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Studies on A. Einstein , B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," I: Basic methods

Ruggero Maria Santilli*

Abstract

In 1935, A. Einstein expressed his view, jointly with B. Podolsky and N. Rosen, that "quantum mechanics is not a complete theory" (*EPR argument*). Following decades of preparatory studies, R. M. Santilli published in 1998 a paper showing that the objections against the EPR argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. In a more recent paper, Santilli has shown that quantum uncertainties of extended particles appear to progressively tend to zero when in the interior of hadrons, nuclei and stars, and appear to be identically null at the limit of gravitational collapse, essentially along the EPR argument. In this first paper, we review, upgrade and specialize the basic mathematical, physical and chemical methods for the study of the EPR argument. In two subsequent papers, we review the above results and provide specific illustrations and applications.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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1. INTRODUCTION.

1.1. The EPR argument.

As it is well known, quantum mechanics does not admit classical precision in the measurement of the position or the mutual distance of particles (Figure 1) in view of Heisenberg's uncertainty principle and other physical laws.

Albert Einstein did not accept this uncertainty as being final for all possible conditions existing in the universe and made his famous quote *"God does not play dice with the universe."*

More specifically, Einstein accepted quantum mechanics for atomic structures and other systems of point-like particles in vacuum (conditions known as *exterior dynamical problems*), but believed that quantum mechanics is an "incomplete theory," in the sense that it could admit a "completion" into

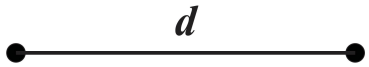

Figure 1: *In this figure, we present a conceptual rendering of the sole representation of particles permitted by the differential calculus underlying quantum mechanics, namely, the representation as isolated points in empty space which particles, being dimensionless, can only be at a distance, with ensuing EPR argument on the need for superluminal interactions to explain quantum entanglement [1].*

such a form to recover classical determinism at least under limit conditions.

Einstein communicated his view to the post doctoral associates, B. Podolsky and N. Rosen at the Institute for Advanced Study, Princeton, NJ, and all three together published in the 1935, May 15th issue of the Physical Review, the paper entitled *"Can Quantum Mechanical Description of Physical reality be Considered Complete?"* which paper became known as the *EPR argument* [1].

Soon after the appearance of paper [1], N. Bohr published paper [2] expressing a negative judgment on the possibility of "completing" quantum mechanics along the EPR argument.

Bohr's paper was followed by a variety of papers essentially supporting Bohr's rejection of the EPR argument, among which we recall *Bell's inequality* [3] establishing that the $SU(2)$ spin algebra does not admit limit values with an identical classical counterpart.

We should also recall *von Neumann theorem* [4] achieving a rejection of the EPR argument via the uniqueness of the eigenvalues of quantum mechanical Hermitean operators under unitary transforms.

The field became known as *local realism* and was centered on the rejection of the EPR argument on claims of *lack of existence of hidden variables* λ *[5] in quantum mechanics* (see the review [6] with a comprehensive literature).

Nowadays, the EPR argument is generally ignored in view of the widespread belief that quantum mechanics is universally valid for whatever conditions may exist in the universe without any scrutiny of the limitations and/or insufficiencies of quantum mechanics in various fields reviewed in this section.

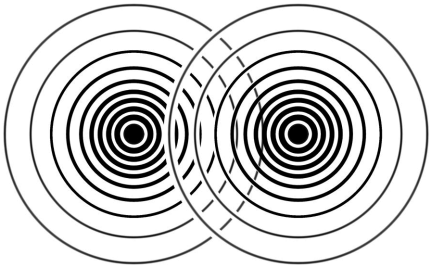

Figure 2: *A conceptual rendering of the main assumption of the apparent proofs [7] [8] of the EPR argument [1], consisting in the representation of particles as extended, deformable and hyperdense in conditions of mutual overlapping with ensuing continuous contact at a distance eliminating the need for superluminal interactions to explains quantum entanglement. Despite its simplicity, the quantitative treatment of said view required decades of studies due to the need for a "completion" of the mathematics underlying quantum mechanics. Intriguingly, the "completion" here considered turned out to be of isotopic/axiom-preserving type, thus being fully admitted by quantum mechanical axioms, merely subjected to a realization broader than that of the Copenhagen school.*

1.2. Apparent proofs of the EPR argument.

In Vol. 50, pages 177-190, 1998, of *Acta Applicandae Mathematica*, R. M. Santilli published paper [7] entitled "Isorepresentation of the Lie-isotopic $SU(2)$ Algebra with Application to Nuclear Physics and Local Realism," which paper appears to confirm Einstein's view on the existence of a "completion" of quantum mechanics into the *isotopic branch of hadronic mechanics*, or *isomechanics* for short and a "completion" of quantum chemistry into a form known as *isochemistry*. These "completions" are based on a broadening of applied mathematics known as *isomathematics* and admit progressive conditions of particles in the interior of hadrons, nuclei, stars and black holes that appear to recover classical determinism.

The proof presented in paper [7] was done via the following three main steps:

1.2.1. The proof that Bell's inequality, von Neumann's theorems and other similar objections of the EPR argument [6] are indeed correct, but under the generally tacit assumptions:

A) The point-like approximation of particles moving in vacuum (Figure 1);

B) The sole admission of Hamiltonian interactions [18];

C) The treatment of assumptions A and B via 20th century applied mathematics, including Lie's theory and the Newton-Leibnitz differential

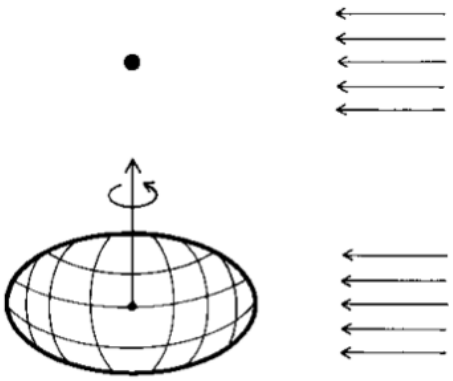

Figure 3: *An illustration of the central objective for the proof of the EPR argument consisting in the transition from the quantum mechanical representation of the Newtonian notion of massive points moving in vacuum under linear, local and potential interactions (top view), to the time invariant representation of extended particles moving within physical media under linear and non-linear, local and non-local and potential as well as non-potential interactions (bottom view).*

calculus;

1.2.2. The proof that the above treatments are not applicable to Einstein's vision on the existence of a "completion" of quantum mechanics based on the following assumptions:

A') The representation of extended, therefore deformable and hyperdense particles under conditions of mutual penetration/entanglement known (Figure 2) as occurring in the structure of hadrons, nuclei, stars and black holes (systems known as *interior dynamical problems*);

B') The emergence under condition A' of Hamiltonian as well as contact non-Hamiltonian interactions of non-linear, non-local and non-potential character;

C') The treatment of assumptions A' and B' via isomathematics that, as we shall see in Section 2, is based on:

i) The axiom-preserving isotopy $ab = ab \to a\star b = a\hat{T}b$ of the associative product ab between generic quantities a, b (numbers, functions, operators, etc.), where $\hat{T}$ is a positive-definite quantity called the *isotopic element* representing the dimension, deformability and density of particles;

ii) The ensuing axiom-preserving "completion" of Lie's theory with isotopic product $[\hat{x,y}] = x \star y - y \star x$ between Hermitean operators x, y;

iii) The reconstruction of the 20th century applied mathematics into a form compatible with isoproduct $a\star b$, including most importantly the isotopic lifting of the Newton-Leibnitz differential calculus from its centuries

Figure 4: *A first illustration of the lack of "completion" of quantum mechanics beyond scientific doubt is the time reversal invariance of the theory with equal probability for events forward and backward in time. Such a time reversibility is acceptable for atomic structures, particles in accelerators, crystals and other reversible systems, but it does not allow a consistent physical or chemical representation of energy-releasing process, such as the coal burning depicted in this figure. In fact, the time reversal image of coal burning implies that smoke must reconstruct coal with evident violation of causality.*

old definition at isolated points to its definition in the volumes of particles represented by $\hat{T}$.

1.2.3. The proof that the Lie-isotopic $\hat{SU}(2)$ algebra with isoproduct $[x\hat{,}y]$ admits limit conditions with an identical classical counterpart.

More recently, R. M. Santilli completed the above proof in paper [8] by showing that, under the above indicated conditions, the standard deviations for coordinates Δr and momenta Δp appear to progressively tend to zero for extended particles within hadrons, nuclei and stars, and appear to be identically null for extended particles within the limit conditions in the interior of gravitational collapse, essentially along Einstein's vision.

It should be noted that the above proofs of the EPR argument are centered in the *preservation* of the basic axioms of quantum mechanics, only submitted to their broadest possible *realization*.

It should be also noted that, under said broadest possible realization, quantum axioms do admit an *explicit and concrete realization of hidden variables* embedded in the structure of the Lie-isotopic product $[x\hat{,}y] = x\hat{T}y - y\hat{T}x$, for instance, via realization $\hat{T} = Diag.(\lambda, 1/\lambda), \;\; Det\hat{T} = 1$ [7].

1.3. Insufficiencies of quantum mechanics for irreversible processes.

One of the insufficiencies of quantum mechanics well known since the 1930's is the inability to represent physical or chemical energy releasing processes, such as nuclear fusions, fuel combustion and other processes.

This is due to the fact that energy releasing processes are *irreversible over time* (namely, the time reversal image of the processes violates causality), while quantum axioms were conceived for and remain solely applicable to *systems reversible over time* (namely, the time reversal image of the systems verifies causality), such as atomic structures, particles in accelerators, crystals and other systems (Figure 4).

It is hoped that the ongoing alarming deterioration of our environment, with the consequential need for *new* clean energies, illustrate the need for a "completion" of quantum mechanics for irreversible processes outlined for completeness in Section 2.

It should be indicated that, except for the short presentation in Section 2, this and the following paper are solely devoted to the apparent proofs of the EPR argument for reversible interior systems, while the study for the broader irreversible interior systems is done elsewhere (see Refs. [9] to [83]).

This is due to the fact that the objections against the EPR argument were formulated for reversible exterior systems. Consequently, proofs [7] and [8] studied in these papers were formulated for reversible interior systems.

1.4. Insufficiencies of quantum mechanics in particle physics.

Quantum mechanics is justly considered to be *exactly valid* for the structure of the hydrogen atom because it achieved a numerically exact representation of *all* experimental data for the system considered.

It is hoped serious scholars will admit that quantum mechanics cannot be considered as being exactly valid for particle physics because of the known inability to achieve an exact representation of *all* experimental data of any given family of particles, despite the admission of a number of hypothetical neutrinos and other *ad hoc* conjectures.

Recall that, with the exception of electrons, protons and the hypothetical neutrinos, *all particles produced by contemporary accelerators are structurally irreversible because unstable, thus being outside the capability of a full representation via quantum mechanics (Section 1.3), e.g., because they require a covering Lie-admissible (rather the Lie) treatment.* [22] [23].

Additionally, quantum mechanics is completely inapplicable (rather than violated) for the most fundamental synthesis in nature, that of the

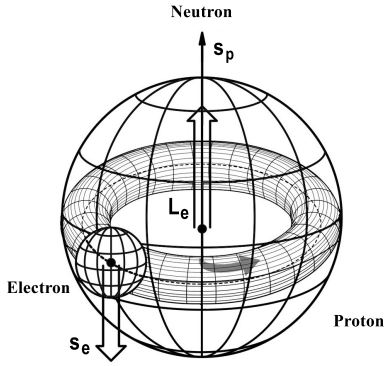

Figure 5: *A conceptual rendering of the synthesis of the neutron as a "compressed hydrogen atom" in the core of stars according to H. Rutherford [84], which synthesis cannot be represented by quantum mechanics due to a mass excess and other reasons (see Section 1.4). By contrast, all characteristics of the neutron in said synthesis have been represented at the non-relativistic and relativistic levels by the "completion" of 20th century applied mathematics and physics studied in this paper (Refs. [85] [95]).*

neutron from a proton and an electron as occurring in the core of stars [84], with ensuing inapplicability to the synthesis of other particles, such as that of the π^0 meson from the positronium [17].

This is due to the fact that quantum mechanical axioms have been conceived for the synthesis of particles in which the mass of the final state is *smaller* than the sum of the masses of the original constituents, resulting in the well known *mass defect* caused by *negative potentials*.

By contrast, the mass of the neutron $E_n = 939.565\ MeV$ is $0.782\ MeV$ *bigger* than the sum of the masses of the proton $E_p = 938.272\ MeV$ and of the electron $E_e = 0.511\ MeV$, resulting in a *mass excess* requiring a *positive potential* for which Schrödinger, Dirac and other quantum mechanical equations admit no physically meaningful solutions, with similar cases occurring for the synthesis of other particles [17].

The inability by quantum mechanics to represent the fundamental synthesis of the neutron in a star is ultimately due to the *point-like characterization of particles* since it is mathematically and physically impossible to fuse together two point-like particles (the proton and the electron) into a third point-like particle (the neutron).

In turn, this insufficiency identified the need for the representation of hadrons as extended, deformable and hyperdense, which representation is at the foundation of the EPR proof [7].

Another insufficiency of quantum mechanics is its sole capability of representing *linear interactions* (i.e., interactions linear in the wave function). By contrast, the sole known possibility of achieving a bound state with excess mass is that via the admission of non-linear, generally non-Hamiltonian interactions for extended particles in conditions of mutual penetration/entanglement, as it is the case for the electron when totally compressed inside the proton [16] [17].

In turn, non-linear interactions are crucial for the "completion of the wavefunction" advocated by Einstein, Podolsky and Rosen [1] as we shall see in Paper III of this series.

It is nowadays known that the above insufficiencies originate from the theory at the foundation of quantum mechanics, Lie's theory, because said theory solely admits Hamiltonian linear interactions [19].

It is hoped that the above insufficiencies illustrate the significance of the "completion" of Lie's theory used for the proof of the EPR argument [7].

In fact, only following the achievement of a "completion" of 20th century mathematical and physical methods for extended, deformable and hyperdense particles in interior dynamical conditions, Santiili achieved a numerically exact representation of *all*— characteristic of the neutron in its synthesis from a proton and an electron at the non-relativistic (Refs. [85] to [87]), relativistic (Refs. [88] to [89]) and experimental (Refs. [90] to [95]) levels (see Sections 2, 3 and Paper II of this series).

1.5. Insufficiencies of quantum mechanics in nuclear physics.

There is no doubt that quantum mechanics has permitted historical achievements in nuclear physics.

However, quantum mechanics is only *approximately valid* in nuclear physics because of the inability to achieve over the last century a representation of the characteristics of the simplest nucleus, the deuteron, with embarrassing deviations of the prediction of the theory from experimental data for heavier nuclei, such as the Zirconium [59].

In Santilli's view, the primary reason for the indicated insufficiency is that the *mathematics* underlying quantum mechanics, with particular reference to the *Newton-Leibnitz differential calculus*, imply the conception of nuclei as ideal spheres with isolated points in its interior (Figure 6) while in the physical reality, nuclei are composed by extended and hyperdense protons and neutrons in conditions of partial mutual penetration established by the comparison of nuclear volumes with the constituent volumes [59] (Figure 7).

The inability to represent nuclei as they are in the physical reality im-

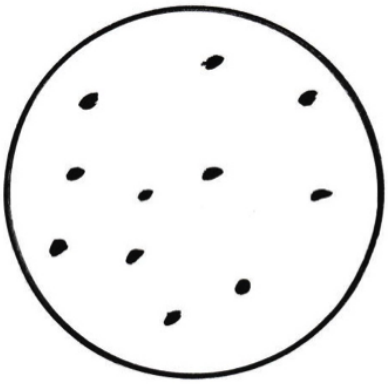

Figure 6: *The mathematics underlying quantum mechanics is local-differential, thus solely admitting a point-like approximation of particles. This figure illustrates the consequential conception of nuclei as ideal spheres with isolated points in their interior with ensuing insufficiencies beyond scientific doubt.*

plies the inability to achieve an exact representation of nuclear magnetic moments. In fact, the quantum mechanical representation of the anomalous magnetic moment of the deuteron still misses $1\ \%$ despite all possible relativistic or quark-based corrections.

Additionally, quantum mechanics misses much bigger percentages of nuclear magnetic moments for heavier nuclei (as illustrated in Figure 8).

In turn, the inability to represent protons and neutrons as extended charge distributions implies the inability to represent deformations under strong nuclear forces, with related deformation of their angular momenta.

In fact, J. M. Blatt and V. F. Weisskopf state on page 31 of their treatise in nuclear physics [96]: *It is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon* (Figure 9).

The representation of nucleons as extended, thus deformable and hyperdense charge distributions via the "completion of 20th century mathematics (Section 2) and physics (Section 3) has permitted the exact representation of the anomalous magnetic moment of the deuteron [97], as well as of heavier nuclei [98].

It should be noted that the representation of nuclear magnetic moments is presented in Ref. [7] as an illustration of the implications of the proof of the EPR arguments for extended particles in interior conditions.

A second insufficiency of quantum mechanics in nuclear physics is the lack of a consistent representation of nuclear spins despite efforts also conducted for about one century.

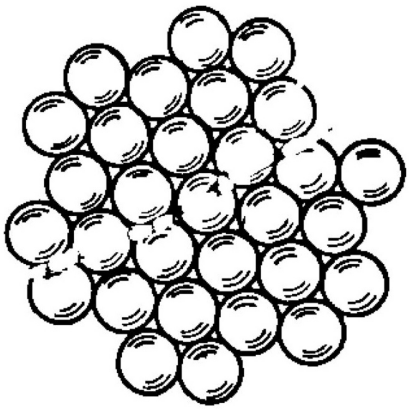

Figure 7: *A conceptual rendering of nuclei as they occur in the physical reality, i.e., a collection of extended and hyperdense protons and neutrons in conditions of partial mutual penetration, with ensuing non-linear, non-local and non-potential interactions beyond any dream of quantitative treatment via quantum mechanics.*

Recall that the proton and the neutron have both spin $1/2$ and that the only stable bound state predicted by quantum mechanics between two particles with spin $1/2$ is the singlet with spin 0.

Therefore, quantum mechanics predicts that the deuteron in its ground state must have spin 0, while experimental data establish that the deuteron has spin 1.

In an attempt of salvaging quantum mechanics, the spin of the deuteron is generally represented via a combination of *excited orbital states* which, even though significant, does not represent the spin 1 of the deuteron *in its ground state*.

The achievement of the synthesis of the neutron (Refs. [85] to [95]) has permitted a resolution of the above impasse because *the deuteron emerges as being a three-body state composed by two protons and one exchange electron, with ensuing spin* 1 *in the ground state* [59].

Subsequent studies by A. A. Bhalekar and R. M. Santilli [100] based on the "completion" of 20th century mathematical and physical methods have achieved a representation of the spin of stable nuclei in their bound state.

It should be finally noted that the biggest insufficiency of quantum mechanics in nuclear physics is given by the inability to achieve a consistent representation of nuclear forces in one century of efforts.

This is due to the fact that the sole forces permitted by quantum mechanics are of *potential,* thus of action- at-a-distance type, which is solely

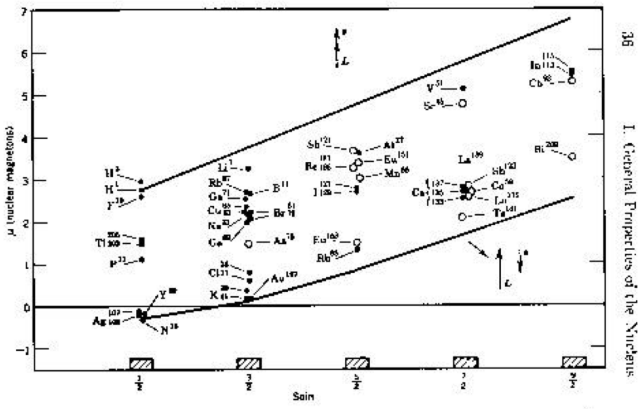

Figure 8: *A view of the experimental data of nuclear magnetic moments that cannot be exactly represented by quantum mechanics. Similar insufficiencies exist for nuclear spins.*

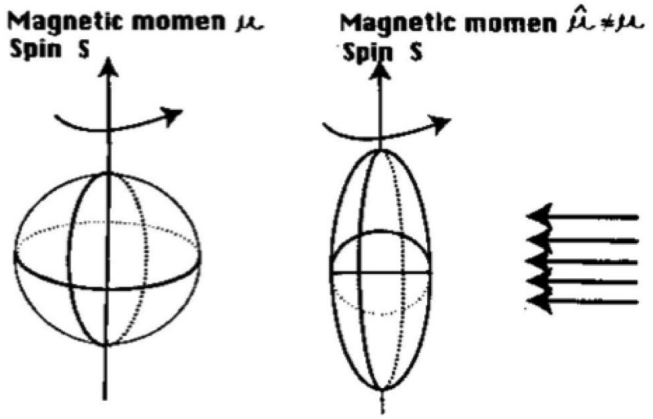

Figure 9: *A conceptual rendering of the deformability of protons and neutrons under strong nuclear interactions predicted by J. M. Blatt and V. F. Weisskopf [96] as being the origin of the inability by quantum mechanics to represent nuclear magnetic moments.*

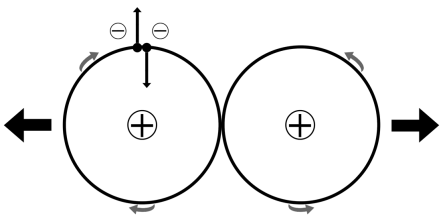

Figure 10: *This picture depicts the hydrogen molecule at absolute zero degree temperature to illustrate the fact that, despite historical achievements, quantum mechanics and chemistry have been unable to identify the force attracting identical electrons in valence bonds, because the sole admitted forces are of potential-Coulomb type, thus implying a repulsion in valence bonds with ensuing lack of a "complete" representation of molecular structures [60].*

possible, mathematical and physical conception for nuclei as ideal spheres with point-like particles in their interior (Figure 6).

One of the most important applications of the new methods studied in this work is that of representing nuclear forces as being non-linear, non-local and non-potential forces due to the mutual penetration/entanglement of the charge distribution of the hyperdense nucleons. The latter forces have emerged as being strongly attractive thus allowing the first known initiation of the understanding of the charge independence of nuclear forces [59].

1.6. Insufficiencies of quantum mechanics in chemistry.

Without doubt, quantum mechanics and chemistry have permitted chemical discoveries of historical proportions. Hence, the historical and scientific value of quantum chemistry is out of question.

Yet, it is the fate of all theories to admit, with the advancement of scientific knowledge, suitable coverings and this is the fate of quantum chemistry as well.

In fact, on strict scientific grounds quantum chemistry is only *approximately valid* in chemistry, thus admitting a suitable "completion," because of the inability in one century of efforts to achieve an exact representation of molecular experimental data from first axiomatic principles without *ad hoc* form factors and other adaptations.

Recall that quantum mechanics has achieved a numerically exact representation of all experimental data of the hydrogen atom.

By contrast, when two hydrogen atoms are bonded into the hydrogen molecule H_2, quantum mechanics and chemistry still miss the represen-

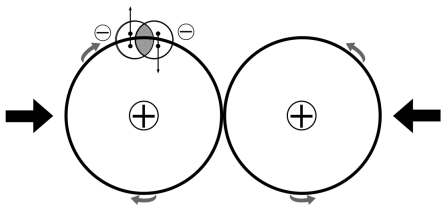

Figure 11: *This picture depicts the hydrogen molecule at absolute zero degree temperature with the strongly attractive force between identical valence electrons permitted by the representation of valence pairs as being composed by electrons with extended wavepackets in conditions of mutual penetration/entanglement with ensuing non-linear, non-local and non-potential interactions that result to be so strongly attractive [17] to overcome repulsive Coulomb forces [60]. The mutual distance d of said valence pair approaches Einstein's classical determinism and achieves it fully when it is in the interior of black holes [7].*

tation of $1\ \%$ of the H_2 binding energy, which is not insignificant since it corresponds to about $940\ kcal/mole$.

An in depth study of the impasse has shown that the above insufficiency is due to *the inability by quantum mechanics and chemistry to represent the attractive force between identical valence electrons in molecular bonds* (Figure 10) because according to basic axioms, *two identical valence electrons must repel each other according to quantum mechanics and chemistry due to their equal charge* [60].

As it had been the case for other problems, the primary difficulty to achieve an attractive force between identical valence electrons was of *mathematical* rather than of physical or chemical character, because quantum mechanics and chemistry solely admit potential forces that, in this case, can only be of repulsive Coulomb type.

By contrast, the sole possibility of resolving the impasse was the representation of electrons as extended wavepackets, that when in conditions of mutual penetration at $1\ fm$ mutual distance, admit contact, non-linear, non-local and non-potential interactions of Hulten type.

These new interactions result to be so strong to "absorb" repulsive Coulomb forces resulting in the needed attraction [60] (Figure 11).

The new valence force was first identified in Table 5 of the 1978 paper [17] as responsible for the birth of strong interactions in the synthesis of the π^0 meson from an electron and a positron.

In 1995, A. O. E. Animalu and R. M. Santilli published paper [101] establishing that the Hulten force of paper [17] is so strong to account for the

bond of the two identical electrons in the Cooper pair of superconductivity.

A generalization of superconductivity based on the new methods was then developed and it is today known as *Animalu isosuperconductivity* [102].

In 2000, R. M. Santilli and D. D. Shillady showed that valence electron pairs with a strongly attractive force, called *isoelectronia*, permit numerically exact representations from first axiomatic principles of experimental data on the hydrogen [103] and water [104] molecule, which representation had escaped quantum chemistry for about one century (see also review [105]).

The achievement of a new model of molecular structures based on the isoelectronium valence bond has permitted novel advances in larger molecules whose study has been initiated by A. A. Bhalekar and R. M. Santilli [106] with intriguing implications, e.g., possible improvements in the combustion of fossil fuels based on a more accurate representation of their molecular structure [53].

Additionally, Santilli and Shillady showed that *perturbative series of the resulting "completion" of quantum chemistry converge at least one thousand times faster than the corresponding series of quantum chemistry* (see Section 4.13).

1.7. Implications of the EPR argument.

It is hoped that the preceding sections have indicated the truly vast implications for all quantitative sciences of Einstein's view that "quantum mechanics is not a complete theory" [1], thus warranting due scientific process.

In this paper we outline the main aspects of the new mathematical and physical methods underlying proof [7], with the understanding that a technical knowledge can be solely achieved via a study of the original literature.

The reader should be aware that the literature accumulated in half a century of research in the field by numerous scientists is rather vast. Consequently, in this paper we can only quote the most important original contributions and provide comprehensive references for interested readers.

2. LIE-ADMISSIBLE "COMPLETION" OF 20TH CENTURY APPLIED MATHEMATICS.

2.1. Foreword.

R. M. Santilli never accepted quantum mechanics as a "complete" theory

beginning with his graduate studies in physics at the University of Torino, Italy, in the mid 1960's because quantum axioms are invariant under time reversal, due to the invariance under anti-Hermiticity of the Lie product between Hermitean operators

$$[x, y] = -[x, y]^\dagger, \tag{1}$$

and other physical laws.

By recalling the fundamental character of Lie's theory, it follows that the *mathematics* (more than physical laws) underlying quantum mechanics does not allow a consistent representation of nuclear fusions and other physical or chemical energy releasing processes, due to their known irreversibility over time (Figure 4).

2.2. The historical teachings by Lagrange and Hamilton.

In view of the above lack of "completeness" of quantum mechanics, R. M. Santilli initiated his Ph. D. studies with the reading of the original works by J. L. Lagrange and studying his true analytic equations, those with external terms [9]

$$\frac{d}{dt}\frac{\partial L(r, v)}{\partial v} - \frac{\partial L(r, v)}{\partial r} = F_{ak}(t, r, v), \tag{2}$$

as well as the true Hamilton's equations, those with external terms [10]

$$\begin{aligned} \tfrac{dr}{dt} &= \tfrac{\partial H(r,p)}{\partial p}, \\ \tfrac{dp}{dt} &= -\tfrac{\partial H(r,p)}{\partial r} + F(t, r, p), \end{aligned} \tag{3}$$

where the Lagrangian L and Hamiltonian H were used to represent conservative and potential, thus notoriously reversible forces, while the irreversibility of nature was represented with their *external forces* F.

2.3. The "No Reduction Theorem."

The external terms have been truncated in the 20th century sciences on claims that irreversible systems can be decomposed into their elementary particle constituents at which level the validity of quantum mechanics is fully recovered.

However, Santilli proved the following theorem as part of his Ph. D. thesis (see Refs. [56] [25]).

THEOREM 2.3.1 (No reduction Theorem): A macroscopic time irreversible system cannot be consistently decomposed into a finite number of quantum mechanical particles and vice versa, a finite collection of quantum mechanical particles

cannot reproduce a macroscopic irreversible system under the correspondence or other principle.

Consequently, a serious study of irreversible systems requires a return to the true Lagrange's and Hamilton's equations, those with external terms.

2.4. The inevitability of the Lie-admissible "completion" of 20th century science.

As it is well known, the true Lagrange and Hamilton equations cannot be assumed as a "completion" of 20th century sciences because they are not derivable from a potential.

Additionally, the brackets of the time evolution of an observable Q represented via Hamilton's equations with external terms

$$\begin{aligned} \tfrac{dA}{dt} = (Q, H, F) = \\ = \tfrac{\partial Q}{\partial r}\tfrac{\partial H}{\partial p} - \tfrac{\partial H}{\partial r}\tfrac{\partial Q}{\partial p} + \tfrac{\partial Q}{\partial r} F, \end{aligned} \tag{4}$$

characterizes the *triple system* (Q, H, F) that, in view of the external terms, violate the right scalar and associative axioms to characterize an algebra as currently understood in mathematics.

In the absence of a consistent algebra in the brackets of the time evolution, it was not possible to achieve a "completion" of quantum mechanics via a covering for irreversible systems.

Hence, Santilli was forced to seek the needed "completion" on algebraic grounds.

Following a year of research in the European mathematics libraries, Santilli did his Ph. D. thesis in 1965 on the "completion" of Lie algebras into A. A. Albert's *Lie-admissible and Jordan-admissible algebras* [11] with product [12]

$$(a, b) = pab - qba, \tag{5}$$

later on known as *(p, q)-deformations [13]*, where $p,\ q, p \pm q$ are non-null scalars, and time irreversibility is assured for $p \neq q$ for which irreversiblity is ensured for $p \neq q$ by the property $(x, y) \neq -(x, y)^\dagger$.

To achieve a first approximation of Hamilton's equations with external terms, Santilli introduced the following *parametric Lie-admissible generalization of Hamilton's equation* [14] [15]

$$\frac{dr}{dt} = p\frac{\partial H(r, p)}{\partial p}, \quad \frac{dp}{dt} = -q\frac{\partial H(r, p)}{\partial r}, \tag{6}$$

with corresponding *parametric Lie-admissible generalization of Heisenberg equation* for the time evolution of a Hermitean operator Q (for $\hbar = 1$)

$$i\frac{dQ}{dt} = (Q, H) = pQH - qHQ. \tag{7}$$

As one can see, all dynamical equations are manifestly irreversible over time, as desired.

2.5. Lie-admissible genomathematics and genomechanics.

In September 1977, Santilli joined the Department of Mathematics of Harvard University under DOE support during which stay he introduced the most general known realization of irreversible Lie-admissible algebras (see Refs. [16] to [23]) based on the generalization and differentiation of the ordinary product ab of arbitrary quantities (numbers, functions, operators, etc.) into the *ordered genomodular product to the right*

$$a > b = a\hat{R}b, \tag{8}$$

and that to the left

$$a < b = a\hat{S}b, \tag{9}$$

where $\hat{R}, S$ and $R \pm S$ are positive-definite operators with an unrestricted functional dependence on wavefunctions $\psi(t, r)$ and any other needed variables.

The operators R and S were called *genotopic element to the right and to the left,* respectively, where the prefix "geno" was suggested by Carla Santilli in the Greek sense of "inducing a new structure" [16].

The new genomodular products permitted the construction of new mathematics known as *genomathematics to the right and to the left,* [19] with corresponding "completion" of quantum mechanics into an irreversible covering known as *genotopic branch of hadronic mechanics* or *genomechanics for short* [19] in which irreversible energy releasing processes are represented with ordered genomodular product to the right, as it is the case for the *geno-Schrödinger equation* or *Schrödinger-Santilli genoequation* [20]

$$H > \psi = H(r, p)R(\psi, ...)\psi = E\psi, \tag{10}$$

Irreversibility is then assured whenever the genotopic elements R and S are not invariant under time reversal.

The time evolution of a Hermitean operator Q is given by the *Lie-admissible generalization of Heisenberg equations,* also known as *Heisenberg-Santilli genoequations* (see, e.g., Ref. [107]) first introduced in Eqs. (4.15.34),

page 746 of Ref. [17] (see the 2006 general treatment [24] and the 2016 update [25]) which can be written in the infinitesimal form

$$\begin{aligned} i\tfrac{dQ}{dt} = (Q.H) = Q < H - H > Q = \\ = QSH - HRQ, \end{aligned} \tag{11}$$

and the finite form

$$Q(t) = e^{HRti} Q(0) e^{-itSH}. \tag{12}$$

The representation of Lagrange's and Hamilton's external terms is provided by the difference $R - S$ at both classical and operator levels [24].

2.6. Universality of Lie-admissible formulations.

The following simple realization of the genotopic elements

$$\begin{aligned} S = 1, \quad R = 1 - \tfrac{1}{H} K(\psi, \partial\psi, ...), \\ i\tfrac{dQ}{dt} = (Q.H) = [Q, H] + QK, \end{aligned} \tag{13}$$

where K is a positive-definite operator representing non-Hamiltonian interactions illustrates that the Lie-admissible generalization (11) of Heisenberg's equations constitute an operator image of Hamilton's equations with external terms (3) [56].

The double infinity of possible realizations of the genotopic elements R and S then allows Lie-admissible equations (11) to be "directly universal" for the representation of all possible (regular) non-linear, non-local and non-Hamiltonian interactions in the sense of representing all of them ("universality") directly in the frame of the experimentalism without the use of the transformation theory ("direct universality") (for details, see Ref. [24]).

It should be finally indicated that the original 1978 proposal [16] established the universality of Lie-admissible algebras because the product $(A, B) = A < B - B > A$ admit as particular case the product of all possible "algebras" as commonly understood in mathematics, including Associative, Lie, Jordan, Lie-isotopic, Jordan-isotopic, alternative, super-associative, super-Lie, super-Jordan, nilpotent, flexible and other possible algebras.

2.7. Prediction of new clean nuclear fusions.

Scientific and industrial applications to search for new clean energies were initiated in the late 1990's only following the achievement of maturity in

the mathematical and physical methods needed for the representation of irreversible processes.

The first application was the conception of the new *Intermediate Controlled Nuclear Fusion* (ICNF) of light, natural and stable elements into light, natural and stable elements with smaller mass which occur without the emission of harmful (e.g., neutron) radiations and without the release of radioactive waste (see Ref. [26] to [32] for originating papers and Refs. [33] to [40] for independent studies). Laboratory analyses [41] to [52] confirmed the *existence* oof ICNF at the particle levels, with the understanding that the related industrial production of new clean energy require additional extensive research.

2.8. Prediction of a new clean combustion of fossil fuels.

To illustrate the implications of the lack of "completion" of quantum mechanics for energy releasing processes, we should note that the current combustion of fossil fuels is essentially that at the dawn of our civilization, because we essentially strike a spark and ignite the fuel with known alarming environmental deterioration of our planet.

The achievement of the Lie-admissible representation of energy releasing processes has permitted the first known conception and initiation of tests for a new principle of combustion called *HyperCombustion* which is based on the conventional combustion of carbon and oxygen dating back to the dawn of our civilization, plus the novel synthesis of a limited number of nuclei C-12 and O-16 into Si-28 to achieve full combustion of fossil fuels as well as a significant increase of energy output [53].

2.9. Literature on hadronic mathematics and mechanics.

Due to the prediction of new clean nuclear energies, the connection between irreversible mechanics and thermodynamics and other features, the literature on the foundations of hadronic mechanics is rather vast.

Ref. [54] provides a summary of the formalism of hadronic mechanics. Refs. [55] to [61] provide general presentations of hadronic mechanics. Vol. I of Refs. [61] contains a comprehensive literature up to 2008 with an upgrade to 2016 in Ref. [25].

Additional references are available in the reprint volumes [62] [63] and in the proceedings of five *Workshops on Lie-Admissible Algebras*,, twenty five *Workshops on Hadronic Mechanics*, and three international conferences on the *Lie-admissible treatment of Irreversible Processes* whose references are available from Ref. [61]. Representative independent papers are available from Refs. [64] to [74] and independent monographs are available from Refs. [75] to [83].

3. LIE-ISOTOPIC "COMPLETION" OF 20TH CENTURY APPLED MATHEMATICS.

3.1. Foreword.

As it is well known, the objections against the EPR argument (Section 1.1) were formulated for *isolated reversible systems of point particles in vacuum with linear, local and Hamiltonian interactions* called *reversible exterior dynamical problems*.

Hence, the proof of the EPR argument had to study *isolated reversible systems of extended, therefore deformable and hyperdense particles in conditions of mutual penetration/entanglement with ensuing linear and non-linear, local and non-local and Hamiltonian as well as non-Hamiltonian internal interactions.* The latter systems are called *reversible interior dynamical problems*, and they occur in the structure of hadrons, nuclei, stars and black holes.

Despite their reversible character, the latter systems could not be studied with 20th century applied mathematics, including Lie's theory, due to its strictly Hamiltonian character. Reversible interior dynamical systems could not be studied with Lie-admissible formulations due to their irreversible character. Hence, the needed new mathematics had to be built.

In this section, we review the foundations of the new mathematics for the consistent representation of reversible interior dynamical systems which was essentially constructed as a reversible particular case of universal Lie-admissible formulations.

3.2. Isoproduct.

In order to achieve a representation of the latter systems, Santilli introduced in the original proposal [16] of 1978 the axiom-preserving particular case of genomathematics called *isomathematics*, which is characterized by the genoproduct to the right being equal to that to the left, $R = S = \hat{T}$, resulting in the time-reversal invariant *isoproduct*, (first introduced in classical realization in Eqs. (3.7.10), page 352 of Ref. [16], introduced in operator form in Eq. (4.15.46), page 751, Ref. [17] and then studied in details in Ref. [19], Eq. (2), page 71 on)

$$a \star b = a \, \hat{T} \, b, \tag{14}$$

where $\hat{T}$, called the *isotopic element*, is a function, matrix or operator solely restricted to be positive-definite, but possesses otherwise an unrestricted functional dependence on all needed local variables, such as: spacetime coordinates $x = (t, r)$; linear momentum p; energy E; frequency ν; density of the medium α; temperature τ; pressure π; wavefunctions ψ; their

derivatives $\partial\psi$; and any other needed variable

$$\hat{T} = \hat{T}(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \tag{15}$$

where the prefix "iso" was also suggested by Carla Santilli in its Greek meaning of preserving the axions.

The *physical* significant of isoproduct (14) is illustrated by nothing that it allows "ab initio" a *direct representation of extended, thus deformable and hyperdense particles and their non-Hamiltonian interactions* illustrated in Figure 3 (Section 1.4). This important task is achieved via simple realizations of the isotopic element of the type (needed for the neutron synthesis from the hydrogen studied in Section 2.4) [55] [56]

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2})e^{-\Gamma}, \tag{16}$$

with subsidiary conditions

$$\begin{gathered} n_\mu = n_\mu(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \mu = 1, 2, 3, 4, \\ \Gamma(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) \geq 0. \end{gathered} \tag{17}$$

The isoproduct also allows a direct representation of nuclei as a collection of extended nucleons in conditions of mutual penetration/entanglement as presented in Figure 7 and Section 1.5 with broader realizations of the type (needed to represent nuclear magnetic moments and spins, or for the achievement of an attractive force between identical valence electron bonds in molecular structures) [57]

$$\begin{gathered} \hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2})e^{-\Gamma}, \\ k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4. \end{gathered} \tag{18}$$

In the above realizations of the isotopic element, n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_1^2 = n_2^2, = n_3^2 = 1$, for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

The *mathematical* significance of basic assumption (14) is that it requires, for consistency, a compatible "completion" of *all* aspects of 20th century applied mathematics without any known exception.

This program was initiated in the 1978 proposal [16], continued in the 1981 monograph [19] and completed in numerous works by various mathematicians (see the 1995 monograph [55] for a comprehensive presentation).

Regrettably we cannot provide a technical review of isomathematics to prevent excessive length. Nevertheless, a rudimentary outline of the main aspects of isomathematics appears to be recommendable for an understanding of proof [7] of the EPR argument.

3.3. Isonumbers.

As it is well known, physical theories are formulated over a field $F(n, \times, 1)$ of real, complex or quaternionic numbers n with product $nm = n \times m$ and multiplicative unit 1. Said field remains invariant under the unitary time evolution of quantum mechanics, thus allowing the prediction of the same numerical values under the same conditions at different times.

But the time evolutions of hadronic mechanics, such as Eqs. (11), are *non-unitary* when formulated on conventional space over a conventional field (not so for isomathematics as shown below).

This implies the loss over time of the multiplicative unit 1, and consequently, of the entire numeric field, with ensuing lack of consistent experimental verifications.

To resolve this impasse, Santilli had no other choice than that of reinspecting the historical classification of numbers, by discovering in this way that *the abstract axioms of a numeric field do not necessarily restrict the multiplicative unit to be the number* 1, *and allow for unit an arbitrary positive-definite quantity* $\hat{I}$ *provided that the multiplication is redefined for* $\hat{I}$ *to verify the unit axiom* [107].

This lead to "completion" of numeric fields $F(n, \times, 1$ into *isofields* $\hat{F}(\hat{n}, \star, \hat{I}$ of *isoreal, isocomplex, and isoquaternionic isonumbers* $\hat{n} = n\hat{I}$ with *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{19}$$

isoproduct (14), $\hat{n} \star \hat{m} = (nm)\hat{I}$, and isounit isoaxiom

$$\hat{I} \star \hat{n} = \hat{n} \star \hat{I} = \hat{n}, \ \ \forall \hat{n} \ \in \ \hat{F}. \tag{20}$$

Isofields are completed by compatible redefinitions of all numeric operations, such as *isoquotient, isosquare, isosquareroot,* etc. [107].

It should be indicated that *isofields verify all axioms of a field. Hence, isonumbers are fully acceptable for experimental verifications* [57].

Isofields are classified into those of the *first kind* (*second kind*) depending on whether the isounit $\hat{I}$ is (is not) an element of the original field.

In this paper, we shall solely consider isofields of the second kind since the representation of extended particles and their non-Hamiltonian interactions is achieved via the isounit or equivalently, the isotopic element.

Paper [107] has stimulated various studies in number theory, among which we mention the study by C-Xu. Jiang [79], A. K. Aringazin [108], C. Corda [109] and others.

3.4. Isofunctions.

As it is well known, a necessary condition for a variable to be measurable is that it is an element of the base field, and the same holds for functions of said variable.

The implementation of the same rule under isotopic "completion" stimulated the construction of the *isofunctional isoanalysis* initiated by: J. V. Kadeisvili [110][111]; A. K. Aringazin, D. A. Kirukhin and R. M. Santilli [112]; Raul M. Falcon Ganfornina and Juan Nunez Valdes [80]; and others.

We here limit ourselves to indicate: the *isotime* $\hat{t} = t\hat{I}_t$, *isospace isocoordinates* $\hat{r} = r\hat{I}_r$, and the *isofunctions of isovariable*,

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}, \tag{21}$$

such as the *isoexponentiation*

$$\hat{e}^{\hat{X}} = [e^{\hat{X}\hat{T}}]\hat{I} = \hat{I}[e^{\hat{T}\hat{X}}]. \tag{22}$$

Similar expressions hold for virtually all conventional functions used in applications [55].

3.5. Isospaces.

The initial construction of isomathematics [19] was formulated via conventional vector or metric spaces over conventional fields.

The consistent need to formulate spaces over isofields triggered the isotopic "completion" of metric spaces into *isospaces* whose study was initiated by the mathematician Gr. Tsagas and his school [113]. In turn, these studies triggered the construction of the *isotopology* by R. M. Falcon Ganfornina and J. Nunez Valdes [114], yielding the first known topology for the characterization of extended particles, known as the *Tsagas, Sourlas, Santilli, Ganfornina and Nunez (TSSGN) isotopology.*

Let $E(r, \delta, I)$ be the conventional Euclidean space with space coordinates $r = (x, y, z)$, metric $\delta = Diag.(1, 1, 1)$, unit $I = Diag(1, 1, 1)$ and invariant

$$r^2 = (x^2 + y^2 + z^2)1, \tag{23}$$

where one should note the trivial multiplication by 1 for compatibility with the isotopic image studied below.

The representation isospace of the non-relativistic proof of the EPR argument is given by the infinite family of *iso-Euclidean isospaces,* $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I})_r$, first formulated in the 1978 Ref. [16] and treated in detail in Ref. [19] , finalized in 1996 Ref. [115] and extensively treated in monograph [55].

For the simple realization of the isotopic element

$$\hat{T} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2), \tag{24}$$

iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I}_r)$ are characterized by: the *isocoordinates* $\hat{r} = r\hat{I}$, the *iso-Euclidean isometric* $\hat{\Delta} = (\hat{T}\delta\hat{I}$, and the *isospace isounit* $\hat{I}_r = 1/\hat{T} > 0$ resulting in the *iso-Euclidean isoinvariant*

$$\begin{aligned} \hat{r}^{\hat{2}} = (\hat{r}^j \star \hat{\Delta}_{jm} \star \hat{r}^m) = (r^j \hat{\delta}_{jm} r^m)\hat{I}_r = \\ = (\tfrac{r_1^2}{n_1^2} + \tfrac{r_2^2}{n_2^2} + \tfrac{r_3^2}{n_3^2})\hat{I}_r, \end{aligned} \tag{25}$$

where we should recall that, for consistency, all scalar quantities have to be elements of an isofield $\hat{F}$.

The above conditions require that: squares must be isosquares $\hat{r}^{\hat{2}} = \hat{r} \star \hat{r} = \hat{r}^2\hat{I}_r$; coordinates have to be isocoordinates $\hat{r} = r\hat{I}_r$; to be isomatrices, isometrics must have the structure $\hat{\Delta} = \hat{\delta}\hat{I}_r$; and the elaboration requires the use of the *isotrigonometric isofunctions* as well as of the *isospherical isocoordinates* (see Ref. [55] for details).

Let $M(x, \eta, I)$ be the conventional Minkowski space with spacetime coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = Diag.(1, 1, 1, -1)$, unit $I = Diag(1, 1, 1, 1)$ and invariant

$$\begin{aligned} x^2 = (x^\mu \eta_{\mu\nu} x^\nu)I = \\ = (x_1^2 + x_2^2 + x_3^3 - c^2t^2)I. \end{aligned} \tag{26}$$

The isospace for relativistic treatments of extended particles is given by the infinite family of *iso-Minkowski spaces* $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I}_x)$ also known as *Minkowski-Santilli isospaces,* (see e.g., Ref. [107]) first introduced in Ref. [116] and then treated in details in Ref. [56].

Iso-Minkowskian isospaces are characterized by the *isospace-time isocoordinates* $\hat{x} = x\hat{I}$; isounit $\hat{I} = 1/\hat{T}$, and *isometric* $\hat{\Omega} = \hat{\eta}\hat{I}_x = (\hat{T}\eta)\hat{I}_x$ formulated on the isoreal isonumbers $\hat{\mathcal{R}}$.

For the simple realization of the isotopic element

$$\hat{T} = Diag(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2), \tag{27}$$

Figure 12: *This picture illustrates the isodifferential calculus [115] via birds flying in close formation without wing interferences, which can be best understood by assuming that birds conceive themselves as a volume encompassing their wings, rather than a mass concentrated in their center of gravity as it would be requested by the Newton-Leibnitz differential calculus.*

we have the infinite family of realizations of the *isospace-time isoinvariant*

$$\begin{aligned}\hat{x}^{\hat{2}} = \hat{x}^{\mu} \star \hat{\Omega}_{\mu\nu} \star \hat{x}^{\nu} = (x^{\mu}\hat{\eta}_{\mu\nu}x^{\nu})\hat{I} = \\ = (\tfrac{x_1^2}{n_1^2} + \tfrac{x_2^2}{n_2^2} + \tfrac{x_3^2}{n_3^2} - t^2\tfrac{c^2}{n_4^2})\hat{I},\end{aligned} \tag{28}$$

where the final multiplication by the isounit is again necessary for the invariant to be an isoscalar.

It should be noted that, in addition to the use of the isospherical isocoordinates, data elaborations in the iso-Minkowskian isospace requires the use of *isohyperbolic isofunctions* (see Ref. [56], Chapters 5 and 6).

Note also that invariant (30) is the most general possible symmetric (non-singular) invariant in $(3+1)$-dimensions, thus including as particular cases all possible Minkowskian, Riemannian, Fynslerian and all other geometries.

3.6. Isodifferential isocalculus.

Despite the above advances, numerical predictions of isomathematics lacked the crucial property of invariance over time.

In addition, isomechanics and hadronic mechanics at large, were incomplete due to the inability to formulate the isotopies and genotopies of the linear and angular momenta (see next section).

In order to resolve this impasses, Santilli had no other choice than that of reinspecting the Newton-Leibnitz differential calculus, by discovering in this way that, contrary to rather popular belief for four centuries, *the differential calculus depends on the multiplicative unit of the base field* because, when the unit depends on the variable of differentiation, said calculus has to be "completed" into the infinite family of *isodifferentials,* e.g., of the isocoordinates $\hat{r}$, first formulated in memoir [115] submitted in 1995 and published in 1996 and then treated in details in Refs. [55] [56]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r,...)] = dr + r\hat{T}d\hat{I}(r,...), \tag{29}$$

with corresponding *isoderivative* [114]

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{30}$$

Following decades of searches, the discovery of the isodifferential isocalculus finally permitted the achievement of the invariance over time of numerical predictions, the formulation of isolinear and genolinear momenta and signaled the achievement of maturity for applications and experimental verifications (see Refs. [56] [57] for details).

All novel applications of isomathematics in physics, chemistry and other fields, including the proof of the EPR argument [7], originate from the extra term $r\hat{T}d\hat{I}(r,...)$ in isodifferential (29), which is absent in the mathematics for point particles.

The covering character of the isodifferential isocalculus over the conventional calculus is illustrated by the fact that whenever the isounit is independent from the differentiation variable or it is a constant, the conventional calculus is recovered uniquely and identically.

It should be indicated that the biggest difficulty in the use of the isodifferential isocalculus is of *conceptual,* rather than of mathematical character, because it requires the transition from the visualization of the calculus at individual points to volumes (or surfaces) represented by the isotopic element $\hat{T}$ (Figure 12).

By looking in retrospect, it appears nowadays evident that a "completion" of quantum mechanics for the representation of extended particles is fundamentally inconsistent when formulated via the conventional differential calculus, because of its sole possible characterization of particles as being point-like.

Consequently, the generalization of the differential calculus into a form defined on volumes represented by $\hat{T}$, rather than defined on coordinate points r, is essential for a consistent representation of extended particles.

Nineteen years following the discovery of the isodifferential isocalculus for extended particles, comprehensive studies in the field have been conducted by the mathematician S. Georgiev in the series of six monographs [83] which consider the broadest possible formulation of the isodifferential calculus, including the case of different isotopic elements for different isovariables.

3.7. Lie-Santilli isotheory.

In Santilli's view, the physically most important part of isomathematics is given by the isotopic "completion" of the various branches of Lie's theory, today known as the *Lie-Santilli isotheory*, which was first formulated in papers [16] [17] of 1978, systematically studied in monograph [19], finalized in Refs. [55] [56] of 1995 following the discovery of the isodifferential calculus [115] and recently studied in paper [117].

In this section, we follow the presentation of Ref. [19] of 1981 upgraded into a formulation on isospaces over isofields and elaborated via the isodifferential isocalculus.

Let L be a n-dimensional Lie algebra with Hermitean generators $X_k,\ k = 1, 2, ...n$ defined on a conventional space over a conventional numeric field. Then, the infinite family of isotopies $\hat{L}$ of L are characterized by the following main theorems:

THEOREM 3.7.1 [19]: (Poincaré-Birkhoff-Witt-Santilli isotheorem): The isocosets of the isounit and of the isostandard isomonomials

$$\hat{I},\ \ \hat{X}_k,\ \ \hat{X}_i \star \hat{X}_j,\ i \leq j,\ \ \hat{X}_i \star \hat{X}_j \star \hat{X}_k,\ i \leq j \leq k,\ \ldots, \tag{31}$$

form an infinite dimensional isobasis of the universal enveloping isoassociative isoalgebra $\hat{E}(\hat{L})$ (also called isoenvelope for short) of a Lie-Santilli isoalgebra $\hat{L}$.

The first illustration of the above theorem is given by isoexponential isofunction (22) whose correct derivation requires infinite basis (31).

The appearance of the non-linear, non-local and non-potential isotopic element $\hat{T}$ in the *exponent* illustrates the non-trivial character of the Lie-Santilli isotheory.

THEOREM 3.7.2 [19]: (Lie-Santilli isoalgebras) The antisymmetric isoalgebras $\hat{L}$ attached to the isoenveloping algebras $\hat{E}(\hat{L})$ verify the isocommutation rules

$$\begin{aligned} [\hat{X}_i,\hat{X}_j] &= \hat{X}_i \star \hat{X}_j - \hat{X}_j \star \hat{X}_i = \\ &= \hat{C}_{ij}^k(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) \star \hat{X}_k, \end{aligned} \tag{32}$$

where the quantities $\hat{C}$ are called the structure quantities.

The above isocommutation rules show the axiom-preserving character of the isotopies in view of their evident verification of the Lie axioms, although via a broader realization.

The Lie-Santilli isotheory is called *regular* or *irregular* depending on whether the structure quantities $\hat{C}_{i,j}^{k}$ are isoscalars or isofunctions, respectively.

This classification is important because as shown in the next section, the regular Lie-Santilli isotheory can be constructed via a well defined transform of corresponding Lie theory, while irregular Lie-Santilli isotheories are truly new theories verifying Lie's axioms without a known map form the conventional formulation.

THEOREM 3.7.3 [19]: (Lie-Santilli isogroups) The isoexponentiated form $\hat{G}$ of isocommutation rules (32) defined on an isospace $\hat{S}$ with local isocoordinates $\hat{x}$ over an isofield $\hat{F}$ with isounit $\hat{I} = 1/\hat{T} > 0$ is a group mapping each element $\hat{x} \in \hat{S}$ into a new element $\hat{x}' \in \hat{S}$ via the isotransformations

$$\hat{x}' = \hat{g}(\hat{w}) \star \hat{x}, \quad \hat{x}, \hat{x}' \in \hat{S}, \quad \hat{w} \in \hat{F}, \tag{33}$$

verifying the following isomodular isoaction to the right:

1) The isomap of $\hat{g} \star \hat{S}$ into $\hat{S}$ is isodifferentiable $\forall \hat{g} \in \hat{G}$;

2) $\hat{I}$ is the left and right isounit of $\hat{G}$,

$$\hat{I} \star \hat{g} = \hat{g} \star \hat{I} \equiv \hat{g}, \quad \forall \hat{g} \in \hat{G}; \tag{34}$$

3) The isomodular isoaction is isoassociative,

$$\hat{g}_1 \star (\hat{g}_2 \star \hat{x}) = (\hat{g}_1 \star \hat{g}_2) \star \hat{x}, \quad \forall \hat{g}_1, \hat{g}_2 \in \hat{G}; \tag{35}$$

4) In correspondence with every element $\hat{g}(\hat{w}) \in \hat{G}$ with $\hat{w} \in \hat{F}$ there exists the inverse element $\hat{g}(-\hat{w})$ such that

$$\hat{g}(\hat{0}) = \hat{g}(\hat{w}) \star \hat{g}(-\hat{w}) = \hat{I}; \tag{36}$$

5) The following composition laws are verified

$$\hat{g}(\hat{w}) \star \hat{g}(\hat{w}') = \hat{g}(\hat{w}') \star \hat{g}(\hat{w}) = \hat{g}(\hat{w} + \hat{w}'), \forall \hat{g} \in \hat{G}, \ \hat{w} \in \hat{F}; \tag{37}$$

with corresponding isomodular action to the left, and general expression

$$\hat{g}(\hat{w}) = \prod_{k} \hat{e}^{\hat{i} \star \hat{w}_k \star \hat{X}_k} \star \hat{g}(0) \star \prod_{k} \hat{e}^{-\hat{i} \star \hat{w}_k \star \hat{X}_k}. \tag{38}$$

Nowadays, *isomathematics* is referred to as the infinite family of isotopies of 20th century applied mathematics, with particular reference to the isotopies of Lie's theory when formulated on isospaces over isofields and elaborated via the isodifferential isocalculus. Isomathematics is then classified into *regular and irregular isomathematics* depending on whether the structure quantities $\hat{C}_{ij}$ are isoscalars or isofunctions of local isovariables, respectively.

It should be indicated that, the proof [7] of the EPR argument uses both regular and irregular Lie-Santilli isoalgebras.

Since Lie's theory is at the foundation of the axiomatic structure and applications of quantum mechanics, the covering Lie-Santilli isotheory predictably stimulated a number of independent contributions, such as the studies by the mathematicians: D. S. Sourlas and Gr. T. Tsagas [76], J. V. Kadeisvili [118], T. Vougiouklis [119] and papers quoted therein.

3.8. Simple construction of regular isomathematics.

A simple method for physicists has been identified in Ref. [120] of 1997 for the construction of regular isomathematics. The method consists of: 1) Selecting the desired representation of extended particles with non-Hamiltonian interactions via isotopic elements $\hat{T}$ of type (16) or (18); 2) Identifying a *non-unitary* transformation representing the selected isounit $\hat{I} = 1\hat{T}$

$$UU^\dagger = \hat{I}; \tag{39}$$

3) Subjecting the *totality* of conventional applied mathematics to the above nonunitary transform with no known exception, resulting in expressions of the type

$$I \to \hat{I} = UIU^\dagger = 1/\hat{T}, \tag{40}$$

$$n \to \hat{n} = UnU^\dagger = nUU^\dagger = n\hat{I},\ n \in F, \tag{41}$$

$$f(r) \to \hat{f}(\hat{r}) = Uf(r)U^\dagger, \tag{42}$$

$$e^A \to Ue^AU^\dagger = \hat{I}e^{\hat{T}\hat{A}} = (e^{\hat{A}\hat{T}})\hat{I}, \tag{43}$$

$$\begin{aligned} AB \to U(AB)U^\dagger = \\ = (UAU^\dagger)(UU^\dagger)^{-1}(UBU^\dagger) = \hat{A}\star\hat{B}. \end{aligned} \tag{44}$$

It should be indicated that the above transformations imply the possibility of constructing the infinite family of Lie-Santilli isoalgebras via nonunitary transforms of the considered Lie algebra. This is possible in view

of the transformation of commutation rules into their covering isocommutator forms,

$$[A, B] \to U[A, B]U^\dagger = [\hat{A}\hat{,}\hat{B}]. \tag{45}$$

This property is evidently important for the construction of the isorepresentation of regular isoalgebras used for physical and chemical applications.

Note that *serious inconsistencies occur, at times without their detection by non-experts, in the event only one single quantity or operation of 20th century applied mathematics is not subjected to the above non-unitary map.*

We should finally indicate that *the proof of the EPR argument used in Ref. [7] is of 'non-unitary' character,* thus implying that the physical conditions for said proof are outside the class of equivalence of quantum mechanics.

3.9. Invariance of regular isomathematics.

An additional contribution of paper [120] is the proof that *the dimension, shape and density of extended particles and their non-Hamiltonian interactions are represented by isomathematics in a form invariant over time.*

Firstly, Ref. [120] showed that, following the construction of regular isomathematics via non-unitary transformations (Section 2.2.8), isomathematics is *not* invariant under additional non-unitary transforms, e.g., because of the lack of invariance of the basic isounit

$$\hat{I} \to \hat{I}' = W\hat{I}W^\dagger \neq \hat{I}, \;\; WW^\dagger \neq I, \tag{46}$$

with consequential physical inconsistencies since any structural change of the isounit implies the transition to a different physical or chemical system.

However, non-unitary transforms can always be identically rewritten as *isounitary isotransforms* according to the rule [56]

$$WW^\dagger = \hat{I}, \quad W = \hat{W}\hat{T}^{1/2}, \tag{47}$$

$$WW^\dagger = \hat{W} \star \hat{W}^\dagger = \hat{W}^\dagger \star \hat{W} = \hat{I}, \tag{48}$$

under which reformulation we have the following *invariance of the isounit of the isotopic element and of the isoproduct of regular isomathematics* [120]

$$\hat{I} \to \hat{I}' = \hat{W} \star \hat{I} \star \hat{W}^\dagger \equiv \hat{I}, \tag{49}$$

$$\hat{A} \star \hat{B} \to \hat{W} \star (\hat{A} \star \hat{B}) \star \hat{W}^\dagger =$$

$$= \hat{A}' \star \hat{B}' = \hat{A}'\hat{T}\hat{B}', \tag{50}$$

$$\hat{A}' = \hat{W} \star A \star \hat{W}^\dagger). \; \hat{B}' = \hat{W} \star \hat{B} \star \hat{W}^\dagger, \; \hat{T} = (W^\dagger \star \hat{W})^{-1}.$$

The invariance of the entire isomathematics follows. Note that the invariance is ensured by the *invariant numeric values of the isounit and therefore, of the isotopic element under isounitary isotransforms,*

$$\hat{I} \to \hat{I}' \equiv \hat{I}, \tag{51}$$

$$A \star B = A\hat{T}B \to A' \star' B' = A'\hat{T}'B' = \hat{A}' \star \hat{B}' = \hat{A}'\hat{T}\hat{B}' \tag{52}$$

$$\hat{T} \to \hat{T}' \equiv \hat{T}. \tag{53}$$

By noting that, as shown in the next section, the time evolution of the isotopic "completion" of quantum mechanics is an isounitary isotransform, paper [120] established that *isomechanics is an axiom-preserving "completion" of quantum mechanics capable of representing extended particles under Hamiltonian as well as non-Hamiltonian interactions in a form invariant over time.*

In closing, we should recall other generalizations of 20th century mathematics and their applications, such as the so-called *deformations.* These generalizations are mathematically correct, but physically inconsistent because they violate causality laws (for brevity, see the *Theorems of Inconsistency of Non- Unitary Theories* in Vol. I of Refs. [61]). These inconsistencies arise from a structural generalization of Lie's algebras and other quantum laws when formulated on conventional spaces over conventional fields, thus preventing their reformulation as isounitary theories.

Note that the invariance of isomathematics reviewed in this section implies the verification of causality on isospaces over isofields in the same way as quantum mechanics verify causality laws.

4. LIE-ISOTOPIC "COMPLETION" OF QUANTUM MECHANICS.

4.1. Foreword.

In this section, we review the foundation of the *isotopic branch of hadronic mechanics,* also known as *isomechanics*, which is used in the proof of the EPR argument [7].

An important difference between preceding works and the presentation in this section is that the previous works generally present isomechanics in its *projection* on conventional spaces over conventional fields.

By contrast, in this section we put the emphasis in the full formulation of isomechanics, that on isospaces over isofields because important to illustrate that, contrary to opposing view (Section 1.1), proof [7] of the EPR

argument is fully compatible with quantum axioms, only subjected to a broader realization.

Isomechanics was first introduced via isoproducts defined on conventional spaces over conventional fields in the 1978 papers [16] [17] and in the 1981 monograph [19]; it first achieved mathematical maturity in the 1996 memoir [115] thanks to the discovery of the isodifferential isocalculus; isomechanics was finally presented in a systematic way in monographs [55] [56] [57] with a 2016 upgrade in Section 2 of memoir [25].

Independent studies are available in Refs. [75] to [83]. A comprehensive list of references up to 2008 is available in Vol. I of Refs. [61] while a 2016 upgrade is available in Ref. [25]. We regret the inability of reviewing all important contributions on isomechanics to prevent an excessive length and are forced to outline only the most salient structural contributions.

4.2. Iso-Newton isoequations.

As it is well known, the fundamental equations of mechanics are the historical *Newton's equations*, representing systems of point-particles with Hamiltonian (that is, *variationally selfadjoint,* SA [18]) and non-Hamiltonian (*variationally non selfadjoint,* NSA [18]) forces [18] defined on a conventional Euclidean space, $E(r,\delta,I)$ over the field of real numbers $\mathcal{R}$

$$\begin{aligned} m\tfrac{dv_{ak}}{dt} - F^{SA}_{ak}(r,v) - F^{NSA}_{ak}(t,r,v) = 0, \\ k = 1,2,3, \quad a = 1,2,...,N, \quad N \geq 2. \end{aligned} \tag{54}$$

It is generally believed that Newton's equations with non-conservative forces can solely represent open, irreversible systems. In Ref. [19] Section 6.3, Page 236, Santilli introduced *closed non-self-adjoint systems,* which are given by systems (54) violating the integrability conditions for their representation via Lagrange's or Hamilton's equations, yet verifying all ten conservation laws of Galileo relativity under the conditions

$$\begin{aligned} \textstyle\sum_{ak} F^{NSA}_{ak} = 0, \\ \textstyle\sum_{ak} \mathbf{r}\cdot\mathbf{F}^{NSA} = 0, \\ \textstyle\sum_{ak} \mathbf{r}\mathbf{F}^{NSA} = 0. \end{aligned} \tag{55}$$

which conditions are evidently applicable only for $N \geq 2$, since the case of one particle $N = 1$ is trivial.

The fundamental equations of isomechanics are given by the isotopic "completion" of Eqs. (54) known as *iso-Newton isoequations*, which were

first introduced in Ref. [115] immediately following the discovery of the isodifferential isocalculus, and they are also known as *Newton-Santilli isoequations,* (see, e.g., Refs. [107] [75] [81]) defined on iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ (Section 2.2.5) over isoreal isonumbers $\hat{\mathcal{R}}$ (Section 2.2.3)

$$\hat{m}_a \star \frac{\hat{d}\hat{v}_{ak}}{\hat{d}\hat{t}} - F^{SA}_{ak}(\hat{r}, \hat{v}) = 0. \tag{56}$$

A first important feature of Eqs. (56) is that of providing the first known consistent representation of the actual shapes and dimensions of the particles considered via the isodifferential calculus, with realization of the isotopic element of type (16).

A second important feature of Eqs. (56) is that of representing all potential-(SA) forces via conventional Newtonian forces $F^{SA}(r, v)$ while representing all non-potential-(NSA) forces via the isodifferential calculus.

This feature is treated in details in Ref. [115] and can be summarized as follows.

Note that the basic isounit of Eqs. (56) is the *isovelocity isounit*, $\hat{I} = \hat{I}_v$. Assume for simplicity that the isotime is equal to the conventional time,

$$\hat{t} = t \;\; \hat{I}_t = 1. \tag{57}$$

Consequently, from isoderivative (30), we have

$$\frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = I_t d\hat{v}/d\hat{t} = d\hat{v}/dt. \tag{58}$$

Consider then the projection of Eqs. (56) in the Euclidean space $E(v, \delta, I)$ and use the various rules of isomathematics (Section 2.2). Then Eqs. (56) can be written in the projected form (where all multiplications are conventional),

$$\begin{aligned} (m\hat{I})\hat{T}\tfrac{\hat{d}(v\hat{I})}{dt} - F^{SA}\hat{I} = \\ = \hat{I}[m\tfrac{dv}{dt} - F^{SA}] + mv\hat{T}\tfrac{d\hat{I}}{dt} = 0. \end{aligned} \tag{59}$$

By dividing the above equation with $\hat{I} > o$, one obtains Newton's equations with the following realizations of the NSA forces

$$F^{NSA}(t, r, v) = mv\hat{T}\frac{d\hat{I}}{dt}. \tag{60}$$

In conclusion, *the iso-Newton isoequations (56) embed all NSA forces into the isoderivatives by therefore allowing the first known consistent operator image of non-Hamiltonian forces studied in subsequent sections.*

4.3. Iso-Lagrangian and iso-Hamiltonian isomechanics.

Iso-Newton isoequations (56) admit a representation in terms of the *iso-Lagrange isoequations* first formulated in memoir [115] via the isodifferential calculus, thus being defined on an iso-Euclidean isospace over the isoreal isofield

$$\frac{\hat{d}}{\hat{dt}}\frac{\hat{\partial}\hat{L}(\hat{r},\hat{v})}{\hat{\partial}\hat{v}_{ak}} - \frac{\hat{\partial}\hat{L}(\hat{r},\hat{v})}{\hat{\partial}\hat{r}_{ak}} = 0, \tag{61}$$

where $\hat{L} = L\hat{I}$ is an *iso-Lagrangian*, namely, a conventional Lagrangian formulated on isospaces over isofields thus being multiplied by the isounit to be an isoscalar.

Eqs. (56) also admit the isocanonically isoequivalent isorepresentation in terms of the *iso-Hamilton equations* first formulated in memoir [115] on an isophase isospaces over an isoreal isofield and also known as *Hamilton-Santilli isoequations* (see, e.g., Ref. [107])

$$\begin{aligned} \frac{\hat{d}\hat{r}_{ak}}{\hat{dt}} &= \frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{p}_{ak}}, \\ \frac{\hat{d}\hat{p}_{ak}}{\hat{dt}} &= -\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{r}_{ak}}, \end{aligned} \tag{62}$$

where $\hat{H} = H\hat{I}$ is the iso-Hamiltonian, that is, a conventional Hamiltonian formulated on an isophase isospace over an isofield.

Note that iso-Hamiltonian isomechanics admits the following time evolution for a quantity $\hat{Q}$

$$\frac{\hat{d}\hat{Q}}{\hat{dt}} = [\hat{Q},\hat{H}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}_{ak}}\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_{ak}} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_{ak}}\frac{\hat{\partial}\hat{Q}}{\hat{\partial}\hat{r}_{ak}}, \tag{63}$$

where the brackets $[\hat{Q},\hat{H}]$ constitute a classical realization of Lie-Santilli isoalgebras.

4.4. Isovariational isoprinciple.

Another important feature of Eqs. (56) is that of permitting the first known representation of variationally non- selfadjoint/non-Hamiltonian systems via an *isovariational isoprinciple* [115],

$$\hat{\delta}\hat{A} = \hat{\delta}\hat{\int}(\hat{p}_{ak} \star \hat{d}\hat{r}_{ak} - \hat{H} \star \hat{dt}) = 0, \tag{64}$$

which representation is notoriously impossible for NSA Newton's equations, with the consequential lack of achievement of consistent operator forms of nonconservative forces.

In view of the universality of Eqs. (56), the above isovariational isoprinciple is *directly universal*, that is, capable of representing all infinitely possible, regular, time reversal invariant Newtonian systems (54) ("universality") directly in the coordinates of the experimenter ("direct universality").

4.5. Iso-Hamilton-Jacobi isoequations.

Much along conventional analytic procedures, it is easy to prove that isovariational isoprinciple (64) implies the following *iso-Hamilton-Jacobi isoequations* also called *Hamilton-Jacobi-Santilli isoequations* [115] [25] that are at the foundation of the *isoquantization* reviewed in the next section

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{t}} + \hat{H} = 0, \tag{65}$$

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}_{ak}} - \hat{p}_{ak} = 0, \tag{66}$$

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_{ak}} = 0. \tag{67}$$

We should recall from Section 2.2.5 that isodynamical isoequations of classical isomechanics require two different isofields, the first being the *isotime isofield* with isounits $\hat{I}_t$ and the second being the *isovelocity isofield* with isounits $\hat{I}_v$.

However, the direct universality is already achieved with the sole use of the isovelocity isounit. Hence, the isotime isounit can be assumed to be 1 without any loss of direct universality.

For non-relativistic formulations, we shall use isotime in the isodynamical equations for completeness, with the tacit understanding that, unless otherwise specified, isotime will be assumed to be equal to the conventional time.

As it will soon be evident, the Hamilton-Jacobi-Santilli isoequations (65)-(67) are truly fundamental for the construction of operator isomechanics, as well as for the proof of the EPR argument because said equations have permitted:

1) The achievement from Eqs. (65) of a unique and unambiguous map of classical into operator isomechanics;

2) The achievement from Eqs. (66) of the first known operator form of the isolinear isomomentum;

3) The achievement, from Eqs. (67) of an operator isomechanics whose *isowave Isofunctions* solely depend on local isocoordinates $\hat{\psi}(\hat{r})$. This feature is necessary for a consistent isotopic "completion" of quantum mechanics since conventional wave functions $\psi(t,r)$ do not depend on linear momenta p. Hence, any "completion" of quantum mechanics whose wavefunctions also depend on linear momenta would not be an axiom-preserving map.

It should be noted that, by comparison, the use of the Birkhoffian mechanics would imply broader Hamilton-Jacobi-Santilli isoequations (see page 205 of Ref. [19]) with 'wave functions' depending also on isomomenta, $\hat{\psi}(\hat{t},\hat{r},\hat{p})$, resulting in an operator mechanics beyond our current knowledge for quantitative treatments.

4.6. Naive isoquantization.

As it is well known, the conventional "naive quantization" of Hamiltonian mechanics into quantum mechanics is based on the following map generated by the conventional Hamilton-Jacobi equations

$$\begin{aligned} A = \int (p_k dx^k - H dt) \to \\ \to -i\hbar log\psi(t,r), \end{aligned} \tag{68}$$

that identifies Planck's constant $\hbar = 1$ as the fundamental unit of the theory.

The isotopic lifting of the naive quantization, called *naive isoquantization* (first identified by A. E. O. Animalu and R. M. Santilli in Ref. [121]), characterizes the following map of (classical) iso-Hamiltonian mechanics into (operator) isomechanics via Hamilton-Jacobi-Santilli isoequation (65) (where the sum over the indices ak is omitted for simplicity)

$$\begin{aligned} \hat{A} = \hat{\int} (\hat{p} \star d\hat{r} - \hat{H} \star d\hat{t}) \ \to \\ \to -i\hat{I} Log\hat{\psi}, \end{aligned} \tag{69}$$

with the following fundamental identification of the *isolinear isomomentum* from Eq. (66)

$$\hat{p} \star \psi = -\hat{i} \star \hat{\partial}_r \hat{\psi}, \tag{70}$$

and the equally fundamental, independence of the isowavefunction from the isolinear isomomentum from Eq. (67)

$$\hat{\partial}_{\hat{p}} \hat{\psi} = 0, \tag{71}$$

where we use the notion of *isolog* $\hat{log}\hat{\psi} = \hat{I}\log\psi$ (see [55]).

The above naive isoquantization identifies the central assumption of isomechanics, namely, *the map of Planck's constant $\hbar$ into an integro-differential operator, the isounit $\hat{I}$*,

$$\hbar \to \hat{I}(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) > 0. \tag{72}$$

that, from Section 2.2.8, can be achieved via a non-unitary transform of Plank's constant selected in such a way to represent the desired systems, e.g., as in model (16),

$$\hbar = 1 \quad \to \hat{\hbar} = U\hbar U^{\dagger} = UU^{\dagger} = \hat{I}. \tag{73}$$

The above transform is then restricted by the subsidiary condition that isomechanics must recover quantum mechanics at mutual distances of particles bigger than their size d

$$Lim_{r>d/2}\hat{I} = 1. \tag{74}$$

Therefore, the studies herein reported assume that isomechanics is solely valid within the volume occupied by hadrons, nuclei or stars, while quantum mechanics is assumed to be exactly valid everywhere else (Figure 13).

Note that the structure of the isotopic element (16) permits a smooth transition from isomechanics to quantum mechanics.

The above condition means that, for the case of the structure of a hadron, isomechanics is solely valid within a sphere with diameter $d \approx 1fm = 10^{-15}\ cn$. For the case of the structure of the deuteron, isomechanics is solely valid within a volume with diameter $d \approx 2.50\ fm$; and the same applies for nuclei, stars and black holes.

The main objective of the "completion" of Planck's constant $\hbar$ into the integro-differential isounit $\hat{I}$ is to represent the expected, generalized, energy exchanges of particles in interior dynamical conditions (as expected for an electron in the core of a star), which exchanges cannot be the same as those occurring when particles moves in vacuum due to the surrounding pressures and other factors (see paper II for details).

4.7. Iso-Hilbert isospaces.

Another basic notion of isomechanics is its formulation on the *iso-Hilbert isospaces $\hat{\mathcal{H}}$*, also called the *Hilbert-Myung-Santilli isospace* (HMS isospace) because first introduced by H.C. Myung and R.M. Santilli in Ref. [122] of 1982 over a conventional field of complex numbers $\mathcal{C}$ and then formulated on an isocomplex isofield $\hat{\mathcal{C}}$ in Ref. [115].

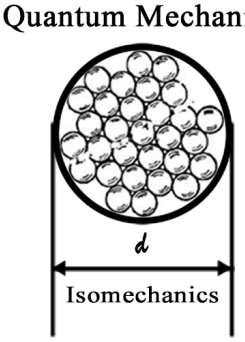

Figure 13: *An illustration of the central assumption according to which isomechanics is solely valid within the volume occupied by hadrons, nuclei, stars or black holes, while quantum mechanics is valid everywhere else thanks to the rapid convergence of isotopic elements, such as Eq. (16), to the unit value* 1.

HMS isospaces are characterized by (see Ref. [55] for details): *isostates* $\hat{\psi}$, with *isonormalization*

$$< \hat{\psi} | \star | \hat{\psi} >= \hat{I}, \tag{75}$$

isoexpectation isovalues of an iso-Hermitean operator $\hat{A}$,

$$\hat{<}\hat{A}\hat{>} =< \hat{\psi} | \star \hat{A} \star | \hat{\psi} >, \tag{76}$$

and basic isoidentity

$$\hat{<}\hat{I}\hat{>} = \hat{I}, \tag{77}$$

where the "hat" denotes definition on isospace over isofields.

It should be recalled from Ref. [122] that the condition of iso-Hermiticity coincides with that of Hermiticity. Therefore, *all quantities that are observable in quantum mechanics remain observable in isomechanics* (see monograph [56] for details).

4.8. Iso-Schrödinger isorepresentation.

Recall that the Schrödinger representation is crucially dependent on the realization of the linear momentum in term of the differential calculus,

$$p\psi(t,r) = -i\hbar\partial_r\psi(t,r). \tag{78}$$

Consequently, the "completion" of quantum mechanics mandated the search for the "completion" of the differential calculus [115] to achieve a consistent formulation of the linear momentum such as that of Eq. (70),

that we rewrite in the more detailed form on the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\imath\hat{cal}C$ (with $\hbar = 1$)

$$\begin{aligned} \hat{p}_k \star |\hat{\psi}(\hat{t},\hat{r}) >= -\hat{i} \star \hat{\partial}_{\hat{r}^k} |\hat{\psi}(\hat{t},\hat{r}) >= \\ = -i\hat{I}\partial_{\hat{r}^k} |\hat{\psi}(\hat{t},\hat{r}) >, \end{aligned} \tag{79}$$

where: $\hat{r} = r\hat{I}$ are the isocoordinates on an iso-Euclidean isospace over an isofield, from which one can derive the *iso-Schrödinger isoequation,* also called *Schrödinger-Santilli isoequation* [17] [115]

$$\begin{aligned} \hat{i} \star \hat{\partial}_{\hat{t}} |\hat{\psi}(\hat{t},\hat{r}) >= \hat{H} \star |\hat{\psi}(\hat{t},\hat{r}) >= \\ = \hat{H}(r,p)\hat{T}(t,r,p,\psi,\partial\psi,...)|\hat{\psi}(\hat{t},\hat{r} >) = \\ = \hat{E} \star |\hat{\psi}(\hat{t},\hat{r}) >= E|\hat{\psi}(\hat{t},\hat{r}) >, \end{aligned} \tag{80}$$

where $\hat{E} = E\hat{I}$ is an isoeigenvalue defined on the isoreal isofield $\hat{\mathcal{R}}$, and E is an ordinary eigenvalue defined on the field of real numbers.

The iso-Schrödinger isorepresentation is completed by the *isocanonical isocommutation rules,* solely definable thanks to the isodifferential realization (79) of the isolinear isomomentum [115]

$$[\hat{r}_i,\hat{p}_j]|\hat{\psi} >= \hat{i} \star \hat{\delta}_{i.j}|\hat{\psi} >= i\delta_{i.j}|\hat{\psi} >, [\hat{r}_i,\hat{r}_j]|\hat{\psi} >= [\hat{p}_i,\hat{p}_j]|\hat{\psi} >= 0. \tag{81}$$

Note that the characterization of extended particles at short mutual distances requires the knowledge of *two isoobservables,* the conventional Hamiltonian H for the representation of SA interactions and the isotopic element $\hat{T}$ for the representation of dimension, shape, density and NSA interactions.

On more technical grounds, Eqs. (80) are referred to as *regular iso-Schrödinger equations* to emphasize, in the sense of Theorem 3.7.2, the fact that they can be derived from the conventional Schrödinger equation via non-unitary transformations. The broader *irregular iso-Schrödinger equation* which cannot be derived via non-unitary transformations due to the addition of strong interactions, are studied in Paper II, Section 4.3., Eqs. (89).

4.9. Iso-Heisenberg isorepresentation.

Non-relativistic isomechanics is additionally based on the *iso-Heisenberg isoequations,* also called *Heisenberg-Santilli isoequations* (first formulated in

Eq. (4.15.49), page 752 of the 1978 paper [17] over conventional fields and formulated via the full use of isomathematics in the 1996 memoir [115]), here written for the iso infinitesimal isotime evolution of an iso-Hermitean operator $\hat{Q}$

$$\hat{i} \star \frac{\hat{d}\hat{Q}}{\hat{dt}} = [\hat{Q},\hat{H}] = \hat{Q} \star \hat{H} - \hat{H} \star \hat{Q} = \\ = \hat{Q}\hat{T}(\psi, ...)\hat{H}(r,p) - \hat{H}(r,p)\hat{T}(\psi, ...)\hat{Q}, \tag{82}$$

and then in their isoexponentiated form

$$\hat{Q}(\hat{t}) = \hat{e}^{\hat{H}\star\hat{t}\star\hat{i}} \star \hat{Q}(0) \star \hat{e}^{-\hat{i}\star\hat{t}\star\hat{H}} = \\ = e^{\hat{H}\hat{T}ti} Q(0) e^{-it\hat{T}\hat{H}}, \tag{83}$$

where we have used isoexponentiation (22).

Note the characterization of isoinfinitesimal isoequations (82) via the Lie-Santilli isoalgebras (Theorem 3.7.2.) and the characterization of their finite form (83) via the Lie-Santilli isogroups (Theorem 3.7.3).

Note also that the Lie-isotopic equations (82) (83) are a particular case of the broader Lie-admissible equations (11) (12), respectively.

4.10. Iso-Klein-Gordon isoequation.

The relativistic isoequations of hadronic mechanics are characterized by the iso-Casimir isoinvariants of the basic symmetry of the iso-Minkowski isospace-time, the *Lorentz-Poincaré-Santilli isosymmetry* studied in paper II.

At this stage of our analysis, we merely consider the following isotopic "completion" of the second order invariant of the *Lorentz-Poincaré symmetry* formulated on iso-Minkowskian isospace $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I}$ over the isoreals $\hat{\mathcal{R}}$ (Section 2.5)

$$\hat{p}^{\hat{2}} = \hat{p}_\mu \star \hat{p}^\mu = (\hat{M} \star \hat{C})^{\hat{2}} = (mC)^2 \hat{I}, \tag{84}$$

where

$$\hat{M} = m\hat{I}, \tag{85}$$

is the *isomass*, and

$$\hat{C} = C\hat{I} = \frac{c}{n_4}\hat{I}, \tag{86}$$

is the *light isospeed* from isoinvariant (26).

By using the isolinear isomomentum (79) isoinvariant (84) characterizes the second order isoequation of isomechanics known as *iso-Klein-Gordon*

isoequation [123]

$$\begin{aligned}\hat{p}_\mu \star \hat{p}^\mu|\hat{\psi}(\hat{x})>= \hat{\Omega}^{\mu\nu} \star \hat{p}_\mu \star \hat{p}_\nu|\hat{\psi}(\hat{x})>= \\ = \hat{\eta}^{\mu\nu}(-i\hat{I}\partial_\mu)\hat{T}(-i\hat{I}\partial_\nu)|\hat{\psi}(\hat{x})>= \\ = -\hat{I}\hat{\eta}^{\mu\nu}\partial_\mu\partial_\nu|\hat{\psi}(\hat{x})>= \hat{I}(mC)^2|\hat{\psi}(\hat{x})>,\end{aligned} \tag{87}$$

also called *Klein-Gordon-Santilli isoequation,* first introduced with the isodifferential isocalculus in Chapter 9 of Refs. [56] and in papers [123] [124], (see also review [25]).

4.11. Iso-Dirac isoequation.

The first-order relativistic isoequation of hadronic mechanics is given by the isolinearization [124] of isoinvariant (84) and it is called the *iso-Dirac isoequation,* or *Dirac-Santilli isoequation* (see Refs. [115] [56] [123] [124])

$$\begin{aligned}[\hat{\Omega}^{\mu\nu} \star \hat{\Gamma}_\mu \star \hat{\partial}_\nu + \hat{M} \star \hat{C}]|\hat{\psi}(\hat{x})>= \\ = (-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_\mu\partial_\nu + mC)|\hat{\psi}(\hat{x})>= 0,\end{aligned} \tag{88}$$

where the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\begin{aligned}\hat{\gamma}_k = \tfrac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 = \tfrac{i}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix},\end{aligned} \tag{89}$$

where $\hat{\sigma}_k$ are the *regular iso-Pauli isomatrices* studied in Section 3.3 of paper II, with *anti-isocommutation rules*

$$\begin{aligned}\{\hat{\gamma}_\mu,\hat{\gamma}_\nu\} = \hat{\gamma}_\mu\hat{T}\hat{\gamma}_\nu + \hat{\gamma}_\nu\hat{T}\hat{\gamma}_\mu = \\ = 2\hat{\eta}_{\mu\nu}.\end{aligned} \tag{90}$$

One should note that the anti isoanticommutators of the Dirac-Santilli isogamma isomatrices yield the isometric $\hat{\eta}_{\mu\nu}$ of the iso-Minkowski isospacetime (Section 2.5).

Recall that the iso-Minkowski isospace-time includes as particular cases all possible, non-singular, symmetric space-times, thus including the Riemannian space-time reformulated with isomathematics. In fact, the iso-Minkowskian isometric $\eta^{\mu\nu}$ admits as a particular case the Schwartzchild

metric via the following simple realization of the isotopic element

$$\hat{T}_{kk} = 1/(1 - 2M/r),$$
$$\hat{T}_{44} = 1 - 2M/r. \tag{91}$$

Additionally, the iso-Minkowskian isometric admits the following combination of the Schwartzchild metric for exterior gravitational problems and that for interior problems (see, Eqs. (9.5.18) page 448, Ref. [56]

$$\hat{T}_{kk} = 1/(1 - 2M/r)n_k^2,$$
$$\hat{T}_{44} = (1 - 2M/r)/n_4^2. \tag{92}$$

Consequently, the Dirac-Santilli isoequation with realization (91) of the isotopic element permits the study of an electron in an *exterior gravitational field,* while realization (92) permits the study of *electrons in interior gravitational fields*.

Relativistic isoequations are far from being mere academic curiosities because, as we shall see in paper II, they have provided the first and only known relativistic representation of *all* characteristics of the neutron in its synthesis from a proton and an electron [124], *none* of which characteristics are representable via quantum mechanics. Additional advances permitted by relativistic isomechanics will be indicated in paper II.

More technically, Eq. (88) is referred to as *regular iso-Dirac equations* to emphasize, in the sense of Theorem 3.7.2, the fact that they can be derived from the conventional Dirac equation via non-unitary transformations. The broader *irregular iso-Dirac equation* which cannot be derived via non-unitary transformations due to the addition of strong interactions, are studied in Paper II, Section 4.4., Eqs. (97).

4.12. Representation of non-linear interactions.

An important insufficiency of quantum mechanics is the inability to characterize individual constituents under non-linear internal forces in view of the inapplicability of the superposition principle.

In fact, the sole possible, quantum mechanical representation of non-linear interactions is that via the Hamiltonian

$$H(r, p, \psi, ...)|\psi(t, r) >= E|\psi(t, r) >, \tag{93}$$

under which the total state $|\psi(t, r) >$ does not admit a consistent decomposition into the individual states.

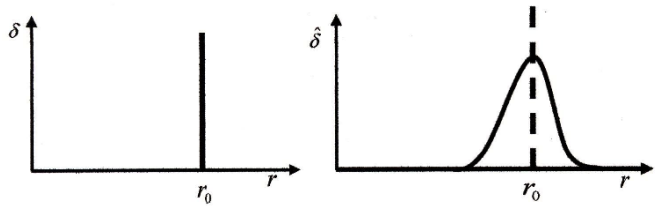

Figure 14: *An illustration on the left of the divergence of the Dirac delta distribution at its origin caused by the point-like approximation of particles, with ensuing divergencies in quantum mechanics. An illustration on the right of the removal of said divergencies by the Dirac-Myung-Santilli isodelta isofunction thanks to the representation of particles as extended, with ensuing lack of divergencies in isomechanics.*

It is easy to see that this insufficiency is resolved by isomechanics thanks to the embedding of all non-linear forces in the basic invariant of the theory, the isounit (or isotopic element).

In fact, the Schrödinger-Santilli isoequation (80) can be explicitly written

$$\hat{H} \star |\hat{\psi} >= \hat{H}(\hat{r}, \hat{p})\hat{T}(\hat{\psi}, ...)|\hat{\psi} >= E|\hat{\psi} >, \tag{94}$$

and its total isostate verifies the factorization

$$\hat{\psi} = \Pi_k \hat{\psi}_k, \;\; k = 1, 2, ..., N, \tag{95}$$

called *isosuperposition isoprinciple* [56].

It is evident that factorization (94) allows the characterization of individual constituents under non-linear internal interactions, thus permitting new structural models of hadrons, nuclei, stars and black holes.

4.13. Isostrong isoconvergence.

In all applications to date, the basic isotopic element (16) resulted to have a numeric value smaller than one

$$||\hat{T}|| \ll 1. \tag{96}$$

This feature has the important consequence that *perturbative and other series that are slowly convergent or divergent in quantum mechanics become strongly convergent under their isotopic "completion"* [56].

To illustrate this important feature, consider a divergent quantum mechanical series, such as the canonical series

$$A(w) = A(0) + (AH - HA)/1! + ... =\rightarrow$$
$$\rightarrow \infty, \;\; w > 1. \tag{97}$$

But the value of the isotopic element is much smaller than the parameter w. Therefore, the isotopic "completion" of the above series

$$A(w) = A(0) + (A\hat{T}H - H\hat{T}A)/1! + ... \rightarrow$$
$$\rightarrow N < \infty, \tag{98}$$

is strongly convergent.

Specific examples of convergence of isoperturbative isoseries much more rapid than corresponding quantum mechanical series have been provided in Refs. [103] [104] (see Section 1.6 for additional comments).

4.14. Removal of quantum divergencies.

As it is well known, the divergencies of quantum mechanics originate from the singularity existing at the origin of the *Dirac delta distribution* (Figure 14) which divergence originates from the point-like approximation of particles.

Another important feature of isomechanics is that of avoiding these singularities as illustrated by the isotopic image of Dirac's delta "distribution", known as *Dirac-Myung-Santilli isodelta isofunction* first introduced in Ref. [122] (see also Nishioka's studies [125] to [128])

$$\hat{\delta}(\hat{r}) = \hat{\int} \hat{e}^{\hat{k}\star\hat{r}} \star \hat{dk} = \int e^{\hat{k}\hat{T}\hat{r}} dk, \tag{99}$$

where we have used isoexponentials and isointegrals [56].

As illustrated in Figure 14, the appearance of the isotopic element in the *exponent* of the integrant changes a sharp singularity at the origin $r = 0$ into a bell-shaped function.

In summary, *the singularities of quantum mechanics are ultimately due to the point-like abstraction of particles or equivalently, to the formulation of the differential calculus at isolated points. Whenever particles are represented with their actual extended size, and the differential calculus is extended to formulations over volumes, quantum singularities no longer hold.*

4.15. Isoscattering isotheory.

Another important application of isomechanics is the isotopic "completion" of the conventional, potential, scattering theory into the covering *isoscattering isotheory* studied by R. Mignani [129] to [131], A. K. Aringazin and D. A. Kirukhin [132], A. O. E. Animalu and R. M. Santilli [133] and others (see Chapter 12 of Ref. [56]).

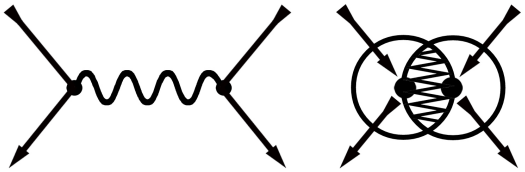

Figure 15: *The left view illustrates the scattering of point-particles, with ensuing realization of interactions via point-particle exchanges, that have been experimentally established for electromagnetic interactions. The right view illustrates the scattering of extended hadrons at high energy implying the presence of contact, non-linear, non-local and non-potential interactions in the scattering region. These conditions imply the impossibility of conventional particle exchanges evidently because of the extremely big density of the scattering regions that may approach the density of black holes, thus requiring broader scattering theories.*

Regrettably, we cannot review these studies to avoid a prohibitive length. We limit ourselves to recall that the potential scattering theory was originally conceived by Feynman and others for the electromagnetic interactions of *point-like* particles in vacuum.

Its dominant notion is the characterization of interactions via the *exchange of point-like particles.*

The historical experimental verifications of the potential scattering theory under the indicated conditions triggered its use for the scattering of *extended* hadrons all the way to their recent very high energies.

The studies herein reported on the covering isoscattering isotheory have indicated that the consistent representation of hadrons as extended, therefore deformable and hyperdense, implies necessary revisions of hadron physics beginning with a "completion" of its mathematical foundations, then passing to the "completion" of quantum laws. These advances then require, for consistency, the "completion" of the potential scattering theory into a covering theory in which the hyperdense character of the scattering region implies the presence of non-linear, non-local and non-potential effects preventing any consistent representation of interactions via the sole exchanges of extended hadrons in favor of covering vistas (Figure 15).

It should be stressed that the isoscattering isotheory has a Lie-Santilli algebraic structure, thus being solely applicable to time reversal invariant collisions generally given by elastic scattering. The study of inelastic scattering of extended hadrons at high energy requires the broader *genoscattering genotheory* with the covering Lie-admissible algebraic structure, which broader theory cannot be considered here for brevity (see Refs. [56] [133]).

To avoid major misconceptions, it should be indicated that *the isoscattering isotheory cannot change the numeric values of scattering angles, cross sections and other experimentally measured quantities.* However, the isoscattering isotheory does require serious revisions of the *theoretical interpretation* of measured quantities.

In closing, we should recall that rather vast studies have been conducted on *non-unitary scattering theories* (see, e.g., Ref. [134] and papers quoted therein).

As it is well known, these studies had to be abandoned because non-unitary theories violate causality laws. By contrast, *the isoscattering isotheory is isounitary on the iso-Hilbert isospace over an isofield,* thus restoring causality (Section 2.9).

This occurrence illustrates again the importance of isomathematics for the verification of the EPR argument and related applications.

4.16. Geno- and Iso-chemistry.

The Lie-admissible mathematical and physical methods of Section 2 have allowed the "completion" of quantum chemistry into *Lie-admissible hadronic chemistry,* also known as *genochemistry* [60], which is the first known chemical formulation specifically built for the consistent treatment of chemical reactions at large and energy releasing chemical processes in particular.

As it is well known but generally ignored, chemical reactions are generally *irreversible over time,* while quantum chemistry is strictly reversible. Hence, the EPR argument on the "lack of completion of quantum mechanics" does indeed apply to quantum chemistry.

In addition to a "completion" for chemical reactions, quantum chemistry needs an additional "completion," this time, for the achievement of an *attractive* force between the *identical* electrons of valence couplings in molecular structures (Section 1.6).

Since isolated molecules existing in nature are stable, thus being *reversible— over time,* and so are their valence electron bonds, there was the need of building the *Lie-isotopic particularization of the Lie-admissible hadronic chemistry* which became known as *isochemistry* because based on the isomathematics of Section 3.

Isochemistry did indeed achieve the first known attractive force between identical electrons in valence couplings [60] in a form permitting the exact representation of experimental data on the hydrogen [103] and the water [104] molecules.

As reviewed in details in Paper II, these studies essentially established that *the contact, non-potential interactions occurring in deep mutual penetration/entanglement of the wavepackets of particles (Figure 2) are strongly attrac-*

tive, thus being responsible for the neutron and other hadron syntheses (Section 1.4), as well as for the attractive force between identical electrons in valence pairs (Section 1.6), thus illustrating the truly fundamental character of short range, contact, non-potential interactions in the ultimate structure of nature.

5. CONCLUDING REMARKS.

In 1935, A. Einstein, B. Podolsky and N. Rosen presented the view that *quantum mechanics is not a complete theory* [1].

Following decades of preparatory works, R. M. Santilli published in 1998 paper [7] that:

1) Confirmed the validity of the objections against the EPR argument by Bohr [2], Bell [3], von Neumann [4] and others for point-particles in vacuum under linear, local and potential/Hamiltoian interactions *exterior dynamical problems;*

2) Proved the inapplicability (rather than the violation) of said objections for extended, thus deformable particles within hyperdense physical media with ensuing linear and non-linear, local and non-local and potential as well as non-potential/non-Hamiltonian interactions expected in the structure of hadrons, nuclei, stars and black holes (*interior dynamical systems*);

3) Proved the existence of *hidden variables* [5] for interior dynamical systems when represented via the isotopic branch of hadronic mathematics and mechanics;

4) Achieved a consequential exact representation of nuclear magnetic moments; and

5) Showed the apparent existence of identical classical counterparts for extended particles in interior dynamical conditions.

More recently, Santilli completed the above proof in paper [8] by showing that the standard deviations for coordinates Δr and momenta Δp appear to progressively tend to zero for extended particles within hadrons, nuclei and stars, and appear to be identically null for extended particles at the limit conditions in the interior of gravitational collapse, essentially along Einstein's vision.

In this paper, we have reviewed and upgraded the mathematical, physical and chemical methods used for proofs [7] [8], with particular reference to the following aspects:

1) Review and upgrade the Lie-admissible and Lie-isotopic "completions" of 20th century applied mathematics for the representation of time irreversible and reversible interior systems, respectively,, which "completions" were initiated by Santilli in 1978 [16] when at Harvard University

with DOE support under the names of *geno- and iso-mathematics,* respectively, and waere subsequently studied by various authors [55];

2) Review and upgrade the "completions" of quantum mechanics and chemistry into the Lie-admissible and Lie-isotopic branches of hadronic mechanics and chemistry that were also initiated by Santilli in 1978 [17] under the names of *genotopic and isotopic branches of hadronic mechanics and chemistry,* which "completions" were then studied by various authors [56];

3) Review and upgrade the main aspect of the studies herein considered, namely, the "completions" of the Newton-Leibnitz differential calculus into forms applicable to irreversible and reversible interior systems of extended particles, which "completions" were initiated by Santilli in the 1996 paper [115] under the name of *geno- and iso-differential isocalculus* and studied in details by S. Georgiev [83] and other mathematicians [55].

In a second paper of this series, we review and upgrade *geno- and iso-symmetries* for interior systems and then review the apparent proofs [7] [8] of the EPR argument.

In the third paper of this series, we study specific cases of interior dynamical systems in particle physics, nuclear physics and chemistry that progressively approach classical determinism, and present applications to new clean energies that are prohibited by quantum mechanics, yet fully admitted by its "completion."

Acknowledgments. The writing of these papers has been possible thanks to pioneering contributions by numerous scientists we regret not having been able to quote in this paper (see Vol. I, Refs. [61] for comprehensive literature), including:

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I would like to express my deepest appreciation for all the above courageous contributions written at a point in time of the history of science dominated by organized oppositions against the pursuit of new scientific knowledge due to the trillions of dollars of public funds released by governmental agencies for research on pre-existing theories.

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Studies on A. Einstein, B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," II: Apparent confirmation of the EPR argument

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Abstract

In 1935, A. Einstein expressed his historical view, jointly with B. Podolsky and N. Rosen, that quantum mechanics could be "completed" into a form recovering classical determinism at least under limit conditions (*EPR argument*). In the preceding Paper I, we have outlined the novel methods underlying the "completion" of quantum mechanics into hadronic mechanics for the representation of extended, thus deformable particles within physical media. In this Paper II, we study the *isosymmetries* for interior dynamical systems; we confirm the 1998 apparent proof that interior dynamical systems admit a classical counterpart; we confirm the 2019 apparent proof that Einstein's determinism is progressively approached for extended particles in the interior of hadrons, nuclei and stars, while being fully verified in the interior of gravitational collapse; and we show for the first time that the recovering of Einstein's determinism in interior systems implies the apparent removal of quantum mechanical divergencies. In the subsequent Paper III, we present a number of illustrative examples and novel applications in mathematics, physics and chemistry.
Keywords: EPR argument, isomathematics, isomechanics.
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3. CONCLUDING REMARKS.

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1. INTRODUCTION

1.1. The EPR argument.

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein believed that "quantum mechanics is not a complete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

In view of the rather widespread belief that quantum mechanics is a final theory valid for all conceivable conditions existing in the universe, objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3] [4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature). The field became known as *local realism* and included the dismissal of the EPR argument based on claims that quantum axioms do not admit *hidden variables* λ [7] [8].

1.2. Outline of Paper I.

This paper, and the preceding Ref. [9] (hereinafter referred to as Paper I), are dedicated to the review and upgrade of decades of studies by mathematicians, physicists, and chemists (see Refs. [10] to [71] and papers quoted therein) on the apparent proof of the EPR argument via the "completion," also called *isotopic lifting,* of quantum mechanics into the axiom-preserving *hadronic mechanics* (see the 1995 monographs [29] [30] [31] and literature quoted therein).

More specifically, in Section I-1.1, we have outlined the EPR argument [1] jointly with representative objections [2] to [6].

In Section I-1.2, we have outlined the apparent proof by R. M. Santilli [10] (See also the detailed study in monograph [30], particularly Chapter 4 and Appendix 4C, page 166) that interior dynamical systems represented

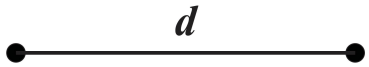

Figure 1: *In this figure, we present a conceptual rendering of the tacit assumption underlying the objections against the EPR argument [2] - [6], namely, the representation of particles as being point-like because mandated by the Newton-Leibnitz differential calculus underlying quantum mechanics, namely, the representation of particles as isolated points in empty space. A first consequence is that, being dimensionless, particles can only be at a distance, with ensuing Einstein's argument on the need for superluminal interactions to explain quantum entanglement [1]. A second consequence is that, being at a distance, the sole possible interactions are of linear, local and potential type, under which assumptions the objections against the EPR argument are indeed valid.*

with hadronic mathematics and mechanics admit classical counterparts.

In the same Section I-1.2, we have outlined the apparent second proof by Santilli [11] that classical determinism is progressively approached in the interior of hadrons, nuclei, stars and gravitational collapse as predicted by Einstein.

In support of the plausibility of the EPR argument, in the subsequent Sections I-1.3 to I-1.7, we have outlined insufficiencies of quantum mechanics for time-irreversible processes, particle physics, nuclear physics, chemistry, and other fields. We have also provided various references indicating the apparent resolution of said insufficiencies by hadronic mechanics.

In Section I-2, we have outlined the *Lie-admissible covering of Lie's theory* [12] [13], with ensuing time-irreversible *Lie-admissible brach of hadronic mechanics*, also known as *genomechanics*, [12] [14] allowing studies on the compatibility of mechanics with thermodynamics, said compatibility being notoriously impossible for quantum mechanics.

Quantum mechanics and the objections against the EPR argument are formulated for time-reversal invariant systems of exterior dynamical systems. Therefore, in preparation for the proof of the EPR argument studied in Section 3, we have outlined and upgraded in Section I-3 the time-reversal invariant *Lie-isotopic subclass of Lie-admissible mathematics*, also known as *isomathematics*, [15] [18] which is used for the representation of time-reversible invariant interior dynamical systems.

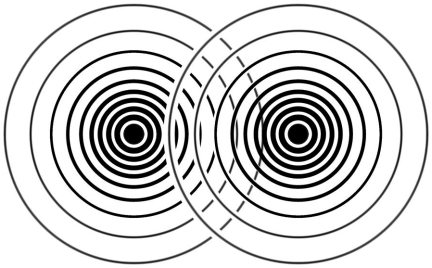

Figure 2: *A conceptual rendering of the main assumption of the apparent proofs [10] [11] of the EPR argument [1], is the representation of particles as extended, deformable and hyperdense in conditions of mutual overlapping/entanglement with ensuing continuous contact at a distance which eliminates the need for superluminal interactions to explain quantum entanglement. A first implication is the need, for consistency, of generalizing Newton-Leibnitz differential calculus from its historical form solely definable at isolated points, to a covering form definable in volumes [21]. Another implication is the emergence of contact, non-linear, non-local and non-potential interactions that, being not representable by Hamiltonians are Lagrangians, require a structural lifting of the Lie algebra of quantum mechanics under which the objections against the EPR argument are inapplicable (Section 3). Intriguingly, the "completions" here considered turned out to be of isotopic/axiom-preserving type, thus being fully admitted by quantum mechanical axioms, merely subjected to a realization broader than that of the Copenhagen school. The apparent proofs of the EPR argument [10] [11] become an unavoidable consequence of the indicated "completions" (Section 3).*

In the same Section I-3, we have devoted particular attention to the "completion" of conventional Hilbert spaces [19], numeric fields [20] and Newton-Leibnitz differential calculus [21] into forms defined on *volumes*, rather than points.

In the same Section I-1.3, we have provided particular attention to the main methods for the proofs of the EPR argument, namely, the *axiom-preserving, isotopic lifting of Lie's theory* [26], today known as the *Lie-Santilli isotheory* [38].

Finally, in Section I-4, we have outlined and upgraded the time-reversal invariant *isotopic branch of hadronic mechanics*, also known as *isomechanics* [30] which provides the dynamical foundations of the proofs of the EPR argument [10] [11].

1.3. Basic assumptions.

The most dominant aspects underlying the studies here considered are:

1) The validity of quantum mechanics for point-like particles in vacuum with ensuing linear, local and action-at-a-distance/potential interactions (*exterior dynamical problems*) occurring in atomic structures, particles in accelerators, crystals and numerous other systems in nature (Figure 1);

2) The "completion" of quantum mechanics into hadronic mechanics for the representation of extended, therefore deformable and hyperdense particles within physical media with ensuing, additional, non-linear, non-local and contact/non-potential interactions (*interior dynamical problems*), occurring in the structure of hadrons, nuclei and stars, with limit conditions occurring in the interior of gravitational collapse where the inapplicability (rather than the violation) of quantum mechanics is already accepted by the majority of serious scholars (Figure 2, 3).

The central assumption of these studies is the axiom-preserving lifting of the conventional associative product $ab = a \times b$ between *all* possible quantum mechanical quantities (numbers, functions, matrices, etc.) into the *isoproduct* [14] [26] (Section 3)

$$a \star b = a \, \hat{T} \, b, \tag{1}$$

where $\hat{T}$, called the *isotopic element*, is restricted to be positive-definite, $\hat{T} > 0$, but possesses otherwise an unrestricted functional dependence on all needed local variables.

Refs. [14] [26] constructed an axiom-preserving isotopy of the various branches of Lie's theory, resulting in a theory today known as the *Lie-Santilli isotheory* [38] (Section I-3.7) with isotopic lifting of lie algebras of the type [10]

$$[X_i \hat{,} X_j] = X_i \star X_j - X_j \star X_i = C_{ij}^k X_k. \;\; i, j = 1, 2, ..., N. \tag{2}$$

Following laborious efforts for the achievement of mathematical maturity, Ref. [10] applied the Lie-Santilli isotheory to the isotopy $\hat{SU}(2)$ of the $SU(2)$ spin with three-dimensional isoalgebras of type (2) and introduced the realization of hidden variables [7] [8] of the type

$$\hat{T} = Diag.(1/\lambda, \lambda), \;\; Det\hat{T} = 1. \tag{3}$$

Ref. [10], therefore establishing that, contrary to objections [2] to [6], *the abstract axioms of quantum mechanics do indeed admit explicit and concrete realizations of hidden variables.*

The proof in Ref. [10] that interior systems admit identical classical counterparts was consequential (Section 3).

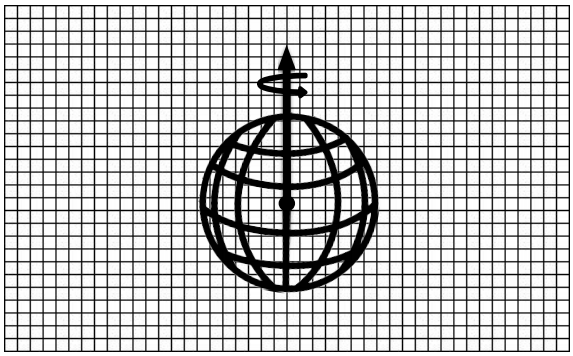

Figure 3: *A conceptual rendering of the central notion used for the study of the EPR argument, namely, a mathematically consistent representation invariant over time of extended, deformable and hyperdense particles in interior conditions, such as protons in the interior of a star, thus being under the most general known (non-singular) non-linear, non- local and non-potential interactions fully representable via the isotopic element of isoproduct (1).*

Isoproduct (1) also allows a direct and immediate representation of extended particles in conditions of mutual penetration with realizations of the type (Figure 3) [33]

$$\hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2}) e^{-\Gamma},$$
$$k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4, \tag{4}$$

where n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_k^2 = 1, `k = 1, 2, 3$ for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

The smaller than 1 absolute value of the isotopic element $\hat{T}$ occurring in all known applications [26]-[36]

$$| \hat{T} | \leq 1, \tag{5}$$

permitted Ref. [**?**] to show that *the standard deviations Δr and Δp appear to progressively tend to zero with the increase of the density of the medium, and appear to achieve full classical determinism in the interior of gravitational collapse, as originally conceived by Einstein.*

The initial construction of the isotopies of 20th century applied mathematics with isoproduct (1) defined over conventional numeric fields $F(n, \times, 1)$ [26] turned out to be inconsistent because the underlying time evolution is *non-unitary,* thus causing the lack of invariance over time of the traditional basic unit 1, with ensuing inapplicability over time of the entire field $F(n, \times, 1)$.

The above occurrence mandated the construction of *isofields* $\hat{F}(\hat{n}, \star, \hat{I})$ [20] [41](Section I-3.3) with basic *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{6}$$

and *isonumbers* $\hat{n} = n\hat{I}$ equipped with isoproduct (1).

Ref. [20] essentially established that *the abstract axioms of a numeric field do not require that the multiplicative unit of the field be the trivial number* 1*, since said unit can be an arbitrary quantity with an unrestricted functional dependence on local variables, provided that said multiplicative unit is positive definite and the field is lifted into a compatible form.*

Despite all the above efforts, the ensuing isomathematics was still inapplicable to the proof of the EPR argument because it lacked the crucial *invariance over time,* namely, the prediction of the same interior dynamical systems under the same conditions but at different times.

The above occurrence forced the construction of the covering of the Newton-Leibnitz differential calculus into the covering *isodifferential isocalculus* [21] [44] (Section I-3.6) with basic *isodifferential* (Figure 2) [**?**]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \tag{7}$$

and corresponding *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{8}$$

In essence, Ref. [21] established the inapplicability of the conventional differential calculus whenever the axioms of numeric fields admit multiplicative units with a dependence on the differentiation variable, with ensuing inapplicability of quantum mechanics, as well as of the objections against the EPR argument, for interior dynamical systems.

The "completion" of the differential calculus into an isotopic form compatible with basic isoproduct (1) finally allowed the achievement of invariance over time (Section I-3.9), thus signaling the achievement of maturity for the apparent proof of the EPR argument reviewed.

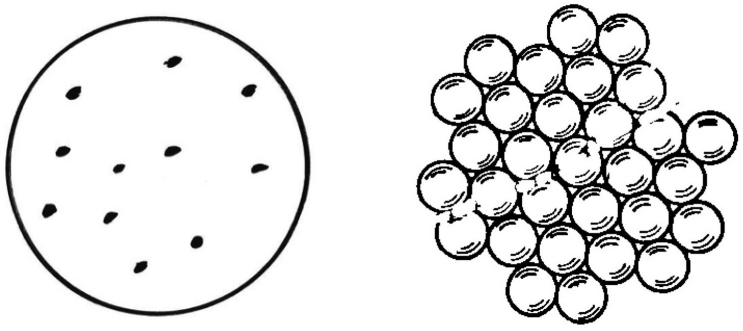

Figure 4: *In the l.h.s. of this picture, we present a conceptual rendering of the structure of nuclei as ideal spheres with isolated point-like particles in their interior. This view is an inevitable consequence of the elaboration of quantum mechanics via the conventional differential calculus, resulting in rather serious insufficiencies in nuclear physics outlined in Section I-1.5. In the r.h.s. of this picture, we present a conceptual rendering of the representation of nuclei as occurring in the physical reality, namely, as a collection of extended, therefore deformable charge distributions in condition of partial mutual penetration according to Eq. (4) of isomathematics and related isomechanics, thus permitting the resolution of at least some of the insufficiencies of quantum mechanics in nuclear physics reviewed in Section I-1.5.*

In Section 2 of this paper, we complete the methodological needs by outlining and upgrading the time-reversal invariant coverings of conventional spacetime symmetries, known as *isosymmetries*, for systems of extended particles in interior conditions; in Section 3, we review and upgrade the Lie-isotopic $SU(2)$-spin symmetry and related proofs [10] [11] of the EPR argument.

A few comments on terminologies appear to be recommendable.

The word "completion" is used in these studies to honor the memory of Albert Einstein and should not be intended to indicate "final" theories. In fact, isomathematics and isomechanics admit coverings of Lie-admissible character [12] (Section I-2) that, in turn, admits coverings of hyperstructural character [43], with additional coverings remaining possible in due time.

The terms "non-Hamiltonian interactions" are intended to indicate interactions that are not representable with a Hamiltonian, and are technically identified as interactions violating the integrability conditions for the existence of a Hamiltonian, namely, the *conditions of variational self-adjointness* [25].

When dealing with stable and isolated interior dynamical systems, the

terms "non-conservative forces" are strictly referred to *internal* non-Hamiltonian exchanges verifying conditions (1-55) for the verification of the ten conventional total conservation laws for the total energy, momentum, angular momentum and the uniform motion of the center of mass.

The terms "physical media" refer to media composed by matter in its various states, and are often referred to as *hadronic media,* in the sense that the media are *not* composed by empty space, thus requiring the use of hadronic mathematics and mechanics for their quantitative treatment.

The terms "extended particles" refer to: the wavepacket of elementary particles such as the electron assumed to be of about $1\ fm = 10^{-15}\ cm$; extended charge distributions for protons and neutrons when members of a nuclear structure, also assumed to have a diameter of about $1\ fm$; and stable nuclei when considering the structure of stars. Due to its crucial significance for the structure of interior systems, a technical definition of the notion of "extended particles" will be given in Section 3 via the notion of *isoparticle* as isorepresentations of space-time isosymmetries.

2. ISOSYMMETRIES

2.1. Foreword.

In this section, we study the axiom-preserving "completion" (or isotopic lifting) of conventional space-time symmetries, known as *Lie-isotopic symmetries,* or *isosymmetries* for short, which provide the invariance of stable and isolated (thus time reversible) interior dynamical systems of extended particles at mutual distances smaller than their size as occurring, e.g., in nuclear structures (Figure 3).

Lie-isotopic symmetries were first introduced by Santilli in the 1978 Harvard University paper [13] as a particular case of the broader *Lie-admissible symmetries* for irreversible, non-conservative systems [14]. Isosymmetries were then studied in various subsequent works quoted in this section.

The understanding of this section requires a knowledge of the *Lie-Santilli isotheory* (Section I-3.7), which was first formulated in monographs [25] [26] over the field of real numbers. Isosymmetries were then formulated in monographs [29] [30] with the full use of isomathematics, including the use isofields [20] [41] and the isodifferential calculus [21] [44] (see Refs. [38] [46] [47] [48] for works on the Lie-Santilli isotheory, and Ref. [45] for a general review with applications and experimental verifications).

The assumption at the foundation of isosymmetries is *the preservation of the abstract axioms of 20th century space-time symmetries, and the mere construction of their broadest possible realization permitted by isomathematics.*

Consequently, criticisms of isosymmetries and their novel implications

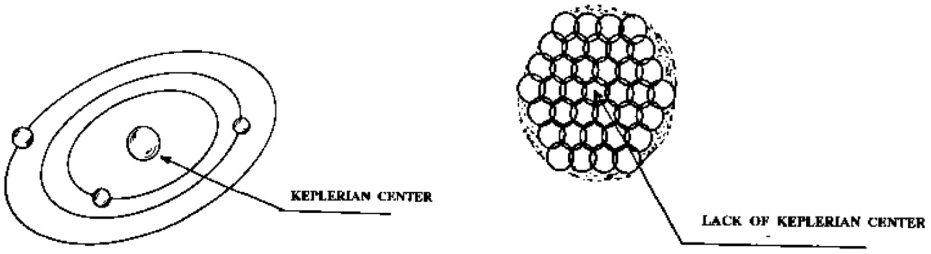

Figure 5: *The l.h.s. of this figure illustrates Keplerian systems for which space-time symmetries have been constructed, namely, exterior dynamical systems of point-like masses orbiting in vacuum around a heavier point-like mass known as the Keplerian center. The r.h.s. of this figure illustrates interior systems for which isosymmetries have been built, namely, systems of extended particles in conditions of mutual penetration without any Keplerian center.*

are *de facto* criticisms on 20th century space-time symmetries and their implications

2.2. Inapplicability of Lie symmetries for interior systems.

A rather widespread view of 20th century physics is the lack of any difference between exterior and interior dynamical systems on grounds that the latter can be reduced to their elementary constituents, by therefore recovering exterior conditions.

The above view was disproved by R. M. Santilli in his Ph. D. thesis (see the review in Ref. [30]) on numerous grounds, the first being the notorious incompatibility of quantum mechanics with thermodynamics whose resolution motivated the Lie-admissible generalization of Lie algebras and related physical theories [29] [30].

The absence of structural differences between exterior and interior systems was dismissed more directly with the following [26] [28]:

NO REDUCTION THEOREM 2.2.1: A classical dynamical system with non-conservative interior forces cannot be consistently reduced to a finite number of isolated particles all in conservative conditions and, vice- versa, the latter system cannot reproduce the former under the correspondence or other principles.

The first direct consequence of the above No Reduction Theorem is the "inapplicability" (rather than the "violation") for interior dynamical systems of conventional space-time symmetries that have been proved to be so effective for exterior dynamical systems.

Said inapplicability was also proved [*loc. cit.*] from the fact that the

Galileo and the Lorentz-Poincaré symmetries can only provide a non-relativistic and relativistic characterization, respectively, of *Keplerian systems,* namely, systems of point-like masses orbiting in vacuum around a heavier mass called the *Keplerian nucleus* [26].

However, interior dynamical systems do not admit a Keplerian structure because *nuclei have no nuclei* [33] and the same happens for hadrons, stars and gravitational collapse (Figure 5).

It is then possible to prove, e.g., via the imprimitivity theorem, that the lack of existence of a Keplerian structure implies the lack of exact validity of conventional space-time symmetries [26] [30].

On more technical grounds, Lie's theory is known to be solely applicable to exterior systems of point-like particles in vacuum with ensuing sole possible, linear, local and Hamiltonian interactions.

Experimental evidence on interior dynamical systems, e.g., on nuclear volumes compared to the volumes of individual nucleons, establishes that nuclei are composed of extended charge distributions in conditions of partial mutual penetration/entanglement with the ensuing existence of additional, non-linear, non-local and non-Hamiltonian interactions under which Lie's theory is inapplicable.

Hence, the transition of particles from exterior to interior conditions implies the inapplicability of the $SU(2)$-spin symmetry with consequential inapplicability of Bell's inequality [3] and other objections against the EPR argument [6] in favor of suitable covering vistas [10] [11].

In any case, the $SU(2)$ symmetry, while unquestionable effective for exterior dynamical systems, has been unable to provide a consistent representation of the spin of particles and nuclei, thus warranting the search for a suitable "completion."

2.3. The fundamental theorem on isosymmetries.

The construction of isosymmetries requires the full use of isomathematics with particular reference to the Lie-Santilli isotheory formulated on isospaces over isofield and elaborated via the isodifferential calculus (Section I-2.7).

Said construction can be done with the following theorem (for brevity, see the proof in Section 1.2 , Vol. I of Refs. [36]):

THEOREM 2.3.1: Let G be an N-dimensional Lie symmetry of the line element of a k-dimensional metric or pseudo-metric space $S(x, m, I)$ over a numeric field F

with coordinates x, metric m over a numeric field F with conventional unit I,

$$G: \quad x' = \Lambda(w)x, \quad y' = \Lambda(w)y, \quad x, y \in S,$$

$$(x' - y')^\dagger \Lambda^\dagger m \Lambda (x' - y') \equiv (x - y)^\dagger m (x - y), \tag{9}$$

$$\Lambda^\dagger(w) m \Lambda(w) \equiv m. \quad w \in F.$$

Then, all infinitely possible (non-singular) Lie-Santilli isotopies $\hat{G}$ of G on isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$ with isocoordinates

$$\hat{x} = xI, \tag{10}$$

isometric

$$\hat{M} = \hat{m}\hat{I} = (\hat{T}_i^k m_{kj})\hat{I}, \tag{11}$$

and isounit

$$\hat{I} = 1/\hat{T} > 0, \tag{12}$$

over an isofield $\hat{F}$ with isounit $v\hat{I}$ leave invariant the isoline element of the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$:

$$\hat{G}: \quad \hat{x}' = \hat{\Lambda}(\hat{w}) \star \hat{x}, \quad \hat{y}' = \hat{\Lambda}(\hat{w}) \star \hat{y}, \quad \hat{x}, \hat{y} \in \hat{S},$$

$$(\hat{x}' - \hat{y}')^\dagger \star \hat{\Lambda}^\dagger \star \hat{M} \star \hat{\Lambda} \star (\hat{x}' - \hat{y}') \equiv (x - y)^\dagger \hat{m} (x - y), \tag{13}$$

$$\hat{\Lambda}^\dagger(\hat{w}) \star \hat{M} \star \hat{\Lambda}(\hat{w}) \equiv \hat{M}.$$

All infinitely possible so constructed isosymmetries $\hat{G}$ are locally isomorphic to the original symmetry G.

The reader should note that, while a given Lie symmetry G is unique as well known, there can be an infinite number of covering isosymmetries $\hat{G}$ with generally different explicit forms o the isotransformations due to the infinite number of possible isotopic elements representing the infinitely different internal interactions of extended particles within physical media.

Note also that all possible isotopic images of a given Lie symmetry can be explicitly and uniquely constructed via the sole knowledge of the original Lie symmetry and of the isotopic element $\hat{T} > 0$, or of the isounit $\hat{I} = 1/\hat{T}$, which property shall be hereon tacitly assumed.

2.4. Isospaces and isogeometries.

As it is well known, the fundamental representation space of relativistic space-time symmetries is the conventional *Minkowski space* $M(x,\eta,I)$ formulated on the field of real numbers $\mathcal{R}$ with coordinates $x=(x^1,x^2,x^3,x^4=ct)$, metric $\eta=Diag.(1,1,1,-1)$, unit $I=Diag(1,1,1,1)$ and invariant

$$\begin{aligned} x^2=(x^\mu\eta_{\mu\nu}x^\nu)I= \\ =(x_1^2+x_2^2+x_3^3-c^2t^2)I, \end{aligned} \tag{14}$$

where the trivial multiplication by the conventional unit $I=Diag.(1,1,1,1)$ is done for compatibility with isomathematics.

The fundamental isospaces of space-time isosymmetries are given by the infinite family of *iso-Minkowski isospaces,* also called *Minkowski-Santilli isospaces,* $\hat{M}(\hat{x},\hat{\Omega},\hat{I})$ formulated on the isofield of isoreal isonumbers $\hat{\mathcal{R}}$. (Section I-3.9), which isospaces were first introduced by R. M. Santilli in Ref. [23] of 1983 and then treated in details in works [29] [30].

Iso-Minkowskian isospaces are characterized by *space-time isocoordinates* $\hat{x}=x\hat{I}$; *isounit* $\hat{I}=1/\hat{T}$, *isometric*

$$\hat{\Gamma}=(\hat{T}_\mu^\rho\eta_{\rho\nu})\hat{I}, \tag{15}$$

(where one should note the necessary structure of an isomatrix [29]), positive-definite *isotopic element* (4) representing a system of extended particles in interior dynamical conditions with a restricted functional dependence on local quantities such as coordinates x, momenta p, energy E, frequency ν, density α, temperature τ, pressure pi, wavefunction ψ, etc., under the conditions

$$n_\mu=n_\mu(x,p,E,\nu,\alpha,\tau,\pi,\psi,\partial\psi,...)>0,\ \mu=1,2,3,4, \tag{16}$$

$$\Gamma(x,p,E,\nu,\alpha,\tau,\pi,\psi,\partial\psi,...)\geq 0, \tag{17}$$

$$\hat{T}=e^{-\Gamma}\ll 1. \tag{18}$$

Iso-Minkowskian isospaces are characterized by the infinite family of isoinvariants (I-28) with isotopic element (4) that, for the case of one single extended particle can be written

$$\begin{aligned} \hat{x}^{\hat{2}}=\hat{x}^\mu\star\hat{\Omega}_{\mu\nu}\star\hat{x}^\nu=(x^\mu\hat{\eta}_{\mu\nu}x^\nu)\hat{I}= \\ =(\tfrac{x_1^2}{n_1^2}+\tfrac{x_2^2}{n_2^2}+\tfrac{x_3^2}{n_3^2}-t^2\tfrac{c^2}{n_4^2})\hat{I}, \end{aligned} \tag{19}$$

where the exponential $exp{-}\Gamma$ has been absorbed in the characteristic quantities n_μ, and the final multiplication by the isounit is necessary for the

isoinvariant to be an isoscalar, namely, an element of the isoreal isofield [20] (Section I-3-5).

The following aspects treated in Paper I are important for the understanding of the apparent proof of the EPR argument:

1. The characteristic quantities n_1^2, n_2^2, n_3^2, admit the first interpretation as representing the deformable semi-axes of elementary or composite particles normalized to the values for the sphere $n_1^2 = n_2^2, = n_3^2 = 1$,.

2. The characteristic quantity n_4^2 admit the first interpretation as representing the *density* of the hadronic medium normalized to the value $n_4 = 1$ for the vacuum.

3. The function $\Gamma \geq 0$ provides an invariant representation (Section I-3-9) of all non-linear, non-local and non-Hamiltonian interactions.

4. Property (18) is verified for all applications of isosymmetries to date [10] to [68].

5. The correct elaboration of iso-Minkowskian isospaces requires the use of the *isospherical and isohyperbolic isocoordinates* (see Refs. [29] [30]).

6. Isoinvariant (19) provides a unified representation of both exterior and interior gravitational problems. In fact, K. Schwartzchild wrote in 1916 *two* important papers, the first paper [49] on the *exterior gravitational problem* which became world famous for its initiation of gravitational singularities, and the second paper [50] in the *interior gravitational problem* which has been vastly ignored, except rare studies (such as that in Section 23.2, page 609, Ref. [52]). Such an oblivion is essentially due to the fact that Schwartzchild's second paper is not aligned with the widespread tendency of reducing masses to point-like constituents, in which case all differences between exterior and interior gravitational problems disappear to the detriment of the depth of the gravitational analysis. Readers should keep in mind the full parallelism between exterior and interior dynamical problems for *particles and gravitation.*

7. The *exterior gravitational interpretation* of isoinvariant (19) is given by the following identical representation of Schwartzchild's exterior metric [50]

$$\hat{T}_{kk} = \frac{1}{1 - \frac{2M}{r}}, \quad \hat{T}_{44} = 1 - \frac{2M}{r}. \tag{20}$$

The corresponding *interior gravitational representation* is given by the fol-

lowing isotopy of Schwartzchild exterior metric

$$\hat{T}_{kk} = \frac{1}{(1 - \frac{2M}{r})n_k^2}, \quad \hat{T}_{44} = (1 - \frac{2M}{r})/n_4^2. \tag{21}$$

In view of the arbitrariness of the functional dependence of the characteristic quantities n_μ, it is easy to prove that Schwartzchild's interior metric [50] is a particular case of the much broader class of interior gravitational models (21).

8. The geometry of the iso-Minkowskian isospaces, first presented by Santilli in Ref. [24] under the name of *iso-Minkowskian isogeometry*, contains the machinery of the Riemannian geometry (due to the dependence of the isometric $\hat{\eta}$ on the local coordinates x), although such a machinery is formulated for consistency over isofields [20] and elaborated via the isodifferential isocalculus [21] (Section I-3.5). Hence, *the isominkowskian isogeometry can unify exterior and interior problems for both particles and gravitation.*

9. Recall that iso-Minkowskian isospaces are locally isomorphic to the conventional Minkowski space (Refs. [23] [24] and Theorem 2.3.1). Therefore, *the iso-Minkowskian isogeometry has a null curvature.* This is due to the fact that, under isotopic lifting, the conventional Minkowski metric $\eta = Diag.(1, 1, 1, -1)$ is lifted into a coordinate-dependent isometric $\hat{T}(x)\eta = \hat{\eta}(x)$ which is *identical* to any given Riemannian metric

$$\eta \ \rightarrow \ \hat{\eta}(x) = \hat{T}(x)\eta = g(x). \tag{22}$$

Jointly, the original unit of the Minkowski space $\hat{I} = Diag.(1, 1, 1, 1)$ is lifted by the *inverse* amount

$$I > 0 \ \rightarrow \ \hat{I}(x) = 1/\hat{T}(x) > 0, \tag{23}$$

resulting in no actual curvature. The above features have suggested the introduction of the new notion of *isoflat isospace*, referred to an isospace that has null curvature when formulated on isofields, while recovering conventional curvature when formulated on conventional fields. Readers should be aware that the achievement of the universal symmetry of (non–singular) Riemannian line elements studied in the next sections are due precisely to the isoflatness of the iso-Minkowski isospace since no such symmetry is possible for a convemtional Riemannian space, as well known.

Recall that the fundamental representation space of symmetries in 3-space dimensions is the conventional Euclidean space $E(r, \times, I$ with coor-

dinates $r = (x^1, x^2, x^3)$, metric $\delta = Diag.(1, 1, 1)$ and unit $I = Diag.(1, 1, 1)$ on the conventional field of real numbers.

Similarly, the fundamental representation space of isosymmetries in 3-dimensions is the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$, also called *Euclid-Santilli isospace* (Refs. [14] [26] [29] and Section I-3.5) which is the space component of the iso-Minkowskian isospace. As such, the iso-Euclidean isospace is hereon tacitly assumed to be known.

2.5. Lorentz-Poincaré-Santilli isosymmetries.

2.5.1. Main references. Following, and only following the construction of the isotopies of Lie's theory, Santilli conducted systematic studies on the isotopies of the various aspects of the Lorentz-Poincaré symmetry for the achievement of the universal invariance of spacetime isoinvariant (19), including:

1) The classical isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [53];

2) The operator isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [54];

3) The isotopies $\hat{SO}(3)$ of the rotational symmetry $SO(3)$ [55] [56] [57];

4) The isotopies $\hat{SU}(2)$ of the $SU(2)$ spin symmetry [10] [58];

5) The isotopies $\hat{P}(3.1)$ of the Poincaré symmetry $P(3.1)$ [59] [60], which included the universal symmetry of (non-singular) Riemannian line elements;

6) The isotopies $\hat{\mathcal{P}}(3.1)$ of the spinorial covering $\mathcal{P}(3.1)$ of the Poincaré symmetry [61] [62];

7) The isotopies $\hat{M}(3.1)$ of the Minkowskian geometry $M(3.1)$ [24].

A general presentation is available in the 1995 monographs [29] [30] with the full use of isomathematics, including isofields and isodifferential calculus, with upgrades in the 2008 monographs [36].

The resulting infinite family of isosymmetries $\hat{SO}(3.1)$ are known as the *Lorentz-Santilli (LS) isosymmetries* while the broader isosymmetries $\hat{P}(3.1)$ and $\hat{\mathcal{P}}(3.1)$ are known as *Lorentz-Poincaré-Santilli isosymmetries* (see Refs. [37] [42] [45] and papers quoted therein).

Experimental verifications of LPS isosymmetries for interior dynamical systems are available in monographs [31] and in Section 3 of the more recent review [63].

In inspecting the subsequent sections, the reader should be aware of the "direct universality" of the LPS isosymmetries for the considered infinite family of interior dynamical systems [64], including the treatment of exterior and interior, particle and gravitational problems (Section 4).

2.5.2. Basic definitions. As it is well known, the conventional Lorentz-Poincaré (LP) symmetry is the symmetry of line element (14) which we rewrite in the form

$$(x-y)^2 = (x^\mu - y^\mu)\eta_{\mu\nu}(x^\nu - y^\nu)I =$$

$$= [(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 - (t_1-t_2)^2c^2)]\, I, \tag{24}$$

$$\eta = Diag.(1,1,1,-c^2),, \;\; I = Diag.(1,1,1,1),$$

where the exponential component $exp{-}\Gamma$ is again embedded for simplicity in the characteristic quantities $n\mu^2$.

The LPS isosymmetry is the universal symmetry of the isoline element (19) in the iso-Minkowski isospace $\hat{M}(\hat{x},\hat{\Omega},\hat{I})$ over the isoreal isonumbers $\hat{\mathcal{R}}$ rewritten in the form

$$(\hat{x}-\hat{y})^{\hat{2}} = \left[(\hat{x}^\mu - \hat{y}^\mu)\star\hat{\Omega}_{\mu\nu}\star(\hat{x}^\nu - \hat{y}^\nu)\right] =$$

$$= [x^\mu - y^\mu)\hat{\eta}_{\mu\nu}(x^\nu - y^\nu)]\,\hat{I} =$$

$$= \left[\tfrac{(x_1-y_1)^2}{n_1^2} + \tfrac{(x_2-y_2)^2}{n_2^2} + \tfrac{(x_3-y_3)^2}{n_3^2} - (t_1-t_2)^2\tfrac{c^2}{n_4^2}\right]\hat{I}, \tag{25}$$

$$\hat{\eta} = \hat{T}\eta, \;\; \hat{T} = Diag((\tfrac{1}{n_1^2},\tfrac{1}{n_2^2},\tfrac{1}{n_3^2},\tfrac{1}{n_4^2}),$$

$$n_\mu = n_\mu(x,v,a,E,d,\omega,\tau,\psi,\partial\psi,...) > O, \;\; \hat{I} = 1/\hat{T} > 0.$$

2.5.3. Isotransformations. By following Theorem 2.3.1, the isotransformations of the LPS isosymmetries can be written

$$\hat{x}' = \hat{\Lambda}(\hat{w})\star\hat{x}, \tag{26}$$

where $\hat{\Lambda}(\hat{w}) = \Lambda(\hat{w})\hat{I}$, resulting in generally *non-linear* isotransformations, including isotranslations of the type

$$\hat{x}' = \hat{x} + \hat{A}(\hat{x},...), \tag{27}$$

verifying the following property

$$\hat{\Lambda}^\dagger\star\hat{\eta}\star\hat{\Lambda} = \Lambda\hat{\eta}\Lambda^\dagger. \tag{28}$$

Under the *condition of isomodularity*

$$\hat{Det}\,(\hat{\Lambda}) = +\hat{I}, \tag{29}$$

we have the *isoconnected LS isosymmetries* $\hat{SO}^0(3.1)$ and the *isoconnected LPS isosymmetries* $\hat{P}^0(3.1)$.

Consider the conventional generators of the Poincaré symmetry

$$(J_k) = (J_{\mu\nu}),\ P_\mu, k = 1,2,3,4,5,6,\ \mu,\nu = 1,2,3,4. \tag{30}$$

By keeping in mind isoexponentiation (I-16), the isotransformations of $\hat{SO}^0(3.1)$ can be written [60]

$$\begin{aligned} \hat{x}' &= (\hat{e}^{iJ_k w_k}) \star \hat{x} \star (\hat{e}^{-iJ_k w_k}) = \\ &= \left[(e^{iJ_k \hat{T} w_k}) x (e^{-iw_k \hat{T} J_k})\right] \hat{I}, \end{aligned} \tag{31}$$

and the isotranslations $\hat{A}(3.1)$ can be written

$$\begin{aligned} \hat{x}' &= (\hat{e}^{iP_\mu a_\mu}) \star \hat{x} \star (\hat{e}^{-iP_\mu a_\mu}) = \\ &= \left[(e^{iP_\mu \hat{T} a_\mu}) x (e^{-ia_\mu \hat{T} P_\mu})\right] \hat{I}. \end{aligned} \tag{32}$$

It is evident that the above isotransformations do constitute Lie-Santilli isogroups according to Theorem I-2.7.3.

2.5.4. Isocommutation rules. As recalled earlier, the total quantities of an isolated, stable, interior system must be conserved for consistency.

In order to represent this evidence, the Lie-Santilli isotheory was constructed [26] in such a way to preserve conventional generators, because they represent total conservation laws, and isotopically lift their product.

By expanding the preceding finite isotransforms in terms of the isounit, the *LPS isoalgebra* $\hat{so}^0(3.1)$ is characterized by the conventional generators of the LP algebra and the isocommutation rules [30] [60] (here written in their projection on conventional spaces over conventional fields)

$$\begin{aligned} [J_{\mu\nu}\hat{,}J_{\alpha\beta}] &= \\ &= 1(\hat{\eta}_{\nu\alpha} J_{\beta\mu} - \hat{\eta}_{\mu\alpha} J_{\beta\nu} - \hat{\eta}_{\nu\beta} J_{\alpha\mu} + \hat{\eta}_{\mu\beta} J_{\alpha\nu}), \end{aligned} \tag{33}$$

$$[J_{\mu\nu}\hat{,}P_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu)$$

$$[P_\mu\hat{,}P_\nu] = 0, \tag{34}$$

$$\hat{\eta}_{\mu\nu} = \hat{T}\eta = (\hat{T}^\rho_\mu \eta_{\rho\nu}). \tag{35}$$

where one should note the appearance of the *structure functions* $\hat{\eta}(x, p,$-$E, \nu, \alpha, \tau, \psi, ...)$, rather than the traditional structure constants (Theorem I-2.7.2).

The presence of structure functions $\hat{\eta}$ in isocommutation rules (33)-(35), Theorem I-3.7.2 and the analysis of Section I-3.8 imply the following important property (Section I-3.8):

LEMMA 2.5.1: LPS isosymmetries cannot be derived via non-unitary transformations of the conventional LP symmetry.

Despite the above non-equivalence, the property $\hat{T} > 0$, the topological structure $(+1, +1, +1, -1)$ of the isometric $\hat{\eta} = \hat{T}\eta$ and Theorem 2.3.1 imply that:

LEMMA 2.5.2. *All LPS isosymmetries are locally isomorphic to the conventional LP symmetry.*

Recall from Section I-1 that an important limitation of quantum mechanics for the study of the EPR argument is the inability to achieve a consistent and effective treatment of non-linear interactions that are expected in the structure of hadrons, nuclei and stars. In Section I-4.12, we have shown that the isotopic "completion" of quantum mechanics into hadronic mechanics does indeed allow a consistent and effective treatment of non-linear interactions via their embedding in the isotopic element $\hat{T}$.

Due to the unrestricted functional dependence of the isotopic element $\hat{T}$ and, therefore, of the isometric $\hat{\eta} = \hat{T}\eta$, it is easy to see that the LPS isosymmetries are indeed non-linear as a necessary condition to provide the invariance of non-linear dynamical equations.

Note that *isolinear isomomenta $\hat{P}_\mu$ isocommute on isospaces over isofields, but they do not commute on conventional spaces over conventional fields,* Eqs. (35), thus confirming that *the LPS isosymmetry is isolinear, that is, linear on isospaces over isofields but generally non-linear in their projection on conventional spaces over conventional fields.*

This important property can be illustrated by recalling the isolinear isomomentum (I-79) on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $\hat{\psi} >$ over the isocomplex isonumbers $\hat{\mathcal{C}}$

$$\hat{P}_\mu \star |\hat{\psi} >= -i\hat{I}\partial_\mu |\hat{\psi} > . \tag{36}$$

Isocommutators (35) on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ can then be explicitly written

$$\begin{aligned}
&[\hat{P}_\mu,\hat{P}_\nu] \star |\hat{\psi}>= (\hat{P}_\mu \star \hat{P}_\nu - \hat{P}_\nu \star \hat{P}_\mu) \star |\hat{\psi}>= \\
&= (-i\hat{I}\partial_\mu)\hat{T}(-i\hat{I}\partial_\nu) - (-i\hat{I}\partial_\nu)\hat{T}(-i\hat{I}\partial_\mu)\hat{T}|\hat{\psi}>= \\
&= (i\hat{I}\partial_\mu\partial_\nu - i\hat{I}\partial_\nu\partial_\mu)|\hat{\psi}>= 0.
\end{aligned} \tag{37}$$

By contrast, the projection of the same isocommutators (35) on a conventional Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ no longer commutes,

$$\begin{aligned}
&[\hat{P}_\mu, \hat{P}_\nu]|\hat{\psi}>= (\hat{P}_\mu\hat{P}_\nu - \hat{P}_\nu\hat{P}_\mu)|\hat{\psi}>= \\
&= (-i\hat{I}\partial_\mu)(-i\hat{I}\partial_\nu) - (-i\hat{I}\partial_\nu)(-i\hat{I}\partial_\mu)|\hat{\psi}>\neq 0.
\end{aligned} \tag{38}$$

because, in general, $\partial_\mu\hat{I} \neq \partial_\nu\hat{I}$, and this proves the isolinear character of the isomomentum.

Besides a direct relevance for the structure of hadrons, nuclei and stars, the above isolinearity has important implications, such as a new consistent operator form of gravitation, a new grand unification and other advances [35].

The presence of the structure functions in the isocommutation rules, the capability to provide the invariance under non-linear interactions and other features and applications outlined in Section 4 illustrate the non-triviality of the Lie-Santilli isotheory.

2.5.5. Iso-Casimir Isoinvariants. The simple direct use of isocommutation rules (33)-(35) establishes that the *iso-Casimir-isoinvariants* of $\hat{p}^0 0(3.1)$ are given by [60]

$$\begin{aligned}
&\hat{C}_1 = \hat{I}((t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) > 0, \\
&\hat{C}_2 = \hat{P}^{\hat{2}} = \hat{P}_\mu \star \hat{P}^\mu = (\hat{\eta}_{\mu\nu}P^\mu P^\nu)\hat{I} = \\
&= (\textstyle\sum_{k=1,2,3} \frac{1}{n_k^2}P_k^2 - \frac{c^2}{n_4^2}p_4^2)\hat{I}, \\
&\hat{C}_3 = \hat{W}^{\hat{2}} = \hat{W}_\mu \star \hat{W}^\mu, \;\; \hat{W} = W\hat{I}, \\
&\hat{W}_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \star J^{\alpha\beta} \star P^\rho,
\end{aligned} \tag{39}$$

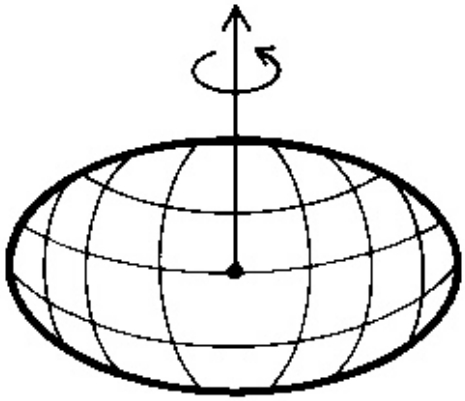

Figure 6: *It was generally believed in the 20th century physics that the rotational symmetry is broken for ellipsoids. Santilli isorotational isosymmetry has restored the exact character of the rotational symmetry for all possible (topology preserving) deformations of the sphere [30].*

and they are at the foundation of classical and operator *relativistic isomechanics* (Section I-4) with deep implications for structure models of interior dynamical systems [31].

2.5.6. Isorotations. By using isotransforms (32), the explicit form of the isorotations $\hat{SO}(3)$, first derived in Refs. [55] [56], can be written in the isoplane $(\hat{x},^1 , \hat{x}^2)$ of iso-Euclidean isospaces $\hat{E}(\hat{x}, \hat{\Delta}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, here formulated for simplicity in their projection on the conventional Euclidean space (see Ref. [30] for the general case)

$$\begin{aligned} x^{1'} &= x^1 \cos[\theta(n_1 n_2)^{-1}] - x^2 \tfrac{n_1^2}{n_2^2} \sin[\theta(n_1 n_2)^{-1}], \\ x^{2'} &= x^1 \tfrac{n_2^2}{n_1^2} \sin[\theta(n_1 n_2)^{-1}] + x^2 \cos[\theta(n_1 n_2)^{-1}]. \end{aligned} \tag{40}$$

It was generally believed in the 20th century that the $SO(3)$ symmetry is broken for ellipsoid deformations of the sphere. By contrast, as shown by isotransforms (40) the $\hat{SO}(3)$ isosymmetry achieves the invariance of ellipsoids (Figure 6). But $SO(3)$ and $\hat{SO}(3)$ are locally isomorphic (Theorem 2.3.1). We therefore have the following property [55] [56]:

LEMMA 2.5.3: The Lie-Santilli $\hat{SO}(3)$ isosymmetry restores the exact character of the rotational symmetry for all ellipsoid deformations of the sphere.

This property is due to the fact that the mutation of the semiaxes of the sphere occur jointly with the *inverse*, mutation of the related units, thus

maintaining the perfect spherical shape in isospaces over isofields

$$Radius \quad 1_k \to 1/n_k^2, \quad Unit \quad 1_k \to n_k^2. \tag{41}$$

Note the crucial role of isonumbers for the reconstruction of the exact rotational symmetry because said reconstruction occurs thanks to the isoinvariant by the isounit.

2.5.7. Lorentz-Santilli isotransforms. The infinite family of isoconnected Lorentz-Santilli (LS) isotransforms $\hat{SO}^0(3.1)$ on iso-Minkowskian isospaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, first derived by in Ref. [53] of 1983, can be written in the $(\hat{x}^3, \hat{x}^4)$-isoplane in their projection in the conventional l Minkowski space $M(x, \eta, I)$, as follows (see Ref. [30] for the general case):

$$\begin{aligned} x^{1'} &= x^1, \quad x^{2'} = x^2, \\ x^{3'} &= \hat{\gamma}(x^3 - \hat{\beta}\tfrac{n_3}{n_4}x^4), \\ x^{4'} &= \hat{\gamma}(x^4 - \hat{\beta}\tfrac{n_4}{n_3}x^3), \end{aligned} \tag{42}$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \quad \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}. \tag{43}$$

A significant aspect of Ref. [53] is the solution of the *historical Lorentz problem*, namely, the invariance of locally varying speeds of light within physical media

$$C = \frac{c}{n_4}. \tag{44}$$

In fact, Lorentz first attempted the invariance of the speed of light $C = c/n_4$, but had to restrict his study to the invariance of the constant speed of light in vacuum c, due to insurmountable technical difficulties. Santilli has shown that Lorentz's difficulties were due to the use of Lie's theory, because, under the use of the covering Lie-Santilli isotheory, the invariance of $C = c/n_4$ was achieved in two pages of the 1983 letter [53].

A second significant aspect of Ref. [53] is the achievement of the first invariant formulation of extended, thus deformable and hyperdense particles, as stated beginning with the title of the quoted paper.

It was generally believed in the 20th century that the Lorentz symmetry $SO^0(3.1)$ is broken for locally varying speed of light within physical media represented with the wiggly circle of Figure 7. Ref. [53] proved that the isosymmetry $\hat{SO}^0(3.1)$ achieves the invariance of $C = c/n_4$. But $SO^0(3.1)$

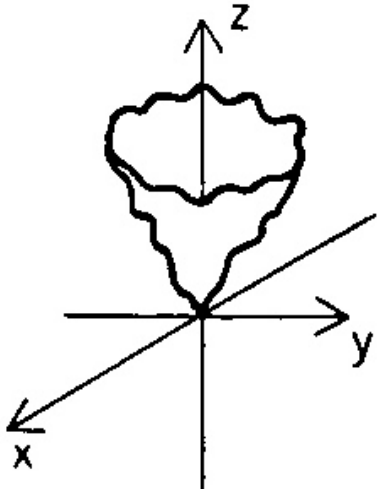

Figure 7: *It was generally believed in the 20th century physics that the Lorentz symmetry is broken for locally varying speeds of light within physical media (here represented with a wiggly light cone). The Lorentz-Santilli isosymmetry has restored the exact validity of the Lorentz symmetry for interior dynamical problems [53] [30].*

and $\hat{SO}^0(3.1)$ are locally isomorphic, thus restoring the exact character of the abstract axioms of the Lorentz for all possible values $C = c/n_4$. We therefore have the following important property [30]:

LEMMA 2.3.5: The Lie-Santilli $\hat{SO}^0(3.1)$ isosymmetry restores the exact validity of Lorentz's axioms for locally varying speeds of light.

This property is due to the reconstruction of the exact light cone on the iso-Minkowskian isospace over isofields with maximal causal value c, called the *light isocone,*

$$\hat{x}^{\hat{2}} = \hat{x}_3^2 + \hat{x}_4^2 = 0, \tag{45}$$

while its projection on the conventional Minkowski space over conventional fields represents a locally varying speed

$$\hat{x}^2 = (\frac{x_3^2}{n_3^2} - t^2\frac{c^2}{n^4})\hat{I} = 0. \tag{46}$$

This property is due to the fact that the mutation of the $\hat{x}^3$ and $\hat{x}^4$ isocoordinates occurs jointly with the *inverse* mutation of the corresponding isounits, by therefore preserving the original perfect light cone with c as

the maximal causal speed (see the 1966 monograph [30] for details)

$$\begin{aligned} x^3 \ &\rightarrow \ \tfrac{x^3}{n_3}, \quad I_3 = 1 \ \rightarrow \ \hat{I}_3 = n_3 \\ x^4 = tc \ &\rightarrow \tfrac{x^4}{n_4} = t\tfrac{c}{n_4}, \quad I_4 = 1 \ \rightarrow \ \hat{I}_4 = n_4. \end{aligned} \tag{47}$$

Another significant aspect of Ref. [53] is the achievement of the first known invariance of non-linear, non-local and non-Hamiltonian interactions thanks to their embedding in the characteristic n-quantities of the isoinvariant (25).

2.5.8. Isotranslations. In view of their non-linearity, isotranslations in four parameters a_μ can be written in their projection in the conventional Minkowski space [30]

$$x'^\mu = x^\mu + A^\mu(a, x, \ldots), \tag{48}$$

and can be written via a power series expansion of the general expression

$$A^\mu = a^\mu(n_\mu^{-2} + a^\alpha[n_\mu^{-2}\hat{,}P_\alpha]/1! + \ldots), \tag{49}$$

The understanding of the isotopic completion of 20th century space-time symmetries requires the knowledge that, when properly written on iso-Minkowskian isospace over isofields, isotranslations recover their conventional form . [30].

2.5.9. Isodilatations. Santilli introduced in Ref. [60] a novel one-dimensional isoinvariance denoted $\hat{D}$ which is given by the dilatation of the isometric caused by its multiplication by as parameter w, while the isounit is jointly subjected to the *inverse* dilatation

$$\begin{aligned} \hat{\Omega} = \hat{\eta}\hat{I} \ &\rightarrow \ \hat{w} \star \hat{\Omega} = w\hat{\eta}\hat{I}' \\ \hat{I} \ &\rightarrow \ \hat{I}' = \tfrac{1}{w}\hat{I}, \end{aligned} \tag{50}$$

under which isoinvariant (25) remain manifestly unchanged.

In essence, the new symmetry originates from the fact that, for mathematical consistency, isoinvariants must be elements of t isofields, thus having structure (25), namely, isoinvariants must be given by a conventional invariant multiplied by the isounit.

Ref. [60] showed that, by writing conventional invariants with the multiplication, in this case, by the trivial unit 1, the new dilatation symmetry persists for conventional space-time symmetries,

$$\eta \rightarrow \eta' = w\eta, \quad 1 \ \rightarrow \ 1' = \frac{1}{w}1. \tag{51}$$

The above properties imply the following:

LEMMA 2.5.5: The conventional Lorentz-Poincaré symmetry is eleven-dimensional with structure

$$P^0(3.1) = so^0(3.1) \times A(3.1) \times D, \tag{52}$$

and, consequently, the Lorentz-Poincaré-Santilli isosymmetry is also eleven-dimensional with the structure

$$\hat{P}^0(3.1) = \hat{so}^0(3.1) \star \hat{A}(3.1) \star \hat{D}. \tag{53}$$

The above seemingly trivial property has permitted Santilli the study of a new grand unification of electroweak and gravitational interactions based on the embedding of gravitation in the isotopic degree of freedom of the theory [35].

2.5.10. Isoinversions. The isotopic "completion" of conventional inversions has been studied in details in Refs. [30] and consists of the *isotime isoinversions*

$$\hat{\tau}\hat{t} = (\tau\hat{t})\hat{I} \tag{54}$$

plus the *isospace isoinversions*

$$\hat{\pi}\hat{r} = (\pi\hat{r})\hat{I} \tag{55}$$

where τ and π are conventional time and space inversions, respectively.

Despite their simplicity, Santilli has shown in Ref. [30] that *not only continuous, but also discrete space-time symmetries can be reconstructed as being exact on isospaces over isofields when assumed to be broken on conventional spaces over conventional fields.*

2.5.11. Isospinorial LPS isosymmetry. Recall that the spinorial covering $\mathcal{P}^0(3.1)$ of the connected component of the LP symmetry $P^0(3.1)$ is constructed via the use of the Dirac gamma matrices. In fact, the conventional generators are realized via suitable combination of Dirac gamma matrices.

By following the same historical pattern, Santilli proposed in the 1995 communication [61] of the *Joint Institute for Nuclear Research,* Dubna, Russia (see also the subsequent paper [62]) the following eleven-dimensional isotopic "completion" $\hat{\mathcal{P}}^0(3.1)$ of $\mathcal{P}^0(3.1)$

$$\hat{\mathcal{P}}(3.1) = \hat{SL}(2.\hat{C}) \star \hat{\mathcal{A}}(3.1) \star \hat{\mathcal{D}}, \tag{56}$$

with realization of the generators in terms of the Dirac-Santilli isogamma isomatrices $\hat{\Gamma}_\mu = \hat{\gamma}_\mu \hat{I}$, Eqs. (I-89),

$$\hat{SL}(2.\hat{C}): \quad \hat{R}_k = \tfrac{1}{2}\epsilon_{kij}\hat{\Gamma}_i \star \hat{\Gamma}_j, \quad \hat{S}_k = \tfrac{1}{2}\hat{\Gamma}_k \star \hat{\Gamma}_4,$$

$$\hat{\mathcal{A}}(3.1): \quad \hat{P}_\mu, \tag{57}$$

$$k = 1, 2, 3, 4, 5, 6, \quad \mu = 1, 2, 3, 4.$$

The verification by the above isogenerators of isocommutation rules (33)-(35) is an instructive exercise for the interested reader. The proof that the Dirac-Santilli isoequations (I-88) transform isocovariantly under $\hat{\mathcal{P}}^0(3.1)$ is equally instructive.

2.5.12. Galilean isosymmetries.

As it is well known, the Galileo symmetry $G(3.1)$ characterizes the non-relativistic motion of point particles in vacuum, with consequential absence of resistive or non-potential forces (see the vertical line of Figure 8).

The isotopies of the Galileo symmetry are intended to characterize the non-relativistic motion of extended particles within physical media, by therefore experiencing resistive non-potential forces (see the wiggly line of Figure 8).

The resulting infinite family of isosymmteries $\hat{G}(3.1)$ are here called *Galilean isosymmetries* to stress the preservation of the basic axioms of the Galileo symmetry and the mere construction of the broadest possible realizations permitted by isomathematics.

The Lie-isotopic lifting of the Galileo symmetry were introduced by Santilli in the 1978 paper [12] as a particular case of the covering *Lie-Admissible symmetries*, also called *genosymmetries*, which are intended fo characterize the *time rate of variation of physical quantities*.

The first direct study of Galilean isosymmetries was done in Section 5.3, pages 225 on, of the 1981 monograph [26] formulated over conventional fields. These isotopies were then systematically studies and upgraded in the two 1991 volumes [27] [28]. The formulation of Galilean isosymmetries with the full use of isomathemaics was done in the 1995 monographs [29] [30] with a final study presented in Ref. [32].

The above studies attracted the attention of Abdus Salam, founder and president of the *International Center for Theoretical Physics* (ICTP), Trieste, Italy, who invited Santilli in 1991 to deliver at his Center a series of lectures

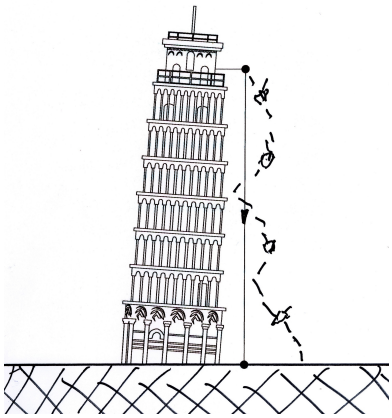

Figure 8: *This figure presents a conceptual rendering of the free fall of point-masses in vacuum studied by Galileo (represented with a straight line), and the free fall of extended masses experiencing resistive forces from our atmosphere studied by Santilli (represented with a wiggling line) [26] [30] [37]. It is symptomatic to note that the achievement of the symmetry for extended masses required the construction of a covering of the mathematics used for the point masses with particular reference to the generalization of Newton-Leibnitz differential calculus, from its historical formulation for isolated point, to a covering formulation for volumes [21].*

in the isotopies of the Galileo symmetry and relativity, said invitation being apparently the last by Salam prior to his death.

During his visit at the ICTP, Santilli wrote papers [65] through [71]. The notes from Santilli's lectures were collected by A. K. Aringazin, A. Jannussis, F. Lopez, M. Nishioka and B. Vel-janosky and published in volume [37] of 1992.

This work is primarily intended for relativistic isosymmetries. Additionally, all primary applications require relativistic treatments. Therefore, we regret to be unable to review Galilean isosymmetries to prevent a prohibitive length.

Nevertheless, the reader should be aware that an introductory knowledge of the Galilean isosymmetries is suggested, e.g., from the reading of the ICTP papers [65] to [71].

3. APPARENT PROOFS OF THE EPR ARGUMENT

3.1. Foreword.

As it is well known, the conventional Pauli matrices $\sigma_k,\ k = 1, 2, 3$, are the fundamental (also called adjoint), irreducible unitary representation of the $SU(2)$-spin symmetry and play a crucial role for the objections against the

EPR argument [2] - [6] .

In this section, we review the isotopic "completion'" of Pauli's matrices into isomatrices

$$\hat{\Sigma}_k = \hat{\sigma}_k \hat{I} \tag{58}$$

which constitute the isofundamental, isoirreducible, isounitary isorepresentation of the Lie-Santilli $\hat{SU}(2)$ isosymmetry and play a crucial role in the apparent proof of the EPR argument for extended particles within physical media studied later on in this section.

By recalling that the $SU(2)$ symmetry characterizes the spin of point-particles in vacuum, the "completed $\hat{SU}(2)$ isosymmetry is intended to characterize the spin of extended particles within hyperdense media called *hadronic spin*, such as the spin of an electron in the core of a star.

The isotopic "completion" of Pauli's matrices was introduced by Santilli in 1993 while visiting the *Joint Institute for Nuclear Research*, Dubna, Russia [58]. Said "completion" was presented systematically in Refs. [29] [30], used for the apparent proofs of the EPR argument [10] [11], and they are nowadays known as the *Pauli-Santilli isomatrices* [45].

In particular, the preceding studies have shown that, unlike the case for the $SU(2)$ symmetry, the isotopic $\hat{SU}(2)$ isosymmetry admits an explicit and concrete realization of hidden variables λ [3] [4] via realizations of the isotopic element of type $\hat{T} = Diag.(1.\lambda, \lambda)$ Eq. (3).

In this section, we shall review the construction of $\hat{SU}(2)$ isosymmetry and of Pauli-Santilli isomatrices of regular and irregular type with hidden variables. We shall then use the methods acquired in this and in the preceding paper [9], for the proof that *interior dynamical systems represented via isomathematics and isomechanics appear to admit identical classical counterparts* [10] (Section 3.7), and to *progressively approach the classical EPR determinism [1] in the structure of hadrons, nuclei and stars, while achieving the EPR determinism in the interior of gravitational collapse* [11] (Section 3.8).

A first understanding of this section requires a knowledge of the Lie-Santilli isotheory (Section I-2.7) [26] [30] [38] [46] [47] [49]. A technical understanding of this section requires a technical knowledge of hadronic mechanics [29]- [31].

3.2. Pauli matrices.

As it is well known (see, e.g., Ref. [72]), the carrier space of the two-dimensional group of special unitary transformations $SU(2)$ is the two-dimensional complex Euclidean space $E(z, \delta, I)$ with coordinates $z = (z_1, z_2)$, metric $\delta = Diag.(1, 1)$ and unit $I = Diag/(1, 1)$.

The two-dimensional, fundamental (also called adjoint), irreducible,

unitary representation of the special unitary Lie algebra $su(2)$ of the $SU(2)$-spin symmetry is given by the celebrated *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{59}$$

with commutations rules

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = i2\epsilon_{ijk}\sigma_k, \tag{60}$$

and eigenvalues on a Hilbert space $calH$ over the field of complex numbers $\mathcal{C}$ with basis $|b>$

$$\begin{gathered} \sigma^2|b>=(\sigma_1\sigma_1+\sigma_2\sigma_2+\sigma_3\sigma_3)|b>=3|b>, \\ \sigma_3|b>=\pm 1|b> . \end{gathered} \tag{61}$$

Among the various properties of Pauli's matrices, we should recall their uniqueness in the sense that their expression is invariant under the class of equivalence admitted by quantum mechanics, that under unitary transformation.

We should also recall that Pauli's matrices are also fundamental for the structure of Dirac's equation, Eq. (I-9) since they appear in the very definition of Dirac's gamma matrices, Eqs. (I-89).

3.3. Regular Pauli-Santilli isomatrices.

By following Ref. [58], the carrier isospace of the two-dimensional Lie-Santilli isogroup of isospecial isounitary isotransformations $\hat{SU}(2)$ is the isocomplex iso-Euclidean isospace $\hat{E}(\hat{z}, \hat{\Delta}, \hat{I})$ with isocoordinates

$$\hat{z} = z\hat{I} = (z_1, z_2) = (z_1, z_2)\hat{I}; \tag{62}$$

isounit and isotopic element

$$\hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix} = 1/\hat{T} > o, \tag{63}$$

$$\hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2}; \end{pmatrix} \tag{64}$$

isometric

$$\hat{\Delta} = \hat{\delta}\hat{I} = (\hat{T}_i^k\delta_{ki})\hat{I} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}\hat{I}; \tag{65}$$

positive-definite characteristic quantities n_k with unrestricted functional dependence on the variables for interior dynamical problems

$$n_k = n_k(z, \bar{z}, E, \mu, \alpha, \tau, \psi, \partial\psi, ...) > 0, \quad k = 1, 2; \tag{66}$$

and basic isoinvariant

$$\begin{gathered} \hat{z}_i \star \hat{\Delta}_{ij} \star \hat{\bar{z}}_j = (z_i \hat{\delta}_{ij} z_j)\hat{I} = \\ = (\tfrac{z_1\bar{z}_1}{n_1^2} + \tfrac{z_2\bar{z}_2}{n_2^2})\hat{I}. \end{gathered} \tag{67}$$

By also following Refs. [29] [58], the isogroup of regular, isospecial, isounitary, isotransformations $\hat{SU}(2)$ leaving invariant isoline element (67), is characterized by the isotransforms

$$\hat{z}' = \hat{U}(\hat{\theta}) \star z = \hat{U}(\hat{\theta})\hat{T}\hat{z}, \tag{68}$$

verifying the following conditions [30]:

1. Isounitarity

$$\hat{U}(\hat{\theta}) \star \hat{U}^{\dagger}(\hat{\theta}) = \hat{U}^{\dagger}(\hat{\theta}) \star \hat{U}(\hat{\theta}) = \hat{I}; \tag{69}$$

2. Isogroup isoaxioms

$$\begin{gathered} \hat{U}(\hat{\theta}_1) \star \hat{U}(\hat{\theta}_2) = \hat{U}(\hat{\theta}_1 + \hat{\theta}_2), \\ \hat{U}(\hat{\theta}) \star \hat{U}(-\hat{\theta}) = \hat{U}(0) = \hat{I}, \quad k = 1, 2, 3; \end{gathered} \tag{70}$$

and

3. Isospecial isounitarity

$$IsoDet\hat{U}(\hat{\theta}) = \hat{I}, \ Det(U\hat{T}) = 1. \tag{71}$$

The latter condition essentially restricts the isogroup $\hat{SU}(2)$ to its isoconnected component $\hat{SU}^0(2)$, which is hereon tacitly assumed.

The above conditions imply the local isomorphism

$$\hat{SU}(2) \approx SU(2), \tag{72}$$

and the following explicit realization in terms of isoexponential (I-22)

$$\begin{gathered} \hat{U}(\hat{\theta}) = \Pi_k U_k(\theta_k)\hat{I} = \Pi_k \hat{e}^{\hat{i}\star\hat{J}_k\star\hat{\theta}_k} = \Pi_k(e^{iJ_k\hat{T}\theta_k})\hat{I}, \\ U_k(\theta_k) = e^{iJ_k\hat{T}\theta_k} \quad , k = 1, 2, 3, \end{gathered} \tag{73}$$

where $\hat{J}_k$ represents the isogenerators of the Lie-Santilli isoalgebra $\hat{su}(2)$ verifying the conditions

$$IsoTr\hat{J}_k = 0, \;\; Tr(\hat{J}_k\hat{T}) = 0, \tag{74}$$

and the isocommutation rules

$$\begin{aligned}[\hat{J}_i\hat{,}\hat{J}_j] = \hat{J}_i \star \hat{J}_j - \hat{J}_j \star \hat{J}_i = \\ = \hat{J}_i\hat{T}\hat{J}_j - \hat{J}_j\hat{T}\hat{J}_i = \epsilon_{ijk}\hat{J}_k.\end{aligned} \tag{75}$$

Note that, in accordance with Theorem I-2.7.2, the isorepresentations here considered are called *regular* because they can be constructed via non-unitary transformations of the conventional $su(2)$ algebra, resulting in the preservation of the conventional structure *constants* ϵ_{ijk}.

However, as we shall see in the next section, the isotopies of the $su(2)$ algebra imply realizations called *irregular* that cannot be constructed via non-unitary representations of $su(2)$ [58], in which case the structure constants ϵ_{ijk} are replaced by *structure functions* with an arbitrary (non-singular) functional dependence on local variables,

$$\hat{C}_{ijk} = C_{ijk}(z, \bar{z}, E, \nu, \alpha, \tau, \psi, \partial\psi, ...)\hat{I}. \tag{76}$$

As one can verify, $\hat{su}(2)$ admits the following iso-Casimir isoinvariant

$$\begin{aligned}\hat{J}^{\hat{2}} = \Sigma_k \hat{J}_k \star \hat{J}_k = \\ = \hat{J}_1\hat{T}J_1 + \hat{J}_2\hat{T}J_2 + \hat{J}_3\hat{T}J_3.\end{aligned} \tag{77}$$

The maximal set of isocommuting isooperators is then given by $\hat{J}_3$ and $\hat{J}^{\hat{2}}$.

By again following Ref. [58], in order to compute the explicit form of the isorepresentations of $\hat{su}(2)$, we introduce the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$[19] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [20] with d-dimensional isobasis $|\hat{b}_k^d >$ verifying isonormalization (I-75),

$$\begin{aligned}< \hat{b}_k^d| \star |\hat{b}_k^d >=< \hat{b}_k^d|\hat{T}|\hat{b}_k^d >= \hat{I}, \\ d = 1, 2, 3, ...N, , \quad k = 1, 2, , 3.\end{aligned} \tag{78}$$

From the local isomorphism $\hat{su}(2) \approx su(2)$ we know that the isoeigenvalue equations have the structure

$$\begin{aligned}\hat{J}_k \star |\hat{b}_k^d >= b_k^d|\hat{b}_k^d >, \\ \hat{J}^{\hat{2}} \star |\hat{b}_k^d >= \Sigma_k b_k^d(b_k^d + W)|\hat{b}_k^d > \;\; , \\ W = Det\hat{T} = 1/n_1^2 n_2^2,\end{aligned} \tag{79}$$

where $W = 1$ for regular isorepresentation, otherwise W is an arbitrary function of local quantities to be identified via subsidiary constraints from the medium in which extended particles are immersed.

The explicit form of the isorepresentations of $\hat{su}(2)$ is then given by the simple isotopy of the conventional case [72]

$$\begin{aligned} \hat{J}_{\pm} &= \hat{J}_1 \pm \hat{J}_2, \\ (\hat{J}_1)_{ij} &= i\tfrac{1}{2} < \hat{b}_k^d| \star (\hat{J}_- - \hat{J}_+) \star |\hat{b}_k^d >, \\ (\hat{J}_2)_{ij} &= i\tfrac{1}{2} < \hat{b}_k^d| \star (\hat{J}_- + \hat{J}_+) \star |\hat{b}_k^d >, \\ (J_3)_{ij} &=< \hat{b}_k^d| \star \hat{J}_3 \star |\hat{b}_k^d > . \end{aligned} \tag{80}$$

By continuing to follow Ref. [58], we now restrict our attention to the two-dimensional isofundamental (isoadjoint) isorepresentation of $\hat{su}(2)$ occurring for $d = 2$, in which case we assume

$$\hat{J}_k = \frac{1}{2}\hat{\sigma}_k, \quad k = 1, 2, 3, \tag{81}$$

and select the basic isounitary isotransform according to Sections I-2.8 and I-2.9

$$UU^{\dagger} = f(W) > 0, \;\; W = Det.\hat{I} = n_1^2 n_2^2, \tag{82}$$

where $f(W)$ is a smooth function such that $f(1) = 1$.

By using the above procedure, we have the following *regular Pauli-Santilli isomatrices* first introduced by Santilli in Ref. [58], Eqs. (3.2) (where the isometric elements are denoted $g_{kk} = n_k^{-2}, \; k - 1, 2,$

$$\begin{aligned} \hat{\Sigma}_k &= \hat{\sigma}_k \hat{I}, \\ \hat{\sigma}_1 = (n_1 n_2) \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 &= (n_1 n_2) \begin{pmatrix} 0 & -\dot{\imath} n_1^{-2} \\ \dot{\imath} n_2^{-2} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 &= (n_1 n_2) \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2}, \end{pmatrix}. \end{aligned} \tag{83}$$

with isocommutation rules

$$[\hat{\sigma}_i, \hat{\sigma}_j] = i2\epsilon_{ijk}\hat{\sigma}_k, \tag{84}$$

in which the 'regular' character of the isomatrices is established by the presence of the conventional (constant) structure constants.

We then have the isoeigenvalues isoequations

$$\hat{\sigma}_3 \star |\hat{b}_m^2 >= \hat{\sigma}_3 \hat{T} |\hat{b}_m^2 >= \pm \tfrac{1}{n_1 n_2} |\hat{b}_m^2 >$$

$$\hat{\sigma}^{\hat{2}} = (\sigma_1 \hat{T} \hat{\sigma}_1 + \sigma_2 \hat{T} \hat{\sigma}_2^+ \sigma_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}_m^2 >= \tag{85}$$

$$= 3 \tfrac{1}{n_1^2 n_2^2} |\hat{b}_m^2 >,$$

showing that the regular Pauli-Santilli isomatrices preserve the conventional structure constants ϵ_{ijk} of Pauli matrices, but exhibit structure (84) with generalized isoeigenvalues containing two characteristic quantities n_1^2, n_2^2.

It is evident that, under isounimodularity condition (71),

$$Det\hat{T} = 1, \;\; n_1 = 1/n_2, \tag{86}$$

isomatrices (83) reduce to

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -\dot{\imath} n_1^{-2} \\ \dot{\imath} n_2^{-2} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{87}$$

by verifying conventional commutation rules (84) and conventional eigenvalues

$$\hat{\sigma}_3 \star |\hat{b}_m^2 >= \pm |\hat{b}_m^2 >$$

$$\hat{\sigma}^{\hat{2}} \star |\hat{b}_m^2 >= 3 |\hat{b}_m^2 > . \tag{88}$$

In order to search for additional realizations of regular Pauli-Santilli isomatrices, we now assume the following non-unitary transform

$$U = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} = U^\dagger, \tag{89}$$

under which we have the following second realization of regular Pauli-Santilli isomatrices

$$\hat{\sigma}_k = U \sigma_k U^\dagger,$$

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -\dot{\imath} n_1 n_2 \\ \dot{\imath} n_1 n_2 & 0 \end{pmatrix}, \tag{90}$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}.$$

It is an instructive exercise for the interested reader to verify that the above isomatrices verify the isocommutation rules (84) and conventional isoeigenvalue (99), namely, *the second realization of the Pauli-Santilli isomatrices, Eqs. (83), also admit conventional structure constants and eigenvalues despite the degrees of freedom permitted by the two characteristic quantities* $n_1^2,\ n_2^2$.

We now assume the following non-diagonal realization of the non-unitary transform

$$U = \begin{pmatrix} 0 & n_1 \\ \\ n_2 & 0 \end{pmatrix}, U^\dagger = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \tag{91}$$

$$UU^\dagger = \hat{I} = 1/\hat{T} > 0,$$

which characterizes the following third realization of the regular Pauli-Santilli isomatrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix},\ \hat{\sigma}_2 = \begin{pmatrix} 0 & -\dot{m} n_1 n_2 \\ \dot{m} n_1 n_2 & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} -n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}. \tag{92}$$

It is easy to see that the above third realization of the regular Pauli-Santilli isomatrices also verify conventional commutation rules (84) and eigenvalues (88).

Note that, while Pauli's matrices are invariant under unitary transforms, there exist a number of Pauli-Santilli isomatrices each of which is invariant under isounitary isotransforms (Section I-3.9).

3.4. Irregular Pauli-Santilli isomatrices.

3.4.1. Historical notes. One of the most fundamental, yet unresolved processes in nature is the synthesis of the neutron from the hydrogen in the core of stars, which is a pre-requisite for the production of light and, therefore, for the existence of life itself.

In this section, we would like to outline the main historical aspects on the synthesis of the neutron and identify the open problems because truly fundamental for the construction of the new mathematics needed for their solution.

Recall that stars initiate their life as an aggregate of hydrogen and grow via the accretion of hydrogen existing in interstellar spaces.

In 1910, H. Rutherford [73] conjectured that, when the pressure and temperature at the core of the star reaches certain values, the hydrogen

atom is "compressed" into a neutral particle n which is called the *neutron* according to the reaction (see Section I-4)

$$e^- + p^+ \ \rightarrow \ n. \tag{93}$$

Rutherford hypothesis was experimentally confirmed by J. Chadwick in 1932 [74], and the neutron became part of scientific history.

Following the experimental verification that the neutron has the same spin $1/2$ of the electron and of the proton, in an attempt at maintaining the conservation of the angular momentum, E. Fermi [75] suggested that the synthesis of the neutron occurs with the emission of a hypothetical, massless and chargeless particle ν with spin $1/2$ which he called the *neutrino* (meaning "little neutron" in Italian), according to the reaction widely accepted by the scientific communioty for about one century

$$e^- + p^+ \ \rightarrow \ n + \nu. \tag{94}$$

After joining Harvard University in September 1977 under DOE support, R. M. Santilli [15]-[17] noted that, despite the salvaging of space-time symmetries and related conservation laws, reaction (93) is not compatible with quantum mechanical laws because the rest energy of the neutron E_n is $0.782 \ MeV$ *bigger* than the sum of the rest energies of the proton E_p and of the electron E_e,

$$\begin{aligned} E_p = 938.272 \ MeV, \ \ E_e = 0.511 \ MeV, \ \ E_n = 939.565 \ MeV, \\ E_n - (E_p + E_e) = 0.782 \ MeV > 0. \end{aligned} \tag{95}$$

Therefore, Santilli presented a number of arguments according to which the synthesis of the neutron is clear evidence of Einstein's view on the lack of "completion" of quantum mechanics (see Einstein's name in the title of the 1981 paper [17] released from the Department of Mathematics of Harvard University). Subsequently, Santilli achieved in Ref. [76] (see the independent review [**?**] the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen representation, with a relativistic representation subsequently achieved in Ref. [62] (see the independent review in ref. [45]).

The technically most difficult problem of the above representation was the identification of the spin-orbit coupling for the electron when totally immersed inside the proton which was first solved non-relativistically in Ref. [76] and relativistically in Ref. [62] (see the review in Ref. [30], Chapter 6 in particular).

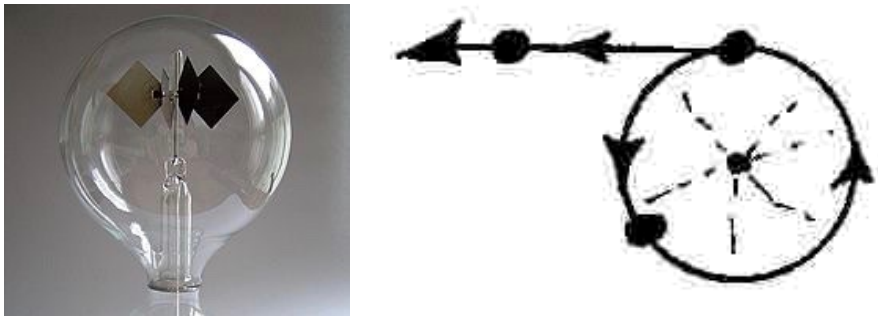

Figure 9: *To illustrate some of isosymmetries for interior systems, in the left we show the conversion of linear momentum into angular momentum for the case of photons causing the rotation of a small propeller in a vacuum chamber. In the right, we show the opposite conversion of angular momentum into linear momentum, as it is the case for the sling shot. The left view illustrates that the neutrino hypothesis is not necessary for the synthesis of the neutron when particles are represented as being extended. The right view illustrates that the emission of an antineutrino is not necessary in the neutron decay because the internal angular momentum of the electron can be converted into its external linear momentum without any violation of physical laws.*

Recall that quantum mechanics has an excellent consistency for bound states with *negative potentials* causing a *mass defect.* Santilli's first argument is that a representation of experimental data (95) via quantum mechanics is impossible because it would require a *positive potential* capable of producing a *mass excess,* which features imply the loss of physical consistency of Schrödinger equation *for bound states* (and not for free particles with positive kinetic energy) because the indicial equation of Schrödinger equation admits no consistent solutions for positive potential energies, (see Section I-4).

The inability of Dirac's and other quantum mechanical equations for the representation of experimental data (95) then followed.

Various conjectures, aimed at maintaining for the neutron structure the theory so effective for the hydrogen atom, were proved not to be consistent. For instance, the hypothesis that the missing energy of $0.782\ MeV$ is provided by the star via a relative energy between the electron and the proton had to be dismissed because the cross section $e-p$ at $0.782\ MeV$ is essentially null, thus preventing any fusion between the electron and the proton.

Similarly, the hypothesis that the missing energy is provided by the

antineutrino $\bar{\nu}$ via reactions of the type

$$e^- + \bar{\nu} + p^+ \ \rightarrow \ n, \tag{96}$$

had to be equally abandoned because the cross section between neutrinos or antineutrinos and individual particles is identically null.

As it is well known, the neutrino hypothesis is necessary for the quantum mechanical treatment of synthesis (94), namely, for the point-like characterization of the proton, the electron and the neutron. Ref. [17] indicated that the neutrino hypothesis cannot any longer be consistently applied for the neutron synthesis whenever particles are represented as being extended.

Independently from that, there exist no known conventional possibility of identifying the *energy* needed for the creation of the neutrino since synthesis (94) already misses $0.782 \ MeV$ for the synthesis of the neutron.

Another argument of Ref. [17] is that the conservation of the angular momentum is necessary for the synthesis of bound states with a Keplerian center under the validity of conventional space-time symmetries, such as for the synthesis of the hydrogen atom form an electron and a proton.

However, said conservation is no longer necessary for bound states at short distances without a Keplerian center, since in that case we have the validity of space-time isosymmetries for which the angular momentum can be transformed into linear momentum and vice-versa without any violation of physical laws (see Section 2.2 and Figure 9).

But *the neutron has no Keplerian center*, with the consequential lack of applicability of the Lorentz-Poincaré symmetry, and the ensuing lack of necessary conservation of the angular momentum in favor of alternative hypotheses.

In view of the above (and other) insufficiencies of the neutrino hypothesis, Santilli suggested in Ref. [78] the introduction in the *l.h.s.* (rather than the r.h.s.) of the synthesis of the mass-less, charge-less and spin-less particle called the *etherino* and denoted with the symbol a (from the Latin *aether*)

$$p^+ + a + e^- \ \rightarrow \ n, \tag{97}$$

whose scope is to represent the delivery of the missing $0.782 \ MeV$ to the neutron.

An intriguing aspect is that *the etherino hypothesis can be shown to be compatible with the experimental data of the so-called "neutrino experiments," of course, under the condition of abandoning point-like abstractions of hadrons and representing them as they are in the physical reality, i.e., extended, deformable and hyperdense.*

By recalling that there is no known consistent way of accounting for the missing $0.782\ MeV$ as originating from the star, Ref. [**?**] submitted the hypothesis that the missing energy may originate from the ether as a universal medium of extremely high energy density for the characterization and propagation of particles and electromagnetic waves.

However, it should be stressed that *the etherino is not intended to be a particle, but to be an "impulse" representing the mechanism of supplying the missing* $0.782\ MeV$ *for the neutron, the origin of the missing energy from the ether being only one among other possibilities.*

In view of the above insufficiencies of quantum mechanics for the synthesis of the neutron, Santilli initiated in Refs. [15] - [17] the search for a "completion" of quantum mechanics into hadronic mechanics with particular reference to the "completion"of Lie's theory at large, and the $SU(2)$-spin symmetry in particular, for the characterization of the spin of the electron when "compressed" inside the hyperdense proton.

It should also be recalled that, during the same period, Santilli conducted a post Ph. D. Seminar Course at the Lyman Laboratory of Physics of Harvard University with a technical treatment of the insufficiency of quantum mechanics for the neutron synthesis via the *conditions of variational self-adjointness* for the existence of a Lagrangian or a Hamiltonian. This Seminar Course was eventually published by Springer-Verlag in monographs [25] [26] whose primary aim is the first known presentation of the axiom-preserving Lie-Santilli isotheory and the axiom-inducing Lie-Santilli genotheory.

In fact, possible representations of experimental data (95) for the neutron synthesis violate the conditions of variational self-adjointness, thus mandating the search for a covering theory.

The subsequent 1995 papers [55] [56] [58] achieved the regular isotopies $\hat{SU}(2)$ of the spin symmetry (reviewed in the preceding section).

However, *regular isotopies of the* $SU(2)$ *spin symmetry are insufficient for the neutron synthesis because it requires alterations (called mutations) of conventional eigenvalues that can be solely represented via irregular isorepresentations.*

The irregular isorepresentations of the $SU(2)$-spin symmetry were identified, apparently for the first time, by Santilli in the 1990 paper [78] and used to achieve the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen.

In the 1995 paper [62], Santilli presented a relativistic study of $\hat{SU}(2)$ as an isosubalgebra of the irregular isospinorial covering of the Lorentz-Poincaré symmetry (Section 2.5.11) and used the results to achieve a relativistic representation of the neutron synthesis.

The indicated irregular representations of the $SU(2)$-spin symmetry were then instrumental for the apparent confirmations of the EPR argument in Refs.[10] [11] reviewed in this section.

3.4.2. Non-relativistic formulation. The first irregular isotopies of Pauli's matrices, today known as *irregular Pauli-Santilli isomatrices* [45], have been introduced in Eqs. (2.32) of Ref. [78] via the use of the isorepresentations of $\hat{SU}(2)$ worked out in the preceding papers [55] [56], and are given by

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & -n_1 \\ n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_1 \\ in_2 & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \frac{1}{n)1n_2} \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}, \tag{98}$$

with irregular isocommutation rules

$$[\hat{\sigma}_i,\hat{\sigma}_j] = 2i\frac{1}{n_1 n_2}\epsilon_{ijk}\hat{\sigma}_k, \ \ i,j,k,= 1,2,3 \tag{99}$$

and isoeigenvalues

$$\hat{\sigma}_3 \star |\hat{u}>= \pm\tfrac{1}{n_1 n_2}|\hat{u}>,$$
$$\hat{\sigma}^{\hat{2}} \star |\hat{u}>= \tfrac{1}{n_1 n_2}(\tfrac{1}{n_1 n_2}+2)|\hat{u}> . \tag{100}$$

It is easy to see that, when the hyperdense medium surrounding the immersed particle is homogeneous and isotropic, the characteristics quantities can be normalized to the values $n_1 = n_2 = n_3 = 1$, in which case isoeigenvalues (100) are conventional. We therefore have the following

LEMMA 3.1: Irregular isorepresentations of the Lie-Santilli isosymmetry $\hat{su}(2)$ represent the inhomogeneity and anisotropy of media in which extended particles are immersed.

Among a number of additional irregular Pauli-Santilli isomatrices with isotopic element $\hat{T}$ in Eq. (64) we quote Eqs. (3.2) of Ref. [10]

$$\hat{\sigma}_1 = n_1 n_2 \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \ \hat{\sigma}_2 = n_1 n_2 \begin{pmatrix} 0 & -in_1^{-2} \\ in_2^{-2} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = n_1 n_2 \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{101}$$

with irregular isocommutation rules

$$[\hat{\sigma}_i,\hat{\sigma}_j] = 2i\frac{1}{n_1 n_2}\hat{\sigma}_k, \tag{102}$$

and isoeigenvalues

$$\begin{aligned} \hat{\sigma}_3 \star |\hat{u}> &= \pm\tfrac{1}{n_1 n_2}|\hat{u}>, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u}> &= 3\tfrac{1}{n_1^2 n_2^2}|\hat{u}>, \end{aligned} \tag{103}$$

The above isorepresentation appears to be significant when the medium causes a proportional alteration/mutation of both the third component as well as the total value of the spin of a particle having the value $1/2$ in vacuum.

Another example of irregular Pauli-Santilli isomatrices is given by Eqs. (3.7) of Ref. [10]

$$\begin{aligned} \hat{\sigma}_1 = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_2 \\ in_1 & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & \\ 0 & -n_2^2 \end{pmatrix}, \end{aligned} \tag{104}$$

with irregular isocommutation rules

$$\begin{aligned} [\hat{\sigma}_1,\hat{\sigma}_2] = 2i\tfrac{1}{n_1^2 n_2^2}\hat{\sigma}_3, \quad [\hat{\sigma}_2,\hat{\sigma}_3] = 2i\hat{\sigma}_1, \\ [\hat{\sigma}_3,\hat{\sigma}_1] = 2i\hat{\sigma}_2, \end{aligned} \tag{105}$$

and mutated isoeigenvalues

$$\begin{aligned} \hat{\sigma}_3 \star |\hat{u}> &= \pm|\hat{u}>, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u}> &= \tfrac{2}{n_1^2 n_2^2}|\hat{u}> . \end{aligned} \tag{106}$$

The above isorepresentation may be useful when the anisotropy and inhomogeneity of the medium maintain the spin value $1/2$ along the third axis, yet they are such to deform the remaining components.

Additional example of irregular Pauli-Santilli isomatrices are available from Refs. [10] and [58], and can be readily constructed by interested readers.

3.4.3. Relativistic formulation. Consider the iso-Minkowskian isospace $\hat{M}(\hat{x}, \bar{\Omega}, \hat{I})$ with isometric

$$\hat{\Omega} = \hat{\eta}\hat{I}, \;\; \bar{\eta} = \hat{T}\eta,$$

$$\hat{T} = Diag.(\tfrac{1}{m_1^2}, \tfrac{1}{m_2^2}, \tfrac{1}{m_3^2}, \tfrac{1}{m_4^2}), \tag{107}$$

$$m_\mu = m_\mu(r, p, E, \nu, \alpha, \tau, \pi, \psi, ...) > 0. \;\; \mu = 1, 2, 3, 4,$$

where the new characteristic quantities m_μ have been introduced to avoid confusion with the previously used symbols n_μ.

The relativistic formulation of the irregular isorepresentation of $\hat{SU}(2)$ were derived, apparently for the first time, in Eqs. (6.4c)-(6.4d) of Ref. [62], and can be written

$$J_k = \frac{1}{2}\epsilon_{kij}\hat{\gamma}_i \star \gamma_j, \tag{108}$$

where $\hat{\gamma}$ are the regular Dirac-Santilli isomatrices (I-89), i.e.,

$$\begin{aligned} \hat{\gamma}_k &= \tfrac{1}{m_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 &= \tfrac{i}{m_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \end{aligned} \tag{109}$$

and $\hat{\sigma}_k$ are the *regular Pauli-Santilli isomatrices.*

The irregular character of isorepresentation (108) is established by the presence of structure functions in the isocommutation rules, Eqs, (6.4c) of Ref. [62],

$$[J_i\hat{,}J_j] = \epsilon_{ijk}\frac{1}{m_k^2}J_k, \tag{110}$$

and in the irregular isoeigenvalues

$$\begin{aligned} J_3 \star |\hat{\psi}> &= \pm\tfrac{1}{2}\tfrac{1}{m_1 m_2}|\hat{\psi}>, \\ J^{\hat{2}} \star |\hat{\psi}> &= \tfrac{1}{4}(\tfrac{1}{m_1 m_2} + \tfrac{1}{m_2 m_3} + \tfrac{1}{m_3 m_1})|\hat{\psi}>, \end{aligned} \tag{111}$$

that, as shown in Ref. [62], permit a relativistic representation of the spin of the neutron in its synthesis from the hydrogen.

Again one should note that, when the medium is homogeneous and isotropic, isoeigenvalues (101) are conventional.

Note that the assumption of mutated spin for an extended particle within a hyperdense medium implies the *inapplicability (rather than the violation) of the Fermi-Dirac statistics, Pauli's exclusion principle and other quantum mechanical laws* with the understanding that said mutations are internal, thus solely testable under *external* strong interactions, as indicated beginning with the *title* of Harvard's 1978 paper [15].

3.5. Isotopies of hadronic spin and angular momentum.

3.5.1. Historical notes. An electron orbiting in vacuum around the proton in the hydrogen atom experiences no resistive forces, thus verifying known symmetries and conservation laws.

When the same electron has been "compressed" inside the proton according to Rutherford [73], Santilli [78] argued that the sole possible angular moment is that permitted by *constraints* exercised on the electron by the internal medium.

Since the electron is about $2,000$ times lighter than the proton, *the most stable configuration is that for which the electron is "constrained" to orbit with a value of the angular momentum equal to the proton spin*, since any different configuration would imply big resistive forces (Figure 9).

Needless to say, fractional angular momenta are anathema for the quantum mechanical description of point-particles in vacuum.

However, the angular momentum of extended particles immersed within hyperdense hadronic media can acquire values other than integers, depending on the local physical conditions of the medium surrounding the particle, such as pressure, density, anisotropy, inhomogeneity, etc.

The first known quantitative study of *constrained angular momenta* of extended particles within hyperdense hadronic media was done at the non-relativistic level by Santilli in Ref. [78] of 1990 following the preceding isotopies of the rotational symmetry, Refs. [55] [56]. The study was then extended to the relativistic level in Ref. [62] of 1990.

These studies are crucial for quantitative representations of the synthesis of hadrons providing apparent verifications of the EPR argument, and can be summarized as follows.

3.5.2. Non-relativistic representation. Recall the central assumption of isosymmetries according to which conventional generators are preserved (because representing conventional total conservation laws), and only their product is lifted into the isotopic form (1) (to represent the extended character of the particles and their non-Hamiltonian interactions).

Hence, the definition of the *isoangular isomomentum*, also called *hadronic angular momentum*, on an iso-Euclidean isospace is the same as that of

quantum mechanics

$$L_k = \epsilon_{ijk}\hat{r}_i \star \hat{p}_j = \epsilon_{ijk} r_i p_j, \tag{112}$$

although it is defined on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $|\hat{\psi} >$ on an isocomplex isofield $\hat{\mathcal{C}}$, with *isolinear isomomentum* Eqs, (I-79), and isocommutation rules are then given by Eqs. (I-81).

It is then easy to verify the following isocommutation rules for the hadronic angular isomomentum, Eqs. (2.22b) [78]

$$[L_i \hat{,} L_j] = i\hat{I}\epsilon_{ijk} L_k, \tag{113}$$

where, as one can see, the characteristics of the medium, represented by the isounit $\hat{I}$, enter directly in the isocommutation rules.

The use of the *isosperical isoharmonic isofunctions* (see page 240 of Ref. [30] for details)

$$\begin{aligned} \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = UY(\theta, \phi)U^\dagger = \hat{T}^{-1} Y_{\ell m}(\theta, \phi), \\ UU^\dagger = \hat{I} = 1/\hat{T} \neq I, \end{aligned} \tag{114}$$

where $Y_{\ell m}(\theta, \phi$ are the conventional spherical harmonic functions, yields the following isoeigenvalues, Eqs. (2.25), Ref. [78],

$$\begin{aligned} L_3 \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I} m \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ L^{\hat{2}} \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I}\ell(\hat{I}\ell + 1)\hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ m = \ell, \ell - 1, ..., -\ell, \;\; m = 1, 2, 3, ... \end{aligned} \tag{115}$$

where one can see again the mutation of the eigenvalues caused by the surrounding medium.

Applications to particle physics then require specific realizations of the isounit $\hat{I}$, such as the simple assumption of expressions (4) used in Ref. [78]

$$\rho = |\hat{I}| = \approx |e^\gamma|, \tag{116}$$

where ρ is a function of all possible or otherwise needed local variables of the medium.

3.5.3. Isotopies of non-relativistic spin-orbit coupling. As one can see, isoeigenvalues (115) do not allow a representation of the constrained hadronic angular momentum of the electron when compressed inside the proton (Figure 9).

In view of this insufficiency, Santilli conducted in Ref. [76] (see Also Ref. [30], Chapter 6) a study of the eigenvalues of the combined spin and angular momentum of the electron in the indicated interior conditions.

We consider then the *total hadronic momentum*

$$J_{tot} = L_\ell \hat{\otimes} J_s, \tag{117}$$

with corresponding basis $|\hat{Y}\hat{\otimes}\hat{u}>$ and isoexpectation values, Eqs. (2.34), Ref. [78],

$$\begin{gathered} J_{3,tot}|\hat{Y}\hat{\otimes}\hat{u}>=(\rho_m(\ell)\pm\tfrac{m(s)}{n_1 n_2})|\hat{Y}\hat{\otimes}\hat{u}> \\ J^{\hat{2}}_{tot}\star|\hat{Y}\hat{\otimes}\hat{u}>=(\rho\ell\pm\tfrac{s}{n_1 n_2})(\rho\ell\pm\tfrac{s}{n_1 n_2}+1)|\hat{Y}\hat{\otimes}\hat{u}> \\ \ell=0,1,2,3,...\ \ s=0,\tfrac{1}{2}1,\tfrac{3}{2},..., \\ m(\ell)=\ell,\ell-1,...,-\ell,\ \ m(s)=s,s-1,...,-s. \end{gathered} \tag{118}$$

Following a laborious journey initiated in 1977, isoeigenvalues (118) finally permitted Santilli to achieve the desired solution for $\ell = 1$ and $s = \frac{1}{2}$, Eq. (2.36), Ref. [78],

$$\rho = \frac{1}{2}\frac{1}{n_1 n_2}, \tag{119}$$

for which *the total hadronic angular momentum of the electron in the synthesis of the neutron is identically null, $J_{tot} = 0$, and the spin of the neutron coincides with that of the proton.*

More detailed studies pertaining to electric and magnetic dipoles excluded the alternative $J = 1$ of eigenvalues (118), as well as total hadronic angular momenta of the electron other than zero.

The preceding studies permitted a quantitative non-relativistic representation of the spin of the neutron in its synthesis from the hydrogen atom. A representation of the remaining characteristics of the neutron (mass, radius, charge, dipole moments, etc.) is reviewed in Section 4.5.

3.5.4. Isotopies of relativistic spin-orbit couplings. The hadronic spin $\hat{S} = S\hat{I}$ is a realization of the $\hat{SU}(2)$ isosubalgebra of $\hat{\mathcal{P}}(3.1)$ with generators (57), while the hadronic angular momentum $\hat{L} = L\hat{I}$ is a realization of the isorotational $\hat{SO}(3)$ isosubalgebra. Their relativistic formulation on iso-Minkowskian isospace (107) has been first derived in Eqs. (6.4a) (6.4b), Ref. [62] and are given by

$$\begin{aligned} S_k &= 2\epsilon_{kij}\hat{\gamma}_i \star \hat{\gamma}_j, \\ L_k &= \epsilon_{kij} r_i \star p_j, \end{aligned} \tag{120}$$

where $\hat{\gamma}_k$ are the Dirac-Santilli isomatrices.

We then have the irregularisocommutation rules

$$\begin{aligned} [S_i\hat{,}S_j] &= \epsilon_{kij} m_k^2 \hat{S}_k, \\ [L_i\hat{,}L_j] &= \epsilon_{ijk} m_k^2 L_k, \end{aligned} \tag{121}$$

and isoeigenvalues, Eqs, (6.4d) Ref. [62]

$$\begin{aligned} \hat{S}_3 \star |\hat{\psi}> &= \pm \tfrac{1}{m_1 m_2} |\hat{\psi}>, \\ \hat{S}^{\hat{2}} \star |\hat{\psi}> &= (m_1^{-2} m_2^{-2} + m_2^{-2} m_3^{-2} + m_3^{-2} m_1^{-2}) |\hat{\psi}>, \\ \hat{L}_3 \star |\hat{\psi}> &= \pm m_1 m_2 |\hat{\psi}>, \\ \hat{L}^{\hat{2}} \star |\hat{\psi}> &= (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) |\hat{\psi}> . \end{aligned} \tag{122}$$

The most salient difference between relativistic isoeigenvalues (122) and their non-relativistic counterparts (155) is that *the former admit fractional hadronic angular momenta while the latter do not.*

In fact, for the following values admitted by a homogeneous and isotropic medium [62]

$$m_1 = m_2 = m_3 = \frac{1}{\sqrt{2}}, \tag{123}$$

isoeigenvalues (122) become

$$\begin{aligned} \hat{S}_3 \star |\hat{\psi}> &= \pm \tfrac{1}{2} |\hat{\psi}>, \\ \hat{S}^{\hat{2}} \star |\hat{\psi}> &= \tfrac{3}{4} |\hat{\psi}>, \\ \hat{L}_3 \star |\hat{\psi}> &= \pm \tfrac{1}{2} |\hat{\psi}>, \\ \hat{L}^{\hat{2}} \star |\hat{\psi}> &= \tfrac{3}{4} |\hat{\psi}> . \end{aligned} \tag{124}$$

Consequently, *isoeigenvalues (122) permit a quantitative representation of the hadronic angular momentum of the electron as being constrained to be equal to the proton spin* [61] [62] (Figure zzzz).

In this case too, the total hadronic angular momentum of the electron is null because the only stable hadronic spin-orbit coupling is in singlet, and the spin of the electron can be assumed in first good approximation not to be mutated since the electron is about $2,000$-times lighter than the proton.

Hadronic spins, hadronic angular momenta and hadronic spin-orbit couplings were studied in detail Chapter 6 of Ref. [30] resulting in Lemma 6.12.1 here reproduced without proof:

LEMMA 3.2: When immersed within hadrons or nuclei with spin $1/2$*, an elementary particle having spin* $1/2$ *in vacuum can only have a null total hadronic angular momentum.*

As we shall see in Section 4.6, the above configuration of the synthesis of the neutron from the hydrogen is an apparent verification of the EPR argument.

3.6. Realization of hidden variables.

As recalled in Section 1.1, the conventional quantum mechanical realization of the Lie symmetry $SU(2)$ does not allow a consistent representation of hidden variables λ [3] [4].

It is easy to see that, despite the local isomorphism $\hat{SU}(2) \approx SU(2)$, the Lie-Santilli isosymmetry $\hat{SU}(2)$ does indeed allow explicit and concrete realizations of hidden variables thanks to the degree of freedom permitted by the isotopic element (1) in the structure of the Lie-Santilli isoproduct (2) with realizations of the isotopic element of type (3).

In this section, we review the explicit and concrete realization of *regular hidden variables,* namely, realizations that can be derived via non-unitary transforms of the Lie algebra $su(2)$, and then review *irregular hidden variables,* namely, realizations that do not admit such a simple derivation.

Regular and irregular realizations of hidden variables have been first identified by Santilli in Ref. [58] of 1993, and then used for the proof of the EPR argument [10] reviewed in Section 3.7.

Realizations of regular hidden variables are easily provided by Pauli-Santilli isomatrices (83) with the identifications

$$n_1^2 = \lambda_1, \;\; n_2^2 = \lambda_2, \tag{125}$$

yielding the desired realization, Eqs. (3.9), Ref. [58],

$$\begin{aligned} \hat{\sigma}_1 = (\lambda_1\lambda_2) \begin{pmatrix} 0 & \lambda_1^{-1} \\ \lambda_2^{-1} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = (\lambda_1\lambda_2) \begin{pmatrix} 0 & -i\lambda_1^{-1} \\ i\lambda_2^{-1} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = (\lambda_1\lambda_2) \begin{pmatrix} \lambda_2^{-1} & 0 \\ 0 & -\lambda_1^{-1} \end{pmatrix} \end{aligned} \tag{126}$$

verifying isocommutation rules

$$[\hat{\sigma}_i,\hat{\sigma}_j] = i\epsilon_{ijk}\hat{\sigma}_k, \tag{127}$$

and isoeigenvalue isoequations

$$\hat{\sigma}_3 \star |\hat{b}>= \pm(\lambda_1\lambda_2)|\hat{b}>$$
$$\hat{\sigma}^{\hat{2}} = 3(\lambda_1\lambda_2)^2|\hat{b}> . \tag{128}$$

We consider now the particular case of Eq. (3), i.e.,

$$Det.\hat{T} = 1, \;\; n_1^2 = 1/n_2^2 = \lambda, \tag{129}$$

derivable via the basic non-unitary transformation

$$\hat{T} = (UU^\dagger)^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{130}$$

In this case, isomatrices (83) become (Eqs. (3.9) of [58])

$$\hat{\sigma}_1(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \;\; \hat{\sigma}_2(\lambda) = \begin{pmatrix} 0 & -i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix},$$
$$\hat{\sigma}_3(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}. \tag{131}$$

It is an instructive exercise for the interested reader to verify that the above realization of the regular Pauli-Santilli isomatrices verifies isocommutation rules with the same stricture constants of the $SU(2)$ algebra

$$[\hat{\sigma}_i(\lambda)\hat{,}\hat{\sigma}_j(\lambda)] = i2\epsilon_{ijk}\hat{\sigma}_k(\lambda), \tag{132}$$

and admit conventional eigenvalues

$$\hat{\sigma}_3(\lambda) \star |\hat{b}>= \pm|\hat{b}>$$
$$\hat{\sigma}(\lambda)^{\hat{2}} = 3|\hat{b}> . \tag{133}$$

Consequently, we have the following property [58]:

LEMMA 3.3. Regular Pauli-Santilli isomatrices provide an explicit and concrete realization of regular hidden variables directly in the spin $1/2$ *algebra.*

Note that, besides being positive-definite, hidden variables have an unrestricted functional dependence on all needed local variables, Eqs. (66).

An example of irregular hidden variables is provided by the $\hat{SU}(2)$ component of the spinorial covering of the Lorentz-Poincaré-Santilli isosymmetry.

To illustrate this realization, introduce *three* additional hidden variables for the characterization of isospace (107)

$$m_\mu = \lambda_\mu, \;\; \mu = 1, 2, 3, 4. \tag{134}$$

Realization (108) then implies the following irregular Dirac-Santilli isomatrices

$$\begin{aligned} \hat{\gamma}_k(\lambda) &= \tfrac{1}{\gamma_k} \begin{pmatrix} 0 & \hat{\sigma}_k(\lambda) \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 &= \tfrac{i}{m_4} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \end{aligned} \tag{135}$$

where $\hat{\sigma}_k$ are the *regular or irregular Pauli-Santilli isomatrices,* with isocommutation rules

$$[S_i(\lambda)\hat{,}S_j(\lambda)] = \epsilon_{ijk}\frac{1}{\lambda_k}S_k, \tag{136}$$

and isoeigenvalues

$$\begin{aligned} S_3 \star |\hat{\psi}> &= \pm\tfrac{1}{2}\tfrac{1}{\sqrt{\lambda_1\lambda_2}}|\hat{\psi}>, \\ S^{\hat{2}} \star |\hat{\psi}> &= \tfrac{1}{4}(\tfrac{1}{\sqrt{\lambda_1\lambda_2}} + \tfrac{1}{\sqrt{\lambda_2\lambda_3}} + \tfrac{1}{\sqrt{\lambda_3\lambda_1}})|\hat{\psi}> . \end{aligned} \tag{137}$$

Consequently, we have the following property [62]

LEMMA 3.4: The axioms of Dirac's equation admit up to five generally different, regular or irregular hidden variables.

Additional realizations of irregular hidden variables can be found in Eqs. (3.11) of Ref. [58] or can be easily derived from the preceding realization of the Pauli-Santilli isomatrices.

3.7. Apparent admission of classical counterparts.

As it is well known, Bell's inequality [3] [4], von Neumann's theorem [5], and the theory of local realism at large (see review [6] with a comprehensive literature) are generally assumed to be evidence of the impossibility of "completing" quantum mechanics into a broader theory, with ensuing rejection of the EPR argument [1].

Following decades of preparatory works reviewed in Paper I [9] and in the preceding sections of this paper, Santilli proved in Ref. [10] of 1998 (see also the detailed study in Ref. [30], particularly Chapter 4 and Appendix 4C, page 166) that:

1) Bell's inequality, von Neumann's theorem and related studies are indeed valid, but under the *tacit* assumption of representing particles as being point-like, with ensuing sole admission of linear, local and potential interactions (exterior dynamical problems).

2) Bell's inequality, von Neumann's theorem and related studies are *inapplicable* (rather than being violated) for extended particles within physical media, due to the presence of additional non-liner, non-local and non-potential interactions (interior dynamical systems).

3) The latter systems represented with the axiom-preserving "completion" of 20th century applied mathematics into isomathematics and the ensuing "completion" of quantum mechanics into hadronic mechanics [29]-[31] verify Statement 2 and admit well defined classical counterparts.

To review the preceding advances, consider two quantum mechanical particles with spin $1/2$ denoted 1 and 2 which verify the $SU(2)$ spin symmetry.

Assume that, as a result of some interaction, the two particles have antiparallel spins represented in the Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$. The total state in $c\hat{a}lH$ is then given by

$$|S_{1-2}>=\frac{1}{\sqrt{2}}(|S_{1\uparrow}>\times|S_{2\downarrow}>-|S_{1\downarrow}>\times|S_{2\uparrow}>), \tag{138}$$

with conventional l normalization

$$<S_{1-2}|S_{1-2}>=1, \tag{139}$$

where $\times$ is the conventional associative product.

Let a_1, b_1 and a_2, b_2 be unit vectors along the z-axis of a conventional Euclidean space $E(r, \delta, I)$ for particle 1 and 2, respectively. Introduce the quantum mechanical probability

$$P(a_1, b_1) =< S_{1-2}|(\sigma_1 \otimes a_1) \times (\sigma_2 \otimes b_1)|S_{1-2}>= -a_1 \otimes b_1, \tag{140}$$

where $\otimes$ is the conventional scalar product.

Then, Bell's inequality can be written [4] (see Ref. [6] for numerous equivalent formulations)

$$D_{Bell}^{QM} = Max|P(a_1, b_1) - P(a_1, b_2) + P(a_2, b_1) + P(a_2, b_2)| \leq 2, \tag{141}$$

and implies the following property:

LEMMA 3.5: Particles in vacuum verifying the Lie symmetry $SU(2)$ admit no classical counterparts.

PROOF: The classical counterpart of Bell's inequality is given by

$$D_{Max}^{Classical} = Max|a_1 \otimes b_1 - a_1 \otimes b_2| + |a_2 \otimes b_1 + a_2 \otimes b_2| = 2\sqrt{2}. \quad (142)$$

But the quantum mechanical value of D_{Bell}^{QM} is *always* smaller than its classical counterpart $D_{Max}^{Classical}$,

$$D_{Bell}^{QM} < D_{Max}^{Classical}, \quad (143)$$

by therefore establishing the impossibility for an $SU(2)$-invariant system to admit identical classical images. Q. E. D.

Santilli [10] has shown that inequality (141) is inapplicable for the same particles when they are in interior dynamical conditions, e.g., when they are in the core of a star, or at the limit, when they are in the interior of a gravitational collapse.

Considers two extended particles also denoted 1 and 2. Suppose that said particles verify the regular $\hat{SU}(2)$ isosymmetry with spin $1/2$ (Section 3.3), thus implying the elaboration via isomathematics (Section I-3) and the verification of the isotopic branch of hadronic mechanics (Section I-4).

Suppose that the two extended particles with spin $1/2$ are characterized by the following isotopic elements:

$$\begin{aligned} Particle\ 1:\ \hat{T}_1 = Diag(\lambda_1, 1/\lambda_1), \\ Particle\ 2:\ \hat{T}_2 = Diag(\lambda_2, 1/\lambda_2), \end{aligned} \quad (144)$$

with realization (83) of the Pauli-Santilli isomatrices.

Suppose that, due to preceding interactions, the two extended particles are in single overlapping/entanglement thus having opposite spins.

Let $\hat{I}_1$ and $\hat{I}_2$ be the isounits for particles 1 and 2, respectively. The systems of the assumed two isoparticles is then characterized by the total isounit

$$\hat{I}_{tot} = \hat{I}_1 \times \hat{I}_2 = \frac{1}{\hat{T}_{tot}} = \frac{1}{\hat{T}_1 \times \hat{T}_2}. \quad (145)$$

In this case, the total isostate on the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ [19] over the isofield of isocomplex isonumbers $\hat{calC}$ [20] is given by

$$|\hat{S}_{1-2}>= \frac{1}{\sqrt{2}}(|\hat{S}_{1\uparrow}> \hat{\times} |\hat{S}_{2\downarrow}> - |\hat{S}_{1\downarrow}> \times |\hat{S}_{2\uparrow}>). \quad (146)$$

The lack of validity of inequality (141) for irregular isorepresentations of $\hat{SU}(2)$ is evident (e.g., because of the anomalous spin isoeigenvalues) and, as such, it is ignored.

A significant aspect of Ref. [10] is the proof of the inapplicability of inequality (141), not only for regular isorepresentation of $\hat{SU}(2)$, but also when such isorepresentations are isounimodular, Eqs. (144).

Let a_1, b_1, a_2, b_2 be unit vectors along the z-axis of an iso-Euclidean isospace. Introduce the isoprobability (Eq. (32.39), page 99, Ref. [30])

$$\begin{aligned}\hat{P}(a,b) =< \hat{S}_{1-2}| \star (\hat{\Sigma}_1 \hat{\otimes}_1 a) \times (\hat{\Sigma}_2 \hat{\otimes}_2 b)|\hat{S}_{1-2} > \hat{I}_{tot} = \\ =< \hat{S}_{1-2}| \star (\hat{\sigma}_1 \otimes a) \times (\hat{\sigma}_2 \otimes b)|\hat{S}_{1-2} > \hat{I}_{tot},\end{aligned} \tag{147}$$

with isonormalization (here referred to individual diagonal elements of isotopic elements and isounits)

$$< \hat{S}_{1-2}| \star |\hat{S}_{1-2} >=< \hat{S}_{1-2}|\hat{T}_{tot}|\hat{S}_{1-2} >= \hat{I}_{tot} \tag{148}$$

where: $\star$ is the total isoproduct; $\hat{\otimes}_k,\ k = 1, 2,$ is the isoscalar isoproduct; and we have used simplifications of the type

$$\hat{\Sigma}_1 \hat{\otimes}_1 a = (\hat{\sigma}_1 \hat{I}_1)(\hat{T}_1 \otimes) a = \hat{\sigma}_1 \otimes a. \tag{149}$$

An isotopy of the conventional case yields the following isobasis, Eq. (6.5) of Ref. [10],

$$|S_{1-2} >= \frac{1}{2}\left\{ \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \begin{pmatrix} o \\ \lambda_2^{1/2} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda_2^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \right\}. \tag{150}$$

The appropriate use of products and isoproducts then yield expression (5.6) Ref. [10], i.e.,

$$\begin{aligned}< \hat{S}_{1-2}|\hat{T}_{tot}(\hat{\sigma}_1 \otimes_1 a) \times (\hat{\sigma}_2 \otimes_2 b)\hat{T}_{tot}|\hat{S}_{1-2} >= \\ = -a_x b_x - a_y b_y - \tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) a_z b_z.\end{aligned} \tag{151}$$

The continuation of the isotopy of the conventional case, yields the main result, Eq. (5.8) of Ref. [10], which provides the following *isotopic "completion" of Bell's inequality*,

$$\begin{aligned}\hat{D}_{Max}^{HM} = D_{Max}^{HM} \hat{I}_{tot} = \\ Max|\hat{P}(a_1, b_1) - \hat{P}(a_1, b_2) + \hat{P}(a_2, b_1) + \hat{P}(a_2, b_2)| = \\ = [\tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) D_{Bell}^{QM}\ \hat{I}_{tot},\end{aligned} \tag{152}$$

with consequential:

LEMMA 3.6. Extended particles within physical media that are invariant under the Lie-Santilli isosymmetry $\hat{SU}(2)$ admit identical classical counterparts.

PROOF: Isoinequality (141) establishes the lack of universal validity of Bell's inequality (128) because the factor $\frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2)$ can have values *bigger* than one, thus implying

$$D^{HM}_{Max} \geq D^{QM}_{Bll}. \tag{153}$$

Consider then a classical iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ representing motion of classical extended particles 1 and 2 within physical media [30] with isometric elements

$$\hat{\delta}_{11} = 1, \;\; \hat{\delta}_{22} = 1, \;\; \hat{\delta}_{33} = \frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2) = 2, \tag{154}$$

in which case

$$D^{HM}_{Max} \equiv htD^{Classical}_{Max}, \tag{155}$$

by therefore establishing that systems of extended particles within physical media verifying the $\hat{SU}(2)$ isosymmetry admits an identical classical counterpart along the EPR argument. Q.E.D.

It is an instructive exercise for the interested reader to prove that the above lemma also holds for different isorenormalizations, e.g., Eqs. (171) of next section, with the understanding that different isorenormalizations imply different isobasis and different hidden variable terms in Eqs. (151).

Note the crucial role of hidden variables for the proof of Lemma 3.6. It is an instructive exercise for interested readers to prove that Lemma 3.6 holds for any other regular, isounimodular isorepresentation of the isotopic $\hat{SU}(2)$ symmetry in terms of hidden variables presented in Section 3.3.

The proof of the lack of applicability of von Neumann's theorem [5] for extended particles in interior conditions is elementary. Recall that von Neumann's theorem is based on the uniqueness of the eigenvalues E of a Hermitean operator $H, \; H|\psi >= E|\psi >$ under unitary transformation on $\mathcal{H}$,

$$UH|\psi > U^\dagger = UE|\psi > U^\dagger = EU|\psi > U^\dagger, \;\; UU^\dagger = U^\dagger U = I, \tag{156}$$

under the tacit assumption of point particles in vacuum.

By contrast, when the same particles is in interior conditions, it is subjected to an infinite number of different physical different interactions with

the medium represented by the isotopic element $\hat{T}$ with ensuing isoeigenvalue equation (Section I-4), [9],

$$abel1H \star |\hat{\psi}_{\hat{T}} r >= HT|\hat{\psi}_{\hat{T}} >= E_{\hat{T}}|\hat{p}si_{\hat{T}} >, \tag{157}$$

thus establishing that a given quantum mechanical operator H representing the energy of an extended particle in interior conditions has an infinite number of *generally different* isoeigenvalues $E_{\hat{T}}$ depending on the infinite number of different interior conditions.

Note that, for each given $\hat{T}$ the isoeigenvalue $E_{\hat{T}}$ is invariant under isounitary isotransformations (Section I-3-9).

3.8. Apparent admission of classical determinism.

Consider a point-like particle in empty space represented in the 3-dimensional Euclidean space $E(r, \delta, I)$, where r represents coordinates, $\delta = Diag.(1, 1, 1)$ represents the Euclidean metric and $I = Diag.(1, 1, 1,)$ represents the space unit.

Let the operator representation of said point-like particle be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\psi(r)$ and familiar normalization

$$< \psi(r)| \, |\psi(r) >= \int_{-\infty}^{+\infty} \psi(r)^{\dagger}\psi(r)dr = 1. \tag{158}$$

As it is well known, the primary objections against the EPR argument [2]- [6] were based on Heisenberg uncertainty principle according to which *the position r and the momentum p of said particle cannot both be measured exactly at the same time.*

By introducing the *standard deviations* Δr and Δp, the uncertainty principle is generally written in the form

$$\Delta r \Delta p \; \geq \; \frac{1}{2}\hbar, \tag{159}$$

which is easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p \; \geq \; |\frac{1}{2i} < \psi| \, [r, p] \, |\psi > | = \frac{1}{2}\hbar. \tag{160}$$

Standard deviations have the known form (see, e.g., Ref. [79]) with $\hbar = 1$

$$\begin{aligned} \Delta r &= \sqrt{< \psi(r)|[\, r - (< \psi(r)| \, r \, |\psi(r) >)]^2 |\psi(r) >}, \\ \Delta p &= \sqrt{< \psi(p)| \, [p - (< \psi(p)| \, p \, |\psi(p) >)]^2 |\psi(p) >}, \end{aligned} \tag{161}$$

where $\psi(r)$ and $\psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

We consider now an extended particle, this time, in interior conditions, e.g., in the core of a star, classically represented by the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ with isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{162}$$

and isotopic element (4)) under conditions (5).

For simplicity, we assume that the extended particle has no Hamiltonian interactions due to the dominance of the latter interactions over the former.

Consequently, we can represent the extended particle in the isospace $\hat{\mathcal{H}}$ over the isofield $\hat{\mathcal{C}}$ and introduce the time independent *isoplanewave* [18]

$$\begin{aligned} \hat{\psi}(\hat{r}) = \tilde{\psi}(\hat{r})\hat{I} = \\ = \hat{N} \star (\hat{e}^{\hat{i}\star\hat{k}\star\hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{163}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar*, $\hat{k} = k\hat{I}$ is the *isowavenumber*, and the isoexponentiation is given by Eq. (I-22) [26].

The corresponding representation in isomomentum isospace is given by

$$\tilde{\psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i}\star\hat{n}\star\hat{p}}, \tag{164}$$

where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

The *isopropability isofunction* is then given by (Ref. [30] page 99)

$$\hat{\mathcal{P}} = \hat{<}| \star |\hat{>} =< \hat{\psi}(\hat{r})| \, T \, |\hat{\psi}(\hat{r}) > I, \tag{165}$$

that, written in terms of isointegrals (Ref. [29] page 354), becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \star \hat{\psi}(\hat{r}) \star \hat{dr} = \\ = \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^\dagger \tilde{\psi}(\hat{r})(dr + r\hat{T}d\hat{I}), \end{aligned} \tag{166}$$

where one should keep in mind that the isodifferential $\hat{dr}$ given by Eqs. (I-29).

The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [30]

$$\hat{<}|\star\hat{Q}\star|\hat{>}=<\hat{\psi}(\hat{r})|\star\hat{Q}\star|\hat{\psi}(\hat{r})>=$$

$$=\hat{\int}_{-\infty}^{+\infty}\hat{\psi}(\hat{r})^{\dagger}\star\hat{Q}\star\hat{\psi}(\hat{r})\hat{dr}= \tag{167}$$

$$=\hat{\int}_{-\infty}^{+\infty}\tilde{\psi}(\hat{r})^{\dagger}\hat{Q}\tilde{\psi}(\hat{r})\hat{dr},$$

with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

Santilli then introduced apparently for the first time in Ref. [11] the *isotopic operator*

$$\hat{\mathcal{T}}=\hat{T}\hat{I}=I, \tag{168}$$

that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the "completion" of the isoproduct from its scalar form (1) to the isoscalar form

$$\hat{n}^{\hat{2}}=\hat{n}\star\hat{n}=\hat{n}\star\hat{\mathcal{T}}\star\hat{n}=\ n^2I. \tag{169}$$

In Sections 3.6, 3.7, we have shown that the Lie-Santilli isosymmetry $\hat{SU}(2)$ admits an explicit and concrete realization of hidden variables that allowed the construction of identical classical counterparts for interior dynamical systems.

Ref. [11] introduced the isoexpectation isovalue of the isotopic operator

$$\hat{<}|\star\hat{\mathcal{T}}\star|\hat{>}=<\hat{\psi}(\hat{r})|\star\hat{\mathcal{T}}\star|\hat{\psi}(\hat{r})>\hat{I}= \tag{170}$$

$$=\hat{\int}_{-\infty}^{+\infty}\tilde{\psi}(\hat{r})^{\dagger}\hat{T}\tilde{\psi}(\hat{r})\hat{dr},$$

and assumed the isonormalization (again, intended for diagonal matrix elements)

$$\hat{<}|\star\hat{\mathcal{T}}\star|\hat{>}= \tag{171}$$

$$=\hat{\int}_{-\infty}^{+\infty}\hat{\psi}(\hat{r})^{\dagger}\hat{T}\hat{\psi}(\hat{r})\hat{dr}=\hat{T}.$$

Consider then the *isostandard isodeviation* for isocoordinates $\Delta\hat{r}=\Delta r\hat{I}$ and isomomenta $\Delta\hat{p}=\Delta p\hat{I}$, where Δr and Δp are the standard deviations in our space.

By using isocanonical isocommutation rules (I-81), we obtain the expression

$$\Delta\hat{r}\star\Delta\hat{p}=\Delta r\Delta p\hat{I}\ \approx\ \tfrac{1}{2}|<\hat{\psi}(\hat{r})|\star[\hat{r},\hat{p}]\star\hat{\psi}(\hat{r})>|\hat{I}= \tag{172}$$

$$=\tfrac{1}{2}|<\hat{\psi}(\hat{r})|\hat{T}\,[\hat{r},\hat{p}]\,\hat{T}|\hat{\psi}(\hat{r})>\hat{I},$$

One should note the replacement of the symbol $\geq$ in Eq. (160) with the symbol $\approx$ in Eq. (172). This is due to the fact that the historical arguments applying for a point-like particle in vacuum no longer apply for an interior system because the pressure exercised by the medium on the particle (Figure 4) reduce the lower limit of Eq. (160) to the approximate value of Eq. (172).

Under the above assumptions, by eliminating the common isounit $\hat{I}$, Ref. [11] achieved the desired result here called *isodeterministic isoprinciple*

$$\begin{aligned} \Delta r \Delta p &\approx \tfrac{1}{2} | < \hat{\psi}(\hat{r}) | \star [\hat{r}, \hat{p}] \star | \hat{\psi}(\hat{r}) > = \\ &= \tfrac{1}{2} | < \hat{\psi}(\hat{r}) | \hat{T} \, [\hat{r}, \hat{p}] \, \hat{T} | \hat{\psi}(\hat{r}) > = \\ &\tfrac{1}{2} \textstyle\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) \hat{d} \hat{r} = \tfrac{1}{2} T \ll 1 \end{aligned} \tag{173}$$

where the property $\Delta r \Delta p \ll 1$ follows from the fact that the isotopic element $\hat{T}$ has values smaller than 1 in the fitting of all experimental data dealing with hadronic media such as hadrons, nuclei and stars, and null value for gravitational collapse [31].

In the event Eq. (35), page 14 of Ref. [11] should be compatible with Eq. (173) above, it is sufficient to turn into a comma the sign $=$ in the right of the central expression of Eq. (35), or absorb the factor $1/2$ of Eq. (173) into the isorenormalization.

In this way, thanks to a laborious scientific journey initiated at Harvard University in late 1977, and thanks to contributions by numerous mathematicians, theoreticians and experimentalists, Santilli reached the following verification of the EPR argument [11]:

LEMMA 3.7 (ISODETERMINISTIC PRINCIPLE): The isostandard isodeviations for isocoordinates $\Delta \hat{r}$ and isomomenta $\Delta \hat{p}$, as well as their product, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

PROOF: Define the isostandard isodeviations via the following isotopy of quantum mechanical expressions (161) (where we ignore the common multiplication by the isounit)

$$\begin{aligned} \Delta r &= \sqrt{< \hat{\psi}(\hat{r}) | [\, \hat{r} - < \hat{\psi}(\hat{r}) | \star \hat{r} \star | \hat{\psi}(\hat{r}) >]^{\hat{2}} \hat{|} \psi(\hat{r}) >}, \\ \Delta p &= \sqrt{< \hat{\psi}(\hat{p}) | \, [\hat{p} - < \hat{\psi}(\hat{p}) | \star \hat{p} \star | \hat{\psi}(\hat{p}) >]^{\hat{2}} | \hat{\psi}(\hat{p}) >}, \end{aligned} \tag{174}$$

where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity. But the isotopic element represents the interactions of the particle with the physical medium and tends toward null values for gravitational collapse, Eqs. (I-91) (I-92). Therefore, isosquare in expression (171) implies the expressions

$$\begin{aligned} \Delta r &= \sqrt{\hat{T} < \hat{\psi}(\hat{r})|[\,\hat{r} - < \hat{\psi}(\hat{r})| \star \hat{r} \star |\hat{\psi}(\hat{r}) >]^2 |\hat{\psi}(\hat{r}) >}, \\ \Delta p &= \sqrt{\hat{T} < \hat{\psi}(\hat{p})|\,[\hat{p} - < \hat{\psi}(\hat{p})| \star \hat{p} \star |\hat{\psi}(\hat{p}) >]^2 |\hat{\psi}(\hat{p}) >}, \end{aligned} \tag{175}$$

that approach indeed null value under the indicated limit conditions of gravitational collapse

$$\begin{aligned} Lim_{\hat{T}=0} \Delta r = 0, \\ Lim_{\hat{T}=0} \Delta p = 0, \end{aligned} \tag{176}$$

Q.E.D.

3.9. Apparent removal of quantum divergencies.

Recall from Section I-4.13 that, under condition (I-96), corresponding to condition (173), there is a rapid convergence of isoseries (I-97), as well as the removal of the singularity of Dirac's delta distribution, Eq. (I-98) (Figure I-14).

The above properties can be now formalized according to the following:

COROLLARY 3.7.1. Einstein's determinism according to Lemma 3.7 implies the removal of quantum mechanical divergencies.

PROOF. Lemma 3.7 is based on values of the isotopic element $\hat{T}$ being smaller than 1, which values imply in turn the rapid convergence of perturbative series without divergencies (Section I-4-13).
Q.E.D.

The removal of quantum divergencies, that have been cause of controversies for about one century, illustrates the far reaching implications of Einstein's determinism for interior dynamical systems.

3. CONCLUDING REMARKS.

Following the study of basic methods in Paper I, in this paper we have provided an apparent confirmation of proofs [10] [11] of the EPR argument [1] for extended, thus deformable particles within hyperdense media with

ensuing linear and non-linear, local and non-local and potential as well as non-potential/non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics admitting a conventional Hamiltonian H or Lagrangian L for the invariant representation of linear, local and potential interactions, plus the isotopic element $\hat{T}$ of isoproducts $A \star B = A\hat{T}B$, Eq. (1), for the invariant representation of non-linear, non-local and non-Hamiltonian interactions.

Following the outline and upgrade of isosymmetries for time-reversal invariant interior systems (Section 3), we have apparently confirmed the proof of Ref. [10] according to which extended particles in interior dynamical conditions admit identical classical counterpart.

We have then apparently confirmed the proof of Ref. [11] according to which extended particles progressively approach classical determinism when in the interior of hadrons, nuclei and stars, and achieve full determinism at the limit of gravitational collapse, essentially as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

To illustrate the far reaching implications of what appears to be Einstein's most important legacy, we have shown for the first time that the recovering of Einstein's determinism for interior conditions appears to imply the removal of quantum divergencies due to the rapid convergence of the isoseries of hadronic mechanics, the removal of the singularity in Dirac's delta distribution and other features.

A number of illustrations and novel applications in mathematics, physics and chemistry are presented in the forthcoming Paper III.

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Studies on A. Einstein, B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," III: Illustrative examples and applications

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Abstract

In the preceding Papers I and II of this series, we have presented a review and upgrade of novel mathematical, physical and chemical methods, and shown their use for a confirmation of the apparent proof of the EPR argument that extended particles within physical media admit classical counterparts, while Einstein's determinism appears to be progressively verified with the increase of the density of the medium. In this third paper, we have additionally shown, apparently for the first time, the validity of the EPR final statement to the effect that *the wavefunction [of quantum mechanics] does not provide a complete description of the physical reality.*" In fact, we have studied the axiom-preserving "completion" of the quantum mechanical wavefunction due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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3. CONCLUDING REMARKS.

Acknowledgments.

References.

1. INTRODUCTION.

1.1. The EPR argument.

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe."

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

Objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3] [4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature).

The field became known as *local realism* and included the dismissal of the EPR argument based on claims that quantum axioms do not admit *hidden variables* λ [7] [8].

1.2. Outline of preceding works.

Following various preparatory works, in Ref. [9] of 1998, R. M. Santilli:

1) Assumed the validity of quantum mechanics, with consequential validity of the objections against the EPR argument [2] - [6], for point-like particles in empty space under linear, local and potential interactions (*exterior dynamical problems*);

2) Proved the inapplicability (and not their violation) of said objections for the broader class of extended, deformable and hyperdense particles within physical media under the most general known linear and non-linear, local and non-local and potential as well as non-potential interactions (*interior dynamical problems*); and

3) Provided the apparent proof that *interior dynamical systems admit classical counterparts in full accordance with the EPR argument* via the representation of interior systems with of *isomathematics* also called *isotopoic branch of hadronic mathematics*, and *isomechanics*, also called *isotopic branch of hadronic mechanics*.

In the 2019 paper [10], Santilli provided the apparent proof that *Einstein's determinism is progressively approached in the interior of hadrons, nuclei and stars and it is fully achieved in the interior of gravitational collapse.*

In Ref. [11], herein referred to as Paper I, we provided a review of isomathematics and isomechanics.

In the subsequent Ref. [12], hereinafter referred as Paper II, we provided a review of isosymmetries, with particular reference to the isotopies of the spin and rotational symmetries, and provided an apparent confirmation of the proofs [9] and [10] of the EPR argument.

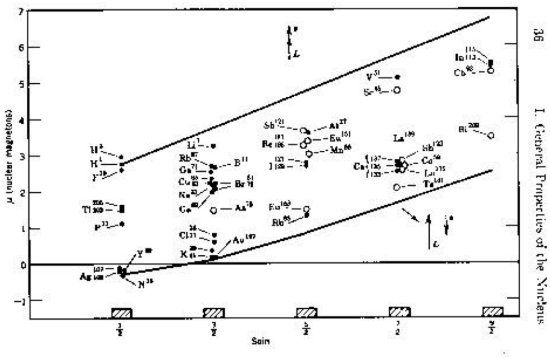

Figure 1: *In this figure, we present the so-called Schmidt limits essentially representing the deviations of experimental data on nuclear magnetic moments from the predictions of quantum mechanics. In Santilli's view [11], this occurrence is clear evidence on the need for a "completion" of quantum mechanics along Einstein's legacy [1], beginning with a "completion" of applied mathematics into a form suitable to represent extended and deformable nucleons according to the founders of nuclear physics, such as E. Fermi [13] and V. F. Weisskopf who states in page 31 of his treatise in nuclear physics with J. M. Blatt [14]: " it is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon. " For penetrating appraisals of the insufficiencies of 20th century science, one should also see K. R. Popper [15]. J. Dunning-Davies [16], J. Horgan [17] and others. In this paper, we outline and update the achievement by hadronic mathematics and mechanics of exact representations of nuclear experimental data along the indicated historical legacies, and point out important implications for much needed new, clean energies and fuels (Sections 2.6.5, 2.7.3, 2.8.3).*

In this third and final paper of this series: we present apparent experimental verifications of the EPR argument in physics and chemistry; we outline expected industrial applications; and identify the most salient implications of the EPR argument.

1.3. Basic notions.

The basic assumptions underlying the apparent proofs of the EPR argument are the following::

1) The axiom-preserving lifting of the conventional associative product $ab = a \times b$ between *all* possible quantum mechanical quantities (numbers, functions, matrices, etc.) into the *isoproduct*, first introduced in the 1978 Harvard university paper [18] (see also Refs. [19] to [23])

$$a \star b = a \, \hat{T} \, b, \tag{1}$$

where $\hat{T}$, called the *isotopic element*, is restricted to be positive-definite,

$\hat{T} > 0$, but possesses otherwise an unrestricted functional dependence on all needed local variables, including wavefunctions and their derivatives.

20th century applied mathematics and quantum mechanics are reformulated in an axiom-preserving, thus isotopic form, via isoproduct (1) (Paper I and Section II-2).

3) The axiom-preserving isotopy of the various branches of Lie's theory, first achieved by Santilli in Refs. [18] [23] and are today known as the *Lie-Santilli isotheory* [24] (Section I-3.7) with Lie-Santilli algebras of the type [9]

$$[X_i\hat{,}X_j] = X_i \star X_j - X_j \star X_i = C_{ij}^k X_k. \;\; i,j = 1,2,...,N. \tag{2}$$

and ensuing systematic isotopy of space-time and internal symmetries (Section II-2).

3) Immediate explicit and concrete realizations of "hidden variables" [7] [8] of the type

$$\hat{T} = Diag.(1/\lambda, \lambda), \;\; Det\hat{T} = 1. \tag{3}$$

Ref. [9], therefore establishing that, contrary to objections [2] to [6], *the abstract axioms of quantum mechanics do indeed admit explicit and concrete realizations of hidden variables.*

4) Representation of extended particles in conditions of mutual penetration via realizations of the isotopic element $\hat{T}$ of isoproduct (1) of the type [25]

$$\begin{gathered} \hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2}) e^{-\Gamma}, \\ k = 1,2,...,N, \quad \mu = 1,2,3,4, \end{gathered} \tag{4}$$

where n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_k^2 = 1, `k = 1,2,3$ for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

In particular, the isotopic element $\hat{T}$ has resulted to have numeric values *smaller* than 1 in all known applications [23]-

$$|\,\hat{T}\,| \;\leq\; 1, \tag{5}$$

which property permitted Ref. [10] to show that *the standard deviations Δr and Δp progressively tend to zero with the increase of the density of the medium of interior problems.*

5) Isosymmetries do not preserve over time the basic unit 1 of conventional numeric fields $F(n, \times, 1)$ with consequential lack of experimental

verifications. This occurrence mandated Santilli to formulate isomathematics on *isofields* $\hat{F}(\hat{n}, \star, \hat{I})$ [26] [27](Section I-3.3) with *isonumbers* $\hat{n} = n\hat{I}$ equipped with isoproduct (1) and basic *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{6}$$

which remains numerically invariant under isosymmetries as necessary for consistency (Sections I-3.8, I-3.9 and II-2).

5) Recall that the Newton-Leibnitz differential calculus provides the ultimate characterization of the point-like character of particles admitted by quantum mechanics due to the known feature that said calculus can be solely defined as a finite number of isolated points.

This occurrence mandated Santilli to construct the covering of the Newton-Leibnitz differential calculus into the *isodifferential isocalculus* (Section I-3.6) with basic *isodifferential* [28] [29]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \tag{7}$$

and corresponding *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}, \tag{8}$$

which are defined on *volumes* represented by the isotopic element $\hat{T}$, rather than points.

Recall that Bell's inequality [3] D^{qm}_{Bell} does not admit a classical counterpart D^{class} because always smaller than the classical counterpart,

$$D^{qm}_{Bell} < D^{class}. \tag{9}$$

In Ref. [9], Eqs. (5.8), page 189, proved that, thanks to the existence in hadronic mechanics of hidden variables of type (3), the corresponding inequality in hadronic mechanics (hm), D^{hm} is bigger than Bell's inequality, and always admits a classical counterpart

$$D^{hm} \equiv D^{class}, \tag{10}$$

resulting in the following:

LEMMA II-3.6 (EXISTENCE OF CLASSICAL COUNTERPART). Extended particles within physical media that are invariant under the Lie-Santilli isosymmetry $\hat{SU}(2)$ *admit identical classical counterparts.*

The main result of Ref. [10], Eqs. (35), page 14,

$$\Delta r \Delta p \approx \tfrac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r},\hat{p}] \star |\hat{\psi}(\hat{r}) >=$$

$$= \tfrac{1}{2}| < \hat{\psi}(\hat{r})|\hat{T}\,[\hat{r},\hat{p}]\,\hat{T}|\hat{\psi}(\hat{r}) >= \tag{11}$$

$$\textstyle\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger}\hat{T}\hat{\psi}(\hat{r})\hat{d}\hat{r} = T \ll 1,$$

verifying Einstein's determinism, was expressed with the following:

LEMMA II-3.7 (EINSTEIN DETERMINISM): The isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as their product, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

By recalling that isoperturbative series of hadronic mechanics has based on isoproduct (1) (Section I-4),

$$A(0) = \hat{I} + (A\hat{T}H - H\hat{T}A)/1! + ... \tag{12}$$

and that $\hat{T} \ll 1$ from Lemma II-3.7, the following consequential property was introduced, apparently for the first time in Paper II:

COROLLARY 3.7.1 (LACK OF DIVERGENCIES) Einstein's determinism according to Lemma II-3.7 implies the lack of quantum mechanical divergencies in hadronic mechanics.

It should be indicated that the technical understanding of the verifications and applications presented in this paper require a knowledge of isomathematics and isomechanics.

2. VERIFICATIONS AND APPLICATIONS.

2.1. Foreword.

Being dimensionless, *Newtonian massive points cannot experience resistive or contact force of any type.* By recalling that Newton's equation have been the foundations of physics for the past four centuries, 20th century mainstream particle physics has been developed without the notion of resistive force, with ensuing lack of treatment of the *pressure* exercised by a medium on a particle in its interior, contrary to clear evidence that a proton in the core of a star is exposed to extremely big pressures (Figure 1).

In this section, we shall illustrate the fact that, once admitted, the pressure exercised on extended particles characterizes in their interior characterizes standard deviations $\Delta\hat{r}$ and $\Delta\hat{p}$ that, being *constrained* by said pressure, verify the isodeterministic principle of Lemma II-3.7 as well as the rapid convergence of isoperturbative series according to Corollary II-3.7.1, by progressively approaching classical determinism with the increase of the pressure, up to the apparent achievement of classical determinism for the interior of gravitational collapse as predicted by Einstein, Podolsky and Rosen [1].

A physically important notion emerging from the examples provided below is that *the EPR argument appears to be verified by strong interactions* because, as indicated by Santilli in the 1978 paper [19], contact non-Hamiltonian interactions responsible for the synthesis of the neutron and other hadrons are short range, strongly attractive, charge independent and non-Hamiltonian (technically identified as *variationally non-selfadjoint interactions* [22]), thus providing a conceivable, first known, explicit and concrete representation of strong interactions.

The models outlined in this section were first proposed by Santilli in Ref. [19] in their time irreversible form, as requested for decaying bound states, thus being elaborated with Lie-admissible genomathematics, in which case, the need for a "completion" of quantum mechanics is beyond scientific doubt (Section I-1.3).

However, the objections against the EPR argument [2] - [6] have been formulated for conventional quantum axioms, thus implying the sole consideration of time-reversal invariant states. In this section, we illustrate the need for a "completion" of quantum mechanics also for time-reversal invariant systems of extended particles in interior conditions.

Therefore, unstable strongly interacting particles (hadrons) are hereon studied for such a small period of time to allow their time-reversal invariant approximation.

2.2. Particles under pressure.

One of the simplest illustrations of Lemma II-3.7 is given by a particle in the center of a star, thus being under extreme pressures π from the surrounding hadronic medium in all radial directions (Figure 1).

By ignoring particle reactions in first approximation, the conditions here considered can be rudimentarily represented for very short periods of time by assuming that the function $\Gamma > 0$ in the exponent of the isotopic element (4) is linearly dependent on the pressure π, resulting in a

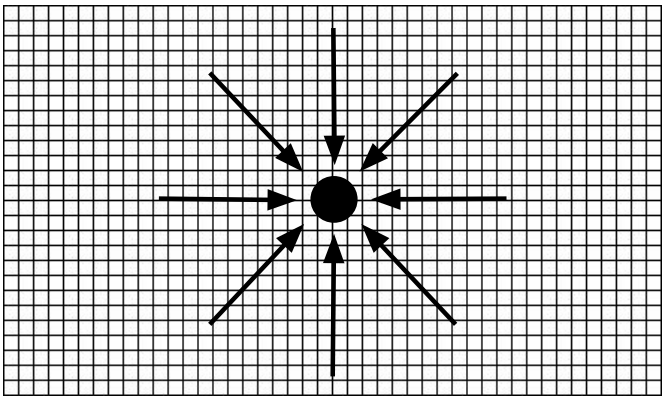

Figure 2: *In this figure, we present a conceptual rendering of the central notion used for the verification of the EPR argument, namely, the pressure experienced by extended particles immersed in hyperdense media, such as a proton in the core of a star, which pressure evidently restricts uncertainties in favor of Einstein's determinism (Lemma II-3.7). Note that the notion of pressure does not exist in 20th century physics due to the approximation of particles as being point-like.*

realization of the isotopic element of the simple type

$$\hat{T} = e^{-w\pi} \ll 1, \quad \hat{I} = e^{+w\pi} \gg 1, \tag{13}$$

where w is a positive constant.

Isodeterministic principle (11) for the considered particle is then given by

$$\Delta r \Delta p \approx \frac{1}{2} e^{-w\pi} \ll 1, \tag{14}$$

and tends to null values for diverging pressures.

The above example illustrates the consistency of isorenormalization (II-171) because a constant isotopic element verifies the isonormalization

$$\begin{aligned} &\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} = \\ &= T < \hat{\psi}(\hat{r})| \, |\hat{\psi}(\hat{r}) > \hat{I} = \\ &=< \hat{\psi}(\hat{r})| \, |\hat{\psi}(\hat{r}) >, \end{aligned} \tag{15}$$

but not necessarily other isorenormalizations.

Note that we have considered an individual extended particle immersed in a hadronic medium, rather than the bound state of extended particles in condition of mutual penetration that are studied in the next sections.

Consequently, isotopic element (14) represents a *subsidiary constraint on standard deviations* caused, as indicated, by the pressure of the surrounding hadronic medium on the particle considered.

It is easy to see that, since $\Gamma(\pi) > 0$, more complex functional dependence on the pressure π continue to verify Lemma II-3.7 as well as Corollary II-3.7.1.

2.3. Non-relativistic hadronic bound states.

As recalled in Paper I, the dynamical equations of quantum mechanics, such as the Schrödinger and the Dirac equation, are characterized by the conventional differential calculus which can solely be defined at a finite set of isolated points on a given representation space. Consequently, quantum mechanical bound states are solely possible for *point-like constituents* under linear, local and potential interactions (technically identified as *variationally self-adjoint interactions* [22]). This it is the case for the familiar Schrödinger equation for the bound state of two point-like particles of mass m with Coulomb potential $V(r)$ in a Euclidean space $E(r, \delta, I)$ formulated on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$

$$
\begin{aligned}
i\tfrac{\partial}{\partial t}\psi(t,r) = H\psi(t,r) = \left[\tfrac{\hbar^2}{m}\Sigma_k p_k p_k - V(r)\right]\psi(t,r) = \\
= \left[\tfrac{1}{m}\Sigma_k(-i\hbar\partial_k)(-i\hbar\partial_k) - V(r)\right]\psi(t,r) = \\
= \left[-\tfrac{\hbar^2}{m}\Delta_r - V(r)\right]\psi(t,r) = E\psi(t,r).
\end{aligned}
\tag{16}
$$

By contrast, *hadronic bound states* are bound states of *extended particles* at mutual distances smaller or equal to the *hadronic horizon* (Figure I-13)

$$
R = \frac{1}{b} \approx 10^{-13}\, cm. \tag{17}
$$

In such a region, bound states verify hadronic mechanics, can be represented with isomathematics (Section I-3) and isomechanics (Section I-4) for the case of time-reversal invariant bound states (such as the deuteron), or genomathematics and genomechanics for time irreversible bound states (such as all hadrons produced in particle physics laboratories).

By using the methods outlined in Paper I and in the preceding sections, when assumed to be stable in first approximation (thus being time reversal invariant), hadronic bound states are characterized by the following main features:

1) The bound states occur between *isoparticles*, namely, isoirreducible, isounitary isorepresentations of the isospinorial covering of the Galileo-Santilli isosymmetry $\hat{\mathcal{G}}$ for non-relativistic treatments (Sections II-2.5.1 and II-3.9) or of the Lorentz-Poincaré-Santilli isosymmetry $\hat{\mathcal{P}}$ for relativistic treatments (Sections II-2.5.11 and II-3.9) represented by hadronic mechanics including the verification of Einstein's determinism (Lemma II-3.7) and the absence of divergencies (Corollary II-3.7.1).

2) The representation of the extended character of the isoparticles is done with isoproduct (1) and isotopic element (4), resulting in iso-Schrödinger equations of type (I-80), while the deep mutual penetration of the wavepackets and/or charge distributions of isoparticles generates novel non-linear, non-local and non-potential interactions represented by the exponent of the the isotopic element (4) and other means. Note that the latter interactions are short range, strongly attractive, charge independent, and non-Hamiltonian according to all studies conducted to date in the field, thus allowing an initial yet explicit and concrete realization of strong interactions [19].

3) By recalling that isosymmetries $\hat{\mathcal{G}}$ and $\hat{\mathcal{P}}$ are all *irregular* realizations of the Lie-Santilli isotheory (Sections I-2.7 and II-2.5.4), a necessary condition for the invariance of hadronic dynamical equations under isosymmetries is that contact interactions *cannot* be derived via non-unitary transforms of quantum mechanical potentials, thus being *basically new interactions*. The physically equivalent property is that, as it is well known, *strong interaction cannot be derived via non-unitary or other known transformations of electromagnetic interactions*, thus confirming the necessary use of the irregular Lie-Santilli isotheory and hadronic dynamical equations.

The notion of hadronic bound states was proposed, apparently for the first time, by Santilli in the 1978 Ref. [19] and was extensively studied hereafter in various works by various authors (see the 2001 monograph [30], the 2011 review [31], and papers quoted therein).

It is important to review the derivation of the basic non-relativistic and relativistic isoequations for hadronic bound states to show their apparent verification of the isodeterministic principle of Lemma 3.7.

The fundamental *non-relativistic, irregular isoequation of a time-reversal invariant hadronic bound state* of two isoparticles of mass m at mutual distances of the order of the hadronic horizon $R = 10^{-13}\, cm$ in an iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ formulated on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ over

the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ can be written

$$i\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\hat{\psi}(\hat{t},\hat{r}) = \left[(\hat{1}/\hat{m}) \star \Sigma_k \hat{p}_k \star \hat{p}_k \pm \hat{V}(\hat{r}) - \hat{S}(\hat{\psi})\right] \star \hat{\psi}(\hat{t},\hat{r}) =$$

$$= \left[(1/m)\Sigma_k \hat{p}_k \hat{T} \hat{p}_k \hat{T} \pm V(\hat{r}) - S(\hat{\psi})\right] \hat{\psi}(\hat{t},\hat{r}) = \tag{18}$$

$$\hat{E} \star \hat{\psi}(\hat{t},\hat{r}) = E\hat{\psi}(\hat{t},\hat{r}),$$

where $\hat{V}(\hat{r}) = V(\hat{r})\hat{I}$; $\hat{S}(\hat{\psi}) = S(\hat{\psi})\hat{I}$ represents the novel short range, strongly attractive force; the value $-\hat{V}(\hat{r})$ occurs for bound states with opposite charge (as it is the case for the synthesis of hadrons reviewed below); the value $+\hat{V}(\hat{r})$ occurs for isoparticles with the same charge (as occurring for valence electron bonds reviewed below); and one should note that isoeigenvalues can always be reduced to conventional eigenvalues, thus allowing experimental verifications.

Due to the large representational capabilities of isoequations (182), we use the following simplifying assumptions:

1) The isotime is equal to the conventional time, $\hat{t} = t\hat{I}_t = t,\ \hat{I}_t = 1$;

2) Being extremely small, the orbits of the isoparticles are assumed to be nearly constant circles, thus implying that the n_k characteristic quantities of then isotopic element (4) can be normalized to the sphere, $n_k = 1,\ k = 1,2,3$;

3) The isotopic element is assumed to be given by the exponential term of Eq. (4) with realizations of the non-linear, non-local and non-potential interactions of the type (Eq. (4.7), page 170 Ref. [30])

$$\hat{T} = e^{-\Gamma} = e^{-N\psi/\hat{\psi}} \approx 1 - N\psi/\hat{\psi}, \tag{19}$$

where ψ behaves like the solution of quantum equation (180),

$$\psi(r) \approx W_1 e^{-br}, \tag{20}$$

and $\hat{\psi}$ behaves like the solution of the hadronic equation expected to be of the type

$$\hat{\psi} \approx W_2(1 - e^{\frac{(1--br)}{r}}, \tag{21}$$

where W_1 and W_2 are positive normalization constants.

We therefore have the following explicit form of the isotopic element

$$\hat{T} = e^{-W_2 \frac{e^{-br}}{(1-e^{-br})/r}} \approx 1 - W\frac{e^{-br}}{(1-e^{-br})/r}, \tag{22}$$

exhibiting the *Hulten potential*

$$V_{Hult} = W_2 \frac{e^{-br}}{1 - e^{-br}}, \tag{23}$$

directly in the exponent of the isotopic element, where W is normalization constants.

It should be recalled that Santilli suggested the use of the Hulten potential in the 1978 paper [19], Eq. (5.1.6), page 833, as an *initial yet explicit and concrete representation of strong interactions.*

Under the above assumptions, isotopic element (186) verifies the central condition for the validity of the isodeterministic principle as well as the rapid convergence of isoperturbative series inside the hadronic horizon (Lemma II-3.7 and Corollary II-3.7.1), in a way fully compatible with the validity of conventional uncertainties as well as divergence of perturbative series outside said horizon

$$\begin{aligned} |\hat{T}| \ &\ll \ 1, \\ Lim_{r \gg R} \hat{T} \ &= \ 1. \end{aligned} \tag{24}$$

The use of the isolinear isomomentum (I-79), the projection of isodynamical equation (II-182) into our Euclidean space can be written in the form first derived in Eq. (5.1.9), page 833, Ref [19]

$$\begin{aligned} i\tfrac{\partial}{\partial t}\hat{\psi}(t,r) &= \left[\tfrac{1}{m}\Sigma_k \hat{p}_k \star \hat{p}_k \star \pm V\tfrac{e^2}{r} - W\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = \\ &= \left[\tfrac{1}{m}\Sigma_k(-i\hat{I}\partial_k)(-i\hat{I}\partial_k) \pm W_1\tfrac{e^2}{r} - W_2\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = \\ &= \left[-\tfrac{1}{\bar{m}}\Delta_r \pm V\tfrac{e^2}{r} - W\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = E\hat{\psi}(t,r), \end{aligned} \tag{25}$$

where: W_1' and W_2 are renormalization constants; E is the binding energy of the hadronic bound state; the total energy E_{tot} is given by

$$E_{tot} = E_1 + E_2 + -E, \tag{26}$$

where $E_k, k = 1, 2$ are the total energies of particles 1 and 2, respectively, and

$$\bar{m} = \frac{m}{\hat{I}^2}. \tag{27}$$

The nonrelativistic expression of the mean-life of the hadronic bound state can be derived via an isotopy of the quantum mechanical form, yielding the expression (Ref. [19], Eq. (5.1.13), page 835)

$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_k}{\hbar},\ k = 1, 2. \tag{28}$$

The angular component of Eqs. (85) has been studied in detail in Ref. [33] via the isospherical isoharmonics.

The radial component of the non-relativistic, irregular, hadronic isoequation for the characterization of the total energy E_{tot}, mean life τ and charge radius R of a time-reversal invariant hadronic bound state can be written (Ref. [19], Eqs. (5.1.40, page 835)

$$\begin{aligned}
&\left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2\tfrac{e^{-br}}{1-e^{-br}})\right] = 0,\\
&E_{tot} = E_1 + E_2 - E,\ \ \bar{m} = \tfrac{m}{I^2}\\
&\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_k}{\hbar},\ k = 1, 2,\\
&R = b^{-1},
\end{aligned} \tag{29}$$

where the last two equations are subsidiary constraints on the first two.

The analytic solution of the above equations has been studied in detail in Section 5, Ref. [19], including boundary conditions requested by the subsidiary constraints, and we cannot review it here for brevity. We merely limit ourselves to the indication that said analytic solution was reduced to the solution of the following two algebraic equations on the parameters k_1 and k_2 (Ref. [19], Eq. (5.1.32), page 840)

$$\begin{aligned}
&k_1[k_1 - (k_2 - 1)^3] = \tfrac{1}{2c}E_{tot}R,\\
&\tfrac{(k_2-1)^3}{k_1} = \tfrac{9\times 10^6}{3\pi c}\tfrac{R}{\tau}.
\end{aligned} \tag{30}$$

It is now important to evaluate the numeric value of the binding energy E in Eq. (198). For this purpose, we recall that the Hulten potential behaves like the Coulomb potential at short distances

$$V_{hp} \approx K\frac{1}{r}, \tag{31}$$

where K is a positive constant. Consequently, *the Hulten potential can absorb the Coulomb potential resulting in a short range strongly attractive force irrespective of whether the Coulomb force is attractive or repulsive.*

The radial equation can then be reduced in first approximation to the expression (Eq. (5.1.14a), page 836, Ref. [19])

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E + W_2\frac{e^{-br}}{1-e^{-br}})\right] = 0, \tag{32}$$

and its energy spectrum results to be the typical *finite* spectrum of the Hulten potential

$$|E_{hp}| = \frac{1}{4R^2\bar{m}}(k_2\frac{1}{n} - n)^2, \;\; n = 1, 2, 3, ... \tag{33}$$

An important feature of hadronic bound states is that, since we can ignore Coulomb interactions and solely assume *contact* interactions represented with the exponent of the isotopic element (4), *the value of the binding energy E is expected to be small or null,*

$$|E| = \frac{1}{4R^2\bar{m}}(k_2\frac{1}{n} - n)^2 = 0. \tag{34}$$

This is due to the fact that *contact interactions do not carry potential energy* (this is classically the case for a balloon moved by winds in our atmosphere).

Therefore, *the Hulten potential for consistent hadronic bound states is expected to admit one single energy level, the ground state* since all possible excited states imply radial distances bigger than the hadronic horizon R, with consequential recovering of quantum mechanics.

In particular, property (34) is solely possible for the following values of the k-[arameters

$$k_1 \; > \; 0, \;\; k_2 \; \geq \; 1. \tag{35}$$

The absence of a spectrum of energies was called in Section 5 of Ref. [19] the *hadronic suppression of quantum mechanical energy spectra* in order to differentiate the *quantum mechanical classification of hadrons into families* (which is characterized by energy spectra) from the *structure of individual hadrons of a given classification family* (which is expected to require different constituents for different particles due to the general difference of spontaneous decays with the lowest mode).

2.4. Relativistic hadronic bound states.

The relativistic counterpart of Eqs. (25) was identified, apparently for the first time, in Refs. [35] [36] and was formulated in the isoproduct of a real-valued iso-Minkowski isospace for orbital motions and a complex valued iso-Euclidean isospace for the hadronic spin

$$\hat{S}_{tot} = \hat{M}(\hat{x}, \hat{\eta}, \hat{I}_{orb}) \star \hat{R}(\hat{z}\hat{\delta}, \hat{I}_{spin}), \tag{36}$$

resulting in the following *irregular extension of the Dirac-Santilli isoequation (I-88)* ,

$$[\hat{\Omega}^{\mu\nu} \star \hat{\Gamma}_\mu \star \hat{\partial}_\nu + \hat{M} \star \hat{C} - \hat{V}_{hp}]\hat{\psi}(\hat{x}) = \\ = (-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_\mu\partial_\nu + mC - V_{hp})\hat{\psi}(\hat{x}) = 0, \tag{37}$$

where $\hat{S} = S\hat{I}_{orb}$ represents strong interactions, and the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\hat{\gamma}_k = \frac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 = \frac{i}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}; \tag{38}$$

where $\hat{\sigma}_k$ are the *irregular Pauli-Santilli isomatrices* studied in Section II-3.4 with the following *anti-isocommutation rules*

$$\{\hat{\gamma}_\mu,\hat{\gamma}_\nu\} = \hat{\gamma}_\mu\hat{T}\hat{\gamma}_\nu + \hat{\gamma}_\nu\hat{T}\hat{\gamma}_\mu = \\ = 2\hat{\eta}_{\mu\nu}. \tag{39}$$

where $\hat{\eta}$ is the isometric of the orbital iso-Minkowskian isospace.

2.5. Einstein's determinism in the structure of mesons.

2.5.1. Insufficiencies of quark conjectures. While the classification of hadrons into families has received a rather large consensus since its initiation by M. Gell-Mann in the 1960's [75], the conjecture that the hypothetical quarks are the actual physical constituents of hadrons has been controversial since its inception. These problematic aspects were reviewed in detail in the 1979 paper [20] (written at the Department of Mathematics of Harvard University under DOE support), and can be summarized as follows:

1) The quantum mechanical classification of point-like particles permitted by the $SU(3)$ model and, more recently, by the standard model, has indeed achieved a satisfactory *classification* of hadrons into families.

2) Quarks are purely mathematical representations of a purely mathematical unitary symmetry defined on a purely mathematical complex-valued internal space and, as such, quarks cannot be the actual physical constituents of hadrons for numerous insufficiencies or sheer inconsistencies, such as:

2A) By recalling that quarks have to be point-like as a necessary condition to maintain the validity of quantum mechanics in the interior of hadrons, the ensuing conception of the hyperdense hadrons as ideal spheres with point-particles in their interior is not realistic;

2B) Quarks cannot be physical particles in our spacetime, and the same holds for their masses, because they cannot be defined as unitary irreducible representations of the Lorentz-Poincaré symmetry (Section 3.9);

2C) Quarks cannot be rigorously confined inside hadrons (i.e,. confined with a rigorously proved, identically null probability of tunnel effect) due to the uncertainty principle;

2D) Quarks have not been directly detected under collisions at the extremely high energies achieved at CERN and at other particle physics laboratories;

2E) The wavepackets of all particles are of the same order of magnitude of the size of all hadrons. Hence, the hyperdense character of hadrons is due to the total mutual penetration of the wavepackets of their constituents, resulting in non-linear, non-local and non-Hamiltonian internal interactions under which the SU(3) and other symmetries cannot be consistently defined (Sections 3, 4).

3) History has thought that the study of atoms (as well as of other natural systems) required *two* different yet compatible models, one for the *classification* of atoms into family, and a different model for the *structure* of each atom of a given classification family. Particularly significant is the fact that the classification of atoms could be achieved via the use of *pre-existing mathematics*, while the structure of atoms required *new mathematics*, such as the Hilbert spaces, that are unnecessary for the classification of atoms.

In order to resolve the insufficiencies of the conjecture that quarks are physical particles, Santilli [20] suggested to follow the teaching of the history of science, and study hadrons via *two* different models, the standard model for the classification of hadrons and a different, yet compatible, model for the structure of individual hadrons of a given classification multiplet.

In particular, the classification of hadrons can be effectively done via quantum mechanics because individual hadrons can be well approximated as being point-like particles in vacuum. By contrast, the structure of hadrons requires a necessary "completion" of quantum mechanics into a covering theory suggested beginning with the title of Ref. [20] in view of the unavoidable, internal, non-linear, non-local, and non-potential interactions.

2.5.2. Hadronic structure model of mesons. A primary aim of papers [18] [19] [20] and monographs [22] [23] of 1978-1979 was the "completion" of quantum mechanics (qm) into the covering hadronic mechanics (hm) for the specific intent of achieving a representation of *all* characteristics of mesons as hadronic bound states of actual, massive, physical particles

Table I. The Technical Origin of Some of the Controversies in Hadron Physics*

$\{\pi^0 : 1^-, 0^-, +, 135, 0.8 \times 10^{-16}, \sim 1\ F\}$, $\{\pi^\pm : 1^-, 0^-, \ldots, 139, 2.6 \times 10^{-8}, \sim 1\ F\}$
$\{K^\pm : \frac{1}{2}, 0^-, \ldots, 439, 1.2 \times 10^{-8}, \sim 1\ F\}$, $\{K_S^0 : \frac{1}{2}, 0^-, \ldots, 498, 0.9 \times 10^{-10}, \sim 1\ F\}$
$\{K_L^0 : \frac{1}{2}, 0^-, \ldots, 498, 5.1 \times 10^{-8}, \sim 1\ F\}$, $\{\eta : 0^+, 0^-, +, 549, \Gamma = 0.8\ \text{keV}, \sim 1\ F\}$

$\pi^0 \to \gamma\gamma$, 98.8 %
$\pi^0 \to \gamma e^+ e^-$, 1.15 %
$\pi^0 \to \gamma\gamma\gamma$, $<5 \times 10^{-6}$
$\pi^0 \to e^+e^-e^+e^-$, 3.3×10^{-5}
$\pi^0 \to \gamma\gamma\gamma\gamma$, $<6 \times 10^{-5}$
$\pi^0 \to e^+e^-$, $<2 \times 10^{-5}$
$\pi^\pm \to \mu\nu$, 100 %
$\pi^\pm \to e\nu$, 1.2×10^{-4}
$\pi^\pm \to \mu\nu\gamma$, 1.2×10^{-4}
$\pi^\pm \to \pi^0 e\nu$, 1.02×10^{-5}
$\pi^\pm \to e\nu\gamma$, 3×10^{-4}
$\pi^\pm \to e^+e^-\nu$, $<3.4 \times 10^{-5}$
$K^\pm \to \mu\nu$, 63.6 %
$K^\pm \to \pi\pi^0$, 21.05 %
$K^\pm \to \pi\pi^-\pi^+$, 5.6 %
$K^\pm \to \pi\pi^0\pi^0$, 1.7 %
$K^\pm \to \mu\pi^0\nu$, 3.2 %
$K^\pm \to e\pi^0\nu$, 4.8 %
$K^\pm \to \mu\nu\gamma$, 5.8×10^{-6}
$K^\pm \to e\pi^0\pi^0\nu$, 1.8×10^{-5}
$K^\pm \to \pi\pi^+e^-\nu$, 3.7×10^{-5}
$K^\pm \to \pi\pi^+e^+\nu$, $<5 \times 10^{-7}$
$K^\pm \to \pi\pi^\pm\mu^\pm\nu$, 0.9×10^{-6}
$K^\pm \to \pi\pi^\pm\mu^\pm\nu$, $<3.0 \times 10^{-6}$
$K^\pm \to e\nu$, 1.5×10^{-5}
$K^\pm \to e\nu\gamma$, 1.6×10^{-5}
$K^\pm \to \pi\pi^0\gamma$, 2.7×10^{-6}
$K^\pm \to \pi\pi^+\pi^-\gamma$, 1×10^{-6}
$K^\pm \to \mu\pi^0\nu\gamma$, $<6 \times 10^{-5}$
$K_S^0 \to \pi^0\gamma\gamma$, $<2.4 \times 10^{-6}$
$K_S^0 \to \gamma\gamma$, 4.9×10^{-6}
$K_L^0 \to e\mu$, $<2.0 \times 10^{-8}$
$K_L^0 \to \mu^+\mu^-$, 1.0×10^{-8}
$K_L^0 \to \mu^+\mu^-\gamma$, $<7.8 \times 10^{-6}$
$K_L^0 \to \mu^+\mu^-\pi^0$, $<5.7 \times 10^{-7}$
$K_L^0 \to e^+e^-$, $<2.0 \times 10^{-9}$
$K_L^0 \to e^+e^-$, $<2.8 \times 10^{-5}$
$K_L^0 \to \pi^0\pi^0e^0\nu$, $<2.2 \times 10^{-5}$

$K^\pm \to e\pi^0\nu\gamma$, 3.7×10^{-4}
$K^\pm \to \pi e^+e^-$, 2.6×10^{-7}
$K^\pm \to \pi\nu e^+e^-$, $<1.5 \times 10^{-5}$
$K^\pm \to \pi\mu^+\mu^-$, $<2.4 \times 10^{-8}$
$K^\pm \to \pi\gamma\gamma$, $<3.5 \times 10^{-6}$
$K^\pm \to \pi\gamma\gamma\gamma$, $<3.0 \times 10^{-6}$
$K^\pm \to \pi\nu\bar{\nu}$, $<0.6 \times 10^{-6}$
$K^\pm \to \pi\gamma$, $<4 \times 10^{-6}$
$K^\pm \to e\pi^0\mu^\pm$, $<2.8 \times 10^{-6}$
$K^\pm \to e\pi^0\mu^\mp$, $<1.4 \times 10^{-6}$
$K^\pm \to \mu\nu\nu$, $<6 \times 10^{-6}$
$K_S^0 \to \pi^+\pi^-$, 68.7 %
$K_S^0 \to \pi^0\pi^0$, 31.3 %
$K_S^0 \to \mu^+\mu^-$, $<3.2 \times 10^{-7}$
$K_S^0 \to e^+e^-$, $<3.4 \times 10^{-6}$
$K_S^0 \to \pi^+\pi^-\gamma$, 2.0×10^{-6}
$K_S^0 \to \gamma\gamma$, $<0.4 \times 10^{-6}$
$K_L^0 \to \pi^0\pi^0\pi^0$, 21.4 %
$K_L^0 \to \pi^+\pi^-\pi^0$, 12.2 %
$K_L^0 \to \pi\mu\nu$, 27.1 %
$K_L^0 \to \pi e\nu$, 39.0 %
$K_L^0 \to \pi e\nu\gamma$, 1.3 %
$K_L^0 \to \pi^+\pi^-$, 0.2 %
$K_L^0 \to \pi^0\pi^0$, 0.09 %
$K_L^0 \to \pi^+\pi^-\gamma$, 6.0×10^{-4}
$\eta \to \gamma\gamma$, 38.0 %
$\eta \to \pi^0\gamma\gamma$, 3.1 %
$\eta \to 3\pi^0$, 29.9 %
$\eta \to \pi^+\pi^-\pi^0$, 23.6 %
$\eta \to \pi^+\pi^-\gamma$, 4.9 %
$\eta \to e^+e^-\gamma$, 0.5 %
$\eta \to \pi^0e^+e^-$, <0.04 %
$\eta \to \pi^+\pi^-$, <0.15 %
$\eta \to \pi^+\pi^-e^+e^-$, 0.1 %
$\eta \to \pi^+\pi^-\pi^0\gamma$, $<6 \times 10^{-4}$
$\eta \to \pi^+\pi^-\gamma\gamma$, <0.2 %
$\eta \to \mu^+\mu^-$, 2.2×10^{-5}

* In this table we have listed most of the experimental data on the octet of light mesons from the particle data[80] with only one addition, the inclusion of the (approximate) charge radius of the particles. The data are divided into two groups: those for total

Figure 3: *A reproduction of Table 1, page 429, Ref. [20] used to identify the physical constituents of mesons in the massive particles produced free in the spontaneous decays, generally those with the lowest mode. The "completion" of quantum mechanics into hadronic mechanics then becomes recommendable for any quantitative treatment of the indicated structure model.*

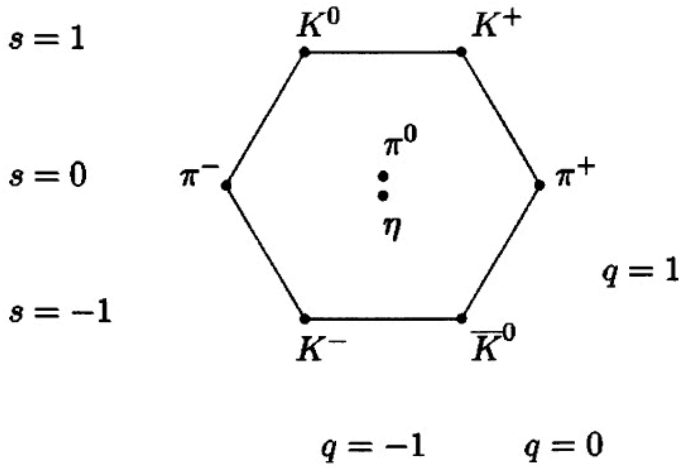

Figure 4: *An illustration of the compatibility between the hadronic structure model of mesons, Eqs. (40)-(45), with physical constituents and the conventional $SU(3)$ classification. This compatibility is achieved via the identification of the isounits or isotopic elements per each meson, Eq. (46), and their use as the hyperunit of the $\tilde{SU}(3)$ hypersymmetry in view of its local isomorphism to the conventional symmetry.*

produced free in the spontaneous decays , generally those with the lowest mode illustrated in Table 1, page 429, Ref. [20] (reproduced in Figure 2).

The above view suggested the following new structure models for the octet of mesons:

$$\pi^0 = (\tilde{e}^-, \tilde{e}^+)_{hm}, \tag{40}$$

$$\pi^\pm = (\tilde{\pi}^0, \tilde{e}^\pm)_{hm}, \tag{41}$$

$$K^0 = (\tilde{\pi}^0, \tilde{\pi}^0)_{hm}, \tag{42}$$

$$K^\pm = (\tilde{\pi}^0, \tilde{\pi}^\pm)_{hm}, \tag{43}$$

$$K_S = (\tilde{\pi}^-, \tilde{\pi}^+)_{hm}, \tag{44}$$

$$K_L = (\tilde{K}^-, \tilde{\pi}^+)_{hm}. \tag{45}$$

In the above models, the role of positrons as physical constituents of mesons provides the first known numerical representation of their very short meanlives, while all other characteristics are numerically represented via the hadronic bound states of Sections 2.3 and 2.4.

Structure models (40) - (45) are incompatible with quantum mechanics because the rest energy of all particles are *bigger* than the sum of the rest energies of the constituents (thus requiring *positive binding energies,*

with ensuing *mass excesses* that are anathema for quantum mechanics as discussed in Section I.3), and for other reasons.

Necessary conditions to prevent misrepresentations is that models (40) - (45) are treated with hadronic mechanics and their constituents are *isoparticles and anti-isoparticles* (Section II-3.9) hereon denoted with an upper tilde.

Therefore, the elementary constituents of the π^0 in Eq.(40) are *one iso-electron* $\tilde{e}^-$ *and one isopositron* $\tilde{e}^+$ (called *eletons* in Ref. [19]); the constituents of the $\pi^\pm$, Eq. (51) are *one iso-meson* $\tilde{\pi}^0$ and *one isoelectron or isopositron;* etc.

Following their emission in the spontaneous via isotunnel effects (i.e., tunnel effects in the iso-Hilbert isospace), the isoconstituents assume conventional quantum mechanical characteristics plus possible secondary effects with the emissions of massless particles.

Note that *all models (40)-(45) either are or can be reduced to two-body hadronic bound states,* thus admitting analytic solutions in their representation via Eqs. (29). Note also that models (40)-(45) have a kind of bootstrap structures, since a given meson appears in the structure of heavier mesons.

Note additionally that *models (40)-(45) imply the increase of the number of "elementary" constituents with the increase of the rest energy.* In fact, the π^0 has only two elementary constituents while K_L has eight elementary constituents.

Note finally that the role of isopositrons as actual physical constituents of mesons provides the *only* known mechanism via particle-antiparticle annihilation for the *quantitative representation* of the very small mean-lives of mesons, by keeping in mind that the rest energy of electron-positron bound states is positive in our world and negative in the antimatter world (Section 2.5.3).

Since different structure models are characterized by numerically different isounits, the compatibility of the above hadronic structure model of mesons with their classification is readily achieved at the higher level of the *hyperstructural branch of hadronic mechanics* [33] [37], with the following total multi-valued *hyperunit*

$$\hat{I}_{tot} \ = \ \{\hat{I}_{\pi^0}, \hat{I}_{\pi^\pm}, \hat{I}_{K^0}, \hat{I}_{K^\pm}, \hat{I}_{K_S}, \hat{I}_{K_L}\}. \tag{46}$$

2.5.3. Positive energy of particle-antiparticle bound states. The *positive* total energy of particle-antiparticle hadronic bound states has been studied in detail in monograph [38] (see, e.g., Section 2.3.14, page 131), and in various additional works, including all historical references.

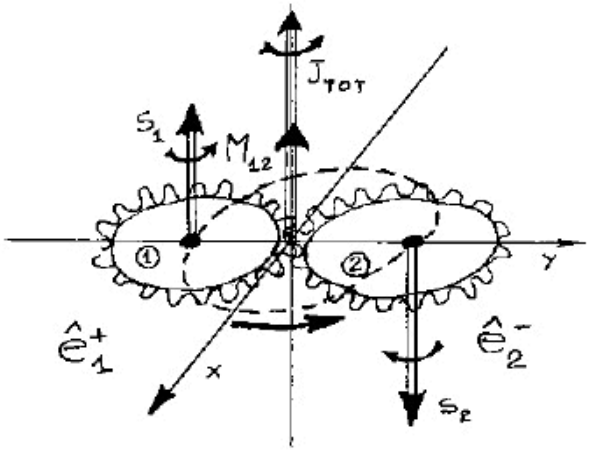

Figure 5: *A reproduction of "gear model" used in Section 5, page 852, Ref. [19] to illustrate the strongly attractive character of contact non-Hamiltonian forces when the constituents are in singlet coupling and their strongly repulsive character when in triplet couplings.*

We cannot possibly review here the underlying *isodual theory of antimatter*, but an indication of the following main notions appears recommendable to avoid basic misconceptions.

Recall that *negative energies violate causality*, for which reason P. A. M. Dirac was forced to work out his "hole theory."

Santilli resolved the problem of causality by referring all physical quantities, thus including energies, to *isodual units defined on isodual fields with a 'negative' unit.*

Hence, the terms "negative energy of antiparticles" have no meaning in the field here considered, because the correct statement is "negative energies of antiparticles measured to negative units of energy."

Within the above setting, Ref. [38], Eqs. (2.3.77), page 131, shows that *the total energy of a particle-antiparticle bound state is positive when measured in our world (with positive units) and negative when measured in the antimatter word (with negative units).*

The above property is truly fundamental for a consistent quantitative representation of all characteristics of unstable particles.

In fact, the presence of antiparticles in the structure of mesons and (unstable) baryons, first proposed by Santilli in 1978 [19], appears to be the most plausible origin of the extreme instability of mesons and (unstable) baryons with mean lives such as $\tau = 10^{-16}\ s$ or less, with no equally effec-

tive or plausible alternative known to this day.

2.5.4. Hadronic structure model of the π^0 meson. The characteristics of the π^0 meson are:

1. Rest energy $E = 134.96MeV$,
2) Mean-life $\tau = 0.828 \times 10^{-16}\ s$,
3) Charge radius $R = 10^{-13}\ cm$,
4) Null charge and spin.
5) Null electric and magnetic moments,
6) Negative parity; and
7) Primary decay

$$\pi^0 \to \gamma + \gamma,\ \ 98.85\ \%. \tag{47}$$

The above characteristics are *all* numerically represented by hadronic bound state (29)) as a "compressed" form of the positronium (Pos), resulting in hadronic bound state (40) of one isoelectron and one isopositron (Ref. [19], Section 5, page 828 on)

$$Pos = (e^-, e^+)_{qm} \ \ \to \ \ \pi^0 = (e^-, e^+)_{hm}, \tag{48}$$

with hadronic structure equations

$$\begin{aligned} &\left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2 \tfrac{e^{-br}}{1-e^{-br}})\right] = 0, \\ &E_{tot} = E_{\tilde{e}^-} + E_{\tilde{e}^+} - E = 135\ MeV\ \ \bar{m} = \tfrac{m}{I^2} \\ &\tau^{-1} = 2\pi\lambda^2 |\hat{\psi}(0)|^2 \tfrac{\alpha^2 E_{\tilde{e}}}{\hbar} = 10^{-16} s \\ &R = b^{-1} = 10^{-13}\ cm = 1\ fm, \end{aligned} \tag{49}$$

and numeric solution

$$k_1 = 0.34,\ \ k_2 = 1 + 4.27 \times 10^{-2}, \tag{50}$$

verifying conditions (35) as expected.

In appraising representation (48), the reader should keep in mind the assumption of Section 2.3 of '*constrained, nearly constant and circular orbits of the constituents*' under which the isounit can be considered independent from local coordinates.

The above results confirm all expectations indicated in preceding sections, namely,

1) Hadronic spectrum (33) admits one and only one energy level, the π^0, since all excited states are those of the positronium;

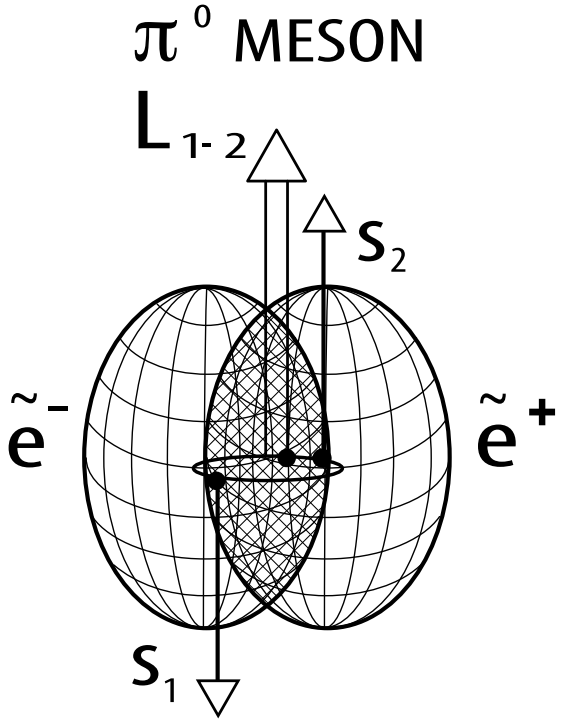

Figure 6: *A conceptual rendering of the structure model of the π^0 meson as a hadronic bound state of one isoelectron $\tilde{e}^-$ and one isopositron $\tilde{e}^+$ in singlet coupling with unmutated spins $s_1 = -s_2 = 1/2$ and orbital hadronic momentum in the ground state $L_{1-2} = 0$. The dashed area represents the contact, non-linear, non-local and non-potential interactions responsible for the bound state that, being not representable with a Hamiltonian H, are representable via the isotopic element $\hat{T}$ in the iso-Schrödinger equation $H \star \hat{\psi} = H\hat{T}\hat{\psi} = E\hat{\psi}$.*

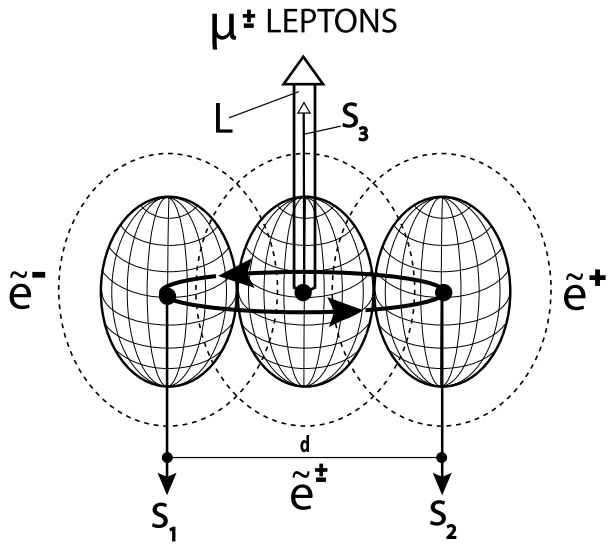

Figure 7: *To achieve compatibility with the standard model, it is generally believed that muons are elementary particles in contrast with the experimental evidence that they are naturally unstable with various decay modes (51) that strongly suggest a composite structure. In this figure, we present a conceptual rendering of the structure of the* $\mu^{\pm}$ *leptons predicted by hadronic mechanics as a three-body of elementary isoconstituents identified in the muon decay with the lowest mode. The weak interaction character of the particles is derived from a weak hadronic bound state essentially given by the overlapping of the classical radius of the constituents indicated by dashed ellipses. By recalling that all three constituents have a point-like charge, the dimension* d *of the charge radius of the two peripheral constituents is purely nominal.*

2) The presence of the antiparticle $\tilde{e}^+$ in the π^0 structure (40) explains its very short mean-life as well as its main decay (47);

3) Isodeterministic conditions (11) are verified by model (40), as a result of which standard deviations Δr and Δp have individual values smaller than one (Lemma II-3.7) and the isoperturbation series have no divergences (Corollary II-3.7.1).

2.5.5. Hadronic structure model of the remaining mesons. Recall the main characteristics of the muons $\mu^\pm$ are:

1) Rest energy $105,658\ MeV$;

2) Charge radius $R = 10^{-13}\ cm$;

3) Mean-life $\tau = 2.19703 \times 10^{-6}\ s$;

4) Spin $1/2$ and elementary charge;

5) Spontaneous decay

$$\begin{aligned} \mu^\pm &\to e^- + \nu + \bar{\nu}, \\ \mu^\pm &\to e^- + \gamma, \\ \mu^\pm &\to e^- + 2\gamma, \\ \mu^\pm &\to e^- + e^\pm + e^+. \end{aligned} \tag{51}$$

For the intent of achieving compatibility with the standard model, the $\mu^\pm$ leptons are considered to be "elementary particles." Santilli [19] cannot accept such a view because in contrast with the experimental evidence that *the muons are naturally unstable particles with decays (51).*

Therefore, Ref. [19], Section 5, proposed the structure model of the muons with the physical constituents identified by the decay mode (51) with the smallest mode $< 10 \times 10^{-12}$ (Figure 5)

$$\mu^\pm = (e^-, e^\pm, e^+)_{hm}. \tag{52}$$

The birth of weak interactions was suggested as being due to a weak form of wave-overlapping essentially that of the classical size of the constituents

$$R_{e^\pm} = 2.28 \times 10^{-13}\ cm, \tag{53}$$

which said weak contact interactions can be quantitatively represented by hadronic mechanics via an isotopic element $\hat{T}$ with values close to 1.

A main aspect is that there is the birth of strong interactions in model (52) because we do not have a wave-overlapping inside the hadronic horizon, as it is the case for the π^0.

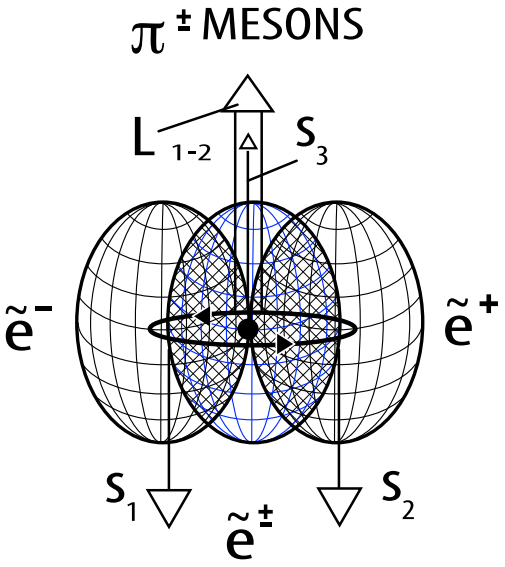

Figure 8: *A conceptual rendering of the structure of the $\pi^{\pm}$ mesons as a three-body hadronic bound states (57) of elementary isoparticles which can be interpreted as a "compressed $\mu^{\pm}$ lepton" (Figure 6). Note that individual pairs of isoconstituents are coupled in singlet as necessary for consistency. Note also that the isoconstituents have a point-like charge structure. Therefore, the charge diameter of the structure is given by the distance between the centers of the two peripheral isoconstituents.*

It is easy to see that model (52) can represent the muon spin $1/2$ and decay (51) with the lowest mode as a tunnel effect of the constituents.

The remaining characteristics can be represented via a direct analytic solution of model (52) as a restricted three-body problem.

Ref. [19] suggested the approximation of the model into a two-body structure with weakly bounded constituents and the structure equations

$$\begin{gathered}\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr})+\bar{m}(E\pm W_1e^{\frac{2}{r}}+W_2\frac{e^{-br}}{1-e^{-br}})\right]=0,\\ E_{tot}=E_{100\ Mev}+E_{e^\pm}-E,=105\ MeV\ \ \bar{m}=\frac{m}{I^2}\\ \tau^{-1}=2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2E_e}{\hbar}=10^{-6}\ s\\ R=b^{-1}=10^{-13\ cm},\end{gathered}\tag{54}$$

admitting the following values for the the k-parameters

$$k_1=0.93,\ \ k_2=1+8.47\times10^{-2},\tag{55}$$

that also verify conditions (35).

The main characteristics of the $\pi^\pm$ mesons are:

1) Rest energy $139.570\ MeV$;

2) Charge radius $R=10^{-15}\ cm$;

3) Mean-life $\tau=2.603\times10^{-8}\ s$;

4) Spin $J=0$ and elementary charge;

5) Decay with lowest mode $<3.2\times10^{-9}$)

$$\pi^\pm\rightarrow e^-+e^\pm+e^++\nu\pi^\pm\tag{56}$$

.

The above characteristics are all represented by the hadronic structure model of the $\pi^\pm$ mesons first proposed in Section 5, Ref. [19]

$$\pi^\pm\ =(\tilde{e}^-,\tilde{e}^\pm,\tilde{e}^+)_{hm}\approx(\tilde{\pi}^0,\tilde{e}^\pm)_{hm},\tag{57}$$

for which, unlike the case of the muons, the constituents are elementary isoparticles in condition of deep mutual penetration inside the hadronic horizon (Figure 7).

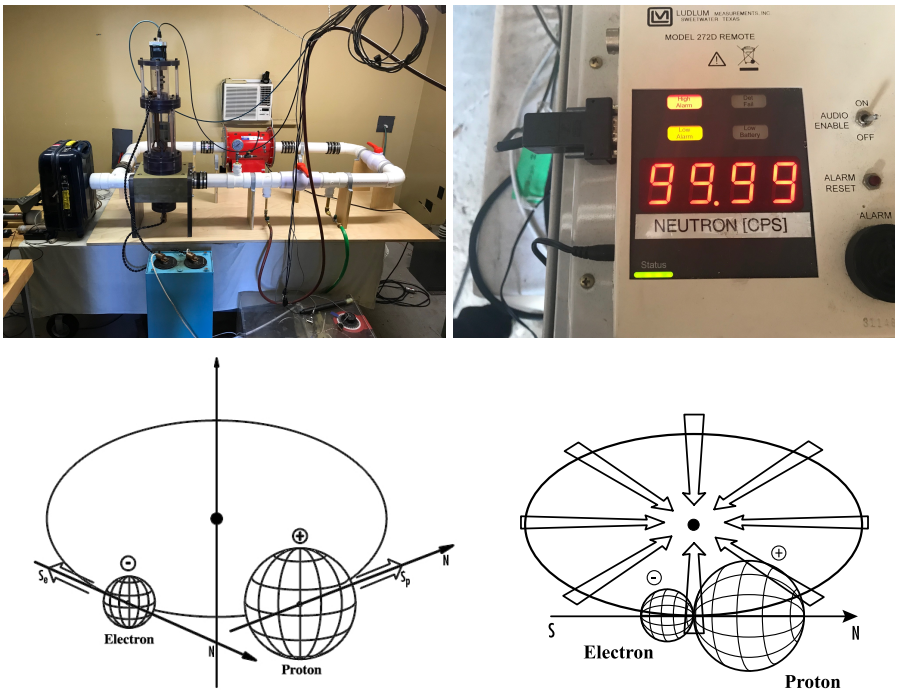

Figure 9: *In this figure we show: 1) At the top left a picture of the Directional Neutron Source (DNS) developed by Santilli at Thunder Energies Corporation, now Hadronic Technologies Corporation [39] which synthesizes on demand neutral and negatively charged hadrons from a hydrogen gas with preferred directionality, energy and flux; 2) At the top right a typical case of neutron CPS; 3) At the bottom left a conceptual rendering of the ionization of the hydrogen gas by the electric arc during its activation and the proper axial alignment of the proton and electron with opposite charge and magnetic polarities; and 4) At the bottom right a conceptual rendering of the "compression" of the electron inside the proton by the electric arc during its disconnection.*

The rest energy, mean-life and charge radius of the $\pi^{\pm}$ are readily represented by model (57) with the hadronic structure equations

$$\begin{gathered}\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2\frac{e^{-br}}{1-e^{-br}})\right] = 0,\\ E_{tot} = E_{\tilde{\pi}^{\pm}} + E_{\tilde{e}^{\pm}} - E = 139\ MeV,\ \ \bar{m} = \frac{m}{I^2}\\ \tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_{\tilde{e}^{\pm}}}{\hbar} = 2.603\times 10^{-8}\ s,\\ R = b^{-1} = 10^{-13}\ cm,\end{gathered} \tag{58}$$

admitting solutions for the k-parameters

$$k_1 = 0.34,\ \ k_2 = 1 + 3.67\times 10^{-3}, \tag{59}$$

that again verify conditions (35).

The main recent advance since the above 1978 proposal [19] is the achievement of a consistent representation of the total angular momentum $J = 0$ of the $\pi^{\pm}$ meson thanks to the irregular $SU(2)$ isosymmetry (Section 3.5). In essence, when the three isoparticles are all compressed inside the hadronic horizon, the orbital motion of the two peripheral isoconstituents L_{1-2} is constrained, for stability, to be equal to the spin $S_{\tilde{e}^{\pm}}$ of the central isoparticle, thus having value $L_{1-2} = S_{tildee^{\pm}} = 1/2$ with total angular momentum (Sections II-3.4 and II-3.5, Section II-3.5.3 in particular).

$$J_{tot} = s_1 + s_2 + s_3 + L_{1-2} = -1/2 + 1/2 - 1/2 - 1/2 = 0. \tag{60}$$

Note that model (57) can be interpreted as a form of "compressed muons" (52) (see Figures 6 and 7). The orbital motion of the peripheral electrons is unrestricted in model (52), yielding a total angular momentum $1/2$. By contrast, the peripheral isoelectrons and isopositron in model (57) are *constrained to orbit inside the central isoelectron or isopositron*, thus being forced to have the orbital value equal to the spin of the isoparticle. In fact, orbital values different than $1/2$ would imply extreme resistive forces with ensuing excessive instabilities.

In Santilli's view, an important aspect of model (57) is its apparent verification of the isodeterminism of Lemma II-3.7 as well as of the lack of divergencies in the isoperturbation seres (Corollary II-3.7.1) due to the small value of the isotopic element assured by values (59).

Interested readers can verify that values (35) remain verified under the use of more accurate experimental values for the characteristics of muons and mesons.

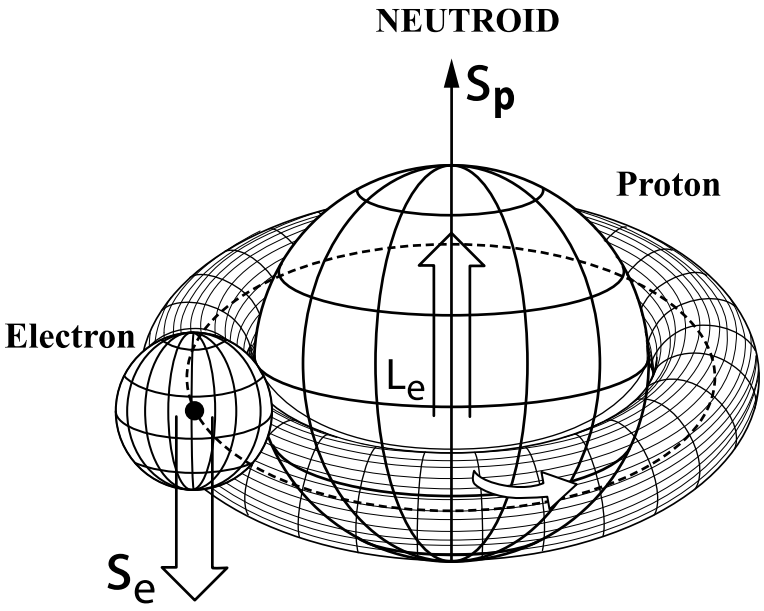

Figure 10: *A conceptual rendering of the structure of a particle called the neutroid* $\tilde{n} = (e^-, p^+)_{hm}$*, Eq. (61), with spin* $J = 1$*, charge radius* $R = 10^{-13}$ *cm, mass essentially that of the proton and mean-life of the order of* 5 *s which has been undetected in all experiments to date [I-90]-[I-95] by neutron detectors, yet has caused nuclear transmutations typical of neutron irradiation. The bound state is created by the extremely strong,* $r - p$ *Coulomb attraction which is of the order of* 10^{24} *N plus weak contact interactions implying ignorable mutations of the electron and the proton. The* $e - p$ *singlet coupling and the orbital motion in the ground state imply the spin* $J = 1$ *explaining the lack of detection of neutroids by neutron detectors. Note that the neutroid is impossible for quantum mechanics.*

It is an instructive exercise for the interested reader to work out the hadronic structure models of the remaining mesons, Eqs. (42)-(45) showing the increase of the k_1 value and the decrease of the k_2 value with the increase of the rest energy (see Section 6.2 of Ref. [31] for an independent review).

2.6. Einstein's determinism in the structure of baryons.

2.6.1. Structure of the neutroid. Stars initiate their lives as being composed of hydrogen; they grow in time via the accretion of interstellar-intergalactic hydrogen; and eventually reach such pressures and tempera-

tures in their core to synthesize the neutron as a "compressed" hydrogen atom according to H. Rutherford [40].

A main aspect of the studies herein considered is that, in Santilli's view [20], the synthesis of the neutron from the hydrogen in the core of stars is one of the best illustrations of the validity of the EPR argument [1] because said synthesis cannot be represented with quantum mechanics for various technical reasons reviewed in Section I-1, in denial of the experimental evidence that, *having opposite charges, the proton and the electron at mutual distances of* $10^{-13}\ cm$ *are attracted to each other with a Coulomb force of about* $230\ N$ *(See Eq, (107) for its calculation). Such an attractive force is so big for particle standards to dismiss as inapplicable any theory unable of producing a bound state under such an anormous attractive force.*

Consequently, Santilli dedicated decades of mathematical, theoretical, experimental and industrial research on the neutron synthesis due to its truly fundamental character for all quantitative sciences (see Refs. (I-85] to [I-95] and independent reviews [31] [41]).

Besides a number of preceding attempts without neutron detection, the first indirect experimental detection of the synthesis of the neutron from the hydrogen was initiated by in the 1960's by Don Carlo Borghi [76], confirmed in subsequent tests [I-90] - [I-95], and routinely verifiable with the Directional Neutron Source developed by Hadronic Technologies Corporation [39] (Figure 8), indicate the existence of an unstable, neutral, intermediate state called *neutroid* (denoted with the symbol $\tilde{n}$) which is unidentified by neutron detectors, while causing nuclear transmutations typically triggered by neutron irradiation.

Hadronic mechanics suggests the following representation of the structure of the neutroid (Figure 9)

$$\tilde{n} = (\tilde{e}^-_\downarrow, p^+_\uparrow)_{hm}, \tag{61}$$

consisting of an isoelectron $\tilde{e}^-$ with spin $s_1 = 1/2$ in singlet contact coupling with a standard proton p^+ with spin $s_2 = 1/2$ and orbital motion $L_{1-2} = 0$ in the ground state.

These assumptions suggest the following predicted features of the neutroid:

1) Rest energy estimated to be about $940\ MeV$,

2) Mean-life estimated to be of at least $\tau = 5\ s$,

3) Charge radius estimated to be of about $R \approx 10^{-13}\ cm$,

4) Spin predicted to be $J = 0$, and

5) Spontaneous decay

$$\tilde{n} \to e^- + p^. \tag{62}$$

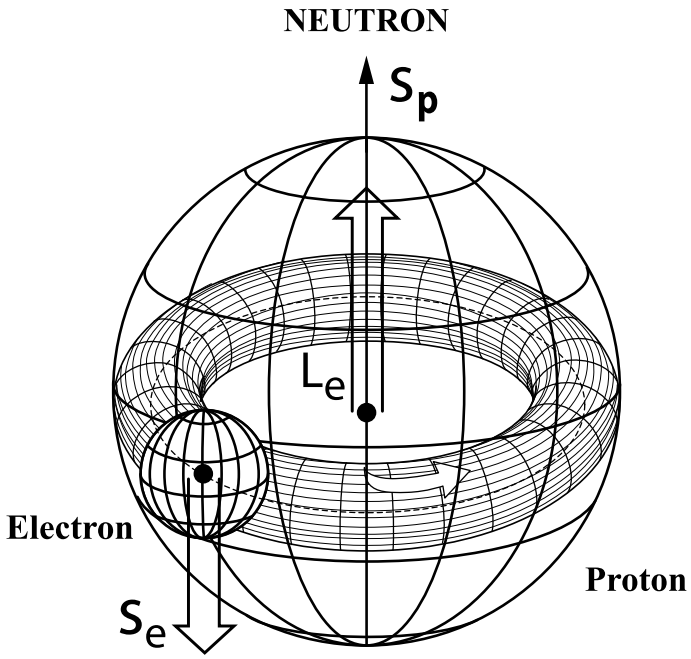

Figure 11: *A conceptual rendering of the structure of the neutron as "compressed" hydrogen atoms in the core of stars according to H. Rutherford [40]. When the proton is represented as an extended particle, there is the emergence of a constrained orbital motion of the electron inside the proton verifying the isodeterminism of Lemma 3.7, which allows the first known exact representation of all the characteristics of the neutron at the nonrelativistic [42] and relativistic [36] levels without need for the neutrino hypothesis.*

The above features can be represented with hadronic bound state (29) specialized to the indicated values

$$\begin{gathered}\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr})+\bar{m}(E\pm W_1\frac{e^2}{r}+W_2\frac{e^{-br}}{1-e^{-br}})\right]=0,\\ E_{tot}=E_{\tilde{e}}+E_p-E=940\ MeV\\ \tau^{-1}=2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2E_1}{\hbar}=5\ s\\ R=b^{-1}=10^{-13}\ cm,\end{gathered}\tag{63}$$

with values of the k-parameters

$$k_1\approx 1,\ \ k_2\approx 1+10^{-6},\tag{64}$$

verifying the crucial condition (35).

Once absorbed by a stable nucleus $N(A,Z,J)$, the neutroid is transformed by strong interactions into a neutron plus secondary emissions, resulting in a generally untabulated unstable nucleus $\tilde{N}(A+1,Z,J+1/2)$ that decays into stable nuclei plus radiations.

The spin 0 of the neutroid explains the impossibility of its detection via conventional neutron detectors.

The decay of the nucleus $\tilde{N}(A+1,Z,J+1/2)$ explains the triggering by undetectable particles of conventional nuclear transmutations normally triggered by neutron irradiations.

Hadronic bound state (61) is clearly impossible for quantum mechanics, but readily possible for hadronic mechanics via structure model (29) due to the combination of strongly attractive Coulomb and contact forces at mutual distances of the order of $10^{-13}\ cm$ with isotopic elements of the simple type (19) verifying isodeterministic Lemma II-3.7. and the rapid convergence of isoperturbation series of Corollary II-3.7.1.

2.6.2. Nonrelativistic representation of the neutron synthesis. We consider now the synthesis of the neutron on demand from a hydrogen gas in the needed directionality, (low) energy and CPS.

Such a synthesis has been first achieved by the Directional Neutron Source (DNS) developed by Santilli at Thunder Energies Corporation, now Hadronic Technologies Corporation [39].

Recall that, when seen in an oscilloscope set for milliseconds, electric arcs between carbon electrodes, are continuously connected and disconnected to seek the shortest distance for the discharge.

With reference to Figure 9, the electric arc ionizes the hydrogen gas during its connection and creates an axial alignment along a magnetic line of electrons and protons with opposite charge and magnetic polarities.

By contrast, during their disconnection electric arcs "compress" the electron inside the proton by synthesizing in this way the neutron in a way crucially dependent on the shape of the arc, its power and other factors.

With reference to Figure 11, when compressed inside the hyperdense proton, the much lighter electron is "constrained" to have an orbital motion equal to the proton spin so as to avoid extreme resistive forces with ensuing high instabilities.

Note that the neutroid (figure 10) appears to be an unavoidable intermediate step prior to the full synthesis of the neutron (Figure 11).

The well known main characteristics of the neutron are the following:

1) Rest energy $939.565\ MeV$;

2) Charge radius $R = 1.73 \times 10^{-13}\ cm$;

3) Mean-life $\tau = 881\ s(about\ 15\ m)$;

4) Spin $1/2$;

5) Charge ;0

6) Anomalous magnetic moment $\mu_n = -1.9\frac{e}{2m_pc}$ and null electric dipole moment;

7) Decay

$$n\ \rightarrow\ p^+ + e^- + \bar{\nu}. \tag{65}$$

The nonrelativistic representation of *all*— characteristics of the neutron in its synthesis from the hydrogen was first achieved by Santilli in the 1990 paper [43] via hadronic bound state

$$n\ \ =\ \ (\tilde{e}^-_{\downarrow}, \tilde{p}^+_{\uparrow})_{hm}, \tag{66}$$

where one should note that, unlike the case of the neutroid (61), both the electron and the proton are mutated into isoparticles.

Note also that model (66) represents a *compressed neutroid* in a way much similar to the structure of the $\pi^{\pm}$ mesons as compressed muons , $\mu^{\pm}$ (Section 2.5.4).

The representation of the rest energy, charge radius, mean-life, charge, parity and tunnel effect decay of the neutron have been first achieved in

Ref. [43] Eqs. (2.19), page 521, via hadronic two-body bound state

$$\begin{gathered}\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr})+\bar{m}(E\pm W_1\frac{e^2}{r}+W_2\frac{e^{-br}}{1-e^{-br}})\right]=0,\\ E_{tot}=E_{\tilde{e}}+E_{\tilde{p}}-E=939\ MeV\\ \tau^{-1}=2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2E_{\tilde{e}}}{\hbar}=881\ s\ (about 15m)\\ R=b^{-1}=10^{-13}cm,\end{gathered}\tag{67}$$

with values of the k-parameters

$$k_1=2.6,\ \ k_2=1+0.81\times10^{-8},\tag{68}$$

verifying conditions (35) for the validity of isodeterminism inside the neutron and the validity of conventional uncertainties in its outside.

The representation of the spin $1/2$ of the neutron was also achieved for the first time in Ref. [43], Eqs. (2.22)-(2.37), thanks to the appearance of a constrained *orbital* motion of the isoelectron when totally "compressed" inside the proton (which orbital motion is completely non-existent for quantum mechanics).

The representation of the spin $1/2$ is additionally permitted by the *isotopies of spin-orbit couplings* (see Chapter 6, page 209 on, Ref. [33], for a detailed treatment), which the hadronic angular momentum of the isoelectron is constrained to be equal to the spin of the isoproton, as a necessary condition to avoid big instabilities according to Eqs. (iI-118) under constraint (II-119). Hence, $J^n_{tot}=s_p+s_e+L_e=1/2$, namely, *the spin of the neutron coincides with that of the proton.*

Recall the following experimental values of magnetic moments in nuclear units $\mu_N=\frac{e}{2m_pc}$

$$\begin{gathered}\mu_n^{exp}=-1.91\ \mu_N\\ \mu_p^{exp}=2.7\ \mu_N\\ \mu_e^{exp}=1\ \mu_B=5\times10^{-4}\ \mu_N,\end{gathered}\tag{69}$$

where one can see that the magnetic moment of the neutron is "anomalous" because outside the prediction of quantum mechanics both in its direction and numeric value.

The first known exact representation of the anomalous magnetic moment of the neutron was achieved in Ref. [43], Eqs. (2.39)-(2.41), page 526, via the novel contribution of the internal orbital motion of the isoelectron.

By noting from values (233) that the magnetic moment of the electron is very small for nuclear standard, thus being ignorable in first approximation, the main assumption of Ref. [43] is that *the magnetic moment of the neutron of model (66) is given by the magnetic moment of the proton plus the orbital magnetic moment ot the isoelectron inside the proton.*

Note that said contribution is completely absent in quantum mechanics due to its point-like approximation of particles as being point-like when at mutual distances *smaller* than their charge diameter.

Note that the electron is negatively charged and, consequently, the contribution to the magnetic moment of the neutron from its rotation inside the proton in the direction of the proton spin is *negative,* thus providing the first known representation of the *anomalous direction* of the neutron magnetic moment.

Calculations done in Ref. [43] for the magnetic moment of the isoelectron orbiting inside the proton yield the value $\mu_{\tilde{e}}^{orb} \approx -4.6\ \mu_N$, by therefore reaching the first known representation of the *anomalous value* of the neutron magnetic moment (Eqs. (2.40), page 526, Ref. [43], see also Ref. [31], Section 6.3.D and paper [41], Section 4, page 41)

$$\mu_n = \mu_{\tilde{p}} + \mu_{\tilde{e}}^{orb} + \mu_{\tilde{e}}^{intr} \approx$$

$$\approx \mu_{\tilde{p}} + \mu_{\tilde{e}}^{orb} = \tag{70}$$

$$= 2.7\ \mu_N - 4.6\ \mu_N = -1.91\ \mu_N = \mu_n^{exp}.$$

Note that *the negative sign of the neutron magnetic moment can be considered as direct evidence of the presence of an electron in singlet internal coupling in the proton,* since no other known particle besides the electron can provide the internal, rather large contribution $\mu_{\tilde{e}}^{orb} = -4.6\ \mu_N$.

In Santilli's view, *the inability by quantum mechanics to represent both the anomalous direction and value of the neutron magnetic moment constitutes an additional evidence beyond scientific doubt supporting Einstein's vision on the lack of "completion" of quantum mechanics.*

Furthermore, *the impossibility for quantum mechanics to characterize the motion of the electron compressed inside the proton, constitutes evidence that the most important "completion" of quantum mechanics needed to represent experimental values is the characterization of particles with their actual shape and density.*

The parity and null value of the electric dipole moment were represented via a isotopy of conventional lines.

For historical comments on the birth of the neutrino hypothesis in quantum mechanics [13] and its possible replacement with the etherino hypoth-

esis [42] in hadronic mechanics, one may consult Section II-3.4.1 on Irregular Pauli-Santilli isomatrices.

The reader should keep in mind that the neutron is naturally unstable (when isolated). Consequently, the synthesis of the neutron is a time-irreversible process. It then follows that the Lie-isotopic isomechanical treatment presented in this section is an approximation of the broader treatment via the covering Lie-admissible genomechanics outlined in Section 1-2 (see Refs. [32] [33] for extensive treatments).

With reference to the geno-Schrödinger and geno-Heisenberg equations (I-10) and (I-11), respectively, the genomechanical treatment can be studied via the extension of isotopic element (19) into a time-dependent form characterizing the genotopic elements for motions forward and backward in time.

2.6.3. Relativistic representation of the neutron synthesis. The relativistic representation of *all* characteristics of the neutron in synthesis (66) was first achieved by Santilli in Refs. [35] [36] (see independent review [31], Section 6.3, page 342 on) via the isosymmetry of the irregular Dirac-Santilli isoequation (37), namely, the isotopy $\hat{\mathcal{P}}(3.1)$ of the spinorial covering of the Poincaré symmetry (Section 2.5.11) and can be outlined as follows.

Recall that the non-potential hadronic representation of strong interactions via the isotopic element (1) (Section 2.3) implies the "absorption" of the Coulomb potential by the Hulten potential, as essentially implied by the charge independence of strong interactions.

This feature permits to ignore Coulomb binding energies in first approximation, resulting in a *weakly bounded* relativistic hadronic structure of the neutron, namely, a state with a small binding energy typical of all non-potential interactions.

Consider first synthesis (61) of the neutroid, and assume its representation in the iso-Minkowski isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{I})$ with isospacetime (II-19). Assume in first approximation that the proton is perfectly spherical for which $n_1 = n_2 = n_3 = 1$ and assume that the density of the region of neutroid-proton overlapping is close to that of the vacuum, thus implying the value $n_4 = 1$. These assumptions imply the simplified isometric

$$\hat{\eta} = Diag.(1, 1, 1, -1)e^{-K}, \quad K > 0. \tag{71}$$

The relativistic version of the synthesis (61) with the representation of all characteristics of the neutroid then follows via a simply isotopy of the relativistic treatment of the hydrogen atom.

In the transition to the relativistic treatment of the structure of the neutron, Eq. (66), the isoproton cannot any longer be assumed to be perfectly

spherical and the density of the overlapping region becomes dominant, resulting in values of the characteristic quantities $n_\mu \neq 1,\ \mu = 1, 2, 3, 4$.

In Refs. [35] [36], Santilli assumes that the value of the characteristic quantity n_4 representing the density of the neutron is equal to the density of the proton-antiproton fireball of the Bose-Einstein correlation [46] [47], resulting in the values

$$n_4 = 0.62, \quad b_4 = \frac{1}{n_4} = 1.62, \tag{72}$$

where $b_4 = 1/n_4$ is the notation used in Ref. [36].

The Lorentz-Santilli isotransforms (II-42) then imply the following *isorenormalization of the rest energy of the electron*, namely a renormalization caused by non-potential interactions (Ref. [36], Eqs. (7.1), page 191)

$$E_e = m_e c^2 = 0.511\ MeV \quad \rightarrow \quad E_{\tilde{e}} = m_e \frac{c^2}{n_4^2} = 1.341\ MeV. \tag{73}$$

As one can see, the above isorenormalization removes the problem of the missing $0.782\ MeV$ energy in the neutron synthesis when represented on the iso-Minkowskian isospace over an isofield, thus rendering consistent the needed isorelativistic isoequations.

It should be stressed that the above isorenormalization continues to be based on the etherino mechanism for the delivery of the missing energy to the neutron [42].

The relativistic representation of the spin of the neutron in synthesis (66) was first achieved in Refs. [35] [36]. Recall from Figure 10 that the spin S_1 of isoelectron is opposite to the spin S_2 of the isoproton. The relativistic spin-orbit coupling implies the *constraint* that the orbital angular momentum L_1 of the isoelectron inside the isoproton be equal to the isoproton spin,

$$L_1 = S_2. \tag{74}$$

The above identity is manifestly impossible for the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1)$ and relativistic quantum mechanics, but it is indeed possible for the covering isosymmetry $\hat{\mathcal{P}}(3.1)$. In fact, with reference to Section II-3.5.3, Eqs. (II-122), identity (II- 241) implies the con-

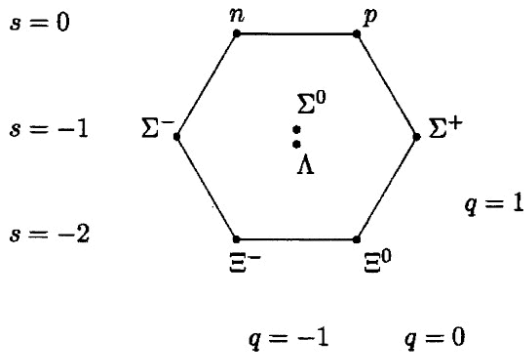

Figure 12: *An illustration of the compatibility of the quantum mechanical $SU(3)$ and subsequent classifications of baryons unto families, and the structure of individual baryons as a hadronic bound state with physical constituents generally emitted free in the spontaneous decays with the lowest mode, Eqs. (241). Said compatibility is achieved via the multi-valued hyperunit characterized by the isounits or isotopic elements of the individual baryons, Eqs. (242), and its use to build the hyperstructural image of the applicable Lie symmetry that turns out to be locally isomorphic to the original symmetry due to the positive-definite character of the isounit.*

ditions (Ref. [36], Eqs. (7.2), page 192)

$$\begin{aligned} \hat{L}_3 \star |\hat{\psi}> = \pm n_1 n_2 |\hat{\psi}> = \\ = \hat{S}_3 \star |\hat{\psi}> = \tfrac{1}{2}\tfrac{1}{n_1 n_2}|\hat{\psi}>, \\ \hat{L}^{\hat{2}} \star |\hat{\psi}> = (n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2)|\hat{\psi}> = \\ = \hat{S}^{\hat{2}} \star |\hat{\psi}> = \tfrac{1}{4}(n_1^{-2} n_2^{-2} + n_2^{-2} n_3^{-2} + n_3^{-2} n_1^{-2})|\hat{\psi}>, \end{aligned} \tag{75}$$

admitting the simple solution

$$\begin{aligned} n_k^2 = \tfrac{1}{\sqrt{2}} = 0.706, \\ b_k^2 = \tfrac{1}{n_k^2} = \sqrt{2} = 1.415, \;\; k = 1, 2, 3, \end{aligned} \tag{76}$$

where $b_k^2 = 1/n_k^2$ in the notation of Ref. [36].

The relativistic representation of the anomalous magnetic moment of the neutron was also achieved for the first time in Ref. [36], Eqs. (7.4), page 192. The representation is again permitted by the contribution from

the orbital motion of the isoelectron inside the isoproton, and it is given by non-relativistic expression (234) with a more accurate representation of the orbital contribution.

It should be indicated that the non-relativistic and relativistic structure models of the neutron outlined in this section are mere approximation of a much more complex reality in which all characteristics of the constituents are mutated, thus including the charge.

2.6.4. Remaining baryons. A central requirement for the consistency of model (66) is that *the excited states of the neutron are the conventional states of the hydrogen atom.* This illustrates again the hadronic suppression of quantum mechanical energy spectra, since the latter are typical for the classification, rather than the structure of hadrons.

By recalling the condition that the number of elementary constituents of hadrons increases with the increase of the rest energy, the hadronic structure of the remaining baryons is reducible to two isoparticle structures derived from the spontaneous decays generally those with the lowest mode, much along the structure of mesons [21] (see [31], Section 6.3.J, page 366 for an independent review)

$$\begin{aligned}
p^+(938\ MeV) &= stable,\\
n(940\ MeV) &= (\tilde{p}^+, \tilde{e}^-)_{hm},\\
\Lambda(1115\ MeV) &= (\tilde{p}^+, \tilde{\pi}^-)_{hm},\\
\Sigma^+(1189\ MeV) &= (\tilde{p}^+, \tilde{\pi}^0)_{hm},\\
\Sigma^0(1192\ MeV) &= (\tilde{n}, \tilde{\pi}^0)_{hm},\\
\Sigma^-(1197\ MeV) &= (\tilde{n}, \tilde{\pi}^-)_{hm},\\
\Sigma^0(1314\ MeV) &= (\tilde{\Lambda}, \tilde{\pi}^0,\\
\Xi^-(1321\ MeV) &= (\tilde{\Lambda}, \tilde{\pi}^-)_{hm},
\end{aligned} \tag{77}$$

by keeping in mind that numerous alternative internal exchanges of isoparticles do indeed occur while keeping constant the total characteristics.

It is an instructive exercise for the interested reader to see that all the above models verify condition (II-199) on the lack of hadronic excited states, as well as conditions (II-188) for the validity of isodeterministic Lemma II-3.7 and related rapid convergence of isoseries.

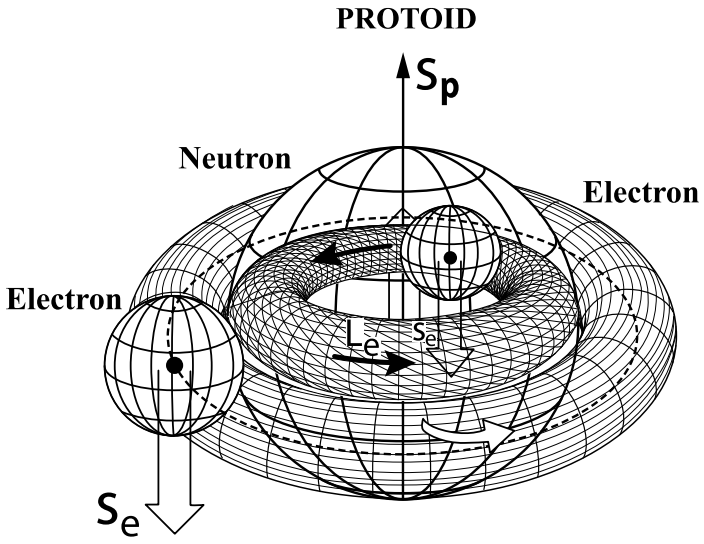

Figure 13: *A conceptual rendering of the protoid as weakly bounded electron and neutron in singlet contact coupling, resulting in a negatively charged hadron with spin $S = 0$ with intriguing applications.*

The compatibility of hadronic structure models (77) with the $SU(3)$-color or more recent classifications is achieved at the hyperstructural level [33] [37] with the following ordered hyperunit [21] (Figure 11)

$$\hat{I}_{tot} = \{\hat{I}_p, \hat{I}_n, \hat{I}_\Lambda, \hat{I}_{\Sigma^+}, \hat{I}_{\Sigma^0}, \hat{I}_{\sigma^-}, \hat{I}_{\Xi^0}, \hat{I}_{\Xi^-}\} \tag{78}$$

Said compatibility is ultimately due to the positive-definiteness of the individual isounits that, in turn, implies the local isomorphism between the hypersymmetry $\tilde{SU}(3)$ and the conventional $SU(3)$ symmetry.

2.6.5. Industrial applications. There is no doubt that the "completion" of quantum mechanics is, by far, Einstein's most important legacy because of its *basic* implications for mathematics, physics, chemistry and other quantitative sciences, with expected industrial applications generally beyond our expectation at this writing.

As an illustration, we outline in this section the industrial applications expected from the technology underlying the synthesis of the neutron from the hydrogen which technology has been possible thanks to the "completion" of quantum into hadronic mechanics.

Besides the synthesis of neutroids (Figure 10) and neutrons (11), the

Directional Neutron Source (DNS) [39] (Figure 9) permits the study of the synthesis of a *negatively charged, strongly interacting particles* [48] (patent pending) with rest energy of about $940\ MeV$, spin 0, charge radius $R = 1\ fm$, and mean-life of about $3\ s$ which is called the *protoid*, and denoted $\hat{p}_1$ (Figure 13).

Recall that contact non-Hamiltonian interactions solely depend on wave-overlapping, thus being charge independent. The protoid is predicted to be given by a singlet coupling of an electron essentially at contact with a neutron, thus yielding total angular momentum 0, with the following structure model:

$$\hat{p}_1 = (e^-_\downarrow, n_\uparrow)_{hm}. \tag{79}$$

Hadronic mechanics predicts a second negatively charged proton called *pseudo-proton*, and denoted $\hat{p}_2$, with essentially the same rest energy, charge radius and mean-life of the neutroid but with spin 1 (Figure 14).

Recall that an electric arc submerged within a hydrogen gas creates a plasma in its surrounding comprising: protons, electrons, neutrons and valence electron pairs in singlet couplings known as *isoelectronia* with rest energy of about $1\ MeV$, null spin and magnetic moment, and a mean-life expected to be of the order of $1\ s$ or fractions thereof (Section 2.8.3).

Following the synthesis of the neutron via the "compression" of an electron inside the proton (Section 2.6.2), the pseudo-proton is predicted to be generated by a "compression" of an electron, this time, inside the neutron, or equivalently, by the "compression" of an isoelectronium inside the proton, with structure model

$$\hat{p}_2 = (\tilde{e}, \tilde{n})_{hm} \approx [\tilde{p}, (\tilde{e}^-_\uparrow, \tilde{e}^-_\downarrow)]_{hm}. \tag{80}$$

The spin $\S_{\hat{p}2} = 1$ is due to the sum of the proton spin $S_p = 1/2$ plus the constrained orbital motion of the electron pair along the proton spin $L_{1,2} = 1/2$.

Recall that protons are *repelled* by nuclei. Hence, the industrially significant feature of negatively charged hadrons is that they are *attracted* by nuclei, thus initiating a basically new approach toward the controlled nuclear fusions that resolves the extremely large Coulomb repulsion that has opposed nuclear fusions to date.

In view of the above features, the technologies that can be developed with the DNS are the following:

1. The detection of fissionable nuclear material that can be concealed in suitcases, containers or underground, since the neutron irradiation of fissionable material under controlled directionality, energy and flux, triggers the decay of some of the nuclei with a shower of clearly detectable

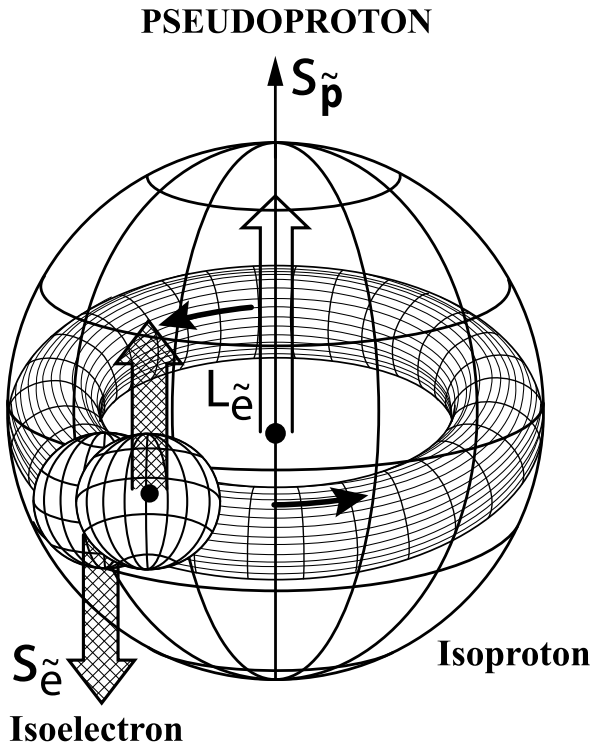

Figure 14: *A conceptual rendering of the pseudo-proton created by the "compression" of an electron, this time, inside a neutron, or equivalently, of a valence electron pair in singlet coupling, resulting in a negatively charged hadron with spin* 1.

radiations. By comparison, most nuclear material are detected as ordinary metals under currently available X-ray, microwave and other scans.

2. The detection of the presence and concentration of precious metals in mining operations, since the neutron irradiation, for instance, of the walls of mining tunnel triggers known nuclear transmutations with the emission of sharp, clearly detectable photons, while statistical data identify the concentration.

3. The control of large metal welds in civilian and military naval construction, which control is rendered particularly effective by the easy mobility of the DNS.

4. Cancer treatments via pseudo-proton irradiation which is expected to be less invasive and more localized than currently available proton irradiation since, in the former case, pseudo-protons are *attracted* by cancerous nuclei, while, in the latter case, protons are *repelled*.

5. Initiation of the much overdue research and development toward the recycling of nuclear waste via its stimulated decay while delivering energy, which recycling is possible in a number of ways, including the irradiation of nuclear waste pellets with a sufficiently intense pseudo-proton flux resulting in a deficiency of the nuclear charge Z for a given atomic number A and ensuing decay.

Note that the above applications are possible thanks to the *mean-life of DNS synthesized particle of the order of seconds or fractions thereof,* which are quite large for particle standard.

By comparison, negatively charged strongly interacting particles synthesized in contemporary physics laboratory with mean-lives of the order of $10^{-30}\ s$ or so have no known industrial or medical applications.

2.7. Einstein's determinism in nuclear structures.

2.7.1. Historical notes. As it is well known, nuclei were first assumed to be bound states of protons and electrons under their mutual Coulomb attraction which force, being inversely proportional to the square of the distance, is extremely big at nuclear distances.

This assumption was soon dismissed because it was considered inconsistent with quantum mechanics and unable to represent experimental data.

Consequently, nuclei were assumed to be bound states of protons and neutrons under a strong nuclear force that remains unknown to this day.

In reality, nuclei provide additional, rather clear evidence on the "lack of completeness" of quantum mechanics beyond scientific doubt, due to

the well known inability by quantum mechanics to achieve an exact representation of nuclear experimental data.

As an illustration, in about one century of efforts and the use of large public funds, quantum mechanics has been unable to achieve a representation of the characteristics of the simplest nucleus, the *deuteron,* with embarrassing deviations between the prediction of the theory and the experimental data for heavier nuclei such as the Zirconium [25] (see also K. R. Popper [15], J. Dunning-Davies [16], J. Horgan [17] and quoted literature):

1. Quantum mechanics has been unable to represent the spin $J = 1$ of the deuteron in its ground state $L = 0$. This is due to the fact that the sole stable quantum mechanical bound state of two particles with spin $1/2$ is the singlet state, in which case the spin of the deuteron should be $J = 0$. For the intent of preserving the validity of quantum mechanics in nuclear physics, the spin $J = 1$ of the deuteron is represented via the assumption of a *combination of excited orbital spaces,* $L = 0, 1, 2$ contrary to the evidence that isolated deuterons are in their ground state, with the consequential inability to represent the deuteron positive parity $P = (-1)^L - 0$, and other insufficiencies.

2. Quantum mechanics has been unable to achieve an exact representation of the deuteron magnetic moment, because the representation of about 1% of the experimental value is still missing despite all possible relativistic (as well as quark-inspired) corrections, with large deviations occurring for heavier nuclei.

3. Quantum mechanics has been unable to represent the stability of the deuteron, since neutrons are well know to be naturally unstable, thus mandating a quantitative representation of the mechanism turning the neutron into a permanently stable particle when bonded to a proton.

4. Quantum mechanics has been unable to represent the proton-neutron exchange in the deuteron structure.

5. Quantum mechanics has been unable to achieve an explicit and concrete representation of the attractive strong nuclear force bonding together the proton and the neutron. Due to the intent of preserving the exact validity of quantum mechanics in nuclear physics, with the consequential sole representation of strong interactions via a potential in a Hamiltonian, the representation of strong nuclear forces has been attempted by adding potentials in the Hamiltonian, up to fifty additive potentials (sic) without the achievement of the needed representation of experimental data.

In a variety of works outlined in these papers (see monograph [25] and

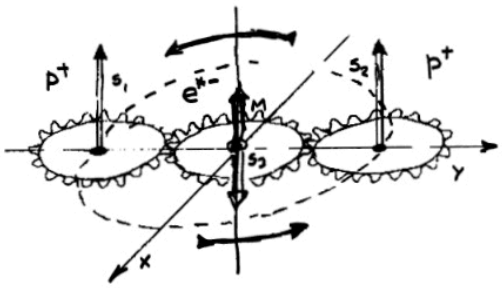

Figure 15: *The "gear model" used by Santilli [25], page 180, to illustrate the prediction from the spin $J_D = 1$ that the deuteron is a three-body hadronic bound state of one isoelectron and two isoprotons, with the individual couplings $p - e$ and $e - p$ being in singlet as necessary for stability. The spin $J_D = 1$ is then achieved from the "constrained angular momentum" of the bound state studied in Sections II-3.5.3 and II-3.5.4.*

papers quoted therein), Santilli has expressed the view that the "completion" of quantum into hadronic mechanics permits the apparent resolution of the above insufficiencies or sheer inconsistencies via a return to the original conception of nuclei as bound state of electrons and protons, although under non-linear, non-local and non-potential interactions caused by deep mutual overlapping.

2.7.2. Deuteron three-body model. Following the achievement of mathematical, theoretical, experimental and industrial advances in structure model (66) of the neutron as a hadronic bound state of one isoelectron and one isoproton, Santilli proposed in Section IV of Ref. [25], page 143 on (see also independent reviews [31] and [49]), the structure model of the deuteron D as a *restricted three-body hadronic bound state of two isoprotons and one isoelectron* (Figures 13, 14) which can be reduced in first approximation to a two-body hadronic bound state of one isoproton and one isoneutron

$$D = (\tilde{p}^+_\uparrow, \tilde{e}_\downarrow, \tilde{p}^+_\uparrow)_{hm} \approx (\tilde{p}^\uparrow, \tilde{n}^\uparrow)_{hm}. \tag{81}$$

Ref. [25] then presented various arguments showing that model (81) apparently resolves quantum mechanical insufficiencies (1 - 5) of the preceding section.

The analysis of model (81) was conducted in the 1999 monograph [25] following the publication in 1998 of the first proof of the EPR argument [9], as well as its detailed study in the 1995 monograph [33], Chapter 4, particularly Appendix 4C, page 166. Therefore, Santilli was aware that, unlike the case for atomic structures, deuteron model (81) does admit a classical

counterpart (Section II-3.7) due to the inapplicability of Bell's inequality (9) [3] in favor of isoidentity (10).

However, at the time of writing the 1999 monograph [25], Santilli was unaware of the additional apparent proof of the progressive validity of Einstein's determinism in the interior of hadrons, nuclei and stars [10], and its apparent full achievement in the interior of gravitational collapse.

It is, therefore, important to review the main aspects of model (81) to indicate, apparently for the first time, its verification of Einstein's determinism we have called isodeterministic according to Lemma II-3.7 and Corollary II-3.7.1, and verify the apparent resolution of historical problems (1 - 5) of the preceding section.

1. NONRELATIVISTIC REPRESENTATION OF THE SPIN $J_D = 1$ IN THE GROUND STATE $L_D = 0$. With reference to Figure 24, model (81) includes *five* angular momenta: 1) The parallel spins S_1 and S_2 of the isoprotons; 2) The angular momentum L_{1-2} of the two isoprotons; 3) The spin of the isoelectron S_3; and 4) The angular momentum L_3 of the isoelectron inside one of the two isoprotons.

Recall that, according to experimental data on nuclear dimensions, the isoproton and the isoneutron are expected to have a diameter of $1.73\ fm$, while the diameter of the deuteron is $4,26\ fm$.

Consequently, the two isoprotons are separated by about $0.886\ fm$. In model (81), this space is occupied by the isoelectron acting like a "gluon" of the two isoprotons, while permitting their triplet alignment necessary for the spin $J_D = 1$.

In fact, the isoelectron is in the singlet coupling with each of the two isoprotons necessary for stability according to hadronic mechanics (Figure 14).

Recall also that, in semiclassical approximation, the wavepacket of the electron is of the order of $2.2\ fm$.

The above data provide not only a considerable synchronicity in model (81), but also a rather strong bond caused by the nuclear forces, the Coulomb attraction between the isoelectron and the isoprotons, as well as the strong attractive force originating from deep wave-overlapping in singlet coupling.

Recall finally that the diameter of the horizon for the full applicability of hadronic mechanics has been selected in Paper I to be $d = 2\times 10^{-13}\ cm = 2fm$, while the deuteron is about double that size.

The above data on dimensions suggest the use of the following isosymmetries for the characterization of the angular momentum and spin of the deuteron:

Use the regular isosymmetries $\hat{O}(3)$ and $\hat{SU}(2)$ for the characterization of the angular momentum and spin outside the hadronic horizon. We are here referring to isosymmetries which can be constructed via a non-unitary transform of the corresponding Lie symmetries, thus preserving conventional eigenvalues for angular momentum and spin (Section I-3.8 and I-3.9).

Use the irrreegular isosymmetries $\hat{O}(3)$ and $\hat{SU}(2)$ for the characterization of the angular momentum and spin inside the hadronic horizon. We are here referring to isosymmetries which cannot be constructed via non-unitary transforms o the corresponding Lie symmetries, by therefore having anomalous values of the angular momentum and spin (Section II-3.4).

It is then easy to see that the diameter $D = 2.59\ fm$ of the orbital motion of the two isoprotons is close to, but bigger than the hadronic horizon $d = 2\ fm$. We can therefore see the regular isorotational symmetry $\hat{O}3)$ admitting conventional angular momentum eigenvalues with ground state $L_{1-2} = 0$.

By contrast we have to use the irregular $\hat{O}(3)$ isosymmetry for the rotation of the isoelectron in the interior of the isoproton, resulting in the constrained value $L_3 = 1/2$ for the neutron (Sections 2.6.2 and 2.6.3).

In this way, Ref. [25] achieved the first known representation of the spin of the deuteron in its true ground state according to values:

$$\begin{aligned} J_D = S_1 + S_2 + S_3 + L_{1-2} + L_3 = \\ = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} + 0 + \tfrac{1}{2} = 1. \end{aligned} \tag{82}$$

By comparing the hadronic structure model of the neutron, Figure 10, and that of the deuteron, Figure 15, it is evident that the isoelectron is entirely compressed inside the isoproton in the former case, but only partially compressed in the latter case.

This occurrence is expected due to the presence in model (81) of the second isoproton with ensuing strongly attractive Coulomb force responsible for the partial extraction of the isoelectron from the isoproton, although this possibility has not been investigated to date.

Similarly, structure (81) appears to be particularly suited for the representation of the proton-neutron exchange in the deuteron structure, although this possibility has also not been subjected to a quantitative study to our best knowledge.

2. RELATIVISTIC REPRESENTATION OF THE SPIN $J_D = 1$ IN THE GROUND STATE $L_D = 0$. Ref. [25] was written following the 1995 publication of *Elements*

of Hadronic Mechanics, Vols. I [32] and II [33]. Therefore, Ref. [25] studied model (81) with the entire sue of said Volumes whose knowledge, particularly that reviewed in Section II-3, is herein tacitly assumed.

The study of model (81) was done in Ref. [25], Section IV-2.3.4, page 161, via the use of the irregular iso-Dirac equation (II-201) defined on iso-semi-product (II-200) of an iso-Minkowskian space $\hat{M}(\hat{x}, \hat{\eta}, \hat{I}_{orb})$ for the relativistic description of the hadronic orbital motion, multiplied by a three-dimensional complex-valued isounitary isospace for the characterization of the hadronic spin $\hat{E}(\hat{z}, \hat{\eta}, \hat{I}_{spin}$.

It should be recalled that, in principle, *all* conventional intrinsic characteristics of the proton and the electron, including their charges, are expected to be mutated when said particles are in conditions of total mutual penetration. These mutations are quantitatively represented by the Lorentz-Poincaré-Santilli isosymmetry of the irregular iso-Dirac equation (Section II-2.5.11) and its irregular Pauli-Santilli isomatrices (Section II-3.4).

However, Ref. [25], Section 2.3.5, page 162, has shown that, for the case of the deuteron (but not necessarily so for heavier nuclei), the spins and changes of the isoprotons and of the isoelectron can be assumed in first approximation to have conventional values. In this case, the sole mutations achieving the representation of experimental data of the deuteron are those for shapes and constrained angular momenta.

The first known relativistic representation of the spin $J_D = 1$ of the deuteron in its ground state $L_D - 0$ was achieved in Ref. [25], Section 3.5.6, thanks to constrained spin-orbit coupling inside the isoneutron of model (81).

The latter coupling has been reviewed at the non-relativistic level in Section II-3.5.3 and at the relativistic level in Section II-3.5.4, see in particular Eqs. (II-118), (II-119), with a detailed description available in Ref. [33], Chapter 6.

The resulting relativistic characterization of hadronic angular momenta and spins is given by

$$\begin{aligned} S_{k\alpha} &= \epsilon_{kij}\hat{\gamma}_{i\alpha} \star \hat{\gamma}_{j\alpha}, \quad \alpha = \tilde{p}1, \tilde{p}2, \tilde{e}, \\ L_{k,p1-p2} &= \epsilon_{kij} r_{i,p1-p2} \star p_{j,p1-p2}, \end{aligned} \tag{83}$$

where $\hat{\gamma}_k$ are the irregular Dirac-Santilli isomatrices.

We then have the irregular isocommutation rules Ref. [36], Eqs. (6.4),

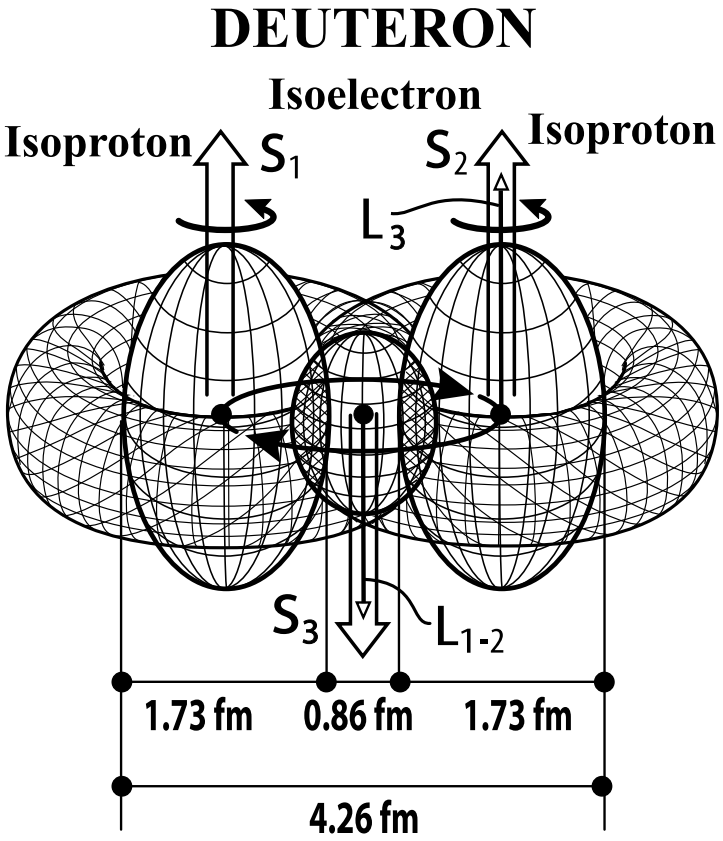

Figure 16: *A conceptual rendering of the deuteron model (91) illustrating the representation for the first time of spin $J_D = 1$ with angular momentum $L_D = 0$, where: the two isoprotons have the charge diameter $D_{\tilde{p}} = 1.73\ fm$; the deuteron has the charge diameter $D_D = 4.26\ fm$; the interspace between the isoprotons is then of $0.86\ fm$; the isoelectron has a point-like charge, the hadronic diameter $D_{\tilde{e}}^{hm} = 1\ fm$, and a semi-classical wavepacket with $D_{\tilde{e}}^{cl} = 2.2\ fm$. The above data illustrate the stability of model (81) with the central isoelectron allowing the isoprotons in the triplet coupling needed to achieve the spin $J_D = 1$ in the true ground state (Figure 14). The doughnuts around the two isoprotons are used to represent the proton-neutron exchange forces.*

page 190,

$$[S_{i,\alpha},\hat{S}_{j\alpha}] = 2\epsilon_{kij}m^2_{k\alpha}\hat{S}_{k\alpha},$$
$$[L_{i,1-2},\hat{L}_{j,1-2}] = \epsilon_{ijk}m^2_{k,1-2}L_{k,1-2}, \tag{84}$$

and isoeigenvalues, Eqs, [36], Eqs. (6.4d), page 190,

$$\hat{S}_{3\alpha}\star|\hat{\psi}>=\pm\tfrac{1}{m_{1\alpha}m_{2\alpha}}|\hat{\psi}>,$$
$$\hat{S}^{\hat{2}}_{\alpha}\star|\hat{\psi}>=(m^{-2}_{1\alpha}m^{-2}_{2\alpha}+m^{-2}_{2\alpha}m^{-2}_{3\alpha}+m^{-2}_{3\alpha}m^{-2}_{1\alpha})|\hat{\psi}>,$$
$$\hat{L}_{3,1-2}\star|\hat{\psi}>=\pm m_{1,1-2}m_{2,1-2}|\hat{\psi}>,$$
$$\hat{L}^{\hat{2}}_{\beta}1-2\star|\hat{\psi}>=(m^2_{1,1-2}m^2_{2,1-2}+m^2_{2,1-2}m^2_{3,1-2}+m^2_{3,1-2}m^2_{1,1-2})|\hat{\psi}>. \tag{85}$$

Under the assumption that the hadronic medium in the interior of the proton is homogeneous, we can assume the values [36]

$$m_{1\alpha}=m_{1,1-2}=m_{2\alpha}=m_{2,1-2}=m_{3\alpha}=m_{3,1-2}=\frac{1}{\sqrt{2}}=0.842, \tag{86}$$

under which the isoeigenvalues become

$$\hat{S}_{3\alpha}\star|\hat{\psi}>=\pm\tfrac{1}{2}|\hat{\psi}>,$$
$$\hat{S}^{\hat{2}}_{\alpha}\star|\hat{\psi}>=\tfrac{3}{4}|\hat{\psi}>,$$
$$\hat{L}_{3,1-2}\star|\hat{\psi}>=\pm\tfrac{1}{2}|\hat{\psi}>,$$
$$\hat{L}^{\hat{2}}_{1-2}\star|\hat{\psi}>=\tfrac{3}{4}|\hat{\psi}>. \tag{87}$$

It then follows that the total angular momentum of the isoneutron is given by

$$J_{\tilde{n}}=S_2+S^{spin}_{\tilde{e}}+L^{orb}_{\tilde{e}}=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=\frac{1}{2}. \tag{88}$$

The total value of the deuteron spin $J_D=1$ in its ground state $L_D=0$ then follows from the fact indicated above that the orbital diameter is bigger than the hadronic horizon.

3. REPRESENTATION OF THE DEUTERON REST ENERGY, STABILITY AND SIZE. The hadronic representation of the rest energy, stability and charge radius of the deuteron were first achieved in Ref. [25], Eqs. (5.2.16), page 179,

via the use of structural equations (29) (see also independent reviews [31], [49])

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2\frac{e^{-br}}{1-e^{-br}})\right] = 0,$$

$$E_{tot} = E_1 + E_2 - E = 1875.7\ MeV,\ \ \bar{m} = \frac{m}{I^2}$$
$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_1}{\hbar} = \infty, \tag{89}$$

$$R = b^{-1} = 2.13 \times 10^{-13}\ cm = 2.13\ fm,$$

with solutions for the k-parameters

$$k_1 = 1\ \ k_2 = 2.5. \tag{90}$$

The total mass/rest energy of the deuteron is then given by

$$M_D = M_p + M_n - E = 1875.7\ MeV, \tag{91}$$

where the binding energy is given by $E = 2.2\ MeV$.

Value (90) should be compared to the corresponding values for the neutron, Eqs. (69). As one can see, values (90) verify conditions (35) on the validity of the isodeterminism of Lemma II-3.7 as well as with the increase of the rapid convergence of isopperturbative series of Corollary II-3.7.1.

Note that, according to our assumption, values (81) verify the crucial condition (35) for the lack of existence of excited states, because excited states would imply the existing of the deuteron constituents out of the hadronic horizon with its consequential disintegration.

Note also that the stability of the deuteron is intrinsic in model (81) since the deuteron is reduced to the only known, massive, permanently stable particles existing in the universe, the proton and the electron.

Needless to say, the above representation should be considered as a first approximation, with a number of improvements being possible such as the inclusion of Coulomb interactions with possible excited states within the hadronic horizon.

4. REPRESENTATION OF THE DEUTERON MAGNETIC MOMENT. As it is well known, the experimental value of the deuteron magnetic moment is given by [13] [14]

$$\mu_D^{exp} = 0.857\ \mu_N. \tag{92}$$

For the quantum mechanical ground state with $L_D = 0$, in case consistent, one would obtains

$$\mu_D^{qm} = \frac{1}{2}(g_p^S + g_n^S) = 2.792 - 1.913 = 0.879\ \mu_N, \tag{93}$$

(where the g's are tabulated g-factors), by therefore missing about 3% of experimental value (256), which deviation cannot be entirely resolved via relativistic or other corrections.

The first, exact, non-relativistic representation of the deuteron magnetic moment was achieved by Santilli for the approximate two-body version of model (81) in paper [50] of 1994 (written at the Joint Institute for Nuclear Research, Dubna, Russia) under the sole assumption that the charge distributions of protons and neutrons are deformed under strong nuclear forces, resulting in a corresponding mutation of their intrinsic magnetic moment as predicted by E. Fermi [13], J.M. Blatt and V. F. Weisskopf [14], and other founders of nuclear physics (see their recollection in Ref. [25], page 158-159, Section IV-2.3.1, under the title *The Historical Hypothesis*).

Santilli achieved the first, exact, relativistic representation of magnetic moment of the two-body deuteron model in paper [51] of 1996 via the use of the iso-Dirac equation.

The 1998 Ref. [25], Section IV-2.3.6, page 163, presents the first known exact representation of the deuteron magnetic moment for the full three-body model (81).

It is important to review and upgrade the latter representation to show its compatibility with Einstein's determinism according to Lemma II-3.7 and rapid convergence according to Corollary II-3.7.1.

Assume, in first approximation, that the isoparticles of model (81) have the same spheroidal shape with semiaxes $n_1^2,\ n_2^2,\ n_3^2$ under the condition of preserving the volume of the original particles assumed for simplicity to be normalized to 1

$$n_1^2 \times n_2^2 \times n_3^2 = 1. \tag{94}$$

Represents with n_4^2 the density of the two isoprotons (denoted with the subindeces 1 and 2), resulting in the isotopic element for the two isoprotons

$$\hat{T}_{1,2} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2). \tag{95}$$

Then, the Lorentz-Santilli isotransforms (II-42) imply the following mutation of the intrinsic magnetic moment of the proton (see Ref. [36], page 190 for a detailed derivation with the notation $b_\mu = 1/n_\mu,\ \mu = 1, 2, 3, 4$)

$$\begin{aligned} \mu_{1,2}^{qm} \ \ &\rightarrow \ \ \tilde{\mu}_{1,2}^{hm} = \tfrac{n_4}{n_3}\mu_{1,2}^{qm} = \tilde{g}^S \mu_N, \\ &\tilde{g} = \tfrac{n_4}{n_3} g. \end{aligned} \tag{96}$$

Recall that the isoprotons have Spin $S_{1,2} = 1/2$ and $L_D = 0$.

In order to compute the magnetic moment of the three-body model (81), we first compute *the intrinsic and orbital contributions of the isoprotons*

via a simple isotopy of the corresponding quantum mechanical expression [45]

$$\tilde{\mu}_k^{hm} =$$

$$\frac{1}{J+1} < \hat{Y}_{L,S} | \frac{1}{2} \tilde{g}_k^S S + \frac{1}{2} \tilde{g}_k^L L) | \hat{Y}(L,S) >= \tag{97}$$

$$= \frac{1}{2(J+1)} \tilde{g}_k^S \left[J(J+1) - L(L+1) + S(S+1), \right], \;\; k = \tilde{p}1, \tilde{p}2,$$

where $\hat{Y}_{L,S}$ are the isospherical isoharmonics (Ref. [33], Section 6.4, page 240) and $J_k = S_k + L_k, \; k = 1, 2$ is the total angular momentum.

Simple calculations show that, for $L_k = 0$ and $\mu_p^{exp} = 2.79 \; \mu_N$, expression (261) yields the value

$$\begin{gathered} \tilde{\mu}_k^{hm} = \\ = \frac{1}{2} \tilde{g}_k^s = \frac{n_4}{2n_3} g_k^S = \frac{n_4}{n_3} 2.79 \; \mu_N. \end{gathered} \tag{98}$$

To compute *the intrinsic and orbital contribution from the isoelectron*, we recall from Eq. (69) that its intrinsic value is ignorable in first approximation, the sole expected contribution being that from its constrained orbital motion inside the proton.

By recalling that the electron has a point-like charge which cannot be mutated by isotopies, then the density of the electron remains the conventional one for the vacuum with $n_{4,\tilde{e}} = 1$.

Under the approximation that the deformation of the isoelectron is the same as those of the two isoprotons, we have the mutation of the value (70) for the neutron,

$$\mu_{\tilde{e},orb}^{hm} = -\frac{1}{n_3} \mu_{\tilde{e}}^{orb} = -\frac{1}{n_3} 4.6 \; \mu_N. \tag{99}$$

The quantum mechanical magnetic moment of the three-body model (81) of the deuteron, in case consistent, would be given by

$$\mu_D^{qm} = 2\mu_p + \mu_e = 2 \times 2.79 - 4.6 \; \mu_N = 0.98 \mu_N, \tag{100}$$

thus lacking an exact representation of value (92) because in *excess* for about 8%.

By comparison, the magnetic moment for model (81) according to hadronic mechanics is given by

$$\mu_D^{hm} = 2\frac{n_4}{n_3} \mu_{\tilde{p}} + \frac{1}{n_3} \mu_{\tilde{e}} = \frac{n_4}{n_3} 5.78 - \frac{1}{n_3} 4.6 \; \mu_N = 0.857 \; \mu_N. \tag{101}$$

By assuming the value $n_4 = 0.62$ from the density of the proton-antiproton fireball in the Bose-Einstein correlation [46] [47], we can write the *hadronic representation of the deuteron magnetic moment*

$$\mu_D = \frac{0.62 \times 5.78}{n_3} - \frac{4.6}{n_3} = 0.857\ \mu_N, \tag{102}$$

which is verified for

$$n_3 = 1.20. \tag{103}$$

The use of the volume-preserving condition (94) then yields the remaining ellipsoidal values

$$n_1 = n_2 = 0.91. \tag{104}$$

The above values imply that the shape deformation of the deuteron constituents caused by the internal strong nuclear force turns charge distributions and/or wavepackets from a spherical to a *prolate* spheroidal ellipsoid with semiaxes

$$n_1^2 = n_2^2 = 0.83, \ \ n_3^2 = 1.44, \tag{105}$$

which prolate deformation was predicted by all preceding works by Santilli in the field.

Note that *the magnetic moment of the deuteron is represented exactly not only in its value, but also in its sign,* that of being parallel to the spins of the isoprotons.

It then follows that *representation (101) can be construed as evidence on the presence of two protons with parallel spins inside the deuteron,* which presence mandates three-body model (81) with a central exchanged electron for stability (Figures 14, 15).

Note also that the *prolate* deformation of shape is necessary because the quantum mechanical expression (93) is in *excess* of experimental value (92). By comparison, an *oblate* deformation would *increase* said excess value due to the increase of the rotating charge distribution at the equator.

Almost needless to say, representation (101) is intended to illustrate the *capability* by hadronic mechanics to achieve an exact representation of the deuteron magnetic moment.

A number of improvement of representation (101) are possible, among which we mention a more accurate value of the density n_4 of the proton, a more accurate value of the orbital contribution of the isoelectron due to its geometric difference with that for the neutron indicated above, and other improvements.

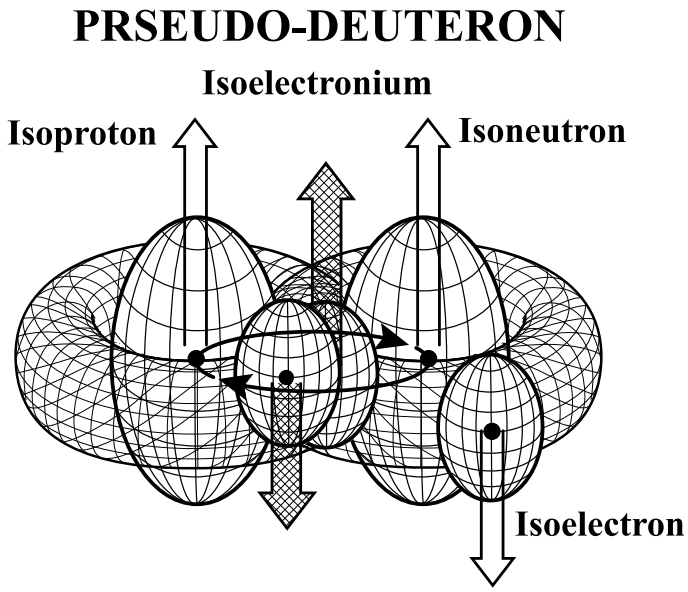

Figure 17: *An illustration of the negatively charged pseudo-deuteron predicted by hadronic mechanics via the use of a deuterium gas in the Directional Neutron Source of Figure 8. Due to its negative charge, the pseudo-deuteron can potentially initiate a basically new type of nuclear fusions, here called hyperfusions, based on the natural attraction of the pseudo-deuteron by natural deuteron resulting in the helium without production of harmful radiations or release of radioactive waste.*

5. REPRESENTATION OF NUCLEAR FORCES. Hadronic structure model (81) achieves the first known explicit concrete representation of strong nuclear forces thanks to the structural lifting of the mathematics underlying quantum mechanics into the covering isomathematics.

Such a generalization resulted to be necessary due to the non-Hamiltonian character of strong interactions, with particular reference to the generalization of the associative product into isoproduct (1) and the representation of strong nuclear forces via isotopic element (4), with ensuing lifting of the Schrödinger equation, from its historical form $H\psi = E\psi$, into the isotopic form

$$H \star \psi = H(r,p)\hat{T}(\psi,...)\hat{\psi} = E\hat{\psi}. \tag{106}$$

6. REDUCTION OF STABLE NUCLEI TO ISOELECTRONS AND ISOPROTONS. The reduction of neutrons to isoprotons and isoelectrons clearly implies the possible reduction of all stable matter in the universe to the only known, massive and stable particles, protons and electrons [52].

This important possibility has been confirmed by the reduction of the deuteron to isoprotons and isoelectrons [25], and it is under study by A. A. Bhalekar and R. M. Santilli [54].

2.7.3. Industrial applications. In Section I-1.4 we have indicated the societal duty of seeking basically *new* forms of clean energies, while continuing research along conventional lines.

This is due to the inability of identifying industrially viable nuclear fusions in about three quarters of a century and the investment of billions of dollars of public funds.

Recall that the primary obstacle opposing the achievement of the controlled nuclear fusion of two deuterons $D(1,2,1)$ into the Helium $He(2.4.0)$ is the *extreme repulsive Coulomb force between nuclei* that, the distance of $1\ fm = 10^{-15}\ m$ needed to activate strong nuclear forces is given by

$$\begin{aligned} F = k\frac{e^2}{r^2} = \\ = (8.99 \times 10^9)\frac{(1.60\times 10^{-19})^2}{(10^{-15})^2} = 230\ N, \end{aligned} \tag{107}$$

which force is extremely large for nuclear standards.

As it is well known, the efforts done to date for overcoming such a large repulsive force have been the use of high energies resulting, as expected, in uncontrollable instabilities at the initiation of the fusion itself.

Another problem that has prevented the achievement of the controlled

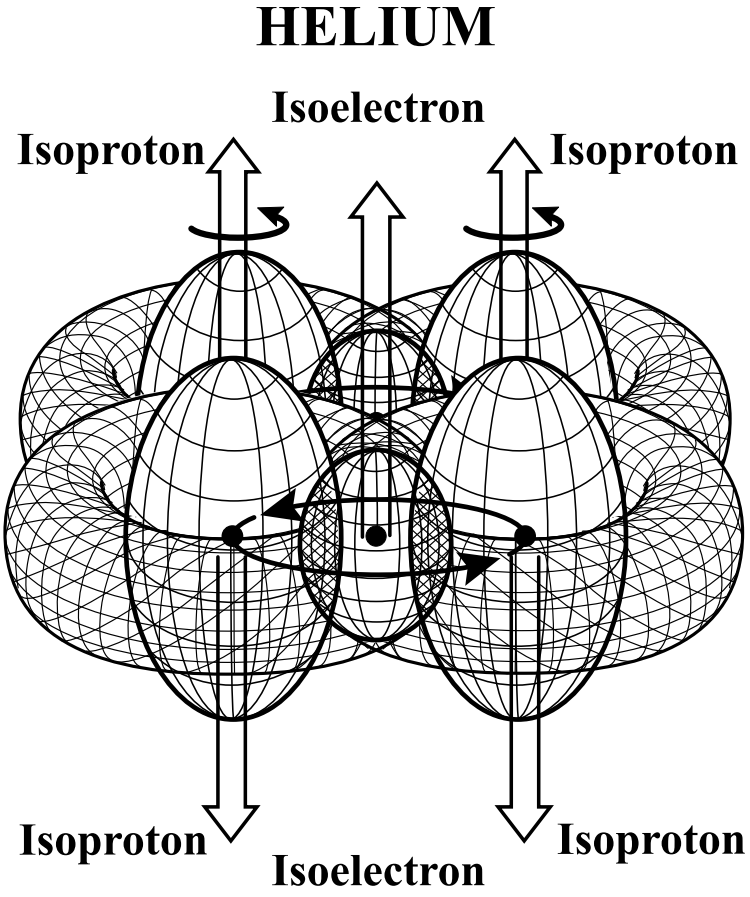

Figure 18: *An illustration of the helium nucleus as predicted by hadronic mechanics and consisting of four isoprotons and two isoelectrons. Note that the illustration depicts the conventional conception of the Helium as being composed by two protons and two neutrons, with the sole replacement of the neutrons with their physical constituents permitted by hadronic mechanics, the proton and the electron. Note finally that the uncertainties of the constituents of the helium follow Einstein's determinism per Lemma II-3.7 and Corollary II-3.7.1.*

fusion of deuterons into helium is the need for their *singlet coupling*

$$D(1,2,1_{\downarrow}) + D(1,2,1_{\uparrow}) \ \rightarrow \ He(2,4,0), \tag{108}$$

which is of extremely difficult engineering realization at low energies, and virtually impossible at high energies, resulting in nuclear fusions that can at best be at random.

For additional technical requirements needed to achieve the fusion of deuterons into helium, interested readers may consult the *hadronic laws for nuclear fusions,* which have been systematically studied in Ref. [25], Chapter 4 and Section 4.2, page 188, in particular.

The Directional Neutron Source (DNS, Figure 8) is produced and sold by Thunder Energies Corporation, now called Hadronic Technologies Corporation, for the synthesis of neutrons (Section 2.6.3) and pseudo-protons (Section 2.6.5) from a commercially available hydrogen gas.

However, the same DNS is produced for the use, without any modification, of a commercially available *deuterium gas* as a basic feedstock, in which case, hadronic mechanics predicts the synthesis of a *negatively charged deuteron* called *pseudo-deuteron* [48] (patent pending) which is predicted to have a structure of the type (Figure 17)

$$\hat{D} = \left[\tilde{p}_{\uparrow}, (\tilde{e}^{-}_{\uparrow}, \tilde{e}^{-}_{\downarrow}), \tilde{n}_{\uparrow})\right]_{hm}. \tag{109}$$

The mechanism for the synthesis of the pseudo-deuteron (which we write with nuclear symbols $\hat{D}(-1,2,1)$) from a deuterium gas stems from the main feature of the DNS, that of actually "compressing" electrons in the interior hadrons (Figure 9) thanks to a specially conceived electric arc and other engineering features. Consequently, pseudo-deuteria are expected to be synthesized in the DNF operating on deuterium gas, following molecular separation with the ensuing presence of valence electron pairs called *isoelectronia* (Section 2.8.2 and Figure 20), of course, in a way dependent on power, pressure, flow, and other factors.

Preliminary calculations via hadronic mechanics indicate that the pseudo-deuteron should have a mean-life of the order of a second, thus being sufficient for industrial applications.

The importance of the research under consideration is that, contrary to current trends, *pseudo-deuterons are attracted, rather than being repelled by natural deuterons.*

This basic feature provides plausible means to search for basically *new* nuclear fusions called *hyper-fusions,* here referred to fusions of natural, positively charged nuclei and synthesized negatively charged nuclei, such as

the hyper-fusion of a pseudo-deuteron and a deuteron into the helium

$$\hat{D}(-1,2,1_{\uparrow}) + D(1,2,1_{\downarrow}) \ \rightarrow \ He(2,4,0) + 2e^{-}, \tag{110}$$

where the singlet coupling is naturally achieved because the pseudo-deuteron has not only the charge, but also the magnetic moment opposite that of the natural deuteron.

The energy expected to be released by each hyperfusion (110) is given by

$$\Delta E = E_{He} - (E_{\hat{D}} + E_D) == 23.8 \ MeV = 3.81 \times 10^{-12} \ J, \tag{111}$$

and occurs *without any emission of harmful radiation (since electrons can be stopped with a metal shield), and without the release of radioactive waste.*

As an illustration, the possible achievement of the rather realistic number of 10^{18} controlled fusions (110) per hour would yield the significant release of about 10^6 J of clean energy per hour without harmful radiations or waste.

In regard to the possible achievement of controlled hyper-fusions, we should mention that the DNS is considered in this section for the mere intent of achieving experimental measurements on the sole *existence* of pseudo-deuterons. Their production in the needed number and energy, and the hadronic reactor needed for the utilization of the produced heat, evidently require specialized equipment under proper funding.

By recalling that conventional nuclear fusions have requested billions of dollars of investments of public funds without any industrially viable result to date, readers should be warned against any expectation prior to the investment of equally large funds

2.8. Einstein's determinism in molecular structures.

2.8.1. Insufficiencies of quantum chemistry. There is no doubt that quantum chemistry has permitted truly historical advances in the 20th century. However, as it is the case for nuclear physics, serious scholars are expected to admit that quantum chemistry cannot be *exactly valid* for chemical structures and processes because of a number of insufficiencies, such as [30] (for dissident views, see also Refs. [15] [16] [17]):

1. Even though possessing excellent practical values, *the quantum chemical notion of valence is a 'nomenclature' without quantitative treatment because, due to their equal charge, the Schrödinger equation predicts that valence electron pairs 'repel' (rather than attract) each other,* due to the necessary sign $+$ of the Coulomb potential in the equation for the electron pair

$$i\frac{\partial}{\partial t}\psi(t,r) = \left[-\frac{\hbar^2}{m}\Delta_r + \frac{e^2}{r}\right]\psi(t,r). \tag{112}$$

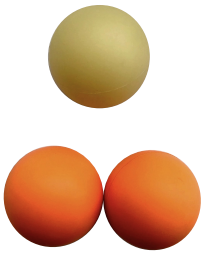

Figure 19: *Lectures in hadronic chemistry are generally initiated by showing a ball in representation of a hydrogen atom with the recollection that quantum mechanics achieved an exact representation of all its experimental data. The speaker then shows two joined balls representing a hydrogen molecule, with the recollection that, this time, quantum mechanics and chemistry have not achieved an exact representation of molecular data. The reason indicated for this dichotomy is that mutual distances in the atomic structure, that are of the order of* $10^{-8}\ cm$*, are such to allow an effective point-like approximation of the constituents. By contrast, mutual distances for valence electron pairs in molecular structures, which are of the order of* $10^{-13}\ cm$*, are smaller than the size of the electron wavepackets, which is of about* $2.2 \times 10^{-13}\ cm$*, thus prohibiting an effective point-like approximation of particles in favor of the representation of their actual size and as well as of the ensuing contact non-potential interactions due to deep wave overlapping.*

In particular, the *repulsive* force between the two identical valence electrons has the extremely big value of $230\ N$ at mutual distances of $1\ fm$, Eq. (107), with repulsive values remaining big at atomic distances. Consequently, *quantum chemistry misses a quantitative model of molecular structures.*

2. According to quantum chemistry (see, e.g., Ref. [55]), valence electron bonds between two atoms are created by the overlap of atomic orbitals. Even though approximately valid, such a model is "incomplete" because, due to the point-like approximation of the electrons, said orbitals remain mostly independent, thus allowing their polarizations under a sufficiently strong electric or magnetic field, with the consequential prediction that substances are generally ferromagnetic.

3. Quantum mechanics has achieved an exact representation of all experimental data of the *hydrogen atom* H, but in the transition to the *hydrogen molecule* $H_2 = H - H$ (where $-$ represents valence bond), quantum mechanics and chemistry still miss $1\ \%$ of the H_2 binding energy which corresponds to the rather significant value of $950\ BTU$.

4. Quantum chemistry still misses to this day the quantitative identi-

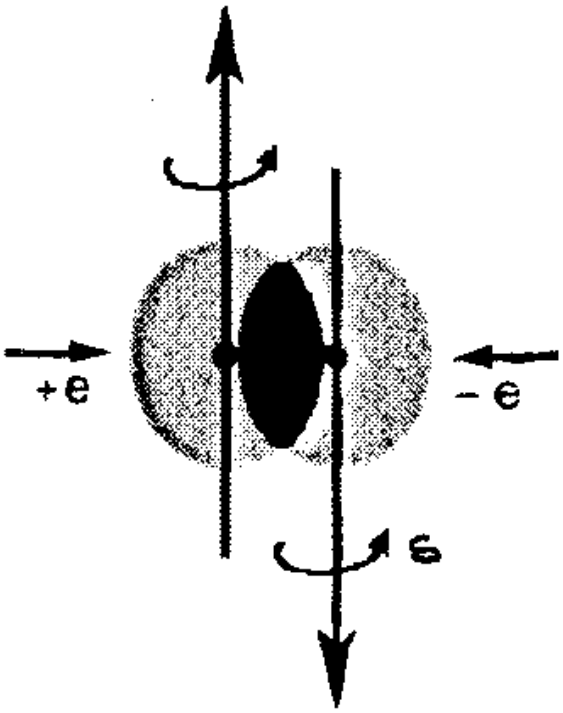

Figure 20: *According to the basic principles of quantum mechanics and chemistry, electron valence pairs in molecular structures should "repel" (rather than "attract") each other with a force of the order of* $230\ N$*, Eq. (107), which is extremely big for particle standards. Hadronic mathematics, mechanics and chemistry have achieved an attractive force between valence electron pairs so strong to turn them into a quasi-stable particle called the isoelectronium [30], that allowed the first known achievement of exact representation of experimental data on the hydrogen [52] and water [53] molecules that are not possible with quantum chemistry. In Santilli's view, the impossibility in quantum chemistry to achieve an attractive force between valence electron pairs as occurring in nature, is additional evidence supporting the need for the "completion" of quantum principles into covering vistas according to Einstein's legacy.*

fication of the *attractive force* bonding together neutral, dielectric and diamagnetic molecules in their liquid state, with ensuing lack of quantitative representation of the liquid state for fuels.

5. The basic axioms of quantum chemistry are reversible over time. Consequently, quantum chemistry is structurally unable to provide an exact representation of combustion, as well as all energy-releasing processes with ensuing implications for the solution of problems of large societal values, such as the improvement of fossil fuel combustion.

We cannot possibly review in this paper the ongoing process toward the solution of the above insufficiencies of quantum chemistry via the covering hadronic chemistry [30]. Nevertheless, we believe it is important to indicate the main evidence supporting the validity for molecular structures of the "completion" of quantum chemistry into a suitable covering theory according to Einstein-Podolsky-Rosen argument [1].

Insufficiencies (1 - 5) above are known to chemists, as attested by the widespread distribution of works in the field by Santilli and other scholars. For this reason, said insufficiencies are referred to as *the best kept secrets in the best Ph. D. courses in chemistry around the world.*

2.8.2. Attractive force in valence electron bonds. Recall that in the 1978 paper [19], Santilli: 1) Identified the non-potential, strongly attractive force created by the deep overlapping of the wavepackets of particles in singlet coupling; 2) Constructed the foundations of hadronic mechanics with particular reference to the iso-Schrödinger equation for quantitative treatments of non-potential interactions; and 3) Applied the emerging new methods to achieve an exact representation of all characteristics of mesons as bound states of electrons and positrons (Sections 2.5.3 to 2.5.5).

In the 1995 paper [95], A. O. E. Animalu and R. M. Santilli showed that the indicated non-potential force is essentially charge independent since it remains strongly attractive also for electron-electron pairs in superconductivity thanks to the "absorption" of the Coulomb potential by the Hulten potential irrespective of its sign, Eq. (5.1.15), page 836, Ref. [19] (reviewed in Eq. (31) above).

Thanks to the "completion" of quantum into hadronic chemistry (hc), in the 2001 monograph [30] Santilli worked out a new notion of valence bonds, today known as *strong valence bond,* which consists of valence isoelectron pairs, known as *isoelectronium* and denoted $\mathcal{I}$, with a fully identified *attractive* force with the structure (see the independent reviews [31]

[96] [97])

$$\mathcal{I} = (\tilde{e}_\uparrow, \tilde{e}_\downarrow)_{hc}. \tag{113}$$

The achievement of the attractive force between the identical valence electrons, apparently done for the first time in Ref. [30], Chapter 4, is described in detail in Section 2.3, and it is essentially based on the "completion" of all products into isoproducts (1) with consequential "completion" of the local Newton's differential calculus into the non-local isodifferential calculus (7) (8), resulting in the iso-Schrödinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(t,r) = \left[-\frac{\hbar^2}{m}\Delta_r + \frac{e^2}{r}\right]\psi(t,r) \ \rightarrow \\ \rightarrow i\hat{\frac{\partial}{\partial t}}\psi(t,r) = \left[-\frac{\hbar^2}{m}\hat{\Delta}_r + \frac{e^2}{r}\right]\hat{T}(\psi,\hat{\psi},...)\hat{\psi}(t,r) \approx \\ \approx i\hat{\frac{\partial}{\partial t}}\hat{\psi}(t,r) = \left[-\frac{\hbar^2}{m}\hat{\Delta}_r - W\frac{e^{-br}}{1-e^{-br}}\right]\hat{T}(\psi,\hat{\psi},...)\hat{\psi}(t,r), \end{aligned} \tag{114}$$

with a strong attraction characterized by the Hulten potential.

In Santilli's view, *the achievement of an attractive force in valence electron pairs via the "completion" of the quantum wavefunction $\psi(t,r)$ into the hadronic isowavefunction $\hat{\psi}(t,r)$ is a strong confirmation of the final statement by Einstein, Podolski and Rosen [1] according to which quantum wavefunctions do not represent the entire physical or chemical reality.*

The strength of the internal bond allows the reduction of four-body molecules, such as the hydrogen $H_2 = H - H$, down to a *restricted three-body system* that, as such, admit full analytic solutions with major simplification of otherwise notoriously complex elaborations.

It should be noted that, besides the existence of molecules, the biggest evidence on the existence of the isoelectronium as a particle is provided by experimental data on the photoionization of the hydrogen and helium molecules (see, e.g., Ref. [56]) in which bonded electron pairs in singlet have been systematically detected to survive molecular separation.

For comparison with preceding hadronic structures, let us recall that Ref. [30], Eqs. (4.5) to (4.23), pages 169 to 173, provided the following representation via model (29) of the rest energy $E = 1\ MeV$ and change radius $R = 6.84\times10^{-11}\ cm$ of the isoelectronium for the case of full stability

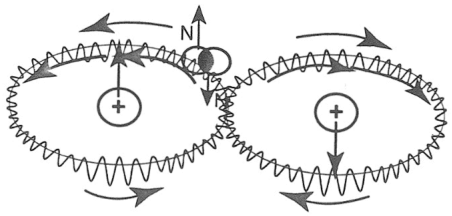

Figure 21: *An illustration of the isochemical model of the hydrogen molecule achieved by R. M. Santilli and D. D. Shillady [52] with the following main features: 1) Valence electron pairs are bonded into isoelectronia according to the principles of hadronic chemistry; 2) Due to their strong internal forces, isoelectronia are forced to have oo orbits around corresponding nuclei resulting in opposite angular momenta that represent the diamagnetic character of the hydrogen and other molecules; 3) The conventional four-body equations for the hydrogen molecule are reduced to a restricted three-body form admitting a full analytic solution.*

$$\tau = \infty$$

$$\begin{aligned}
&\left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2\tfrac{e^{-br}}{1-e^{-br}})\right] = 0,\\
&E_{tot} = E_1 + E_2 - E = 1\ MeV,\ \ \bar{m} = \tfrac{m}{I^2}\\
&\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_1}{\hbar} = \infty,\\
&R = b^{-1} = 6.84 \times 10^{-11}\ cm,
\end{aligned} \tag{115}$$

with solutions for the k-parameters

$$k_1 = 0.19\ \ k_2 = 1, \tag{116}$$

for which the binding energy is null (in first approximation) as expected from the sole use of contact non-potential interactions, resulting in the total energy of the isoelectronium $E = E_1 + E_2 = 1.022\ MeV$.

Note that, as expected from the charge independence of contact interactions, values (114) for the isoelectronium $(\tilde{e}_1, \tilde{e}_2)_{hc}$ are rather close to the corresponding values (50) for f the π^0 meson $(\tilde{e}^-, \tilde{e}^+)_{hm}$.

Interested readers should meditate a moment on the capability of contact interactions to achieve a strongly attractive force by overcoming the extremely big repulsive Coulomb force of $230\ N$.

This occurrence emphasizes the need in hadronic structure models such as (112) of the notion of *isoparticles* in which *all* intrinsic characteristics are mutated, thus including the charge.

This occurrence also suggests the *study of structure models of the elementary charge,* such as that of Ref. [44], capable of such a mutation in condition of deep mutual overlapping to represent the physical evidence of a strong attraction in singlet valence couplings.

In 1999, D. D. Shillady and R. M. Santilli proved that the isotopic branch of hadronic chemistry achieves the first known, numerically exact representation of the binding energy and other characteristics of the hydrogen [52] (Figure 23) and water [53] (Figure 24) molecules via isoperturbative isoseries (namely isoseries based on isoproduct $A\hat{T}B$ with $\hat{T} \ll 1$) whose convergence is at least *one thousand times faster* than the convergence of conventional quantum chemical series, thus providing an important experimental confirmation in molecular structures of both Einstein's determinism per Lemma II-3.7 as well as Corollary II-3.7.1.

In regard to the rather complex case in which the isoelectronium is partially unstable, the numerically exact representation of H_2 and H_2O data, and other aspects, we refer the interested reader to the above quoted literature.

The strong valence bond of isoelectronia evidently resolves Insufficiencies 1) and 2) of the preceding section.

Insufficiency 3) is resolved by the fact that, due to the strength of its bond, isoelectronium (112) is forced to have an oo-orbit around the respective nuclei (Figure 22) with ensuing opposite angular momenta and opposite magnetic polarities that permit a quantitative representation of the diamagnetic character of the hydrogen, water and other molecules.

The apparent resolution of Insufficiency 4) was provided by the new chemical species of *magnecules* [30] (see the U. S. patent [57], the latest experimental verification [58] and Figure 25) that allowed the identification of an actual attractive force between water molecules in their liquid state [98].

The resolution of Insufficiency 5) can be studied via the *Lie-admissible branch of hadronic chemistry,* also called *genochemistry* [30], although it does not appear that, at this writing, a consistent representation of energy releasing processes is of interest to contemporary chemists at large.

2.8.3. Industrial applications. It is a truism to state that the combustion of fossil fuels occurring in the ongoing disproportionate number of civilian, industrial and military vehicles is the same as it was at the dawn of our civilization some $50,000$ years ago because, in all cases, we strike a spark and lit the fuel.

To be "True Researchers" in Einstein's words, it is our societal duty to

Species	H_2	H_2O	HF
SCF-energy (DH) (a.u.)	-1.132800^a	-76.051524	-100.057186
Hartree-Fock[d] (a.u.)			-100.07185^d
Iso-energy (a.u.)	-1.174441^c	-76.398229^c	-100.459500^c
Horizon R_c (Å)	0.00671	0.00038	0.00030
QMC energy[d,e] (a.u.)	-1.17447	-76.430020^e	-100.44296^d
Exact non-rel. (a.u.)	-1.174474^f		-100.4595^d
Corellation (%)	99.9^b	91.6^b	103.8
SCF-dipole (D)	0.0	1.996828	1.946698
Iso-dipole (D)	0.0	1.847437	1.841378
Exp. dipole (D)	0.0	1.85^g	1.82^g
Time[h] (min:s)	0:15.49	10:08.31	6:28.48

(DH+) Dunning-Huzinaga (10S/6P), [6,2,1,1,1/4,1,1]+H_2P_1+3D1.
[a]LEAO-6G1S + optimized GLO-2S and GLO-2P.
[b]Relative to the basis set used here, not quite HF-limit.
[c]Iso-energy calibrated to give exact energy for HF.
[d]Hartree-Fock and QMC energies from Luchow and Anderson [33].
[e]QMC energies from Hammond *et al.* [30].
[f]First 7 sig. fig. from Kolos and Wolniewicz [34].
[g]Data from Chemical Rubber Handbook, 61st ed., p. E60.
[h]Run times on an O2 Silicon Graphics workstation (100 MFLOPS max.).

Figure 22: *A reproduction of Table 4.2 of Ref. [52] presenting the first known exact representation of the experimental data of the hydrogen molecule via isomathematics and isochemistry, by providing an experimental confirmation of the validity in molecular structres of Einstein's determinism according Lemma II-3.7, with a convergence of the isoperturbative series at least one thousand times faster than conventional series, by therefore providing an experimental confirmation of Corollary II-3.7.1.*

initiate the laborious process of trial and errors in seeking a clean combustion of fossil fuels that, to be as such, as to be *new*, namely based on new mathematical, physical and chemical principles.

In the hope of initiating the expectedly long and laborious process of trials and errors, Santilli has submitted in Ref. [59] the study of the new form of combustion of carbon and oxygen, known as *hypercombustion*, intended to achieve the full combustion of fossil fuels via a small percentage of nuclear fusions of the novel magnecular forms of $6-C-12$ and $8-O-16$ into the stable $14-Si-28$, thus without the emission of harmful radiations and without the release of radioactive waste [69] [70] (Figure 24).

We should mention in this respect the gaseous fuel *magnegas* [30] which is synthesized by hadronic reactors from a mixture of oil and water, in production and sale world wide by Magnegas Corporation, now called Taronis Corporation, whose combustion in air shows no appreciable carbon monoxide CO and hydrocarbons (HC) (Figure 25).

This environmental result was achieved via a combustion temperature more than double that of commercially available fuels, which temperature is permitted by magnecular bonds that are *weaker* than molecular bonds, and at which temperature CO and HC cannot remain unburned.

HyerCombustion aims at achieving the needed higher combustion temperature of petroleum fuels via the use of the indicated nuclear processes that are solely possible under the"completion" of quantum into hadronic mechanics.

It appears advisable to indicate that, besides the expected environmental advances, the new chemical species of magnecules appears to have significant medical applications, such as the possible killing of the corona virus in lungs (and other organs) of patients via ventilators releasing the new polarized species of *magneoxygen*, rather than conventional oxygen [62] - [65].

2.9. Einstein's determinism in gravitational collapse.

In the recent paper [10], Santilli recalls that iso-space-time metrics contain as particular cases all possible symmetric metrics in $(3+1)$-dimensions, thus including the Riemannian metric.

Ref. [10] then factorizes the space component of the Schwartzchild metric $g_s(r)$ according to isotopic rule introduced in Refs. [66] [67]

$$g_s(r) = \hat{T}(r)\delta, \tag{117}$$

where δ is the Euclidean metric.

In this way, Santilli reaches the following realization of the isotopic

	OH^+	OH^-	H_2O	HF
SCF-Energy[a]	-74.860377	-75.396624	-76.058000	-100.060379
Hartree-Fock[b]				-100.07185[b]
Iso-Energy[c]	-75.056678	-75.554299	-76.388340	-100.448029
Horizon R_c (Å)	0.00038	0.00038	0.00038	0.00030
QMC Energy[b,d]	-76.430020[d]			-100.44296[b]
Exact non-rel.				-100.4595
Iso-Dipole (D)	5.552581	8.638473	1.847437	1.8413778
Exper. Dipole			1.84	1.82

[a] Dunning-Huzinaga (10*s*/6*p*), (6,2,1,1,1/4,1,1)+H2*s*1+H2*p*1+3*d*1.
[b] Iso-Energy calibrated to give maximum correlation for HF.
[c] Hartree-Fock and QMC energies from Luchow and Anderson [22].
[d] QMC energies from Hammond, Lester and Reynolds [21].

Figure 23: *A reproduction of Table 5.1. of Ref. [53] presenting the first exact representation of the experimental data of the water molecule, including its electric and magnetic moments, via isomathematics and isochemistry, by therefore providing a second experimental confirmation of the validity in molecular structures of Einstein's determinism according to Lemma II-3.7, with a convergence of the isoperturbative series at least one thousand time faster than conventional series, thus providing an experimental confirmation of Corollary II-3.7.1.*

element

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \tag{118}$$

where M is the gravitational mass of the body considered with ensuing isodeterministic isoprinciple, Ref. [10], Eq. (46), page 16,

$$\Delta\hat{r}\Delta\hat{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r\to 0} = 0, \tag{119}$$

which confirms the possible recovering of full classical determinism in the interior of gravitational collapse essentially as predicted by Einstein (see Ref. [16], Chapter 6 in particular, for a critical analysis of black holes).

It should perhaps be indicated that the 1993 paper [100] identified the universal isosymmetry of all possible (non-singular) Riemannian line elements in $3 + 1)$-dimensions formulated on iso-Minkowskian isospaces [78] over isofields. Papers [66] [67] introduced the factorization of a full Riemannian metric $g(x),\ x = (r, t)$ in $(3 + 1)$-dimensions

$$g(x) = \hat{T}_{gr}(x)\eta, \tag{120}$$

where $\hat{T}_{gr}$ is the *gravitational isotopic element*, and η is the Minkowski metric $\eta = Diag.(1, 1, 1, -1)$.

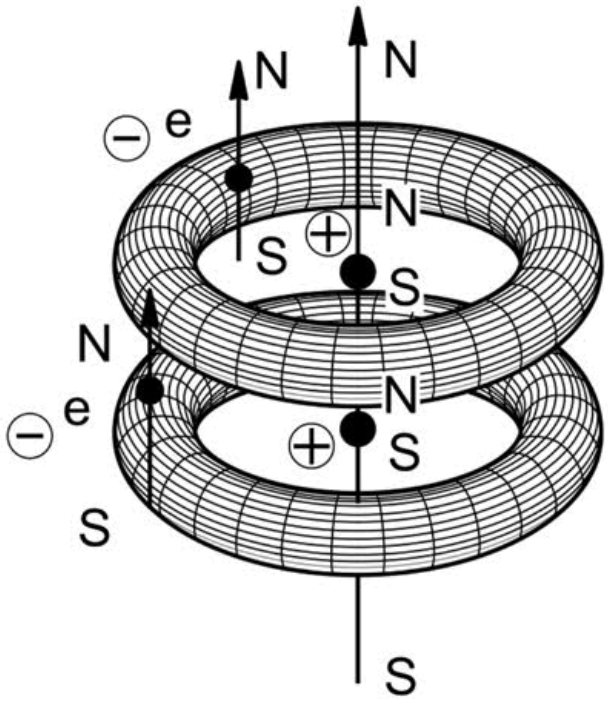

Figure 24: *In this figure, we present a conceptual rendering of the new chemical species of Santilli magnecules [30] for the case of the elementary hydrogen magnecules* $\tilde{H}_2 = H \times H$ *(where* $\times$ *denotes magnecular bond). The species is obtained via the use of special electric arcs producing electric and magnetic fields over certain minimum values [***?** *], which arcs ionize the hydrogen molecule and polarize the electron orbits into toroids by therefore creating a new magnetic dipole moment which does not exist in the natural state. Polarized atoms bond together in the configuration of the figure which is stable at ambient temperatures by creating the new species of magnehydrogen (see Refs. [62] to [65]). It should be indicated that elementary magnecules between other selected atoms verify all hadronic laws for controlled nuclear fusions ([25], Section 4), for instance, of* $\hat{CO} = 6 - C - 12 \times 8 - O - 16$ *into* $14 - Si - 28$ *without the emission of harmful radiations and without the release of radioactive waste [69] [70].*

Refs. [66] [67] then reformulated the Riemannian geometry via the transition from a formulation over the field of real numbers $\mathcal{R}$ to that over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ where the *gravitational isounit* is evidently given by

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x). \tag{121}$$

The above reformulation turns the Riemannian geometry into a new

geometry called iso-Minkowskian isogeometry [78] which is locally isomorphic to the *Minkowskian* geometry, while maintaining the mathematical machinery of the Riemannian geometry (covariant derivative, connection, geodesics, etc.) although reformulated in terms of the isodifferential isocalculus.

The following advantages should be mentioned for the *identical* iso-Minkowskian reformulation of general relativity, including Einstein's field equations, known as *isogravitation* [83]:

1) The achievement of a consistent operator form of gravity via the axiom-preserving embedding of the gravitational isounit $\hat{I}_{gr}(x)$ in the *unit* of relativistic quantum mechanics [33] [21];

2) The achievement of the universal *LPS isosymmetry* of *all* non-singular Riemannian metrics [100], which symmetry is locally isomorphic to the LP symmetry, (Section II-2), while being notoriously impossible in a conventional Riemannian space over the reals;

3) The achievement of clear compatibility, actually a *true isounification,* of general and special relativity since the latter can be identically recovered with the simple limit

$$\hat{I}_{gr} \to I = Diag.(1, 1, 1, 1), \tag{122}$$

implying the transition from the universal LPS isosymmetry to the LP symmetry of special relativity with ensuing recovering of conservation and other special relativity laws;

4) The possibility of initiating systematic studies on *interior gravitational problems* along the forgotten Schwartzchild's second paper [73];

5) The achievement of axiomatic compatibility between gravitation and electroweak interactions thanks to the replacement of curvature into the covering notion of *isoflatness,* while offering realistic hopes to achieve a grand unification [38]; and other intriguing advances.

3. CONCLUDING REMARKS.

At the end of their historical paper [1], Einstein, Podolsky and Rosen state:

While we have thus shown that the wavefunction [of quantum mechanic] does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

In the preceding Papers I and II we have provided a review and upgrade of the apparent proof of the existence of classical counterparts for

Figure 25: *A picture of the 300 kW mobile magnegas refinery built by Santilli when Chief Scientist of Magnegas Corporation (now Taronis Corporation) to process a mixture of fossil oil and water into magnegas whose combustion with the proper stochiometric oxygen has no detectable CO or HC [57]. The oxygen content of magnegas appears to have special polarizations [62] - [65] deserving tests for its possible use in ventilators to kill Corona viruses due to the emission of strong UV light in its depolarization in human lungs.*

extended particles within physical media [9] (Section II-3.7), and of the apparent progressive verification of Einstein determinism in the interior of hadrons, nuclei and stars [10] (Section II-3.8).

In this third paper, it appears we have additionally proved, apparently for the first time, the above quoted, concluding EPR statement. In fact, the "completion" of the quantum wavefunction $\psi(t, r)$ of the Schrödinger equation into the hadronic isowavefunction $\hat{\psi}(\hat{t}, \hat{r})$ of the Schrödinger-Santilli isoequation allows the achievement, otherwise impossible via quantum mechanics, of an exact representation of all characteristics of: the neutron in its synthesis from the hydrogen atom (Section 2.6); the deuteron (Section 2.7); the attractive force between the identical electrons in valence couplings (Section2.8); and the progressive achievement of Einstein's determinism in interior dynamical conditions (Section 2.9).

Acknowledgments. The writing of these papers has been possible thanks to pioneering contributions by numerous scientists we regret not having been able to quote in this paper (see Vol. I, Refs. [**?**] for comprehensive literature), including:

The mathematicians: H. C. Myung, for the initiation of iso-Hilbert spaces; S. Okubo, for pioneering work in non-associative algebras; N. Kamiya, for basic advances on isofields; M. Tomber, for an important bibliography on non-associative algebras; R. Ohemke, for advances in Lie-admissible algebtas; J. V. Kadeisvili, for the initiation of isofunctional isoanalysis; Chun-Xuan Jiang, for advances in isofield theory; Gr. Tsagas, for advances in the Lie-Santilli theory; A. U. Klimyk, for the initiation of Lie-admissible deformations; Raul M. Falcon Ganfornina and Juan Nunez Valdes, for advances in isomanifolds and isotopology; Svetlin Georgiev, for advances in the isodifferential isocalculus; A. S. Muktibodh, for advances in the isorepresentations of Lie-Santilli isoalgebras; Thomas Vougiouklis, for advances on Lie-admissible hypermathematics; and numerous other mathematicians.

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tween Lie-admissible mechanics and Nambu mechanics; Yu. Arestov, for the experimental verification of the iso-Minkowskian structure of hadrons; A. Ahmar, for experimental verification of the iso-Minkowskian character of Earth's atmosphere; L. Ying, for the experimental verification of nuclear fusions of light elements without harmful radiation or waste; Y. Yang, for the experimental confirmation of magnecules; D. D. Shillady, for the isorepresentation of molecular binding energies; R. L. Norman, for advances in the laboratory synthesis of the neutron from the hydrogen; I. Gandzha for important advances on isorelativities; A. A. Bhalekar, for the exact representation of nuclear spins; Simone Beghella Bartoli important contributions in experimental verifications of hadronic mechanics; and numerous other physicists.

I would like to express my deepest appreciation for all the above courageous contributions written at a point in time of the history of science dominated by organized oppositions against the pursuit of new scientific knowledge due to the trillions of dollars of public funds released by governmental agencies for research on pre-existing theories.

Special thanks are due to A. A. Bhalekar, J. Dunning-Davies, S. Georgiev, R. Norman, T. Vougiouklis and other colleague for deep critical comments.

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Pairwise Paracompactness

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Abstract

The purpose of this paper is to introduce and study a new paracompactness in bitopological spaces using (τ_i, τ_j)-$g^*\omega\alpha$-closed sets. Further, the properties of (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous functions and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps and (τ_i, τ_j)-$g^*\omega\alpha$-paracompact spaces are discussed in bitopological spaces.

Keywords: (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-open sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps, (τ_i, τ_j)-$g^*\omega\alpha$-paracompact spaces.
2010 AMS subject classifications: 54E55. [1]

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1 Introduction

The research in topology over last two decades has reached a high level in many directions. Topological methods are widely used in many other branches of modern mathematics such as differential equation, functional analysis, classical mechanics, general theory of relativity, mathematical economics, quantum theory, biology etc.

Bitopological space is a triplet (X, τ_1, τ_2), where X is a non empty set and τ_1 and τ_2 are topologies on a space X. In 1963, J. C. Kelly [8] initiated the study of bitopological spaces. In 1985, Fututake [5] studied the concept of generalized closed (briefly g-closed) sets in bitopological spaces. After that, several authors turned their attention towards the generalizations of various concepts in topology by considering bitopological spaces.

In this paper, (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous functions and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps are defined and studied in bitopological spaces. Also, the concept of (τ_i, τ_j)-$g^*\omega\alpha$-paracompactness in bitopological spaces is introduced and studied.

2 Preliminaries

Throughout this present paper, let X, Y and Z always represents non-empty bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, γ_1, γ_2) on which no separation axioms are assumed unless explicitly mentioned and the integers $i, j, k \in \{1, 2\}$.

Definition 2.1. *[13] A space X is said to be $g^*\omega\alpha$-paracompact if every open cover of X has a $g^*\omega\alpha$-locally finite $g^*\omega\alpha$-refinement.*

Definition 2.2. *Let $A \subseteq X$. Then A is said to be a*
(a) $\omega\alpha$-closed [2] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is ω-open in X.
(b) $g^*\omega\alpha$-closed[12] if cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is $\omega\alpha$-open in X.

Definition 2.3. *A subset A of a bitopological space (X, τ_1, τ_2) is called a*
(a) (τ_i, τ_j)-g-closed [5] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in τ_i.
(b) (τ_i, τ_j)-rg-closed [1] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i.
(c) (τ_i, τ_j)-αg-closed [3] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in τ_i.
(d) (τ_i, τ_j)-gα-closed [3] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in τ_i.
(e) (τ_i, τ_j)-gpr-closed [7] if τ_j-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i.
(f) (τ_i, τ_j)-g^*-closed [14] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in τ_i.

(g) (τ_i, τ_j)-$\omega\alpha$-closed [11] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in τ_i.
In all the above definitions $i \neq j$.

Definition 2.4. *A map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is called a*
(a) τ_j-μ_k-continuous [10] if $f^{-1}(G) \in \tau_j$ for every open set G in μ_k.
(b) $D(\tau_i, \tau_j)$-μ_k-continuous [10] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-g-closed in (X, τ_1, τ_2).
(c) $D_r(\tau_i, \tau_j)$-μ_k-continuous [1] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-rg-closed in (X, τ_1, τ_2).
(d) $C(\tau_i, \tau_j)$-μ_k-continuous [6] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-ω-closed in (X, τ_1, τ_2).
(e) $D^*(\tau_i, \tau_j)$-μ_k-continuous [14] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-g*-closed in (X, τ_1, τ_2).
(f) (τ_i, τ_j)-αg-continuous [4] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-αg-closed in (X, τ_1, τ_2).

Definition 2.5. *[8] A bitopological space (X, τ_1, τ_2) is said to be pairwise Hausdorff if for each pair of distinct points x and y of X, there exist $U \in P_i$ and $V \in P_j$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.*

3 (τ_i, τ_j)-g$^*\omega\alpha$-Closed Sets

This section deals with the concept of $g^*\omega\alpha$-closed sets in bitopological spaces and some of their properties.

Definition 3.1. *Let $(i, j) \in \{1, 2\}$ where $i \neq j$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-g$^*\omega\alpha$-closed if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$-$\omega\alpha$-open in X.*

Example 3.1. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Consider a set in the space (X, τ_1, τ_2), $A = \{n, p\}$ which is (τ_1, τ_2)-g$^*\omega\alpha$-closed.*

Remark 3.1. *If $\tau_1 = \tau_2 = \tau$ in Definition 3.1, then (τ_i, τ_j)-g$^*\omega\alpha$-closed set in (X, τ_1, τ_2) is same as g$^*\omega\alpha$-closed [12] in (X, τ).*

The family of all (τ_i, τ_j)-g$^*\omega\alpha$-closed sets in (X, τ_1, τ_2) is denoted by $P(\tau_i, \tau_j)$.

Theorem 3.1. *Every τ_j-closed (resp. (τ_i, τ_j)-regular closed) is (τ_i, τ_j)-g$^*\omega\alpha$-closed.*
However the converse need not be true in general as shown in the following example.

Example 3.2. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}\}$. *Consider the set,* $A = \{m, p\}$ *is* (τ_1, τ_2)-$g^*\omega\alpha$-*closed but not* τ_2-*closed (resp.* (τ_i, τ_j)-*regular closed).*

We have the following implification:
τ_j-closed $\rightarrow$ (τ_i, τ_j)-$g^*\omega\alpha$-closed $\rightarrow$ (τ_i, τ_j)-αg-closed

Remark 3.2. *If A and B are* (τ_i, τ_j)-$g^*\omega\alpha$-*closed in* (X, τ_1, τ_2) *then* $A \cup B$ *is also* (τ_i, τ_j)-$g^*\omega\alpha$-*closed.*

Theorem 3.2. *If a subset A of* (X, τ_1, τ_2) *is* (τ_i, τ_j)-$g^*\omega\alpha$-*closed then* τ_j-$cl(A) - A$ *does not contain any non empty* $\omega\alpha$-*closed set in* τ_i.

Proof. Let $A \subseteq (\tau_i, \tau_j)$-$g^*\omega\alpha$-closed and $F \subseteq \tau_i$-$\omega\alpha$-closed set such that $F \subseteq \tau_j$-$cl(A) - A$. Now $F \subseteq \tau_j$-$cl(A)$ and $F \subseteq X - A$. Then $A \subseteq X - F$ and by hypothesis A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $X - F$ is $\tau_i - \omega\alpha$-open. Thus from Definition 3.1, τ_j-$cl(A) \subseteq X - F$, that is $F \subseteq (X - \tau_j$-$cl(A))$. Then $F \subseteq (\tau_j$-$cl(A)) \cap (X - \tau_j$-$cl(A)) = \phi$ and so $F = \phi$ which is a contradiction. Hence τ_j-$cl(A) - A$ does not contain any non empty $\omega\alpha$-closed set. □

Remark 3.3. *A* (τ_i, τ_j)-$g^*\omega\alpha$-*closed set need not be* τ_i-$g^*\omega\alpha$-*closed or* τ_j-$g^*\omega\alpha$-*closed.*

Example 3.3. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}\}$ *and* $\tau_2 = \{X, \phi\}$. *Then the set* $A = \{n, p\}$ *is* (τ_i, τ_j)-$g^*\omega\alpha$-*closed but not* τ_2-$g^*\omega\alpha$-*closed.*
Also, if $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$ *be topology on X. Then the set* $A = \{m, p\}$ *is*(τ_i, τ_j)-$g^*\omega\alpha$-*closed but not* τ_1-$g^*\omega\alpha$-*closed in* (X, τ_1, τ_2).

Remark 3.4. *In general* $P(\tau_i, \tau_j) \neq P(\tau_j, \tau_i)$.

Example 3.4. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. *Then* $g^*\omega\alpha C(\tau_1, \tau_2) = \{X, \phi, \{m, n\}\}$ *and* $g^*\omega\alpha C(\tau_2, \tau_1) = \{X, \phi, \{n, p\}, \{p\}\}$. *Hence we can observe that* $g^*\omega\alpha C(\tau_1, \tau_2) \neq g^*\omega\alpha C(\tau_2, \tau_1)$.

Remark 3.5. *If* $\tau_1 \subseteq \tau_2$, *then* $P(\tau_2, \tau_1) \subseteq P(\tau_1, \tau_2)$ *but converse is not true.*

Example 3.5. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. *Then* $P(\tau_2, \tau_1) = \{X\}, \phi, \{m, n\}, \{n, p\}\}$ *and* $P(\tau_1, \tau_2) = \{X, \phi, \{m, n\}, \{n, p\}, \{p\}\}$. *Then* $P(\tau_2, \tau_1) \subseteq G(\tau_1, \tau_2)$ *but* $\tau_1 \nsubseteq \tau_2$.

Theorem 3.3. *A* τ_i-$\omega\alpha$-*open and* (τ_i, τ_j)-$g^*\omega\alpha$-*closed set is* τ_j-*closed.*

Proof. Now $A \subseteq A$. Then τ_j-$cl(A) \subseteq A$ and $A \subseteq \tau_j$-$cl(A)$. Therefore τ_j-$cl(A) = A$ and hence $A \in \tau_j$-closed. □

Theorem 3.4. *Let A be τ_i-$\omega\alpha$-open and (τ_i, τ_j)-$g^*\omega\alpha$-closed. Suppose F is τ_j-closed, then $A \cap F$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed.*

Proof. Let A be τ_i-$\omega\alpha$-open and A be (τ_i, τ_j)-$g^*\omega\alpha$-closed and F be τ_j-closed. Then from Theorem 3.3, A is τ_j-closed. So $A \cap F$ is τ_j-closed and hence (τ_i, τ_j)-$g^*\omega\alpha$-closed. □

Theorem 3.5. *If A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$, then τ_j-$cl(B) - B$ contains no non empty τ_i-closed set.*

Proof. Let A be (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$. Then B is (τ_i, τ_j)-$g^*\omega\alpha$-closed follows from Theorem 3.19 [12]. Hence τ_j-$cl(B) - B$ contains no non empty τ_i-closed set. □

Corolary 3.1. *If A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$, then τ_j-$cl(B) - B$ contains no non empty τ_i-$\omega\alpha$-closed set.*

Theorem 3.6. *Arbitrary union of (τ_i, τ_j)-$g^*\omega\alpha$-closed sets $\{A_i : i \in I\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed if the family $\{A_i : i \in I\}$ is τ_j-locally finite.*

Proof. Let $\{A_i : i \in I\}$ is τ_j-locally finite and $\{A_i : i \in I\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. Let $\cup A_i \subseteq U$ where $U \in \tau_i$-$\omega\alpha$-open. Then $A_i \subseteq U$ and $\omega\alpha$-open in τ_i. Since A is (τ_i, τ_j)-$g^*\omega\alpha$-closed the for each $i \in I$, τ_j-$cl(A_i) \subseteq U$. Consequently $\cup\tau_j$-$cl(A_i) \subseteq U$. Since the family, $\{A_i : i \in I\}$ is τ_j-locally finite τ_j-$cl(\cup A_i) = \cup(\tau_j$-$cl(A_i)) \subseteq U$. Therefore $\cup A_i$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. □

Theorem 3.7. *For an element x in X, the set $X - \{x\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed or $X - \{x\}$ is $\omega\alpha$-open in τ_i.*

Proof. Suppose $X - \{x\}$ is not $\omega\alpha$-open in τ_i, then X is the only $\omega\alpha$-open set containing $X - \{x\}$, that is τ_j-$cl(X - \{x\}) \subseteq \tau_j$-$cl(\{x\}) = X$. Hence τ_j-$cl(X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is $\tau_i, \tau_j)$-$g^*\omega\alpha$-closed. □

Definition 3.2. *A subset A of a bitopological space (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-open if its complement is (τ_i, τ_j)-$g^*\omega\alpha$-closed.*

Definition 3.3. *For a subset A of a bitopological space (X, τ_1, τ_2), (τ_i, τ_j)-$g^*\omega\alpha$-interior of A is denoted by (τ_i, τ_j)-$g^*\omega\alpha$-int(A) and is defined as (τ_i, τ_j)-$g^*\omega\alpha$-$int(A) = \cup\{F : F \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-open and $F \subseteq A\}$.*

Theorem 3.8. *Let A be (τ_i, τ_j)-$g^*\omega\alpha$-open. Then $P = X$ whenever G is τ_i-$\omega\alpha$-open and τ_j-$g^*\omega\alpha$-$int(A) \cup A^c \subseteq G$.*

Theorem 3.9. *A set A is (τ_i, τ_j)-$g^*\omega\alpha$-open if and only if $F \subseteq \tau_j$-$int(A)$ whenever F is τ_i-closed and $F \subseteq A$.*

Theorem 3.10. *If A and B are separated (τ_i, τ_j)-$g^*\omega\alpha$-open sets then $A \cup B$ is also (τ_i, τ_j)-$g^*\omega\alpha$-open.*

Proof. Suppose A and B are (τ_i, τ_j)-$g^*\omega\alpha$-open sets. Let F be an τ_i-closed set such that $F \subseteq A \cup B$. Since A and B are separated, τ_i-cl(A) $\cap B = A \cap \tau_i$-cl(B) = ϕ and τ_j-$cl(A) \cap B = A \cap \tau_j$-$cl(B) = \phi$. Then $F \cap \tau_j$-$cl(A) \subseteq (A \cup B) \cap \tau_j$-$cl(A) = A$. Similarly, $F \cap \tau_j$-$cl(B) \subseteq B$. Since F is τ_i-closed, we have $F \cap \tau_i$-$cl(A), F \cap \tau_i$-$cl(B)$ are also τ_i-closed and from hypothesis A and B are (τ_i, τ_j)-$g^*\omega\alpha$-open sets, $F \cap \tau_j$-$cl(A) \subseteq \tau_j$-$int(A)$ and $F \cap \tau_j$-$cl(B) \subseteq \tau_j$-$int(B)$. Now $F = F \cap (A \cup B) \subseteq (F \cap \tau_j$-$cl(A)) \cup (F \cap \tau_j$-$cl(B)) \subseteq \tau_j$-$int(A \cup B)$. Hence $A \cup B$ is (τ_i, τ_j)-$g^*\omega\alpha$-open. □

Definition 3.4. *A bitopological space (X, τ_1, τ_2) is said to be a (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space if every (τ_i, τ_j)-$g^*\omega\alpha$-closed set is τ_j-closed.*

Example 3.6. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Then (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Theorem 3.11. *If a bitopological space (X, τ_1, τ_2) is (τ_i, τ_j)-$T_{g^*\omega\alpha}$ space, then for each $x \in X$, $\{x\}$ is τ_i-$\omega\alpha$-closed or τ_j-open.*

Proof. Suppose $\{x\}$ is not (τ_i, τ_j)-$g^*\omega\alpha$-open, then $\{x\}^c$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. As X is is (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space, $\{x\}^c$ is τ_j-closed and hence $\{x\}$ is τ_j-open. □

Remark 3.6. *Every singleton subset of (X, τ_1, τ_2) is τ_j-closed or τ_i-$\omega\alpha$-closed but (X, τ_1, τ_2) is not (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space.*

Example 3.7. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Then every singleton set $\{x\}$ of X is either τ_2-open or τ_1-$\omega\alpha$-closed. However, (X, τ_1, τ_2) is not (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Remark 3.7. *If (X, τ_1) and (X, τ_2) are both $T_{g^*\omega\alpha}$-space, then it need not imply (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Example 3.8. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{n\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. Then (X, τ_1) and (X, τ_2) are $T_{g^*\omega\alpha}$-space, but (X, τ_1, τ_2) is not (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Remark 3.8. *The space (X, τ_1) is not generally $T_{g^*\omega\alpha}$-space if (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Example 3.9. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Then (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space, but (X, τ_1) is not $T_{g^*\omega\alpha}$-space.*

4 (τ_i, τ_j)-$g^*\omega\alpha$-Continuous and (τ_i, τ_j)-$g^*\omega\alpha$-Irresolute Maps

Definition 4.1. *A map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is called $P(\tau_i, \tau_j)$-μ_k-continuous (pairwise $g^*\omega\alpha$-continuous) if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-closed in (X, τ_1, τ_2).*

Theorem 4.1. *Every is τ_j-μ_k-continuous function is $P(\tau_i, \tau_j)$-μ_k-continuous.*

Proof. Follows from Theorem 3.1. □
The converse need not be true as seen from the following example.

Example 4.1. *Let $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{n\}\}$ and $\mu_2 = \{Y, \phi, \{m\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be the identity map. Then f is $P(\tau_1, \tau_2)$-μ_1-continuous but not τ_2-μ_1-continuous, since for the μ_1-closed set $A = \{m, p\}$ in Y, $f^{-1}(\{m, p\}) = \{m, p\}$ is not τ_2-closed in X.*

Remark 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be $P(\tau_i, \tau_j)$-μ_k-continuous $g : (Y, \mu_1, \mu_2) \to (Z, \gamma_1, \gamma_2)$ be $P(\mu_1, \mu_2)$-γ_m-continuous but their composition need not be $P(\tau_i, \tau_j)$-γ_m-continuous.*

Example 4.2. *Let $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{m\}, \{n, p\}\}$, $\mu_2 = \{Y, \phi, \{m\}\}$, $\gamma_1 = \{Z, \phi, \{m\}, \{m, p\}\}$ and $\gamma_2 = \{Z, \phi, \{m\}, \{m, n\}, \{m, p\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be identity map and define a map $g : (Y, \mu_1, \mu_2) \to (Z, \gamma_1, \gamma_2)$ by $g(m) = n, g(n) = m, g(p) = p$. Then f and g are pairwise $g^*\omega\alpha$-continuous maps but their composition is not pairwise $g^*\omega\alpha$-continuous, since for the γ_1-closed set $\{n, p\}$ in (Z, γ_1, γ_2), $(gof)^{-1}(\{n, p\}) = f^{-1}(g^{-1}(\{n, p\})) = f^{-1}(\{m, p\}) = \{m, p\}$ is not (τ_1, τ_2)-$g^*\omega\alpha$-closed in (X, τ_1, τ_2).*

Definition 4.2. *A map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is called pairwise $g^*\omega\alpha$-irresolute if for every $A \in P(\mu_k, \mu_e)$ in (Y, μ_1, μ_2), $f^{-1}(A) \in P(\tau_i, \tau_j)$ in (X, τ_1, τ_2).*

Theorem 4.2. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ pairwise $g^*\omega\alpha$-irresolute if f is $P(\tau_i, \tau_j)$-μ_e-continuous.*

Proof. Let F be μ_e-closed, then F is (μ_k, μ_e)-$g^*\omega\alpha$-closed in (Y, μ_1, μ_2). From Theorem 3.1, $F \in P(\mu_k, \mu_e)$. Since f is pairwise $g^*\omega\alpha$-irresolute, $f^{-1}(F) \in P(\tau_i, \tau_j)$. Therefore f is $P(\tau_i, \tau_j)$-μ_e-continuous. □
The converse of this theorem need not be true as seen from the following example.

Example 4.3. *Let* $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{n\}, \{n, p\}\}$ *and* $\mu_2 = \{Y, \phi, \{m\}, \{m, p\}\}$. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ *be the identity map. Then f is* $P(\tau_1, \tau_2)$-μ_1-*continuous map but not pairwise* $g^*\omega\alpha$-*irresolute map, since for the* (μ_1, μ_2)-$g^*\omega\alpha$-*closed set* $\{m, p\}$ *in* (Y, μ_1, μ_2), $f^{-1}(\{m, p\}) = \{m, p\}$ *is not* (τ_i, τ_j)-$g^*\omega\alpha$-*closed set in* (X, τ_1, τ_2).

Theorem 4.3. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ *be a map and* (Y, μ_1, μ_2) *be* (μ_k, μ_e)-$T_{g^*\omega\alpha}$-*space. Then f is pairwise* $g^*\omega\alpha$-*irresolute if and only if f is* $P(\tau_i, \tau_j)$-μ_e-*continuous.*

Proof. Suppose f is pairwise $g^*\omega\alpha$-irresolute. From Theorem 4.2, f is $P(\tau_i, \tau_j)$-μ_e-continuous.
Conversely, let f be $P(\tau_i, \tau_j)$-μ_e-continuous map. Let F be (μ_k, μ_e)-$g^*\omega\alpha$-closed in (Y, μ_1, μ_2). By hypothesis (Y, μ_1, μ_2) is (μ_k, μ_e)-$T_{g^*\omega\alpha}$-space, F is μ_e-closed set in (Y, μ_1, μ_2). Again, since f is $P(\tau_i, \tau_j)$-μ_e-continuous, $f^{-1}(F)$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed set in (X, τ_1, τ_2). Hence f is pairwise $g^*\omega\alpha$-irresolute. □

5 (τ_i, τ_j)-$g^*\omega\alpha$-Paracompact Spaces

We recall that, a collection $\xi = \{F_\lambda : \lambda \in \Gamma\}$ of subsets of a space X is called a locally finite with respect to the topology τ_i, if for each $x \in X$ there exists $U_x \in \tau_i$ containing x and U_x which intersects at most finitely many members of ξ.

Definition 5.1. *A collection* $\xi = \{F_\lambda : \lambda \in \Gamma\}$ *of subsets of a space X is called* (τ_i, τ_j)-*P-locally finite if for each* $x \in X$ *there exist* (τ_i, τ_j)-$g^*\omega\alpha$-*open* U_x *in X and* U_x *intersects at most finitely many members of* ξ.

Theorem 5.1. *Let* $\xi = \{F_\lambda : \lambda \in \Gamma\}$ *be a collection of subsets of* (X, τ_1, τ_2) *then*
(a) ξ *is* (τ_i, τ_j)-$g^*\omega\alpha$-*locally finite if and only if* $\{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda) : \lambda \in \Gamma\}$ *is* (τ_i, τ_j)-*P-locally finite.*
(b) if ξ *is* (τ_i, τ_j)-*P-locally finite, then* $\cup(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda) = (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(\cup F_\lambda)$.
(c) ξ *is locally finite with respect to the topology* τ_i *if and only if the collection* $\{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda : \lambda \in \Gamma)\}$ *is locally finite with respect to the topology* τ_i.

Proof. (a) Suppose ξ is (τ_i, τ_j)-P-locally finite. Then for each $x \in X$, there exists (τ_i, τ_j)-$g^*\omega\alpha$-open set U_x containing x, which meets only finitely many of the sets F_λ, say $F_{\lambda_1}, F_{\lambda_2}, ..., F_{\lambda_n}$. Since $F_{\lambda_K} \subseteq (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_{\lambda_K})$ for each k = 1,2,...,n and U_x meets (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_1}), ..., (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_{\lambda_n})$. Therefore $g^*\omega\alpha\text{-}cl(F_\lambda)$ where $\lambda \in \gamma$ is (τ_i, τ_j)-P-locally finite.
Conversely, let $x \in X$. Then there exists (τ_i, τ_j)-$g^*\omega\alpha$-open U_x, which meets only finitely many of the sets (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_n})$, say (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_1}), ...,$

$(\tau_i, \tau_j)g^*\omega\alpha$-$cl(F_{\lambda_n})$. Then $U_x \cap (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k}) \neq \phi$. Let $q \in U_x$ and $q \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$, implies that for every (τ_i, τ_j)-$g^*\omega\alpha$-open set V_q, we have $V_q \cap F_{\lambda_k} \neq \phi$. But, we have U_x is (τ_i, τ_j)-$g^*\omega\alpha$-open set containing q and so $U_x \cap F_{\lambda_k} \neq \phi$ for each k=1,2,...,n. Thus ξ is (τ_i, τ_j)-P locally finite.
(b) Suppose ξ is (τ_i, τ_j)-P-locally finite, then $\cup(\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda) \subseteq (\tau_i, \tau_j)g^*\omega\alpha$-$cl(\cup F_\lambda)$. On the other hand, let $q \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(\cup F_\lambda)$. Then for every (τ_i, τ_j)-$g^*\omega\alpha$-open set V_q such that $V_q \cap (\cup F_\lambda) \neq \phi$. But from the hypothesis, there exists (τ_i, τ_j)-$g^*\omega\alpha$-open set U_q such that U_q meets only finitely many of the sets F_λ, say $F_{\lambda_1}, F_{\lambda_2}, ..., F_{\lambda_n}$. Thus for each (τ_i, τ_j)-$g^*\omega\alpha$-open set V_q containing q, we have $V_q \cap (\cup F_{\lambda_k}) \neq \phi$ where k=1,2,...,n. That is, for each $q \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(\cup F_{\lambda_k})$, there exists h such that $q \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_h})$. Therefore $q \in \cup(\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda)$ and hence (τ_i, τ_j)-$g^*\omega\alpha$-$cl(\cup F_\lambda) = \cup(\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda)$.
(c) Suppose ξ is locally finite with respect to the topology τ_i, then for each $x \in X$ there exists τ_i-open set U_x which meets only finitely many set F_λ, say $F_{\lambda 1}, F_{\lambda 2}, ...$, F_{λ_n}, but $F_{\lambda k} \subseteq (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$. Then U_x meets (τ_i, τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_1}), ...,$ (τ_i, τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_n})$. Thus (τ_i, τ_j)-$g^*\omega\alpha$-$cl(F_\lambda : \lambda \in \Gamma)$ is locally finite with respect to the topology τ_i.
Conversely, let $x \in X$. Then there exists τ_1-open set U_x which meets only finitely many of the sets (τ_i, τ_j)-$g^*\omega\alpha$-$cl(F_\lambda)$, that is (τ_i, τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_1}), ..., (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_n})$. Let $q \in U_x$ and $q \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$ where k=1,2,...,n. Then for each (τ_i, τ_j)-$g^*\omega\alpha$-open set V_q containing q such that $V_q \cap F_{\lambda_k} \neq \phi$. But $q \in U_x$ and so U_x meets only finitely many of the sets F_λ. Hence ξ is locally finite with respect to the topology τ_i. □

Lemma 5.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is (τ_i, τ_j)-$g^*\omega\alpha$-closed if and only if for every $y \in Y$ and $U \in \tau_1 O(X)$ which contains $f^{-1}(y)$ there exists $V \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-open set in (Y, σ_1, σ_2) such that $y \in Y$ and $f^{-1}(V) \subseteq U$.*

Theorem 5.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (τ_i, τ_j)-$g^*\omega\alpha$-irresolute. If $\xi = \{F_\lambda : \lambda \in \Gamma\}$ be a (τ_i, τ_j)-P-locally finite collection in Y, then $f^{-1}(\xi) = \{f^{-1}(F_\lambda) : \lambda \in \Gamma\}$ is (τ_i, τ_j)-P locally finite collection in X.*

Theorem 5.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (τ_i, τ_j)-$g^*\omega\alpha$-continuous. If $\xi = \{F_\lambda : \lambda \in \Gamma\}$ is (τ_i, τ_j)-P locally finite collection in Y, then $f^{-1}(\xi) = \{f^{-1}(F_\lambda) : \lambda \in \Gamma\}$ is locally finite collection with respect to the topology τ_i.*

Definition 5.2. *A non empty collection $\xi = \{A_i, i \in I$, an index set$\}$ is called a (τ_i, τ_j)-$g^*\omega\alpha$-open cover of a bitopological space (X, τ_1, τ_2) if $X = \cup A_i$ and $\xi \subseteq \tau_1$-$g^*\omega\alpha O(X, \tau_1, \tau_2) \cup \tau_2$-$g^*\omega\alpha O(X, \tau_1, \tau_2)$ and ξ contains at least one member of τ_1-$g^*\omega\alpha O(X, \tau_1, \tau_2)$ and one member of τ_2-$g^*\omega\alpha O(X, \tau_1, \tau_2)$.*

Definition 5.3. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-$g^*\omega\alpha$-compact if every cover of A by (τ_i, τ_j)-$g^*\omega\alpha$-open sets has a finite subcover.*

Example 5.1. *Let* $X = \{m, n, p, q\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ *and* $\tau_2 = \{X, \phi, \{m, n\}, \{m, n, p\}, \{m, p, q\}\}$. *Let* $\xi = \{\{m\}, \{m, n\}, \{m, n, p\}, \{m, p, q\}\}$ *be a* $g^*\omega\alpha$*-open cover of* (X, τ_1, τ_2). *Then* (X, τ_1, τ_2) *is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact.*

Definition 5.4. *A set A of a bitopological space* (X, τ_1, τ_2) *is said to be* (τ_i, τ_j)-$g^*\omega\alpha$*-compact relative to X if every* (τ_i, τ_j)-$g^*\omega\alpha$*-open cover of A has a finite subcover as a subspace.*

Theorem 5.4. *Every* (τ_i, τ_j)-$g^*\omega\alpha$*-compact space is* (τ_i, τ_j) *compact.*
Proof: *Let* (X, τ_1, τ_2) *be* (τ_i, τ_j)-$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i : i \in I\}$ *be* (τ_i, τ_j) *open cover of X. Then* $X = \cup A_i$ *and* $\xi \subseteq \tau_i \cup \tau_j$, *so* ξ *contains at least one member of* τ_i *and one member of* τ_j. *Since, every* τ_i*-open set is* τ_i-$g^*\omega\alpha$*-open, we have* $X = \cup A_i$ *and* $\xi \subseteq \tau_i$-$g^*\omega\alpha O(X) \cup \tau_j$-$g^*\omega\alpha O(X)$ *and by definition* ξ *contains at least one member of* τ_i-$g^*\omega\alpha O(X, \tau_1, \tau_2)$ *and one member of* τ_j-$g^*\omega\alpha O(X, \tau_1, \tau_2)$. *Therefore* ξ *is* (τ_i, τ_j)-$g^*\omega\alpha$*-open cover of X. As X is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact then* ξ *has the finite subcover and hence X is* (τ_i, τ_j) *compact.*

Theorem 5.5. *If Y is* τ_i-$g^*\omega\alpha$ *closed subset of a* (τ_i, τ_j)-$g^*\omega\alpha$*-compact space* (X, τ_1, τ_2) *then Y is* τ_j-$g^*\omega\alpha$ *compact.*
Proof: *Let* (X, τ_1, τ_2) *be* (τ_i, τ_j)-$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i : i \in I\}$ *be a* τ_j-$g^*\omega\alpha$ *open cover of Y. As Y is* τ_i-$g^*\omega\alpha$ *closed,* Y^c *is* τ_i-$g^*\omega\alpha$ *open. Also,* $\xi \cup Y^c = Y^c \cup \{A_i : i \in I\}$ *be a* (τ_i, τ_j)-$g^*\omega\alpha$*-open cover of X. Since X is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact, we have* $X = Y^c \cup A_1 \cup ... \cup A_n$, *so* $Y = A_1 \cup .... \cup A_n$. *Hence, Y is* τ_j-$g^*\omega\alpha$ *compact.*

Theorem 5.6. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* (τ_i, τ_j) *continuous, bijective and* (τ_i, τ_j)-$g^*\omega\alpha$*-irresolute. Then the image of a* (τ_i, τ_j)-$g^*\omega\alpha$*-compact space under f is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact.*
Proof: *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* (τ_i, τ_j) *continuous surjective and* (τ_i, τ_j)-$g^*\omega\alpha$*-closed. Let X be* (τ_i, τ_j)-$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i, i \in I\}$ *be a* (τ_i, τ_j)-$g^*\omega\alpha$*-open cover of Y. Then* $Y = \cup A_i$ *and* $\xi \subseteq \sigma_1$-$g^*\omega\alpha O(Y) \cup \sigma_2$-$g^*\omega\alpha O(Y)$ *and* ξ *contains at least one member of* σ_1-$g^*\omega\alpha O(Y)$ *and one member of* σ_2-$g^*\omega\alpha O(Y)$. *Therefore* $X = f^{-1}(\cup(A_i)) = \cup f^{-1}(A_i)$ *and* $f^{-1}(\xi) \subseteq \tau_1$-$g^*\omega\alpha O(X) \cup \tau_2$-$g^*\omega\alpha O(X)$ *and* $f^{-1}(\xi)$ *contains at least one member of* τ_1-$g^*\omega\alpha O(X)$ *and one member of* τ_2-$g^*\omega\alpha O(X)$. *Therefore* $f^{-1}(\xi)$ *is the* (τ_i, τ_j)-$g^*\omega\alpha$*-open cover of X. Since X is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact, we have* $X = \cup f^{-1}(A_i)$ *for each i = 1,...,n, that is* $Y = f(X) = \cup(A_i)$, *i=1,...,n. Hence,* ξ *has the finite subcover. Therefore Y is* (τ_i, τ_j)-$g^*\omega\alpha$*-compact.*

Definition 5.5. *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_i, τ_j)-$g^*\omega\alpha$*-paracompact (pairwise* $g^*\omega\alpha$ *paracompact) if every* τ_i*-open cover of X has a* (τ_i, τ_j)*-P-locally finite* (τ_i, τ_j)-$g^*\omega\alpha$*-open refinement.*

Example 5.2. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. *Let* $\xi = \{\{m\}, \{m, n\}\}$. *Then the space* (X, τ_1, τ_2) *is* (τ_i, τ_j) $g^*\omega\alpha$*-paracompact.*

Definition 5.6. *Let* (X, τ_1, τ_2) *be a bitopological space. Then X is said to be:*
(a) (τ_i, τ_j)*-*$g^*\omega\alpha$*-regular if for each* τ_i*-closed set F and* $x \in X$ *there exist* τ_i*-*$g^*\omega\alpha$*-open set U and* τ_j*-*$g^*\omega\alpha$*-open set V such that* $x \in U$ *and* $F \subseteq V$.
(b) (τ_i, τ_j)*-*$g^*\omega\alpha$*-normal if there exist two disjoint* τ_i*-closed sets A and B , there exist disjoint* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open sets U and V such that* $A \subseteq U$ *and* $B \subseteq V$.

Example 5.3. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. *Then* (X, τ_1, τ_2) *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-regular.*

Example 5.4. *Let* $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ *and* $\tau_2 = \{X, \phi, \{m\}\}$. *Then* (X, τ_1, τ_2) *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-normal.*

Proposition 5.1. *A space* (X, τ_1, τ_2) *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-regular if and only if for each* τ_i*-open set U and* $x \in U$ *there exists* $V \in (\tau_i, \tau_j)$*-*$g^*\omega\alpha$*O(X) such that* $x \in V \in (\tau_i, \tau_j)g^*\omega\alpha$*-*$cl(V) \subseteq U$.

Theorem 5.7. *Every* (τ_i, τ_j)*-*$g^*\omega\alpha$*-paracompact pairwise Hausdorff bitopological space is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-regular.*

Proof. Let A be τ_i-closed set with $x \notin A$. Then for each $y \in A$, choose τ_i-open sets U_y and H_x such that $y \in U_y$, $x \in H_x$ and $U_y \cap H_x = \phi$, that is $x \notin (\tau_i, \tau_j)g^*\omega\alpha$-$cl(U_y)$. Therefore the family $U = \{U_y : y \in A\} \cup \{X - A\}$ is an τ_i-open cover of X and so it has a (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement say ρ. Let $V = \{H \in \rho : H \cap A \notin \phi\}$, then V is a (τ_i, τ_j)-$g^*\omega\alpha$-open set containing A and (τ_i, τ_j)-$g^*\omega\alpha$-$cl(V) = \cup\{(\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(H) : H \in \rho$ and $H \cap A \notin \phi\}$. Therefore $U = X - (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(V)$ is a (τ_i, τ_j)-$g^*\omega\alpha$-open set containing x such that U and V are disjoint subsets of X. Thus X is (τ_i, τ_j)-$g^*\omega\alpha$-regular. □

Corolary 5.1. *Every* (τ_i, τ_j)*-*$g^*\omega\alpha$*-paracompact pairwise Hausdorff bitopologcal space* (τ_i, τ_j)*-*$g^*\omega\alpha$*-normal.*

Theorem 5.8. *Let* (X, τ_1) *and* (X, τ_2) *be two regular spaces. Then* (X, τ_1, τ_2) *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-paracompact if and only if every* τ_i*-open cover* ξ *of X has a* (τ_i, τ_j)*-P locally finite* (τ_i, τ_j)*-*$g^*\omega\alpha$*-closed refinement say* ρ.

Proof. Necessity: Let ξ be an τ_1-open cover of X. Then for each $x \in X$ choose a member $U_x \in \xi$. Since (X, τ_1) and (X, τ_2) are τ_i-regular, there exists τ_i-open set V_x containing x such that $\tau_i \subseteq cl(V_x) \subseteq U_x$. Therefore $\Psi = \{V_x : x \in X\}$ is an τ_i-open cover of X and by hypothesis Ψ has a (τ_i, τ_j)-P-locally

finite (τ_i, τ_j)-$g^*\omega\alpha$-refinement say $\Omega = \{W_\lambda : \lambda \in \Gamma\}$. Consider the collection $(\tau_i, \tau_j)g^*\omega\alpha - \Omega = \{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(W_\lambda) : \lambda \in \Gamma\}$ is a (τ_i, τ_j)-P-locally finite of (τ_i, τ_j)-$g^*\omega\alpha$-closed subsets of (X, τ_1, τ_2). Since for every $\lambda \in \Gamma$, (τ_i, τ_j)-$g^*\omega\alpha$-$cl(W_\lambda) \subseteq (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(V_x) \subseteq \tau_1\text{-}cl(V_x) \subseteq U_x$ for some $U_x \in \xi$, therefore (τ_i, τ_j)-$g^*\omega\alpha$-$cl(\Omega)$ is a refinement of ξ.
Sufficiency: Let ξ be an τ_i-open cover of X and Ψ be a (τ_i, τ_j)-P locally finite (τ_i, τ_j)-$g^*\omega\alpha$-closed refinement of ξ. Then for each $x \in X$ choose $W_x \in (\tau_i, \tau_j)$-$g^*\omega\alpha$O(X) such that $x \in W_x$ and W_x intersects at most finitely many member of Ψ. Let Σ be (τ_i, τ_j)-$g^*\omega\alpha$-closed (τ_i, τ_j)-P-locally finite refinement of $\Omega = \{W_x : x \in X\}$. Then for each $V \in \Psi$, $V^1 = X - H$, where $H \in \Sigma$ and $H \cap V = \phi$. Then $\{V^1 : V \in \Psi\}$ is a (τ_i, τ_j)-$g^*\omega\alpha$-open cover of X. Now for each $V \in \Psi$, let us choose $U_v \in \Xi$ such that $V \subseteq U_v$. Hence the collection $\{U_v \cap V^1 : V \in \Psi\}$ is a (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement of ξ. Thus (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact. □

Theorem 5.9. *Let A be a $(i, j)regular$ closed subset of a bitopological space (X, τ_1, τ_2). Then $(A, \tau_{i_A}, \tau_{j_A})$ is (i, j)-$g^*\omega\alpha$-paracompact.*

Proof. Let $\Sigma = \{V_\lambda : \lambda \in \Gamma\}$ is an τ_i-open cover of A in $(A, \tau_{i_A}, \tau_{j_A})$. Then for each $\lambda \in \Gamma$, choose an $U_\lambda \in \tau_i$ such that $V_\lambda = A \cap U_\lambda$. Then the collection $\xi = \{U_\lambda : \lambda \in \Gamma\} \cup \{X - A\}$ which is an τ_i-open cover of the (i, j)-$g^*\omega\alpha$-paracompact space X and so it has a (i, j)-P-locally finite (i, j)-$g^*\omega\alpha$-open refinement say $\Sigma = \{W_\delta : \delta \in \Delta\}$. But we have $(i, j) - RO(X) \subseteq (i, j) - O(X)$, then the collection $\{A \cap W_\delta : \delta \in \Delta\}$ is a (i, j)-P-locally finite (i, j)-$g^*\omega\alpha$-open refinement of Σ in $(A, \tau_{i_A}, \tau_{j_A})$. □

Theorem 5.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (τ_1, σ_1) and (τ_2, σ_2) closed (τ_i, τ_j)-$g^*\omega\alpha$-irresolute surjective function such that $f^{-1}(y)$ is τ_i-compact in (X, τ_1) for each $y \in Y$. If (Y, σ_1, σ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact, then (X, τ_1, τ_2) is also (τ_i, τ_j)-$g^*\omega\alpha$-paracompact.*

Proof. Let $\xi = \{U_\lambda : \lambda \in \Gamma\}$ be an τ_i-open cover of a bitopological space (X, τ_1, τ_2). Then for each $y \in Y$, ξ is an τ_i-open cover of the τ_i-compact subspace $f^{-1}(y)$. So there exist a finite subcover Γ_y of Γ such that $f^{-1}(y) \subseteq \cup U_\lambda$ for each $\lambda \in \Xi_\lambda$. Let $U_\lambda = \cup U_\lambda$ which is an τ_i-open in (X, τ_1). As f is (τ_1, σ_1)-closed, then for each $y \in Y$ there exists σ_1-open set V_y in Y such that $y \in V_y$ and $f^{-1}(V_y) \subseteq U_\lambda$. Then the collection $\Psi = \{V_y : y \in Y\}$ is an σ_1-open cover of the (τ_i, τ_j)-$g^*\omega\alpha$-paracompact space (Y, σ_1, σ_2) and so it has a (τ_i, τ_j)-P locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement say $\Omega = \{W_\gamma : \gamma \in \Delta\}$. As f is (τ_i, τ_j)-$g^*\omega\alpha$-irresolute, the collection $f^{-1}(\Omega) = \{f^{-1}(W_\gamma) : \gamma \in \Delta\}$ which is an (τ_i, τ_j)-$g^*\omega\alpha$-open (τ_i, τ_j)-P-locally finite cover of (X, τ_1, τ_2) such that for each $\gamma \in \delta$, $f^{-1}(W_\gamma) \subseteq U_y$ for some $y \in Y$. Then the collection $\{f^{-1}(W_\gamma) \cap U_\gamma : \gamma \in$

$\delta, \lambda \in \Gamma_y\}$ is an (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement of ξ. Thus (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact. □

Conclusion

The notions of sets and functions in topological spaces are extensively developed and used in many fields such as particle physics, computational topology, quantum physics. By researching generalizations of closed sets, some new Paracompact spaces have been founded and they turn out to be useful in the study of digital topology.

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Some characterizations of fuzzy comultisets and quotient fuzzy multigroups

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Abstract

The idea of fuzzy multisets has been applied to some group theoretic notions. Nonetheless, the notions of cosets and quotient groups have not been substantiated in fuzzy multigroup environment. The aim of this paper is to present the concepts of cosets and quotient groups in fuzzy multigroup context with some related results. To start with, the connection between fuzzy comultisets of fuzzy multigroups and the cosets of groups is established. Some characterizations of fuzzy comultisets are outlined with proofs. In addition, quotient fuzzy multigroup is proposed and some of its properties are explored. It is proven that a normal fuzzy submultigroup, $\tilde{H}$ of a fuzzy multigroup, $\tilde{G}$ is commutative if and only if the quotient fuzzy multigroup, $\frac{\tilde{G}}{\tilde{H}}$ of $\tilde{G}$ by $\tilde{H}$ is commutative. Finally, group theoretic isomorphism theorems are established in fuzzy multigroup setting.

Keywords: fuzzy multiset; fuzzy multigroup; fuzzy comultiset; quotient/factor fuzzy multigroup.

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1 Introduction

Fuzzy set theory proposed in 1965 by Zadeh (1965), although with vehement opposition as at then, has been extensively researched with applicative expressions ranging from engineering and computer science to medical diagnosis and social behavior, etc. In a way of extending the application of fuzzy sets to group theory, Rosenfeld (1971) proposed the notion of fuzzy groups as an extension of group theory and some number of results were obtained. Several studies have been carried out on some group theoretic notions in fuzzy group setting (see Ajmal and Prajapati, 1992; Bhattacharya and Mukherjee, 1987; Ejegwa and Otuwe, 2019; Mukherjee and Bhattacharya, 1986, 1984; Onasanya and Ilori, 2013).

With the interest derived from fuzzy sets and multisets (see Blizard, 1989), the idea of fuzzy multisets or fuzzy bags was proposed in (Yager, 1986) as a generalization of fuzzy sets in multiset framework. Myriad of works have been carried out on the fundamentals and properties of fuzzy multisets (see Biswas, 1999; Ejegwa, 2014, 2019a; Miyamoto, 1996; Miyamoto and Mizutani, 2004; Onasanya and Sholabomi, 2019). In recent times, the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory (Shinoj et al., 2015). The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied with some results (Baby et al., 2015; Ejegwa, 2018b), and the notions of center and centralizer in fuzzy multigroup context were established (Ejegwa, 2018b). In the same vein, the concept of fuzzy multigroupoids was introduced and the idea of fuzzy submultigroups was explored with a number of results (Ejegwa, 2018d). The concepts of normal subgroups, characteristic subgroups and Frattini subgroups have been established in fuzzy multigroup setting with some results (Ejegwa, 2018a; Ejegwa et al., 2020; Ejegwa, 2020c). In (Ejegwa, 2018c), the idea of homomorphism in the environment of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups were elaborated. Subsequently, the idea of direct product of fuzzy multigroups was proposed and a number of results were established (Ejegwa, 2019b). The idea of alpha-cuts of fuzzy multigroups and its homomorphic properties have been studied (Ejegwa, 2020b,a). The concept of fuzzy multigroups was redefined by Rasuli (2020) as an extension of the work in (Anthony and Sherwood, 1979).

The present paper is a further study of fuzzy multigroups in group theoretic analogs. Motivated by the researches done in fuzzy multigroups so far, it is expedient to investigate the notions of cosets and quotient groups in the light of fuzzy multigroups to strengthen the theory of fuzzy multigroups. Establishing the ideas of fuzzy comultisets and quotient/factor multigroups shall enhance the plausibility of studying nilpotency and solvability in fuzzy multigroup setting. Actually, this work is an application of fuzzy multisets to cosets and factor groups. In so doing, this paper assay to introduce fuzzy comultisets and quotient fuzzy multi-

groups with some analog results. The relationship between fuzzy comultisets of fuzzy multigroups and that of cosets of groups is examined, and the isomorphism theorems are duly established. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy multisets and fuzzy multigroups. In Section 3, the idea of fuzzy comultisets is proposed and some of its properties are discussed. Section 4 discusses the concept of quotient or factor fuzzy multigroups with some results. Finally, Section 5 concludes the paper and provides direction for future studies.

2 Preliminaries

In this section, we present some existing definitions and results to be used in the sequel.

2.1 Fuzzy multisets

Definition 2.1. (Yager, 1986) Assume X is a set of elements. Then, a fuzzy bag/multiset, $\tilde{G}$ drawn from X can be characterized by a count membership function, $CM_{\tilde{G}}$ such that

$$CM_{\tilde{G}} : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval, $I = [0, 1]$.

A fuzzy multiset, $\tilde{G}$ can be characterized by a function

$$CM_{\tilde{G}} : X \to N^I \text{ or } CM_{\tilde{G}} : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By Miyamoto and Mizutani (2004), it implies that $CM_{\tilde{G}}(x)$ for $x \in X$ is given as

$$CM_{\tilde{G}}(x) = \{\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x), ...\},$$

where $\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x), ... \in [0, 1]$ such that $\mu^1_{\tilde{G}}(x) \geq \mu^2_{\tilde{G}}(x) \geq ... \geq \mu^n_{\tilde{G}}(x) \geq ...$, whereas in a finite case, we write

$$CM_{\tilde{G}}(x) = \{\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x)\},$$

for $\mu^1_{\tilde{G}}(x) \geq \mu^2_{\tilde{G}}(x) \geq ... \geq \mu^n_{\tilde{G}}(x)$.

A fuzzy multiset, $\tilde{G}$ can be represented in the form $\tilde{G} = \{\frac{\langle CM_{\tilde{G}}(x)\rangle}{x} \mid x \in X\}$.

We denote the set of all fuzzy multisets by $FMS(X)$.

Definition 2.2. (Yager, 1986) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, $\tilde{H}$ is called a fuzzy submultiset of $\tilde{G}$ written as $\tilde{H} \subseteq \tilde{G}$ if $CM_{\tilde{H}}(x) \leq CM_{\tilde{G}}(x),\ \forall x \in X$. Also, if $\tilde{H} \subseteq \tilde{G}$ and $\tilde{H} \neq \tilde{G}$, then $\tilde{H}$ is called a proper fuzzy submultiset of $\tilde{G}$ and denoted as $\tilde{H} \subset \tilde{G}$.

Definition 2.3. (Yager, 1986) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, $\tilde{G}$ and $\tilde{H}$ are comparable to each other if and only if $\tilde{H} \subseteq \tilde{G}$ or $\tilde{G} \subseteq \tilde{H}$, and $\tilde{G} = \tilde{H}$ if and only if $CM_{\tilde{G}}(x) = CM_{\tilde{H}}(x),\ \forall x \in X$.

Definition 2.4. (Miyamoto, 1996) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, the intersection and union of $\tilde{G}$ and $\tilde{H}$, denoted by $\tilde{G} \cap \tilde{H}$ and $\tilde{G} \cup \tilde{H}$ are defined by the rules that for any object $x \in X$,

(i) $CM_{\tilde{G}\cap\tilde{H}}(x) = CM_{\tilde{G}}(x) \wedge CM_{\tilde{H}}(x)$,

(ii) $CM_{\tilde{G}\cup\tilde{H}}(x) = CM_{\tilde{G}}(x) \vee CM_{\tilde{H}}(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum operations, respectively.

Before finding the intersection and union of $\tilde{G}$ and $\tilde{H}$, the membership sequences of $\tilde{G}$ and $\tilde{H}$ should be equal. If not, it could be completed by affixing zero(s).

2.2 Fuzzy multigroups

We denote group by X and assume that all fuzzy multigroups are drawn from $FMG(X)$, which is the set of all fuzzy multigroups of X.

Definition 2.5. (Shinoj et al., 2015) A fuzzy multiset $\tilde{G}$ of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_{\tilde{G}}(xy) \geq CM_{\tilde{G}}(x) \wedge CM_{\tilde{G}}(y),\ \forall x, y \in X$,

(ii) $CM_{\tilde{G}}(x^{-1}) = CM_{\tilde{G}}(x),\ \forall x \in X$.

It can be easily verified that if $\tilde{G}$ is a fuzzy multigroup of X, then

$$CM_{\tilde{G}}(e) = \bigvee_{x\in X} CM_{\tilde{G}}(x),$$

that is, $CM_{\tilde{G}}(e)$ is the tip of $\tilde{G}$.

Remark 2.1. (Shinoj et al., 2015) We notice that a fuzzy multiset, $\tilde{G}$ of a group X is a fuzzy multigroup if $\forall x, y \in X$,

$$CM_{\tilde{G}}(xy^{-1}) \geq CM_{\tilde{G}}(x) \wedge CM_{\tilde{G}}(y)$$

holds.

Definition 2.6. (Shinoj et al., 2015) Let $\tilde{G}$ be a fuzzy multigroup of a group X. Then $\tilde{G}^{-1}$ is defined by $CM_{\tilde{G}^{-1}}(x) = CM_{\tilde{G}}(x^{-1}), \forall x \in X$.

By Definition 2.5, we get $CM_{\tilde{G}^{-1}}(x) = CM_{\tilde{G}}(x^{-1}) = CM_{\tilde{G}}(x)$. That is, $\tilde{G}^{-1} = \tilde{G}$. Thus, $\tilde{G} \in FMG(X) \Leftrightarrow \tilde{G}^{-1} \in FMG(X)$.

Proposition 2.1. (Shinoj et al., 2015) *Let* $\tilde{G}, \tilde{H} \in FMG(X)$. *Then,* $\tilde{G} \cap \tilde{H} \in FMG(X)$.

Definition 2.7. (Ejegwa, 2018d) Let $\{\tilde{G}_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of fuzzy multigroups of X. Then,

$$CM_{\bigcap_{i\in I}\tilde{G}_i}(x) = \bigwedge_{i\in I} CM_{\tilde{G}_i}(x),\ \forall x \in X$$

and

$$CM_{\bigcup_{i\in I}\tilde{G}_i}(x) = \bigvee_{i\in I} CM_{\tilde{G}_i}(x),\ \forall x \in X.$$

The family of fuzzy multigroups $\{\tilde{G}_i\}_{i\in I}$ of X is said to have inf or sup assuming chain if either $\tilde{G}_1 \subseteq \tilde{G}_2 \subseteq ... \subseteq \tilde{G}_n$ or $\tilde{G}_1 \supseteq \tilde{G}_2 \supseteq ... \supseteq \tilde{G}_n$, respectively.

Definition 2.8. (Baby et al., 2015) Let $\tilde{G} \in FMG(X)$. Then, $\tilde{G}$ is said to be commutative if for all $x, y \in X$,

$$CM_{\tilde{G}}(xy) = CM_{\tilde{G}}(yx).$$

Definition 2.9. (Ejegwa, 2018a) Let $\tilde{G}, \tilde{H} \in FMG(X)$. Then, the product, $\tilde{G} \circ \tilde{H}$ of $\tilde{G}$ and $\tilde{H}$ is defined to be a fuzzy multiset of X as follows:

$$CM_{\tilde{G}\circ\tilde{H}}(x) = \begin{cases} \bigvee_{x=yz}[CM_{\tilde{G}}(y) \wedge CM_{\tilde{H}}(z)], & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.10. (Ejegwa, 2018d) Let $\tilde{G} \in FMG(X)$. A fuzzy submultiset, $\tilde{H}$ of $\tilde{G}$ is called a fuzzy submultigroup of $\tilde{G}$ denoted by $\tilde{H} \subseteq \tilde{G}$ if $\tilde{H}$ is a fuzzy multigroup. A fuzzy submultigroup, $\tilde{H}$ of $\tilde{G}$ is a proper fuzzy submultigroup denoted by $\tilde{H} \subset \tilde{G}$, if $\tilde{H} \subseteq \tilde{G}$ and $\tilde{H} \neq \tilde{G}$.

Definition 2.11. (Ejegwa, 2018a) Let $\tilde{H}, \tilde{G} \in FMG(X)$ such that $\tilde{H} \subseteq \tilde{G}$. Then, $\tilde{H}$ is called a normal fuzzy submultigroup of $\tilde{G}$ if

$$CM_{\tilde{H}}(xyx^{-1}) = CM_{\tilde{H}}(y),\ \forall x, y \in X.$$

Definition 2.12. (Ejegwa, 2018a) Let $\tilde{H}$ be a fuzzy submultiset of $\tilde{G} \in FMG(X)$. Then, the normalizer of $\tilde{H}$ in $\tilde{G}$ is the set given by

$$N(\tilde{H}) = \{g \in X \mid CM_{\tilde{H}}(gy) = CM_{\tilde{H}}(yg),\ \forall y \in X\}.$$

Theorem 2.1. (Ejegwa, 2018a) *Let X be a finite group and $\tilde{H}$ be a fuzzy sub-multigroup of $\tilde{G} \in FMG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid CM_{\tilde{H}}(g) = CM_{\tilde{H}}(e)\}, \\ K &= \{x \in X \mid CM_{\tilde{H}x}(y) = CM_{\tilde{H}e}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again, $H = K$.

Definition 2.13. (Ejegwa, 2018d; Shinoj et al., 2015) Let $\tilde{G} \in FMG(X)$. Then, the set $\tilde{G}_*$ defined by

$$\tilde{G}_* = \{x \in X \mid CM_{\tilde{G}}(x) > 0\}$$

is the level set or support of $\tilde{G}$. It follows that $\tilde{G}_*$ is a subgroup of X.

Also, the set $\tilde{G}^*$ defined by

$$\tilde{G}^* = \{x \in X \mid CM_{\tilde{G}}(x) = CM_{\tilde{G}}(e)\}$$

is a subgroup of X.

Definition 2.14. (Ejegwa, 2018c) Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose $\tilde{G}$ and $\tilde{H}$ are fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from $\tilde{G}$ to $\tilde{H}$ which satisfies

(i) $CM_{\tilde{G}}(f^{-1}(y_1y_2)) \geq CM_{\tilde{G}}(f^{-1}(y_1)) \wedge CM_{\tilde{G}}(f^{-1}(y_2)),\ \forall y_1, y_2 \in Y,$

(ii) $CM_{\tilde{H}}(f(x_1x_2)) \geq CM_{\tilde{H}}(f(x_1)) \wedge CM_{\tilde{H}}(f(x_2)),\ \forall x_1, x_2 \in X,$

where

(i) the image of $\tilde{G}$ under f, denoted by $f(\tilde{G})$, is a fuzzy multiset over Y defined by

$$CM_{f(\tilde{G})}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_{\tilde{G}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of $\tilde{H}$ under f, denoted by $f^{-1}(\tilde{H})$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(\tilde{H})}(x) = CM_{\tilde{H}}(f(x)),\ \forall x \in X.$$

Proposition 2.2. (Ejegwa, 2018c) *Let $f : X \to Y$ be a homomorphism and $\tilde{G} \in FMG(X)$. If f is injective, then $f^{-1}(f(\tilde{G})) = \tilde{G}$.*

Theorem 2.2. (Ejegwa, 2018c) *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $\tilde{G} \in FMG(X)$ if and only if $f(\tilde{G}) \in FMG(Y)$.

(ii) $\tilde{H} \in FMG(Y)$ if and only if $f^{-1}(\tilde{H}) \in FMG(X)$.

3 Fuzzy comultiset and some of its properties

In this section, we define fuzzy comultiset and characterize some of its properties.

Definition 3.1. Suppose $\tilde{H}$ is a fuzzy submultigroup of a fuzzy multigroup $\tilde{G}$ of X. Then, the fuzzy submultiset, $y\tilde{H}$ of $\tilde{G}$ for $y \in X$ defined by

$$CM_{y\tilde{H}}(x) = CM_{\tilde{H}}(y^{-1}x),\ \forall x \in X$$

is called the left fuzzy comultiset of $\tilde{H}$. Similarly, the fuzzy submultiset, $\tilde{H}y$ of $\tilde{G}$ for $y \in X$ defined by

$$CM_{\tilde{H}y}(x) = CM_{\tilde{H}}(xy^{-1}),\ \forall x \in X$$

is called the right fuzzy comultiset of $\tilde{H}$.

The following result proves that the right and left fuzzy comultisets of a fuzzy submultigroup in a fuzzy multigroup are equal.

Proposition 3.1. *If $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G} \in FMG(X)$, then the right and left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ are identical.*

Proof. Let $x, y, \in X$. Assume $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G}$. Then, we have

$$\begin{aligned} CM_{y\tilde{H}}(x) = CM_{\tilde{H}}(y^{-1}x) &\geq CM_{\tilde{H}}(y) \wedge CM_{\tilde{H}}(x) \\ &= CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y) \\ &= CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y^{-1}). \end{aligned}$$

Suppose by hypothesis, $CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y^{-1}) = CM_{\tilde{H}}(xy^{-1})$. Then, we have

$$CM_{y\tilde{H}}(x) \geq CM_{\tilde{H}y}(x).$$

Again,

$$\begin{aligned} CM_{\tilde{H}y}(x) = CM_{\tilde{H}}(xy^{-1}) &\geq CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y) \\ &= CM_{\tilde{H}}(y) \wedge CM_{\tilde{H}}(x) \\ &= CM_{\tilde{H}}(y^{-1}) \wedge CM_{\tilde{H}}(x). \end{aligned}$$

By the same hypothesis, we get

$$CM_{\tilde{H}y}(x) \geq CM_{y\tilde{H}}(x).$$

Hence, $CM_{y\tilde{H}}(x) = CM_{\tilde{H}y}(x) \Rightarrow y\tilde{H} = \tilde{H}y$. □

Remark 3.1. Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. We notice that

(i) the right and left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ are fuzzy submultigroups of $\tilde{G}$.

(ii) $x\tilde{H} = y\tilde{H} = z\tilde{H} = \tilde{H},\ \forall x, y, z \in X$. This is not applicable in the conventional case.

(iii) there is a one-to-one correspondence between the set of right fuzzy comultisets and the set of left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$.

(iv) the number of fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ equals the cardinality of $\tilde{H}_*$.

(v) $x\tilde{H} \cap y\tilde{H} \cap z\tilde{H} = \tilde{H} = x\tilde{H} \cup y\tilde{H} \cup z\tilde{H}, \forall x, y, z \in X$.

Theorem 3.1. *Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then, $g\tilde{H} = h\tilde{H}$ for $g, h \in X$ if and only if*

$$CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g) = CM_{\tilde{H}}(e).$$

Proof. Let $g\tilde{H} = h\tilde{H}$. Then, $CM_{g\tilde{H}}(g) = CM_{h\tilde{H}}(g)$ and $CM_{g\tilde{H}}(h) = CM_{h\tilde{H}}(h)$ $\forall g, h \in X$. Hence,

$$CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g) = CM_{\tilde{H}}(e).$$

Conversely, let $CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g)\ \forall g, h \in X$. For every $x \in X$, we have

$$\begin{aligned} CM_{g\tilde{H}}(x) = CM_{\tilde{H}}(g^{-1}x) &= CM_{\tilde{H}}(g^{-1}hh^{-1}x) \\ &\geq CM_{\tilde{H}}(g^{-1}h) \wedge CM_{\tilde{H}}(h^{-1}x) \\ &= CM_{\tilde{H}}(h^{-1}x) \\ &= CM_{h\tilde{H}}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CM_{h\tilde{H}}(x) = CM_{\tilde{H}}(h^{-1}x) &= CM_{\tilde{H}}(h^{-1}gg^{-1}x) \\ &\geq CM_{\tilde{H}}(h^{-1}g) \wedge CM_{\tilde{H}}(g^{-1}x) \\ &= CM_{\tilde{H}}(g^{-1}x) \\ &= CM_{g\tilde{H}}(x). \end{aligned}$$

Hence, $CM_{g\tilde{H}}(x) = CM_{h\tilde{H}}(x) \Rightarrow g\tilde{H} = h\tilde{H}$. □

Corolary 3.1. *Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then $\tilde{H}g = \tilde{H}h$ for $g, h \in X$ if and only if*

$$CM_{\tilde{H}}(gh^{-1}) = CM_{\tilde{H}}(hg^{-1}) = CM_{\tilde{H}}(e).$$

Proof. Straightforward from Theorem 3.1. □

Proposition 3.2. *Let $\tilde{H}, \tilde{G} \in FMG(X)$ such that $\tilde{H} \subseteq \tilde{G}$. If $g\tilde{H} = h\tilde{H}$, then $CM_{\tilde{H}}(g) = CM_{\tilde{H}}(h)$, $\forall g, h \in X$.*

Proof. Let $g, h \in X$. Suppose $g\tilde{H} = h\tilde{H}$, then we have

$$\begin{aligned} CM_{g\tilde{H}}(g) = CM_{h\tilde{H}}(g) &\Rightarrow CM_{\tilde{H}}(g^{-1}g) = CM_{\tilde{H}}(h^{-1}g) \\ &\Rightarrow CM_{\tilde{H}}(e) = CM_{\tilde{H}}(h^{-1}g) \end{aligned}$$

$\forall g, h \in X$. The fact that,

$$CM_{\tilde{H}}(e) = CM_{\tilde{H}}(h^{-1}g) \Rightarrow CM_{\tilde{H}}(h) = CM_{\tilde{H}}(g),$$

the result follows.

Alternatively, suppose $z \in X$, we get

$$\begin{aligned} CM_{g\tilde{H}}(z) = CM_{h\tilde{H}}(z) &\Rightarrow CM_{\tilde{H}}(g^{-1}z) = CM_{\tilde{H}}(h^{-1}z) \\ &\Rightarrow CM_{\tilde{H}z^{-1}}(g) = CM_{\tilde{H}z^{-1}}(h) \\ &\Rightarrow CM_{\tilde{H}}(g) = CM_{\tilde{H}}(h), \end{aligned}$$

because $\tilde{H}z^{-1} = \tilde{H}$. □

Theorem 3.2. *Let $\tilde{G} \in FMG(X)$. Any fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ and for any $z \in X$, the fuzzy submultiset, $z\tilde{H}z^{-1}$, where $CM_{z\tilde{H}z^{-1}}(x) = CM_{\tilde{H}}(z^{-1}xz)$ for each $x \in X$ is a fuzzy submultigroup of $\tilde{G}$.*

Proof. Let $x, y \in X$ and $\tilde{H} \subseteq \tilde{G}$. We prove that $z\tilde{H}z^{-1}$ is a fuzzy submultigroup of $\tilde{G}$. Now

$$\begin{aligned} CM_{z\tilde{H}z^{-1}}(xy^{-1}) &= CM_{\tilde{H}}(z^{-1}xy^{-1}z) \\ &= CM_{\tilde{H}}(z^{-1}xzz^{-1}y^{-1}z) \\ &\geq CM_{\tilde{H}}(z^{-1}xz) \wedge CM_{\tilde{H}}(z^{-1}y^{-1}z) \\ &= CM_{z\tilde{H}z^{-1}}(x) \wedge CM_{z\tilde{H}z^{-1}}(y^{-1}) \\ &= CM_{z\tilde{H}z^{-1}}(x) \wedge CM_{z\tilde{H}z^{-1}}(y), \ \forall z \in X. \end{aligned}$$

Hence, $z\tilde{H}z^{-1}$ is a fuzzy submultigroup of $\tilde{G}$. □

Corolary 3.2. *Let $\{\tilde{H}_i\}_{i\in I} \in FMG(X)$, then*

(i) $\bigcap_{i\in I} z\tilde{H}_i z^{-1} \in FMG(X)$, $\forall z \in X$,

(ii) $\bigcup_{i\in I} z\tilde{H}_i z^{-1} \in FMG(X)$, $\forall z \in X$ *provided* $\{\tilde{H}_i\}_{i\in I}$ *have sup/inf assuming chain.*

Proof. The results follow from Theorem 3.2. □

The following results are the application of product of fuzzy multigroups to the idea of fuzzy comultisets.

Proposition 3.3. *Suppose $\tilde{H}$ is a fuzzy submultigroup of $\tilde{H} \in FMG(X)$. Then;*

(i) $\tilde{H}g \circ \tilde{H}g = \tilde{H}g$.

(ii) $\tilde{H}g \circ \tilde{H}h = \tilde{H}h \circ \tilde{H}g$.

(iii) $(\tilde{H}g \circ \tilde{H}h)^{-1} = (\tilde{H}h)^{-1} \circ (\tilde{H}g)^{-1}$.

(iv) $(\tilde{H}g \circ \tilde{H}h)^{-1} = \tilde{H}g \circ \tilde{H}h$.

Proof. Using Definition 2.9, the results follow. □

Remark 3.2. Proposition 3.3 also holds for left fuzzy comultisets.

Proposition 3.4. *If $\tilde{H}$ is a fuzzy submultigroup of a commutative fuzzy multigroup $\tilde{G}$ of X, then*

(i) $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$, $\forall y, z \in X$,

(ii) $y\tilde{H} \circ z\tilde{H} = yz\tilde{H}$, $\forall y, z \in X$.

Proof. Let $\tilde{H} \in FMG(X)$ and $x, y, z \in X$, then we have

(i)

$$\begin{aligned} CM_{\tilde{H}y\circ\tilde{H}z}(x) &= \bigvee_{x=zy} [CM_{\tilde{H}y}(z) \wedge CM_{\tilde{H}z}(y)],\ \forall y, z \in X \\ &= \bigvee_{x=zy} [CM_{\tilde{H}}(zy^{-1}) \wedge CM_{\tilde{H}}(yz^{-1})],\ \forall y, z \in X \\ &= \bigvee_{x=zy} [CM_{\tilde{H}\cap\tilde{H}}((zy^{-1})(yz^{-1}))],\ \forall y, z \in X \\ &= CM_{\tilde{H}}(xz^{-1}y^{-1}) \\ &= CM_{\tilde{H}yz}(x). \end{aligned}$$

Hence, $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$.

(ii) Similar to (i).

□

Corolary 3.3. *Suppose $\tilde{H}$ is a fuzzy submultigroup of a commutative fuzzy multigroup $\tilde{G}$ of X. Then, the following statements are equivalent.*

(i) $(\tilde{H}y \circ \tilde{H}z)^{-1} = \tilde{H}y \circ \tilde{H}z$.

(ii) $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$.

Proof. The result is easy to see by combining Definition 2.9 and Proposition 3.4. □

Remark 3.3. If $(\tilde{H}y \circ \tilde{H}y)^{-1} = \tilde{H}y \circ \tilde{H}z$ and $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$, then $(\tilde{H}y \circ \tilde{H}z)^{-1} = \tilde{H}yz$.

Theorem 3.3. *If $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G} \in FMG(X)$ such that $\tilde{G}$ is commutative, then $\tilde{H}g \circ \tilde{H}h = \tilde{H}gh$ if and only if $g\tilde{H} \circ h\tilde{H} = gh\tilde{H}, \forall\, g, h \in X$. Consequently, $\tilde{H}gh = gh\tilde{H}$.*

Proof. Suppose $\tilde{H}g \circ \tilde{H}h = \tilde{H}gh$. By Definition 2.9, we get

$$\begin{aligned}
CM_{\tilde{H}gh}(x) &= CM_{\tilde{H}g \circ \tilde{H}h}(x) \\
&= \bigvee_{y \in X} [CM_{\tilde{H}g}(y) \wedge CM_{\tilde{H}h}(y^{-1}x)] \\
&= \bigvee_{y \in X} [CM_{\tilde{H}}(yg^{-1}) \wedge CM_{\tilde{H}}(y^{-1}xh^{-1})] \\
&= \bigvee_{y \in X} [CM_{\tilde{H}}(g^{-1}y) \wedge CM_{\tilde{H}}(h^{-1}y^{-1}x)] \\
&= \bigvee_{y \in X} [CM_{g\tilde{H}}(y) \wedge CM_{h\tilde{H}}(y^{-1}x)] \\
&= CM_{g\tilde{H} \circ h\tilde{H}}(x) \\
&= CM_{gh\tilde{H}}(x)
\end{aligned}$$

$\Rightarrow g\tilde{H} \circ h\tilde{H} = gh\tilde{H}$.

Conversely, assuming $g\tilde{H} \circ h\tilde{H} = gh\tilde{H}$. Then

$$\begin{aligned} CM_{gh\tilde{H}}(x) &= CM_{g\tilde{H}\circ h\tilde{H}}(x) \\ &= \bigvee_{y\in X}[CM_{g\tilde{H}}(y) \wedge CM_{h\tilde{H}}(y^{-1}x)] \\ &= \bigvee_{y\in X}[CM_{\tilde{H}}(g^{-1}y) \wedge CM_{\tilde{H}}(h^{-1}y^{-1}x)] \\ &= \bigvee_{y\in X}[CM_{\tilde{H}}(yg^{-1}) \wedge CM_{\tilde{H}}(y^{-1}xh^{-1})] \\ &= \bigvee_{y\in X}[CM_{\tilde{H}g}(y) \wedge CM_{\tilde{H}h}(y^{-1}x)] \\ &= CM_{\tilde{H}g\circ\tilde{H}h}(x) \\ &= CM_{\tilde{H}gh}(x) \end{aligned}$$

$\Rightarrow \tilde{H}g \circ \tilde{H}h = \tilde{H}gh$. Hence, the result follow. □

Theorem 3.4. *Suppose $\tilde{G} \in FMG(X)$ and $\tilde{H}$ a fuzzy submultigroup of $\tilde{G}$. Define*

$$H = \{g \in X \mid CM_{\tilde{H}}(g) = CM_{\tilde{H}}(e)\}.$$

Then $Hx = Hy \Leftrightarrow \tilde{H}x = \tilde{H}y, \forall x, y \in X$. Similarly, $xH = yH \Leftrightarrow x\tilde{H} = y\tilde{H}$.

Proof. This result gives a relationship between fuzzy comultisets of a fuzzy submultigroup of a fuzzy multigroup and the cosets of a subgroup of a given group.

By Theorem 2.1, we know that H is a subgroup of X and

$$H = \{x \in X \mid CM_{\tilde{H}x}(z) = CM_{\tilde{H}e}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. Thus

$$CM_{\tilde{H}xy^{-1}}(z) = CM_{\tilde{H}e}(z)\ \forall z \in X \text{ and so } CM_{\tilde{H}}(zyx^{-1}) = CM_{\tilde{H}}(z).$$

Put $z = zy^{-1}$, we get

$$\begin{aligned} CM_{\tilde{H}}(zy^{-1}yx^{-1}) = CM_{\tilde{H}}(zy^{-1}) &\Rightarrow CM_{\tilde{H}}(zx^{-1}) = CM_{\tilde{H}}(zy^{-1}) \\ &\Rightarrow CM_{\tilde{H}x}(z) = CM_{\tilde{H}y}(z) \end{aligned}$$

and so, $\tilde{H}x = \tilde{H}y$.

Conversely, suppose that $\tilde{H}x = \tilde{H}y$, that is $CM_{\tilde{H}x}(z) = CM_{\tilde{H}y}(z),\ \forall z \in X$. This implies that

$$CM_{\tilde{H}}(zx^{-1}) = CM_{\tilde{H}}(zy^{-1}).$$

Put $z = y$, we get

$$CM_{\tilde{H}}(yx^{-1}) = CM_{\tilde{H}}(e).$$

So, $yx^{-1} \in H$. Thus, $Hx = Hy$.

The proof of $xH = yH \Leftrightarrow x\tilde{H} = y\tilde{H}$ is similar. □

4 Quotient fuzzy multigroups

In this section, we present the notion of quotient groups in fuzzy multigroup setting and establish the isomorphism theorems.

Definition 4.1. Suppose $\tilde{G}$ is a fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, the union of the set of left/right fuzzy comultisets of $\tilde{H}$ such that the fuzzy comultisets satisfy

$$x\tilde{H} \circ y\tilde{H} = xy\tilde{H},\ \forall x, y \in X$$

is called quotient or factor fuzzy multigroup of $\tilde{G}$ by $\tilde{H}$, denoted by $\frac{\tilde{G}}{\tilde{H}}$.

Remark 4.1. Suppose $\frac{\tilde{G}}{\tilde{H}}$ is a factor fuzzy multigroup of $\tilde{G}$ by $\tilde{H}$, it implies that $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$ and $\frac{\tilde{G}}{\tilde{H}} = e\tilde{H} = \tilde{H}$. This property is not applicable in classical case.

Remark 4.2. Suppose $\tilde{G}$ is a fuzzy multigroup of X, and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then

(i) if $\tilde{I}$ is a fuzzy submultigroup of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$, then $\frac{\tilde{I}}{\tilde{H}}$ is a fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$.

(ii) every fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is of the form $\frac{\tilde{I}}{\tilde{H}}$, for some fuzzy submultigroup $\tilde{I}$ of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$.

Theorem 4.1. *If $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then $\tilde{H}$ is commutative if and only if $\frac{\tilde{G}}{\tilde{H}}$ is commutative.*

Proof. Let $x, y \in X$. Suppose $\tilde{H}$ is commutative, then

$$CM_{\tilde{H}}(xyx^{-1}y^{-1}) = CM_{\tilde{H}}(e),$$

and hence,

$$CM_{\tilde{H}}(xy) = CM_{\tilde{H}}(yx).$$

Consequently, $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$ by Definition 2.11. Thus, since

$$CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(xyx^{-1}y^{-1}) = CM_{\tilde{H}}(e),$$

we have

$$\begin{aligned} CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(e) &\Rightarrow CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(xy(xy)^{-1}) \\ &\Rightarrow CM_{\tilde{H}yx}(xy) = CM_{\tilde{H}xy}(xy). \end{aligned}$$

Thus, $\tilde{H}xy = \tilde{H}yx$. It follows that, $\tilde{H}x \circ \tilde{H}y = \tilde{H}y \circ \tilde{H}x$ since $\tilde{H}x \circ \tilde{H}y = \tilde{H}xy$ and $\tilde{H}y \circ \tilde{H}x = \tilde{H}yx$ by Proposition 3.4. Hence, $\frac{\tilde{G}}{\tilde{H}}$ is commutative.

Conversely, assume $\frac{\tilde{G}}{\tilde{H}}$ is commutative, then

$$\tilde{H}x \circ \tilde{H}y = \tilde{H}y \circ \tilde{H}x \Rightarrow \tilde{H}xy = \tilde{H}yx.$$

Thus,

$$CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(e) \Rightarrow CM_{\tilde{H}}(xy) = CM_{\tilde{H}}(yx),$$

completes the proof. □

Theorem 4.2. *Suppose $\tilde{G} \in FMG(X)$ and $\tilde{H}, \tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$ and $\tilde{H} \subseteq \tilde{I}$, then $\frac{\tilde{I}}{\tilde{H}}$ is a normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$.*

Proof. Let $x \in X$. Then $CM_{\frac{\tilde{I}}{\tilde{H}}}(x) \leq CM_{\frac{\tilde{G}}{\tilde{H}}}(x)$ since $\tilde{H} \subseteq \tilde{I}$ and, $\tilde{H}$ and $\tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$. So, $\frac{\tilde{I}}{\tilde{H}}$ is a fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$. Subsequently,

$$CM_{\frac{\tilde{I}}{\tilde{H}}}(yxy^{-1}) = CM_{\frac{\tilde{I}}{\tilde{H}}}(x)\ \forall x, y \in X.$$

Hence, $\frac{\tilde{I}}{\tilde{H}}$ is a normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ by Definition 2.11. □

Remark 4.3. Let $\tilde{G}$ be a fuzzy multigroup of X, and $\tilde{I}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, every normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is of the form $\frac{\tilde{I}}{\tilde{H}}$, for some normal fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$.

Theorem 4.3. *Suppose $\tilde{G}, \tilde{H} \in FMG(X)$ and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$.*

Proof. By Definition 2.13, $\tilde{H}_*$ is a subgroup of X and $\tilde{H} \cap \tilde{G} \in FMG(X)$ by Proposition 2.1. So, $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a fuzzy multigroup of X. Since $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$, then $\tilde{H} \cap \tilde{G}$ is a fuzzy submultigroup of $\tilde{G}$ and $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a fuzzy submultigroup of $\tilde{H}$. We show that $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$. Let $x, y \in \tilde{H}_*$. Then $xyx^{-1} \in \tilde{H}_*$ since

$$CM_{\tilde{H}}(xyx^{-1}) = CM_{\tilde{H}}(y) > 0$$

by definition of $\tilde{H}_*$. This proves that $\tilde{H}_*$ is a normal subgroup of X.

It is easy to see that $\tilde{H} \cap \tilde{G}$ is normal since

$$\begin{aligned} CM_{\tilde{H} \cap \tilde{G}}(xyx^{-1}) &= CM_{\tilde{H}}(xyx^{-1}) \wedge CM_{\tilde{G}}(xyx^{-1}) \\ &= CM_{\tilde{H}}(y) \wedge CM_{\tilde{G}}(y) \\ &= CM_{\tilde{H} \cap \tilde{G}}(y). \end{aligned}$$

In fact, $\tilde{H} \cap \tilde{G}$ is a normal fuzzy submultigroup since $\tilde{H} \cap \tilde{G} = \tilde{H}$, and $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$. Hence, $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$. □

Theorem 4.4. *Suppose $\tilde{G}$ is a fuzzy multigroup of X, $\tilde{H} \subseteq \tilde{G}$ and $N(\tilde{H})$ is a normalizer. Then $N(\tilde{H})$ is a subgroup of X and $\frac{\tilde{H}}{N(\tilde{H})}$ is a normal fuzzy submultigroup of $\tilde{G}$.*

Proof. Clearly, $e \in N(\tilde{H})$. Let $x, y \in N(\tilde{H})$. Then for any $z \in X$, we have

$$\begin{aligned} CM_{\tilde{H}}((xy^{-1})z) &= CM_{\tilde{H}}(x(y^{-1}z)) \\ &= CM_{\tilde{H}}((y^{-1}z)x) \\ &= CM_{\tilde{H}}(y^{-1}(zx)) \\ &= CM_{\tilde{H}}(y(zx)^{-1}) \\ &= CM_{\tilde{H}}(y(x^{-1}z^{-1})) \\ &= CM_{\tilde{H}}(z(xy^{-1})). \end{aligned}$$

Hence, $xy^{-1} \in N(\tilde{H})$. Therefore, $N(\tilde{H})$ is a subgroup of X. By Definition 4.1, it follows that $\frac{\tilde{H}}{N(\tilde{H})} \in FMG(N(\tilde{H}))$ and clearly, $\frac{\tilde{H}}{N(\tilde{H})}$ is a fuzzy submultigroup of $\tilde{H}$. Since, $CM_{\frac{\tilde{H}}{N(\tilde{H})}}(xyx^{-1}) = CM_{\frac{\tilde{H}}{N(\tilde{H})}}(y)$, $\forall x, y \in X$, it implies that $\frac{\tilde{H}}{N(\tilde{H})}$ is a normal fuzzy submultigroup of $\tilde{H}$. □

Theorem 4.5. *Suppose $\tilde{G}$ is a commutative fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, there exists a natural homomorphism $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ defined by $CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y)$, $\forall\, x, y \in X$.*

Proof. Let $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ be a mapping defined by

$$CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y),\ \forall\, x, y \in X.$$

That is, $CM_{f(\tilde{G})}(y) = CM_{x\tilde{H}}(y) \Rightarrow f(\tilde{G}) = x\tilde{H}$ (consequently, $f(\tilde{G}_*) = x\tilde{H}_*$). Since $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ is derived from $f : \tilde{G}_* \to \frac{\tilde{G}_*}{\tilde{H}_*}$ such that $\tilde{H}_*$ is a normal subgroup of $\tilde{G}_*$, then to prove that f is a homomorphism, we show that

$$CM_{xy\tilde{H}}(z) = CM_{x\tilde{H} \circ y\tilde{H}}(z),\ \forall z \in X \Rightarrow f(xy) = f(x)f(y).$$

Since $\tilde{H}$ is commutative, then

$$CM_{\tilde{H}}(xz) = CM_{\tilde{H}}(zx) \Rightarrow CM_{\tilde{H}}(z^{-1}xz) = CM_{\tilde{H}}(x),\ \forall z \in X.$$

It is certain that,

$$CM_{x\tilde{H}}(z) = CM_{\tilde{H}}(x^{-1}z) \text{ and } CM_{y\tilde{H}}(z) = CM_{\tilde{H}}(y^{-1}z).$$

Then

$$CM_{xy\tilde{H}}(z) = CM_{\tilde{H}}((xy)^{-1}z).$$

Now,

$$\begin{aligned} CM_{x\tilde{H}\circ y\tilde{H}}(z) &= \bigvee_{z=rs}[CM_{x\tilde{H}}(r) \wedge CM_{y\tilde{H}}(s)] \\ &= \bigvee_{z=rs}[CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)]. \end{aligned}$$

Similarly,

$$\begin{aligned} CM_{xy\tilde{H}}(z) = CM_{\tilde{H}}((xy)^{-1}z) &= CM_{\tilde{H}}(y^{-1}x^{-1}z) \\ &\geq \bigvee_{z=rs}[CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)]. \end{aligned}$$

Suppose by hypothesis,

$$CM_{\tilde{H}}(y^{-1}x^{-1}z) = \bigvee_{z=rs}[CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)],$$

then it follows that $CM_{xy\tilde{H}}(z) = CM_{x\tilde{H}\circ y\tilde{H}}(z),\ \forall z \in X$. Consequently, we have $f(xy) = f(x)f(y),\ \forall x, y \in X$. Hence, f is a homomorphism. □

Corolary 4.1. *Let $\tilde{G}, \tilde{H} \in FMG(X)$ such that $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y),\ \forall x, y \in X$ and $CM_{\tilde{G}}(e) \geq CM_{\tilde{H}}(x),\ \forall x \in X$. If $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ is a natural homomorphism defined by $CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y),\ \forall\, x, y \in X$, then $f^{-1}(f(\tilde{H})) = \tilde{G} \circ \tilde{H}$.*

Proof. Let $x \in X$. To proof the result, we assume that $f(x) = f(y),\ \forall x, y \in X$. Thus,

$$\begin{aligned} CM_{f^{-1}(f(\tilde{H}))}(x) &= \bigvee_{x\in X}[CM_{f(\tilde{H})}(f(x))] \\ &= \bigvee_{x\in X}[CM_{\tilde{H}}(f^{-1}(f(y))] \\ &= CM_{\tilde{H}}(y). \end{aligned}$$

Again,

$$\begin{aligned} CM_{\tilde{G}\circ\tilde{H}}(x) &= \bigvee_{x=zy}[CM_{\tilde{G}}(z) \wedge CM_{\tilde{H}}(y)] \\ &= \bigvee_{x\in X}[CM_{\tilde{G}}(xy^{-1}) \wedge CM_{\tilde{H}}(y)] \\ &= \bigvee_{x\in X}[CM_{\tilde{G}}(e) \wedge CM_{\tilde{H}}(y)] \\ &= CM_{\tilde{H}}(y). \end{aligned}$$

Hence, the proof follows. □

Remark 4.4. Assuming there is a bijective correspondence between every (normal) fuzzy submultigroup of $\tilde{G}$ that contains $\tilde{H}$ and the (normal) fuzzy submultigroups of $\frac{\tilde{G}}{\tilde{H}}$. That is, if $\tilde{I}$ is a (normal) fuzzy submultigroup of $\tilde{G}$ containing $\tilde{H}$, then the corresponding (normal) fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is $f(\tilde{I})$.

Theorem 4.6. *Let X and Y be groups, $f : X \to Y$ be an isomorphism and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G} \in FMG(X)$ such that $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y) \forall x, y \in X$ with $kerf = \{e\}$. Then $\frac{\tilde{G}}{\tilde{H}} \cong \frac{f(\tilde{G})}{f(\tilde{H})}$.*

Proof. By Theorem 2.2 and Definition 4.1, $\frac{\tilde{G}}{\tilde{H}}$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ are fuzzy multigroups, respectively.

Let $h : \frac{\tilde{G}}{\tilde{H}} \to \frac{f(\tilde{G})}{f(\tilde{H})}$ be defined by $h(\tilde{H}x) = f(\tilde{H})(f(x)),\ \forall x \in X$. If $\tilde{H}x = \tilde{H}y$, then $CM_{\tilde{H}}(xy^{-1}) = CM_{\tilde{H}}(e)$. Since $kerf = \{e\}$ meaning $kerf \subseteq A^*$, then $f^{-1}(f(\tilde{H})) = \tilde{H}$ by Proposition 2.2. Thus,

$$CM_{f^{-1}(f(\tilde{H}))}(xy^{-1}) = CM_{f^{-1}(f(\tilde{H}))}(e)$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(xy^{-1})) = CM_{f(\tilde{H})}(f(e))$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(x)(f(y))^{-1}) = CM_{f(\tilde{H})}(f(e))$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(x)) = CM_{f(\tilde{H})}(f(y)e') \text{ (where } f(e) = e').$$

Hence,

$$CM_{f(\tilde{H})}(f(x)) = CM_{f(\tilde{H})}(f(y)) \Rightarrow f(\tilde{H})(f(x)) = f(\tilde{H})(f(y)).$$

Hence, h is well-defined. It is also a homomorphism because

$$\begin{aligned} h(\tilde{H}x\tilde{H}y) = h(\tilde{H}xy) &= f(\tilde{H})(f(xy)) \\ &= f(\tilde{H})(f(x)f(y)) \\ &= f(\tilde{H})(f(x))f(\tilde{H})(f(y)) \\ &= h(\tilde{H}x)h(\tilde{H}y). \end{aligned}$$

Suppose f is an epimorphism, then $\exists\, x \in X$ such that $f(x) = y$. Thus,

$$h(\tilde{H}x) = f(\tilde{H})(f(x)) = f(\tilde{H})(y).$$

Moreover,

$$f(\tilde{H})(f(x)) = f(\tilde{H})(f(y)) \Rightarrow CM_{f(\tilde{H})}(f(x)(f(y))^{-1}) = CM_{f(\tilde{H})}(e') \Rightarrow$$

$$CM_{f(\tilde{H})}(f(xy^{-1})) = CM_{f(\tilde{H})}(f(e)) \Rightarrow CM_{f^{-1}(f(\tilde{H}))}(xy^{-1}) = CM_{f^{-1}(f(\tilde{H}))}(e)$$

implies $CM_{\tilde{H}}(xy^{-1}) = CM_{\tilde{H}}(e) \Rightarrow \tilde{H}x = \tilde{H}y$, which proves that h is an isomorphism. Hence, the result follows. □

Corolary 4.2. *Let $f : X \to Y$ be an isomorphism and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G} \in FMG(Y)$ such that $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y),\ \forall x, y \in Y$. Then $\frac{f(\tilde{G})}{f(\tilde{H})} \cong \frac{\tilde{G}}{\tilde{H}}$.*

Proof. By Theorem 2.2, $f(\tilde{G}), f(\tilde{H}) \in FMG(X)$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ are fuzzy multigroups by Definition 4.1.

Again, since $\tilde{H} \in FMG(Y)$, then

$$f(f^{-1}(\tilde{H})) = \tilde{H}.$$

If $x \in kerf$, then $f(x) = e' = f(e)$, and so

$$CM_{\tilde{H}}(f(x)) = CM_{\tilde{H}}(f(e)) \Rightarrow CM_{f^{-1}(\tilde{H})}(x) = CM_{f^{-1}(\tilde{H})}(e).$$

Hence, $kerf \subseteq f^{-1}(\tilde{H}^*)$. The proof is completed following the same logic as in Theorem 4.6. □

Theorem 4.7. *Let $\tilde{H}, \tilde{G} \in FMG(X)$ and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{G}}{\tilde{G}_*} \approx \frac{\tilde{G}}{\tilde{H}}$.*

Proof. Let f be a natural homomorphism from $\tilde{G}_*$ onto $\frac{\tilde{G}_*}{\tilde{H}_*}$ defined by $f(x\tilde{H}_*) = x\tilde{H}_*\ \forall x \in \tilde{G}_*$. Then, we have

$$CM_{f(\frac{\tilde{G}}{\tilde{G}_*})}(x\tilde{H}_*) = \vee[CM_{\frac{\tilde{G}}{\tilde{G}_*}}(z)],\ \forall z \in \tilde{G}_*, f(z) = x\tilde{H}_*.$$

Since $\frac{\tilde{G}}{\tilde{G}_*}$ and $\tilde{G}$ are bijective correspondence to each other (by Remark 4.4) and $z = f^{-1}(x\tilde{H}_*) = x\tilde{H}_*$, it follows that

$$\begin{aligned} CM_{f(\frac{\tilde{G}}{\tilde{G}_*})}(x\tilde{H}_*) &= \vee[CM_{\frac{\tilde{G}}{\tilde{G}_*}}(z)],\ \forall z \in \tilde{G}_*, f(z) = x\tilde{H}_* \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in x\tilde{H}_* \\ &= CM_{\frac{\tilde{G}}{\tilde{H}}}(x\tilde{H}_*),\ \forall x \in \tilde{G}_*, \end{aligned}$$

because $\frac{\tilde{G}}{\tilde{H}}$ and $\tilde{G}$ are bijective correspondence to each other. Hence, it follows that $\frac{\tilde{G}}{\tilde{G}_*} \approx \frac{\tilde{G}}{\tilde{H}}$. □

Lemma 4.1. *Suppose $f : X \to Y$ and $\tilde{G} \in FMG(X)$, then $(f(\tilde{G}))_* = f(\tilde{G}_*)$.*

Proof. Straightforward. □

Theorem 4.8. *Let $\tilde{G} \in FMG(X)$. Suppose Y is a group and $\tilde{I} \in FMG(Y)$ such that $\tilde{G} \approx \tilde{I}$. Then, there exists a normal fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ such that $\frac{\tilde{G}}{\tilde{H}} \cong \frac{\tilde{I}}{\tilde{I}_*}$.*

Proof. Since $\tilde{G} \approx \tilde{I}, \exists$ an epimorphism f of X onto Y such that $f(\tilde{G}) = \tilde{I}$. Define $\tilde{H} \in FMG(X)$ as follows: $\forall\, x \in X$,

$$CM_{\tilde{H}}(x) = \begin{cases} CM_{\tilde{G}}(x) & \text{if } x \in kerf \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\tilde{H} \subseteq \tilde{G}$. If $x \in kerf$, then $yxy^{-1} \in kerf,\ \forall y \in X$, and so

$$CM_{\tilde{H}}(yxy^{-1}) = CM_{\tilde{G}}(yxy^{-1}) = CM_{\tilde{G}}(x) = CM_{\tilde{H}}(x),\ \forall y \in X.$$

If $x \notin kerf$, then $CM_{\tilde{H}}(x) = 0$ and so

$$CM_{\tilde{H}}(yxy^{-1}) = CM_{\tilde{H}}(x) = 0,\ \forall y \in X.$$

Hence, $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$. Also, $\tilde{G} \approx \tilde{I} \Rightarrow f(\tilde{G}) = \tilde{I}$ which further implies $(f(\tilde{G}))_* = \tilde{I}_*$ and $f(\tilde{G}_*) = \tilde{I}_*$ by Lemma 4.1. Let $f = g$. Then g is a homomorphism of $\tilde{G}_*$ onto $\tilde{I}_*$ and $ker\, g = \tilde{H}_*$. Thus, there exists an isomorphism h of $\frac{\tilde{G}_*}{\tilde{H}_*}$ onto $\tilde{I}_*$ such that $h(x\tilde{H}_*) = g(x) = f(x),\ \forall x \in \tilde{G}_*$. For such an h, we have

$$\begin{aligned} CM_{h(\frac{\tilde{G}}{\tilde{H}})}(z) &= \vee[CM_{\frac{\tilde{G}}{\tilde{H}}}(x\tilde{H}_*)],\ \forall x \in \tilde{G}_*, h(x\tilde{H}_*) = z \\ &= \vee(\vee[CM_{\tilde{G}}(y)],\ \forall y \in x\tilde{H}_*), \forall x \in \tilde{G}_*, g(x) = z \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in \tilde{G}_*, g(y) = z \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in X, f(y) = z \\ &= CM_{\tilde{G}}(f^{-1}(z)) = CM_{f(\tilde{G})}(z) = CM_{\tilde{I}}(z),\ \forall z \in \tilde{I}_*. \end{aligned}$$

Therefore, $\frac{\tilde{G}}{\tilde{H}} \cong \frac{\tilde{I}}{\tilde{I}_*}$. □

Theorem 4.9. *Suppose $\tilde{G}$ is a fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})} \simeq \frac{(\tilde{H}\circ\tilde{G})}{\tilde{H}}$.*

Proof. From Definition 2.13, it is easy to infer that $\tilde{H}_*$ is also a normal subgroup of X. By the Second Isomorphism Theorem for groups, we deduce

$$\frac{\tilde{G}_*}{\tilde{H}_*\cap\tilde{G}_*} \cong \frac{\tilde{H}_*\tilde{G}_*}{\tilde{H}_*}.$$

Assume that

$$(\tilde{H}\cap\tilde{G})_* = \tilde{H}_*\cap\tilde{G}_* \text{ and } (\tilde{H}\circ\tilde{G})_* = \tilde{H}_*\tilde{G}_*.$$

Consequently, we have

$$\frac{\tilde{G}_*}{(\tilde{H}\cap\tilde{G})_*} \cong \frac{(\tilde{H}\circ\tilde{G})_*}{\tilde{H}_*},$$

where f is given by

$$f(x(\tilde{H}\cap\tilde{G})_*) = x\tilde{H}_*,\ \forall x\in\tilde{G}_*.$$

Thus,

$$\begin{aligned} CM_{f(\frac{\tilde{G}}{\tilde{H}\cap\tilde{G}})}(y\tilde{H}_*) &= C_{\frac{\tilde{G}}{\tilde{H}\cap\tilde{G}}}(y(\tilde{H}\cap\tilde{G})_*) \\ &= \vee[CM_{\tilde{G}}(z)],\ \forall z\in y(\tilde{H}\cap\tilde{G})_* \\ &\leq \vee[CM_{\tilde{H}\circ\tilde{G}}(z)],\ \forall z\in y(\tilde{H}_*\cap\tilde{G}_*) \\ &\leq \vee[CM_{\tilde{H}\circ\tilde{G}}(z)],\ \forall z\in y\tilde{H}_* \\ &= CM_{\frac{\tilde{H}\circ\tilde{G}}{\tilde{H}}}(y\tilde{H}_*),\ \forall y\in\tilde{G}_*. \end{aligned}$$

Hence, $f(\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})}) \subseteq \frac{\tilde{H}\circ\tilde{G}}{\tilde{H}}$. Therefore, $\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})} \simeq \frac{(\tilde{H}\circ\tilde{G})}{\tilde{H}}$. □

Theorem 4.10. *Let $\tilde{H},\tilde{I},\tilde{G}\in FMG(X)$ such that $\tilde{H}\subseteq\tilde{I}$, and $\tilde{H}$ and $\tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$. Then $(\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}}) \cong \frac{\tilde{G}}{\tilde{I}}$.*

Proof. If $\tilde{H},\tilde{I}\in FMG(X)$ and $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{I}$, then $\tilde{H}_*$ is a normal subgroup of $\tilde{I}_*$ and both $\tilde{H}_*$ and $\tilde{I}_*$ are normal subgroups of $\tilde{G}_*$. From the principle of Third Isomorphism Theorem for groups, it follows that

$$(\frac{\tilde{G}_*}{\tilde{H}_*})/(\frac{\tilde{I}_*}{\tilde{H}_*}) \cong (\frac{\tilde{G}_*}{\tilde{I}_*}),$$

where f is given by

$$f(x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*})) = x\tilde{I}_*,\ \forall x\in\tilde{G}_*.$$

Then

$$\begin{aligned}
CM_{f((\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}}))}(x\tilde{I}_*) &= CM_{(\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}})}(x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*})) \\
&= \vee[CM_{\frac{\tilde{G}}{\tilde{H}}}(y\tilde{H}_*)],\ \forall y \in \tilde{G}_*, y\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[\vee(CM_{\tilde{G}}(z)), \forall z \in y\tilde{H}_*],\ \forall y \in \tilde{G}_*,\ y\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, z\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, f(z) \in x\tilde{I}_* \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, f(z) = z \\
&= CM_{\frac{\tilde{G}}{\tilde{I}}}(x\tilde{I}_*),
\end{aligned}$$

$\forall x \in \tilde{G}_*$, where the equalities hold since f is one-to-one. Hence, the result follows. □

5 Conclusions

In this paper, the ideas of fuzzy comultisets and quotient fuzzy multigroups have been proposed in an attempt to further strengthen the theory of fuzzy multigroups. A number of some related results were duly discussed in details. The connection between fuzzy comultisets of fuzzy multigroups and the cosets of groups has been proven. Some characterizations of fuzzy comultisets were delineated with verifications. In addition, quotient fuzzy multigroup was explicated and some of its properties were explored. Finally, group theoretic isomorphism theorems were substantiated in fuzzy multigroup setting. Nevertheless, some group analog concepts could still be exploited in fuzzy multigroup context.

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Geometrical foundations of the sampling design with fixed sample size

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Abstract

We study the sampling design with fixed sample size from a geometric point of view. The first-order and second-order inclusion probabilities are chosen by the statistician. They are subjective probabilities. It is possible to study them inside of linear spaces provided with a quadratic and linear metric. We define particular random quantities whose logically possible values are all logically possible samples of a given size. In particular, we define random quantities which are complementary to the Horvitz-Thompson estimator. We identify a quadratic and linear metric with regard to two univariate random quantities representing deviations. We use the α-criterion of concordance introduced by Gini in order to identify it. We innovatively apply to probability this statistical criterion.

Keywords: tensor product; linear map; bilinear map; quadratic and linear metric; α-product; α-norm

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1 Introduction

Given a finite population having N elements, we are only interested in considering samples containing units of this population where no element of the population under consideration can be selected more than once in the same sample (Basu [1971]). We are not interested in considering ordered samples of a given size selected from a finite population (Basu [1958]). On the other hand, when we consider not ordered samples where repetitions are not allowed we have no loss of information about a given parameter of the population under consideration (Conti and Marella [2015]). All logically possible samples of a given size belong to a given set. We suppose that we are always able to number them. It is known that if the number of all logically possible samples of a given set is very large then it could be a very hard or impossible work to give to them a number (Godambe and Joshi [1965]). A sampling design is characterized by a pair of elements (Joshi [1971]). The first element of this pair represents the set of all logically possible samples selected from a finite population. The second element of this pair represents all probabilities assigned to the samples of the set of all logically possible samples of a given size. We consider a distribution of probability in this way (Hartley and Rao [1962]). Each element of the set of all logically possible samples of a given size can be viewed as a logically possible event of a finite partition of incompatible and exhaustive elementary events. It is then possible to assign a subjective probability to each logically possible event of this partition (Good [1962]). A probability subjectively assigned to each logically possible event of a finite partition of events must only be coherent. It is inadmissible when it is not coherent. A probability is subjectively assigned to each logically possible event of a finite partition of events even when it is an equal probability assigned to each of them. An equal probability assigned to each logically possible event of a finite partition of events is always a subjective judgment. We have to note a very important point: when we say that it is possible to assign a subjective and coherent probability to every logically possible event of a given set of events we mean that the choice of any value in the interval from 0 to 1 is allowed. Such an interval includes both endpoints. It would therefore be possible to assign to every logically possible event of a given set of events a probability equal to 0. This choice is absolutely coherent. We will however introduce a restriction that is concerned with this point. We have to note another very important point: we methodologically distinguish what it is logically possible from what it is subjectively probable. What it is logically possible at a given instant it is not either certainly true or certainly false. One and only one element of the elements belonging to the set containing all logically possible elements at a given instant will be true a posteriori. A subjective probability is then assigned to each element of the set containing all logically possible elements before knowing this thing.

2 Events as points in the space of random quantities

We consider a finite set of vectors denoted by $\mathcal{S}$ into the field $\mathbb{R}$ of real numbers. We enumerate them. We consequently write

$$s_1, \ldots, s_N, \tag{1}$$

where it turns out to be $s_i \in \mathcal{S}$, $i = 1, \ldots, N$. We consider a linear space over $\mathbb{R}$ of all linear combinations of elements of $\mathcal{S}$ expressed in the form

$$c_1 s_1 + \ldots + c_N s_N, \tag{2}$$

where every c_i, $i = 1, \ldots, N$, is a real number. We observe that (2) is completely determined by the real numbers $c_1, \ldots, c_N$. Each number c_i is associated with the element s_i of the set $\mathcal{S}$. It is known that an association is exactly a function. For each $s_i \in \mathcal{S}$ and $c \in \mathbb{R}$ we then consider

$$cs_i \tag{3}$$

to be the function that associates c with s_i and 0 with s_j, where we have $j \neq i$. Given $a \in \mathbb{R}$, we obtain

$$a(cs_i) = (ac)s_i. \tag{4}$$

Given $c' \in \mathbb{R}$, we obtain

$$(c + c')s_i = cs_i + c's_i. \tag{5}$$

Thus, it is possible to consider a linear space over $\mathbb{R}$. It is the set of all functions of $\mathcal{S}$ into $\mathbb{R}$. These functions can be written in the form given by (2). The functions

$$1s_1, \ldots, 1s_N \tag{6}$$

are linearly independent so they represent a basis of the linear space under consideration. We have then to suppose that $c_1, \ldots, c_N$ are elements of $\mathbb{R}$ such that it is possible to obtain the zero function given by

$$c_1 s_1 + \ldots + c_N s_N = 0. \tag{7}$$

This means that we have $c_i = 0$ for every c_i, $i = 1, \ldots, N$. This thing consequently proves the linear independence under consideration. Moreover, it is always possible to write s_i instead of $1s_i$. A sample belonging to the set of all logically possible samples of a given size is then expressed by the vector

$$\delta(s') = \begin{bmatrix} \delta(1; s') \\ \delta(2; s') \\ \vdots \\ \delta(N; s') \end{bmatrix} \tag{8}$$

having N components, where s' is a sample of the set of all logically possible samples denoted by $\mathcal{S}'$ (Godambe [1955]). We will always consider vectors viewed as ordered lists of real numbers within this context. A sample can be expressed by the real numbers of a linear combination of N-dimensional vectors by means of which another N-dimensional vector is obtained. If a sample is identified with an N-dimensional vector then its components express the real numbers of a linear combination of the elements of a basis of the linear space under consideration. This linear space is denoted by $\mathbb{R}^N$. Its basis is denoted by $\mathcal{S} = \{\mathbf{e}_j\}$, $j = 1, \ldots, N$. We always consider orthonormal bases within this context. We therefore write

$$\delta(1; s')\mathbf{e}_1 + \delta(2; s')\mathbf{e}_2 + \ldots + \delta(N; s')\mathbf{e}_N = \mathbf{y}, \tag{9}$$

where we have $\mathbf{y} \in \mathbb{R}^N$. We consider as many linear combinations of the elements of $\mathcal{S} = \{\mathbf{e}_j\}$, $j = 1, \ldots, N$, as logically possible samples there are into the set of all logically possible samples of a given size denoted by $\mathcal{S}'$. We note that the real numbers of every linear combination of the elements of $\mathcal{S} = \{\mathbf{e}_j\}$, $j = 1, \ldots, N$, represent one of the logically possible samples of $\mathcal{S}'$. We have evidently

$$\delta(i; s') = \begin{cases} 1 & \text{if } i \in s' \\ 0 & \text{if } i \notin s' \end{cases} \tag{10}$$

for every $i = 1, \ldots, N$, where the elements of the population under consideration are overall N. We consider all logically possible samples of $\mathcal{S}'$ having the same size denoted by n. Since the population has got N elements we observe that the number of n-combinations is equal to the binomial coefficient denoted by $\binom{N}{n}$. We observe that $\mathcal{S}'$ whose elements are elementary events is a subset of $\mathbb{R}^N$. We say that $\mathcal{S}'$ is embedded in $\mathbb{R}^N$.

3 Finite partitions of logically possible elementary events

Given N, all logically possible samples whose size is equal to n belong to the set denoted by $\mathcal{S}'$. We have

$$n = \sum_{i=1}^{N} \delta(i; s') \tag{11}$$

for every $s' \in \mathcal{S}'$. Every sample of the set of all logically possible samples corresponds to a vertex denoted by $\delta(s')$ of an N-dimensional unit hypercube denoted by $[0, 1]^N$. All logically possible samples of $\mathcal{S}'$ can be viewed as possible events

of a finite partition of incompatible and exhaustive elementary events (de Finetti [1982b]). We are consequently able to define a univariate random quantity whose logically possible values are represented by all logically possible samples of $\mathcal{S}'$. The logically possible values of it are not real numbers but they are N-dimensional vectors of an N-dimensional linear space over $\mathbb{R}$. Every logically possible sample belonging to $\mathcal{S}'$ has a subjective probability of being selected (de Finetti [1989]. It represents the degree of belief in the selection of a logically possible sample assigned by a given individual (the statistician) at a certain instant with a given set of information. An evaluation of probability known over a set of possible and elementary events coinciding with all logically possible samples of $\mathcal{S}'$ is admissible when it is coherent. This means that it must be

$$\sum_{s' \in \mathcal{S}'} p(s') = 1. \tag{12}$$

It is essential to note a very important point: we have to introduce an unusual restriction with regard to the coherence because we exclude of choosing a subjective probability equal to 0 with respect to any possible and elementary event. This implies that any logically possible sample of $\mathcal{S}'$ has always a probability greater than zero of being selected. We have consequently

$$0 < p(s') \leq 1 \tag{13}$$

for every $s' \in \mathcal{S}'$ (Coletti et al. [2015]). Thus, conditions of coherence coincide with positivity of each probability of a random event and finite additivity of probabilities of incompatible and exhaustive events (Gilio and Sanfilippo [2014]). We will also consider bivariate random quantities whose components are two univariate random quantities (de Finetti [2011]). If the logically possible values of these univariate random quantities are the same vectors of the same N-dimensional linear space over $\mathbb{R}$ then these random quantities have the same marginal distributions of probability. They represent the same finite partition of incompatible and exhaustive elementary events. Putting them into a two-way table we observe that it is always a table having the same number of rows and columns.

4 First-order inclusion probabilities viewed as a coherent prevision of a univariate random quantity

We consider a univariate random quantity denoted by S whose logically possible values are vectors of $\mathbb{R}^N$. Given N and n, the number of the logically possible values of S coincides with the binomial coefficient expressed by

$$\binom{N}{n} = k. \tag{14}$$

The set of the logically possible values of S is then given by $I(S) = \{s'_1, \ldots, s'_k\}$, with $s'_i \in \mathcal{S}'$, $i = 1, \ldots, k$. A nonzero probability is assigned to each sample of the set of all logically possible samples. Let $p(s'_1), \ldots, p(s'_k)$ be these probabilities. It must therefore be

$$\sum_{i=1}^{k} p(s'_i) = 1, \tag{15}$$

with

$$0 < p(s'_i) \leq 1 \tag{16}$$

for every $i = 1, \ldots, k$. It is possible to obtain an N-dimensional vector after assigning a nonzero probability to each sample of $\mathcal{S}'$. We denote it with π. It represents the first-order inclusion probabilities of all units of the population under consideration. Thus, we write

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} = p(s'_1) \begin{bmatrix} \delta(1; s'_1) \\ \delta(2; s'_1) \\ \vdots \\ \delta(N; s'_1) \end{bmatrix} + \ \ldots \ + p(s'_k) \begin{bmatrix} \delta(1; s'_k) \\ \delta(2; s'_k) \\ \vdots \\ \delta(N; s'_k) \end{bmatrix}, \tag{17}$$

where we have $\pi_i > 0$ for every $i = 1, \ldots, N$. We have evidently written a convex combination of the vertices of the N-dimensional unit hypercube $[0, 1]^N$ corresponding to the samples of $\mathcal{S}'$. Each vertex is a sample having a nonzero weight representing a subjective probability. It is essential to note that π is a coherent prevision of S denoted by $\mathbf{P}(S)$. We therefore write

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} = \mathbf{P}(S) = \sum_{i=1}^{k} \delta(s'_i) p(s'_i). \tag{18}$$

We observe that the logically possible values of S are represented by vectors having N components so its coherent prevision must also be represented by a vector having N components. The logically possible values of S belong to the set denoted by $I(S)$. Each element of $I(S)$ contains first-order inclusion "a posteriori" probabilities. This implies that π must contain first-order inclusion "a priori" probabilities based on the degree of belief in the selection of all logically possible samples attributed by the statistician at a certain instant with a given set of information. An "a posteriori" probability of a unit of the population of being included in a given sample is always predetermined. If a unit of the population is contained "a posteriori" in the sample that has been selected then its probability is equal to 1. If a unit of the population does not belong "a posteriori" to the sample that

has been selected then its probability is equal to 0. A convex combination coinciding with $\mathbf{P}(S)$ has conveniently been taken under consideration because the logically possible values of S are incompatible and exhaustive elementary events of a finite partition of random events. In general, if we consider an event divided into two or more than two incompatible events then we obtain that its coherent probability is the sum of two or more than two coherent probabilities. This sum is a linear combination of probabilities (de Finetti [1980]). We evidently consider a convex combination coinciding with $\mathbf{P}(S)$ within this context, where its weights or coefficients are "a priori" subjective probabilities connected with the samples of $\mathcal{S}'$ (de Finetti [1981]). This convex combination is characterized by k column vectors viewed as k matrices. Each row of every $N \times 1$ matrix is a first-order inclusion "a posteriori" probability. We therefore consider a linear combination of probabilities (de Finetti [1982a]).

5 First-order inclusion probabilities obtained by means of linear maps

We consider all logically possible samples belonging to the set $\mathcal{S}'$. Given N and n, let k be the number of all elements of $\mathcal{S}'$. We are consequently able to determine an $N \times k$ matrix in $\mathbb{R}$. We denote it by B. It is therefore possible to define a linear map expressed by

$$L_B \colon \mathbb{R}^k \to \mathbb{R}^N. \tag{19}$$

It depends on B. Moreover, it also depends on the choice of bases for $\mathbb{R}^k$ and $\mathbb{R}^N$. We choose standard bases for $\mathbb{R}^k$ and $\mathbb{R}^N$. We consider all probabilities assigned to the logically possible samples of $\mathcal{S}'$ whose size is equal to n. They can be viewed as a column vector. We denote it by Q. We have then

$$Q = \begin{bmatrix} p(s'_1) \\ p(s'_2) \\ \vdots \\ p(s'_k) \end{bmatrix}. \tag{20}$$

It therefore turns out to be

$$L_B(Q) = BQ = \pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix}. \tag{21}$$

We note that if $k = N$ then we are able to define a linear map expressed by

$$L_B \colon \mathbb{R}^N \to \mathbb{R}^N. \tag{22}$$

We observe that B is a square matrix. This linear map is an endomorphism. It is also an isomorphism. It is then an automorphism, so we write

$$B^{-1}\pi = \begin{bmatrix} p(s'_1) \\ p(s'_2) \\ \vdots \\ p(s'_k) \end{bmatrix}. \tag{23}$$

Given B, each row of Q can subjectively vary because an evaluation of probability known over a set of logically possible events must only be coherent. This means that the sum of all probabilities of the samples of $\mathcal{S}'$ must be equal to 1. We consequently observe that there are infinite ways of choosing all probabilities of the samples of $\mathcal{S}'$. They are conveniently caught by L_B. It is hence possible to obtain π as a multiplication of matrices according to a linear map depending on B and the standard bases of the linear spaces under consideration. Also, we always obtain

$$\sum_{i=1}^{N} \pi_i = n. \tag{24}$$

6 First-order and second-order inclusion probabilities obtained by means of tensor products

We consider a bivariate random quantity denoted by S_{12} whose components are two univariate random quantities denoted by ${}_1S$ and ${}_2S$. We therefore write $S_{12} = \{{}_1S, {}_2S\}$. Given N and n, the logically possible values of each univariate random quantity coincide with k samples belonging to the set $\mathcal{S}'$. They are all logically possible samples of $\mathcal{S}'$ whose size is equal to n. Each sample of $\mathcal{S}'$ is a vector of $\mathbb{R}^N$. We have to note a very important point: we suppose that the logically possible values of ${}_1S$ and ${}_2S$ are the same N-dimensional vectors of the same N-dimensional linear space over $\mathbb{R}$. These univariate random quantities have then the same marginal distributions of probability. Putting them into a two-way table we observe that it is always a square table. We observe that all probabilities of the joint distribution of probability outside of the main diagonal of this table are always equal to 0. The nonzero probabilities of the joint distribution of probability coincide with $p(s'_1), \ldots, p(s'_k)$. They are on the main diagonal of the table under consideration. A coherent prevision of S_{12} denoted by $\mathbf{P}(S_{12})$

is obtained by means of the sum of k square matrices. The number of rows and columns of every square matrix of this sum is equal to N. Each square matrix of this sum derives from a tensor product belonging to the same linear space denoted by $\mathbb{R}^N \otimes \mathbb{R}^N$. It is an N^2-dimensional linear space over $\mathbb{R}$. We always consider as many tensor products as joint probabilities are associated with the samples of $\mathcal{S}'$. We have then

$$p(s'_i)\left(\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix},\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\right)\mapsto p(s'_i)\left(\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\otimes\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\right) \tag{25}$$

for every $i = 1, \ldots, k$. We note that it turns out to be

$$p(s'_i)\left(\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\otimes\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\right)=p(s'_i)\begin{bmatrix}\delta(1;s'_i)\\ \delta(2;s'_i)\\ \vdots\\ \delta(N;s'_i)\end{bmatrix}\begin{bmatrix}\delta(1;s'_i) & \delta(2;s'_i) & \ldots & \delta(N;s'_i)\end{bmatrix}. \tag{26}$$

If we consider a coherent prevision of S_{12} then we deal with a bilinear map expressed by $\mathbb{R}^N \times \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$, where the linear space over $\mathbb{R}$ of the $N \times N$ matrices in $\mathbb{R}$ is denoted by $\mathcal{M}_{N,N}(\mathbb{R})$. This linear space is isomorphic to $\mathbb{R}^{N^2}$. The matrix product resulting from this bilinear map is factorized by means of the tensor product of vectors of $\mathbb{R}^N$. It is also factorized by means of a unique linear map whose domain coincides with $\mathbb{R}^N \otimes \mathbb{R}^N$. This is because we are able to know a basis of $\mathbb{R}^N \otimes \mathbb{R}^N$ as well as the value of the linear map under consideration on basis elements. We suppose that a basis of $\mathbb{R}^N \otimes \mathbb{R}^N$ results from the standard basis of $\mathbb{R}^N$, where $\mathbb{R}^N$ is evidently considered two times. It is therefore possible to say that there exists a unique linear map given by $\mathbb{R}^N \otimes \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$. It coincides with the product of a joint probability viewed as a scalar and a square matrix. We consider k products of a joint probability and a square matrix. We obtain k square matrices in this way. We consider the sum of these k square matrices in order to obtain a coherent prevision of S_{12}. We observe that $\mathbb{R}^N \times \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$ and $\mathbb{R}^N \otimes \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$ have the same codomain. A factorization of $\mathbb{R}^N \times \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$ is then realized by means of a bilinear map given by $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^N$ and a linear map given by $\mathbb{R}^N \otimes \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$. These two maps are connected, so we obtain a composition of functions identified with $\mathbb{R}^N \times \mathbb{R}^N \to \mathcal{M}_{N,N}(\mathbb{R})$. The following

commutative diagram

$$\begin{array}{ccc} \mathbb{R}^N \times \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \otimes \mathbb{R}^N \\ & \searrow & \downarrow \\ & & \mathcal{M}_{N,N}(\mathbb{R}) \end{array}$$

permits of visualizing what we have said. A coherent prevision of S_{12} is then bilinear and homogeneous. It is given by

$$\mathbf{P}(S_{12}) = \Pi = \begin{bmatrix} \pi_1 & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_2 & \dots & \pi_{2N} \\ \dots & \dots & \dots & \dots \\ \pi_{N1} & \pi_{N2} & \dots & \pi_N \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_{12} & \dots & \pi_{1N} \\ \pi_{12} & \pi_2 & \dots & \pi_{2N} \\ \dots & \dots & \dots & \dots \\ \pi_{1N} & \pi_{2N} & \dots & \pi_N \end{bmatrix}. \tag{27}$$

It coincides with the symmetric matrix of the first-order and second-order inclusion probabilities. The trace of this matrix is evidently equal to n (Angelini [2020]).

7 The covariance of two univariate random quantities obtained by considering two bilinear maps

Given $S_{12} = \{{}_1S, {}_2S\}$, the covariance of ${}_1S$ and ${}_2S$ is expressed by

$$\mathrm{C}({}_1S, {}_2S) = \mathbf{P}(S_{12}) - \mathbf{P}({}_1S)\mathbf{P}({}_2S), \tag{28}$$

where $\mathbf{P}(S_{12})$ represents the prevision or mathematical expectation or expected value of S_{12}, while $\mathbf{P}({}_1S)$ and $\mathbf{P}({}_2S)$ represent the prevision or mathematical expectation or expected value of ${}_1S$ and ${}_2S$. We note that a coherent prevision of S_{12} derives from a bilinear map because we have

$$\mathbf{P}(S_{12}) = \begin{bmatrix} \pi_1 & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_2 & \dots & \pi_{2N} \\ \dots & \dots & \dots & \dots \\ \pi_{N1} & \pi_{N2} & \dots & \pi_N \end{bmatrix}. \tag{29}$$

Moreover, since we have

$$\mathbf{P}({}_1S) = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} \tag{30}$$

as well as

$$\mathbf{P}({}_2S) = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix}, \tag{31}$$

we note that the product of these two linear maps is evidently bilinear. Such a product is expressed in the form

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_N \end{bmatrix} = \begin{bmatrix} \pi_1\pi_1 & \pi_1\pi_2 & \dots & \pi_1\pi_N \\ \pi_2\pi_1 & \pi_2\pi_2 & \dots & \pi_2\pi_N \\ \dots & \dots & \dots & \dots \\ \pi_N\pi_1 & \pi_N\pi_2 & \dots & \pi_N\pi_N \end{bmatrix}. \tag{32}$$

It is then evident that the covariance of ${}_1S$ and ${}_2S$ derives from two bilinear maps because we can write

$$\mathrm{C}({}_1S, {}_2S) = \begin{bmatrix} \pi_1 & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_2 & \dots & \pi_{2N} \\ \dots & \dots & \dots & \dots \\ \pi_{N1} & \pi_{N2} & \dots & \pi_N \end{bmatrix} - \begin{bmatrix} \pi_1\pi_1 & \pi_1\pi_2 & \dots & \pi_1\pi_N \\ \pi_2\pi_1 & \pi_2\pi_2 & \dots & \pi_2\pi_N \\ \dots & \dots & \dots & \dots \\ \pi_N\pi_1 & \pi_N\pi_2 & \dots & \pi_N\pi_N \end{bmatrix}. \tag{33}$$

By writing

$$\mathrm{C}({}_1S, {}_2S) = \begin{bmatrix} (\pi_1 - \pi_1\pi_1) & (\pi_{12} - \pi_1\pi_2) & \dots & (\pi_{1N} - \pi_1\pi_N) \\ (\pi_{21} - \pi_2\pi_1) & (\pi_2 - \pi_2\pi_2) & \dots & (\pi_{2N} - \pi_2\pi_N) \\ \dots & \dots & \dots & \dots \\ (\pi_{N1} - \pi_N\pi_1) & (\pi_{N2} - \pi_N\pi_2) & \dots & (\pi_N - \pi_N\pi_N) \end{bmatrix} \tag{34}$$

we note that it is possible to consider as many random components as inclusion probabilities are studied. A unit of the population under consideration can be included, or not, in a given sample (Bondesson [2010]). This thing is uncertain until a given sample is selected (Hájek [1958]). Two different units of the population under consideration can be included, or not, in the same sample (Deville and Tillé [1998]). This thing is uncertain until a given sample is selected. A component associated with one or two different units of the population under consideration is evidently random for this reason (Connor [1966]). This means that each random component is characterized by a subjective probability. It is an "a priori" probability. It is also characterized by an "a posteriori" probability coinciding with one of the two logically possible values of a random event, 0 and 1. One and only one of these two logically possible values of a random event will be true "a posteriori". On the other hand, it is known that the notion of probability basically

deals with an aspect that is included between two extreme aspects. The first extreme aspect deals with situations of non-knowledge or ignorance or uncertainty determining the set of all logically possible samples of a given size viewed as elementary events. They are evidently all logically possible alternatives that can be considered. The second extreme aspect deals with definitive certainty expressed in the form of what it is certainly true or certainly false. Thus, every logically possible sample of a given size definitively becomes true or false. Probability is subjectively distributed by the statistician as a mass over the domain of all logically possible samples of a given size before knowing which is the true sample to be selected "a posteriori". Having said that, the variance of every random component as well as the covariance of two random components are dealt with by means of the first-order and second-order inclusion probabilities. The variance of each random component is represented by every element on the main diagonal of the symmetric matrix given by (34). The covariance of two random components is represented by every element outside of the main diagonal of the square matrix given by (34).

8 A univariate random quantity representing deviations

We define a univariate random quantity representing deviations. We denote it by D. We firstly consider S whose values are all logically possible samples of a given size viewed as elementary events belonging to the set $\mathcal{S}'$. Given N and n, the number of the logically possible values of S is equal to $\binom{N}{n} = k$. The set of the logically possible values of S is then given by $I(S) = \{s'_1, \ldots, s'_k\}$, with $s'_i \in \mathcal{S}'$, $i = 1, \ldots, k$. A nonzero probability denoted by $p(s'_i)$, $i = 1, \ldots, k$, is assigned to each sample of $\mathcal{S}'$. We therefore obtain an N-dimensional vector denoted by π. It represents the first-order inclusion probabilities of all units of the population under consideration. They are all greater than zero. This vector is always independent of the origin of the coordinate system that we could consider. We note that the number of the logically possible values of D is equal to k. It is the same of the one of S. The set of the logically possible values of D is given by $I(D) = \{d'_1, \ldots, d'_k\}$, with

$$d'_i = \left(\begin{bmatrix} \delta(1; s'_i) \\ \delta(2; s'_i) \\ \vdots \\ \delta(N; s'_i) \end{bmatrix} - \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} \right), \tag{35}$$

where we have $i = 1, \ldots, k$. It follows that we have

$$p(s'_1)d'_1 + \ldots + p(s'_k)d'_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{36}$$

This means that $\mathbf{P}(S)$ is an N-dimensional vector such that all deviations from it that are multiplied by the corresponding probabilities represent N-dimensional vectors whose sum coincides with the zero vector of $\mathbb{R}^N$. We are now able to calculate the variance of S by using D. We refer to the α-criterion of concordance introduced by Gini. It is a statistical criterion that we innovatively apply to probability viewed as a mass. An absolute maximum of concordance is then realized when each d'_i, $i = 1, \ldots, k$, is multiplied by itself. If each d'_i, $i = 1, \ldots, k$, is multiplied by itself then we obtain k square matrices. Every multiplication that we consider is a tensor product of two vectors of $\mathbb{R}^N$. These two vectors represent two deviations which are the same. The components of these two vectors are then the same. Hence, the variance of S coincides with the sum of k traces of k square matrices. Each trace of the square matrix under consideration is an inner product viewed as an α-product. An α-product is a bilinear form. We consider each $p(s'_i)$, $i = 1, \ldots, k$, as a scalar. Each $p(s'_i)$, $i = 1, \ldots, k$, is firstly a subjective probability. Thus, it always characterizes a random quantity. It is nevertheless viewed as a scalar within this context. We can therefore multiply all components of d'_i by $p(s'_i)$, $i = 1, \ldots, k$. We note that the components of each d'_i, $i = 1, \ldots, k$, are always independent of the origin of the coordinate system that we could consider. We therefore write

$$\sigma_S^2 = \operatorname{tr}\left({d'_1}^T(p(s'_1)d'_1)\right) + \ldots + \operatorname{tr}\left({d'_k}^T(p(s'_k)d'_k)\right). \tag{37}$$

We have evidently introduced a quadratic and linear metric in this way. We therefore note that σ_S^2 is the sum of the squares of k α-norms. It is possible to verify that every trace of a square matrix is an α-product which is an α-commutative product, an α-associative product, an α-distributive product and an α-orthogonal product. We have to note a very important point: S and D are two different quantities from a geometric point of view because they are represented by different sets of N-dimensional vectors. They are nevertheless the same quantity from a randomness point of view. They are characterized by the same probabilities. We therefore observe the same events because we consider only a change of origin.

9 Intrinsic properties of a univariate random quantity representing deviations

Translations and rotations of vectors identifying a given univariate random quantity representing deviations are intrinsic properties of it. They do not depend on the choice of a basis of a given linear space. We say that all vectors of $\mathcal{S}'$ are subjected to the same translation when we consider k sums of two vectors. We consider k sums of two vectors because the number of the elements of $\mathcal{S}'$ is equal to k. The first vector of each sum of them is given by s'_i, $i = 1, \ldots, k$. The second vector of each sum of them is given by an arbitrary N-dimensional vector which is always the same. We say that all vectors of $\mathcal{S}'$ are then subjected to the same change of origin. It follows that σ_S^2 is invariant with respect to a translation of all vectors of $\mathcal{S}'$. We say that a quadratic and linear metric is invariant with respect to a translation of all vectors of $\mathcal{S}'$. Concerning a rotation, let $A = (a_j^{i'})$ be an $N \times N$ orthogonal matrix. Each element of this matrix is denoted by two indices. We use contravariant and covariant indices without loss of generality. The contravariant indices represent the rows of the matrix. We have $i' = 1, \ldots, N$. The covariant indices represent the columns of the matrix. We have $j = 1, \ldots, N$. We observe that rotations of all vectors contained in $I(D) = \{d'_1, \ldots, d'_k\}$ are characterized by A. We write

$$\mathcal{R}_A(d'_i) \colon d'_i \Rightarrow Ad'_i = (d'_i)^*, \tag{38}$$

where we have $i = 1, \ldots, k$. We evidently denote by $(d'_i)^*$ the result of the rotation of the vector d'_i obtained by means of the orthogonal matrix denoted by A. The vector $(d'_i)^*$ is an N-dimensional vector. Its components are originated by N linear and homogeneous relationships. We have to note a very important point: $\mathbf{P}(S)$ is an N-dimensional vector such that all rotated deviations from it that are multiplied by the corresponding probabilities represent N-dimensional vectors whose sum coincides with the zero vector of $\mathbb{R}^N$. We have then

$$p(s'_1)(d'_1)^* + \ldots + p(s'_k)(d'_k)^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{39}$$

If we consider rotated deviations then we write

$$\sigma_{S^*}^2 = \operatorname{tr}\left((d'_1)^{*T}(p(s'_1)(d'_1)^*)\right) + \ldots + \operatorname{tr}\left((d'_k)^{*T}(p(s'_k)(d'_k)^*)\right), \tag{40}$$

where S^* represents a univariate random quantity connected with rotated deviations. Since it turns out to be

$$\sigma_S^2 = \sigma_{S^*}^2, \tag{41}$$

we say that the variance of S is invariant with respect to all rotated deviations obtained by means of the same orthogonal matrix denoted by A. We have therefore introduced a quadratic and linear metric which is invariant with respect to translations and rotations of vectors identifying a univariate random quantity representing deviations.

10 A univariate random quantity representing variations and its intrinsic properties

We define a univariate random quantity representing variations. We denote it by V. Given D, the set of the logically possible values of V is expressed by $I(V) = \{v'_1, \ldots, v'_k\}$, with

$$v'_i = d'_i \frac{1}{\sqrt{\sigma_S^2}}, \tag{42}$$

where we have $i = 1, \ldots, k$. We therefore note that S, D and V are different quantities from a geometric point of view. They are conversely the same quantity from a randomness point of view. It is possible to verify that it turns out to be

$$\sigma_V^2 = 1. \tag{43}$$

This index is always equal to 1 independently of the components of d'_i, $i = 1, \ldots, k$. It is evident that these components identify σ_S^2, so we say that $\sigma_V^2 = 1$ is also independent of σ_S^2. We observe that rotations of all vectors belonging to $I(V) = \{v'_1, \ldots, v'_k\}$ are always characterized by an $N \times N$ orthogonal matrix. We write

$$(v'_i)^* = (d'_i)^* \frac{1}{\sqrt{\sigma_S^2}}, \tag{44}$$

where we have $i = 1, \ldots, k$. If we consider translations and rotations of vectors identifying a univariate random quantity representing variations then we observe intrinsic properties that we have already considered. We note that V can be subjected to an affine transformation. If V is subjected to an affine transformation then we write

$$V \Rightarrow aV + b, \tag{45}$$

where we have $a \neq 0$. We therefore observe that each vector of $I(aV + b)$ is equal to the corresponding vector of $I(V)$. This means that the components of each vector of $I(aV + b)$ are the same of the ones of the corresponding vector of $I(V)$. Hence, we say that univariate random quantities representing variations are invariant with respect to an affine transformation. Given $S_{12} = \{{}_1S, {}_2S\}$, we note

that we have ${}_1V = {}_2V = V$ if and only if it turns out to be ${}_1S = {}_2S = S$. It is possible to verify that the covariance of ${}_1V$ and ${}_2V$ is an α-product. It is always equal to 1. On the other hand, it coincides with the Bravais-Pearson correlation coefficient in the case of a perfect direct linear relationship between two quantities. It is possible to verify that the Bravais-Pearson correlation coefficient is invariant with respect to rotations of vectors belonging to $I(V)$. It is therefore invariant with respect to an affine transformation of V. We have to note a very important point: intrinsic properties that we have considered can be related to the random quantities themselves or to specific metric indices based on these quantities. Specific metric indices are evidently based on random quantities representing deviations or variations because we calculate them after taking such random quantities into account. We have to note another very important point: we are not interested in translating or rotating a geometric object in real terms but we are interested in studying its intrinsic properties because these properties are a fundamental consequence of its geometric representation.

11 Metric aspects of an estimate of the population mean

We want to wonder what happens from a metric point of view when we study one or more than one attribute with respect to each element of the population under consideration. We suppose of observing three different and independent characteristics of each element of the population under consideration. We admit this thing without loss of generality. We therefore consider three different and independent variables denoted by X, Y and Z. We note that X is the variable concerning the first attribute of each element of the population under consideration. The variable concerning the second attribute of each element of the population under consideration is denoted by Y. The variable concerning the third attribute of each element of the population under consideration is denoted by Z. If we study only one attribute of each element of the population under consideration then we estimate the population mean by using the univariate Horvitz-Thompson estimator. It is defined by

$$t_{HT}^{(x)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\pi_i} \delta(i; s') x_i, \tag{46}$$

where we have $s' \in \mathcal{S}'$. It is linear and homogeneous (Horvitz and Thompson [1952]). We note that s' is one of the logically possible samples of $\mathcal{S}'$. Also, the weight of the generic unit i of the population under consideration never depends on s'. It is obtained beginning from (17). We have conversely considered all

logically possible samples of $\mathcal{S}'$ when we have defined S, D and V. We did not consider only one of them. These random quantities are complementary to the univariate Horvitz-Thompson estimator for this reason. Also, we have always taken $\mathbf{P}(S) = \pi$ into account when we have defined S, D and V. On the other hand, a coherent prevision of S is itself linear and homogeneous. The expected value of the univariate Horvitz-Thompson estimator is given by

$$\mathrm{E}[t_{HT}^{(x)}] = \mu_x. \tag{47}$$

It is equal to the population mean denoted by μ_x for any vector $(x_1\ x_2\ \ldots\ x_N)^T \in \mathbb{R}^N$. We have

$$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{48}$$

The variance of the univariate Horvitz-Thompson estimator is given by

$$\mathrm{V}(t_{HT}^{(x)}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{x_i}{\pi_i}\frac{x_j}{\pi_j}\Delta_{ij}, \tag{49}$$

where we have $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, with $i, j = 1, \ldots, N$. We note that Δ_{ij}, $i, j = 1, \ldots, N$, is obtained by means of (34). Since we consider all logically possible samples whose size is equal to n we can also write

$$\mathrm{V}(t_{HT}^{(x)}) = -\frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{x_i}{\pi_i} - \frac{x_j}{\pi_j}\right)^2\Delta_{ij}, \tag{50}$$

where we have again $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, with $i, j = 1, \ldots, N$ (Yates and Grundy [1953]). This variance is estimated by the univariate Yates-Grundy estimator given by

$$\hat{\mathrm{V}}_{YG}(t_{HT}^{(x)}) = \frac{1}{2N^2}\sum_{i\in s'}\sum_{j\in s'}\left(\frac{x_i}{\pi_i} - \frac{x_j}{\pi_j}\right)^2\frac{\pi_i\pi_j - \pi_{ij}}{\pi_{ij}}, \tag{51}$$

where we have $\pi_{ij} > 0$ because we assume that the sampling design is measurable and $\pi_{ij} \leq \pi_i\pi_j$, with $i, j = 1, \ldots, N$. The same thing goes when we consider Y and Z. We have to note a very important point: the variance of S denoted by σ_S^2 coincides with the variance of the univariate Horvitz-Thompson estimator given by (50) when the absolute values of each deviation of x_i from x_j, with $i \neq j = 1, \ldots, N$, are multiples of N. In addition to this thing, the variance of S coincides with the variance of the univariate Horvitz-Thompson estimator given by (50) when the entropy H of the sampling design with fixed sample size is maximum (Tillé and Wilhelm [2017]), where we have

$$H = -\sum_{s'\in\mathcal{S}'} p(s')\log p(s'). \tag{52}$$

We note that H is maximum when we have

$$p(s'_1) = p(s'_2) = \ldots = p(s'_k), \tag{53}$$

with $\sum_{i=1}^{k} p(s'_i) = 1$. It does not turn out to be $p(s') = 0$ within this context. However, if we observe $p(s') = 0$ with regard to (52) then it turns out to be $[0 \log 0] = 0$ by convention. We therefore say that the weights of the univariate Horvitz-Thompson estimator are based on a coherent prevision of S. We have obtained a linear and quadratic metric by considering two univariate random quantities representing deviations. We have obtained the variance of S by using this metric. The same thing goes when we consider Y and Z. We have to note another very important point: by studying three different and independent attributes of each element of the population under consideration we do not jointly consider three variables but we jointly consider two variables at a time. This is because it is not appropriate to use a trilinear form when we deal with metric relationships. If we jointly study two attributes of each element of the population under consideration then we estimate the bivariate population mean by using the bivariate Horvitz-Thompson estimator. We write

$$t_{HT}^{(xy)} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{\pi_i} \delta(i; s') x_i \frac{1}{\pi_j} \delta(j; s') y_j \tag{54}$$

when we jointly consider X and Y, where all first-order inclusion probabilities are greater than zero. They are obtained by means of (17). We write

$$t_{HT}^{(xz)} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{\pi_i} \delta(i; s') x_i \frac{1}{\pi_j} \delta(j; s') z_j \tag{55}$$

when we jointly consider X and Z, where all first-order inclusion probabilities are greater than zero. They are obtained by means of (17). We write

$$t_{HT}^{(yz)} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{\pi_i} \delta(i; s') y_i \frac{1}{\pi_j} \delta(j; s') z_j \tag{56}$$

when we jointly consider Y and Z, where all first-order inclusion probabilities are greater than zero. They are obtained by means of (17). The bivariate Horvitz-Thompson estimator is obtained by multiplying two linear and homogeneous expressions. This means that what we have said concerning the weights of the univariate Horvitz-Thompson estimator does not change. The expected value of the bivariate Horvitz-Thompson estimator concerning X and Y is given by

$$\mathrm{E}[t_{HT}^{(xy)}] = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{\pi_i} \mathrm{E}[\delta(i; s')] x_i \frac{1}{\pi_j} \mathrm{E}[\delta(j; s')] y_j. \tag{57}$$

We observe that it turns out to be $\mathrm{E}[\delta(i;s')] = \pi_i$ as well as $\mathrm{E}[\delta(j;s')] = \pi_j$ for every $s' \in \mathcal{S}'$, $i, j = 1, \ldots, N$. It is therefore evident that (57) is equal to the population mean denoted by $\mu_{(xy)}$ for any vector $(x_1 \ x_2 \ \ldots \ x_N)^T \in \mathbb{R}^N$ and $(y_1 \ y_2 \ \ldots \ y_N)^T \in \mathbb{R}^N$, where we have

$$\mu_{(xy)} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \, y_j. \tag{58}$$

The same thing goes when we consider the expected value of the bivariate Horvitz-Thompson estimator concerning X and Z as well as the expected value of the bivariate Horvitz-Thompson estimator concerning Y and Z. We consider an auxiliary variable denoted by X' related to X when the values of X given by x_i, $i = 1, \ldots, N$, are unknown. We consider an auxiliary variable denoted by Y' related to Y when the values of Y given by y_i, $i = 1, \ldots, N$, are unknown. We consider an auxiliary variable denoted by Z' related to Z when the values of Z given by z_i, $i = 1, \ldots, N$, are unknown. The known values of X' are given by x'_i, $i = 1, \ldots, N$. We write

$$\mu_{x'} = \frac{1}{N} \sum_{i=1}^{N} x'_i. \tag{59}$$

If X and X' are approximately proportional then it turns out to be

$$\frac{x_i}{x'_i} \approx \text{constant}, \tag{60}$$

where we have $i = 1, \ldots, N$. The first-order inclusion probabilities chosen by the statistician are then given by

$$\pi_i = \frac{n x'_i}{N \mu_{x'}}, \tag{61}$$

where we have $i = 1, \ldots, N$. We note that such probabilities are used into (23) in order to obtain $p(s'_i)$, $i = 1, \ldots, k$, when we have $k = N$. We observe that $p(s'_i)$, $i = 1, \ldots, k$, are used in order to obtain a coherent prevision of S. If we have $k \neq N$ then we consider a system of N linear equations with k unknowns, where $\pi_1, \ldots, \pi_N$ are constant terms. We evidently refer to (21). We therefore observe that $\pi_1, \ldots, \pi_N$ represent a coherent prevision of S obtained beginning from $p(s'_i)$, $i = 1, \ldots, k$. We observe that α-products and α-norms use $p(s'_i)$, $i = 1, \ldots, k$, as scalars. Also the second-order inclusion probabilities characterize our metric structure. They are obtained by means of tensor products having $p(s'_i)$, $i = 1, \ldots, k$, as scalars. They are chosen by the statistician because he subjectively chooses $p(s'_i)$, $i = 1, \ldots, k$. He is consequently able to observe $\pi_{ij} > 0$, $i, j = 1, \ldots, N$. We have established them in (27). The same thing goes when we consider Y' and Z'.

12 A metric homoscedasticity of different variables identifying different and independent attributes of the units of the population

We have jointly to consider two variables at a time for a metric reason. When we jointly consider X and Y we have firstly to disaggregate $t_{HT}^{(xy)}$. Given

$$t_{HT}^{(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\pi_i}\delta(i;s')x_i \tag{62}$$

and

$$t_{HT}^{(y)} = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\pi_j}\delta(j;s')y_j, \tag{63}$$

the covariance of these two univariate Horvitz-Thompson estimators is therefore expressed by

$$\mathrm{C}(t_{HT}^{(x)},\, t_{HT}^{(y)}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{x_i}{\pi_i}\frac{y_j}{\pi_j}\Delta_{ij}, \tag{64}$$

where we have $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, with $i, j = 1, \ldots, N$. We note that Δ_{ij}, $i, j = 1, \ldots, N$, is obtained by means of (34). The same thing goes when we jointly consider X and Z as well as Y and Z. We note that

$$\mathrm{C}(t_{HT}^{(x)},\, t_{HT}^{(x)}) = \mathrm{V}(t_{HT}^{(x)}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{x_i}{\pi_i}\frac{x_j}{\pi_j}\Delta_{ij}, \tag{65}$$

where we have $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, $i, j = 1, \ldots, N$. We observe that Δ_{ij}, $i, j = 1, \ldots, N$, is obtained by means of (34). We note that

$$\mathrm{C}(t_{HT}^{(y)},\, t_{HT}^{(y)}) = \mathrm{V}(t_{HT}^{(y)}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}\Delta_{ij}, \tag{66}$$

where we have $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, with $i, j = 1, \ldots, N$. We observe that Δ_{ij}, $i, j = 1, \ldots, N$, is obtained by means of (34). It is also possible to write

$$\mathrm{C}(t_{HT}^{(z)},\, t_{HT}^{(z)}) = \mathrm{V}(t_{HT}^{(z)}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{z_i}{\pi_i}\frac{z_j}{\pi_j}\Delta_{ij}, \tag{67}$$

where we have $\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$, with $i, j = 1, \ldots, N$. We observe that Δ_{ij}, $i, j = 1, \ldots, N$, is obtained by means of (34). We are interested in knowing

what happens from a metric point of view when we study three different and independent attributes with respect to each element of the population under consideration. We have defined S, D and V. In particular, we consider a bivariate random quantity representing deviations. It is expressed by $D_{12} = \{{}_1D, {}_2D\}$. Its components are two univariate random quantities, ${}_1D$ and ${}_2D$, identifying two sets of N-dimensional vectors. Each vector of a set of N-dimensional vectors is equal to the corresponding vector of the other set of N-dimensional vectors. We have consequently $I({}_1D) = I({}_2D) = \{d'_1, \ldots, d'_k\}$. Given $p(s'_i)$, $i = 1, \ldots, k$, we observe that ${}_1D$ is equal to ${}_2D$, so the covariance of ${}_1D$ and ${}_2D$ is equal to the variance of S denoted by σ^2_S. We observe this thing regardless of any pair of variables that we consider. We could indifferently consider X and Y or X and Z or Y and Z. On the other hand, if we take ${}_1V$ and ${}_2V$ into account then we note that their covariance is equal to 1. Since it turns out to be ${}_1V = {}_2V = V$ we say that the variance of V is equal to 1. We observe this thing regardless of any pair of variables that we consider. We could indifferently consider X and Y or X and Z or Y and Z. We therefore say that X, Y and Z are homoscedastic from a metric point of view. We say this thing after considering all logically possible samples having a given size belonging to $\mathcal{S}'$. We say this thing after defining S with respect to X, Y, Z. We say this thing because, given $p(s'_i)$, $i = 1, \ldots, k$, the variance of S is always the same. It is obtained by virtue of the metric structure that we have introduced.

13 What is all this for?

All the first-order inclusion probabilities derive from a coherent prevision of S. A coherent prevision of S always depends on $p(s'_i)$, $i = 1, \ldots, k$, where these probabilities are coherently chosen by the statistician. All the second-order inclusion probabilities derive from a coherent prevision of S_{12}. A coherent prevision of S_{12} always depends on $p(s'_i)$, $i = 1, \ldots, k$. A coherent prevision of S is linear and homogeneous. A coherent prevision of S_{12} is bilinear and homogeneous. The bivariate Horvitz-Thompson estimator is obtained by multiplying two linear and homogeneous expressions. This means that what we are going to say concerning the weights of the univariate Horvitz-Thompson estimator continues to be valid even when we make reference to the bivariate Horvitz-Thompson estimator. We therefore make reference to the first-order inclusion probabilities. If there exists a direct linear relationship between X' and X then the statistician chooses high inclusion probabilities denoted by π_i with respect to the units of the population under consideration having high attributes of X' denoted by x'_i, $i = 1, \ldots, N$. This is because they are likely associated with high attributes of X denoted by x_i, $i = 1, \ldots, N$. The same thing goes when we consider a direct linear relationship

between Y' and Y as well as between Z' and Z. If X and X' are approximately proportional then the first-order inclusion probabilities chosen by the statistician are given by

$$\pi_i = \frac{nx'_i}{\sum_{j=1}^{N} x'_j}, \tag{68}$$

where we have $i = 1, \ldots, N$. If it turns out to be $\pi_i > 1$ for some unit of the population under consideration then we have $\pi_i = 1$ for all units of the population under consideration having i as a label and such that it turns out to be $nx'_i \geq \sum_{j=1}^{N} x'_j$ because x'_i is high. We consider $n > 1$ within this context. The statistician consequently chooses

$$\pi_i = (n - n_A)\frac{x'_i}{\sum_{\substack{j=1 \\ j \notin A}}^{N} x'_j}, \tag{69}$$

where we have $i = 1, \ldots, N$, $i \notin A$, concerning the remaining units of the population under consideration. The set of the units of the population under consideration such that it turns out to be $nx'_i \geq \sum_{j=1}^{N} x'_j$ is denoted by A, while their number is denoted by n_A. The same thing goes when we consider Y' and Y as well as Z' and Z. Having said that, we evidently establish a linear relationship between $p(s'_i)$, $i = 1, \ldots, k$, and π_i, $i = 1, \ldots, N$. If the statistician chooses $p(s'_i)$, $i = 1, \ldots, k$, with $\sum_{i=1}^{k} p(s'_i) = 1$, then it is possible to get π_i, $i = 1, \ldots, N$, with $\sum_{i=1}^{N} \pi_i = n$. We write

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} = \sum_{i=1}^{k} \delta(s'_i)p(s'_i). \tag{70}$$

He is consequently able to obtain $\pi_i > 0$ for every $i = 1, \ldots, N$. Conversely, if the statistician chooses π_i, $i = 1, \ldots, N$, then it is possible to get $p(s'_i)$, $i = 1, \ldots, k$. We observe that α-products and α-norms use $p(s'_i)$, $i = 1, \ldots, k$, as scalars. We obtain different metric relationships by using α-norms whose scalars are $p(s'_i)$, $i = 1, \ldots, k$. We note that $\pi_1, \ldots, \pi_N$ are used into

$$B^{-1}\mathbf{P}(S) = \begin{bmatrix} p(s'_1) \\ p(s'_2) \\ \vdots \\ p(s'_k) \end{bmatrix} \tag{71}$$

in order to obtain $p(s'_i)$, $i = 1, \ldots, k$, when we have $k = N$. We note that B is a square matrix, while B^{-1} is its inverse. If we have $k \neq N$ then we consider

a system of N linear equations with k unknowns, where $\pi_1, \ldots, \pi_N$ are constant terms. We evidently refer to

$$L_B(Q) = B \begin{bmatrix} p(s'_1) \\ p(s'_2) \\ \vdots \\ p(s'_k) \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{bmatrix} = \mathbf{P}(S). \tag{72}$$

It is known that if the statistician chooses appropriate inclusion probabilities then he is able to obtain a more efficient estimator of the population mean.

14 Conclusions

We have considered random quantities whose logically possible values are all logically possible samples of a given size belonging to a given set. Every logically possible sample belonging to a given set has a subjective probability of being selected. We have obtained the first-order inclusion probabilities by means of coherent previsions of univariate random quantities. We have defined bivariate random quantities whose components are two univariate random quantities having all logically possible samples of a given size as their logically possible values. All univariate random quantities which we have defined are complementary to the univariate Horvitz-Thompson estimator. It is linear and homogeneous like a coherent prevision of a univariate random quantity whose logically possible values are all logically possible samples of a given size belonging to a given set. A univariate random quantity representing deviations as well as a univariate random quantity representing variations are defined on the basis of a coherent prevision of a given univariate random quantity. These random quantities are the same quantity from a randomness point of view. We have identified a quadratic and linear metric with regard to two univariate random quantities representing deviations. We have used the α-criterion of concordance introduced by Gini in order to identify it.

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Analytical and numerical solution of differential equations with generalized fuzzy derivative

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Abstract

The aim of this work is to present a novel approach based on the fuzzy neural network for finding the numerical solution of the first order fuzzy differential equations under generalized H-derivation.The differentiability concept that used in this paper is the generalized differentiability since a first order fuzzy differential equation under this differentiability can have two solutions.The fuzzy trial solution of the fuzzy initial value problem is written as a sum of two parts. The first part satisfies the fuzzy condition, it contains no fuzzy adjustable parameters. The second part involves fuzzy feed-forward neural networks containing fuzzy adjustable parameters. This method, in comparison with existing numerical methods and the analytical solutions, shows that the use of fuzzy neural networks provides solutions with good generalization and high accuracy.

Keywords: Fuzzy differential equations; generalized H-derivation; fuzzy neural network; fuzzy trial solution; error function.

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Received on December 10th, 2019. Accepted on March 3rd, 2020. Published on June 30th, 2020. doi: 10.23755/rm.v38i0.519. ISSN: 1592-7415. eISSN: 2282-8214. ©Basim Nasih Abood. This paper is published under the CC-BY licence agreement.

1 Introduction

Nowadays, fuzzy differential equations (FDEs) is a popular topic studied by many researchers since it is utilized widely for the purpose of modeling problems in science and engineering. Most of the practical problems require the solution of a FDE which satisfies fuzzy initial or fuzzy boundary conditions, therefore, a fuzzy initial or fuzzy boundary problem should be solved. However, many fuzzy initial or fuzzy boundary value problems could not be solved exactly, sometimes it is even impossible to find their analytical solutions. Thus, considering their approximate solutions is becoming more important [1].

The theory of FDE was first formulated by Kaleva and Seikkala. Kaleva had formulated FDE in terms of the Hukuhara derivative (H-derivative). Buckley and feuring have given a very general formulation of a first-order fuzzy initial value problem. They first find the crisp solution, make it fuzzy and then check if it satisfies the fuzzy differential equation [2].

In recent years artificial neural network (ANN) for estimation of the ordinary differential equation (ODE) and partial differential equation (PDE) has been used. We briefly review some articles in the literature concerning the differential equations. Lee and Kang in [3] used parallel processor computers to solve a first order differential equation with Hopfield neural network models. Meade and Fernandez in [4,5] solved linear and non-linear ODEs by using feed-forward neural networks (FFNN) architecture and B-splines of degree one. Lagaris and Likas in [6,7] used ANN for solving ODEs and PDEs with the initial / boundary value problems. Liu and Jammes in [8] developed some properties of the trial solution to solve the ODEs by using ANN. Ali and Ucar in [9] solved the vibration control problems by using ANN. Tawfiq in [10] presented and developed supervised and unsupervised algorithms for solving ODE and PDE. Malek and shekari in [11] presented numerical method based on ANN and optimization techniques which the higher-order ODE answers approximates by finding a package form analytical of specific functions. Pattanaik and Mishra in [12] applied and developed some properties of ANN for solution of PDE in RF Engineering**.** Baymani and Kerayechian in [13] proposed ANN approach for solving stokes problems. Oraibi in [14] designed FFNN for solving ordinary initial value problem. Ali in [15]

designed fast FFNN to solve two-point boundary value problems. Hussein in [16] designed fast FFNN to solve singular boundary value problems. Tawfiq and Al-Abrahemee in [17] designed ANN to solve singular perturbation problems, and other researchers.

Numerical solution of FDE by using ANN is the subject of a very modern because it only goes back to 2010. Effati and pakdaman in [18] used ANN for solving FDE, they used for the first time the ANN to approximate fuzzy initial value problems. Mosleh and Otadi in [19] used ANN for solving fuzzy Fredholm integro-differential equations. Ezadi and Parandin in [20] used ANN based on semi-Taylor series to solve first order FDE.

Numerical solution of FDE by using fuzzy artificial neural network (FANN) is more modern than the previous subject, where it goes back to 2012. Mosleh and Otadi in [21] used FANN for solving first order FDE, they used for the first time FANN to approximate fuzzy initial value problems. Mosleh in [22] used FANN for solving a system of FDE. Mosleh and Otadi in [23] used FANN for solving second order FDE. Suhhiem in [24] developed and used FANN for solving fuzzy and non-fuzzy differential equations.

In 2008, the concept of the generalized Hukuhara – differentiability is studied by Chalco-Cano and Roman Flores [25,26] to solve FDE.

In this work**,** for solving FDE Under Generalized H – Derivation, we present modified method which relies on the function approximation capabilities of fuzzy FFNN and results in the construction of a solution written in a differentiable, closed analytic form. This form employs fuzzy FFNN as the basic approximation element, whose fuzzy parameters (weights and biases) are adjusted to minimize an appropriate error function. To train the FANN which we design, we employ optimization techniques, which in turn require the computation of the gradient of the error with respect to the network parameters. In this proposed approach the model function is expressed as the sum of the two terms: the first term satisfies the fuzzy initial / fuzzy boundary conditions and contains no fuzzy adjustable parameters. The second term can be found by using fuzzy FFNN, which is trained so as to satisfy the FDE.

2 Basic Definitions

2.1 Fuzzy Concepts

In this subsection, the basic notations which are used in fuzzy calculus are introduced.

Definition (2.1.1), [2] : The r - level (or r - cut) set of a fuzzy set $\widetilde{A}$ labeled by A_r, is the crisp set of all x in X (universal set) such that : $\mu_{\widetilde{A}}(x) \geq r$; i. e.

$$A_r = \{x \in X : \mu_{\widetilde{A}}(x) \geq r, r \in [0,1] \}. \quad (1)$$

Definition(2.1.2),[2]: Extension Principle

Let X be the Cartesian product of universes $X_1, X_2, \ldots, X_m$ and $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m$ be m - fuzzy subset in $X_1, X_2, \ldots, X_m$ respectively, with Cartesian product $\widetilde{A} = \widetilde{A}_1 \times \widetilde{A}_2 \times \ldots \times \widetilde{A}_m$ and f is a function from X to a universe Y, ($y = f(x_1, x_2, \ldots, x_m)$). Then, the extension principle allows to define a fuzzy subset $\widetilde{B} = f(\widetilde{A})$ in Y by $\widetilde{B} = \{(y, \mu_{\widetilde{B}}(y)) : y = f(x_1, x_2, \ldots, x_m), (x_1, x_2, \ldots, x_m) \in X\}$, where

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_m)\in f^{-1}(y)} \operatorname{Min}\left\{\mu_{\widetilde{A}_1}(x_1),\ldots, \mu_{\widetilde{A}_m}(x_m)\right\}, & \text{if } f^{-1}(y)\neq\emptyset \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

and f^{-1} is the inverse image of f.

For m = 1, the extension principle will be :

$\widetilde{B} = f(\widetilde{A}) = \{\left(y, \mu_{\widetilde{B}}(y)\right) : y = f(x), x \in X\}$, where

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\widetilde{A}}(x), & \text{if } f^{-1}(y)\neq\emptyset \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Definition (2.1.3), [24]: Fuzzy Number

A fuzzy number $\tilde{u}$ is completely determined by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \le r \le 1$, which satisfy the following requirements:

1) $\underline{u}(r)$ is a bounded left continuous and non-decreasing function on $[0,1]$.

2) $\overline{u}(r)$ is a bounded left continuous and non-increasing function on $[0,1]$.

3) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

The crisp number a is simply represented by:

$\underline{u}(r) = \overline{u}(r) = a$, $0 \le r \le 1$.

The set of all the fuzzy numbers is denoted by E^1.

Remark (1), [24]: For arbitrary $\tilde{u} = \left(\underline{u}, \overline{u}\right)$, $\tilde{v} = \left(\underline{v}, \overline{v}\right)$ and $K \in R$, the addition and multiplication by K can be defined as :

1) $\underline{(u+v)}\ (r) = \underline{u}(r) + \underline{v}(r)$ (4)

2) $\overline{(u+v)}\ (r) = \overline{u}(r) + \overline{v}(r)$ (5)

3) $\underline{(Ku)}\ (r) = K\,\underline{u}(r)$, $\overline{(Ku)}\ (r) = K\,\overline{u}(r)$, if $K \ge 0$ (6)

4) $\underline{(Ku)}\ (r) = K\,\overline{u}(r)$, $\overline{(Ku)}\ (r) = K\,\underline{u}(r)$, if $K < 0$. (7)

For all $r \in [0,1]$.

Remark (2), [1]:

The distance between two arbitrary fuzzy numbers $\tilde{u} = \left(\underline{u}, \overline{u}\right)$ and $\tilde{v} = \left(\underline{v}, \overline{v}\right)$ is given as:

$$D(\tilde{u}, \tilde{v}) = \left[\int_0^1 \left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 \left(\overline{u}(r) - \overline{v}(r)\right)^2 dr\right]^{\frac{1}{2}} \quad (8)$$

Remark (3), [1]: (E^1, D) is a complete metric space.

Remark (4), [2]: The operations of fuzzy numbers (in parametric form) can be generalized from that of crisp intervals. Let us have a look at the operations of intervals. $\forall\ a_1, b_1, a_2, b_2 \in R$, $A = [a_1, b_1]$ and $B = [a_2, b_2]$.

Assuming A and B numbers expressed as interval, main operations of intervals are :

1) Addition: $A + B = [a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$.

2) Subtraction: $A - B = [a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$.

3) Multiplication:

$A.\ B = [\min\{a_1 a_2, a_1 b_2, b_1\ a_2, b_1 b_2\}, \max\{a_1 a_2, a_1 b_2, b_1\ a_2, b_1 b_2\}]$

4) Division : $A/B = [\min\{a_1 / a_2, a_1 / b_2, b_1 / a_2, b_1 / b_2\}, \max\{a_1 / a_2, a_1 / b_2, b_1 / a_2, b_1 / b_2\}]$ excluding the case $a_2 = 0$ or $b_2 = 0$.

5) Inverse : $A^{-1} = [a_1, b_1]^{-1} = \left[\min\left\{\frac{1}{a_1}, \frac{1}{b_1}\right\}, \max\left\{\frac{1}{a_1}, \frac{1}{b_1}\right\}\right]$

excluding the case $a_1 = 0$ or $b_1 = 0$.

In the case of $0 \leq a_2 \leq b_2$, multiplication operation can be simplified as:

$A.\ B = [\min\{a_1 a_2, a_1 b_2\}, \max\{b_1\ a_2, b_1 b_2\}]$

when previous sets A and B is defined in the positive real number R^+, the operations of multiplication, division and inverse are written as :

3′) Multiplication: $A.\ B = [a_1, b_1].\ [a_2, b_2] = [a_1 a_2, b_1 b_2]$

4′) Division: $A / B = [a_1, b_1] / [a_2, b_2] = \left[\frac{a_1}{b_2}, \frac{b_1}{a_2}\right]$.

5′) Inverse: $A^{-1} = [a_1, b_1]^{-1} = \left[\frac{1}{b_1}, \frac{1}{a_1}\right]$.

Definition (2.1.4), [24] : Triangular Fuzzy Number

Among the various shapes of fuzzy numbers, triangular fuzzy numbers is the most popular one. A triangular fuzzy number is a fuzzy number represented with three points as follows: $\widetilde{A} = (a_1, a_2, a_3)$, where $a_1 \leq a_2 \leq a_3$.

This representation is interpreted as membership functions:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & , \text{if} \quad x < a_1 \\ \frac{x - a_1}{a_2 - a_1} & , \text{if} \quad a_1 \le x \le a_2 \\ \frac{a_3 - x}{a_3 - a_2} & , \text{if} \quad a_2 \le x \le a_3 \\ 0 & , \text{if} \quad x > a_3 \end{cases} \tag{9}$$

Now if you get crisp interval by r - cut operation, interval $[A]_r$ shall be obtained as follows $\forall\, r \in [0,1]$ from: $\frac{\underline{A} - a_1}{a_2 - a_1} = r$, $\frac{a_3 - \overline{A}}{a_3 - a_2} = r$,

We get: $\underline{A} = (a_2 - a_1)r + a_1$, $\overline{A} = (a_2 - a_3)\, r + a_3$.

Thus: $[A]_r = \left[\underline{A}, \overline{A}\right] = [(a_2 - a_1)r + a_1,\ (a_2 - a_3)\, r + a_3]$ (10)

which is the parametric form of triangular fuzzy number $\widetilde{A}$.

Definition $(2.1.5)$, $[18]$: Fuzzy Function

A classical function $F : X \longrightarrow Y$ maps from a fuzzy domain $\widetilde{A} \subseteq X$ into a fuzzy range $\widetilde{B} \subseteq Y$ if and only if $\forall\, x \in X$, $\mu_{\widetilde{B}}(F(x)) \ge \mu_{\widetilde{A}}(x)$.

Remark (5) , $[18]$:

(1) The function $F : R \longrightarrow E^1$ is called a fuzzy function .

(2) We call every function defined in set $\widetilde{A} \subseteq E^1$ to $\widetilde{B} \subseteq E^1$ a fuzzy function.

Definition $(2.1.6)$, $[18]$: The fuzzy function $F : R \longrightarrow E^1$ is said to be continuous if :

For an arbitrary $t_1 \in R$ and $\epsilon > 0$ there exists a $\delta > 0$ such that:

$|t - t_1| < \delta \Rightarrow D(F(t), F(t_1)) < \epsilon$, where D is the distance between two fuzzy numbers.

Definition (2.1.7),[25]: Let I be a real interval. The r-level set of the fuzzy function $y : I \to E^1$ can be denoted by :

$$[y(t)]^r = [y_1^r(t), y_2^r(t)] \quad t \in I \tag{11}$$

The Seikkala derivative $y'(t)$ of the fuzzy function $y(t)$ is defined by:

$$[y'(t)]^r = [(y_1^r)'(t), (y_2^r)'(t)] \quad t \in I \tag{12}$$

2.2 **H – Differentiability**

In this subsection, the basic definitions which are used in H – differentiability are introduced.

Definition (2.2.1), [**18**]**:** let u, v ∈ E^1 · If there exist w ∈ E^1 such that

u = v+w then w is called the H-difference (Hukuhara-difference) of u, v and it is denoted by w= u ⊖ v.

In this work the ⊖ sign stands always for H-difference, and let us remark that u ⊖ v ≠ u + (-1) v.

Definition (2.2.2), [**24**]

Let F : (a,b) → E^1 and t_0 ∈ (a,b).We say that F is H-differential (Hukuhara-differential) at t_0, if there exists an element $F'(t_0) \in E^1$ such that for all h> 0 (sufficiently small), ∃ F (t_0 +h)⊖$F(t_0)$, $F(t_0)$ ⊖ F (t_0 - h) and the limits (in the metric D)

$$\lim_{h\to 0} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0) \tag{13}$$

Then $F'(t_0)$ is called fuzzy derivative (H-derivative) of F at t_0.

where D is the distance between two fuzzy numbers.

It is necessary to note that the definition (2.2.2) is the classical definition of the H-derivative (or differentiability in the sense of Hukuhara).

Definition (2.2.3), [**25**, **26**]**:**

Let $F : T \to E^1$ and $t_0 \in T \subseteq R$. F is differentiable at t_0, if

(1) there exist an element $F'(t_0) \in E^1$, such that for all $h > 0$ sufficiently small, there are $F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0) \tag{14}$$

(in this case, F is called (1)-differentiable)

or

(2) there exist an element $F'(t_0) \in E^1$, such that for all $h > 0$ sufficiently small, there are $F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h\to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0) \qquad (15)$$

(in this case, F is called (2)-differentiable)

Where the relation (1) is the classical definition of the H-derivative.

Theorem (1) : Let $F : I \to E^1$ be a function and denote $[F(t)]^r = [f_r(t), g_r(t)]$, for each $r \in [0,1]$. Then

(i) If F is differentiable in the first form **(1)** of definition (2.2.3.), then f_r and g_r are differentiable functions and

$$[F'(t)]^r = [f_r'(t), g_r'(t)]$$

(ii) If F is differentiable in the second form **(2)** of definition (2.2.3), then f_r and g_r are differentiable functions and

$$[F'(t)]^r = [g_r'(t), f_r'(t)]$$

Proof: see [25] □

3 Fuzzy Neural Network [24]

A fuzzy neural network (FNN) or neuro – fuzzy system is a learning machine that finds the parameters of a fuzzy system (i.e., fuzzy set, fuzzy rules) by exploiting approximation techniques from neural networks. Combining fuzzy systems with neural networks. Both neural networks and fuzzy systems have some things in common. They can be used for solving a problem (e. g. fuzzy differential equations, fuzzy integral equations, etc.).

Before 2005 FNN called fuzzy weight neural networks was developed by Pabisek, Jakubek and et al. Membership functions of FWNN were formulated by the multi – layered perceptron (MLP) network training, separately for each learning pattern and then the interval arithmetic was applied to process crisp or fuzzy data.

3.1 Input – Output Relations of Each Unit [21,22]

Let us consider a three-layer fuzzy FFNN with n input units, m hidden units and s output units. Target vector, connection weights and biases are fuzzy numbers and input vector is real numbers. For convenience in this discussion, FNN with an input layer, a single hidden layer, and an output layer in Fig. (1) is represented as a basic structural architecture. Here, the dimension of FNN is denoted by the number of neurons in each layer, that is $n \times m \times s$, where n, m and s are the number of the neurons in the input layer, the hidden layer and the output layer, respectively.

The architecture of the model shows how the FNN transforms the n inputs $(x_1, x_2, \dots, x_i, \dots, x_n)$ into the s fuzzy outputs $([y_1]_r, [y_2]_r, \dots, [y_k]_r, \dots, [y_s]_r)$ throughout the m hidden fuzzy neurons $\left([z_1]_r, [z_2]_r, \dots [z_j]_r, \dots [z_m]_r\right)$, where the cycles represent the neurons in each layer. Let $[b_j]_r$ be the fuzzy bias for the fuzzy neuron $[z_j]_r$, $[c_k]_r$ be the fuzzy bias for the fuzzy neuron $[y_k]_r$, $[w_{ji}]_r$ be the fuzzy weight connecting crisp neuron x_i to fuzzy neuron $[z_j]_r$, and $[w_{kj}]_r$ be the fuzzy weight connecting fuzzy neuron $[z_j]_r$ to fuzzy neuron $[y_k]_r$.

When an n – dimensional input vector $(x_1, x_2, \dots, x_i, \dots, x_n)$ is presented to our FNN, its input – output relations can be written as follows, where $H : R^n \longrightarrow E^s$:

Input units: $o_i = x_i$, i = 1,2,3, …n (16)

Hidden units: $[z_j]_r = H\left([net_j]_r\right)$, j = 1,2,3, …,m, (17)

where

$[net_j]_r = \sum_{i=1}^{n} o_i\, [w_{ji}]_r + [b_j]_r$ (18)

Output units:

$$[y_k]_r = H([net_k]_r),\ k = 1,2,3,\ldots,s, \tag{19}$$

where

$$[net_k]_r = \sum_{j=1}^{m} [w_{kj}]_r\,[z_j]_r + [c_k]_r \tag{20}$$

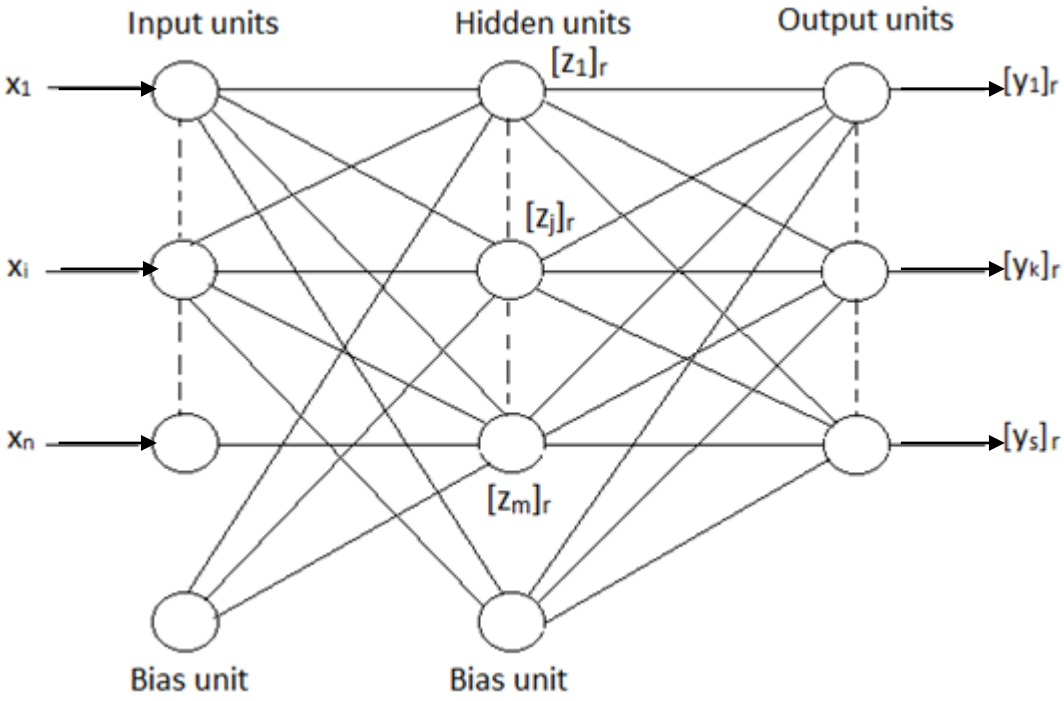

Fig. (1) Three-layer feed forward fuzzy neural network.

The architecture of our fuzzy neural network is shown in Fig. (1), where connection weights, biases, and targets are fuzzy numbers and inputs are real numbers.

From the operations of fuzzy numbers (which we have described in section two), the above relations are rewritten as follows when the inputs x_i´s are non – negative, i.e., $x_i \geq 0$:

Input units: $o_i = x_i$ (21)

Hidden units:

$$[z_j]_r = H\left([net_j]_r\right) = \left[[z_j]_r^L,\ [z_j]_r^U\right] = \left[H\left([net_j]_r^L\right), H\left([net_j]_r^U\right)\right] \quad (22)$$ where

$$[net_j]_r^L = \sum_{i=1}^{n} o_i\,[w_{ji}]_r^L + [b_j]_r^L \quad (23)$$

$$[net_j]_r^U = \sum_{i=1}^{n} o_i\,[w_{ji}]_r^U + [b_j]_r^U \quad (24)$$

Output units:

$$[y_k]_r = H\left([net_k]_r\right) = [[y_k]_r^L,\ [y_k]_r^U] = [H([net_k]_r^L), H([net_k]_r^U)] \quad (25)$$

where

$$[net_k]_r^L = \sum_{j\in a}[w_{kj}]_r^L\ [z_j]_r^L + \sum_{j\in b}[w_{kj}]_r^L\ [z_j]_r^U + [c_k]_r^L \quad (26)$$

$$[net_k]_r^U = \sum_{j\in c}[w_{kj}]_r^U\ [z_j]_r^U + \sum_{j\in d}[w_{kj}]_r^U\ [z_j]_r^L + [c_k]_r^U \quad (27)$$

For $[z_j]_r^U \geq [z_j]_r^L \geq 0$,

where

$$a = \left\{j :\ [w_{kj}]_r^L \geq 0\right\},\ b = \left\{j :\ [w_{kj}]_r^L < 0\right\}$$

$$c = \left\{j :\ [w_{kj}]_r^U \geq 0\right\},\ d = \left\{j :\ [w_{kj}]_r^U < 0\right\},$$

$a \cup b = \{1,2,3,\ldots,m\}$ and $c \cup d = \{1,2,3,\ldots,m\}$.

4 Technique of The Proposed Method

4.1 First Order Fuzzy Differential Equation

To solve any fuzzy ordinary differential equation we consider a three – layered fuzzy FFNN with one unit entry x, one hidden layer consisting of m activation functions and one unit output $N(x,p)$. The activation function for the hidden units of our FNN is the hyperbolic tangent function. Here, the dimension of FNNM is $(1 \times m \times 1)$.

For every entry x (where $x \geq 0$) equations (21-27) will be :

Input unit: o = x, (28)

Hidden units :

$$[z_j]_r = \left[[z_j]_r^L,\ [z_j]_r^U\right] = \left[s\left([net_j]_r^L\right), s\left([net_j]_r^U\right)\right] \quad (29)$$

where

$$[net_j]_r^L = o\,[w_j]_r^L + [b_j]_r^L \quad (30)$$

$$[net_j]_r^U = o\,[w_j]_r^U + [b_j]_r^U \quad (31)$$

Output unit:

$$[N]_r = [[N]_r^L,\ [N]_r^U] \quad (32)$$

where

$$[N]_r^L = \sum_{j\in a}[v_j]_r^L\,[z_j]_r^L + \sum_{j\in b}[v_j]_r^L\,[z_j]_r^U \quad (33)$$

$$[N]_r^U = \sum_{j\in c}[v_j]_r^U\,[z_j]_r^U + \sum_{j\in d}[v_j]_r^U\,[z_j]_r^L \quad (34)$$

For illustration the solution steps of our proposed method, we will consider the first order fuzzy differential equation :

$$\frac{dy(x)}{dx} = F(x,y) \quad ,\ x \in [a,b]\ ,\ y\ (a) = A \quad (35)$$

where A is a fuzzy number in E^1 with r – level sets:

$$[A]_r = [[A]_r^L,\ [A]_r^U]\ ,\ r \in [0,1].$$

The fuzzy trial solution for this problem is:

$$[y_t(x,p)]_r^L = [A]_r^L + (x-a)[N(x,p)]_r^U$$

$$[y_t(x,p)]_r^U = [A]_r^U + (x-a)[N(x,p)]_r^L \quad (36)$$

This fuzzy solution by intention satisfies the fuzzy initial condition in (35).

In this work we use the error function : $E = \sum_{i=1}^{g}\left(E_{ir}^{L} + E_{ir}^{U}\right)$, where E_{ir}^{L} and E_{ir}^{U} can be viewed as the squared errors for the lower limits and the upper limits of the r-level sets, respectively.

Therefore, The error function that must be minimized for the problem (35) is in the form [23] :

$$E= \sum_{i=1}^{g}\left(E_{ir}^{L} + E_{ir}^{U}\right) \tag{37}$$

where

$$E_{ir}^{L} = \left[\left[\frac{d\, y_t\,(x_i,p)}{dx}\right]_r^{L} - \left[F\left(x_i, y_t\,(x_i,p)\right)\right]_r^{L} \right]^2$$

$$E_{ir}^{U} = \left[\left[\frac{d\, y_t\,(x_i,p)}{dx}\right]_r^{U} - \left[F\left(x_i, y_t\,(x_i,p)\right)\right]_r^{L} \right]^2 \tag{38}$$

where $\{x_i\}_{i=1}^{g}$ are discrete points belonging to the interval $[a, b]$ (training set) and in the cost function (37), E_r^L and E_r^U can be viewed as the squared errors for the lower limits and the upper limits of the r – level sets, respectively.

It is easy to express the first derivative of $[N(x,p)]_r^U$ and $[N(x,p)]_r^L$ in terms of the derivative of the hyperbolic tangent activation function, i.e.,

$$\frac{\partial\,[N]_r^L}{\partial x} = \sum_a \left[v_j\right]_r^L \frac{\partial\,[z_j]_r^L}{\partial\,[net_j]_r^L}\ \frac{\partial\,[net_j]_r^L}{\partial x} + \sum_b \left[v_j\right]_r^L \frac{\partial\,[z_j]_r^U}{\partial\,[net_j]_r^U}\ \frac{\partial\,[net_j]_r^U}{\partial x} \tag{39}$$

$$\frac{\partial\,[N]_r^U}{\partial x} = \sum_c \left[v_j\right]_r^U \frac{\partial\,[z_j]_r^U}{\partial\,[net_j]_r^U}\ \frac{\partial\,[net_j]_r^U}{\partial x} + \sum_d \left[v_j\right]_r^U \frac{\partial\,[z_j]_r^L}{\partial\,[net_j]_r^L}\ \frac{\partial\,[net_j]_r^L}{\partial x} \tag{40}$$

From (29-31) we can get

$$\frac{\partial [net_j]_r^L}{\partial x} = \left[w_j\right]_r^L \tag{41}$$

$$\frac{\partial [z_j]_r^L}{\partial\,[net_j]_r^L} = 1 - \left(\left[z_j\right]_r^L\right)^2 \tag{42}$$

$$\frac{\partial [net_j]_r^U}{\partial x} = \left[w_j\right]_r^U \tag{43}$$

$$\frac{\partial \left[z_j\right]_r^U}{\partial \left[net_j\right]_r^U} = 1 - \left(\left[z_j\right]_r^U\right)^2 \tag{44}$$

From (36) we can get

$$\frac{\partial [y_t\,(x,p)]_r^L}{\partial x} = [N\,(x,p)]_r^U + (x - a)\,\frac{\partial\,[N\,(x,p)]_r^U}{\partial x} \tag{45}$$

$$\frac{\partial [y_t\,(x,p)]_r^U}{\partial x} = [N\,(x,p)]_r^L + (x - a)\,\frac{\partial\,[N\,(x,p)]_r^L}{\partial x} \tag{46}$$

Then we have

$$E_{ir}^U = [\,\sum_a \left[v_j\right]_r^L \left[z_j\right]_r^L + \sum_b \left[v_j\right]_r^L \left[z_j\right]_r^U + (x_i - a)\,(\sum_a \left[v_j\right]_r^L \left[w_j\right]_r^L (1 - \left(\left[z_j\right]_r^L\right)^2) + \sum_b \left[v_j\right]_r^L \left[w_j\right]_r^U (1 - \left(\left[z_j\right]_r^U\right)^2)) - F(x_i, [A]_r^U + (x_i - a)\,(\sum_a \left[v_j\right]_r^L \left[z_j\right]_r^L + \sum_b \left[v_j\right]_r^L \left[z_j\right]_r^U))\,]^2 \tag{47}$$

$$E_{ir}^L = [\,\sum_c \left[v_j\right]_r^U \left[z_j\right]_r^U + \sum_d \left[v_j\right]_r^U \left[z_j\right]_r^L + (x_i - a)\,(\sum_c \left[v_j\right]_r^U \left[w_j\right]_r^U (1 - \left(\left[z_j\right]_r^U\right)^2) + \sum_d \left[v_j\right]_r^U \left[w_j\right]_r^L (1 - \left(\left[z_j\right]_r^L\right)^2)) - F(x_i, [A]_r^L + (x_i - a)\,(\sum_c \left[v_j\right]_r^U \left[z_j\right]_r^U + \sum_d \left[v_j\right]_r^U \left[z_j\right]_r^L))\,]^2 \tag{48}$$

Then we substitute (47) and (48) in (37) to find the error function that must be minimized for problem (35).

Note : For reducing the complexity of the learning algorithm in (37), we have used partially fuzzy neural network (PFNN) architecture where connection weights to the output unit are fuzzy numbers while connection weights and biases to the hidden units are real numbers .

The input – output relation of each unit of the PFNN can be rewritten for r – level sets as follows:

Input unit: $o = x$ (49)

Hidden units: $z_j = s\left(net_j\right)$, $j = 1,2,3, \ldots m$ (50)

where $net_j = o\, w_j + b_j$ (51)

Output unit:

$$[N]_r=[[N]_r^L,\ [N]_r^U]=\left[\sum_{j=1}^{m}[v_j]_r^L z_j\ ,\sum_{j=1}^{m}[v_j]_r^U z_j\ \right] \quad (52)$$

Now, to find the minimized error function for problem (35) :

$$\frac{\partial\,[N]_r^L}{\partial x}=\sum_{j=1}^{m}[v_j]_r^L\ \frac{\partial\,z_j}{\partial\,net_j}\ \frac{\partial\,net_j}{\partial x}=\sum_{j=1}^{m}[v_j]_r^L\ w_j\left(1-\ {z_j}^2\right) \quad (53)$$

$$\frac{\partial\,[N]_r^U}{\partial x}=\sum_{j=1}^{m}[v_j]_r^U\ \frac{\partial\,z_j}{\partial\,net_j}\ \frac{\partial\,net_j}{\partial x}=\sum_{j=1}^{m}[v_j]_r^U\ w_j\left(1-\ {z_j}^2\right) \quad (54)$$

Then we obtain :

$$E_{ir}^U=[\sum_{j=1}^{m}z_j\left[v_j\right]_r^L+(x_i-a)\ \sum_{j=1}^{m}\ w_j\,(1-\ {z_j}^2)\left[v_j\right]_r^L-F(x_i,[A]_r^U+(x_i-a)\ \sum_{j=1}^{m}z_j\ \left[v_j\right]_r^L)\ ]^2 \quad (55)$$

$$E_{ir}^L=[\sum_{j=1}^{m}z_j\left[v_j\right]_r^U+(x_i-a)\ \sum_{j=1}^{m}\ w_j\,(1-\ {z_j}^2)\left[v_j\right]_r^U-F(x_i,[A]_r^L+(x_i-a)\ \sum_{j=1}^{m}z_j\ \left[v_j\right]_r^U)\ ]^2 \quad (56)$$

Then we substitute (55) and (56) in (37) to find the error function that must be minimized for problem (35) under PFNN.

4.2 Reducing a FDE to a System of ODEs [24,25]

The solution of the fuzzy differential equation (35) is depend on the choice of the derivative (in the first form or in the second form of definition (10)).

Let us explain the proposed method, if we denote

$$[y(x)]^r=[y_1^r(x),y_2^r(x)],\qquad [y_0]^r=[y_{01}^r\,,\,y_{02}^r]$$

and

$$[F(x,y(x))]^r=\left[\ F_1^r\left(x,y_1^r(x),y_2^r(x)\right),F_2^r\left(x,y_1^r(x),y_2^r(x)\right)\ \right] \quad (57)$$

we have the following results:

Case I. If we consider $y'(x)$ in the first form **(1)** of definition (2.2.3), then we have to solve the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = F_1^r\left(x, y_1^r(x), y_2^r(x)\right) \qquad y_1^r(a) = y_{01}^r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = F_2^r\left(x, y_1^r(x), y_2^r(x)\right) \qquad y_2^r(a) = y_{02}^r$$

Case II. If we consider $y'(t)$ in the second form **(2)** of definition (2.2.3) then we have to solve the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = F_2^r\left(x, y_1^r(x), y_2^r(x)\right) \qquad y_1^r(a) = y_{01}^r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = F_1^r\left(x, y_1^r(x), y_2^r(x)\right) \qquad y_2^r(a) = y_{02}^r$$

The existence and uniqueness of the two solutions (for problem (35)) which described above are given by the following theorem

Theorem (2) : Let $F : I \times E^1 \rightarrow E^1$ be a continuous fuzzy function such that there exists $k > 0$ such that

$D\left(F(x,w), F(x,z)\right) \leq k\, D(w,z)$ for all $t \in I$ and $w, z \in E^1$ then the problem (35) has two solutions (one (1)-differentiable and the other one (2)-differentiable) on I, where $I = [a,b]$.

Proof: see [26] □

To illustrate how we can find the two solutions for a fuzzy differential equation under generalized H-derivation, we present the following problems:

Problem (1): Consider the fuzzy initial value problem

$$y' = -y(x) \ , \quad y(0) = [\,0.96 + 0.04r, 1.01 - 0.01r\,]$$

(1) According to subsection (4.2), **Case I.**, after reducing the above problem , we have the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = -y_1^r(x), \qquad y_1^r(0) = 0.96 + 0.04r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = -y_2^r(x), \qquad y_2^r(0) = 1.01 - 0.01r$$

Which gives the following fuzzy analytical solution

$y(x,r) = [\,(0.96 + 0.04r)e^{-x}, (1.01 - 0.01r)e^{-x}\,]$

(2) According to subsection (4.2), **Case II.**, after reducing the above problem , we have the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = -y_2^r(x), \qquad y_1^r(0) = 0.96 + 0.04r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = -y_1^r(x), \qquad y_2^r(0) = 1.01 - 0.01r$$

Which gives the following fuzzy analytical solution

$y(x,r) = [\,(0.985 + 0.015r)e^{-x} - (1-r)0.025e^{x}, (0.985 + 0.015r)e^{-x} + (1-r)0.025e^{x}\,]$.

Problem (2): Consider the fuzzy initial value problem

$$y' = -3y(x) + e^{2x}, \quad y(0) = [\,0.75 + 0.25r, 1.25 - 0.25r\,]$$

(1) According to subsection (4.2), **Case I.**, after reducing the above problem , we have the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = -3y_1^r(x) + e^{2x}, \qquad y_1^r(0) = 0.75 + 0.25r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = -3y_2^r(x) + e^{2x}, \qquad y_2^r(0) = 1.25 - 0.25r$$

Which gives the following fuzzy analytical solution

$y(x,r) = [\,(0.55 + 0.25r)e^{-3x} + 0.2e^{2x}, (1.05 - 0.25r)e^{-3x} + 0.2e^{2x}\,]$

(2) According to subsection (4.2), **Case II.**, after reducing the above problem , we have the following system of ODEs

$$\frac{d}{dx}\left(y_1^r(x)\right) = -3y_2^r(x) + e^{2x}, \qquad y_1^r(0) = 0.75 + 0.25r$$

$$\frac{d}{dx}\left(y_2^r(x)\right) = -3y_1^r(x) + e^{2x}, \qquad y_2^r(0) = 1.25 - 0.25r$$

Which gives the following fuzzy analytical solution

$y(x,r) = [\ (0.75 + 0.25r)\cosh 3x - (1.25 - 0.25r)\sinh 3x + 0.2e^{2x} - 0.2e^{-3x},\ (1.25 - 0.25r)\cosh 3x - (0.75 + 0.25r)\sinh 3x + 0.2e^{2x} - 0.2e^{-3x}\ ]$.

5 Numerical Examples

To show the behavior and properties of the proposed method, two problem will be solved in this section. We have used a multilayer perceptron having one hidden layer with ten hidden units and one output unit. The activation function of each hidden unit is hyperbolic tangent activation function. The analytical solution $[y_a(x)]_r^L$ and $[y_a(x)]_r^U$ has been known in advance. Therefore, we test the accuracy of the obtained solutions by computing the deviation (absolute error):

$\overline{e}(x, r) = |[y_a(x)]_r^U - [y_t(x)]_r^U|$, $\underline{e}(x, r) = |[y_a(x)]_r^L - [y_t(x)]_r^L|$

In order to obtain better results, more hidden units or training points may be used. To minimize the error function we have used BFGS quasi-Newton method (For more details, see [24]). The computer programs which we have used in this work are coded in MATLAB 2015.

Example 1: Consider the following linear fuzzy initial value problem:

$y' = -y + x + 1$, with $x \in [0, 1]$

$y(0) = [0.96 + 0.04r, 1.01 - 0.01r]$, where $r \in [0, 1]$.

The analytical solution (According to subsection (4.2), Case II.) are :

$[y_a(x)]_r^L = x + (0.985 + 0.015r)e^{-x} - (1 - r)0.025e^{x}$

$[y_a(x)]_r^U = x + (0.985 + 0.015r)e^{-x} + (1 - r)0.025e^{x}$

The trial solution (According to the proposed method in this work) are:

$[y_t(x)]_r^L = (0.96 + 0.04r) + x\,[N(x, p)]_r^U$

$[y_t(x)]_r^U = (1.01 - 0.01r) + x\,[N(x, p)]_r^L$

The ANN trained using a grid of ten equidistant points in [0, 1].

The error function that must be minimized for this problem will be:

$$E = \sum_{i=1}^{11}([x_i \sum_{j=1}^{10}[v_j]_r^L w_j s'(x_i w_j + b_j) + (1 + x_i) \sum_{j=1}^{10} [v_j]_r^L s(x_i w_j + b_j) - x_i - 0.01r + 0.01]^2 + [x_i \sum_{j=1}^{10}[v_j]_r^U w_j s'(x_i w_j + b_j) + (1 + x_i) \sum_{j=1}^{10}[v_j]_r^U s(x_i w_j + b_j) - x_i + 0.04 r - 0.04]^2) \quad (58)$$

Where s′ is first derivative of hyperbolic tangent activation function.

Then we use (58) to update the weights and biases.

Analytical and trial(numerical) solutions for this problem can be found in table (1) and table (2).

Example 2: Consider the following non-linear fuzzy initial value problem:

$y' = \cos(x^2 + y^2)$, with $x \in [0, 1]$

$y(0) = [0.75 + 0.25r, 1.25 - 0.25r]$, where $r \in [0, 1]$.

The trial solutions (According to the proposed method in this work) for this problem are :

$$[y_t(x)]_r^L = (0.75 + 0.25r) + x\,[N(x, p)]_r^U$$

$$[y_t(x)]_r^U = (1.25 - 0.25r) + x\,[N(x, p)]_r^L$$

The ANN trained using a grid of ten equidistant points in [0, 1].

The error function that must be minimized for this problem will be $E = \sum_{i=1}^{11}([x_i \sum_{j=1}^{10}[v_j]_r^L w_j s'(x_i w_j + b_j) + \sum_{j=1}^{10} [v_j]_r^L s(x_i w_j + b_j) - \cos(x_i^2 + (1.25 - 0.25r + x_i \sum_{j=1}^{10} [v_j]_r^L s(x_i w_j + b_j))^2)]^2 + [$

$$x_i \sum_{j=1}^{10} [v_j]_r^U \, w_j \, s'(x_i w_j + b_j) + \sum_{j=1}^{10} [v_j]_r^U \, s(x_i w_j + b_j) - \cos(x_i^2 + (0.75 + 0.25r + x_i \sum_{j=1}^{10} [v_j]_r^U \, s(x_i w_j + b_j))^2)]^2) \quad (59)$$

Then we use (59) to update the weights and biases.

Trial (numerical) solutions for this problem can be found in table (3).

r	$[y_t(x)]_r^L$	$\underline{e}(x, r)$	$[y_t(x)]_r^U$	$\overline{e}(x, r)$
0	1.948579274	0.000000422	2.318032417	0.000000761
0.1	1.967254967	0.000000472	2.299762662	0.000000643
0.2	1.985930525	0.000000387	2.281492466	0.000000085
0.3	2.004605986	0.000000205	2.263223455	0.000000711
0.4	2.023282053	0.000000629	2.244953513	0.000000406
0.5	2.041957703	0.000000635	2.226683566	0.000000096
0.6	2.060632805	0.000000094	2.208413946	0.000000114
0.7	2.079308402	0.000000048	2.190144557	0.000000362
0.8	2.097984071	0.000000074	2.171875271	0.000000713
0.9	2.116660365	0.000000725	2.153604974	0.000000054
1	2.135335791	0.000000508	2.135336111	0.000000828

Table (1): Trial solutions for example (1), x = 2.

x	$[y_t(x)]_r^L$	$\underline{e}(x, r)$	$[y_t(x)]_r^U$	$\overline{e}(x, r)$
0	0.98	0	1.005	0
0.1	0.984236574	0.000000074	1.011865864	0.000000090
0.2	0.997322850	0.000000112	1.027858634	0.000000827
0.3	1.018388968	0.000000119	1.052136129	0.000000810
0.4	1.046645251	0.000000414	1.083941124	0.000000670
0.5	1.081372866	0.000000202	1.122591654	0.000000958
0.6	1.121919133	0.000000069	1.167472118	0.000000084
0.7	1.167689342	0.000000337	1.218032891	0.000000068
0.8	1.218140638	0.000000903	1.273778484	0.000000226
0.9	1.272775429	0.000000081	1.334265671	0.000000245
1	1.331142019	0.000000196	1.399099381	0.000000513

Table (2): Trial solutions for example (1), r = 0.5.

x	$[y_t(x)]_r^L$	$[y_t(x)]_r^U$
0	0.8125	1.1875
0.1	0.814603338	1.216504331
0.2	0.814821036	1.244468346
0.3	0.812924175	1.270982832
0.4	0.808261184	1.295659825
0.5	0.799681021	1.318165427
0.6	0.785827891	1.338238511
0.7	0.765526122	1.355644313
0.8	0.737875907	1.370043258
0.9	0.702086306	1.380817214
1	0.657325883	1.386926291

Table (3): Trial solutions for example (2), r = 0.25.

6 Conclusion

In this paper, we have presented numerical method based on fuzzy neural network for solving first order fuzzy initial value problem under generalized H-derivation. we have demonstrated the ability of the fuzzy neural network to approximate the solution of the fuzzy differential equations. Therefore, we can conclude that the method which we proposed can handle effectively all types of the fuzzy differential equations and provide accurate approximate solution throughout the whole domain and not only at the training set. As well, one can use the interpolation techniques to find the approximate solution at points between the training points or at points outside the training set. Further research is in progress to apply and extend this method to solve higher order fuzzy differential equations and partial fuzzy differential equations.

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Minimal H_v-fields

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Abstract

Hyperstructures have applications in mathematics and in other sciences, which range from biology, hadronic physics, leptons, linguistics, sociology, to mention but a few. For this, the largest class of the hyperstructures, the H_v-structures, is used. They satisfy the weak axioms where the non-empty intersection replaces equality. The fundamental relations connect, by quotients, the H_v-structures with the classical ones. H_v-numbers are elements of H_v-field, and they are used in representation theory. We focus on minimal H_v-fields.

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1 Introduction

The class of hyperstructures called H_v-*structures* introduced in 1990 [Vougiouklis, 1991a], [Vougiouklis, 1994] by Vougiouklis, satisfy the *weak axioms* where the non-empty intersection replaces equality.

Algebraic hyperstructure $(H, \cdot)$ is a set H equipped with a *hyperoperation* (abbreviated: **hope**) $\cdot : H \times H \to P(H) - \{\varnothing\}$. We abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$. $(H, \cdot)$ is an H_v-*semigroup* if it is *WASS*, it is called H_v-**group** if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H$.

Motivation. The quotient of a group by an invariant subgroup, is a group. The quotient of a group by a subgroup is a hypergroup, Marty 1934. The quotient of a group by any partition (equivalence) is an H_v-group, Vougiouklis 1990.

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In an H_v-semigroup the *powers* are: $h^1=\{h\}, h^2=h\cdot h, ..., h^n=h\circ h\circ ...\circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $(H,\cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and the minimum s, such that $H=h^1\cup h^2...\cup h^s$. Analogously the cyclicity for the infinite period is defined. If thereare h and s, the minimum one, such that $H=h^s$, then $(H,\cdot)$ is a *single-power cyclic of period s.*

$(R,+,\cdot)$ is called H_v-**ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\ (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be an H_v-ring,a COW H_v-group $(M,+)$ is called H_v-**module** over R, if there is an external hope

$$\cdot: R\times M\to P(M):(a,x)\to ax$$

such that $\forall a,b\in R$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

For more definitions and applications on H_v-structures one can see in books and papers as [Corsini, 1993],[Corsini and Leoreanu, 2003],[Davvaz, 2003],[Davvaz and Leoreanu, 2007],[Davvaz and Vougiouklis, 2018],[Vougiouklis, 1994],[Vougiouklis, 1995],[Vougiouklis, 1999b].

Let $(H,\cdot),(H,*)$ H_v-semigroups, the hope $(\cdot)$ is **smaller** than $(*)$, and $(*)$ **greater** than $(\cdot)$, iff there exists an automorphism

$$f\in Aut(H,*) \text{ such that } xy\subset f(x*y),\ \forall x,y\in H.$$

We write $\cdot\leq *$ and say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is *basic structure* and $(H,*)$ is $H_b-structure$.

Minimal is called an H_v-group which contains no other H_v-group defined on the same set. We extend this definition to any H_v-structures with any more properties.

Theorem 1.1. ***(The Little Theorem).*** *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

The little theorem leads to a partial order on H_v-structures and to posets.

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h\in H$, if we take the restriction of $(\cdot)$ in $H-\{h\}$. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$. $\underline{h}\in H$ merges with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ a class with representative $\underline{h}$.

M. Koskas in 1970, introduced in hypergroups the relation β^*, which connects hypergroups with groups and it is defined in H_v-groups as well. Vougiouklis [Vougiouklis, 1985], [Vougiouklis, 1988], [Vougiouklis, 1991a], [Vougiouklis, 1994], [Vougiouklis, 1995], [Vougiouklis, 2016] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental*.

Definition 1.1. *The **fundamental relations** β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Remark: Let $(G, \cdot)$ be group and R a partition in G, then $(G/R, \cdot)$ is an H_v-group, therefore the quotient $(G/R, \cdot)/\beta^*$ is a group, the *fundamental* one.

Theorem 1.2. *Let $(H, \cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. Define the relation β in H by: $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β.*

Analogous theorems are for H_v-rings, H_v-vector spaces and so on [Vougiouklis, 1994].

Theorem 1.3. *Let $(R, +, \cdot)$ be H_v-ring. Denote U all finite polynomials of elements of R. Define the relation γ in R by:*

$$x\gamma y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation γ^* is the transitive closure of the relation γ.

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of a. First, we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}^*(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}^*(a) \cdot \underline{\gamma}(b)\}, \ \forall a, b \in R.$$

Take $a' \in \underline{\gamma}(a)$, $b' \in \underline{\gamma}(b)$. Then we have

$$a'\underline{\gamma}a \text{ iff } \exists x_1, ..., x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, ..., u_m \in U$$

$$\text{such that } \{x_i, x_{i+1}\} \subset u_i, i = 1, ..., m, \text{ and}$$

$$b'\underline{\gamma}b \text{ iff } \exists y_1, ..., y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, ..., v_n \in U$$

$$\text{such that } \{y_j, y_{j+1}\} \subset v_j, i = 1, ..., n.$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1,\ i = 1, ..., m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j,\ j = 1, ..., n.$$

The sums

$$u_i + v_1 = t_i,\ i = 1, ...m-1 \text{ and } u_m + v_j = t_{im+j-1},\ j = 1, ..., n$$

are also polynomials, thus, $t_k \in U \forall k \in \{1, ..., m+n-1\}$.

Now, pick up $z_1, ..., z_{m+n}$ such that

$$zi \in x_i + y_1,\ i = 1, ..., n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1, ..., n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k,\ k = 1, ..., m+n-1$.

Thus, every $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. So $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c), \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b)$$

In a similar way we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d), \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$$

The WASS and the weak distributivity on R guarantee that associativity and distributivity are valid for R/γ^*. Therefore R/γ^* is a ring.

Let σ be an equivalence relation in R such that R/σ is a ring and $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons $\forall a, b \in R$, i.e.

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b),\ \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b).$$

Therefore we write, for every $a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B),\ \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$$

By induction, we extend these relations on finite sums and products. Thus, $\forall u \in U$, we have the relation $\sigma(x) = \sigma(u)\ \forall x \in u$. Consequently $x \in \gamma(a) => x \in \sigma(a), \forall x \in R$. But σ is transitively closed, so we obtain: $x \in \underline{\gamma}(x) => x \in \sigma(a)$. That $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. □

An element is called **single** if its fundamental class is singleton [Vougiouklis, 1994].

General structures can be defined using fundamental structures. From 1990 there is the following [Vougiouklis, 1991a], [Vougiouklis, 1994]:

Definition 1.2. *An H_v-ring $(R,+,\cdot)$ is called H_v-**field** if R/γ^* is a field. An H_v-module over an H_v-field F, it is called H_v-**vector space**.*

The analogous to Theorem 1.3, on H_v-vector spaces, can be proved:

Theorem 1.4. *Let $(V,+)$ be H_v-vector space over the H_v-field F. Denote U the set of all expressions of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. Define the relation ϵ in V as follows: $x\epsilon y$ iff $\{x,y\}\subset u$ where $u\in U$. Then ϵ^* is the transitive closure of the relation ϵ.*

Definition 1.3. *Let $(L,+)$ be H_v-vector space over the H_v-field F, $\phi: F\to F/\gamma^*$ canonical; $\omega_F=\{x\in F:\phi(x)=0\}$, the core, 0 is the zero of F/γ. Let ω_L be the core of $\phi': L\to L/\epsilon^*$ and denote by 0 the zero of L/ϵ^*, as well. Take the bracket (commutator) hope:*

$$[,]: L\times L\to P(L):(x,y)\to[x,y]$$

*then **L** is an H_v-**Lie algebra** over F if the following axioms are satisfied:*

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$
$$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,$$
$$\forall x,x_1,x_2,y,y_1,y_2\in L,\lambda_1,\lambda_2\in F$$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\ \ \forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \ \forall x,y,z\in L$

Definition 1.4. *The H_v-semigroup $(H\cdot)$ is h/v-**group** if H/β^* is a group [Vougiouklis, 2003].*

The h/v-group is a generalization of H_v-group where a *reproductivity of classes* is valid: if $\sigma(x),\forall x\in H$ equivalence classes then $x\sigma(y)=\sigma(xy)=\sigma(x)y,\forall x,y\in H$. Similarly *$h/v$-rings*, *$h/v$-fields*, *$h/v$-vector spaces* etc, are defined.

The **uniting elements** method introduced by Corsini & Vougiouklis in 1989, is the following [Corsini and Vougiouklis, 1989]: Let G be a structure and a not valid property d, described by a set of equations. Take the partition in **G** for which put in the same class, all pairs of elements that causes the non-validity of d. The quotient by this partition G/d is an H_v-structure. Then, the quotient out G/d by β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid.

Theorem 1.5. *Let $(R,+,\cdot)$ be a ring, and $F=\{f_1,...,f_m,f_{m+1},...,f_{m+n}\}$ be a system of equations on **R** consisting of subsystems $F_m=\{f_1,...,f_m\}$ and $F_n=\{f_{m+1},...,f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using F and F_m respectively, and σ_n the equivalence defined on F_n on the ring $\mathbf{R}_m=(\mathbf{R}/\sigma_m)/\gamma^*$. Then*

$$(\mathbf{R}/\sigma)/\gamma^*\cong(\mathbf{R}_m/\sigma_n)/\gamma^*$$

2 Large classes of hopes

A class of H_v-structures, introduced in [Vougiouklis, 1991b], [Vougiouklis, 1994], [Vougiouklis, 2014b], is the following:

Definition 2.1. *An H_v-structure is called **very thin** if there exists a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products are singletons.*

From the very thin hopes the Attach Construction is obtained [Vougiouklis, 1999a], [Vougiouklis, 2014b], [Vougiouklis, 2017]: Let $(H,\cdot)$ be an H_v-semigroup and $v \notin H$. We extend the hope $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H},\cdot)$ is an H_v-group, where $(\underline{H},\cdot)/\beta^* \cong \mathbf{Z}_2$ and v is a single.
Let $(H,\cdot)$ H_v-semigroup, and [x] the fundamental class of $\forall x \in H$. Unit class is [e] if

$$([e]\cdot[x]) \cap [x] \neq \varnothing \text{ and } ([x]\cdot[e]) \cap [x] \neq \varnothing, \forall x \in H,$$

and $\forall x \in H$, we call inverse class of [x], the class $[x]^{-1}$, if

$$([x]\cdot[x]^{-1}) \cap [e] \neq \varnothing \text{ and } ([x]^{-1}\cdot[x]) \cap [e] \neq \varnothing.$$

Enlarged hopes are the ones where a new element appears in one result. The useful cases are those h/v-structures with the same fundamental structure.

Construction 2.1. *(a) Let $(H,\cdot)$ be an H_v-semigroup, $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:*

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

*The $(\underline{H},\cdot)$ is an h/v-group, called **attach**, where $(\underline{H},\cdot)/\beta^* \cong Z_2$ and v is single. Scalars and units of $(H,\cdot)$ are scalars and units in $(\underline{H},\cdot)$. If $(H,\cdot)$ is COW then $(\underline{H},\cdot)$ is COW.*

(b) $(H,\cdot)$ Hv-semigroup, $v \notin H$, $(\underline{H},\cdot)$ its attached h/v-group. Take $0 \notin H$ and define in $\underline{H}_\circ = H \cup \{v, 0\}$ two hopes:

$$\textit{hypersum}(+) : 0 + 0 = x + v = v + x = 0,$$

$$0 + v = v + 0 = x + y = v, 0 + x = x + 0 = v + v = H, \forall x, y \in H$$

$$\textit{hyperproduct}(\cdot) : \text{ remains the same as in } \underline{H} \text{ moreover}$$

$$0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \forall x \in \underline{H}$$

Then $(\underline{H}_o, +, \cdot)$ is h/v-field with $(\underline{H}_o, +, \cdot)/\gamma^* \cong \mathbf{Z}_3$. $(+)$ is associative, $(\cdot)$ is WASS and weak distributive to $(+)$. 0 is zero absorbing in $(+)$. $(\underline{H}_o, +, \cdot)$ is the **attached h/v-field** of $(H, \cdot)$.

Definition 2.2. *[Vougiouklis, 2008], [Vougiouklis, 2016] Let $(G, \cdot)$ be groupoid and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write ∂-hope, on G as follows*

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \ \ \forall x, y \in G.$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

If $(G, \cdot)$ is groupoid and $f : G \to P(G) - \{\varnothing\}$ be multivalued map. We define the ∂-hope on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \ \ \forall x, y \in G$$

.

Motivation for the ∂-hope is the derivative where only the product of functions is used.

Basic property: if $(G, \cdot)$ is semigroup then $\forall f$, the ∂-hope is WASS.

Examples (a) In integers $(\mathbf{Z}, +, \cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial_+, \partial_\cdot)$, where ∂_+ and $\partial_\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbf{Z}, \partial_+, \partial_\cdot)/\gamma^* \cong \mathbf{Z}_n.$$

(b) In $(\mathbf{Z}, +, \cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial_\cdot)$ is an H_v-ring, moreover, $(\mathbf{Z}, \partial_+, \partial_\cdot)/\gamma^* \cong \mathbf{Z}_n$.

Special case of the above is for n=p, prime, then $(\mathbf{Z}, \partial_+, \partial_\cdot)$ is an H_v-field.

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [Vougiouklis, 1999a], [Vougiouklis, 2014b], [Vougiouklis, 2017].

Theorem 2.1. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ms$ we enlarge the multiplication only in the product of the elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot)$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m + 2\}$, then the result remains the same. In this case 0 and 1 are scalars.

Corolary 2.1. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$, where p is prime , we enlarge only the product $0 \cdot p$ by $0 \oplus p = \{0, p\}$ and the rest results remain the same. Then $(\mathbf{Z}_n, +, \oplus)$ is a very thin H_v-field.*

Now we focus on **Very Thin minimal H_v-fields** obtained by a classical field.

Theorem 2.2. *In a field $(F, +, \cdot)$, we enlarge only in the product of the special elements a and b, by setting $a \otimes b = \{ab, c\}$, where $c \neq ab$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin, H_v-field*

$$(F, +, \otimes)/\gamma^* \cong \{0\}$$

Thus, there is no non-degenerate H_v-field obtained by a field by enlarging any product.

Proof. Take any $x \in F - \{0\}$, then from $a \otimes b = \{ab, c\}$ we obtain

$$(a \otimes b) - ab = \{0, c - ab\} \text{ and then } (x(c - ab)^{-1}) \otimes ((a \otimes b) - ab) = \{0, x\}$$

thus, $0 \gamma x, x \in F - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus, $(F, +, \otimes)/\gamma^* \cong \{0\}$.

□

Theorem 2.3. *In a field $(F, +, \cdot)$, we enlarge only in the sum of the special elements a and b, by setting $a \oplus b = \{a + b, c\}$, where $c \neq a + b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin, H_v-field $(F, +, \oplus)/\gamma^* \cong \{0\}$. Thus, there is no non-degenerate H_v-field obtained by a field by enlarging any sum.*

Proof. Take any $x \in F - \{0\}$, then from $a \oplus b = \{a + b, c\}$ we obtain

$$(a \oplus b) - a + b = \{0, c - a + b\} \text{ and then } (x(c - a + b)^{-1}) \cdot ((a \oplus b) - a + b) = \{0, x\}$$

thus, $0 \gamma x, x \in F - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus,

$$(\mathbf{F}, +, \oplus)/\gamma^* \cong \{0\}.$$

□

The above two theorems state that there is no non-degenerate H_v-field obtained by a field by enlarging any sum or product.

Hopes defined on classical structures are the following [Corsini, 1993], [Corsini and Leoreanu, 2003], [Vougiouklis, 1987], [Vougiouklis, 1994] :

Definition 2.3. *Let $(G,\cdot)$ be groupoid then for every $P\subset G$, $P\neq\varnothing$, we define the following hopes called **P-hopes**: $\forall x,y\in G$*

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),$$

$$\underline{P}_r: x\underline{P}_r y=(xy)P\cup x(yP),\ \ \underline{P}_l: x\underline{P}_l y=(Px)y\cup P(xy).$$

*The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called **P-hyperstructures**. If $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup.*

3 Representations and Applications

H_v-structures used in Representation (abbr. rep) Theory of H_v-groups can be achieved by generalized permutations [Vougiouklis, 1992] or by H_v-matrices [Vougiouklis, 1985], [Vougiouklis, 1994], [Vougiouklis, 1999b].

$\mathbf{H_v}$-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices (a_{ij}) and (b_{ij}), of type $m\times n$ and $n\times r$, respectively, is defined in the usual manner, and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the *n-ary circle hope* on the hyper-sum.

The problem of the H_v-matrix (or h/v-group) reps is the following:

Definition 3.1. *Let $(H,\cdot)$ be H_v-group. Find an H_v-ring $(R,+,\cdot)$, a set M_R=$\{(a_{ij})|a_{ij}$∈$R\}$, and a map $\boldsymbol{T}: H\to \boldsymbol{M}_R: h\mapsto T(h)$ such that*

$$T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\forall h_1,h_2\in H.$$

*T is an **$\mathbf{H_v}$-matrix rep**. If $T(h_1h_2)\subset T(h_1)T(h_2),\forall h_1,h_2\in H$, then **T** is an inclusion rep. If $T(h_1h_2)=T(h_1)T(h_2),\forall h_1,h_2\in H$, then **T** is a good rep. If T is a good rep and one to one then it is a faithful rep.*

The rep problem is simplified in cases such as if the h/v-rings have scalars 0 and 1. The main theorem of the theory of reps is the following:

Theorem 3.1. *A necessary condition in order to have an inclusion rep T of an h/v-group $(H,\cdot)$ by $n\times n$ h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following: $\forall\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H, i,j\in\{1,...,n\}$ such that*

$$T(\beta^*(a))\subset\{A=(a'_{ij})|a'_{ij}\in\gamma^*(a_{ij}),i,j\in\{1,...,n\}\}$$

*The inclusion rep $T: H\to M_R: a\mapsto T(a)=(a_{ij})$ induces an homomorphic T^* of H/β^* on R/γ^* by $T^*(\beta^*(a))=[\gamma^*(a_{ij})],\beta^*(a)H/\beta^*$, where $\gamma^*(a_{ij})R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$. T^* is called **fundamental induced rep** of T.*

In reps we need small H_v-fields with results of few elements.

An important hope on non-square matrices is defined [Vougiouklis, 2009], [Vougiouklis and Vougiouklis, 2005]:

Definition 3.2. *Let $A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, such that $1 \leq s \leq m$, $1 \leq t \leq n$. Define a mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ with entries the sets*

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in N, i + \kappa s \leq m, j + \lambda t \leq n\}.$$

*The map $\underline{st} : M_{m\times n} M_{s\times t} : A \to A\underline{st}(\underline{a}_{ij})$, is called **helix-projection** of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in a_{ij}, \forall i, j$. Obviously $A\underline{mn} = A$.*

Let$A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, $1 \leq s \leq m$, $1 \leq t \leq n$. We apply the helix-projection first on the columns and then on the rows and the result is the same: $(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}$.

Definition 3.3. *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be matrices. Denote s=min(m,u), t=min(n,u), then we define the **helix-sum** by*

$$\oplus : M_{m\times n} M_{u\times v} P(M_{s\times t}) :$$

$$(A, B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$. Denote s=min(n,u), then we define the helix-product by

$$\otimes : M_{m\times n} M_{u\times v} P(M_{s\times t}) :$$

$$(A, B) \to A \otimes B = A\underline{ms} + B\underline{sv} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{m\times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = \sum (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$..

Remark. The definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined, as well.

Last decades H_v-structures have applications in mathematics and in other sciences. Applications range from biology and hadronic physics or leptons to mention but a few. The hyperstructure theory is related to fuzzy one; consequently, can be widely applicable in industry and production, too [Corsini and Leoreanu, 2003], [Davvaz and Leoreanu, 2007], [Davvaz et al., 2015], [Davvaz and Vougiouklis, 2018], [Santilli and Vougiouklis, 1996], [Vougiouklis, 2014a], [Vougiouklis, 2020], [Vougiouklis and Kambaki-Vougioukli, 2013].

An application, which combines H_v-hyperstructures and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis **(V & V bar)** [Vougiouklis and Kambaki-Vougioukli, 2013]. They suggest the following:

Definition 3.4. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 _____ *1*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of V & V bar bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

The Lie-Santilli theory on isotopies was born to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix into an appropriate new matrix. The original theory is reconstructed to admit the new matrix as left and right unit. The **isofields** needed in this theory correspond into the hyperstructures called e-hyperfields, introduced by [Santilli and Vougiouklis, 1996, Davvaz et al., 2015].

Definition 3.5. *A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

Definition 3.6. *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called **e-hyperfield** if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and for all $x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 3.7. *The Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

g_1, g_2,... are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ becomes an H_v-group, actually is an H_b-group which contains the $(G, \cdot)$. The H_v-group $(G, \otimes)$ is an e-hypergroup.

Example. Consider the quaternions $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = -1, ij = -ji = k$ and denote $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}, k * i = \underline{j}$ and $i * j = \underline{k}$. Then $(Q, *)$ is strong e-hypergroup.

A **generalization of P-hopes** used in Santilli's isotheory, is [Davvaz et al., 2015], [Vougiouklis, 2016] : Let $(G, \cdot)$ be abelian group, $P \subset G$ with $\#P < 1$. We define the hope $\times_p$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is abelian H_v-group.

4 Small hypernumbers. Minimal h/v-fields

The small non-degenerate h/v-fields on $(\mathbf{Z}_n, +, \cdot)$ in iso-theory, satisfy the following:

1. very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. if an element has inverse element, this is unique.

Therefore, we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement.

Theorem 4.1. *[Vougiouklis, 2017] All multiplicative h/v-fields defined on* $(\mathbf{Z}_4, +, \cdot)$*, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic cases: The only product which is set is* $2 \otimes 3 = \{0, 2\}$ *or* $3 \otimes 2 = \{0, 2\}$*. Fundamental classes: [0]=0,2, [1]=1,3 and we have*

$$(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$$

Example. Denote E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Take the 2×2 upper triangular h/v-matrices on the above h/v-field $(\mathbf{Z}_4, +, \otimes)$ of the case that only 2⊗3={0,2} is a hyperproduct:

$$I = E_{11} + E_{22}, a = E_{11} + E_{12} + E_{22}, b = E_{11} + 2E_{12} + E_{22}, c = E_{11} + 3E_{12} + E_{22},$$

$$d = E_{11} + 3E_{22}, e = E_{11} + E_{12} + 3E_{22}, f = E_{11} + 2E_{12} + 3E_{22}, g = E_{11} + 3E_{12} + 3E_{22},$$

then, we obtain for X=$\{$I,a,b,c,d,e,f,g$\}$, that $(X,\otimes)$ is non-COW, H_v-group where the fundamental classes are $\underline{a}=\{a,c\},\underline{d}=\{d,f\},\underline{e}=\{e,g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2\times\mathbf{Z}_2,+)$. There is only one unit and every element has unique double inverse. Only f has one more right inverse element d, since $f\otimes d=\{I,b\}$. $(X,\otimes)$ is not cyclic.

Theorem 4.2. *All multiplicative h/v-fields on $(\mathbf{Z}_6,+,\cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,*

(I) $2\otimes 3=\{0,3\},2\otimes 4=\{2,5\},3\otimes 4=\{0,3\},3\otimes 5=\{0,3\},4\otimes 5=\{2,5\}$. *The fundamental classes are [0]=0,3, [1]=1,4, [2]=2,5 and we have*

$$(\mathbf{Z}_6,+,\otimes)/\gamma^*\cong(\mathbf{Z}_3,+,\cdot).$$

(II) $2\otimes 3=\{0,2\}$ *or* $2\otimes 3=\{0,4\}$, $2\otimes 4=\{0,2\}$ *or* $\{2,4\},2\otimes 5=\{0,4\}$ *or* $2\otimes 5=\{2,4\}$, $3\otimes 4=\{0,2\}$ *or* $\{0,4\},3\otimes 5=\{3,5\},4\otimes 5=\{0,2\}$ *or* $\{2,4\}$. *In all these cases the fundamental classes are [0]=0,2,4, [1]=1,3,5 and we have*

$$(\mathbf{Z}_6,+,\otimes)/\gamma^*\cong(\mathbf{Z}_2,+,\cdot).$$

Example. In the h/v-field $(\mathbf{Z}_6,+,\otimes)$ where only the hyperproduct is $2\otimes 4=\{2,5\}$ take the h/v-matrices of type $\underline{i}=E_{11}+iE_{12}+4E_{22}$, where i=0,1,...,5, then the multiplicative table of the hyperproduct of those h/v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{0,3}$	$\underline{1,4}$	$\underline{2,5}$	$\underline{0,3}$	$\underline{1,4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

The fundamental classes are $(0)=\underline{0,3},(1)=\underline{1,4},(2)=\underline{2,5}$ and the fundamental group is isomorphic to $(\mathbf{Z}_3,+)$. The $(\mathbf{Z}_6,\otimes)$ is h/v-group which is cyclic where 2 and 4 are generators of period 4.

Example. Consider the h/v-field $(\mathbf{Z}_10,+,\otimes)$ where only $3\times 8=\{4,9\}$ is a hyperproduct. Let us take the h/v-matrix

$$A=3E_{11}+E_{22}+2E_{33}+6E_{12}+2E_{13}+9E_{23}$$

Then from the above formulas we obtain that the set of inverse h/v-matrices is

$$A^{-1}=[2]E_{11}+[1]E_{22}+[3]E_{33}+[3]E_{12}+[2]E_{13}+[3]E_{23}$$

So, for example, if we take the h/v-matrix

$$A^{-1} = 7E_{11} + 6E_{22} + 8E_{33} + 8E_{12} + 2E_{13} + 3E_{23}$$

we obtain that

$$A \cdot A^{-1} = E_{11} + E_{22} + E_{33} + \{0, 5\}E_{12} + 5E_{23}$$

therefore, it contains a unit h/v-matrix.

Theorem 4.3. *All multiplicative h/v-fields defined on* $(\mathbf{Z}_9, +, \cdot)$*, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,* $2 \otimes 3 = 0, 6$ *or* $3, 6, 2 \otimes 4 = 2, 8$ *or* $5, 8, 2 \otimes 6 = 0, 3$ *or* $3, 6, 2 \otimes 7 = 2, 5$ *or* $5, 8, 2 \otimes 8 = 1, 7$ *or* $4, 7, 3 \otimes 4 = 0, 3$ *or* $3, 6, 3 \otimes 5 = 0, 6$ *or* $3, 6, 3 \otimes 6 = 0, 3$ *or* $0, 6, 3 \otimes 7 = 0, 3$ *or* $3, 6, 3 \otimes 8 = 0, 6$ *or* $3, 6, 4 \otimes 5 = 2, 5$ *or* $2, 8, 4 \otimes 6 = 0, 6$ *or* $3, 6, 4 \otimes 8 = 2, 5$ *or* $5, 8, 5 \otimes 6 = 0, 3$ *or* $3, 6, 5 \otimes 7 = 2, 8$ *or* $5, 8, 5 \otimes 8 = 1, 4$ *or* $4, 7, 6 \otimes 7 = 0, 6$ *or* $3, 6, 6 \otimes 8 = 0, 3$ *or* $3, 6, 7 \otimes 8 = 2, 5$ *or* $2, 8$*. In all the above cases the fundamental classes are*

$$[0] = \{0, 3, 6\}, [1] = \{1, 4, 7\}, [2] = \{2, 5, 8\}, and we have (\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot).$$

Theorem 4.4. *All multiplicative h/v-fields on* $(\mathbf{Z}_{10}, +, \cdot)$*, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

(I) We have the only one hyperproduct,
$2 \otimes 4 = \{3, 8\}, 2 \otimes 5 = \{0, 5\}, 2 \otimes 6 = \{2, 7\}, 2 \otimes 7 = \{4, 9\},$
$2 \otimes 9 = \{3, 8\}, 3 \otimes 4 = \{2, 7\}, 3 \otimes 5 = \{0, 5\}, 3 \otimes 6 = \{3, 8\}, 3 \otimes 8 = \{4, 9\},$
$3 \otimes 9 = \{2, 7\}, 4 \otimes 5 = \{0, 5\}, 4 \otimes 6 = \{4, 9\},$
$4 \otimes 7 = \{3, 8\}, 4 \otimes 8 = \{2, 7\}, 5 \otimes 6 = \{0, 5\}, 5 \otimes 7 = \{0, 5\},$
$5 \otimes 8 = \{0, 5\}, 5 \otimes 9 = \{0, 5\}, 6 \otimes 7 = \{2, 7\}, 6 \otimes 8 = \{3, 8\},$
$6 \otimes 9 = \{4, 9\}, 7 \otimes 9 = \{3, 8\}, 8 \otimes 9 = \{2, 7\}.$

In all the above cases the fundamental classes are $[0] = \{0, 3, 6\}, [1] = \{1, 4, 7\}, [2] = \{2, 5, 8\},$
and we have

$$(\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot).$$

(II) The cases with classes $[0] = \{0, 2, 4, 6, 8\}$ *and* $[1] = \{1, 3, 5, 7, 9\}$*, and fundamental field*

$$(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot).$$

are described as follows: In the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting 1, and we cannot enlarge only the $3 \otimes 7 = 1$*. The number of those h/v-fields is 103.*

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On the planarity of line Mycielskian graph of a graph

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Abstract

The line Mycielskian graph of a graph G, denoted by $L_\mu(G)$ is defined as the graph obtained from $L(G)$ by adding $q+1$ new vertices $E^{'} = e_i^{'} : 1 \leq i \leq q$ and e, then for $1 \leq i \leq q$, joining $e_i^{'}$ to the neighbours of e_i and to e. The vertex e is called the root of $L_\mu(G)$. This research paper deals with the characterization of planarity of line Mycielskian Graph $L_\mu(G)$ of a graph. Further, we also obtain the characterization on outerplanar, maximal planar, maximal outerplanar, minimally nonouterplanar and 1-planar of $L_\mu(G)$.

Keywords: Planar graph, Outerplanar, Maximal planar, Maximal outerplanar, Minimally nonouterplanar and 1-planar.
2010 AMS subject classifications: 05C07, 05C10, 05C38, 05C60, 05C76. [1]

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1 Introduction

All graphs considered in this paper are finite, undirected and without loops. The phraseologies are referred from [Harary, 1969].

The graph G is said to be embedded in a surface S when its vertices are represented by points in S and each edge by a curve joining corresponding points in S, in such a way that no curve intersects itself and two curves intersect each other only at a common vertex.

A graph G is said to be planar if it can be embedded in the plane. A plane representation of a planar graph divides the plane into number of plane areas called regions or faces. The regions enclosed by the planar graph are called interior faces of the graph. The region surrounding the planar graph is called the exterior face of the graph. A planar graph G is called maximal planar if the addition of any edge to G creates a nonplanar graph. A maximal planar graph is a planar graph in which every face (including the exterior face) is bounded by a triangle [Dillencourt, 1991]. A planar graph is called outerplanar if it can be embedded in the plane so that all its points lie on the same face. An outerplanar graph is called maximal outerplanar if no line can be added without losing outerplanarity.

The inner vertex number $i(G)$ of a planar graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be k - minimally nonouterplanar if $i(G) = k, k \geq 1$. An 1-minimally nonouterplnar graph is called minimally nonouterplanar [Kulli and Basavanagoud, 2004].

Two graphs are said to be homeomorphic if one graph can be obtained from the other by insertion of vertex of degree two into its edges or by the merger of adjacent edges, where the incident vertex is of degree two.

Crossing number $cr(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane.

[Ringel, 1965] introduced the concept of 1-planarity. A graph G is called 1-planar if it can be drawn in the plane so that all or any edge is crossed by at most one other edge.

The line Mycielskian graph of a graph G, denoted by $L_\mu(G)$ is defined as the graph obtained from $L(G)$ by adding $q+1$ new vertices $E^{'} = e_i^{'} : 1 \leq i \leq q$ and e, then for $1 \leq i \leq q$, joining $e_i^{'}$ to the neighbours of e_i and to e. The vertex e is called the root of $L_\mu(G)$ [Mirajkar and Mathad, 2019].

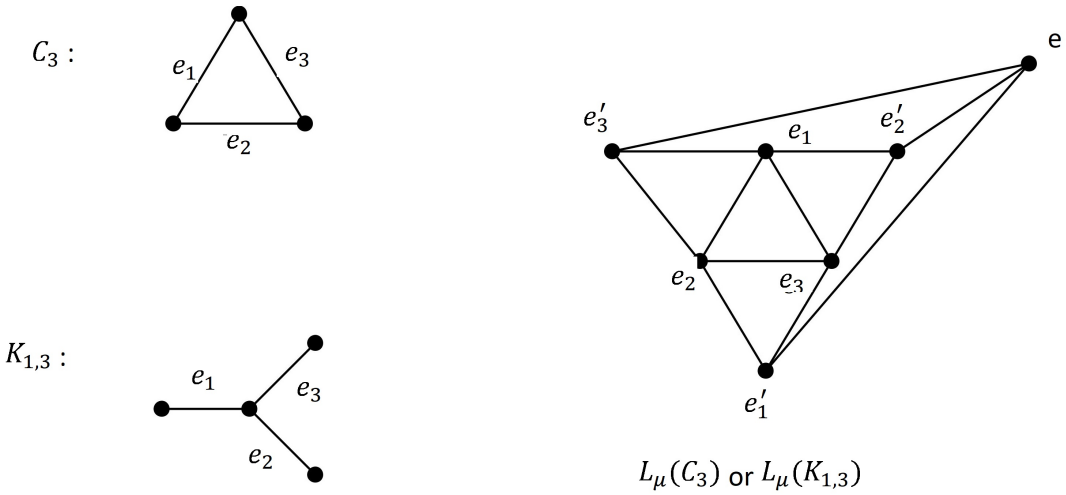

Figure 1. C_3, $K_{1,3}$ and their line Mycielskian graphs $L_\mu(C_{1,3})$ and $L_\mu(K_{1,3})$

Motivated by the the research work [Mirajkar and Mathad, 2019], the present problem is initiated. Further it is extended with the objective of obtaining the characterization results on planarity, outerplanar, maximal planar, maximal outerplanar, minimal nonouterplanar and 1-planar of $L_\mu(G)$.

2 Prelimnaries

The following important theorems and remark are used for proving further results.

Theorem 2.1. *[Kuratowski, 1930] A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 2.2. *[Harary, 1969] A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$.*

Theorem 2.3. *[Czap and Hudák, 2012] The complete graph K_{α_1}, $\alpha_1 \leq 6$ is 1-planar.*

Theorem 2.4. *[Mirajkar et al., 2019] The line Mycielskian graph $L_\mu(G)$ of a graph G is disconnected iff $G = K_2$.*

Remark 2.1. *[Mirajkar and Mathad, 2019] $L(G)$ is subgraph of $L_\mu(G)$.*

3 Results

Theorem 3.1. *The line Mycielskian graph $L_\mu(G)$ is planar if and only if the graph G is C_n, $n = 3$ or C_3.*

Proof. Suppose $L_\mu(G)$ is planar and $G = C_n$.

We consider the following cases.

Case 1. Suppose $n = 5$, then $G = C_5$. $L(C_5)$ is C_5 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. The construction of $L_\mu(G)$ with five newly introduced vertices $e_1^{'}, e_2^{'}, e_3^{'}$, $e_4^{'}$ and $e_5^{'}$ corresponding to vertices of $L(C_5)$ and root vertex e produces five mutually adjacent vertices with degree four which is a subgraph homeomorphic to K_5. By theorm 2.1, a contradiction.

Case 2. Suppose $n = 4$, then $G = C_4$. $L(C_4)$ is C_4 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

Here $L_\mu(G)$ is constructed by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}$, and $e_4^{'}$ corresponding to the vertices e_1, e_2, e_3, and e_4 of $L(C_4)$ and root vertex e. Newly introduced vertices are connected with the vertices e_1, e_2, e_3 and e_4 of C_4 in such a way that the, $e_1^{'}$ and $e_3^{'}$ are adjacent to the two opposite vertices e_2 and e_4 and $e_2^{'}$ and $e_4^{'}$ are adjacent to e_1 and e_3. Root vertex e is connected to $e_1^{'}, e_2^{'}, e_3^{'}$, $e_4^{'}$. This construction produces the five mutually adjacent vertices with degree four and thus contains a subgraph homeomorphic to K_5, a contradiction (From theorem 2.1).

Case 3. Suppose $n = 3$, then $G = C_3$. $L(C_3)$ is C_3 by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. To construct line Mycielskian graph $L_\mu(G)$, three new vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ corresponding to the edges of G and root vertex e are introduced. The edges between the vertices are drawn in such a way that the newly introduced vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ are adjacent to the corresponding adjacent edges of e_1, e_2 and e_3 of G respectively. The root vertex e joins the vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ such that no crossing of edges occur as shown in Figure 1. This implies that line graph $L_\mu(G)$ is planar for $G = C_3$.

From the above cases, it is observed that $L_\mu(G)$, for all $G = C_n$, $n \geq 4$, contains a subgraph homeomorphic to K_5, a contradiction. Thus $L_\mu(G)$ is planar only if G is C_n, $n = 3$.

Conversely, Suppose $G = C_3$, then the construction of $L_\mu(G)$ as discussed in above case 3 which results into planar graph.

Theorem 3.2. *The line Mycielskian graph $L_\mu(G)$ is planar if and only if the graph G is a path graph P_n, $n \geq 3$.*

Proof. Suppose $L_\mu(G)$ is planar and $G = P_n$.

We consider the following cases.

Case 1. Suppose $n = 2$, then $G = P_2$, then from theorem 2.4, $L_\mu(G)$ is disconnected graph, a contradiction.

Case 2. Suppose $G = P_n, n \geq 3$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. $L(P_n)$ is P_{n-1}.

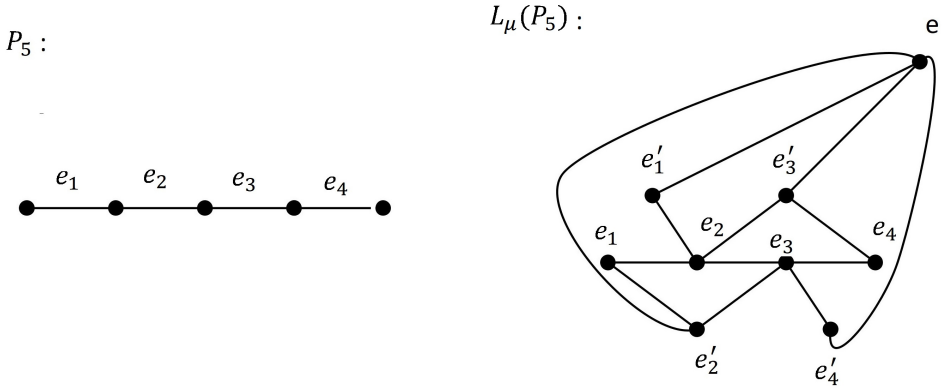

Figure 2. P_5 and its line Mycielskian graph $L_\mu(P_5)$

For the construction of $L_\mu(G)$, new vertices $e_1^{'}, e_2^{'}$,.., $e_{n-1}^{'}$ corresponding to edges of G and root vertex e are introduced. In $L_\mu(G)$, the edges between the vertices are connected in such a way that, the newly introduced vertices $e_1^{'}$ and $e_{n-1}^{'}$ are adjacent to e_2 and e_{n-2} of $L(G)$ respectively and the other vertices $e_2^{'}, e_3^{'}$,.. , $e_{n-2}^{'}$ are adjacent to two vertices e_{i-1} and e_{i+1}, $i = 2, 3, .., (n-2)$. The root vertex e is adjacent to the vertices $e_1^{'}, e_2^{'}$,.., $e_{n-1}^{'}$. All the faces are polygons without any crossings as shown in Figure 2. i.e., $L_\mu(G)$ is planar.
It is clear from the above two cases that $L_\mu(G)$ is planar only for $G = P_n, n \geq 3$.

Conversely, Suppose $G = P_n$, $n \geq 3$. Then the construction of $L_\mu(G)$ results into planar graph (as discussed above).

Theorem 3.3. *The line Mycielskian graph $L_\mu(G)$ of a graph G is planar if and only if one of the follwing conditions hold*
(i) $\Delta(G) = 2$, except for C_n, $n \geq 4$
(ii) $G = K_{1,3}$

Proof. Suppose $L_\mu(G)$ of G is planar.

We discuss the following cases based on the maximum degree of G.

Case 1. Suppose G is a graph with $\Delta(G) = 5$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. Then the line Mycielskian graph $L_\mu(G)$ contains K_5 as its subgraph. By Theorem 2.1, $L_\mu(G)$ is nonplanar, a contradiction.

Case 2. Next suppose G is a graph with $\Delta(G) = 4$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$ and contains K_4 as its subgraph. $L_\mu(G)$ is obtained by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}$ and $e_4^{'}$ corresponding to the edges of subgraph K_4 of $L(G)$ and root vertex e by the definition. Vertices of K_4 and newly introduced vertices are connected. The edge from root vertex e is drawn in such a way that it is adjacent to the vertices $e_1^{'}, e_2^{'}, e_3^{'}$ and $e_4^{'}$. This construction produces subgraph K_4 with each vertex degree ≥ 4. One of the newly introduced vertices of is $L_\mu(G)$ is adjacent to the three vertices of K_4 with path length 1. It is also adjacent to the remaining one vertex of K_4 with path length 2. This produces the subgraph homeomorphic to K_5, a contradiction.

Case 3. Suppose $\Delta(G) = 3$, G contains $K_{1,3}$ as its subgraph. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. Since $L(K_{1,3})$ is C_3, the construction of $L_\mu(K_{1,3})$ is same as $L_\mu(C_3)$ as shown in Figure 1. From theorem 3.1, $L_\mu(C_3)$ is planar. Thus $L_\mu(K_{1,3})$ is also planar.

Further in the construction of $L_\mu(K_{1,3})$, presence of additional edge in $K_{1,3}$ to any vertex leads to increase in the number of vertices and edges (2 vertices and 3 edges) in $L_\mu(G)$ and thus contains K_4. On constructing $L_\mu(G)$, newly introduced vertices, root vertex and vertices of K_4 produces five mutually adjacent vertices with degree 4 which is a subgraph homeomorphic to K_5. By theorem 2.1, $L_\mu(G)$ is nonplanar, a contradiction.

Case 4. Suppose $\Delta(G) = 2$, Obiviously G is either path graph P_n or cycle C_n. By theorem 3.1 and theorem 3.2, $L_\mu(C_n)$, $n = 3$ and $L_\mu(P_n)$, $n \geq 3$ are planar.

From all the above cases, it is noted that, $L_\mu(G)$ contains subgraph homeomorphic to K_5 for $\Delta(G) \geq 3$, a contradiction and is planar only for $\Delta(G) = 2$.

Converse is obivious.

Theorem 3.4. *The line Mycielskian graph $L_\mu(G)$ is outerplanar if and only if G is P_3.*

Proof. Suppose $L_\mu(G)$ is outerplanar. Then $L_\mu(G)$ is planar. By theorems 3.1, 3.2 and 3.3, $L_\mu(G)$ is planar only for the graphs $C_3, K_{1,3}$ and P_n, $n \geq 3$.
Suppose G=C_3 or $K_{1,3}$. From theorem 3.1 and Figure 1, $L_\mu(G)$ contains a subgraph homeomorphic to $K_{2,3}$. From theorem 2.2, $L_\mu(G)$ is not outerplanar, a contradiction.

Assume $G = P_4$. line graph of P_4 is P_3. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. On constructing $L_\mu(G)$ with newly introduced vertices and root vertex (as explained in case 2 of theroem 3.2) forms a subgraph homeomorphic to $K_{2,3}$. By theorem 2.2 $L_\mu(P_4)$ is not outerplanar. Which is contradiction. i.e., G cannot be P_4.

Similarly, for higher values of n i.e., for $G = P_4$, $n \geq 5$, the same construction (case 2 of theorem 3.2) repeates and generates a subgraph with every four vertices of $L_\mu(G)$ which is homeomorphic to $K_{2,3}$, a contradiction.

Next assume $G = P_3$. Line graph of P_3 is P_2 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. $L_\mu(G)$ is constructed by introducing root vertex e and new vertices $e_1^{'}$ and $e_2^{'}$ corresponding to the edges of e_1 and e_2 of G respectively. The edges from new vertices are drawn in such a way that $e_1^{'}$ is adjacent to e_2 and $e_2^{'}$ to e_1 which forms C_5, as shown in Figure 3, which is outerplanar. Therefore, $G = P_3$.

By observing the above cases , it can be stated that $L_\mu(G)$ for $G = P_n$, $n \geq 4$ contains subgraph homeomorphic to $K_{2,3}$, a contradiction and hence is outerplanar only for $G = P_n$, $n = 3$.

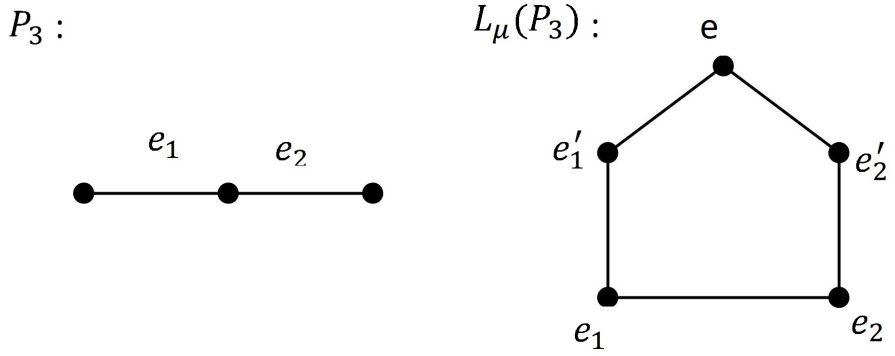

Figure 3. P_3 and its line Mycielskian graph $L_\mu(P_3)$

Conversely, suppose $G = P_3$. From theorem 3.2, $L_\mu(G)$ is planar for G=P_n, $n \geq 3$. i.e., $L_\mu(P_3)$ produces C_5 and is planar (as discussed above). Clearly which is outerplanar.

Theorem 3.5. *The line Mycielskian graph $L_\mu(G)$ is maximal planar if and only if G is C_3 or $K_{1,3}$ or P_n, $n \geq 4$.*

Proof. Suppose $L_\mu(G)$ is maximal planar, then it is planar. From theorem 3.1, theorem 3.2 and theorem 3.3, $L_\mu(G)$ is planar if G is C_3 or $K_{1,3}$ or P_n, $n \geq 4$.

Suppose G is C_3 or $K_{1,3}$. On the construction of $L_\mu(C_3)$ or $L_\mu(K_{1,3})$ as shown in Figure 1, it can be observed that, three of its interior faces are triangles and by the definition of homeomorphic graph, the other two faces can be triangularised by merging adjacent edges if the incident vertex is of degree two. i.e., $L_\mu(G)$ has triangulation plane. $L_\mu(G)$ is maximal planar.

Suppose G is P_n, $n \geq 3$. From theorem 3.2, $L_\mu(G)$ is planar.

We consider the following cases.

Suppose $G = P_3$. From theorem 3.4, the construction of $L_\mu(P_3)$ forms C_5 and does not contain any triangle faces as shown in Figure 3. Thus $L_\mu(P_3)$ is not maximal planar, a contradiction.

Next, suppose $G = P_n$, $n \geq 4$. The construction of $L_\mu(G)$ contains some of its interior faces as C_4's and some as C_5's. By definition of homeomorphic graph, all the faces can be triangularised by merging the adjacent edges. Thus $L_\mu(G)$ has triangulation plane and is maximal planar for $G = P_n$, $n \geq 4$.

Conversely, suppose $G = C_3$ or $K_{1,3}$ or P_n, $n \geq 4$. From theorem 3.1, theorem 3.3 and theorem 3.2, $L_\mu(G)$ is planar.

Let us first consider $G = C_3$ or $K_{1,3}$.

Three interior faces of $L_\mu(G)$ are triangles and two faces are C_4's and by the definition of homeomorphic graph, they can be triangularized by merging adjacent edges. i.e., $L_\mu(G)$ has triangulation plane. Hence, $L_\mu(G)$ is maximal planar.

Next suppose $G = P_n$, $n \geq 4$. From Figure 2, all the faces of $L_\mu(G)$ can be triangularized by merging adjacent edges and thus $L_\mu(G)$ has triangulation plane and it is maximal planar.

□

Theorem 3.6. *For any graph G, The line Mycielskian graph $L_\mu(G)$ of a graph G is not maximal outerplanar.*

Proof. Suppose $L_\mu(G)$ is maximal outerplanar, then it is outerplanar. From theorem 3.4, $L_\mu(G)$ is outerplanar only for $G = P_3$. From Figure 3, it is obvious that addition of an edge between any two non-adjacent vertices does not violate the property of planarity, a contradiction.

□

Theorem 3.7. *For any graph G, the line Mycielskian graph $L_\mu(G)$ of a graph G is not minimally nonouterplanar.*

Proof. Suppose $L_\mu(G)$ is minimally nonouterplanar, then $L_\mu(G)$ is planar. From theorem 3.1, theorem 3.2 and theorem 3.3, $L_\mu(G)$ is planar only if G is C_n, $n = 3$ or P_n, $n \geq 3$ or $K_{1,3}$.

We consider the following three cases.

Case 1. Suppose $G = C_n$, $n = 3$ or $G = K_{1,3}$. From Figure 1, $L_\mu(G)$ contains at least three points belong to interior region, a contradiction.

Case 2. Suppose $G = P_n$, $n \geq 3$.

Subcase 2.1 Suppose $n = 3$ then $G = P_3$. From Figure 3, in $L_\mu(G)$ all the points belong to exterior region. No point belong to interior region, a contradiction.

Subcase 2.2. Suppose $G = P_n$, $n \geq 4$, then $L_\mu(G)$ is constructed with newly introduced vertices $e_1^{'}, e_2^{'}$,.., $e_{n-1}^{'}$ corresponding to edges of G and root vertex e (construction explained in case 2 of theorem 3.2).This construction generates subgraph with every four vertices of $L_\mu(G)$ which is homeomorphic to $K_{2,3}$. This process continues for the successive value of n in G and thus $L_\mu(G)$ contains more than one subgraph homeomorphic to $K_{2,3}$, a contradiction.

In either cases, $L_\mu(G)$ is not minimally nonouterplanar.

□

Theorem 3.8. *If the graph G is a cycle C_n, $n \geq 4$, then line Mycielskian graph $L_\mu(G)$ is 1-planar with n crossings .*

Proof. Let G be C_n, $n \geq 4$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

From theorem 3.1, $L_\mu(C_n)$, $n \geq 4$ is nonplanar.

Here $L_\mu(G)$ is constructed by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}$, $e_4^{'}$,.., $e_n^{'}$ and root vertex e. Root vertex, newly introduced vertices and vertices of $L(C_n)$ are connected as explained in the theorem 3.1. This construction produces the five mutually adjacent vertices with degree ≥ 4 and thus contains a subgraph homeomorphic to K_5. From theorem 2.3, $L_\mu(G)$ is 1-planar.

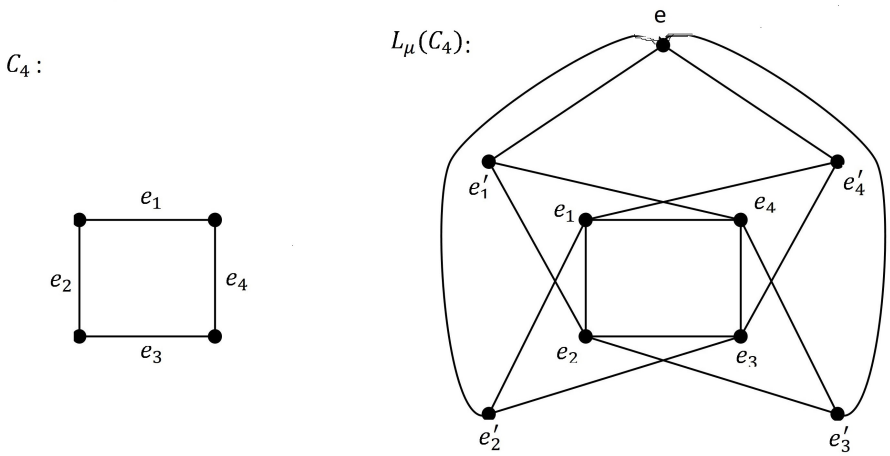

Figure 4. C_4 and its line Mycielskian graph $L_\mu(C_4)$

In $L_\mu(G)$, every crossing arises from the edges drawn between the vertices $L(C_n)$ and newly introduced vertices. Each newly introduced vertex is adjacent with two vertices of $L(C_n)$ and each crosing occurs from the two adjacent vertices of $L(C_n)$. As there are n vertices in $L(C_n)$, the number of crossings are n.

□

Theorem 3.9. *If G is $K_{1,3}\bullet K_2$, then the line Mycielskian graph $L_\mu(G)$ is 1-planar with crossing number* 1.

Proof . Let G be $K_{1,3} \bullet K_2$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

From theorem 3.3, $L_\mu(K_{1,3})$ is planar. But inclusion of K_2 to $K_{1,3}$ increases number of vertices and edges in $L_\mu(G)$ for which the construction is explained in theorem 3.3 case 3. This construction produces five mutually adjacent vertices with degree 4. Thus $L_\mu(G)$ contains a subgraph homeomorphic to K_5. From theorem 2.3, $L_\mu(G)$ is 1-planar.

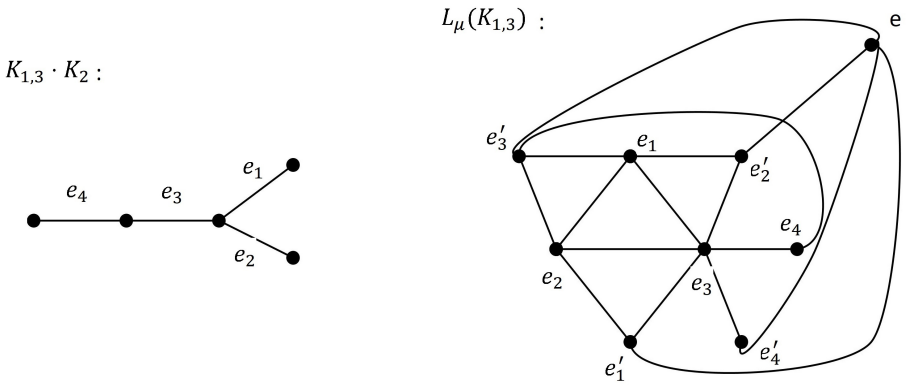

Figure 5. $K_{1,3} \bullet K_2$ and its line Mycielskian graph $L_\mu((K_{1,3} \bullet K_2)$

In graph $G = K_{1,3} \bullet K_2)$ shown in Figure 5, the number of edges are four. Since only one vertex e_3 of G is adjacent to three edges, in $L_\mu(G)$ the degree of newly introduced vertex corresponding to this edge is four as shown in Figure 5, where as the degree of other newly introduced vertices $e_1^{'}, e_2^{'}$, and $e_4^{'}$ is ≤ 3. Thus, this results the graph $L_\mu(G)$ to have only one crossing from the edges drawn between the root vertex e and newly introduced vertices $e_2^{'}, e_3^{'}$ and $e_4^{'}$.

4 Conclusion

In this research paper we obtained the characterization results on planarity, outerplanar, maximal planar, maximal outerplanar, minimal nonouterplanar and 1-planar of line Mycielskian graph $L_\mu(G)$. The results revealed that $L_\mu(G)$ is planar only for the graphs C_3, P_n,$n \geq 3$, $K_{1,3}$ and maximal planar if G is C_3, P_n,$n \geq 4$ and $K_{1,3}$. It is outerplanar only if G= P_3.There is no existance of any graph whose $L_\mu(G)$ is maximal outerplanar and not minimally nonouterplanar. Further we also obtained $L_\mu(G)$ for 1-planar when G= C_n, $n \geq 4$ and $(K_{1,3} \bullet K_2)$ with n crossings and 1 crossing respectively.

Future line of this research can be extended to study some more properties of planarity such as crossing numbers, genus, thickness and coarseness of line Mycielskian graph of a graph.

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Approximation of functions by (C,2)(E,1) product summability method of Fourier series

Jitendra Kumar Kushwaha*

Abstract

Various investigators such as Leindler [10], Chandra [1], Mishra et al. [7], Khan [11], Kushwaha [6] have determined the degree of approximation of 2π *-periodic* functions belonging to classes $Lip\alpha$, $Lip(\alpha,r)$, $Lip(\xi(t),r)$ of functions through trigonometric Fourier approximation using different summability means. Recently Nigam [12] has determined that the Fourier series is summable under the summability means (C,2)(E,1) but he did not find the degree of approximation of function belonging to various classes. In this paper a theorem concerning the degree of approximation of function f belonging to $Lip(\xi(t),r)$ class by (C,2)(E,1) product summability method of Fourier series has been established which in turn generalizes the result of H. K. Nigam [12].

Keywords: Degree of approximation; Fourier series; Pruduct summability methods.
2010 AMS subject classification: 42B05; 42B08.†

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1. Introduction

The study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. Broadly speaking, signals are treated as function of single variable and images are represented by function of two variables. The study of these concepts is directly related to the emerging area of information technology. Studies on trigonometric approximation of functions in L_p -norm using different linear operators such Hölder,Nörlund, Euler, Riesz, Borel etc. were made by several researchers like Mohapatra & Chandra [9], Holland, Mohapatra & sahney [8], Chandra [2]. The degree o approximation of a function belonging to different class of functions by product summability methods were made by Lal & Singh [5], Lal & Kushwaha [6]. The aim of this paper is to study Fourier series and conjugate series by product operators. The advantage of considering product operators over linear operators can be understood with the observation that the infinite series, which is neither summable by left linear operators nor by right linear operators individually, is summable to some number by the product operators obtained from the same linear operators placed in the same sequential order. Moreover , in studies of error estimates $E_n(f)$ through Trigonometric Fourier Approximation, product operators give better approximation than individual linear operators. Generalizing the result of Nigam [12], the degree of approximation of function f belonging to $Lip(\xi(t),r)$ class by (C,2)(E,1) product summability method of Fourier series has been established.

Therefore, in this paper, (C,2)(E,1) product summability method is introduced and a theorem on the approximation of functions belonging to $L(\xi(t),r)$ class has been established.

Let $\sum_{n=0}^{\infty} u_n$ be given infinite series with s_n for its n^{th} partial sum.

Let $\{t_n^{E_1}\}$ denote the sequence of (E,1) mean of the sequence $\{s_n\}$. If the (E,1) transform of s_n is defined

$$\text{as } t_n^{E_1}(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k(f;x) \to s \text{ as } n \to \infty \qquad (1.1)$$

The series $\sum_{n=0}^{\infty} u_n$ is said to summable to the number s by the (E,1) method (Hardy [14]).

Let $\{t_n^{C_2}\}$ denote the sequence of (C, 2) mean of the sequence $\{s_n\}$. If the (C, 2) transform of s_n is defined as

$$t_n^{C_2}(f;x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\,s_k(f;x)\to s \text{ as } n\to\infty \tag{1.2}$$

the series $\sum_{n=0}^{\infty}u_n$ is said to be summable to the number s by (C, 2) method (Cesàro method).

Thus if

$$t_n^{C_2\cdot E_1}(f;x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\frac{1}{2^k}\sum_{v=0}^{n}\binom{n}{v}s_v(f;x)\to s \text{ as } n\to\infty \tag{1.3}$$

Where $t_n^{C_2\cdot E_1}$ denotes the sequence of (C,2)(E,1) product mean of the sequence s_n. The series $\sum_{n=0}^{\infty}u_n$ is said to summable to the number s by (C,2)(E,1) method. We observe that (C,2)(E,1) method is regular.

Let f be 2π *-periodic* and Lebesgue integrable function. The Fourier series associated with f at a point x is defined by

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x) \tag{1.4}$$

with partial sum $s_n(f;x)$.

Throughout this paper, we use following notations:

$$\phi(t)=\phi(x,t)=f(x+t)+f(x-t)-f(x)$$

$$M_n(t)=\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{v=0}^{k}\left\{\binom{k}{v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right\}\right].$$

2. Main Theorem

We prove the following theorem

Theorem . If $f:R\to R$ is 2π -periodic, Lebesgue integrable on $[-\pi,\pi]$ and belonging to $Lip(\xi(t),r)$ class then the degree of approximation of f by the (C,2)(E,1) product means

$t_n^{C_2\cdot E_1}(f;x)=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\frac{1}{2^k}\sum_{v=0}^{n}\binom{n}{v}s_v(f;x)$ of its Fourier series (1.4) is given by

$$\left\| t_n^{C_2.E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1} \right) \right).$$

3. Lemmas

3.1 Lemma 1

For $0 < t < 1/(n+1)$, $|K_n(t)| = O(n+1)$.

Proof

For $0 < t < 1/(n+1)$, $\sin nt \le n \sin t$

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\sin(\nu+1/2)}{\sin(t/2)} \right] \right|$$

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{(2\nu+1)\sin(t/2)}{\sin(t/2)} \right] \right|$$

$$\le \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} (2k+1) \sum_{\nu=0}^{k} \binom{k}{\nu} \right] \right|$$

$$= \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} [(n-k+1)(2k+1)]$$

$$= \frac{n+1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} (2k+1) - \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} [k(2k+1)]$$

$$= \frac{1}{\pi(n+2)} \sum_{k=0}^{n} (2k+1) - \frac{1}{\pi(n+1)(n+2)} \left[2\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k \right]$$

$$= \frac{(n+1)^2}{\pi(n+2)} - \frac{1}{\pi(n+1)(n+2)} \left[\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} \right]$$

$$= O(n+1).$$

3.2 Lemma 2

For $1/(n+1) \le t \le \pi$, $|K_n(t)| = O(1/t)$.

Proof

For $1/(n+1) \le t \le \pi$, applying Jordan's lemma, $\sin(t/2) \ge t/\pi$ and $\sin nt \le 1$.

$$|M_n(t)| \le \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\sin(\nu+1/2)}{\sin(t/2)} \right] \right|$$

$$\le \frac{(n+1)}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{(t/\pi)} \right] \right|$$

$$- \frac{1}{\pi(n+1)(n+2)} \left| \sum_{k=0}^{n} \left[\frac{k}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{(t/\pi)} \right] \right|$$
$$= \frac{1}{t(n+2)} \sum_{k=0}^{n} 1 - \frac{1}{t(n+1)(n+2)} \sum_{k=0}^{n} k$$
$$= O(1/t).$$

4. Proof of the Theorem

Following Titchmarsh [13] and using Riemann Lebesgue theorem, $s_n(f;x)$ of the series (1.4) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{\sin(t/2)} dt$$

Using (1), the (E,1) transforms of $s_n(f;x)$ is given by

$$t_n^{(E,1)} - f(x) = \frac{1}{\pi 2^{n+1}} \int_0^{\pi} \phi(t) \left(\sum_{k=0}^{n} \binom{n}{k} \frac{\sin(n+1/2)t}{\sin(t/2)} \right) dt$$

The (C,2) (E,1) transform of $s_n(f;x)$ is given by

$$t_n^{C_2 \cdot E_1} - f(x) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{(n-k+1)}{2^k} \int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \left\{ \sum_{\nu=0}^{k} \binom{k}{\nu} \sin(\nu+1/2)t \right\} dt \right]$$

$$= \int_0^{\pi} \phi(t) K_n(t)\, dt$$
$$= \int_0^{1/(n+1)} \phi(t) K_n(t)\, dt + \int_{1/(n+1)}^{\pi} \phi(t)\, K_n(t)\, dt$$
$$= I_1 + I_2 \qquad (4.1)$$

Now, $I_1 = \int_0^{1/(n+1)} \xi(t) \left| K_n(t) \right| dt$

$$= O\left[\int_0^{1/(n+1)} \xi(t)(n+1)\, dt \right], \text{ by Lemma (1)}$$
$$= O(n+1) \left[\int_0^{1/(n+1)} \xi(t)\, dt \right]$$

$$= \left(O(n+1)\xi\left(\frac{1}{n+1}\right)\right)\int_{\varepsilon}^{1/(n+1)} dt \text{ , where } 0<\varepsilon<1/(n+1)$$

by first mean value theorem of calculus

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right). \tag{4.2}$$

Lastly, $I_2 = O\left[\int_{1/(n+1)}^{\pi} \phi(t)\, K_n(t)\, dt\right]$

$$= O\left[\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{(n+1)t}\, dt\right], \text{ by lemma (2)}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right). \tag{4.3}$$

Combining (4.1)-(4.3), we get

$$\left\| t_n^{C_2 \cdot E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

5. Conclusions

The result of main theorem is

$$\left\| t_n^{C_2 \cdot E_1} - f \right\|_r = O\left(\xi\left(\frac{1}{n+1}\right)\right).$$

from which the results of H.K. Nigam [12] can be derived directly.

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Uniqueness of an entire function sharing fixed points with its derivatives

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Abstract

The uniqueness problems of an entire functions that share a nonzero finite value have been studied and many results on this topic have been obtained. In this paper we prove a uniqueness theorem for an entire function, which share a linear polynomial, in particular fixed points, with its higher order derivatives.

Keywords: Uniqueness; Entire functions; Fixed points; Sharing; Derivatives
2010 AMS subject classifications: 30D35. [1]

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1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. A meromorphic function $a = a(z)$ is called a small function of f if $T(r, a) = S(r, f)$, where $T(r, f)$ is the Nevanlinna characteristic function of f and $S(r, f) = \circ\{T(r, f)\}$, as $r \to \infty$, possibly outside a set of finite linear measure.

Let f and g be two non-constant meromorphic functions and $a = a(z)$ be a polynomial. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with same multiplicities. On the other hand, we say that f and g share a IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities. We express the CM sharing and IM sharing respectively by the notations $f = a \rightleftharpoons g = a$ and $f = a \Leftrightarrow g = a$.

Let $z_k(k = 1, 2, \ldots)$ be zeros of $f - a$ and t_k be the multiplicity of the zero z_k. If $z_k(k = 1, 2, \ldots)$ are also zeros of $g - a$ and the multiplicity of the zero z_k is at least t_k then we use the notation $f = a \to g = a$.

For standared definitions and notations of the distribution theory we refer the reader to Hayman [1964].

The problem of uniqueness of meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory of meromorphic function. There are some results related to value sharing.

In the begining, Jank, Mues and Volkmann Jank et al. [1986] considered the situation that an entire function shares a nonzero value with its derivatives and they prove the following result.

Theorem A. *Jank et al. [1986]. Let f be a non-constant entire function and a be a non-zero finite value. If f, $f^{(1)}$ and $f^{(2)}$ share a CM, then $f \equiv f^{(1)}$.*

Following example shows that in Theorem A the second derivative cannot be replaced by any higher order derivatives.

Example 1.1. *Let $k(\geq 3)$ be an integer and $\omega(\neq 1)$ is a $(k-1)^{th}$ root of unity. We put $f = e^{\omega z} + \omega - 1$. Then f, $f^{(1)}$ and $f^{(k)}$ share the value ω CM, but $f \not\equiv f^{(1)}$.*

On the basis of this example, Zhong improved Theorem A by considering higher order derivetives in the following way.

Theorem B. *Let f be a non-constant entire function and a be a non-zero finite number. Also let $n(\geq 1)$ be a positive integer. If f and $f^{(1)}$ share the value a CM, and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f \equiv f^{(n)}$.*

In 2002, Chang and Fang [2002] extendeed Theorem A by considering shared fixed points.

Theorem C. *Chang and Fang [2002]. Let f be a non-constant entire function. If f, $f^{(1)}$ and $f^{(2)}$ share z CM, then $f \equiv f^{(1)}$.*

Later in 2003, Wang and Yi [2003] improved Theorem A and generalize Theorem B by considering higher order derivatives in the following way.

Theorem D. *Wang and Yi [2003]. Let f be a non-constant entire function and a be a non-zero finite constant. Also let m and n be positive integers satisfying $m > n$. If f and $f^{(1)}$ share the value a CM, and if $f^{(m)}(z) = f^{(n)}(z) = a$ whenever $f(z) = a$, then*

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},$$

where $A(\neq 0)$ and λ are constants satisfying $\lambda^{n-1} = 1$ and $\lambda^{m-1} = 1$.

In this paper we improve Theorem D by considering the situation when a non-constant entire function f shares a linear plynomial $a(z) = \alpha z + \beta$, $\alpha(\neq 0)$ and β are constants, with higher order derivatives. The main result of the paper is the following theorem.

Theorem 1.1. *Let f be a non-constant entire function and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constants. Also let m amd n be two positive integers satisfying $m > n > 1$. If*

$$f(z) = a(z) \rightleftharpoons f^{(1)}(z) = a(z)$$

and

$$f(z) = a(z) \to f^{(m)}(z) = f^{(n)}(z) = a(z),$$

then

$$f(z) = Ce^z$$

or

$$f(z) = Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2},$$

where C and λ are non-zero constants.

2 Lemmas

In this section we state some necessary lemmas.

Lemma 2.1. *Ngoan and Ostrovskii [1965]. Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(\log r),$$

as $r \to \infty$.

The above lemma motivates us to prove the following:

Lemma 2.2. *Let f be an entire function of finite order and k be a positive integer. Then for any small function $a(z)$ with respect to $f(z)$,*

$$m\left(r, \frac{f^{(k)}(z) - a^{(k)}(z)}{f(z) - a(z)}\right) = o(\log r),$$

as $r \to \infty$.

Proof. Let $g(z) = f(z) - a(z)$. Then

$$g^{(k)}(z) = f^{(k)}(z) - a^{(k)}(z).$$

Now by Lemma 2.1 and using above equality, we have

$$m\left(r, \frac{g^{(k)}(z)}{g(z)}\right) = o(\log r),$$

as $r \to \infty$. This implies

$$m\left(r, \frac{f^{(k)}(z) - a^{(k)}(z)}{f(z) - a(z)}\right) = o(\log r),$$

as $r \to \infty$. This proves the lemma. □

Lemma 2.3. *Clunie [1962]. Let f be a transcendental meromorphic solution of the equation*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are polynomials in f and its derivatives with meromorphic coefficients a_j (say). If the total degree of $Q(f)$ is at most n, then

$$m(r, P(f)) \leq \sum_j m(r, a_j) + S(r, f).$$

Lemma 2.4. *Chen and Li [2014]. Let $a(z)$ be an entire function of finite order and $Q(z)$ be a non-constant polynomial. If f is an entire solution of the equation*

$$f^{(k)} - e^{Q(z)}f = a(z)$$

such that $\rho(f) > \rho(a)$, then $\rho(f) = \infty$.

We use this Lemma to prove the following one.

Lemma 2.5. *Let f be a non-constant entire function of finite order and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constant. Also let k be a positive integer. If $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, then*

$$\frac{f^{(k)}(z) - a(z)}{f(z) - a(z)} \equiv c, \tag{2.1}$$

for some nonzero constant c.

Proof. Since f has finite order and since $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, it follows from the Hadamard factorization theorem that

$$\frac{f^{(k)}(z) - a(z)}{f(z) - a(z)} \equiv e^{Q(z)}, \tag{2.2}$$

where $Q(z)$ is a polynomial.

Suppose that $F(z) = f(z) - a(z)$. Then $F^{(k)}(z) = f^{(k)}(z)$.

From (2.2) and above equality, we have

$$F^{(k)}(z) - e^{Q(z)}F(z) = a(z).$$

If $Q(z)$ is non-constant, then from above equality and by Lemma 2.4, we get F has infinite order. Since f has finite order, this is impossible. Hence $Q(z)$ is a constant. Therefore from (2.2), we obtain (2.1) for a non-zero constant c. This proves the lemma. □

Lemma 2.6. *Let f be a transcendental entire function of finite order and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constants. Also let m be a positive integer. If*

(i) $m\left(r, \frac{1}{f(z)-a(z)}\right) = S(r, f)$,

(ii) $f(z) = a(z) \rightleftharpoons f^{(1)}(z) = a(z)$

and

(iii) $f(z) = a(z) \to f^{(m)}(z) = a(z)$,

then

$$f(z) = Ce^z,$$

where C is a non-zero constant.

Proof. Let

$$h(z) = \frac{f^{(1)}(z) - a(z)}{f(z) - a(z)}. \tag{2.3}$$

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, we see that $h(z)$ is an entire function.

Now by Lemma 2.1, Lemma 2.2 and from the hypothesis of Lemma 2.6, we deduce that

$$\begin{aligned} T(r,h(z)) &= m(r,h(z)) \\ &= m\left(r, \frac{f^{(1)}(z) - a(z)}{f(z) - a(z)}\right) \\ &\leq m\left(r, \frac{f^{(1)}(z) - a^{(1)}(z)}{f(z) - a(z)}\right) + m\left(r, \frac{a^{(1)}(z) - a(z)}{f(z) - a(z)}\right) + \log 2 \\ &= S(r,f). \end{aligned} \tag{2.4}$$

We rewrite (2.3), as

$$\begin{aligned} f^{(1)}(z) &= h(z)f(z) + a(z)(1 - h(z)) \\ &= \xi_1(z)f(z) + \eta_1(z), \end{aligned} \tag{2.5}$$

where $\xi_1(z)$ and $\eta_1(z)$ are defined by

$$\xi_1(z) = h(z), \;\; \eta_1(z) = a(z)(1 - h(z)).$$

By (2.5), we have

$$\begin{aligned} f^{(2)}(z) &= \xi_1(z)f^{(1)}(z) + \xi_1^{(1)}(z)f(z) + \eta_1^{(1)}(z) \\ &= \xi_1(z)[\xi_1(z)f(z) + \eta_1(z)] + \xi_1^{(1)}(z)f(z) + \eta_1^{(1)}(z) \\ &= [\xi_1^{(1)}(z) + \xi_1(z)\xi_1(z)]f(z) + \eta_1^{(1)}(z) + \eta_1(z)\xi_1(z) \\ &= \xi_2(z)f(z) + \eta_2(z), \end{aligned}$$

where

$$\xi_2(z) = \xi_1^{(1)}(z) + \xi_1(z)\xi_1(z) \;\; and \;\; \eta_2(z) = \eta_1^{(1)}(z) + \eta_1(z)\xi_1(z).$$

Now from above equality and using (2.5), we get

$$\begin{aligned} f^{(3)}(z) &= \xi_2(z)f^{(1)}(z) + \xi_2^{(1)}(z)f(z) + \eta_2^{(1)}(z) \\ &= [\xi_2^{(1)}(z) + \xi_1(z)\xi_2(z)]f(z) + \eta_2^{(1)}(z) + \eta_1(z)\xi_2(z) \\ &= \xi_3(z)f(z) + \eta_3(z), \end{aligned}$$

where

$$\xi_3(z) = \xi_2^{(1)}(z) + \xi_1(z)\xi_2(z) \ \ and \ \ \eta_3(z) = \eta_2^{(1)}(z) + \eta_1(z)\xi_2(z).$$

Similarly,

$$f^{(k)}(z) \ = \ \xi_k(z)f(z) + \eta_k(z), \tag{2.6}$$

where

$$\xi_{k+1}(z) = \xi_k^{(1)}(z) + \xi_1(z)\xi_k(z) \tag{2.7}$$

and

$$\eta_{k+1}(z) \ = \ \eta_k^{(1)}(z) + \eta_1(z)\xi_k(z). \tag{2.8}$$

Puting $k = 1$ in (2.7), we have

$$\begin{aligned} \xi_2(z) &= \xi_1^{(1)}(z) + \xi_1(z)\xi_1(z) \\ &= h^2(z) + h^{(1)}(z). \end{aligned}$$

Again puting $k = 2$ in (2.7), we get

$$\begin{aligned} \xi_3(z) &= \xi_2^{(1)}(z) + \xi_1(z)\xi_2(z) \\ &= \left[h^2(z) + h^{(1)}(z)\right]^{(1)} + h(z)[h^2(z) + h^{(1)}(z)] \\ &= h^3(z) + h^{(2)}(z) + 3h(z)h^{(1)}(z). \end{aligned}$$

Similarly,

$$\xi_4(z) = h^4(z) + h^{(3)}(z) + 4h(z)h^{(2)}(z) + 3\left[2h^2(z) + h^{(1)}(z)\right]h^{(1)}(z).$$

Hence using mathematical induction, one can easily check

$$\xi_k(z) \ = \ h^k(z) + P_{k-1}(z, h(z)), \tag{2.9}$$

where $P_{k-1}(z, h(z))$ is a polynomial such that total degree $degP_{k-1}(z, h(z)) \leq k-1$ in $h(z)$ and its derivatives, and all coefficients in $P_{k-1}(z, h(z))$ are constants.

Now putting $k = 1$ in (2.8), we have

$$\begin{aligned} \eta_2(z) &= \eta_1^{(1)}(z) + \eta_1(z)\xi_1(z) \\ &= [a(z)(1 - h(z))]^{(1)} + a(z)(1 - h(z))h(z) \\ &= -a(z)h^2(z) - a(z)h^{(1)}(z) + (a(z) - \alpha)h(z) + \alpha. \end{aligned}$$

Again putting $k = 2$ in (2.8), we get

$$\begin{aligned}\eta_3(z) &= \eta_2^{(1)}(z) + \eta_1(z)\xi_2(z)\\ &= \left[-a(z)h^2(z) - a(z)h^{(1)}(z) + (a(z)-\alpha)h(z) + \alpha\right]^{(1)}\\ &\quad +a(z)(1-h(z))(h^2(z) + h^{(1)}(z))\\ &= -a(z)h^3(z) - a(z)h^{(2)}(z) + [2a(z) - 3a(z)h(z) - 2\alpha]\, h^{(1)}(z)\\ &\quad +(a(z)-\alpha)h^2(z) + \alpha h(z).\end{aligned}$$

Similarly,

$$\begin{aligned}&= -a(z)h^4(z) - a(z)h^{(3)}(z) + [3a(z) - 4a(z)h(z) - 3\alpha]\, h^{(2)}(z)\\ &\quad + \left[5a(z)h(z) - 5\alpha h(z) - 6a(z)h^2(z) - 3a(z)h^{(1)}(z) + 3\alpha\right] h^{(1)}(z)\\ &\quad +(a(z)-\alpha)h^3(z) + \alpha h^2(z).\end{aligned}$$

Like the previous one, it can be easily verified that

$$\eta_k(z) = -a(z)h^k(z) + Q_{k-1}(z, h(z)), \tag{2.10}$$

where $Q_{k-1}(z, h(z))$ is a polynomial such that total degree $degQ_{k-1}(z, h(z)) \leq k-1$ in $h(z)$ and its derivatives, and all coefficients in $Q_{k-1}(z, h(z))$ are either constants or polynomial $a(z)$.

From (2.4) and (2.9), for $k = 1, 2, \cdots$, we have

$$\begin{aligned}T(r, \xi_k(z)) &= T(r, h^k(z) + P_{k-1}(z, h(z)))\\ &\leq T(r, h^k(z)) + T(r, P_{k-1}(z, h(z))) + \log 2\\ &= S(r, f).\end{aligned}$$

Similarly,

$$T(r, \eta_k(z)) = S(r, f).$$

From hypothesis of Lemma 2.6, we have

$$\begin{aligned}N\left(r, \frac{1}{f(z)-a(z)}\right) &= T(r, f(z)) - m\left(r, \frac{1}{f(z)-a(z)}\right) + O(1)\\ &= T(r, f(z)) + S(r, f),\end{aligned} \tag{2.11}$$

which implies that $f(z) - a(z)$ must have zeros.

Let z_j be a zero of $f(z) - a(z)$ with multiplicity $\delta(j)$. Since $f(z) = a(z) \to f^{(m)}(z) = a(z)$, we see that z_j is also a zero of $f^{(m)}(z) - a(z)$ with multiplicity at least $\delta(j)$. Hence $f(z_j) = a(z_j)$ and $f^{(m)}(z_j) = a(z_j)$.

It follows from (2.6) that, for $k = m$,

$$f^{(m)}(z) = \xi_m(z)f(z) + \eta_m(z) \tag{2.12}$$

and then

$$a(z_j) \;=\; a(z_j)\xi_m(z_j) + \eta_m(z_j).$$

Now we shall prove that,

$$a(z) \equiv a(z)\xi_m(z) + \eta_m(z). \tag{2.13}$$

Otherwise,

$$a(z)\xi_m(z) + \eta_m(z) - a(z) \not\equiv 0.$$

From (2.12), we have

$$a(z)\xi_m(z) + \eta_m(z) - a(z) \;=\; (f^{(m)}(z) - a(z)) - \xi_m(z)(f(z) - a(z)).$$

By the reasoning as mentioned above, we deduce that z_j is a zero of $(f^{(m)}(z) - a(z)) - \xi_m(z)(f(z) - a(z))$, that is, a zero of $a(z)\xi_m(z) + \eta_m(z) - a(z)$ with multiplicity at least $\delta(j)$. It follows from this and the fact that $\xi_m(z)$ and $\eta_m(z)$ are small functions of $f(z)$,

$$\begin{aligned} N\left(r, \frac{1}{f(z) - a(z)}\right) &\leq N\left(r, \frac{1}{a(z)\xi_m(z) + \eta_m(z) - a(z)}\right) \\ &\leq T\left(r, \frac{1}{a(z)\xi_m(z) + \eta_m(z) - a(z)}\right) \\ &= S(r, f), \end{aligned}$$

which contradicts (2.11). Thus

$$a(z) \;\equiv\; a(z)\xi_m(z) + \eta_m(z),$$

which is (2.13).

Now by induction we prove that

$$\eta_{k+1}(z) + a(z)\xi_{k+1}(z) \;=\; (a(z) - \alpha)h^k(z) + R_{k-1}(z, h(z)), \tag{2.14}$$

where $R_{k-1}(z, h(z))$ is a polynomial such that $deg R_{k-1}(z, h(z)) \leq k - 1$ in $h(z)$ and its derivatives, and all the coefficients in $R_{k-1}(z, h(z))$ are constants or polynomial $a(z)$.

Firstly, from (2.7), (2.8) and for $k = 1$, we have

$$\begin{aligned}\eta_2(z) + a(z)\xi_2(z) &= \eta_1^{(1)}(z) + \eta_1(z)\xi_1(z) + a(z)\left[\xi_1^{(1)}(z) + \xi_1(z)\xi_1(z)\right] \\ &= [a(z)(1-h(z))]^{(1)} + a(z)(1-h(z))h(z) + a(z)h^{(1)}(z) \\ &\quad +a(z)h^2(z) \\ &= a(z)(-h^{(1)}(z)) + \alpha(1-h(z)) + a(z)h(z) - a(z)h^2(z) \\ &\quad +a(z)h^{(1)}(z) + a(z)h^2(z) \\ &= (a(z)-\alpha)h(z) + \alpha.\end{aligned}$$

Secondly, we suppose that the following equation holds

$$\eta_k(z) + a(z)\xi_k(z) = (a(z)-\alpha)h^{k-1}(z) + R_{k-2}(z, h(z)).$$

Now, by (2.7)–(2.10), we deduce that

$$\begin{aligned}\eta_{k+1}(z) + a(z)\xi_{k+1}(z) &= \eta_k^{(1)}(z) + \eta_1(z)\xi_k(z) + a(z)(\xi_k^{(1)}(z) + \xi_1(z)\xi_k(z)) \\ &= \left[-a(z)h^k(z) + Q_{k-1}(z, h(z))\right]^{(1)} + a(z)(1-h(z))\xi_k(z) \\ &\quad +a(z)\left[h^k(z) + P_{k-1}(z, h(z))\right]^{(1)} + a(z)h(z)\xi_k(z) \\ &= -ka(z)h^{k-1}(z) - \alpha h^k(z) + [Q_{k-1}(z, h(z))]^{(1)} + a(z)\xi_k(z) \\ &\quad -a(z)h(z)\xi_k(z) + ka(z)h^{k-1}(z) + a(z)\left[P_{k-1}(z, h(z))\right]^{(1)} \\ &\quad +a(z)h(z)\xi_k(z) \\ &= a(z)\left[h^k(z) + (P_{k-1}(z, h(z)))\right] - \alpha h^k(z) + [Q_{k-1}(z, h(z))]^{(1)} \\ &\quad +a(z)\left[P_{k-1}(z, h(z))\right]^{(1)} \\ &= (a(z)-\alpha)h^k(z) + R_{k-1}(z, h(z)),\end{aligned}$$

which proves (2.14).

From (2.13) and (2.14), we obtain

$$(a(z)-\alpha)h^{m-1}(z) + R_{k-2}(z, h(z)) \equiv a(z). \tag{2.15}$$

Clearly, $R_{k-2}(z, h(z)) \not\equiv a(z)$. Othewise, from (2.3), (2.15) and the hypothesis of Lemma 2.6, we have a contradiction. Hence by Lemma 2.3 and from (2.15), we can deduce that $h(z)$ must be constant.

From (2.7) and $\xi_1(z) = h(z)$, we have

$$\xi_2(z) = h^2(z), \;\; \xi_3(z) = h^3(z), \;\; \xi_4(z) = h^4(z).$$

Similarly,

$$\xi_k(z) = h^k(z), \;\; for \;\; k = 1, 2, \ldots. \tag{2.16}$$

Now, from (2.8) and $\eta_1(z) = a(z)(1 - h(z))$, we get

$$\begin{aligned}\eta_2(z) &= (1 - h(z))(\alpha + a(z)h(z)),\\ \eta_3(z) &= (1 - h(z))(\alpha + a(z)h(z))h(z),\\ \eta_4(z) &= (1 - h(z))(\alpha + a(z)h(z))h^2(z).\end{aligned}$$

Similarly,

$$\eta_k(z) = (1 - h(z))(\alpha + a(z)h(z))h^{k-2}(z), \ \ for \ \ k = 2, 3, \ldots. \quad (2.17)$$

From (2.13), (2.16) and (2.17), we have

$$\begin{aligned}a(z) &\equiv a(z)h^m(z) + (1 - h(z))(\alpha + a(z)h(z))h^{m-2}(z)\\ &\equiv h^{m-2}(z)\left[a(z)h^2(z) + \alpha(1 - h(z)) + a(z)h(z) - a(z)h^2(z)\right]\\ &\equiv h^{m-2}(z)\left[a(z)h(z) + \alpha(1 - h(z))\right],\end{aligned}$$

which implies that $h(z) = 1$.

Hence from (2.3) and $h(z) = 1$, we can obtain

$$f^{(1)}(z) = f(z).$$

This implies

$$f(z) = Ce^z.$$

where $C(\neq 0)$ ia a constant. This proof the Lemma 2.6.

□

3 Proof of the theorem 1.1

First we verify that $f(z)$ cannot be a polynomial. Let $f(z)$ be a polynomial of degree 1. Suppose that $f(z) = A_1z + B_1$, where $A_1(\neq 0)$ and B_1 are constants. Then $f^{(1)}(z) = A_1$, $f^{(m)}(z) \equiv 0 \equiv f^{(n)}(z)$. Now $\frac{\beta - B_1}{A_1 - \alpha}$ is the only zero of $f(z) - a(z)$, $\frac{A_1 - \beta}{\alpha}$ is the only zero of $f^{(1)}(z) - a(z)$ and $-\frac{\beta}{\alpha}$ is the only zero of $f^{(m)}(z) - a(z)$. Since $f(z)$ and $f^{(1)}(z)$ share polynomial $a(z)$ CM and the zeros of $f(z) - a(z)$ are the zeros of $f^{(m)}(z) - a(z)$, we have $\frac{A_1 - \beta}{\alpha} = -\frac{\beta}{\alpha}$ and so $A_1 = 0$, a contradiction.

Now let $f(z)$ be a polynomial of degree greater than 1. Suppose that $\deg(f(z)) = p$. Then $\deg(f(z) - a(z)) = p$ and $\deg(f^{(1)}(z) - a(z)) = p - 1$, it contradicts the fact that $f(z)$ and $f^{(1)}(z)$ share polynomial $a(z)$ CM.

Hence $f(z)$ is a transcendental entire function. Thus $T(r, a(z)) = S(r, f)$.

To prove the theorem let us consider two functions defined as follows.

$$\Phi(z) = \frac{(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))}{f(z) - a(z)} \tag{3.1}$$

and

$$\Psi(z) = \frac{(a(z) - a^{(1)}(z))f^{(n)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))}{f(z) - a(z)}. \tag{3.2}$$

Then $\Phi(z) \not\equiv \Psi(z)$.

We know from the hypothesis of Theorem 1.1 that $\Phi(z)$ and $\Psi(z)$ are entire functions. Then, by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} T(r, \Phi(z)) &= m(r, \Phi(z)) \\ &= m\left(r, \frac{(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))}{f(z) - a(z)}\right) \\ &\leq m\left(r, (a(z) - a^{(1)}(z))\frac{f^{(m)}(z)}{f(z) - a(z)}\right) + m\left(r, a(z)\frac{(f^{(1)}(z) - a^{(1)}(z))}{f(z) - a(z)}\right) \\ &\quad + \log 2 \\ &= S(r, f). \end{aligned}$$

Similarly,

$$T(r, \Psi(z)) = S(r, f).$$

We shall the following three cases.

Case 1. First we suppose that $\Phi(z) \not\equiv 0$. Then by (3.1), we have

$$f(z) = a(z) + \frac{1}{\Phi(z)}\{(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))\}. \tag{3.3}$$

From (3.1) and (3.2), we get

$$f^{(1)}(z) = \frac{(a(z) - a^{(1)}(z))}{a(z)(\Phi(z) - \Psi(z))}(\Phi(z)f^{(n)}(z) - \Psi(z)f^{(m)}(z)) + a^{(1)}(z).$$

Therefore

$$\begin{aligned} f^{(1)}(z) - a(z) &= \frac{(a(z) - a^{(1)}(z))}{a(z)(\Phi(z) - \Psi(z))}(\Phi(z)f^{(n)}(z) - \Psi(z)f^{(m)}(z)) \\ &\quad + a^{(1)}(z) - a(z). \end{aligned} \tag{3.4}$$

First we suppose that $m > n > 2$. Then from (3.4), we get

$$\begin{aligned}\frac{1}{f^{(1)}(z)-a(z)} &= \frac{1}{a(z)(\Phi(z)-\Psi(z))}\frac{(\Phi(z)f^{(n)}(z)-\Psi(z)f^{(m)}(z))}{f^{(1)}(z)-a(z)} \\ &\quad +\frac{1}{a^{(1)}(z)-a(z)}. \end{aligned} \tag{3.5}$$

Using Lemma 2.1 and from (3.5), we have

$$m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)=S(r,f). \tag{3.6}$$

Next we suppose $m > n = 2$. Then from (3.4), we get

$$\begin{aligned}(f^{(1)}(z)-a(z))(\Phi(z)-\Psi(z))a(z) &= \gamma(z)+(a(z)-a^{(1)}(z))\Phi(z)(f^{(2)}(z)-a^{(1)}(z)) \\ &\quad -(a(z)-a^{(1)}(z))\Psi(z)f^{(m)}(z), \end{aligned} \tag{3.7}$$

where

$$\gamma(z)=(a(z)-a^{(1)}(z))(\Phi(z)a^{(1)}(z)-(\Phi(z)-\Psi(z))a(z)).$$

Clearly $\gamma(z)\not\equiv 0$.
If $\gamma(z)\equiv 0$, then

$$\Phi(z)\equiv\frac{a(z)}{a(z)-a^{(1)}(z)}\Psi(z),$$

which is a contradiction because $\Phi(z)$ and $\Psi(z)$ are entire functions and $\Psi(z)\neq 0$ when $a(z)-a^{(1)}(z)=0$.

Now from (3.7) we get

$$\begin{aligned}\frac{1}{f^{(1)}(z)-a(z)} &= \frac{(\Phi(z)-\Psi(z))a(z)}{\gamma(z)}-\frac{a(z)-a^{(1)}(z)}{\gamma(z)}\Phi(z)\frac{f^{(2)}(z)-a^{(1)}(z)}{f^{(1)}(z)-a(z)} \\ &\quad +\frac{a(z)-a^{(1)}(z)}{\gamma(z)}\Psi(z)\frac{f^{(m)}(z)}{f^{(1)}(z)-a(z)}. \end{aligned} \tag{3.8}$$

Again using Lemma 2.1 and from (3.8), we get

$$m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)=S(r,f),$$

which is (3.6).

Since $\Phi(z) \not\equiv 0$, it follows from (3.3) and Lemma 2.1 that

$$\begin{aligned}
T(r, f(z)) &= m(r, f(z)) \\
&= m\left(r, a(z) + \frac{1}{\Phi(z)}\{(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))\}\right) \\
&= m\left(r, a(z) + \frac{(a(z) - \alpha)f^{(m)}(z) - a(z)f^{(1)}(z) + a(z)\alpha}{\Phi(z)}\right) \\
&\leq m(r, a(z)) + m\left(r, \frac{(a(z) - \alpha)f^{(m)}(z) - a(z)f^{(1)}(z)}{\Phi(z)}\right) + m\left(r, \frac{\alpha a(z)}{\Phi(z)}\right) \\
&\quad + \log 3 \\
&= m\left(r, a(z)f^{(1)}(z)\frac{\frac{(a(z)-\alpha)}{a(z)}\frac{f^{(m)}(z)}{f^{(1)}(z)} - 1}{\Phi(z)}\right) + S(r, f) \\
&\leq m\left(r, \frac{\frac{(a(z)-\alpha)}{a(z)}\frac{f^{(m)}(z)}{f^{(1)}(z)} - 1}{\Phi(z)}\right) + m(r, f^{(1)}(z)) + S(r, f) \\
&\leq m\left(r, \frac{f^{(m)}(z)}{f^{(1)}(z)} - 1\right) + m(r, f^{(1)}(z)) + S(r, f) \\
&= T(r, f^{(1)}(z)) + S(r, f). \qquad (3.9)
\end{aligned}$$

Applying Lemma 2.1, We can easily see that

$$\begin{aligned}
T(r, f^{(1)}(z)) &= m(r, f^{(1)}(z)) \\
&= m\left(r, \frac{f^{(1)}(z)}{f(z)} \cdot f(z)\right) \\
&\leq m\left(r, \frac{f^{(1)}(z)}{f(z)}\right) + m(r, f(z)) \\
&= m(r, f(z)) + S(r, f) \\
&\leq T(r, f(z)) + S(r, f). \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10), we have

$$T(r, f^{(1)}(z)) = T(r, f(z)) + S(r, f). \qquad (3.11)$$

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, by using (3.6) and (3.11) together with

the First Fundamental Theorem, we obtain

$$\begin{aligned} m\left(r,\frac{1}{f(z)-a(z)}\right) &= T(r,f(z))-N\left(r,\frac{1}{f(z)-a(z)}\right)+O(1) \\ &= T(r,f^{(1)}(z))-N\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)+S(r,f) \\ &= m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)+N\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right) \\ &\quad -N\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)+S(r,f) \\ &= m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)+S(r,f) \\ &= S(r,f). \end{aligned}$$

Hence by Lemma 2.6, we have

$$f(z)=Ce^{z},$$

where $C(\neq 0)$ is a constant.

Case 2. Now we suppose that $\Psi(z)\not\equiv 0$. Then following the similar arguments of Case-1 and using Lemma 2.6, we have

$$f(z)=Ce^{z}.$$

where $C(\neq 0)$ is a constant.

Case 3. Finally we suppose that $\Phi(z)\equiv 0$ and $\Psi(z)\equiv 0$. Then from (3.1) and (3.2), we get

$$(a(z)-a^{(1)}(z))f^{(m)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))\equiv 0 \tag{3.12}$$

and

$$(a(z)-a^{(1)}(z))f^{(n)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))\equiv 0. \tag{3.13}$$

Now subtracting (3.13) from (3.12), we have

$$(a(z)-a^{(1)}(z))(f^{(m)}(z)-f^{(n)}(z))\equiv 0.$$

Since $a(z)\not\equiv a^{(1)}(z)$, we get

$$f^{(m)}(z)\equiv f^{(n)}(z).$$

Solving this we have

$$f(z) = p_0 + p_1 e^{t_1 z} + p_2 e^{t_2 z} + \cdots + p_{m-n} e^{t_{m-n} z},$$

where t_1, t_2, $\cdots$, t_{m-n} are distinct $(m-n)^{th}$ roots of unity and p_0, p_1, p_2, $\cdots$ p_{m-n} are constants.

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, applying Lemma 2.5, we get

$$\frac{f^{(1)}(z) - a(z)}{f(z) - a(z)} = \lambda,$$

for some nonzero constant λ.

Solving above equality, we obtain

$$f(z) = Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2},$$

where $C(\neq 0)$ is a constant. This completes the proof of Theorem 1.1.

4 Conclusions

After the above discussion we arrive at the conclusion that if an entire function and its first derivative share a linear polynomial with counting multiplicity and it partially shares the linear polynomial with its two higher order derivatives then the funtion is either one of the following two forms.
(i) $f(z) = Ce^z$,
(ii) $f(z) = Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2}$,
where C and λ are non-zero constants.

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On commutativity of prime and semiprime rings with generalized derivations

Md Hamidur Rahaman*

Abstract

Let R be a prime ring, extended centroid C and $m, n, k \geq 1$ are fixed integers. If R admits a generalized derivation F associated with a derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ or $(F(x) \circ_m y)^k + x \circ_n d(y)$=0 for all $x, y \in I$, where I is a nonzero ideal of R, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$. Moreover, we also examine the case R is a semiprime ring.

Keywords: Prime rings, Semiprime rings, Extended centroid, Utumi quotient rings, Generalized derivations.
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1 Introduction

R is always an associative ring with centre $Z(R)$, extended centroid C and Utumi quotient ring U. For further information, definitions and properties on these concepts refer to Beidar and Martindale [1996]. For any $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and Jordan commutator $xy + yx$ respectively. Given $x, y \in R$, we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$ and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m \geqslant 2$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies that $a = 0$. Every prime ring is a semiprime ring but semiprime ring need not be prime ring. The socle of a ring R denoted by $Soc(R)$ is the sum of the minimal left (right) ideals of R, if R has minimal left

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(right) ideals, otherwise $Soc(R) = (0)$. The goal of this paper is to establish that there is a relationship between the structure of the ring R and the behaviour of suitable additive mappings defined on R that satisfy certain special identities. In particular we study the case when the map is a generalized derivation of R. We recall that an additive map $d : R \to R$ is a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $I_a(x) = [a, x]$ for all $x \in R$ and d is outer derivation if it is not inner derivation. An additive map $F : R \to R$ is said to be a generalized derivation if there is a derivation d of R such that for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. All derivations are generalized derivations but generalized derivation need to be derivation. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see Ashraf and Rehman [2002] , where further references can be found). In Argaç and Inceboz [2009], Argaç and Inceboz proved that: If R is a prime ring, I a nonzero ideal of R, k a fixed positive integer and R admits a nonzero derivation d with the property $(d(x \circ y))^k = x \circ y$ for all $x, y \in I$, then R is commutative. In [Quadri et al., 2003, Theorem 2.3], Quadri et al. discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al. proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In Ashraf and Rehman [2002], Ashraf and Rehman proved that if R is a 2-torsion free prime ring, I a nonzero ideal of R and d a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative.

The present paper is motivated by the previous results and we here generalized the result obtained in Ashraf and Rehman [2002] . Moreover, we continue this line of investigation by examining what happens if a ring R satisfies the following identities:

$$(i)(F(x) \circ y)^m + (x \circ d(y))^n = 0, (ii)(F(x) \circ_m y)^k + x \circ_n d(y) = 0$$

for all $x, y \in I$, a nonzero ideal of R. We obtain some analogous results for semiprime ring in the case $I = R$. More precisely, we shall prove the following theorems:

Theorem 1.1. Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Theorem 1.2. Let R be a prime ring, I a nonzero ideal of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Theorem 1.3. Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in R$, then R is commutative.

Theorem 1.4. Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in R$, then R is commutative.

2 Preliminary results

Before starting our results, we state some well known facts which are very crucial for developing the proof of our main result. In particular, we will make frequent use of the following facts.

Fact 2.1. *[Lee, 1992, Theorem 2] If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities.*

Fact 2.2. *[Lee, 1999, Theorem 4] Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R is uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Moreover, a and d are uniquely determined by the generalized derivation F.*

Fact 2.3. *[Chuang, 1988, Theorem 2] If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities with coefficients in U.*

Fact 2.4. *Let R be a prime ring and d a nonzero derivation on R and I be a nonzero ideal of R. By Kharchenko's Theorem [Kharchenko, 1978, Theorem 2], if I satisfies the differential polynomial identity $P(x_1, x_2, ..., x_n, d(x_1), d(x_2), ..., d(x_n)) = 0$, then either d is an inner derivation or d is outer derivation and I satisfies the generalized polynomial identity $P(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) = 0$.*

Fact 2.5. *[Chuang, 1994, Page no. 38] If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 2.6. *[Lee, 1992, Lemma 2] Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.*

Fact 2.7. *[Chuang, 1994, Page no. 42] Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a $B - algebra$ orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.*

3 Main Results

Theorem 1.1 Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a generalized derivation F associated with a derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$, then either R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Proof. By hypothesis

$$(F(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in I. \tag{1}$$

By the Fact 2.1 I, R and U satisfy the same generalized polynomial identity (GPI), we have

$$(F(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{2}$$

Since R is a prime ring and F a generalized derivation of R, by Fact 2.2, $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Then U satisfies

$$((ax + d(x)) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{3}$$

That is

$$(ax \circ y + d(x) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{4}$$

In the light of Kharchenko's theorem [Kharchenko, 1978, Theorem 2], we divide the proof into two cases:-
Case I Let d be an inner derivation of U, that is, $d(x) = [q, x]$ for all $x \in U$ and for some $q \in U$. Then U satisfies

$$F(x, y) = (ax \circ y + [q, x] \circ y)^m + (x \circ [q, y])^n = 0 \text{ for all } x, y \in U. \tag{5}$$

In case C is infinite, we have $F(x, y) = 0$ for all $x, y \in U \bigotimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \bar{C}$ are prime and centrally

closed Erickson et al. [1975], we may replace R by U or $U \bigotimes_C \bar{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $F(x, y) = 0$ for all $x, y \in R$. By Martindale's theorem [Martindale, 1969, Theorem 3], R is then a primitive ring having nonzero socle, $soc(R)$ with D as associative division ring. Hence by Jacobson's theorem [Jacobson, 1956, p.75], R is isomorphic to dense ring of linear transformations of vector space V over C. Then the density of R on V implies that $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Suppose that there exists $v \in V$ such that v and qv are linearly D-independent. Since $dim_D V \geqslant 2$, then there exists $w \in V$ such that $\{v, qv, w\}$ is linearly independent over D. By density of R there exist $x, y \in R$ such that

$$xv = 0, xqv = w, yw = v, xw = 0, yv = 0, yqv = v. \tag{6}$$

Multiplying equation (5) by v from right and using conditions in equation (6), we get $(-1)^m v = 0$, a contradiction. Now we want to show that $qv = v\beta$ for some $\beta \in D$. Let v, w be linearly independent. Then by the precedant argument, there exist β_v, β_w, $\beta_{v+w} \in D$, such that $qv = v\beta_v$, $qw = w\beta_w$, $q(v+w) = (v+w)\beta_{v+w}$. Moreover, $v\beta_v + w\beta_w = (v+w)\beta_{v+w}$ and hence $v(\beta_v - \beta_{v+w}) + w(\beta_w - \beta_{v+w}) = 0$. Since v, w are linearly independent, we have $\beta_v = \beta_w = \beta_{v+w}$ that is β does not depend on the choice of v.

Let now for $r \in R$, $v \in V$, by precedant calculation, $qv = v\alpha$, $r(qv) = r(v\alpha)$ and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$ for any $v \in V$, that is, $[q, r]V = 0$. Since V is a left faithful irreducible R -module, $[q, r] = 0$ for all $r \in R$ i.e., $q \in Z(R)$ and $d = 0$.

Case 2 Let d be an outer derivation of R. Then by Fact 2.4, I satisfies the generalized polynomial identity

$$(ax \circ y + t \circ y)^m + (x \circ t)^n = 0 \text{ for all } x, y, t \in I. \tag{7}$$

In particular, for $y = 0$, I satisfies $(xt + tx)^n = 0$. By Chuang [Chuang, 1988, Theorem 2], this polynomial identity is also satisfied by Q and hence R as well. By Lemma 1 Lanski [1993], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over a field F, where $k \geq 1$. Moreover, R and $M_k(F)$ satisfy the same polynomial identity, that is $(xt + tx)^n = 0$ for all $t, x \in M_k(F)$. Let e_{ij} be the usual matrix unit with 1 in the (i, j) entry and zero elsewhere. By choosing $t = e_{12}, x = e_{21}$, we see that $(xt + tx)^n = (e_{11} + e_{22})^n \neq 0$, a contradiction.

Theorem 1.2 Let R be a prime ring, I a nonzero ideal of R and m, n, k are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$, then either

R is commutative or there exist $b \in U$, Utumi ring of quotient of R such that $F(x) = bx$ for all $x \in R$.

Proof. By hypothesis

$$(F(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in I. \tag{8}$$

By the Fact 2.1 I, R and U satisfy the same generalized polynomial identity (GPI), we have

$$(F(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{9}$$

By Fact 2.2, $F(x) = ax + d(x)$ for some $a \in U$ and derivation d on U. Then U satisfies

$$((ax + d(x)) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{10}$$

That is

$$(ax \circ_m y + d(x) \circ_m y)^k + x \circ_n d(y) = 0 \text{ for all } x, y \in U. \tag{11}$$

In the light of Kharchenko's theorem [Kharchenko, 1978, Theorem 2], we divide the proof into two cases:-

Case I Let d be an inner derivation of U that is $d(x) = [q, x]$ for all $x \in U$ and for some $q \in U$. Then U satisfies

$$(ax \circ_m y + [q, x] \circ_m y)^k + x \circ_n [q, y] = 0 \text{ for all } x, y \in U. \tag{12}$$

As in the proof of Theorem 1.1, we have

$$(ax \circ_m y + [q, x] \circ_m y)^k + x \circ_n [q, y] = 0 \text{ for all } x, y \in R. \tag{13}$$

where R is a primitive ring with D as the associated division ring. If V is finite dimensional over D, then the density of R implies that $R \cong M_k(D)$, where $k = dim_D V$. Assume that $dim_D V \geqslant 2$, otherwise we are done. Suppose that there exists $v \in V$ such that v and qv are linearly D-independent. Since $dim_D V \geqslant 2$, then there exists $w \in V$ such that $\{v, qv, w\}$ is linearly independent over D. By density of R there exist $x, y \in R$ such that

$$xv = 0, xqv = w, yw = v, xw = 0, yv = 0, yqv = v.$$

Multiplying equation (13) by v from right, we get $(-1)^{mk}v = 0$ which is a contradiction to the linearly independent of the set $\{v, qv\}$. Therefore, $\{v, qv\}$ is linearly

dependent and so $q \in Z(R)$, i.e, $d = 0$.
Case 2 Let d be an outer derivation. Then

$$(ax \circ_m y + t \circ_m y)^k + x \circ_n s = 0 \text{ for all } x, y, s, t \in I. \tag{14}$$

In particular, choosing $y = 0$, we have $s \circ_n x = 0$. By Chuang [Chuang, 1988, Theorem 2], this polynomial identity is also satisfied by Q and hence R as well. By Lemma 1 Lanski [1993], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over a field F, where $k \geq 1$. Moreover, R and $M_k(F)$ satisfy the same polynomial identity, that is, $s \circ_n x = 0$ for all $s, x \in M_k(F)$. Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By choosing $s = e_{12}, x = e_{11}$, we see that $s \circ_n x = e_{12} \neq 0$, a contradiction.

The following examples demonstrate that R to be $prime$ can not be omitted in the hypothesis of Theorem 1.1 and Theorem 1.2.

Example 3.1. *For any ring S, let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in S \right\}$. Then R is a ring under usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, d : R \to R$ by $F(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation on R associated with the nonzero derivation d satisfying $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in I$. However R is not commutative as well as F can not be written as $F(x) = bx$ for all $x \in R$ as d is nonzero. Hence Theorem 1.1 is not true for arbitrary rings.*

Example 3.2. *Let $R = \left\{ \begin{pmatrix} x & y \\ o & z \end{pmatrix} \mid x, y, z \in Z_2 \right\}$ and $I = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in Z_2 \right\}$. Then R is a ring under usual addition and multiplication of matrices and I is a nonzero ideal of R. Define maps $F, d : R \to R$ by $F(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $d(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation on R associated with the nonzero derivation d satisfying $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in I$. However R is not commutative as well as F can not be written as $F(x) = bx$ for all $x \in R$ as d is nonzero . Hence Theorem 1.2 is not true for arbitrary rings.*

Theorem 1.3 Let R be a semiprime ring, U the left Utumi quotient ring of R and m, n are fixed positive integers. If R admits a generalized derivation F associated

with a nonzero derivation d such that $(F(x) \circ y)^m + (x \circ d(y))^n = 0$ for all $x, y \in R$, then R is commutative.

Proof. By Fact 2.6, any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U. Moreover, by Fact 2.3 I, R and U satisfy the same GPIs and by Fact 2.1 I, R, U satisfy same differential identities. Also by Fact 2.2, we have $F(x) = ax + d(x)$ for some $a \in U$ and derivation d of U. Then

$$((ax + d(x)) \circ y)^m + (x \circ d(y))^n = 0 \text{ for all } x, y \in U. \tag{15}$$

By Fact 2.5, we have $Z(U) = C$. Let $M(C)$ be the set of all maximal ideals of C and $P \in M(C)$. By Fact 2.7, we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P \in M(C)\} = 0$. Set $\overline{U} = U/PU$. Then derivation d canonically induce derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in \bar{U}$. Therefore,

$$((\bar{a}\bar{x} + \overline{d(x)}) \circ \bar{y})^m + (\overline{x} \circ \overline{d(y)})^n = 0$$

for all $\overline{x}, \overline{y} \in \overline{U}$. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 1.1, we have for each $P \in M(C)$ either $[U, U] \subseteq PU$ or $d(U) \subseteq PU$. In any case $d(U)[U, U] \subseteq PU$ for all $P \in M(C)$. Thus $d(U)[U, U] \subseteq \bigcap\{PU \mid P \in M(C)\} = 0$, we obtain $d(U)[U, U] = 0$. Therefore, $[U, U] = 0$ since $\bigcap\{PU \mid P \in M(C)\} = 0$ and $d \neq 0$. Since R is subring of U, so in particular $[R, R] = 0$. Hence R is commutative. This completes the proof of the theorem.

Using the similar arguments as used in the proof of the above theorem, we can prove the following theorem.

Theorem 1.4 Let R be a semiprime ring, U the left Utumi quotient ring of R and k, m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ_m y)^k + x \circ_n d(y) = 0$ for all $x, y \in R$, then R is commutative.

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A note on $\alpha-$irresolute topological rings

Madhu Ram*

Abstract

In [4], we introduced the notion of $\alpha-$irresolute topological rings in Mathematics. This notion is independent of topological rings. In this note, we point out that under certain conditions an $\alpha-$irresolute topological ring is topological ring and vice versa. We prove that the Minkowski sum $\mathfrak{A} + \mathfrak{B}$ of an $\alpha-$compact subset $\mathfrak{A} \subseteq \mathcal{R}$ and an $\alpha-$closed subset $\mathfrak{B} \subseteq \mathcal{R}$ of an $\alpha-$irresolute topological ring $(\mathcal{R}, \Im)$ is actually a closed subset of $\mathcal{R}$. In the twilight of this note, we pose several natural questions which are noteworthy.

Keywords: $\alpha-$open sets, $\alpha-$closed sets, $\alpha-$irresolute topological rings.

2010 AMS subject classifications: 16W80, 16W99. [1]

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1 Introduction

Let's first introduce some notations. We denote a topological space by $(X,\ \Im)$ (or simply X) where $\Im$ is a non-trivial topology on X. For every topological space $(X,\ \Im)$, there is a finer topology on X which is called the $\alpha-$topology on X and is denoted by $\Im^{\alpha}$. In fact, $\Im^{\alpha}$ is the family of all $\alpha-$open sets in X (with respect to $\Im$). Njastad [3] showed that $\Im^{\alpha}$ is a topology on X.

In 2019 ([4]), we introduced a category of $\alpha-$regular spaces called $\alpha-$irresolute topological rings. An $\alpha-$irresolute topological ring, denoted by $(\mathcal{R},\ \Im)$, is a ring $\mathcal{R}$ that is endowed with a topology $\Im$ such that the following mappings

$$(1) \qquad (\mathcal{R},\ \Im^{\alpha}) \times (\mathcal{R},\ \Im^{\alpha}) \longmapsto (\mathcal{R},\ \Im^{\alpha})$$

$$(\varsigma,\ \xi) \longrightarrow \varsigma - \xi, \text{ for all } \varsigma,\ \xi \in \mathcal{R}$$

and

$$(2) \qquad (\mathcal{R},\ \Im^{\alpha}) \times (\mathcal{R},\ \Im^{\alpha}) \longmapsto (\mathcal{R},\ \Im^{\alpha})$$

$$(\varsigma,\ \xi) \longrightarrow \varsigma.\xi \text{ for all } \varsigma,\ \xi \in \mathcal{R}$$

are continuous.

The notion of $\alpha-$irresolute topological rings has structural nuances and conceptual niceties with the notion of topological rings. Structurally, it sounds that there may be a strong relationship between these concepts. In this note, we point out some conditions with which an $\alpha-$irresolute topological ring is topological ring and vice versa. For subsets $\mathfrak{A}$, $\mathfrak{B}$ of $\mathcal{R}$, we can define the so-called Minkowski addition of two sets as

$$\mathfrak{A} + \mathfrak{B} = \{a + b : a \in \mathfrak{A},\ b \in \mathfrak{B}\}$$

We describe that the Minkowski sum $\mathfrak{A}+\mathfrak{B}$ of $\mathfrak{A} \subseteq \mathcal{R}$, an $\alpha-$compact subset, and $\mathfrak{A} \subseteq \mathcal{R}$, an $\alpha-$closed subset, is closed subset of $\mathcal{R}$.

Definition 1.1. *A subset U of a topological space $(X,\ \Im)$ is said to be*

(1) $\alpha-$open [3] if $U \subseteq Int(Cl(Int(U)))$.
(2) semi-open [2] if $U \subseteq Cl(Int(U))$.

The complement of an $\alpha-$open (resp. semi-open) set is called $\alpha-$closed (resp. semi-closed).

2 Main results

For an $\alpha-$irresolute topological ring $(\mathcal{R},\ \Im)$, let Γ denotes the collection of all $\alpha-$open subsets of $\mathcal{R}$ containing the additive identity 0 of $\mathcal{R}$. In the sequel, we use the following lemma:

Lemma 2.1. *([4, Corollary 3.9.1]) Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Then for every $\mathfrak{V} \in \Gamma$, there exists a symmetric $\sigma \in \Gamma$ such that $\sigma + \sigma \subseteq \mathfrak{V}$.*

Definition 2.1. *A subset U of a topological space $(X,\ \Im)$ is said to be $\alpha-$compact [1] if every cover of U by $\alpha-$open subsets of X has a finite subcover.*

Theorem 2.1. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any subsets of $\mathcal{R}$ such that $\mathfrak{A}$ is $\alpha-$compact and $\mathfrak{B}$ is $\alpha-$closed satisfying $\mathfrak{A} \cap \mathfrak{B} = \emptyset$. Then there exists a set $\sigma \in \Gamma$ with the property $(\mathfrak{A}+\sigma) \cap (\mathfrak{B}+\sigma) = \emptyset$.*

Proof. Let ς be an element of $\mathfrak{A}$. By Lemma 2.1, there exists a symmetric $\sigma_\varsigma \in \Gamma$ such that

$$\varsigma + \sigma_\varsigma + \sigma_\varsigma + \sigma_\varsigma \cap \mathfrak{B} = \emptyset$$

or that

$$\varsigma + \sigma_\varsigma + \sigma_\varsigma \cap \mathfrak{B} + \sigma_\varsigma = \emptyset. \qquad (*)$$

Continuing in a similar vein, we obtain a family of $\alpha-$open subsets,

$$\pi = \{\varsigma + \sigma_\varsigma : \sigma_\varsigma \in \Gamma,\ \varsigma \in \mathfrak{A}\}.$$

Since $\mathfrak{A}$ is $\alpha-$compact, we have a finite subset $\mathfrak{J} \subseteq \mathfrak{A}$ such that

$$\mathfrak{A} \subseteq \bigcup\{\varsigma + \sigma_\varsigma : \sigma_\varsigma \in \Gamma,\ \varsigma \in \mathfrak{J}\}.$$

Consider the set

$$\sigma = \bigcap\{\sigma_\varsigma : \varsigma \in \mathfrak{J}\}.$$

Then σ is $\alpha-$open subset of $\mathcal{R}$.

By virtue of $(*)$,

$$(\mathfrak{A} + \sigma) \cap (\mathfrak{B} + \sigma) = \emptyset. \qquad \square$$

Theorem 2.2. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\mathfrak{A} \subseteq \mathcal{R}$ an $\alpha-$compact, and $\mathfrak{B} \subseteq \mathcal{R}$ an $\alpha-$closed. Then $\mathfrak{A} + \mathfrak{B}$ is $\alpha-$closed subset of $\mathcal{R}$.*

Proof. We prove this theorem by contrapositive. Let $\varsigma \notin \mathfrak{A} + \mathfrak{B}$.

Then, by Theorem 2.1, for each $\nu \in \mathfrak{A}$, there exists a set $\mathfrak{U}_\nu \in \Gamma$ such that

$$(\varsigma + \mathfrak{U}_\nu) \cap (\nu + \mathfrak{B} + \mathfrak{U}_\nu) = \emptyset. \qquad (*)$$

Consider the collection of sets obtained form this fixed ς,

$$\pi = \{\nu + \mathfrak{U}_\nu : \nu \in \mathfrak{A}\}$$

By Theorem 3.1 in [4], π is a cover of $\mathfrak{A}$ by $\alpha-$open subsets of $\mathcal{R}$. Therefore, it has a finite subcover

$$\pi' = \{\nu_i + \mathfrak{U}_{\nu_i} : \nu_i \in \mathfrak{A},\ i = 1,\ 2, ...,\ n\}. \qquad (**)$$

To obtain the sets,

$$\mathfrak{P} = \bigcap\{\mathfrak{U}_{\nu_i} : i = 1,\ 2,\ ....,\ n\},\ \mathfrak{Q} = \bigcup\{\mathfrak{U}_{\nu_i} : i = 1,\ 2,\ ....,\ n\}. \qquad (@)$$

By construction, $\mathfrak{P}$ and $\mathfrak{Q}$ are $\alpha-$open subsets of $\mathcal{R}$.

Because of (*) and (**), we have

$$(\varsigma + \mathfrak{U}_{\nu_i}) \cap (\nu_i + \mathfrak{B} + \mathfrak{U}_{\nu_i}) = \emptyset, \forall\ i = 1,\ 2,\ ..... n. \qquad (@@)$$

Then (@) and (@@) together complete the proof. □

Theorem 2.3. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\mathfrak{A} \subseteq \mathcal{R}$ an $\alpha-$compact, and $\mathfrak{B} \subseteq \mathcal{R}$ an $\alpha-$closed. Then $\mathfrak{A} + \mathfrak{B}$ is closed subset of $\mathcal{R}$.*

To prove Theorem 2.3, we need the following fact:

Theorem 2.4. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\sigma \subseteq \mathcal{R}$ an $\alpha-$compact, and $\nabla \subseteq \mathcal{R}$ an $\alpha-$closed satisfying $\sigma \cap \nabla = \emptyset$. Then there exist disjoint open subsets $\mathfrak{P}$ and $\mathfrak{Q}$ of $\mathcal{R}$ such that $\sigma \subseteq \mathfrak{P}$ and $\nabla \subseteq \mathfrak{Q}$.*

Proof. In light of Theorem 2.1, there exist $\alpha-$open subsets $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathcal{R}$ such that

$$\sigma \subseteq \mathfrak{U},\ \nabla \subseteq \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset.$$

Take $\mathfrak{P} = Int(Cl(Int(\mathfrak{U})))$ and $\mathfrak{Q} = Int(Cl(Int(\mathfrak{V})))$.

Then $\mathfrak{P},\ \mathfrak{Q} \in \Im$ with $\mathfrak{P} \cap \mathfrak{Q} = \emptyset$. It finishes the proof. □

Proof of Theorem 2.3. Let $\varsigma \notin \mathfrak{A} + \mathfrak{B}$. By Theorem 2.2, $\mathfrak{A} + \mathfrak{B}$ is $\alpha-$closed subset of $\mathcal{R}$.

In view of Theorem 2.4, there exists an open subset $\mathfrak{P} \subseteq \mathcal{R}$ such that

$$\varsigma \in \mathfrak{P} \text{ and } \mathfrak{P} \cap (\mathfrak{A} + \mathfrak{B}) = \emptyset.$$

This proves that $\mathfrak{A} + \mathfrak{B}$ is closed subset of $\mathcal{R}$. Proof is over.

We now give two theorems, Theorem 2.5 and Theorem 2.6, having conceptual niceties or that indicate the relationship between the topological rings and the $\alpha-$irresolute topological rings.

Theorem 2.5. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring such that every nowhere dense subset of $\mathcal{R}$ is $\alpha-$closed. Then $(\mathcal{R},\ \Im)$ is a topological ring.*

Proof. Let $\mathfrak{B}$ be any $\alpha-$open subset of $\mathcal{R}$. Then $\mathfrak{B} = \mathfrak{O} - \nabla$ where $\mathfrak{O}$ is an open subset of $\mathcal{R}$ and ∇ is nowhere dense subset of $\mathcal{R}$. Now,

Let $\varsigma \in \mathfrak{B}$ be any element. Then $\varsigma \notin \nabla$. By hypothesis, ∇ is $\alpha-$closed subset of $\mathcal{R}$.

By dint of Theorem 2.4, there exist open subsets $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathcal{R}$ such that

$$\varsigma \in \mathfrak{U},\ \nabla \subseteq \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset.$$

Whence we easily conclude that $\mathfrak{B}$ is open subset of $\mathcal{R}$. Hence $(\mathcal{R},\ \Im)$ is a topological ring. □

Theorem 2.6. *Let $(\mathcal{R},\ \Im)$ be a topological ring such that every nowhere dense subset of $\mathcal{R}$ is closed. Then $(\mathcal{R},\ \Im)$ is $\alpha-$irresolute topological ring.*

Proof. Let $\mathfrak{B}$ be any $\alpha-$open subset of $\mathcal{R}$. Then $\mathfrak{B} = \mathfrak{P} - \nabla$ for some open subset $\mathfrak{P} \subseteq \mathcal{R}$ and nowhere subset $\nabla \subseteq \mathcal{R}$. By given condition, ∇ is a closed subset of $\mathcal{R}$.

Thus, for any $\varsigma \in \mathfrak{B}$, there exist $\mathfrak{U},\ \mathfrak{V} \in \Im$ with the property that

$$\varsigma \in \mathfrak{U},\ \nabla \subseteq \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset.$$

Let $\mathfrak{O} = \mathfrak{P} \cap \mathfrak{U}$. Then evidently, $\mathfrak{O}$ is open subset of $\mathcal{R}$ containing ς such that $\mathfrak{O} \subseteq \mathfrak{B}$. Thereby it follows that $(\mathcal{R},\ \Im)$ is an $\alpha-$irresolute topological ring. □

Theorem 2.7. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Then, for every $\mathfrak{O} \in \Gamma$, there exists $\sigma \in \Gamma$ such that $\alpha Cl(\sigma) \subseteq \mathfrak{O}$.*

Proof. Let $\mathfrak{O} \in \Gamma$. By Theorem 2.1, there exists a set $\sigma \in \Gamma$ such that

$$\sigma \cap (\mathfrak{O}^c + \sigma) = \emptyset.$$

This implies that

$$\sigma \subseteq (\mathfrak{O}^c + \sigma)^c.$$

Consequently, $\alpha Cl(\sigma) \subseteq \mathfrak{O}$. □

Definition 2.2. *A topological space $(X,\ \Im)$ is said to be*

(1) $\alpha - T_0$ if for distinct points x and y in X, there exists an $\alpha-$open set U in X such that either $x \in U,\ y \notin U$ or $y \in U,\ x \notin U$.

(2) $\alpha - T_1$ if for distinct points x and y in X, there exist $\alpha-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

(3) $\alpha - T_2$ if for distinct points x and y in X, there exist disjoint $\alpha-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

Theorem 2.8. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Then the following are equivalent:*

(1) $(\mathcal{R},\ \Im)$ is $\alpha - T_0$ space.
(2) $(\mathcal{R},\ \Im)$ is $\alpha - T_1$ space.
(3) $(\mathcal{R},\ \Im)$ is $\alpha - T_2$ space.

Proof. We prove only *(1) implies (3)*. Other implications are obvious.

(1) implies (3): Let $\varsigma \neq 0$. Without loss of generality, we may assume that there is $\sigma \in \Gamma$ such that $\varsigma \notin \sigma$. Then, by Theorem 2.7, there exists $\mathfrak{O} \in \Gamma$ such that $\varsigma \notin \alpha Cl(\mathfrak{O})$.

By Theorem 2.1, there exists $\mathfrak{U} \in \Gamma$ such that

$$(\varsigma + \mathfrak{U}) \cap (\alpha Cl(\mathfrak{O}) + \mathfrak{U}) = \emptyset.$$

Thence we infer that $\varsigma + \mathfrak{U}$ and $\mathfrak{U}$ are disjoint $\alpha-$open subsets of $\mathcal{R}$ containing ς and 0 respectively. Hence $(\mathcal{R},\ \Im)$ is $\alpha-$Hausdorff space. □

We end this note with some open questions. Imbibing theorems, Theorem 2.3, Theorem 2.5, and Theorem 2.6, we point out several interesting and pertinent questions which are worthy to the healthy discussion about the interconnection between the $\alpha-$irresolute topological rings and the topological rings.

Question 1 . Does there exist an $\alpha-$irresolute topological ring $(\mathcal{R},\ \Im)$ which is also a topological ring but $\Im \neq \Im^{\alpha}$?

Question 2 . Can we have a finite $\alpha-$irresolute topological ring which is not a topological ring?

Question 3 . Is there any $\alpha-$irresolute topological ring which is not a topological ring showing the Essentiality of each condition in Theorem 2.3?

Question 4 . Does there exist a topological ring which is not an $\alpha-$irresolute topological ring satisfying Theorem 2.3?

Question 5 . Can we replace '$\alpha-$closedness' in the statement of Theorem 2.5 by a weaker form of it like semi-closedness, pre-closedness, etc.?

Question 6 . Can we replace 'closedness' in the hypothesis of Theorem 2.6 by a weaker form of it like $\alpha-$closedness, semi-closedness, etc.?

Question 7 . Does there exist an $\alpha-$regular topological ring which is not an $\alpha-$irresolute topological ring?

3 Conclusions

In the literature it is a well known result that the algebraic sum of a compact set and a closed set in a topological ring is closed set. In this note, we showed that the algebraic sum $\mathfrak{A} + \mathfrak{B}$ of an $\alpha-$compact subset $\mathfrak{A} \subseteq \mathcal{R}$, and an $\alpha-$closed subset $\mathfrak{B} \subseteq \mathcal{R}$ of an $\alpha-$irresolute topological ring $\mathcal{R}$ is a closed subset of $\mathcal{R}$. In addition, we have shown the equivalence of $\alpha - T_i$ spaces in $\alpha-$irresolute topological rings for $i = 0,\ 1,\ 2$.

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Properties of Quasinormal Groups (PQG)

Behnam Razzaghmanesshi [*]

Abstract

A subgroup H of a group G is termed permutable (or quasi normal) in G if it satisfies the following equivalent conditions:
For any subgroup K of G, HK (the product of subgroups H and K) is a group. For any subgroup K of G, HK= KH, i.e., H and K are permuting subgroups. For every g in G, H permutes with the cyclic subgroup generated by g. Also we say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B. In this paper we prove the following theorem.
Let G=AB be the mutually permutable product of the super soluble subgroups A and B. If CoreG(A∩B)=1, then G is super soluble.
Keywords: quasinormal; permutable product; super soluble; etc.
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1 Introduction

All groups considered in this paper are finite. It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in [3] that if a group G is the product of two normal supersoluble groups and G′ is nilpotent, then G is super soluble. The search for generalisations of Baer's result has been a fruitful topic of investigation recently (see [5,7]).

Most of the generalisations centre around replacing normality of the factors by different permutability conditions. In [2], Asaad and Shaalan considered products satisfying one of the following conditions. We will follow Carocca [6], and say that G=AB is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B. Essentially, the results by Asaad and Shaalan are devoted to obtaining sufficient conditions for a mutually permutable product of two supersoluble subgroups to be supersoluble. They prove in [2, Theorem 3.8] the following generalisation of Baer's theorem:

Let G be the mutually permutable product of the supersoluble subgroups A and B. If G′ is nilpotent, then G is supersoluble.They also show that the result remains true if "G′ nilpotent" is replaced by "Bnilpotent"[2, Theorem 3.2]. In addition, they prove [2, Theorem 3.1]: If G is the totally permutable product of the supersoluble subgroups A and B, then G is supersoluble. It is well known that if G=AB is a mutually permutable product of two supersoluble subgroups A and B such that A∩B=1, then the product is in fact totally permutable [6,Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalisation of this last property.

Theorem 1.

Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. If CoreG(A∩B) =1, then G is supersoluble.

The second aim of the present paper has been to obtain more complete information about the structure of mutually permutable products of two supersoluble groups. As a straightforward consequence of Theorem 1, we have that, in the notation used above, G/CoreG(A∩B) is always supersoluble. Therefore, every mutually permutable product of two supersoluble subgroups is metasupersoluble. It is possible to obtain more precise information about its structure, as our second main theorem shows.

Theorem 2. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then G/F (G)is supersoluble and

metabelian.This last theorem can not be improved easily, as the following example shows.

Example. Let S3 be the symmetric group of degree 3, given by S3=⟨α, β:α2=β3=1;βα=β2⟩ and let T7 be the non-abelian group of order 73 and exponent 7. Write T7= ⟨a,b⟩ with a7=b7=[a,b]7=1 and let c=[a,b]. We have that S3 acts on T7 in the following way: aα=b, bα=a, cα=c−1, aβ=a2, bβ=b4, cβ=c. Thus, we can consider the semidirect product G=[T7] S3. Take now the subgroups

A= T7 ⟨β⟩ and B=T7 ⟨α⟩ of G. Clearly both A and B are supersoluble, and it is easy to check that G=AB is the mutually permutable product of A and B. Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results: The following four lemmas are needed to prove Theorem 1.

Lemma 1[4, Theorem 2]. If G=AB is the mutually permutable product of the supersoluble subgroups A and B, then G is soluble.

Lemma 2. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then, either G is supersoluble or NA < G and NB < G for every minimal normal subgroup N of G.

Proof. Assume that G is not supersoluble. Then both A and B are proper subgroups of G. Let N be a minimal normal subgroup of G and for contradiction assume that NA=G. Then, as N is abelian, N∩A is a normal subgroup of ⟨N,A⟩ = G. Since N is a minimal normal subgroup of G and A<G, we have that N∩A=1 and consequently A is a maximal subgroup of G. Clearly, we can also assume that B is not contained in A. It is not difficult to argue that we can choose an element b of B\A such that bq∈A for some prime q. Since the product G=AB is mutually permutable, A ⟨b⟩ is a subgroup of G and the maximality of A implies that G=A ⟨b⟩ . We now take orders to reach our final contradiction:

|A||N|=|G|=|A|| ⟨b⟩ ||A∩ ⟨b⟩ |=q|A|. Consequently, we have that |N|=q and then G is supersoluble, a contradiction.

Lemma 3. Let G=AB be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G. Then either

N∩A=N∩B=1 or N=(N∩A)(N∩B).

Proof. Let N be a minimal normal subgroup of G. Clearly A(N∩B) and (N∩A)B are both subgroups of G. Note that A normalizes N∩(A(N∩B))=(N∩A)(N∩B) and B normalizes N∩((A∩N)B)= (N∩A)(N∩B).

Therefore (N∩A)(N∩B) is a normal subgroup of G and the minimality of N yields the result.

Lemma 4. Let G be a group, and N a minimal normal subgroup of G such that |N|=pn, where p is a prime and n>1. Denote C=CG(N) and assume that G/Cis supersoluble. Then, if Q/Cis a subgroup of G/C containing Op′(G/C), we have that Q is normal in G and N=∏ti=1Ni, where Ni are non-cyclic minimal normal subgroups of NQ for i=1,...,t.

Proof. Since by [8, Lemma A.13.6], we have that Op(G/C)=1 and the commutator subgroup (G/C)′ of G/C is nilpotent because e G/C is supersoluble, it follows that (G/C)′is a p′-group. Therefore (G/C)′is contained in Op′(G/C) and thus Op′(G/C)is a Hall p′-subgroup of G/C. Consequently, Q/Cis a normal subgroup of G/C and hence Q is normal in G. Consider now N as a G-module over GF (p)by conjugation. Then, by Clifford's Theorem [8, Theorem B.7.3], N viewed as a Q-module is a direct sum N=∏ti=1Ni, where Ni are irreducible Q-modules for i=1,...,t. Suppose that there exists i∈{1,...,t}such that |Ni|=p. Then clearly |Nj|= p for all j. Therefore Q/CQ(Ni) is abelian of exponent dividing p−1, and the same is true for Q/C. In particular, Q/C=Op′(G/C) is a Hall p′-subgroup of G/C. Since N is not cyclic, it follows that Q = G and thus p divides |G/C|. Hence p is the largest prime dividing |G/C|. From the supersolubility of G/C, we obtain that 1= Op(G/C) is a Sylow subgroup of G/C, a contradiction. Consequently, Ni is a non-cyclic minimal normal subgroup of NQ for all i∈{1,t},as we wanted to prove.

Proof of Theorem 1. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B, with CoreG(A∩B)=1, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU. Note that N is an elementary abelian p-group for some prime p. Applying Lemma 2, we have that both NA and NB are proper subgroups of G. Moreover, using Lemma 3, we have that either N=(N∩A)(N∩B) or N∩A=N∩B=1. Assume first that N=(N∩A)(N∩B).

(i) If N∩A=1, then N is cyclic. Assume that N∩A=1. It follows that N is contained in B. Let N0 be a non-trivial cyclic subgroup of N. Since AN0 is a subgroup of G, we have that N0 =AN0∩N is anormal subgroup of AN0. Hence every cyclic subgroup of N is normalised by A. Now let N1 be a minimal normal subgroup of B contained in N. Since B is supersoluble, it follows

That N1 is cyclic and thus normalised by A. Hence N1 is a normal subgroup of G. The minimality of N implies that N=N1 and consequently N is cyclic.

(ii) N∩A=1 and N∩B=1.On the contrary, assume that N∩A=1. From (i), we know that N is cyclic. Moreover, Nis contained in B. Hence AN∩B= (A∩B)N.

Let L=CoreG(A∩B)N). Clearly, N is contained in Land L=L∩((A∩B)N)=(L∩A∩B)N. It is clear that G/L=(AL/L)(BL/L)is a mutually permutable product of AL/L and BL/L suchthat CoreG/L((AL/L)∩(BL/L))=1. By the minimality of G, it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that G/CG(N) is abelian. Hence G/CL(N) is supersoluble and GUCL(N)=C. Note that C=N×(C∩A∩B). Therefore C∩A∩B contains a Hall p′-subgroup of C. Since CoreG(A∩B)=1 and Op′(C) is a normal subgroup of G contained in C∩A∩B, we have that Op′(C)=1. Moreover, C′=(C∩A∩B)′ is a normal subgroup of G contained in A∩B. Consequently, C′=1 and C is an abelian p-group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G, by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8,Theorem V.3.2.e], Dcovers N and thus N is contained in D. It follows ND∩GU=1, a contradiction.

(iii) Either N=N∩A or N=N∩B. If we have N=N∩A=N∩B, then N is contained in A∩B, contradicting the factthat CoreG(A∩B)=1. We may assume without loss of generality that N∩A=N.

(iv) AN and BN are both supersoluble. Since N=(N∩A)(N∩B) and N=N∩A, it follows that N∩B is not contained in N∩A. Let n be any element of N∩B such that n/∈N∩A, and write N0 = ⟨n⟩ . Note that AN0 is a subgroup of G, and AN0∩N=(N∩A)N0. Therefore N0(N∩A) is a normal subgroup of AN0, and consequently A normalizes (A∩N)N0. This yields that A/CA(N/N∩A) acts as a power automorphism group on N/N∩A. This means that AN is supersoluble. If N∩B=N, then BN=B is supersoluble. On the contrary, if N∩B=N, we can argue as above and we obtain that BN is supersoluble. Consequently, ACG(N)/CG(N) and BCG(N)/CG(N) are both abelian groups of exponent dividing p−1. But then G/CG(N)=(ACG(N)/CG(N))(BCG(N)/CG(N)) is a π-group for some set of primes π such that if q∈π, then q divides p−1.

(v) Let B0 be a Hall π-subgroup of B. Then AB0∩N= A∩N.

This follows just by observing that AB0∩Nis contained in each Hall π′-subgroup of AB0 and every Hall π′-subgroup of A is a Hall π′-subgroup of AB0. Note that |G/CG(N)| is a π-number and AB0 contains a Hall π-subgroup of G. Therefore G=(AB0)CG(N). But then A∩N is a normal subgroup of G. The minimality of G yields either A∩N= 1or A∩N= N. This contradicts our assumption 1=N∩A=N, and so we cannot have N=(A∩N)(B∩N). Thus, by Lemma 3, we may assume N∩A=N∩B=1. Let M= CoreG(AN∩BN). Then N∩M= N and G/M is supersoluble by the minimality of G. Again, we reach a contradiction after several steps.

(vi) M=N. Suppose that M=N. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write C=CG(N), and consider the quotient group G/C. It is clear that G/C is supersoluble. Let Q/C=Op(G/C). Since Op(G/C)=1 and

(G/C)′is nilpotent, it follows that Q/C is a normal Hall p′-subgroup of G/C. Let Bp′ be a Hall p′-subgroup of B. Since |N| divides |B:A∩B|, we have that (A∩B)Bp′ is a proper subgroup of B. Let T be a maximal subgroup of B containing (A∩B)Bp′. Then AT is a maximal subgroup of G and |G:AT|= p= |B:T|. If N is not contained in AT, we have G=(AT)N and AT∩N=1. Then |N|=p, a contradiction. Therefore, N is contained in AT. In particular, the family S={X:X is a proper subgroup of B, (A∩B)Bp′X and NAX} is non-empty. Let R be an element of S of minimal order. Observe that AR has p-power index in G and thus ARC/C contains Op′(G/C). Regarding N as a AR-module over GF (p), we know, by Lemma 4, that N is a direct sum N=∏ti=1Ni, where Ni is an irreducible AR-module whose dimension is greater than 1, for all i∈{1,...,t}. Assume that (A∩B)Bp′=R. Then AR=ABp′ and thus N is contained in A, a contradiction. Therefore ABp′∩B=(A∩B)Bp′ is a proper subgroup of R. Let S be a maximal subgroup of R containing (A∩B)Bp′. From the minimality of R, we know that N is not contained in AS. Consequently, there exists some i∈{1,...,t} such that Ni is not contained in AS, which is a maximal subgroup of AR. Hence AR=(AS)Ni. Since Ni is a minimal normal subgroup of AR, it follows that AS∩Ni= 1and |Ni|= |AR:AS|= |R:S|= p, a contradiction.

(vii) M is an elementary abelian p-group. Note that M=N(M∩A)=N(M∩B) and |M∩A|=|M∩B|=|M|/|N|. Moreover, A(M∩B)is a subgroup of G such that A(M∩B)∩M= (M∩A)(M∩B). Hence (M∩A)(M∩B) is also a subgroup of G. If M∩A= M∩B, then M∩A is a normal subgroup of G contained in A∩B. This implies that M∩A=1 and consequently M=N, a contradiction. It yields that M∩A=M∩B. Next we see that (M∩A)(M∩B) is a normal subgroup of G. Since (M∩A)(M∩B)= M∩A(M∩B), we have that A normalizes (M∩A)(M∩B). Similarly, B normalises

(M∩A)(M∩B) since (M∩A)(M∩B)= M∩B(M∩A). This implies normality of (M∩A)(M∩B) in G. Let X=(M∩A)(M∩B). Since we cannot have M∩A= M∩B, M∩A must be strictly contained in X. Thus X=X∩M=(X∩N)(M∩A) > M∩A gives us X∩N=1. But then X∩N=N, giving NX. Suppose that Q is a Hall p′-subgroup of M∩B. Then QA is a subgroup and so QA∩M=Q(M∩A) is also a subgroup which contains Q. Hence, as |M:M∩A|=pk for some k, we have that QM∩A∩B. Thus QB∩MM∩A∩B and similarly QA∩MM∩A∩B. Consequently, QM is contained in M∩A∩B. Since QM=Op(M), it follows that Op(M) is a normal subgroup of G contained in A∩B. Hence Op(M)=1, a contradiction, and consequently Q=1andMis a p-group. Hence N is contained in Z(M) and M=N×(M∩A)=N×(M∩B). Thus φ(M)=φ(M∩A)=φ(M∩B) is a normal subgroup of G contained in A∩B. This implies that φ(M)=1 and M is an elementary abelian p-group, as claimed. (viii) Final contradiction. We have from the previous steps that M∩A is not contained in M∩B and that M∩B is not contained in M∩A because otherwise, since |M∩A|=|M∩B|, it follows that

M∩A=M∩B is a normal subgroup of G contained in A∩B. This would imply M∩A=M∩B=1, and M=(M∩A)N=N. This fact contradicts step (vi).

Let x be an element of M∩B such that x/∈M∩A. Then A 〈x〉 is a subgroup of G, and so is M0=A 〈x〉 ∩M=(A∩M) 〈x〉 . Therefore, M0 is an A-invariant subgroup of G. In particular, since M=(M∩A)(M∩B), we have that every subgroup of M/M∩A is A-invariant; that is, A/CA(M/M∩A) acts as a group of power automorphisms on M/M∩A. It is clear that M/M∩A is A-isomorphic to N. Consequently, A/CA(N) acts as a group of power automorphisms on N. This implies that A normalises each subgroup of N. A nalogously, B normalises each subgroup of N. It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus NGU∩D=1, a contradiction. Proof of Theorem 2.

Let M=GU denote the supersoluble residual of G. Theorem 1yields that G/CoreG(A∩B) is supersoluble. Therefore, M is contained in CoreG(A∩B). In particular, M is supersoluble. Let F(M) be the Fitting subgroup of M. Since A and Bare supersoluble, we have that [M,A]F(A)∩MF(M) and [M,B]F(B)∩MF(M). Consequently, [M,G] is contained in F(M). Note now that the chief factors of G between F(M) and Mare cyclic,and recall that G/M is supersoluble. Therefore, we have that G/F (M) is supersoluble. This implies that M=F(M) and thus M is nilpotent. Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A′ and B′ both centralise every chief factor of G. Let H/K be a chief factor of G. If H/K is cyclic, then as G′ centralizes H/K, so do A′ an dB′. Hence we may assume that H/K is a non-cyclic p-chief factor of G for some prime p. Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider K=1. Since F(G) centralizes H [8, Theorem A.13.8.b], G/CG(H) is supersoluble. Let Ap′ be a Hall p′-subgroup of A. By Maschke's theorem [8, Theorem A.11.5],H is a completely reducible Ap′-module and HAp′ is supersoluble because H is contained in A. Therefore Ap′/CAp′(H) is abelian of exponent dividing p−1. This implies that the primes involved in |A/CA(H)| can only be p or divisors of p−1.The same is true for |B/CB(H)|. This implies that if p divides |G/CG(H)|, then p is the largest prime dividing |G/CG(H)|. But since Op(G/CG(H))=1 and G/CG(H) is supersoluble, it follows that G/CG(H) must be a p′-group. Consider H as A-module over GF (p). Since ACG(H)/CG(H) is a p′-group, we have that H is a completely reducible A-module and every irreducible A-submodule of H is cyclic. Consequently A′ centralizes H, and the same is true for B′. Let now U/V be a chief factor of G. Then G/CG(U/V)is the product of the abelian subgroups ACG(U/V)/CG(U/V) and BCG(U/V)/CG(U/V). By Itô's theorem [9], we

have that G/CG(U/V)is metabelian. Since F(G)is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that G/F (G) is metabelian.3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

Corollary 1[2, Theorem 3.2]. Let G=AB be the mutually permutable product of the subgroups A and B. If A is supersoluble and B is nilpotent, then G is supersoluble.

Proof. Assume that the assertion is false, and let G be a minimal counterexample. We have that G is a primitive group, and so G has a unique minimal normal subgroup, N say, with N=CG(N) a p-group for some prime p. Since G is not supersoluble, applying Theorem 1, we know that CoreG(A∩B)=1. This yields that N is contained in A∩B. Now, since N is contained in B, which is nilpotent, it follows that any p′-element of B must centralize N. Since CG(N)=N, we have that B itself is a p-group. Consequently, A must contain a Hall p′-subgroup of G. Now let T/N=Op′(G/N). The previous argument yields that T/N is contained in A/N. Note that if B=N, then G =AN= A is supersoluble, a contradiction. Thus, N is a proper subgroup of B. This implies that p must divide |G:T|. Since G/N is supersoluble, p must divideq−1 for some prime q∈π(T/N). It is clear then that q can not divide p−1. Therefore, there exists a Sylow q-subgroup Aq of A which centralizes N. Using that CG(N)=N, it yields that Aq=1 and thus q does not divide |G|,a contradiction.

Corollary 2[2, Theorem 3.8]. Let G=AB be the mutually permutable product of thes upersoluble subgroups A and B. If G′ is nilpotent, then G is supersoluble.

Proof. We assume the result to be false, and choose a minimal counterexample G. Thus, G is a primitive group with unique minimal normal subgroup N. We also have that G=NM, where M is a maximal subgroup of G,N∩M=1 and N=F(G)=Op(G) for some prime p. Now G′ is nilpotent and thus G′=F(G)=N. Therefore, M is an abelian group. Since N is self-centralising, arguing as we did in the previous corollary, we have that N is contained in A∩B. Note that M∼=G/N, and thus Op(M)=1. Since M is abelian, this yields that M is a p′-group. Thus M is in fact a Hall p′-subgroup of G. Applying [1, Theorem 1.3.2], wehave that there exist a Hall p′-subgroup Ap′ of A and a Hall p′-subgroup Bp′ of B suchthat M=Ap′Bp′. Since NA∩B, it follows that both Ap′ and Bp′ must have exponent dividing p−1.Regarding N as a M-module, it is easy to see that M must be a cyclic group. Now, since M=Ap′Bp′ has exponent

dividing p−1, it follows that N is a cyclic group as well. This implies that G is supersoluble, a contradiction.

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