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Anti-fuzzy Bi-ideals in Hypersemigroups

M. Vasu*
S. Dhanasekaran†

Abstract

This study aims at defining the concept of anti-fuzzy bi-ideals of hypersemigroups. This study also defines the hypersemigroup bi-ideals in terms of anti-fuzzy bi-ideals additionally.

Keywords: Hypergroupoid, hypersemigroup, anti-fuzzy bi-ideal, regular, intra-regular.

2020 AMS subject classifications: 20M99, 08A72, 06F05.^

1 Introduction

In 1965, Zadeh [7] has presented the idea of fuzzy set which has seen a great success in a variety of domains. Good and Hughes [2] are the first two authors who have proposed the notion of the bi-ideal of the fuzzy set. Kuroki [5] was the first one to suggest fuzzy ideals in semigroups. According to Biswas [1], the idea of anti-fuzzy subgroup was first proposed. Shabir and Nawaz [6] has carried out the research on anti-fuzzy principles of semi-groups and gave some intriguing findings. Kehayopulu [4] was the first one to examine fuzzy sets in hypersemigroups. In this study, the bi-ideals of hypersemigroups in terms of anti-fuzzy bi-ideals is defined and the concept of anti-fuzzy bi-ideals of a hypersemigroup is established.

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2 Preliminaries

An hypergroupoid is a nonempty group $\mathcal{S}^{\mathcal{H}}$ with an hyperoperation $\circ : \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}} \rightarrow \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) | (f, g) \rightarrow f \circ g$ on $\mathcal{S}^{\mathcal{H}}$ and an operation $* : \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \rightarrow \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) | (\mathcal{R}, \mathcal{P}) \rightarrow \mathcal{R} * \mathcal{P}$ on $\mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ (introduced by the operation on $\mathcal{S}^{\mathcal{H}}$) such that $\mathcal{R} * \mathcal{P} = \bigcup_{(f,g) \in \mathcal{R} \times \mathcal{P}} (f \circ g)$ for every $\mathcal{R}, \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ ($\mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ being the set of nonempty subsets of $\mathcal{S}^{\mathcal{H}}$).

The purpose of the operation $*$ is clearly stated. If $(\mathcal{R}, \mathcal{P}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$, then $\mathcal{R} * \mathcal{P} = \bigcup_{(f,g) \in \mathcal{R} \times \mathcal{P}} (f \circ g)$. For every $(f, g) \in \mathcal{R} \times \mathcal{P}$, we get $(f, g) \in \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}}$, then $(\{f\} \circ \{g\}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. Thus we get $\mathcal{R} * \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. If $(\mathcal{R}, \mathcal{P}), (\mathcal{F}, \mathcal{G}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ such that $(\mathcal{R}, \mathcal{P}) = (\mathcal{F}, \mathcal{G})$, then $\mathcal{R} * \mathcal{P} = \bigcup_{(f,g) \in \mathcal{R} \times \mathcal{P}} (f \circ g) = \bigcup_{(f,g) \in \mathcal{F} \times \mathcal{G}} (f \circ g) = \mathcal{F} * \mathcal{G}$.

Since the hyperoperation ' $\circ$ ' and the operation ' $*$ ' are interdependent, hypergroupoids may alternatively be expressed by $(\mathcal{S}^{\mathcal{H}}, \circ)$ rather than $(\mathcal{S}^{\mathcal{H}}, \circ, *)$. Suppose $\mathcal{S}^{\mathcal{H}}$ is an hypergroupoid, then $\forall m, n \in \mathcal{S}^{\mathcal{H}}$, we get $m \circ n = \{m\} * \{n\}$. That is, $\{m\} * \{n\} = \bigcup_{x \in \{m\}, y \in \{n\}} x \circ y = m \circ n$.

An hypergroupoid $\mathcal{S}^{\mathcal{H}}$ is said to be hypersemigroup if $(m \circ n) * \{o\} = \{m\} * (n \circ o)$ for every $m, n, o \in \mathcal{S}^{\mathcal{H}}$. Since $m \circ n = \{m\} * \{n\}$ for any $m, n \in \mathcal{S}^{\mathcal{H}}$, an hypergroupoid $\mathcal{S}^{\mathcal{H}}$ is an hypersemigroup if and only if for any $m, n, o \in \mathcal{S}^{\mathcal{H}}$, we have $(\{m\} * \{n\}) * \{o\} = \{m\} * (\{n\} * \{o\})$.

According to Zadeh [7], we claim that the hypergroupoid $(\mathcal{S}^{\mathcal{H}}, \circ)$'s fuzzy subset (or fuzzy set) f if ψ maps $\mathcal{S}^{\mathcal{H}}$ into the real closed interval $[0, 1]$ of real integers, i.e., $\psi : \mathcal{S}^{\mathcal{H}} \rightarrow [0, 1]$.

The subset of $\mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}}$ classified as follows is denoted by $\mathcal{A}_{\mathfrak{h}}$:

$$\mathcal{A}_{\mathfrak{h}} = \{(a, b) \in \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}} | \mathfrak{h} \in a \circ b\} \text{ for any element } \mathfrak{h} \text{ of } \mathcal{S}^{\mathcal{H}}.$$

We designate by $\psi \circ \mathfrak{q}$, the fuzzy subset of $\mathcal{S}^{\mathcal{H}}$ defined by any two fuzzy subsets of ψ and $\mathfrak{q}$,

$$\psi \circ \mathfrak{q} : \mathcal{S}^{\mathcal{H}} \rightarrow [0, 1] \quad | \quad \mathfrak{h} \rightarrow \begin{cases} \bigwedge_{(a,b) \in \mathcal{A}_{\mathfrak{h}}} \max\{f(a), g(b)\} & \text{if } \mathcal{A}_{\mathfrak{h}} \neq \phi \\ 1 & \text{if } \mathcal{A}_{\mathfrak{h}} = \phi. \end{cases}$$

The collection of all fuzzy subsets of $\mathcal{S}^{\mathcal{H}}$ is indicated by $\mathcal{F}(\mathcal{S}^{\mathcal{H}})$ and the power relation $\leq$ on $\mathcal{F}(\mathcal{S}^{\mathcal{H}})$ classified by: $\psi \leq \Pi$ if and only if $\psi(m) \leq \Pi(m)$ for every $m \in \mathcal{S}^{\mathcal{H}}$.

For an hypersemigroup $\mathcal{S}^{\mathcal{H}}$, the following definitions apply to the fuzzy sets "0" and "1":

$$0 : \mathcal{S}^{\mathcal{H}} \rightarrow [0, 1] \quad | \quad m \rightarrow 0(m) = 0 \text{ and } 1 : \mathcal{S}^{\mathcal{H}} \rightarrow [0, 1] \quad | \quad m \rightarrow 1(m) = 1.$$

An hypersemigroup $\mathcal{S}^{\mathcal{H}}$ is allegedly regular suppose $\forall \mathfrak{h} \in \mathcal{S}^{\mathcal{H}} \exists u \in \mathcal{S}^{\mathcal{H}} \ni \mathfrak{h} \in (\mathfrak{h} \circ u) * \{\mathfrak{h}\}$.

An hypersemigroup $\mathcal{S}^{\mathcal{H}}$ is allegedly intra-regular suppose $\forall \mathfrak{h} \in \mathcal{S}^{\mathcal{H}} \exists a, b \in \mathcal{S}^{\mathcal{H}} \ni \mathfrak{h} \in (a \circ \mathfrak{h}) * (\mathfrak{h} \circ b)$.

The same symbol is used to denote both the hyperoperation on $\mathcal{S}^{\mathcal{H}}$ and the multiplication among its two fuzzy subsets (no confusion is possible).

Proposition 2.1. [3] Assume $(\mathcal{S}^{\mathcal{H}}, \circ)$ be an hypergroupoid, $x \in \mathcal{S}^{\mathcal{H}}$ and $\mathcal{R}, \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. Then we obtain the following:

- (1) $x \in \mathcal{R} * \mathcal{P} \iff x \in \mathfrak{f} \circ \mathfrak{g}$ for some $\mathfrak{f} \in \mathcal{R}, \mathfrak{g} \in \mathcal{P}$.
- (2) If $\mathfrak{f} \in \mathcal{R}$ and $\mathfrak{g} \in \mathcal{P}$, then $\mathfrak{f} \circ \mathfrak{g} \subseteq \mathcal{R} * \mathcal{P}$.

3 Anti-fuzzy bi-ideals

Unless otherwise stated, let $\mathcal{S}^{\mathcal{H}}$ denote a hypersemigroup in this section.

Definition 3.1. An anti-fuzzy subsemigroup of $\mathcal{S}^{\mathcal{H}}$ is a fuzzy subset of ψ of $\mathcal{S}^{\mathcal{H}}$ suppose $\psi(\gamma \circ \zeta) \leq \max \{\psi(\gamma), \psi(\zeta)\}$ for all $\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Definition 3.2. An anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is a fuzzy subset of ψ of $\mathcal{S}^{\mathcal{H}}$ if

- (i) $\psi(\gamma \circ \zeta) \leq \max \{\psi(\gamma), \psi(\zeta)\}$
- (ii) $\psi((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi(\gamma), \psi(k)\}$ for all $\gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}}$.

Lemma 3.1. A bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is not an empty subset of $\mathcal{A}$ of $\mathcal{S}^{\mathcal{H}}$ iff the fuzzy subset $\psi_{\mathcal{A}}$ of $\mathcal{S}^{\mathcal{H}}$ classified as

$$\psi_{\mathcal{A}} : \mathcal{S}^{\mathcal{H}} \rightarrow \{\mathfrak{r}, \rho\}, \iota \rightarrow \begin{cases} \mathfrak{r} & \text{if } \iota \in \mathcal{A} \\ \rho & \text{if } \iota \notin \mathcal{A} \end{cases}$$

where $0 \leq \mathfrak{r}, \rho \leq 1$ and $\mathfrak{r} \leq \rho$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Proof: Assume $\mathcal{A}$ is a bi-ideals of $\mathcal{S}$. Let $\gamma, \zeta \in \mathcal{A}$. Then $\gamma \circ \zeta \in \mathcal{A}$.

- (1) Suppose $\gamma, \zeta \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}(\gamma \circ \zeta) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.
- (2) If $\gamma \in \mathcal{A}, \zeta \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \mathfrak{r}, \psi_{\mathcal{A}}(\zeta) = \rho$ and then $\psi_{\mathcal{A}}(\gamma \circ \zeta) = \rho \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.
- (3) If $\gamma \notin \mathcal{A}, \zeta \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \rho, \psi_{\mathcal{A}}(\zeta) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}(\gamma \circ \zeta) = \rho \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.

- (4) If $\gamma, \zeta \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \rho$ and then $\psi_{\mathcal{A}}(\gamma \circ \zeta) = \rho \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$ for all $\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Let $\gamma, k \in \mathcal{A}$ and $\zeta \in \mathcal{S}^{\mathcal{H}}$. Then $(\gamma \circ \zeta) * \{k\} \in \mathcal{A}$.

- (1) If $\gamma, k \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(k) = \tau$ and so $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.
- (2) If $\gamma \in \mathcal{A}, k \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \tau, \psi_{\mathcal{A}}(k) = \rho$ and then $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.
- (3) If $\gamma \notin \mathcal{A}, k \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \rho, \psi_{\mathcal{A}}(k) = \tau$ and so $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.
- (4) If $\gamma, k \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(k) = \rho$ and then $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$ for all $\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Thus, an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is $\psi_{\mathcal{A}}$.

Alternatively, assume $\psi_{\mathcal{A}}$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Consider $\gamma, \zeta, k \in \mathcal{A}$.

Then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \psi_{\mathcal{A}}(k) = \tau$.

Therefore, $\psi_{\mathcal{A}}$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$, we obtain

$$\psi_{\mathcal{A}}(\gamma \circ \zeta) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\} = \max \{\tau, \tau\} = \tau.$$

This implies $\gamma \circ \zeta \in \mathcal{A}$.

$$\text{Also } \psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\} = \max \{\tau, \tau\} = \tau.$$

This implies $(\gamma \circ \zeta) * \{k\} \in \mathcal{A}$.

Hence $\mathcal{A}$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}}$. □

Definition 3.3. Assume ψ be a fuzzy subset of $\mathcal{S}^{\mathcal{H}}$.

Then $\mathcal{L}(\psi; \rho) = \{\gamma \in \mathcal{S}^{\mathcal{H}} | \psi(\gamma) \leq \rho\}$ where $0 < \rho \leq 1$ is said to be the lower level cut of ψ .

The characterization of an anti-fuzzy bi-ideal is then provided in terms of its level bi-ideals.

Theorem 3.1. A fuzzy subset ψ of $\mathcal{S}^{\mathcal{H}}$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ iff the collection $\mathcal{L}(\psi; \rho) = \{\gamma \in \mathcal{S}^{\mathcal{H}} | \psi(\gamma) \leq \rho\}$ where $0 < \rho \leq 1$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Proof: Assume ψ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Consider $\gamma, \zeta \in \mathcal{L}(\psi; \rho)$.

Then $\psi(\gamma) \leq \rho$ and $\psi(\zeta) \leq \rho$.

$$\text{As } \psi \text{ is the anti-fuzzy bi-ideal of the } \mathcal{S}^{\mathcal{H}}, \psi(\gamma \circ \zeta) \leq \max \{\psi(\gamma), \psi(\zeta)\} \leq \max \{\rho, \rho\} = \rho.$$

This suggests $\gamma \circ \zeta \in \mathcal{L}(\psi; \rho)$.

Let $\gamma, k \in \mathcal{L}(\psi; \rho)$ and $\zeta \in \mathcal{S}^{\mathcal{H}}$.

Then $\psi(\gamma) \leq \rho$ and $\psi(k) \leq \rho$.

Now $\psi((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi(\gamma), \psi(k)\} \leq \max \{\rho, \rho\} = \rho$.

This suggests $(\gamma \circ \zeta) * \{k\} \in \mathcal{L}(\psi; \rho)$.

Therefore, $\mathcal{L}(\psi; \rho)$ a bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Alternatively, suppose $\mathcal{L}(\psi; \rho)$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}} \forall 0 < \rho \leq 1$.

If $\exists \gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$ such that $\psi(\gamma \circ \zeta) > \max \{\psi(\gamma), \psi(\zeta)\} = \rho$, then $\rho \in (0, 1]$, $\gamma, \zeta \in \mathcal{L}(\psi; \rho)$ but $\gamma \circ \zeta \notin \mathcal{L}(\psi; \rho)$, which is incongruous.

Thus $\psi(\gamma \circ \zeta) \leq \max \{\psi(\gamma), \psi(\zeta)\} \forall \gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Suppose $\exists \gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}} \ni \psi((\gamma \circ \zeta) * \{k\}) > \max \{\psi(\gamma), \psi(k)\} = \rho_0$, then $\rho_0 \in (0, 1]$, $\gamma, \zeta \in \mathcal{L}(\psi; \rho_0)$ but $(\gamma \circ \zeta) * \{k\} \notin \mathcal{L}(\psi; \rho)$, which is incongruous.

Thus $\psi((\gamma \circ \zeta) * \{k\}) \leq \max \{\psi(\gamma), \psi(k)\}$ for all $\gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}}$.

Hence ψ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$. $\square$

Lemma 3.2. A fuzzy subset of $\mathcal{S}^{\mathcal{H}}$ is ψ . Then ψ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ iff $\psi \leq \psi \circ 0 \circ \psi$.

Proof: Let $h \in \mathcal{S}^{\mathcal{H}}$.

Then $\psi(h) \leq (\psi \circ 0 \circ \psi)(h)$.

If $\mathcal{A}_h = \phi$, then $(\psi \circ 0 \circ \psi)(h) = ((\psi \circ 0) \circ \psi)(h) = 1 \geq \psi(h)$.

If $\mathcal{A}_h \neq \phi$, then $(\psi \circ 0 \circ \psi)(h) = \bigwedge_{(\zeta, k) \in \mathcal{A}_h} \max \{(\psi \circ 0)(\zeta), \psi(k)\}$.

It suffices to demonstrate this $\max \{(\psi \circ 0)(\zeta), \psi(k)\} \geq \psi(h)$ for every $(\zeta, k) \in \mathcal{A}_h$. (1)

For this purpose, let us assume $(\zeta, k) \in \mathcal{A}_h$.

If $\mathcal{A}_\zeta = \phi$, then $(\psi \circ 0)(\zeta) = 1 \geq \psi(k)$ and $\max \{(\psi \circ 0)(\zeta), \psi(k)\} = 1 \geq \psi(h)$.

Now assume that $\mathcal{A}_\zeta \neq \phi$.

Then $(\psi \circ 0)(\zeta) = \bigwedge_{(r, s) \in \mathcal{A}_\zeta} \max \{\psi(r), 0(s)\} = \bigwedge_{(r, s) \in \mathcal{A}_\zeta} \psi(r)$. (2)

Let's think about the following scenarios:

(i) Let $\psi(h) \leq (\psi \circ 0)(\zeta)$. Then $\psi(h) \leq (\psi \circ 0)(\zeta) \leq \max \{(\psi \circ 0)(\zeta), \psi(k)\}$.
Thus (1) is proved.

(ii) Let $\psi(h) > (\psi \circ 0)(\zeta)$.

Then there exists $(r, s) \in \mathcal{A}_\zeta$ such that $\psi(h) > \psi(r)$. (3)

If $\psi(h) \leq \psi(r)$ for every $(r, s) \in \mathcal{A}_\zeta$, then $\psi(h) \leq \bigwedge_{(r, s) \in \mathcal{A}_\zeta} \psi(r)$.

Then by (2), we get $\psi(h) \leq (\psi \circ 0)(\zeta)$ it is unattainable.

Since $(\zeta, k) \in \mathcal{A}_h$, we have $\zeta, k \in \mathcal{S}^{\mathcal{H}}$ and $h \in \zeta \circ k$.

Since $(\mathfrak{r}, \mathfrak{s}) \in \mathcal{A}_\zeta$, we have $\mathfrak{r}, \mathfrak{s} \in \mathcal{S}^{\mathcal{H}}$ and $\zeta \in \mathfrak{r} \circ \mathfrak{s}$.

Then we get $h \in \zeta \circ k = \{\zeta\} * \{k\} \subseteq (\mathfrak{r} \circ \mathfrak{s}) * \{k\}$.

Because ψ is a anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$, we obtain

$\psi(h) \leq \max \{\psi(r), \psi(k)\}$. (4)

If $\psi(r) \geq \psi(k)$, then $\psi(h) \leq \psi(r)$ which is not possible by (3).

Thus we have $\psi(r) < \psi(k)$. (5)

Then $\max\{\psi(r), \psi(k)\} = \psi(k)$ and by (4), we have $\psi(h) \leq \psi(k)$. (6)

Since $(r, s) \in \mathcal{A}_\zeta$ and by (2), we have $\psi(r) \geq (\psi \circ 0)(\zeta)$. (7)

By (5) and (7), we have $\psi(k) > \psi(r) \geq (\psi \circ 0)(\zeta)$.

Then $\max\{(\psi \circ 0)(\zeta), \psi(k)\} = \psi(k) \geq \psi(h)$ [by (6)] and (1) is proved.

Conversely, assume that $a, b, c \in \mathcal{S}^H$ and $p \in (a \circ b) * \{c\}$.

Then $\psi(p) \leq \max\{\psi(a), \psi(c)\}$.

Since $p \in (a \circ b) * \{c\}$, by Proposition 2.1, there exists $\gamma \in a \circ b$ such that $p \in \gamma \circ c$.

Since $\gamma \in a \circ b$, we have $(a, b) \in \mathcal{A}_\gamma$.

Since $p \in \gamma \circ c$, we have $(\gamma, c) \in \mathcal{A}_p$.

Since $(\gamma, c) \in \mathcal{A}_p$, $\mathcal{A}_p$ is a non-empty set and we also have

$$\begin{aligned} (\psi \circ 0 \circ \psi)(p) &= \bigwedge_{(u,v) \in \mathcal{A}_p} \max\{(\psi \circ 0)(u), \psi(v)\} \\ &\leq \max\{(\psi \circ 0)(\gamma), \psi(c)\}. \end{aligned}$$

Since $(a, b) \in \mathcal{A}_\gamma$, we have $\mathcal{A}_\gamma \neq \phi$ and we also have

$$\begin{aligned} (\psi \circ 0)(\gamma) &= \bigwedge_{(c,w) \in \mathcal{A}_\gamma} \max\{\psi(c), 0(w)\} \\ &\leq \max\{\psi(a), 0(b)\} = \psi(a). \end{aligned}$$

Then we get $(\psi \circ 0 \circ \psi)(p) \leq \max\{\psi(a), \psi(c)\}$.

On the otherhand, since $p \in (a \circ b) * \{c\} \subseteq \mathcal{S}^H * \mathcal{S}^H \subseteq \mathcal{S}^H$ and $\psi \leq \psi \circ 0 \circ \psi$, we obtain $\psi(p) \leq (\psi \circ 0 \circ \psi)(p)$.

Thus we have $\psi(p) \leq (\psi \circ 0 \circ \psi)(p) \leq \max\{\psi(a), \psi(c)\}$. □

Theorem 3.2. *If a hypersemigroup $\mathcal{S}^H$ is regular, then we get $\psi \circ 0 \circ \psi = \psi$ for any anti-fuzzy bi-ideal ψ of $\mathcal{S}^H$.*

Proof: Assume $\mathcal{S}^H$ is regular and ψ is its anti-fuzzy bi-ideal.

Let $h \in \mathcal{S}^H$.

Given that $\mathcal{S}^H$ is regular, $u \in \mathcal{S}^H$ exists like that $h \in (h \circ u) * \{h\}$.

This implies $(h, uh) \in \mathcal{A}_h$ and then $\mathcal{A}_h \neq \phi$.

$$\begin{aligned} (\psi \circ 0 \circ \psi)(h) &= \bigwedge_{(p,q) \in \mathcal{A}_h} \max\{(\psi \circ 0)(p), \psi(q)\} \\ &\leq \max\{\psi(h), (\psi \circ 0)(uh)\} \\ &= \max\left\{\psi(h), \bigwedge_{(r,s) \in \mathcal{A}_{uh}} \max\{0(r), \psi(s)\}\right\} \\ &\leq \max\left\{\psi(h), \max\{0(u), \psi(h)\}\right\} \\ &= \max\left\{\psi(h), \max\{0, \psi(h)\}\right\} \\ &= \max\{\psi(h), \psi(h)\} \\ &= \psi(h). \end{aligned}$$

Thus $(\psi \circ 0 \circ \psi)(h) \leq \psi(h)$.

By the above lemma, we obtain $\psi(h) \leq (\psi \circ 0 \circ \psi)(h)$.

Hence $\psi \circ 0 \circ \psi = \psi$. □

4 Conclusion

In this work, we defined anti-fuzzy bi-ideal in hypersemigroup and discussed about the characterization of an anti-fuzzy bi-ideals in hypersemigroup. Also we defined the lower level cut of the fuzzy subset and we analysed the characterization of an anti-fuzzy bi-ideal in terms of lower level cut of the fuzzy subset. Moreover, in this study we proved an anti-fuzzy bi-ideal is equivalently denoted by $\psi \leq \psi \circ 0 \circ \psi$. Also we proved $\psi = \psi \circ 0 \circ \psi$ if the hypersemigroup is regular. In the forthcoming work, the regular and intra-regular hypersemigroups will be defined using anti-fuzzy bi-ideal concepts.

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On Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Convergence of Double Sequences in Neutrosophic Metric Spaces

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Abstract

The concept of Wijsman $\mathfrak{I}_2$ - Statistical Convergence ($\mathfrak{W}\mathfrak{I}_2\mathfrak{St}\mathfrak{C}$), Wijsman $\mathfrak{I}_2$ - Lacunary Statistical Convergence ($\mathfrak{W}\mathfrak{I}_2\mathfrak{LSt}\mathfrak{C}$), Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}$) and Wijsman Strongly $\mathfrak{I}_2$ - Cesaro Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{Ce}\mathfrak{C}$) of double sequences in the Neutrosophic Metric Spaces ($\mathfrak{NM}\mathfrak{S}$) are examined in this paper. Additionally, we introduce the concepts of Wijsman Strongly $\mathfrak{I}_2^*$ - Lacunary Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2^*\mathfrak{L}\mathfrak{C}$), Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Cauchy ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{Ca}$), and Wijsman Strongly $\mathfrak{I}_2^*$ - Lacunary Cauchy ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2^*\mathfrak{L}\mathfrak{Ca}$) sequence in $\mathfrak{NM}\mathfrak{S}$ and establish impressive results.

Keywords: Fixed point; Neutrosophic Metric Spaces; Wijsman strongly $\mathfrak{I}_2$ - lacunary convergent and lacunary Cauchy.

2020 AMS subject classifications: 54H25, 47H10 ¹

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1 Introduction

Fuzzy sets were initially described by Zadeh [20]. Various article's publishing has far-reaching consequences throughout scientific disciplines. The concept has real-world relevance, yet it doesn't offer satisfactory answers for various issues. These difficulties inspire creative investigations. Atanassov [1] looked the study of intuitionistic fuzzy sets and found that they work well in this kind of scenario. The idea of intuitionistic fuzzy metric space has been presented by Park [14]. Jeyaraman et. al and Sowndararajan et. al proposed the Neutrosophic Metric Spaces concept and outlined several fixed-point solutions [8,9,10,16,17,18]. Das et al. [4] investigated I and I* convergence sequences, while Ulusu and Nuray [19] presented Wijsman Lacunary Statistical Convergence of sequences. Numerous authors had a significant role in ideal and Wijsman ideal convergence sequence[7,13]. Mursaleen et. al. [12] were described the seperability concept. Fridy and Orhan [6] developed the idea of lacunary Statistical convergence via Lacunary sequence. Major article's publishing had a significant impact across all disciplines of science. There are several lacunary statistical convergence sequence [2,3,5,11,15] had a significant impact across all disciplines of mathematics and science.

We have indicated through this entire work $\mathfrak{I}_2$ - to be the admissible ideal in $\mathbb{N} \times \mathbb{N}$, $\omega_2 = \{(j_u, k_s)\}$ to be a double lacunary sequence, $(\Omega, \psi, \varrho, \varphi, *, \diamond, \otimes)$ to be the $\mathfrak{NM}\mathfrak{S}$ and $\{F_{wq}\}$ to be nonempty closed subsets of Ω .

In the present paper, we define the concept of $\mathfrak{WI}_2\mathfrak{StC}$, $\mathfrak{WI}_2\mathfrak{LStC}$, $\mathfrak{WSI}_2\mathfrak{Lc}$ and $\mathfrak{WSI}_2\mathfrak{LcC}$ of double sequences in the $\mathfrak{NM}\mathfrak{S}$ are examined. Also, we give the notions of $\mathfrak{WSI}_2^*\mathfrak{Lc}$, $\mathfrak{WSI}_2\mathfrak{Lca}$, and $\mathfrak{WSI}_2^*\mathfrak{Lca}$ set sequence in $\mathfrak{NM}\mathfrak{S}$ and establish results. Also $\mathfrak{I}_2$ and $\mathfrak{I}_2^*$ -convergence of double sequences in the setting of $\mathfrak{NM}\mathfrak{S}$ and established some relationship between these types of convergence.

2 Preliminaries

Definition 2.1. A sequence $\Upsilon_{w\lambda}$ of nonempty closed subsets of Ω is known as $\mathfrak{WI}_2\mathfrak{StC}$ to Υ or $\mathfrak{S}(\mathfrak{I}_{\mathfrak{W}_2}^{\psi, \varrho, \varphi})$ - convergent to Υ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$, if for every $\varepsilon \in (0, 1)$, $\tau > 0$, for each $\xi \in \Omega$ and for every $\varpi > 0$,

$$\left\{ \frac{1}{st} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\} \in \mathfrak{I}_2$$

We demonstrate this symbolically by

$$\Upsilon_{w\lambda} \mathfrak{S} \left(\mathfrak{I}_{\mathfrak{W}_2}^{\psi, \varrho, \varphi} \right) \Upsilon \text{ or } \Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left(\mathfrak{I}_{\mathfrak{W}_2}^{\psi, \varrho, \varphi} \right) \right).$$

The set of all $\mathfrak{WI}_2\mathfrak{StC}$ sequences in $\mathfrak{NM}\mathfrak{S}$ is indicated by $\mathfrak{S} \left(\mathfrak{I}_{\mathfrak{W}_2}^{\psi, \varrho, \varphi} \right)$.

On Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Convergence of Double Sequences in Neutrosophic Metric Spaces

Example 2.1. Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:

$$\Upsilon_{w\lambda} = \begin{cases} (a, b) \in \mathfrak{R}^2 : (a + w)^2 + (b + \lambda)^2 = 1, & \text{if } w \text{ and } \lambda \text{ are square integers,} \\ \{(1, 1)\}, & \text{otherwise.} \end{cases}$$

If $\mathfrak{I}_2 = \mathfrak{I}_2^\delta \mathfrak{I}_2^\delta$ is the class of $K \subset \mathbb{N} \times \mathbb{N}$ (with density of ζ equal to 0), then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{I}_2\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to $\Upsilon = \{(1, 1)\}$ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$.

Definition 2.2. A sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{C}\mathfrak{e}\mathfrak{S}$ to Υ or $\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right]$ -summable to Υ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$, if for every $\varepsilon \in (0, 1)$, for each $\xi \in \Omega$ and for all $\varpi > 0$,

$$\left\{ \begin{array}{l} \frac{1}{st} \sum_{w, \lambda=1,1}^{s,t} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{st} \sum_{w, \lambda=1,1}^{s,t} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{st} \sum_{w, \lambda=1,1}^{s,t} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathfrak{I}_2.$$

We write $\Upsilon_{w\lambda} \rightarrow \mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \Upsilon$ or $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Example 2.2. Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:

$$\Upsilon_{w\lambda} = \begin{cases} (a, b) \in \mathbb{R}^2 : (a + 1)^2 + b^2 = \frac{1}{w\lambda}; & \text{if } w \text{ and } \lambda \text{ are square integers,} \\ \{(0, 1)\}; & \text{otherwise.} \end{cases}$$

If $\mathfrak{I}_2 = \mathfrak{I}_2^f \left(\mathfrak{I}_2^f \right)$ is the class of finite subsets of $\mathbb{N} \times \mathbb{N}$, then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{C}\mathfrak{e}\mathfrak{S}$ to $\Upsilon = \{(0, 1)\}$ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$.

Definition 2.3. The sequence $\{\Upsilon_{w\lambda}\}$ is known as $\mathfrak{W}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to Υ or $\mathfrak{S}_{\omega_2} \left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right)$ -convergent to Υ with regard to (ψ, ϱ, φ) , if for every $\varepsilon \in (0, 1)$, $\tau > 0$, for each $\xi \in \Omega$ and for all $\varpi > 0$,

$$\left\{ \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\} \in \mathfrak{I}_2.$$

We write $\Upsilon_{w\lambda} \rightarrow \mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \Upsilon$ or $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Example 2.3. Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:

$$\Upsilon_{w\lambda} = \begin{cases} (a, b) \in \mathfrak{R}^2 : (a - w)^2 + (b + \lambda)^2 = 1, & \text{if } (w, \lambda) \in \mathfrak{I}_{us}; \\ \{(-1, 1)\}, & \text{otherwise.} \end{cases}$$

If we take $\mathfrak{I}_2 = \mathfrak{I}_2^\delta$, then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to $\Upsilon = \{(-1, 1)\}$ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$.

Definition 2.4. A sequence $\{\Upsilon_{w\lambda}\}$ is Wijsman Strong $\mathfrak{I}_2$ -Lacunary Summable $(\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{S})$ to Υ or $\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right]$ -summable to Υ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$,

if for every $\varepsilon \in (0, 1)$, for all $\varpi > 0$ and for each $\xi \in \Omega$.

$$\left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

We write $\Upsilon_{w\lambda} \xrightarrow{\mathfrak{N}_{\omega_2}[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)}]} \Upsilon$ or $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Example 2.4. Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:

$$\Upsilon_{w\lambda} = \begin{cases} (a, b) \in \mathfrak{R}^2 : a^2 + (b-1)^2 = \frac{1}{w\lambda}; & \text{if } (w, \lambda) \in \mathcal{I}_{us}; w, \\ \{(1, 0)\}; & \text{otherwise.} \end{cases}$$

If $\mathcal{I}_2 = \mathcal{I}_2^f$, then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2\mathcal{LS}$ to $\Upsilon = \{(1, 0)\}$ with regard to $\mathfrak{MM}(\psi, \varrho, \varphi)$.

3 Main Results

Theorem 3.1. Let $\omega_2 = \{(j_u, k_s)\}$ be a Double Lacunary Sequence ($\mathcal{DL}\mathcal{S}$). Then $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right) \Rightarrow \Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Proof. Let $\varepsilon \in (0, 1)$ and $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. At that time, for every $\xi \in \Omega$, we get

$$\begin{aligned} & \sum_{(w,\lambda) \in \mathcal{I}_{us}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\} \\ & \geq \sum_{\substack{(w,\lambda) \in \mathcal{I}_{us} : |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}, \\ & \geq \varepsilon \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \end{aligned}$$

and so

$$\begin{aligned} & \frac{1}{\varepsilon h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\} \\ & \geq \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \text{ and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \end{aligned}$$

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Then, for any $\tau > 0$, for each $\xi \in \Omega$,

$$\left\{ \left\{ \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, p)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, p) - \varphi(\xi, \Upsilon, p)| \geq \varepsilon \end{array} \right| \right\} \geq \tau \right\} \\ \subseteq \left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, p) - \psi(\xi, \Upsilon, p)| \leq 1 - \varepsilon \cdot \tau \\ \text{or } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, p) - \varrho(\xi, \Upsilon, p)| \geq \varepsilon \cdot \tau \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, p) - \varphi(\xi, \Upsilon, p)| \geq \varepsilon \cdot \tau \end{array} \right\}. \quad \square$$

Theorem 3.2. *Let $\omega_2 = \{(j_u, k_s)\}$ be a $\mathfrak{DL}\mathfrak{S}$. Then, $\{\Upsilon_{w\lambda}\}$ is bounded ($\{\Upsilon_{w\lambda}\} \in L_\infty^2(\Omega)$) and $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right) \Rightarrow \Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. The set of all bounded double sequences of sets in $\mathfrak{NM}\mathfrak{S}$ is indicated by $\mathfrak{L}_\infty^2(\Omega)$.*

Proof. Assume that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ and $\{\Upsilon_{w\lambda}\} \in \mathfrak{L}_\infty^2(\Omega)$. To be noted at this point, there is an $\mathfrak{K} > 0$ such that $|\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \geq 1 - \mathfrak{K}$ or $|\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \leq \mathfrak{K}$ and $|\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \leq \mathfrak{K}$ for every $\xi \in \Omega$ and $w, \lambda \in \mathbb{N}$. Given $\varepsilon \in (0, 1)$, we obtain

$$\begin{aligned} & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\} \\ &= \frac{1}{h_{us}} \sum_{\substack{(w,\lambda) \in \mathfrak{I}_{us}: |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon/2 \\ |\varrho(\xi, \Upsilon, p) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon/2 \\ |\varphi(\xi, \Upsilon, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon/2}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\} \\ &+ \frac{1}{h_{us}} \sum_{\substack{(w,\lambda) \in \mathfrak{I}_{us}: |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon/2 \\ |\varrho(\xi, \Upsilon, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon/2 \\ |\varphi(\xi, \Upsilon, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon/2}} \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\} \\ &\leq \frac{\mathfrak{K}}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \frac{\varepsilon}{2} \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \end{array} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

As a consequence, for each $\xi \in \Omega$, we get

$$\begin{aligned} & \left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right\} \\ & \subseteq \left\{ \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \frac{\varepsilon}{2} \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \end{array} \right| \geq \frac{\varepsilon}{2R} \right\} \in \mathcal{I}_2. \end{aligned}$$

□

Corollary 3.1. *We have the following result:*

$$\left\{ \mathfrak{S}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right\} \cap \mathfrak{L}_{\infty}^2(\Omega) = \left\{ \mathfrak{N}_{\omega_2} \left[\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right\} \cap \mathfrak{L}_{\infty}^2(\Omega).$$

Theorem 3.3. *If $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$, then $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left(\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right) \right)$ implies $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left(\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right) \right)$.*

Proof. Assume that $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$.

Then, there are $\eta > 0, \vartheta > 0$ such that $\lambda_u \geq 1 + \eta$ and $\lambda_s \geq 1 + \vartheta$.

For sufficiently large u, s which gives that $\frac{h_{us}}{j_u k_s} \geq \frac{\eta \vartheta}{(1+\eta)(1+\vartheta)}$.

Assume that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left(\mathcal{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right) \right)$.

For each $\varepsilon \in (0, 1)$, for all $\varpi > 0$, and for each $\xi \in \Omega$, we have

$$\begin{aligned} & \frac{1}{j_u k_s} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \\ & \geq \frac{1}{j_u k_s} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \\ & = \frac{h_{us}}{j_u k_s} \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \\ & \geq \frac{\eta \vartheta}{(1+\eta)(1+\vartheta)} \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right|. \end{aligned}$$

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Thus, for any $\tau > 0$,

$$\left\{ \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\}$$

$$\left\{ \frac{1}{j_u k_s} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \frac{\eta\vartheta\tau}{(1+\eta)(1+\vartheta)} \right\}$$

Consequently, by our notion, the set on the right side belongs to $\mathfrak{I}_2$, and obviously the set on the left side belongs to $\mathfrak{I}_2$.

As a result, we obtain $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. □

Theorem 3.4. *If $\limsup \lambda_u < \infty$ and $\limsup \lambda_s < \infty$, then $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ implies $\mathfrak{u}_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.*

Proof. Presume that $\limsup \lambda_u < \infty$ and $\limsup \lambda_s < \infty$. Then, there are $\mathfrak{P}, \mathfrak{R} > 0$ such that $\lambda_u < \mathfrak{P}$ and $\lambda_s < \mathfrak{R}$ for all u and s .

Assume that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ and let

$$\mathfrak{K}_{us} = \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right|.$$

Since $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$, it holds for each $\varepsilon \in (0, 1), \tau > 0$, for each $\xi \in \Omega$ and for all $\varpi > 0$,

$$\left\{ \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\} = \left\{ \frac{\mathfrak{K}_{us}}{h_{us}} \geq \tau \right\} \in \mathfrak{I}_2.$$

So, we can select positive integers $u_0, s_0 \in \mathbb{N}$ such that $\frac{\mathfrak{K}_{us}}{h_{us}} < \tau$ for all $u \geq u_0, s \geq s_0$.

Now, take $\mathfrak{D} = \max\{\mathfrak{K}_{us} : 1 \leq u \leq u_0, 1 \leq s \leq s_0\}$, and let m and n be integers providing $j_{u-1} < m \leq j_u$ and $k_{s-1} < n \leq k_s$.

Then, for every $\varepsilon > 0$ and each $\xi \in \Omega$, we get

$$\begin{aligned}
& \frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - (\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \\
& \leq \frac{1}{j_{u-1}k_{s-1}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - (\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \\
& = \frac{1}{j_{u-1}k_{s-1}} \{ \mathfrak{K}_{11} + \mathfrak{K}_{12} + \mathfrak{K}_{21} + \mathfrak{K}_{22} + \cdots + \mathfrak{K}_{u_0 s_0} + \cdots + \mathfrak{K}_{us} \} \leq \frac{1}{j_{u-1}k_{s-1}} \\
& \leq \frac{u_0 s_0}{j_{u-1}k_{s-1}} \left(\max_{\substack{1 \leq w \leq u_0 \\ 1 \leq \lambda \leq s_0}} \{ \mathfrak{K}_{w\lambda} \} \right) + \frac{1}{j_{u-1}k_{s-1}} \left\{ \begin{array}{l} \mathfrak{h}_{u_0(s_0+1)} \frac{\mathfrak{K}_{u_0(s_0+1)}}{\mathfrak{h}_{u_0(s_0+1)}} + \mathfrak{h}_{(u_0+1)s_0} \frac{\mathfrak{K}_{(u_0+1)s_0}}{\mathfrak{h}_{(u_0+1)s_0}} \\ + \mathfrak{h}_{(u_0+1)(s_0+1)} \frac{\mathfrak{K}_{(u_0+1)(s_0+1)}}{\mathfrak{h}_{(u_0+1)(s_0+1)}} + \cdots + \mathfrak{h}_{us} \frac{\mathfrak{K}_{us}}{\mathfrak{h}_{us}} \end{array} \right\} \\
& \leq \frac{u_0 s_0 \mathfrak{D}}{j_{u-1}k_{s-1}} + \frac{1}{j_{u-1}k_{s-1}} \left(\max_{\substack{u > u_0 \\ s > s_0}} \frac{\mathfrak{H}_{us}}{\mathfrak{h}_{us}} \right) \left(u, s \sum_{\substack{u > u_0 \\ s > s_0}} \mathfrak{h}_{w\lambda} \right) \\
& \leq \frac{u_0 s_0 \mathfrak{D}}{j_{u-1}k_{s-1}} + \tau \frac{(j_u - j_{u_0})(k_s - k_{s_0})}{j_{u-1}k_{s-1}} \leq \frac{u_0 s_0 \mathfrak{D}}{j_{u-1}k_{s-1}} + \tau \lambda_u \lambda_s \leq \frac{u_0 s_0 \mathfrak{D}}{j_{u-1}k_{s-1}} + \tau \mathfrak{P}\mathfrak{R}.
\end{aligned}$$

Since $j_{u-1}k_{s-1} \rightarrow \infty$ as $m, n \rightarrow \infty$, it concludes that for each $\xi \in \Omega$,

$$\frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \rightarrow 0$$

and as a result, for any $\tau_1 > 0$, the set

$$\left\{ \frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - (\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau_1 \right\} \in \mathfrak{I}_2.$$

It gives that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. $\square$

Theorem 3.5. Let ω_2 be a $\mathfrak{DLS}$. If $1 < \liminf_u \lambda_u < \limsup_u u\lambda < \infty$ and $1 < \liminf_s \lambda_s < \limsup_s s\lambda < \infty$, then $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\Upsilon_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ if $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{S} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Proof. It is clearly understood from Theorem (3.3) and theorem (3.4). $\square$

Theorem 3.6. Let $\mathfrak{I}_2$ be a Strongly Admissible Ideal ($\mathfrak{SAI}$) providing feature $(\mathfrak{AP}_2)$, $\omega_2 \in \Upsilon(\mathfrak{I}_2)$. If $\{\Upsilon_{w\lambda}\} \in \mathfrak{S} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \cap \mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right]$, then $\mathfrak{S} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim_{w, \lambda \rightarrow \infty} \Upsilon_{w\lambda} = \mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim_{w, \lambda \rightarrow \infty} \Upsilon_{w\lambda}$

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Proof. Assume that $\mathfrak{S} \left[\mathcal{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim_{w, \lambda \rightarrow \infty} \Upsilon_{w\lambda} = \mathfrak{U}$ and

$\mathfrak{S}_{\omega_2} \left[\mathcal{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim_{w, \lambda \rightarrow \infty} \Upsilon_{w\lambda} = \mathfrak{Z}$ and $\mathfrak{Y} \neq \mathfrak{Z}$.

Let $0 < \varepsilon < \frac{1}{2} |\psi(\xi, \mathfrak{Y}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)|$, $0 < \varepsilon < \frac{1}{2} |\varrho(\xi, \mathfrak{Y}, \varpi) - \varrho(\xi, \mathfrak{Z}, \varpi)|$ and $0 < \varepsilon < \frac{1}{2} |\varphi(\xi, \mathfrak{Y}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)|$, for every $\xi \in \Omega$.

Since $\mathcal{I}_2$ provides the feature $(\mathfrak{A}\mathfrak{P}_2)$, then there is $\Omega \in \Upsilon(\mathcal{I}_2)$ such that for every $\xi \in \Omega$ and for $(m, n) \in \Omega$.

$$\text{Let } \lim_{m, n \rightarrow \infty} \frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon, \varpi) - \psi(\xi, \mathfrak{Y}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \mathfrak{Y}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Y}, \varpi)| \geq \varepsilon \end{array} \right| = 0$$

$$\mathfrak{D} = \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Y}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \mathfrak{Y}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right\}$$

$$\mathfrak{S} = \left\{ \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right\}$$

Then, $mn = |\mathfrak{D} \cup \mathfrak{S}| \leq |\mathfrak{D}| + |\mathfrak{S}|$. This gives that $1 \leq \left(\frac{|\mathfrak{D}|}{mn} \right) + \left(\frac{|\mathfrak{S}|}{mn} \right)$.

Since $\left(\frac{|\mathfrak{S}|}{mn} \right) \leq 1$ and $\lim_{m, n \rightarrow \infty} \frac{|\mathfrak{D}|}{mn} = 0$, we have to get $\lim_{m, n \rightarrow \infty} \frac{|\mathfrak{S}|}{mn} = 1$.

Let $\Omega^* = \Omega \cap \omega_2 \in \Upsilon(\mathcal{I}_2)$.

Then, for every $\xi \in \Omega$ and $(w_k, \lambda_j) \in \Omega^*$, the $w_k \lambda_j^{\text{th}}$ term of the statistical limit expression

$$\frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right|, \text{ is}$$

$$\frac{1}{w_k \lambda_j} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right|$$

$$= \frac{1}{\bigcup_{u, s=1, 1}^{k, j} h_{us}} \bigcup_{u, s=1, 1}^{k, j} \mathfrak{V}_{us} h_{us}, \text{ where}$$

$$\mathfrak{V}_{us} = \frac{1}{h_{us}} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \xrightarrow{\mathcal{I}_2} 0 \quad (1)$$

because $\left(\mathfrak{S}_{\omega_2} \left[\mathcal{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right) - \lim_{w, \lambda} \Upsilon_{w\lambda} = \mathfrak{Z}$.

Since ω_2 is a lacunary sequence, (1) is a regular weighted mean transform of $\mathfrak{V}_{us}$'s and as a result, it is $\mathcal{I}_2$ -convergent to 0 as $k, j \rightarrow \infty$ and also it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ provides the feature $(\mathfrak{A}\mathfrak{P}_2)$.

Anyway, because this is a sequence of

$$\left\{ \frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon, \varpi) - (\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right| \right\}_{(m,n) \in \mathfrak{M}'}$$

We conclude that

$$\left\{ \frac{1}{mn} \left| \begin{array}{l} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \mathfrak{Z}, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon, \varpi) - (\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \mathfrak{Z}, \varpi)| \geq \varepsilon \end{array} \right| \right\}_{(m,n) \in \mathfrak{M}'}$$

which is not convergent to 1. The contradiction here shows that we cannot have $\mathfrak{Y} \neq \mathfrak{Z}$. $\square$

Theorem 3.7. *If $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$ then*

$$\left(\mathfrak{C}_1 \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right) \subseteq \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right).$$

Proof. Let $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$.

Then, there are $\eta, \vartheta > 0$ such that $\lambda_u \geq 1 + \eta$ and $\lambda_s \geq 1 + \vartheta$, for all u and s which gives that $\frac{j_u k_s}{h_{us}} \leq \frac{(1+\eta)(1+\vartheta)}{\eta\vartheta}$ and $\frac{j_{u-1} k_{s-1}}{h_{us}} \leq \frac{1}{\eta\vartheta}$.

Assume that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{C}_1 \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. For each $\xi \in \Omega$, we get

$$\begin{aligned} & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \\ &= \frac{1}{h_{us}} \sum_{w,\lambda=1,1}^{j_u, k_s} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ & \quad - \frac{1}{h_{us}} \sum_{w,\lambda=1,1}^{j_{u-1}, k_{s-1}} |\psi(\xi, \Upsilon, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \\ &= \frac{j_u k_s}{h_{us}} \left[\frac{1}{j_u k_s} \sum_{w,\lambda=1,1}^{j_u, k_s} |\psi(\xi, \Upsilon, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \right] \\ & \quad - \frac{j_{u-1} k_{s-1}}{h_{us}} \left[\frac{1}{j_{u-1} k_{s-1}} \sum_{w,\lambda=1,1}^{j_{u-1}, k_{s-1}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \right]. \end{aligned}$$

Since $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{C}_1 \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$, then for each

$$\frac{1}{j_u k_s} \sum_{w,\lambda=1,1}^{j_u, k_s} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \xrightarrow{\mathfrak{J}_2} 0 \quad \text{and}$$

$$\frac{1}{j_{u-1} k_{s-1}} \sum_{w,\lambda=1,1}^{j_{u-1}, k_{s-1}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \xrightarrow{\mathfrak{J}_2} 0.$$

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As a result, when the above equality is checked, for every $\xi \in \Omega$, we have

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \xrightarrow{\mathfrak{I}_2} 0.$$

Similarly, we obtain

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0,$$

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0.$$

That is, $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

As a result, we obtain $\left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right) \subseteq \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. □

Theorem 3.8. *If $\liminf_u \lambda_u = 1$ and $\liminf_s \lambda_s = 1$ then*

$$\left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right) \subseteq \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right).$$

Proof. Take $\liminf_u \lambda_u = 1$ and $\liminf_s \lambda_s = 1$, and $\Upsilon_{w\lambda} \in \mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right]$.

Then for every $\varpi > 0$, we acquire

$$\left. \begin{aligned} \mathfrak{h}_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 1, \\ \mathfrak{h}'_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0, \\ \mathfrak{h}''_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0 \end{aligned} \right\} \text{ as } u, s \rightarrow \infty.$$

Then for $\varepsilon > 0$, there are $u_0, s_0 \in \mathfrak{N}$ such that $\mathfrak{h}_{us} < 1 + \varepsilon$ for all $u > u_0, s > s_0$.

Also, we can find $\zeta > 0$ such that $\mathfrak{h}_{us} < \zeta$, $\mathfrak{h}'_{us} < \zeta$ and $\mathfrak{h}''_{us} < \zeta$, $u, s = 1, 2, \dots$

Let m and n be an integer with $j_{u-1} < m \leq j_u$ and $k_{s-1} \leq n \leq k_s$. Then,

$$\begin{aligned} & \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ & \leq \frac{1}{j_{u-1}k_{s-1}} \sum_{w,\lambda=1,1}^{j_{u-1},k_{s-1}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ & = \frac{1}{j_{u-1}k_{s-1}} \left[\sum_{(w,\lambda) \in \mathfrak{I}_{11}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| + \dots + \right. \\ & \quad \left. \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \right] \\ & = \sup_{\substack{1 \leq u \leq u_0 \\ 1 \leq s \leq s_0}} \mathfrak{h}_{us} \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + \frac{h_{(u_0+1)(s_0+1)}}{j_{u-1}k_{s-1}} \mathfrak{R}_{(u_0+1)(s_0+1)} + \dots + \frac{\mathfrak{h}_{us}}{j_{u-1}k_{s-1}} \mathfrak{h}_{us} \end{aligned}$$

$$< \zeta \frac{j_{u_0} k_{s_0}}{j_{u-1} k_{s-1}} + (1 + \varepsilon) \frac{j_{u_0} k_{s_0}}{j_{u-1} k_{s-1}}.$$

Since $j_{u-1} k_{s-1} \rightarrow \infty$ as $m, n \rightarrow \infty$, it follows that

$$\frac{1}{mn} \sum_{\substack{m,n \\ w,\lambda=1,1}} |\psi(\xi, \Upsilon_{w\lambda}, p) - \psi(\xi, \Upsilon, p)| \xrightarrow{\mathfrak{I}_2} 1.$$

Similarly, we can show that

$$\sum_{\substack{m,n \\ w,\lambda=1,1}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0$$

$$\text{and } \sum_{\substack{m,n \\ w,\lambda=1,1}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0.$$

Hence $\{\Upsilon_{w\lambda}\} \in \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. □

Theorem 3.9. *If $\{\Upsilon_{w\lambda}\} \in \mathfrak{N}_{\omega_2} \left[I_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \cap \mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right]$, then $\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim \Upsilon_{w\lambda} = \mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] - \lim \Upsilon_{w\lambda}$.*

Proof. Let $\Upsilon_{w\lambda} \rightarrow \Upsilon_1 \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ and $\Upsilon_{w\lambda} \rightarrow \Upsilon_2 \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Assume $r \in \mathbb{N}$ and $\varepsilon > 0$ in such a way that $r > \frac{2}{\varepsilon}$. Then, for any $p > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\left. \begin{aligned} & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \psi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| > 1 - \frac{1}{r}, \\ & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \varrho \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r} \\ & \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \varphi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r}, \end{aligned} \right\}$$

for all $u > u_0, s > s_0$.

Also, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\left. \begin{aligned} & \frac{1}{mn} \sum_{\substack{m,n \\ w,\lambda=1,1}} \left| \psi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| > 1 - \frac{1}{r}, \\ & \frac{1}{mn} \sum_{\substack{m,n \\ w,\lambda=1,1}} \left| \varrho \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r} \\ & \text{and } \frac{1}{mn} \sum_{\substack{m,n \\ w,\lambda=1,1}} \left| \varphi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r}, \end{aligned} \right\}$$

for all $m > m_0, n > n_0$.

Take $r_1 = \max\{u_0, m_0\}$ and $r_2 = \max\{s_0, n_0\}$. Then we take $k, t \in \mathbb{N}$ such

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that

$$\begin{aligned}
& \left| \psi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| \\
& \geq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \psi \left(\xi, \Upsilon_{w\lambda}, \frac{p}{2} \right) - \psi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| > 1 - \frac{1}{r} \\
& \left| \psi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| \\
& \geq \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left| \psi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| > 1 - \frac{1}{r} \\
& \left| \varrho \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| \\
& \leq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \varrho \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r} \\
& \left| \varrho \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| \\
& \leq \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left| \varrho \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| < \frac{1}{r} \text{ and} \\
& \left| \varphi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| \\
& \leq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left| \varphi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| < \frac{1}{r} \\
& \left| \varphi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| \\
& \leq \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left| \varphi \left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2} \right) - \varphi \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| < \frac{1}{r}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& |\psi(\xi, \Upsilon_1, \varpi) - \psi(\xi, \Upsilon_2, \varpi)| \\
& \geq \left| \psi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| + \left| \psi \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \psi \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| \\
& > \left(1 - \frac{1}{r} \right) + \left(1 - \frac{1}{r} \right) > 1 - \varepsilon, \\
& |\varrho(\xi, \Upsilon_1, \varpi) - \varrho(\xi, \Upsilon_2, \varpi)| \\
& \leq \left| \varrho \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_1, \frac{\varpi}{2} \right) \right| + \left| \varrho \left(\xi, \Upsilon_{kt}, \frac{\varpi}{2} \right) - \varrho \left(\xi, \Upsilon_2, \frac{\varpi}{2} \right) \right| \\
& < \left(\frac{1}{r} \right) + \left(\frac{1}{r} \right) < \varepsilon \text{ and}
\end{aligned}$$

$$\begin{aligned}
 & |\varphi(\xi, \Upsilon_1, \varpi) - \varphi(\xi, \Upsilon_2, \varpi)| \\
 & \leq \left| \varphi\left(\xi, \Upsilon_{kt}, \frac{\varpi}{2}\right) - \varphi\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right) \right| + \left| \varphi\left(\xi, \Upsilon_{kt}, \frac{\varpi}{2}\right) - \varphi\left(\xi, \Upsilon_2, \frac{\varpi}{2}\right) \right| \\
 & < \left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) < \varepsilon.
 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\begin{aligned}
 & |\psi(\xi, \Upsilon_1, \varpi) - \psi(\xi, \Upsilon_2, \varpi)| = 1, |\varrho(\xi, \Upsilon_1, \varpi) - \varrho(\xi, \Upsilon_2, \varpi)| = 0 \quad \text{and} \\
 & |\varphi(\xi, \Upsilon_1, \varpi) - \varphi(\xi, \Upsilon_2, \varpi)| = 0, \text{ for all } \varpi > 0, \text{ which yields } \Upsilon_1 = \Upsilon_2.
 \end{aligned}$$

As we go through the definitions and theorems that follow,

let us consider $(\Omega, \psi, \varrho, \varphi, *, \diamond, \otimes)$ to be a separable $\mathfrak{NM}\mathfrak{S}$ and $\mathfrak{I}_2$ to be $\mathfrak{SA}\mathfrak{I}$. $\square$

Definition 3.1. The sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WS}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}\mathfrak{a}$ if for each $\varepsilon \in (0, 1)$ for each $\xi \in \Omega$ and for all $\varpi > 0$, there are $s = s(\varepsilon, \xi), t = t(\varepsilon, \xi) \in \mathbb{N}$ such that

$$\mathfrak{Y}(\varepsilon, \xi, \varpi) = \left\{ \begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathfrak{I}_2$$

Theorem 3.10. Every $\mathfrak{WS}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}$ sequence of closed sets $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WS}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}\mathfrak{a}$ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$.

Proof. Let $\Upsilon_{w\lambda} \rightarrow \mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi, \varrho, \varphi)} \right] \Upsilon$. At that case, for each $\varepsilon \in (0, 1)$, for every $\xi \in \Omega$ and for all $\varpi > 0$,

$$\mathfrak{Y}(\varepsilon, \xi, \varpi) = \left\{ \begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathfrak{I}_2.$$

Since $\mathfrak{I}_2$ is $\mathfrak{SA}\mathfrak{I}$, the set

$$\mathfrak{Y}^c(\varepsilon, \xi, \varpi) = \left\{ \begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon \end{array} \right\} \in \mathfrak{I}_2$$

is nonempty and belongs to $\Upsilon(\mathfrak{I}_2)$. So, we select positive integers u and s such that $(u, s) \neq \mathfrak{Y}(\varepsilon, \xi, \varpi)$ and we get

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$$\left. \begin{array}{l} \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon, \\ \text{and } \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w_0 \lambda_0}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon, \\ \text{and } \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon. \end{array} \right\}$$

Now, presume that

$$\mathfrak{Z}(\varepsilon, \xi, \varpi) = \left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w \lambda}, \varpi) - \psi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \leq 1 - 2\varepsilon \\ \text{or } \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w \lambda}, \varpi) - \varrho(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \geq 2\varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w \lambda}, \varpi) - \varphi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \geq 2\varepsilon. \end{array} \right\}$$

Consider the inequality

$$\begin{aligned} & \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w \lambda}, \varpi) - \psi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \\ & \leq \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w \lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ & \quad + \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w \lambda}, \varpi) - \psi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)|, \\ & \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w \lambda}, \varpi) - \varrho(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \\ & \geq \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w \lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ & \quad + \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w_0 \lambda_0}, \varpi) - \varrho(\xi, \Upsilon, \varpi)|, \\ & \frac{1}{h_{us}} \sum_{(w, \lambda), (w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w \lambda}, \varpi) - \varphi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi)| \\ & \geq \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w \lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \\ & \quad + \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w_0 \lambda_0}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{aligned}$$

Notice this if $(u, s) \in \mathfrak{Z}(\varepsilon, \xi, \varpi)$, therefore,

$$\begin{aligned}
 & \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\
 & \quad + \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{w_0\lambda_0}, \varpi)| \leq 1 - 2\varepsilon, \\
 & \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| + \\
 & \quad \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{w_0\lambda_0}, \varpi)| \geq 2\varepsilon, \\
 & \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| + \\
 & \quad \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{w_0\lambda_0}, \varpi)| \geq 2\varepsilon.
 \end{aligned}$$

From another point of view, since $(u, s) \neq \mathfrak{Y}(\varepsilon, \xi, \varpi)$, we get

$$\left. \begin{aligned}
 & \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon, \\
 & \text{or } \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w_0\lambda_0}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \\
 & \text{and } \frac{1}{h_{us}} \sum_{(w_0, \lambda_0) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w_0\lambda_0}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon
 \end{aligned} \right\}$$

$$\left. \begin{aligned}
 & \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\
 & \text{or } \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\
 & \text{and } \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon.
 \end{aligned} \right\}$$

Hence, $(u, s) \in \mathfrak{Y}(\varepsilon, \xi, \varpi)$. This gives that $\mathfrak{Z}(\varepsilon, \xi, \varpi) \subset \mathfrak{Y}(\varepsilon, \xi, \varpi) \in \mathfrak{I}_2$, so the sequence is Wijsman strongly $\mathfrak{I}_2$ -lacunary sequence. $\square$

Definition 3.2. The sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2\mathfrak{Lcto}\Upsilon$ iff there is a set $\Omega = \{(w, \lambda) \in \mathbb{N} \times \mathbb{N}\}$ such that $\mathfrak{M}' = \{(w, \lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$ for each $\xi \in \Omega$,

$$\lim_{u, s \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| = 1,$$

$$\lim_{u, s \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| = 0,$$

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$$\lim_{u,s \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| = 0.$$

In this case, we write $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Theorem 3.11. *If the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2^* \mathfrak{LCE}$ to Υ , then $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2 \mathfrak{LCE}$ to Υ .*

Proof. Assume that $\Upsilon_{w\lambda} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$.

Then, there is a set $\Omega = \{(w, \lambda) \in \mathbb{N} \times \mathbb{N}\}$ such that

$\mathfrak{M}' = \{(w, \lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$, for each $\xi \in \Omega$,

$$\left. \begin{aligned} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| &> 1 - \varepsilon, \\ \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| &< \varepsilon, \\ \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| &< \varepsilon, \end{aligned} \right\}$$

for every $\varepsilon > 0$ and for all $w, \lambda \geq k_0 = k_0(\varepsilon, \xi) \in \mathbb{N}$.

Hereby for each $\varepsilon > 0$ and $\xi \in \Omega$, we get

$$\chi(\varepsilon, \xi, p) = \left\{ \begin{aligned} &\frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ &\text{or } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ &\text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{aligned} \right\}$$

$\subset \mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\})))$.

For $\mathbb{N} \times \mathbb{N} \setminus \mathfrak{M}' = \mathfrak{K} \in \mathfrak{I}_2$. Since $\mathfrak{I}_2$ is an $\mathfrak{AI}$, we obtain

$\mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}))) \in \mathfrak{I}_2$

and so $\chi(\varepsilon, \xi, \varpi) \in \mathfrak{I}_2$. Hence $\{\Upsilon_{w\lambda}\} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$. $\square$

Theorem 3.12. *Let $\mathfrak{I}_2$ be a $\mathfrak{SAI}$ involving feature $(\mathfrak{AP}_2)$. Then $\{\Upsilon_{w\lambda}\} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{(\psi, \varrho, \varphi)} \right] \right)$ implies $\{\Upsilon_{w\lambda}\} \rightarrow \Upsilon \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{J}_{\mathfrak{M}_2}^{*(\psi, \varrho, \varphi)} \right] \right)$.*

Definition 3.3. *The sequence $\{\Upsilon_{w\lambda}\}$ is known as $\mathfrak{WSI}_2^* \mathfrak{LCE}$ a sequence if for each $\varepsilon \in (0, 1)$ for all $\xi \in \Omega$ and for all $\varpi > 0$, there is a set $\Omega = \{(w, \lambda) \in \mathbb{N} \times \mathbb{N}\}$ such that $\mathfrak{M}' = \{(w, \lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$ and $\mathbb{N} = \mathbb{N}(\varepsilon, \xi) \in \mathbb{N}$ such that*

$$\left. \begin{aligned} \frac{1}{h_{us}} \sum_{(w,\lambda), (s,t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| &> 1 - \varepsilon \\ \frac{1}{h_{us}} \sum_{(w,\lambda), (u,s) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| &< \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda), (u,s) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| &< \varepsilon, \end{aligned} \right\}$$

for every $w, \lambda, s, t \geq \mathbb{N}$.

Theorem 3.13. Every $\mathfrak{WSI}_2^* \mathfrak{LCA}$ sequence of closed sets is $\mathfrak{WSI}_2 \mathfrak{LCA}$ in $\mathfrak{NM}(\psi, \varrho, \varphi)$.

Proof. If the hypothesis is provided, then for each $\varepsilon \in (0, 1)$, for each $\xi \in \Omega$, and for all $\varpi > 0$, there is a set $\mathfrak{Q} = \{(w, \lambda) \in \mathbb{N} \times \mathbb{N}\}$ such that

$\mathfrak{M}' = \{(w, \lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$ and $\mathbb{N} = \mathbb{N}(\varepsilon, \xi) \in \mathbb{N}$ such that

$$\left. \begin{aligned} & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (s, t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| > 1 - \varepsilon \\ & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (u, s) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \text{ and} \\ & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (u, s) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| < \varepsilon, \end{aligned} \right\} \text{ for each } w, \lambda, s, t \geq$$

$\mathbb{N}$.

Let $\mathfrak{K} = \mathbb{N} \times \mathfrak{NM}'$. It is clear that $\mathfrak{K} \in \mathfrak{I}_2$ and

$$\chi(\varepsilon, \xi, \varpi) = \left\{ \left(\begin{aligned} & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (s, t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| > 1 - \varepsilon \\ \text{or } & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (u, s) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \\ \text{and } & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda), (u, s) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \end{aligned} \right) \right\}$$

$\subset \mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \dots, (\mathbb{N} - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (\mathbb{N} - 1)\})))$.

As $\mathfrak{I}_2$ be a $\mathfrak{SAI}$, then

$\mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \dots, (\mathfrak{N} - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (\mathfrak{N} - 1)\}))) \in \mathfrak{I}_2$.

Therefore, we obtain $\chi(\varepsilon, \xi, \varpi) \in \mathfrak{I}_2$; that is $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2 \mathfrak{LCA}$ with regard to (ψ, ϱ, φ) . $\square$

Theorem 3.14. Let $\mathfrak{I}_2$ be an $\mathfrak{AI}$ involving property $(\mathfrak{AP}_2)$. Then, the concept of $\mathfrak{WSI}_2 \mathfrak{LCA}$ of sets coincides with $\mathfrak{WSI}_2^* \mathfrak{LCA}$ s of sets.

Proof. If a set sequence is $\mathfrak{WSI}_2^* \mathfrak{LCA}$, then it is $\mathfrak{WSI}_2 \mathfrak{LCA}$ sequence according to theorem (3.13), where $\mathfrak{I}_2$ need not have the feature $(\mathfrak{AP}_2)$.

Now it is adequate to demonstrate that a sequence $\{\Upsilon_{w\lambda}\}$ in Ω is $\mathfrak{WSI}_2^* \mathfrak{LCA}$ sequence under assumption that it is a $\mathfrak{WSI}_2 \mathfrak{LCA}$. Let $\{\Upsilon_{w\lambda}\}$ be a $\mathfrak{WSI}_2 \mathfrak{LCA}$ sequence. In this case, for each $\varepsilon \in (0, 1)$, for all $\xi \in \Omega$, there is a number $s = s(\varepsilon, \xi), t = t(\varepsilon, \xi) \in \mathbb{N}$ such that

$$\mathfrak{Y}(\varepsilon, \xi, \varpi) = \left\{ \begin{aligned} & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| \leq 1 - \varepsilon \\ \text{or } & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \\ \text{and } & \frac{1}{\mathfrak{h}_{us}} \sum_{(w, \lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \end{aligned} \right\} \in \mathfrak{I}_2$$

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Let

$$\chi_j(\varepsilon, \xi, \varpi) = \left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda), (s,t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| > 1 - \frac{1}{j}, \\ \text{or } \frac{1}{h_{us}} \sum_{(w,\lambda), (u,s) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| < \frac{1}{j} \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda), (u,s) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| < \frac{1}{j} \end{array} \right\}$$

where $s(j) = s\left(\frac{1}{j}\right)$ and $t(j) = t\left(\frac{1}{j}\right)$, $j = 1, 2, \dots$

Clearly, for $j = 1, 2, \dots$, $\chi_j(\varepsilon, \xi, \varpi) \in \Upsilon(\mathfrak{I}_2)$. Since $\mathfrak{I}_2$ has the property $(\mathfrak{A}\mathfrak{P}_2)$, there is $\chi \subset \mathbb{N} \times \mathbb{N}$ so that $\chi \in \Upsilon(\mathfrak{I}_2)$ and $\chi \setminus \chi_j$ is finite for all j .

Now, we demonstrate that

$$\begin{aligned} \lim_{w, \lambda, s, t \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}, (s,t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| &= 1, \\ \lim_{w, \lambda, s, t \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}, (s,t) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| &= 0, \\ \lim_{w, \lambda, s, t \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}, (s,t) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| &= 0, \end{aligned}$$

for all $w, \lambda, s, t > u(r)$.

So, it follows that for each $\xi \in \Omega$ and $(w, \lambda), (s, t) \in \chi$. To show these, let $\varepsilon \in (0, 1)$ and $r \in \mathbb{N}$ such that $\frac{2}{\varepsilon}$. If $(w, \lambda), (s, t) \in \chi$, then $\chi \setminus \chi_r$ is a finite set, therefore, there is $u = u(r)$ so that

$$\left. \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{s_r t_r}, \varpi)| > 1 - \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(s,t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{s_r t_r}, \varpi)| > 1 - \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{s_r t_r}, \varpi)| < \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(s,t) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{s_r t_r}, \varpi)| < \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{s_r t_r}, \varpi)| < \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(s,t) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{s_r t_r}, \varpi)| < \frac{1}{r}, \end{array} \right\}$$

$$\frac{1}{h_{us}} \sum_{(w,\lambda), (s,t) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)|$$

$$\begin{aligned}
&\geq \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&\quad + \frac{1}{h_{us}} \sum_{(s,t) \in \mathcal{I}_{us}} |\psi(\xi, \Upsilon, \varpi) - \psi(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&> \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \varepsilon, \\
&\frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| \\
&\leq \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&\quad + \frac{1}{h_{us}} \sum_{(s,t) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&< \frac{1}{r} + \frac{1}{r} < \varepsilon. \\
&\frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| \\
&\leq \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&\quad + \frac{1}{h_{us}} \sum_{(s,t) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{s_r t_r}, \varpi)| \\
&< \frac{1}{r} + \frac{1}{r} < \varepsilon.
\end{aligned}$$

Therefore, for each $\varepsilon \in (0, 1)$, there exists $u = u(\varepsilon)$, and $(w, \lambda), (s, t) \in \chi \in \Upsilon(\mathcal{I}_2)$, we get

$$\left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathcal{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

This implies that $\{\Upsilon_{w\lambda}\}$ is $\mathcal{WSI}_2^* \mathcal{LCA}$ sequence. $\square$

Definition 3.4. A sequence $\{\Upsilon_{w\lambda}\}$ in $\mathcal{NM}\mathcal{S}$ is called to be Wijsman lacunary convergent to Υ with regard to $\mathcal{NM}(\psi, \varrho, \varphi)$ if, for every $\varpi > 0$ and $\varepsilon \in (0, 1)$, there is $m_0, n_0 \in \mathbb{N}$ such that

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$$\begin{aligned} & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon, \\ & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \text{ and} \\ & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon, \text{ for all } u > m_0 \text{ and } \mathfrak{S} > n_0. \end{aligned}$$

We write $(\psi, \varrho, \varphi)^{\omega_2} - \lim \Upsilon_{w\lambda} = \Upsilon$.

Definition 3.5. Take $(\Omega, \psi, \varrho, \varphi, *, \diamond, \otimes)$ as a separable $\mathfrak{NMS}$ and take $\{\Upsilon_{w\lambda}\} \in \Omega$.

(i) $\Upsilon \in \Omega$ is known as Wijsman Lacunary $\mathfrak{I}_2$ ($\mathfrak{WL}\mathfrak{I}_2$)-limit point of $\{\Upsilon_{w\lambda}\}$ if there is set $\mathfrak{Q} = \{(w_1, \lambda_1) < (w_2, \lambda_2) < \dots (w_u, \lambda_s) < \dots\} \subset \mathbb{N} \times \mathbb{N}$ such that the set $\mathfrak{M}' = \{(w_u, \lambda_s) \in \mathfrak{I}_{us}\} \neq \mathfrak{I}_2$ and $(\psi, \varrho, \varphi)^{\omega_2} - \lim \Upsilon_{w_u\lambda_s} = \Upsilon$.

(ii) $\Upsilon \in \Omega$ is known as $\mathfrak{WL}\mathfrak{I}_2$ -cluster point of $\{\Upsilon_{w\lambda}\}$ if, for every $\varpi > 0$ and $\varepsilon \in (0, 1)$, we get

$$\left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon \end{array} \right\} \notin \mathfrak{I}_2.$$

Here, $\wedge_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda})$ denotes the set of all $\mathfrak{WL}\mathfrak{I}_2$ -limit points and $\Gamma_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda})$ indicates the set of all $\mathfrak{WL}\mathfrak{I}_2$ -cluster points in $\mathfrak{NMS}$.

Theorem 3.15. For each sequence $\{\Upsilon_{w\lambda}\}$ in $\mathfrak{NMS}$, we have,

$$\wedge_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda}) \subseteq \Gamma_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda}).$$

Proof. Let $\Upsilon \in \wedge_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda})$. So, there is a set $\mathfrak{Q} \subset \mathbb{N} \times \mathbb{N}$ such that $\mathfrak{M}' \neq \mathfrak{I}_2$, where $\mathfrak{Q}$ and $\mathfrak{M}'$ are as in Definition (3.5), satisfying $(\psi, \varrho, \varphi)^{\omega_2} - \lim \Upsilon_{w_u\lambda_s} = \Upsilon$. Hence, for every $\varpi > 0$ and $\varepsilon \in (0, 1)$, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w_u\lambda_s}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon, \\ & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w_u\lambda_s}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \text{ and} \\ & \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w_u\lambda_s}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon, \text{ for all } u \geq m_0 \text{ and } s \geq n_0. \end{aligned}$$

Therefore,

$$\mathfrak{J} = \left\{ \begin{array}{l} \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon, \\ \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon \end{array} \right\}$$

$\supseteq \mathfrak{M}'\{(w_1, \lambda_1), (w_2, \lambda_2), \dots, (w_{m_0}, \lambda_{n_0})\}$.

Now, with $\mathfrak{I}_2$ being admissible, we must have

$\mathfrak{M}'\{(w_1, \lambda_1), (w_2, \lambda_2), \dots, (w_{m_0}, \lambda_{n_0})\} \neq \mathfrak{I}_2$ and as such $\neq \mathfrak{I}_2$.

Hence $\Upsilon \in \Gamma_{(\psi, \varrho, \varphi)}^{\mathfrak{I}_{\omega_2}}(\Upsilon_{w\lambda})$. □

4 Conclusion

In this investigation, researchers looked at the Wijsman lacunary ideal combination of the double sets collections, a kind of ideal union. We looked at several novel $\mathfrak{MM}\mathfrak{S}$ concepts for two-set groups, and we got some verifying results. Binary sets recurrence in $\mathfrak{MM}\mathfrak{S}$ have been characterised, together with their corresponding Wijsman lacunary $\mathfrak{I}_2$ - limit as well as cluster foci. While confirmation typically employ an alternate strategy, a few of the findings given in the current work have almost similar to the research focused on the pertinent topic. Only when $\mathfrak{I}$ and $\mathfrak{I}^*$ are admissible Ideals some of the results are true. We can apply all the results of the current paper and introduce new theories in different spaces like neutrosophic normed linear space, locally solid Riesz space and so on. Once we have proved the completeness of the space, easily we can obtain a fixed point theories in the respective space.

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Weighted Norlund-Euler Statistical Convergence In Neutrosophic Normed Linear Spaces

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Abstract

This paper explores the neutrosophic Norlund $\mathcal{I}$ -statistically convergent sequence space. In Norlund convergent spaces, we introduce some neutrosophic normed spaces ($\mathcal{NN}\mathcal{S}$ s). The topological and algebraic characteristics of these convergent sequence spaces are also examined. Results are gathered from many angles, and fresh instances are created to support the equivalents and show the validity of the new ideas. The findings of this study provide a thorough foundation for $\mathcal{NN}\mathcal{S}$ and significantly advance the literature's theoretical understanding of the phenomenon. This study's distinguishing characteristic is the first comprehensive analysis of the properties and use of neutrosophic Norlund $\mathcal{I}$ -statistically convergent sequences in $\mathcal{NN}\mathcal{S}$, based on the accepted definition.

Keywords: Statistical convergence, weighted statistical convergence, neutrosophic normed linear space.

2020 AMS subject classifications: 46S40, 54H25¹

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1 Introduction

Fuzzy theory has significantly advanced the mathematical foundations of fuzzy set ($\mathfrak{F}\mathfrak{S}$) theory, which was developed by Zadeh [15] in 1965. According to Zadeh [15], a $\mathfrak{F}\mathfrak{S}$ gives each component of a given crisp universe set from $[0,1]$ a membership value. Lack of membership degree knowledge is a problem that $\mathfrak{F}\mathfrak{S}$ s frequently struggle to solve. As a result, Atanassov [1] investigated the intuitionistic $\mathfrak{F}\mathfrak{S}$ ($\mathfrak{I}\mathfrak{F}\mathfrak{S}$), a development of $\mathfrak{F}\mathfrak{S}$.

Neutrosophy is the idea of objective knowledge of thinking, and neutral describes the fundamental distinction between neutral, fuzzy, intuitive $\mathfrak{F}\mathfrak{S}$ s and logic. Following the development of the neutrosophic set ($\mathfrak{N}\mathfrak{S}$), which is a generalisation of the classical set, $\mathfrak{F}\mathfrak{S}$, and $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ by Smarandache [14]. Smarandache implemented the $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ theory by defining a new component, called the indeterminacy membership function, after taking everything into consideration.

Every element of the universe has a degree of $\mathfrak{T}$, $\mathfrak{F}$, and $\mathfrak{I}$ in $\mathfrak{N}\mathfrak{S}$, which is defined as a set. The degree of non-belongingness in $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ s is dependent on the degree of belongingness rather than being independent of it. The degree of non-belongingness of an element in $\mathfrak{F}\mathfrak{S}$ s is exactly equivalent to one degree of belongingness, which makes it a notable example of an $\mathfrak{I}\mathfrak{F}\mathfrak{S}$. In $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ s, uncertainty is predicated on the degree of belongingness, whereas in $\mathfrak{N}\mathfrak{S}$, uncertainty is taken into account independently of $\mathfrak{T}$ and $\mathfrak{F}$ values. $\mathfrak{N}\mathfrak{S}$ s are really more general than $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ because there are no limitations on the degree of $\mathfrak{T}$, $\mathfrak{F}$, and $\mathfrak{I}$. This has led to a variety of mathematicians producing their research in various mathematical forms.

For weak contractions on neutrosophic normed spaces, Jeyaraman, Ramachandran and Shakila [6] demonstrated approximate fixed point theorems in 2022. In this paper, we study the neutrosophic deferred statistical convergence of double sequences in the neutrosophic normed space. Along with being compared to itself under various limitations, this novel method is also contrasted with neutrosophic statistical convergence of double sequences. The fundamental theory of mathematics makes use of the notion of sequence convergence. Summability theory contains a large number of convergence 3 concepts, including classical measure theory, fuzzy theory, approximation theory, and probability theory. The connections between these concepts are examined.

2 Preliminary

Definition 2.1. [2] Let $\mathfrak{K} \subseteq \mathbb{N}$. The number $\delta_{\mathfrak{N}\mathfrak{E}}^q(\mathfrak{K}) = \lim_{n \rightarrow \infty} \frac{1}{\mathfrak{N}_n} |\{\mathfrak{k} \leq \mathfrak{N}_n : \mathfrak{k} \in \mathfrak{K}\}|$ is said to be weighted Norlund-Euler density of $\mathfrak{K}$. A sequence $x = (x_{\mathfrak{k}})$ is

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said to be generalized weighted Norlund-Euler statistically convergent (or $\mathfrak{S}_{\mathfrak{NE}}^q$ -convergent) to $\mathfrak{L}$ if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{N}_n} \left| \left\{ \mathfrak{k} \leq \mathfrak{N}_n : \mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}_{\mathfrak{k}-v} |x_v - \mathfrak{L}| \geq \varepsilon \right\} \right| = 0.$$

In this case, we write $\mathfrak{S}_{\mathfrak{NE}}^q - \lim x = \mathfrak{L}$.

Definition 2.2. [10] The 7-tuple $(\mathfrak{V}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ is said to be Neutrosophic Normed Linear Space ($\mathfrak{NNLS}$) where $\mathfrak{V}$ is a linear space over a field $\mathfrak{F}$, $*$ is a continuous t -norm, $\diamond$ and $\otimes$ are continuous t -conorm, ζ, η, ϑ are fuzzy sets on $\mathfrak{V} \times (0, \infty)$, ζ denotes the degree of membership and η, ϑ denotes the degree of nonmembership of $(x, \lambda) \in \mathfrak{V} \times (0, \infty)$ satisfying the following conditions for every $x, z \in \mathfrak{V}$ and $s, \lambda > 0$:

- (i) $0 \leq \zeta(x, \lambda) \leq 1, 0 \leq \eta(x, \lambda) \leq 1, 0 \leq \vartheta(x, \lambda) \leq 1$
- (ii) $\zeta(x, \lambda) + \eta(x, \lambda) + \vartheta(x, \lambda) \leq 3$,
- (iii) $\zeta(x, \lambda) > 0$,
- (iv) $\zeta(x, \lambda) = 1$ if and only if $x = 0$,
- (v) $\zeta(\alpha x, \lambda) = \zeta\left(x, \frac{\lambda}{|\alpha|}\right)$ if $\alpha \neq 0$,
- (vi) $\zeta(x, \lambda) * \zeta(z, s) \leq \zeta(x + z, \lambda + s)$,
- (vii) $\zeta(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (viii) $\lim_{\lambda \rightarrow \infty} \zeta(x, \lambda) = 1$ and $\lim_{t \rightarrow 0} \zeta(x, \lambda) = 0$,
- (ix) $\eta(x, \lambda) < 1$,
- (x) $\eta(x, \lambda) = 0$ if and only if $x = 0$,
- (xi) $\eta(\alpha x, \lambda) = \eta\left(x, \frac{\lambda}{|\alpha|}\right)$ if $\alpha \neq 0$,
- (xii) $\eta(x, \lambda) \diamond \eta(z, s) \geq \eta(x + z, s + \lambda)$,
- (xiii) $\eta(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (xiv) $\lim_{\lambda \rightarrow \infty} \eta(x, \lambda) = 0$ and $\lim_{t \rightarrow 0} \eta(x, \lambda) = 1$.
- (xv) $\vartheta(x, \lambda) < 1$,

(xvi) $\vartheta(x, \lambda) = 0$ if and only if $x = 0$,

(xvii) $\vartheta(\alpha x, \lambda) = \vartheta\left(x, \frac{\lambda}{|\alpha|}\right)$ if $\alpha \neq 0$,

(xviii) $\vartheta(x, \lambda) \otimes \omega(z, s) \geq \vartheta(x + z, s + \lambda)$,

(xix) $\vartheta(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,

(xx) $\lim_{t \rightarrow \infty} \vartheta(x, \lambda) = 0$ and $\lim_{\lambda \rightarrow 0} \vartheta(x, \lambda) = 1$.

In this case (ζ, η, ϑ) is called Neutrosophic fuzzy linear norm.

Example 2.1. [10] Let $(\mathfrak{Y}, ||\cdot||)$ be a normed linear space, and let $p * q = pq$, $p \diamond q = \min\{p + q, 1\}$ and $p \otimes q = \max\{p + q, 1\}$ for all $p, q \in [0, 1]$. For all $x \in \mathfrak{Y}$ and every $\lambda > 0$, consider $\zeta(x, \lambda) := \frac{\lambda}{\lambda + ||x||}$, $\eta(x, \lambda) := \frac{||x||}{\lambda + ||x||}$ and $\vartheta(x, \lambda) := \frac{||x||}{||\lambda + x||}$. Then $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ is an $\mathfrak{NNLS}$.

Definition 2.3. [10] Let $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$. A sequence $x = (x_{\mathfrak{k}})$ in $\mathfrak{Y}$ is convergent to $\mathfrak{L} \in \mathfrak{Y}$ with respect to the Neutrosophic Normed linear norm (ζ, η, ϑ) if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that $\zeta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) > 1 - \varepsilon$, $\eta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) < \varepsilon$ and $\vartheta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) < \varepsilon$ for all $\mathfrak{k} \geq \mathfrak{k}_0$ where $\mathfrak{k} \in \mathbb{N}$. It is denoted by $(\zeta, \eta, \vartheta) - \lim x = \mathfrak{L}$.

Theorem 2.1. [11] Let $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$. Then, a sequence $x = (x_{\mathfrak{k}})$ in $\mathfrak{Y}$ is convergent to $\mathfrak{L} \in \mathfrak{Y}$ if and only if $\lim_{\mathfrak{k} \rightarrow \infty} \zeta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) = 1$, $\lim_{\mathfrak{k} \rightarrow \infty} \eta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) = 0$ and $\lim_{\mathfrak{k} \rightarrow \infty} \vartheta(x_{\mathfrak{k}} - \mathfrak{L}, \lambda) = 0$.

3 Main results

In this section, we introduce a new summability method named as $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)}$ -summability. We introduce a new concept of statistical convergence called weighted Norlund-Euler statistical convergence with regard to the Neutrosophic fuzzy norm (ζ, η, ϑ) as a result of this definition. We also look into some relationships between these ideas.

Definition 3.1. Let $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$. A sequence $x = (x_{\mathfrak{k}})$ in $\mathfrak{Y}$ is said to be $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})$ -summable to $\mathfrak{L} \in \mathfrak{Y}$ with respect to the Neutrosophic fuzzy norm (ζ, η, ω) (or $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)}$ -summable to $\mathfrak{L} \in \mathfrak{Y}$) if, for every

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$\varepsilon > 0$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &> 1 - \varepsilon, \\ \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &< \varepsilon \text{ and} \\ \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &< \varepsilon, \end{aligned}$$

for all $n \geq n_0$. In this case, we write $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}$.

Definition 3.2. Let $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$. A sequence $x = (x_{\mathfrak{k}})$ in $\mathfrak{Y}$ is said to be weighted Norlund-Euler statistical convergent to $\mathfrak{L} \in \mathfrak{Y}$ with respect to the Neutrosophic fuzzy norm (ζ, η, ϑ) (or $\mathfrak{S}_{\mathfrak{NE}}^{q(\zeta, \eta, \vartheta)}$ -convergent to $\mathfrak{L} \in \mathfrak{Y}$) if, for every $\varepsilon > 0$ and $\lambda > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{R}_n} \left| \left\{ \begin{array}{l} \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \leq 1 - \varepsilon \text{ or} \\ \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \text{ and} \\ \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \end{array} \right\} \right| = 0.$$

In this case, we write $\mathfrak{S}_{\mathfrak{NE}}^{q(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}$.

In the following theorems, we establish the relation between weighted Norlund Euler statistical convergence and $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})$ -summability in Neutrosophic Normed Linear spaces.

Theorem 3.1. Let $(\mathfrak{Y}, \zeta, \eta, \omega, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$, $x = (x_{\mathfrak{k}})$ be a sequence in $\mathfrak{Y}$ and $\frac{\mathfrak{R}_n}{n} \geq 1$ for all $n \in \mathbb{N}$. If $\mathfrak{S}_{\mathfrak{NE}}^{q(\zeta, \eta, \omega)} - \lim x = \mathfrak{L}$,

$$\begin{aligned} \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &\geq 1 - M, \\ \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &\leq M \text{ and} \\ \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) &\leq M, \end{aligned}$$

for all $\mathfrak{k} \in \mathbb{N}$, then $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}$.

Proof. Suppose that $\mathfrak{S}_{\mathfrak{N}\mathfrak{E}}^{q(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}$. Then, for every $\varepsilon > 0$ and $\lambda > 0$, we define the sets $\mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)$ and $\mathfrak{K}_{\mathfrak{N}\mathfrak{N}}^c(\varepsilon)$ such that

$$\mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon) = \left\{ \begin{array}{l} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \leq 1 - \varepsilon \text{ or} \\ \eta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \text{ and} \\ \vartheta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \end{array} \right\}$$

and

$$\mathfrak{K}_{\mathfrak{N}\mathfrak{N}}^c(\varepsilon) = \left\{ \begin{array}{l} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) > 1 - \varepsilon \text{ or} \\ \eta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) < \varepsilon \text{ and} \\ \vartheta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) < \varepsilon \end{array} \right\}.$$

Hence

$$\begin{aligned} & \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\mathfrak{k}=0}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &= \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)}}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &+ \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}^c(\varepsilon)}}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &= \mathfrak{S}_1(\mathfrak{n}) + \mathfrak{S}_2(\mathfrak{n}) \end{aligned}$$

where

$$\mathfrak{S}_1(\mathfrak{n}) = \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)}}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \text{ and} \quad (1)$$

$$\mathfrak{S}_2(\mathfrak{n}) = \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}^c(\varepsilon)}}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right). \quad (2)$$

If $\mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)$, then

$$\mathfrak{S}_1(\mathfrak{n}) = \frac{1}{\mathfrak{N}_{\mathfrak{n}}} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)}}^{\mathfrak{n}} \zeta \left(\mathfrak{p}_{\mathfrak{n}-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \frac{|\mathfrak{K}_{\mathfrak{N}\mathfrak{N}}(\varepsilon)|}{\mathfrak{N}_{\mathfrak{n}}} (1-M). \quad (3)$$

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Since $\mathfrak{S}_{\mathfrak{N}\mathfrak{E}}^{q(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}$.

$$\lim_{n \rightarrow \infty} \mathfrak{S}_1(n) \geq 0. \quad (4)$$

If $\mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}_n}^c(\varepsilon)$, then we have

$$\mathfrak{S}_2(n) = \frac{1}{\mathfrak{N}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{N}_n}^c(\varepsilon)}}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) > \frac{|\mathfrak{K}_{\mathfrak{N}_n}^c(\varepsilon)|}{\mathfrak{N}_n} (1-\varepsilon). \quad (5)$$

which yields that

$$\lim_{n \rightarrow \infty} \mathfrak{S}_2(n) > (1 - \varepsilon). \quad (6)$$

Using equalities (1)-(2) and inequalities (3)-(6), we get

$$\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{N}_n} \sum_{\mathfrak{k}=0}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) = 1. \quad (7)$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{N}_n} \sum_{\mathfrak{k}=0}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) = 0. \quad (8)$$

$$\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{N}_n} \sum_{\mathfrak{k}=0}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) = 0. \quad (9)$$

As a result of equalities (8), (9) and (10), we have

$$(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)} - \lim x = \mathfrak{L}. \quad (10)$$

□

Theorem 3.2. Let $(\mathfrak{Y}, \zeta, \eta, \vartheta, *, \diamond, \otimes)$ be an $\mathfrak{NNLS}$ and $\frac{\mathfrak{N}_n}{n} \geq 1$ for all $n \in \mathbb{N}$. If a sequence $x = (\mathfrak{x}_{\mathfrak{k}})$ in $\mathfrak{Y}$ is $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)}$ -summable to $\mathfrak{L} \in \mathfrak{Y}$, then a sequence $x = (\mathfrak{x}_{\mathfrak{k}})$ is $\mathfrak{S}_{\mathfrak{N}\mathfrak{E}}^{q(\zeta, \eta, \vartheta)}$ -convergent to $\mathfrak{L} \in \mathfrak{Y}$.

Proof. Suppose that $x = (\mathfrak{x}_{\mathfrak{k}})$ is $(\mathfrak{N}, \mathfrak{p}, \mathfrak{q})(\mathfrak{E}, \mathfrak{q})^{(\zeta, \eta, \vartheta)}$ -summable to $\mathfrak{L} \in \mathfrak{Y}$. For every $\varepsilon > 0$ and $\lambda > 0$, define the sets $\mathfrak{K}_{\mathfrak{N}_n}(\varepsilon)$ and $\mathfrak{K}_{\mathfrak{N}_n}^c(\varepsilon)$ such that

$$\mathfrak{K}_{\mathfrak{N}_n}(\varepsilon) = \left\{ \begin{array}{l} \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \leq 1 - \varepsilon \text{ or} \\ \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \text{ and} \\ \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \geq \varepsilon \end{array} \right\}$$

and

$$\mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon) = \left\{ \begin{array}{l} \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) > 1 - \varepsilon \text{ or} \\ \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) < \varepsilon \text{ and} \\ \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) < \varepsilon \end{array} \right\}.$$

Then

$$\begin{aligned} & \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &= \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon)}}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &+ \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon)}}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &\geq \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon)}}^n \zeta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\ &> \frac{1}{\mathfrak{R}_n} |\mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon)| (1 - \varepsilon) \end{aligned} \tag{11}$$

As a consequence of inequality (11), we get $\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{R}_n} |\mathfrak{K}_{\mathfrak{R}_n}^c(\varepsilon)| = 1$. Similarly, for

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every $\varepsilon > 0$ and $\lambda > 0$,

$$\begin{aligned}
& \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&= \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&+ \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}^c(\varepsilon)}}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&\leq \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \eta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&\leq \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \varepsilon = \frac{1}{\mathfrak{R}_n} |\mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)| \varepsilon.
\end{aligned} \tag{12}$$

$$\begin{aligned}
& \frac{1}{\mathfrak{R}_n} \sum_{\mathfrak{k}=0}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&= \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&+ \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}^c(\varepsilon)}}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&\leq \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \vartheta \left(\mathfrak{p}_{n-\mathfrak{k}} \mathfrak{q}_{\mathfrak{k}} \frac{1}{(1+\mathfrak{q})^{\mathfrak{k}}} \sum_{v=0}^{\mathfrak{k}} \binom{\mathfrak{k}}{v} \mathfrak{q}^{\mathfrak{k}-v} (x_v - \mathfrak{L}), \lambda \right) \\
&\leq \frac{1}{\mathfrak{R}_n} \sum_{\substack{\mathfrak{k}=0 \\ \mathfrak{k} \in \mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)}}^n \varepsilon = \frac{1}{\mathfrak{R}_n} |\mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)| \varepsilon.
\end{aligned} \tag{13}$$

By inequality (12) and (13), we have $\lim_{n \rightarrow \infty} \frac{1}{\mathfrak{R}_n} |\mathfrak{R}_{\mathfrak{R}_n}(\varepsilon)| = 0$. This completes the proof. $\square$

4 Conclusion

Smarandache first presented $\mathfrak{NS}$ in 1998 [14]. Maji has created a unique concept for the $\mathfrak{NS}$. The concept of $\mathfrak{NSNS}$ was subsequently investigated, and some of its characteristics were proposed. Kirisi and Simsek used continuous t-norms and continuous t-conorms to study the $\mathfrak{MNS}$. Kirisci and Simsek proposed $\mathfrak{MNS}$ as well as statistical convergence in $\mathfrak{MNS}$. Although several aspects of neutrosophic Norlund convergent sequence spaces have been studied.

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Numerical solution for fuzzy fractional differential equations using fuzzy fractional fourth order Runge-Kutta method based on root mean square and contraharmonic mean

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Abstract

The objective of this research is to determine the approximate solution to Fuzzy Fractional Differential Equations (FFDEs). For Fuzzy Fractional Initial Value Problems (FFIVPs), the methods called Fuzzy Fractional Fourth Order Runge-Kutta method based on Root Mean Square (FFRK4RM) and called Fuzzy Fractional Fourth Order Runge-Kutta method based on Contraharmonic Mean (FFRK4CoM) is developed. In this paper, both linear and nonlinear FFDEs can be solved using triangular and trapezoidal fuzzy numbers. FFRK4RM and FFRK4CoM can be compared. The tables gives the absolute error between the exact and approximate solutions. From the graphical results, the approximate solution approaches the exact result very closely as the step size gets smaller. The outcomes show that the suggested approach is easy to use, accurate, clear, and convenient for solving both linear and nonlinear FFIVP.

Keywords: FFRK4RM, FFRK4CoM, FFDEs, Triangular fuzzy number, Trapezoidal fuzzy number.

2020 AMS subject classifications: 34A07, 26A33, 65L06, 34A12. ¹

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1 Introduction

Fractional calculus is expanded upon in ordinary calculus. This involves computing a function's derivative in any order. The memory and heredity characteristics of numerous substances and processes in porous media, electrical circuits, control, biology, electromagnetic processes, biomechanics, and electro chemistry have been documented using fractional differential operators Agarwal [2010], Baleanu [2012], Tarasov [2010]. Over the past few decades, fractional calculus and its applications have gained popularity, mostly because it has proven to be a helpful tool for modeling a number of complex processes in a wide range of seemingly unrelated fields of science and engineering Podlubny [1999], Kenneth [1993]. Measurement uncertainty is represented by a fuzzy number. Since its invention by Lotfi Zadeh in 1965, fuzzy sets have found utility in a variety of contexts Diamond [1994], Geotschel [1986], O.Kaleva [1987].

The solution for fractional differential equations in uncertain settings was first introduced by Agarwal [2010]. In applied sciences and the engineering, Fuzzy Fractional Differential Equations (FFDEs) is a crucial topic. Consider the general FFDE with the fuzzy initial conditions of the form:

$${}^c D^\delta \tilde{p}(\zeta, \mu) + L\tilde{p}(\zeta, \mu) + N\tilde{p}(\zeta, \mu) = g(\zeta, \mu), \quad n-1 < \delta < n, \quad n \in \mathbb{N} \quad (1)$$

with the fuzzy initial condition $D^k \tilde{p}(0, \mu) = \tilde{h}^k(\zeta)$, $k = 0, 1, 2, \dots, n-1$, where $D^\delta \tilde{p}(\zeta, \mu)$ represents the Caputo fractional derivative of $\tilde{p}(\zeta, \mu)$ and $\tilde{p}(\zeta, \mu)$ is the fuzzy function. The general linear and nonlinear differential operators are denoted by L and N . The source term is $g(\zeta, \mu)$.

Numerous applications of FFDEs are made in the field of biology and other engineering disciplines O.Kaleva [1987]. There have been several implementations in the FFDE sector to find exact and approximative solutions Chakraverty [2016]. Lydia [2020] proposed the numerical solution of fractional RC and RL electrical circuits by fractional Runge-Kutta method. Nirmala [2011] studied numerical solution of fuzzy differential equation by RK4 approach with higher order derivative approximations. Sharmila [2013a] introduced the numerical solution of N^{th} order fuzzy initial value problems by RK4 approach based on Centroidal mean. Sharmila [2013b] proposed the numerical solution of fuzzy initial value problems by RK4 approach based on Contraharmonic mean. The aforementioned articles assisted in the development of the FFRK4RM and FFRK4CoM for FFDEs.

The format of the paper is as follows: Some fundamental tools utilising the fuzzy number, Mittag-Leffler function, and Caputo fractional derivative are provided in Section 2. In Section 3, the Fuzzy Fractional Initial Value Problem (FFIVP) is defined. In Section 4, the Runge-Kutta method for FFIVP is illustrated. It is also clearly detailed how to solve the FFIVP using the fourth order Runge-Kutta approach based on root mean square and contraharmonic mean. In

Section 5, numerical illustrations demonstrate the manner in which FFRK4RM and FFRK4CoM work to solve both linear and nonlinear FFDEs. The conclusion offered in Section 6.

2 Basic Tools

Definition 2.1. *The fractional derivative of $f(\zeta)$ in Caputo sense, is defined as follows:*

$${}^c D_t^\delta f(\zeta) = \frac{1}{\Gamma(m-\delta)} \int_0^\zeta (\zeta-\Phi)^{m-\delta-1} f^{(m)}(\Phi) d\Phi, \quad m-1 < \delta \leq m, \quad \zeta > 0, \quad m \in \mathbb{N}$$

Definition 2.2. *The Mittag-Leffler function is defined as follows,*

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\delta + 1)}, \quad \delta \in \mathbb{C}, \quad \text{Re}(\delta) > 0$$

A further generalization of the above equation is defined in the form

$$E_{\delta,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\delta + \beta)}, \quad \delta, \beta \in \mathbb{C}, \quad \text{Re}(\delta) > 0, \quad \text{Re}(\beta) > 0$$

Definition 2.3. *(Triangular fuzzy number)*

It's a three-pointed fuzzy number represented by $N = (\theta_1, \theta_2, \theta_3)$. N 's membership function is as follows:

$$\mu_N(\chi) = \begin{cases} 0, & \chi < \theta_1 \\ \frac{\chi-\theta_1}{\theta_2-\theta_1}, & \theta_1 \leq \chi \leq \theta_2 \\ \frac{\theta_3-\chi}{\theta_3-\theta_2}, & \theta_2 \leq \chi \leq \theta_3 \\ 0, & \chi > \theta_3 \end{cases}$$

Definition 2.4. *(Trapezoidal fuzzy number)*

It's a four-pointed fuzzy number represented by $M = (\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$. M 's membership function is as follows:

$$\mu_M(\chi) = \begin{cases} 0, & \chi < \Upsilon_1 \\ \frac{\chi-\Upsilon_1}{\Upsilon_2-\Upsilon_1}, & \Upsilon_1 \leq \chi \leq \Upsilon_2 \\ 1, & \Upsilon_2 \leq \chi \leq \Upsilon_3 \\ \frac{\Upsilon_4-\chi}{\Upsilon_4-\Upsilon_3}, & \Upsilon_3 \leq \chi \leq \Upsilon_4 \\ 0, & \chi > \Upsilon_4 \end{cases}$$

3 Fuzzy Fractional Initial Value Problem

Consider the following fuzzy differential equation of fractional order

$$\begin{cases} {}^c D^\delta \tilde{p}(\zeta, \mu) = f(\zeta, \tilde{p}) \\ \tilde{p}(\zeta_0) = \tilde{y}_0 \in \mathbb{R}_F, \quad \delta \in (0, 1] \end{cases} \quad (2)$$

Applying Caputo fractional derivative of order $(1 - \delta)$ to equation (2) as in Lydia [2020],

$$D^1 \tilde{p}(\zeta, \mu) = {}^c D^{1-\delta} f(\zeta, \tilde{p}) \quad (3)$$

where $f(\zeta, \tilde{p})$ be a fuzzy continuous function and δ times differentiable in the independent variable ζ over the range of $[\zeta_0, \zeta_m]$. Let the interval $[\zeta_0, \zeta_m]$ be subdivided into N sub-intervals $[\zeta_j, \zeta_{j+1}]$ of step size $h = (\zeta_m - \zeta_0)/N$ for $j = 0, 1, \dots, N$.

Fuzzy function $\tilde{p}$ is denoted by $\tilde{p} = [\underline{p}, \overline{p}]$. It means that the μ -level set of $\tilde{p}(\zeta)$ for $\zeta \in [\zeta_0, T]$ is $[\tilde{p}(\zeta)]_\mu = [\underline{p}(\zeta, \mu), \overline{p}(\zeta, \mu)]$, $[\tilde{p}(\zeta_0)]_\mu = [\underline{p}(\zeta_0, \mu), \overline{p}(\zeta_0, \mu)]$, $\mu \in (0, 1]$. We write

$${}^c D^{1-\delta} f(\zeta, \tilde{p}) = [{}^c D^{1-\delta} \underline{f}(\zeta, p), {}^c D^{1-\delta} \overline{f}(\zeta, p)]$$

$${}^c D^{1-\delta} \underline{f}(\zeta, p) = F[\zeta, \underline{p}, \overline{p}], \quad {}^c D^{1-\delta} \overline{f}(\zeta, p) = G[\zeta, \underline{p}, \overline{p}]$$

because of $\tilde{p}'(\zeta, \mu) = {}^c D^{1-\delta} f(\zeta, \tilde{p})$.

$${}^c D^{1-\delta} \underline{f}(\zeta, p(\zeta, \mu)) = F[\zeta, \underline{p}(\zeta, \mu), \overline{p}(\zeta, \mu)]$$

$${}^c D^{1-\delta} \overline{f}(\zeta, p(\zeta, \mu)) = G[\zeta, \underline{p}(\zeta, \mu), \overline{p}(\zeta, \mu)]$$

The extension principle of Zadeh leads to the following definition of $f(\zeta, p(\zeta))$ when $p(\zeta)$ is a fuzzy number

$$f(\zeta, p(\zeta))(s) = \sup\{p(\zeta)(\tau)/s = f(\zeta, \tau)\}, s \in \mathbb{R}$$

It follows that

$$[{}^c D^{1-\delta} f(\zeta, p(\zeta))]_r = [{}^c D^{1-\delta} \underline{f}(\zeta, p(\zeta, \mu)), {}^c D^{1-\delta} \overline{f}(\zeta, p(\zeta, \mu))], \quad \mu \in (0, 1]$$

where

$${}^c D^{1-\delta} \underline{f}(\zeta, p(\zeta, \mu)) = \min\{{}^c D^{1-\delta} f(\zeta, p) | p \in [\tilde{p}(\zeta)]_\mu\}$$

$${}^c D^{1-\delta} \overline{f}(\zeta, p(\zeta, \mu)) = \max\{{}^c D^{1-\delta} f(\zeta, p) | p \in [\tilde{p}(\zeta)]_\mu\}$$

4 Runge-Kutta Method for Fuzzy Fractional Initial Value Problems

Consider the following FFDE

$$\begin{cases} {}^c D^\delta \tilde{p}(\zeta, \mu) = f(\zeta, \tilde{p}) \\ \tilde{p}(\zeta_0) = \tilde{p}_0 \in \mathbb{R}_F, \quad \delta \in (0, 1] \end{cases} \quad (4)$$

The difference between the value of y at time ζ_{m+1} and time ζ_m is expressed as the basis for all Runge-Kutta methods.

$$\tilde{p}_{m+1} - \tilde{p}_m = \sum_{i=1}^q w_i \tilde{k}_i \quad (5)$$

where for $i = 1, 2, \dots, m$. w_i 's are constants and

$$\tilde{k}_i = h D^{1-\delta} f \left(\zeta_m + c_i h, \tilde{p}_m + \sum_{j=1}^{i-1} a_{ij} \tilde{k}_j \right) \quad (6)$$

where $\tilde{k}_i = [\underline{k}_i, \bar{k}_i]$. Equations (4) is to be exact for powers of h through h^q , because it is match with Taylor series of order q .

4.1 The fourth order Runge- Kutta method based on Root Mean Square for Solving FFIVP

Jansirani [2020] proposed the composite RK4 method for based on variety of means by using intuitionistic fuzzy differential equations. Root mean square formula is as follows:

$$\begin{aligned} \underline{p}_{m+1} &= \underline{p}_m + \frac{1}{3}h \left[\sqrt{\frac{\underline{k}_1^2 + \underline{k}_2^2}{2}} + \sqrt{\frac{\underline{k}_2^2 + \underline{k}_3^2}{2}} + \sqrt{\frac{\underline{k}_3^2 + \underline{k}_4^2}{2}} \right] \\ \bar{p}_{m+1} &= \bar{p}_m + \frac{1}{3}h \left[\sqrt{\frac{\bar{k}_1^2 + \bar{k}_2^2}{2}} + \sqrt{\frac{\bar{k}_2^2 + \bar{k}_3^2}{2}} + \sqrt{\frac{\bar{k}_3^2 + \bar{k}_4^2}{2}} \right] \end{aligned}$$

where

$$\begin{aligned} \underline{k}_1 &= \underline{f}(\zeta_m, \underline{p}_m, \bar{p}_m) \\ \bar{k}_1 &= \bar{f}(\zeta_m, \underline{p}_m, \bar{p}_m) \\ \underline{k}_2 &= \underline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1 \underline{k}_1, \bar{p}_m + a_1 \bar{k}_1) \\ \bar{k}_2 &= \bar{f}(\zeta_m + a_1 h, \underline{p}_m + a_1 \underline{k}_1, \bar{p}_m + a_1 \bar{k}_1) \end{aligned}$$

$$\begin{aligned}\underline{k}_3 &= \underline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \bar{p}_m + a_2\bar{k}_1 + a_3\bar{k}_2) \\ \bar{k}_3 &= \bar{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \bar{p}_m + a_2\bar{k}_1 + a_3\bar{k}_2) \\ \underline{k}_4 &= \underline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \bar{p}_m + a_4\bar{k}_1 + a_5\bar{k}_2 + a_6\bar{k}_3) \\ \bar{k}_4 &= \bar{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \bar{p}_m + a_4\bar{k}_1 + a_5\bar{k}_2 + a_6\bar{k}_3)\end{aligned}$$

The parameters are: $a_1 = \frac{1}{2}, a_2 = \frac{1}{6}, a_3 = \frac{7}{16}, a_4 = \frac{1}{8}, a_5 = \frac{-17}{56}, a_6 = \frac{33}{28}$
The parametric form of (2) is,

$$\begin{cases} {}^c D^\delta \underline{p}(\zeta, \mu) = \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ {}^c D^\delta \bar{p}(\zeta, \mu) = \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ \underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \quad \bar{p}(\zeta_0) = \bar{p}_0(0, \mu) \\ \delta \in (0, 1), \quad \mu \in [0, 1] \end{cases} \quad (7)$$

Applying Caputo fractional derivative of order $(1 - \delta)$ to equation (12) as in Lydia [2020],

$$\begin{cases} D^1 \underline{p}(\zeta, \mu) = {}^c D^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ D^1 \bar{p}(\zeta, \mu) = {}^c D^{1-\delta} \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \end{cases} \quad (8)$$

with the same initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \quad \bar{p}(\zeta_0) = \bar{p}_0(0, \mu)$.

Let the exact solution $[\tilde{P}(\zeta)]_\mu = [\underline{P}(\zeta, \mu), \bar{P}(\zeta, \mu)]$ is approximated by some $[\tilde{p}(\zeta)]_\mu = [\underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)]$ From (5), (6),

$$\begin{aligned}\underline{p}(\zeta_{m+1}, \mu) - \underline{p}(\zeta_m, \mu) &= \sum_{i=1}^4 w_i \underline{k}_i(\zeta_m, p(\zeta_m, \mu)) \\ \bar{p}(\zeta_{m+1}, \mu) - \bar{p}(\zeta_m, \mu) &= \sum_{i=1}^4 w_i \bar{k}_i(\zeta_m, p(\zeta_m, \mu))\end{aligned} \quad (9)$$

where for $i = 1, 2, \dots, m$. $\tilde{k}_i(\zeta, p(\zeta, \mu)) = [\underline{k}_i(t, p(\zeta, \mu)), \bar{k}_i(t, p(\zeta, \mu))]$ and w_i 's are constants.

$$\begin{aligned}\underline{k}_i(\zeta_m, p(\zeta_m, \mu)) &= h D^{1-\delta} \underline{f} \left(\zeta_m + c_i h, \underline{p}(\zeta_m, p(\zeta_m, \mu)) + \sum_{j=1}^{i-1} a_{ij} \underline{k}_j(\zeta_m, p(\zeta_m, \mu)) \right) \\ \bar{k}_i(\zeta_m, p(\zeta_m, \mu)) &= h D^{1-\delta} \bar{f} \left(\zeta_m + c_i h, \bar{p}(\zeta_m, p(\zeta_m, \mu)) + \sum_{j=1}^{i-1} a_{ij} \bar{k}_j(\zeta_m, p(\zeta_m, \mu)) \right)\end{aligned} \quad (10)$$

and

$$\begin{aligned}\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} \underline{f}(\zeta_m, p) / p \in (\underline{p}(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)) \right\} \\ \bar{k}_1(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} \bar{f}(\zeta_m, p) / p \in (\underline{p}(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)) \right\}\end{aligned}$$

$$\begin{aligned}
 \underline{k}_2(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m, \mu)), \bar{s}_1(\zeta_m, p(\zeta_m, \mu))) \right\} \\
 \bar{k}_2(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m, \mu)), \bar{s}_1(\zeta_m, p(\zeta_m, \mu))) \right\} \\
 \underline{k}_3(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m, \mu)), \bar{s}_2(\zeta_m, p(\zeta_m, \mu))) \right\} \\
 \bar{k}_3(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m, \mu)), \bar{s}_2(\zeta_m, p(\zeta_m, \mu))) \right\} \\
 \underline{k}_4(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m, \mu)), \bar{s}_3(\zeta_m, p(\zeta_m, \mu))) \right\} \\
 \bar{k}_4(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m, \mu)), \bar{s}_3(\zeta_m, p(\zeta_m, \mu))) \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 \underline{s}_1(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{2}\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) \\
 \bar{s}_1(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{2}\bar{k}_1(\zeta_m, p(\zeta_m, \mu)) \\
 \underline{s}_2(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{16}\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) + \frac{7}{16}\underline{k}_2(\zeta_m, p(\zeta_m, \mu)) \\
 \bar{s}_2(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{16}\bar{k}_1(\zeta_m, p(\zeta_m, \mu)) + \frac{7}{16}\bar{k}_2(\zeta_m, p(\zeta_m, \mu)) \\
 \underline{s}_3(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{8}\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) - \frac{17}{56}\underline{k}_2(\zeta_m, p(\zeta_m, \mu)) + \frac{33}{28}\underline{k}_3(\zeta_m, p(\zeta_m, \mu)) \\
 \bar{s}_3(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{8}\bar{k}_1(\zeta_m, p(\zeta_m, \mu)) - \frac{17}{56}\bar{k}_2(\zeta_m, p(\zeta_m, \mu)) + \frac{33}{28}\bar{k}_3(\zeta_m, p(\zeta_m, \mu))
 \end{aligned}$$

$$\begin{aligned}
 \underline{F}[\zeta_m, p(\zeta_m, \mu)] &= \left[\sqrt{\frac{\underline{k}_1^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_2^2(\zeta_m, p(\zeta_m, \mu))}{2}} \right. \\
 &\quad + \sqrt{\frac{\underline{k}_2^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_3^2(\zeta_m, p(\zeta_m, \mu))}{2}} \\
 &\quad \left. + \sqrt{\frac{\underline{k}_3^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_4^2(\zeta_m, p(\zeta_m, \mu))}{2}} \right] \\
 \bar{F}[\zeta_m, p(\zeta_m, \mu)] &= \left[\sqrt{\frac{\bar{k}_1^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_2^2(\zeta_m, p(\zeta_m, \mu))}{2}} \right. \\
 &\quad + \sqrt{\frac{\bar{k}_2^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_3^2(\zeta_m, p(\zeta_m, \mu))}{2}}
 \end{aligned}$$

$$+ \sqrt{\frac{\bar{k}_3^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_4^2(\zeta_m, p(\zeta_m, \mu))}{2}} \Big]$$

The exact and approximate solutions at $\zeta_m, 0 \leq m \leq N$ are denoted by $[\tilde{P}(\zeta_m)]_\mu = [P(\zeta_m, \mu), \bar{P}(\zeta_m, \mu)]$ and $[\tilde{p}(\zeta_m)]_\mu = [p(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)]$ respectively. The solution is calculated by grid points at $a = \zeta_0 < \zeta_1 < \dots < \zeta_m = b$ and $h = \frac{(b-a)}{N} = \zeta_{i+1} - \zeta_i$.

Then, using the initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \bar{p}(\zeta_0) = \bar{p}_0(0, \mu)$ and the fourth order Runge - Kutta formula based on root mean square for (2) is as follows:

The exact solutions at ζ_{m+1} is given by

$$\underline{P}(\zeta_{m+1}, \mu) \approx \underline{P}(\zeta_m, \mu) + \frac{1}{3} \underline{F}[\zeta_m, P(\zeta_m, \mu)]$$

$$\bar{P}(\zeta_{m+1}, \mu) \approx \bar{P}(\zeta_m, \mu) + \frac{1}{3} \bar{F}[\zeta_m, P(\zeta_m, \mu)]$$

The approximate solutions at t_{n+1} is given by

$$\underline{p}(\zeta_{m+1}, \mu) = \underline{p}(\zeta_m, \mu) + \frac{1}{3} \underline{F}[\zeta_m, p(\zeta_m, \mu)]$$

$$\bar{p}(\zeta_{m+1}, \mu) = \bar{p}(\zeta_m, \mu) + \frac{1}{3} \bar{F}[\zeta_m, p(\zeta_m, \mu)]$$

To implement this method, numerical approximation is needed to the fractional derivative operator $D^{1-\delta}$ as in Lydia [2020] fuzzify it, then it becomes

$${}^c D^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^m c_{jm} \underline{f}(\zeta_j, \underline{p}(\zeta_j, \mu), \bar{p}(\zeta_j, \mu))$$

$${}^c D^{1-\delta} \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^m c_{jm} \bar{f}(\zeta_j, \underline{p}(\zeta_j, \mu), \bar{p}(\zeta_j, \mu))$$

where the coefficients $c_{jm}, j = 0, 1, 2, \dots, m$ are obtained using the quadrature rule Lydia [2020]

$$c_{jm} = \begin{cases} \delta m^{\delta-1} - m^\delta + (m-1)^\delta, & j = 0 \\ (m-j+1)^\delta - 2(m-j)^\delta + (m-j-1)^\delta, & j = 1, 2, \dots, m-1 \\ 1, & j = m \end{cases} \quad (11)$$

Clearly $\underline{p}(\zeta, \mu)$ and $\bar{p}(\zeta, \mu)$ converges to $\underline{P}(\zeta, \mu)$ and $\bar{P}(\zeta, \mu)$ whenever $h \rightarrow 0$.

4.2 The fourth order Runge- Kutta method based on Contra-harmonic Mean for FFIVP

Sharmila [2013a] proposed the fourth order RK method based on Contraharmonic Mean for fuzzy initial value problem as follows:

$$\begin{aligned}\underline{p}_{m+1} &= \underline{p}_m + \frac{1}{3}h \left[\frac{\underline{k}_1^2 + \underline{k}_2^2}{\underline{k}_1 + \underline{k}_2} + \frac{\underline{k}_2^2 + \underline{k}_3^2}{\underline{k}_2 + \underline{k}_3} + \frac{\underline{k}_3^2 + \underline{k}_4^2}{\underline{k}_3 + \underline{k}_4} \right] \\ \bar{p}_{m+1} &= \bar{p}_m + \frac{1}{3}h \left[\frac{\bar{k}_1^2 + \bar{k}_2^2}{\bar{k}_1 + \bar{k}_2} + \frac{\bar{k}_2^2 + \bar{k}_3^2}{\bar{k}_2 + \bar{k}_3} + \frac{\bar{k}_3^2 + \bar{k}_4^2}{\bar{k}_3 + \bar{k}_4} \right]\end{aligned}$$

where

$$\begin{aligned}\underline{k}_1 &= \underline{f}(\zeta_m, \underline{p}_m, \bar{p}_m) \\ \bar{k}_1 &= \bar{f}(\zeta_m, \underline{p}_m, \bar{p}_m) \\ \underline{k}_2 &= \underline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1 \underline{k}_1, \bar{p}_m + a_1 \bar{k}_1) \\ \bar{k}_2 &= \bar{f}(\zeta_m + a_1 h, \underline{p}_m + a_1 \underline{k}_1, \bar{p}_m + a_1 \bar{k}_1) \\ \underline{k}_3 &= \underline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2 \underline{k}_1 + a_3 \underline{k}_2, \bar{p}_m + a_2 \bar{k}_1 + a_3 \bar{k}_2) \\ \bar{k}_3 &= \bar{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2 \underline{k}_1 + a_3 \underline{k}_2, \bar{p}_m + a_2 \bar{k}_1 + a_3 \bar{k}_2) \\ \underline{k}_4 &= \underline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4 \underline{k}_1 + a_5 \underline{k}_2 + a_6 \underline{k}_3, \bar{p}_m + a_4 \bar{k}_1 + a_5 \bar{k}_2 + a_6 \bar{k}_3) \\ \bar{k}_4 &= \bar{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4 \underline{k}_1 + a_5 \underline{k}_2 + a_6 \underline{k}_3, \bar{p}_m + a_4 \bar{k}_1 + a_5 \bar{k}_2 + a_6 \bar{k}_3)\end{aligned}$$

The parameters are: $a_1 = \frac{1}{2}, a_2 = \frac{1}{8}, a_3 = \frac{3}{8}, a_4 = \frac{1}{4}, a_5 = \frac{-3}{4}, a_6 = \frac{3}{2}$
The parametric form of (2) is,

$$\begin{cases} {}^c D^\delta \underline{p}(\zeta, \mu) = \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ {}^c D^\delta \bar{p}(\zeta, \mu) = \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ \underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \quad \bar{p}(\zeta_0) = \bar{p}_0(0, \mu) \\ \delta \in (0, 1), \quad \mu \in [0, 1] \end{cases} \quad (12)$$

Applying Caputo fractional derivative of order $(1 - \delta)$ to equation (12) as in Lydia [2020],

$$\begin{cases} D^1 \underline{p}(\zeta, \mu) = {}^c D^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)), \\ D^1 \bar{p}(\zeta, \mu) = {}^c D^{1-\delta} \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \end{cases} \quad (13)$$

with the same initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \quad \bar{p}(\zeta_0) = \bar{p}_0(0, \mu)$

Let the exact solution $[\tilde{p}(\zeta)]_r = [\underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)]$ is approximated by some $[\tilde{p}(\zeta)]_r =$

$[p(\zeta, \mu), \bar{p}(\zeta, \mu)]$. From (5), (6),

$$\begin{aligned} \underline{p}(\zeta_{m+1}, \mu) - \underline{p}(\zeta_m, \mu) &= \sum_{i=1}^4 w_i \underline{k}_i(\zeta_m, p(\zeta_m, \mu)) \\ \bar{p}(\zeta_{m+1}, \mu) - \bar{p}(\zeta_m, \mu) &= \sum_{i=1}^4 w_i \bar{k}_i(\zeta_m, p(\zeta_m, \mu)) \end{aligned} \quad (14)$$

where for $i = 1, 2, \dots, m$. $\tilde{k}_i(\zeta, p(\zeta, \mu)) = [\underline{k}_i(\zeta, p(\zeta, \mu)), \bar{k}_i(\zeta, p(\zeta, \mu))]$ and w_i 's are constants.

$$\begin{aligned} \underline{k}_i(\zeta_m, p(\zeta_m, \mu)) &= hD^{1-\delta} f \left(\zeta_m + c_i h, \underline{p}(\zeta_m, p(\zeta_m, \mu)) + \sum_{j=1}^{i-1} a_{ij} \underline{k}_j(\zeta_m, p(\zeta_m, \mu)) \right) \\ \bar{k}_i(\zeta_m, p(\zeta_m, \mu)) &= hD^{1-\delta} f \left(\zeta_m + c_i h, \bar{p}(\zeta_m, p(\zeta_m, \mu)) + \sum_{j=1}^{i-1} a_{ij} \bar{k}_j(\zeta_m, p(\zeta_m, \mu)) \right) \end{aligned} \quad (15)$$

and

$$\begin{aligned} \underline{k}_1(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m, p)/p \in (\underline{p}(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)) \right\} \\ \bar{k}_1(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m, p)/p \in (\underline{p}(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)) \right\} \\ \underline{k}_2(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m, \mu)), \bar{s}_1(\zeta_m, p(\zeta_m, \mu))) \right\} \\ \bar{k}_2(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m, \mu)), \bar{s}_1(\zeta_m, p(\zeta_m, \mu))) \right\} \\ \underline{k}_3(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m, \mu)), \bar{s}_2(\zeta_m, p(\zeta_m, \mu))) \right\} \\ \bar{k}_3(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m, \mu)), \bar{s}_2(\zeta_m, p(\zeta_m, \mu))) \right\} \\ \underline{k}_4(\zeta_m, p(\zeta_m, \mu)) &= \min \left\{ h {}^c D^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m, \mu)), \bar{s}_3(\zeta_m, p(\zeta_m, \mu))) \right\} \\ \bar{k}_4(\zeta_m, p(\zeta_m, \mu)) &= \max \left\{ h {}^c D^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m, \mu)), \bar{s}_3(\zeta_m, p(\zeta_m, \mu))) \right\} \end{aligned}$$

where

$$\begin{aligned} \underline{s}_1(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{2} \underline{k}_1(\zeta_m, p(\zeta_m, \mu)) \\ \bar{s}_1(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{2} \bar{k}_1(\zeta_m, p(\zeta_m, \mu)) \\ \underline{s}_2(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{8} \underline{k}_1(\zeta_m, p(\zeta_m, \mu)) + \frac{3}{8} \underline{k}_2(\zeta_m, p(\zeta_m, \mu)) \\ \bar{s}_2(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{8} \bar{k}_1(\zeta_m, p(\zeta_m, \mu)) + \frac{3}{8} \bar{k}_2(\zeta_m, p(\zeta_m, \mu)) \end{aligned}$$

Numerical solution for FFDEs using FFRK4RM and FFRK4CoM

$$\begin{aligned}
 \underline{s}_3(\zeta_m, p(\zeta_m, \mu)) &= \underline{p}(\zeta_m, \mu) + \frac{1}{4}\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) - \frac{3}{4}\underline{k}_2(\zeta_m, p(\zeta_m, \mu)) + \frac{3}{2}\underline{k}_3(\zeta_m, p(\zeta_m, \mu)) \\
 \bar{s}_3(\zeta_m, p(\zeta_m, \mu)) &= \bar{p}(\zeta_m, \mu) + \frac{1}{4}\bar{k}_1(\zeta_m, p(\zeta_m, \mu)) - \frac{3}{4}\bar{k}_2(\zeta_m, p(\zeta_m, \mu)) + \frac{3}{2}\bar{k}_3(\zeta_m, p(\zeta_m, \mu)) \\
 \underline{F}[\zeta_m, p(\zeta_m, \mu)] &= \left[\frac{\underline{k}_1^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_2^2(\zeta_m, p(\zeta_m, \mu))}{\underline{k}_1(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_2(\zeta_m, p(\zeta_m, \mu))} \right. \\
 &\quad + \frac{\underline{k}_2^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_3^2(\zeta_m, p(\zeta_m, \mu))}{\underline{k}_2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_3(\zeta_m, p(\zeta_m, \mu))} \\
 &\quad \left. + \frac{\underline{k}_3^2(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_4^2(\zeta_m, p(\zeta_m, \mu))}{\underline{k}_3(\zeta_m, p(\zeta_m, \mu)) + \underline{k}_4(\zeta_m, p(\zeta_m, \mu))} \right] \\
 \bar{F}[\zeta_m, p(\zeta_m, \mu)] &= \left[\frac{\bar{k}_1^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_2^2(\zeta_m, p(\zeta_m, \mu))}{\bar{k}_1(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_2(\zeta_m, p(\zeta_m, \mu))} \right. \\
 &\quad + \frac{\bar{k}_2^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_3^2(\zeta_m, p(\zeta_m, \mu))}{\bar{k}_2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_3(\zeta_m, p(\zeta_m, \mu))} \\
 &\quad \left. + \frac{\bar{k}_3^2(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_4^2(\zeta_m, p(\zeta_m, \mu))}{\bar{k}_3(\zeta_m, p(\zeta_m, \mu)) + \bar{k}_4(\zeta_m, p(\zeta_m, \mu))} \right]
 \end{aligned}$$

The exact and approximate solutions at $\zeta_m, 0 \leq m \leq N$ are denoted by $[\tilde{P}(\zeta_m)]_\mu = [\underline{P}(\zeta_m, \mu), \bar{P}(\zeta_m, \mu)]$ and $[\tilde{p}(\zeta_m)]_\mu = [\underline{p}(\zeta_m, \mu), \bar{p}(\zeta_m, \mu)]$ respectively. The solution is calculated by grid points at $a = \zeta_0 < \zeta_1 < \dots < \zeta_m = b$ and $h = \frac{(b-a)}{N} = \zeta_{i+1} - \zeta_i$.

Then, using the initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0, \mu), \bar{p}(\zeta_0) = \bar{p}_0(0, \mu)$ and the fourth order Runge - Kutta formula based on Contraharmonic Mean for (2) is as follows: The exact solutions at ζ_{m+1} is given by

$$\begin{aligned}
 \underline{P}(\zeta_{m+1}) &\approx \underline{P}(\zeta_m) + \frac{1}{3}\underline{F}[\zeta_m, P(\zeta_m, \mu)] \\
 \bar{P}(\zeta_{m+1}) &\approx \bar{P}(\zeta_m) + \frac{1}{3}\bar{F}[\zeta_m, P(\zeta_m, \mu)]
 \end{aligned}$$

The approximate solutions at ζ_{m+1} is given by

$$\begin{aligned}
 \underline{p}(\zeta_{m+1}) &= \underline{p}(\zeta_m) + \frac{1}{3}\underline{F}[\zeta_m, p(\zeta_m, \mu)] \\
 \bar{p}(\zeta_{m+1}) &= \bar{p}(\zeta_m) + \frac{1}{3}\bar{F}[\zeta_m, p(\zeta_m, \mu)]
 \end{aligned}$$

To implement this method, numerical approximation is needed to the fractional derivative operator $D^{1-\delta}$ as in Lydia [2020] fuzzify it, then it becomes

$${}^c D^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^m c_{jm} \underline{f}(\zeta_j, \underline{p}(\zeta_j, \mu), \bar{p}(\zeta_j, \mu))$$

$${}^c D^{1-\delta} \bar{f}(\zeta, \underline{p}(\zeta, \mu), \bar{p}(\zeta, \mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^m c_{jm} \bar{f}(\zeta_j, \underline{p}(\zeta_j, \mu), \bar{p}(\zeta_j, \mu))$$

where the coefficients c_{jm} , $j = 0, 1, 2, \dots, m$ are obtained using the quadrature rule

$$c_{jm} = \begin{cases} \delta m^{\delta-1} - m^\delta + (m-1)^\delta, & j = 0 \\ (m-j+1)^\delta - 2(m-j)^\delta + (m-j-1)^\delta, & j = 1, 2, \dots, m-1 \\ 1, & j = m \end{cases}$$

Clearly $\underline{p}(\zeta, \mu)$ and $\bar{p}(\zeta, \mu)$ converges to $\underline{P}(\zeta, \mu)$ and $\bar{P}(\zeta, \mu)$ whenever $h \rightarrow 0$.

5 Numerical Examples

In this section, examples for linear and nonlinear FFDE is discussed using the methodology of FFRK4RM and FFRK4CoM.

Example 5.1. Consider the linear triangular FIVP for fractional order

$$D^\delta \tilde{p}(\zeta, \mu) = \tilde{p}(\zeta, \mu) + 1, \quad 0 < \delta \leq 1, \quad \mu \in [0, 1] \quad (16)$$

subject to the initial condition, $\tilde{p}(0, \mu) = [0.2\mu - 0.2, 0.2 - 0.2\mu]$

The Exact solution of (16) is

$$\underline{p}(\zeta, \mu) = (0.2\mu - 0.2)E_\delta(\zeta^\delta) + \zeta^\delta E_{\delta, \delta+1}(\zeta^\delta)$$

$$\bar{p}(\zeta, \mu) = (0.2 - 0.2\mu)E_\delta(\zeta^\delta) + \zeta^\delta E_{\delta, \delta+1}(\zeta^\delta)$$

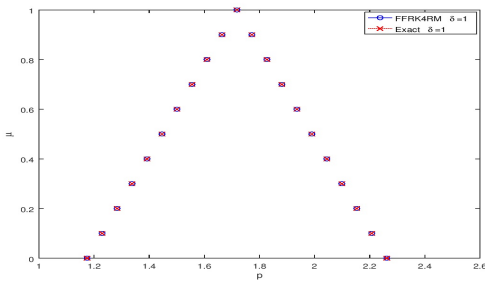


Figure 1: Exact and FFRK4RM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

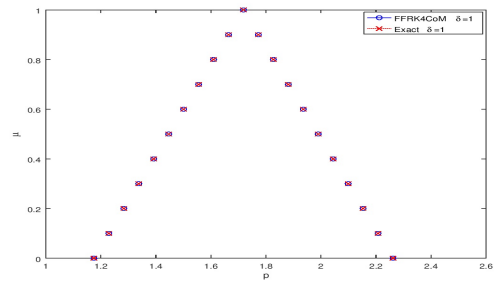


Figure 2: Exact and FFRK4CoM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

Numerical solution for FFDEs using FFRK4RM and FFRK4CoM

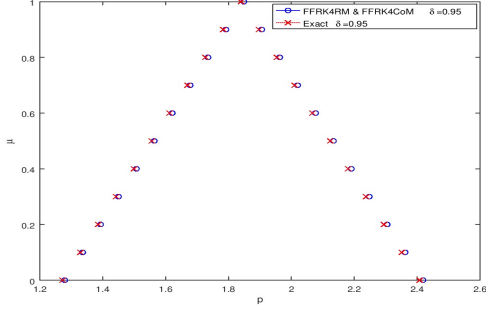


Figure 3: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.1, \zeta = 1$ and $\delta = 0.95$

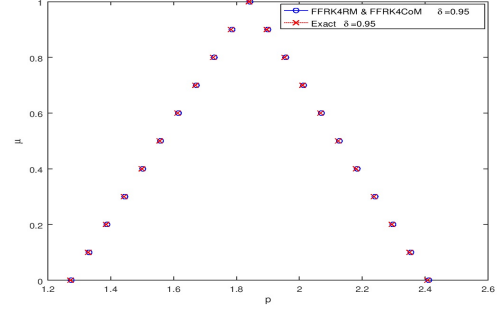


Figure 4: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.01, \zeta = 1$ and $\delta = 0.95$

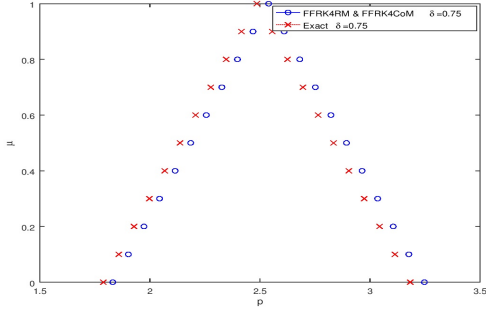


Figure 5: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.1, \zeta = 1$ and $\delta = 0.75$

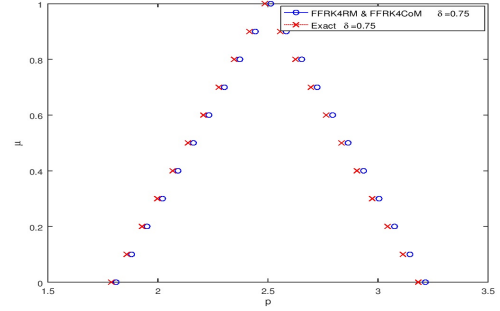


Figure 6: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.01, \zeta = 1$ and $\delta = 0.75$

μ	FFRK4RM		FFRK4CoM	
	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$
0	3.503367e-007	5.255050e-007	2.993491e-006	4.490236e-006
0.1	3.590951e-007	5.167466e-007	3.068328e-006	4.415399e-006
0.2	3.678535e-007	5.079882e-007	3.143165e-006	4.340561e-006
0.3	3.766119e-007	4.992298e-007	3.218002e-006	4.265724e-006
0.4	3.853703e-007	4.904713e-007	3.292840e-006	4.190887e-006
0.5	3.941288e-007	4.817129e-007	3.367677e-006	4.116050e-006
0.6	4.028872e-007	4.729545e-007	3.442514e-006	4.041212e-006
0.7	4.116456e-007	4.641961e-007	3.517351e-006	3.966375e-006
0.8	4.204040e-007	4.554377e-007	3.592189e-006	3.891538e-006
0.9	4.291624e-007	4.466793e-007	3.667026e-006	3.816701e-006
1	4.379208e-007	4.379208e-007	3.741863e-006	3.741863e-006

Table 1: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1, \zeta = 1$ & $\delta = 1$

	FFRK4RM		FFRK4CoM	
μ	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$
0	7.016838e-003	1.052526e-002	7.020171e-003	1.053026e-002
0.1	7.192259e-003	1.034984e-002	7.195676e-003	1.035475e-002
0.2	7.367679e-003	1.017441e-002	7.371180e-003	1.017925e-002
0.3	7.543100e-003	9.998994e-003	7.546684e-003	1.000374e-002
0.4	7.718521e-003	9.823573e-003	7.722189e-003	9.828240e-003
0.5	7.893942e-003	9.648152e-003	7.897693e-003	9.652736e-003
0.6	8.069363e-003	9.472731e-003	8.073197e-003	9.477231e-003
0.7	8.244784e-003	9.297310e-003	8.248701e-003	9.301727e-003
0.8	8.420205e-003	9.121889e-003	8.424206e-003	9.126223e-003
0.9	8.595626e-003	8.946468e-003	8.599710e-003	8.950719e-003
1	8.771047e-003	8.771047e-003	8.775214e-003	8.775214e-003

Table 2: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1$, $\zeta = 1$ & $\delta = 0.95$

	FFRK4RM		FFRK4CoM	
μ	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$
0	4.280237e-002	6.420356e-002	4.281712e-002	6.422568e-002
0.1	4.387243e-002	6.313350e-002	4.388755e-002	6.315525e-002
0.2	4.494249e-002	6.206344e-002	4.495798e-002	6.208483e-002
0.3	4.601255e-002	6.099338e-002	4.602841e-002	6.101440e-002
0.4	4.708261e-002	5.992332e-002	4.709883e-002	5.994397e-002
0.5	4.815267e-002	5.885326e-002	4.816926e-002	5.887354e-002
0.6	4.922273e-002	5.778320e-002	4.923969e-002	5.780311e-002
0.7	5.029279e-002	5.671314e-002	5.031012e-002	5.673269e-002
0.8	5.136285e-002	5.564308e-002	5.138055e-002	5.566226e-002
0.9	5.243291e-002	5.457302e-002	5.245097e-002	5.459183e-002
1	5.350297e-002	5.350297e-002	5.352140e-002	5.352140e-002

Table 3: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1$, $\zeta = 1$ & $\delta = 0.75$

Example 5.2. Consider a nonlinear trapezoidal FIVP for fractional order

$$D_{\zeta}^{\delta} \tilde{p}(\zeta) = 2\tilde{p}(\zeta) - \tilde{p}^2(\zeta) + 1, \quad (17)$$

subject to the initial condition $\tilde{p}(0) = [0.75 + 0.1\mu, 1.125 - 0.125\mu]$

The exact solution for (17) is $\tilde{p}(t) = \frac{\tilde{p}(0)((1+\sqrt{2})e^{2\sqrt{2}\zeta} + \sqrt{2}-1) + e^{2\sqrt{2}\zeta}-1}{\tilde{p}(0)(e^{2\sqrt{2}\zeta}-1) + (\sqrt{2}-1)e^{2\sqrt{2}\zeta} + \sqrt{2}+1}$ at $\delta = 1$

μ	FFRK4RM		FFRK4CoM	
	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$
0	1.376584e-006	2.304300e-007	1.007102e-005	1.212671e-005
0.1	1.341864e-006	2.372080e-007	1.015498e-005	1.212002e-005
0.2	1.307424e-006	2.439957e-007	1.023728e-005	1.211324e-005
0.3	1.273258e-006	2.507932e-007	1.031792e-005	1.210638e-005
0.4	1.239362e-006	2.576005e-007	1.039691e-005	1.209943e-005
0.5	1.205730e-006	2.644176e-007	1.047423e-005	1.209238e-005
0.6	1.172358e-006	2.712445e-007	1.054991e-005	1.208525e-005
0.7	1.139241e-006	2.780813e-007	1.062394e-005	1.207803e-005
0.8	1.106374e-006	2.849278e-007	1.069632e-005	1.207073e-005
0.9	1.073753e-006	2.917843e-007	1.076707e-005	1.206333e-005
1	1.041373e-006	2.986507e-007	1.083618e-005	1.205584e-005

Table 4: Absolute Error between FFRK4RM and FFRK4CoM of Example 2 for $h = 0.1$, $\zeta = 1$ & $\delta = 1$

μ	$\delta = 0.95$				$\delta = 0.75$			
	FFRK4RM		FFRK4CoM		FFRK4RM		FFRK4CoM	
	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$	$p(\zeta, \mu)$	$\bar{p}(\zeta, \mu)$
0	2.184525	2.267603	2.184537	2.267615	2.134656	2.214349	2.134673	2.214366
0.1	2.187265	2.267155	2.187277	2.267167	2.137047	2.213872	2.137064	2.213889
0.2	2.189969	2.266706	2.189981	2.266718	2.139421	2.213395	2.139438	2.213412
0.3	2.192637	2.266255	2.192649	2.266267	2.141777	2.212917	2.141794	2.212934
0.4	2.195271	2.265803	2.195282	2.265815	2.144115	2.212439	2.144132	2.212456
0.5	2.19787	2.26535	2.197882	2.265363	2.146436	2.211959	2.146453	2.211977
0.6	2.200436	2.264897	2.200449	2.264909	2.148741	2.21148	2.148758	2.211497
0.7	2.20297	2.264441	2.202982	2.264454	2.151029	2.21100	2.151046	2.211017
0.8	2.205472	2.263985	2.205485	2.263997	2.153301	2.210519	2.153318	2.210536
0.9	2.207943	2.263528	2.207955	2.26354	2.155558	2.210037	2.155575	2.210054
1	2.210384	2.263069	2.210396	2.263081	2.157799	2.209555	2.157816	2.209572

Table 5: Approximate solution for FFRK4RM and FFRK4CoM of Example 1 for $\delta = 0.95$, $\delta = 0.75$ at $\zeta = 1$.

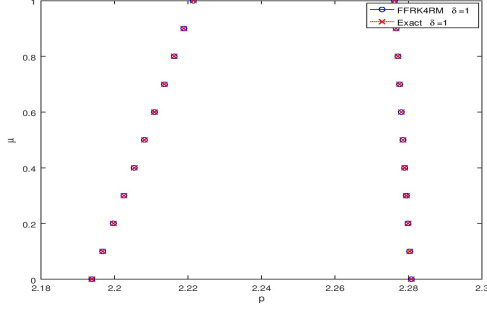


Figure 7: Exact and FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

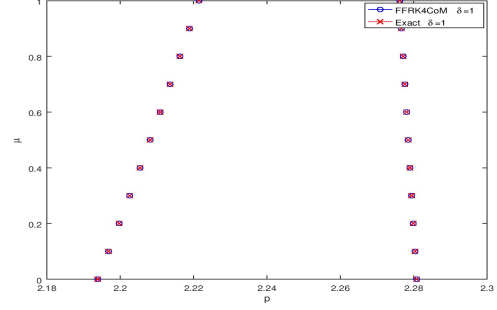


Figure 8: Exact and FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

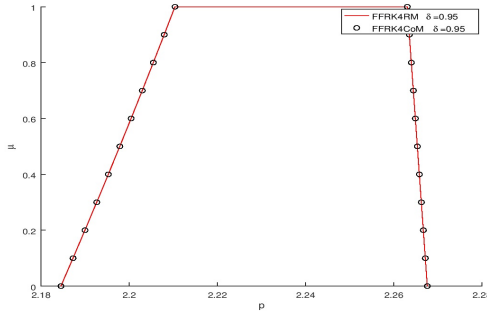


Figure 9: FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.95$

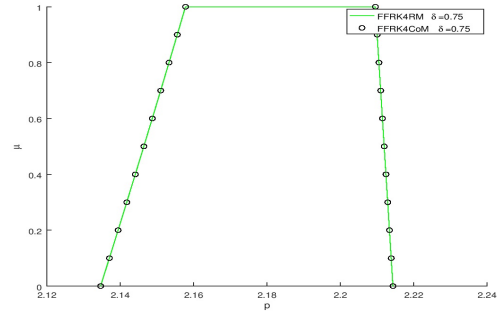


Figure 10: FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.75$

6 Conclusion

FFRK4RM and FFRK4CoM have been effectively utilized in this paper for both linear and nonlinear FFIVPs. When the derivative is in fractional order, FFRK4RM and FFRK4CoM almost offer the same result. However, when the derivative order is an integer, FFRK4RM performs better than FFRK4CoM in terms of result. FFRK4RM and FFRK4CoM both methods are given the best result compare with the max-min improved Euler methods. The computational load will be reduced by using the Octave program. The tables and figures show the proposed method is how effectively approaches the exact result very closely as the step size gets smaller. The absolute errors of linear and nonlinear FFIVPs are investigated graphically and numerically for various values of δ in order to

evaluate the dependability and efficiency of the suggested approach. The analysis of the 2D plots and tables for various values of δ confirms that the approximate solution converges quickly to the exact solution.

The benefits of using these techniques are:

- The problem does not need any small or big parametric assumptions.
- Both weak and highly nonlinear problems can be solved using these techniques.
- The suggested approaches generate solutions without perturbation, linearization, or discretization, as opposed to other analytical approximation methods.
- He's and Adomian polynomials are not required for solving nonlinear problems.

It may be inferred that the suggested approaches are rapid, exact, and simple to apply and produce excellent outcomes. As fractional order differential systems in uncertain environments, these methods have been used in the future for many significant applications in engineering and science.

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New approximate fixed point results for rational contraction mappings

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Abstract

In this paper, we investigate approximate fixed point results for rational contraction mappings in a metric space. This manuscript's intention is to demonstrate approximate fixed point results and the diameter of the approximate fixed point results on metric spaces. Particularly, we use some rational contraction mappings, which were mainly discussed in Dass and Gupta [1975] and Jaggi [1977]. A few examples are included to illustrate our results. Also, we discuss some applications of approximate fixed point results in the field of mathematics rigorously.

Keywords: fixed point, approximate fixed point, rational contraction, diameter approximate fixed point.

2020 AMS subject classifications: 47H10 54H25 ¹

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1 Introduction

Researchers from several fields have contributed to the growth of science and technology by employing fixed point theory. Large-scale problems requiring fixed point theory are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing fixed point theory approaches and have provided various useful techniques for discovering fixed points in complex issues. These are currently crucial in many mathematics related areas and applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of fixed point theory, mathematician Brouwer [1911], proposed fixed point theorems for continuous mappings on finite dimensional spaces. In 1922, Banach [1922] established and confirmed the renowned Banach contraction principle. Then, using contractive mappings on metric spaces, various experts extended the Banach principle and proved more complicated fixed point results. Kannan (refer, Kannan [1968], Kannan [1969]) established a fixed point results for operators that are not required to be continuous. Chatterjea [1972] has investigated and presented a similar type of fixed point results. The author, Zamfirescu [1972], combined the above two contractions and derived his fixed point results. Ćirić [1971] invented generalized contractions and found some fixed point theorems by using them. Further, Reich [1971], introduced his contraction operator and proposed fixed point findings. Subsequently, authors Hardy and Rogers [1973] introduced a new contraction known as the Hardy and Rogers contraction, and they derived some fixed point results. Similarly, Bianchini [1972] proved fixed point theorem by using another contraction mapping. The unique fixed point theorem for weakly B-contraction mapping was proved by Marudai and Bright [2015]. Additionally, some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters were proposed by Istratescu [1982] and Istratescu [1983]. Recently, many authors expanded these results and proved some innovative theorems (see, Mitrović et al. [2019], Shahrazad et al. [2011]). The first scholars to investigate a generalisation of the Banach fixed point theorem while simultaneously using a contractive condition of the rational type were Dass and Gupta [1975]. In the similar manner, Jaggi [1977] used a contractive condition of the rational type to prove a fixed point theorem in complete metric space. Then, Harjani et al. [2010] expanded Jaggi's findings to partially ordered metric spaces. Later, some results that focused on coupled random fixed point theorems for mappings satisfying a contraction condition of rational type on partially ordered metric spaces were derived by Ćirić et al. [2012]. Rational contraction conditions have been heavily employed in both the fixed point and common fixed point locations. Due to the complexity of finding an exact fixed point, one can show the results of an approximate fixed point. Because an exact fixed point has overly strict limitations. This is the main rea-

son to find an approximate fixed point (ϵ -fixed point). Assume that a selfmap, $T : K \rightarrow K$, has an approximate fixed point (called, t_0). In which case, the point Kt_0 is "very near" to the point t_0 . Here, the distance is less than ϵ , that is., $d(Kt_0, t_0) < \epsilon$. An approximate fixed point is a point that is nearly located at its respective fixed point. Originally, Tis et al. [2003] established the existence of approximate fixed points, which turn out to be still guaranteed under various weakened versions of the well-known fixed point theorems of Brouwer, Kakutani, and Banach. Moreover, they studied approximate fixed point results for contraction and nonexpansive mappings (refer to Theorems 2.1, 2.2, 3.1 and 4.1, respectively). Following that, Berinde [2006] proposed approximate fixed point results (Qualitative theorems) by using various operators on metric spaces (not necessarily complete). Further, he derived the diameter of the approximate fixed point results (Quantitative theorems) by using two main lemmas (refer, Berinde [2002] and Berinde [2003]). Also, Dey and Saha [2012] extended these results, and they explained that the diameter of approximate fixed points for the Reich operator tends to zero when ϵ approaches zero. Further, the authors (refer, Mohsenialhosseini [2017], Mohsenialhosseini et al. [2011]) considered some new approximate fixed point results for cyclical contraction mappings. Additionally, Mohsenialhosseini and Saheli [2021] extended these results for a family of contraction mappings and showed the existence of a common fixed point for Mohseni-Saheli mapping. Also, the authours Tijani and Olayemi [2021] proved approximate fixed point results by using some rational contraction mappings. In this manuscript, we derive some results, which include approximate fixed point theorems and diameters of approximate fixed point theorems on metric spaces (not necessarily complete), by using some rational contraction mappings, which were discussed mainly in Dass and Gupta [1975] and Jaggi [1977].

The remaining parts of this manuscript are presented as follows: In Section 2, some definitions and lemmas are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and diameter of approximate fixed points are rigorously discussed. In Section 4, we discuss and prove some applications related to approximate fixed points in the field of applied mathematics. Finally, in Section 5, we present some conclusions.

2 Preliminaries

In this section, we present some fundamental definitions and lemmas of approximate fixed point results, which are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. Berinde [2006] Let (K, d) be a metric space and $T : K \rightarrow K$, $\epsilon > 0$. Then $t \in K$ is said to be an approximate fixed point (ϵ -fixed point) of K if

$$d(t, Kt) < \epsilon.$$

Let $F_\epsilon(K) = \{t \in K : d(t, Kt) < \epsilon\}$ denotes the set of all ϵ -fixed points of K for a given $\epsilon > 0$.

Definition 2.2. Berinde [2006] Let $T : K \rightarrow K$. Then K has the approximate fixed point property (a.f.p.p) if for all $\epsilon > 0$,

$$F_\epsilon(M) \neq \emptyset.$$

Lemma 2.1. Berinde [2006] Let (K, d) be a metric space, $T : K \rightarrow K$ such that K is asymptotic regular, that is., $d(K^n(t), K^{n+1}(t)) \rightarrow 0$ as $n \rightarrow \infty$, for all $t \in K$. Then, for all $\epsilon > 0$,

$$F_\epsilon(K) \neq \emptyset.$$

Definition 2.3. Berinde [2006] Let (K, d) be a metric space, $T : K \rightarrow K$ a operator and $\epsilon > 0$. We define the diameter of the set $F_\epsilon(K)$, that is.,

$$\Delta(F_\epsilon(K)) = \sup\{d(t, r) : t, r \in F_\epsilon(K)\}.$$

Lemma 2.2. Berinde [2006] Let (K, d) be a metric space, $T : K \rightarrow K$ an operator and $\epsilon > 0$. We assume that:

- (i) $F_\epsilon(K) \neq \emptyset$;
- (ii) for all $\theta > 0$, there exists $\phi(\theta) > 0$ such that $d(t, r) - d(Kt, Kr) \leq \theta$ implies that $d(t, r) \leq \phi(\theta)$, for all $t, r \in F_\epsilon(K)$.

Then;

$$\Delta(F_\epsilon(K)) \leq \phi(2\epsilon).$$

Definition 2.4. Tijani and Olayemi [2021] Consider a selfmap $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then φ is said to be a comparison mapping if it satisfies the following conditions:

- (i) φ is monotone increasing mapping, and
- (ii) $\varphi^n(t)$ converges to 0 as $n \rightarrow +\infty$, for all $t \in \mathbb{R}_+$.

Furthermore, we prove the following result, which is necessary to demonstrate the main findings of this manuscript:

Theorem 2.1. Let (K, d) be a metric space and let $T : K \rightarrow K$ be a contraction. Then K has an approximate fixed point (ϵ -fixed point).

Proof. Fix $k_0 \in K$ and a sequence $\{k_n\}$ is defined by $k_{n+1} = Kk_n$, for all $n \geq 0$. This implies $\{k_n\}$ is a Cauchy sequence. Thus, for all $\epsilon > 0$ there exist $k_0 \in \mathbb{N}$ such that for every $t, r \geq k_0$ implies $d(k_t, k_r) < \epsilon$. In particular, if $n \geq k_0$, $d(k_n, k_{n+1}) < \epsilon$. That is, $d(k_n, Kk_n) < \epsilon$. Therefore, $k_n \in F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. Hence, K has an approximate fixed point (ϵ -fixed point). $\square$

Example 2.1. Let $K = (0, 1)$ and a selfmap $T : K \rightarrow K$ is defined by $Kk = k/2$. Then, K is a contraction. Since $(0, 1)$ is not complete and hence K does not have a fixed point. But $\{k_n\}$ is defined by $k_{n+1} = Kk_n$ with $k_0 = 1/2$. Note that $k_{n+1} = 1/2^{n+2}$. Then, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. Hence, K has an ϵ -fixed point.

3 Main Results

In this section, we prove some approximate fixed point results for various rational type contraction mappings on metric spaces.

Theorem 3.1. Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu, \mu \in [0, 1)$ with $\nu + \mu < 1$ such that $d(t, Kt) + d(r, Kr) \neq 0$ and for all $t, r \in K$ such that

$$d(Kt, Kr) \leq \frac{\nu[d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)]}{d(t, Kr) + d(r, Kt)} + \mu d(t, r).$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1 - \mu)}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$, $t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\ &\leq \nu \left[\frac{d(K^{n-1} t, K^n t)d(K^{n-1} t, K^{n+1} t) + d(K^n t, K^{n+1} t)d(K^n t, K^{n-1} t)}{d(K^{n-1} t, K^{n+1} t) + d(K^n t, K^{n-1} t)} \right] \\ &\quad + \mu d(K^{n-1} t, K^n t) \\ &\leq \nu d(K^{n-1} t, K^n t) + \mu d(K^{n-1} t, K^n t) \\ &\leq \lambda d(K^{n-1} t, K^n t), \text{ where } \lambda = (\nu + \mu) \\ &\leq \lambda^2 d(K^{n-2} t, K^{n-1} t) \\ &\dots \\ &\leq \lambda^n d(t, Kt) \end{aligned}$$

Since $d(K^n t, K^{n+1} t) \rightarrow 0$ as $n \rightarrow +\infty$, for all $t, r \in K$. This implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ -fixed point. Clearly condition (i) of Lemma 2.2 is proved. Now only to prove, the condition (ii) of Lemma 2.2. For that, take $\theta > 0$ and $t, r \in F_\epsilon(K)$. Assume also that $d(t, r) - d(Kt, Kr) \leq \theta$. Show that $\phi(\theta) > 0$ exists. Consider,

$$\begin{aligned}
 d(t, r) &\leq d(Kt, Kr) + \theta \\
 &\leq \nu \left[\frac{d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)}{d(t, Kr) + d(r, Kt)} \right] + \mu d(t, r) + 2\epsilon \\
 &= \nu \left[\frac{d(t, Kt)[d(t, r) + d(r, Kr)] + d(r, Kr)[d(t, r) + d(t, Kt)]}{d(t, r) + d(r, Kr) + d(t, r) + d(t, Kt)} \right] + \mu d(t, r) + 2\epsilon \\
 &= \nu \left[\frac{\epsilon[d(t, r) + \epsilon] + \epsilon[d(t, r) + \epsilon]}{2d(t, r) + 2\epsilon} \right] + \mu d(t, r) + 2\epsilon \\
 &= \frac{2\nu\epsilon d(t, r) + 2\nu\epsilon^2 + 2\mu\epsilon d(t, r) + 2\mu[d(t, r)]^2 + 4\epsilon d(t, r) + 4\epsilon^2}{2d(t, r) + 2\epsilon}
 \end{aligned}$$

Which implies that

$$\begin{aligned}
 2[d(t, r)]^2 + 2\epsilon d(t, r) \\
 = 2\nu\epsilon d(t, r) + 2\nu\epsilon^2 + 2\mu\epsilon d(t, r) + 2\mu[d(t, r)]^2 + 4\epsilon^2 + 4\epsilon d(t, r)
 \end{aligned}$$

That is,

$$\begin{aligned}
 2[d(t, r)]^2 - 2\mu[d(t, r)]^2 + 2\epsilon d(t, r) - 2\nu\epsilon d(t, r) - 2\mu\epsilon d(t, r) - 4\epsilon d(t, r) \\
 = 2\nu\epsilon^2 + 4\epsilon^2
 \end{aligned}$$

$$(2 - 2\mu)[d(t, r)]^2 - (2\epsilon + 2\mu\epsilon + 2\nu\epsilon)d(t, r) = 2\epsilon^2(\nu + 2)$$

Thus, we have $a = (2 - 2\mu)$, $b = -(2\epsilon + 2\mu\epsilon + 2\nu\epsilon)$ and $c = -2\epsilon^2(\nu + 2)$. Therefore,

$$\begin{aligned}
 d(t, r) &\leq \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon \pm \sqrt{(2\epsilon + 2\mu\epsilon + 2\nu\epsilon)^2 + 4(2 - 2\mu)(2\nu\epsilon^2 + 4\epsilon^2)}}{4 - 4\mu} \\
 &= \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon + \sqrt{4\epsilon^2 + 4\mu^2\epsilon^2 + 4\nu^2\epsilon^2 + 8\mu\epsilon^2 + 8\nu\mu\epsilon^2 + 8\nu\epsilon^2 + 16\nu\epsilon^2 + 32\epsilon^2 - 16\nu\mu\epsilon^2 - 32\mu\epsilon^2}}{4(1 - \mu)} \\
 &= \frac{2\epsilon + 2\mu\epsilon + 2\nu\epsilon + \sqrt{36\epsilon^2 + 4\mu^2\epsilon^2 + 4\nu^2\epsilon^2 - 24\mu\epsilon^2 - 8\nu\mu\epsilon^2 + 24\nu\epsilon^2}}{4(1 - \mu)} \\
 &= \frac{2(\epsilon + \mu\epsilon + \nu\epsilon) + 2\sqrt{9\epsilon^2 + \mu^2\epsilon^2 + \nu^2\epsilon^2 - 6\mu\epsilon^2 - 2\nu\mu\epsilon^2 + 6\nu\epsilon^2}}{4(1 - \mu)}
 \end{aligned}$$

Approximate fixed point results

$$\begin{aligned}
 &< \frac{\epsilon + \mu\epsilon + \nu\epsilon + 9\epsilon^2 + \mu^2\epsilon^2 + \nu^2\epsilon^2 - 6\mu\epsilon^2 - 2\nu\mu\epsilon^2 + 6\nu\epsilon^2}{2(1 - \mu)} \\
 &= \frac{\epsilon^2(9 + \mu^2 + \nu^2 - 6\mu - 2\nu\mu + 6\nu) + \epsilon(\nu + \mu + 1)}{2(1 - \mu)}
 \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1 - \mu)}, \text{ for all } \epsilon > 0.$$

□

Example 3.1. Let (K, d) be a metric space. Let $K = (0, 1/2]$ be endowed with usual metric. Let $T : K \rightarrow K$ be defined by $Kt = t/2$, for all $t \in K$. For that, choose $\nu = 0.25, \mu = 0.2$ and $\epsilon = 0.5$. Also $t = 0.5, r = 0.25 \in K$. Then, $d(t, Kt) = 0.333 < 0.5$, $d(r, Kr) = 0.125 < 0.5$. We have $d(t, r) = 0.25$ and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + \mu^2 + 6\nu - 6\mu - 2\nu\mu + 9) + \epsilon(\nu + \mu + 1)}{2(1 - \mu)} = 1.907.$$

Hence, Theorem 3.1 is satisfied.

Theorem 3.2. Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu, \mu \in [0, 1)$ with $\nu + \mu < 1$ such that $1 + d(t, r) \neq 0$ and for all $t, r \in K$ implies that

$$d(Kt, Kr) \leq \frac{\nu d(r, Kr)[1 + d(t, Kt)]}{1 + d(t, r)} + \mu d(t, r).$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{2\epsilon^2(\nu - \nu\mu + 1) + \epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{1 - \mu}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0, t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
 d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\
 &\leq \nu \left[\frac{d(K^n t, K^{n+1} t)[1 + d(K^{n-1} t, K^n t)]}{1 + d(K^{n-1} t, K^n t)} \right] + \mu d(K^{n-1} t, K^n t) \\
 &\leq \nu d(K^n t, K^{n+1} t) + \mu d(K^{n-1} t, K^n t) \\
 &\leq \lambda d(K^{n-1} t, K^n t), \text{ where } \lambda = \frac{\mu}{1 - \nu} \\
 &\leq \lambda^2 d(K^{n-2} t, K^{n-1} t) \\
 &\dots \\
 &\leq \lambda^n d(t, Kt)
 \end{aligned}$$

Since $d(K^n t, K^{n+1} t) \rightarrow 0$ as $n \rightarrow +\infty$, for all $t, r \in K$. This implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ -fixed point. Here, as in the previous Theorem 3.1, we have

$$\begin{aligned} d(t, r) &\leq d(Kt, Kr) + \theta \\ &= \frac{\nu d(r, Kr)[1 + d(t, Kt)]}{1 + d(t, r)} + \mu d(t, r) + 2\theta \\ &= \frac{\nu\epsilon[1 + \epsilon]}{1 + d(t, r)} + \mu d(t, r) + 2\epsilon \\ &= \frac{\nu\epsilon + \alpha\epsilon^2 + \mu d(t, r) + \mu[d(t, r)]^2 + 2\epsilon d(t, r) + d(t, r)}{1 + d(t, r)} \end{aligned}$$

Which implies that

$$d(t, r) + [d(t, r)]^2 \leq \nu\epsilon + \nu\epsilon^2 + \mu d(t, r) + \mu[d(t, r)]^2 + 2\epsilon d(t, r) + 2\epsilon$$

$$d(t, r) + [d(t, r)]^2 - \mu d(t, r) - \mu[d(t, r)]^2 - 2\epsilon d(t, r) \leq \nu\epsilon^2 + \nu\epsilon + 2\epsilon$$

$$(1 - \mu)[d(t, r)]^2 + (1 - \mu - 2\epsilon)d(t, r) \leq \nu\epsilon^2 + \nu\epsilon + 2\epsilon$$

Thus, we have $a = (1 - \mu)$, $b = (1 - \mu - 2\epsilon)$ and $c = -(\nu\epsilon^2 + \nu\epsilon + 2\epsilon)$. Therefore,

$$\begin{aligned} d(t, r) &\leq \frac{-(1 - \mu - 2\epsilon) \pm \sqrt{(1 - \mu - 2\epsilon)^2 + 4(1 - \mu)(\nu\epsilon^2 + \nu\epsilon + 2\epsilon)}}{2(1 - \mu)} \\ &\leq \frac{-1 + \mu + 2\epsilon + \sqrt{1 - \mu^2 + 4\epsilon^2 - 2\mu + 4\mu\epsilon - 4\epsilon + 4\nu\epsilon^2 + 4\nu\epsilon + 8\epsilon - 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 8\mu\epsilon}}{2(1 - \mu)} \\ &\leq \frac{-1 + \mu + 2\epsilon + \sqrt{1 - \mu^2 - 2\mu + 4\epsilon^2 + 4\nu\epsilon^2 + 4\epsilon + 4\nu\epsilon + 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 4\mu\epsilon}}{2(1 - \mu)} \\ &< \frac{-1 + \mu + 2\epsilon + 1 + \mu^2 - 2\mu + 4\epsilon^2 + 4\nu\epsilon^2 + 4\epsilon + 4\nu\epsilon - 4\nu\mu\epsilon^2 - 4\nu\mu\epsilon - 4\mu\epsilon}{2(1 - \mu)} \\ &= \frac{\mu^2 - \mu + 4\epsilon^2(1 + \nu - \nu\mu) + 2\epsilon(3 + 2\nu - 2\nu\mu - 2\mu)}{2(1 - \mu)} \\ &< \frac{4\epsilon^2(1 + \nu - \nu\mu) + 2\epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{2(1 - \mu)} \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{2\epsilon^2(\nu - \nu\mu + 1) + \epsilon(2\nu - 2\mu - 2\nu\mu + 3)}{1 - \mu}, \text{ for all } \epsilon > 0.$$

□

Theorem 3.3. *Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu \in [0, 1)$ such that $d(t, Kt) + d(r, Kr) \neq 0$ and for all $t, r \in K$ implies that*

$$d(Kt, Kr) \leq \frac{d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)}{d(t, Kr) + d(r, Kt)}.$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$, $t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\ &\leq \nu \frac{d(K^{n-1} t, K^n t)d(K^{n-1} t, K^{n+1} t) + d(K^n t, K^{n+1} t)d(K^n t, K^{n-1} t)}{d(K^{n-1} t, K^{n+1} t) + d(K^n t, K^{n-1} t)} \\ &= \nu \frac{d(K^{n-1} t, K^n t)d(K^{n-1} t, K^{n+1} t)}{d(K^{n-1} t, K^{n+1} t)} \\ &= \nu d(K^{n-1} t, K^n t) \\ &\leq \nu^2 d(K^{n-2} t, K^{n-1} t) \\ &\dots \\ &\leq \nu^n d(t, Kt) \end{aligned}$$

Since $d(K^n t, K^{n+1} t) \rightarrow 0$ as $n \rightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ -fixed point. Here, as same as in previous Theorem 3.1, we have

$$\begin{aligned} d(t, r) &\leq d(Kt, Kr) + \theta \\ &\leq \frac{\nu[d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)]}{d(t, Kr) + d(r, Kt)} + \theta \\ &= \frac{\nu\epsilon[\epsilon + d(t, r)] + \nu\epsilon[\epsilon + d(t, r)]}{2\epsilon + 2d(t, r)} + 2\epsilon \\ &= \frac{\nu\epsilon^2 + \nu\epsilon d(t, r) + \nu\epsilon^2 + \nu d(t, r) + 4\epsilon^2 + 4\epsilon d(t, r)}{2\epsilon + 2d(t, r)} \end{aligned}$$

Which implies that

$$2[d(t, r)]^2 + 2\epsilon d(t, r) - 2\nu\epsilon d(t, r) - 4\epsilon d(t, r) = 2\nu\epsilon^2 + 4\epsilon^2$$

$$2[d(t, r)]^2 - 2\epsilon(\nu + 1)d(t, r) = 2\epsilon^2(\nu + 2)$$

Thus, we have $a = 2$, $b = -2\epsilon(\nu + 1)$ and $c = -2\epsilon^2(\nu + 2)$. Therefore,

$$\begin{aligned} d(t, r) &\leq \frac{2\epsilon(\nu + 1) \pm \sqrt{4\epsilon^2(\nu + 1)^2 + 16\epsilon^2(\nu + 2)}}{4} \\ &= \frac{2\epsilon(\nu + 1) + \sqrt{4\epsilon^2(\nu^2 + 1 + 2\nu) + 16\epsilon^2\nu + 32\epsilon^2}}{4} \\ &= \frac{2\epsilon(\nu + 1) + \sqrt{4\epsilon^2\nu^2 + 4\epsilon^2 + 8\epsilon^2\nu + 16\epsilon^2\nu + 32\epsilon^2}}{4} \\ &= \frac{2\epsilon(\nu + 1) + \sqrt{4\epsilon^2\nu^2 + 36\epsilon^2 + 24\epsilon^2\nu}}{4} \\ &= \frac{2\epsilon(\nu + 1) + 2\sqrt{\epsilon^2\nu^2 + 9\epsilon^2 + 6\epsilon^2\nu}}{4} \\ &< \frac{\epsilon\nu + \epsilon + \epsilon^2\nu^2 + 9\epsilon^2 + 6\epsilon^2\nu}{2} \\ &= \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2} \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2}, \text{ for all } \epsilon > 0.$$

□

Corollary 3.1. *Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu \in [0, 1)$ such that $d(t, Kt) + d(r, Kr) \neq 0$ and for all $t, r \in K$ implies that*

$$d(Kt, Kr) \leq \nu \cdot \max \left\{ \frac{d(t, Kt)d(t, Kr) + d(r, Kr)d(r, Kt)}{d(t, Kr) + d(r, Kt)}, d(t, r) \right\}.$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \max \left\{ \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2}, \frac{2\epsilon}{1 - \nu} \right\}, \text{ for all } \epsilon > 0.$$

Proof. The same proof of above Theorem 3.3 completes this corollary. □

Example 3.2. Let (K, d) be a metric space. Let $K = (0, 1/2]$ be endowed with usual metric. Let $T : K \rightarrow K$ be defined by $Kt = t/2$, for all $t \in K$. To show that the corollary 3.1 is satisfied. For that choose $\nu = 0.25$ and $\epsilon = 0.5$. Also $t = 0.5, r = 0.25 \in M$. Then, $d(t, Kt) = 0.333 < 0.5$; $d(r, Kr) = 0.125 < 0.5$. We have $d(t, r) = 0.25$ and

$$\Delta(F_\epsilon(K)) = 0.25 < \frac{\epsilon^2(\nu^2 + 6\nu + 9) + \epsilon(\nu + 1)}{2} = 1.633.$$

Hence, Theorem 3.3 and Corollary 3.1 are satisfied.

Theorem 3.4. Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu, \mu \in [0, 1)$ and for all $t, r \in K$ such that

$$d(Kt, Kr) \leq \nu d(t, r) + \mu \frac{d(t, Kt)d(r, Kr)}{d(t, r)}.$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\mu - \nu\mu + 1) + \epsilon}{1 - \nu}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$, $t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(K^n t, K^{n+1} t) &= d(K(K^{n-1} t), K(K^n t)) \\ &\leq \nu d(K^{n-1} t, K^n t) + \frac{\mu d(K^{n-1} t, K^n t)(K^n t, K^{n+1} t)}{d(K^{n-1} t, K^n t)} \\ &= \nu d(K^{n-1} t, K^n t) + \mu d(K^n t, K^{n+1} t) \\ &= \lambda d(K^{n-1} t, K^n t), \text{ where } \lambda = \frac{\nu}{1 - \mu} \\ &\leq \lambda^2 d(K^{n-2} t, K^{n-1} t) \\ &\dots \\ &\leq \lambda^n d(t, Kt) \end{aligned}$$

Since $d(K^n t, K^{n+1} t) \rightarrow 0$ as $n \rightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ -fixed point. Here, as in the previous Theorem 3.1, we have

$$\begin{aligned} d(t, r) &\leq d(Kt, Kr) + \theta \\ &= \nu d(t, r) + \mu \frac{d(t, Kt)d(r, Kr)}{d(t, r)} + \theta \\ &= \nu d(t, r) + \frac{\mu \epsilon^2}{d(t, r)} + 2\epsilon \\ &= \frac{\nu [d(t, r)]^2 + \mu \epsilon^2 + 2\epsilon d(t, r)}{d(t, r)} \end{aligned}$$

On simplifying,

$$[d(t, r)]^2 = \nu[d(t, r)]^2 + \mu\epsilon^2 + 2\epsilon d(t, r)$$

$$[d(t, r)]^2 - \nu[d(t, r)]^2 - 2\epsilon d(t, r) = \mu\epsilon^2$$

$$(1 - \nu)[d(t, r)]^2 - 2\epsilon d(t, r) = \mu\epsilon^2$$

Which implies that $a = (1 - \nu)$, $b = -2\epsilon$ and $c = -\mu\epsilon^2$. Therefore,

$$\begin{aligned} d(t, r) &= \frac{2\epsilon \pm \sqrt{4\epsilon^2 + 4(1 - \nu)\mu\epsilon^2}}{2(1 - \nu)} \\ &= \frac{2\epsilon + \sqrt{4\epsilon^2 + 4\mu\epsilon^2 - 4\nu\mu\epsilon^2}}{2(1 - \nu)} \\ &= \frac{2\epsilon + 2\sqrt{\epsilon^2 + \mu\epsilon^2 - \nu\mu\epsilon^2}}{2(1 - \nu)} \\ &< \frac{\epsilon + \epsilon^2 + \mu\epsilon^2 - \nu\mu\epsilon^2}{(1 - \nu)} \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^2(\mu - \nu\mu + 1) + \epsilon}{1 - \nu}, \text{ for all } \epsilon > 0.$$

□

Corollary 3.2. Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu \in [0, 1)$ and for all $t, r \in K$ such that

$$d(Kt, Kr) \leq \frac{\mu d(t, Kt)d(r, Kr)}{d(t, r)}.$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \epsilon^2(1 + \mu) + \epsilon, \text{ for all } \epsilon > 0.$$

Proof. The same proof of Theorem 3.4 completes this corollary when $\nu = 0$. □

Theorem 3.5. Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exists $\nu, \mu \in [0, 1)$ with $d(r, Kr) + d(t, r) \neq 0$ and for all $t, r \in K$ such that

$$d(Kt, Kr) \leq \nu \frac{d(t, Kt)d(t, Kr)d(r, Kr)}{d(r, Kr) + d(t, r)} + \mu d(t, r).$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1 - \mu)}, \forall \epsilon > 0.$$

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Proof. Let $\epsilon > 0$, $t_0 \in K$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Kt_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
 d(K^{n+1}t, K^n t) &= d(K(K^n t), K(K^{n-1}t)) \\
 &\leq \nu \frac{d(K^n t, K^{n+1}t)d(K^n t, K^n t)d(K^{n-1}t, K^n t)}{d(K^{n-1}, K^{n+1}t)} + \mu d(K^n t, K^{n-1}t) \\
 &= \mu d(K^n t, K^{n-1}t) \\
 &\leq \mu^2 d(K^{n-1}t, K^{n-2}t) \\
 &\dots \\
 &\leq \mu^n d(Kt, t)
 \end{aligned}$$

Since $d(K^n t, K^{n+1}t) \rightarrow 0$ as $n \rightarrow +\infty$, for all $t, r \in K$. Which implies that $\{t_n\}$ is a Cauchy sequence. By Theorem 2.1, $F_\epsilon(K) \neq \emptyset$, for all $\epsilon > 0$. That is, K has an ϵ -fixed point. Here, as in the previous Theorem 3.1, we have

$$\begin{aligned}
 d(t, r) &\leq d(Kt, Kr) + \theta \\
 &\leq \frac{d(t, Kt)d(t, Kr)d(r, Kr)}{d(r, Kr) + d(t, r)} + \mu d(t, r) + \theta \\
 &= \frac{\nu \epsilon d(t, r) + \nu \epsilon^3}{\epsilon + d(t, r)} + \mu d(t, r) + 2\epsilon \\
 &= \frac{\nu \epsilon^2 d(t, r) + \nu \epsilon^3 + \mu d(t, r) + \mu [d(t, r)]^2 + 2\epsilon^3 + 2\epsilon d(t, r)}{\epsilon + d(t, r)}
 \end{aligned}$$

That is,

$$(1 - \mu)[d(t, r)]^2 = (\nu \epsilon^2 + \mu \epsilon + \epsilon)d(t, r) + \nu \epsilon^3 + 2\epsilon^3$$

Which implies that $a = (1 - \mu)$, $b = -(\nu \epsilon^2 + \mu \epsilon + \epsilon)$ and $c = -(\mu \epsilon^3 + 2\epsilon^3)$. Therefore,

$$\begin{aligned}
 d(t, r) &= \frac{\nu \epsilon^2 + \mu \epsilon + \epsilon \pm \sqrt{(\nu \epsilon^2 + \mu \epsilon + \epsilon)^2 + 4(1 - \mu)(\nu \epsilon^3 + 2\epsilon^3)}}{2(1 - \mu)} \\
 &= \frac{\nu \epsilon^2 + \mu \epsilon + \epsilon + \sqrt{\nu^2 \epsilon^4 + \mu^2 \epsilon^2 + \epsilon^2 + 2\nu \mu \epsilon^3 + 2\mu \epsilon^2 + 2\nu \epsilon^3 + 4\nu \epsilon^3 + 8\epsilon^3 - 4\nu \mu \epsilon^3 - 8\mu \epsilon^3}}{2(1 - \mu)} \\
 &= \frac{\nu \epsilon^2 + \mu \epsilon + \epsilon + \sqrt{\nu^2 \epsilon^4 + \mu^2 \epsilon^2 + \epsilon^2 - 2\nu \mu \epsilon^3 - 6\mu \epsilon^3 + 6\nu \epsilon^3 + 8\epsilon^3}}{2(1 - \mu)} \\
 &< \frac{\nu \epsilon^2 + \mu \epsilon + \epsilon + \nu^2 \epsilon^4 + \mu^2 \epsilon^2 + \epsilon^2 - 2\nu \mu \epsilon^3 - 6\mu \epsilon^3 + 6\nu \epsilon^3 + 8\epsilon^3}{2(1 - \mu)}
 \end{aligned}$$

Hence,

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1 - \mu)}, \forall \epsilon > 0.$$

□

Corolary 3.3. *Let (K, d) be a metric space and $T : K \rightarrow K$. Then there exist $\nu \in [0, 1)$ and φ , a comparison mapping satisfying $\varphi(t) < t$, for all $t > 0$. Also, $d(r, Kr) + d(t, r) \neq 0$ and for all $t, r \in K$ such that*

$$d(Kt, Kr) \leq \nu \frac{d(t, Kt)d(t, Kr)d(r, Kr)}{d(r, Kr) + d(t, r)} + \varphi(\mu d(t, r)).$$

Prove that K has an approximate fixed point (ϵ -fixed point) and

$$\Delta(F_\epsilon(K)) < \frac{\epsilon^4\nu^2 + \epsilon^3(6\nu - 6\mu - 2\nu\mu + 8) + \epsilon^2(\mu^2 + \nu + 1) + \epsilon(\mu + 1)}{2(1 - \mu)}, \forall \epsilon > 0.$$

Proof. The same proof of Theorem 3.5 completes this corollary because comparison mapping $\varphi(t) < t$, for all $t > 0$. □

4 Applications

Approximate fixed point theory covers a wide range of applications in mathematics, particularly differential geometry, numerical analysis, and so on. By reading Debnath et al. [2021], Khuri and Louhichi [2018] and references therein, one can find a variety of applications involving approximate fixed point results in the field of mathematics. The examples below demonstrate how to apply approximate fixed point findings in differential equations.

Example 4.1. *Consider $l''(k) = 6l^2(k)$, $0 \leq k \leq 1$ subect to $l(0) = 1/4$, $l(1) = 1/9$. Exact solution is $l_0(k) = -5k/36 + 1/4$; Consider a mapping $K : [0, 1] \rightarrow [0, 1]$ is defined by*

$$\begin{aligned} K(l) &= l + \int_0^1 G(k, s)[l''(s) - \phi(s, l(s), l'(s))]ds \\ &= \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k, s)\phi(s, l(s), l'(s))ds \\ &== \frac{-5k}{36} + \frac{1}{4} - \int_0^1 G(k, s)6l''(s)ds \end{aligned}$$

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Consider,

$$\begin{aligned}
 |K(l_1) - K(l_2)| &= 6 \left| - \int_0^1 G(k, s) l_1^2(s) ds + \int_0^1 G(k, s) l_2^2(s) ds \right| \\
 &= 6 \left(\int_0^1 |G(k, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right)^{\frac{1}{2}} \\
 &\leq \frac{1}{4\sqrt{3}} \left(\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right)^{\frac{1}{2}} \\
 &< \sup_{[0,1]} |l_1(s) - l_2(s)|
 \end{aligned}$$

Hence, K is a contraction operator. So, by Theorem 2.1, K has an ϵ -fixed point.

Example 4.2. Consider $l''(s) = \frac{3l^2(s)}{2}$, $0 \leq k \leq 1$ subect to $l(0) = 4, l(1) = 1$.
Exact solution is $l(s) = \frac{4}{(1+s)^2}$; Consider a mapping $K : [0, 1] \rightarrow [0, 1]$ by

$$K(l) = l + \int_0^1 G(k, s) [l''(s) - \phi(s, l(s))] ds \quad (1)$$

Consider, $l''(k) = 0$ which implies

$$l(k) = c_1 k + c_2 \quad (2)$$

By initial condition we have $c_2 = 4$ and $c_1 = -3$. Then (2) becomes $l(k) = -3k + 4$. Therefore, from (1), we get

$$\begin{aligned}
 K(l) &= -3k + 4 + \int_0^1 G(k, s) [l''(s) - \phi(s, l(s))] ds \\
 &= -3k + 4 + \int_0^1 G(k, s) l''(s) ds - \int_0^1 G(k, s) \phi(s, l(s)) ds \\
 &= -3k + 4 + \int_0^1 G(k, s) \frac{3}{2} l^2(s) ds
 \end{aligned}$$

Consider,

$$\begin{aligned}
 |K(l_1) - K(l_2)| &= \left| - \int_0^1 G(k, s) \frac{3}{2} l_1^2(s) ds + \int_0^1 G(k, s) \frac{3}{2} l_2^2(s) ds \right| \\
 &= \frac{3}{2} \left| \int_0^1 G(k, s) [l_2^2(s) - l_1^2(s)] ds \right| \\
 &\leq \frac{3}{2} \left(\int_0^1 |G(k, s)|^2 ds \right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{3}{2} \left(\int_0^k s^2 (1-k)^2 ds + \int_k^1 k^2 (1-s)^2 ds \right)^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{3}{2} \left\{ \frac{(1-k)^2 k^3}{3} + \frac{k^2 (1-k)^3}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{3}{2} \left\{ \frac{(1-k)^2}{3} [k^3 + k^2(1-k)] \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{3}{2} \left\{ \frac{(1-k)^2 k^2}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{3}{8\sqrt{3}} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{3}}{8} \left[\int_0^1 |l_2^2(s) - l_1^2(s)|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{3}}{8} \sup_{[0,1]} |l_2(s) - l_1(s)| \\
 &\leq \sup_{[0,1]} |l_2(s) - l_1(s)|
 \end{aligned}$$

Hence, K is a contraction operator. So, by Theorem 2.1, K has an ϵ -fixed point.

5 Conclusion

In this paper, some approximate fixed point theorems are established in metric space by using various types rational contractions mappings. It is worth observing that in the limiting case $\epsilon \rightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. Since different findings delivered in the future might be shown in a smaller setting to ensure the existence of the approximate fixed points. The same time the idea of an approximate fixed points (ϵ -fixed points) are therefore no less important than that of exact fixed points.

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Extended b -metric spaces and the related approximate fixed point results

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Abstract

In this paper, we investigate some approximate fixed point results in extended b -metric spaces using several contraction mappings such as the Mohseni-Saheli contraction, the B -contraction, and their consequences. Additionally, we prove some ϵ -fixed point results by using rational type contraction mappings, which were discussed mainly in Dass and Gupta [1975] and Jaggi [1977]. Also, a few examples are included to illustrate the results. Finally, we discuss some applications that support our main results in the field of applied mathematics.

Keywords: fixed point, b -metric space, approximate fixed point, rational contraction.

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1 Introduction

By using fixed point theory ($f.p.t$), researchers from many fields have contributed to the progress of science and technology. Large scale problems requiring $f.p.t$ are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing $f.p.t$ approaches and have provided various useful techniques for discovering $f.p$'s in complex issues. These are currently crucial in many mathematics related areas and its applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of $f.p.t$, mathematician Brouwer [1908], proposed $f.p$ theorems ($f.p.t$'s) for continuous mappings on finite dimensional spaces. Further, Banach [1922] established and confirmed the renowned Banach contraction principle. Several authors used the Banach contraction principle in numerous ways and presented numerous fixed point results ($f.p.r$'s) [see, Bianchini [1972], Chatterjea [1972], Ćirić [1971], Kannan [1968], Kannan [1969], Karapinar [2011], Khan [1976], Reich [1971] & Zamfirescu [1972]]. On the other hand, the concept of b -metric space was initiated by Bakhtin [1989], which was an interesting expansion of metric spaces. Recently, more and more extensions of b -metric spaces, such as $b_v(s)$ -metric spaces and b -rectangular metric spaces, were introduced, and some $f.p.t$'s are shown on these spaces [see, Mitrović and Radenović [2017], Roshan et al. [2016]].

Due to the complexity of finding an exact $f.p$, one can show the results of an approximate $f.p$ ($a.f.p$). Because an exact $f.p$ has overly strict limitations. This is the main reason to find an $a.f.p$ (ϵ -fixed point). Assume that a selfmap, $T : M \rightarrow M$, has an $a.f.p$ (called, p_0). In which case, the point Mp_0 is "very near" to the point p_0 . Here, the distance is less than ϵ , that is., $d(Mp_0, p_0) < \epsilon$. An $a.f.p$ is a point that is nearly located at its respective $f.p$. Originally, Tis et al. [2003] established the existence of $a.f.p$'s, which turn out to be still guaranteed under various weakened versions of the well-known $f.p.t$'s of Brouwer, Kakutani, and Banach. Moreover, they studied $a.f.p.r$'s for contraction and nonexpansive mappings (refer to Theorems 2.1, 2.2, 3.1 and 4.1, respectively). Following that, Berinde [2006] proposed $a.f.p.r$'s (Qualitative theorems) by using various contraction operators on metric spaces (not necessarily complete). Further, he derived the diameter of the $a.f.p.r$'s (Quantitative theorems) by using two main lemmas (also, refer Berinde [2002] and Berinde [2003]). In addition to that, Dey and Saha [2012] extended these results, and they explained that the diameter of $a.f.p$ for the Reich operator tends to zero when ϵ approaches zero. Further, S. A. M. Mohsenialhosseini (refer, Mohsenialhosseini [2017], Mohsenialhosseini et al. [2011]) considered some new $a.f.p.r$'s for cyclical contraction mappings. Moreover, Mohsenialhosseini and Saheli [2021] extended these results for a family of contraction mappings and showed the existence of a common $f.p$ for Mohseni-

Saheli mapping. Also, the authours Tijani and Olayemi [2021] proved $a.f.p.r$'s by using some rational contraction mappings. In this manuscript, we derive some results, which include $a.f.p.t$'s and diameters of $a.f.p.t$'s on b -metric spaces, by using some rational contraction mappings.

On the other hand, the first researchers to investigate a generalization of the Banach $f.p$ theorem while simultaneously using a contraction condition of the rational type were Dass and Gupta [1975]. Later, Jaggi [1977], used a contraction condition of the rational type to prove a $f.p$ theorems in complete metric spaces. Moreover, rational contraction conditions have been heavily employed in both the $f.p$ and common $f.p$ locations. In this, we first provide $f.p$ solutions utilizing two contraction mappings, B -contraction mapping (refer to Marudai and Bright [2015]), the Hardy and Rogers [1973] contraction mapping, and their associated consequents. Secondly, we demonstrate the $f.p.r$'s in the setting of rational contraction mappings, which were discussed mostly in Dass and Gupta [1975] and Jaggi [1977].

The remaining parts of this manuscript are displayed as follows: In Section 2, some basic notions, such as notations, definitions and lemmas, are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and uniqueness of $a.f.p.r$'s are rigorously discussed in b -metric space. In Section 4, we go one step forward and provide some applications of $a.f.p.$ in the field of applied mathematics. Finally, in Section 5, we present some conclusions.

2 Preliminaries

In this section, some notations and basic notions, such as fundamental definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. Boriceanu [2009] Let M be a non-empty set and $b \geq 1$ be a given real number. A function $d : M \times M \longrightarrow \mathbb{R}_+$ is called a b -metric provided that for all $p, q, r \in M$ satisfies the following conditions.

- (i) $d(p, q) = 0$ iff $p = q$;
- (ii) $d(p, q) = d(q, p)$;
- (iii) $d(p, q) \leq b[d(p, r) + (r, q)]$

The pair (M, d) is called a b -metric space. Immediately from the notion of b -metric space we have the result, every metric space is a b -metric space with $b = 1$. But the converse does not hold.

Example 2.1. Boriceanu [2009] Let $M = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$
 $d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and
 $d(0, 0) = d(1, 1) = d(2, 2) = 0$
 Then, $d(p, q) \leq \frac{m}{2}[d(p, r) + d(r, q)]$ for all $p, q, r \in M$.
 if $m > 2$ then the triangle inequality does not hold.

Definition 2.2. Berinde [2006] Let (M, d) be a metric space and $T : M \rightarrow M$.
 Then $p \in M$ is said to be an ϵ -fixed point (approximate fixed point) of T , if for
 every $\epsilon > 0$ implies

$$d(p, Tp) < \epsilon.$$

Let $F_\epsilon(T) = \{p \in M : d(p, Tp) < \epsilon\}$ denotes the set of all ϵ -fixed point of T for
 a given $\epsilon > 0$.

Definition 2.3. Berinde [2006] Let $T : M \rightarrow M$. Then T has the approximate
 fixed point property (a.f.p.p) if for every $\epsilon > 0$,

$$F_\epsilon(T) \neq \emptyset.$$

Lemma 2.1. Berinde [2006] Let A be a closed subset of a metric space (M, d)
 and $T : A \rightarrow M$ be a compact map. Then T has a fixed point if and only if it has
 the approximate fixed point property.

Remark 2.1. Berinde [2006] In the following, by $\delta(A)$ for a set $A \neq \emptyset$ we will
 understand the diameter of the set A , i.e.,

$$\delta(A) = \sup\{d(p, q) : p, q \in A\}.$$

Definition 2.4. Berinde [2006] Let (M, d) be a metric space, $T : M \rightarrow M$ a
 operator and $\epsilon > 0$. We define the diameter of the set $F_\epsilon(T)$, i.e.,

$$F_\epsilon(T) = \sup\{d(p, q) : p, q \in F_\epsilon(T)\}.$$

Lemma 2.2. Berinde [2006] Let (M, d) be a metric space, $T : M \rightarrow M$ an
 operator and $\epsilon > 0$. We assume that:

(i) $F_\epsilon(T) \neq \emptyset$; and

(ii) for all $\theta > 0$, there exists $\phi(\theta) > 0$ such that $d(p, q) - d(Tp, Tq) \leq \theta$
 implies that $d(p, q) \leq \phi(\theta)$, for all $p, q \in F_\epsilon(T)$.

Then;

$$\delta(F_\epsilon(T)) \leq \phi(2\epsilon).$$

Definition 2.5. *Mohsenalhosseini and Saheli [2021] Let (M, d) be a b -metric space with coefficient $b \geq 1$. A selfmap $T : M \rightarrow M$ is said to be a Mohseni-saheli type contraction if there exists $k_1 \in [0, \frac{1}{2})$ with $k_1(b+1) < 1$ such that*

$$d(Tp, Tq) \leq k_1[d(p, q) + d(Tp, Tq)], \text{ for all } p, q \in M.$$

Definition 2.6. *Marudai and Bright [2015] Let (M, d) be a b -metric space with coefficient $b \geq 1$. A selfmap $T : M \rightarrow M$ is said to be a B -contraction if there exists $k_1, k_2, k_3 \in [0, 1)$ with $k_1(b+1) + bk_2 + b(b+1)k_3 < 1$ such that*

$$\begin{aligned} d(Tp, Tq) &\leq k_1[d(p, Tp) + d(q, Tq)] + k_2d(p, q) \\ &\quad + k_3[d(p, Tq) + d(q, Tp)], \text{ for all } p, q \in M. \end{aligned}$$

Definition 2.7. *Bianchini [1972] Let (M, d) be a b -metric space. A selfmap $T : M \rightarrow M$ is said to be a Bianchini contraction if there exists $k \in (0, 1)$ with $kb < 1$ such that*

$$d(Tp, Tq) \leq kB(p, q),$$

where

$$B(p, q) = \max \{d(p, Tp), d(q, Tq)\}, \text{ for all } p, q \in M.$$

3 Main Results

In this section, we prove the approximate fixed point theorems using various types of contraction mappings, including the B -contraction, the Mohseni-Saheli contraction, and their consequences on b -metric spaces.

Theorem 3.1. *Let (M, d) be a metric space and let $T : M \rightarrow M$ be a contraction. Then T has an approximate fixed point (ϵ -fixed point).*

Proof. Fix $p_0 \in M$. Define a sequence $\{p_n\}$ by $p_{n+1} = Tp_n$, for all $n \geq 0$. Then $\{p_n\}$ is a Cauchy sequence. Thus for all $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ implies $d(p_n, p_m) < \epsilon$. In particular, if $n \geq n_0$, $d(p_n, p_{n+1}) < \epsilon$. That is, $d(p_n, Tp_n) < \epsilon$. Therefore, $p_n \in F_\epsilon(T) \neq \emptyset$, for all $\epsilon > 0$. Hence T has an approximate fixed point (ϵ -fixed point). $\square$

Example 3.1. *Let $M = (0, 1)$. A selfmap $T : M \rightarrow M$ is defined by $Tp = p/2$. Then T is a contraction. Since $(0, 1)$ is not complete and hence T does not have a fixed point. But $\{p_n\}$ is defined by $p_{n+1} = Tp_n$ with $p_0 = 1/2$. Note that $p_{n+1} = 1/2^{n+2}$. Then, $F_\epsilon(T) \neq \emptyset$, for all $\epsilon > 0$. Hence T has an ϵ -fixed point.*

Following is the Mohseni-saheli contraction mapping considered here to prove our first a.f.p theorem in a b -metric space.

Theorem 3.2. *Let $T : M \rightarrow M$ be a Mohseni-saheli contractive type mapping in a b -metric space with $b \geq 1$. Then T has an ϵ -fixed point and*

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(1 + k_1b)}{1 - k_1 - k_1b}, \text{ for all } \epsilon > 0.$$

Proof. Given that T is a Mohseni-saheli type contraction mapping. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(T^n p, T^{n+1} p) &= d(T(T^{n-1} p), T(T^n p)) \\ &\leq k_1 d(T^{n-1} p, T^n p) + k_1 d(T^n p, T^{n+1} p) \\ &\leq \left(\frac{k_1}{1 - k_1} \right) d(T^{n-1} p, T^n p) \\ &= \lambda d(T^{n-1} p, T^n p), \text{ where } \lambda = \frac{k_1}{1 - k_1} \\ &\leq \lambda^2 d(T^{n-2} p, T^{n-1} p) \\ &\dots \\ &\leq \lambda^n d(p, Tp) \end{aligned}$$

Since $d(T^n p, T^{n+1} p) \rightarrow 0$ as $n \rightarrow \infty$, for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ -fixed point. It shows that the condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p, q) - d(Tp, Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned} d(p, q) &\leq d(Tp, Tq) + \theta \\ &\leq k_1 d(p, q) + k_1 d(Tp, Tq) + \theta \\ &\leq k_1 d(p, q) + k_1 b [d(Tp, p) + d(p, q) + d(q, Tq)] + \theta \\ &= \left(\frac{2k_1 b \epsilon + \theta}{1 - k_1 - k_1 b} \right) \\ &= \gamma \end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p, q) - d(Tp, Tq) \leq \theta$ implies $d(p, q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(1 + k_1b)}{1 - k_1 - k_1b}, \text{ for all } \epsilon > 0.$$

□

Following is the B -contraction mapping considered here to prove our second a.f.p.t in a b -metric space.

Theorem 3.3. *Let $T : M \rightarrow M$ be a B -contraction in a b -metric space with $b \geq 1$. Then T has an ϵ -fixed point and $\delta(F_\epsilon(T)) \leq \frac{2\epsilon(k_1 + k_3b + 1)}{1 - k_2 - 2k_3b}$, for all $\epsilon > 0$.*

Proof. Given that T is a B -contraction operator. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
 d(T^n p, T^{n+1} p) &= d(T(T^{n-1} p), T(T^n p)) \\
 &\leq k_1[d(T^{n-1} p, T^n p) + d(T^n p, T^{n+1} p)] + k_2 d(T^n p, T^{n-1} p) \\
 &\quad + k_3[d(T^{n-1} p, T^{n+1} p) + d(T^n p, T^n p)] \\
 &\leq k_1[d(T^{n-1} p, T^n p) + d(T^n p, T^{n+1} p)] + k_2 d(T^{n-1} p, T^n p) \\
 &\quad + k_3 b[d(T^{n-1} p, T^n p) + d(T^n p, T^{n+1} p)] \\
 &\leq \left(\frac{k_1 + k_2 + k_3 b}{1 - k_1 - k_3 b} \right) d(T^{n-1} p, T^n p) \\
 &= \lambda d(T^{n-1} p, T^n p), \text{ where } \lambda = \frac{k_1 + k_2 + k_3 b}{1 - k_1 - k_3 b} \\
 &\leq \lambda^2 d(T^{n-2} p, T^{n-1} p) \\
 &\dots \\
 &\leq \lambda^n d(p, Tp)
 \end{aligned}$$

Since $d(T^n p, T^{n+1} p) \rightarrow 0$ as $n \rightarrow \infty$, for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ -fixed point. It shows that condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p, q) - d(Tp, Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned}
 d(p, q) &\leq d(Tp, Tq) + \theta \\
 &\leq k_1[d(p, Tp) + d(q, Tq)] + k_2 d(p, q) + k_3[d(p, Tq) + d(q, Tp)] + \theta \\
 &= 2k_1\epsilon + k_2 d(p, q) + 2bk_3\epsilon + 2k_3bd(p, q) + \theta \\
 &= \left(\frac{2k_1\epsilon + 2k_3b\epsilon + \theta}{1 - k_2 - 2k_3b} \right) \\
 &= \gamma
 \end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p, q) - d(Tp, Tq) \leq \theta$ implies $d(p, q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(k_1 + k_3b + 1)}{1 - k_2 - 2k_3b}, \text{ for all } \epsilon > 0.$$

□

Following is the Bianchini contraction mapping considered here to prove our third *a.f.p.t* in a *b*-metric space.

Theorem 3.4. *Let $T : M \rightarrow M$ be a Bianchini contraction in a *b*-metric space with $b \geq 1$. Then T has an ϵ -fixed point and $\delta(F_\epsilon(T)) \leq \epsilon(k + 2)$, for all $\epsilon > 0$.*

Proof. Given T is Bianchini contraction. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n$, for all $n \geq 0$.

Case 1. If $B(p, q) = d(p, Tp)$. Then, the Definition 2.7 becomes:

$$\begin{aligned} d(Tp, Tq) &\leq kd(p, Tp) \\ \text{Substitute } q = Tp, d(Tp, T^2p) &\leq kd(p, Tp) \\ \text{Again substitute } p = Tp, d(T^2p, T^3p) &\leq kd(Tp, T^2p) \\ &= k^2d(p, Tp) \\ &\vdots \\ d(T^n p, T^{n+1} p) &\leq k^n d(p, Tp). \end{aligned}$$

Case 2. If $B(p, q) = d(q, Tq)$. Then, the Definition 2.7 becomes:

$$\begin{aligned} d(Tp, Tq) &\leq kd(q, Tq) \\ \text{Substitute } q = Tp, d(Tp, T^2p) &\leq kd(Tp, T^2p) \end{aligned}$$

This is impossible because $k \in (0, 1)$. Therefore, **Case 2** does not exist. Now by **Case 1**, $d(T^n p, T^{n+1} p) \rightarrow 0$ as $n \rightarrow \infty$, for all $p, q \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ -fixed point. It shows that the condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(K)$. Assume that $d(p, q) - d(Kp, Kq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exist.

$$\begin{aligned} d(p, q) &\leq d(Tp, Tq) + \theta \\ &\leq kB(p, q) + \theta \\ &= kd(p, Tp) + \theta \\ &= k\epsilon + \theta \\ &= \gamma \end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p, q) - d(Tp, Tq) \leq \theta$ implies $d(p, q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon)$, for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \epsilon(k + 2), \text{ for all } \epsilon > 0.$$

□

Remark 3.1. We have proved many approximate a.f.p.r's by using various contraction mappings in b -metric space. The following results shows the diameters of various contraction mappings.

- (i) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1 \in (0, 1)$ with $bk_1 < 1$ such that

$$d(Tp, Tq) \leq k_1 d(p, q), \text{ for all } p, q \in M.$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon}{1 - k_1}, \text{ for all } \epsilon > 0.$$

This is called contraction mapping in b -metric space.

- (ii) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1 \in [0, \frac{1}{2})$ with $k_1(b + 1) < 1$ such that

$$d(Tp, Tq) \leq k_1[d(p, Tp) + d(q, Tq)], \text{ for all } p, q \in M.$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq 2\epsilon(k_1 + 1), \text{ for all } \epsilon > 0.$$

This is called Kannan contraction mapping in b -metric space.

- (iii) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1 \in [0, \frac{1}{2})$ with $k_1 b(b + 1) < 1$ such that

$$d(Tp, Tq) \leq k_1[d(q, Tp) + d(p, Tq)], \text{ for all } p, q \in M.$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(k_1 b + 1)}{1 - 2k_1 b}, \text{ for all } \epsilon > 0.$$

This is called Chatterjea contraction mapping in b -metric space.

- (iv) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1, k_2, k_3, k_4, k_5 \in [0, 1)$ with $(k_1 + k_2)b + k_3 + (b^2 + b)k_4 < 1$ such that

$$\begin{aligned} d(Tp, Tq) &\leq k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq) \\ &\quad + k_4 d(p, Tq) + k_5 d(q, Tp), \text{ for all } p, q \in M. \end{aligned}$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + k_4 b + k_5 b + 2)}{1 - k_1 - k_4 b - k_5 b}, \text{ for all } \epsilon > 0.$$

This is called Hardy-Rogers contraction mapping in b -metric space.

- (v) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1, k_2, k_3, k_4 \in [0, 1)$ with $b(k_1 + k_2) + k_3 + (b^2 + b)k_4 < 1$ such that

$$d(Tp, Tq) \leq k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq) + k_4 [d(p, Tq) + d(q, Tp)], \text{ for all } p, q \in M.$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + 2k_4b + 2)}{1 - k_1 - 2k_4b}, \text{ for all } \epsilon > 0.$$

This is called Ciric-type contraction mapping in b -metric space.

- (vi) Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k_1, k_2, k_3 \in [0, 1)$ with $b(k_1 + k_2) + k_3 < 1$ such that

$$d(Tp, Tq) \leq k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq), \text{ for all } p, q \in M.$$

Hence T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + 2)}{1 - k_1}, \text{ for all } \epsilon > 0.$$

This is called Reich-type contraction mapping in b -metric space.

Next we prove some $a.f.p.t'$ s for rational type contraction mappings in the settings of b -metric spaces.

Theorem 3.5. Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exists $k \in [0, 1)$ such that $d(p, Tq) + d(q, Tp) \neq 0$ and

$$d(Tp, Tq) \leq \frac{k[d(p, Tp)d(p, Tq) + d(q, Tq)d(q, Tp)]}{d(p, Tq) + d(q, Tp)}, \text{ for all } p, q \in M.$$

Then T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k + 1)}{2}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} d(T^n p, T^{n+1} p) &= d(T(T^{n-1} p), T(T^n p)) \\ &\leq k \left[\frac{d(T^{n-1} p, T^n p)d(T^{n-1} p, T^{n+1} p) + d(T^n p, T^{n+1} p)d(T^n p, T^n p)}{d(T^{n-1} p, T^{n+1} p) + d(T^n p, T^n p)} \right] \\ &\leq kd(T^{n-1} p, T^n p) \\ &\dots \\ &\leq (k)^n d(p, Tp) \end{aligned}$$

Since $d(T^n p, T^{n+1} p) \rightarrow 0$ as $n \rightarrow \infty$ for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. It shows that condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p, q) - d(Tp, Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned} d(p, q) &\leq k \left[\frac{d(p, Tp)d(p, Tq) + d(q, Tq)d(q, Tp)}{d(p, Tq) + d(q, Tp)} \right] + 2\epsilon \\ &= k \left[\frac{d(p, Tp)[bd(p, q) + bd(q, Tq)] + d(q, Tq)[bd(p, q) + bd(p, Tp)]}{bd(p, q) + bd(q, Tq) + bd(p, q) + bd(p, Tp)} \right] + 2\epsilon \\ &= k \left[\frac{\epsilon[bd(p, q) + b\epsilon] + \epsilon[bd(p, q) + b\epsilon]}{2bd(p, q) + 2b\epsilon} \right] + 2\epsilon \\ &= \frac{2bk\epsilon d(p, q) + 2kb\epsilon^2 + 4b\epsilon d(p, q) + 4b\epsilon^2}{2bd(p, q) + 2b\epsilon} \end{aligned}$$

On simplyfying, we get

$$2b[d(p, q)]^2 \leq 2b\epsilon(1 + k)d(p, q) + 2b\epsilon^2(k + 2)$$

Which implies $a = 2b, b = -2b\epsilon(1 + k)$ and $c = -2b\epsilon^2(k + 2)$. Therefore,

$$\begin{aligned} d(p, q) &\leq \frac{2b\epsilon(1 + k) + \sqrt{4b^2\epsilon^2(1 + k)^2 + 16b^2\epsilon^2(k + 2)}}{4b} \\ &= \frac{2b\epsilon(1 + k) + \sqrt{36b^2\epsilon^2 + 24b^2\epsilon^2k + 4k^2b^2\epsilon^2}}{4b} \\ &= \frac{b\epsilon(1 + k) + \sqrt{9b^2\epsilon^2 + 6b^2\epsilon^2k + k^2b^2\epsilon^2}}{2b} \\ &< \frac{b\epsilon + b\epsilon k + 9b^2\epsilon^2 + 6b^2\epsilon^2k + k^2b^2\epsilon^2}{2b} \\ &= \frac{\epsilon + \epsilon k + 9b\epsilon^2 + 6b\epsilon^2k + bk^2\epsilon^2}{2} \end{aligned}$$

Hence,

$$\delta(F_\epsilon(T)) < \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k + 1)}{2}, \text{ for all } \epsilon > 0.$$

□

Example 3.2. Let (M, d) be a b -metric space with the coefficient $b = 1$. Let $M = (0, 1/2]$ be endowed with usual metric. Let $T : M \rightarrow M$ be defined by $Tp = p/2$, for all $p \in M$. To show that the equation is satisfied. For that choose

$k = 1/4$ and $\epsilon = 1/2$. Also $p = 1/2, q = 1/4 \in M$. Then $d(p, Tp) = 1/3 < 1/2$; $d(q, Tq) = 1/8 < 1/2$. We have $d(p, q) = 1/4$ and

$$\delta(F_\epsilon(T)) < \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k + 1)}{2} = \frac{193}{128}.$$

Therefore, it satisfies all the conditions of Theorem 3.5. Hence, it has an ϵ -fixed point.

Corolary 3.1. Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exist $k_1, k_2 \in [0, 1)$ with $k_1 + bk_2 < 1$ such that

$$d(Tp, Tq) \leq \frac{k_1 d(q, Tq)[1 + d(p, Tp)]}{1 + d(p, q)} + k_2 d(p, q), \text{ for all } p, q \in M.$$

Then T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) < \frac{k_2^2 - k_2 + 6\epsilon + 4\epsilon^2(1 + k_1 - k_1 k_2) + 4\epsilon(k_1 - k_2 - k_1 k_2)}{2(1 - k_2)}, \forall \epsilon > 0.$$

Proof. Complete this corollary by following the same procedure as in Theorem 3.5. $\square$

Example 3.3. Let (M, d) be a b -metric space with the coefficient $b = 1$. Let $M = (0, 1/2]$ be endowed with usual metric. Let $T : M \rightarrow M$ be defined by $Tp = p/2$, for all $p \in M$. For that choose $k_1 = 1/4, k_2 = 1/5$ and $\epsilon = 1/2$. Also $p = 1/2, q = 1/4 \in M$. Then $d(p, Tp) = 1/3 < 1/2$; $d(q, Tq) = 1/8 < 1/2$. We have $d(p, q) = 1/4$ and

$$\frac{k_2^2 - k_2 + 6\epsilon + 4\epsilon^2(1 + k_1 - k_1 k_2) + 4\epsilon(k_1 - k_2 - k_1 k_2)}{2(1 - k_2)} \leq \frac{207}{80}.$$

Therefore, it satisfies all the conditions of corollary 3.1. Hence, it has an ϵ -fixed point.

Corolary 3.2. Let (M, d) be a b -metric space with the coefficient $b \geq 1$ and $T : M \rightarrow M$. Then there exist $k_1, k_2 \in [0, 1)$ with $k_1 + bk_2 < 1$ and for all $p, q \in M$ with $p \neq q$ such that

$$d(Tp, Tq) \leq \frac{k_1 d(p, Tp)d(q, Tq)}{d(p, q)} + k_2 d(p, q), \text{ for all } \epsilon > 0.$$

Then T has an ϵ -fixed point and

$$\delta(F_\epsilon(T)) < \epsilon \left(\frac{2}{1 - k_2} + k_1 \right), \text{ for all } \epsilon > 0.$$

Proof. Complete this corollary by following the same procedure as in Theorem 3.5. $\square$

4 Applications

The $a.f.p.t$ covers a wide range of applications in the field of mathematics, particularly differential geometry, numerical analysis, and so on. By reading Khuri and Louhichi [2018] and the references therein, one can find a variety of applications involving $a.f.p.r$'s in the field of applied mathematics. The examples below demonstrate how to apply $a.f.p$ findings in differential equations.

Example 4.1. Let $T = C([0, 1], \mathbb{R})$ and T is complete extended b -metric space defined by $d(p, q) = \sup_{t \in [0, 1]} |p - q|^2$. Also, consider $y''(t) = 3y^2(t)/2$, $0 \leq t \leq 1$ and the initial conditions $y(0) = 4$, $y(1) = 1$. Here, the exact solution is $y(t) = 4/(1+t)^2$. We have, $y_0(t) = c_1t + c_2$. By using the initial conditions, we get $y_0(t) = 4 - 3t$. Now, define the integral operator,

$$A(y) = y(t) + \int_0^1 G(t, s)[y'' - f(s, y, y')]ds \quad (1)$$

where

$$G(t, s) = \begin{cases} s(1-t) & 0 \leq s \leq t \\ t(1-s) & t \leq s \leq 1 \end{cases}$$

Then, the equation (1) becomes

$$\begin{aligned} A(y) &= y(t) + \int_0^1 G(t, s)y''(s)ds - \int_0^1 G(t, s)f(s, y, y')ds \\ &= (4 - 3t) - \int_0^1 G(t, s)[(-3/2)y^2(s)]ds \\ &= 4 - 3t + \frac{3}{2} \left\{ \int_0^1 G(t, s)y^2(s)ds \right\} \end{aligned}$$

Consider,

$$\begin{aligned} d(Ap, Aq) &= \sup_{t \in [0, 1]} |Ap - Aq|^2 \\ &= \sup_{t \in [0, 1]} \left| \frac{3}{2} \int_0^1 G(t, s)p^2(s)ds - \frac{3}{2} \int_0^1 G(t, s)q^2(s)ds \right|^2 \\ &\leq \frac{9}{4} \left(\int_0^1 |G(t, s)|^2 ds \right) \left(\int_0^1 |p^2(s) - q^2(s)|^2 ds \right) \\ &\leq \frac{9}{4} \frac{t^2(1-t)^2}{3} \int_0^1 |p^2(s) - q^2(s)|^2 ds \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{3}{4} \left(\frac{1}{4}\right) \left(\frac{1}{4}\right) \int_0^1 |p^2(s) - q^2(s)|^2 ds \\
 &\leq \frac{3}{64} \sup_{t \in [0,1]} |p(s) - q(s)|^2 \\
 &\leq \frac{3}{64} d(p, q)
 \end{aligned}$$

Then, Definition 2.6 gives

$$d(Ap, Aq) \leq (3/64)d(p, q) + k_1[d(p, Ap) + d(q, Aq)] + k_3[d(p, Aq) + d(q, Ap)].$$

Thus, $k_2 = 3/64$ and $k_1 = k_3 = 0$ satisfies all the conditions of Theorem 3.3. Also, by Theorem 3.1, A has a.f.p in $T = C([0, 1], \mathbb{R})$. Therefore, the given bounded value problem has a.f.p in T .

5 Conclusion

In this paper, some approximate fixed point theorems are established in a b -metric space by utilizing various types of contraction mappings. Further, some approximate fixed point theorems are newly developed for rational type contractions in a b -metric space. It is worth observing that in the limiting case $\epsilon \rightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. The notion of approximate fixed points is consequently not less essential than that of exact fixed points. Results that will be given in the future can be demonstrated in a lesser setting to ensure the existence and uniqueness of the approximate fixed points.

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Isomorphism on Interval-Valued Complex Fuzzy Graphs

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Abstract

This research paper delves definition of isomorphism, weak isomorphism and co-weak isomorphism within the realm of highly edge improper and neighbourly edge improper interval-valued complex fuzzy graphs as well as their complementary counterparts. Additionally we establish the isomorphism of μ -complement of highly edge improper interval-valued complex fuzzy graphs.

Key words: Interval-valued complex fuzzy graph, order, size, degree of an edge, neighbourly edge improper, highly edge improper interval-valued complex fuzzy graph, complement, isomorphism.

AMS subject classification: 05C72.

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1 Introduction

Isomorphism is one of the area in graph theory that many researchers focussing today. The notion of Interval valued fuzzy sets was introduced by Zadeh(9) in 1975, as an extension of fuzzy sets in which the values of the membership degrees are instead of the numbers, the intervals of numbers. The Interval valued fuzzy sets supply more satisfactory description of ambiguity than traditional fuzzy sets. Hence, the applications of Interval valued fuzzy sets are very much important to use in fuzzy control. Isomorphism on Fuzzy Graphs was discussed by A. Nagoor Gani, J. Malarvizhi(3). Isomorphism on Interval Valued Fuzzy Graph was studied by M. Naga Maruthi Kumari., R. Chandrasekhar(2). S. Ravi Narayanan and N.S. Santhi Maheswari discussed the isomorphic properties of irregular interval valued fuzzy graph (4). R.Sridevi., R. Venkateshwara was discussed the concept of edge improper interval-valued complex fuzzy graphs(8). In this article we are considering isomorphism on interval-valued complex fuzzy graphs. Though isomorphism on interval-valued fuzzy graph has already exist. This isomorphic concept is concentrated on these case which involve time periodic phenomena. We have discussed weak isomorphism, co-weak isomorphism, isomorphism and its related results.

2 Preliminaries

Definition 2.1. (6) A Interval-Valued Complex Fuzzy Graph(ivcfg) with an underlying set F is defined to be a pair $G : (S, T)$, where $S = [\mu_S^-, \mu_S^+]$ is an ivcf set on F and $T = [\mu_T^-, \mu_T^+]$ is a ivcf set on $I \subseteq F \times F$ such that

$$\begin{aligned} \mu_T^-(jl)e^{i\alpha_T(jl)} &\leq \min\{\mu_S^-(j), \mu_S^-(l)\}e^{i\min\{\alpha_S^-(j), \alpha_S^-(l)\}} \\ \mu_T^+(jl)e^{i\beta_T(jl)} &\leq \min\{\mu_S^+(j), \mu_S^+(l)\}e^{i\min\{\beta_S^+(j), \beta_S^+(l)\}} \text{ for all } j, l \in F. \end{aligned}$$

Definition 2.2. (6) Let $G : (S, T)$ be an ivcf graph, where $S = (\mu_S^-, \mu_S^+)$ and $T = (\mu_T^-, \mu_T^+)$ be two ivcf sets on a non empty set F . The order of an ivcf graph is defined by $O_I(G) = (O_I^-(G), O_I^+(G))$, where $O_I^-(G) = \sum_{j \in F} \mu_S^-(j)e^{\sum_{j \in F} \alpha_S^-(j)}$ and $O_I^+(G) = \sum_{j \in F} \mu_S^+(j)e^{\sum_{j \in F} \beta_S^+(j)}$ for all $j \in F$.

Definition 2.3. (6) Let $G : (S, T)$ be an ivcf graph. The positive degree of a vertex is defined by $d_I^+(j) = \sum_{jl \in I} \mu_T^+(jl)e^{\sum_{jl \in I} \beta_T(jl)}$ and the negative degree of a vertex is defined by $d_I^-(j) = \sum_{jl \in I} \mu_T^-(jl)e^{\sum_{jl \in I} \alpha_T(jl)}$ for all $jl \in I$. Then degree of a vertex j is defined as $d_I(x) = (d_I^-(j), d_I^+(j))$.

Definition 2.4. (6) Let $G : (S, T)$ be an ivcf graph, where $S = (\mu_S^-, \mu_S^+)$ and $T = (\mu_T^-, \mu_T^+)$ be two ivf sets on a non empty set P . The size of an ivcf graph is

defined by $S_I(G) = (S_I^-(G), S_I^+(G))$, where $S_I^-(G) = \sum_{j \in I} \mu_T^-(jl) e^{\sum_{jl \in I} \alpha_T(jl)}$ and $S_I^+(G) = \sum_{j \in I} \mu_T^+(jl) e^{\sum_{jl \in I} \beta_T(jl)}$ for all $jl \in I$.

3 Isomorphism on Edge Improper Interval-Valued Complex Fuzzy Graphs

In this section, homomorphism, weak isomorphism, co-weak isomorphism, isomorphism of two ivcf graphs are defined and some of its properties are discussed.

Definition 3.1. A homomorphism p of neighbourly edge improper ivcf graph (highly edge improper ivcf graph) G_1 and G_2 is a mapping $p: F_1 \rightarrow F_2$ which fulfills the following criteria

- (i) $\mu_S^+(x) e^{i\beta_S^+(x)} \leq \mu_S^+(p(x)) e^{i\beta_S^+(p(x))}$ and $\mu_S^-(x) e^{i\alpha_S^-(x)} \leq \mu_S^-(p(x)) e^{i\alpha_S^-(p(x))}$
- (ii) $\mu_T^+(xy) e^{i\beta_T^+(xy)} \leq \mu_T^+(p(xy)) e^{i\beta_T^+(p(xy))}$ and $\mu_T^-(xy) e^{i\alpha_T^-(xy)} \leq \mu_T^-(p(xy)) e^{i\alpha_T^-(p(xy))}$.

Example 3.1. Consider neighbourly edge improper interval-valued complex fuzzy graph on $G^*(F, I)$.

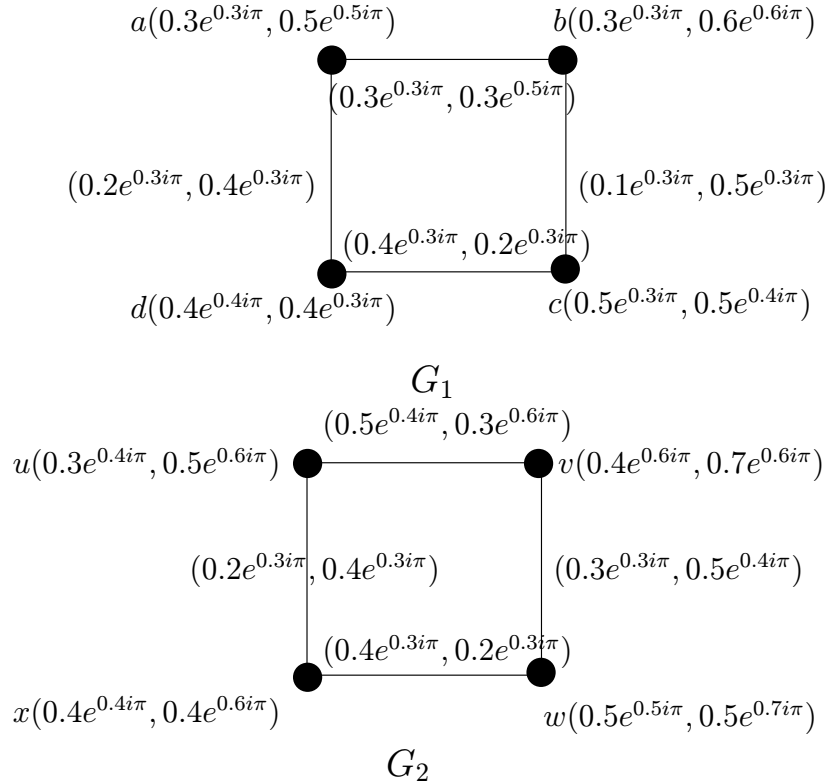


Figure 1: Homomorphism of neighbourly edge improper ivcf graph

In Figure 1, there is a homomorphism $p: F_1 \rightarrow F_2$ such that $p(a) = u, p(b) = v, p(c) = w, p(d) = x$.

Definition 3.2. A weak isomorphism p of neighbourly edge improper ivcf graph (highly edge improper ivcf graph) G_1 and G_2 is a bijective mapping $p: F_1 \rightarrow F_2$ which fulfills the following criteria

(i) p is homomorphism (ii) $\mu_S^+(x)e^{i\beta_S^+(x)} = \mu_S^+(p(x))e^{i\beta_S^+(p(x))}$ and $\mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}$.

Example 3.2. A neighbourly edge improper ivcf graph is considered on $G^*(F, I)$.

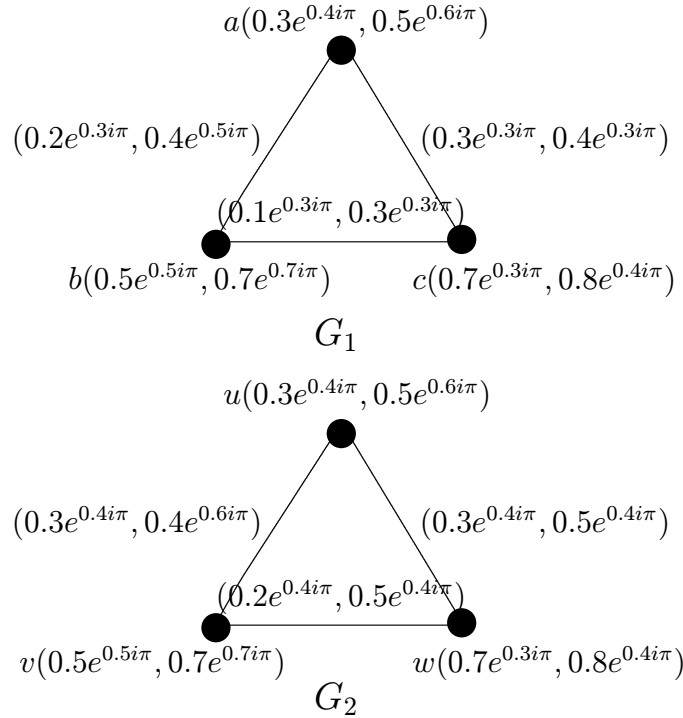


Figure 2: Weak isomorphism of neighbourly edge improper ivcf graph

In Figure 2, there is a weak isomorphism $p: F_1 \rightarrow F_2$ such that $p(a) = u, p(b) = v, p(c) = w$.

Definition 3.3. A co-weak isomorphism of neighbourly edge improper ivcf graph (highly edge improper ivcf graph) G_1 and G_2 is a bijective mapping $p: F_1 \rightarrow F_2$ which fulfills the following criteria

(i) p is homomorphism
(ii) $\mu_T^+(xy)e^{i\beta_T^+(x)} = \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}$ and $\mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}$.

Example 3.3. A highly edge improper ivcf graph is considered on $G^*(F, I)$.

Isomorphism on Interval-Valued Complex Fuzzy Graphs

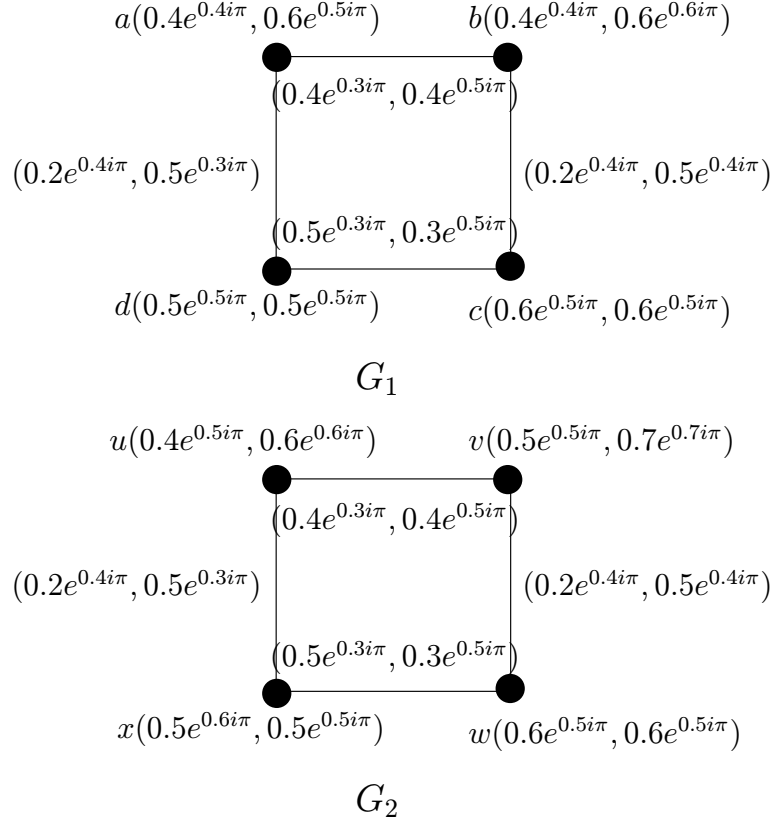


Figure 3: Co-weak isomorphism of neighbourly edge improper ivcf graph

In Figure 3, there is a homomorphism $p: F_1 \rightarrow F_2$ such that $p(a) = u$, $p(b) = v$, $p(c) = w$, $p(d) = x$.

Definition 3.4. An isomorphism p of neighbourly edge improper ivcf graph (highly edge improper ivcf graph) G_1 and G_2 is a bijective mapping $p: F_1 \rightarrow F_2$ which fulfills the following criteria

- (i) $\mu_S^+(x)e^{i\beta_S^+(x)} = \mu_S^+(p(x))e^{i\beta_S^+(p(x))}$ and $\mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}$.
- (ii) $\mu_T^+(xy)e^{i\beta_T^+(x)} = \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}$ and $\mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}$.

Theorem 3.1. If any two neighbourly (highly) edge improper ivcf graphs are isomorphic then their order and size are same.

Proof. If $p: G_1 \rightarrow G_2$ be an isomorphism between neighbourly edge improper ivcf graph then,

$$\mu_S^+(x)e^{i\beta_S^+(x)} = \mu_S^+(p(x))e^{i\beta_S^+(p(x))} \text{ and } \mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}.$$

$$\mu_T^+(xy)e^{i\beta_T^+(x)} = \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))} \text{ and } \mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}.$$

$$\begin{aligned}
 O_I(G_1) &= (\sum \mu_S^+(u)e^{i\beta_S^+(u)}, \sum \mu_S^-(u)e^{i\alpha_S^-(u)}) \\
 &= (\sum \mu_S^+(p(u)e^{i\beta_S^+(p(u))}), \sum \mu_S^-(p(u)e^{i\alpha_S^-(p(u))})) \\
 &= (\sum \mu_S^+(w)e^{i\beta_S^+(w)}, \sum \mu_S^-(w)e^{i\alpha_S^-(w)}) = O_I(G_2) \\
 S_I(G_1) &= (\sum \mu_T^+(uv)e^{i\beta_T^+(uv)}, \sum \mu_T^-(uv)e^{i\alpha_T^-(uv)}) \\
 &= (\sum \mu_T^+(p(uv)e^{i\beta_T^+(p(uv))}), \sum \mu_T^-(p(uv)e^{i\alpha_T^-(p(uv))})) \\
 &= (\sum \mu_T^+(wz)e^{i\beta_T^+(wz)}, \sum \mu_T^-(wz)e^{i\alpha_T^-(wz)}) = S_I(G_2).
 \end{aligned}$$

□

Remark 3.1. The converse of above Theorem 3.1 need not be true for both neighbourly edge improper ivcf graph and highly edge improper ivcf graph.

Fact 3.1. If neighbourly edge improper ivcf graph(highly edge improper ivcf graph) are weak isomorphic then their order are same.

Remark 3.2. The converse of the above fact is need not be true.

Fact 3.2. If neighbourly edge improper ivcf graph(highly edge improper ivcf graph)are co-weak isomorphic then their size are same.

Remark 3.3. The converse of the above fact is need not be true.

Theorem 3.2. If G_1 and G_2 are neighbourly(highly) edge improper ivcf graphs which are isomorphic then the degrees of corresponding vertices u and $p(u)$ are preserved.

Proof. If $p: G_1 \rightarrow G_2$ is an isomorphism between neighbourly edge improper ivcf graph then

$$\begin{aligned}
 \mu_T^+(uv)e^{i\beta_T^+(uv)} &= \mu_T^+(p(uv))e^{i\beta_T^+(p(uv))} \text{ and} \\
 \mu_T^-(uv)e^{i\alpha_T^-(uv)} &= \mu_T^-(p(uv))e^{i\alpha_T^-(p(uv))}.
 \end{aligned}$$

$$d_I^+(u) = \sum \mu_T^+(uv)e^{\sum i\beta_T^+(uv)} = \sum \mu_T^+(p(uv))e^{\sum i\beta_T^+(p(uv))} = d_I^+(p(u)).$$

$$d_I^-(u) = \sum \mu_T^-(uv)e^{\sum i\alpha_T^-(uv)} = \sum \mu_T^-(p(uv))e^{\sum i\alpha_T^-(p(uv))} = d_I^-(p(u)).$$

Thus the degrees of corresponding vertices of G_1 and G_2 are the same. □

Remark 3.4. The converse of Theorem 3.2 need not be true.

Theorem 3.3. Any two highly edge improper ivcf graphs are isomorphic if and only if their complement are isomorphic.

Proof. Let G_1 and G_2 be two highly edge improper ivcf graphs.

Assume that G_1 and G_2 are isomorphic. There exists a bijective map $p: F_1 \rightarrow F_2$ satisfying

$$\begin{aligned}\mu_S^+(x)e^{i\beta_S^+(x)} &= \mu_S^+(p(x))e^{i\beta_S^+(p(x))}, \mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}, \\ \mu_T^+(xy)e^{i\beta_T^+(xy)} &= \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}, \mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}, \\ \bar{\mu}_T^+(xy)e^{i\bar{\beta}_T^+(xy)} &= \mu_S^+(x)e^{i\beta_S^+(x)} \wedge \mu_S^+(y)e^{i\beta_S^+(y)} - \mu_T^+(xy)e^{i\beta_T^+(xy)} \\ &= \mu_S^+(p(x))e^{i\beta_S^+(p(x))} \wedge \mu_S^+(p(y))e^{i\beta_S^+(p(y))} - \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))} \\ \bar{\mu}_T^-(xy)e^{i\bar{\alpha}_T^-(xy)} &= \mu_S^-(x)e^{i\alpha_S^-(x)} \wedge \mu_S^-(y)e^{i\alpha_S^-(y)} - \mu_T^-(xy)e^{i\alpha_T^-(xy)} \\ &= \mu_S^-(p(x))e^{i\alpha_S^-(p(x))} \wedge \mu_S^-(p(y))e^{i\alpha_S^-(p(y))} - \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}\end{aligned}$$

Hence $\bar{G}_1 \cong \bar{G}_2$.

Conversely assume $\bar{G}_1 \cong \bar{G}_2$

There exists a bijective map $p: F_1 \rightarrow F_2$ satisfying

$$\begin{aligned}\mu_S^+(x)e^{i\beta_S^+(x)} &= \mu_S^+(p(x))e^{i\beta_S^+(p(x))}, \mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}, \\ \bar{\mu}_T^+(xy)e^{i\bar{\beta}_T^+(xy)} &= \bar{\mu}_T^+(p(xy))e^{i\bar{\beta}_T^+(p(xy))}, \bar{\mu}_T^-(xy)e^{i\bar{\alpha}_T^-(xy)} = \bar{\mu}_T^-(p(xy))e^{i\bar{\alpha}_T^-(p(xy))}\end{aligned}$$

using definition of complement

$$\begin{aligned}\bar{\mu}_T^+(xy)e^{i\bar{\beta}_T^+(xy)} &= \mu_S^+(x)e^{i\beta_S^+(x)} \wedge \mu_S^+(y)e^{i\beta_S^+(y)} - \mu_T^+(xy)e^{i\beta_T^+(xy)} \\ \bar{\mu}_T^-(xy)e^{i\bar{\alpha}_T^-(xy)} &= \mu_S^-(x)e^{i\alpha_S^-(x)} \wedge \mu_S^-(y)e^{i\alpha_S^-(y)} - \mu_T^-(xy)e^{i\alpha_T^-(xy)}\end{aligned}$$

using the above equations

$$\mu_T^+(xy)e^{i\beta_T^+(xy)} = \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}, \mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}$$

Hence $G_1 \cong G_2$. □

Theorem 3.4. Let G_1 and G_2 be two highly edge improper ivcf graphs. If G_1 is weak isomorphism with G_2 , then $\bar{G}_1$ is weak isomorphic with $\bar{G}_2$.

Proof. If p is weak isomorphism between G_1 and G_2 then $p: F_1 \rightarrow F_2$ is a bijective function such that, $\mu_S^+(x)e^{i\beta_S^+(x)} = \mu_S^+(p(x))e^{i\beta_S^+(p(x))}$, $\mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))}$

$$\mu_T^+(xy)e^{i\beta_T^+(xy)} \leq \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}, \mu_T^-(xy)e^{i\alpha_T^-(xy)} \leq \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}$$

As, $p^{-1}: F_2 \rightarrow F_1$ is also bijective, for every $w \in F_2$, there exists $x \in F_1$ such that $p^{-1}(w) = m_i$

$$\mu_S^+(x)e^{i\beta_S^+(x)} = \mu_S^+(p^{-1}(w))e^{i\beta_S^+(p^{-1}(w))}, \mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p^{-1}(w))e^{i\alpha_S^-(p^{-1}(w))}$$

$$\bar{\mu}_T^+(xy)e^{i\bar{\beta}_T^+(xy)} = \mu_S^+(x)e^{i\alpha_S^-(x)} \wedge \mu_S^+(y)e^{i\alpha_S^-(y)} - \mu_T^+(xy)e^{i\alpha_T^-(xy)}$$

$$\bar{\mu}_T^+(p^{-1}(wz))e^{i\bar{\beta}_T^+(p^{-1}(wz))} \geq \mu_S^+(p(x))e^{i\beta_S^+(p(x))} \wedge \mu_S^+(p(y))e^{i\beta_S^+(p(y))} - \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}$$

$$= \mu_S^+(w)e^{i\beta_S^+(w)} \wedge \mu_S^+(z)e^{i\beta_S^+(z)} - \mu_S^+(wz)e^{i\beta_S^+(wz)} = \bar{\mu}_T^+(wz)e^{i\beta_T^+(wz)}$$

$$\bar{\mu}_T^+(wz)e^{i\beta_T^+(wz)} \leq \bar{\mu}_T^+(p^{-1}(wz))e^{i\bar{\beta}_T^+(p^{-1}(wz))}$$

$$\bar{\mu}_T^-(xy)e^{i\bar{\alpha}_T^-(xy)} = \mu_S^-(x)e^{i\alpha_S^-(x)} \wedge \mu_S^-(y)e^{i\alpha_S^-(y)} - \mu_S^-(xy)e^{i\alpha_S^-(xy)}$$

$$\bar{\mu}_T^-(p^{-1}(wz))e^{i\bar{\alpha}_T^-(p^{-1}(wz))} \leq \mu_S^-(p(x))e^{i\alpha_S^-(p(x))} \wedge \mu_S^-(p(y))e^{i\alpha_S^-(p(y))} - \mu_S^-(p(xy))e^{i\alpha_S^-(p(xy))}$$

$$= \mu_S^-(w)e^{i\alpha_S^-(w)} \wedge \mu_S^-(z)e^{i\alpha_S^-(z)} - \mu_S^-(wz)e^{i\alpha_S^-(wz)} = \bar{\mu}_T^-(wz)e^{i\alpha_T^-(wz)}$$

$$\bar{\mu}_T^-(wz)e^{i\alpha_T^-(wz)} \geq \bar{\mu}_T^-(p^{-1}(wz))e^{i\bar{\alpha}_T^-(p^{-1}(wz))}.$$

So, $p^{-1}: F_1 \rightarrow F_2$ is weak isomorphism between $\bar{G}_1$ and $\bar{G}_2$. $\square$

4 μ -Complement of Highly Edge improper Interval-valued complex Fuzzy Graphs

Definition 4.1. Let $G: (S, T)$ be a ivcf graph. The μ -complement of G is defined as $G^\mu: (S, T^\mu)$ where $T^\mu = (\mu_T^{+\mu}, \mu_T^{-\mu})$

$$\mu_T^{+\mu}(xy)e^{i\beta_T^{+\mu}(xy)} = \begin{cases} \mu_S^+(x)e^{i\beta_S^+(x)} \wedge \mu_S^+(y)e^{i\beta_S^+(y)} & \mu_T^+(xy)e^{i\beta_T^+(xy)} > 0 \\ -\mu_T^+(xy)e^{i\beta_T^+(xy)} & \mu_T^+(xy)e^{i\beta_T^+(xy)} = 0 \\ 0 & \mu_T^+(xy)e^{i\beta_T^+(xy)} = 0 \end{cases}$$

$$\mu_T^{-\mu}(xy)e^{i\alpha_T^{-\mu}(xy)} = \begin{cases} \mu_S^-(x)e^{i\alpha_S^-(x)} \wedge \mu_S^-(y)e^{i\alpha_S^-(y)} & \mu_T^-(xy)e^{i\alpha_T^-(xy)} > 0 \\ -\mu_T^-(xy)e^{i\alpha_T^-(xy)} & \mu_T^-(xy)e^{i\alpha_T^-(xy)} = 0 \\ 0 & \mu_T^-(xy)e^{i\alpha_T^-(xy)} = 0. \end{cases}$$

Theorem 4.1. Let G_1 and G_2 be two highly edge improper isomorphic ivcf graphs. Then μ -complement of G_1 and G_2 are also isomorphic. But the μ complements need not be highly edge improper ivcf graphs.

Proof. Assume that G_1 and G_2 are isomorphic. There exists bijective map $p: F_1 \rightarrow F_2$ satisfying

$$\begin{aligned}\mu_S^+(x)e^{i\beta_S^+(x)} &= \mu_S^+(p(x))e^{i\beta_S^+(p(x))}, \mu_S^-(x)e^{i\alpha_S^-(x)} = \mu_S^-(p(x))e^{i\alpha_S^-(p(x))} \\ \mu_T^+(xy)e^{i\beta_T^+(xy)} &= \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}, \mu_T^-(xy)e^{i\alpha_T^-(xy)} = \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}\end{aligned}$$

By definition of μ -complement,

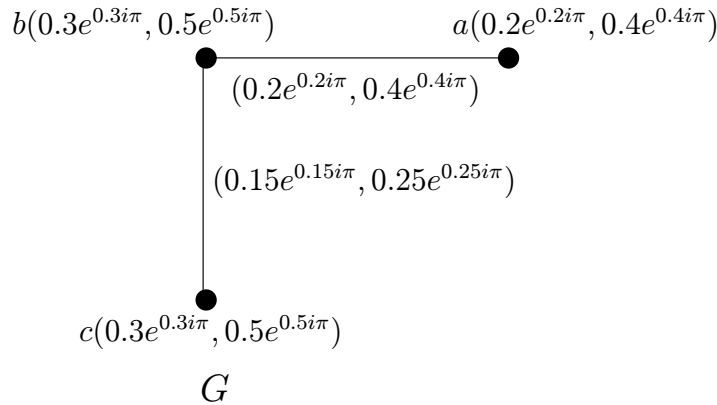
$$\begin{aligned}\mu_T^{+\mu}(xy)e^{i\beta_T^{+\mu}(xy)} &= \mu_S^+(x)e^{i\beta_S^+(x)} \wedge \mu_S^+(y)e^{i\beta_S^+(y)} - \mu_T^+(xy)e^{i\beta_T^+(xy)} \\ &= \mu_T^+(p(x))e^{i\beta_T^+(p(x))} \wedge \mu_T^+(p(y))e^{i\beta_T^+(p(y))} - \mu_T^+(p(xy))e^{i\beta_T^+(p(xy))}\end{aligned}$$

$$\begin{aligned}\mu_T^{-\mu}(xy)e^{i\alpha_T^{-\mu}(xy)} &= \mu_S^-(x)e^{i\alpha_S^-(x)} \wedge \mu_S^-(y)e^{i\alpha_S^-(y)} - \mu_T^-(xy)e^{i\alpha_T^-(xy)} \\ &= \mu_T^-(p(x))e^{i\alpha_T^-(p(x))} \wedge \mu_T^-(p(y))e^{i\alpha_T^-(p(y))} - \mu_T^-(p(xy))e^{i\alpha_T^-(p(xy))}\end{aligned}$$

Hence $\bar{G}_1 \sim \bar{G}_2$. $\square$

Definition 4.2. An ivcf graph G is said to be self μ -complementary if $G \sim G^\mu$.

Example 4.1. An ivcf graph is considered on $G^*(F, I)$.



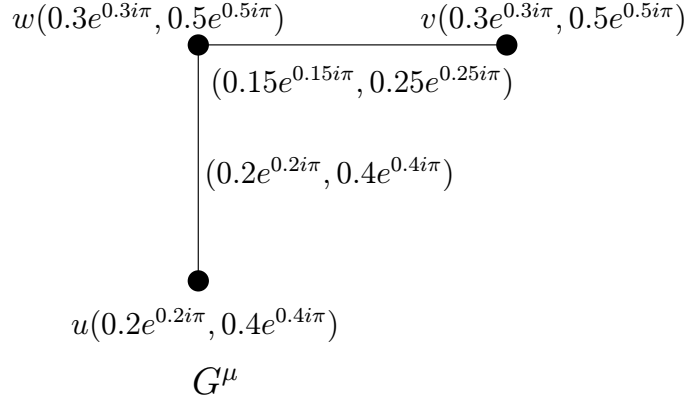


Figure 4: Self μ -complementary

In Figure 4, the two graphs are isomorphic. Hence G is Self μ -complementary ivcf graph.

Theorem 4.2. Let G be self μ -complementary highly edge improper ivcf graph, then

$$\sum \mu_T^+(uv)e^{i\beta_T^+(uv)} = \frac{1}{2} \sum \mu_S^+(u)e^{i\beta_S^+(u)} \wedge \mu_S^+(v)e^{i\beta_S^+(v)} \text{ and } \sum \mu_T^-(uv)e^{i\alpha_T^-(uv)} = \frac{1}{2} \sum \mu_S^-(u)e^{i\alpha_S^-(u)} \wedge \mu_S^-(v)e^{i\alpha_S^-(v)}.$$

Proof. Let G be self μ -complementary highly edge improper ivcf graph. Since $G \sim G^\mu$, there exists a bijective map $p: V \rightarrow V$ such that

$$\mu_S^+(u)e^{i\beta_S^+(u)} = \mu_S^{+\mu}(p(u))e^{i\beta_S^{+\mu}(p(u))} = \mu_S^+(p(u))e^{i\beta_S^+(p(u))} \text{ and } \mu_S^-(u)e^{i\alpha_S^-(u)} = \mu_S^{-\mu}(p(u))e^{i\alpha_S^{-\mu}(p(u))} = \mu_S^-(p(u))e^{i\alpha_S^-(p(u))}$$

$$\mu_T^+(uv)e^{i\beta_T^+(uv)} = \mu_T^{+\mu}(p(uv))e^{i\beta_T^{+\mu}(p(uv))} \text{ and } \mu_T^-(uv)e^{i\alpha_T^-(uv)} = \mu_T^{-\mu}(p(uv))e^{i\alpha_T^{-\mu}(p(uv))}$$

By definition of μ -complement, we have

$$\mu_T^{+\mu}(xy)e^{i\beta_T^{+\mu}(xy)} = \begin{cases} \mu_S^+(x)e^{i\beta_S^+(x)} \wedge \mu_S^+(y)e^{i\beta_S^+(y)} & \mu_T^+(xy)e^{i\beta_T^+(xy)} > 0 \\ -\mu_T^+(xy)e^{i\beta_T^+(xy)} & \mu_T^+(xy)e^{i\beta_T^+(xy)} = 0 \\ 0 & \mu_T^+(xy)e^{i\beta_T^+(xy)} = 0 \end{cases}$$

$$\mu_T^{+\mu}(p(uv))e^{i\beta_T^{+\mu}(p(uv))} = \mu_S^+(p(u))e^{i\beta_S^+(p(u))} \wedge \mu_S^+(p(v))e^{i\beta_S^+(p(v))} - \mu_T^+(p(uv))e^{i\beta_T^+(p(uv))}$$

$$\begin{aligned}
 & \text{Now, } \mu_T^+(uv)e^{i\beta_T^+(uv)} \\
 &= \mu_S^+(p(u))e^{i\beta_S^+(p(u))} \wedge \mu_S^+(p(v))e^{i\beta_S^+(p(v))} - \mu_T^+(p(uv))e^{i\beta_T^+(p(uv))} \\
 & \mu_T^+(uv)e^{i\beta_T^+(uv)} + \mu_T^+(p(uv))e^{i\beta_T^+(p(uv))} = \mu_S^+(p(u))e^{i\beta_S^+(p(u))} \wedge \mu_S^+(p(v))e^{i\beta_S^+(p(v))} \\
 & \Rightarrow 2\mu_T^+(uv)e^{i\beta_T^+(uv)} = \mu_S^+(u)e^{i\beta_S^+(u)} \wedge \mu_S^+(v)e^{i\beta_S^+(v)} \\
 & \Rightarrow 2 \sum \mu_T^+(uv)e^{i\beta_T^+(uv)} = \sum \mu_S^+(u)e^{i\beta_S^+(u)} \wedge \mu_S^+(v)e^{i\beta_S^+(v)} \\
 & \Rightarrow \sum \mu_T^+(uv)e^{i\beta_T^+(uv)} = \frac{1}{2} \sum \mu_S^+(u)e^{i\beta_S^+(u)} \wedge \mu_S^+(v)e^{i\beta_S^+(v)}
 \end{aligned}$$

Similarly we can show that $\sum \mu_T^-(uv)e^{i\alpha_T^-(uv)} = \frac{1}{2} \sum \mu_S^-(u)e^{i\alpha_S^-(u)} \wedge \mu_S^-(v)e^{i\alpha_S^-(v)}$. □

Remark 4.1. If G is highly edge improper ivcf graph with $\sum \mu_T^+(uv)e^{i\beta_T^+(uv)} = \frac{1}{2} \sum \mu_S^+(u)e^{i\beta_S^+(u)} \wedge \mu_S^+(v)e^{i\beta_S^+(v)}$ and $\sum \mu_T^-(uv)e^{i\alpha_T^-(uv)} = \frac{1}{2} \sum \mu_S^-(u)e^{i\alpha_S^-(u)} \wedge \mu_S^-(v)e^{i\alpha_S^-(v)}$ then G need not be self μ -complementary.

5 Conclusion

In this paper we have introduced the notions of isomorphic, weak isomorphic, co-weak isomorphic of neighbourly(highly) edge improper ivcf graphs. In future we study the properties for some standard graphs and have an idea to extend the concept of μ -complement of neighbourly(highly) edge improper ivcf graphs. Though isomorphism exists on several fuzzy graphs isomorphic on interval-valued complex fuzzy graph is limited only on the case which involves time phenomena.

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Topology on BP-Algebras

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Abstract

In this paper we will define topological BP-algebras and find some properties of these algebras. Also we show how to connect certain topologies with ideals of BP-algebras. Furthermore, we derived certain standard properties using topological concepts and we give a characterization of topological BP-algebras in terms of neighborhoods.

Keywords: BP-algebras, Topology, Topological BP-algebras Hausdroff.

2010 AMS subject classification: 54A05, 46H05, 22A26, 06F35, 03G25.

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1. Introduction

The two classes of abstract algebras namely BCK-algebras and BCI-algebras are introduced by Imai Y and Iseki K [5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Hu Q.P and Li X [4] introduced a wide class of abstract algebras: BCH-algebras. Also it is known that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Ahn S.S and Han J.S [1] introduced the concepts of BP-algebras and they discussed some relations with several algebras. Moreover they have discussed quadratic BP-algebras and shown that the quadratic BP-algebras is equivalent to several quadratic algebras. Alo R and Deeba E [3] attempted to study the topological concepts of the BCK-structure. Theories of topological groups, topological rings and topological modules are well known and still investigated by many Mathematicians. Topological universal algebraic structures have been studied by some authors.

Motivated by this, in this paper, we study the connection between BP-Algebras and topology. We will define topological BP-algebras (TBP-algebras) and find some properties of TBP-algebras. Also we show how to connect certain topologies with ideals of BP-algebras. Furthermore, we derive certain standard properties using topological concepts and we give a characterization of topological BP-algebras in terms of neighborhoods. Finally, we show that a TBP-algebra B is Hausdroff if and only if $\{0\}$ is closed in B.

2. Preliminaries

Definition 2.1 [1]. Let X be a set with a binary operation $*$ and a constant 0 .

Then $(X, *, 0)$ is called a BP-algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * (x * y) = y$
3. $(x * z) * (y * z) = x * y$ for any $x, y, z \in X$.

Example 2.2 [1]. Let $X = \{0, a, b, c\}$ be a set with following table

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 1

Then $(X, *, 0)$ is a BP-algebra.

Theorem 2.3 [1]. If $(X, *, 0)$ is a BP-algebra, then the following results hold: for any $x, y \in X$

1. $x * 0 = x$.
2. $x * y = 0$ implies $y * x = 0$.
3. $0 * x = 0 * y$ implies $x = y$.
4. $0 * x = y$ implies $0 * y = x$.
5. $0 * x = x$ implies $x * y = y * x$

Proposition 2.4 [1]. If $(X, *, 0)$ is a BP-algebra with $(x * y) * z = x * (z * y)$ for any $x, y, z \in X$, then $0 * x = x$ for any $x \in X$.

Theorem 2.5 [1]. If $(X, *, 0)$ is a BP-algebra with $x * y = 0$ and $y * x = 0$, then $x = y$.

Definition 2.6 [10]. Let S be a non-empty subset of a BP-algebra X , then S is called BP-subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2.7 [10]. Let $(X, *, 0)$ be a BP-algebra and I be a non-empty subset of X . Then I is called an ideal of X , if it satisfies the following conditions.

1. $0 \in I$.
2. $x * y \in I$ and $y \in I \Rightarrow x \in I$.

Definition 2.8 [9]. Let X be a non-empty set. A Collection τ of subsets of X is called a Topology if it satisfies the following conditions.

1. ϕ and X is in τ .
2. Arbitrary union of the members of any finite sub collection of τ is in τ .
3. Intersection of the members of any finite sub collection of τ is in τ .

The tuple (X, τ) is called a topological space.

Definition 2.9 [9]. Let (X, τ) is a Topological space, we say that a subset U of X is an open set of X if U belongs to the collection τ .

Definition 2.10 [9]. The complement of an open set is called a closed set.

Definition 2.11 [9]. A subset A of X is said to be neighborhood of $x \in A$ if there exist an open set U such that $x \in U \subseteq A$.

Definition 2.12[9]. Let (X, τ) be a topological space. (X, τ) is called a T_i -space $i \in \{0, 1, 2\}$ if it satisfies the following axioms.

T_0 : For every $x, y \in X$ and $x \neq y$, there exist at least one open neighbourhood U that contains one point of the pair (either x or y) excluding the other point.

T_1 : For each $x, y \in X$ and $x \neq y$, there exist open neighbourhoods U_1 and U_2 such that U_1 contains x but not y and U_2 contains y but not x .

T_2 (Hausdroff): For each $x, y \in X$ and $x \neq y$, both have disjoint open neighbourhoods U, V such that $x \in U$ and $y \in V$

Definition 2.13 [9]. Let (X, τ) be a topological space and $A \subseteq X$.

Define $\tau_A = \{ U \cap A / U \in \tau \}$. Then (A, τ_A) is a topological space and the topology τ_A is called the subspace topology on A .

Definition 2.14 [9]. Let (X, τ) be a topological space and $A \subseteq X$. Let $a \in A$ is called an interior point of A , if there exist an open set U of ‘ a ’ such that $U \subseteq A$.

Remark 2.15 ([9]). 1. Restriction function of a continuous function is a continuous.
2. Every T_2 -space is a T_1 -space. Furthermore, every T_1 -space is a T_0 -space.

3. Topological BP-algebras

Definition 3.1: Let B be a BP-algebra and X, Y be any two non-empty subsets of B . We define a subset $X * Y$ of B as follows. $X * Y = \{a * b / a \in X, b \in Y\}$.

Example 3.2: Let $B = (\{0, p, q, r\}, *, 0)$ be a BP-algebra with the binary operation $*$ defined by the Cayley table.

$*$	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	p	q	0

Table 2

Let $X = \{0, p\}$ and $Y = \{0, q\}$ be two subsets of X ,

$$X * Y = \{0 * 0, 0 * q, p * 0, p * q\}$$

$$= \{0, p, p, r\}$$

$$= \{0, p, r\} \subseteq B.$$

Topology on BP-Algebras

Definition 3.3: Let B be a BP-algebra. Let T be the collection of subsets of B , T is said to be a topology on B if the following conditions are satisfied.

1. $B, \emptyset \in T$
2. Arbitrary union of members of T is in T .
3. Finite intersection of members of T is in T .

Definition 3.4: Let $(B, *, 0)$ be a BP-algebra and T be a topology on B . Then

$(B, *, T)$ is called a topological BP-algebra if the operation $*$ is continuous.

Equivalently, for $a, b \in B$ if there exist an open set U of $a*b$ then there exist two open sets X and Y of a and b respectively such that $X * Y \subseteq U$.

Example 3.5: Let $B = (\{0, p, q, r\}, *, 0)$ be a BP-algebra with the binary operation $*$ defined by as on example 3.2.

Define a topology $T = \{B, \emptyset, \{0\}, \{0, p, q\}\}$

Let $0, p \in B$ and $U = \{0, p, q\}$

Now $a * b = p * 0 = p \in U$.

Then there exist an open set $X = \{0, p, q\}$, $Y = \{0\}$ if p and 0 respectively such that $X * Y = \{0 * 0, p * 0, q * 0\}$

$$= \{0, p, q\} \subseteq U.$$

Therefore every open set U for $a * b$ then there exist an open sets X and Y of a and b respectively such that $X * Y \subseteq U$ for all $a, b \in B$.

Hence $(B, *, T)$ is a Topological BP-algebra and it is denoted by TBP-algebra.

Example 3.6: Let $B = (\{0, p, q, r\}, *, 0)$ be a BP-algebra with the binary operation $*$ defined in Example 3.2.

Define a topology $T = \{B, \emptyset, \{q\}, \{r\}\}$

Let $0, q \in B$, $q * 0 = q = \{q\} = U$.

Let $X = \{q\}$ and $Y = \{B\}$, then $X * Y = \{q * 0, q * p, q * q, q * r\} = \{q, r, 0, p\} \not\subseteq U$.

Hence $(B, *, T)$ is not a Topological BP-algebra.

Remark 3.7: 1. The member of T is known as open sets in B .

2. An open set of an element $a \in B$ is a member of T containing a .

3. The complement of $C \in T$ that is $B \setminus C$ is called a closed set in B .

Example 3.8: Let $B = (\{0, p, q, r\}, *, 0)$ be a BP-algebra with the binary operation $*$ defined in Example 3.5. Define a topology $T = \{B, \emptyset, \{0, p, q\}, \{r\}\}$.

Here $\{0, p, q\}$ is open in B . Now $\{r\}$ is closed in B . An open set of $p \in B$ is $\{0, p, q\}$.

Theorem 3.9: Let $(B, *, T)$ be a TBP-algebra. If $\{0\}$ is an open set in TBP-algebra, then T is a discrete topology in TBP-algebra.

Proof. Let $(B, *, T)$ be a TBP-algebra. For every $a \in B$, $a*a = 0$.

Since $\{0\}$ is an open in TBP-algebra, there exist two neighborhoods X and Y of a such that $X * Y \subseteq \{0\} \Rightarrow X * Y = \{0\}$ -----(1)

Let $U = X \cap Y$. Clearly $U * U = \{0\}$ by (1)

Since $a * b = 0$ and $b * a = 0 \Rightarrow a = b$, which implies $U = \{a\}$. Therefore T is discrete.

4. Properties of topological BP-algebras

Definition 4.1: Let $(B, *, T)$ be a TBP-algebra. Then B is said to be T_0 -space

BP-algebra if it satisfies the following axiom:

For every $a, b \in B$ and $a \neq b$, there exist at least one open neighborhood U that contains one point of the pair (either a or b) excluding the other point.

Definition 4.2: Let $(B, *, T)$ be a TBP-algebra. Then B is said to be T_1 -space

BP-algebra if it satisfies the axiom:

For each $a, b \in B$ and $a \neq b$, there exist an open neighborhoods X_1 and

X_2 such that X_1 contains a but not b and X_2 contains b but not a .

Definition 4.3: Let $(B, *, T)$ be a TBP-algebra. Then B is said to be T_2 -space

(Hausdroff) BP-algebra if it satisfies the axiom: For each $a, b \in B$ and $a \neq b$, both have disjoint open neighborhoods X, Y such that $a \in X$ and $b \in Y$.

Example 4.4: Consider the TBP-algebra $(B = \{0, p, q\}, *, T)$ with the following Cayley table.

$*$	0	p	q
0	0	q	p
p	p	0	q
q	q	p	0

Table 3

Consider the topology $T = \{\emptyset, B, \{0, p\}, \{q\}, \{q, 0\}, \{0\}\}$.

Then $(B, *, T)$ is called a T_0 -space BP-algebra.

Example 4.5: From the TBP-algebra given in Example 3.13.

Consider the topology T on B is given by $T = \{\emptyset, B, \{0, p\}, \{q\}, \{q, 0\}, \{0\}, \{p\},$

$\{q, p\}\}$. Here T satisfies the T_1 and T_2 axioms.

Thus $(B, *, T)$ is both T_1 -space BP-algebra and T_2 -space BP-algebra.

Theorem 4.6: Let B be a TBP-algebra and $\{0\}$ is closed set on B if and only if B is Hausdroff.

Proof. Let $(B, *, T)$ be a TBP-algebra.

Let $a, b \in B$. Assume that a and b are different in B , then $a * b \neq 0$ or $b * a \neq 0$

We can assume that $a * b \neq 0$, then there exist some open sets X and Y of a and b respectively such that $X * Y \subseteq B \setminus \{0\}$, since $\{0\}$ is closed and B is a TBP-algebra.

Suppose, let $a \in X \cap Y$. Then $a \in X$ and $a \in Y$ implies $a * a = 0 \in X * Y$.

This contradicts to the fact that $X * Y \subseteq B \setminus \{0\} \Rightarrow X \cap Y = \emptyset$.

Therefore B is Hausdroff.

Conversely, Assume that B is Hausdroff

Claim: $\{0\}$ is closed in TBP-algebra

We want to prove $B \setminus \{0\}$, there exist two neighborhoods X and Y of a and 0 respectively such that $X \cap Y = \emptyset$.

Since B is Hausdroff, then $X \subseteq B \setminus \{0\}$

$\Rightarrow B \setminus \{0\}$ is an open so that $\{0\}$ is closed in TBP-algebra.

Theorem 4.7: Let B be a TBP-algebra and $\{0\}$ be closed, then $\bigcap N_0 = \{0\}$

where N_0 is the neighborhood system of 0 .

Proof. Let B be a TBP-algebra

Since $\{0\}$ is closed. By the above theorem 3.15, B is Hausdroff.

Given an element $a \in B \setminus \{0\}$, 0 has a neighborhood X such that $a \notin X$ and

$a \notin \bigcap N_0$. Hence $\bigcap N_0 = \{0\}$.

Definition 4.8: Let $(B, *, T)$ be a TBP-algebra and I be a non-empty subset of

B . Then I is called an ideal of B if it satisfies the following conditions.

1. $0 \in I$.
2. If $a * b \in I$ and $b \in I \Rightarrow a \in I$.

Example 4.9: Let $(B = \{0, p, q, r\}, *, T)$ be a TBP-algebra with the binary operation $*$ defined by the example 3.2. $I = \{0\}$ is an ideal.

Theorem 4.10: Let B be a TBP-algebra. If D is an open set in B which is also a BP-ideal of B , then it is closed set in B .

Proof. Let D be a BP-ideal which is an open set in B and let $a \in B \setminus D$. Then there exist a neighborhood X of a such that $X * X \subseteq D$.

Since $a * a = 0 \in D$ and D is open.

Claim: $X \subseteq B \setminus D$.

If $X \not\subseteq B \setminus D$, then there exists $b \in X \cap D$.

Note that $c * b \in X * X \subseteq D$ for all $c \in X$

Since $b \in D$ and D is a BP-ideal, it follows that $c \in D$.

Which show that $X \subseteq D$, $\Rightarrow$ a contradiction. Hence D is closed.

Definition 4.11: Let $(B, *, T)$ be a TBP-algebra. Let $A \subseteq B$, $x \in A$ is called an interior point of A , if there exist a neighborhood X of x contained in A .

Example 4.12: Let $(B = \{0, p, q\}, *, T)$ be a TBP-algebra with the binary operation $*$ defined by the example 4.4.

Define a topology $T = \{B, \emptyset, \{p, 0\}, \{q\}\}$.

Let $A = \{p, q\}$. Here, r is an interior point of A .

Theorem 4.13: Let A be a BP-ideal of a TBP-algebra B . If 0 is an interior point of A , then A is an open.

Proof. Let $a \in A$.

Since $a * a = 0 \in A$ and 0 is an interior point of A , there exist a neighborhood X of 0 which is contained in A .

Then there exist neighborhoods G and H of a such that $G * H \subseteq X \subseteq A$

On the other hand for every $b \in G$, $b * a \in G * H \subseteq A$.

Since A is a BP-ideal and $a \in A$, it follows that $b \in A$ so that $a \in G \subseteq A$

Hence A is open.

Definition 4.14: Let B and C be a TBP-algebra. A map $f: B \rightarrow C$ is continuous.

If for every $a \in B$ and any open set Y of $f(a)$ there exist an open set X containing a such that $f(X) \subseteq Y$.

Example 4.15: Consider a TBP-algebra $(B = \{0, p, q\}, *, T)$ as an example 3.13.

Define a topology $T = \{B, \emptyset, \{0, p\}\}$.

Define a function $f: B \rightarrow B$ by $f(a) = a$ for all $a \in B$.

Clearly, f is continuous.

Notation 4.16: Let B be TBP-algebra and $a, b \in B$, then $\{b \in A / b \equiv a \pmod I\}$

$$= \{b \in A / a * b, b * a \in I\}.$$

Theorem 4.17: Let B be a TBP-algebra. If I is an (closed) open ideal, then for each $a \in B$, then $\{b \in B / b \equiv a \pmod I\}$ is closed (open).

Proof. Define a map $G_a: B \rightarrow B$ such that $G_a(b) = a * b$.

Since B is TBP-algebra, then G_a is continuous.

Define a map $H_a: B \rightarrow B$ such that $H_a(b) = b * a$.

Since B is a TBP-algebra, then H_a is continuous.

$$G_a^{-1}(I) \cap H_a^{-1}(I) = \{b \in B / a * b, b * a \in I\}.$$

$\Rightarrow$ Inverse image of an (closed) open set is an closed (open)

$\Rightarrow G_a^{-1}(I) \cap H_a^{-1}(I)$ is an closed (open)

Therefore $\{b \in B / b \equiv a \pmod I\}$ is an closed (open).

Theorem 4.18: Suppose that $\{0\}$ is closed (open), then $\{a\}$ is closed (open) for all $x \in B$.

Proof. Define a map $g: B \times B \rightarrow B$ such that $g(x, y) = x * y$.

Since B is a TBP-algebra ' $*$ ' is continuous.

which implies g is continuous.

Define a map $h: B \times B \times B \rightarrow B \times B$ such that $h(x, y, z) = (x * y, y * z)$

which implies h is continuous. (Since g is continuous and Remark 2.15)

Since $\{0\}$ is closed, then $\{(0, 0)\} \in B \times B$ is closed.

Let $x \in B$ and let $f: B \rightarrow B \times B$ such that $f(y) = h(x, y, x) = (x * y, y * x)$

Here f is the restriction of h to $\{x\}$ (since h to $\{x\} \times B \times \{x\}$)

and hence f is continuous (since h is continuous and Remark 2.15)

$f^{-1}(0, 0) = \{y / x * y = 0 \text{ and } y * x = 0\} = \{x\}$. Thus $\{x\}$ is also closed.

Similarly, for the other case where $\{0\}$ is an open.

Theorem 4.19: B is T_1 -space BP-algebra if and only if B is T_2 -space BP-algebra.

Proof. Let B be a T_1 -space BP-algebra.

Let $a, b \in A$ and $a \neq b$, there exist an open neighborhoods X_1 and X_2 such that

X_1 contains a but not b and X_2 contains b but not a .

$\Rightarrow \{a\}$ is closed, for all $a \in B$

$\Rightarrow \{0\}$ is closed.

By theorem 3.15, we have B is a TBP-algebra.

Conversely, Let B be a T_2 -space BP-algebra.

Let $a, b \in B$ and $a \neq b$ both have disjoint open neighborhoods X and Y such that $a \in X$ and $b \in Y$ and $a \notin Y$ and $b \notin X \implies B$ is T_1 -space BP-algebra.

Theorem 4.20: Let $(B, *, T)$ be a TBP-algebra. If S is a subset of B such that for each $a, b \in S$, $a * b \in S$, then $(S, *, T_S)$ is a TBP-algebra, where T_S is the subspace topology from B .

Proof. Let $a, b \in S \implies a * b \in S$

Let U be an open set of $a * b$ in B .

$\implies U \cap S$ is an open set for $a * b$ in S .

Since B is a TBP-algebra, there exist an open sets X and Y of a and b respectively such that $X * Y \in U$

$\implies$ There exist an open sets $X \cap S$ and $Y \cap S$ of a and b in S such that

$(X \cap S) * (Y \cap S) \subseteq U \cap S \implies (S, *, T_S)$ is a TBP-algebra.

4 Conclusion and Discussion

In this paper, we have introduced the topological concepts on BP-algebra, which is induced by the binary operation $*$ of the BP-algebra and provided some important properties. We have studied the connection of the topologies of the BP-algebra with the ideals. In addition, we have shown that if 0 is an interior point of a BP-ideal of a TBP-algebra B , then the BP-ideal of a TBP-algebra B is open.

Finally, studying a topological properties on other algebras (BF/BH/TM-algebras) is an interesting open problem.

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Some Fixed Point Results in Neutrosophic b-Metric Spaces

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Abstract

Several fixed point theorems in the Neutrosophic b-Metric Spaces are presented in this study. We provide a necessary condition for a sequence to be Cauchy in the Neutrosophic b-Metric Space, which is a significant result. Additionally, we present a large number of fixed point theorems in the Neutrosophic b-Metric Spaces under the well-known contraction conditions.

Keywords: Fixed point; Neutrosophic b-Metric Space.

2020 AMS subject classifications: 54H25; 05C40. ¹

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1 Introduction

Due to the wide range of applications it has in science, economics and game theory, fixed point theory has grown in importance in mathematics. Since the publication of Banach's fundamental finding in 1922, metric fixed point theory has been the subject of active scientific research. It is commonly known that the fixed point theory's main conclusion-the Banach contraction principle-has been applied and developed in a wide range of ways. Bakhtin presented the idea of b-metric space in 1989 as a generalization of metric space and demonstrated a b-metric space equivalent to the Banach contraction principle. Czerwik[5] expanded the results of the b-metric space in 1993. Since then, other articles, including [4], have addressed the generalization of the well-known Banach fixed point theorem in the b-metric space. A countable of the t-norm and contraction condition were used in 2021 by Jeyaraman and Poovaragavan[8] to show that the sequence in generalized fuzzy b-metric spaces is a Cauchy sequence. Since the paper's release, Zadeh's [13] Fuzzy Sets(FSs) theory has a significant impact on all scientific disciplines. Although this idea is crucial for real-life circumstances, there weren't enough timely solutions for some issues. Such issues have recently become the subject of new quests. For these situations, Atanassov[1] pioneered Intuitionistic Fuzzy Sets(IFS). The term "Neutrosophic Set" (NS) refers to a novel interpretation of Smarandache's definition of the term "Classical Set". Interesting and helpful for problem-solving is Atanassov's [1] IFS theory. The fuzzy set described by Zadeh[13] is really extended by the IFS. The IFS theory has been used in a variety of fields recently, including logic programming [2], decision-making issues, medical diagnosis[3] etc. The idea of Contractive mappings in Neutrosophic b-metric spaces in fixed point theorems was used in 2022 by Shakila and Jeyaraman[9].

The main idea of this paper is to describe a contraction for Neutrosophic b-Metric Space and show a very helpful lemma that guarantees that a sequence $\{x_n\}$ is a Cauchy sequence using the idea of a countable extension of the t-norm. The proofs of numerous well-known fixed-point theorems are made simpler using this lemma. Some of them are discussed in the paper's main section.

2 Preliminaries

Proposition 2.1. *Let $(\varsigma_n)_{n \in \mathbb{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n \rightarrow \infty} \varsigma_n = 1$, and let $*$ be a t-norm of Ω -type.*
Then $\lim_{n \rightarrow \infty} *_{i=n}^{\infty} \varsigma_i = \lim_{n \rightarrow \infty} *_{i=1}^{\infty} \varsigma_{n+i} = 1$.

Definition 2.1. [9] A 7-tuple $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is said to be a Neutrosophic b-metric space (NbMS), if Λ is an arbitrary set, $b \geq 1$ is a given real number, $*$ is

Some Fixed Point Results in Neutrosophic b-Metric Spaces

a continuous triangular norm, $\diamond$ is a continuous triangular conorm, Ξ, Ω and Υ are nonempty sets on $\Lambda^2 \times [0, \infty)$ meeting the aforementioned prerequisites: For all $\varsigma, \vartheta, \delta \in \Lambda$ and $\varpi, \mu > 0$:

- (a) $\Xi(\varsigma, \vartheta, \varpi) + \Omega(\varsigma, \vartheta, \varpi) + \Upsilon(\varsigma, \vartheta, \varpi) \leq 3$;
- (b) $0 \leq \Xi(\varsigma, \vartheta, \varpi) \leq 1; 0 \leq \Omega(\varsigma, \vartheta, \varpi) \leq 1; 0 \leq \Upsilon(\varsigma, \vartheta, \varpi) \leq 1$;
- (c) $\Xi(\varsigma, \vartheta, 0) > 0$;
- (d) $\Xi(\varsigma, \vartheta, \varpi) = 1$ iff $\varsigma = \vartheta$;
- (e) $\Xi(\varsigma, \vartheta, \varpi) = \Xi(\vartheta, \varsigma, \varpi)$;
- (f) $\Xi(\varsigma, \delta, b(\varpi + \mu)) \geq * \left(\Xi(\varsigma, \vartheta, \frac{\varpi}{b}), \Xi(\vartheta, \delta, \frac{\mu}{b}) \right)$;
- (g) $\Xi(\varsigma, \vartheta, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (h) $\Omega(\varsigma, \vartheta, 0) < 1$;
- (i) $\Omega(\varsigma, \vartheta, \varpi) = 0$ iff $\varsigma = \vartheta$;
- (j) $\Omega(\varsigma, \vartheta, \varpi) = \Omega(\vartheta, \varsigma, \varpi)$;
- (k) $\Omega(\varsigma, \delta, b(\varpi + \mu)) \leq \diamond \left(\Omega(\varsigma, \vartheta, \frac{\varpi}{b}), \Omega(\vartheta, \delta, \frac{\mu}{b}) \right)$;
- (l) $\Omega(\varsigma, \vartheta, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (m) $\Upsilon(\varsigma, \vartheta, 0) < 1$;
- (n) $\Upsilon(\varsigma, \vartheta, \varpi) = 0$ iff $\varsigma = \vartheta$;
- (o) $\Upsilon(\varsigma, \vartheta, \varpi) = \Upsilon(\vartheta, \varsigma, \varpi)$;
- (p) $\Upsilon(\varsigma, \delta, b(\varpi + \mu)) \leq \diamond \left(\Upsilon(\varsigma, \vartheta, \frac{\varpi}{b}), \Upsilon(\vartheta, \delta, \frac{\mu}{b}) \right)$;
- (q) $\Upsilon(\varsigma, \vartheta, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;

Here, $\Xi(\varsigma, \vartheta, \varpi)$, $\Omega(\varsigma, \vartheta, \varpi)$ and $\Upsilon(\varsigma, \vartheta, \varpi)$ denote the degree of nearness, the degree of non-nearness and the degree of neutralness between ς and ϑ with respect to ϖ respectively.

Since a neutrosophic b-metric is a neutrosophic metric when $b = 1$, the class of NbMS is essentially bigger than that of neutrosophic metric spaces.

The following illustration demonstrates that a neutrosophic b-metric space on Λ does not necessarily have to be a neutrosophic metric space on Λ .

Example 2.1. [6] Let $\Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-|\varsigma-\vartheta|^\theta}{\varpi}}$, $\Omega(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{-|\varsigma-\vartheta|^\theta}{\varpi}}$, $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{|\varsigma-\vartheta|^\theta}{\varpi}}$ where $\theta > 1$. Then Ξ, Ω, Υ is a neutrosophic b-metric with $b = 2^{\theta-1}$. For $\theta = 2$, it is clear from the prior example that $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is not a neutrosophic metric space.

Example 2.2. [6] Let $\Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-d(\varsigma, \vartheta)}{\varpi}}$ or $\Xi(\varsigma, \vartheta, \varpi) = \frac{\varpi}{\varpi + d(\varsigma, \vartheta)}$, $\Omega(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{-d(\varsigma, \vartheta)}{\varpi}}$ or $\Omega(\varsigma, \vartheta, \varpi) = 1 - \frac{\varpi}{\varpi + d(\varsigma, \vartheta)}$ and $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{d(\varsigma, \vartheta)}{\varpi}}$ or $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - \frac{\varpi + d(\varsigma, \vartheta)}{\varpi}$ where d is a b-metric on Λ . Then it is easy to show that Ξ, Ω and Υ is a neutrosophic b-metric.

Definition 2.2. [6] A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is called b-nondecreasing if $\varsigma > b\vartheta$ implies $g(\varsigma) \geq g(\vartheta)$ for all $\varsigma, \vartheta \in \mathbb{R}$.

Lemma 2.1. [9] Let $\Xi(\varsigma, \vartheta, \cdot), \Omega(\varsigma, \vartheta, \cdot)$ and $\Upsilon(\varsigma, \vartheta, \cdot)$ be a NbMS. Then $\Xi(\varsigma, \vartheta, \varpi)$ is b -nondecreasing with respect to $\varpi, \Omega(\varsigma, \vartheta, \varpi)$ and $\Upsilon(\varsigma, \vartheta, \varpi)$ are b -non-increasing with respect to ϖ , for all $\varsigma, \vartheta \in \Lambda$.

Definition 2.3. [9] Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ be a neutrosophic b -metric space. For $\varpi > 0$, the open ball $\Psi(\varsigma, r, \varpi)$ with centre $\varsigma \in \Lambda$ and radius $0 < r < 1$ is defined as

$$\Psi(\varsigma, r, \varpi) = \{\vartheta \in \Lambda : \Xi(\varsigma, \vartheta, \varpi) > 1 - r, \Omega(\varsigma, \vartheta, \varpi) < r \text{ and } \Upsilon(\varsigma, \vartheta, \varpi) < r\}.$$

A sequence $\{\varsigma_n\}$:

(a) Converges to ς if $\Xi(\varsigma_n, \vartheta, \varpi) \rightarrow 1, \Omega(\varsigma_n, \vartheta, \varpi) \rightarrow 0$ and $\Upsilon(\varsigma_n, \vartheta, \varpi) \rightarrow 0$ as $n \rightarrow \infty$, for each $\varpi > 0$. In this case we write $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma$;

(b) is called a Cauchy sequence if for all $0 < \epsilon < 1$ and $\varpi > 0$, there exists $n_0 \in \mathbb{N}$ such that $\Xi(\varsigma_n, \varsigma_m, \varpi) > 1 - \epsilon, \Omega(\varsigma_n, \varsigma_m, \varpi) < \epsilon$ and $\Upsilon(\varsigma_n, \varsigma_m, \varpi) < \epsilon$ for all $n, m \geq n_0$.

Definition 2.4. [9] If every Cauchy sequence in NbMS is convergent, NbMS is complete.

Lemma 2.2. [9] In a NbMS $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ we have:

- (i) If a sequence $\{\varsigma_n\}$ in Λ converges to ς , then ς is unique.
- (ii) If a sequence $\{\varsigma_n\}$ in Λ converges to ς , then it is a Cauchy sequence.

Proposition 2.2. Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ be a NbMS and suppose that $\{\varsigma_n\}$ Converges to ς . Then we have

$$\begin{aligned} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\leq \limsup_{n \rightarrow \infty} \Xi(\varsigma_n, \vartheta, \varpi) \leq \Xi(\varsigma, \vartheta, b\varpi), \\ \Xi\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\leq \liminf_{n \rightarrow \infty} \Xi(\varsigma_n, \vartheta, \varpi) \leq \Xi(\varsigma, \vartheta, b\varpi), \\ \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\geq \liminf_{n \rightarrow \infty} \Omega(\varsigma_n, \vartheta, \varpi) \geq \Omega(\varsigma, \vartheta, b\varpi), \\ \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\geq \limsup_{n \rightarrow \infty} \Omega(\varsigma_n, \vartheta, \varpi) \geq \Omega(\varsigma, \vartheta, b\varpi) \text{ and} \\ \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\geq \liminf_{n \rightarrow \infty} \Upsilon(\varsigma_n, \vartheta, \varpi) \geq \Upsilon(\varsigma, \vartheta, b\varpi), \\ \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b}\right) &\geq \limsup_{n \rightarrow \infty} \Upsilon(\varsigma_n, \vartheta, \varpi) \geq \Upsilon(\varsigma, \vartheta, b\varpi) \end{aligned}$$

Example 2.3. Let $\Lambda = [0, \infty), \Xi(\varsigma, \vartheta, \varpi) = e^{-\frac{d(\varsigma, \vartheta)}{\varpi}}, \Omega(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{d(\varsigma, \vartheta)}{\varpi}}$ and $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{d(\varsigma, \vartheta)}{\varpi}} * = *_\theta, \diamond = \diamond_\theta$ and $d(\varsigma, \vartheta) = \begin{cases} 0, & \varsigma = \vartheta, \\ 2|\varsigma - \vartheta|, & \varsigma, \vartheta \in [0, 1) \\ \frac{1}{2}|\varsigma - \vartheta| & \text{otherwise.} \end{cases}$
Then $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is a NbMS with $b = 4$. The b -metric d in this example is

taken from [12]. Note that the Neutrosophic b-metric Ξ, Ω, Υ is not continuous. Let us observe that

$$\begin{aligned}\lim_{n \rightarrow \infty} \Xi \left(1, 1 - \frac{1}{n}, \varpi \right) &= \lim_{n \rightarrow \infty} e^{-\frac{1}{2n\varpi}} = 1, \Xi(1, 1, \varpi) = 1, \\ \lim_{n \rightarrow \infty} \Omega \left(0, \frac{1}{n}, \varpi \right) &= \lim_{n \rightarrow \infty} \left(1 - e^{-\frac{2}{n\varpi}} \right) = 0, \Omega(0, 0, \varpi) = 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon \left(0, \frac{1}{n-1}, \varpi \right) &= \lim_{n \rightarrow \infty} \left(1 - e^{-\frac{1}{\varpi} \left(\frac{2}{n-1} \right)} \right) = 0, \Upsilon(0, 0, \varpi) = 0.\end{aligned}$$

But,

$$\begin{aligned}\lim_{n \rightarrow \infty} \Xi \left(0, 1 - \frac{1}{n}, \varpi \right) &= \lim_{n \rightarrow \infty} e^{-\frac{1}{\varpi} (2) \left(\frac{1}{n} - 1 \right)} = e^{-\frac{2}{\varpi}}, \\ \Xi(0, 1, \varpi) &= e^{-\frac{1}{2\varpi}}, \\ \lim_{n \rightarrow \infty} \Omega \left(1, \frac{1}{n}, \varpi \right) &= \lim_{n \rightarrow \infty} \left(1 - e^{-\frac{1}{\varpi} (2) \left(1 - \frac{1}{n} \right)} \right) = 1 - e^{-\frac{2}{\varpi}}, \\ \Omega(1, 0, \varpi) &= 1 - e^{-\frac{1}{2\varpi}} \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon \left(1, \frac{1}{n-1}, \varpi \right) &= \lim_{n \rightarrow \infty} \left(1 - e^{-\frac{1}{\varpi} (2) \left(\frac{1}{1-\frac{1}{n}} \right)} \right) = 1 - e^{\frac{2}{\varpi}}, \\ \Upsilon(1, 0, \varpi) &= 1 - e^{\frac{1}{2\varpi}}.\end{aligned}$$

$\Xi(\varsigma, \vartheta, \varpi), \Omega(\varsigma, \vartheta, \varpi)$ and $\Upsilon(\varsigma, \vartheta, \varpi)$ is not continuous.

3 Main results

With extra conditions, we shall continue to employ a NbMS in the sense of Definition (2.1) $\lim_{\varpi \rightarrow \infty} \Xi(\varsigma, \vartheta, \varpi) = 1, \lim_{\varpi \rightarrow \infty} \Omega(\varsigma, \vartheta, \varpi) = 0$ and $\lim_{\varpi \rightarrow \infty} \Upsilon(\varsigma, \vartheta, \varpi) = 0$.

Lemma 3.1. Let $\{\varsigma_n\}$ be a sequence in a NbMS $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$. Let's say that there is $\mu \in (0, \frac{1}{b})$ such that

$$\begin{aligned}\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) &\geq \Xi \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right), \\ \Omega(\varsigma_n, \varsigma_{n+1}, \varpi) &\leq \Omega \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right) \text{ and} \\ \Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) &\leq \Upsilon \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0\end{aligned}\tag{1}$$

and there exists $\varsigma_0, \varsigma_1 \in \Lambda$ and $\rho \in (0, 1)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} *_{i=n}^{\infty} \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right) &= 1, \\ \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right) &= 0 \text{ and} \\ \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right) &= 0, \varpi > 0. \end{aligned} \quad (2)$$

Then $\{\varsigma_n\}$ is a Cauchy sequence.

Proof. Let $\tau \in (\mu b, 1)$. Clearly, $\sum_{i=1}^{\infty} \tau^i$ is convergent and a thing exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \tau^i < 1$ for all $n > n_0$. Let $n > m > n_0$. Since Ξ is b-nondecreasing, Ω and Υ are non-increasing also by (f), (k) and (p) of (Definition: 2.1) for every $\varpi > 0$, we have

$$\begin{aligned} \Xi(\varsigma_n, \varsigma_{n+m}, \varpi) &\geq \Xi \left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b} \right) \\ &\geq * \left(\Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), \Xi \left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b^2} \right) \right) \\ &\geq * \left(\Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), * \left(\Xi \left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi \tau^{n+1}}{b^3} \right), \dots \right) \right. \\ &\quad \left. \Xi \left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi \tau^{n+m-1}}{b^{m+1}} \right) \dots \right) \end{aligned}$$

By (1), it is evident that $\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) \geq \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n} \right)$, $n \in \mathbb{N}$, $\varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned} \Xi(\varsigma_n, \varsigma_{n+m}, \varpi) &\geq * \left(\Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^n}{b^2 \mu^n} \right), * \left(\Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+1}}{b^3 \mu^{n+1}} \right), \dots \right) \right. \\ &\quad \left. \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+m-1}}{b^{m+1} \mu^{n+m-1}} \right) \right) \\ &\geq *_{i=n}^{n+m-1} \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^{i-n+2} \mu^i} \right) \\ &\geq *_{i=n}^{n+m-1} \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^i \mu^i} \right) \geq *_{i=n}^{\infty} \Xi \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right), \text{ where } \rho = \frac{b\mu}{\tau}. \\ \Omega(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \Omega \left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b} \right) \\ &\leq \diamond \left(\Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), \Omega \left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b^2} \right) \right) \end{aligned}$$

$$\leq \diamond \left(\begin{array}{c} \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), \diamond \left(\Omega \left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi \tau^{n+1}}{b^3} \right), \dots \right) \\ \Omega \left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi \tau^{n+m-1}}{b^{m+1}} \right) \dots \end{array} \right)$$

By (1), it is evident that $\Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n} \right)$, $n \in \mathbb{N}$, $\varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned} \Omega(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \diamond \left(\begin{array}{c} \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^n}{b^2 \mu^n} \right), \diamond \left(\Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+1}}{b^3 \mu^{n+1}} \right), \dots \right) \\ \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+m-1}}{b^{m+1} \mu^{n+m-1}} \right) \dots \end{array} \right) \\ &\leq \diamond_{i=n}^{n+m-1} \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^{i-n+2} \mu^i} \right) \\ &\leq \diamond_{i=n}^{n+m-1} \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^i \mu^i} \right) \\ &\leq \diamond_{i=n}^{\infty} \Omega \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right), \text{ where } \rho = \frac{b\mu}{\tau}. \\ \Upsilon(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \Upsilon \left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b} \right) \\ &\leq \diamond \left(\Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), \Upsilon \left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi \sum_{i=n}^{n+m-1} \tau^i}{b^2} \right) \right) \\ &\leq \diamond \left(\begin{array}{c} \Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau^n}{b^2} \right), \diamond \left(\Upsilon \left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi \tau^{n+1}}{b^3} \right), \dots \right) \\ \Upsilon \left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi \tau^{n+m-1}}{b^{m+1}} \right) \dots \end{array} \right) \end{aligned}$$

By (1) it follows that $\Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n} \right)$, $n \in \mathbb{N}$, $\varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned} \Upsilon(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \diamond \left(\begin{array}{c} \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^n}{b^2 \mu^n} \right), \diamond \left(\Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+1}}{b^3 \mu^{n+1}} \right), \dots \right) \\ \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^{n+m-1}}{b^{m+1} \mu^{n+m-1}} \right) \dots \end{array} \right) \\ &\leq \diamond_{i=n}^{n+m-1} \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^{i-n+2} \mu^i} \right) \\ &\leq \diamond_{i=n}^{n+m-1} \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi \tau^i}{b^i \mu^i} \right) \\ &\leq \diamond_{i=n}^{\infty} \Upsilon \left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i} \right), \text{ where } \rho = \frac{b\mu}{\tau}. \end{aligned}$$

Since $\rho \in (0, 1)$, by (2) follows, it is inferred that $\{\varsigma_n\}$ is a Cauchy sequence. $\square$

Corolary 3.1. Let $\{\varsigma_n\}$ be a sequence in a NbMS $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$, and let $*$ be of Ω - type. If it's there $\mu \in (0, \frac{1}{b})$ like that

$$\begin{aligned}\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) &\geq \Xi\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), \\ \Omega(\varsigma_n, \varsigma_{n+1}, \varpi) &\leq \Omega\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right) \text{ and} \\ \Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) &\leq \Upsilon\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0,\end{aligned}\quad (3)$$

then $\{\varsigma_n\}$ is a Cauchy sequence.

Lemma 3.2. If for some $\mu \in (0, 1)$ and $\varsigma, \vartheta \in \Lambda$,

$$\begin{aligned}\Xi(\varsigma, \vartheta, \varpi) &\geq \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\ \Omega(\varsigma, \vartheta, \varpi) &\leq \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \text{ and} \\ \Upsilon(\varsigma, \vartheta, \varpi) &\leq \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right),\end{aligned}\quad (4)$$

where $\varpi > 0$, then $\varsigma = \vartheta$

Proof. Condition (4) implies that

$$\begin{aligned}\Xi(\varsigma, \vartheta, \varpi) &\geq \lim_{n \rightarrow \infty} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 1, \Omega(\varsigma, \vartheta, \varpi) \leq \lim_{n \rightarrow \infty} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 0 \text{ and} \\ \Upsilon(\varsigma, \vartheta, \varpi) &\leq \lim_{n \rightarrow \infty} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 0, \text{ where } n \in \mathbb{N}, \varpi > 0 \text{ and by (d), (i) and (n)} \\ \text{of Definition (2.1) it follows that } \varsigma &= \vartheta. \quad \square\end{aligned}$$

Theorem 3.1. Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ be a Complete Neutrosophic b-Metric space (CNbMS) and let $g : \Lambda \rightarrow \Lambda$. Let's say that there is $\mu \in (0, \frac{1}{b})$ such that

$$\begin{aligned}\Xi(g\varsigma, g\vartheta, \varpi) &\geq \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\ \Omega(g\varsigma, g\vartheta, \varpi) &\leq \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \text{ and} \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)\end{aligned}\quad (5)$$

where $\varsigma, \vartheta \in \Lambda, \varpi > 0$ and there exists $\varsigma_0 \in \Lambda$ and $\rho \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} *_{i=n}^{\infty} \Xi\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 1,$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Omega \left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i} \right) &= 0 \text{ and} \\ \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Upsilon \left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i} \right) &= 0 \text{ where } \varpi > 0. \end{aligned} \quad (6)$$

Then g has a one and only fixed point in Λ .

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. If we take $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$ in (5), then we have

$$\begin{aligned} \Xi(\varsigma_n, \varsigma_{n+1}, \varpi) &\geq \Xi \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right), \Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right) \text{ and} \\ \Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) &\leq \Upsilon \left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu} \right) \end{aligned}$$

where $n \in \mathbb{N}, \varpi > 0$, and Lemma (3.1) implies that $\{\varsigma_n\}$ is a Cauchy sequence. Since $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is complete, a thing exists $\varsigma \in \Lambda$ like that $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\varsigma, \varsigma_n, \varpi) &= 1, \lim_{n \rightarrow \infty} \Omega(\varsigma, \varsigma_n, \varpi) = 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\varsigma, \varsigma_n, \varpi) &= 0, \text{ where } \varpi > 0. \end{aligned} \quad (7)$$

Conditions (5) and by (f), (k) and (p) of Definition (2.1) are used to show that ς is a fixed point for g .

$$\begin{aligned} \Xi(g\varsigma, \varsigma, \varpi) &\geq * \left(\Xi \left(g\varsigma, \varsigma_n, \frac{\varpi}{2b} \right), \Xi \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right) \\ &\geq * \left(\Xi \left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu} \right), \Xi \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right), \\ \Omega(g\varsigma, \varsigma, \varpi) &\leq \diamond \left(\Omega \left(g\varsigma, \varsigma_n, \frac{\varpi}{2b} \right), \Omega \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right) \\ &\leq \diamond \left(\Omega \left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu} \right), \Omega \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right) \text{ and} \\ \Upsilon(g\varsigma, \varsigma, \varpi) &\leq \diamond \left(\Upsilon \left(g\varsigma, \varsigma_n, \frac{\varpi}{2b} \right), \Upsilon \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right) \\ &\leq \diamond \left(\Upsilon \left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu} \right), \Upsilon \left(\varsigma_n, \varsigma, \frac{\varpi}{2b} \right) \right) \text{ for all } \varpi > 0. \end{aligned}$$

By (7), as $n \rightarrow \infty$, we get

$$\Xi(g\varsigma, \varsigma, \varpi) \geq *(1, 1) = 1, \Omega(g\varsigma, \varsigma, \varpi) \leq \diamond(0, 0) = 0 \text{ and}$$

$$\Upsilon(g\varsigma, \varsigma, \varpi) \leq \diamond(0, 0) = 0.$$

Suppose that ς and ϑ are fixed points for g . By (5) we have

$\Xi(\varsigma, \vartheta, \varpi) = \Xi(g\varsigma, g\vartheta, \varpi) \geq \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right),$
 $\Omega(\varsigma, \vartheta, \varpi) = \Omega(g\varsigma, g\vartheta, \varpi) \leq \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)$ and
 $\Upsilon(\varsigma, \vartheta, \varpi) = \Upsilon(g\varsigma, g\vartheta, \varpi) \leq \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \varpi > 0,$ and through Lemma (3.2) suggests that $\varsigma = \vartheta$. $\square$

Example 3.1. Let $\Lambda = [0, 1]$. By Example (2.1), for $\theta = 2$. Thus, it follows $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is a NbMS with $b = 2$ and Neutrosophic b -Metric $\Xi(\varsigma, \vartheta, \varpi) = e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}, \Omega(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}, \Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}, \varsigma, \vartheta \in \Lambda, \varpi > 0$. Let $g(\varsigma) = \omega\varsigma, \omega < \frac{\sqrt{2}}{2}, \varsigma \in \Lambda$. Then

$$\begin{aligned}
 \Xi(g\varsigma, g\vartheta, \varpi) &= e^{-\frac{\omega^2(\varsigma-\vartheta)^2}{\varpi}} \geq e^{-\frac{\mu(\varsigma-\vartheta)^2}{\varpi}} = \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\
 \Omega(g\varsigma, g\vartheta, \varpi) &= 1 - e^{-\frac{\omega^2(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{-\frac{\mu(\varsigma-\vartheta)^2}{\varpi}} = \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \text{ and} \\
 \Upsilon(g\varsigma, g\vartheta, \varpi) &= 1 - e^{-\frac{\omega^2(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{-\frac{\mu(\varsigma-\vartheta)^2}{\varpi}} = \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \varsigma, \vartheta \in \Lambda, \varpi > 0,
 \end{aligned}$$

for $\frac{1}{b} > \mu > \omega^2$. As a result, Theorem (3.1)'s condition (5) is met and g has a singular fixed point in Λ .

Theorem 3.2. Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is a CNbMS and let $g : \Lambda \rightarrow \Lambda$. Let's say that there is $\mu \in (0, \frac{1}{b})$ like that

$$\begin{aligned}
 \Xi(g\varsigma, g\vartheta, \varpi) &\geq \min \left\{ \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right) \right\}, \\
 \Omega(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right) \right\} \text{ and} \\
 \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right) \right\} \quad (8)
 \end{aligned}$$

for all $\varsigma, \vartheta \in \Lambda, \varpi > 0$ and there exists $\varsigma_0 \in \Lambda$, and $\rho \in (0, 1)$ such that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} *_{i=n}^{\infty} \Xi\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) &= 1, \\
 \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Omega\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) &= 0 \text{ and} \\
 \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Upsilon\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) &= 0 \text{ for all } \varpi > 0. \quad (9)
 \end{aligned}$$

Then g has a one and only fixed point in Λ .

Some Fixed Point Results in Neutrosophic b-Metric Spaces

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. By (8) with $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$, for every $n \in \mathbb{N}$ and every $\varpi > 0$, we have

$$\begin{aligned}\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) &\geq \min \left\{ \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right) \right\} \\ &\geq \min \left\{ \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right) \right\}\end{aligned}$$

If $\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \Xi \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.

So, $\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.

$$\begin{aligned}\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \Omega \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Omega \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), \Omega \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right) \right\} \\ &\leq \max \left\{ \Omega \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Omega \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right) \right\}\end{aligned}$$

If $\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Omega \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.

So, $\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Omega \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.

$$\begin{aligned}\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \Upsilon \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Upsilon \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), \Upsilon \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right) \right\} \\ &\leq \max \left\{ \Upsilon \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Upsilon \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right) \right\}\end{aligned}$$

If $\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Upsilon \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.

So, $\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Upsilon \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.

Consequently, there is $\varsigma \in \Lambda$ like that

$$\begin{aligned}\lim_{n \rightarrow \infty} \varsigma_n &= \varsigma \text{ and } \lim_{n \rightarrow \infty} \Xi(\varsigma, \varsigma_n, \varpi) = 1, \\ \lim_{n \rightarrow \infty} \Omega(\varsigma, \varsigma_n, \varpi) &= 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\varsigma, \varsigma_n, \varpi) &= 0, \text{ where } \varpi > 0.\end{aligned}\tag{10}$$

Let's demonstrate that ς is g 's fixed point. Let $\tau_1 \in (\mu b, 1)$ and $\tau_2 = 1 - \tau_1$. By (8) we have

$$\begin{aligned}\Xi(g\varsigma, \varsigma, \varpi) &\geq * \left(\Xi \left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b} \right), \Xi \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right) \\ &\geq * \left(\min \left\{ \begin{array}{c} \Xi \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right) \end{array} \right\}, \Xi \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right)\end{aligned}$$

Taking $n \rightarrow \infty$ and using (10), we get

$$\begin{aligned}\Xi(g\varsigma, \varsigma, \varpi) &\geq * \left(\min \left\{ 1, \Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 1 \right\}, 1 \right) \\ &= * \left(\Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 1 \right) = \Xi \left(\varsigma, g\varsigma, \frac{\varpi}{\rho} \right), \varpi > 0\end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0, 1)$. So, $\Xi(g\varsigma, \varsigma, \varpi) \geq \Xi \left(g\varsigma, \varsigma, \frac{\varpi}{\rho} \right)$, $\varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$. Imagine that ς and ϑ are fixed points for g , that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned}\Xi(g\varsigma, g\vartheta, \varpi) &\geq \min \left\{ \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma, g\varsigma, \frac{\varpi}{\mu} \right), \Xi \left(\vartheta, g\vartheta, \frac{\varpi}{\mu} \right) \right\} \\ &= \min \left\{ \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), 1, 1 \right\} = \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right) = \Xi \left(g\varsigma, g\vartheta, \frac{\varpi}{\mu} \right) \\ \Omega(g\varsigma, \varsigma, \varpi) &\leq \diamond \left(\Omega \left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b} \right), \Omega \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right) \\ &\leq \diamond \left(\max \left\{ \begin{array}{c} \Omega \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right) \end{array} \right\}, \Omega \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right)\end{aligned}$$

Taking $n \rightarrow \infty$ and using (10), we get

$$\begin{aligned}\Omega(g\varsigma, \varsigma, \varpi) &\leq \diamond \left(\max \left\{ 0, \Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 0 \right\}, 0 \right) \\ &= \diamond \left(\Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 0 \right) \\ &= \Omega \left(\varsigma, g\varsigma, \frac{\varpi}{\rho} \right), \varpi > 0,\end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0, 1)$. So, $\Omega(g\varsigma, \varsigma, \varpi) \leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\rho}\right)$, $\varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$.

Imagine that ς and ϑ are fixed points for g , that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned} & \Omega(g\varsigma, g\vartheta, \varpi) \\ & \leq \max \left\{ \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi}{\mu}\right), \Omega\left(\vartheta, g\vartheta, \frac{\varpi}{\mu}\right) \right\} \\ & = \max \left\{ \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0 \right\} = \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) = \Omega\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \\ & \Upsilon(g\varsigma, \varsigma, \varpi) \\ & \leq \diamond \left(\Upsilon\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right) \\ & \leq \diamond \left(\max \left\{ \begin{array}{c} \Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array} \right\}, \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right) \end{aligned}$$

Taking $n \rightarrow \infty$ and using (10), we get

$$\begin{aligned} \Upsilon(g\varsigma, \varsigma, \varpi) & \leq \diamond \left(\max \left\{ 0, \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0 \right\}, 0 \right) \\ & = \diamond \left(\Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0 \right) \\ & = \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi}{\rho}\right), \varpi > 0 \end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0, 1)$. So, $\Upsilon(g\varsigma, \varsigma, \varpi) \leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\rho}\right)$, $\varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$.

Imagine that ς and ϑ are fixed points for g , that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned} \Upsilon(g\varsigma, g\vartheta, \varpi) & \leq \max \left\{ \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(\vartheta, g\vartheta, \frac{\varpi}{\mu}\right) \right\} \\ & = \max \left\{ \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0 \right\} = \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) = \Upsilon\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{aligned}$$

for $\varpi > 0$, and it is implied from Lemma (3.2) that $g\varsigma = g\vartheta$, that is $\varsigma = \vartheta$. $\square$

Example 3.2. Let $\Lambda = (0, 2)$, $\Xi(\varsigma, \vartheta, \varpi) = e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}$, $\Omega(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}$, $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{-\frac{(\varsigma-\vartheta)^2}{\varpi}}$, $*$ $=$ $*_{\theta}$ and $\diamond = \diamond_{\theta}$. Then $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is a

CNbMS with $b = 2$. Let $g(\varsigma) = \begin{cases} 2 - \varsigma, & \varsigma \in (0, 1) \\ 1, & \varsigma \in [1, 2) \end{cases}$

Case 1: If $\varsigma, \vartheta \in [1, 2)$ then $\Xi(g\varsigma, g\vartheta, \varpi) = 1, \Omega(g\varsigma, g\vartheta, \varpi) = 0$ and $\Upsilon(g\varsigma, g\vartheta, \varpi) = 0, \varpi > 0$, and conditions (8) are met.

Case 2: If $\varsigma \in [1, 2)$ and $\vartheta \in (0, 1)$, then, for $\mu \in (\frac{1}{4}, \frac{1}{2})$, we have

$$\begin{aligned} \Xi(g\varsigma, g\vartheta, \varpi) &= e^{\frac{-(1-\vartheta)^2}{\varpi}} \geq e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} = \Xi\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\ \Omega(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{-(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} = \Omega\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{4\mu(1-\vartheta)^2}{\varpi}} = \Upsilon\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \varpi > 0. \end{aligned}$$

Case 3: As in the earlier instance, for $\mu \in (\frac{1}{4}, \frac{1}{2})$, we have

$$\begin{aligned} \Xi(g\varsigma, g\vartheta, \varpi) &\geq \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \\ \Omega(g\varsigma, g\vartheta, \varpi) &\leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right) \text{ and} \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma \in (0, 1), \vartheta \in [1, 2), \varpi > 0. \end{aligned}$$

Case 4: If $\varsigma, \vartheta \in (0, 1)$ then for $\mu \in (\frac{1}{4}, \frac{1}{2})$,

$$\begin{aligned} \Xi(g\varsigma, g\vartheta, \varpi) &\geq e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}} \geq e^{\frac{-(1-\vartheta)^2}{\varpi}} \geq e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} \\ &= \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\ \Xi(g\varsigma, g\vartheta, \varpi) &\geq \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0, \\ \Omega(g\varsigma, g\vartheta, \varpi) &\leq 1 - e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} \\ &= \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\ \Omega(g\varsigma, g\vartheta, \varpi) &\leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0 \text{ and} \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq 1 - e^{\frac{(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{4\mu(1-\vartheta)^2}{\varpi}} \\ &= \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0. \end{aligned}$$

So conditions (8) are satisfied for all $\varsigma, \vartheta \in \Lambda, \varpi > 0$, and it follows from Theorem (3.2) that $\varsigma = 1$ is a singular fixed point for g .

Theorem 3.3. Let $(\Lambda, \Xi, \Omega, \Upsilon, *_{\min}, \diamond_{\max})$ be a CNbMS, and let $g : \Lambda \rightarrow \Lambda$. If for some $\mu \in (0, \frac{1}{b^2})$,

$$\begin{aligned} \Xi(g\varsigma, g\vartheta, \varpi) &\geq \min \left\{ \begin{array}{l} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Xi\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Xi\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array} \right\}, \\ \Omega(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \begin{array}{l} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Omega\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Omega\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array} \right\} \text{ and} \\ \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \begin{array}{l} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Upsilon\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Upsilon\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array} \right\}, \quad (11) \end{aligned}$$

where $\varsigma, \vartheta \in \Lambda, \varpi > 0$ then g has a unique fixed point in Λ .

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. By (11) with $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$, using (f, k and p) and the assumption that $* = *_{\min}$ and $\diamond = \diamond_{\max}$ we have

$$\begin{aligned} &\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \\ &\geq \min \left\{ \begin{array}{l} \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \min \left\{ \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\}, \Xi\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array} \right\} \\ &\geq \min \left\{ \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) \right\}, n \in \mathbb{N}, \varpi > 0. \\ &\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \\ &\leq \max \left\{ \begin{array}{l} \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \max \left\{ \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\}, \Omega\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array} \right\} \\ &\leq \max \left\{ \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) \right\}, n \in \mathbb{N}, \varpi > 0 \text{ and} \\ &\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \\ &\leq \max \left\{ \begin{array}{l} \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \max \left\{ \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\}, \Upsilon\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array} \right\} \\ &\leq \max \left\{ \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) \right\}, \text{ where } n \in \mathbb{N}, \varpi > 0. \end{aligned}$$

As in the proof of Theorem (3.2), it is inferred from Lemma (3.2) and Corollary (3.1) that

$$\begin{aligned} \Xi(\varsigma_{n+1}, \varsigma_n, \varpi) &\geq \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \text{ and} \\ \Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \quad n \in \mathbb{N}, \varpi > 0, \text{ and } \{\varsigma_n\} \text{ is a Cauchy sequence.} \end{aligned}$$

Thus, there is $\varsigma \in \Lambda$ like that $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \Xi(\varsigma, \varsigma_n, \varpi) &= 1, \lim_{n \rightarrow \infty} \Omega(\varsigma, \varsigma_n, \varpi) = 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\varsigma, \varsigma_n, \varpi) &= 0, \varpi > 0. \end{aligned} \quad (12)$$

Let $\tau_1 \in (b^2\mu, 1)$ and $\tau_2 = 1 - \tau_1$. By (11) and (f, k and p) for $*$ = $*_{\min}$, $\diamond$ = $\diamond_{\max}$ we have

$$\begin{aligned} &\Xi(g\varsigma, \varsigma, \varpi) \\ &\geq \min \left\{ \Xi\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Xi\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \\ &\geq \min \left\{ \min \left\{ \begin{aligned} &\Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ &\Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ &\min \left\{ \Xi\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right) \right\}, \\ &\Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{aligned} \right\}, \Xi\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \\ &\Omega(g\varsigma, \varsigma, \varpi) \\ &\leq \max \left\{ \Omega\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Omega\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \\ &\leq \max \left\{ \max \left\{ \begin{aligned} &\Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ &\Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ &\max \left\{ \Omega\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right) \right\}, \\ &\Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{aligned} \right\}, \Omega\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \\ &\Upsilon(g\varsigma, \varsigma, \varpi) \\ &\leq \max \left\{ \Upsilon\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Upsilon\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \\ &\leq \max \left\{ \max \left\{ \begin{aligned} &\Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ &\Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ &\max \left\{ \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right) \right\}, \\ &\Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{aligned} \right\}, \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right\} \end{aligned}$$

for all $n \in \mathbb{N}$, $\varpi > 0$. Taking $n \rightarrow \infty$ and using (12), we get

$$\begin{aligned}
 & \Xi(g\varsigma, \varsigma, \varpi) \\
 & \geq \min \left\{ \min \left\{ 1, \Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 1, \min \left\{ \Xi \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), 1 \right\}, 1 \right\}, 1 \right\} \\
 & = \Xi \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), \varpi > 0, \\
 & \Omega(g\varsigma, \varsigma, \varpi) \\
 & \leq \max \left\{ \max \left\{ 0, \Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 0, \max \left\{ \Omega \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), 0 \right\}, 0 \right\}, 0 \right\} \\
 & = \Omega \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), \varpi > 0, \\
 & \Upsilon(g\varsigma, \varsigma, \varpi) \\
 & \leq \max \left\{ \max \left\{ 0, \Upsilon \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), 0, \max \left\{ \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), 0 \right\}, 0 \right\}, 0 \right\} \\
 & = \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), \varpi > 0,
 \end{aligned}$$

and by lemma (3.2) with $\rho = \frac{b^2\mu}{\tau_1} \in (0, 1)$. Thus, it follows $g\varsigma = \varsigma$.

With reference to condition (11), two fixed points $\varsigma = g\varsigma$ and $\vartheta = g\vartheta$, we have

$$\begin{aligned}
 & \Xi(g\varsigma, g\vartheta, \varpi) \\
 & \geq \min \left\{ \begin{array}{l} \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Xi \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Xi \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \min \left\{ \Xi \left(g\varsigma, \varsigma, \frac{\varpi}{b\mu} \right), \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Xi \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right) \end{array} \right\} \\
 & = \min \left\{ \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), 1, 1, \min \left\{ 1, \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right) \right\} \\
 & = \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) = \Xi \left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu} \right), \varpi > 0, \\
 & \Omega(g\varsigma, g\vartheta, \varpi) \\
 & \leq \max \left\{ \begin{array}{l} \Omega \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Omega \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Omega \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \max \left\{ \Omega \left(g\varsigma, \varsigma, \frac{\varpi}{b\mu} \right), \Omega \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Omega \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right) \end{array} \right\} \\
 & = \max \left\{ \Omega \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), 0, 0, \max \left\{ 0, \Omega \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Omega \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right) \right\} \\
 & = \Omega \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) = \Omega \left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu} \right), \varpi > 0, \\
 & \Upsilon(g\varsigma, g\vartheta, \varpi)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \max \left\{ \begin{array}{l} \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Upsilon \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \max \left\{ \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi}{b\mu} \right), \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Upsilon \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right) \end{array} \right\} \\
 &= \max \left\{ \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), 0, 0, \max \left\{ 0, \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right) \right\} \\
 &= \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) = \Upsilon \left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu} \right), \varpi > 0,
 \end{aligned}$$

and it is implied from Lemma (3.2) that $\varsigma = \vartheta$. □

Theorem 3.4. Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond), * \geq *_{\theta}, \diamond \leq \diamond_{\theta}$, be a CNbMS, and let $g : \Lambda \rightarrow \Lambda$. Suppose that for some $\mu \in (0, \frac{1}{b^2})$,

$$\begin{aligned}
 \Xi(g\varsigma, g\vartheta, \varpi) &\geq \min \left\{ \begin{array}{l} \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Xi \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Xi \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \sqrt{\Xi \left(g\varsigma, \vartheta, \frac{2\varpi}{\mu} \right), \Xi \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right)} \end{array} \right\} \\
 \Omega(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \begin{array}{l} \Omega \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Omega \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Omega \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \sqrt{\Omega \left(g\varsigma, \vartheta, \frac{2\varpi}{\mu} \right), \Omega \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right)} \end{array} \right\} \\
 \Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \max \left\{ \begin{array}{l} \Upsilon \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Upsilon \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \sqrt{\Upsilon \left(g\varsigma, \vartheta, \frac{2\varpi}{\mu} \right), \Upsilon \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right)} \end{array} \right\}, \quad (13)
 \end{aligned}$$

$\varsigma, \vartheta \in \Lambda, \varpi > 0$ and there exists $\varsigma_0 \in \Lambda$ and $\rho \in (0, 1)$ such that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} *_{i=n}^{\infty} \Xi \left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i} \right) &= 1, \\
 \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Omega \left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i} \right) &= 0 \text{ and} \\
 \lim_{n \rightarrow \infty} \diamond_{i=n}^{\infty} \Omega \left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i} \right) &= 0 \text{ for all } \varpi > 0. \quad (14)
 \end{aligned}$$

Then g has a unique fixed point in Λ .

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. Taking $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$ in condition (13), by using (f, k and p) and $* \geq *_{\theta}, \diamond \leq \diamond_{\theta}$ we have

$$\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \min \left\{ \begin{array}{l} \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu} \right), \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu} \right), \\ \sqrt{\Xi \left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu} \right) \cdot \Xi \left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu} \right)}, \Xi \left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu} \right) \end{array} \right\},$$

$$\begin{aligned}\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \begin{array}{l} \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \sqrt{\Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) \cdot \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right)}, \Omega\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array} \right\}, \\ \Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \begin{array}{l} \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \sqrt{\Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) \cdot \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right)}, \Upsilon\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array} \right\},\end{aligned}$$

$n \in \mathbb{N}$, $\varpi > 0$. Since $\Xi(\varsigma, \vartheta, \varpi)$ is b -nondecreasing in ϖ and $\sqrt{(a.b)} \geq \min\{a, b\}$, $\Omega(\varsigma, \vartheta, \varpi)$ and $\Upsilon(\varsigma, \vartheta, \varpi)$ is b -nonincreasing in ϖ and $\sqrt{(a.b)} \leq \max\{a, b\}$ we obtain that

$$\begin{aligned}\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) &\geq \min \left\{ \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\}, \\ \Omega(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\} \text{ and} \\ \Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max \left\{ \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \right\}.\end{aligned}$$

By Lemma (3.2) and (3.1) it follows that

$$\begin{aligned}\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) &\geq \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \\ \Omega(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right) \text{ and} \\ \Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right).\end{aligned}$$

where $n \in \mathbb{N}$, $\varpi > 0$, and $\{\varsigma_n\}$ is a Cauchy sequence. Since $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond)$ is a complete space, there exists $\varsigma \in \Lambda$ such that $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma$ and

$$\begin{aligned}\lim_{n \rightarrow \infty} \Xi(\varsigma, \varsigma_n, \varpi) &= 1, \\ \lim_{n \rightarrow \infty} \Omega(\varsigma, \varsigma_n, \varpi) &= 0 \text{ and} \\ \lim_{n \rightarrow \infty} \Upsilon(\varsigma, \varsigma_n, \varpi) &= 0, \varpi > 0.\end{aligned}\tag{15}$$

Let $\tau_1 \in (b^2\mu, 1)$ and $\tau_2 = 1 - \tau_1$. By (13) and (f, k and p) for $*$ and $\diamond \leq \diamond_\theta$ we have

$$\begin{aligned}\Xi(g\varsigma, \varsigma, \varpi) \\ \geq * \left(\Xi\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Xi\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right) \right)\end{aligned}$$

$$\begin{aligned}
&\geq * \left(\min \left\{ \begin{array}{c} \Xi \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \\ \sqrt{\Xi \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right) \cdot \Xi \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu} \right)}, \\ \Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right) \end{array} \right\}, \Xi \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right) \\
&\geq * \left(\min \left\{ \begin{array}{c} \Xi \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Xi \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \\ \min \left\{ \Xi \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), \Xi \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu} \right) \right\}, \\ \Xi \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \Xi \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \end{array} \right\} \right), \\
&\Omega(g\varsigma, \varsigma, \varpi) \\
&\leq \diamond \left(\Omega \left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b} \right), \Omega \left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b} \right) \right) \\
&\leq \diamond \left(\max \left\{ \begin{array}{c} \Omega \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \\ \sqrt{\Omega \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right) \cdot \Omega \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu} \right)}, \\ \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right) \end{array} \right\}, \Omega \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right), \\
&\leq \diamond \left(\max \left\{ \begin{array}{c} \Omega \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Omega \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \\ \max \left\{ \Omega \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right), \Omega \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu} \right) \right\}, \\ \Omega \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \Omega \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \end{array} \right\} \right) \text{ and} \\
&\Upsilon(g\varsigma, \varsigma, \varpi) \\
&\leq \diamond \left(\Upsilon \left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b} \right), \Upsilon \left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b} \right) \right) \\
&\leq \diamond \left(\max \left\{ \begin{array}{c} \Upsilon \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu} \right), \Upsilon \left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu} \right), \\ \Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right), \\ \sqrt{\Upsilon \left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu} \right) \cdot \Upsilon \left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu} \right)}, \\ \Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu} \right) \end{array} \right\}, \Upsilon \left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b} \right) \right),
\end{aligned}$$

$$\leq \diamond \left(\max \left\{ \begin{array}{l} \Upsilon \left(\varsigma, \varsigma_n, \frac{\varpi \tau_1}{b\mu} \right), \Upsilon \left(\varsigma, g\varsigma, \frac{\varpi \tau_1}{b\mu} \right), \\ \Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau_1}{b\mu} \right), \\ \max \left\{ \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), \Upsilon \left(\varsigma, \varsigma_n, \frac{\varpi \tau_1}{b^2\mu} \right) \right\}, \\ \Upsilon \left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi \tau_1}{b\mu} \right), \Upsilon \left(\varsigma_{n+1}, \varsigma, \frac{\varpi \tau_2}{b} \right) \end{array} \right\} \right),$$

for all $\vartheta \in \mathbb{N}$ and $\varpi > 0$. Taking $n \rightarrow \infty$ and using (15), we get

$$\begin{aligned} & \Xi(g\varsigma, \varsigma, \varpi) \\ & \geq * \left(\min \left\{ 1, \Xi \left(\varsigma, g\varsigma, \frac{\varpi \tau_1}{b\mu} \right), 1, \min \left\{ \Xi \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), 1 \right\}, 1 \right\}, 1 \right) \\ & = \Xi \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), \\ & \Omega(g\varsigma, \varsigma, \varpi) \\ & \leq \diamond \left(\max \left\{ 0, \Omega \left(\varsigma, g\varsigma, \frac{\varpi \tau_1}{b\mu} \right), 0, \max \left\{ \Omega \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), 0 \right\}, 0 \right\}, 0 \right) \\ & = \Omega \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right) \text{ and} \\ & \Upsilon(g\varsigma, \varsigma, \varpi) \\ & \leq \diamond \left(\max \left\{ 0, \Upsilon \left(\varsigma, g\varsigma, \frac{\varpi \tau_1}{b\mu} \right), 0, \max \left\{ \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), 0 \right\}, 0 \right\}, 0 \right) \\ & = \Upsilon \left(g\varsigma, \varsigma, \frac{\varpi \tau_1}{b^2\mu} \right), \varpi > 0 \end{aligned}$$

and by Lemma (3.2) with $\rho = \frac{b^2\mu}{\tau_1} \in (0, 1)$ as a result, $g\varsigma = \varsigma$.

Imagine that ς and ϑ are fixed points for g . By condition (13) we have

$$\begin{aligned} & \Xi(g\varsigma, g\vartheta, \varpi) \\ & \geq * \left(\begin{array}{l} \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), \Xi \left(g\varsigma, \varsigma, \frac{\varpi}{\mu} \right), \Xi \left(g\vartheta, \vartheta, \frac{\varpi}{\mu} \right), \\ \sqrt{\Xi \left(g\varsigma, \varsigma, \frac{\varpi}{b\mu} \right), \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right), \Xi \left(\varsigma, g\vartheta, \frac{\varpi}{\mu} \right)} \end{array} \right) \\ & \geq * \left(\Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right), 1, 1, \min \left\{ 1, \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) \right\}, \Xi \left(\varsigma, \vartheta, \frac{\varpi}{\mu} \right) \right) \\ & = \Xi \left(\varsigma, \vartheta, \frac{\varpi}{b\mu} \right) = \Xi \left(g\varsigma, g\vartheta, \frac{\varpi}{\mu} \right), \\ & \Omega(g\varsigma, g\vartheta, \varpi) \end{aligned}$$

$$\begin{aligned}
&\leq \diamond \left(\frac{\Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right)}{\sqrt{\Omega\left(g\varsigma, \varsigma, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)}} \right) \\
&\leq \diamond \left(\Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0, \max\left\{0, \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \right) \\
&= \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right) = \Omega\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right), \text{ and} \\
&\Upsilon(g\varsigma, g\vartheta, \varpi) \\
&\leq \diamond \left(\frac{\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right)}{\sqrt{\Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)}} \right) \\
&\leq \diamond \left(\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0, \max\left\{0, \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \right) \\
&= \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right) = \Upsilon\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right), \varpi > 0.
\end{aligned}$$

Therefore, it follows from Lemma (3.2) that $\varsigma = \vartheta$. $\square$

Example 3.3. Let $\Lambda = \{0, 1, 3\}$, $\Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}$, $\Omega(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}$ and $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{(\varsigma-\vartheta)^2}{\varpi}}$, $*$ $= *_{\theta}$, $\diamond = \diamond_{\theta}$. Then $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond)$ is a CNbMS with $b = 2$. Define the function $g : \Lambda \rightarrow \Lambda$ as follows: $g_0 = g_1 = 1, g_3 = 0$. Observe that if $\varsigma = \vartheta$ or $\varsigma, \vartheta \in [0, 1]$, then $\Xi(g\varsigma, g\vartheta, \varpi) = 1, \Omega(g\varsigma, g\vartheta, \varpi) = 0$ and $\Upsilon(g\varsigma, g\vartheta, \varpi) = 0, \varpi > 0$ and condition (13) is fulfilled.

Case: 1 Let $\varsigma = 0$ and $\vartheta = 1$. Then since $\mu \in (\frac{1}{9}, \frac{1}{4})$, we have that

$$\begin{aligned}
\Xi(g\varsigma, g\vartheta, \varpi) &= e^{\frac{-1}{\varpi}} \geq \min\left\{e^{\frac{-\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}, 1, 1, e^{\frac{-\mu}{\varpi}}\right\}, \\
\Omega(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{-1}{\varpi}} \leq \max\left\{1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 0, 0, 1 - e^{\frac{-\mu}{\varpi}}\right\}, \text{ and} \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{1}{\varpi}} \leq \max\left\{1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 0, 0, 1 - e^{\frac{-\mu}{\varpi}}\right\}.
\end{aligned}$$

Case: 2 Let $\varsigma = 0$ and $\vartheta = 3$. Then since $\mu \in (\frac{1}{9}, \frac{1}{4})$, we have that

$$\begin{aligned}
\Xi(g\varsigma, g\vartheta, \varpi) &= e^{\frac{-1}{\varpi}} \geq \min\left\{e^{\frac{-9\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}, e^{\frac{-9\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}, 1\right\}, \\
\Omega(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{-1}{\varpi}} \leq \max\left\{1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 0\right\}, \text{ and} \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{1}{\varpi}} \leq \max\left\{1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 0\right\}.
\end{aligned}$$

Case: 3 Let $\varsigma = 1$ and $\vartheta = 3$. Then selecting $\mu \in (\frac{1}{9}, \frac{1}{4})$, we have that

$$\begin{aligned}\Xi(g_{\varsigma}, g_{\vartheta}, \varpi) &= e^{\frac{-1}{\varpi}} \geq \min \left\{ e^{\frac{-4\mu}{\varpi}}, 1, e^{\frac{-9\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}} \right\}, \\ \Omega(g_{\varsigma}, g_{\vartheta}, \varpi) &= 1 - e^{\frac{-1}{\varpi}} \leq \max \left\{ 1 - e^{\frac{-4\mu}{\varpi}}, 0, 1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}} \right\} \text{ and} \\ \Upsilon(g_{\varsigma}, g_{\vartheta}, \varpi) &= 1 - e^{\frac{1}{\varpi}} \leq \max \left\{ 1 - e^{\frac{4\mu}{\varpi}}, 0, 1 - e^{\frac{9\mu}{\varpi}}, 1 - e^{\frac{\mu}{\varpi}}, 1 - e^{\frac{\mu}{\varpi}} \right\}.\end{aligned}$$

Therefore we conclude that $\mu \in (\frac{1}{9}, \frac{1}{4})$, condition (13) is fulfilled for $\varsigma, \vartheta \in \Lambda$ and $\varpi > 0$.

Therefore, $\varsigma = 1$ is a one and only fixed point and all conditions of Theorem (3.4) are met.

4 Conclusion

An extension of the t-norm that is countable is used in this paper. Using this condition of expansion and contraction, we can show that the sequence in NbMS is a Cauchy sequence. By using our findings, some fixed point theorems are additionally established.

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Existence Results for ψ -Caputo Fractional Differential Equations with Boundary Conditions

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Abstract

In this paper, we examine the existence and uniqueness of arrangements of a limit esteem problem for a fractional differential equation of order $\alpha \in (2, 3)$, including a general type of fractional derivative. initially, we demonstrate a comparability between the Cauchy problem and the Volterra condition. Then, at that point, two outcomes on the existence of arrangements are demonstrated, and we end for certain illustrative models.

Keywords: fractional differential equations, Cauchy problem, Volterra condition

2020 AMS subject Classifications: 34A08; 34K37; 34K40; 58C30.

1. Introduction

Fractional differential conditions (FDEs) have obtained impressive importance since it has shifted applications in various applied sciences and designing [1-3]. There are different meanings of fractional calculus (FC) that created the FDEs whether in demonstrating or descript the memory precisely. The field of fragmentary differential conditions has been exposed to an escalated improvement of the hypothesis and

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applications in numerical physics, finance, astronomy ,cosmology, combination model for low light upgrade, and in as of late applied the subjective examination in the hour of Coronavirus 19[5].

One of the notable subordinates with partial orders are Riemann-Liouville, Caputo, Weyl Hadamard and Grunwald-Letnikov, and so on, Partial administrators are assumed an indispensable part for the mathematical demonstrating and genuine applications.. As of late, some essential hypothesis for starting worth issues for fragmentary differential conditions including the Riemann-Liouville differential administrator of request $\alpha \in (0, 1]$ as talked about by Lakshmikantham and , Vatsala [9,10,13]. Here we extend the work in [1] with mixed boundary conditions dependence on the lipschitz conditions. The motive of this paper is to explore the existence and uniqueness of solutions for a nonlinear partial differential equation (FDE) including the Caputo fractional derivative of a function u concerning another function ψ

$$\left\{ \begin{array}{l} ({}^c D_{a+}^{\alpha, \psi} u)(t) = f(t, u(t), u'(t, u(t))), \quad t \in [a, b] \\ u(a) = u_a \\ u'(a) = u_a^1 \\ u(b) = MI_{a+}^{\alpha, \psi} u(x) \end{array} \right. \quad (1)$$

Where $\alpha \in (2, 3)$, u_a , u_a^1 , M , $x \in \mathbb{R}$ with $x \in (a, b)$, $u \in C^2[a, b]$, $({}^c D_{a+}^{\alpha, \psi} u)(t)$ is the ψ – caputo fractional derivative of u [1] and consider the continuous function $f: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$.

If $f \in [a, b]$, $f(t, u(t), u'(t, u(t))) = 0$. Consider $\mathfrak{D}u(t) = u'(t, u(t))$

Then (1) changes to

$$\left\{ \begin{array}{l} ({}^c D_{a+; \psi}^{\alpha} u)(t) = f(t, u(t), \mathfrak{D}u(t)), \quad t \in [a, b] \\ u(a) = u_a \\ u'(a) = u_a^1 \\ u(b) = MI_{a+; \psi}^{\alpha} u(x) \end{array} \right. \quad (2)$$

Here we postulate to the limitation.

$$K = \frac{2M}{\Gamma(\alpha+3)} (\psi(x) - \psi(a))^{\alpha+2} - (\psi(b) - \psi(a))^2 \neq 0 \quad (3)$$

is satisfied

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In this article we organized the basic needed results, definitions, lemmas and theorems. In section 3 we present the existence and uniqueness results using Banach and Leray Schauder fixed point theorems based on our system of equation (2). In section 4 we demonstrate one example based on our result.

2. Preliminaries

We presently review the a few documentations, definitions fundamental outcome that are utilized all through this article. By $C(J, \mathbb{R})$, we mean the Banach space of all persistent capabilities from J to $\mathbb{R}$ with the norm $\|u\|_\infty = \sup\{|u(t)|: t \in J\}$. and an increasing differentiable function $\psi: [a, b] \rightarrow \mathbb{R} \ni \psi'(t) \neq 0, \forall t \in [a, b]$

Definition 2.1([11]) Let us defined the function $f: [0, T] \rightarrow \mathbb{R}$ be an integrable function. The ψ -R-L necessary of request $\alpha > 0$ of f for $0 < t < T < +\infty$, is characterized by

$$I_{0+;\psi}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s) ds \quad (5)$$

Definition 2.2([6]) Considering $0 < t < T < +\infty$, the ψ -R-L fractional derivative of order $\alpha > 0$ of a function $f \in AC_\psi^r([0, T])$ is defined by

$$\begin{aligned} D_{0+;\psi}^\alpha f(t) &= I_{0+;\psi}^{r-\alpha} (\omega_\psi^r f)(t) + \sum_{n=0}^{r-1} \frac{(\omega_\psi^n f)(0)}{\Gamma(n-\alpha+1)} (\psi(t) - \psi(0))^{n-\alpha} \\ &= \frac{1}{\Gamma(r-\alpha)} \int_0^t (\psi(t) - \psi(s))^{r-\alpha-1} \psi'(s) \omega_\psi^r f(s) ds \\ &\quad + \sum_{n=0}^{r-1} \frac{(\omega_\psi^n f)(0)}{\Gamma(n-\alpha+1)} (\psi(t) - \psi(0))^{n-\alpha} \end{aligned} \quad (6)$$

In the event that the basic exists, where $n = [\alpha] + 1$, $\Gamma(\cdot)$ is the Euler's gamma capability characterized by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Note:2.3 The fractional integral is satisfied the semigroup law, (i.e).,

$$I_{a+;\psi}^\alpha I_{a+;\psi}^\beta u(t) = I_{a+;\psi}^{\alpha+\beta} u(t), \quad \forall \alpha, \beta > 0.$$

the fractional derivative that we will bargain in our work is a Caputo type operator, and it is characterized by the articulation

$$({}^c D_{a+;\psi}^\alpha u)(t) = D_{a+;\psi}^\alpha \left[u(t) - \sum_{n=0}^{k-1} \frac{u_\psi^{[n]}(a)}{n!} (\psi(t) - \psi(a))^n \right] \quad (7)$$

Where $u \in C^{n-1}[a, b]$, $n = [\alpha] + 1$ for $\alpha \notin N$, $n = \alpha$ for $\alpha \in N$, and

$$u_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t)$$

Now if $\alpha = h \in \mathbb{N}$, we have $({}^c D_{a+;\psi}^\alpha u)(t) = u_\psi^{[h]}(t)$ And for $\alpha \notin \mathbb{N}$, we have $({}^c D_{a+;\psi}^\alpha u)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(\xi) (\psi(t) - \psi(\xi))^{n-\alpha-1} u_\psi^{[n]}(\xi) d\xi$

A significant function in fractional calculus is the Mittag-Leffler function, which is a speculation of the factorial function to real numbers, and it is given by

$$E_\alpha(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(\alpha n + 1)}.$$

The power function derivative is given by

$$({}^c D_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{\gamma-1}) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} (\psi(t) - \psi(a))^{\gamma-\alpha-1}.$$

Thus the mittag leffler function is given by

$({}^c D_{a+;\psi}^\alpha E_\alpha(\delta(\psi(t) - \psi(a))^\alpha)) = \delta E_\alpha(\delta(\psi(t) - \psi(a))^\alpha)$ n the accompanying theorem, gives the relation between fractional integrals and fractional derivatives.

Theorem:2.4 [4] Let $u: [a, b] \rightarrow \mathbb{R}$ be a function

- a) If u is continuous, then $({}^c D_{a+;\psi}^\alpha I_{a+;\psi}^\alpha u)(t) = u(t)$
- b) If u is of class C^{n-1} , then

$$I_{a+;\psi}^\alpha {}^c D_{a+;\psi}^\alpha u(t) = u(t) - \sum_{n=0}^{k-1} \frac{u_\psi^{[n]}(a)}{n!} (\psi(t) - \psi(a))^n.$$

In [2.5] theorem is workout for a further class of function. □

Theorem:2.5 For $\gamma > 0$ and $\delta \in \mathbb{R} - \{0\}$, we have the subsequent couple of formulas

$$I_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\psi(t) - \psi(a))^{\gamma+\alpha-1}.$$

And $I_{a+;\psi}^\alpha E_\alpha(\delta(\psi(t) - \psi(a))^\alpha) = \frac{1}{\delta} (E_\alpha(\delta(\psi(t) - \psi(a))^\alpha) - 1)$

Proof: For the first formula

$$\begin{aligned} I_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{\gamma-1} &= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\xi) (\psi(t) - \psi(\xi))^{\alpha-1} (\psi(t) - \psi(\xi))^{\gamma-1} d\xi \\ &= \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\alpha-1} \int_a^t \psi'(\xi) \left(1 - \frac{\psi(\xi) - \psi(a)}{\psi(t) - \psi(a)}\right)^{\alpha-1} (\psi(\xi) - \psi(a))^{\gamma-1} d\xi \\ &= \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\gamma+\alpha-1} \int_0^1 (1-p)^{\alpha-1} p^{\gamma-1} dp \\ &= \frac{1}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\gamma+\alpha-1} \frac{\Gamma(\gamma) \Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \end{aligned}$$

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$$I_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} (\psi(t) - \psi(a))^{\gamma+\alpha-1}.$$

Next,

$$\begin{aligned} I_{a+;\psi}^\alpha E_\alpha(\delta(\psi(t) - \psi(a))^\alpha) &= \sum_{n=0}^{\infty} \frac{\delta^n}{\Gamma(n\alpha+1)} I_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{\delta^n}{\Gamma(n\alpha + \alpha + 1)} I_{a+;\psi}^\alpha (\psi(t) - \psi(a))^{n\alpha+\alpha} \\ &= \frac{1}{\delta} \sum_{n=0}^{\infty} \frac{\delta^n}{\Gamma(n\alpha + 1)} (\psi(t) - \psi(a))^{n\alpha} \\ I_{a+;\psi}^\alpha E_\alpha(\delta(\psi(t) - \psi(a))^\alpha) &= \frac{1}{\delta} (E_\alpha(\delta(\psi(t) - \psi(a))^\alpha) - 1) \end{aligned}$$

□

3. Existence and uniqueness

Our next outcome lays out an equality between issue (2) and Volterra integral equation.

Theorem:3.1 The given system of equation is equivalent to the integral equation

$$u(t) = I_{a+;\psi}^\alpha f(t, u(t), \mathfrak{D}u(t)) + u_a + \frac{u_a^1}{\psi'(a)} (\psi(t) - \psi(a)) + \tau_u (\psi(t) - \psi(a))^2, \quad t \in [a, b] \quad (6)$$

Where

$$\tau_u = \left[I_{a+;\psi}^\alpha f(b, u(b), \mathfrak{D}u(b)) - M I_{a+;\psi}^{2\alpha} f(x, u(x), \mathfrak{D}u(x)) + u_a \left(1 - \frac{M}{\Gamma(1+\alpha)} (\psi(x) - \psi(a))^\alpha \right) \right] \quad (7)$$

Proof: By utilize the theorem 1 applying the fractional integral both sides on $({}^c D_{a+;\psi}^\alpha u)(t) = f(t, u(t), \mathfrak{D}u(t))$ and we attain the relation.

$$u(t) = I_{a+;\psi}^\alpha f(t, u(t), \mathfrak{D}u(t)) + \tau_0 + \tau_1 (\psi(t) - \psi(a)) + \tau_2 (\psi(t) - \psi(a))^2$$

Here τ_0, τ_1, τ_2 are the constants.

Since $u(a) = u_a$; $u'(a) = u_a^1$ we deduce that $\tau_0 = u_a$; $\tau_1 = \frac{u_a^1}{\psi'(a)}$. Next we find the value of τ_2

$$\begin{aligned} I_{a+;\psi}^\alpha u(t) &= I_{a+;\psi}^\alpha \left(I_{a+;\psi}^\alpha f(t, u(t), \mathfrak{D}u(t)) + u_a + \frac{u_a^1}{\psi'(a)} (\psi(t) - \psi(a)) \right. \\ &\quad \left. + \tau_2 (\psi(t) - \psi(a))^2 \right) \end{aligned}$$

=

$$I_{a+;\psi}^{2\alpha} f(t, u(t), \mathfrak{D}u(t)) + \frac{u_a}{\Gamma(\alpha+1)} (\psi(t) - \psi(a))^\alpha + \frac{u_a^1}{\psi'(a) \Gamma(\alpha+2)} (\psi(t) - \psi(a))^{\alpha+1} + \frac{2\tau_2}{\Gamma(\alpha+3)} (\psi(t) - \psi(a))^{\alpha+2}$$

Since $u(b) = MI_{a+;\psi}^\alpha u(x)$, we decide the value of τ_u , in this manner finishing the initial segment of the proof. For the converse, if u satisfies condition (5), applying the fractional derivative to the two sides of the equation and going to that

$$({}^c D_{a+;\psi}^\alpha u_a 0 = ({}^c D_{a+;\psi}^\alpha (\psi(t) - \psi(a)) = ({}^c D_{a+;\psi}^\alpha (\psi(t) - \psi(a)))^2 = 0$$

□

We attain that $({}^c D_{a+;\psi}^\alpha u(t) = f(t, u(t), \mathfrak{D}u(t))$.

Note 3.2: Now we define the functional $\mathbb{F}$ based on the theorem 3

$$\begin{aligned} \mathbb{F}(u)(t) &= I_{a+;\psi}^\alpha f(t, u(t), \mathfrak{D}u(t)) + u_a + \frac{u_a^1}{\psi'(a)} (\psi(t) - \psi(a)) \\ &\quad + \tau_u (\psi(t) - \psi(a))^2 \end{aligned} \quad (8)$$

From the equation (7) we attain the value of τ_u .

(A1) Let us define a continuous function $f \in (C[a, b] \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R})$ and $u \in C[a, b]$ and there exists a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \mathfrak{M}_1 (\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for each u_1, u_2, v_1, v_2 in Y , $\mathfrak{M}_2 = \max_{t \in \mathbb{R}} \|f(t, 0, 0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$.

Let $Y = C[\mathbb{R}, X]$ be the set of continuous functions on $\mathbb{R}$ with in the Banach space X values.

(A2) Let $u' \in C[a, b]$ satisfy the Lipschitz condition. i.e., There exists a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and N such that

$$\|\mathfrak{D}(t, u) - \mathfrak{D}(t, v)\| \leq \mathfrak{N}_1 (\|u - v\|),$$

for all u, v in Y . $\mathfrak{N}_2 = \max_{t \in \mathbb{D}} \|\mathfrak{D}(t, 0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

Lemma 3.3 If (A1) and (A2) are satisfied, then the estimate $\|\mathfrak{D}u(t)\| \leq t(\mathfrak{N}_1 \|u\| + \mathfrak{N}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{N}t \|u - v\|$ are satisfied for any $t \in \mathbb{R}$ and $u, v \in Y$ and consider $\mathfrak{B} = \mathfrak{M} + \mathfrak{N}t$

Theorem:3.4 If $\exists$ an integrable function $\mathfrak{B}: [a, b] \rightarrow \mathbb{R}_0^+$ $\ni$: from the assumption there exists then a real $\mathfrak{h} > 0$ ($\mathfrak{h} \leq b - a$) then the given system (1) has a unique solution in $[a, a + \mathfrak{h}]$,

Proof: Let $\mathfrak{h} > 0$ be such that

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$$\sup_{t \in [a, a+h]} \left\{ I_{a+}^{\alpha} \mathfrak{B}(t) + [I_{a+}^{\alpha} \mathfrak{B}(b) + MI_{a+}^{2\alpha} \mathfrak{B}(x)] \frac{(\psi(t) - \psi(a))^2}{K} \right\} < 1 \quad (9)$$

And Y the set $Y = \{u \in C^2[a, a+h] : {}^c D_{a+}^{\alpha} \mathfrak{B}(t) \text{ exists and is continuous in } [a, a+h]\}.$

We prove that the function $\mathbb{F}$ is a contraction. Allow us to see that $\mathbb{F}$ is well defined. (i.e.), $\mathbb{F}(Y) \subseteq Y$. Given a function $u \in Y$, then we have the map $t \mapsto \mathbb{F}(u)(t)$ is of class C^2 and the below function is continuous. $({}^c D_{a+}^{\alpha} \mathbb{F}(u))(t) = f(t, u(t), \mathfrak{D}u(t))$

Let $u_1, u_2 \in Y$ be arbitrary, then

$$\begin{aligned} & \|\mathbb{F}(u_1) - \mathbb{F}(u_2)\| \\ & \leq \sup_{t \in [a, a+h]} \left\{ I_{a+}^{\alpha} |f(t, u_1(t), \mathfrak{D}u_1(t)) - f(t, u_2(t), \mathfrak{D}u_2(t))| \right. \\ & \quad + [I_{a+}^{\alpha} |f(t, u_1(b), \mathfrak{D}u_1(b)) - f(t, u_2(b), \mathfrak{D}u_2(b))| \\ & \quad + MI_{a+}^{2\alpha} |f(t, u_1(x), \mathfrak{D}u_1(x)) \\ & \quad \left. - f(t, u_2(x), \mathfrak{D}u_2(x))|] \frac{(\psi(t) - \psi(a))^2}{K} \right\} \\ & \leq \sup_{t \in [a, a+h]} \left\{ I_{a+}^{\alpha} (\mathfrak{M} + \mathfrak{N}t) \|u_1 - u_2\| + [I_{a+}^{\alpha} (\mathfrak{M} + \mathfrak{N}b) \|u_1 - u_2\| + \right. \\ & \quad \left. MI_{a+}^{2\alpha} (\mathfrak{M} + \mathfrak{N}x) \|u_1 - u_2\| \frac{(\psi(t) - \psi(a))^2}{K} \right\} \\ & \leq \sup_{t \in [a, a+h]} \left\{ I_{a+}^{\alpha} \mathfrak{B}(t) + [I_{a+}^{\alpha} \mathfrak{B}(b) + MI_{a+}^{2\alpha} \mathfrak{B}(x)] \frac{(\psi(t) - \psi(a))^2}{K} \right\} \|u_1 \\ & \quad - u_2\| \end{aligned}$$

Hence, $\mathbb{F}$ is a contraction, By using the Banach Fixed point theorem the given system of equation has unique solution. □

Review that a group of functions ϑ is equicontinuous if for each $\epsilon > 0$, $\exists$ some $\delta > 0$ to such an extent that $d(u(t_1), u(t_2)) < \epsilon$ for all $u \in \vartheta$ and all t_1, t_2 to such an extent that $d(t_1, t_2) < \delta$.

Theorem:3.5 Suppose that $\exists$ the following assumption hold

(A3) $\exists$ two continuous function $H_1, H_2: [a, b] \rightarrow \mathbb{R}$ and a non-decreasing function $Y: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ $\exists: \forall t \in [a, b]$ and $u \in \mathbb{R}$ we have

$$|f(t, u(t), \mathfrak{D}u(t))| \leq H_1(t)Y(\mathfrak{B}|u|) + H_2(t) \leq \mathfrak{B}H_1(t)Y(|u|) + H_2(t)$$

Then the real system $\mathbb{R} > 0$ such that

$$\begin{aligned} & \lambda_1 \mathfrak{B}Y(\mathbb{R}) + \lambda_2 + |u_a| + \frac{|u_a^1|}{\psi'(a)} (\psi(b) - \psi(a)) \\ & + \left| u_a \left[1 - \frac{M}{\Gamma(\alpha + 1)} (\psi(x) - \psi(a))^\alpha \right] \right. \\ & + \frac{|u_a^1|}{\psi'(a)} \left[\psi(b) - \psi(a) \right. \\ & \left. \left. - \frac{M}{\Gamma(\alpha + 2)} (\psi(x) - \psi(a))^{\alpha+1} \right] \right| \left| \frac{(\psi(b) - \psi(a))^2}{K} \right| \end{aligned}$$

$\leq \mathbb{R}$.

Where

$$\lambda_i = \sup_{t \in [a,b]} I_{a+;\psi}^\alpha |H_i(t)| + (I_{a+;\psi}^\alpha |H_i(b)| + MI_{a+;\psi}^{2\alpha} |H_i(x)|) \frac{(\psi(b) - \psi(a))^2}{K}, \quad i = 1, 2$$

Then by the Cauchy problem (2) has atleast one solution in $[a, b]$.

Proof: By the equation (8) $\mathbb{F}$ is given by, consider the closed ball

$$\mathbb{B}_{\mathbb{R}} = \{u \in C^2[a, b] : \|u\| < R\}$$

Step 1: $\mathbb{F}(\mathbb{B}_{\mathbb{R}}) \subseteq \mathbb{B}_{\mathbb{R}}$. Given $u \in \mathbb{B}_{\mathbb{R}}$ then

$\|\mathbb{F}(x)\|$

$$\begin{aligned} & \leq \sup_{t \in [a,b]} \left\{ I_{a+;\psi}^\alpha |f(t, u(t), \mathfrak{D}u(t))| + |u_a| + \frac{|u_a^1|}{\psi'(a)} (\psi(t) - \psi(a)) \right. \\ & \quad \left. + |\tau_u| (\psi(t) - \psi(a))^2 \right\} \\ & \leq \sup_{t \in [a,b]} \left\{ I_{a+;\psi}^\alpha |\mathfrak{B}H_1(t)Y(|u(t)|) + H_2(t)| + |u_a| + \frac{|u_a^1|}{\psi'(a)} (\psi(t) - \psi(a)) \right. \\ & \quad + [I_{a+;\psi}^\alpha |\mathfrak{B}H_1(b)Y(|u(b)|) + H_2(b)|] + MI_{a+;\psi}^{2\alpha} |\mathfrak{B}H_1(x)Y(|u(x)|) \\ & \quad + H_2(x) \\ & \quad + \left| u_a \left[1 - \frac{M}{\Gamma(\alpha + 1)} (\psi(x) - \psi(a))^\alpha \right] \right. \\ & \quad + \frac{|u_a^1|}{\psi'(a)} [\psi(b) - \psi(a) \\ & \quad \left. \left. - \frac{M}{\Gamma(\alpha + 2)} (\psi(x) - \psi(a))^{\alpha+1} \right] \right| \left| \frac{(\psi(b) - \psi(a))^2}{K} \right\} \end{aligned}$$

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$$\begin{aligned}
 &\leq \lambda_1 \mathfrak{B}(\mathbb{R}) + \lambda_2 + |u_a| + \frac{|u_a^1|}{\psi'(a)} (\psi(b) - \psi(a)) \\
 &\quad + \left| u_a \left[1 - \frac{M}{\Gamma(\alpha + 1)} (\psi(x) - \psi(a))^\alpha \right] \right. \\
 &\quad + \frac{|u_a^1|}{\psi'(a)} \left[\psi(b) - \psi(a) \right. \\
 &\quad \left. \left. - \frac{M}{\Gamma(\alpha + 2)} (\psi(x) - \psi(a))^{\alpha+1} \right] \right| \left| \frac{(\psi(b) - \psi(a))^2}{K} \right| \\
 &\leq \mathbb{R}.
 \end{aligned}$$

Step :2 $\mathbb{F}(\mathfrak{B}_{\mathbb{R}})$ is equicontinuous. Let $t_1, t_2 \in [a, b]$ with $t_1 > t_2$ and

$$L = \max_{(t,x) \in [a,b] \times \mathfrak{B}_{\mathbb{R}}} |f(t, u(t), \mathfrak{D}u(t))|$$

$$\begin{aligned}
 \text{Thus } &|\mathbb{F}(u)(t_1) - \mathbb{F}(u)(t_2)| \\
 &\leq \frac{L}{\Gamma(\alpha + 1)} [(\psi(t_1) - \psi(a))^\alpha - (\psi(t_2) - \psi(a))^\alpha] + \\
 &\quad \frac{|u_a^1|}{\psi'(a)} (\psi(t_1) - \psi(t_2)) + |\tau_u| [(\psi(t_1) - \psi(a))^2 - (\psi(t_2) - \psi(a))^2]
 \end{aligned}$$

As $t_1 \rightarrow t_2$ and which combines to zero. By the Ascoli-Arzela Hypothesis, $\mathbb{F}(\mathbb{B}_{\mathbb{R}})$ is contained in a compact set. At last, we presently demonstrate that there are no $u \in \partial \mathbb{B}_{\mathbb{R}}$ and $\rho \in (0,1) \ni u = \rho \mathbb{F}(u)$. For that reason, assume that such u and ρ exist. For this situation,

$$\mathbb{R} = \|u\| < \|\mathbb{F}(u)\| \leq \mathbb{R}$$

Now we have seen previously, which show our case. All in all, by the Leray-Schauder alternative, we demonstrate the presence of a fixed point to $\mathbb{F}$. $\square$

4. Example

In this section we discuss the problem which is related to the main result. Consider the fractional differential equation

$$\left\{ \begin{array}{l} ({}^c D_{a+;\psi}^\alpha u)(t) = \frac{t}{3\sqrt{\pi}} \sin(u(t) + u'(t)), \quad t \in [a, b] \\ u(0) = u_0 \\ u'(0) = u_0^1 \\ u(1) = MI_{0+;\psi}^{2.5} u(0.5) \end{array} \right.$$

Now consider Where $\alpha \in (2,3)$, u_a , u_a^1 , M , $x \in \mathbb{R}$. From [2] we consider the value of M . Let $u, v \in [0, \infty)$ and $t \in J$, thus we have

$$\begin{aligned} |f(t, u_1, v_1) - f(t, u_2, v_2)| &\leq \frac{t}{3\sqrt{\pi}} (|u_1 - u_2| + |v_1 - v_2|), \\ &\leq \frac{1}{3\sqrt{\pi}} |u - v| \leq \frac{1}{5} |u - v| \end{aligned}$$

Hence from the assumption $\mathfrak{B}(t) = \frac{1}{5}$, $u, v \in [0, \infty)$, then we have by the theorem 4 we conclude that

$$I_{a+}^\alpha \mathfrak{B}(t) + [I_{a+}^\alpha \mathfrak{B}(b) + MI_{a+}^{2\alpha} \mathfrak{B}(x)] \frac{(\psi(t) - \psi(a))^2}{K} < 1$$

Thus, the existence and uniqueness of the solution in the interval $[0, 1]$.

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New existence results on Caputo Fractional derivative with non-linear Integral conditions via Fixed Point theorems

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Abstract

In this article, we investigate the sufficient conditions for the existence of solutions to a Caputo fractional derivative with a class of boundary value problem dependence on the lipschitz first derivative conditions in Banach Space. Our main tool is a fixed point theorem. An numerical example is given to clarify the results.

Keywords: Fractional Calculus; fractional integral ; BVP; Caputo derivative; Lipschitz first derivatives; Existence; integral conditions; fixed point

2020 AMS subject classifications: 34A08; 34K37; 34K40; 58C30.

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1 Introduction

In the past decades, Fractional differential equations extensively appear in the study of many real world phenomena. They have used to evolve variety of different areas of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. we refer the reader to the articles (7; 8). Recently, most of the researchers have applying various fixed point theorems and get many interesting results of the existence of solutions for fractional differential equations (9; 11). Hilal et al. (12) studied the existence of an impulsive fractional integro-differential equations in a non-compact semigroup. In (13), Zhang et al. gave the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions.

Motivated by the above mentioned works, this article investigate the existence of solutions for the boundary value problem with Caputo fractional differential equations and nonlinear integral conditions in Banach space as follows

$$\begin{cases} {}^c D^\alpha u(t) = f(t, u(t), u'(t, u(t))), \text{ for all } t \in J = [0, \mathfrak{T}], 1 < \alpha \leq 2, \\ u(0) - u'(0) = \int_0^{\mathfrak{T}} p(s, u(s)) ds, \\ u(\mathfrak{T}) + u'(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s, u(s)) ds, \end{cases} \quad (1)$$

Where ${}^c D^\alpha u(t)$ is the Caputo derivative, $f \in (J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $p, q \in (J \times \mathbb{R}, \mathbb{R})$ are given continuous functions. $f(t, u(t), u'(t, u(t))) = 0$ Consider $\mathfrak{D}u(t) = u'(t, u(t))$. Then (1) becomes

$$\begin{cases} {}^c D^\alpha u(t) = f(t, u(t), \mathfrak{D}u(t)), \text{ for all } t \in J = [0, \mathfrak{T}], 1 < \alpha \leq 2, \\ u(0) - u'(0) = \int_0^{\mathfrak{T}} p(s, u(s)) ds, \\ u(\mathfrak{T}) + u'(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s, u(s)) ds, \end{cases} \quad (2)$$

The main aim of the paper is to explore the existence of solutions to the equation (2) by using various fixed point theorems. This paper is organized as five parts. In Section 2, we review few preliminaries and lemmas of fractional calculus that we require in the sequel. The existence of solutions for initial value problem for an fractional derivative results relying on various fixed point theorems for the problem (2) are proved in Section 3. Finally we present an illustrative example.

2 Facts

Here, we familiarize the some notation, definitions and basic lemmas and theorems. These are used in the further section in this paper. We denote $C(J, \mathbb{R})$ which means set of all continuous functions from J into $\mathbb{R}$ in the Banach space with the norm

$$\|u\|_{\infty} = \sup \{|u(t)| : t \in J\}$$

Let $L^1(J, R)$ denote the Banach space of functions $u: J \rightarrow R$ which is the Lebesgue integrable with norm

$$\|u\|_{L^1} = \int_0^{\mathfrak{T}} |u(t)| dt,$$

Also the second derivative u'' of the differentiable function u is continuous which is belongs to the space $C^2(J, R)$. (4; 5) Integral of the Function $f \in L^1([a, b], \mathbb{R}_+)$ with the Fractional order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

When $a = 0$, we get $I^{\alpha} f(t) = [f * \varphi_{\alpha}](t)$, where $\varphi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_{\alpha}(t) = 0$ for $t \leq 0$, and $\varphi_{\alpha} \rightarrow \delta(t)$ as $\alpha \rightarrow 0$, where δ is the delta function. (4; 5) Function $f \in [a, b]$, the α th R-L fractional order derivative of f , is given as

$$(D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α . (3) Function $f \in [a, b]$, the Caputo fractional derivative of order α of f , is defined as

$$({}^C D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s) ds,$$

where $n = [\alpha] + 1$.

Proposition 2.4 ((1; 2)) $f'(u) \in D$ satisfy the Lipschitz condition. i.e., $\exists$ a +ve constant η such that

$$\|f'(u) - f'(v)\| \leq \eta (\|u - v\|), \quad \forall u, v \in D.$$

3 Main Results

In this section, we give the existence of solutions via fixed point theorems for the problem (2)

Function $u \in C^2(J, R)$ whose α -derivative exists on J is said to be a solution of (2) if u satisfies the equation ${}^c D^\alpha u(t) = f(t, u(t), \mathfrak{D}u(t))$ on J and the conditions

$$u(0) - u'(0) = \int_0^{\mathfrak{T}} p(s, u(s)) ds, \quad \text{and} \quad u(\mathfrak{T}) + u'(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s, u(s)) ds,$$

we use the following auxiliary lemmas to prove the existence solution for the equation (2). (6) Let $\alpha > 0$; then the differential equation

$${}^c D^\alpha g(t) = 0$$

has solutions $g(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_0 t^{n-1}$, $c_i \in R$, $i = 0, 1, 2, \dots, n$, $n = [\alpha] + 1$ (6) Let $\alpha > 0$, then

$$I^{\alpha c} D^\alpha g(t) = g(t) c_0 + c_1 t + c_2 t^2 + \dots + c_0 t^{n-1}$$

for some $c_i \in R$, $i = 0, 1, 2, \dots, n$, $n = [\alpha] + 1$,

Follow up the Lemma (3) and (??) the result is useful in pursuing result. Let $1 < \alpha \leq 2$ and $\eta, \Theta_1, \Theta_2 : J \rightarrow R$ be continuous. A function u is a solution of the fractional integral equation

$$u(t) = \mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t, s) \eta(s) ds \quad (3)$$

with

$$\mathfrak{H}(t) = \frac{(\mathfrak{T} + 1 - t)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} \Theta_1(s) ds + \frac{(t + 1)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} \Theta_2(s) ds \quad (4)$$

and

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)} & t \leq s \leq \mathfrak{T}. \end{cases} \quad (5)$$

if and only if u is a solution of the fractional BVP

$${}^c D^\alpha u(t) = \eta(t), t \in J \quad (6)$$

$$u(0) - u'(0) = \int_0^{\mathfrak{T}} \Theta_1(s) ds \quad (7)$$

$$u(\mathfrak{T}) + u'(\mathfrak{T}) = \int_0^{\mathfrak{T}} \Theta_2(s) ds \quad (8)$$

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Proof Assume that (6); and Lemma (??) we get that

$$u(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) ds. \quad (9)$$

From (7) and (8), we obtain

$$c_0 - c_1 = \int_0^{\mathfrak{T}} \Theta_1(s) ds, \quad (10)$$

and

$$c_0 + c_1(\mathfrak{T}+1) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-1} \eta(s) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-2} \eta(s) ds = \int_0^{\mathfrak{T}} \Theta_2(s) ds. \quad (11)$$

From (10) and (11) implies that

$$\begin{aligned} c_1 = & \frac{1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_2(s) ds - \frac{1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_1(s) ds - \frac{1}{(\mathfrak{T}+2)\Gamma(\alpha)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-1} \eta(s) ds \\ & - \frac{1}{(\mathfrak{T}+2)\Gamma(\alpha-1)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-2} \eta(s) ds \end{aligned} \quad (12)$$

and

$$\begin{aligned} c_0 = & \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_1(s) ds + \frac{1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_2(s) ds - \frac{1}{(\mathfrak{T}+2)\Gamma(\alpha)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-1} \eta(s) ds \\ & - \frac{1}{(\mathfrak{T}+2)\Gamma(\alpha-1)} \int_0^{\mathfrak{T}} (\mathfrak{T}-s)^{\alpha-2} \eta(s) ds \end{aligned} \quad (13)$$

From (9), (12) and (13) and we utilize the fact that $\int_0^{\mathfrak{T}} = \int_0^t + \int_t^{\mathfrak{T}}$, we obtain

$$u(t) = \mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t, s) \eta(s) ds \quad (14)$$

Where

$$\mathfrak{H}(t) = \frac{(\mathfrak{T}+1-t)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_1(s) ds + \frac{(t+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \Theta_2(s) ds \quad (15)$$

and

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)} & t \leq s \leq \mathfrak{T}. \end{cases}$$

Hence we have (3). Conversely, it is clear that if u satisfies Equation (3), then (6)-(8) hold.

We need the following assumptions to prove Banach fixed point theorem. Let

$$G^* = \sup_{(t,s) \in J \times J} |G(t, s)|.$$

A₁ Let $u \in C[J, R]$ and $g \in (C[a, b] \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R})$ is continuous function and there exist a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \mathfrak{M}_1(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for all u_1, v_1, u_2, v_2 in Y , $\mathfrak{M}_2 = \max_{t \in \mathfrak{R}} \|f(t, 0, 0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[\mathfrak{R}, X]$ be the set continuous functions on $\mathfrak{R}$ with values in the Banach spaces X .

A₂ Let $u' \in C[a, b]$ satisfy the Lipschitz condition. i.e., There exist a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|\mathfrak{D}(t, u) - \mathfrak{D}(t, v)\| \leq \mathfrak{N}_1(\|u - v\|),$$

for all u, v in Y . $\mathfrak{N}_2 = \max_{t \in D} \|\mathfrak{D}(t, 0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

Assume If **A₁** and **A₂** are hold, then the estimate

$\|\mathfrak{D}u(t)\| \leq t(\mathfrak{N}_1\|u\| + \mathfrak{N}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{N}t\|u - v\|$, $\forall t \in \mathbb{R}$, and $u, v \in Y$ are satisfied. Take ϱ is $(\mathfrak{M} + \mathfrak{N})$.

Assume that

$\exists$ a +ve constant ϱ_0 such that

$$\|p(t, u_1) - p(t, u_2)\| \leq \varrho_0\|u_1 - u_2\|, \forall t \in J, \forall u_1, u_2 \in \mathbb{R}.$$

$\exists$ a +ve constant ϱ_1 such that

$$\|q(t, u_1) - q(t, u_2)\| \leq \varrho_1\|u_1 - u_2\|, \forall t \in J, \forall u_1, u_2 \in \mathbb{R}.$$

if

$$\left[\frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2}(\varrho_0 + \varrho_1) + \mathfrak{T}\varrho G^* \right] < 1 \quad (16)$$

then the BVP (2) has a unique solution on J .

Proof For the fixed point problem we reform the problem (2). Now let us consider the operator

$$F : C(J, R) \rightarrow C(J, R)$$

defined by

$$(Fu)(t) = \mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t, s)f(s, u(s), \mathfrak{D}(u(s)))ds. \quad (17)$$

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where

$$\mathfrak{H}(t) = \frac{(\mathfrak{T} + 1 - t)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} p(s, u(s)) ds + \frac{(t + 1)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} q(s, u(s)) ds$$

we have the value of the function $G(t,s)$ by (5). the soluton of the problem (2) is the fixed point operator F . we have to prove that the operator F has a fixed point for that we will use the Banach contraction principle. Here we have to prove that F is a contraction.

Take $u, v \in C(J, \mathbb{R})$. For every $t \in J$ we have

$$\begin{aligned} |(Fu)(t) - (Fv)(t)| &\leq \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |p(s, u(s)) - p(s, v(s))| ds \\ &\quad + \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |q(s, u(s)) - q(s, v(s))| ds \\ &\quad + \int_0^{\mathfrak{T}} G(s, t) |f(s, u(s), \mathfrak{D}(u(s))) - f(s, v(s), \mathfrak{D}(v(s)))| ds \\ &\leq \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_0 \|u - v\|_{\infty} + \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_1 \|u - v\|_{\infty} \\ &\quad + \mathfrak{T} G^* (\mathfrak{M} \|u - v\|_{\infty} + \mathfrak{N}_t \|u - v\|_{\infty}) \\ &\leq \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_0 \|u - v\|_{\infty} + \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_1 \|u - v\|_{\infty} \\ &\quad + \mathfrak{T} G^* \varrho \|u - v\|_{\infty} \\ &\leq \left[\frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_0 + \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} \varrho_1 + \mathfrak{T} \varrho G^* \right] \|u - v\|_{\infty} \end{aligned}$$

Consequently

$$\|F(u) - F(v)\|_{\infty} \leq \left[\frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} (\varrho_0 + \varrho_1) + \mathfrak{T} \varrho G^* \right] \|u - v\|_{\infty}$$

Hence, by (17) F is a contraction. Here we conclude that based on the fixed point theorem, the operator F has a fixed point. Hence, we get the solution of the problem (2). $\square$

Assume that

The function $f: J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

$\exists$ a +ve constant N such that

$$|f(t, u(t), \mathfrak{D}(u(t)))| \leq N \forall t \in J, \forall u \in \mathbb{R}$$

$\exists$ a +ve constant N_0 such that

$$|p(t, u(t))| \leq N_0 \forall t \in J, \forall u \in \mathbb{R}$$

$\exists$ a +ve constant N_1 such that

$$|q(t, u(t))| \leq N_1 \forall t \in J, \forall u \in \mathbb{R}$$

Then the BVP (2) has atleast one solution on J.

Proof We shall prove $\mathcal{F}$ has a fixed point using Schaefer's fixed point theorem. This proof have four steps.

Step 1 : F is continuous. Let u_n be a sequence such that $u_n \rightarrow u$ in $C(J, \mathbb{R})$. Then for each $t \in J$,

$$\begin{aligned} |(Fu_n)(t) - (Fu)(t)| &\leq \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |p(s, u(s)) - p(s, v(s))| ds \\ &\quad + \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |q(s, u(s)) - q(s, v(s))| ds \\ &\quad + G^* \int_0^{\mathfrak{T}} |f(s, u(s), \mathfrak{D}(u(s))) - f(s, v(s), \mathfrak{D}(v(s)))| ds \end{aligned}$$

Here the functions f, p and q are continuous, we have

$$\|F(u_n) - F(u)\|_{\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 2 : The function operator $\mathcal{F}$ maps into bounded sets in $J \times \mathbb{R} \times \mathbb{R}$.

It suffices to prove that for any $\gamma > 0$, $\exists$ a + constant Λ such that for each value $u \in B_{\gamma} = \{u \in C(J, \mathbb{R}) : \|u\|_{\infty} \leq \gamma\}$ we have $\|F(u)\|_{\infty} \leq \Lambda$ by (A_6) - A_8 we have

$$\begin{aligned} |(Fu)(t)| &\leq \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |p(s, u(s))| ds + \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |q(s, u(s))| ds \\ &\quad + \int_0^{\mathfrak{T}} G(s, t) |f(s, u(s), \mathfrak{D}(u(s)))| ds \\ &\leq \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} N_0 + \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} N_1 + N \mathfrak{T} G^*. \end{aligned}$$

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for all $t \in J$, thus

$$\|F(u)\|_{\infty} \leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2} \right] [N_0 + N_1] + N\mathfrak{T}G^* = \Lambda$$

Step 3: The operator F maps bounded sets into equicontinuous sets in $C(J, \mathbb{R})$.

Consider $t_1, t_2 \in J, t_1 < t_2$, and let B_{γ} be a bounded set in $C(J, \mathbb{R})$ as in Step 2. If $u \in B_{\gamma}$, then

$$\begin{aligned} & |(Fu)(t_2) - (Fu)(t_1)| \\ &= \frac{(t_2 - t_1)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |p(s, u(s))| ds + \frac{(t_2 - t_1)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |q(s, u(s))| ds \\ &+ \int_0^{\mathfrak{T}} |G(t_2, s) - G(t_1, s)| |f(s, u(s), \mathfrak{D}u(s))| ds. \\ &\leq \frac{(t_2 - t_1)\mathfrak{T}}{\mathfrak{T} + 2} N_0 + \frac{(t_2 - t_1)\mathfrak{T}}{\mathfrak{T} + 2} N_1 + \mathfrak{T}N \sup_{s \in J} |G(t_2, s) - G(t_1, s)|. \end{aligned}$$

since $t_1 \rightarrow t_2$, the above inequality tends to zero. Therefore the operator $F : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ is continuous and completely continuous, since the result of Step 1 to Step 3 with the Arzela-Ascoli theorem.

Step 4 A priori bounds.

Here we have to prove the below set is bounded.

$$\lambda = \{u \in C(J, \mathbb{R}) : u = \omega F(u) \text{ for any } 0 < \omega < 1\}$$

Consider $u \in \lambda$; then $u = \omega F(u)$ for any $0 < \omega < 1$. Thus, for all $t \in J$ we have

$$u(t) = \omega \left[\mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t, s) g(s, u(s), \mathfrak{D}u(s)) ds \right].$$

from the assumption $(A_7) - (A_9)$ implies that

$$\begin{aligned} |u(t)| &\leq \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |p(s, u(s))| ds + \frac{\mathfrak{T} + 1}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} |q(s, u(s))| ds \\ &+ \int_0^{\mathfrak{T}} G(t, s) |f(s, u(s), \mathfrak{D}(u(s)))| ds \\ &\leq \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} N_0 + \frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2} N_1 + N\mathfrak{T}G^* \end{aligned}$$

for all $t \in J$, we have

$$\|u\|_{\infty} \leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2} \right] [N_0 + N_1] + N\mathfrak{T}G^* = \Lambda$$

Therefore the set λ is bounded. As a conclusion of Schaefer's fixed point theorem. F has a fixed point that is a solution of the problem (2) $\square$

Next we confer an existence result for the problem (2) based on application of the non-linear alternative of Leray-Schauder type. Here we can cripple the conditions (A_7) - (A_9) . If the following conditions and (A_6) hold

There exists $\psi_f \in L^1(J, R^+)$ and a continuous and a non-decreasing functions $\tau : [0, \infty) \rightarrow (0, \infty)$ and $\theta : [0, \infty) \rightarrow (0, \infty)$ such that

$$|f(t, u, \mathfrak{D}(u(t)))| \leq \psi_f(t)\tau(|u|)\theta(|\mathfrak{D}u|), \quad \forall t \in J, u \in \mathbb{R}.$$

$\exists \psi_p \in L^1(J, R^+)$ and a continuous and a non-decreasing function $\tau_0 : [0, \infty) \rightarrow (0, \infty)$ such that

$$|p(t, u| \leq \psi_p(t)\tau_0(|u|), \quad \forall t \in J, u \in \mathbb{R}.$$

$\exists \psi_q \in L^1(J, R^+)$ and a continuous and a non-decreasing function $\tau_1 : [0, \infty) \rightarrow (0, \infty)$ such that

$$|q(t, u| \leq \psi_q(t)\tau_1(|u|), \quad \forall t \in J, u \in \mathbb{R}.$$

$\exists$ a number $N > 0$

$$\frac{\|u\|_{\infty}}{\mathbf{x}\tau_0(\|u\|_{\infty}) + \mathbf{y}\tau_1(\|u\|_{\infty}) + \mathbf{z}\mathfrak{N}_1G^*\tau(\|u\|_{\infty})\theta(\|u\|_{\infty}) + \mathbf{z}\mathfrak{N}_2G^*\theta^0\tau(\|u\|_{\infty})} > 1 \quad (18)$$

where

$$\mathbf{x} = \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_p(s)ds \quad \mathbf{y} = \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_q(s)ds$$

$$\mathbf{z} = \int_0^{\mathfrak{T}} \psi_f(s)ds$$

let $\theta^0 = \sup \{\theta(t) : t \in J\}$ then the BVP (2) has atleast one solution on J.

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Proof It is easy to shown that the operator defined by (17) is continuous and completely continuous. For $\omega \in [0, 1]$ and let the value u satisfy $u(t) = \omega(Fu)(t)$ for all $t \in J$. by the assumption A_9 - A_{11} , we have

$$\begin{aligned}
 |y(t)| &\leq \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_p(s) \tau_0(|u(s)|) ds + \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_q(s) \tau_1(|u(s)|) ds \\
 &\quad \int_0^{\mathfrak{T}} G(t,s) \psi_f(s) \tau(|u(s)|) \theta(|\mathfrak{D}u(s)|) ds \\
 &\leq \tau_0(\|u\|_\infty) \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_p(s) ds + \tau_1(\|u\|_\infty) \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_q(s) ds + \\
 &\quad \tau(\|u\|_\infty) \theta(t(\mathfrak{N}_1\|u\|_\infty + \mathfrak{N}_2)) G^* \int_0^{\mathfrak{T}} \psi_f(s) ds \\
 &\leq \tau_0(\|u\|_\infty) \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_p(s) ds + \tau_1(\|u\|_\infty) \frac{\mathfrak{T}+1}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \psi_q(s) ds + \\
 &\quad \mathfrak{N}_1 \theta \tau(\|u\|_\infty) \theta(\|u\|_\infty) G^* \int_0^{\mathfrak{T}} \psi_f(s) ds + \mathfrak{N}_2 \theta^0 \tau(\|u\|_\infty) G^* \int_0^{\mathfrak{T}} \psi_f(s) ds
 \end{aligned}$$

for all $t \in J$, thus,

$$\frac{\|u\|_\infty}{x\tau_0(\|u\|_\infty) + y\tau_1(\|u\|_\infty) + z\mathfrak{N}_1 G^* \tau(\|u\|_\infty) \theta(\|u\|_\infty) + z\mathfrak{N}_2 G^* \theta^0 \tau(\|u\|_\infty)} \leq 1$$

by condition (18), $\exists N$ such that $\|u\| \neq N$.

Let

$$Y = \{u \in C(J, R) : \|u\|_\infty < N\}$$

The operator $F: \overline{Y} \rightarrow C(J, R)$ is continuous and completely continuous. From the choice of Y , there is no $u \in \partial Y$ such that $u = \omega F(u)$ for some $\omega \in (0, 1)$. As a consequence of the non-linear alternative of Leray-Schauder type (?), we see that F has a fixed point u in Y that is a solution of the problem (2). $\square$ The last existence result is based on the following fixed point theorem due to Burton and Kirk (?). Let X be a Banach space, and let $C, D : X \rightarrow X$ be two operators satisfying:

- (i) C is contraction
- (ii) D is completely continuous

then either

1. the operator equation $u = C(u) + D(u)$ has a solution, or
2. the set $\lambda = \{l \in X : l = \omega C(l/\omega) + \omega D(l)\}$ is unbounded for $\omega \in (0, 1)$

Assume that (A_4) , (A_5) and (A_{10}) hold if

$$\frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2}(\varrho_0 + \varrho_1) < 1 \quad (19)$$

and

$$\limsup_{l \rightarrow +\infty} \frac{(1 - (\mathfrak{T}(\mathfrak{T} + 1)(\varrho_0 + \varrho_1)/(\mathfrak{T} + 2)))l}{\mathfrak{N}_1 \mathfrak{z} G^* \tau(l) \theta(l) + \mathfrak{N}_2 \mathfrak{z} G^* \theta^0 \tau(l) + ((\mathfrak{T}(\mathfrak{T} + 1)(\bar{p} + \bar{q}))/\mathfrak{T} + 2)} > 1 \quad (20)$$

hold, where $\bar{p} = \sup_{s \in J} |p(s, 0)|$ and $\bar{q} = \sup_{s \in J} |q(s, 0)|$, then the BVP has atleast one solution on J .

Proof Consider the operators $C, D : C(J, R) \rightarrow C(J, R)$ as defined by

$$(Cu)(t) = \frac{(\mathfrak{T} + 1 - t)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} p(s, u(s)) ds + \frac{(t + 1)}{\mathfrak{T} + 2} \int_0^{\mathfrak{T}} q(s, u(s)) ds$$

and

$$(Du)(t) = \int_0^{\mathfrak{T}} G(t, s) f(s, u(s), \mathfrak{D}(u(s))) ds,$$

where G is defined in equation (6). From A_4, A_5 and (19) we can simply prove that C is a contraction. From the assumption A_{10} the operator D is continuous. To complete the existence of a fixed point of the operator C, D , it suffices to prove that the set λ in Theorem (3) is bounded. Let $u \in \lambda$, then for each $t \in J$.

$$u(t) = \omega C\left(\frac{l}{\omega}\right)(t) + \omega D(l)(t).$$

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from A_4, A_5 and A_{10} we have

$$\begin{aligned}
 |u(t)| &\leq \frac{\omega(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} |p(s, \frac{u(s)}{\omega})| ds + \frac{\omega(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} |q(s, \frac{u(s)}{\omega})| ds \\
 &\quad + \omega \int_0^{\mathfrak{T}} |G(t, s)| f(s, u(s), \mathfrak{D}(u(t))) ds \\
 &\leq \frac{(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \varrho_0 |u(s)| ds + \frac{(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} |p(s, 0)| ds \\
 &\quad + \frac{(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} \varrho_1 |u(s)| ds + \frac{(\mathfrak{T}+1)}{\mathfrak{T}+2} \int_0^{\mathfrak{T}} |q(s, 0)| ds \\
 &\quad + G^* \int_0^{\mathfrak{T}} \psi_g(s) \tau(|u(s)|) \theta(|\mathfrak{D}u(s)|) ds \\
 &\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0 + \varrho_1)}{\mathfrak{T}+2} \|u\|_{\infty} + \frac{\mathfrak{T}(\mathfrak{T}+1)(\bar{p} + \bar{q})}{\mathfrak{T}+2} \\
 &\quad + zG^* \tau(\|u\|_{\infty}) \theta(\|\mathfrak{D}u(t)\|_{\infty}) \\
 &\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0 + \varrho_1)}{\mathfrak{T}+2} \|u\|_{\infty} + \frac{\mathfrak{T}(\mathfrak{T}+1)(\bar{p} + \bar{q})}{\mathfrak{T}+2} \\
 &\quad + zG^* \tau(\|u\|_{\infty}) \theta(t(\mathfrak{N}_1 \|u\|_{\infty} + \mathfrak{N}_2)) \\
 &\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0 + \varrho_1)}{\mathfrak{T}+2} \|u\|_{\infty} + \frac{\mathfrak{T}(\mathfrak{T}+1)(\bar{p} + \bar{q})}{\mathfrak{T}+2} \\
 &\quad + \mathfrak{N}_1 zG^* \tau(\|u\|_{\infty}) \theta(\|u\|_{\infty}) + \mathfrak{N}_2 zG^* \theta^0 \tau(\|u\|_{\infty})
 \end{aligned}$$

thus

$$\frac{(1 - (\mathfrak{T}(\mathfrak{T}+1)(\varrho_0 + \varrho_1)/(\mathfrak{T}+2)))\|u\|_{\infty}}{\mathfrak{N}_1 zG^* \theta^0 \tau(\|u\|_{\infty}) \theta(\|u\|_{\infty}) + \mathfrak{N}_2 zG^* \theta^0 \tau(\|u\|_{\infty}) + ((\mathfrak{T}(\mathfrak{T}+1)(\bar{p} + \bar{q}))/(\mathfrak{T}+2))} \leq 1 \quad (21)$$

From (20) it follows that, $\exists$ an +ve constant Λ such that $\|u\|_{\infty} > \Lambda, \forall u \in \lambda$, the above condition (21) is violated. Hence $\|u\|_{\infty} \leq \Lambda, \forall u \in \lambda$ in order that the set λ is bounded. From theorem (3), we can conclude. $\square$

4 An Example

Let us consider the following fractional BVP to illustrate our main results.

$$\begin{cases} {}^c D^{\alpha} u(t) = \frac{t}{3\sqrt{(\pi)}} \sin(u(t) + u'(t)), \forall t \in [0, 1], 1 < \alpha \leq 2 \\ u(0) - u'(0) = \sum_{i=0}^{\infty} c_i u(t_i) \\ u(T) + u'(T) = \sum_{j=0}^{\infty} d_j u(t_j) \end{cases} \quad (22)$$

where $0 < t_0 < t_1 < t_2 < \dots < 1$, $0 < \bar{t}_0 < \bar{t}_1 < \bar{t}_2 < \dots < 1$, and c_i and d_j , $i, j = 0, \dots$, are given positive constants with

$$\sum_{i=0}^{\infty} c_i < \infty \quad \sum_{j=0}^{\infty} d_j < \infty$$

Let

$$f(t, u(t), \mathfrak{D}u(t)) = \frac{t}{3\sqrt{(\pi)}} \sin(u(t) + u'(t))$$

Let $u, v \in [0, \infty)$ and $t \in [0, 1]$. then we have

$$\begin{aligned} |f(t, u_1, v_1) - f(t, u_2, v_2)| &\leq \frac{t}{3\sqrt{(\pi)}} (|u_1 - u_2| + |v_1 - v_2|), \\ &\leq \frac{1}{3\sqrt{(\pi)}} |u - v| \\ &\leq \frac{1}{5} |u - v|. \end{aligned}$$

hence the condition A_1 and A_2 holds with $\varrho = (1/5)$ clearly A_4 and A_5 now we check the condition (16) is satisfied with $\mathfrak{T} = 1$, $\varrho_0 = \sum_{i=0}^{\infty} c_i$, $\varrho_1 = \sum_{i=0}^{\infty} d_i$, and G is given by

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)} & t \leq s \leq 1. \end{cases} \quad (23)$$

from above equation we have

$$G^* < \frac{5}{3\Gamma(\alpha)} + \frac{2}{3\Gamma(\alpha-1)}$$

Then,

$$\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2} \varrho_0 + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2} \varrho_1 + \mathfrak{T} \varrho G^* = \frac{2}{3} \left(\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j \right) + \frac{5}{3\Gamma(\alpha)} + \frac{2}{15\Gamma(\alpha-1)} < 1 \quad (24)$$

the equation is contented for relevent values of c_i, d_j , and $\alpha \in (1, 2]$. We get from Theorem (3) that the problem (22) has a unique solution on $[0, 1]$ for such values of $\alpha \in (1, 2]$

Remark 1 we can choose, for instance the constants c_i and d_j to be

$$c_i = \frac{2}{25} \left(\frac{1}{5}\right)^i \quad d_j = \frac{2}{75} \left(\frac{1}{5}\right)^j$$

in this case

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$$\sum_{i=0}^{\infty} c_i = \frac{1}{10} \qquad \sum_{j=0}^{\infty} d_j = \frac{1}{30}$$

then the equation gives

$$\frac{1}{3\Gamma(\alpha)} + \frac{2}{15(\Gamma(\alpha - 1))} < \frac{41}{45}$$

which is satisfied for each $\alpha \in (1, 2]$. Indeed the numerical calculation shows that the function

$$W(\alpha) = \frac{1}{3\Gamma(\alpha)} + \frac{2}{15\Gamma(\alpha - 1)} - \frac{41}{45}$$

is negative for all $\alpha \in (1, 2]$.

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Separation Axioms on Soft $S\mathcal{I}_{g\delta s}$ -Closed Sets

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Abstract

In this paper, we have introduce and study weak separation axioms such as soft $\mathcal{I}_{g\delta s}$ - T_0 , soft $\mathcal{I}_{g\delta s}$ - T_1 , soft $\mathcal{I}_{g\delta s}$ - T_2 , δ - $T_{3/4}$, $\mathcal{I}_{g\delta s}$ -regular space in soft ideal topological spaces and also investigate some of its characterizations and properties with the help of general topology concepts. Also give relationships between the above new regular spaces. Then we give sufficient and necessity condition between soft $S\mathcal{I}_{g\delta s}$ -closed sets and soft singleton sets in soft $\mathcal{I}_{g\delta s} - T_1$ space.

Keywords: soft ideal topological space, soft $\mathcal{I}_{g\delta s}$ - T_0 space, soft $\mathcal{I}_{g\delta s}$ - T_1 space, soft $\mathcal{I}_{g\delta s}$ - T_2 space, δ - $T_{3/4}$ space, $\mathcal{I}_{g\delta s}$ -regular space.

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1 Introduction

Ideal is a key idea in topological space and has a significant impact on research into topological spaces. The notion of the ideal topological spaces introduced and studied by Kuratowski[11] and Vaidyanathaswamy[23]. In 1999, Molodtsov[15] introduced the new notion soft sets and Shabir and Naz[20] introduced soft topological spaces using soft sets in 2011. After that many researchers working with soft topological spaces. The properties and applications of soft set theory have been studied [3, 10, 16]. In recent years, many interesting applications of soft set theory have been expanded by incorporating fuzzy set concepts [1, 2, 4, 12, 13, 14, 16].

In this paper, we have introduce and study weak separation axioms such as soft $\mathcal{I}_{g\delta s}$ - T_0 , soft $\mathcal{I}_{g\delta s}$ - T_1 , soft $\mathcal{I}_{g\delta s}$ - T_2 , δ - $T_{3/4}$, $\mathcal{I}_{g\delta s}$ -regular space in soft ideal topological spaces and also investigate some of its characterizations and properties with the help of general topology concepts. Also give relationships between the above new regular spaces. Then we give sufficient and necessity condition between soft $\mathcal{I}_{g\delta s}$ -closed sets and soft singleton sets in soft $\mathcal{I}_{g\delta s}$ - T_1 space.

2 Preliminaries

Definition 2.1. [15] A soft set F_A over the universe X is defined by the set of ordered pairs

$$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in \wp(X)\}$$

where $F_A : E \rightarrow \wp(X)$, such that $F_A(e) \neq \emptyset$, if $e \in A \subset E$ and $F_A(e) = \emptyset$ if $e \notin A$. The family of all soft sets over X is denoted by $SS(X)$.

Definition 2.2. [14] The soft set $F_\emptyset$ over a common universe set X is said to be null soft set, denoted by $\emptyset$. Here $F_\emptyset(e) = \emptyset$, $\forall e \in E$.

Definition 2.3. [14] A soft set F_A over X is called an absolute soft set, denoted by $\tilde{A}$, if $e \in A$, $F_A(e) = X$.

Definition 2.4. [24] A soft set F_A in a soft topological space (X, τ, E) is said to be a soft regular open (resp. soft regular closed) if $F_A = \text{int}(\text{cl}(F_A))$ (resp. $F_A = \text{cl}(\text{int}(F_A))$).

Definition 2.5. Let $\mathcal{I}$ be a non-null collection of soft sets over a universe X with the same set of parameters E . Then $\mathcal{I} \subset SS(X)$ is called a soft ideal on X with the same set E if

- (1) $F_A \in \mathcal{I}$ and $G_A \in \mathcal{I} \Rightarrow F_A \cup G_A \in \mathcal{I}$.
- (2) $F_A \in \mathcal{I}$ and $G_A \subset F_A \Rightarrow G_A \in \mathcal{I}$

Definition 2.6. Let (X, τ, E) be a soft topological space and $\mathcal{I}$ be a soft ideal over X with the same set of parameters E . Then $F_A^* = \bigcup \{x_e \in X : O_{x_e} \cap F_A \notin \mathcal{I}, \text{ for all } O_{x_e} \in \tau\}$ is called the soft local function of F_A with respect to $\mathcal{I}$ and τ , where O_{x_e} is a τ -open set containing x_e .

Definition 2.7. [8] Let F_A be a soft subset of soft topological space (X, τ, E) . Then

1. x_e is called a soft δ -cluster point of F_A if $F_A \cap \text{int}(\text{cl}(U_A)) \neq \emptyset$ for every soft open set U_A containing x_e .
2. The family of all soft δ -cluster point of F_A is called the soft δ -closure of F_A and is denoted by $\text{cl}_\delta(F_A)$.
3. A soft subset F_A is said to be soft δ -closed if $\text{cl}_\delta(F_A) = F_A$. The complement of a soft δ -closed set of X is said to be soft δ -open.

Definition 2.8. A soft subset F_A of a soft topological space (X, τ, E) is said to be

1. soft preopen if $F_A \subset \text{int}(\text{cl}(F_A))$,
2. soft semiopen if $F_A \subset \text{cl}(\text{int}(F_A))$,
3. soft α -open if $F_A \subset \text{int}(\text{cl}(\text{int}(F_A)))$.

The complement of a soft preopen (resp. soft semiopen, soft α -open) set is called a soft preclosed (resp. soft semiclosed, soft α -closed) set. The set of all soft preopen (resp. soft semiopen, soft α -open, soft preclosed, soft semiclosed, soft α -closed) sets of (X, τ, E) is denoted by $\text{SPO}(X)$ (resp. $\text{SSO}(X)$, $\text{S}\alpha\text{O}(X)$, $\text{SPC}(X)$, $\text{SSC}(X)$, $\text{S}\alpha\text{C}(X)$).

$\text{int}_S(F_A) = \{\bigcup U_A : U_A \subset F_A \text{ and } U_A \text{ is soft semiopen sets}\}$ and $\text{cl}_S(F_A) = \{\bigcap G_A : F_A \subset G_A \text{ and } G_A \text{ is soft semiclosed}\}$.

Definition 2.9. [8] A soft subset F_A of a soft ideal topological space $(X, \tau, E, \mathcal{I})$ is said to be

1. soft pre- $\mathcal{I}$ -open if $F_A \subset \text{int}(\text{cl}(F_A))$,
2. soft semi- $\mathcal{I}$ -open if $F_A \subset \text{cl}(\text{int}(F_A))$,
3. soft α - $\mathcal{I}$ -open if $F_A \subset \text{int}(\text{cl}(\text{int}(F_A)))$.

The set of all soft semi- $\mathcal{I}$ -open sets is denoted by $\text{SSI}\mathcal{O}(X)$

The complement of soft pre- $\mathcal{I}$ -open (resp. soft semi- $\mathcal{I}$ -open, soft α - $\mathcal{I}$ -open) set is called a soft pre- $\mathcal{I}$ -closed (resp. soft semi- $\mathcal{I}$ -closed, soft α - $\mathcal{I}$ -closed).

Definition 2.10. The soft semi- $\mathcal{I}$ -closure of F_A is defined by the intersection of all soft semi- $\mathcal{I}$ -closed sets containing F_A and is denoted by $S\mathcal{I}_{scl}(F_A)$

Definition 2.11. [18] A soft set F_A of soft ideal topological space X is called soft generalized δ semi-closed (briefly soft $\mathcal{I}_{g\delta s}$ -closed) set if $S\mathcal{I}_{scl}(F_A) \subset G_A$ whenever $F_A \subset G_A$ and G_A is soft δ -open over X .

A soft set F_A of X is called soft generalized δ semi-open (briefly soft $S\mathcal{I}_{g\delta s}$ -open) set if F_A^c is soft $S\mathcal{I}_{g\delta s}$ -closed.

The family of all soft $S\mathcal{I}_{g\delta s}$ -closed subsets of the space X is denoted by $S\mathcal{I}_{g\delta s}\text{-}C(X)$ and soft $S\mathcal{I}_{g\delta s}$ -open subsets of the space X is denoted by $S\mathcal{I}_{g\delta s}\text{-}O(X)$.

Definition 2.12. [18] Let X be a soft ideal topological space and F_A be a soft subset over X . Then soft $S\mathcal{I}_{g\delta s}$ -closure of F_A denoted by $S\mathcal{I}_{g\delta s}\text{-}cl(F_A)$ and defined as the intersection of all soft $S\mathcal{I}_{g\delta s}$ -closed sets over X containing F_A .

Definition 2.13. [9] Let (X, τ, E) and (Y, σ, K) be soft topological spaces and soft function $f: (X, \tau, E) \rightarrow (Y, \sigma, K)$ is called

1. soft continuous if $f^{-1}(F_B)$ is soft open over X for every soft open set F_B over Y .
2. soft open (resp. soft closed) if $f(F_A)$ is soft open over Y for every soft open set F_A over X (resp. $f(F_A)$ is soft closed over Y for every soft closed set F_A over X).
3. soft semiopen (resp. soft semiclosed) if $f(F_A)$ is soft semiopen over Y for every soft open set F_A over X (resp. $f(F_A)$ is soft semiclosed over Y for every soft closed set F_A over X).
4. soft semicontinuous if $f^{-1}(F_B)$ is soft semiclosed over X for every soft closed set F_B over Y .
5. soft irresolute if $f^{-1}(F_B)$ is soft semiopen over X for every soft semiopen set F_B over Y .

Definition 2.14. A function $f : (X, \tau, E) \rightarrow (Y, \sigma, K, \mathcal{I})$ is said to be soft $S\mathcal{I}_{pg\delta s}$ closed if $f(V_A)$ is soft $S\mathcal{I}_{g\delta s}$ -closed over Y for every soft semiclosed set V_A over X .

Definition 2.15. 1. A function $f : (X, \tau, E, \mathcal{I}) \rightarrow (Y, \sigma, K, \mathcal{J})$ is soft $S\mathcal{I}_{g\delta s}$ -irresolute if $f^{-1}(V_B)$ is soft $S\mathcal{I}_{g\delta s}$ -closed over X for every soft $S\mathcal{I}_{g\delta s}$ -closed set V_B over Y .

2. A function $f : (X, \tau, E, \mathcal{I}) \rightarrow (Y, K, \sigma)$ is soft $S\mathcal{I}_{g\delta s}$ -continuous if $f^{-1}(V_B)$ is soft $S\mathcal{I}_{g\delta s}$ -closed over X for every soft closed set V_B over Y .

Definition 2.16. [7]

1. Two soft subsets G_A, H_A over X are said to be soft distinct, written $G_A \cap H_A = \emptyset$ if $G(e) \cap H(e) = \emptyset$, for all $e \in A$.
2. Two soft points x_e, y_e over X are distinct, written $x_e \neq y_e$, if there soft sets G_A and H_A are disjoint.
3. A soft ideal topological space X is said to be soft T_0 space if for each pair of distinct soft points x_e and y_e over X , there exists a soft open set containing one soft point but not the other.
4. A soft ideal topological space X is said to be soft T_1 space if for any pair of distinct soft points x_e and y_e , there exist soft open sets G_A and H_A such that $x_e \in G_A, y_e \notin G_A$ and $x_e \notin H_A, y_e \in H_A$.
5. A soft ideal topological space X is said to be soft T_2 space if for any pair of distinct soft points x_e and y_e , there exist disjoint soft open sets G_A and H_A such that $x_e \in G_A$ and $y_e \in H_A$.

Definition 2.17. [25] A soft topological space (X, τ, E) is said to be soft regular if for all soft closed sets F_A over X and each soft point x_e such that $x_e \notin F_A$, there exists soft open sets U_A and V_A over X such that $x_e \in U_A, F_A \subset V_A$ and $U_A \cap V_A = \emptyset$.

Definition 2.18. A soft ideal topological space X is said to be soft $TL_{g\delta s}$ -space if every soft $SL_{g\delta s}$ -closed set is soft closed over X .

Definition 2.19. 1. A function $f : (X, \tau, E) \rightarrow (Y, \sigma, K)$ is said to be perfectly soft continuous, if $f^{-1}(V_B)$ is soft clopen over X for every soft open set V_B over Y .

2. A function $f : (X, \tau, E) \rightarrow (Y, \sigma, K, \mathcal{I})$ is said to be perfectly soft $SL_{g\delta s}$ -continuous (resp. totally soft semicontinuous), if $f^{-1}(V_B)$ is soft clopen over X for every soft $SL_{g\delta s}$ -open (resp. soft semiopen) set V_B over Y .

Definition 2.20. A function $f : (X, \tau, E) \rightarrow (Y, \sigma, K, \mathcal{I})$ is said to be strongly soft $SL_{g\delta s}$ -closed (resp. strongly soft $SL_{g\delta s}$ -open), if $f(F_A)$ is soft $SL_{g\delta s}$ -closed (resp. soft $SL_{g\delta s}$ -open) set over Y for every soft $SL_{g\delta s}$ -closed (resp. soft $SL_{g\delta s}$ -open) set F_A over X .

Definition 2.21. A function $f : (X, \tau, E) \rightarrow (Y, \sigma, K, \mathcal{I})$ is said to be quasi soft $SL_{g\delta s}$ -closed (resp. quasi soft $SL_{g\delta s}$ -open) if for each soft $SL_{g\delta s}$ -closed (resp. soft $SL_{g\delta s}$ -open) set F_A over X , $f(F_A)$ is soft closed (resp. open) set over Y .

3 soft Separation Axioms

Definition 3.1. A soft ideal topological space X is said to be soft $\mathcal{I}_{g\delta s}$ - T_0 space if for each pair of distinct soft points x_e and y_e over X , there exists a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set containing one soft point but not the other.

Theorem 3.1. A soft ideal topological space X is a soft $\mathcal{I}_{g\delta s}$ - T_0 space if and only if soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closures of distinct soft points are distinct.

Proof: Let x_e and y_e be distinct soft points over X . Since X is soft $\mathcal{I}_{g\delta s}$ - T_0 space, there exists a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set G_A such that $x_e \in G_A$ and $y_e \notin G_A$. Consequently, $X - G_A$ is a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closed set containing y_e but not x_e . But $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ is the intersection of all soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closed sets containing y_e . Hence $y_e \in \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ but $x_e \notin \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ as $x_e \notin X - G_A$. Therefore, $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{x_e\} \neq \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$.

Conversely, let $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{x_e\} \neq \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ for $x_e \neq y_e$. Then there exists at least one soft point z_e over X such that $z_e \in \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{x_e\}$ but $z_e \notin \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$. We claim $x_e \notin \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$, because if $x_e \in \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ then $\{x_e\} \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$ implies $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{x_e\} \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$. So $z_e \in \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$, which is a contradiction. Hence $x_e \notin \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$, which implies $x_e \in X - \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}\{y_e\}$, which is a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set containing x_e but not y_e . Hence X is soft $\mathcal{I}_{g\delta s}$ - T_0 space.

Theorem 3.2. If $f : X \rightarrow Y$ is a bijection, strongly soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open and X is soft $\mathcal{I}_{g\delta s}$ - T_0 space, then Y is also soft $\mathcal{I}_{g\delta s}$ - T_0 space.

Proof: Let y_e and z_e be two distinct soft points over Y . Since f is bijective there exist distinct soft points o_e and x_e over X such that $f(o_e) = y_e$ and $f(x_e) = z_e$. Since X is soft $\mathcal{I}_{g\delta s}$ - T_0 space there exists a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set G_A such that $o_e \in G_A$ and $x_e \notin G_A$. Therefore $y_e = f(o_e) \in f(G_A)$ and $z_e = f(x_e) \notin f(G_A)$. Since f being strongly soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open function, $f(G_A)$ is soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open over Y . Thus, there exists a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set $f(G_A)$ over Y such that $y_e \in f(G_A)$ and $z_e \notin f(G_A)$. Therefore Y is soft $\mathcal{I}_{g\delta s}$ - T_0 space.

Definition 3.2. A soft ideal topological space X is said to be soft $\mathcal{I}_{g\delta s}$ - T_1 space if for any pair of distinct soft points x_e and y_e , there exist soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open sets G_A and H_A such that $x_e \in G_A$, $y_e \notin G_A$ and $x_e \notin H_A$, $y_e \in H_A$.

Theorem 3.3. A soft ideal topological space X is soft $\mathcal{I}_{g\delta s}$ - T_1 space if and only if soft singleton are soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closed sets.

Proof: Let X be a soft $\mathcal{I}_{g\delta s}$ - T_1 space and $x_e \in K_A$. Let $y_e \in K_A - \{x_e\}$. Then for $x_e \neq y_e$, there exists soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set U_A such that $y_e \in U_A$ and $x_e \notin U_A$.

Separation Axioms on Soft $SL_{g\delta s}$ -Closed Sets

Consequently, $y_e \in U_A \subset X - \{x_e\}$. That is $X - \{x_e\} = \cup \{U_A : y_e \in K_A - \{x_e\}\}$, which is soft $SL_{g\delta s}$ -open set. Hence $\{x_e\}$ is soft $SL_{g\delta s}$ -closed set.

Conversely, suppose $\{x_e\}$ is soft $SL_{g\delta s}$ -closed set for every soft point x_e over X . Let x_e and y_e soft point over X with $x_e \neq y_e$. Now $x_e \neq y_e$ implies $y_e \in X - \{x_e\}$. Hence $X - \{x_e\}$ is soft $SL_{g\delta s}$ -open set containing y_e but not x_e . Similarly, $X - \{y_e\}$ is soft $SL_{g\delta s}$ -open set containing x_e but not y_e . Therefore X is soft $IL_{g\delta s}$ - T_1 space.

Theorem 3.4. *The property being soft $IL_{g\delta s}$ - T_1 space is preserved under bijection and strongly soft $SL_{g\delta s}$ -open function.*

Proof: Let $f : X \rightarrow Y$ be bijective and strongly soft $SL_{g\delta s}$ -open function. Let X be a soft $IL_{g\delta s}$ - T_1 space and y_e, z_e be any two distinct soft points over Y . Since f is bijective there exist distinct soft points o_e, x_e over X such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Now X being a soft $IL_{g\delta s}$ - T_1 space, there exist soft $SL_{g\delta s}$ -open sets G_A and H_A such that $o_e \in G_A, x_e \notin G_A$ and $o_e \notin H_A, x_e \in H_A$. Therefore $y_e = f(o_e) \in f(G_A)$ but $z_e = f(x_e) \notin f(G_A)$ and $z_e = f(x_e) \in f(H_A)$ and $y_e = f(o_e) \notin f(H_A)$. Now f being strongly soft $SL_{g\delta s}$ -open, $f(G_A)$ and $f(H_A)$ are soft $SL_{g\delta s}$ -open subsets over Y such that $y_e \in f(G_A)$ but $z_e \notin f(G_A)$ and $z_e \in f(H_A)$ and $y_e \notin f(H_A)$. Hence Y is soft $IL_{g\delta s}$ - T_1 space.

Theorem 3.5. *Let $f : X \rightarrow Y$ be bijective and soft $SL_{g\delta s}$ -open function. If X is soft $IL_{g\delta s}$ - T_1 and soft $TL_{g\delta s}$ -space, then Y is soft $IL_{g\delta s}$ - T_1 space.*

Proof: Let y_e, z_e be any two distinct soft points over Y . Since f is bijective there exist distinct soft points o_e, x_e over X such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Now X being a soft $IL_{g\delta s}$ - T_1 space, there exist soft $SL_{g\delta s}$ -open sets G_A and H_A such that $o_e \in G_A, x_e \notin G_A$ and $o_e \notin H_A, x_e \in H_A$. Therefore $y_e = f(o_e) \in f(G_A)$ but $z_e = f(x_e) \notin f(G_A)$ and $z_e = f(x_e) \in f(H_A)$ and $y_e = f(o_e) \notin f(H_A)$. Now X is soft $TL_{g\delta s}$ -space which implies G_A and H_A are soft open sets over X and f is soft $SL_{g\delta s}$ -open function, $f(G_A)$ and $f(H_A)$ are soft $SL_{g\delta s}$ -open subsets over Y . Thus there exist soft $SL_{g\delta s}$ -open sets such that $y_e \in f(G_A)$ but $z_e \notin f(G_A)$ and $z_e \in f(H_A)$ and $y_e \notin f(H_A)$. Hence Y is soft $IL_{g\delta s}$ - T_1 space.

Theorem 3.6. *If $f : X \rightarrow Y$ is soft $SL_{g\delta s}$ -continuous injection and Y is soft T_1 space then X is soft $IL_{g\delta s}$ - T_1 space.*

Proof: Let $f : X \rightarrow Y$ be soft $SL_{g\delta s}$ -continuous injection and Y be soft T_1 . For any two distinct soft points o_e, x_e over X there exist distinct soft points y_e, z_e over Y such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Since Y is soft T_1 space there exist soft open sets G_A and H_A over Y such that $y_e \in G_A, z_e \notin G_A$ and $y_e \notin H_A, z_e \in H_A$. That is $o_e \in f^{-1}(G_A), o_e \notin f^{-1}(H_A)$ and $x_e \in f^{-1}(H_A), x_e \notin f^{-1}(G_A)$. Since f is soft $SL_{g\delta s}$ -continuous $f^{-1}(G_A), f^{-1}(H_A)$ are soft $SL_{g\delta s}$ -open sets over X . Thus, for two distinct soft points o_e, x_e over X there exist soft

$S\mathcal{I}_{g\delta s}$ -open sets $f^{-1}(G_A)$ and $f^{-1}(H_A)$ such that $o_e \in f^{-1}(G_A)$, $o_e \notin f^{-1}(H_A)$ and $x_e \in f^{-1}(H_A)$, $x_e \notin f^{-1}(G_A)$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_1 space.

Theorem 3.7. *If $f : X \rightarrow Y$ is soft $S\mathcal{I}_{g\delta s}$ -irresolute injective function and Y is soft $\mathcal{I}_{g\delta s}$ - T_1 space then X is soft $\mathcal{I}_{g\delta s}$ - T_1 space.*

Proof: Let o_e, x_e be pair of distinct soft points over X . Since f is injective there exist distinct soft points y_e, z_e over Y such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Since Y is soft $\mathcal{I}_{g\delta s}$ - T_1 space there exist soft $S\mathcal{I}_{g\delta s}$ -open sets G_A and H_A over Y such that $y_e \in G_A$, $z_e \notin G_A$ and $y_e \notin H_A$, $z_e \in H_A$. That is $o_e \in f^{-1}(G_A)$, $o_e \notin f^{-1}(H_A)$ and $x_e \in f^{-1}(H_A)$, $x_e \notin f^{-1}(G_A)$. Since f is soft $S\mathcal{I}_{g\delta s}$ -irresolute $f^{-1}(G_A)$, $f^{-1}(H_A)$ are soft $S\mathcal{I}_{g\delta s}$ -open sets over X . Thus, for two distinct soft points o_e, x_e over X there exist soft $S\mathcal{I}_{g\delta s}$ -open sets $f^{-1}(G_A)$ and $f^{-1}(H_A)$ such that $o_e \in f^{-1}(G_A)$, $o_e \notin f^{-1}(H_A)$ and $x_e \in f^{-1}(H_A)$, $x_e \notin f^{-1}(G_A)$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_1 space.

Definition 3.3. *A soft ideal topological space X is said to be soft $\mathcal{I}_{g\delta s}$ - T_2 space if for any pair of distinct soft points x_e and y_e , there exist disjoint soft $S\mathcal{I}_{g\delta s}$ -open sets G_A and H_A such that $x_e \in G_A$ and $y_e \in H_A$.*

Theorem 3.8. *If $f : X \rightarrow Y$ is soft $S\mathcal{I}_{g\delta s}$ -continuous injection and Y is soft T_2 then X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.*

Proof: Let $f : X \rightarrow Y$ be soft $S\mathcal{I}_{g\delta s}$ -continuous injection and Y be soft T_2 . For any two distinct soft points o_e, x_e over X there exist distinct soft points y_e, z_e over Y such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Since Y is soft T_2 space there exist disjoint soft open sets G_A and H_A over Y such that $y_e \in G_A$ and $z_e \in H_A$. That is $o_e \in f^{-1}(G_A)$ and $x_e \in f^{-1}(H_A)$. Since f is soft $S\mathcal{I}_{g\delta s}$ -continuous $f^{-1}(G_A)$, $f^{-1}(H_A)$ are soft $S\mathcal{I}_{g\delta s}$ -open sets over X . Further f is injective, $f^{-1}(G_A) \cap f^{-1}(H_A) = f^{-1}(G_A \cap H_A) = f^{-1}(\emptyset) = \emptyset$. Thus, for two disjoint soft points o_e, x_e over X there exist disjoint soft $S\mathcal{I}_{g\delta s}$ -open sets $f^{-1}(G_A)$ and $f^{-1}(H_A)$ such that $o_e \in f^{-1}(G_A)$ and $x_e \in f^{-1}(H_A)$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.

Theorem 3.9. *If $f : X \rightarrow Y$ is soft $S\mathcal{I}_{g\delta s}$ -irresolute injective function and Y is soft $\mathcal{I}_{g\delta s}$ - T_2 space then X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.*

Proof: Let o_e, x_e be pair of distinct soft points over X . Since f is injective there exist distinct soft points y_e, z_e over Y such that $y_e = f(o_e)$ and $z_e = f(x_e)$. Since Y is soft $\mathcal{I}_{g\delta s}$ - T_2 space there exist disjoint soft $S\mathcal{I}_{g\delta s}$ -open sets G_A and H_A over Y such that $y_e \in G_A$ and $z_e \in H_A$. That is $o_e \in f^{-1}(G_A)$ and $x_e \in f^{-1}(H_A)$. Since f is soft $S\mathcal{I}_{g\delta s}$ -irresolute injective $f^{-1}(G_A)$, $f^{-1}(H_A)$ are distinct soft $S\mathcal{I}_{g\delta s}$ -open sets over X . Thus, for two disjoint soft points o_e, x_e over X there exist disjoint soft $S\mathcal{I}_{g\delta s}$ -open sets $f^{-1}(G_A)$ and $f^{-1}(H_A)$ such that $o_e \in f^{-1}(G_A)$ and $x_e \in f^{-1}(H_A)$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.

Definition 3.4. A soft ideal topological space X is called soft δ - $T_{3/4}$ if every soft $ST_{g\delta s}$ -closed set is soft δ - $\mathcal{I}$ -closed.

Theorem 3.10. For a soft ideal topological space X , if X is a soft δ - $T_{3/4}$ -space, then every singleton soft set $\{x_e\}$ is soft δ - $\mathcal{I}$ -open or soft δ - $\mathcal{I}$ -closed.

Proof: Suppose X is a soft δ - $T_{3/4}$ space. If $\{x_e\}$ is not soft δ - $\mathcal{I}$ -closed, then $X - \{x_e\}$ is not soft δ - $\mathcal{I}$ -open. Then the only soft δ - $\mathcal{I}$ -open set containing $X - \{x_e\}$ is X . Therefore $X - \{x_e\}$ is soft $ST_{g\delta s}$ -closed set over X . Since X is a soft δ - $T_{3/4}$ space, $X - \{x_e\}$ is soft δ - $\mathcal{I}$ -closed, which implies $\{x_e\}$ is soft δ - $\mathcal{I}$ -open.

Definition 3.5. A soft subset F_A over X is soft δ - $\mathcal{I}$ -nowhere dense if $\text{int}(cl_\delta^*(F_A)) = \emptyset$.

Lemma 3.1. Let X be a soft ideal topological space then, every singleton soft set is soft δ - $\mathcal{I}$ -nowhere dense or soft δ -pre- $\mathcal{I}$ -open over X .

Proof: Suppose $\{x_e\}$ is not a soft δ - $\mathcal{I}$ -nowhere dense over X . Therefore $\text{int}(cl_\delta^*(F_A)) \neq \emptyset$ which implies that $\{x_e\} \subset \text{int}(cl_\delta^*(\{x_e\}))$. Hence $\{x_e\}$ is a soft δ -pre- $\mathcal{I}$ -open over X .

Definition 3.6. A soft ideal topological space X is called soft δ - $T_{1/2}$ space if every soft $ST_{g\delta s}$ -closed set is soft semi- $\mathcal{I}$ -closed.

Theorem 3.11. For a soft ideal topological space X the following conditions are equivalent

1. X is soft δ - $T_{1/2}$ space.
2. Every soft singleton set is either soft δ - $\mathcal{I}$ -closed or soft semi- $\mathcal{I}$ -open.

Proof: (1) $\Rightarrow$ (2) If $\{x_e\}$ is not soft δ - $\mathcal{I}$ -closed, then $X - \{x_e\}$ is not soft δ - $\mathcal{I}$ -open. Then the only soft δ - $\mathcal{I}$ -open set containing $X - \{x_e\}$ is X . Therefore $X - \{x_e\}$ is soft $ST_{g\delta s}$ -closed set over X . By (1), $X - \{x_e\}$ is soft semi- $\mathcal{I}$ -closed, which implies $\{x_e\}$ is soft semi- $\mathcal{I}$ -open.

(2) $\Rightarrow$ (1) Let F_A be soft subest over X and $x_e \in ST_{scl}(F_A)$. Then consider the following cases

Case (1): Let $\{x_e\}$ be soft semi- $\mathcal{I}$ -open. Since $x_e \in ST_{scl}(F_A)$, then $\{x_e\} \cap F_A \neq \emptyset$. This implies $x_e \in F_A$.

Case (2): Let $\{x_e\}$ be soft δ - $\mathcal{I}$ -closed. Assume that $x_e \notin F_A$, then $x_e \in ST_{scl}(F_A) - F_A$, which implies $\{x_e\} \subset ST_{scl}(F_A) - F_A$. This is not possible according to Theorem ???. This shows that, $x_e \in F_A$.

So in both cases, $ST_{scl}(F_A) \subset F_A$. Since the reverse inclusion is trivial, implies $ST_{scl}(F_A) = F_A$. Therefore F_A is soft semi- $\mathcal{I}$ -closed.

Theorem 3.12. *Every soft δ - $T_{3/4}$ space is soft δ - $T_{1/2}$ space.*

Proof: Let X be a soft δ - $T_{3/4}$ space. Then by Theorem 3.10, every soft singleton set over X is soft δ - $\mathcal{I}$ -open or soft δ - $\mathcal{I}$ -closed. But every soft δ - $\mathcal{I}$ -open set is soft semi- $\mathcal{I}$ -open set. Thus every soft singleton set over X is soft semi- $\mathcal{I}$ -open or soft δ - $\mathcal{I}$ -closed. By theorem 3.11, X is soft δ - $T_{1/2}$ space.

Theorem 3.13. *For any soft ideal topological space X*

1. $SSIO(X) \subset S\mathcal{I}_{g\delta s}\text{-}O(X)$.
2. *A space X is soft δ - $T_{1/2}$ -space if and only if $SSIO(X) = S\mathcal{I}_{g\delta s}\text{-}O(X)$.*

Proof: (1) If F_A is soft semi- $\mathcal{I}$ -open, then $X - F_A$ is soft semi- $\mathcal{I}$ -closed. So $X - F_A$ is soft $S\mathcal{I}_{g\delta s}$ -closed, this implies F_A is soft $S\mathcal{I}_{g\delta s}$ -open. Hence $SSIO(X) \subset S\mathcal{I}_{g\delta s}\text{-}O(X)$.

(2) Let F_A be a soft δ - $T_{1/2}$ space and $F_A \in S\mathcal{I}_{g\delta s}\text{-}O(X)$. Then $X - F_A$ is soft $S\mathcal{I}_{g\delta s}$ -closed set. By hypothesis, $X - F_A$ is soft semi- $\mathcal{I}$ -closed and hence $F_A \in SSIO(X)$. Therefore $S\mathcal{I}_{g\delta s}\text{-}O(X) \subset SSIO(X)$. By (1), $SSIO(X) \subset S\mathcal{I}_{g\delta s}\text{-}O(X)$. Therefore $SSIO(X) = S\mathcal{I}_{g\delta s}\text{-}O(X)$.

Conversely, let $SSIO(X) = S\mathcal{I}_{g\delta s}\text{-}O(X)$ and F_A be a soft $S\mathcal{I}_{g\delta s}$ -closed set. Then $X - F_A$ is soft $S\mathcal{I}_{g\delta s}$ -open. Hence $X - F_A$ is soft semi- $\mathcal{I}$ -open, which implies F_A is soft semi- $\mathcal{I}$ -closed. Thus every soft $S\mathcal{I}_{g\delta s}$ -closed set is soft semi- $\mathcal{I}$ -closed. Therefore X is soft δ - $T_{1/2}$ -space.

Lemma 3.2. *For a soft ideal topological space X the following are equivalent*

1. *Every soft δ -pre- $\mathcal{I}$ -open singleton set is soft δ - $\mathcal{I}$ -closed.*
2. *Every soft singleton set is soft δ - $\mathcal{I}$ -nowhere dense or soft δ - $\mathcal{I}$ -closed.*

Proof: (1) $\Rightarrow$ (2) By Lemma 3.1, every soft singleton set is either soft δ - $\mathcal{I}$ -nowhere dense or soft δ -pre- $\mathcal{I}$ -open. In the first case we are done and in the second case soft δ - $\mathcal{I}$ -closedness follows from the assumption.

(2) $\Rightarrow$ (1) Let $\{x_e\}$ be soft δ -pre- $\mathcal{I}$ -open. Assume that $\{x_e\}$ is not soft δ - $\mathcal{I}$ -closed. Then by (2), it is soft δ - $\mathcal{I}$ -nowhere dense. Thus $\{x_e\} \subset \text{int}(cl_\delta^*(\{x_e\})) = \emptyset$, which is not possible. Hence $\{x_e\}$ is soft δ - $\mathcal{I}$ -closed.

Theorem 3.14. *For a soft ideal topological space X the following are equivalent*

1. *X is soft δ - T_1 space.*
2. *X is soft δ - $T_{3/4}$ space and every soft singleton set is soft δ - $\mathcal{I}$ -nowhere dense or soft δ - $\mathcal{I}$ -closed.*

3. X is soft δ - $T_{3/4}$ space and every soft δ -pre- $\mathcal{I}$ -open singleton set is soft δ - $\mathcal{I}$ -closed.

Proof: (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) If soft singleton set is not soft δ - $\mathcal{I}$ -closed, then it must be soft δ - $\mathcal{I}$ -open, since X is soft δ - $T_{3/4}$ -space. But soft singleton set is soft δ - $\mathcal{I}$ -open if and only if it is soft regular- $\mathcal{I}$ -open, which implies soft singleton set is soft regular- $\mathcal{I}$ -open. Moreover, by rest of assumption X is soft δ - $\mathcal{I}$ -nowhere dense at the same time, X must be soft δ - T_1 -space.

(2) $\Leftrightarrow$ (3) Follows from lemma 3.2

Theorem 3.15. *In any soft ideal topological space the following are equivalent*

1. X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.
2. For each $x_e \neq y_e$, there exists a soft $ST_{g\delta s}$ -open set G_A such that $x_e \in G_A$ and $y_e \notin ST_{g\delta s}\text{-cl}(G_A)$.
3. For each $x_e \in G_A$, $\{x_e\} = \cap \{ST_{g\delta s}\text{-cl}(G_A) : G_A \text{ is a soft } ST_{g\delta s}\text{-open set over } X \text{ and } x_e \in G_A\}$.

Proof: (1) $\Rightarrow$ (2) Assume (1) holds. Let $x_e \in G_A$ and $x_e \neq y_e$, then there exist disjoint soft $ST_{g\delta s}$ -open sets G_A and H_A such that $x_e \in G_A$ and $y_e \in H_A$. Clearly, $X - G_A$ is soft $ST_{g\delta s}$ -closed set. Since $G_A \cap H_A = \emptyset$, $G_A \subset X - H_A$. Therefore $ST_{g\delta s}\text{-cl}(G_A) \subset ST_{g\delta s}\text{-cl}(X - H_A) = X - H_A$. Now $y_e \notin X - H_A$ implies $y_e \notin ST_{g\delta s}\text{-cl}(G_A)$.

(2) $\Rightarrow$ (3) For each $x_e \neq y_e$, there exists a soft $ST_{g\delta s}$ -open set G_A such that $x_e \in G_A$ and $y_e \notin ST_{g\delta s}\text{-cl}(G_A)$. So $y_e \notin \cap \{ST_{g\delta s}\text{-cl}(G_A) : G_A \text{ is a soft } ST_{g\delta s}\text{-open set over } X \text{ and } x_e \in G_A\} = \{x_e\}$.

(3) $\Rightarrow$ (1) Let x_e, y_e be soft sets over X and $x_e \neq y_e$. By hypothesis there exists a soft $ST_{g\delta s}$ -open set G_A such that $x_e \in G_A$ and $y_e \notin ST_{g\delta s}\text{-cl}(G_A)$. This implies there exists a soft $ST_{g\delta s}$ -closed set H_A such that $y_e \notin H_A$. Therefore $y_e \in X - H_A$ and $X - H_A$ is soft $ST_{g\delta s}$ -open set. Thus, there exist two disjoint soft $ST_{g\delta s}$ -open sets G_A and $X - H_A$ such that $x_e \in G_A$ and $y_e \in X - H_A$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.

4 soft $\mathcal{I}_{g\delta s}$ -regular spaces

Definition 4.1. *A soft ideal topological space X is said to be soft $\mathcal{I}_{g\delta s}$ -regular if for each soft closed set F_A and each soft point $x_e \in X - F_A$, there exist disjoint soft $ST_{g\delta s}$ -open sets U_A and W_A such that $x_e \in U_A$ and $F_A \subset W_A$.*

Remark 4.1. *Every soft regular space is soft $\mathcal{I}_{g\delta s}$ -regular space.*

Theorem 4.1. *Every soft $\mathcal{I}_{g\delta s}$ -regular and soft T_0 space is soft $\mathcal{I}_{g\delta s}$ - T_2 .*

Proof: Let X be a soft $\mathcal{I}_{g\delta s}$ -regular space and x_e, y_e be soft point over X such that $x_e \neq y_e$. Since X being soft T_0 space, there exists a soft open set U_A over X such that $x_e \in U_A$ and $y_e \notin U_A$. Then $X - U_A$ is a soft closed set containing y_e but not x_e . Since X is soft $\mathcal{I}_{g\delta s}$ -regular, there exist disjoint soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open sets V_A and W_A such that $X - U_A \subset W_A$ and $x_e \in V_A$. Now $y_e \in X - U_A$ implies $y_e \in W_A$. Thus for x_e, y_e soft set over X such that $x_e \neq y_e$, there exist disjoint soft open sets V_A and W_A such that $x_e \in V_A$ and $y_e \in W_A$. Therefore X is soft $\mathcal{I}_{g\delta s}$ - T_2 space.

Theorem 4.2. *For a soft ideal topological space X , the following conditions are equivalent.*

1. X is soft $\mathcal{I}_{g\delta s}$ -regular.
2. For every soft point x_e over X and soft open set U_A containing x_e there exists a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set W_A such that $x_e \in W_A \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset U_A$.
3. For every soft closed set F_A , $F_A = \cap \{\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(U_A) : F_A \subset U_A \text{ and } U_A \text{ is soft } \mathcal{S}\mathcal{I}_{g\delta s}\text{-open set over } X\}$.
4. For every soft set F_A and a soft open set H_A such that $F_A \cap H_A \neq \emptyset$, there exists soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set O_A such that $F_A \cap O_A \neq \emptyset$ and $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(O_A) \subset H_A$.
5. For every non empty soft set F_A and soft closed set H_A such that $F_A \cap H_A \neq \emptyset$, there exist disjoint soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open sets L_A and M_A such that $F_A \cap L_A \neq \emptyset$ and $H_A \subset M_A$.

Proof: (1) $\Rightarrow$ (2) Let U_A be a soft open set containing x_e . Then $X - U_A$ is soft closed set not containing x_e . Since X is soft $\mathcal{I}_{g\delta s}$ -regular, there exist soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open sets L_A and W_A such that $x_e \in W_A$, $X - U_A \subset L_A$ and $W_A \cap L_A = \emptyset$. This implies $W_A \subset X - L_A$. Therefore, $\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(X - L_A) = X - L_A$, because $X - L_A$ is soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closed. Hence $x_e \in W_A \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset X - L_A \subset U_A$. That is $x_e \in W_A \subset \mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset U_A$.

(2) $\Rightarrow$ (3) Let F_A be a soft closed set and $x_e \notin F_A$. Then $X - F_A$ is a soft open set containing x_e . By (2), there is a soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set W_A such that $x_e \in W_A \subset \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset X - F_A$. And so, $F_A \subset X - \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A) \subset X - W_A$. Consequently $X - W_A$ is soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -closed set not containing x_e . Put $U_A = X - \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(W_A)$. This implies $F_A \subset U_A$ and U_A is soft $\mathcal{S}\mathcal{I}_{g\delta s}$ -open set over X and $x_e \notin \mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(U_A)$, implies $\cap \{\mathcal{S}\mathcal{I}_{g\delta s}\text{-cl}(U_A) : F_A \subset U_A \text{ and } U_A \text{ is soft } \mathcal{S}\mathcal{I}_{g\delta s}\text{-open set over } X\} \subset F_A$(1) But F_A is soft closed and every soft

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closed set is soft $SL_{g\delta s}$ -closed. Therefore $F_A \subset \cap \{SL_{g\delta s}\text{-cl}(U_A) : F_A \subset U_A \text{ and } U_A \text{ is soft } SL_{g\delta s}\text{-open set over } X\}$ (2) is always true. From (1) and (2), $F_A = \cap \{SL_{g\delta s}\text{-cl}(U_A) : F_A \subset U_A \text{ and } U_A \text{ is soft } SL_{g\delta s}\text{-open set over } X\}$.

(3) $\Rightarrow$ (4) Let $F_A \cap H_A \neq \emptyset$ and H_A is soft open. Let $x_e \in F_A \cap H_A$. Then $X - H_A$ is a soft closed set not containing x_e . By (3), there exists a soft $SL_{g\delta s}$ -open set U_A over X such that $X - H_A \subset U_A$ and $x_e \notin SL_{g\delta s}\text{-cl}(U_A)$. Put $O_A = X - SL_{g\delta s}\text{-cl}(U_A)$, then O_A is soft $SL_{g\delta s}$ -open set over X , $x_e \in F_A \cap O_A$ and $SL_{g\delta s}\text{-cl}(O_A) \subset SL_{g\delta s}\text{-cl}(X - U_A) = X - U_A \subset H_A$. Hence $SL_{g\delta s}\text{-cl}(O_A) \subset H_A$.

(4) $\Rightarrow$ (5) If $F_A \cap H_A \neq \emptyset$, where F_A is non empty and H_A is soft closed, then $F_A \cap (X - H_A) \neq \emptyset$ and $(X - H_A)$ is soft open. Therefore by (4), there exists soft $SL_{g\delta s}$ -open set L_A such that $F_A \cap L_A \neq \emptyset$ and $L_A \subset SL_{g\delta s}\text{-cl}(L_A) \subset X - H_A$.

(5) $\Rightarrow$ (1) Let F_A be a soft closed set such that $x_e \notin F_A$, then $\{x_e\} \cap F_A = \emptyset$. By (5), there exist disjoint soft open sets L_A and M_A such that $\{x_e\} \cap L_A \neq \emptyset$ and $F_A \subset M_A$, which implies $x_e \in L_A$ and $F_A \subset M_A$. Hence, X is soft $SL_{g\delta s}$ -regular.

Theorem 4.3. *If $f : X \rightarrow Y$ is soft continuous bijective, soft $SL_{g\delta s}$ -open function and X is a soft regular space, then Y is soft $SL_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over Y and $y_e \notin F_A$. Take $y_e = f(x_e)$ for some x_e over X . Since f is soft continuous $f^{-1}(F_A)$ is soft closed set over X such that $x_e \notin f^{-1}(F_A)$. Now X is soft regular, there exist disjoint soft open sets U_A and V_A such that $x_e \in U_A$ and $f^{-1}(F_A) \subset V_A$. That is, $y_e = f(x_e) \in f(U_A)$ and $F_A \subset f(V_A)$. Since f is soft $SL_{g\delta s}$ -open function $f(U_A)$ and $f(V_A)$ are soft $SL_{g\delta s}$ -open sets over Y and f is bijective $f(U_A) \cap f(V_A) = f(U_A \cap V_A) = f(\emptyset) = \emptyset$. Therefore, Y is soft $SL_{g\delta s}$ -regular.

Theorem 4.4. *If $f : X \rightarrow Y$ is soft continuous surjective, strongly soft $SL_{g\delta s}$ -open (resp. quasi soft $SL_{g\delta s}$ -open) function and X is soft $SL_{g\delta s}$ -regular space, then Y is soft $SL_{g\delta s}$ -regular (resp. soft regular).*

Proof: Let F_A be a soft closed set over Y and $y_e \notin F_A$. Take $y_e = f(x_e)$ for some x_e over X . Since f is soft continuous surjective $f^{-1}(F_A)$ is soft closed set over X and $x_e \notin f^{-1}(F_A)$. Now since X is soft $SL_{g\delta s}$ -regular, there exist disjoint soft $SL_{g\delta s}$ -open sets U_A and V_A such that $x_e \in U_A$ and $f^{-1}(F_A) \subset V_A$. That is, $y_e = f(x_e) \in f(U_A)$ and $F_A \subset f(V_A)$. Since f is strongly soft $SL_{g\delta s}$ -open (resp. quasi soft $SL_{g\delta s}$ -open) and bijective, $f(U_A)$ and $f(V_A)$ are disjoint soft $SL_{g\delta s}$ -open (resp. soft open) sets over Y . Therefore, Y is soft $SL_{g\delta s}$ -regular (resp. soft regular).

Theorem 4.5. *If $f : X \rightarrow Y$ is soft $SL_{g\delta s}$ -continuous, soft closed, injection and Y is soft regular, then X is soft $SL_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over X and $x_e \notin F_A$. Since f is soft closed injection $f(F_A)$ is soft closed set over Y such that $f(x_e) \notin f(F_A)$. Now Y is

soft regular, there exist disjoint soft open sets G_A and H_A such that $f(x_e) \in G_A$ and $f(F_A) \subset H_A$. This implies $x_e \in f^{-1}(G_A)$ and $F_A \subset f^{-1}(H_A)$. Since f is soft $SL_{g\delta s}$ -continuous, $f^{-1}(G_A)$ and $f^{-1}(H_A)$ are soft $SL_{g\delta s}$ -open sets over X . Further $f^{-1}(G_A) \cap f^{-1}(H_A) = \emptyset$. Hence X is soft $IL_{g\delta s}$ -regular.

Theorem 4.6. *If $f : X \rightarrow Y$ is soft semi $IL_{g\delta s}$ -continuous, soft semi-closed, injection and Y is soft semi-regular then X is soft $IL_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over X and $x_e \notin F_A$. Since f is soft semi-closed injection $f(F_A)$ is soft semi-closed set over Y such that $f(x_e) \notin f(F_A)$. Now Y is soft semi-regular, there exist disjoint soft semi-open sets G_A and H_A such that $f(x_e) \in G_A$ and $f(F_A) \subset H_A$. This implies $x_e \in f^{-1}(G_A)$ and $F_A \subset f^{-1}(H_A)$. Since f is soft semi $IL_{g\delta s}$ -continuous $f^{-1}(G_A)$ and $f^{-1}(H_A)$ are soft $SL_{g\delta s}$ -open sets over X . Further $f^{-1}(G_A) \cap f^{-1}(H_A) = \emptyset$. Hence X is soft $IL_{g\delta s}$ -regular.

Theorem 4.7. *If $f : X \rightarrow Y$ is strongly soft $SL_{g\delta s}$ -continuous, soft closed, injection and Y is soft $IL_{g\delta s}$ -regular then X is soft regular.*

Proof: Let F_A be a soft closed set over X and $x_e \notin F_A$. Since f is soft closed injection $f(F_A)$ is soft closed set over Y such that $f(x_e) \notin f(F_A)$. Now Y is soft $IL_{g\delta s}$ -regular, there exist disjoint soft $SL_{g\delta s}$ -open sets G_A and H_A such that $f(x_e) \in G_A$ and $f(F_A) \subset H_A$. This implies $x_e \in f^{-1}(G_A)$ and $F_A \subset f^{-1}(H_A)$. Since f is strongly soft $SL_{g\delta s}$ -continuous $f^{-1}(G_A)$ and $f^{-1}(H_A)$ are soft open sets over X . Further $f^{-1}(G_A) \cap f^{-1}(H_A) = \emptyset$. Hence X is soft regular.

Theorem 4.8. *If $f : X \rightarrow Y$ is soft $SL_{g\delta s}$ -irresolute, soft closed, injection and Y is soft $IL_{g\delta s}$ -regular then X is soft $IL_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over X and $x_e \notin F_A$. Since f is soft closed injection $f(F_A)$ is soft closed set over Y such that $f(x_e) \notin f(F_A)$. Now Y is soft $IL_{g\delta s}$ -regular, there exist disjoint soft $IL_{g\delta s}$ -open sets G_A and H_A such that $f(x_e) \in G_A$ and $f(F_A) \subset H_A$. This implies $x_e \in f^{-1}(G_A)$ and $F_A \subset f^{-1}(H_A)$. Since f is soft $SL_{g\delta s}$ -irresolute, $f^{-1}(G_A)$ and $f^{-1}(H_A)$ are soft $SL_{g\delta s}$ -open sets over X . Further $f^{-1}(G_A) \cap f^{-1}(H_A) = \emptyset$. Hence X is soft $IL_{g\delta s}$ -regular.

Corolary 4.1. *If $f : X \rightarrow Y$ is perfectly soft $SL_{g\delta s}$ -continuous injective, soft closed function and Y is soft $IL_{g\delta s}$ -regular space, then X is soft regular.*

Proof: Obvious, because every perfectly soft $SL_{g\delta s}$ -continuous is strongly soft $SL_{g\delta s}$ -continuous.

Theorem 4.9. *If $f : X \rightarrow Y$ is perfectly soft $SL_{g\delta s}$ -continuous injective, soft closed function and Y is soft $IL_{g\delta s}$ -regular space, then X is soft ultra regular.*

Proof: Let F_A be a soft closed set over X and $x_e \notin F_A$. Since f is soft closed injective, $f(F_A)$ is soft closed over Y such that $f(x_e) \notin f(F_A)$. By soft $\mathcal{I}_{g\delta s}$ -regularity over Y , there exist disjoint soft $SL_{g\delta s}$ -open sets U_A and V_A such that $f(x_e) \in U_A$ and $f(F_A) \subset V_A$. That is $x_e \in f^{-1}(U_A)$ and $F_A \subset f^{-1}(V_A)$. Since f is perfectly soft $SL_{g\delta s}$ -continuous, $f^{-1}(U_A)$ and $f^{-1}(V_A)$ are disjoint soft clopen sets over X . Therefore, X is soft ultra regular.

Theorem 4.10. *If $f : X \rightarrow Y$ is totally soft continuous bijective, soft $SL_{g\delta s}$ -open function and X is soft clopen-regular space, then Y is soft $\mathcal{I}_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over Y and $y_e \notin F_A$. Take $y_e = f(x_e)$ for some soft point x_e over X . Since f is totally soft continuous bijective, $f^{-1}(F_A)$ is soft clopen over X such that $x_e \notin f^{-1}(F_A)$. By soft clopen-regularity of X , there exist disjoint soft open sets U_A and V_A such that $x_e \in U_A$ and $f^{-1}(F_A) \subset V_A$. That is $y_e = f(x_e) \in f(U_A)$ and $F_A \subset f(V_A)$. Since f is soft $SL_{g\delta s}$ -open bijective $f(U_A)$ and $f(V_A)$ are disjoint soft $SL_{g\delta s}$ -open sets over Y . Therefore, Y is soft $\mathcal{I}_{g\delta s}$ -regular.

Theorem 4.11. *If $f : X \rightarrow Y$ is completely soft continuous injective, soft $SL_{g\delta s}$ -open function and X is almost soft regular space, then Y is soft $\mathcal{I}_{g\delta s}$ -regular.*

Proof: Let F_A be a soft closed set over Y and $y_e \notin F_A$. Take $y_e = f(x_e)$ for some soft point x_e over X . Since f is completely soft continuous, $f^{-1}(F_A)$ is soft regular open over X such that $x_e \notin f^{-1}(F_A)$. By almost soft regularity over X , there exist disjoint soft open sets U_A and V_A such that $x_e \in U_A$ and $f^{-1}(F_A) \subset V_A$. That is $y_e = f(x_e) \in f(U_A)$ and $F_A \subset f(V_A)$. Since f is soft $SL_{g\delta s}$ -open and injective, $f(U_A)$ and $f(V_A)$ are disjoint soft $SL_{g\delta s}$ -open sets over Y . Therefore, Y is soft $\mathcal{I}_{g\delta s}$ -regular.

5 Conclusion

We concluded that using the above separation axioms concepts on soft $SL_{g\delta s}$ -closed sets in the ideal soft topological space may extended to Urysohn Theorem and then we will prove extension theorem. This findings would assist the budding researchers to engage in the future research in the real life applications.

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