin R. A., Poorten A. J., Williams H. C. Halfway to a solution of $x^2 - Dy^2 = -3$. Journal de Theorie des Nombres Bordeaux. 6, 421 – 457, 1994.
- [8] Shah Devbhadra V. On the solutions of Diophantine Equations $x^2 - Dy^2 = \pm 4N$. Journal of V.N.S.G. University. 4 – B, 56 – 65, 2007.
- [9] Steuding Jörn. Diophantine Analysis. Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton. 2005.
- [10] Stevenhagen P. A density conjecture for the negative Pell equation. Computational Algebra and Number Theory, Mathematics and its Applications. 325, 1995.
- [11] Stolt Bengt. On the Diophantine equation $u^2 - Dv^2 = \pm 4N$, Part I. Arkiv för Matematik. 2 (1), 1 – 23, 1951.
- [12] Stolt Bengt. On the Diophantine equation $u^2 - Dv^2 = \pm 4N$, Part II. Arkiv för Matematik. 2 (10), 251 – 268, 1951.

- [13] Tekcan A. The Pell Equation $x^2 - Dy^2 = \pm 4$. Applied Mathematical Sciences. 1 (8), 363 – 369, 2007.
- [14] Telang S. G. Number Theory. Tata Mc Graw–Hill Pub. Co. Ltd., New Delhi. 2004.

On the traversability of near common neighborhood graph of a graph

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Abstract

The near common neighborhood graph of a graph G , denoted by $ncn(G)$ is defined as the graph on the same vertices of G , two vertices are adjacent in $ncn(G)$, if there is at least one vertex in G not adjacent to both the vertices. In this research paper, the conditions for $ncn(G)$ to be disconnected are discussed and characterization for graph $ncn(G)$ to be hamiltonian and eulerian are obtained.

Keywords: Near common neighborhood graph; Hamiltonian graph; Eulerian graph.

2020 AMS subject classifications: 05C07, 05C10, 05C38, 05C60, 05C76. ¹

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¹Received on July 9, 2021. Accepted on November 23, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.626 . ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Let G be a graph. The near common neighborhood graph of G denoted by $ncn(G)$ is a graph with the same vertices as G in which two vertices u and v are adjacent if there exists at least one vertex $w \in V(G)$ not adjacent to both of u and v [Al-Kenani et al., 2016].

A cycle in a graph G that contains every vertex of G is called spanning cycle of G . Thus a hamiltonian cycle of G is a spanning cycle of G . A hamiltonian graph is a graph that contains a hamiltonian cycle.

An euler trail in a graph G is a trail that contains every edge of that graph. An euler tour is a closed euler trail. An eulerian graph is a graph that has an euler tour.

The graphs considered in this paper are simple, undirected and connected with vertex set $v_i \in V(G)$, $i = 1, 2, 3, \dots, n$. Let $deg(v_i)$ be the degree of vertices of G . Basic terminologies are referred from [Harary, 1969].

The common neighborhood graph (congraph) of G [Zadeh et al., 2014] which is exactly the opposite of near common neighborhood is denoted by $con(G)$ is a graph with vertex set, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G . Here the common neighborhood of some composite graphs are computed and also the relation between hamiltonicity of graph G and $con(G)$ is investigated. [Hamzeh et al., 2018] computed the congraphs of some composite graphs and also results have been calculated for graph valued functions. [Sedghi et al., 2020] obtained the characteristics of congraphs under graph operations and relations between Cayley graphs and its congraphs.

[Al-Kenani et al., 2016] studied near common neighborhood of a graph and obtained results for paths, cycles and complete graphs. Motivated by the results on [Zadeh et al., 2014], [Hamzeh et al., 2018] and [Sedghi et al., 2020], in this paper, the conditions for $ncn(G)$ to be disconnected are discussed and also theorems stating necessary and sufficient conditions for a graph $ncn(G)$ to possess hamiltonian and eulerian cycle are studied.

2 Preliminaries

Below mentioned some important results are used through out the paper for proving the theorems.

Proposition 2.1. [Al-Kenani et al., 2016] For any path P_n ,

$$ncn(P_n) = \begin{cases} \overline{K_n}, & \text{if } n = 2, 3 \\ 2K_2, & \text{if } n = 4 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.2. [Al-Kenani et al., 2016] For any path C_n ,

$$ncn(C_n) = \begin{cases} \overline{K_n}, & \text{if } n = 3, 4 \\ C_5, & \text{if } n = 5 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.3. [Al-Kenani et al., 2016]. For any complete graph K_n and totally disconnected graph $\overline{K_n}$, we have

$$\begin{aligned} 1. ncn(K_n) &= \overline{K_n} \\ 2. ncn(\overline{K_n}) &= K_n, n \geq 3 \end{aligned}$$

Theorem 2.1. [Singh, 2010]. Let G be a simple graph with $p \geq 3$ and $\delta \geq p/2$. Then G is Hamiltonian.

Theorem 2.2. [Singh, 2010] A nonempty connected graph is Eulerian if and only if it has no vertices of odd degree.

Remark 2.1. [Singh, 2010] The complete graph K_p , for $p \geq 3$, is always Hamiltonian.

3 Results

Theorem 3.1. If G is a graph with n vertices, then $ncn(G)$ is disconnected if any one of the following conditions holds

1. G is $P_n, C_n, n \leq 4$
2. G is $K_n, n \geq 3$
3. G has $\Delta(G) = n - 1$
4. G is a graph with two complete graphs connected by bridge
5. G is $K_n \bullet P_t, n \geq 3, t \leq 3$

Proof. The proof of the theorem is constructed by considering the following cases.

Case 1. Suppose $G=P_n$ or $C_n, n \leq 4$.

We consider the following two subcases.

Subcase 1.1. Suppose $G=P_n, n \leq 4$.

If $n = 2, 3, 4$, then by the proposition 2.1, $ncn(G)$ is disconnected.

Subcase 1.2. Suppose $G=C_n, n \leq 4$.

If $n = 3, 4$, then from the proposition 2.2, $ncn(G)$ is disconnected.

Case 2. Suppose $G=K_n, n \geq 3$.

Then from the proposition 2.3, $ncn(G)$ is disconnected.

Case 3. Let G be a graph with vertex set $V(G) = \{v_i | i \in N\}$ and $\Delta(G) = n - 1$. Near common neighbourhood graph $ncn(G)$ is a graph with same vertices v_i as

G . The vertices v_i and v_j , $j = 1, 2, 3, \dots, n, i \neq j$ of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both v_i and v_j .

Since $\Delta(G) = n - 1$ in G (that is v_i is adjacent to all other vertices of G), there does not exist any nonadjacent vertex for v_i and thus v_i cannot be connected to any vertex of $ncn(G)$. This results $ncn(G)$ into disconnected.

Case 4. Let G be a graph with two complete graphs K_m and K_n connected by bridge. Let $v_i \in V(G)$, $i = 1, 2, 3, \dots, m$ be the vertex set of K_m and $v_j \in V(G)$, $j = m + 1, m + 2, m + 3, \dots, n$ be the vertex set of K_n , where m and n are the vertices of bridge.

As G consists of two complete graphs, vertices v_i of K_m and v_j of K_n are respectively mutually adjacent. Nonadjacent vertices for all the vertices $v_i \in K_m$ exists in K_n and for $v_j \in K_n$ exists in K_m . Thus the vertices v_i of K_m and v_j of K_n are mutually connected in $ncn(G)$. This produces the disconnected graph with two components. Further, the end points of bridge are also mutually adjacent to all the vertices of K_m and K_n respectively. Hence nonadjacent vertex does not exist for end points of bridge. This produces the graph $ncn(G)$ into disconnected.

Case 5. Let G be a $K_n \bullet P_t$, $n \geq 3$ and $t \leq 3$.

$ncn(G)$ has the same vertices as G . Two vertices of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both the vertices.

Let v_i , $i = 1, 2, 3, \dots, n, n + 1, n + 2$ be the vertex set of $K_n \bullet P_t$. The vertices of K_n are $v_1, v_2, v_3, \dots, v_n$ and vertices of P_t are v_n, v_{n+1}, v_{n+2} . The vertex v_n is the common vertex which connects K_n and P_t .

We consider the following subcases.

Subcase 5.1 Suppose $t = 2$ that is P_t is path with two vertices, then $G = K_n \bullet P_2$. Since $\Delta(G) = n - 1$, there exists at least one vertex which is adjacent to all the other vertices of G . From theorem 3.1 (Case 3), $ncn(G)$ is disconnected.

Subcase 5.2 Suppose $t = 3$ that is $t = n, n + 1, n + 2$.

In G , all the pairs of vertices of K_n have the nonadjacent vertex as v_{n+2} and can be mutually connected in $ncn(G)$. Also the vertices of P_t , v_{n+1} and v_{n+2} have the nonadjacent vertices in K_n and can be connected in $ncn(G)$. As there does not exist any nonadjacent vertices for the pair of common vertex n and the vertices of P_t in G , they cannot be connected in $ncn(G)$. This produces the the graph $ncn(G)$ with two components. Thus $ncn(G)$ is disconnected.

□.

Theorem 3.2. For any graph G , $ncn(G)$ is hamiltonian if and only if

1. G contains all the vertices of $\deg(v_i) < n - 1$ except for $C_4 \bullet P_2$ and G is a graph with two complete graphs connected by bridge.
2. $G = P_n$ or C_n , $n \geq 5$.
3. G is $K_n \bullet P_t$, $n \geq 3$, $t \geq 4$.

Proof. Let G be graph with vertex set $V(G)$. Suppose $ncn(G)$ is hamiltonian.

In light of the above theorem 3.1 that $ncn(G)$ is disconnected if G is $P_n, C_n, n \leq 4, K_n, n \geq 3, G$ has $\Delta(G) = n - 1, G$ is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t, n \geq 3, t \leq 3$.

Now we consider the graphs for which $ncn(G)$ is connected.

Case 1. Suppose G contains all the vertices of $deg(v_i) < n - 1$.

Let G be a graph $v_i \in V(G)$ vertices with $degree(v_i) < n - 1$ (v_i is not adjacent to all the vertices) and $ncn(G)$ be the graph with same set of vertices as G .

As $deg(v_i) < n - 1$ in G , there exists at least one nonadjacent vertex for any pair of vertices of G . Hence by definition of $ncn(G)$, those vertices in $ncn(G)$ can be connected which produces connected $ncn(G)$ graph.

Further, since for each pair of vertices of G there exists a nonadjacent vertex, $ncn(G)$ contains a cycle and $\delta \geq n/2$. From the theorem 2.1 $ncn(G)$ is hamiltonian.

Next suppose $G = C_4 \bullet P_2$. Let $v_i, i = 1, 2, 3, 4, 5$ be the vertex set of $C_4 \bullet P_2$ with one common vertex between C_4 and P_2 . Among the four vertices of C_4 of G , three vertices (except the common vertex) can be connected mutually adjacent as they have nonadjacent vertex (not common vertex) in P_2 .

Similarly, a vertex of P_2 which is not common can be connected with only three vertices of C_4 in $ncn(G)$ as there exists a nonadjacent vertex for these each pair of vertices. Whereas the common vertex can be connected only with a vertex of P_2 in $ncn(G)$ which is not common, since there exists a nonadjacent vertex in C_4 for this pair and there does not exist nonadjacent vertex for the pair of vertices with C_4 .

This results $ncn(G)$ into connected graph with one pendent vertex and consequently does not contain hamiltonian cycle.

Thus, $ncn(C_4 \bullet P_2)$ is connected but not hamiltonian.

Case 2. Suppose G is P_n or $C_n, n \geq 5$.

Let G be a P_n or $C_n, n \geq 5$. From the propositions 2.1 and 2.2, $ncn(P_n)$ and $ncn(C_n), n = 5, 6$ are connected which contains a cycle and $\delta \geq n/2$. For $n \geq 7$, $ncn(G)$ is K_n . From the theorem 2.1 and remark 2.1, $ncn(G)$ is hamiltonian.

Case 3. Suppose G is $K_n \bullet P_t, n \geq 3, t \geq 4$.

Let G be a $K_n \bullet P_t, n \geq 3, t \geq 4$, where n is the number of vertices of K_n and t is the number of vertices of P_t .

Let $v_i \in V(G), i = 1, 2, 3, \dots, n$ be the vertex set of K_n and $v_j \in V(G), j = n, n + 1, n + 2, \dots, t$ be the vertex set of P_t , where n is the common vertex of K_n and P_t .

As there is increase in the number of vertices (path length) in P_t of G , there exists a nonadjacent vertex for each pair of vertices of K_n and vertices of P_t , which produces connected graph $ncn(G)$ with cycles and also $\delta \geq n/2$. From the theorem 2.1, $ncn(G)$ is hamiltonian.

Converse is obvious.

□

Theorem 3.3. For any graph G , $ncn(G)$ is eulerian if only if G is

1. $P_n, n \geq 7$.
2. $K_n \bullet P_t, n \geq 3, t \geq 5$.
3. $G = C_n, n = 5, 6$.

Proof. Let G is a graph with vertex set $v_i \in G, i = 1, 2, 3, \dots, n$.

Suppose $ncn(G)$ is eulerian, then degree of each v_i of $ncn(G)$ is even. From the theorems 3.1 and 3.2, $ncn(G)$ is disconnected if G is $P_n, C_n, n \leq 4, K_n, n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge, G is $K_n \bullet P_t, n \geq 3, t \leq 3$ and is connected only if G contains all the vertices of $\deg(v_i) < n - 1$, $G = P_n$ or $C_n, n \geq 5$ and G is $K_n \bullet P_t, n \geq 3, t \geq 4$.

From the proposition 2.1, $ncn(G = P_n)$ is K_n with even degree; $n \geq 7$, where $n = 2s + 1, s = 2, 3, 4, \dots$

From the proposition 2.2, $ncn(G = C_n), n = 5, 6$ is K_n of even degree.

From the theorem 3.2, $ncn(G = K_n \bullet P_t)$ is K_n with even degree; $n \geq 3, t \geq 5$, where $n = 2s + 1, s = 1, 2, 3, \dots$

Hence from the theorem 2.2, $ncn(G)$ is eulerian.

Conversely, $ncn(G)$ is a graph with same vertices as G .

From theorem 3.1, $ncn(G)$ is disconnected if G is $P_n, C_n; n \leq 4, K_n; n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t; n \geq 3, t \leq 3$ in all other cases it is connected.

We consider the following cases.

Case 1. Suppose $ncn(G)$ is $K_n, n = 1, 2, 3, \dots, n$, then from the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $G = P_n, C_n, n \geq 5$ and $K_n \bullet P_t; n \geq 3, t \geq 5$.

Subcase 1.1 Suppose $G = P_n$ or C_n , then from the propositions 2.1 and 2.2, $ncn(G)$ is $K_n, n \geq 7$. The degree of each vertex of K_n is even and $n = 2s + 1, s = 1, 2, 3, \dots, n$. From theorem 2.2, $ncn(G)$ is eulerian.

Subcase 1.2. Suppose $G = K_n \bullet P_t, n \geq 3, t \geq 5$, then from the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3, t \geq 5$. The degree of K_n is even if n is odd.

From the theorem 2.2, $ncn(G)$ is eulerian.

Case 2. Suppose $G = C_n, n = 5, 6$.

Subcase 2.1 Suppose $G = C_n, n = 5$, then from the proposition 2.2, $ncn(G)$ is C_5 or 2-regular graph. From the theorem 2.2 $ncn(G)$ is eulerian.

Subcase 2.2 Suppose $G = C_n, n = 6$, then from the proposition 2.2, $ncn(G)$ is 2-regular graph. Hence from the theorem 2.2, $ncn(G)$ is eulerian.

Case 3. Suppose $G = K_n \bullet P_t, n \geq 3, t \geq 5$.

From the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3, t \geq 5$.

The degree of K_n is even if n is odd. From the theorem 2.2, $ncn(G)$ is eulerian.

4 Conclusions

In this paper the study on near common neighborhood graph of a graph is extended and various general conditions for which $ncn(G)$ to be disconnected are discussed. It is disconnected for the graphs $P_n, C_n; n \leq 4, K_n; n \geq 3, G$ has $\Delta(G) = n - 1, G$ is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t; n \geq 3, t \leq 3$ in all other cases it is connected. We have also obtained the necessary and sufficient condition for $ncn(G)$ to possess hamiltonian and eulerian cycles. It contains hamiltonian cycle whenever it is connected except for $C_4 \bullet P_2$. It contains eulerian cycle if $ncn(G)$ is K_n with odd vertices and G is $C_n, n = 5, 6$.

References

- A.N Al-Kenani, A. Alwardi, and O.A. Al-Attas. On the near-common neighborhood graph of a graph. *International Journal of Computer Applications*, 146 (1):1–4, 2016.
- A. Hamzeh, A. Iranmanesh, S.H. Zadeh, M.A. Hosseinzadeh, and I. Gutman. On common neighborhood graphs ii. *Iranian Journal of Mathematical Chemistry*, 9(1):37–46, 2018.
- F. Harary. *Graph Theory*. Addison-Wesley Mass, Reading, New York, 1969.
- S. Sedghi, D.W. Lee, and N. Shobe. Characteristics of common neighborhood graph under graph operations and on cayley graphs. *Iranian Journal of Mathematical Sciences and Informatics*, 15(2):13–20, 2020.
- G.S. Singh. *Graph Theory*. PHI Learning Private Limited, New York, 2010.
- S.H Zadeh, A. Iranmanesh, A. Hamzeh, and M.A Hosseinzadeh. On the common neighborhood graphs. *Electronic Notes in Discrete Mathematics*, 45:51–56, 2014.

Properties of an anti-vague filter in *BL*- algebras

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Abstract

In this paper, the concept of an anti-vague filter of a *BL*-algebra is introduced with suitable illustration, and also obtained some related properties. Further, we have investigated some more equivalent conditions of anti-vague filter.

Keywords: *BL*-algebra; filter; implicative filter; vague set; vague filter; anti-vague filter

2010 AMS subject classification‡: 03B50; 03B52; 03E72; 06D35.

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‡ Received on August 28, 2021. Accepted on November 13, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.650. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Hájek [5] introduced the idea of *BL*-algebras as the algebraic structure for his Basic Logic. The interval $[0, 1]$ endowed with the structure induced by a continuous t - norm is a well-known example of *BL*- algebra. The *MV*-algebras, on the other hand, are one of the most well-known groups of *BL*-algebras, having been introduced by Chang [2] in 1958. In 1965, Zadeh [12] introduced the concept of a fuzzy set. The flaw in fuzzy sets is that they only have one feature, which means they cannot convey supporting and opposing data. Gau and Buehrer [4] introduced the principle of vague set in 1993 as a result of this. The authors [7, 8, 9, 10] discussed the vague filter, implicative filter, prime, and Boolean implicative filters of *BL*- algebras, as well as some of their properties.

The frame work of this study is constructed as follow: some basic observations connected to anti-vague filter are provided in “Preliminaries”. “Anti-vague filter” presents the new notions of anti-vague filter in *BL*-algebra and investigated some related properties, also derived some equivalent conditions for an anti-vague filter to be a vague filter. Finally, the conclusion is presented in “Conclusion”.

2. Preliminaries

In this section, we will go through some basic *BL*-algebra, filter, and vague set concepts, as well as their properties, which will help in the development of the main results.

Definition 2.1[5] A *BL*-algebra is an algebra $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

- (i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
- (ii) $(A, *, 1)$ is a commutative monoid,
- (iii) $*$ and $\rightarrow$ form an adjoint pair, that is, $z \leq x \rightarrow y$ if and only if $x * z \leq y$ for all $x, y, z \in A$,
- (iv) $x \wedge y = x * (x \rightarrow y)$,
- (v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Proposition 2.2[6] In a *BL*- algebra A , the following properties are hold for all $x, y, z \in A$,

- (i) $y \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$,
- (ii) $1 \rightarrow x = x$,
- (iii) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (iv) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
- (v) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$,

- (vi) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$,
- (vii) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
- (viii) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (ix) $x \leq (x \rightarrow y) \rightarrow y$,
- (x) $x * (x \rightarrow y) = x \wedge y$,
- (xi) $x * y \leq x \wedge y$
- (xii) $x \rightarrow y \leq (x * z) \rightarrow (y * z)$,
- (xiii) $x * (y \rightarrow z) \leq y \rightarrow (x * z)$,
- (xiv) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$,
- (xv) $(x * x^-) = 0$.

Note. In the sequel, we shall use A to denote as BL - algebras and the operation $\vee, \wedge, *$ have priority towards the operations " $\rightarrow$ ".

Note. In a BL - algebra A , we can define $x^- = x \rightarrow 0$ for all $x \in A$.

Definition 2.3[13] A filter of a BL - algebra A is a non-empty subset F of A such that for all $x, y \in A$,

- (i) If $x, y \in F$, then $x * y \in F$,
- (ii) If $x \in F$ and $x \leq y$, then $y \in F$.

Proposition 2.4[13] Let F be a non-empty subset of a BL - algebra A . Then, F is a filter of A if and only if the following conditions are hold

- (i) $1 \in F$,
- (ii) $x, x \rightarrow y \in F$ implies $y \in F$.

A filter F of a BL -algebra A is proper if $F \neq A$.

Definition 2.5[1, 3, 4] A vague set S in the universe of discourse X is characterized by two membership functions given by

- (i) A truth membership function $t_S: X \rightarrow [0, 1]$,
- (ii) A false membership function $f_S: X \rightarrow [0, 1]$.

Where $t_S(x)$ is lower bound of the grade of membership of x derived from the 'evidence for x ', and $f_S(x)$ is a lower bound of the negation of x derived from the 'evidence against x ' and $t_S(x) + f_S(x) \leq 1$. Thus the grade of membership of x in the vague set S is bounded by a subinterval $[t_S(x), 1 - f_S(x)]$ of $[0, 1]$. The vague set S is written as $S = \{(x, [t_S(x), f_S(x)]) / x \in X\}$, where the interval $[t_S(x), 1 - f_S(x)]$ is called the value of x in the vague set S and denoted by $V_S(x)$.

Definition 2.6[4] A vague set S of a set X is called

- (i) the zero vague set of X if $t_S(x) = 0$ and $f_S(x) = 1$ for all $x \in X$,
- (ii) the unit vague set of X if $t_S(x) = 1$ and $f_S(x) = 0$ for all $x \in X$,

- (iii) the α - vague set of X if $t_S(x) = \alpha$ and $f_S(x) = 1 - \alpha$ for all $x \in X$, where $\alpha \in (0, 1)$.

Definition 2.7[4] Let S be a vague set of X with truth membership function t_S and the false membership function f_S . For all $\alpha, \beta \in [0, 1]$, the (α, β) -cut of the vague set X is crisp subset $S_{(\alpha, \beta)}$ of the set X by $S_{(\alpha, \beta)} = \{V(x) \geq [\alpha, \beta] / x \in X\}$. Obviously, $S_{(0,0)} = X$.

Definition 2.8[4] Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. Now, we define refined maximum ($rmax$) and " $\geq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0, 1]$ as $rmax(D_1, D_2) = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$. Similarly, we can define $\leq, =$ and $rmin$.

3. Anti-Vague Filter

In this section, we introduce the notion of an anti-vague filter of BL -algebra with illustration. Moreover, we discuss some related properties.

Definition 3.1 Let S be vague set of a BL -algebra A is called an anti vague filter of A if it satisfies the following axioms

- (i) $V_S(1) \leq V_S(x)$,
- (ii) $V_S(y) \leq rmax\{V_S(x \rightarrow y), V_S(x)\}$ for all $x, y \in A$.

Proposition 3.2 Let S be vague set of BL -algebra A . S is an anti vague filter of A if and only if the following hold if for all $x, y \in A$,

- (i) $t_S(1) \leq t_S(x)$ and $1 - f_S(1) \leq 1 - f_S(x)$,
- (ii) $t_S(y) \leq \max\{t_S(x \rightarrow y), t_S(x)\}$ and $1 - f_S(y) \leq \max\{1 - f_S(x \rightarrow y), 1 - f_S(y)\}$.

Proof: Let S be an anti-vague filter of A . Then from (i) of definition 3.1 and the definition of V_S , we have (i) straight forward. From (ii) of definition 3.1 and the definition of V_S , (ii) is obvious. ■

The following is the example of definition 3.1 and proposition 3.2.

Example 3.3 Let $A = \{0, a, b, 1\}$. The binary operations ' $*$ ' and ' $\rightarrow$ ' give by the following tables 3.1 and 3.2:

Table3.1: ‘ $*$ ’ Operator

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	b
b	0	a	b	b
1	0	a	b	1

Table 3.2: ‘ $\rightarrow$ ’ Operator

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a BL - algebra. Define a vague set S of A as follows:

$$S = \{(1, [0.2, 0.7]), (a, [0.3, 0.5]), (b, [0.3, 0.5]), (0, [0.2, 0.7])\}.$$

It is easily verified that S is an anti-vague filter of A and satisfy the conditions (i) and (ii) of proposition 3.2.

Proposition 3.4 Every anti-vague filter S of BL - algebra A is order preserving.

Proof: Let S be an anti-vague filter of BL -algebra A .

Then, we prove that if $x \leq y$, then $V_S(x) \geq V_S(y)$ for all $x, y \in A$.

From (ii) of the proposition 3.2, we have,

$$t_S(y) \leq \max\{t_S(x \rightarrow y), t_S(x)\}$$

$$= \max \{t_S(1), t_S(x)\},$$

[From (iii) of proposition 2.2]

Also, we have $1 - f_S(y) \leq \max\{1 - f_S(x \rightarrow y), 1 - f_S(y)\}$.

From (i) of the proposition 3.2, we have $t_S(1) \leq t_S(x)$ and $1 - f_S(1) \leq 1 - f_S(x)$. Thus, $t_S(y) \leq \max\{t_S(x), 1 - f_S(y)\}$

$\leq 1 - f_S(y)$, and so

$$\begin{aligned} V_S(y) &= [t_S(y), 1 - f_S(y)] \\ &\leq [t_S(x), 1 - f_S(x)] \\ &= V_S(x). \end{aligned}$$

Hence $V_S(x) \geq V_S(y)$. ■

Proposition 3.5 Let S be a vague set of BL- algebra A , S be an anti-vague filter of A if and only if $x \rightarrow (y \rightarrow z) = 1$ implies $V_S(z) \leq rmax\{V_S(x), V_S(y)\}$ for all $x, y, z \in A$.

Proof: Let S be an anti-vague filter of BL-algebra A .

Then, from (ii) of the definition 3.1, we have

$$V_S(z) \geq rmax\{V_S(z \rightarrow y), V_S(y)\} \text{ for all } x, y, z \in A.$$

$$\text{Now, } V_S(z \rightarrow y) \leq rmax\{V_S(x \rightarrow (y \rightarrow z)), V_S(x)\}.$$

If $x \rightarrow (y \rightarrow z) = 1$, then we have

$$V_S(z \rightarrow y) \leq rmax\{V_S(1), V_S(x)\} = V_S(x).$$

$$\text{So, } V_S(z) \leq rmax\{V_S(x), V_S(y)\}.$$

Conversely, since $x \rightarrow (x \rightarrow 1) = 1$ for all $x \in A$.

$$\begin{aligned} \text{Then } V_S(1) &\leq rmax\{V_S(x), V_S(x)\} \\ &= V_S(x). \end{aligned}$$

On the other hand, from $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$.

$$\text{It follows that } V_S(y) \leq rmax\{V_S(x \rightarrow y), V_S(x)\}.$$

From the definition 3. 1, S is the anti vague filter of A . ■

From (i) of the proposition 2.2, and the proposition 3.5, we have the following.

Corollary 3.6 Let S be vague set of BL - algebra A , S be an anti vague filter of A if and only if $x * y \leq z$ or $y * x \leq z$ implies $V_S(z) \leq rmax\{V_S(x), V_S(y)\}$ for all $x, y, z \in A$.

Proposition 3.7 Let S be a vague set of BL - algebra A , S be an anti vague filter of A if and only if (i) $x \leq y$, then $V_S(x) \geq V_S(y)$,

(ii) $V_S(x * y) \leq rmax\{V_S(x), V_S(y)\}$ for all $x, y \in A$.

Proof: Let S be an anti vague filter of BL - algebra A . Then, from the proposition 3.4, we have $x \leq y$, $V_S(x) \geq V_S(y)$.

Since $x * y \leq x * y$ and corollary 3.6, we have $V_S(x * y) \leq rmax\{V_S(x), V_S(y)\}$.

Conversely, let S be a vague set and satisfies (i) and (ii). For all $x, y, z \in A$, if $x * y \leq z$, then from (i) and (ii), we get $V_S(z) \leq rmax\{V_S(x), V_S(y)\}$.

From corollary 3.6, we have S is an anti vague filter. ■

Proposition 3.8 Let S be a vague set of BL - algebra A . Let S be an anti vague filter of A . The following holds for all $x, y, z \in A$,

(i) If $V_S(x \rightarrow y) = V_S(1)$, then $V_S(x) \geq V_S(y)$,

(ii) $V_S(x \vee y) = rmax\{V_S(x), V_S(y)\}$,

(iii) $V_S(x * y) = rmax\{V_S(x), V_S(y)\}$,

(iv) $V_S(1) = rmax\{V_S(x), V_S(x^-)\}$,

(v) $V_S(x \rightarrow z) \leq rmax\{V_S(x \rightarrow y), V_S(y \rightarrow z)\}$,

(vi) $V_S(x \rightarrow y) \geq V_S(x * z \rightarrow y * z)$,

(vii) $V_S(x \rightarrow y) \geq V_S((y \rightarrow z) \rightarrow (x \rightarrow z))$,

(viii) $V_S(x \rightarrow y) \geq V_S((z \rightarrow x) \rightarrow (z \rightarrow y))$.

Proof: (i) Let S be an anti vague filter of BL - algebra A .

Then, from the definition 3.1, and since $V_S(x \rightarrow y) = V_S(1)$.

We have $V_S(y) \leq rmax\{V_S(x), V_S(x \rightarrow y)\}$

$$= rmax\{V_S(x), V_S(1)\}$$

$$= V_S(x).$$

Thus, $V_S(x) \geq V_S(y)$.

(ii) Since $x \vee y \geq x$ and $x \vee y \geq y$.

From the proposition 3.4, we get $V_S(x \vee y) \leq rmax\{V_S(x), V_S(y)\}$.

From the definition 3.1 we have

$$\begin{aligned} V_S(x \vee y) &\leq rmax\{V_S(x \rightarrow (x \vee y)), V_S(x)\} \\ &= rmax\{V_S((x \rightarrow x) \vee (x \rightarrow y)), V_S(x)\} \\ &= rmax\{V_S(x \rightarrow y), V_S(x)\} \\ &\leq rmax\{rmax\{V_S(y \rightarrow (x \rightarrow y)), V_S(y)\}, V_S(x)\} \\ &= rmax\{rmax\{V_S(1), V_S(y)\}, V_S(x)\} \\ &= rmax\{V_S(y), V_S(x)\} \\ &= rmax\{V_S(x), V_S(y)\} \end{aligned}$$

Hence, $V_S(x \vee y) = rmax\{V_S(x), V_S(y)\}$.

(iii) From (ii) of proposition 3.7, we have

$$V_S(x * y) \leq rmax\{V_S(x), V_S(y)\}.$$

Since $x * y \geq x \vee y$, proposition 3.4, and (ii), we have

$$\begin{aligned} V_S(x * y) &\geq V_S(x \vee y) \\ &= rmax\{V_S(x), V_S(y)\}. \end{aligned}$$

Thus, $V_S(x * y) = rmax\{V_S(x), V_S(y)\}$.

(iv) From (iii), we have $rmax\{V_S(x), V_S(x^-)\} = V_S(x * x^-) = V_S(1)$.

Therefore, $V_S(1) = rmax\{V_S(x), V_S(x^-)\}$.

(v) From (iii) and proposition 3.4, since $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$,

we get $V_S((x \rightarrow y) * (y \rightarrow z)) \geq V_S((x \rightarrow z))$,

$$rmax\{V_S((x \rightarrow y)), V_S(y \rightarrow z)\} \geq V_S((x \rightarrow z)).$$

Therefore, we have $V_S(x \rightarrow z) \leq rmax\{V_S(x \rightarrow y), V_S(y \rightarrow z)\}$.

From the proposition 2.2 and (i) of proposition 3.7 we can prove (vi), (vii) and

(viii) easily. ■

Proposition 3.9 Let S be a vague set of BL - algebra A , S be an anti vague filter of A if and only if (i) $V_S(1) \leq V_S(x)$,

$$(ii) V_S((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq rmax\{V_S(x), V_S(y)\}$$

for all $x, y, \in A$.

Proof: Let S be an anti vague filter of A . By the definition 3.1, (i) is straight forward.

$$\text{Since, } V_S((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq rmax\{V_S((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)), V_S(y)\}. \quad (3.1)$$

Now, we have $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z) = x \vee (y \rightarrow z) \geq x$.

$$V_S((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \leq V_S(x). \quad [\text{from the proposition 3.4}] \quad (3.2)$$

Using (3.2) in (3.1), we have $V_S((x \rightarrow (y \rightarrow z)) \rightarrow z) \leq rmax\{V_S(x), V_S(y)\}$.

Conversely, suppose (i) and (ii) hold.

$$\begin{aligned} \text{Since } V_S(y) &= V_S(1 \rightarrow y) \\ &= V_S(((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow y) \\ &\leq rmax\{V_S(x \rightarrow y), V_S(y)\}. \end{aligned}$$

From (i), S is an anti vague filter of A . ■

Proposition 3.10 Intersection of two anti vague filters of A is also an anti vague filter of A .

Proof: Let U and W be two anti vague filters of A .

To Prove: $U \cap W$ is an anti vague filter of A .

For all $x, y, z \in A$ such that $z \leq x \rightarrow y$, then $z \rightarrow (x \rightarrow y) = 1$.

Since, U, W are two anti vague filters A , we have $V_U(y) \leq rmax\{V_U(z), V_U(x)\}$ and $V_W(y) \leq rmax\{V_W(z), V_W(x)\}$.

That is, $t_U(y) \leq \max\{t_U(z), t_U(x)\}$ and

$1 - f_U(y) \leq \max\{1 - f_U(z), 1 - f_U(x)\}$, $t_W(y) \leq \max\{t_W(z), t_W(x)\}$ and

$$1 - f_W(y) \leq \max\{1 - f_W(z), 1 - f_W(x)\}.$$

Since, $t_{U \cap W}(y) = \min\{t_U(y), t_W(y)\}$

$$\begin{aligned} &\leq \max \{ \max \{ t_U(z), t_U(x) \}, \max \{ t_W(z), t_W(x) \} \} \\ &= \max \{ \max \{ t_U(z), t_W(z) \}, \max \{ t_U(x), t_W(x) \} \} \\ &= \max \{ \max \{ t_{U \cap W}(z), t_{U \cap W}(x) \} \} \end{aligned}$$

$$\begin{aligned} \text{and } 1 - f_{U \cap W}(y) &= \max \{ 1 - f_U(y), 1 - f_W(y) \} \\ &\leq \max \{ \max \{ 1 - f_U(z), 1 - f_U(x) \}, \max \{ 1 - f_W(z), 1 - f_W(x) \} \} \\ &= \max \{ \max \{ 1 - f_U(z), 1 - f_W(z) \}, \max \{ 1 - f_U(x), 1 - f_W(x) \} \} \\ &= \max \{ \max \{ 1 - f_{U \cap W}(z), 1 - f_{U \cap W}(x) \} \}. \end{aligned}$$

Hence, $V_{U \cap W}(y) = [t_{U \cap W}(y), 1 - f_{U \cap W}(y)] \leq r\max\{V_{U \cap W}(z), V_{U \cap W}(x)\}$.

Thus $U \cap W$ is an anti vague filter of A . ■

Corollary 3.11 Let R_j be a family of anti vague filters of A , where $j \in I$, I is a index set, then $\bigcap_{j \in I} R_j$ is an anti vague filter of A .

Note: Union two anti vague filters of BL - algebra A need not be an anti vague filter of A .

Proposition 3.12 A ρ - vague set and zero vague set of a BL -algebra A are anti vague filters of A .

Proof: Let S be a ρ -vague set of BL -algebra A , and S be an anti vague filter of A .

Then, from the proposition 3.4, we have if $x \leq y$, then $V_S(x) \geq V_S(y)$ for all $x, y, \in A$.

To prove: $V_S(x * y) \leq r\max\{V_S(x), V_S(y)\}$ for all $x, y, \in A$.

$$\text{Now, } t_S(x * y) = \rho = \max\{\rho, \rho\} = \max \{ t_S(x), t_S(y) \} \quad (3.3)$$

$$\text{and } 1 - f_S(x * y) = \rho = \max\{\rho, \rho\} = \max \{ 1 - f_S(x), 1 - f_S(y) \} \text{ for all } x, y, \in A \quad (3.4)$$

From (3.3) and (3.4), we have $V_S(x * y) \leq r\max\{V_S(x), V_S(y)\}$.

Thus, ρ - vague set is an anti vague filter of A .

Similarly, we prove zero vague set is an anti vague of A . ■

Theorem 3.13 Let S be a vague set of BL -algebra A , S be an anti vague filter of A if and only if the set $S_{(\rho,\sigma)}$ is either empty or a filter of A for all $\rho, \sigma \in [0, 1]$, where $\rho \leq \sigma$.

Proof: Let S be an anti vague filter of BL -algebra A and $S_{(\rho,\sigma)} \neq \emptyset$ for all $\rho, \sigma \in [0, 1]$.

To prove: $S_{(\rho,\sigma)}$ is a filter of A .

If $x \leq y$ and $x \in S_{(\rho,\sigma)}$. From the proposition 3.12, we have $V_S(y) \leq V_S(x) \leq [\rho, \sigma]$ for all $x, y \in A$.

Thus, $y \in S_{(\rho,\sigma)}$.

If $x, y \in S_{(\rho,\sigma)}$, then $V_S(x)$ and $V_S(y) \leq [\rho, \sigma]$.

From (ii) of the proposition 3.7, we have $V_S(x * y) \leq rmax\{V_S(x), V_S(y)\} \leq [\rho, \sigma]$.

Thus, $x * y \in S_{(\rho,\sigma)}$. Hence $S_{(\rho,\sigma)}$ is a filter of A .

Conversely, if for all $\rho, \sigma \in [0, 1]$, the set $S_{(\rho,\sigma)}$ is either empty or a filter of A .

Let $t_S(x) = \rho_1$, $t_S(y) = \rho_2$, $1 - f_S(x) = \sigma_1$ and $1 - f_S(y) = \sigma_2$.

Put $\rho = \max\{\rho_1, \rho_2\}$ and $\sigma = \max\{1 - \sigma_1, 1 - \sigma_2\}$.

Then, $t_S(x), t_S(y) \leq \rho$ and $1 - f_S(x), 1 - f_S(y) \leq \sigma$.

Thus, $V_S(x)$ and $V_S(y) \leq [\rho, \sigma]$, that is $x, y \in S_{(\rho,\sigma)}$.

Thus, $S_{(\rho,\sigma)} \neq \emptyset$.

Hence, by the assumption $S_{(\rho,\sigma)}$ is a filter of A .

To prove: S is an anti vague filter of A .

If $x \leq y$, $t_S(x) = \rho$ and $1 - f_S(x) = \sigma$.

Then $x \in S_{(\rho,\sigma)}$.

Since, $S_{(\rho,\sigma)}$ is a filter, $y \in S_{(\rho,\sigma)}$, that is, $V_S(y) \leq [\rho, \sigma]$. (3.5)

Since, $S_{(\rho,\sigma)}$ is filter of A , $x * y \in S_{(\rho,\sigma)}$.

That is, $\vartheta_S(x * y) \leq [\rho, \sigma]$ for all $x, y \in A$

$$\begin{aligned}
 &= [\max\{\rho_1, \rho_2\}, \max\{1 - \sigma_1, 1 - \sigma_2\}] \\
 &= rmax\{[t_S(x), 1 - f_S(x)], [t_S(y), 1 - f_S(y)]\} \\
 &= rmax\{V_S(x), V_S(y)\} \text{ for all } x, y \in A. \quad (3.6)
 \end{aligned}$$

From (3.5) and (3.6), S is an anti vague filter of A . ■

Note. The filter $S_{(\rho, \sigma)}$ is called a vague-cut filter of BL - algebra A .

Proposition 3.14 Let S be an anti vague filter of BL -algebra A . Then S_ρ is either empty or a filter of A for all $\rho \in [0, 1]$.

Proof: Let S be an anti vague filter of BL -algebra $\mathcal{B}$. Then from the theorem 3.13, the proof is obvious. ■

4. Conclusion

In the present paper the notion of an anti-vague filter in BL - algebra with suitable examples are studied. Also investigated some related properties with the help of more implication of an anti-vague filter of BL -algebra.

References

- [1]. R.Biswas, *Vague groups*, International Journal of Computational Cognition, Vol. 4(2) (2006), 20-23.
- [2].C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. 88 (1958), 467- 490.
- [3].T. Eswarlal, *Vague ideals and normal vague ideals in semirings*, International Journal of Computational Cognition, Vol. 6, (2008), 60-65.
- [4].W. L. Gau, D. J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics, Vol. 23 (2), (1993), 610-614.
- [5].P. Hájek, *Metamathematics of fuzzy logic*, Klower Academic Publishers, Dordrecht, 1999.
- [6].L. Z. Liu, K. T. Li, *Fuzzy filters of BL-algebras*, Information Sciences, 173 (2005), 141-154.

- [7]. S. Yahya Mohamed and P. Umamaheswari, *Vague Filter of BL- Algebras*, Journal of Computer and Mathematical Sciences, 9(8), (2018), 914-920.
- [8]. S. Yahya Mohamed and P. Umamaheswari, *Vague prime and Boolean filters of BL- Algebras*, Journal of Applied Science and Computations, 5(11), (2018), 470-474.
- [9]. S. Yahya Mohamed and P. Umamaheswari, *Vague implicative filters of BL- algebras*, American International Journal of Research in Science, Technology, Engineering and & Mathematics, Conference Proceeding of ICOMAC-2019, 295-299.
- [10]. S. Yahya Mohamed and P. Umamaheswari, *Vague Positive Implicative filter of BL- algebras*, Malaya Journal of Matematik, 8(1), (2020), 166-170.
- [11]. E. Turunen, *Boolean deductive systems of BL-algebras*, Arch. Math. Logic 40 (2001), 467-473.
- [12]. L. A. Zadeh, *Fuzzy sets*, Inform. Control 8 (1965), 338-353.
- [13]. X. H. Zhang, *Fuzzy logic and its algebraic analysis*, Science Press, Beijing (2008).

Some studies on products of fuzzy soft graphs

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Abstract

In this paper, alpha, beta and gamma product of two fuzzy soft graphs are defined. The degree of a vertex in these product fuzzy soft graphs are determined and its regular properties are studied.

Keywords: alpha, beta and gamma product of fuzzy soft graphs, degree of a vertex, regular properties.

2010 AMS subject classification: 05C72, 05C99.‡

1. Introduction

Graph theory is used to model various types of relations that exist in different fields of physics, chemistry, medicine, electrical network, computer science such as networking, image processing etc. Whenever the information provided is imprecise, uncertainty exists. Molodtsov [1] initiated the concept of soft sets to deal with uncertainty. A. Rosenfeld [2] developed the theory of fuzzy graphs in 1975 based on fuzzy sets which were initiated by Zadeh [3] in 1965. Maji et al. [5,7] presented the definition of fuzzy soft sets and applied it in decision making problems. Later many researchers progressively worked on these concepts and developed it. Operations on fuzzy graphs were demonstrated by J. N. Mordeson and C. S. Peng [6]. Akram and Saira Nawaz [8,11] introduced fuzzy soft graphs, studied some of its properties and applied these concepts in social network and road network. Shashikala S and Anil P N [12,15] discussed connectivity in fuzzy soft graphs and studied hamiltonian fuzzy soft cycles. A. Pouhassani and H. Doostie [13] studied degree, total degree, regularity and total regularity of fuzzy soft graph and its properties.

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‡Received on August 19, 2021. Accepted on December 1, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.644. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

Regular fuzzy soft graphs and its related properties are studied by B Akhilandeswari [16]. Shovan Dogra [10] studied some types of fuzzy graph products such as modular product, homomorphic product and determined its degree of vertices. Union and intersection of fuzzy soft graphs and some of its properties are studied by Mohinta and Samanta [9]. Fuzzy soft theory provides a clear picture of the problems that allows parameterization finds applications in many areas. Recently, it is used to represent the oligopolistic competition among the wireless internet connection providers in Malaysia by Akram and Saba Nawaz [17].

In this paper, some products of fuzzy soft graphs namely alpha, beta and gamma products are defined and degree of a vertex in these products are determined and its regular properties are studied.

2. Preliminaries

Definition 2.1: [11] A fuzzy soft graph $\tilde{G}$ over a graph $G^* : (V, E)$ is a triple $(\tilde{F}, \tilde{K}, A)$ where:

- a) A is a nonempty set of parameters
- b) $(\tilde{F}, A)$ is a fuzzy soft set over V
- c) $(\tilde{K}, A)$ is a fuzzy soft set over E
- d) $(\tilde{F}(e_i), \tilde{K}(e_i))$ is a fuzzy graph on $G^* \forall e_i \in A$
- i.e. $\tilde{K}(e_i)(xy) \leq \min\{\tilde{F}(e_i)(x), \tilde{F}(e_i)(y)\}$ for all $e_i \in A$ and $x, y \in V$.

Definition 2.2: [14] The underlying crisp graph of a fuzzy soft graph $\tilde{G}$ is denoted by $G^* = (F^*, K^*)$ where $F^* = \{x \in V : \tilde{F}(e_i)(x) > 0 \text{ for some } e_i \in A\}$, $K^* = \{(x, y) \in V \times V : \tilde{K}(e_i)(x, y) > 0 \text{ for some } e_i \in A\}$.

Definition 2.3: [8] Let $\tilde{G}$ be a fuzzy soft graph on G^* . The degree of a vertex x is defined as $\deg_{\tilde{G}}(x) = \sum_{e_i \in A} \sum_{y \in V, y \neq x} \tilde{K}(e_i)(xy)$.

Definition 2.4: [8] $\tilde{G}$ is said to be a regular fuzzy soft graph if $(\tilde{F}(e_i), \tilde{K}(e_i))$ is regular fuzzy graph for all $e_i \in A$. If $(\tilde{F}(e_i), \tilde{K}(e_i))$ is a regular fuzzy graph of degree k for all $e_i \in A$ then $\tilde{G}$ is a k -regular fuzzy soft graph.

Definition 2.5: [4] The degree $d_{G^*}(x)$ of a vertex x in G^* is the number of edges incident with x .

In this paper, we assume that $G^* : (V, E)$ of any fuzzy soft graph $\tilde{G}$ is finite and simple.

Notation: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs. The relation $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ for all $e_i \in A_1, e_j \in A_2$ means that $\tilde{F}_1(e_i)(x) \geq \tilde{K}_2(e_j)(e) \forall x \in V_1, e \in E_2$ where $\tilde{F}_1$ is a fuzzy soft subset of V_1 and $\tilde{K}_2$ is a fuzzy soft subset of E_2 .

3. Alpha product (α – product), Beta product (β – product) and Gamma product (γ – product) of Fuzzy soft graph

Definition 3.1: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The α – product $\tilde{G}_1 \times_{\alpha} \tilde{G}_2 : (\tilde{F}_1 \times_{\alpha} \tilde{F}_2, \tilde{K}_1 \times_{\alpha} \tilde{K}_2, A_1 \times A_2)$ is defined as follows:

$(\tilde{F}_1 \times_{\alpha} \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by

$(\tilde{F}_1 \times_{\alpha} \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l) \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and

$(\tilde{K}_1 \times_{\alpha} \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by

$$(\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \begin{cases} \tilde{F}_1(e_i)(x_k) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k = x_m, y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{K}_1(e_i)(x_k x_m) & \text{if } y_l = y_n, x_k x_m \in E_1 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{F}_2(e_j)(y_n) \wedge \tilde{K}_1(e_i)(x_k x_m) & \text{if } x_k x_m \in E_1, \\ & y_l y_n \notin E_2 \\ \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_1(e_i)(x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k x_m \notin E_1, \\ & y_l y_n \in E_2 \end{cases}$$

Definition 3.2: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The β – product $\tilde{G}_1 \times_{\beta} \tilde{G}_2 : (\tilde{F}_1 \times_{\beta} \tilde{F}_2, \tilde{K}_1 \times_{\beta} \tilde{K}_2, A_1 \times A_2)$ is defined as follows:

$(\tilde{F}_1 \times_{\beta} \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by

$(\tilde{F}_1 \times_{\beta} \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l) \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and

$(\tilde{K}_1 \times_{\beta} \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by

$$(\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \begin{cases} \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_1(e_i)(x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k \neq x_m, \\ & y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{F}_2(e_j)(y_n) \wedge \tilde{K}_1(e_i)(x_k x_m) & \text{if } y_l \neq y_n, \\ & x_k x_m \in E_1 \\ \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k x_m \in E_1, y_l y_n \in E_2 \end{cases}$$

Definition 3.3: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The γ -product $\tilde{G}_1 \times_{\gamma} \tilde{G}_2 : (\tilde{F}_1 \times_{\gamma} \tilde{F}_2, \tilde{K}_1 \times_{\gamma} \tilde{K}_2, A_1 \times A_2)$ is defined as follows:

$(\tilde{F}_1 \times_{\gamma} \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by

$(\tilde{F}_1 \times_{\gamma} \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l) \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and

$(\tilde{K}_1 \times_{\gamma} \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by

$$(\tilde{K}_1 \times_{\gamma} \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \begin{cases} \tilde{F}_1(e_i)(x_k) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k = x_m, y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{K}_1(e_i)(x_k x_m) & \text{if } y_l = y_n, x_k x_m \in E_1 \\ \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_1(e_i)(x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k \neq x_m, \\ & y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{F}_2(e_j)(y_n) \wedge \tilde{K}_1(e_i)(x_k x_m) & \text{if } y_l \neq y_n, \\ & x_k x_m \in E_1 \\ \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & \text{if } x_k x_m \in E_1, y_l y_n \in E_2 \end{cases}$$

Example 3.4: Consider two fuzzy soft graphs $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $V_1 = \{x_1, x_2\}$, $E_1 = \{x_1 x_2\}$, $V_2 = \{y_1, y_2, y_3\}$, $E_2 = \{y_1 y_2, y_2 y_3\}$, $A_1 = \{e_i\}$ where $i=1,2$ and $A_2 = \{e_j\}$ where $j=3,4$. Let $(\tilde{F}_1, A_1), (\tilde{F}_2, A_2), (\tilde{K}_1, A_1)$ and $(\tilde{K}_2, A_2)$ be represented by the following Table 1.

$\tilde{F}_1$	x_1	x_2	$\tilde{F}_2$	y_1	y_2	y_3
e_1	0.4	0.6	e_3	0.3	0.5	0.8
e_2	0.7	0.5	e_4	0.5	0.6	0.7
$\tilde{K}_1$	$x_1 x_2$		$\tilde{K}_2$	$y_1 y_2$		$y_2 y_3$
e_1	0.1		e_3	0.3		0.4
e_2	0.3		e_4	0.4		0.5

Table 1 : Tabular representation of two fuzzy soft graphs

Some studies on products of fuzzy soft graphs

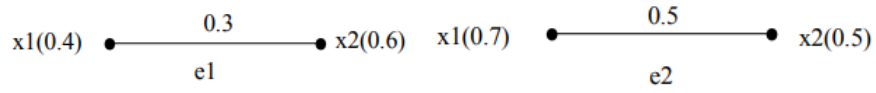


Fig.1. $\tilde{G}_1$

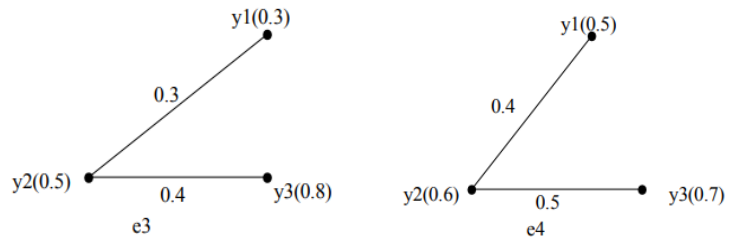


Fig.2. $\tilde{G}_2$

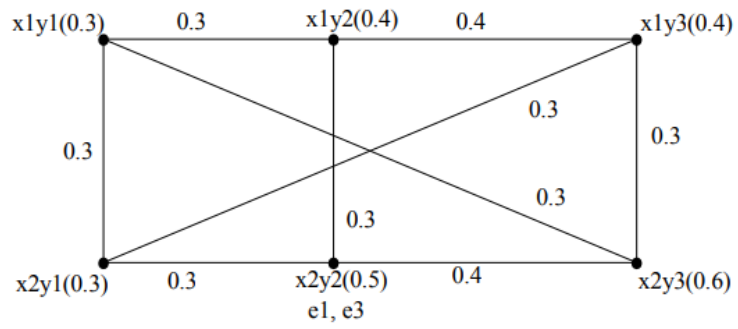


Fig.3. $\tilde{G}_1 \times \tilde{G}_2$

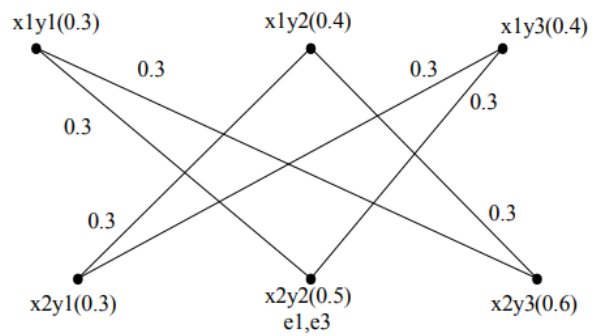


Fig.4. $\tilde{G}_1 \times \tilde{G}_2$

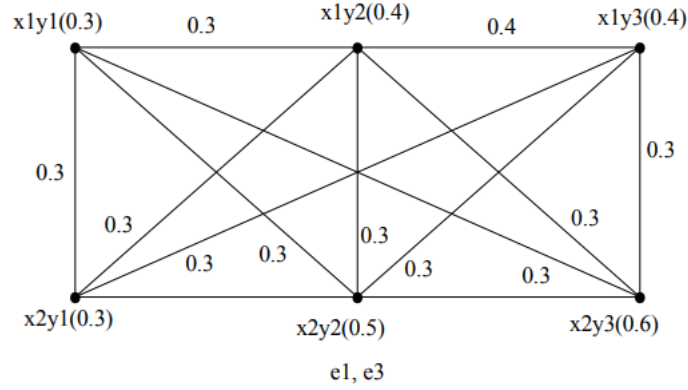


Fig.5. $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$

4. Degree of a vertex in Alpha product (α -product) of two fuzzy soft graphs and its regular properties

Theorem 4.1: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = [1 + d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x_1)$

$$\begin{aligned}
 \text{Proof: } \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\
 &= |e_i| d_{\tilde{G}_2}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1) \\
 &= [1 + d_{G_1^{*c}}(x_1)] |e_i| d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^{*c}}(y_1)] |e_j| d_{\tilde{G}_1}(x_1)
 \end{aligned}$$

This is true for any vertex (x_1, y_1) in $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ with $\tilde{F}_1 \geq \tilde{K}_2$ and $\tilde{F}_2 \geq \tilde{K}_1$.

Theorem 4.2 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$ then

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = c |e_i| |e_j| d_{G_2^*}(y_1) [1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1)]$$

Proof: Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$, for any $(x_k, y_l) \in V_1 \times V_2$

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \\
 &= |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + \\
 &\quad |e_j| d_{G_1^{*c}}(x_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) d_{G_2^*}(y_1)
 \end{aligned}$$

$$\begin{aligned}
 &= |e_j| d_{G_2^*}(y_1) c|e_i| + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{G_1^{*c}}(x_1) c|e_i| d_{G_2^*}(y_1) \\
 &= c|e_i| |e_j| d_{G_2^*}(y_1) [1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1)]
 \end{aligned}$$

Theorem 4.3 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_i)(y_l) = m \quad \forall y_l \in V_2$ then

$$\deg_{\tilde{G}_1 \times \tilde{G}_2}(x_1, y_1) = m|e_i| |e_j| d_{G_1^*}(x_1) [1 + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^{*c}}(x_1)]$$

Proof: Given $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ then $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) + \\
 &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\
 &= |e_i| d_{\tilde{G}_2}(y_1) + |e_i| d_{G_1^*}(x_1) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_1) + |e_i| d_{G_2^{*c}}(y_1) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_1) d_{G_1^*}(x_1) + \\
 &\quad |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1)
 \end{aligned}$$

$$= m|e_i| |e_j| d_{G_1^*}(x_1) [1 + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^{*c}}(x_1)]$$

Theorem 4.4 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\deg_{\tilde{G}_1 \times \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) + |e_j| d_{\tilde{G}_1}(x_1)$

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) +
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\
 & \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \\
 & = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \\
 & \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\
 & = [1 + d_{G_1^*}(x_1)]|e_i|d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^*}(y_1)]|e_j|d_{\tilde{G}_1}(x_1) \\
 & = |e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) \text{ (since } G_1^* \text{ and } G_2^* \text{ are complete graphs)}
 \end{aligned}$$

Theorem 4.5 : Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on regular graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular fuzzy soft graph if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular fuzzy soft graphs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular fuzzy soft graphs of degree k_1 and k_2 respectively. For any vertex $(x_1 y_1) \in V_1 \times V_2$,

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = [1 + d_{G_1^*}(x_1)]|e_i|d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^*}(y_1)]|e_j|d_{\tilde{G}_1}(x_1) \quad (\text{From}$$

Theorem 4.1)

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = [1 + |E_1^c|]|e_i|k_2 + [1 + |E_2^c|]|e_j|k_1$$

This is true $\forall (x_1 y_1) \in V_1 \times V_2$

Hence, $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular fuzzy soft graph.

Conversely, Let $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ be a regular fuzzy soft graph. For any two vertices $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2)$$

$$[1 + d_{G_1^*}(x_1)]|e_i|d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^*}(y_1)]|e_j|d_{\tilde{G}_1}(x_1)$$

$$= [1 + d_{G_1^*}(x_2)]|e_i|d_{\tilde{G}_2}(y_2) + [1 + d_{G_2^*}(y_2)]|e_j|d_{\tilde{G}_1}(x_2) \quad (\text{From Theorem 4.1})$$

Fix $x \in V_1$, consider $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$\begin{aligned}
 & [1 + d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x) \\
 &= [1 + d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_2) + [1 + d_{G_2^{*c}}(y_2)]|e_j|d_{\tilde{G}_1}(x) \\
 &\Rightarrow [1 + d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_1) = [1 + d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_2) \\
 &\Rightarrow d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2)
 \end{aligned}$$

This is true for all vertices of $V_2 \therefore \tilde{G}_2$ is a regular fuzzy soft graph.

Fix $y \in V_2$, consider (x_1, y_1) and (x_2, y_2) in $V_1 \times V_2$,

$$\begin{aligned}
 & [1 + d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y) + [1 + d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_1) \\
 &= [1 + d_{G_1^{*c}}(x_2)]|e_i|d_{\tilde{G}_2}(y) + [1 + d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_2) \\
 &\Rightarrow [1 + d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_1) = [1 + d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_2) \\
 &\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)
 \end{aligned}$$

This is true for all vertices of $V_1 \therefore \tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.6 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is regular if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular fuzzy soft graphs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular fuzzy soft graphs of degree k_1 and k_2 respectively. Let G_1^* and G_2^* are complete graphs.

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = |e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) \text{ (From Theorem 4.4)}$$

This is true $\forall (x_1, y_1) \in V_1 \times V_2$

Hence, $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular fuzzy soft graph.

Conversely, Let $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ be a regular fuzzy soft graph. For any two vertices

(x_1, y_1) and (x_2, y_2) in $V_1 \times V_2$,

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2)$$

$$|e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) = |e_i|d_{\tilde{G}_2}(y_2) + |e_j|d_{\tilde{G}_1}(x_2)$$

Fix $x \in V_1$, consider (x_1, y_1) and (x_2, y_2) in $V_1 \times V_2$,

$$|e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x) = |e_i|d_{\tilde{G}_2}(y_2) + |e_j|d_{\tilde{G}_1}(x)$$

$$\Rightarrow d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2)$$

This is true for all vertices of $V_2 \therefore \tilde{G}_2$ is a regular fuzzy soft graph.

Fix $y \in V_2$, consider $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$\begin{aligned} |e_i|d_{\tilde{G}_2}(y) + |e_j|d_{\tilde{G}_1}(x_1) &= |e_i|d_{\tilde{G}_2}(y) + |e_j|d_{\tilde{G}_1}(x_2) \\ \Rightarrow d_{\tilde{G}_1}(x_1) &= d_{\tilde{G}_1}(x_2) \end{aligned}$$

This is true for all vertices of $V_1 \therefore \tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.7 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs G_1^* and G_2^* respectively then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ becomes a Cartesian product of fuzzy soft graphs.

Proof: By the definition of alpha product of fuzzy soft graphs, for any two vertices $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$(\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) = \begin{cases} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) & \text{if } x_1 = x_2, y_1 y_2 \in E_2 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) & \text{if } y_1 = y_2, x_1 x_2 \in E_1 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) & \text{if } x_1 x_2 \in E_1, \\ & y_1 y_2 \notin E_2 \\ \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) & \text{if } x_1 x_2 \notin E_1, \\ & y_1 y_2 \in E_2 \end{cases}$$

Since G_1^* and G_2^* are complete graphs,

$$\begin{aligned} (\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) &= \begin{cases} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) & \text{if } x_1 = x_2, y_1 y_2 \in E_2 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) & \text{if } y_1 = y_2, x_1 x_2 \in E_1 \end{cases} \\ \Rightarrow (\tilde{K}_1 \times_{\alpha} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) &= (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \end{aligned}$$

is a Cartesian product of $\tilde{G}_1$ and $\tilde{G}_2$.

Theorem 4.8 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$. Let $\tilde{F}_1(e_i)$ be a constant and G_1^* is a complete graph then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is regular fuzzy soft graph if and only if $\tilde{G}_1$ is a regular fuzzy soft graph and G_2^* is a regular graph.

Proof: Let $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$

Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. Let G_1^* be a complete graph.

From Theorem 4.2,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= c|e_i||e_j|d_{G_2^*}(y_1)[1 + d_{G_1^*}(x_1)] + |e_j|d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1)] \\ &= c|e_i||e_j|d_{G_2^*}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) + |e_j|d_{G_2^*}(y_1)d_{\tilde{G}_1}(x_1) \end{aligned}$$

Let $\tilde{G}_1$ be a regular fuzzy soft graph with degree m and G_2^* is a regular graph of degree n.

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= c|e_i||e_j|n + |e_j|m + |e_j|d_{G_2^{*c}}(y_1)m \\ &= |e_i||e_j|cn + |e_j|m[1 + |E_2^c|] \end{aligned}$$

Hence, $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is regular fuzzy soft graph.

Conversely, assume that $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is regular fuzzy soft graph.

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2) \\ c|e_i||e_j|d_{G_2^*}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) + |e_j|d_{G_2^{*c}}(y_1)d_{\tilde{G}_1}(x_1) + |e_j|d_{G_1^{*c}}(x_1)c|e_i|d_{G_2^*}(y_1) &= \\ c|e_i||e_j|d_{G_2^*}(y_2) + |e_j|d_{\tilde{G}_1}(x_2) + |e_j|d_{G_2^{*c}}(y_2)d_{\tilde{G}_1}(x_2) + |e_j|d_{G_1^{*c}}(x_2)c|e_i|d_{G_2^*}(y_2) & \end{aligned}$$

Fix $x \in V_1$, consider (xy_1) and (xy_2) in $V_1 \times V_2$,

$$c|e_i||e_j|d_{G_2^*}(y_1) + |e_j|d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_1)] = c|e_i||e_j|d_{G_2^*}(y_2) + |e_j|d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_2)]$$

This is true when degree of all vertices in G_2^* as well as in its complement are equal. Hence, G_2^* is regular.

Fix $y \in V_2$,

$$\begin{aligned} c|e_i||e_j|d_{G_2^*}(y) + |e_j|d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y)] &= c|e_i||e_j|d_{G_2^*}(y) + |e_j|d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y)] \\ \Rightarrow d_{\tilde{G}_1}(x_1) &= d_{\tilde{G}_1}(x_2) \end{aligned}$$

Hence, $\tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.9 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$. If $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$, $\tilde{G}_1$ are regular fuzzy soft graphs, G_1^* and G_2^* are regular graphs then $\sum_{e_i \in A_k} \tilde{F}_1(e_i)(x_k)$ is same for all $k=1,2,3,\dots$

Proof: Let $\tilde{G}_1$ be a regular fuzzy soft graph of degree m, G_1^* and G_2^* are regular graphs of degree s and n respectively.

Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. Using the definition of $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$,

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$\begin{aligned}
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1=x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1=y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \\
 &\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \\
 &= |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k) \\
 &= |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) [1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1)]
 \end{aligned}$$

Since $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular fuzzy soft graph,

$$\begin{aligned}
 &\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2) \\
 &|e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) [1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1)] = \\
 &|e_j| d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2) [1 + d_{G_1^{*c}}(x_2)] + |e_j| d_{\tilde{G}_1}(x_2) [1 + d_{G_2^{*c}}(y_2)] \\
 &|e_j| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) [1 + d_{G_1^{*c}}(x_1)] + |e_j| m [1 + d_{G_2^{*c}}(y_1)] = \\
 &|e_j| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2) [1 + d_{G_1^{*c}}(x_2)] + |e_j| m [1 + d_{G_2^{*c}}(y_2)] \\
 &\Rightarrow |e_j| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) [1 + d_{G_1^{*c}}(x_1)] = |e_j| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2) [1 + d_{G_1^{*c}}(x_2)] \text{ (Since } G_1^{*c} \text{ is} \\
 &\text{regular)} \\
 &\Rightarrow \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) = \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)
 \end{aligned}$$

Theorem 4.10 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$. If $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$, $\tilde{G}_2$ are regular fuzzy soft graphs, G_1^* and G_2^* are regular graphs then $\sum_{e_j \in A_{21}} \tilde{F}_1(e_j)(y_l)$ is same for all $l = 1, 2, 3, \dots$

Proof: Proof is similar to the proof of theorem 4.9.

Theorem 4.11 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs with $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j) = c$ then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular FSG if and only if G_1^* and G_2^* is a regular graph and $\tilde{G}_2$ is a regular FSG.

Proof: Given $\tilde{F}_2(e_j) = c$, Since $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$.

Let $\tilde{G}_2$ be a regular FSG of degree m , G_1^* and G_2^* are regular graphs of degree n and p respectively.

From Theorem 4.3,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= c|e_i||e_j|d_{G_1^*}(x_1)[1+d_{G_2^{*c}}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^{*c}}(x_1)] \\ &= nc|e_i||e_j|[1+d_{G_2^{*c}}(y_1)]+m|e_i|[1+d_{G_1^{*c}}(x_1)] \\ &\Rightarrow \tilde{G}_1 \times_{\alpha} \tilde{G}_2 \text{ is a regular FSG.} \end{aligned}$$

Conversely, Let $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ be a regular FSG.

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2) \\ c|e_i||e_j|d_{G_1^*}(x_1)[1+d_{G_2^{*c}}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^{*c}}(x_1)] &= \\ c|e_i||e_j|d_{G_1^*}(x_2)[1+d_{G_2^{*c}}(y_2)]+|e_i|d_{\tilde{G}_2}(y_2)[1+d_{G_1^{*c}}(x_2)] & \end{aligned}$$

Fix $x \in V_1$,

$$\begin{aligned} c|e_i||e_j|d_{G_1^*}(x)[1+d_{G_2^{*c}}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^{*c}}(x)] &= \\ c|e_i||e_j|d_{G_1^*}(x)[1+d_{G_2^{*c}}(y_2)]+|e_i|d_{\tilde{G}_2}(y_2)[1+d_{G_1^{*c}}(x)] & \\ \Rightarrow d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2) \text{ and } d_{G_2^{*c}}(y_1) = d_{G_2^{*c}}(y_2) & \\ \Rightarrow \tilde{G}_2 \text{ is a regular FSG and } G_2^* \text{ is regular.} & \end{aligned}$$

Fix $y \in V_2$,

$$\begin{aligned} c|e_i||e_j|d_{G_1^*}(x_1)[1+d_{G_2^{*c}}(y)]+|e_i|d_{\tilde{G}_2}(y)[1+d_{G_1^{*c}}(x_1)] &= \\ c|e_i||e_j|d_{G_1^*}(x_2)[1+d_{G_2^{*c}}(y)]+|e_i|d_{\tilde{G}_2}(y)[1+d_{G_1^{*c}}(x_2)] & \\ \Rightarrow d_{G_1^*}(x_1) = d_{G_1^*}(x_2) \text{ and } d_{G_1^{*c}}(x_1) = d_{G_1^{*c}}(x_2) & \end{aligned}$$

This is true when G_1^* is regular.

Theorem 4.12 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs where G_1^* is a complete graph. If $\tilde{F}_1(e_i)$ and $\tilde{K}_1(e_i)$ are constant and $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is regular and G_2^* is a regular graph.

Proof: Let $\tilde{F}_1(e_i) = \tilde{K}_1(e_i) = c$ and $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ be regular.

Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$.

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2)$$

Using the result of theorem 4.9,

$$|e_j|d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1+d_{G_1^{*c}}(x_1)]+|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)] =$$

$$\begin{aligned}
 & |e_j|d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)[1+d_{G_1^{*c}}(x_2)] + |e_j|d_{\tilde{G}_1}(x_2)[1+d_{G_2^{*c}}(y_2)] \\
 & |e_j|d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)] = \\
 & |e_j|d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2) + |e_j|d_{\tilde{G}_1}(x_2)[1+d_{G_2^{*c}}(y_2)] \quad (\text{Since } G_1^* \text{ is a complete}
 \end{aligned}$$

graph)

Given $\tilde{F}_1(e_i) = c$

$$c|e_i|d_{G_2^*}(y_1) + d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)] = c|e_i|d_{G_2^*}(y_2) + d_{\tilde{G}_1}(x_2)[1+d_{G_2^{*c}}(y_2)]$$

Fix $x \in V_1$,

$$c|e_i|d_{G_2^*}(y_1) + d_{\tilde{G}_1}(x)[1+d_{G_2^{*c}}(y_1)] = c|e_i|d_{G_2^*}(y_2) + d_{\tilde{G}_1}(x)[1+d_{G_2^{*c}}(y_2)]$$

$$\Rightarrow d_{G_2^*}(y_1) = d_{G_2^*}(y_2) \text{ and } d_{G_2^{*c}}(y_1) = d_{G_2^{*c}}(y_2)$$

This is true only when G_2^* is a regular graph.

Similarly, fix $y \in V_2$,

$$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$$

$\Rightarrow \tilde{G}_1$ is regular FSG.

Conversely, Let $\tilde{G}_1$ and G_2^* be regular with degree m and n respectively.

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) &= |e_j|d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1+d_{G_1^{*c}}(x_1)] + |e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)] \\
 &= |e_j|n|e_i|c + |e_j|m(1+p)
 \end{aligned}$$

Therefore, $\tilde{G}_1 \times_{\alpha} \tilde{G}_2$ is a regular FSG.

5. Degree of a vertex in Beta product (β -product) of two fuzzy soft graphs and its regular properties

Theorem 5.1: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively.

$$\text{i) If } \tilde{K}_1(e_i) \leq \tilde{K}_2(e_j) \text{ then } \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_j|d_{\tilde{G}_1}(x_1)d_{G_2^*}(y_1)$$

$$\text{ii) If } \tilde{K}_2(e_j) \leq \tilde{K}_1(e_i) \text{ then } \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_i|d_{\tilde{G}_2}(y_1)d_{G_1^*}(x_1)$$

$$\begin{aligned}
 \text{Proof: } \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) +
 \end{aligned}$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

i) For any vertex $(x_1 y_1) \in V_1 \times V_2$

Since G_1^* and G_2^* are complete graphs, we have

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

Given $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ = |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

$$\text{ii) } \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

Since $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\ = |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1)$$

Theorem 5.2: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If

$\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_1}(y_1)$$

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) [d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1)$$

$$\textbf{Proof: } \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\ \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

i) Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. Also $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) \\ + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$= |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1) + |e_j| [d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_i| d_{\tilde{G}_1}(x_1) d_{G_2^{*}}(y_1)] \\ = |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^{*c}}(y_1) + d_{G_2^{*}}(y_1)] + |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_1}(y_1)$$

ii) Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ & $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) \\ + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)$$

$$= |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1) + |e_j| [d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^{*}}(x_1)] \\ = |e_i| d_{\tilde{G}_2}(y_1) [d_{G_1^{*c}}(x_1) + d_{G_1^{*}}(x_1)] + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1)$$

Theorem 5.3: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^{*} : (V_1, E_1)$ and $G_2^{*} : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$ then

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^{*c}}(y_1) + d_{G_2^{*}}(y_1)] + c |e_i| |e_j| d_{G_1^{*c}}(x_1) d_{G_2^{*}}(y_1)$$

Proof: $\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$

Using the given conditions,

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$\begin{aligned}\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= |e_i| c d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1) |e_j| + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\ &= |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c |e_i| |e_j| d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1)\end{aligned}$$

Theorem 5.4: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_j)(y_1) = m \quad \forall y_1 \in V_2$ then

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) [d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_i| |e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1)$$

$$\textbf{Proof:} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

Using the given conditions,

$$\begin{aligned}&= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\ &= |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^{*c}}(x_1) + |e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1) + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1) \\ &= |e_i| d_{\tilde{G}_2}(y_1) [d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_i| |e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1)\end{aligned}$$

Theorem 5.5: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively. $\tilde{G}_1 \times_{\beta} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs with degrees m and n respectively. Let G_1^* and G_2^* be complete graphs with degrees p and q respectively.

$$\begin{aligned}\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)\end{aligned}$$

Case (i): If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\begin{aligned}\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ &= |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\ &= |e_j| m q\end{aligned}$$

$\therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2$ is a regular FSG.

Case (i): If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1, x_2 \in E_1 \\ y_1, y_2 \in E_2}} \tilde{K}_2(e_j)(y_1, y_2) \\ &= |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1) \\ &= |e_i| n p \end{aligned}$$

$\therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \times_{\beta} \tilde{G}_2$ be a regular FSG.

$$\deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2}(x_2, y_2)$$

Case (i): If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$|e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) = |e_j| d_{\tilde{G}_1}(x_2) d_{G_2^*}(y_2)$$

Fix $x \in V_1$,

$$d_{\tilde{G}_1}(x) d_{G_2^*}(y_1) = d_{\tilde{G}_1}(x) d_{G_2^*}(y_2)$$

$$\Rightarrow d_{G_2^*}(y_1) = d_{G_2^*}(y_2)$$

$\therefore G_2^*$ is a regular graph.

Fix $y \in V_2$,

$$d_{\tilde{G}_1}(x_1) d_{G_2^*}(y) = d_{\tilde{G}_1}(x_2) d_{G_2^*}(y)$$

$$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$$

$\therefore \tilde{G}_1$ is regular.

Case (ii): Similarly for $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, we get G_1^* and $\tilde{G}_2$ as regular.

Theorem 5.6: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs and G_1^* is a complete graph and G_2^* is a regular graph. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) = \tilde{K}_2(e_j)$ then $\tilde{G}_1 \times_{\beta} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is a regular FSG.

Proof: Let G_2^* be a regular graph of degree p and G_1^* is a complete graph. Let $\tilde{K}_1(e_i) = \tilde{K}_2(e_j) = c$, $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$.

Let us assume that $\tilde{G}_1$ is a regular FSG of degree m.

Using the definition of degree of a vertex in beta product of FSGs and the above conditions, we get

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2} (x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\
 &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \text{ (Since } G_1^* \text{ is a complete graph)} \\
 &= |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\
 &= |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)] \\
 &= |e_j| m[p + s] \\
 \therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2 \text{ is a regular FSG.}
 \end{aligned}$$

Conversely, Let $\tilde{G}_1 \times_{\beta} \tilde{G}_2$ be a regular FSG, G_2^* is a regular graph of degree p and G_1^* is a complete graph.

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2} (x_1, y_1) &= \deg_{\tilde{G}_1 \times_{\alpha} \tilde{G}_2} (x_2, y_2) \\
 \Rightarrow |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)] &= |e_j| d_{\tilde{G}_1}(x_2) [d_{G_2^*}(y_2) + d_{G_2^{*c}}(y_2)] \\
 \Rightarrow |e_j| d_{\tilde{G}_1}(x_1) [p + t] &= |e_j| d_{\tilde{G}_1}(x_2) [p + t] \text{ where } d_{G_2^{*c}}(y) = t \\
 \Rightarrow d_{\tilde{G}_1}(x_1) &= d_{\tilde{G}_1}(x_2) \\
 \therefore \tilde{G}_1 \text{ is regular.}
 \end{aligned}$$

Theorem 5.7: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs and G_2^* is a complete graph and G_1^* is a regular graph. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) = \tilde{K}_2(e_j)$ then $\tilde{G}_1 \times_{\beta} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is a regular FSG.

Proof: Proof is similar to the proof of Theorem 5.6.

Theorem 5.8: If $\tilde{G}_1$ and $\tilde{G}_2$ are two regular FSGs with $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, G_1^* and G_2^* are regular but not complete graphs, then beta product of two FSGs $\tilde{G}_1$ and $\tilde{G}_2$ is a regular FSG.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs with degrees m and n respectively. Let G_1^* and G_2^* be regular graphs of degrees p and q respectively and suppose that G_1^* and G_2^* are not complete graphs. Let $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$.

$$\deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\beta} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

Case (i) : Assume that G_1^* and G_2^* are isomorphic graphs. Let $\tilde{K}_1(e_i) = \tilde{K}_2(e_j) = c$

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ &= |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\ &= |e_j| m(q+s) + |e_i| sn \quad (\text{Since } G_1^* \text{ and } G_2^* \text{ are regular graphs of degree } p \text{ and } q \\ &\text{and are isomorphic, } d_{G_1^{*c}}(x_1) = d_{G_2^{*c}}(y_1) = s) \\ &\therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2 \text{ is regular.} \end{aligned}$$

Case (ii) : Assume that G_1^* and G_2^* are not isomorphic but are regular graphs of degrees p and q .

By the definition,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) \end{aligned}$$

If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ &= |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^{*c}}(x_1) \\ &= |e_j| m[q+t] + |e_i| ns \end{aligned}$$

$\therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2$ is regular.

If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\beta} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\ &= |e_i| d_{\tilde{G}_2}(y_1) [d_{G_1^*}(x_1) + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1) \\ &= |e_i| n [p + s] + |e_j| m t \\ \therefore \tilde{G}_1 \times_{\beta} \tilde{G}_2 &\text{ is regular.} \end{aligned}$$

6. Degree of a vertex in Gamma product (γ – product) of two fuzzy soft graphs and its regular properties

Theorem 6.1: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively and $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j), \tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)$$

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)$$

Proof: Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j), \tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. For any vertex $(x_1, y_1) \in V_1 \times V_2$,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_j)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) \end{aligned}$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

i) Given $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$,

$$\begin{aligned} \deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 y_2 \in E_1 \\ x_1 x_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \\ &\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 y_2 \in E_1 \\ x_1 x_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \end{aligned}$$

Since G_1^* and G_2^* are complete graphs,

$$\begin{aligned} &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 y_2 \in E_1 \\ x_1 x_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ \deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) &= |e_j| d_{\tilde{G}_1}(x_1) + |e_i| d_{\tilde{G}_2}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\ &= |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) \end{aligned}$$

ii) Given $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$

$$\begin{aligned} \deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 y_2 \in E_1 \\ x_1 x_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\ \deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) &= |e_j| d_{\tilde{G}_1}(x_1) + |e_i| d_{\tilde{G}_2}(y_1) + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1) \\ &= |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) \end{aligned}$$

Theorem 6.2: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on crisp graphs G_1^* and G_2^* respectively and $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^*}(y_1) + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^*}(x_1)]$$

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \times \tilde{G}_2} (x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^*}(x_1) + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^*}(y_1)]$$

Proof: Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

Using the definition of degree of $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\gamma} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \\ \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_1^*}(y_1) + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^*}(x_1)] \end{aligned}$$

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_{\gamma} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2) \\ &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) \\ \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= |e_j| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^*}(x_1) + d_{G_1^*}(x_1)] + |e_i| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1)] \end{aligned}$$

Theorem 6.3: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$, then

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1) + d_{G_2^*}(y_1)] + c |e_i| |e_j| d_{G_2^*}(y_1)[1 + d_{G_1^*}(x_1)]$$

Proof: Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)(x_1) = c$

By the definition,

$$\begin{aligned} \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \\ &\quad \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \end{aligned}$$

$$\begin{aligned}\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_i| c d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1) |e_j| + \\ &\quad |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) \\ &= |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c |e_i| |e_j| d_{G_2^*}(y_1) [1 + d_{G_1^{*c}}(x_1)]\end{aligned}$$

Theorem 6.4: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. If $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_j)(y_1) = c \quad \forall y_1 \in V_2$, then

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) [1 + d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + c |e_i| |e_j| d_{G_1^*}(x_1) [1 + d_{G_2^{*c}}(y_1)]$$

Proof: Proof is analogues to the proof of Theorem 6.3.

Theorem 6.5 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs with $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$ then $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is regular, G_1^* and G_2^* are regular graphs.

Proof: Given $\tilde{F}_1(e_i)(x_k) = c$, $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$. Let $\tilde{G}_1$ be a regular FSG of degree m , G_1^* and G_2^* are regular graphs of degree p and q respectively.

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

Using Theorem 6.3,

$$\begin{aligned}&= |e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c |e_i| |e_j| d_{G_2^*}(y_1) [1 + d_{G_1^{*c}}(x_1)] \\ &= |e_j| m [1 + |E_2^c| + |E_2|] + c |e_i| |e_j| p [1 + |E_1^c|]\end{aligned}$$

$\therefore \tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$ be a regular FSG.

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_2, y_2)$$

$$\begin{aligned}&|e_j| d_{\tilde{G}_1}(x_1) [1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c |e_i| |e_j| d_{G_2^*}(y_1) [1 + d_{G_1^{*c}}(x_1)] = \\ &= |e_j| d_{\tilde{G}_1}(x_2) [1 + d_{G_2^{*c}}(y_2) + d_{G_2^*}(y_2)] + c |e_i| |e_j| d_{G_2^*}(y_2) [1 + d_{G_1^{*c}}(x_2)]\end{aligned}$$

Fix $y \in V_2$,

$$\begin{aligned}
 & |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y) + d_{G_2^*}(y)] + c|e_i| |e_j| d_{G_2^*}(y)[1 + d_{G_1^{*c}}(x_1)] = \\
 & = |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y) + d_{G_2^*}(y)] + c|e_i| |e_j| d_{G_2^*}(y)[1 + d_{G_1^{*c}}(x_2)] \\
 & \Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2) \text{ and } d_{G_1^{*c}}(x_1) = d_{G_1^{*c}}(x_2)
 \end{aligned}$$

i.e. $\tilde{G}_1$ and G_1^* are regular.

Fix $x \in V_1$,

$$\begin{aligned}
 & |e_j| d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i| |e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x)] = \\
 & = |e_j| d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_2) + d_{G_2^*}(y_2)] + c|e_i| |e_j| d_{G_2^*}(y_2)[1 + d_{G_1^{*c}}(x)]
 \end{aligned}$$

This holds good when $d_{G_2^*}(y_1) = d_{G_2^*}(y_2)$ and $d_{G_2^{*c}}(y_1) = d_{G_2^{*c}}(y_2)$

$\Rightarrow G_2^*$ is regular.

Theorem 6.6: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular FSGs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs of degree m and n respectively. Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, G_1^* and G_2^* are complete graphs of degree p and q respectively.

From Theorem 6.1,

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) \\
 &= |e_j| m[1 + q] + |e_i| n
 \end{aligned}$$

$\therefore \tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG.

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\begin{aligned}
 \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2}(x_1, y_1) &= |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) \\
 &= |e_i| n[1 + p] + |e_j| m
 \end{aligned}$$

$\therefore \tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$ be a regular FSG.

$$\deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2} (x_1, y_1) = \deg_{\tilde{G}_1 \times_{\gamma} \tilde{G}_2} (x_2, y_2)$$

For $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$,

$$|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^*}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)=|e_j|d_{\tilde{G}_1}(x_2)[1+d_{G_2^*}(y_2)]+|e_i|d_{\tilde{G}_2}(y_2)$$

Fix $x \in V_1$,

$$|e_j|d_{\tilde{G}_1}(x)[1+d_{G_2^*}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)=|e_j|d_{\tilde{G}_1}(x)[1+d_{G_2^*}(y_2)]+|e_i|d_{\tilde{G}_2}(y_2)$$

Since G_1^* and G_2^* are complete graphs, we get $d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2)$

$\therefore \tilde{G}_2$ is a regular FSG.

Fix $y \in V_2$,

$$|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^*}(y)]+|e_i|d_{\tilde{G}_2}(y)=|e_j|d_{\tilde{G}_1}(x_2)[1+d_{G_2^*}(y)]+|e_i|d_{\tilde{G}_2}(y)$$

$$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$$

$\therefore \tilde{G}_1$ is a regular FSG.

ii) For $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$, similar process is followed and we get $\tilde{G}_1$ and $\tilde{G}_2$ as regular FSGs.

Theorem 6.7 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs with $\tilde{F}_2(e_j) \leq \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)(y_l) = c \quad \forall y_l \in V_2$ then $\tilde{G}_1 \times_{\gamma} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_2$ is regular, G_1^* and G_2^* are regular graphs.

Proof: Proof is similar to the proof of Theorem 6.5.

7 Conclusions

In this paper, different products of fuzzy soft graphs are defined and demonstrated with some examples. The degree of a vertex in these product FSGs under some conditions and the regular properties associated with it are analyzed. When fuzzy soft graphs are very large these formulas will play a major role which helps in studying some of its properties.

References

- [1] D. Molodtsov, Soft set theory-First results, Computers and Mathematics with Applications, 37 (1999) 19-31.
- [2] A. Rosenfeld, Fuzzy graphs, in: L A Zadeh, K S Fu, M Shimura (eds), Fuzzy sets and their Applications (New York: Academic press 1975), 77-95.
- [3] Lotfi A Zadeh, Fuzzy sets, Information and control, 8 (3) (1965) 338-353.

- [4] Frank Harary, Graph Theory, Narosa/Addison Wesley, Indian Student Edition, (1988).
- [5] P.K. Maji, A. R. Roy, R. Biswas, Fuzzy soft sets, The Journal of Fuzzy Mathematics 9(3) (2001) 589-602.
- [6] J. N. Mordeson and C. S. Peng, Operations on fuzzy graphs, Information Science, 79 (1994) 159-170.
- [7] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, Journal of Computational and Applied Mathematics, 28 (3) (2007) 412-418.
- [8] M. Akram and S. Nawaz, On fuzzy soft graphs, Italian Journal of Pure and Applied Mathematics, 34 (2015) 463-480.
- [9] Sumit Mohanta and T. K. Samanta, An introduction to fuzzy soft graph, Mathematica Moravica, 19(2) (2015) 35-48.
- [10] Shovan Dogra, Different types of product of fuzzy graphs, Progress in Nonlinear Dynamics and Chaos, 3(1) (2015) 41-56.
- [11] M. Akram and S. Nawaz, Fuzzy soft graphs with applications, Journal of Intelligent & Fuzzy System, 30 (2016) 3619-3632.
- [12] Shashikala S and Anil P N, Connectivity in fuzzy soft graph and its complement, IOSR Journal of Mathematics, 12 (5) (2016) 95-99.
- [13] A. Pouhassani and H. Doostie, On the properties of fuzzy soft graphs, Journal of Information and Optimization Sciences, 38: 3-4 (2017) 541-557.
- [14] Anas Al-Masarwah, Majdoleen Abuqamar, Certain types of fuzzy soft graphs, New Mathematics and Natural Computation, (2017) 541-557.
- [15] Shashikala S and Anil P N, Fuzzy soft cycles in Fuzzy soft graphs, Journal of New Results in Science, 8(1) (2019) 26-35.
- [16] B Akhilandeswari, On regular fuzzy soft graphs, International Journal of Scientific Research and Engineering Trends, 5(3) (2019) 831-834.
- [17] Hafiza Saba Nawaz, Muhammad Akram, Oligopolistic competition among the wireless internet service providers of Malaysia using fuzzy soft graphs, Journal of Applied Mathematics and Computing, (2021) 1-36.

A dynamic model of typhoid fever with optimal control analysis

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Abstract

In this study, a deterministic mathematical model of Typhoid fever dynamics with control strategies; vaccination, hygiene practice, sterilization and screening is studied. The model is first analyzed for stability in terms of the control reproduction number, R_c with constant controls. The disease-free equilibrium and endemic equilibrium of the model exist and is shown to be stable whenever $R_c < 1$ and $R_c > 1$, respectively. The model by investigation shows a forward bifurcation and the sensitivity analysis conducted revealed the most biological parameters to be targeted by policy health makers for curtailing the spread of the disease. The optimal control problem is obtained through application of Pontryagin maximum principle with respect to the above-mentioned control strategies. Simulations of the optimal control system and sensitivity of the constant control system confirms that hygiene practice with sterilization could be the best strategy in controlling the disease.

Keywords: Typhoid fever; global stability; sensitivity analysis; optimal control; bifurcation.

2010 AMS subject classification[§]: 92B05, 34D23, 49K15, 37N25, 34D05.

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[§] Received on September 11, 2021. Accepted on December 21, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.657. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1. Introduction

Typhoid fever is a life-threatening infection that is usually caused by *Salmonella enteric serovar Typhi* (S. Typhi) and *Salmonella enteric serovar Paratyphi* (S. Paratyphi, that is Paratyphi A, B, and, uncommonly is S. Paratyphi C) [1]. Typhoid fever has been a public health challenge globally. However, the disease is endemic in most developing countries in Africa and South-East Asia where potable clean water, sanitation and hygiene are either grossly inadequate or non-existent.

The transmission of S. Typhi and S. Paratyphi occur through the consumption of contaminated food or water resulting from inadequate environmental sanitation and hygiene practices [2]. People that are clinically ill from typhoid fever and those who have recovered from it pass out the bacteria in their stools (carriers) and urine [3]. A chronic carrier sheds *Salmonella Typhoid* more than 12 months after onset of illness. Human beings are the only known reservoir of Typhoid and the mode of transmission happens through food and water contaminated by acutely ill or chronic carriers of the bacteria [4].

Vaccine can be taken to prevent Typhoid fever but does not provide long-term immunity [5]. On the other hand, educating travelers moving to typhoid endemic regions on the importance of sanitation and hygiene precautions as well as vaccination will help immensely to preventing the rapid spread of Typhoid disease [4].

Mathematical models of infectious diseases are used to test and compare various intervention strategies especially when there are limited resources [6]. In controlling Typhoid fever, several mathematical models have been formulated. For instance, Mushayabas [7] considered the impact of education campaigns and treatment on the dynamics of Typhoid fever and Abboubakar and Racke [8] carried out a human and bacteria model without considering hygiene practice and individuals protected through vaccination in the population, while Karunditu *et al.* [9], Peter *et al.* [10], Nyerere *et al.* [11], Peter *et al.* [12], Edward and Nyerere [13], Kgosimore and Kelatlehegile [14] and Aji *et al.* [15] considered only human population without factoring in the bacteria concentration in the contaminated food and or water. Tilalum *et al.* [16], Okolo and Abu [17], Peter *et al.* [18], Abboubakar and Racke [19] and Awoke [20] studied the optimal control of typhoid transmission with control measures. None of the aforementioned works studied the combined control measures such as vaccination, hygiene practice, screening of carriers and sterilization of the bacteria in the environment as autonomous or non-autonomous system of equations. This study will bridge these gaps and form a novel contribution to the existing body of knowledge on the subject matter.

Peter *et al.* [10] forms the motivation of this work. They considered Protected, Susceptible, Infected, Treated and Recovery model without the bacteria concentration and the effect of screening of infected carriers and

hygiene practice on transmission dynamics of Typhoid fever. They assumed that the protected class belongs to only individuals that have been vaccinated before entrance into Typhoid endemic population and also optimal control and numerical simulation were not considered in their work. Modifying the work of Peter *et al.* [10], we consider Protected, Susceptible, Infected individuals, Carriers, Recovery and Bacteria concentration model in which some susceptible individuals are protected through vaccination and population practices hygiene which reduces the transmission rate. Hygiene practices which include safe water, sanitation and personal hygiene are crucial in preventing and controlling the spread of Typhoid. In addition, the screening and treating of carriers who are silent spreaders of the disease due to their asymptomatic nature and the sterilization of bacteria concentration in the immediate environment are also important in elimination of typhoid fever in the population. This work will be the first to consider sterilization of the bacteria concentration as a control measure for typhoid fever. Also, the sensitivity analysis for the prediction of appropriate intervention strategies for the control of typhoid fever spread and the optimal control analysis are carried out in this work.

Therefore, a modified version of the work of Peter *et al.* [10] is formulated in Section 2 and a comprehensive mathematical analysis of the model in Section 3. The sensitivity analysis and optimal control strategies of the Typhoid fever dynamics are considered in Section 4 while the numerical simulations and discussion are given in Section 5. Section 6 is the conclusion of the work.

2. Model description and formulation

In this section, the work of Peter *et al.* [10] is modified by considering human population (infected carriers) as well as bacteria concentration. The human population at any time, t is subdivided into five subpopulations namely; protected population, $P(t)$, susceptible population, $S(t)$, infected population, $I(t)$, carrier population, $I_c(t)$ and recovered individuals, $R(t)$. The bacteria concentration is represented by $B_c(t)$. In this study, the protected population, $P(t)$, are susceptible individuals that are vaccinated and individuals coming in from the population that is not at risk of Typhoid fever into typhoid fever endemic population. Infected population, $I(t)$, are infected individuals that are showing symptoms of the disease and are capable of spreading the bacteria in the environment while carrier population, $I_c(t)$, represents asymptomatic infected individuals that are treated but still carrying the Salmonella Typhi. Recovered individuals, $R(t)$, are individuals who have recovered from the disease by treatment or natural immunity.

The protected population, $P(t)$, of the proportion, $\alpha \in (0,1)$ is increased by birth or immigration at a rate, Λ , and also from susceptible individuals that are protected through vaccination at a rate, η . The protected population loses

immunity when the vaccine wanes at a rate, γ . The susceptible population is increased at a rate, $(1 - \alpha)\Lambda$, of the unprotected population through birth or immigration and also from recovered population, $R(t)$, after losing their temporary immunity at a rate, ϕ . Susceptible population contract typhoid disease through food, water or environment contaminated by Salmonella bacteria as a result of inadequate hygiene practice measure at a rate, $(1 - p)\lambda$ and progress to infected population. Here, $\lambda = \frac{\beta B_c}{K + B_c}$ is the force of infection, β is the ingestion or consumption rate of the contaminated food, water or environment, K is the carrying capacity of the bacteria in food, water or environment and $p \in (0,1)$ is the hygiene practice control measure. Infected individuals progress to carrier class at a rate, σ while some infected individuals recovered fully by treatment at rate τ_1 or they die of the disease (bacteria) at a rate, d . Carrier class, $I_c(t)$, recovered by natural immunity at a rate, τ_2 or by early treatment when they are screened at a rate, Ψ with θ as the treatment rate. The natural death rate, μ is assumed for all the human population.

For the bacteria concentration, $B_c(t)$, in the environment, they increased through the shedding from Carriers and symptomatic population, $I_c(t)$ and $I(t)$ at the rates, π_1 and π_2 respectively. The shedding rates, π_1 and π_2 are reduced by p , the level of hygiene practice the infected populations, $I_c(t)$ and $I(t)$, observed. The bacteria decays in the environment at a rate, μ_{B_c} . We assume that there is no human to human transmission but rather human aids in shedding the bacteria in the environment or contaminating the environment; neither there is immigration of infectious humans. Also, disease induced death does not occur in carrier class since they are asymptomatic, that is before the bacteria can cause death, it must have progressed to symptomatic stage. The systematic diagram of model is given in Figure 1.

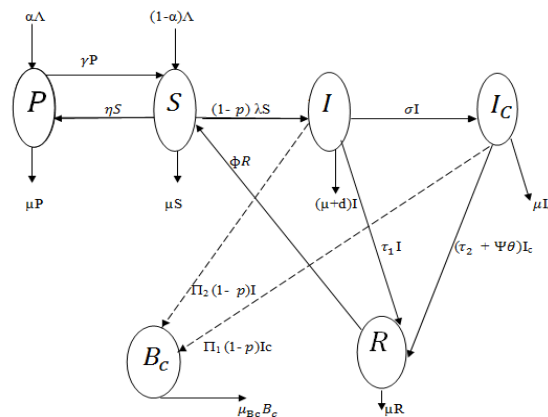


Figure 1. The systematic diagram for Typhoid fever model.

The system of differential equation is derived using the Figure 1 as follows.

$$\left. \begin{aligned} \frac{dP}{dt} &= \alpha\Lambda + \eta S - (\gamma + \mu)P \\ \frac{dS}{dt} &= (1 - \alpha)\Lambda + \gamma P + \phi R - (\eta + \mu + (1 - p)\lambda)S \\ \frac{dI}{dt} &= (1 - p)\lambda S - (\sigma + \tau_1 + \mu + d)I \\ \frac{dI_c}{dt} &= \sigma I - (\tau_2 + \Psi\theta + \mu)I_c \\ \frac{dR}{dt} &= \tau_1 I + (\tau_2 + \Psi\theta)I_c - (\mu + \phi)R \\ \frac{dB_c}{dt} &= \pi_2(1 - p)I + \pi_1(1 - p)I_c - \mu_B B_c \end{aligned} \right\} \quad (1)$$

with initial conditions, $P(0) > 0$, $S(0) > 0$, $I_c(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, $B_c(0) \geq 0$, where $\lambda = \frac{\beta B_c}{(K + B_c)}$ and the model parameters are assumed to be nonnegative.

3. Mathematical Analysis of the Model

3.1 Invariant Region

Invariant region is a region where the model solutions are uniformly bounded.

Theorem 1. All feasible solutions of the model are uniformly bounded in a proper subset $D = D_H \times D_{B_c}$, where $D_H = \{(P, S, I, I_c, R) \in \mathbb{R}_+^5 : N(t) \leq \frac{\Lambda}{\mu}\}$ is a subset for human population and $D_{B_c} = \{B_c \in \mathbb{R}_+ : B_c \leq \frac{[(\pi_2 + \pi_1)(1 - p)]\Lambda}{\mu\mu_B}\}$ is a subset for bacteria concentration in environment.

Proof. The total human population, $N(t)$ is given by $N = P + S + I + I_c + R$ with initial conditions $N(0) = N_0$ and $B_c(0) = B_{c0}$ for the bacteria in the environment. This implies that from equation (1) that $\frac{dN}{dt} = \Lambda - \mu N - dI$. In the absence of disease-induced death rate, that is, $d = 0$, we have $\frac{dN_H}{dt} \leq \Lambda - \mu N$ which by method of integrating factor and the initial condition, $N(0) = N_0$ gives

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right)e^{-\mu t}. \quad (2)$$

As $t \rightarrow \infty$ in equation (2), we have $N(t) \leq \frac{\Lambda}{\mu}$. This means that the feasible solutions of the model for the human population are in the region, $D_H = \{(P, S, I, I_C, R) \in \mathbb{R}_+^5 : N(t) \leq \frac{\Lambda}{\mu}\}$.

For bacteria concentration since $N(t) \leq \frac{\Lambda}{\mu}$, it means that $I \leq \frac{\Lambda}{\mu}$ and $I_C \leq \frac{\Lambda}{\mu}$, we have from the last equation of (1) that

$$\frac{dB_c}{dt} = \pi_2(1-p)N + \pi_1(1-p)N - \mu_B B_c \leq \pi_2(1-p)\frac{\Lambda}{\mu} + \pi_1(1-p)\frac{\Lambda}{\mu} - \mu_B B_c. \quad (3)$$

Solving equation (3) with $B_c(0) = B_{c0}$ as the initial condition yields

$$B_c \leq \frac{(\pi_2 + \pi_1)(1-p)\Lambda}{\mu\mu_B} + \left(B_{c0} - \frac{(\pi_2 + \pi_1)(1-p)\Lambda}{\mu\mu_B}\right)e^{-\mu_B t}. \quad (4)$$

As $t \rightarrow \infty$ in equation (4), we have $B_c \leq \frac{(\pi_2 + \pi_1)(1-p)\Lambda}{\mu\mu_B}$. Therefore, the feasible solution of the bacterial population enters the region $D_{B_c} = \{B_c \in \mathbb{R}_+ : B_c \leq \frac{[(\pi_2 + \pi_1)(1-p)\Lambda]}{\mu\mu_B}\}$. This completes the proof.

Theorem 1 implies that the model is well posed mathematically and epidemiologically. Therefore, it is sufficient enough to study the dynamics of the model (1) in the region $D = D_H \times D_{B_c}$.

3.2 Positivity of the Solutions

Theorem 2. Let $D = \{P, S, I, I_C, R, B_c\} \in \mathbb{R}_+^6$ be solution set such that $P(0) = P_0$, $S(0) = S_0$, $I_C(0) = I_{c0}$, $I(0) = I_0$, $R(0) = R_0$ and $B_c(0) = B_{c0}$ are positive, then the elements of the solution set D are all positive for $t \geq 0$.

Proof. From the first equation of the model equations (1), we have

$$\frac{dP}{dt} = \alpha\Lambda + \eta S - (\gamma + \mu)P \geq -(\gamma + \mu)P. \quad (5)$$

Integrating equation (5) with initial conditions $P(0) = P_0$ yields

$$P(t) \geq P_0 e^{-(\gamma + \mu)t} \geq 0.$$

In a similar way, the rest of the equations of the model equation (1) with initial conditions, $S(0) = S_0$, $I(0) = I_0$, $I_C(0) = I_{c0}$, $R(0) = R_0$ and $B_c(0) = B_{c0}$ give

$$\begin{aligned}
 S(t) &\geq S_0 \exp\left(\int_0^t -(\eta + \mu + (1-p)\lambda) du\right) \geq 0, \\
 I(t) &\geq I_0 \exp\{-(\sigma + \tau_1 + \mu + d)t\} \geq 0, \\
 I_c(t) &\geq I_{c0} \exp\{-(\tau_2 + \Psi\theta + \mu)t\} \geq 0, \\
 R(t) &\geq R_0 \exp\{-(\mu + \phi)t\} \geq 0, \\
 B_c(t) &\geq B_{c0} \exp(-\mu_B t) \geq 0.
 \end{aligned}$$

Therefore, the solution set $\{P(t), S(t), I(t), I_c(t), R(t), B_c(t)\}$, of the system (1) is positive for all $t \geq 0$ since exponential functions and their initial conditions are positive.

3.3 Disease-free equilibrium point and Control Reproduction Number

We compute the control reproduction number, R_c , which is define as the average number of secondary cases reproduced when an infected person is introduced into a population where control measures like vaccination, screening, sanitation and hygiene are in place.

In obtaining this, we apply the next-generation matrix approach [21] at the disease-free disease (DFE). The disease-free equilibrium (DFE) is obtained by equating the right hand side of the equation (1) to zero and solve simultaneously for the disease-free equilibrium, $E_0 = (P^0, S^0, I^0, I_c^0, R^0, B_c^0)$. We have DFE,

$$E_0 = \left(\frac{\Lambda(\alpha\mu+\eta)}{\mu(\gamma+\eta+\mu)}, \frac{\Lambda(\gamma+\mu(1-\alpha))}{\mu(\gamma+\eta+\mu)}, 0, 0, 0, 0 \right).$$

By the principle of next-generation matrix approach, we have

$$F = \begin{pmatrix} 0 & 0 & \frac{a\beta S_0}{K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} k_3 & 0 & 0 \\ -\sigma & k_4 & 0 \\ -\pi_2(1-p) & -\pi_1(1-p) & \mu_B \end{pmatrix}, \quad (6)$$

where

$$a = (1-p), \quad k_1 = (\gamma + \mu), \quad k_2 = (\eta + \mu), \quad k_3 = (\sigma + \tau_1 + \mu + d), \quad k_4 = (\tau_2 + \Psi\theta + \mu), \quad k_5 = (\mu + \phi). \quad (7)$$

Solving for the maximum eigenvalue of the matrix, FV^{-1} , we have

$$R_c = \frac{a\beta S_0[(\sigma\pi_1+\pi_2K_4)(1-p)]}{K\mu_B K_3 K_4}. \quad (8)$$

With the definition of equation (7), we have

$$R_c = \frac{\beta\Lambda(1-p)(\gamma+(1-\alpha)\mu)[(\sigma\pi_1+\pi_2(\tau_2+\Psi\theta+\mu))(1-p)]}{\mu K\mu_B(\gamma+\eta+\mu)(\sigma+\tau_1+\mu+d)(\tau_2+\Psi\theta+\mu)}. \quad (9)$$

The control reproduction number, R_c , can be written as

$$R_c = R_I + R_{IC}, \quad (10)$$

where

$$R_I = \frac{\beta(1-p)^2\pi_2 S_0}{K\mu_B(\sigma+\tau_1+\mu+d)}, \quad R_{IC} = \frac{\beta(1-p)^2\sigma\pi_1 S_0}{K\mu_B(\tau_2 + \Psi\theta+\mu)(\sigma+\tau_1+\mu+d)} \quad (11)$$

denote the reproduction numbers which the infected population and carrier population contributed respectively through their shedding in the environment.

3.4 Local stability of the disease-free equilibrium, E_0

Theorem 3. If E_0 is the DFE of the model, then E_0 is locally asymptotically stable if $R_c < 1$, otherwise it is unstable if $R_c > 1$.

Proof. In proving this theorem, the Jacobian matrix of equation (1) at the disease-free equilibrium, E_0 is given as

$$J(E_0) = \begin{pmatrix} -k_1 & \eta & 0 & 0 & 0 & 0 \\ \gamma & -k_2 & 0 & 0 & \phi & \frac{-a\beta S_0}{K} \\ 0 & 0 & -k_3 & 0 & 0 & \frac{a\beta S_0}{K} \\ 0 & 0 & \sigma & -k_4 & 0 & 0 \\ 0 & 0 & \tau_1 & (\tau_2 + \Psi\theta) & -k_5 & 0 \\ 0 & 0 & \pi_2(1-p) & \pi_1(1-p) & 0 & -\mu_B \end{pmatrix}. \quad (12)$$

The eigenvalues of the Jacobian matrix (12) are $-k_5$ and the solutions of the polynomial

$$\lambda^5 + A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad (13)$$

where

$$\begin{aligned} A &= k_1 + k_2 + k_3 + k_4 + \mu_B, \\ B &= (k_1 + k_2)(k_3 + k_4 + \mu_B) + k_4(k_3 + \mu_B) + \mu(k_2 + \gamma) + k_3\mu_B(1 - R_I), \\ C &= \mu(k_4 + \mu_B)(k_2 + \gamma) + k_1k_2k_3 + k_3\mu_B(k_1 + k_2)(1 - R_I) \\ &\quad + k_3k_4\mu_B(1 - R_c) + k_4(k_1 + k_2)(k_3 + \mu_B), \end{aligned}$$

$$D = \mu k_3 \mu_B (k_2 + \gamma)(1 - R_I) + \mu k_4 (k_3 + \mu_B)(k_2 + \gamma) + k_3 k_4 \mu_B (k_1 + k_2)(1 - R_C),$$

$$E = \mu(k_2 + \gamma)k_3 k_4 \mu_B (1 - R_C).$$

Using the theorem in Heffernan *et al.* [22], the roots of the polynomial (13) have negative real part if $A, B, C, D, E > 0$. With the definition of R_c in equation (10), we have $A, B, C, D, E > 0$ if $R_c < 1$. Therefore, the Jacobian Matrix (12) has negative real eigenvalues if $R_c < 1$. Hence, the disease-free equilibrium, E_0 , is locally asymptotically stable if $R_c < 1$. This ends the proof.

3.5 Global stability of disease-free equilibrium

Theorem 4. The disease-free equilibrium, E_0 , is globally asymptotically stable if $R_c < 1$.

Proof. We construct a Lyapunov function using the infected classes only and this is given by

$$L = \frac{[\sigma\pi_1(1-p)+\pi_2(1-p)k_4]}{\mu_B k_4 k_3} I + \frac{\pi_1(1-p)}{\mu_B k_4} I_C + \frac{1}{\mu_B} B_c. \quad (14)$$

Differentiating (14) with respect to time, t , along the solutions of the model (1) gives

$$\frac{dL}{dt} = \left(\frac{\sigma\pi_1(1-p)+\pi_2(1-p)k_4}{\mu_B k_4 k_3} \right) \left(\frac{a\beta B_c}{(K+B_c)} S - k_3 I \right) + \frac{\pi_1(1-p)}{\mu_B k_4} (\sigma I - k_4 I_C) + \frac{1}{\mu_B} (\pi_2(1-p)I + \pi_1(1-p)I_C - \mu_B B_c). \quad (15)$$

Expanding and simplifying (15) yields

$$\frac{dL}{dt} = \left(\frac{a\beta B_c}{(K+B_c)} \left(\frac{\sigma\pi_1(1-p)+\pi_2(1-p)k_4}{\mu_B k_4 k_3} \right) S - 1 \right) B_c = \left(\frac{R_c K S}{(K+B_c) S_0} - 1 \right) B_c.$$

Since $S \leq S_0$ and $\frac{K}{(K+B_c)} \leq 1$, we have

$$\frac{dL}{dt} \leq B_c (R_c - 1).$$

Clearly, $\frac{dL}{dt} \leq 0$ if $R_c \leq 1$. If $B_c = 0$, $\frac{dL}{dt} = 0$. By virtue of LaSalle's Invariance Principle, the disease-free equilibrium, E_0 , is globally asymptotically stable (GAS) whenever $R_c < 1$.

3.6 Endemic Equilibrium State

The Endemic equilibrium state E^* is a state where the disease is present in the population. At $\frac{dP}{dt} = \frac{dS}{dt} = \frac{dI}{dt} = \frac{dI_c}{dt} = \frac{dR}{dt} = \frac{dB_c}{dt} = 0$, we obtain after solving simultaneously that

$$\begin{aligned} I^* &= \frac{k_4 k_5 (R_c - 1)}{B}, \quad I_c^* = \frac{\sigma k_5 (R_c - 1)}{B}, \quad R^* = \frac{(\tau_1 k_4 + \sigma(\tau_2 + \Psi\theta))(R_c - 1)}{B}, \\ B_c^* &= \frac{[\pi_2(1-p)k_4 + \sigma\pi_1(1-p)]k_5 (R_c - 1)}{B\mu_B}, \\ S^* &= \frac{\Lambda(k_1 - \alpha\mu)B + k_1[\sigma(\tau_2 + \Psi\theta)\phi + \phi k_4 \tau_1 - k_3 k_4 k_5](R_c - 1)}{B(k_1 k_2 - \eta\gamma)}, \\ P^* &= \frac{\alpha\Lambda(B(k_1 k_2 - \eta\gamma)) + \eta[\Lambda(k_1 - \alpha\mu)B + k_1[\sigma(\tau_2 + \Psi\theta)\phi + \phi k_4 \tau_1 - k_3 k_4 k_5](R_c - 1)]}{k_1 B(k_1 k_2 - \eta\gamma)}. \end{aligned}$$

Then, $P^*, S^*, I^*, I_c^*, R^*, B_c^*$ are all positive if and only if $R_c > 1$, which established that the endemic equilibrium state, $E^* = (P^*, S^*, I^*, I_c^*, R^*, B_c^*)$ exists for $R_c > 1$.

3.7 Local Stability of Endemic Equilibrium State

Due to the mathematical complexity of the stability of endemic equilibrium, the Centre manifold theory approach is used to establish the local stability of endemic equilibrium by proving the existence of a forward bifurcation of the system. A forward bifurcation means that the endemic equilibrium is locally asymptotically stable if $R_c > 1$ but near unity.

Theorem 5. The endemic equilibrium is locally asymptotically stable whenever if $R_c > 1$ but near unity.

Proof. Using the approach of Centre manifold theory by Castillo-Chavez and Song [23], let $\beta = \beta^*$ be a bifurcation parameter at $R_c = 1$ so that $\beta = \beta^* = \frac{K\mu_B k_3 k_4}{aS_0[(\sigma\pi_1 + \pi_2 k_4)(1-p)]}$. This implies that the Jacobian matrix of equation (12) has negative eigenvalues and a simple zero eigenvalue.

The left and right eigenvectors associated with the Jacobian matrix (12) are $w = (w_1, w_2, w_3, w_4, w_5, w_6)$ and $v = (v_1, v_2, v_3, v_4, v_5, v_6)$ respectively where

$$\begin{aligned} w_1 &= \frac{\eta(k_3 k_4 k_5 - \phi \tau_1 k_4 - \sigma \phi(\tau_2 + \Psi\theta))w_3}{k_4 k_5 (\eta\gamma - k_1 k_2)}, \quad w_2 = \\ &= \frac{k_1(k_3 k_4 k_5 - \phi \tau_1 k_4 - \sigma \phi(\tau_2 + \Psi\theta))w_3}{k_4 k_5 (\eta\gamma - k_1 k_2)}, \quad w_4 = \frac{\sigma w_3}{k_4}, \quad w_5 = \frac{(\tau_1 k_4 + \sigma(\tau_2 + \Psi\theta))w_3}{k_4 k_5}, \quad w_6 = \\ &= \frac{k_3 w_3}{c}, \quad v_1 = v_2 = v_5 = 0, \quad v_4 = \frac{\pi_1(1-p)cv_3}{k_4 \mu_B}, \quad v_6 = \frac{cv_3}{\mu_B}, \quad w_3, v_3 > 0, \quad c = \frac{a\beta S_0}{K}. \end{aligned}$$

A dynamic model of typhoid fever with optimal control analysis

Representing the state variables $P = x_1$, $S = x_2$, $I = x_3$, $I_c = x_4$, $R = x_5$, $B_c = x_6$ so that the system (1) becomes $\frac{dX}{dt} = F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ with $f_i = f_i(x_1, x_2, x_3, x_4, x_5, x_6)$, we have the non-zero second order partial derivatives at E_0 given as $\frac{\partial^2 f_3(E_0)}{\partial x_2 \partial x_6} = \frac{(1-p)\beta^*}{k}$, $\frac{\partial^2 f_3(E_0)}{\partial x_6^2} = -\frac{2(1-p)\beta^* S_0}{k^2}$, $\frac{\partial^2 f_6(E_0)}{\partial x_6 \partial \beta^*} = \frac{(1-p)S_0}{k}$.

We compute the coefficients, m and n as follows $m = v_3 \left(w_2 w_6 \frac{\partial^2 f_3}{\partial x_2 \partial x_6} + w_6^2 \frac{\partial^2 f_3}{\partial x_6^2} \right)$ and $n = w_3 v_3 \frac{\partial^2 f_3}{\partial x_3 \partial \beta^*} > 0$. Upon substituting, we have

$$m = -\frac{k_3(1-p)\beta^* v_3 w_3^2}{ck} \left(\frac{k_1(k_3 k_4 k_5 - \phi \tau_1 k_4 - \sigma \phi(\tau_2 + \Psi \theta))}{k_4 k_5 (k_1 k_2 - \eta \gamma)} + \frac{2k_3 S_0}{ck} \right) < 0,$$

$$n = \frac{w_3 k_3 v_3 (1-p) S_0}{ck} > 0,$$

which implies that a forward bifurcation exists. Thus, the endemic equilibrium is locally asymptotically stable if $R_c > 1$ but near unity. This is shown graphically in Figure 2.

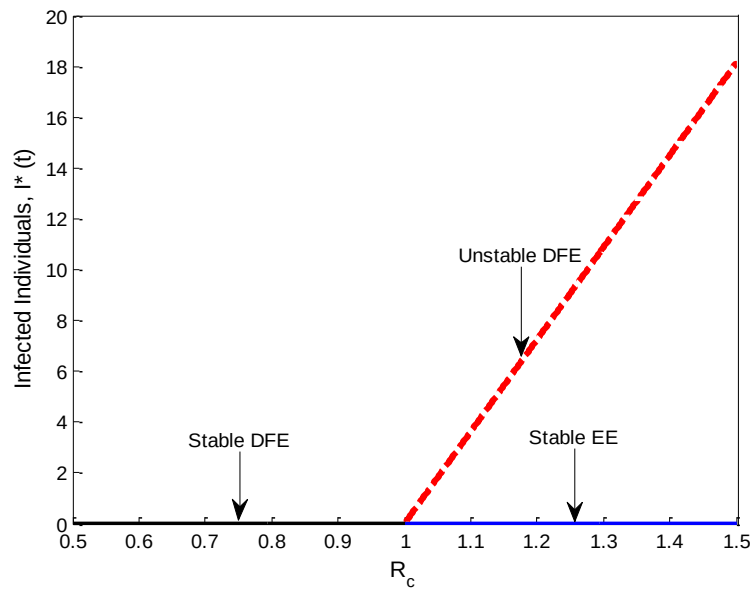


Figure 2. Forward bifurcation for typhoid model.

3.8 Global Stability of Endemic Equilibrium

The global stability of endemic equilibrium, E^* , is established in the absence of disease induced death.

Theorem 4. The endemic equilibrium, E^* , is globally asymptotically stable if $R_c > 1$ and $d = 0$.

Proof. We construct the Lyapunov function given by

$$L = \frac{1}{2}[(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)]^2 + (B_c - B_c^* - B_c^* \ln \frac{B_c}{B_c^*}).$$

Taking the derivative of L along the solutions of equation (1) yields

$$L' = [(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)] \frac{d}{dt} (P + S + I + I_c + R) + \left(1 - \frac{B_c^*}{B_c}\right) \frac{dB_c}{dt}$$

which upon substitution gives

$$L' = [(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)](\Lambda - \mu(P + S + I + I_c + R) - dI) + \left(1 - \frac{B_c^*}{B_c}\right)(\pi_2(1 - p)I + \pi_1(1 - p)I_c - \mu_B B_c). \quad (16)$$

Substituting at endemic equilibrium,

$$\Lambda = \mu(P^* + S^* + I^* + I_c^* + R^*) + dI^*, \mu_B = \frac{\pi_2(1-p)I^*}{B_c^*} + \frac{\pi_1(1-p)I_c^*}{B_c^*}$$

in equation (16) and simplify, we have

$$L' = -\mu[(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)]^2 + \pi_2(1 - p)I^* \left[1 + \frac{I}{I^*} - \frac{B_c}{B_c^*} - \frac{B_c^* I}{I^* B_c}\right] + \pi_1(1 - p)I_c^* \left[1 + \frac{I_c}{I_c^*} - \frac{B_c}{B_c^*} - \frac{B_c^* I_c}{I_c^* B_c}\right] - d(I - I^*)[(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)]. \quad (17)$$

Using the hypothesis that $d = 0$ in (17), we have

$$L' = -\mu[(P - P^*) + (S - S^*) + (I - I^*) + (I_c - I_c^*) + (R - R^*)]^2 + (\pi_2(1 - p)I^* + \pi_1(1 - p)I_c^*) \left[2 - \frac{B_c}{B_c^*} - \frac{B_c^*}{B_c}\right]$$

with $\frac{I_c}{I_c^*} \leq 1$, $\frac{I}{I^*} \leq 1$. This implies that $L' \leq 0$ since $2 - \frac{B_c}{B_c^*} - \frac{B_c^*}{B_c} \leq 0$, by arithmetic and geometric theorem and $L = 0$ if $P = P^*$, $S = S^*$, $I = I^*$, $I_c = I_c^*$, $R = R^*$ and $B_c = B_c^*$. This means that the endemic equilibrium, E^* , is globally asymptotically stable (GAS) whenever $R_c > 1$ and $d = 0$ according to LaSalle's Invariance Principle.

4. Sensitivity Analysis and Optimal Control Analysis

4.1 Sensitivity Analysis of the Model Parameters

Sensitivity analysis is used to examine the connection between uncertain parameters of a mathematical model and a property of the observable output [24]. It is used to determine the model parameters that have a great impact on reproduction number, R_c for the purpose of targeting such by intervention strategies [25]. In carrying out the sensitivity analysis, we adopted normalized forward index method as used by Rodrigues *et al.* [25] and this is given by $S_Y^{R_c} = \frac{\partial R_c}{\partial Y} \times \frac{Y}{R_c}$, where Y is the parameters reflecting in the control reproduction number, R_c . The sensitivity indices of R_c are given in Table 2 using the parameter values in Table 1.

Table 1. Parameter values of the model with their sources

Parameter	Index sign	Sensitivity index values	Parameter	Index sign	Sensitivity index values
Λ	100	[16]	μ	0.0247	[16]
γ	0.33	[26]	d	0.066	[30]
α	0.5	[27]	π_1	0.9	[30]
ϕ	0.000904	[28]	π_2	0.8	[16]
σ	0.03-0.05	[28]	μ_B	0.0345	[31]
τ_1	0.002	[16]	Ψ	0.75	[11]
τ_2	0.0003	[16]	θ	0.2	[16]
p	0.3	Assumed	β	0.9	[16]
η	0.75	[11]	K	500000	[29]

Table 2. Sensitivity index of the parameters values

Parameter	Index sign	Sensitivity index values	Parameter	Index sign	Sensitivity index values
γ	+	0.66520217	d	—	0.53789731
α	—	0.03617419	π_1	+	0.161676646
σ	—	0.08282213	π_2	+	0.838323354
τ_1	—	0.01629918	μ_B	—	0.999999999
τ_2	—	0.00027716	Ψ	—	0.138579983
p	—	1.99999999	θ	—	0.138579983

η	–	0.67891735	β	+	1.00000000

From Table 2, the parameters with positive indices (π_1, π_2, γ) indicate that they have impact on expanding the disease in the population if their values are increasing because the control reproduction number increases as their values increase. Also, the parameters in which their sensitivity indices are negative have influence in reducing the burden of the bacteria in the population as their values increase because the control reproduction number decreases as their values increase, which will lead to reducing the endemicity of the bacteria in the population.

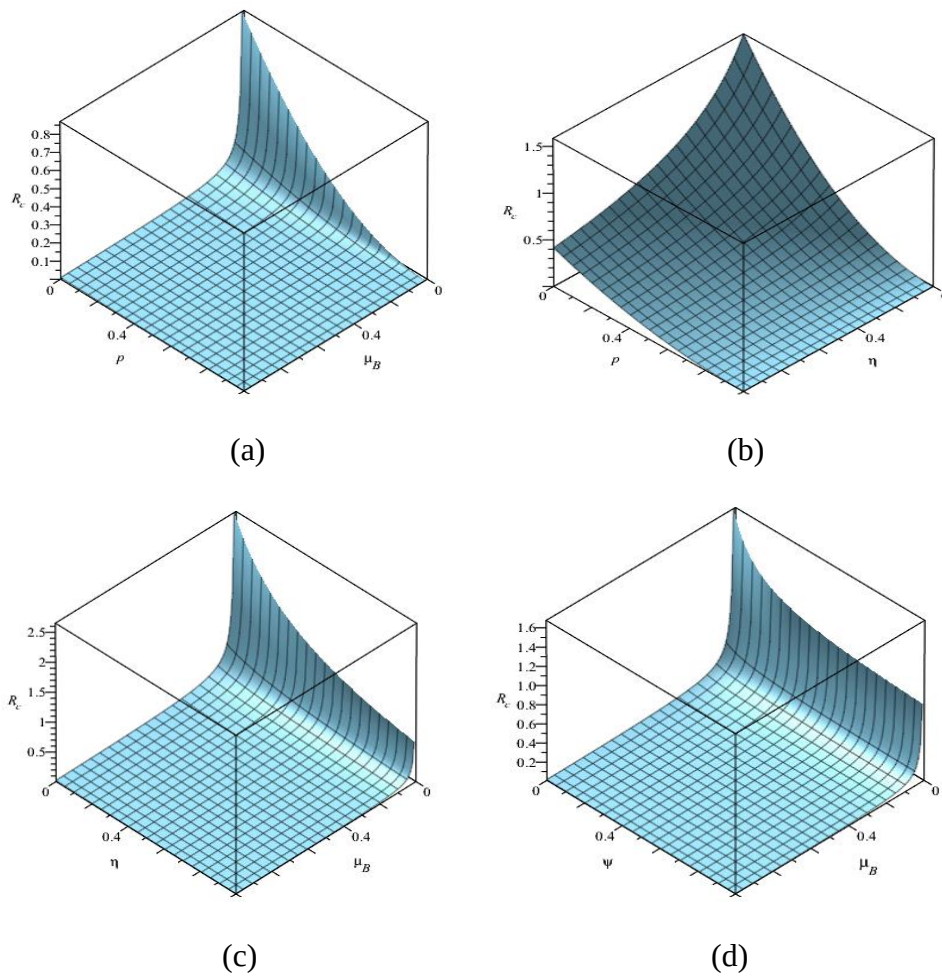


Figure 3. Simulations for the impact of model parameters on control reproduction number.

According to the phase plane (Figure 3a), the value of R_c decreases drastically as ρ and μ_B increases. Also, the value of R_c decreases sharply in

Figure 3b as ρ increases, but the change of η has a significantly lower impact on R_c . The phase planes in Figures 3c and 3d illustrate similar results, which shows that R_c is much sensitive to μ_B than to η and ψ , respectively. Therefore, from all cases, μ_B has shown to be a superior force in reducing the burden of Typhoid fever. However, the combination of ρ and μ_B has proven to be the best control strategy as compared to the rest.

4.2 Optimal Control Analysis

The optimal control model is formulated from system (1) when the constant parameters, η, p, Ψ, μ_B are time dependent that is $\eta(t), p(t), \Psi(t)$ and $\mu_B(t)$ where $\eta(t)$ is vaccination control, $p(t)$ is hygiene practice control, $\Psi(t)$ is the screening control and $\mu_B(t)$ is the sterilization control.

The objective function to be minimized is given as

$$J(\eta(t), p(t), \Psi(t), \mu_B(t)) = \int_0^{t_f} \left(I + I_C + Bc + \frac{m_1 \eta^2(t)}{2} + \frac{m_2 p^2(t)}{2} + \frac{m_3 \Psi^2(t)}{2} + \frac{m_4 \mu_B^2(t)}{2} \right) dt \quad (18)$$

subject to equation (1) with

$$\eta = \eta(t), p = p(t), \Psi = \Psi(t), \mu_B = \mu_B(t). \quad (19)$$

The coefficients, a, b, c , are the weight constants for the infected, carriers and the bacteria concentration respectively whereas $m_i, i = 1, 2, 3, 4$ are cost of implementing these control measures. We assume a quadratic expression for the costs based on literature. The control measures, $\eta(t), p(t), \Psi(t), \mu_B(t)$ are Lebesgue measurable with $0 \leq \eta(t) < 0.9, 0 \leq p(t) < 1, 0 \leq \Psi(t) < 1, 0 \leq \mu_B(t) < 1$ for $0 \leq t \leq t_f$.

We aimed to minimize the number of infectives, carriers, bacteria concentration and their costs of implementations, that is, $J(\eta(t)^*, p(t)^*, \Psi(t)^*, \mu_B(t)^*) = \min J(\eta(t), p(t), \Psi(t), \mu_B(t))$. The optimal control pair is obtained using Pontryagin maximum principle [32]. This principle converts equations (18) and (1) with (19) into a problem of minimizing pointwise a Hamiltonian H with respect to $\eta(t), p(t), \Psi(t), \mu_B(t)$ such that;

$$H(P, S, I, I_C, R, B_C) = \frac{dJ}{dt} + \lambda_1 \frac{dP}{dt} + \lambda_2 \frac{dS}{dt} + \lambda_3 \frac{dI}{dt} + \lambda_4 \frac{dI_C}{dt} + \lambda_5 \frac{dR}{dt} + \lambda_6 \frac{dB_C}{dt}.$$

Thus,

$$H(P, S, I, I_C, R, B_C) = \left(aI + bI_C + \frac{m_1\eta(t)^2}{2} + \frac{m_2p(t)^2}{2} + \frac{m_3\Psi(t)^2}{2} + \frac{m_4\mu_B(t)^2}{2} \right) + \lambda_1 \frac{dP}{dt} + \lambda_2 \frac{dS}{dt} + \lambda_3 \frac{dI}{dt} + \lambda_4 \frac{dI_C}{dt} + \lambda_5 \frac{dR}{dt} + \lambda_6 \frac{dB_C}{dt}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are the adjoint variable functions.

So,

$$\begin{aligned} H = & \left(aI + bI_C + cB_C + \frac{m_1\eta(t)^2}{2} + \frac{m_2p(t)^2}{2} + \frac{m_3\Psi(t)^2}{2} + \frac{m_4\mu_B(t)^2}{2} \right) + \\ & \lambda_1(\alpha\Lambda + \eta(t)S - (\gamma + \mu)P) + \lambda_2 \left((1 - \alpha)\Lambda + \gamma P + \phi R - (\eta(t) + \mu + \right. \\ & \left. (1 - p(t)) \frac{\beta B_C}{(K+B_C)}) S \right) + \lambda_3 \left((1 - p(t)) \frac{\beta B_C}{(K+B_C)} S - (\sigma + \tau_1 + \mu + d)I \right) + \\ & \lambda_4(\sigma I - (\tau_2 + \Psi(t)\theta + \mu)I_C) + \lambda_5(\tau_1 I + (\tau_2 + \Psi(t)\theta)I_C - (\mu + \\ & \phi)R) + \lambda_6(\pi_2(1 - p(t))I + \pi_1(1 - p(t))I_C - \mu_B(t)B_C). \end{aligned} \quad (20)$$

Theorem 5. Given an optimal control $\eta(t)^*, p(t)^*, \Psi(t)^*, \mu_B(t)^*$ and corresponding state variables P, S, I, I_C, R , and B_C that minimize the objective function $J(\eta(t), p(t), \Psi(t), \mu_B(t))$ over U , there exist adjoint functions $\lambda_1, \dots, \lambda_6$ satisfying;

$$\left. \begin{aligned} \frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_2)\gamma + \mu\lambda_2, \\ \frac{d\lambda_2}{dt} &= (\lambda_2 - \lambda_1)\eta(t) + \mu\lambda_2 + (\lambda_2 - \lambda_3)(1 - p(t)) \frac{\beta B_C}{(K+B_C)}, \\ \frac{d\lambda_3}{dt} &= -a + (\lambda_3 - \lambda_4)\sigma + (\lambda_3 - \lambda_5)\tau_1 + \lambda_3(\mu + d) - \pi_2(1 - p(t))\lambda_6, \\ \frac{d\lambda_4}{dt} &= -b + (\lambda_4 - \lambda_5)(\tau_2 + \Psi(t)\theta) + \lambda_4\mu - \pi_1(1 - p(t))\lambda_6, \\ \frac{d\lambda_5}{dt} &= (\lambda_5 - \lambda_4)\phi + \lambda_5\mu, \\ \frac{d\lambda_6}{dt} &= -c + (\lambda_2 - \lambda_3)(1 - p(t)) \frac{\beta}{(K+B_C)} \left(1 - \frac{B_C}{(K+B_C)} \right) + \lambda_6\mu_B(t). \end{aligned} \right\} \quad (21)$$

with the transversality condition, $\lambda_i(t_f) = 0$, for $i = 1(1)6$ and the controls $\eta^*(t), p^*(t), \Psi^*(t)$ and $\mu_B^*(t)$ satisfying the optimality condition;

$$\left. \begin{aligned} \eta^* &= \max \left\{ 0, \min \left(1, \frac{(\lambda_2 - \lambda_1)S}{m_1} \right) \right\}, \\ p^* &= \left\{ 0, \min \left(1, \frac{(\lambda_3 - \lambda_2)\beta B_C S + \lambda_6 \pi_2 I (K+B_C) + \lambda_6 \pi_1 I_C (K+B_C)}{m_2 (K+B_C)} \right) \right\}, \\ \Psi^* &= \max \left\{ 0, \min \left(1, \frac{(\lambda_4 - \lambda_5)\theta I_C}{m_3} \right) \right\}, \\ \mu_B^* &= \max \left\{ 0, \min \left(1, \frac{\lambda_6 B_C}{m_4} \right) \right\}. \end{aligned} \right\} \quad (22)$$

Proof. Using Pontryagin maximum principle, we obtained the adjoint equation and transversality conditions by differentiating the Hamiltonian function with respect to state variables P, S, I, I_c, R and B_c respectively which is evaluated at the optimal control functions $\eta(t)$, $p(t)$, $\Psi(t)$, $\mu_B(t)$. So, the adjoint system (21) is obtained using the following derivatives

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial P}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial S}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I}, \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial I_c}, \frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R}, \frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial B_c}$$

while the interior of the control set of equation (22) is obtained by solving for $\eta(t)$, $p(t)$, $\Psi(t)$, $\mu_B(t)$ in the respective equations

$$\frac{\partial H}{\partial \eta(t)} = 0, \frac{\partial H}{\partial p(t)} = 0, \frac{\partial H}{\partial \Psi(t)} = 0, \frac{\partial H}{\partial \mu_B(t)} = 0.$$

This completes the proof.

The optimality system involves equation (1) with (19), equations (21) and (22).

5. Numerical Simulations and Discussion

The numerical simulations of the optimality system involving equations (1) with (19), (21) and (22) are implemented using Runge-Kutta method with the aid of MATLAB R2007b. The simulations are carried out to examine the impact of the control measures on Typhoid fever. The parameter values used for the simulations are in Table 2 while the initial conditions are from Mushanyu *et al.* (2018) as follows; $S(0) = 10000$, $I(0) = 10$, $I_c(0) = 10$, $R(0) = 0$, $B_c(0) = 100000$. $P(0) = 100$ is assumed. The weight constants for simulation are given as $m_1 = 9 \times 10^{-1}$, $m_2 = 5 \times 10^5$, $m_3 = 7 \times 10^2$ and $m_4 = 4 \times 10^6$.

(a) **Optimal and constant control.** The importance of time-dependent control measures is considered in Figure 3. With optimal control, a typhoid-free population is attained within 200 days compared with constant control which shows the endemicity of the typhoid in the population. This is achieved when u_1 is at the upper bound for 150 days and u_2 , u_3 and u_4 are below a bound of 0.3 for 175 days before they decline to their final time. This implies that control measures should be implemented in time to achieve a typhoid free population.

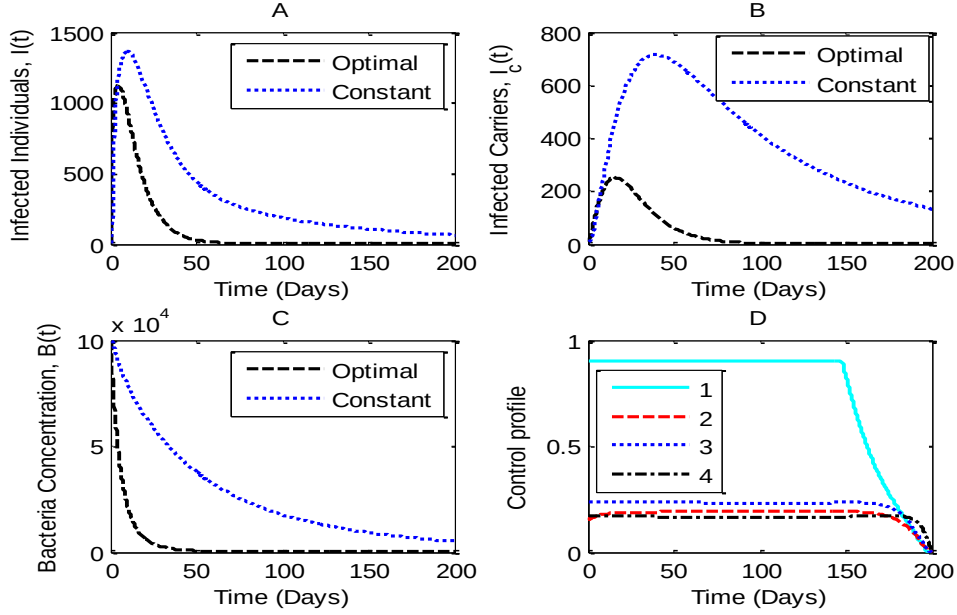


Figure 4. Solutions of Typhoid model for the infected state variables with and without control measures with control profile.

(b) **Vaccination and hygiene practices.** We minimize the objective function for vaccination and hygiene practices ($u_1, u_2 \neq 0, u_3 = u_4 = 0$) to assess their effect on the disease. The number of infected individuals and bacteria concentration are reduced when compared to without control (See Figure 5). This is obtained when u_1 is at its upper bound for all the time 200 days and u_2 attains a bound of 0.9 and decline after 5 days (Figure 5D). However, typhoid disease still remains in the population.

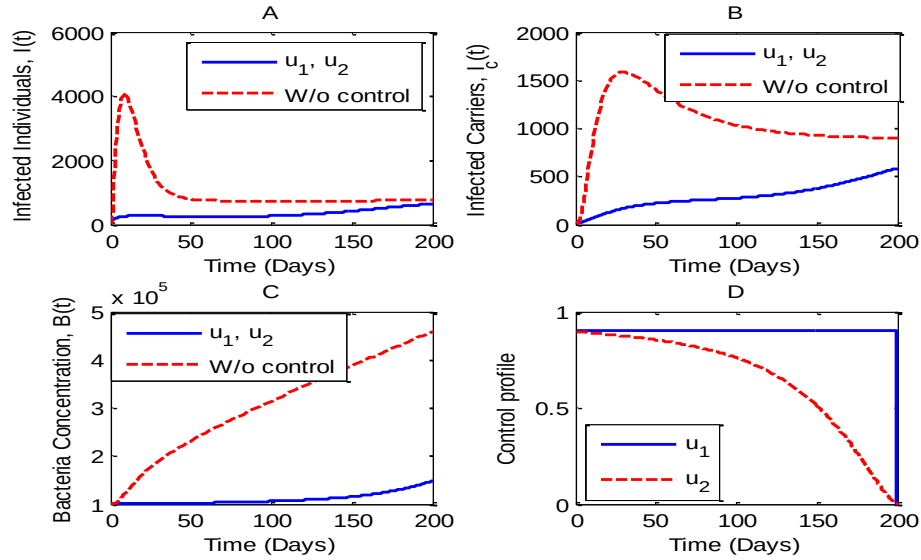


Figure 5. Solutions of Typhoid model for the infected state variables without and with vaccination (u_1) and hygiene practices (u_2) control measures only. W/o means without.

(c) **Vaccination and screening.** We minimize the objective function for vaccination and screening ($u_1, u_3 \neq 0, u_2 = u_4 = 0$). They reduced the number of infected persons and bacteria concentration but not as in case (b) (see Figures 5 and 6) as the number of carriers reduces in Figure 6B than Figure 5B. This may be as a result of screening in the combined control measures. This is achieved when the control, u_1 , is maintain at the upper bound for all time (200 days) while u_3 decline after attaining upper bound for 110 days (Figure 6D).

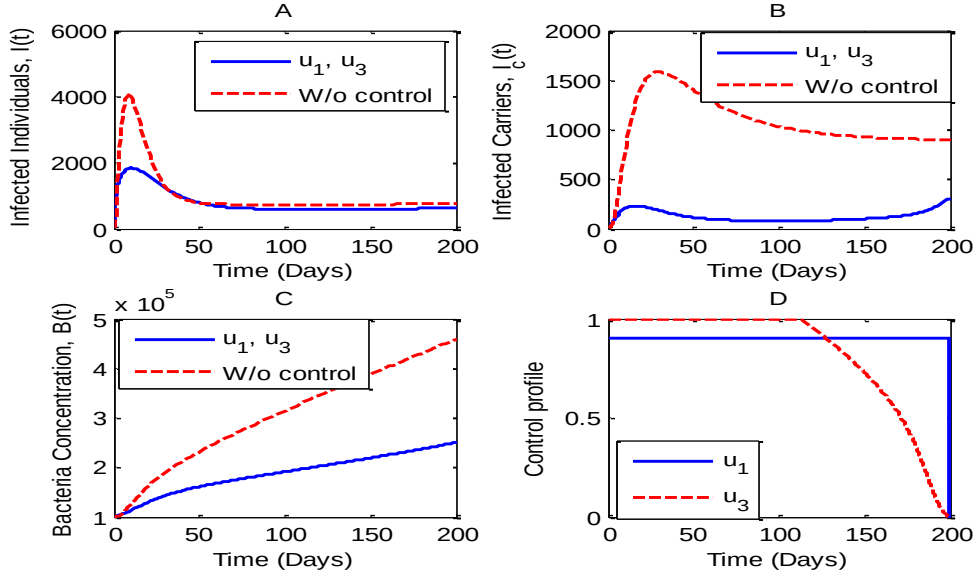


Figure 6. Solutions of Typhoid model for the infected state variables without and with vaccination (u_1) and screening (u_3) control measures only. W/o means without.

(d) **Vaccination and sterilization.** We minimize the objective function for vaccination and sterilization ($u_1, u_4 \neq 0, u_2 = u_3 = 0$). The simultaneous implementation of u_1 and u_4 reduced the number of infected persons and bacteria concentration to zero after 70 days and 30 days respectively while the number of carriers in the population is almost zero as at 200 days. The control, u_1 , maintains an upper bound for 200 days while u_4 attains a bound of 0.2 for 190 days before decline to its final time.

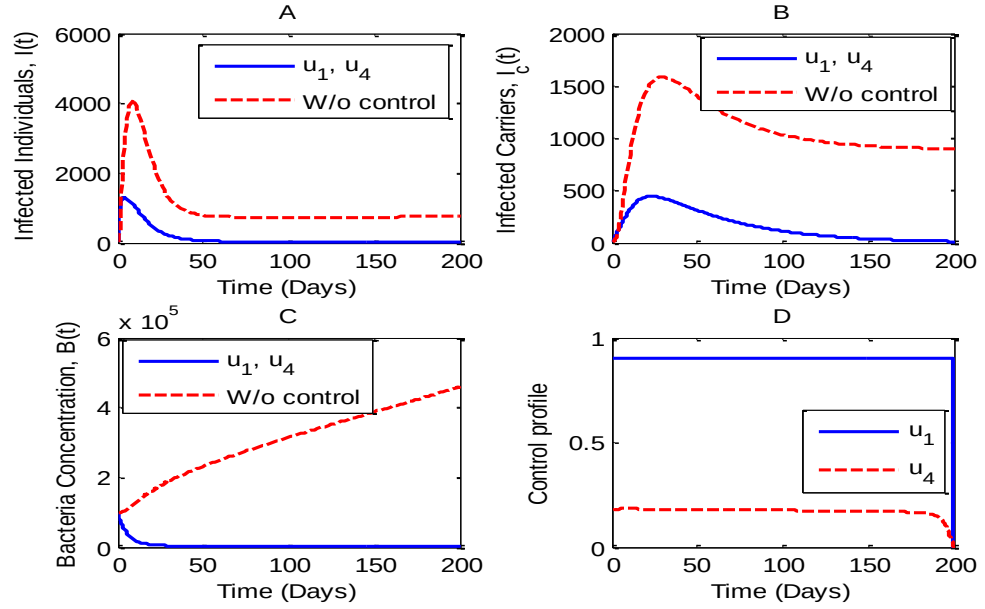


Figure 7. Solutions of Typhoid model for the infected state variables without and with vaccination (u_1) and sterilization (u_4) control measures only. Here, W/o means without.

(e) **Hygiene practices and screening.** We minimize the objective function for hygiene practices and screening ($u_2, u_3 \neq 0, u_1 = u_4 = 0$). The observed effect is similar to case (c) except that u_3 attains an upper bound and declines after 70 days while u_2 of a bound of 0.55 and declines immediately to final time. The disease still remains endemic in the population.

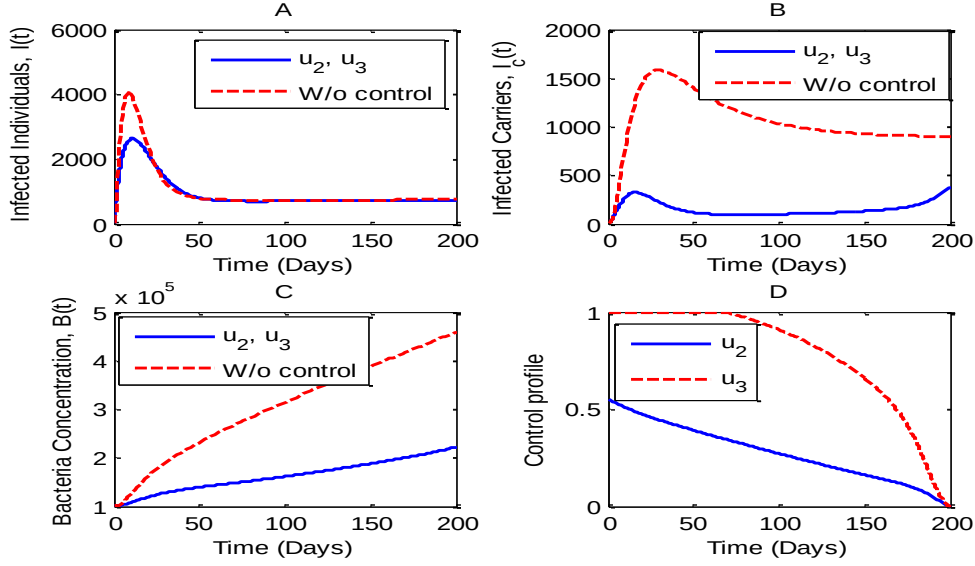


Figure 8. Solutions of Typhoid model for the infected state variables without and with hygiene practices (u_2) and screening (u_3) control measures only. Here, W/o means without.

(f) **Hygiene practices and sterilization.** We minimize the objective function for hygiene practices and sterilization as control measures ($u_2, u_4 \neq 0, u_1 = u_3 = 0$). The combined implementation of u_2 and u_4 reduces the number of infected persons and bacteria concentration to zero after 110 days and 50 days respectively while there is still some infected carriers in the population after 200 days. The hygiene practice u_2 , initially increases from 0.18 to 0.28 bound within 8 days and declines after 120 days while u_4 attains a bound of 0.2 for 195 days before declining to its final time.

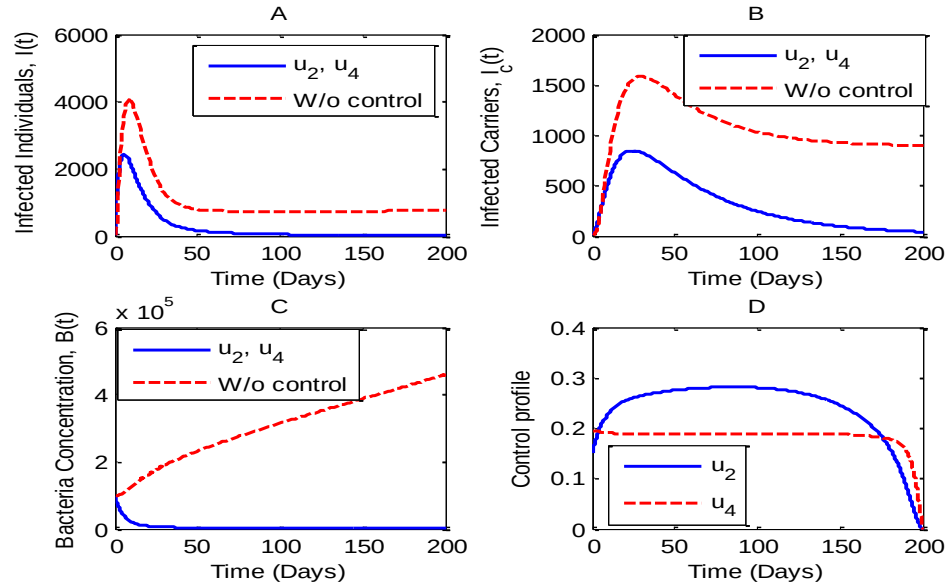


Figure 9. Solutions of Typhoid model for the infected state variables without and with hygiene practices (u_2) and sterilization (u_4) control measures only. Here, W/o means without.

(g) **Screening and sterilization.** We minimize the objective function for screening and sterilization as control measures ($u_3, u_4 \neq 0, u_1 = u_2 = 0$). The simultaneous implementation of u_3 and u_4 behaves similar as cases (e) and (f). Here, the number of infected persons, carriers and bacteria concentration reduce to zero after 75 days, 100 days and 45 days respectively. This is achieved when u_3 and u_4 are at bound 0.28 for 170 days and 0.19 for 190 days respectively before declining to their final time.

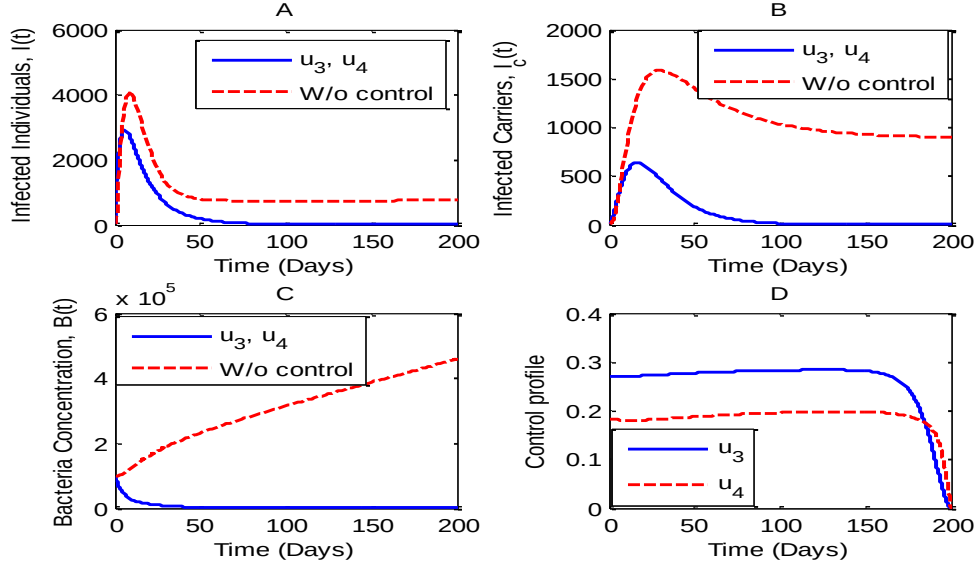


Figure 10. Solutions of Typhoid model for the infected state variables without and with screening (u_3) and sterilization (u_4) control measures only. Here, W/o means without.

(h) **Three combine control measures** We minimize the objective function for three control measures that is $u_1, u_2, u_3, \neq 0, u_4 = 0$ (123), $u_1, u_2, u_4, \neq 0, u_3 = 0$ (124), $u_1, u_3, u_4, \neq 0, u_2 = 0$ (134) and $u_2, u_3, u_4, \neq 0, u_1 = 0$ (234). We notice from Figure (10) that bacteria clearance reduces the number of infected populations ($I(t), I_c(t)$) and bacteria concentration. However, the combine implementation of vaccination, screening and sterilization gives a better result compared to u_1, u_2, u_4 , and u_1, u_3, u_4 , as it achieves a typhoid-free population in shortest period of time than others.

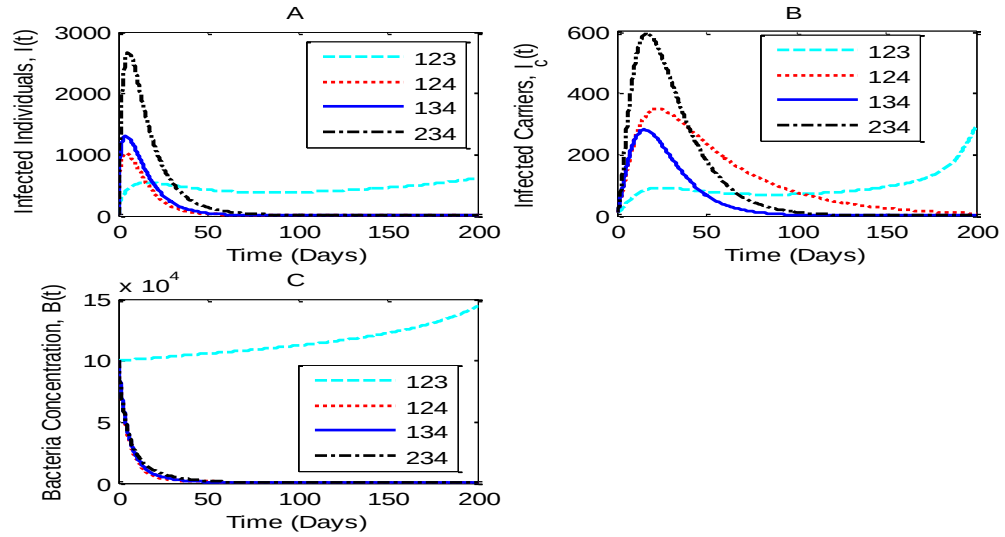


Figure 11. Solutions of Typhoid model for the infected state variables with optimal control. Here, 123 means u_1, u_2, u_3 combine, 124 means u_1, u_2, u_4 combine, 134 means u_1, u_3, u_4 combine, 234 means u_2, u_3, u_4 combine.

7. Conclusion

In this study, the mathematical model of Typhoid fever dynamics with protected human population and bacteria concentration is examined. The control measures such as vaccination, hygiene practice and screening are taken into consideration. The disease-free and endemic equilibrium states are both locally and globally stable whenever $R_c < 1$ and $R_c > 1$ respectively. The local stability of endemic equilibrium state is established using Centre manifold theorem in order to show existence of forward bifurcation while the global stability is done when disease-related death rate is neglected. The sensitivity analysis of the control reproduction number is carried out and the result indicates that the typhoid fever disease will be controlled in the population if susceptible people are vaccinated with high practice of personal hygiene as well as screening of the carriers are screened and also the bacteria in the environment is disinfect or sterilization.

The optimal control analysis is carried out for time-dependent control functions to form non-autonomous system. The Pontryagin maximum principle is used to establish the optimality conditions for the system. This is solved numerically to establish that optimal control implementation achieved infection-free population on time compare to constant control. Considering when there is limited resources to implement all the controls together, screening and bacteria sterilization should be adopted for two combined controls, while vaccination,

screening and bacteria sterilization should be implemented together for three combine controls. However, the combined implementation of all controls is more effective in eradicating the disease from the environment. It is therefore recommended that these preventive measures (vaccination, hygiene practice, screening and sterilization) should be adopted by the policy makers to eliminate the typhoid bacteria from the population.

References

- [1] WHO. Typhoid. www.who.int/news-room/fact-sheets/detail/typhoid, 2020. Accessed on 10th May, 2021.
- [2] M. M. Gibani, M. Voysey, C. Jin, C. Jones, H. Thomaides-Brears, ..., A. J. Pollard. The impact of vaccination and prior exposure on stool shedding of salmonella typhi and salmonella paratyphi in 6 controlled human infection studies. *Clinical Infectious Diseases*, 68(8): 1265-1273, 2019.
- [3] CDC. Typhoid fever. www.nc.cdc.gov/travel/diseases/typhoid, 2020. Accessed on 11th September, 2021.
- [4] A. C. Bradley and S. Eli. Typhoid and paratyphoid fever in travellers. *Lancet Infect Dis.*, 5(10):623-8, 2005.
- [5] S. Cairncross, C. Hunt, S. Boisson, K. Bostoen, V. Curtis, I. C. H. Fung and W. Schmidt. Water, sanitation and hygiene interventions and the prevention of Diarrhoea. *Int J Epidemiol*, Suppl 1(Suppl 1):193-205, 2010.
- [6] I. Bakach, M. R. Just, M. Gambhir and I. C. Fung. Typhoid transmission: A historical perspective on mathematical model development. *Trans R Soc Med Hyg.*, 109(11): 679-89, 2015.
- [7] S. Mushayabasa. A simple epidemiological model for typhoid with saturated incidence rate and treatment effect. *International Journal Mathematical and Computational Sciences*, 6(6): 688-695, 2012.
- [8] H. Abboubakar and R. Racke. Mathematical modelling and optimal control of typhoid fever. *Konstanzer Schriften in Mathematik*, Universitat Konstanz, Konstanzer Online-Publikatins-System (KOPS), Nr. 384, Dezember 2019.
- [9] J. W. Karunditu, G. Kimathi and S. Osman. Mathematical modeling of typhoid fever disease incorporating unprotected humans in the spread dynamics. *Journal of Advances in Mathematical and Computer Science*, 32(3):1-11, 2019.
- [10] O. J. Peter, M. O. Ibrahim, A. Oluwaseun and R. Musa. Mathematical model for the control of typhoid fever. *IOSR Journal of Mathematics*, 13(4): 60-66, 2017.

- [11] N. Nyerere., S. C. Mpeshe and S. Edward. Modeling the impact of screening and treatment on the dynamics of typhoid fever. *World Journal of Modelling and Simulation*. 14(4): 298-306, 2018.
- [12] O. J. Peter, A. Afolabi, F. A. Oguntolu, C. Y. Ishola and A. A. Victor. Solution of a deterministic mathematical model of typhoid fever by variational iteration method. *Science World Journal*, 13(2): 64-68, 2018.
- [13] S. Edward and N. Nyerere. A Mathematical model for the dynamics of cholera with control measure. *Applied and Computational Mathematics*, 4(2): 53-63, 2015.
- [14] M. Kgosimore and G. R. Kelatlehegile. Mathematical analysis of typhoid infection with treatment. *Journal of Mathematical Sciences: Advancse and Applications*, 40: 75-91, 2016.
- [15] B. S. Aji, D. Aldila and B. D. Handari. Modeling the impact of limited treatment resources in the success of typhoid intervention. *AIP Conference Proceedings, International Conference on Science and Applied Science*, 2202(1):020040, 2019. Doi:10.1063/1.5141653
- [16] G. T. Tilahum. O. D. Makinde and D. Malonza. Modelling and optimal control of typhoid fever disease with cost-effective strategies. *Computational and Mathematical Methods in Medicine*, Article ID 2324518, 2017.
- [17] P. N. Okolo and O. Abu. On optimal control and cost-effectiveness analysis for typhoid fever model, *FUDMA Journal of Sciences (FJS)*, 4(3): 437 – 445, 2020.
- [18] O. J. Peter, M. O. Ibrahim, H. O. Edogbanya, F. A. Oguntolu, K. Oshinubi, A. A. Ibrahim, T. A. Ayoola and J. O. Lawal. Direct and indirect transmission of typhoid fever model with optimal control. *Results in Physics*, 27, 104463, 2021.
- [19] H. Abboubakar and R. Racke. Mathematical modelling forecasting and optimal control of typhoid fever transmission dynamics. *Chaos, Solitons and Fractals*, 149, 111074, 2021.
- [20] T. D. Awoke. Optimal control strategy for the transmission dynamics of typhoid fever. *American Journal of Applied Mathematics*, 7(2):37 – 49, 2019.
- [21] J. H. Jones. Notes on R_0 . Department of Anthropological Sciences, Stanford University, May 1, 2007.
- [22] J. M. Heffernan, R. J. Smith and L. M. Wahl. Perspectives on the basic reproductive ratio. *Journal of the Royal Society Interface*. 2(4): 281-293, 2005.
- [23] C. Castillo-Chavez and B. Song. Dynamical model of tuberculosis and their applications. *Mathematical Bioscience and Engineering*, 1(2): 361-404, 2004.

- [24] A. Saltelli, M. Ratto, T. Andres, F. Campolongo, J. Cariboni, D. Gatelli, M. Salsana and S. Tarantola. Global sensitivity analysis - the primer. Wiley, Chichester, 2008.
- [25] H. S. Rodrigues, M T. T. Monteiro and D. F. M. Torres. Sensitivity analysis in a dengue epidemiological model. Conference Papers in Mathematics, 2013: Articles ID: 721406, 2013. doi: 10.1155/2013/721406
- [26] D. T. Lauria, B. Maskery, C. Poulous and D. Whittington. An optimization model for reducing typhoid cases in developing countries without increasing public spending. Vaccine, 27(10): 1609-1621, 2009.
- [27] S. Mushayabasa. Impact of vaccine on controlling typhoid fever in Kassena-Nankana district of Upper East Region of Ghana: Insights from a mathematical model. Journal of Modern Mathematics and Statistics. 5(2): 54-59, 2011.
- [28] I. A. Adetunde. Mathematical models for the dynamics of typhoid fever in Kassena-Nankana district of Upper East Region of Ghana. Journal of Modern Mathematics and Statistics. 2(2): 45-49, 2008.
- [29] M. Gosh, P. Chandra, P. Sinha. and J. B. Shukla. Modelling the spread of bacterial infectious disease with environmental effect in a logistically growing human population. Nonlinear Analysis Real World Applications, 7(3): 341–363, 2006.
- [30] J. Mushanyu, F. Nyabadza, G. Muchatibaya, P. Mafuta and G. Nhawu. Assessing the potential impact of limited public health resources on the spread and control of typhoid. Journal of Mathematical Biology, 77(3):647-670, 2018.
- [31] J. M. Mutual, F. B. Wang and N. K. Vaidya. Modeling malaria and typhoid co-infection dynamics. Mathematical Bioscience, 264(1): 128-144, 2015.
- [32] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko. The mathematical theory of optimal processes, John Wiley & Sons, Lonon, UK, 1962.

Contra N_α -I-Continuity over Nano Ideals

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Abstract

The conceptualization of N_α -I-open sets and N_α -I-continuous functions in nano ideal topology are used to study contra N_α -I-continuity. Also the characteristics and behaviours of contra N_α -I-continuity based on Nano Urysohn Space and Nano Ultra Hausdorff Space are discussed. **Keywords:** CN_α -Cts function, CN_α -I-Cts function, N_α -I- T_2 space, N_α -I-connected.

2020 AMS subject classifications: 54A05, 54B05. ¹

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¹Received on August 7, 2021. Accepted on November 30, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.651. ISSN: 1592-7415. eISSN: 2282-8214. ©Vijaya et al. This paper is published under the CC-BY licence agreement.

1 Introduction

The ideal concept in topology was developed by Kuratowski [Kuratowski, 1966]. The notion of α -I-continuity was introduced in 2004 [A. Acikgoz and Yuksel, 2004]. The conception of nano topology was initiated by L. Thivagar [Thivagar and Richard, 2013a]. In addition to that the concept of continuity, α -continuity, kernal and clopen in Nano topology was introduced by [Karthiksankari and Subbulakshmi, 2019] [Thivagar and Richard, 2013b] and [M. Lellis Thivagar and SuthaDevi, 2017]. Parimala and Jafari [Parimala and Jafari, 2018] had worked on Nano ideals. This work aims the introduction of contra $N\alpha$ -I-continuous functions by applying the concept of $N\alpha$ -I-open and $N\alpha$ -I-continuity in nano ideal topology. Also this contra $N\alpha$ -I-continuity are compared with some existing functions. Moreover, new class of functions are obtained. At every places the new notions have been substantiated with suitable examples. Throughout this article we use the notation NTS, NITS, N-regular, N-open, $N\alpha$ -open, N-clopen, $N\alpha$ -Cts for Nano Topological Spaces, Nano Ideal Topological Spaces, nano regular space, nano open, nano α -open, nano clopen, nano α -continuous respectively. Similar notations are used for their respective closed sets.

2 Preliminaries

Definition 2.1. [M. Lellis Thivagar and SuthaDevi, 2017] Let $(U, \tau_R(X))$ be a NTS and S is a subset of U . The nano kernel of S is defined as $NKer(S) = \cap \{U : S \text{ is a subset of } U, U \in \tau_R(X)\}$.

Theorem 2.1. [M. Lellis Thivagar and SuthaDevi, 2017] Let $(U, \tau_R(X))$ be a NTS and $A_1, A_2 \subseteq U$. We have

1. $x \in NKer(A_1)$ iff for any N-closed set F containing x , A_1 and F are disjoint,
2. If $A_1 \subseteq NKer(A_1)$ and then $A_1 = Ker(A_1)$ if A_1 is N-open in U ,
3. If $A_1 \subseteq A_2$, then $NKer(A_1) \subseteq NKer(A_2)$.

Definition 2.2. [Thivagar and SuthaDevi, 2016] A NTS $(U, \tau_R(X))$ along with an ideal I defined on U is called as a NITS and is denoted by $(U, \tau_R(X), I)$. Throughout this paper U represents a NTS $(U, \tau_R(X))$ and U_I represents a NITS $(U, \tau_R(X), I)$.

Definition 2.3. [Rajasekaran and Nethaji, 2018] Let $(U, \tau_R(X), I)$ be a nano ideal topological space and $A \subseteq U$. Then A is said to be $N\alpha$ -I-open if $A \subseteq Nint(Ncl^*(Nint(A)))$. The complements of $N\alpha$ -I-open is $N\alpha$ -I-closed set.

Theorem 2.2. [V. Inthumathi and Krishnaprakash, 2020] Let $(U_1, \tau_R(X_1), I)$ be a NITS and $(U_2, \tau_R(X_1))$ be a NTS. Then $h : U_1 \rightarrow U_2$ is called $N\alpha$ -I-Cts on U_1 if $h^{-1}(S)$ is $N\alpha$ -I-open in U_1 for any N -open set S in U_2 .

Definition 2.4. Bhuvaneswari and Nagaveni [2018] A NTS $(U, \tau_R(X))$ is called N -regular Space, if for each N -closed set T and each point $x \notin T$, $\exists$ disjoint N -open sets G and H such that $x \in G$ and $T \subset G$.

3 Contra $N\alpha$ -I-Continuity

The notations used are $CN\alpha$ -open, CN -Cts function, $CN\alpha$ -Cts function, $CN\alpha$ -I-Cts function for contra nano α -open, contra nano continuous, contra $N\alpha$ -continuous, contra $N\alpha$ -I-continuous function resp.

Definition 3.1. Let $(U_1, \tau_R(X_1))$ and $(U_2, \tau_{R'}(X_2))$ be NTS. Then $h : U_1 \rightarrow U_2$ is $CN\alpha$ -Cts if $h^{-1}(S)$ is $N\alpha$ -closed in U_1 whenever S is N -open set in U_2 .

Definition 3.2. Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ is $CN\alpha$ -I-Cts if $h^{-1}(S)$ is $N\alpha$ -I-closed in U_1 whenever S is N -open set in U_2 .

Example 3.1. Let $U_1 = \{i, j, k, l\}$, $U_1/R = \{\{i\}, \{j\}, \{k\}, \{l\}\}$ and $X_1 = \{i\}$. Then $\tau_R(X_1) = \{U_1, \phi, \{i\}\}$. Let $I = \{\phi\}$. Here the $N\alpha$ -I-open sets are $\{U_1, \phi, \{i\}, \{i, j\}, \{i, k\}, \{i, l\}, \{i, j, k\}, \{i, j, l\}, \{i, k, l\}\}$. Let $U_2 = \{m, n, o, p\}$ with $U_2/R' = \{\{m\}, \{n\}, \{o, p\}\}$ and $X_2 = \{n, o\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{n\}, \{o, p\}, \{n, o, p\}\}$. We define $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ as $f(i) = m, f(j) = n, f(k) = o$ and $f(l) = p$. Then $h^{-1}(S)$ is $N\alpha$ -I-closed in U_1 whenever S is N -open in U_2 . Therefore h is $CN\alpha$ -I-Cts.

Proposition 3.1. 1. Any $CN\alpha$ -I-Cts function is $CN\alpha$ -Cts.

2. Any CN -Cts function is $CN\alpha$ -I-Cts.

Proof. (i) Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ be a $CN\alpha$ -I-Cts function. Let S be a N -open in U_2 . Since h is $CN\alpha$ -I-Cts, $h^{-1}(S)$ is $N\alpha$ -I-closed in U_1 . We know that each $N\alpha$ -I-closed set is $N\alpha$ -closed. Hence $h^{-1}(S)$ is $N\alpha$ -closed in U_1 . Hence h is $CN\alpha$ -Cts.

(ii) Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ be a CN -Cts function. Let S be a N -open set in U_2 . Since h is CN -Cts, $h^{-1}(S)$ is N -closed in U_1 . It is obvious that every N -closed set is $N\alpha$ -I-closed. Thus $h^{-1}(S)$ is $N\alpha$ -I-closed in U_1 . Which implies h is $CN\alpha$ -I-Cts function. $\square$

Example 3.2. $CN\alpha$ -Cts $\nRightarrow$ $CN\alpha$ -I-Cts

Let $U_1 = \{i, j, k, l\}$ with $U_1/R = \{\{i\}, \{j, k\}, \{l\}\}$ and $X_1 = \{l\}$. Then $\tau_R(X_1) = \{U_1, \phi, \{l\}\}$. Let $I = \{\phi, \{l\}\}$. Here the $N\alpha$ -open sets are $\{U_1, \phi, \{l\}, \{i, l\}, \{j, l\}, \{k, l\}, \{i, j, l\}, \{i, k, l\},$

$\{j, k, l\}$ and $N\alpha$ -I-open sets are $\{U_1, \phi, \{l\}\}$. Let $U_2 = \{m, n, o, p\}$ with $U_2/R' = \{\{m\}, \{n, o\}, \{p\}\}$ and $X_2 = \{m, n\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{m\}, \{n, o\}, \{m, n, o\}\}$. We define $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ as $h(i) = m, h(j) = n, h(k) = o$ and $h(l) = p$. Then $h^{-1}(S)$ is $N\alpha$ -closed in U_1 but not $N\alpha$ -I-closed whenever S is N -open set in U_2 . Hence h is $CN\alpha$ -Cts but not $CN\alpha$ -I-Cts function.

Example 3.3. $CN\alpha$ -I-Cts $\nRightarrow$ CN-Cts

Let $U_1 = \{i, j, k, l\}$ with $U_1/R = \{\{i\}, \{j\}, \{k\}, \{l\}\}$ and $X_1 = \{i\}$. Then $\tau_R(X_1) = \{U_1, \phi, \{i\}\}$. Let $I = \{\phi\}$. Here the $N\alpha$ -I-open sets are $\{U_1, \phi, \{i\}, \{i, j\}, \{i, k\}, \{i, l\}, \{i, j, k\}, \{i, j, l\}, \{i, k, l\}\}$. Let $U_2 = \{m, n, o, p\}$ with $U_2/R' = \{\{m\}, \{o, p\}, \{n\}\}$ and $X_2 = \{n, o\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{n\}, \{o, p\}, \{n, o, p\}\}$. We define $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ as $h(i) = m, h(j) = n, h(k) = o$ and $h(l) = p$. Then $h^{-1}(S)$ is $N\alpha$ -I-closed in U_1 but not N -closed whenever S is N -open set in U_2 . Hence h is $CN\alpha$ -I-Cts but not CN-Cts function.

Theorem 3.1. Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$, then the following statements are equivalent:

1. h is $CN\alpha$ -I-Cts,
2. for each N -closed subset T of U_2 , $h^{-1}(T) \in N\alpha IO(U_1)$,
3. for each $x \in U_1$ and each N -closed set T of U_2 containing $h(x)$, $\exists U \in N\alpha IO(U_1)$ such that $h(U) \subset T$,
4. $h(N\alpha I-cl(V)) \subset NKer(h(V))$ for each $V \subseteq U_1$,
5. $N\alpha I-cl(h^{-1}(W)) \subset h^{-1}(NKer(W))$ for each $W \subseteq U_2$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii) Let T be any N -closed set of U_2 and $x \in h^{-1}(T)$. Then $h(x) \in T$ and $\exists U_x \in N\alpha IO(U_1)$ such that $h(U_x) \subset T$. Therefore, we obtain $h^{-1}(T) = \cup \{U_x : x \in h^{-1}(T)\}$ and hence $h^{-1}(T) \in N\alpha IO(U_1)$.

(ii) $\Rightarrow$ (iv) Let $V \subseteq U_1$. If $y \notin NKer(h(V))$, then by Thm 2.1, $\exists$ a N -closed set T of U_2 containing y such that $h(V) \cap T = \phi$. Therefore $V \cap h^{-1}(T) = \phi$ and $N\alpha I-cl(V) \cap h^{-1}(T) = \phi$. Hence $h(N\alpha I-cl(V)) \cap T = \phi$ and $y \notin h(N\alpha I-cl(V))$. Thus $h(N\alpha I-cl(V)) \subset NKer(h(V))$.

(iv) $\Rightarrow$ (v) Let $W \subseteq U_2$. By the hypothesis and Thm 2.1, $h(N\alpha I-cl(h^{-1}(W))) \subset NKer(h(h^{-1}(W))) \subset NKer(W)$ and $N\alpha I-cl(h^{-1}(W)) \subset h^{-1}(NKer(W))$.

(v) $\Rightarrow$ (i) Let W be a N -open set of U_2 . By Thm 2.1, $N\alpha I-cl(h^{-1}(W)) \subset h^{-1}(NKer(W)) = h^{-1}(W)$ and $N\alpha I-cl(h^{-1}(W)) = h^{-1}(W)$. Therefore $h^{-1}(W)$ is $N\alpha$ -I-closed in $(U_1, \tau_R(X), I)$. $\square$

Theorem 3.2. If a function $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ is $CN\alpha$ -I-Cts and V is N -regular, then h is $N\alpha$ -I-Cts.

Proof. Let $x \in U_1$ and Y a N -open set of U_2 containing $h(x)$. Since U_2 is N -regular, $\exists$ a N -open set Z in U_2 containing $h(x)$ such that $Ncl(Z) \subset Y$. Since h is $CN\alpha$ -I-Cts, by the above theorem, $\exists X \in N\alpha IO(U_1)$ such that $h(X) \subset Ncl(Z)$. Therefore $h(X) \subset Ncl(Z) \subset Y$. Hence h is $N\alpha$ -I-Cts. $\square$

Definition 3.3. A function $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ satisfy the $N\alpha$ -I-interiority rule if $N\alpha I-int(h^{-1}(Ncl(W))) \subset h^{-1}(W)$ Whenever W is N -open set of $(U_2, \tau_{R'}(X_2))$.

Theorem 3.3. If a function $h : (U_1, \tau_R(X_1), I)$ and $(U_2, \tau_{R'}(X_2))$ is $CN\alpha$ -I-Cts and satisfies $N\alpha$ -I-interiority rule, then h is $N\alpha$ -I-Cts.

Proof. Let Y be any N -open set of U_2 . Since h is $CN\alpha$ -I-Cts and $Ncl(Y)$ is N -closed, by Thm 3.1, $h^{-1}(Ncl(Y))$ is $N\alpha$ -I-open in $(U_1, \tau_R(X), I)$. By hypothesis of h , $h^{-1}(Y) \subset h^{-1}(Ncl(Y)) \subset N\alpha I-int(h^{-1}(Ncl(Y))) \subset N\alpha I-int(h^{-1}(Y)) \subset h^{-1}(Y)$. Thus, we obtain $h^{-1}(Y) = N\alpha I-int(h^{-1}(Y))$ and consequently $h^{-1}(Y) \in N\alpha IO(U)$. Therefore h is $N\alpha$ -I-Cts. $\square$

Theorem 3.4. Let $(U_1, \tau_R(X_1), I)$ be any NITS and $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ be a function and $g : U_1 \rightarrow U_1 \times U_2$ be the graph function, given by $g(x) = (x, h(x))$ for every $x \in U_1$. Then f is $CN\alpha$ -I-Cts if and only if g is $N\alpha$ -I-Cts.

Proof. Let $x \in U_1$ and let T be a N -closed subset of $U_1 \times U_2$ containing $g(x)$. Then $T \cap (\{x\} \times U_2)$ is N -closed in $\{x\} \times U_2$ containing $g(x)$. Also $\{x\} \times U_2$ is homeomorphic to U_2 . Hence $\{y \in U_2 : (x, y) \in T\}$ is a N -closed subset of U_2 . Since h is $CN\alpha$ -I-Cts, $\cup \{h^{-1}(Y) \in U_2 : (x, y) \in T\}$ is a $N\alpha$ -I-open subset of $(U_1, \tau_R(X_1), I)$. Further, $x \in \cup \{h^{-1}(Y) \in U_2 : (x, y) \in T\} \subset g^{-1}(T)$. Hence $g^{-1}(T)$ is $N\alpha$ -I-open. Then g is $CN\alpha$ -I-Cts. Conversely, let F be a N -closed subset of U_2 . Then $U_1 \times F$ is a N -closed subset of $U_1 \times U_2$. Since g is $CN\alpha$ -I-Cts, $g^{-1}(U_1 \times F)$ is a $N\alpha$ -I-open subset of U_1 . Also, $g^{-1}(U_1 \times F) = h^{-1}(F)$. Hence h is $CN\alpha$ -I-Cts. $\square$

Definition 3.4. A NITS $(U_1, \tau_R(X_1), I)$ is called $N\alpha$ -I- T_2 if for any distinct two points $x, y \in U_1$, $\exists X, Y \in N\alpha IO(U_1)$ containing x and y , resp., such that $X \cap Y = \phi$.

Definition 3.5. 1. A NTS $(U_1, \tau_R(X_1))$ is termed as a N -Urysohn Space if for any two distinct points $x, y \in U_1$, $\exists$ disjoint N -open subsets $x \in A, y \in B$ such that the N -closures $\overline{A}$ and $\overline{B}$ are disjoint N -closed subsets of U_1 .

2. A NTS $(U_1, \tau_R(X_1))$ is called N -Ultra Hausdorff if any two distinct points of U_1 can be separated by disjoint N -clopen sets.

Theorem 3.5. If $(U_1, \tau_R(X_1), I)$ is an NITS and for any two distinct points $x_1, x_2 \in U_1$, $\exists$ a function h into a N -Urysohn Space $(U_2, \tau_{R'}(X_2))$ such that $h(x_1) \neq h(x_2)$ and h is $CN\alpha$ -I-Cts at x_1, x_2 , then $(U_1, \tau_R(X_1), I)$ is $N\alpha$ -I- T_2 .

Proof. Let x_1, x_2 be any two distinct points of U_1 . Then by hypothesis there is a N-Urysohn Space $(U_2, \tau_{R'}(X_2))$ and a function $h : (U_1, \tau_R(X_1), I)$ and $(U_2, \tau_{R'}(X_2))$, which satisfies the required condition. Let $y_i = h(x_i)$ for $i=1,2$. Then $y_1 \neq y_2$. Since $(U_2, \tau_{R'}(X_2))$ is N-Urysohn, $\exists$ N-open neighbourhoods X_{y_1} and X_{y_2} of y_1, y_2 respectively in U_2 such that $Ncl(X_{y_1}) \cap Ncl(X_{y_2}) = \phi$. Since h is $CN\alpha$ -I-Cts at x_i , $\exists$ $N\alpha$ -I-open neighbourhoods W_{x_i} of x_i in U_1 such that $h(W_{x_i}) \subset Ncl(X_{y_i})$ for $i=1,2$. Hence we get $W_{x_1} \cap W_{x_2} = \phi$ because $Ncl(X_{y_1}) \cap Ncl(X_{y_2}) = \phi$. Therefore $(U_1, \tau_R(X_1), I)$ is $N\alpha$ -I- T_2 . $\square$

Corolary 3.1. *If h is a $CN\alpha$ -I-Cts injective function of a NITS $(U_1, \tau_R(X_1), I)$ into a N-Urysohn space $(U_2, \tau_{R'}(X_2))$, then $(U_1, \tau_R(X_1), I)$ is a $N\alpha$ -I- T_2 space.*

Proof. For any two distinct points x_1, x_2 in U_1 , h is $CN\alpha$ -I-Cts function of U_1 into a N-Urysohn space $(U_2, \tau_{R'}(X_2))$ such that $h(x_1) \neq h(x_2)$ because h is injective. By Thm 3.5, the space $(U_1, \tau_R(X_1), I)$ is $N\alpha$ -I- T_2 . $\square$

Theorem 3.6. *If h is a $CN\alpha$ -I-Cts injective function of a NTS $(U_1, \tau_R(X_1), I)$ into N-Ultra Hausdorff space $(U_2, \tau_{R'}(X_2))$, then $(U_1, \tau_R(X_1), I)$ is a $N\alpha$ -I- T_2 space.*

Proof. Let the pair of distinct points of U_1 be x_1, x_2 . Since f is injective, U_2 is N-Ultra Hausdorff $h(x_1) \neq h(x_2) \exists$ N-clopen sets Z_1, Z_2 such that $h(x_1) \in Z_1, h(x_2) \in Z_2$ and $Z_1 \cap Z_2 = \phi$. Then $x_i \in h^{-1}(Z_i) \in N\alpha IO(U_1)$ for $i=1,2$ and $h^{-1}(Z_1) \cap h^{-1}(Z_2) = \phi$. Therefore $(U_1, \tau_R(X), I)$ is a $N\alpha$ -I- T_2 space. $\square$

Definition 3.6. *Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$. The graph $G(h)$ of the function h is called be $CN\alpha$ -I-closed in $U_1 \times U_2$ if for any $(x_1, x_2) \in (U_1 \times U_2) \setminus G(h)$, $\exists A \in N\alpha IO(U_1)$ and a N-closed set T of U_2 containing x_2 such that $(U_1 \times U_2) \cap G(h) = \phi$.*

Lemma 3.1. *Let $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$. The graph $G(h)$ of the function h is $CN\alpha$ -I-closed in $U_1 \times U_2$ if and only if for each $(x_1, x_2) \in (U_1 \times U_2) \setminus G(h)$, $\exists A \in N\alpha IO(U_1, x_1)$ such that $h(A) \cap Ncl(T) = \phi$ where T is a N-closed subset of $U_1 \times U_2$ containing $g(x_1)$.*

Theorem 3.7. *If $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ is a $CN\alpha$ -I-Cts function and U_2 is a N-Urysohn space, then $G(h)$ is $CN\alpha$ -I-closed in $U_1 \times U_2$.*

Proof. Let $(x_1, x_2) \in (U_1 \times U_2) \setminus G(h)$. Then $x_2 \neq h(x_1)$ and $\exists$ N-open set A, B of U_2 such that $h(x_1) \in A, x_2 \in B$ and $Ncl(A) \cap Ncl(B) = \phi$. Since h is $CN\alpha$ -I-continuous, $\exists U \in N\alpha IO(U_1, x_1)$ such that $h(U) \subset Ncl(A)$. Therefore $h(U) \cap Ncl(B) = \phi$. Hence $G(h)$ is $CN\alpha$ -I-closed. $\square$

Theorem 3.8. *If $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ is a $CN\alpha$ -I-Cts function and $(U_2, \tau_{R'}(X_2))$ is T_2 , then $G(h)$ is $CN\alpha$ -I-closed.*

Proof. Let $(x_1, x_2) \in (U_1 \times U_2) \setminus G(h)$. Then $x_2 \neq h(x_1)$ and $\exists$ N-open set B of U_2 such that $h(x_1) \in B, x_2 \notin B$. Since h is $CN\alpha$ -I-Cts, $\exists U \in N\alpha IO(U_1, x_1)$ such that

$h(U) \subset Ncl(B)$. Therefore $h(U) \cap (U_2 - B) = \phi$ and $U_2 - B$ is a N -closed set of U_2 containing x_2 . Hence $G(h)$ is $CN\alpha$ -I-closed. $\square$

Definition 3.7. A NITS $(U, \tau_R(X), I)$ is called $N\alpha$ -I-connected if there exists $N\alpha$ -I-open sets A and B which form a separation of X .

Proposition 3.2. A $CN\alpha$ -I-Cts image of a $N\alpha$ -I-connected space is connected.

Definition 3.8. A NITS $(U, \tau_R(X), I)$ is called $N\alpha$ -I-normal if given any non-empty disjoint N -closed sets T and F such that $\exists$ $N\alpha$ -I-open sets A of T and B of F such that $A \cap B = \phi$.

Definition 3.9. A NTS $(U, \tau_R(X))$ is called N -Ultra normal if given any non-empty disjoint N -closed sets T and F such that $\exists$ N -clopen sets A of T and B of F such that $A \cap B = \phi$.

Theorem 3.9. If $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ is a $CN\alpha$ -I-Cts closed injective function and $(U_2, \tau_{R'}(X_2))$ is N -Ultra-normal space, then $(U_1, \tau_R(X_1), I)$ is a $N\alpha$ -I-normal space.

Proof. Let the two disjoint N -closed subsets of U_1 be F_1 and F_2 . Since h is N -closed and injective, $h(F_1) \cap h(F_2) = \phi$ where $h(F_1)$ and $h(F_2)$ are N -closed subsets of U_2 . Since U_2 is N -Ultra normal, $\exists$ N -clopen sets Y_1 of $h(F_1)$ and Y_2 of $h(F_2)$ in U_2 such that $Y_1 \cap Y_2 = \phi$. Hence $F_i \subset f^{-1}(Y_i)$, $f^{-1}(Y_i) \in N\alpha IO(U)$ for $i=1,2$ and $f^{-1}(Y_1) \cap f^{-1}(Y_2) = \phi$. Therefore $(U_1, \tau_R(X), I)$ is a $N\alpha$ -I-normal. $\square$

Theorem 3.10. For the functions $h : (U_1, \tau_R(X_1), I) \rightarrow (U_2, \tau_{R'}(X_2))$ and $g : (U_2, \tau_{R'}(X_2), I') \rightarrow (U_3, \tau_{R''}(X_3))$, We have

1. $g \circ h$ is $N\alpha$ -I-Cts, if h is $CN\alpha$ -I-Cts and g is CN -Cts.
2. $g \circ h$ is $CN\alpha$ -I-Cts, if h is $CN\alpha$ -I-Cts and g is N -Cts.

Remark 3.1. In general, $g \circ h$ is not $CN\alpha$ -I-Cts functions if g and f are $CN\alpha$ -I-Cts functions. The below example illustrate this result.

Example 3.4. Let $U_1 = \{i, j, k, l\}$ with $U_1/R = \{\{i, k\}, \{j\}, \{l\}\}$, and $X_1 = \{i, l\}$. Then $\tau_R(X_1) = \{U_1, \phi, \{l\}, \{i, k\}, \{i, k, l\}\}$. Let $I_1 = \{\phi, j\}$. Let $U_2 = \{m, n, o, p\}$ with $U_2/R' = \{\{m, n\}, \{o, p\}\}$ and $Y = \{o, p\}$. Then $\tau_{R'}(X_2) = \{U_2, \phi, \{p\}, \{m, o\}, \{m, o, p\}\}$. Let $I_2 = \{\phi, m\}$. Let $W = \{t, u, v, w\}$ with $W/R'' = \{\{t\}, \{u, v\}, \{w\}\}$ and $Z = \{w\}$. Then $\tau_{R''}(Z) = \{W, \phi, \{w\}\}$. Define $h : (U_1, \tau_R(X), I_1) \rightarrow (U_2, \tau_{R'}(X_2))$ by $h(i) = n$, $h(j) = p$, $h(k) = m$, $h(l) = o$ and $g : (U_2, \tau_{R'}(Y), I_2) \rightarrow (U_3, \tau_{R''}(Z))$ by $g(m) = w$, $g(n) = t$, $g(o) = u$, $g(p) = v$. Then h and g are $CN\alpha$ -I-Cts functions but $(g \circ h)^{-1}(w) = k$ which does not belongs to $N\alpha$ -I-closed in $(U_1, \tau_R(X_1), I)$.

4 Conclusion

Through the above discussions we have summarized the conceptualization of contra $N\alpha$ -I-continuity and its characteristics based on Nano Urysohn Space and Nano Ultra Hausdorff Space. Also, We compared contra $N\alpha$ -I-continuity with some existing functions using suitable examples. Further, this concept may be extended to Frechet Urysohn Space and Completely Hausdorff space in Nano Ideal Topology.

References

- T. Noiri A. Acikgoz and S. Yuksel. On α -i-continuous and α -i-open functions. *Acta Mathematica Hungarica*, 105:27–37, 2004.
- M. Bhuvaneswari and N. Nagaveni. On nwg-normal and nwg-regular spaces. *International Journal of Mathematics Trends and Technology*, 2018.
- P. Karthiksankari and P. Subbulakshmi. On nano totally semi continuous functions in nano topological space. *International Journal of Mathematical Archive*, 10: 14–18, 2019.
- K. Kuratowski. Topology. *Academic Press, New York*, I, 1966.
- Saeid Jafari M. Lellis Thivagar and V. SuthaDevi. On new class of contra continuity in nano topology. *Italian Journal of Pure and Applied Mathematics*, 43: 1–11, 2017.
- M. Parimala and S. Jafari. On some new notions in nano ideal topological space. *Eurasian Bulletin of Mathematics*, 1:51–64, 2018.
- I. Rajasekaran and O. Nethaji. Simple forms of nano open sets in an ideal nano topological spaces. *Journal of New Theory*, 24:35–43, 2018.
- M. Lellis Thivagar and Carmel Richard. On nano forms of weakly open sets. *International Journal of Mathematics and Statistics Invention*, I:31–37, 2013a.
- M. Lellis Thivagar and Carmel Richard. On nano continuity. *Mathematical Theory and Modelling*, 3:32–37, 2013b.
- M. Lellis Thivagar and V. SuthaDevi. New sort of operators in nano ideal topology. *Journal of Ultra Scientist of Physical Sciences*, 28:51–64, 2016.
- S. Narmatha V. Inthumathi and S. Krishnaprakash. Decompositions of nano-r-i-continuity. *International Journal of Advanced Science and Technology*, 29: 2578–2582, 2020.

Study of feedback retrial queueing system with working vacation, setup time and perfect repair

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Abstract

This manuscript analyses a retrial queueing system with working vacation, interruption, feedback, and setup time with the perfect repair. In the proposed model, the server takes vacation whenever the system gets empty but it still serves the customers at a relatively lower rate. To save power, the concept of setup time is included in the model. At vacation completion instant, the server is immediately turned off as soon as the system gets empty. The customer, who arrives during the closed-down state, activates the server and waits for his turn till the server is turned on. The unreliable server may sometimes fail to activate during setup. The failed server will resume service on being repaired. In the paper, explicit expressions for system size, sojourn times, and probabilities of various states of the server are obtained and results are analyzed graphically using MATLAB software.

Keywords: Retrial queue; working vacation; feedback; setup time.

2010 AMS subject classification²: 60K25, 60K30

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² Received on September 4, 2021. Accepted on December 20, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.655. ISSN: 1592-7415. eISSN: 2282-8214. ©Gupta. This paper is published under the CC-BY licence agreement.

1. Introduction

Retrial queueing systems with vacations and feedback of customers have attracted many researchers due to their widespread applications in real-life situations such as cellular networks, call centers, computer systems, inventory systems, and production management. In such systems, the customer, who finds the server busy, joins the orbit (free pool) and after a random period, it retries for the service. The retrials may follow a constant or classical retrial policy. In a constant retrial policy, the customer at the head in retrial orbit can reattempt for the service, but in a classical one, all the customers can retry for their turn independent of all others in the orbit. Many researchers analyzed the applications of retrial queueing systems. Falin and Templeton [6] did pioneering work on retrial queues. A good survey on the retrial queueing system was done by Artalejo et al. [2] and Yang et al. [20].

Queueing systems with vacations play a vital role in many real-life systems. The vacations may be due to many reasons. In classical vacation policy, no service is provided to the customers during vacation. Servi and Finn [18] introduced a new vacation class i.e., working vacation, in which service is provided to customers but at a comparatively lower rate. Readers may refer Do [5], Arivudainambi et al. [1] and Chandrasekaran et al. [3] for reference. Furthermore, the concept of vacation interruption has been widely used in queueing systems. In this policy, at service completion instant, the server interrupts the ongoing vacation and returns to a normal working state on finding waiting customers. Keeping in view the importance of the concept, many researchers have analyzed the queueing systems with vacation interruption. Li and Tian [12] analyzed M/M/1 queueing system with working vacation and interruption using the matrix geometric method. A pioneer work on M/G/1 queueing model with vacation interruption was done by Zhang and Hou [21]. Later, Gupta and Kumar [7, 8, 9] studied retrial queues with different vacation policies, impatient behaviour of customers and obtain closed-form expressions for important performance measures.

The retrial queueing system with Bernoulli feedback of customers is characterized by the feature that the unsatisfied customers may rejoin the system with some probability until they receive satisfactory service. Choi et al. [4] analyzed a retrial queueing model with geometric loss and feedback. Kumar et al. [10,11] studied M/G/1 retrial queueing model with feedback and starting failure. Mokaddis [15] analyzed feedback queueing systems with vacations and system failure. Varalakshmi et al. [19] discussed a single server queue with

immediate feedback and server breakdown. The concept of power saving is very important in today's scenario. To save power, the system should be turned off when not in use. Realizing the need for power saving, many researchers studied queueing model with setup time. Phung- Duc [16, 17] incorporated the concept of setup time to retrial queueing system. We may refer the reader to [13, 14] for the related works.

In this manuscript, a single server retrial queueing system with feedback, setup time, working vacation, and interruption under perfect repair is analyzed. If setup time is taken as zero, the model reduces to M/M/1 feedback retrial queueing system with working vacation and interruption. Further, the model changes to a retrial queueing system with feedback and setup time with perfect repair if vacation time tends to zero. Thus, our model generalizes some of the retrial queueing models existing in queueing literature.

2. Practical application of the model

Consider a manufacturing system consisting of an iron re-rolling mill, a Foreman(server), and a worker(assistant) to operate the mill. The foreman will operate the mill if the raw material is available and produce the products i.e., iron angles, iron rods, etc. If the raw material is not available due to transport issues, an increase in the price of raw material, etc., then the foreman may go on vacation(rest). During the vacation period of the foreman, if raw material becomes available then the worker will operate the mill, but the production will be relatively at a slow speed. When a batch of the product is completed, then the worker will call the foreman to resume the production at a higher speed by interrupting the vacation of the foreman.

In another situation, if the foreman's vacation period completes, he will return to the production to operate the mill. If the raw material is available then he will manage the production at a higher speed otherwise, if the raw material is not available, to save power, he may turn off the mill. Again, the availability of new raw material will initiate the setup of the machine (re-rolling mill) and production starts again if setup occurs successfully otherwise the machine will be sent for repair, and during this period there will be no production.

3. Model description and assumptions

We considered a Markovian retrial queueing system with working vacation, interruption, feedback, and setup time under perfect repair. The following assumptions are taken for the proposed model.

1. The arrival of customers is by the Poisson process with rate λ . The customer who finds the server busy; joins the free pool (orbit) and waits for his turn. The customers in the orbit are assumed to follow classical retrial policy. The retrial time is exponentially distributed with parameter ξ .
2. The service time in the normal state of the server is assumed to be exponentially distributed with rate μ . When all the customers are served, the server goes to a working vacation state in which it still provides service to customers with a slow rate θ . The vacation time and service time in vacations again follow an exponential distribution with vacation rate ϕ .
3. The unsatisfied customers may rejoin the orbit as feedback customers with probability 'f' or may leave the system with complementary probability $\bar{f} = (1-f)$.
4. On completion of vacation, if customers are found still waiting for their turn, the normal service period resumes otherwise, the server is turned off immediately to save power.
5. The customer who arrives in the off-state of the server; waits for his turn in front of the server till it is turned on. The setup time is required to restart the server. The setup time is assumed to follow an exponential distribution with a mean of $1/s$. The customers arriving in the setup state, have to join the orbit.
6. The server is assumed to be unreliable i.e., during the set-up state, activation of the server may fail with probability $\bar{p} = (1-p)$. The failed server is sent for repair and repair time is exponentially distributed with parameter r .
7. The inter-arrival time, service time, vacation time, retrial time, and setup time are all mutually independent.

Taking $N(t)$ as the number of customers in the orbit at time t and $J(t)$ as the state of the server.

where,

$$J(t) = \begin{cases} 0, & \text{the system is busy in a normal state at time } t \\ 1, & \text{the server is free in a normal state at time } t \\ 2, & \text{the server is busy in working vacation state} \\ 3, & \text{the server is free in working vacation state} \\ 4, & \text{the server is in setup or close – down state} \\ 5, & \text{the server is under repair state} \end{cases}$$

$\{N(t), J(t)\}$ represents a Markov process with following state-space $\{(n, j), n \geq 0, j=0,1,2,4,5\} \cup \{(0,3)\}$.

Here $(n, 0), n \geq 0$ represents that server is busy in a regular service period with n customers waiting in the orbit. The state $(0, 1)$ represents that system is in a close-down period. States $(n, 1), n \geq 1$ shows that system is free in the regular service period. States $(n, 2), n \geq 0$ represents the state that the system is busy in the working vacation period. The state $(0, 3)$ represents that the server is free during the vacation period. States $(n, 3), n \geq 1$ do not exist due to the inclusion of the concept of vacation interruption. States $(n, 4), n \geq 0$ represent that system is in a setup state with n customers in the orbit and one customer waiting in the service area for successful set up of the system. States $(n, 5), n \geq 0$, show that the system is under repair due to sudden breakdowns with n customers waiting in the orbit for their turn.

4. Steady-state equations and stationary probabilities

Denoting by p_{nj} , the probability of n customers waiting in the orbit when the system is in state j and using Markov process for the quasi-birth death model, the stationary state equations governing the model are

$$(\lambda + \mu)p_{00} = \phi p_{02} + \xi p_{11} + spp_{04} + rp_{05} \quad (1)$$

$$(\lambda + \mu)p_{n0} = \lambda p_{n-10} + \phi p_{n2} + (n+1)\xi p_{n+11} + \lambda p_{n1} + spp_{n4} + rp_{n5}, \quad n \geq 1 \quad (2)$$

$$\lambda p_{01} = \phi p_{03} \quad (3)$$

$$(\lambda + \xi)p_{n1} = \bar{f}\theta p_{n2} + f\theta p_{n-12} + f\mu p_{n-10} + \bar{f}\mu p_{n0}, \quad n \geq 1 \quad (4)$$

$$(\lambda + \theta + \phi)p_{02} = \lambda p_{03} \quad (5)$$

$$(\lambda + \theta + \phi)p_{n2} = \lambda p_{n-12}, \quad n \geq 1 \quad (6)$$

$$(\lambda + \phi)p_{03} = \mu\bar{f}p_{00} + \bar{f}\theta p_{02} \quad (7)$$

$$(\lambda + s)p_{04} = \lambda p_{01} \quad (8)$$

$$(\lambda + s)p_{n4} = \lambda p_{n-14}, \quad n \geq 1 \quad (9)$$

$$(\lambda + r)p_{05} = s\bar{p}p_{04} \quad (10)$$

$$(\lambda + r)p_{n5} = s\bar{p}p_{n4} + \lambda p_{n-15}, \quad n \geq 1 \quad (11)$$

Defining probability generating functions

$$G_i(z) = \sum_{n=0}^{\infty} p_{ni} z^n, \quad i = 0, 1, 2, 4, 5 \quad (12)$$

Multiplying equations (1) and (2) with appropriate power of z and taking summation for all possible values of n and using above-defined generating functions,

$$\begin{aligned} (\lambda + \mu - \lambda z)G_0(z) \\ = \phi G_2(z) + \lambda G_1(z) + \xi G_1'(z) + spG_4(z) + rG_5(z) - \lambda p_{01} \end{aligned} \quad (13)$$

Multiplying equations (3) and (4) with appropriate power of z and taking summation for all possible values of n and using generating functions we get,

$$(\lambda + \xi)G_1(z) = (fz + \bar{f})\mu G_0(z) + (\bar{f} + fz)\theta G_2(z) - A p_{01} \quad (14)$$

$$\text{where } A = \frac{\bar{f}\theta p_{02} + \mu\bar{f}p_{00} - \xi p_{01} - \phi p_{03}}{p_{01}} \quad (15)$$

$$p_{02} = \frac{\lambda^2}{\phi(\lambda + \theta + \phi)} p_{01} \quad (16)$$

$$p_{03} = \frac{\lambda}{\phi} p_{01} \quad (17)$$

$$p_{00} = \left(\frac{\lambda(\lambda + \phi)}{\mu\phi\bar{f}} - \frac{\lambda^2\theta}{\mu\phi(\lambda + \theta + \phi)} \right) p_{01} \quad (18)$$

Again using probability generating functions along with equations (5) and (6)

$$\begin{aligned} (\lambda + \theta + \phi - \lambda z)G_2(z) &= \lambda p_{03} \\ &= \frac{\lambda^2}{\phi} p_{01} \end{aligned}$$

$$G_2(z) = \frac{\lambda^2}{\phi(\lambda + \theta + \phi - \lambda z)} p_{01} \quad (19)$$

Using equations (8), (9), and (12) together

$$(\lambda + s)G_4(z) = \lambda z G_4(z) + \lambda p_{01}$$

$$G_4(z) = \frac{\lambda}{(\lambda + s - \lambda z)} p_{01} \quad (20)$$

Similar calculations in equations (10) and (11) yield

$$(\lambda + r - \lambda z)G_5(z) = s(1 - p)G_4(z)$$

Makin use of equation (19) in the above equation, we obtain

$$G_5(z) = \frac{\lambda s(1 - p)}{(\lambda + r - \lambda z)(\lambda + s - \lambda z)} p_{01} \quad (21)$$

Using the value of $G_0(z)$ from equation (14) in equation (13), and rearranging the terms we get the following differential equations

$$G_1'(z) + \frac{1}{\xi} \left(\lambda - \frac{(\lambda + \xi)(\lambda + \mu - \lambda z)}{\mu(\bar{f} + fz)} \right) G_1(z) = B(z) \quad (22)$$

$$\text{where } B(z) = \frac{1}{\xi} \left[\left(\lambda + \frac{A(\lambda + \mu - \lambda z)}{\mu(\bar{f} + fz)} \right) p_{01} - \left(\phi + \frac{\theta(\lambda + \mu - \lambda z)}{\mu} \right) G_2(z) - (spG_4(z) + rG_5(z)) \right] \quad (23)$$

To solve the differential equation we first find integrating factor,

$$I.F = e^{\frac{\lambda z}{\xi} \left(1 + \frac{\lambda + \xi}{\mu f} \right)} \left((\bar{f} + fz)^{-\frac{(\lambda + \xi)(\lambda + \mu f)}{\mu \xi f^2}} \right) \quad (24)$$

The solution of the differential equation (23) is

$$G_1(z) = e^{\frac{-\lambda z}{\xi} \left(1 + \frac{\lambda + \xi}{\mu f} \right)} \left((\bar{f} + fz)^{\frac{(\lambda + \xi)(\lambda + \mu f)}{\mu \xi f^2}} \right) \int_0^z e^{\frac{\lambda z}{\xi} \left(1 + \frac{\lambda + \xi}{\mu f} \right)} \left((\bar{f} + fz)^{-\frac{(\lambda + \xi)(\lambda + \mu f)}{\mu \xi f^2}} \right) B(z) dz \quad (25)$$

$G_0(z)$ is obtained from equation (14) as follows

$$G_0(z) = \frac{(\lambda + \xi)G_1(z) - (\bar{f} + fz)\theta G_2(z) + Ap_{01}}{(fz + \bar{f})\mu} \quad (26)$$

Taking limit $z \rightarrow 1$ in equations (19), (20), (21), (25), and (26) we obtain the expressions

$$G_2(1) = \frac{\lambda^2}{\phi(\theta + \phi)} p_{01} \quad (27)$$

$$G_4(1) = \frac{\lambda}{s} p_{01} \quad (28)$$

$$G_5(1) = \frac{\lambda(1-p)}{r} p_{01} \quad (29)$$

$$G_1(1) = e^{\frac{-\lambda}{\xi}\left(1+\frac{\lambda+\xi}{\mu f}\right)} \int_0^1 e^{\frac{\lambda z}{\xi}\left(1+\frac{\lambda+\xi}{\mu f}\right)} \left((\bar{f} + fz)^{-\frac{(\lambda+\xi)(\lambda+\mu f)}{\mu \xi f^2}} \right) B(z) dz \quad (30)$$

$$G_0(1) = \frac{(\lambda + \xi)G_1(1) - \theta G_2(1) + Ap_{01}}{\mu} \quad (31)$$

Differentiating equations (19), (20), (21) and taking limit $z \rightarrow 1$ we get

$$G_2'(1) = \frac{\lambda^3}{\phi(\theta + \phi)^2} p_{01} \quad (32)$$

$$G_4'(1) = \frac{\lambda^2}{s^2} p_{01} \quad (33)$$

$$G_5'(1) = \frac{\lambda^2(1-p)(r+s)}{sr^2} p_{01} \quad (34)$$

Equation (23) on taking limit $z \rightarrow 1$ implies

$$G_1'(1) = \frac{1}{\xi} [\xi G_1(1) - (\theta + \phi)G_2(1) - spG_4(1) - rG_5(1) + (\lambda + A)p_{01}] \quad (35)$$

Similarly differentiating equation (26) and taking limits we obtain

$$G_0'(1) = \frac{1}{\mu} [(\lambda + \xi)G_1'(1) - \theta G_2'(1) - f\theta G_2(1) - \mu f G_0(1)] \quad (36)$$

We observe that all the closed-form expressions for $G_i(z)$ and their derivatives for $i=0, 1, 2, 4, 5$ are implicitly expressed in terms of p_{01} .

p_{01} may be obtained by using the normalization condition

$$G_0(1) + G_1(1) + G_2(1) + G_4(1) + G_5(1) = 1 \quad (37)$$

5. Performance measures

In the present section, we obtain some important system performance measures of our proposed model as follows.

$$\begin{aligned} \text{Expected orbit length } E[L_O] &= G'_0(1) + G'_1(1) + G'_2(1) \\ &\quad + G'_4(1) + G'_5(1) \end{aligned} \quad (38)$$

$$\begin{aligned} \text{Expected sojourn time in orbit } E[W_O] &= E[L_O]/\lambda \\ &= \frac{G'_0(1) + G'_1(1) + G'_2(1) + G'_4(1) + G'_5(1)}{\lambda} \end{aligned} \quad (39)$$

$$\text{Expected system length } E[L_S] = E[L_O] + G_0(1) + G_2(1) \quad (40)$$

$$\begin{aligned} \text{Expected sojourn time in system } E[W_S] &= E[L_S]/\lambda \\ &= \frac{E[L_O] + G_0(1) + G_2(1)}{\lambda} \end{aligned} \quad (41)$$

$$\text{Probability of server being in off state} = p_{01}$$

$$\text{Probability of server being in working vacation state } (Pr_{wv})$$

$$\begin{aligned} &= G_2(1) + \frac{\lambda}{\phi} p_{01} \\ &= \frac{\lambda^2}{\phi(\theta + \phi)} p_{01} + \frac{\lambda}{\phi} p_{01} \end{aligned} \quad (42)$$

$$\text{Probability of server being in setup state } (Pr_S) = G_4(1)$$

$$= \frac{\lambda}{s} p_{01} \quad (43)$$

$$\text{Probability of server in repair state } (Pr_R) = G_5(1)$$

$$= \frac{\lambda(1-p)}{r} p_{01} \quad (44)$$

6. Numerical and graphical analysis

In the present section, the numerical and graphical interpretation of derived closed-form expressions of various system performance measures, for the proposed mathematical model is performed. For this purpose, some of the system parameters are assumed to be fixed as $\lambda=3$, $\mu=7$, $\xi=1.8$, $\phi=2$, $\theta=3$, $f=0.7$, $p=0.7$, $r=0.8$, $s=0.6$, unless otherwise mentioned. The behaviours of important performance measures, for a different set of values of one or more of the parameters is analyzed in the below-plotted graphs.

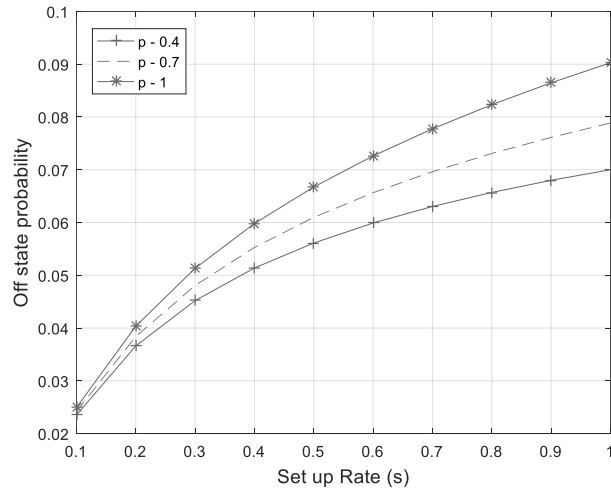


Figure1: Off-state probability versus setup rate for different values of p

From figure 1 we see that with an increase in setup rate, the probability of the server being in off-state increases. This is due to the reason that with an increase in the setup rate, the setup time decreases which causes early return in a normal state of the server hence increasing the probability of the server being in an off state. This probability of off-state increases with an increase in p, for a fixed value of set up rate; this is again due to an increase in chances of successful set up of server that further increases the off state probability.

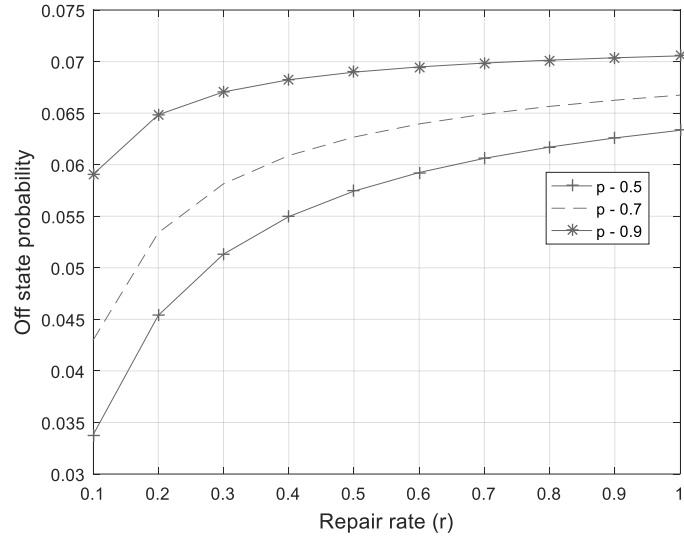


Figure2: Off-state probability versus repair rate for different values of p

We observe from figure 2 that the off-state probability of server increases with repair rate r , for a fixed value of p . This is due to a reduction in repair time with an increase in repair rate which leads to quick repair hence faster return to normal service thereby increasing the off-state probability.

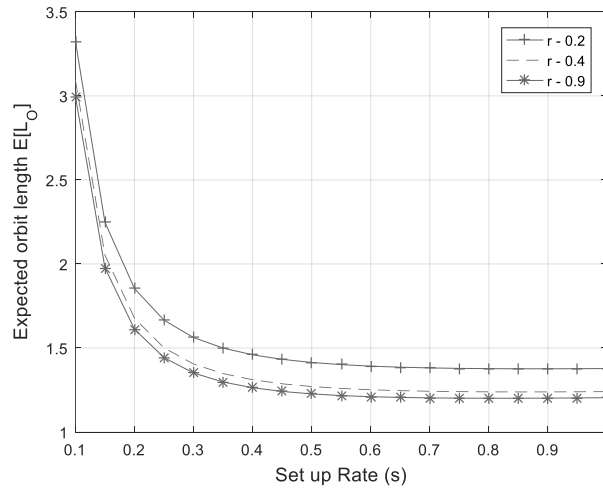


Figure3: Effect of setup rate on mean orbit length for different repair rates

Figure 3 reveals that the expected orbit length decreases with an increase in setup rate. This is because with an increase in setup rate, the time required for set up of server decreases which results in a quick return to normal service

period of server thereby reducing the orbit length. For the same reason, mean orbit length decreases with a decrease in repair time.

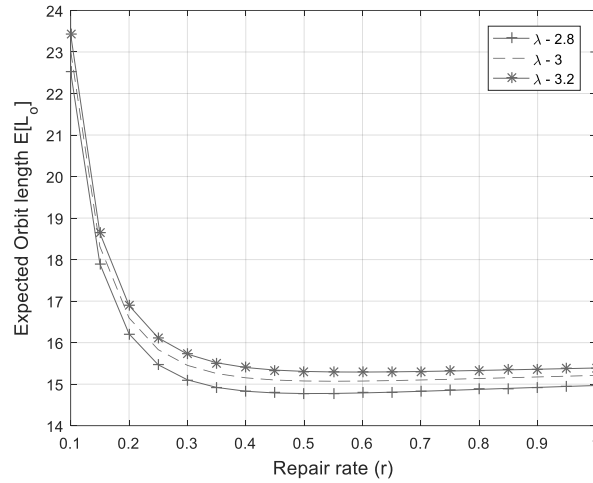


Figure 4: Effect of repair rate on mean orbit length for different arrival rates

We see from figure 4, the expected orbit length decreases with an increase in repair rate. As expected the mean orbit length increases with an increase in arrival rate, for a fixed repair rate. This is due to a reduction in inter-arrival time which increases mean orbit length.

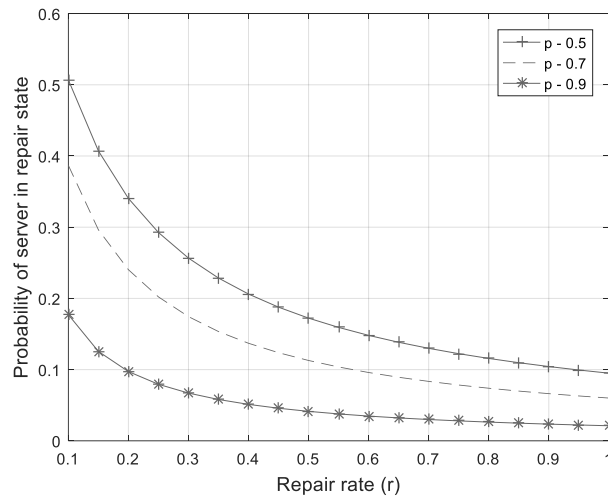


Figure 5: Effect of repair rate on repair state probability of server for different values of p

Figure 5 depicts that with an increase in repair rate, the probability of the server being in repair state decreases. This agrees with our expectations. As the repair rate increases, the repair is done in a lesser time that makes a faster return to the normal state from the repair state hence the probability of the server being in the repair state decreases. Again with an increase in p , for a fixed repair rate, the chances of successful activation (set up) of server raise hence the probability of server being in repair state decreases.

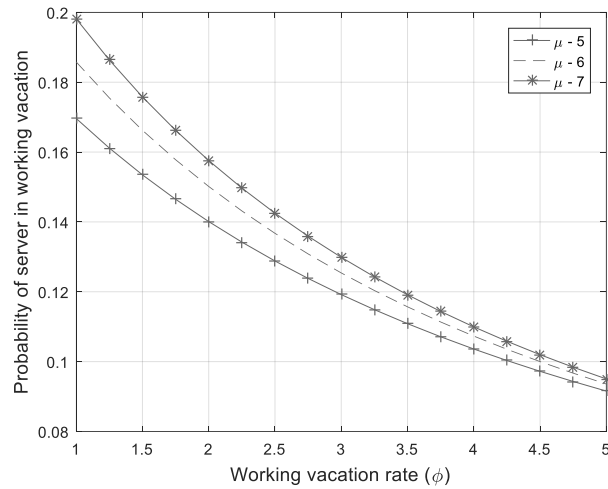


Figure 6: Probability of server in vacation versus vacation rate for different service rates

We observe from figure 6 that the probability of the server being in vacation state decreases with an increase in the rate of working vacation. The reason behind the observation is a decrease in the duration of vacation with an increase in the vacation rate. Further, the probability of the server in vacation state increases with service rate μ ; this is due to faster service which promotes server vacations.

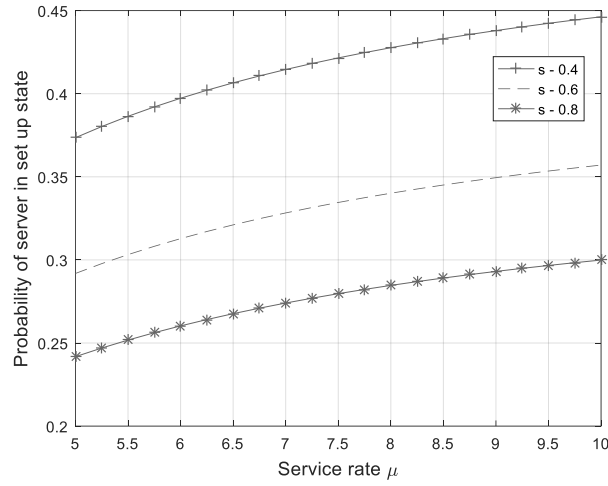


Figure 7: Probability of server in set up versus service rate for different setup rates

Figure 7 depicts that with an increase in service rate, the probability of the server being in setup state increases, this is as expected intuitively. For fixed service rate, as set up rate s increases, the probability decreases; this is due to faster activation of server with reduced setup time.

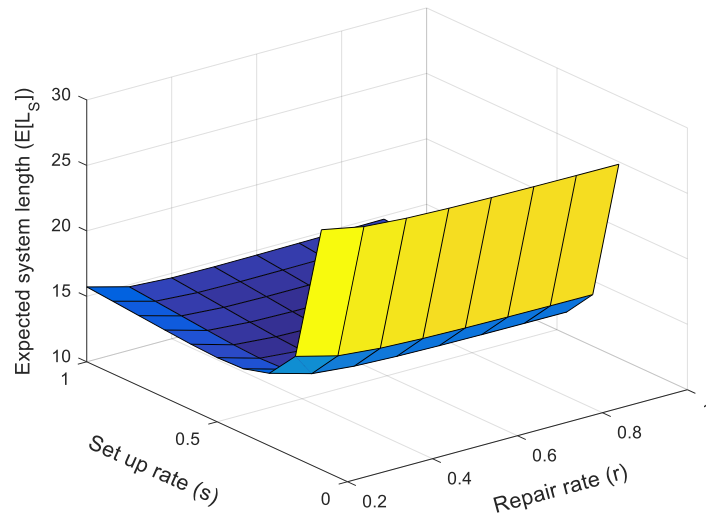


Figure 8: Variation in Expected system length with setup and repair rate

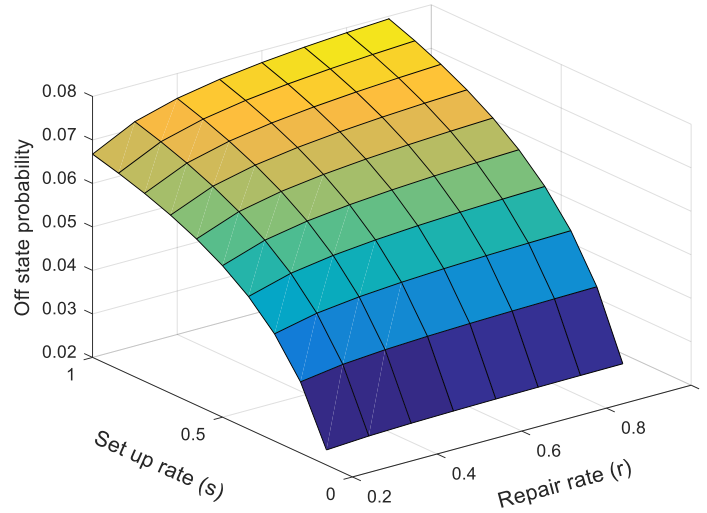


Figure 9: Variation in off state probability of server with setup and repair rate

Figures 8 and 9 represent the graphical behaviour of mean system length and off-state probability with setup and repair rate respectively. As expected, the mean system length decreases whereas off-state probability increases with an increase in the setup rate, for a fixed value of repair rate.

7. Conclusion and future scope

This paper analyses a single server retrial queueing system with working vacation, vacation interruption, Bernoulli feedback and setup time under perfect repair. The closed-form expressions for expected system size along with the probability of various system state probabilities, closed-down state have been obtained via the probability generating functions approach. The variation of the derived expressions against some of the system parameters is graphically studied by using MATLAB software. The observed graphical results are analyzed and are found to agree with the theoretically expected behaviour. The retrial queueing model with imperfect repair and multiple waiting servers can be considered for future investigations.

Conflict of interests

The authors declare that there is no conflict of interest.

References

- [1] Arivudainambi, D., Godhandaraman, P., Rajadurai, P. (2014). Performance analysis of a single server retrial queue with a working vacation. *OPSEARCH*, 51(3), 434–462. DOI 10.1007/s12597-013-0154-1
- [2] Artalejo, J. R., and Gómez-Corral, A. (2008). *Retrial Queueing Systems: A Computational Approach*, Springer, Berlin, Germany.
- [3] Chandrasekaran, V.M., Indhira, K., Saravanarajan, M.C., Rajadurai, P. (2016). A survey on working vacation queueing models. *Int. J. Pure Appl. Math.*, 106, 33–41. DOI: 10.12732/ijpam.v106i6.5
- [4] Choi, B.D., Kim, Y.C. and Lee, Y.W. (1998). The M/M/C retrial queue with geometric loss and feedback. *Computers and mathematics with applications*, 36, 41-52. [https://doi.org/10.1016/S0898-1221\(98\)00160-6](https://doi.org/10.1016/S0898-1221(98)00160-6)
- [5] Do, T.V. (2010). M/M/1 retrial queue with working vacations. *Acta Informatica*, 47, 67–75. <https://doi.org/10.1007/s00236-009-0110-y>
- [6] Falin, G. I., and Templeton, J. G. C. *Retrial Queues*. Chapman & Hall, London. 1997.
- [7] Gupta, P., Kumar, N. (2021). Analysis of classical retrial queue with differentiated vacation and state-dependent arrival rate. *Ratio Mathematica*, 40, 47-66. DOI: 10.23755/rm.v40i1.619.
- [8] Gupta, P., Kumar, N. (2021). Cost Optimization of Single Server Retrial Queueing model with Bernoulli Schedule Working Vacation, Vacation Interruption, and balking. *J. Math. Comput. Sci.*, 11(3), 2508-2523 <https://doi.org/10.28919/jmcs/5552>
- [9] Gupta, P., Kumar, N. (2021). Performance Analysis of Retrial Queueing Model with Working Vacation, Interruption, Waiting Server, Breakdown, and Repair. *Journal of Scientific Research*, 13(3), 833-844. Doi: <http://dx.doi.org/10.3329/jsr.v13i3.52546>
- [10] Kumar, B.K., Madheswari, S.P. and Vijaykumar, A. (2002). The M/G/1 retrial queue with feedback and starting failures. *Applied Mathematical Modelling*. [https://doi.org/10.1016/S0307-904X\(02\)00061-6](https://doi.org/10.1016/S0307-904X(02)00061-6)
- [11] Kumar, B.K., Madheswari, S.P. and Lakshmi, S.R.A. (2013). An M/G/1 Bernoulli feedback retrial queueing system with negative customers. *Operational Research*, 13(2), 187–210.

- [12] Li, J. and Tian, N. (2007). The M/M/1 queue with working vacations and vacation interruptions. *Journal of Systems Science and Systems Engineering*, 16(1) 121–127. <https://doi.org/10.1007/s11518-006-5030-6>
- [13] Manoharan, P. and Jeeva, T. (2019). Impatient customers in an M/M/1 working vacation queue with a waiting server and setup time, *Journal of Computer and Mathematical Sciences*, 10(5), 1189-1196.
- [14] Manoharan, P. and Jeeva, T. (2020). Impatient customers in a Markovian queue with Bernoulli schedule working vacation interruption and setup time. *Applications and Applied Mathematics*, 15(2), 725-739. Available at <http://pvamu.edu/aam>
- [15] Mokaddis, G.S., Metwally, S.A., and Zaki, B.M. (2007). Feedback retrial queueing system with failures and single vacation. *Tamkang journal of science and engineering*, 10, 183-192.
- [16] Phung-Duc, T. (2015). M/M/1/1 retrial queues with setup time. *Advances in Intelligent Systems and Computing*. 383, 93–104. Doi: 10.1007/978-3-319-22267-7_9.
- [17] Phung-Duc, T. (2017). Single server retrial queues with setup time. *Journal of Industrial and Management Optimization*, doi:10.3934/jimo.2016075.
- [18] Servi, L.D. and Finn, S.G. (2002). M/M/1 queues with working vacations (M/M/1/WV). *Performance Evaluation*, 50(1), 41–52. Available at: [https://doi.org/10.1016/S0166-5316\(02\)00057-3](https://doi.org/10.1016/S0166-5316(02)00057-3)
- [19] Varalakshmi, M. Chandrasekaran, V. M., Saravananarajan, MC. (2018). A Single Server Queue with Immediate Feedback, Working Vacation, and Server Breakdown. *International Journal of Engineering and Technology*, 7(4.10), 476-479. DOI: 10.14419/ijet.v7i4.10.21044
- [20] Yang, T., Templeton, J.G.C. (1987). A survey of retrial queues. *Queueing systems*, 2, 201-233.
- [21] Zhang, M. and Hou, Z. (2010). Performance analysis of M/G/1 queue with working vacations and vacation interruption. *Journal of Computational and Applied Mathematics*, 234(10), 2977–2985. <https://doi.org/10.1016/j.cam.2010.04.010>

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